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École doctorale de Sciences Mathématiques de Paris Centre (ED 386)
Institut de Mathématiques de Jussieu-Paris Rive Gauche

Investigation into phenomena surrounding universally Baire sets

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Thèse de doctorat de MATHÉMATIQUES : LOGIQUE ET FONDEMENTS DE
L'INFORMATIQUE

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*Posvećeno mom dedi Dušanu:
Neka ti je večna slava i hvala!*

Résumé

Titre : Enquête sur les phénomènes entourant les ensembles universellement Baire

Mots clef : ensembles universellement Baire, détermination, HOD, souris, souris du type hod, modèle Chang-plus, préservation des ensembles stationnaires, jeu d'existence de modèle, \mathbb{P}_{\max} , absoluité générique, maximalité générique

Résumé : Cette thèse présente mes contributions à divers aspects de la théorie des ensembles universellement Baire. L'un de ces aspects est le plus petit modèle intérieur contenant tous les réels dont tous les ensembles de réels sont universellement Baire (à savoir, $L(\mathbb{R})$) et sa relation avec son modèle intérieur HOD. Nous vérifions ici que $\text{HOD}^{L(\mathbb{R})}$ bénéficie d'une forme de définissabilité locale à l'intérieur de $L(\mathbb{R})$, ce qui justifie sa caractérisation en tant que « modèle de cœur » dans $L(\mathbb{R})$. Nous étudions ensuite une construction « ascendante » d'ensembles universellement Baire plus compliqués (plus généralement, des ensembles déterminés). Cette construction nous permet de donner une description du « type L » du plus petit modèle de $\text{AD}_{\mathbb{R}} + \text{Cof}(\Theta) = \Theta$. Une conséquence de cette description est que ce plus petit modèle est contenu dans le modèle Chang-plus. Lorsque nous combinons notre construction et les travaux de Woodin sur le modèle Chang-plus, il s'ensuit qu'une classe propre de cardinaux de Woodin qui sont des limites de cardinaux de Woodin implique l'existence d'une souris du type hod avec une limite mesurable de cardinaux de Woodin dont la stratégie est universellement Baire.

Un autre aspect de la théorie des ensembles universellement Baire est l'absoluité et la maximalité génériques qui leur sont associées. Nous incluons certains résultats concernant l'absoluité générique de formules qui sont $\Sigma_1^{H(\omega_2)}$ et qui utilisent des ensembles universellement Baire en tant que prédicats ou paramètres, ainsi que la maximalité générique de certaines formules qui sont $\Pi_2^{H(\omega_2)}$ et qui utilisent des ensembles universellement Baire comme prédicats. Dans le second cas, nous sommes amenés à considérer la question générale de savoir quand un modèle d'une formule propositionnelle infinitaire peut être ajouté par un poset préservant les ensembles stationnaires de ω_1 . Nous caractérisons quand cela se produit en termes d'un jeu qui est une variante du jeu d'existence de modèle. Enfin, nous donnons une condition suffisante pour cela en termes de plongements génériques.

Abstract

Title: Investigation into phenomena surrounding universally Baire sets

Keywords: universally Baire sets, determinacy, HOD, mice, hod mice, Chang-plus model, stationary-set-preserving, Model Existence Game, \mathbb{P}_{\max} , generic absoluteness, generic maximality

Abstract: This thesis presents my contributions to various aspects of the theory of universally Baire sets. One of these aspects is the smallest inner model containing all reals whose all sets of reals are universally Baire (viz., $L(\mathbb{R})$) and its relation to its inner model HOD. We verify here that $\text{HOD}^{L(\mathbb{R})}$ enjoys a form of local definability inside $L(\mathbb{R})$, further justifying its characterization as a “core model” in $L(\mathbb{R})$. We then study a “bottom-up” construction of more complicated universally Baire sets (more generally, determined sets). This construction allows us to give an “ L -like” description of the minimum model of $\text{AD}_{\mathbb{R}} + \text{Cof}(\Theta) = \Theta$. A consequence of this description is that this minimum model is contained in the Chang-plus model. Our construction, together with Woodin’s work on the Chang-plus model, shows that a proper class of Woodin cardinals which are limits of Woodin cardinals implies the existence of a hod mouse with a measurable limit of Woodin cardinals whose strategy is universally Baire.

Another aspect of the theory of universally Baire sets is the generic absoluteness and maximality associated with them. We include some results concerning generic $\Sigma_1^{H(\omega_2)}$ -absoluteness with universally Baire sets as predicates or parameters, as well as generic $\Pi_2^{H(\omega_2)}$ -maximality with universally Baire sets as predicates. In the second case, we are led to consider the general question of when a model of an infinitary propositional formula can be added by a stationary-set-preserving poset. We characterize when this happens in terms of a game which is a variant of the Model Existence Game. We then give a sufficient condition for this in terms of generic embeddings.

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Chapter 1

Avant-propos

1.1 Qu'est-ce que les ensembles universellement Baire ?

L'une des questions les plus fondamentales de la théorie des ensembles est de savoir quels ensembles de réels existent. C'est même le cœur de l'*hypothèse du continu* : dans sa forme originale, elle demande s'il existe un ensemble de réels qui ne s'injecte pas dans ω ni ne se surjecte sur \mathbb{R} . L'énorme travail réalisé par les théoriciens des ensembles au cours des dernières décennies nous permet de dire, au moins, qu'il existe deux types d'ensembles de réels : « canonique » et « chaotique ». Cette thèse se concentre sur les ensembles canoniques, également connus sous le nom d'*ensembles universellement Baire*.

Les ensembles universellement Baire ont été introduits dans [1]. Leur définition est basée sur la *propriété de Baire*.

Definition 1.1.1. Soit X un espace topologique et A un sous-ensemble de X . On dit alors que A est **mesurable au sens de Baire** (ou qu'il possède la **propriété de Baire**) s'il existe un ensemble ouvert U tel que $A \Delta U$ est maigre. \square

L'axiome du choix implique qu'il existe des ensembles de réels qui ne sont pas mesurables au sens de Baire ; en revanche, tout ensemble de réels raisonnablement défini l'est.¹ Conscients de ces faits, nous commençons à associer la mesurabilité de Baire aux ensembles canoniques de réels, et son absence aux ensembles chaotiques. Il s'avère que c'est en fait un renforcement de cette propriété qui délimite ces deux types d'ensembles de réels.

Definition 1.1.2. On dit qu'un ensemble de réels A est **universellement Baire** (ou qu'il possède la **propriété de Baire universelle**) si pour tout espace compact de Hausdorff X et toute fonction continue $f : X \rightarrow \mathbb{R}$, l'ensemble $f^{-1}[A]$ est mesurable au sens de Baire dans X . \square

L'une des façons dont les ensembles universellement Baire se comportent bien est le fait qu'ils sont mesurables au sens de Baire. Il s'avère qu'ils sont également mesurables au

¹Voir par exemple Theorem 1.1.9.

sens de Lebesgue. De plus, ils ont la propriété d'ensemble parfait. Pour revenir au lien avec l'hypothèse du continu évoqué au début, le fait que les ensembles universellement Baire jouissent de la propriété d'ensemble parfait signifie qu'un contre-exemple à l'hypothèse du continu ne peut pas être universellement Baire. Bien sûr, on peut se demander quelle est l'importance de cette dernière affirmation, c'est-à-dire quelle est la richesse de la collection des ensembles universellement Baire ; la réponse à cette question deviendra claire : elle est très riche² et encode probablement une part importante du contenu canonique de l'univers³.

Ainsi, les ensembles universellement Baire sont mesurables aux sens de Baire et de Lebesgue et ont la propriété d'ensembles parfait, mais il y a plus. Ces propriétés peuvent être résumées et renforcées en un seul énoncé sur la détermination des jeux à deux joueurs de longueur ω associés à des ensembles universellement Baire.⁴

Theorem 1.1.3 (Martin, Steel, Woodin). *Supposons qu'il existe deux cardinaux de Woodin. Alors tout ensemble universellement Baire de réels est déterminé.*

Proof. Ceci résulte des [3, Théorèmes 3.1, 2.1 et 1.7]. □

Le fait que les ensembles universellement Baire ne puissent pas être témoins de l'échec de l'hypothèse du continu suggère qu'ils sont en quelque sorte immunisés contre le forcing. C'est en effet vrai, et nous allons essayer de le préciser de plusieurs façons. Tout d'abord, les ensembles universellement Baire peuvent être définis simplement comme des projections de certains arbres bien équilibrés.

Definition 1.1.4. Soient T et S des arbres sur $\omega \times \text{Ord}$ et \mathbb{P} un poset. Alors (T, S) sont \mathbb{P} -absolument complémentés si

$$V^{\mathbb{P}} \models p[T] = \mathbb{R} - p[S].$$

□

L'idée est de commencer par T et S qui sont dans V et se projettent sur des compléments. Or, dans toute extension générique, T et S se projettent nécessairement sur des ensembles disjoints. Cependant, il n'y a aucune raison *a priori* pour que leurs projections couvrent l'ensemble de \mathbb{R} ; ceci est le cœur de la définition.

Ces arbres incarnent dans leur définition même une forme d'absoluité générique : il est absolu qu'ils se projettent sur des compléments. Voici leur lien avec les ensembles universellement Baire.

Theorem 1.1.5 (Feng, Magidor, Woodin). *Soit A un ensemble de réels. Alors A est universellement Baire si et seulement si, pour tout poset \mathbb{P} , il existe des arbres \mathbb{P} -absolument complémentés, T et S , tels que $p[T] = A$.⁵* □

²Voir Theorems 1.1.9 and 1.3.8.

³Voir Theorem 1.3.8.

⁴Pour une dérivation des trois propriétés ci-dessus à partir de la détermination, voir [2, Section 27].

⁵Voir [1, Corollaire 2.1].

Il s'avère que les ensembles universellement Baire jouissent d'un degré d'absoluité générique bien plus élevé que ne le laisse supposer la valeur apparente de cette caractérisation. Pour commencer, un ensemble universellement Baire A garantit une manière canonique de réinterpréter génériquement le prédicat « $x \in A$ », défini sur les réels x .

Proposition 1.1.6 (folklore). *Soient \mathbb{P} un poset et T, S et T', S' deux paires d'arbres \mathbb{P} -absolument complémentés satisfaisant $p[T] = p[T']$. Alors $V^{\mathbb{P}} \models p[T] = p[T']$.⁶ \square*

Definition 1.1.7. Soit A un ensemble universellement Baire et g un filtre générique sur V . Alors A_g est l'unique ensemble de réels dans $V[g]$ tel que pour toute paire T, S d'arbres \mathbb{P} -absolument complémentés dans V ,

$$p[T]^V = A \text{ et } p[T]^{V[g]} = A_g.$$

\square

L'intuition est que $A_g \subseteq \mathbb{R}^{V[g]}$ est la réinterprétation canonique dans $V[g]$ de l'ensemble $A \subseteq \mathbb{R}^V$. Ainsi, les ensembles universellement Baire ont des réinterprétations génériques canoniques. Une sorte de converse est également vraie.

Theorem 1.1.8. *Soit $\phi(v_x, v_a)$ une formule de complexité Σ_n , où $n < \omega$, et soit a un ensemble arbitraire. Les énoncés suivants sont équivalents.*

- a. *L'ensemble $A := \{x \in \mathbb{R} : \phi(x, a)\}$ est universellement Baire et pour tous les filtres g génériques sur V ,*

$$A_g = \{x \in \mathbb{R} : \phi(x, a)\}^{V[g]}.$$

- b. *Pour tout $\theta > \text{rank}(a)$ satisfaisant $V_\theta \prec_{\Sigma_{n+5}} V$, il existe $U \subseteq V_\theta$, tel que pour tous les plongements élémentaires dénombrables*

$$\pi : (M, \in, \bar{U}, \bar{a}) \longrightarrow (V_\theta, \in, U, a)$$

avec M transitif, pour tous les filtres g génériques sur M et tous les réels $x \in M[g]$,

$$M[g] \models \phi[x, \bar{a}] \iff V \models \phi[x, a].$$

Proof. Voir [3, Lemma 4.1]. \square

Une fois que nous sommes capables de réinterpréter le prédicat « $x \in A$ », il n'est pas étonnant que des formules plus compliquées puissent être réinterprétées également. Par exemple, il s'avère que si B est projectif depuis un ensemble universellement Baire A , alors B est aussi universellement Baire, et dans $V[g]$ l'ensemble B_g est projectif depuis A_g par la même définition que B depuis A . On peut même dépasser le niveau projectif.

⁶Voir [4, Proposition 1.3-IV].

Theorem 1.1.9 (Woodin). *Supposons qu'il existe une classe propre de cardinaux de Woodin et que A est un ensemble universellement Baire. Alors $(\mathbb{R}, A)^\sharp$ est universellement Baire et pour tout filtre g générique sur V , $((\mathbb{R}^V, A)^\sharp)_g = (\mathbb{R}^{V[g]}, A_g)^\sharp$.*

Proof. Il s'agit d'un résultat non publié, mais comparez avec [5, Theorem 3.3.9]. \square

Ainsi, tout ensemble de réels dans $L(\mathbb{R}, A)$ est universellement Baire, et si g est générique sur V , il existe un plongement élémentaire

$$j : L(\mathbb{R}^V, A) \longrightarrow L(\mathbb{R}^{V[g]}, A_g)$$

tel que pour tout $B \in \mathcal{P}(\mathbb{R}^V) \cap L(\mathbb{R}^V, A)$, $j(B) = B_g$. Ceci généralise clairement le niveau projectif de l'absoluité générique.

Theorem 1.1.9 peut être considéré comme le point de départ de cette thèse. Dans les sections suivantes, nous allons souligner quelques questions que ce théorème soulève naturellement, ce qui était connu à leur sujet, et ce que nous avons démontré.

1.2 Qu'est-ce que la théorie de $L(\mathbb{R})$?

Un corollaire de Theorem 1.1.9 est que la théorie de $L(\mathbb{R})$ est génériquement absolue. Il est donc naturel de se demander quelle est cette théorie. Le théorème cité implique également que tout ensemble de réels dans $L(\mathbb{R})$ est universellement Baire, donc en particulier $L(\mathbb{R})$ satisfait $ZF + AD^+ + DC$. Même si l'axiome du choix échoue dans $L(\mathbb{R})$, son modèle intérieur $\text{HOD}^{L(\mathbb{R})}$ satisfait AC et est proche de $L(\mathbb{R})$.⁷ La théorie de $\text{HOD}^{L(\mathbb{R})}$ fait partie de la théorie de $L(\mathbb{R})$ et est donc génériquement absolue. Cette théorie étend ZFC, mais on peut en dire beaucoup plus : il s'avère que ce modèle est « du type L ».

Concentrons-nous sur cette dernière remarque. Sous nos hypothèses de grands cardinaux, le dièse pour la souris minimale avec ω cardinaux de Woodin, appelé $\mathcal{M}_\omega^\sharp$, existe et possède une stratégie d'itération unique du type (ω_1, ω_1) . Soit Σ^* cette stratégie. Il s'avère qu'elle est universellement Baire et que

$$L(\mathbb{R}, \Sigma^*) = L(\mathbb{R}, \mathbb{R}^\sharp). \quad (1.1)$$

Cela suggère un lien entre les dièses pour les ensembles de réels et les stratégies pour les souris, mais nous nous éloignons du sujet (nous reviendront sur ce lien dans Section 1.3). Nous nous intéressons ici au lien entre $(\mathcal{M}_\omega^\sharp, \Sigma^*)$ et $\text{HOD}^{L(\mathbb{R})}$.

Soit (M, Σ) la paire constituée de $\mathcal{M}_\omega^\sharp$ coupé au successeur de son cardinal de Woodin le plus petit et de la stratégie pour elle obtenue à partir de Σ . Soit s une pile d'arbres normaux sur (M, Σ) ,⁸ avec la dernière paire (N, T) dont la branche principale ne tombe pas. Nous

⁷Une façon de préciser la proximité est d'observer que chaque ensemble de $L(\mathbb{R})$ appartient à HOD_x pour un certain $x \in \mathbb{R}$, alors que d'autre part x est générique sur HOD et que $\text{HOD}_x = \text{HOD}[x]$ (voir [6, Théorème 10.1.2]).

⁸Nous entendons par là que s est sur M et selon Σ .

disons que (N, T) est un *itéré non-tombant* de (M, Σ) . La paire (N, T) peut également être obtenue à partir de (M, Σ) via une autre pile, disons s' . Cependant, les plongements d'itération donnés par s et s' doivent être les mêmes.⁹ Nous désignons ce plongement unique par $i_{(M,\Sigma),(N,T)}$. Au lieu de commencer par (M, Σ) , nous aurions pu commencer par (N, T) , et les mêmes remarques s'appliqueraient à ses itérés. De cette façon, nous obtenons une famille d'itérés non-tombants de (M, Σ) , ainsi que des plongements entre eux : Chaque fois que (N, T) et (P, Υ) sont deux itérés de (M, Σ) tels que (P, Υ) est également un itéré de (N, T) , nous avons le plongement

$$i_{(N,T),(P,\Upsilon)} : (N, T) \longrightarrow (P, \Upsilon).$$

Il s'avère que cette famille est un système inductif. Ignorant pour l'instant les stratégies, nous désignons le modèle limite direct du système par $\mathcal{M}_\infty(M, \Sigma)$.

Theorem 1.2.1 (Steel). *Les énoncés suivants sont vérifiés.*

- a. $\mathcal{M}_\infty(M, \Sigma)$ est bien fondé.
- b. L'image du cardinal de Woodin de M , par le plongement limite $i_{(M,\Sigma),\infty}$, est $\Theta^{L(\mathbb{R})}$.
- c. Le domaine de $\mathcal{M}_\infty(M, \Sigma) \parallel \Theta^{L(\mathbb{R})}$ est $(\text{HOD} \upharpoonright \Theta)^{L(\mathbb{R})}$.

Proof. Une preuve peut être trouvée dans [7]. □

Ainsi $(\text{HOD} \upharpoonright \Theta)^{L(\mathbb{R})}$ a la structure d'une souris. En revanche, $\text{HOD}^{L(\mathbb{R})}$ n'est pas une souris, mais il est très proche d'une. Puisque Σ est universellement Baire, il s'étend canoniquement à une stratégie Σ^+ pour M du type (Ord, Ord) .¹⁰ Par normalisation, il existe un arbre normal \mathcal{T} sur (M, Σ^+) dont le dernier modèle est $\mathcal{M}_\infty(M, \Sigma)$. En particulier, la stratégie de queue $\Sigma_{\mathcal{T}}^+$ est une stratégie d'itération du type (Ord, Ord) pour le modèle $\mathcal{M}_\infty(M, \Sigma)$. Nous avons la caractérisation suivante de $\text{HOD}^{L(\mathbb{R})}$.

Theorem 1.2.2 (Woodin). *Soit \mathcal{T}^* le levage de \mathcal{T} vers \mathcal{M}_ω .¹¹ Il existe alors un sous-ensemble Λ de $\Sigma_{\mathcal{T}}^+$ tel que $\text{HOD}^{L(\mathbb{R})} = \mathcal{M}_\infty^{\mathcal{T}^*}[\Lambda]$.¹²*

Proof. Une démonstration est donnée dans [7]. □

Ainsi, $\text{HOD}^{L(\mathbb{R})}$ diffère d'une souris en ce qu'il est construit à partir d'extenseurs et d'une stratégie d'itération partielle pour son segment initial. Les objets de ce type général figurent en bonne place dans les arguments de Chapter 4. Notons simplement ici que $\text{HOD}^{L(\mathbb{R})}$ possède une structure fine et toutes ses conséquences : par exemple, $\text{HOD}^{L(\mathbb{R})} \models \text{GCH}$.

⁹Ceci résulte du lemme de Dodd-Jensen.

¹⁰Si $A \subseteq H_{\omega_1}$ est codé par un ensemble universellement Baire, on peut alors définir la classe $A^+ \subseteq V$ en affirmant que pour tout X , $X \in A^+$ si et seulement si $V^{\text{Col}(\omega, \text{tc}(X))} \models X \in A_g$. Ainsi, dans le cas de Σ , Σ^+ est défini et s'avère être une stratégie d'itération du type (Ord, Ord) .

¹¹ \mathcal{M}_ω est obtenu à partir de $\mathcal{M}_\omega^\#$ par itération de son extension supérieure hors de l'univers, c'est-à-dire Ord fois.

¹²où $\mathcal{M}_\infty^{\mathcal{T}^*}$ est le modèle final de l'arbre \mathcal{T}^*

Comme il s'agit d'un modèle avec une structure fine proche de $L(\mathbb{R})$, il est tentant d'appeler $\text{HOD}^{L(\mathbb{R})}$ « le modèle de cœur » de $L(\mathbb{R})$, surtout en dessous de $\Theta^{L(\mathbb{R})}$.¹³ Cependant, on attend généralement plus d'un modèle de cœur : par exemple, ce modèle bénéficie généralement d'une certaine définissabilité locale.¹⁴ Dans Chapter 3, nous prouvons un résultat dans cette direction. Pour l'énoncer, nous désignons par μ_X le filtre des clubs sur $[X]^\omega$, pour un ensemble X quelconque.

Theorem 1.2.3. *Supposons qu'il existe ω cardinaux de Woodin avec un cardinal mesurable au-dessus d'eux. L'énoncé suivant est vérifié dans $L(\mathbb{R})$. Pour tous les cardinaux η de HOD dans l'intervalle $[\delta_1^2, \Theta)$, l'ensemble $H(\eta^+)^{\text{HOD}}$ est définissable sans paramètres supplémentaires sur la structure*

$$L_\chi((H_\eta^{\text{HOD}})^\omega)[\mu_\eta],$$

où χ est le deuxième ordinal supérieur à η pour lequel la structure ci-dessus satisfait

$$\text{ZF}^- + \text{« } \mathcal{P}([\eta]^\omega) \text{ existe »}.$$

Proof. Voir Corollary 3.9.13. □

1.3 D'où viennent les ensembles universellement Baire ?

S'il existe une classe propre de cardinaux de Woodin, les ensembles universellement Baire, lorsqu'ils sont étudiés dans le cadre de la réductibilité continue, forment une ligne.

Theorem 1.3.1. *Supposons qu'il existe une classe propre de cardinaux de Woodin. Alors la collection des ensembles universellement Baire forme un ensemble bien pré-ordonné sous la réductibilité de Wadge modifiée¹⁵.*

Proof. Ceci résulte de Theorem 1.1.3 et des [10, Lemme 2.1 et Théorème 2.5]. □

Un rang plus élevé dans ce préordre correspond à une plus grande complexité logique et combinatoire : par exemple, $A <_{\text{W}}^* A^\sharp$. En fait, l'escalade de cette hiérarchie est étroitement liée à l'escalade de l'hiérarchie des grands cardinaux.¹⁶ Bien sûr, on ne peut pas s'attendre à pouvoir faire cet escalade complètement « algorithmiquement », mais si l'on relâche nos exigences, on peut dire beaucoup de choses sur la façon dont les ensembles universellement Baire apparaissent. Par exemple, Theorem 1.1.9 propose une première tentative d'épuisement de l'hiérarchie des ensembles universellement Baire : soit $\mathbb{R}^{\#0} := \mathbb{R}$ et

¹³Voir le titre de [8].

¹⁴Voir [9, Théorème 6.15], où cette propriété est appelée « définissabilité inductive ».

¹⁵La réductibilité de Wadge modifiée, notée \leq_{W}^* , est définie en affirmant que $A \leq_{\text{W}}^* B$ si $A \leq_{\text{W}} B$ ou $A \leq_{\text{W}} B^c$.

¹⁶Par exemple, l'ordre de souris sur les paires du type hod affine l'ordre de Wadge. Voir aussi Conjecture 1.3.2.

soit $\mathbb{R}^{\#(\alpha+1)} := (\mathbb{R}, \mathbb{R}^{\#\alpha})^\#$ chaque fois que $\mathbb{R}^{\#\alpha}$ est défini. Pour que cette procédure soit utile, nous devons donner un sens aux étapes limites, et nous ne savons le faire que pour les ordinaux limites de cofinalité dénombrable.

Soit $(A_n : n < \omega)$ une suite d'ensembles universellement Baire ; nous voulons définir leur jointure, $\bigoplus_{n < \omega} A_n$. Pour $n < \omega$ et $x \in \mathbb{R}$, définissons $\langle n, x \rangle \in \mathbb{R}$ en affirmant

$$\langle n, x \rangle(i) := \begin{cases} n, & i = 0 \\ x(i-1), & i > 0. \end{cases}$$

Nous pouvons maintenant définir

$$\bigoplus_{n < \omega} A_n := \{ \langle n, x \rangle : n < \omega, x \in A_n \} \subseteq \mathbb{R}.$$

Il s'agit également d'un ensemble universellement Baire, et son rang de Wadge est au moins le supremum des rangs des ensembles A_n . Ainsi, pour un ordinal limite γ de cofinalité dénombrable, si $(\mathbb{R}^{\#\alpha} : \alpha < \gamma)$ est défini, on peut définir

$$\mathbb{R}^{\#\gamma} := \bigoplus_{n < \omega} \mathbb{R}^{\#\alpha_n},$$

où $(\alpha_n : n < \omega)$ est une séquence cofinale dans γ . Une subtilité ici est que $\mathbb{R}^{\#\gamma}$ dépend du choix de la séquence $(\alpha_n : n < \omega)$; cependant, tous les ensembles obtenus de cette manière sont équivalents au sens de Wadge, donc la classe d'équivalence de Wadge de $\mathbb{R}^{\#\gamma}$ est unique.

On obtient donc la suite $(\mathbb{R}^{\#\alpha} : \alpha < \omega_1)$. Il y a maintenant deux possibilités : soit cette suite est cofinale parmi les ensembles universellement Baire, soit elle ne l'est pas.¹⁷ Nous excluons la première possibilité en renforçant notre hypothèse de grands cardinaux : nous supposons qu'il existe une classe propre de cardinaux de Woodin qui sont des limites de cardinaux de Woodin.¹⁸ Cela signifie qu'il y a des étapes limites de cofinalité indéénombrable. Ceci pose un problème : nous ne savons pas comment joindre ω_1 ensembles universellement Baire.

Heureusement, il existe une solution : il s'agit de rappeler le lien entre les dièses et les stratégies pour les souris, évoqué dans Eq. (1.1). Ici, il est plus pratique de travailler avec des souris du type hod plutôt qu'avec des souris simples.¹⁹ Il résulte des résultats de Sargsyan²⁰ qu'il existe un ensemble de universellement Baire codant une stratégie pour une souris du type hod et dont le rang de Wadge est supérieur ou égal à

$$\sup\{w(\mathbb{R}^{\#\alpha}) : \alpha < \omega_1\}.$$

Soit Σ une telle stratégie du plus petit rang dans l'ordre de souris ;²¹ cette stratégie n'est pas unique, mais elle l'est à l'équivalence de Wadge près. Soit $\mathbb{R}^{\#\omega_1}$ l'ensemble des réels qui code Σ ; cet ensemble dépend du choix de Σ , mais pas sa classe d'équivalence de Wadge.

¹⁷La première possibilité se présente, par exemple, dans le modèle minimum de ZFC+« il existe une classe propre de cardinaux de Woodin ».

¹⁸La première possibilité est exclue par Theorem 1.3.8 (voir aussi le paragraphe précédent).

¹⁹En raison de leur relation avec les HOD des modèles de la détermination.

²⁰Voir [11, Corollaire 0.1].

²¹On peut facilement reconstruire la souris sous-jacente à partir de sa stratégie. On peut ainsi parler de l'ordre des stratégies, c'est-à-dire de l'ordre des paires de souris impliquées par les stratégies.

Plus généralement, lorsque γ est un ordinal limite de cofinalité indénombrable et que la suite $(\mathbb{R}^{\sharp\alpha} : \alpha < \gamma)$ est définie et n'est pas cofinale au sens de Wadge parmi les ensembles universellement Baire, nous pouvons définir $\mathbb{R}^{\sharp\gamma}$ comme la stratégie d'itération du minimal rang de souris, pour une souris du type hod qui est au-dessus de tout $\mathbb{R}^{\sharp\alpha}$ ($\alpha < \gamma$) au sens de Wadge. Bien sûr, ceci est à condition qu'il existe au moins une telle stratégie. Alors, nous arrivons à la question fondamentale de l'existence des stratégies. La conjecture suivante est une version de *Hod Pair Capturing* pertinente pour notre discussion.

Conjecture 1.3.2. *Supposons qu'il existe une classe propre de cardinaux de Woodin. Alors, soit il existe un ensemble universellement Baire codant une stratégie pour une souris du type hod avec un extenseur long sur sa séquence, soit les ensembles de réels codant des stratégies pour les souris du type hod à extenseurs courts sont cofinals parmi les ensembles universellement Baire.*²² \square

Les souris du type hod à extenseurs courts sont définies avec précision dans [12]. La théorie présentée là peut être ajustée pour donner un sens à une souris du type hod dont les extenseurs sont tous courts, sauf celui en haut. Cela suffit à préciser la première possibilité de la conjecture. Cependant, si cette possibilité se réalise, nous ne savons pas comment poursuivre la description des ensembles universellement Baire au-delà d'une telle stratégie. Plus précisément, en supposant que la conjecture soit vraie, la procédure décrite ci-dessus produit la suite

$$(\mathbb{R}^{\sharp\alpha} : \alpha < \mathfrak{F})$$

avec la propriété suivante : soit \mathfrak{F} est un ordinal limite et la suite est cofinale au sens de Wadge parmi les ensembles universellement Baire, soit \mathfrak{F} est un ordinal successeur et $\mathbb{R}^{\sharp(\mathfrak{F}-1)}$ code une stratégie pour une souris du type hod dont l'extenseur en haut est long. Si le deuxième cas se présente, pour dépasser $\mathbb{R}^{\sharp(\mathfrak{F}-1)}$, nous devons d'abord développer la théorie des souris du type hod avec de longs extenseurs.

Il s'agit d'une description d'une construction ascendante d'ensembles universellement Baire à l'aide de dièses et de souris. Nous souhaitons savoir si ces ensembles peuvent être construits plus simplement, d'une manière combinatoire.

Soit \mathcal{A}_0 l'ensemble des parties de \mathbb{R} calculé dans $L(\mathbb{R})$; \mathcal{A}_0 est un segment initial des ensembles universellement Baire au sens de Wadge. Nous avons déjà montré deux manières d'aller au-delà de \mathcal{A}_0 : nous pouvons prendre \mathbb{R}^{\sharp} ou nous pouvons prendre la stratégie pour $\mathcal{M}_\omega^\sharp$; ces ensembles sont universellement Baire et n'appartiennent pas à \mathcal{A}_0 . Il existe également une troisième voie, qui correspond le mieux à ce que nous souhaitons faire maintenant : tout ensemble de réels de $L(\mathcal{A}_0^\omega)$ est universellement Baire, et un entre eux n'appartient pas à \mathcal{A}_0 . C'est ce qui motive l'introduction de l'opérateur

$$\mathcal{O}_1 : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\omega).$$

Nous pouvons alors énoncer le fait général suivant, exprimé en termes de *point-classes fermées*, c'est-à-dire des collections d'ensembles de réels Γ qui vérifient $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma) = \Gamma$.

Proposition 1.3.3. *Supposons qu'il existe une classe propre de cardinaux de Woodin et que $\Gamma \subsetneq \text{UB}$ soit une point-classe fermée. Alors*

²²Cf. [12, Définition 1.7.1].

a. tout ensemble de réels dans $\mathcal{O}_1(\Gamma)$ est universellement Baire,

b. $\Gamma \subsetneq \mathcal{O}_1(\Gamma)$ si et seulement si $\text{cof}(\Gamma, \leq_W^*) = \omega$.

Proof. Pour la première partie, voir Proposition 2.3.4. Pour la seconde partie, supposons d'abord que $\text{cof}(\Gamma, \leq_W^*) = \omega$. Soit $(A_n : n < \omega)$ une suite cofinale au sens de Wadge dans Γ . Nous avons que

$$\bigoplus_{n < \omega} A_n \in \mathcal{O}_1(\Gamma) - \Gamma.$$

D'un autre côté, supposons que $\text{cof}(\Gamma, \leq_W^*) > \omega$. Alors $\Gamma^\omega \subseteq L(\Gamma)$, donc $\mathcal{O}_1(\Gamma) = \Gamma$. \square

Il s'avère que $L(\mathcal{O}_1(\mathcal{I}_0))$ est de la forme $L(\mathbb{R}, A)$, pour un ensemble A ,²³ de sorte que la cofinalité au sens de Wadge de son ensemble des parties de \mathbb{R} soit à nouveau dénombrable.²⁴ Cela nous incite à essayer d'itérer cet opérateur pour obtenir plus d'ensembles universellement Baire. En général, pour une fonction $\mathcal{O} : \mathcal{P}(\mathcal{P}(\mathbb{R})) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{R}))$, on définit $\mathcal{O}^\alpha : \mathcal{P}(\mathcal{P}(\mathbb{R})) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{R}))$ par récurrence sur α :

- \mathcal{O}^0 est l'identité sur $\mathcal{P}(\mathcal{P}(\mathbb{R}))$,
- $\mathcal{O}^{\alpha+1} := \mathcal{O} \circ \mathcal{O}^\alpha$,
- $\mathcal{O}^{<\gamma} : \mathcal{P}(\mathcal{P}(\mathbb{R})) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \bigcup_{\alpha < \gamma} \mathcal{O}^\alpha(\Gamma)$, pour une limite γ ,
- $\mathcal{O}^\gamma : \mathcal{P}(\mathcal{P}(\mathbb{R})) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \mathcal{P}(\mathbb{R}) \cap L(\mathcal{O}^{<\gamma}(\Gamma))$, pour une limite γ .

Nous définissons ainsi l'étape limite afin de garantir que \mathcal{O}^γ prenne des valeurs dans des point-classes fermées ; de toute façon, seuls les opérateurs possédant cette propriété nous intéressent. Ainsi, les point-classes $\mathcal{O}_1^\alpha(\mathcal{I}_0)$ constitueront une suite croissante de classes de points fermées d'ensembles universellement Baire, jusqu'à sa stabilisation. Ceci se produit avant le petit modèle de $\text{AD}_{\mathbb{R}}$: la point-classe correspondante $\mathcal{O}_1^\alpha(\mathcal{I}_0)$ est la première point-classe fermée Γ qui satisfait $\text{cof}(\Gamma, \leq_W^*) > \omega$. Pour aller au-delà de ce point, un nouvel opérateur est nécessaire.

Le nouvel opérateur provient des travaux non-publiés de Woodin sur le plus petit modèle de $\text{AD}_{\mathbb{R}} + \text{DC}$. L'idée sous-jacente vient d'un théorème de Solovay, qui stipule que sous $\text{AD}_{\mathbb{R}}$, le filtre μ_X des clubs sur $[X]^\omega$ est un ultrafiltre lorsque X est une image surjective des réels. Nous nous intéressons à l'opérateur

$$\mathcal{O}_2 : \mathcal{P}(\mathcal{P}(\mathbb{R})) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\omega)[\mu_\Gamma].$$

Proposition 1.3.4. *Supposons qu'il existe une classe propre de cardinaux de Woodin et que $\Gamma \subsetneq \text{UB}$ soit une point-classe fermée. Alors $\Gamma \subseteq \mathcal{O}_2(\Gamma) \subsetneq \text{UB}$.*

²³On peut simplement prendre $\bigoplus_{n < \omega} A_n$, pour une suite cofinale dans \mathcal{I}_0 .

²⁴Une suite cofinale $(A_n : n < \omega)$ dans l'ensemble des parties de \mathbb{R} de $L(\mathbb{R}, A)$ est donnée par A^\sharp : pour obtenir A_n , il faut prendre les formules de (la théorie de) A^\sharp dans lesquelles seuls les n premiers indiscernables apparaissent.

Proof. D'après [5, Théorème 3.3.19], il existe une point-classe fermée $\Gamma^* \subsetneq \text{UB}$ telle que $\Gamma \subsetneq \Gamma^*$. Il résulte de Proposition 4.3.16 appliqué à $L(\Gamma^*)$ que $\mathcal{O}_2(\Gamma) \subseteq L(\Gamma^*)$, ce qui suffit pour la conclusion. \square

Theorem 1.3.5 (Woodin). *Supposons que*

1. *il existe une classe propre de cardinaux de Woodin,*
2. *A est un ensemble universellement Baire,*
3. *pour toute point-classe fermée $\Gamma \subseteq \{B : B <_{\mathbb{W}}^* A\}$, $L(\Gamma) \not\models \text{AD}_{\mathbb{R}} + \text{DC}$.*

Alors il existe α tel que $A \in \mathcal{O}_2^\alpha(\mathcal{A}_0)$.

Proof. Ceci résulte de Proposition 1.3.4 and Corollary 4.3.18. \square

Sous une hypothèse de grands cardinaux suffisamment forte, le plus petit modèle de $\text{AD}_{\mathbb{R}} + \text{DC}$ existe, et tous les ensembles de réels qu'il contient sont universellement Baire.²⁵ Il s'ensuit que pour un certain α , $L(\mathcal{O}_2^\alpha(\mathcal{A}_0))$ est ce modèle. On ne sait pas si l'ensemble des parties de \mathbb{R} de ce modèle est stable par \mathcal{O}_2 , mais nous savons que la stabilisation arrive avant que le plus petit modèle de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ soit atteint.²⁶ Pour atteindre ce modèle, une modification supplémentaire de l'opérateur est nécessaire. Nous avons trouvé un tel opérateur et démontré qu'il peut atteindre le plus petit modèle de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$.

Notre opérateur est défini comme suit. Tout d'abord, soit

$$\mathcal{O}_3^* : \mathcal{P}(\mathcal{P}(\mathbb{R})) \times \text{Ord} \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : (\Gamma, \kappa) \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\omega, \kappa^\omega)[\mu_\Gamma, \mu_\kappa].$$

Ensuite, pour $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, nous définissons $\mathcal{O}_3(\Gamma)$ comme $\mathcal{O}_3^*(\Gamma, \kappa)$ pour le plus petit $\kappa \leq \text{Ord}$ qui satisfait soit $\Gamma \subsetneq \mathcal{O}_3^*(\Gamma, \kappa)$, soit $\kappa = \text{Ord}$. Voilà ce que nous savons de la mesure dans laquelle \mathcal{O}_3 peut produire des ensembles universellement Baire.

Theorem 1.3.6. *Soit Γ une point-classe fermée telle que $L(\Gamma) \models \text{AD}_{\mathbb{R}}$. Supposons que dans $L(\Gamma)$, il n'existe pas de paire du type hod avec une limite mesurable de cardinaux de Woodin. Alors il existe α tel que $\Gamma = \mathcal{O}_3^\alpha(\mathcal{A}_0)$.*

Proof. Voir Theorem 4.1.1. \square

Nous avons énoncé le théorème en termes de point-classes déterminées, ce qui le rend applicable dans un contexte plus général que celui des ensembles universellement Baire. Nous revenons au contexte des ensembles universellement Baire en rappelant Theorem 1.1.3.

Énonçons maintenant trois corollaires à ce théorème. Nous avons déjà mentionné que le modèle minimum de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ ne peut pas être atteint en itérant l'opérateur \mathcal{O}_2 . Cependant, il peut être atteint en itérant l'opérateur \mathcal{O}_3 : ceci résulte de Theorem 1.3.6 et du fait que le modèle minimum de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ ne contient pas de paire du type hod avec une limite inaccessible de cardinaux de Woodin.

²⁵Par exemple, cela résulte de Theorem 1.3.8 (voir aussi la discussion qui le précède).

²⁶Voir Footnote 10 du Chapter 4.

Corollary 1.3.7. *Supposons qu'il existe un modèle intérieur contenant tous les réels et vérifiant $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$. Alors il existe α tel que $L(\mathcal{O}_3^\alpha(\mathcal{A}_0))$ est le plus petit modèle de ce type.* □

On savait, grâce à des travaux antérieurs de Sargsyan, que le modèle minimum de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ existe, mais il s'agit de la première construction « ascendante » de ce modèle. Pour voir ce qu'apporte une telle construction, considérons le modèle dit de Chang-plus :

$$\text{CM}^+ := L(\text{Ord}^\omega)[(\mu_\alpha^\omega : \alpha \in \text{Ord})].$$

Woodin a montré que les grands cardinaux impliquent que le modèle de Chang-plus satisfaisant $\text{AD}^+ + \text{“}\omega_1 \text{ est supercompact”}$. Il découle des travaux d'Ikegami et Trang que ce modèle satisfait en fait $\text{AD}_{\mathbb{R}} + \text{DC} + \text{“}\omega_1 \text{ est supercompact”}$.²⁷ Nous nous intéressons ici à l'énoncé suivant, plus précis, du résultat de Woodin.

Theorem 1.3.8 (Woodin). *Supposons qu'il existe une classe propre de cardinaux de Woodin qui sont des limites de cardinaux de Woodin. Alors, les énoncés suivants sont vérifiés :*

- a. *il existe un ensemble universellement Baire qui est au-dessus au sens de Wadge de tout ensemble de réels dans le modèle Chang-plus ;*
- b. *Pour tout $X \in \text{CM}^+$, $\mu_X \cap \text{CM}^+$ appartient au modèle Chang-plus et y est un ultrafiltre fin normal.*²⁸

□

En utilisant ce fait, nous pouvons obtenir deux corollaires supplémentaires de notre résultat. Le premier est que le modèle minimum de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ est contenu dans le modèle Chang-plus.

Corollary 1.3.9. *Supposons qu'il existe une classe propre de cardinaux de Woodin qui sont des limites de cardinaux de Woodin. Alors le plus petit modèle de $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ est contenu dans le modèle Chang-plus.* □

L'idée ici est que Theorem 1.3.8 garantit que \mathcal{O}_3 est absolu entre CM^+ et V . Le revers de cette réalité est que l'opérateur \mathcal{O}_3 ne peut aller au-delà de la collection des parties des réels du modèle Chang-plus :

$$\mathcal{O}_3(\mathcal{P}(\mathbb{R}) \cap \text{CM}^+) = \mathcal{P}(\mathbb{R}) \cap \text{CM}^+.$$

Cela conduit à un autre corollaire concernant l'existence de souris du type hod avec des stratégies universellement Baire.

²⁷Voir [13, Théorème 4].

²⁸Voir [14, Théorème 1.3].

Corollary 1.3.10. *Supposons qu'il existe une classe propre de cardinaux de Woodin qui sont des limites de cardinaux de Woodin. Il existe alors une paire du type hod avec la portée H_{ω_1} dont la pré-souris a une limite mesurable de cardinaux de Woodin et dont la stratégie est codée par un ensemble universellement Baire.*

Proof. Soit $\Gamma := \mathcal{P}(\mathbb{R}) \cap \text{CM}^+$. Par Theorem 1.3.8, il existe un ensemble universellement Baire A tel que

$$\Gamma = \{B \subseteq \mathbb{R} : B <_{\mathbb{W}}^* A\}.$$

Par [5, Theorem 3.3.19], il existe une point-classe fermée Γ^* telle que $A \in \Gamma^* \subsetneq \text{UB}$ et $L(\Gamma^*) \models \text{AD}_{\mathbb{R}}$. Puisque $\mathcal{O}_3(\Gamma) = \Gamma$, Theorem 1.3.6 (appliqué à Γ^*) implique qu'il existe une paire du type hod avec la portée H_{ω_1} dans $L(\Gamma^*)$ dont la pré-souris a une limite mesurable de cardinaux de Woodin. \square

1.4 Absoluité et maximalité génériques

Theorem 1.1.9 implique, entre autres, que les vérités qui sont projectives depuis les ensembles universellement Baire sont génériquement absolues.

Corollary 1.4.1. *Supposons que*

1. *il existe une classe propre de cardinaux de Woodin,*
2. *A est un ensemble universellement Baire,*
3. *ϕ est une phrase du langage $\{R_{\in}^2, R_A^1\}$,*
4. *\mathbb{P} est un poset arbitraire.*

Alors $(H_{\omega_1}, \in, A) \models \phi$ si et seulement si $V^{\mathbb{P}} \models ((H_{\omega_1}, \in, A_g) \models \phi)$. \square

On se demande comment aller plus loin. Par exemple, nous pourrions établir ce qui suit.²⁹

Theorem 1.4.2. *Supposons que*

1. *il existe une classe propre de cardinaux de Woodin,*
2. *A est un ensemble universellement Baire,*
3. *ϕ est une phrase de complexité Σ_1 dans le langage $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$,*

²⁹Pour voir que cela est plus fort, notons que tout fait du premier ordre sur (H_{ω_1}, \in, A) est codé dans A^{\sharp} , qui est lui-même universellement Baire.

4. \mathbb{P} est un poset arbitraire.

Alors $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$ si et seulement si $V^{\mathbb{P}} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi)$.

Proof. Voir Corollary 6.6.4. □

Dans ce théorème, les ensembles universellement Baire sont utilisés comme prédicats. Si nous supposons CH, nous pouvons également les utiliser comme paramètres. Nous obtenons ainsi une conjecture qui renforce le théorème de Woodin sur l'absoluité générique conditionnée sur CH pour les énoncés de complexité Σ_1^2 .

Conjecture 1.4.3. *Supposons que*

1. *il existe une classe propre de cardinaux de Woodin,*
2. *CH est vérifiée,*
3. *A est un ensemble universellement Baire,*
4. *$\phi(v_A)$ est une phrase de complexité Σ_1 dans le langage $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1\}$,*
5. *\mathbb{P} est un poset arbitraire qui force CH.*

Alors $(H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A]$ si et seulement si $V^{\mathbb{P}} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A_{\dot{g}}])$. □

La démonstration du théorème de Woodin mentionnée ci-dessus montre que le résultat est vrai si le prédicat NS_{ω_1} est omis.

Theorem 1.4.4. *Supposons que*

1. *il existe une classe propre de cardinaux de Woodin,*
2. *CH est vérifiée,*
3. *A est un ensemble universellement Baire,*
4. *$\phi(v_A)$ est une phrase de complexité Σ_1 ,*
5. *\mathbb{P} est un poset arbitraire qui force CH.*

Alors $(H_{\omega_2}, \in) \models \phi[A]$ si et seulement si $V^{\mathbb{P}} \models ((H_{\omega_2}, \in) \models \phi[A_{\dot{g}}])$.

Proof. Voir Theorem 6.7.9. □

Une autre façon d'aller au-delà de l'absoluité générique établie dans Theorem 1.4.2 est d'examiner les phrases de complexité Π_2 au lieu de ceux de complexité Σ_1 . Dans ce cas, il n'y a aucun espoir d'obtenir une absoluité générique. En effet, $\neg\text{CH}$ peut être exprimé par une phrase de complexité Π_2 sur la structure $(H_{\omega_2}, \in, \mathbb{R})$, tandis que CH et $\neg\text{CH}$ peuvent être forcés tous les deux. Dans ce cas, le mieux que l'on puisse espérer est une forme de maximalité générique. Par exemple, considérons le principe suivant.

Definition 1.4.5. L'énoncé $(\dagger)_{\text{UB}}$ affirme que pour tout ensemble universellement Baire A et toute phrase ϕ de complexité Π_2 du langage $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$, si

$$\exists \mathbb{P}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi),$$

alors $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi$. □

Nous pouvons montrer que cet énoncé est essentiellement équivalent à la forme bornée de MM^{++} suivante (en particulier, elle est vraie sous MM^{++}).

Definition 1.4.6. L'énoncé $\text{UB} - \text{BMM}^{++}$ affirme que pour tout ensemble universellement Baire A , toute formule $\phi(v_a)$ de complexité Σ_1 du langage $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$ et tout $a \in H_{\omega_2}$, si

$$\exists \mathbb{P} \text{ ssp}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi[a]),$$

alors $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi[a]$. □

Theorem 1.4.7 (Asperó, Schindler, Woodin). *Supposons qu'il existe une classe propre de cardinaux de Woodin. Alors :*

- a. $\text{UB} - \text{BMM}^{++}$ implique $(\dagger)_{\text{UB}}$;
- b. s'il existe un cardinal supercompact, alors $(\dagger)_{\text{UB}}$ implique $\text{UB} - \text{BMM}^{++}$.

Proof. Cela résulte des faits bien connus de [15] et [16]. Nous expliquons exactement comment dans Corollary 6.2.13. □

Les principes $\text{UB} - \text{BMM}^{++}$ et $(\dagger)_{\text{UB}}$ peuvent être affaiblis en omettant le prédicat NS_{ω_1} de leurs énoncés. On obtient ainsi respectivement les principes $\text{UB} - \text{BMM}$ et $(\dagger)_{\text{UB}}^-$. L'implication $\text{UB} - \text{BMM}^{++} \implies (\dagger)_{\text{UB}}$ de Theorem 1.4.7 est vérifiée en citant [16] pour obtenir

$$\text{UB} - \text{BMM}^{++} \implies (*_{\text{UB}}$$

et en citant [15] pour obtenir

$$(*_{\text{UB}} \implies (\dagger)_{\text{UB}},$$

où $(*)_{\text{UB}}$ est un renforcement naturel de l'axiome $(*)$ défini dans Definition 6.2.5. Pour montrer que $\text{UB} - \text{BMM} \implies (\dagger)_{\text{UB}}^-$, on ne peut pas simplement passer par $(*)_{\text{UB}}$ comme ci-dessus. Cependant, en passant par les preuves des deux implications et en éliminant les coupures, on obtient cette implication.

Theorem 1.4.8. *Supposons qu'il existe une classe propre de cardinaux de Woodin. Alors $UB - BMM$ implique $(\dagger)_{UB}^-$.*

Proof. Voir Corollary 6.4.5. □

La démonstration de ce théorème montre que l'étape principale est la vérification de la proposition suivante.

Proposition 1.4.9. *Supposons que*

1. *il existe une classe propre de cardinaux de Woodin,*
2. *A est un ensemble universel Baire,*
3. *$\phi(x)$ est une formule de complexité Σ_1 dans le langage $\{R_{\mathbb{C}}, c_{\omega_1}, R_A^1\}$,*
4. *il existe un poset qui force $\langle (H_{\omega_2}, \in, \omega_1, A_g) \models \forall x, \phi(x) \rangle$.*

Alors il existe un poset \mathbb{Q} qui préserve les ensembles stationnaires de ω_1 et qui vérifie que dans $V^{\mathbb{Q}}, H_{\omega_2}^V \in H_{\omega_2}$ et $(H_{\omega_2}, \in, \omega_1, A_{\dot{H}}) \models \forall x \in H_{\omega_2}^V, \phi(x)$.

Proof. Voir Proposition 6.4.4. □

Pour démontrer cette proposition, nous écrivons une formule propositionnelle infinitaire Φ et nous nous demandons si un modèle pour celle-ci peut être ajouté par un poset préservant les ensembles stationnaires de ω_1 . Nous soutenons ensuite que la réponse est affirmative ; tout poset \mathbb{Q} de ce type fonctionne comme requis dans la conclusion de Proposition 1.4.9. Notons également qu'il n'est pas tout à fait simple de trouver la formule correcte Φ , mais les idées nécessaires existent dans [16] (bien que les auteurs y développent un langage *ad hoc* au lieu d'utiliser la logique propositionnelle infinitaire). Quoi qu'il en soit, une fois notre formule établie, la question à laquelle il faut répondre devient assez générale : quand un poset préservant les ensembles stationnaires de ω_1 peut-il ajouter un modèle à une formule propositionnelle infinitaire ? Ceci nous amène au dernier axe de recherche poursuivi dans cette thèse.

1.5 PES-consistance des formules infinitaires

Avant de répondre à la question de savoir quand un modèle de formule propositionnelle infinitaire peut être ajouté par un poset pes (= « préservant (les) ensembles stationnaires (de ω_1) »), considérons d'abord le même problème pour un poset arbitraire. La réponse à ce problème relève essentiellement du folklore, une fois replacée dans son propre contexte.

Soient ϕ et ψ deux formules propositionnelles (ordinaires). Nous disposons d'une notion bien définie de prouvabilité, $\phi \vdash \psi$: elle affirme simplement qu'il existe un arbre de

déduction dans le système de déduction naturelle, NK, dont la conclusion est ψ et dont les seules hypothèses non-déchargées sont contenues dans $\{\phi\}$. Dans ce cas, la relation de prouvabilité est équivalente à la notion standard de conséquence sémantique $\phi \models \psi$ qui affirme que tout modèle de ϕ est aussi un modèle de ψ .

Maintenant, dans le cas où ϕ et ψ sont des formules propositionnelles infinies, la relation $\phi \vdash \psi$ peut être définie de manière relativement simple. L'important est que le système NK soit modifié pour autoriser certaines règles infinies. Par exemple, il devrait y avoir une règle d'introduction pour les conjonctions infinies, ce qui renforce la règle habituelle (finitaire). Une conséquence de cette exigence est que les déductions devraient désormais pouvoir être des arbres infinis ; néanmoins, ce sont toujours des arbres bien fondés. Une fois ces ajustements effectués, la relation de prouvabilité a essentiellement la même définition qu'auparavant.

Quelle est la contrepartie sémantique de cette relation de prouvabilité ? Pour comprendre la réponse, notons que l'assertion $\phi \vdash \psi$ est (1) génériquement absolue, et (2) implique que tout modèle pour ϕ , dans toute extension générique, est également un modèle de ψ . Considérons maintenant la formule naturelle ϕ , dont les modèles codent un sous-ensemble cofinal dénombrable de ω_1^V , d'une part, et la formule $\psi := \perp$, d'autre part. Dans $V^{\text{Col}(\omega, \omega_1)}$, il existe un modèle pour ϕ , tandis qu'il n'en existe pas pour \perp ; donc $\phi \not\vdash \perp$. Ce dernier fait est également vérifié dans V , mais les formules ϕ et \perp ont le même ensemble de modèles là, à savoir l'ensemble vide. Cela montre que l'ensemble pertinent de modèles à considérer pour \models est (au moins) l'ensemble de tous les modèles dans toutes les extensions génériques.

En effet, nous définissons $\phi \models \psi$ comme signifiant « pour tout poset \mathbb{P} et pour tout $\dot{\mu} \in V^{\mathbb{P}}$, $\Vdash_{\mathbb{P}} (\dot{\mu} \models \phi \implies \dot{\mu} \models \psi)$ ». De cette façon, le théorème de complétude est valable : pour toutes formules propositionnelles infinitaires ϕ et ψ , $\phi \vdash \psi$ si et seulement si $\phi \models \psi$.

L'étude de ces deux relations peut être quelque peu simplifiée en prenant la formule de droite comme \perp . Aucune généralité n'est perdue de cette manière : $\phi \vdash \psi$ si et seulement si $\phi \wedge \neg\psi \vdash \perp$, ainsi que $\phi \models \psi$ si et seulement si $\phi \wedge \neg\psi \models \perp$ (c'est-à-dire que $\phi \wedge \neg\psi$ n'a pas de modèle dans aucune extension générique). En rappelant que nous lisons « $\phi \not\vdash \perp$ » comme « ϕ est cohérent » et « $\phi \not\models \perp$ » comme « ϕ est consistant », le théorème de complétude s'écrit alors : une formule propositionnelle infinitaire est cohérente si et seulement si elle est consistant.

Puisque la consistance signifie l'existence d'un modèle dans une extension générique, il est naturel d'appeler « pes-consistance » la propriété d'existence d'un modèle dans une extension générique par un poset préservant les ensembles stationnaires de ω_1 . Notre question motivante est alors de savoir quand une formule propositionnelle infinitaire est pes-consistante. La caractérisation que nous pouvons donner ici n'est pas en termes de la théorie de la démonstration, mais plutôt de la théorie des jeux. Elle trouve sa motivation dans le jeu d'existence de modèle.

Soit ϕ une formule propositionnelle infinitaire sous forme normale de négation. Le jeu d'existence de modèle associé à ϕ est un jeu à deux joueurs de longueur ω , que nous appellerons Héloïse et Abélard. Abélard joue aux *questions*, tandis que Héloïse répond en jouant aux *parties actives*. Ces parties actives sont des ensembles finis de sous-formules de ϕ ; Héloïse doit s'assurer qu'ils ne contiennent pas une paire composée d'une lettre proposi-

tionnelle et de sa négation, sinon elle perd. La partie commence par le choix arbitraire d'une partie active contenant ϕ par Héloïse. Abélard continue avec des questions, puis Héloïse agrandit la partie active afin que celle-ci contienne une réponse à la question.

Abélard peut poser deux types de questions. Dans le premier cas, il choisit une conjonction présente dans la partie active courante, y choisit un conjonctif et demande à Héloïse d'agrandir la partie active afin qu'elle contienne ce conjonctif (et éventuellement d'autres éléments). Dans le second cas, Abélard choisit une disjonction présente dans la partie active courante et demande à Héloïse d'étendre la partie active afin qu'elle contienne au moins une des disjonctions (et éventuellement d'autres éléments). L'idée est que lors de ces agrandissements, Héloïse peut être contraint d'inclure une paire composée d'une lettre propositionnelle et de sa négation. L'idée derrière ce jeu est que Héloïse prétend connaître un modèle pour ϕ , mais Abélard pense qu'elle ment et pose des questions pour la prendre en flagrant délit de mensonge.

Il s'agit du jeu d'existence de modèle associé à ϕ , et Héloïse y a une stratégie gagnante si et seulement si ϕ est consistante. Nous souhaitons savoir s'il existe un jeu qui caractérise la pes-consistance de manière similaire. La réponse est oui, et ce jeu est une variante du jeu que nous venons de décrire ; nous l'appelons $\mathcal{D}_{\phi, \kappa}^{\text{ssp}}$. L'indice κ est un cardinal inaccessible, et sa présence est due au fait que Héloïse doit jouer une partie des conditions secondaires en plus de la partie active. Une *condition secondaire* est un ensemble des ensembles élémentaire dans sa clôture transitive et qui satisfont ZFC^- .³⁰ Un exemple typique est un sous-modèle élémentaire d'un certain H_θ . Une partie de condition secondaire est une \in -chaîne finie de conditions secondaires dénombrables (avec quelques restrictions supplémentaires). Le paramètre κ indique, entre autres, que toutes les conditions secondaires considérées dans le jeu sont dans V_κ .

Une stratégie gagnante pour Héloïse dans le jeu d'existence de modèle, vue comme un poset, peut être utilisée pour forcer une théorie complète (et donc un modèle) pour ϕ . L'idée est que, grâce à ce forcing, nous obtenons un jeu générique d'Abélard dans lequel « toutes les questions » sont posées ; l'union des parties actives de ce jeu constitue une théorie complète. Le jeu $\mathcal{D}_{\phi, \kappa}^{\text{ssp}}$ est une variante du jeu d'existence de modèle dans lequel une stratégie gagnante pour Héloïse, vue comme un poset, ajoute une théorie complète contenant ϕ et préserve en outre les ensembles stationnaires de ω_1 . La préservation des ensembles stationnaires de ω_1 est assurée par la partie des conditions secondaires. Nous conservons les questions d'Abélard qui apparaissent dans le jeu d'existence de modèle afin de garantir que le poset ajoute un modèle pour ϕ , mais nous avons des questions supplémentaires concernant la partie des conditions secondaires qui garantissent que le poset préservera les ensembles stationnaires de ω_1 .

Theorem 1.5.1. *Supposons qu'il existe une classe propre de cardinaux inaccessibles. Soit ϕ une formule propositionnelle infinitaire sous forme normale de négation. Alors les énoncés suivants sont équivalents.*

- a. ϕ est pes-consistante.

³⁰Dans Chapter 5, nous utilisons le terme « modèle virtuel » pour ce qui est appelé ici une condition secondaire.

b. Pour un cardinal inaccessible κ , Héloïse a une stratégie gagnante dans $\mathfrak{D}_{\phi, \kappa}^{\text{ssp}}$.

Proof. Voir Corollary 5.7.8. □

Notre problème motivant, qui consiste à ajouter un modèle à une formule par un poset pes, devient le problème de la description d'une stratégie gagnante pour Héloïse dans $\mathfrak{D}_{\phi, \kappa}^{\text{ssp}}$. Bien sûr, cette stratégie n'est généralement pas évidente, mais elle provient parfois de plongements génériques. Cela suffit à démontrer Theorem 1.4.8, par exemple.

Definition 1.5.2. Supposons que ϕ est une formule propositionnelle infinitaire et que $\kappa > \text{rank}(\phi)$ est inaccessible. Alors ϕ est dit **AS-bonne** en κ ssi ϕ est consistante et pour tout $S \subseteq \omega_1$ stationnaire, il est vérifié dans $V^{\text{Col}(\omega, < \kappa)}$ que pour tout $\mu \models \phi$, il existent un plongement élémentaire $\tau : V \rightarrow W$ et un modèle $\hat{\mu} \models \tau(\phi)$ tels que $\text{crit}(\tau) = \omega_1^V \in \tau(S)$ et pour toute sous-formule ψ de ϕ , $\hat{\mu}(\tau(\psi)) = \mu(\psi)$. □

Theorem 1.5.3. *Supposons que κ est un cardinal inaccessible et que $\phi \in V_\kappa$ est une formule propositionnelle infinitaire qui est AS-bonne à κ . Alors ϕ est pes-consistante.*

Proof. Voir Corollary 5.8.7. □

Chapter 2

Introduction

2.1 Why universally Baire sets?

One of the most fundamental questions in Set Theory is which sets of reals exist. This is even the core of the *Continuum Hypothesis*: in its original form, it asks whether there is a *set of reals* which neither injects into ω nor surjects onto \mathbb{R} . The tremendous work done by set theorists in the last few decades allows us to say, at least, that there are two kinds of sets of reals: “canonical” and “chaotic”. The focus of this dissertation is on the canonical ones, also known as *universally Baire sets*.

Universally Baire sets were introduced in [1]. Their definition is based on the *property of Baire*.

Definition 2.1.1. Let X be a topological space and let A be a subset of X . Then A is said to be **Baire-measurable** (or to have the **property of Baire**) iff there exists an open set U such that $A\Delta U$ is meager. \square

The Axiom of Choice implies that there are sets of reals which are not Baire-measurable. On the other hand, any reasonably defined set of reals is Baire-measurable.¹ Being aware of these facts, we begin to associate Baire measurability with canonical sets of reals, and its failure with the chaotic ones. It turns out that it is actually a strengthening of this property that delimits these two kinds of sets of reals.

Definition 2.1.2. A set of reals A is said to be **universally Baire**² iff for every compact Hausdorff space X and every continuous function $f : X \rightarrow \mathbb{R}$, the set $f^{-1}[A]$ is Baire-measurable in X . \square

One way in which universally Baire sets are “well behaved” is the fact that they are Baire-measurable. It turns out that they are also Lebesgue-measurable. Moreover, they

¹See for example Theorem 2.1.9.

²or, *universally Baire-measurable*

have the Perfect Set Property. Returning to the connection with the Continuum Hypothesis alluded to at the beginning, the fact that universally Baire sets enjoy the Perfect Set Property means that a counterexample to the Continuum Hypothesis cannot be universally Baire. Of course, one may wonder how important this last statement is, i.e., how rich is the collection of universally Baire sets; the answer to this will become clear: it is quite rich³ and probably encodes a significant part of the canonical content of the universe⁴.

Thus, universally Baire sets are Baire- and Lebesgue-measurable and have the Perfect Set Property, but more is true. These properties can be abstracted and strengthened into a statement about the determinacy of length ω , two-player games associated with universally Baire sets.⁵

Theorem 2.1.3 (Martin, Steel, Woodin). *Suppose there are two Woodin cardinals. Then every universally Baire set of reals is determined.*

Proof. This follows from [3, Theorems 3.1, 2.1, and 1.7]. □

The fact that universally Baire sets cannot witness the failure of the Continuum Hypothesis suggests that they are somehow immune to forcing. This is indeed true, and we will try to make it precise in several ways. First, universally Baire sets can be defined simply as projections of certain well-behaved trees.

Definition 2.1.4. Let T and S be trees on $\omega \times \text{Ord}$ and let \mathbb{P} be a poset. Then (T, S) are **\mathbb{P} -absolutely complemented** iff

$$V^{\mathbb{P}} \models p[T] = \mathbb{R} - p[S].$$

□

The point here is that T and S are in V and project to complements, but more is true: in any generic extension, T and S necessarily project to disjoint sets, but there is no reason a priori that their projections cover all of \mathbb{R} - this is what we impose in the definition. Now these trees embody in their very definition a form of generic absoluteness: it is absolute that they project to complements. Here is their connection to universally Baire sets.

Theorem 2.1.5 (Feng, Magidor, Woodin). *Let A be a set of reals. Then A is universally Baire if and only if, for every poset \mathbb{P} , there exist \mathbb{P} -absolutely complemented trees T and S such that $p[T] = A$.*⁶ □

It turns out that universally Baire sets enjoy a much higher degree of generic absoluteness than the face value of this characterization suggests. To begin with, a universally Baire set A ensures a canonical way to generically reinterpret the predicate “ $x \in A$ ”, defined on reals x .

³See Theorems 2.1.9 and 2.3.8.

⁴See Theorem 2.3.8.

⁵For a derivation of the above three properties from determinacy, see [2, Section 27].

⁶See [1, Corollary 2.1].

Proposition 2.1.6 (folklore). *Let \mathbb{P} be a poset and let T, S and T', S' be two pairs of \mathbb{P} -absolutely complemented trees satisfying $p[T] = p[T']$. Then $V^{\mathbb{P}} \models p[T] = p[T']$.⁷ \square*

Definition 2.1.7. Let A be a universally Baire set and let g be a V -generic. Then A_g is the unique set of reals in $V[g]$ such that for some pair T, S of \mathbb{P} -absolutely complemented trees in V ,

$$p[T]^V = A \text{ and } p[T]^{V[g]} = A_g.$$

\square

The intuition here is that $A_g \subseteq \mathbb{R}^{V[g]}$ is the canonical reinterpretation in $V[g]$ of the set $A \subseteq \mathbb{R}^V$. Thus, universally Baire sets have canonical generic reinterpretations. However, a kind of converse is also true.

Theorem 2.1.8. *Let $\phi(v_x, v_a)$ be a Σ_n formula, where $n < \omega$, and let a be an arbitrary set. Then the following are equivalent.*

a. *The set $A := \{x \in \mathbb{R} : \phi(x, a)\}$ is universally Baire and for all V -generic filters g ,*

$$A_g = \{x \in \mathbb{R} : \phi(x, a)\}^{V[g]}.$$

b. *For all $\theta > \text{rank}(a)$ satisfying $V_\theta \prec_{\Sigma_{n+5}} V$, there exists $U \subseteq V_\theta$, such that for all countable elementary embeddings*

$$\pi : (M, \in, \bar{U}, \bar{a}) \longrightarrow (V_\theta, \in, U, a)$$

with M transitive, for all M -generic filters g and all reals $x \in M[g]$,

$$M[g] \models \phi[x, \bar{a}] \iff V \models \phi[x, a].$$

Proof. See [3, Lemma 4.1]. \square

Once we are able to reinterpret the predicate “ $x \in A$ ”, it stands to reason that more complicated formulas can be reinterpreted as well. For example, it turns out that if B is projective in a universally Baire set A , then B is also universally Baire, and in $V[g]$ the set B_g is projective in A_g by the same definition as B from A . We can even go beyond the projective level.

Theorem 2.1.9 (Woodin). *Suppose there exists a proper class of Woodin cardinals and A is a universally Baire set. Then $(\mathbb{R}, A)^\sharp$ is universally Baire and for every V -generic g ,*

$$((\mathbb{R}^V, A)^\sharp)_g = (\mathbb{R}^{V[g]}, A_g)^\sharp.$$

Proof. This is an unpublished result, but compare with [5, Theorem 3.3.9]. \square

⁷See [4, Proposition 1.3-IV].

Thus, every set of reals in $L(\mathbb{R}, A)$ is universally Baire, and if g is V -generic, there exists an elementary embedding

$$j : L(\mathbb{R}^V, A) \longrightarrow L(\mathbb{R}^{V[g]}, A_g)$$

such that for all $B \in \mathcal{P}(\mathbb{R}^V) \cap L(\mathbb{R}^V, A)$, $j(B) = B_g$. This clearly generalizes the projective level of generic absoluteness.

Theorem 2.1.9 can be seen as the starting point for this dissertation. In the following sections, we will point out some questions that this theorem naturally raises, what was known about them, and what we proved.

2.2 What is the Theory of $L(\mathbb{R})$?

A corollary of Theorem 2.1.9 is that the theory of $L(\mathbb{R})$ is generically absolute, so it is natural to ask what this theory is. The quoted theorem also implies that every set of reals in $L(\mathbb{R})$ is universally Baire, so in particular $L(\mathbb{R})$ satisfies $ZF + AD^+ + DC$. Even though the Axiom of Choice fails in $L(\mathbb{R})$, its inner model $\text{HOD}^{L(\mathbb{R})}$ satisfies AC and is close to $L(\mathbb{R})$.⁸ The theory of $\text{HOD}^{L(\mathbb{R})}$ is a part of the theory of $L(\mathbb{R})$ and thus also generically absolute. This theory extends ZFC, but much more can be said about it: it turns out that this model is L -like.

Let us focus on this last remark. Under our large cardinal assumptions, the sharp for the minimal mouse with ω Woodin cardinals, called $\mathcal{M}_\omega^\sharp$, exists and has a unique (ω_1, ω_1) -iteration strategy, say Σ^* . It turns out that this strategy is universally Baire; moreover,

$$L(\mathbb{R}, \Sigma^*) = L(\mathbb{R}, \mathbb{R}^\sharp). \tag{2.1}$$

This suggests a connection between sharps for sets of reals and strategies for mice, but we digress; we will explore this connection further in Section 2.3. Here we are interested in the connection between $(\mathcal{M}_\omega^\sharp, \Sigma^*)$ and $\text{HOD}^{L(\mathbb{R})}$.

Let (M, Σ) denote the pair consisting of $\mathcal{M}_\omega^\sharp$ cut at the successor of its least Woodin cardinal, together with the strategy for it obtained from Σ . Let s be a stack of normal trees on (M, Σ) ,⁹ with the last pair (N, T) whose main branch does not drop. We say that (N, T) is a *non-dropping iterate* of (M, Σ) . The pair (N, T) can also be obtained from (M, Σ) via some other stack, say s' . However, the iteration embeddings given by s and s' must be the same.¹⁰ We denote this unique embedding by $i_{(M, \Sigma), (N, T)}$. Instead of starting with (M, Σ) , we could have started with (N, T) , and the similar remarks would hold for its iterates as well. In this way we get a family of non-dropping iterates of (M, Σ) together with embeddings between them: whenever (N, T) and (P, Υ) are two iterates of (M, Σ)

⁸One way to make the closeness precise is to observe that every set in $L(\mathbb{R})$ belongs to HOD_x for some $x \in \mathbb{R}$, while on the other hand x is generic over HOD and it holds that $\text{HOD}_x = \text{HOD}[x]$ (see [6, Theorem 10.1.2]).

⁹By this we mean that s is on M and according to Σ .

¹⁰This follows from the Dodd-Jensen Lemma.

such that (P, Υ) is also an iterate of (N, T) , we have the embedding

$$i_{(N,T),(P,\Upsilon)} : (N, T) \longrightarrow (P, \Upsilon).$$

It turns out that this family is a directed system. Ignoring the strategies for a moment, we denote the direct limit model of the system by $\mathcal{M}_\infty(M, \Sigma)$.

Theorem 2.2.1 (Steel). *The following holds.*

- a. $\mathcal{M}_\infty(M, \Sigma)$ is well-founded.
- b. The Woodin cardinal of M is mapped to $\Theta^{L(\mathbb{R})}$ by the direct limit embedding $i_{(M,\Sigma),\infty}$.
- c. The domain of $\mathcal{M}_\infty(M, \Sigma) \parallel \Theta^{L(\mathbb{R})}$ is $(\text{HOD} \upharpoonright \Theta)^{L(\mathbb{R})}$.

Proof. A proof can be found in [7]. □

So $(\text{HOD} \upharpoonright \Theta)^{L(\mathbb{R})}$ has the structure of a mouse. On the other hand, full $\text{HOD}^{L(\mathbb{R})}$ is not a mouse, but it is closely related to one; let us explain what it is. Since Σ is universally Baire, it extends canonically to an (Ord, Ord) -strategy Σ^+ for M .¹¹ By normalization, there exists a normal tree \mathcal{T} on (M, Σ^+) whose last model is $\mathcal{M}_\infty(M, \Sigma)$. In particular, the tail strategy $\Sigma_{\mathcal{T}}^+$ is an (Ord, Ord) -iteration strategy for the direct limit model $\mathcal{M}_\infty(M, \Sigma)$. We have the following characterization of $\text{HOD}^{L(\mathbb{R})}$.

Theorem 2.2.2 (Woodin). *Let \mathcal{T}^* be the lifting of \mathcal{T} to \mathcal{M}_ω .¹² Then there exists a subset Λ of $\Sigma_{\mathcal{T}}^+$ such that $\text{HOD}^{L(\mathbb{R})} = \mathcal{M}_\infty^{\mathcal{T}^*}[\Lambda]$.¹³*

Proof. A proof can be found in [7]. □

Thus $\text{HOD}^{L(\mathbb{R})}$ differs from a mouse in that it is built from extenders and a partial iteration strategy for its initial segment. Objects of this general kind figure prominently in the arguments of Chapter 4. Let us just note here that $\text{HOD}^{L(\mathbb{R})}$ has fine structure and all its consequences: for example, $\text{HOD}^{L(\mathbb{R})} \models \text{GCH}$.

Because it is a fine structural model close to $L(\mathbb{R})$, it is tempting to call $\text{HOD}^{L(\mathbb{R})}$ “the core model” of $L(\mathbb{R})$, especially below $\Theta^{L(\mathbb{R})}$.¹⁴ However, one usually expects more from a core model. For example, this model usually enjoys some kind of local definability.¹⁵ In Chapter 3, we prove a result in this direction. To state it, for a set X we denote by μ_X the club filter on $[X]^\omega$.

¹¹If $A \subseteq H_{\omega_1}$ is coded by a universally Baire set, then one can define the class $A^+ \subseteq V$ by asserting that for every X , $X \in A^+$ if and only if $V^{\text{Col}(\omega, \text{tc}(X))} \models X \in A_j$. So in the case of Σ , Σ^+ is defined, and it turns out to be an (Ord, Ord) -iteration strategy.

¹² \mathcal{M}_ω is obtained from $\mathcal{M}_\omega^\sharp$ by iterating its top extender out of the universe, i.e. Ord times.

¹³where $\mathcal{M}_\infty^{\mathcal{T}^*}$ is the final model of the tree \mathcal{T}^*

¹⁴See the title of [8].

¹⁵See [9, Theorem 6.15], where this property is called “inductive definability”. See also [17, Lemma 3.5] which contains a local definition of $K \parallel \kappa^{+K}$ from $K \parallel \kappa$; this kind of description will be one of the steps in our proof of Theorem 2.2.3 (see Proposition 3.8.4).

Theorem 2.2.3. *Suppose there are ω Woodin cardinals with a measurable cardinal above them. The following holds in $L(\mathbb{R})$. For all HOD-cardinals η from the interval $[\delta_1^2, \Theta)$, the set $H(\eta^+)^{\text{HOD}}$ is lightface¹⁶ definable over the structure*

$$L_\chi((H_\eta^{\text{HOD}})^\omega)[\mu_\eta],$$

where χ is the second ordinal greater than η for which the above structure satisfies

$$\text{ZF}^- + \text{“}\mathcal{P}([\eta]^\omega) \text{ exists”}.$$

Proof. See Corollary 3.9.13. □

2.3 Where do universally Baire Sets come from?

If there is a proper class of Woodin cardinals, universally Baire sets, when studied under continuous reducibility, form a line.

Theorem 2.3.1. *Suppose there exists a proper class of Woodin cardinals. Then the collection of all universally Baire sets is pre-wellordered by modified Wadge-reducibility¹⁷.*

Proof. This follows from Theorem 2.1.3 and [10, Lemma 2.1 and Theorem 2.5]. □

Higher rank in this pre-well-ordering corresponds to higher logical and combinatorial complexity: for example, $A <_{\mathbb{W}}^* A^\sharp$. In fact, climbing this hierarchy is closely related to climbing the large cardinal hierarchy.¹⁸ Of course, we cannot expect to be able to do this completely “algorithmically”, but if we relax our requirements, a lot can be said about how universally Baire sets “come to be”. For example, Theorem 2.1.9 suggests a first attempt to exhaust universally Baire hierarchy: let $\mathbb{R}^{\sharp 0} := \mathbb{R}$ and let $\mathbb{R}^{\sharp(\alpha+1)} := (\mathbb{R}, \mathbb{R}^{\sharp\alpha})^\sharp$ whenever $\mathbb{R}^{\sharp\alpha}$ is defined. For this procedure to be useful, we need to make sense of limit stages, and we only know how to do that for limit ordinals of countable cofinality.

Let $(A_n : n < \omega)$ be an ω -sequence of universally Baire sets; we want to define their joint, $\bigoplus_{n < \omega} A_n$. For $n < \omega$ and $x \in \mathbb{R}$, let $\langle n, x \rangle \in \mathbb{R}$ be defined by asserting

$$\langle n, x \rangle(i) := \begin{cases} n, & i = 0 \\ x(i-1), & i > 0. \end{cases}$$

Now we can define

$$\bigoplus_{n < \omega} A_n := \{ \langle n, x \rangle : n < \omega, x \in A_n \} \subseteq \mathbb{R}.$$

¹⁶This means that only the parameters that are part of the structure are used. Of course, the additional predicate that is part of the structure is also used.

¹⁷Modified Wadge-reducibility, denoted by $\leq_{\mathbb{W}}^*$, is defined by asserting that $A \leq_{\mathbb{W}}^* B$ iff $A \leq_{\mathbb{W}} B$ or $A \leq_{\mathbb{W}} B^c$.

¹⁸For example, the mouse order on hod pairs refines the Wadge order. See also Conjecture 2.3.2.

This is also a universally Baire set, and its Wadge rank is at least the supremum of the ranks of A_n 's. Thus, for a limit ordinal γ of countable cofinality, if $(\mathbb{R}^{\sharp\alpha} : \alpha < \gamma)$ is defined, we can set

$$\mathbb{R}^{\sharp\gamma} := \bigoplus_{n < \omega} \mathbb{R}^{\sharp\alpha_n},$$

for some cofinal sequence $(\alpha_n : n < \omega)$ in γ . A subtlety here is that $\mathbb{R}^{\sharp\gamma}$ depends on the choice of the sequence $(\alpha_n : n < \omega)$; however, all sets obtained this way are Wadge-equivalent, so the Wadge equivalence class of $\mathbb{R}^{\sharp\gamma}$ is unique.

So what we get is the sequence $(\mathbb{R}^{\sharp\alpha} : \alpha < \omega_1)$. Now there are two possibilities: either this sequence is cofinal among universally Baire sets, or it is not.¹⁹ We exclude the first possibility by strengthening our large cardinal assumption: we assume that there exists a proper class of Woodin cardinals which are limits of Woodin cardinals.²⁰ This means that there are limit stages of uncountable cofinality. This is a problem: we do not know how to join ω_1 many universally Baire sets.²¹

Fortunately, there is a way to proceed: it involves recalling the connection between sharps and strategies for mice alluded to in Eq. (2.1). Here it is more convenient to work with hod mice instead of mice.²² It follows from results of Sargsyan²³ that there exists a universally Baire set that codes a strategy for a hod mouse and whose Wadge rank is at least

$$\sup\{\mathfrak{w}(\mathbb{R}^{\sharp\alpha}) : \alpha < \omega_1\}.$$

Let Σ be such a strategy of the lowest mouse rank;²⁴ this strategy is not unique, but it is unique up to the mouse equivalence. Let $\mathbb{R}^{\sharp\omega_1}$ be the set of reals that codes Σ ; this set depends on the choice of Σ , but its Wadge equivalence class does not.

More generally, whenever γ is a limit ordinal of uncountable cofinality and the sequence $(\mathbb{R}^{\sharp\alpha} : \alpha < \gamma)$ is defined and not Wadge-cofinal in universally Baire sets, we can define $\mathbb{R}^{\sharp\gamma}$ to be the mouse-order-minimal iteration strategy for a hod mouse that is Wadge-above all $\mathbb{R}^{\sharp\alpha}$ ($\alpha < \gamma$), provided there is at least one such strategy. This brings us to the fundamental question of the existence of strategies. The following conjecture is a version of *Hod Pair Capturing* relevant to our discussion.

¹⁹The first possibility occurs, for example, in the minimum model of ZFC+“there is a proper class of Woodin cardinals”.

²⁰The first possibility is then excluded by Theorem 2.3.8 (see also the preceding paragraph). Let us also note that even smaller large cardinals suffice for this exclusion: see, for example, [18, Theorem 30].

²¹Given a sequence of universally Baire sets $(A_\alpha : \alpha < \omega_1)$, one might try to take as their joint the set

$$\{\langle x, y \rangle : y \in \text{WO} \wedge x \in A_{|y|_{\text{WO}}}\}.$$

However, we cannot expect this set to be universally Baire just because the individual sets are, without saying anything about the sequence itself. It may seem that this problem does not arise for the sequence $(\mathbb{R}^{\sharp\alpha} : \alpha < \omega_1)$: after all, we spoke as if it were canonical. However, this sequence is only canonical when considered as a sequence of Wadge degrees: at limit stages, we must choose a cofinal ω -sequence, which introduces randomness in the construction.

²²Because of their relation to HOD's of determinacy models.

²³See [11, Corollary 0.1].

²⁴One can easily reconstruct the underlying mouse from its strategy. Thus we can speak of the mouse order of strategies, i.e. the mouse order of the mouse pairs implied by strategies.

Conjecture 2.3.2. *Suppose there is a proper class of Woodin cardinals. Then either there is a universally Baire set that codes a strategy for a hod mouse with a long extender on its sequence or sets of reals that code strategies for short-extender hod mice are cofinal among universally Baire sets.*²⁵ \square

Short-extender hod mice are precisely defined in [12]. The theory presented there can be adjusted to make sense of a hod mouse whose extenders are all short except for the top one. This is enough to make the first possibility of the conjecture precise. However, if this possibility occurs, then we do not know how to continue beyond the strategy for such a hod mouse. More precisely, assuming that the conjecture is true, the procedure outlined above produces the sequence

$$(\mathbb{R}^{\sharp\alpha} : \alpha < \mathfrak{F})$$

with the following property: either \mathfrak{F} is a limit ordinal and the sequence is Wadge-cofinal among universally Baire sets or \mathfrak{F} is a successor ordinal and $\mathbb{R}^{\sharp(\mathfrak{F}-1)}$ codes a strategy for a hod mouse whose top extender is long. If the second case occurs, to climb beyond $\mathbb{R}^{\sharp(\mathfrak{F}-1)}$, we must first develop the theory of hod mice with long extenders.

This is a description of a bottom-up construction of universally Baire sets using sharps and mice. We want to know whether these sets can be constructed more simply, in a combinatorial manner.

Let \mathcal{A}_0 be the powerset of reals computed in $L(\mathbb{R})$; \mathcal{A}_0 is a Wadge-initial segment of universally Baire sets. We have already shown two ways to go beyond \mathcal{A}_0 : both \mathbb{R}^{\sharp} and the strategy for $\mathcal{M}_\omega^\sharp$ are universally Baire sets of reals outside \mathcal{A}_0 . There is also a third way, which is most in the spirit of what we want to do now: every set of reals in $L(\mathcal{A}_0^\omega)$ is universally Baire, and not all of them are in \mathcal{A}_0 . This motivates the introduction of the operator

$$\mathcal{O}_1 : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\omega).$$

We can then state the following general fact, which is stated in terms of *closed pointclasses*, i.e., those sets of sets of reals Γ that satisfy $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma) = \Gamma$.

Proposition 2.3.3. *Suppose there exists a proper class of Woodin cardinals and $\Gamma \subsetneq \text{UB}$ is a closed pointclass. Then*

- a. *every set of reals in $\mathcal{O}_1(\Gamma)$ is universally Baire,*
- b. *$\Gamma \subsetneq \mathcal{O}_1(\Gamma)$ if and only if $\text{cof}(\Gamma, \leq_W^*) = \omega$.*

Proof. For the first part, see Proposition 2.3.4. For the second part, first assume that

$$\text{cof}(\Gamma, \leq_W^*) = \omega.$$

Let $(A_n : n < \omega)$ be a Wadge-cofinal sequence in Γ . We have that

$$\bigoplus_{n < \omega} A_n \in \mathcal{O}_1(\Gamma) - \Gamma.$$

On the other hand, suppose $\text{cof}(\Gamma, \leq_W^*) > \omega$. Then $\Gamma^\omega \subseteq L(\Gamma)$, so $\mathcal{O}_1(\Gamma) = \Gamma$. \square

²⁵Cf. [12, Definition 1.7.1].

It turns out that $L(\mathcal{O}_1(\mathcal{I}_0))$ is of the form $L(\mathbb{R}, A)$, for some set A ,²⁶ so that the Wadge-cofinality of its powerset of reals is again countable.²⁷ This motivates us to try to iterate this operator to get more universally Baire sets. In general, for a function

$$\mathcal{O} : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})),$$

we define $\mathcal{O}^\alpha : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R}))$ by recursion on α :

- \mathcal{O}^0 is the identity on $\mathcal{P}(\mathcal{P}(\mathbb{R}))$,
- $\mathcal{O}^{\alpha+1} := \mathcal{O} \circ \mathcal{O}^\alpha$,
- $\mathcal{O}^{<\gamma} : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \bigcup_{\alpha < \gamma} \mathcal{O}^\alpha(\Gamma)$, for a limit γ ,
- $\mathcal{O}^\gamma : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \mathcal{P}(\mathbb{R}) \cap L(\mathcal{O}^{<\gamma}(\Gamma))$, for a limit γ .

We define the limit stage in this way to ensure that \mathcal{O}^γ takes values in closed pointclasses - we are only interested in operators with this property anyway. So the pointclasses $\mathcal{O}_1^\alpha(\mathcal{I}_0)$ will make up an increasing sequence of closed pointclasses of universally Baire sets, up to the point where it stabilizes. This happens before the minimum model of $\text{AD}_{\mathbb{R}}$: the corresponding pointclass $\mathcal{O}_1^\alpha(\mathcal{I}_0)$ is the first closed pointclass Γ that satisfies

$$\text{cof}(\Gamma, \leq_W^*) > \omega.$$

To go beyond this point, a new operator is needed.

The new operator comes from Woodin's unpublished work on the minimum model of $\text{AD}_{\mathbb{R}} + \text{DC}$. The idea behind it is a theorem of Solovay, which states that under $\text{AD}_{\mathbb{R}}$ the club filter on $[X]^\omega$ is an ultrafilter whenever X is a surjective image of the reals. For any set X , let us denote by μ_X the club filter on $[X]^\omega$; we then get the operator

$$\mathcal{O}_2 : \mathcal{P}(\mathcal{P}(\mathbb{R})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : \Gamma \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\omega)[\mu_\Gamma].$$

Proposition 2.3.4. *Suppose there exists a proper class of Woodin cardinals and $\Gamma \subsetneq \text{UB}$ is a closed pointclass. Then $\Gamma \subseteq \mathcal{O}_2(\Gamma) \subsetneq \text{UB}$.*

Proof. By [5, Theorem 3.3.19], there exists a closed pointclass $\Gamma^* \subsetneq \text{UB}$ such that $\Gamma \subsetneq \Gamma^*$. It follows from Proposition 4.3.16 applied to $L(\Gamma^*)$ that $\mathcal{O}_2(\Gamma) \subseteq L(\Gamma^*)$, which is sufficient for the conclusion. \square

Theorem 2.3.5 (Woodin). *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*

²⁶We can simply take $\bigoplus_{n < \omega} A_n$, for a Wadge-cofinal sequence in \mathcal{I}_0 .

²⁷A Wadge-cofinal sequence $(A_n : n < \omega)$ in the powerset of reals of $L(\mathbb{R}, A)$ is given by A^\sharp : to get A_n , take the formulas of (the theory of) A^\sharp in which only the first n indiscernibles appear.

3. for every closed pointclass $\Gamma \subseteq \{B : B <_{\mathbb{W}}^* A\}$, $L(\Gamma) \not\models \text{AD}_{\mathbb{R}} + \text{DC}$.

Then for some α , $A \in \mathcal{O}_2^\alpha(\mathcal{I}_0)$.

Proof. This follows from Proposition 2.3.4 and Corollary 4.3.18. \square

Under a sufficiently strong large-cardinal assumption, the minimum model of $\text{AD}_{\mathbb{R}} + \text{DC}$ exists, and all sets of reals in it are universally Baire.²⁸ It follows that for some α , $L(\mathcal{O}_2^\alpha(\mathcal{I}_0))$ is this model. It is not clear whether further iterations of \mathcal{O}_2 can go beyond this model,²⁹ but we know that they cannot reach the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$.³⁰ To reach this model, a further modification of the operator is required. We have found such an operator and proved that it can reach the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$.

Our operator is defined as follows. First, let

$$\mathcal{O}_3^* : \mathcal{P}(\mathcal{P}(\mathbb{R})) \times \text{Ord} \longrightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})) : (\Gamma, \kappa) \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\omega, \kappa^\omega)[\mu_\Gamma, \mu_\kappa].$$

Then, for $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, we define $\mathcal{O}_3(\Gamma)$ to be $\mathcal{O}_3^*(\Gamma, \kappa)$ for the smallest $\kappa \leq \text{Ord}$ that satisfies either $\Gamma \subsetneq \mathcal{O}_3^*(\Gamma, \kappa)$ or $\kappa = \text{Ord}$. This is what we know about the extent to which \mathcal{O}_3 can produce universally Baire sets.

Theorem 2.3.6. *Let Γ be a closed pointclass such that $L(\Gamma) \models \text{AD}_{\mathbb{R}}$. Suppose that in $L(\Gamma)$, there is no least-branch hod pair with a measurable limit of Woodin cardinals. Then for some α , $\Gamma = \mathcal{O}_3^\alpha(\mathcal{I}_0)$.*

Proof. See Theorem 4.1.1. \square

We stated the theorem in terms of determinacy pointclasses, which makes it applicable in a more general context than just that of universally Baire sets. We return to the universally Baire context by recalling Theorem 2.1.3.

Let us now state three corollaries to this theorem. We have already mentioned that the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ cannot be reached by iterating the operator \mathcal{O}_2 . However, it can be reached by iterating the operator \mathcal{O}_3 : this follows from Theorem 2.3.6 and from the fact that the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ does not contain a hod pair with an inaccessible limit of Woodin cardinals.

Corollary 2.3.7. *Suppose there exists an inner model containing all reals and satisfying $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$. Then for some α , $L(\mathcal{O}_3^\alpha(\mathcal{I}_0))$ is the minimum such model.* \square

²⁸For example, this follows from Theorem 2.3.8 (see also the preceding discussion).

²⁹Let Γ be the powerset of the reals of the minimum inner model containing all reals and satisfying $\text{AD}_{\mathbb{R}} + \text{DC} + \text{“}\mu_{\mathcal{P}(\mathbb{R})}$ is an ultrafilter”}. Then Γ is the \subseteq -minimum pointclass satisfying $\mathcal{O}_2(\Gamma) = \Gamma$. We expect that this pointclass is strictly bigger than the powerset of reals of the minimum model of $\text{AD}_{\mathbb{R}} + \text{DC}$, but this does not seem to be immediate and we could not find a reference in the literature. Let us also mention that we do not know whether the minimum model of $\text{AD}_{\mathbb{R}} + \text{DC} + \text{“}\mu_{\mathcal{P}(\mathbb{R})}$ is an ultrafilter”} is the same as the minimum model of $\text{AD}_{\mathbb{R}} + \text{DC} + \text{“}\omega_1$ is $\mathcal{P}(\mathbb{R})$ -supercompact”}.

³⁰See Footnote 10 of Chapter 4.

It was known from earlier work of Sargsyan that the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ exists, but this is the first “bottom-up” construction of it. To see what is gained by such a construction, consider the so-called Chang-plus model:

$$\text{CM}^+ := L(\text{Ord}^\omega)[(\mu_{\alpha^\omega} : \alpha \in \text{Ord})].$$

Woodin showed that large cardinals imply that the Chang-plus model satisfying $\text{AD}^+ + “\omega_1$ is supercompact” is supercompact. It then follows from work of Ikegami and Trang that this model actually satisfies $\text{AD}_{\mathbb{R}} + \text{DC} + “\omega_1$ is supercompact”.³¹ We are interested here in the following, more precise statement of Woodin’s result.

Theorem 2.3.8 (Woodin). *Suppose there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Then the following holds:*

- a. *there exists a universally Baire set which is Wadge-above every set of reals in the Chang-plus model;*
- b. *for all $X \in \text{CM}^+$, $\mu_X \cap \text{CM}^+$ belongs to the Chang-plus model and is a normal fine ultrafilter there.*³²

□

Using this fact, we can obtain two additional corollaries of our result. The first is that the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ is contained in the Chang-plus model.

Corollary 2.3.9. *Suppose there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Then the minimum model of $\text{AD}_{\mathbb{R}} + \text{cof}(\Theta) = \Theta$ is contained in the Chang-plus model.* □

The point here is that Theorem 2.3.8 also ensures that \mathcal{O}_3 is absolute between CM^+ and V . The flip side of this reality is that the operator \mathcal{O}_3 cannot go beyond the powerset of reals of the Chang-plus model:

$$\mathcal{O}_3(\mathcal{P}(\mathbb{R}) \cap \text{CM}^+) = \mathcal{P}(\mathbb{R}) \cap \text{CM}^+.$$

This leads to another corollary regarding the existence of hod mice with universally Baire strategies.

Corollary 2.3.10. *Suppose there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Then there exists a least-branch hod pair with scope H_{ω_1} whose mouse component has a measurable limit of Woodin cardinals and whose strategy component is coded by a universally Baire set.*

³¹See [13, Theorem 4].

³²See [14, Theorem 1.3].

Proof. Let $\Gamma := \mathcal{P}(\mathbb{R}) \cap \text{CM}^+$. By Theorem 2.3.8, there exists a universally Baire set A such that

$$\Gamma = \{B \subseteq \mathbb{R} : B <_{\mathbb{W}}^* A\}.$$

By [5, Theorem 3.3.19], there exists a closed pointclass Γ^* such that $A \in \Gamma^* \not\subseteq \text{UB}$ and $L(\Gamma^*) \models \text{AD}_{\mathbb{R}}$. Since $\mathcal{O}_3(\Gamma) = \Gamma$, Theorem 2.3.6 (applied to Γ^*) implies that there exists a least-branch hod pair with scope H_{ω_1} in $L(\Gamma^*)$ whose mouse component has a measurable limit of Woodin cardinals. \square

2.4 Generic Absoluteness and Maximality

Theorem 2.1.9 implies, among other things, that truths which are projective in universally Baire sets are generically absolute.

Corollary 2.4.1. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *ϕ is a sentence in the language $\{R_{\in}^2, R_A^1\}$,*
4. *\mathbb{P} is an arbitrary poset.*

Then $(H_{\omega_1}, \in, A) \models \phi$ if and only if $V^{\mathbb{P}} \models ((H_{\omega_1}, \in, A_j) \models \phi)$. \square

We wonder how to go beyond this. For example, we will be able to establish the following.³³

Theorem 2.4.2. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *ϕ is a Σ_1 sentence in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$,³⁴*
4. *\mathbb{P} is an arbitrary poset.*

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$ if and only if $V^{\mathbb{P}} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_j) \models \phi)$.

³³To see that this is stronger, note that every first-order fact over (H_{ω_1}, \in, A) is coded into A^{\sharp} , which is itself universally Baire.

³⁴This means the atomic formulas allowed for building ϕ are of the form $R_{=}^2(x, y)$, $R_{\in}^2(x, y)$, $R_{\text{NS}_{\omega_1}}^1(x)$ or $R_A^1(x)$. These atomic formulas are interpreted as asserting “ $x = y$ ”, “ $x \in y$ ”, “ $x \in \text{NS}_{\omega_1}$ ” and “ $x \in A$ ”, respectively.

Proof. See Corollary 6.6.4. □

In this theorem, universally Baire sets are used as predicates.³⁵ If we assume CH, we can also use them as parameters.³⁶ In this way we arrive at a conjecture which is a strengthening of Woodin's theorem on Generic Σ_1^2 -absoluteness conditioned on CH.

Conjecture 2.4.3. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *CH holds,*
3. *A is a universally Baire set,*
4. *$\phi(v_A)$ is a Σ_1 sentence in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1\}$,*
5. *\mathbb{P} is an arbitrary poset that forces CH.*

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A]$ if and only if $V^{\mathbb{P}} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A_{\dot{g}}])$. □

The proof of Woodin's theorem mentioned above shows that the result is true if the predicate NS_{ω_1} is omitted.

Theorem 2.4.4. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *CH holds,*

³⁵If A is only allowed as a predicate, then the statement “there are x and y in A such that $x \neq y$ ” is not Σ_0 : it is formally equivalent to the Σ_1 formula

$$\exists x, \exists y, R_A^1(x) \wedge R_A^1(y) \wedge \neg R_{=}^2(x, y).$$

One may wonder if the quantification $\exists x \in A$ is bounded. The answer is that this syntax is simply not permitted. Namely, if c is a constant, then $\exists x \in c, \theta$ formally means

$$\exists x, R_c^2(x, c) \wedge \theta;$$

since there is no constant for A in our language, the syntax $\exists x \in A$ does not apply. In other words, if we have some language in which we have a constant symbol c_p and a unary relation R_p^1 for speaking about a set p , then there are two ways to express the idea “ $x \in p$ ”: we can say $R_{\in}^1(x, c_p)$ or we can say $R_p^1(x)$. The first expression can be used to bind a quantifier, whereas the second cannot.

³⁶If CH does not hold, \mathbb{R} cannot be used as a parameter over the domain $H(\omega_2)$ because it does not belong to that domain. On the other hand, if CH holds, \mathbb{R} (and every other set of reals) can be used as a parameter. Using a set A as a parameter means extending the language with a constant symbol and then interpreting it as A ; this by definition requires that A belongs to the domain of interpretation. This is equivalent to maintaining the same language, but considering a formula with an additional free variable v_A for A ; this variable is subsequently evaluated as A .

3. A is a universally Baire set,
4. $\phi(v_A)$ is a Σ_1 sentence in the language of set theory,
5. \mathbb{P} is an arbitrary poset that forces CH.

Then $(H_{\omega_2}, \in) \models \phi[A]$ if and only if $V^{\mathbb{P}} \models ((H_{\omega_2}, \in) \models \phi[A_g])$.

Proof. See Theorem 6.7.9. □

Another way to go beyond the generic absoluteness established in Theorem 2.4.2 is to look at Π_2 sentences instead of Σ_1 . In this case, there is no hope of obtaining generic absoluteness. Namely, $\neg\text{CH}$ can be expressed by a Π_2 sentence over the structure $(H_{\omega_2}, \in, \mathbb{R})$, while both CH and $\neg\text{CH}$ can be forced. In this case, the best one can hope for is generic maximality. For example, consider the following principle.

Definition 2.4.5. The sentence $(\dagger)_{\text{UB}}$ asserts that for every universally Baire set A and every Π_2 sentence ϕ in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$, if

$$\exists \mathbb{P}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_g) \models \phi),$$

then $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi$. □

It can be shown that it is essentially equivalent to the following bounded form of Martin's Maximum⁺⁺ (in particular, it is true under Martin's Maximum⁺⁺).

Definition 2.4.6. The sentence $\text{UB} - \text{BMM}^{++}$ asserts that for every universally Baire set A , every Σ_1 formula $\phi(v_a)$ in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$ and every $a \in H_{\omega_2}$, if

$$\exists \mathbb{P} \text{ ssp}, V^{\mathbb{P}} \models ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_g) \models \phi[a]),$$

then $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi[a]$. □

Theorem 2.4.7 (Asperó, Schindler, Woodin). *Suppose there exists a proper class of Woodin cardinals. Then the following holds.*

- a. $\text{UB} - \text{BMM}^{++}$ implies $(\dagger)_{\text{UB}}$.
- b. If there exists a supercompact cardinal, then $(\dagger)_{\text{UB}}$ implies $\text{UB} - \text{BMM}^{++}$.

Proof. This follows from the well-known facts of [15] and [16]. We spell out exactly how in Corollary 6.2.13. □

The principles $\text{UB} - \text{BMM}^{++}$ and $(\dagger)_{\text{UB}}$ can be weakened by omitting the predicate NS_{ω_1} from their statements. This way we get the principles $\text{UB} - \text{BMM}$ and $(\dagger)_{\text{UB}}^-$, respectively. The implication $\text{UB} - \text{BMM}^{++} \implies (\dagger)_{\text{UB}}$ from Theorem 2.4.7 is verified by citing [16] to get

$$\text{UB} - \text{BMM}^{++} \implies (*)_{\text{UB}}$$

and citing [15] to get

$$(*)_{\text{UB}} \implies (\dagger)_{\text{UB}},$$

where $(*)_{\text{UB}}$ is a natural strengthening of the axiom $(*)$ defined in Definition 6.2.5. To show that $\text{UB} - \text{BMM} \implies (\dagger)_{\text{UB}}^-$, we cannot simply go through $(*)_{\text{UB}}$ as above. However, by going through the proofs of the two implications and eliminating cuts³⁷, we get a version of the proof which can be adjusted to show this new implication.

Theorem 2.4.8. *Suppose there exists a proper class of Woodin cardinals. Then $\text{UB} - \text{BMM}$ implies $(\dagger)_{\text{UB}}^-$.*

Proof. See Corollary 6.4.5. □

From the proof of this theorem, it will be seen that the main step is the verification of the following proposition.

Proposition 2.4.9. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *$\phi(x)$ is a Σ_1 formula in the language $\{R_{\check{c}}^2, c_{\omega_1}, R_A^1\}$,*
4. *there exists a poset that forces “ $(H_{\omega_2}, \in, \omega_1, A_{\dot{H}}) \models \forall x, \phi(x)$ ”.*

Then there exists an ssp poset \mathbb{Q} such that it holds in $V^{\mathbb{Q}}$ that $H_{\omega_2}^V \in H_{\omega_2}$ and

$$(H_{\omega_2}, \in, \omega_1, A_{\dot{H}}) \models \forall x \in H_{\omega_2}^V, \phi(x).$$

Proof. See Proposition 6.4.4. □

To prove this proposition, we write down an infinitary propositional formula Φ and ask whether a model for it can be added by a stationary-set-preserving poset. We then argue that the answer is affirmative; any such poset \mathbb{Q} works as required in the conclusion of Proposition 2.4.9. Note also that it is not entirely straightforward how to find the correct formula Φ , but the necessary ideas exist in [16] (though the authors there develop an ad hoc language instead of using infinitary propositional logic). Be that as it may, once we have our formula, the question to answer becomes quite general: when can a model to an infinitary propositional formula be added by a stationary-set-preserving poset? This leads us to the final line of research pursued in this thesis.

³⁷Assuming the natural translation from informal to formal proofs, applying a lemma within a proof corresponds to a cut; eliminating this cut corresponds to replacing the reference to the lemma by its proof.

2.5 SSP-satisfiability of Infinitary Formulas

Before answering when a model of an infinitary propositional formula can be added by an ssp ³⁸ poset, let us first consider the same problem for an arbitrary poset. The answer to this problem is essentially folklore, once put in the proper context.

Let ϕ and ψ be two (ordinary) propositional formulas. We have a well-defined notion of provability, $\phi \vdash \psi$: this simply asserts that there is a deduction tree in the natural-deduction system, NK, whose conclusion is ψ and whose only non-discharged assumptions are ϕ . In this case, the provability relation is equivalent to the standard notion of semantic consequence, $\phi \models \psi$: the assertion in this case is that every model of ϕ is also a model of ψ .

Now, in the case where ϕ and ψ are infinitary propositional formulas, the relation $\phi \vdash \psi$ can be defined in a relatively straightforward way. The point is that the system NK should be modified to allow certain infinitary rules. For example, there should be an introduction rule for infinite conjunctions, which strengthens the usual (finitary) one. A consequence of this requirement is that deductions should now be allowed to be infinite trees; nevertheless, they are still well-founded trees. Once these adjustments are made, the provability relation has essentially the same definition as before.

What is the semantic counterpart of this provability relation? To understand the answer, note that the assertion $\phi \vdash \psi$ is (1) generically absolute, and (2) implies that every model for ϕ , in every generic extension, is also a model of ψ . Now consider the natural formula ϕ , whose models code a countable cofinal subset of ω_1^V , on the one hand, and the formula $\psi := \perp$, on the other. In $V^{\text{Col}(\omega, \omega_1)}$ there is a model for ϕ , while there is no model for \perp ; so $\phi \not\vdash \perp$. This last fact also holds in V , but the formulas ϕ and \perp have the same set of models, viz., the emptyset. This shows that the relevant set of models to consider for \models is (at least) the set of all models in all generic extensions.

Indeed, we define $\phi \models \psi$ to mean “for all posets \mathbb{P} and for all $\dot{\mu} \in V^{\mathbb{P}}$,

$$\Vdash_{\mathbb{P}} (\dot{\mu} \models \phi \implies \dot{\mu} \models \psi).”$$

In this way, the completeness theorem holds: for all infinitary propositional formulas ϕ and ψ , $\phi \vdash \psi$ if and only if $\phi \models \psi$.

The study of these two relations can be somewhat simplified by taking the formula on the right to be \perp . No generality is lost in this way: $\phi \vdash \psi$ if and only if $\phi \wedge \neg\psi \vdash \perp$, as well as $\phi \models \psi$ if and only if $\phi \wedge \neg\psi \models \perp$ (viz., $\phi \wedge \neg\psi$ has no model in any generic extension). Recalling that we read “ $\phi \not\vdash \perp$ ” as “ ϕ is consistent” and “ $\phi \not\models \perp$ ” as “ ϕ is satisfiable”, the completeness theorem then reads: an infinitary formula is consistent if and only if it is satisfiable.

Since satisfiability means existence of a model in a generic extension, it is natural to call “ssp-satisfiability” the property of existence of a model in a generic extension by a stationary-set-preserving poset. Our motivating question is then when an infinitary propo-

³⁸i.e., “stationary-set-preserving”

sitional formula is ssp-satisfiable. The characterization we are able to give here is not proof-theoretic, but rather game-theoretic. It finds its motivation in the Model Existence Game.

Let ϕ be an infinitary propositional formula in negation normal form. The Model Existence Game associated with ϕ is a two-player game of length ω . Player I plays *questions*, while Player II answers by playing *working parts*. These working parts are finite sets of subformulas of ϕ ; Player II must make sure that they do not contain a pair of a propositional letter and its negation, otherwise she loses. The game starts with Player II choosing an arbitrary working part containing ϕ , and continues with Player I asking questions, and Player II enlarging the working part so that the enlarged working part has an answer to the question.

Player I can ask two kinds of questions. In the first kind, he chooses a conjunction that occurs in the current working part, picks a conjunct in it, and asks Player II to enlarge the working part so that it contains that conjunct (and possibly other things). In the second kind, Player I chooses a disjunction that occurs in the current working part and asks Player II to extend the working part so that it contains at least one of the disjuncts (and possibly other things). The point is that during these enlargements, Player II may be forced to include a pair of a propositional letter and its negation. The intuition behind the game is that Player II pretends to know a model for ϕ , but Player I thinks she is lying and asks questions to catch her in the lie.

This is the Model Existence Game associated with ϕ , and Player II has a winning strategy in it if and only if ϕ is satisfiable. We want to answer the question whether there is a game that characterizes ssp satisfiability in a similar way. The answer is yes, and the game is a variation of the game just described; we call it $\mathcal{D}_{\phi, \kappa}^{\text{ssp}}$. The subscript κ is an inaccessible cardinal, and its presence is due to the fact that Player II has to play a side-condition part in addition to the working part. A *side condition* is a set that is elementary in its transitive closure and satisfies ZFC^- .³⁹ A prototypical example is an elementary submodel of some H_θ . A side-condition part is a finite \in -chain of countable side conditions (with some additional restrictions). The parameter κ indicates, among other things, that all side conditions considered in the game are in V_κ .

A winning strategy for Player II in the Model Existence Game, seen as a poset, can be used to force a complete theory (and hence a model) for ϕ . The point is that forcing with it we get a generic play of Player I in which “all questions” are asked; the union of working parts of this play is a complete theory. Now the game $\mathcal{D}_{\phi, \kappa}^{\text{ssp}}$ is a modification of Model Existence Game in which a winning strategy for Player II, seen as a poset, adds a complete theory containing ϕ and is additionally stationary-set-preserving. The stationary-set preservation is ensured through the side-condition part. We keep Player I’s questions that appear in the Model Existence Game to ensure that the poset adds a model for ϕ , but we have additional questions about the side-condition part that ensure that the poset will be stationary-set-preserving.

Theorem 2.5.1. *Assume there exists a proper class of inaccessible cardinals. Let ϕ be an infinitary propositional formula in negation normal form. Then the following are equivalent.*

³⁹In Chapter 5 we use the term “virtual model” for what is here called a side condition.

a. ϕ is ssp-satisfiable.

b. For some inaccessible cardinal κ , Player II has a winning strategy in $\mathfrak{D}_{\phi, \kappa}^{\text{ssp}}$.

Proof. See Corollary 5.7.8. □

Now our motivating problem of adding a model to a formula by an ssp-poset becomes the problem of describing a winning strategy for Player II in $\mathfrak{D}_{\phi, \kappa}^{\text{ssp}}$. Of course, it is not obvious in general what this strategy is, but sometimes it comes from generic embeddings. This is enough to prove Theorem 2.4.8, for example.

Definition 2.5.2. Suppose ϕ is an infinitary propositional formula and $\kappa > \text{rank}(\phi)$ is inaccessible. Then ϕ is said to be **AS-good** at κ iff ϕ is satisfiable and for all stationary $S \subseteq \omega_1$, it holds in $V^{\text{Col}(\omega, < \kappa)}$ that for all $\mu \models \phi$, there exist elementary $\tau : V \rightarrow W$ and $\hat{\mu} \models \tau(\phi)$ such that $\text{crit}(\tau) = \omega_1^V \in \tau(S)$ and for all subformulas ψ of ϕ , $\hat{\mu}(\tau(\psi)) = \mu(\psi)$. □

Theorem 2.5.3. Suppose κ is an inaccessible cardinal and $\phi \in V_\kappa$ is an infinitary propositional formula which is AS-good at κ . Then ϕ is ssp-satisfiable.

Proof. See Corollary 5.8.7. □

Chapter 3

Local Definability of HOD in $L(\mathbb{R})$

3.1 Introduction

In this chapter, we work in a ZFC universe V with the following large cardinal assumption.

Section Assumptions 3.1.1. We assume that there exists a measurable cardinal with ω Woodins below it. \square

There are two important consequences of this assumption that will play a crucial role here.

Theorem 3.1.2. *The Axiom of Determinacy AD holds in $L(\mathbb{R})$.*

Proof. This is a classical result that came out of the work of Martin, Steel, and Woodin. Its full proof can be found in [19]. \square

The other result, Theorem 3.1.3, concerns the existence and iterability of the mouse $\mathcal{M}_\omega^\sharp$. This theorem is a well-known fact in Inner Model Theory, but we were unable to pinpoint an exact reference. As a courtesy to the reader, we will briefly outline its proof. Regarding its notation, the mouse $\mathcal{M}_\omega^\sharp$ is defined in [20, Definition 7.3]. We will state here its basic properties, but in later sections, we will be mostly concerned by the mouse \mathcal{M}_ω , which is obtained from $\mathcal{M}_\omega^\sharp$ by iterating its top extender out of the universe. At the end of this introduction, we will review the notation concerning mice. We note here that all premice have the degree of soundness, which we denote by $k(-)$. Our convention is that $k(\mathcal{M}_\omega^\sharp) = 0$. This mouse projects to ω and it is sound.

Theorem 3.1.3. *Let λ be the least limit of Woodins. Then $\mathcal{M}_\omega^\sharp$ exists and it has a unique $(\omega_1, \omega_1 + 1)$ -iteration strategy Σ . The strategy $\Sigma \upharpoonright H_{\omega_1}$ is coded by a $< \lambda$ -universally Baire set.*

Proof outline. The existence of $\mathcal{M}_\omega^\sharp$ follows from [20, Theorem 7.2]. To verify the uniqueness, we have to consider Q-structures, which will be reviewed in Section 3.2. Since $\mathcal{M}_\omega^\sharp$

projects to ω , for all normal trees \mathcal{T} on $\mathcal{M}_\omega^\sharp$ of countable limit length and for all cofinal wellfounded branches b through \mathcal{T} , we have that $\mathcal{Q}(\mathcal{T}, b)$ exists. If $\mathcal{T} \hat{\ } b$ is according to some $(\omega_1, \omega_1 + 1)$ -iteration strategy, then $\mathcal{Q}(\mathcal{T}, b)$ is $(\omega_1 + 1)$ -iterable. This means that any such strategy must pick the same branch through \mathcal{T} (cf. [20, Corollary 6.14]). This shows the uniqueness of the restriction of Σ to normal trees. The uniqueness of the full strategy then follows from the normalization (cf. [21]).

To verify that the strategy $\Psi := \Sigma \upharpoonright H_{\omega_1}$ is $< \lambda$ -universally Baire, fix an uncountable cardinal $\kappa < \lambda$. By doing the K^c construction above κ , we reach $\mathcal{M}_\omega^\sharp$ and obtain the strategy for it from the construction. This construction is absolute between V and any generic extension of V by a poset of size $< \kappa$. In the generic extension, we also reach $\mathcal{M}_\omega^\sharp$ and obtain the strategy for it from the construction. Since these strategies come from the realizability into the construction (cf. [20, Theorem 6.6]), it is easily seen that club many hulls are $< \kappa$ -generically correct about Ψ . This implies that Ψ is $< \kappa$ -universally Baire (cf. [3, Lemma 4.1]). \square

We denote by Θ , δ_1^2 , HOD , and HOD_x (for all $x \in \mathbb{R}$) the corresponding objects computed in $L(\mathbb{R})$. It was shown by Steel and Woodin (cf. [7]) that $\text{HOD} \parallel \Theta^1$ is a premouse. They in fact represented this structure as a certain direct limit of mice, which we will describe in Section 3.6. The first paper in this direction was [8], where the author said that “ $\text{HOD}^{L(\mathbb{R})}$ is a core model below Θ ”. One usually expects of a core model more than just the structure of a premouse. For example, one might expect it to be locally definable². Our work can be understood as a step in this direction. We show the following.

Theorem 3.1.4. *Suppose*

1. $\eta \in [\delta_1^2, \Theta)$ is a HOD-cardinal,
2. μ_η is the supercompactness measure³ on $[\eta]^\omega$ in $L(\mathbb{R})$,
3. χ is the second ordinal $> \eta$ satisfying that

$$L_\chi((\text{HOD} \parallel \eta)^\omega)[\mu_\eta] \models \text{ZF}^- + \text{“}\mathcal{P}([\eta]^\omega) \text{ exists”}.$$

Then $\text{HOD} \parallel \eta^{+\text{HOD}}$ is definable over the structure

$$(L_\chi((\text{HOD} \parallel \eta)^\omega)[\mu_\eta], \in, \mu_\eta, (\text{HOD} \parallel \eta)^\omega),$$

with no additional parameters.⁴

Proof. See Corollary 3.9.13. \square

¹Whenever W is a transitive model of ZFC and κ is a cardinal in it, we denote by $W \parallel \kappa$ the model H_κ^W .

²This is elsewhere called “inductively definable”; see for example [9, Theorem 6.15]

³We say *the* supercompactness measure since there exists a unique such measure (cf. Theorem 3.9.2).

⁴For the sake of precision, in the given structure, μ_η is a predicate and we consider its restriction to the domain of the structure, while $(\text{HOD} \parallel \eta)^\omega$ is used as a constant.

In Section 3.2, we review the basics of \mathcal{Q} -structures. Under certain circumstances, these structures determine which branches should be picked by an iteration strategy. They play the crucial role of allowing us to approximate iteration strategies of certain mice inside $L(\mathbb{R})$. These mice will be called super-suitable and they will be our primary focus here. Since $L(\mathbb{R})$ cannot ascertain the super-suitability, we will need to extract some weaker properties from it. One such property is the suitability, introduced in Section 3.3. The super-suitability itself is introduced in Section 3.4, while Section 3.5 analyses one more weakening of it, the so-called short tree iterability. In Section 3.6, we describe the already mentioned result of Steel and Woodin on representing $\text{HOD}\|\Theta$ as a direct limit of mice. This concludes the introductory part of the chapter.

Sections 3.7-3.9 contain the main arguments of the chapter. In Section 3.7, we use known facts on the normalization of stacks of normal trees to see HOD as a normal iterate of super-suitable mice. The normal tree leading from a super-suitable mouse to $\text{HOD}\|\Theta$ is unique and we analyze those trees. In Section 3.8, we use this analysis to give a definition of $\text{HOD}\|\eta^{\text{HOD}}$ over a structure of the form

$$L_\kappa(\mathbb{R}, \mathcal{Q}_\eta),$$

where η is a HOD-cardinal from the interval $[\delta_1^2, \Theta)$, \mathcal{Q}_η is the shortest initial segment of $\text{HOD}\|\Theta$ that knows that η is not Woodin in HOD, and κ is a certain admissible ordinal. Finally, in Section 3.9, we show that \mathcal{Q}_η is definable over a structure of the form

$$L_\chi((\text{HOD}\|\eta)^\omega)[\mu_\eta]$$

and obtain the main result of the chapter, Corollary 3.9.13.

Notation

We review the notation which we will be using. We will try to follow the notation of [12] as closely as possible and note the differences when they arise. We will need to use J -hierarchy above some x , which is defined as follows:

$$\begin{aligned} J_1^E(x) &:= V_\omega \cup \text{trcl}(\{x\}) \\ J_{\alpha+1}^E(x) &:= \text{rud}_E(J_\alpha^E(x) \cup \{J_\alpha^E(x)\}) \\ J_\gamma^E(x) &:= \bigcup_{\xi < \gamma} J_\xi^E(x) \quad (\gamma \text{ limit}). \end{aligned}$$

An x - J -structure M has the form

$$M = (J_\alpha^E(x), \in, E, A, y : y \in \text{trcl}(\{x\})),$$

where A is amenable to $J_\alpha^E(x)$. We denote

$$\hat{o}(M) := \alpha \text{ and } o(M) := \omega\alpha.$$

Following [22], we define the first projectum of M as below.

Definition 3.1.5. Suppose that

1. x is a set,
2. $M = (J_\alpha^E(x), \dots)$ is an x -J-structure.

Then $\rho_1(M)$ is the least $\rho \leq \alpha$ such that there exists $A \in \Sigma_1^M$ satisfying

$$A \cap J_\rho^E(x) \notin M.$$

□

Note that it can (and does) happen, according to our definition, that

$$\rho_1(M) = 1.$$

If $x = 0$, the usual definition would have the projectum be ω , so this is a difference to keep in mind. If $\rho_1(M) > 1$, there are no discrepancies of this kind. Regarding the notions of the standard parameter and soundness, they have their usual description, but w.r.t. the language of x -J-structures. This comment about the language was not necessary when we were defining the projectum: the set A above needs only be boldface definable anyway. However, in the case of the standard parameter and soundness, it is important to know that the elements of $\text{trcl}(\{x\})$ are always allowed to be used as constants.

All premisses are MS-indexed and they have the soundness degree integrated into them. If M is a premiss, then $k(M)$ denotes its soundness degree. For all $(\eta, l) \leq (\hat{o}(M), k(M))$, we denote

$$M|(\eta, l) := (J_\eta^{E^M}, \in, E^M \upharpoonright \eta, E_\eta^M, l),$$

$$M|\eta := M|(\eta, 0),$$

$$M||\eta := (J_\eta^{E^M}, \in, E^M \upharpoonright \eta, \emptyset, 0).$$

An ordinal $\xi \leq o(M)$ is a *strong cutpoint* of M iff for all extenders E on the M -sequence, either $\text{lh}(E) < \xi$ or $\text{crit}(E) > \xi$. In the case that M has a unique Woodin cardinal, that cardinal is denoted by $\delta(M)$. All of this notation relativizes to x -premisses in the obvious way. If we end up talking about $r\Sigma_{\omega+1}$, the reader should understand this as simply talking about $r\Sigma_\omega$.

3.2 Q-structures

In this section, we review basic facts about Q-structures. For an introduction on this, the reader is invited to consult [20]. Our definitions here are based on [23, Definition 3.3].

Definition 3.2.1. Suppose that

1. M is a premiss,

2. $\delta \leq o(M)$,
3. $Q \trianglelefteq M$.

Then Q is a **Q-structure of M at δ** iff both a and b hold, where:

- a. either $\delta = o(Q)$ or $Q \models \text{“}\delta \text{ is Woodin”}$;
- b. one of the conditions i, ii, or iii is met, where:
 - i. $\hat{o}(Q) < \hat{o}(M)$, $k(Q) = \omega$, and δ is not Woodin in $J_1(M)$ ⁵;
 - ii. $\hat{o}(Q) = \hat{o}(M)$, $\rho(Q) = \delta$, and there exists an $r\Sigma_{k(M)+1}$ subset of δ witnessing that δ is not Woodin in $J_1(M)$;
 - iii. $\hat{o}(Q) = \hat{o}(M)$ and $\rho(Q) < \delta$. □

Notation 3.2.2. In the setup of the previous definition, there there exists at most one Q which is a Q-structure for M at δ . If such Q does exist, we call it *the* Q-structure for M at δ and we denote it by $\mathcal{Q}(M, \delta)$. To say that such Q exists, we use the shorthand of saying “ $\mathcal{Q}(M, \delta)$ exists”. □

In an iteration tree of limit length which is not too complicated, cofinal branches come with a naturally assigned Q-structures. These structures can then be used to pick the right branch through that tree.

Definition 3.2.3. Suppose that

1. M is a premouse,
2. \mathcal{T} is a normal tree on M of limit length,
3. b is a cofinal well-founded branch of \mathcal{T} .

Then $\mathcal{Q}(\mathcal{T}, b)$ exists iff $\mathcal{Q}(\mathcal{M}_b^{\mathcal{T}}, \delta(\mathcal{T}))$ exists. In that case, we denote by $\mathcal{Q}(\mathcal{T}, b)$ the structure $\mathcal{Q}(\mathcal{M}_b^{\mathcal{T}}, \delta(\mathcal{T}))$. □

The Q-structures corresponding to two different cofinal branches are mutually in comparable, unless they are of a certain particular type. We isolate this exception in the following definition.

Definition 3.2.4. Suppose that

1. M is a premouse,
2. \mathcal{T} is a normal tree on M of limit length,

⁵Note that the universe of $J_1(M)$ is the same as the universe of $M \parallel (\hat{o}(M) + 1)$.

3. b is a cofinal well-founded branch of \mathcal{T} .

Then (M, \mathcal{T}, b) is an **anomaly** iff all of the following conditions are simultaneously met:

- a. M is not sound,
- b. b does not drop,
- c. $\mathcal{Q}(\mathcal{T}, b)$ exists and is equal to $\mathcal{M}_b^{\mathcal{T}}$. □

Proposition 3.2.5. *Suppose that*

- 1. M is a premouse,
- 2. \mathcal{T} is a normal tree of limit length on M ,
- 3. $b \neq c$ are cofinal well-founded branches through \mathcal{T} ,
- 4. (M, \mathcal{T}, b) and (M, \mathcal{T}, c) are not anomalies.

Then neither $\mathcal{Q}(\mathcal{T}, b) \trianglelefteq \mathcal{Q}(\mathcal{T}, c)$ nor $\mathcal{Q}(\mathcal{T}, c) \trianglelefteq \mathcal{Q}(\mathcal{T}, b)$.

Proof. See [20, Theorem 6.12]. □

If \mathcal{T} is a normal tree on M of limit length and if the next branch to be picked has the \mathcal{Q} -structure, we might try to guess that structure before actually knowing the branch. We now work towards introducing this structure and finally succeed in doing so in Definition 3.2.9.

Definition 3.2.6. Suppose that M is a premouse. Then M is **countably iterable** iff for all countable premice \bar{M} and all elementary $i : \bar{M} \rightarrow M$, it holds that \bar{M} is $(\omega_1 + 1)$ -iterable. □

Lemma 3.2.7. *Suppose that P is a premouse. Then there exists at most one premouse Q satisfying that*

- a. $P \trianglelefteq Q$,
- b. $o(P)$ is a strong cutpoint in Q ,
- c. Q is the \mathcal{Q} -structure of Q at $o(P)$,⁶
- d. Q is sound above $o(P)$,
- e. Q is countably iterable.

⁶Or in other words, $Q = \mathcal{Q}(Q, o(P))$.

Proof. Let us assume otherwise and let $Q_0 \neq Q_1$ be two witnesses. Since we can always take a countable hull of some H_θ , for a large enough θ , we may assume w.l.o.g. that P , Q_0 , and Q_2 are countable. By the argument of [20, Corollary 3.12], we have that either $Q_0 \trianglelefteq Q_1$ or $Q_1 \trianglelefteq Q_0$. However, the minimality which is a part of the definition of a Q-structure would then imply $Q_0 = Q_1$, which is a contradiction. \square

Definition 3.2.8. Suppose that P is a premouse. The Q-structure above P , denoted by $\mathcal{Q}(P)$, is the unique premouse Q satisfying:

- a. $P \trianglelefteq Q$,
- b. $o(P)$ is a strong cutpoint in Q ,
- c. Q is the Q-structure of Q at $o(P)$,
- d. Q is sound above $o(P)$,
- e. Q is countably iterable. \square

Definition 3.2.9. Suppose that

1. M is a premouse,
2. \mathcal{T} is a normal tree on M of limit length.

Then $\mathcal{Q}(\mathcal{T})$ exists iff $\mathcal{Q}(\mathcal{M}(\mathcal{T}))$ exists. In that case, we define

$$\mathcal{Q}(\mathcal{T}) := \mathcal{Q}(\mathcal{M}(\mathcal{T})).$$

\square

If mice do not have extenders overlapping local Woodins, they are said to be tame. Tame mice are simple enough so that Q-structures can be used to identify the right branches through trees on them, i.e. the branches that must be picked by any sufficiently strong strategy for those mice.

Definition 3.2.10. Suppose that M is a premouse. Then M is tame iff for all $\eta < \hat{o}(M)$, if $E_\eta^M \neq \emptyset$, then for all $\delta \in [\text{crit}(E), \eta)$, $M \parallel \eta \models \text{“}\delta \text{ is not Woodin”}$. \square

Proposition 3.2.11. Suppose that

1. M is a tame premouse,
2. \mathcal{T} is a normal tree on M of limit length.

Then there exists at most one cofinal wellfounded branch b through \mathcal{T} such that

- a. (M, \mathcal{T}, b) is not an anomaly,
- b. $\mathcal{Q}(\mathcal{T}, b)$ exists,
- c. $\mathcal{Q}(\mathcal{T}, b)$ is countably iterable.

Proof. Let us assume otherwise and let $b_0 \neq b_1$ be two such branches. Since M is tame, $\delta(\mathcal{T})$ is a strong cutpoint in both $\mathcal{Q}(\mathcal{T}, b_0)$ and $\mathcal{Q}(\mathcal{T}, b_1)$. By the uniqueness of \mathcal{Q} -structures (cf. Lemma 3.2.7), we have that $\mathcal{Q}(\mathcal{T}, b_0) = \mathcal{Q}(\mathcal{T}, b_1)$, which contradicts Proposition 3.2.5. \square

Proposition 3.2.12. *Suppose that*

- 1. M is a tame premouse,
- 2. Σ_M is an $(\omega_1 + 1)$ -iteration strategy for M ,
- 3. \mathcal{T} is a normal tree on M of limit length according to Σ_M ,
- 4. $b := \Sigma_M(\mathcal{T})$,
- 5. $\mathcal{Q}(\mathcal{T}, b)$ exists.

Then $\mathcal{Q}(\mathcal{T})$ exists and is equal to $\mathcal{Q}(\mathcal{T}, b)$.

Proof. The \mathcal{Q} -structure $\mathcal{Q}(\mathcal{T}, b)$ is countably iterable since $\mathcal{M}_b^{\mathcal{T}}$ is countably iterable. The ordinal $\delta(\mathcal{T})$ is a strong cutpoint in $\mathcal{Q}(\mathcal{T}, b)$ because M is tame. This suffices for the conclusion. \square

The ordinal $(\delta_1^2)^{L(\mathbb{R})}$ will figure prominently in the present work. There are many different characterizations of this ordinal, but we choose the one most useful for our purposes (cf. [22, Lemma 1.12]). Since we will not compute this ordinal in any other model except $L(\mathbb{R})$, we omit the superscript.

Definition 3.2.13. The ordinal δ_1^2 is the least ordinal δ satisfying that Σ_1 -formulas with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$ are absolute between $L_\delta(\mathbb{R})$ and $L(\mathbb{R})$. \square

This reflection implies that all countable mice that have ω_1 -iteration strategies in $L(\mathbb{R})$, have such strategies in $L_{\delta_1^2}(\mathbb{R})$. The following proposition is an example of how this fact can be used.

Proposition 3.2.14. *Suppose that*

- 1. M is a countable ω -small^{*l*} premouse,
- 2. for all $\delta < o(M)$, if $M \models \text{“}\delta \text{ is Woodin”}$, then $\mathcal{Q}(M, \delta)$ exists,

⁷See [20, Definition 7.1]

3. M is $(\omega_1 + 1)$ -iterable.

Then $L(\mathbb{R}) \models$ “ M is $(\omega_1 + 1)$ -iterable”. Moreover, there exists a set of reals in $L_{\delta_1^2}(\mathbb{R})$ which canonically codes an ω_1 -iteration strategy for M .

Proof. Since M is $(\omega_1 + 1)$ -iterable, we have that it is weakly ω -iterable⁸. By the proof of [20, Theorem 7.10], we have that M has an $(\omega_1 + 1)$ -iteration strategy in $L(\mathbb{R})$. The moreover part follows from Definition 3.2.13. \square

3.3 Suitable Premice

We are really interested in super-suitable premice. They are the appropriate initial segments of iterates of \mathcal{M}_ω and they are going to be introduced in Section 3.4. However, super-suitable premice cannot be defined internally in $L(\mathbb{R})$, so we need to work with an approximate notion, that of a suitable premouse. Of course, it will be a theorem that super-suitable premice are suitable.

Definition 3.3.1. Suppose that

1. a is countable and transitive,
2. M is an a -premouse.

Then M is **Lp-good** iff all of the following conditions are met:

- a. $k(M) = \omega$,
- b. $\rho_\omega(M) = 1^9$,
- c. M has an ω_1 -iteration strategy in $L(\mathbb{R})$. \square

Lp-good a -premouse extend each other and there is no longest one among them. We will be interested in their supremum.

Lemma 3.3.2. *Suppose that*

1. a is countable and transitive,
2. M, N are Lp-good.

Then either $M \triangleleft N$ or $N \triangleleft M$.

⁸See [20, Definition 7.7]

⁹See Definition 3.1.5 and the comment after it.

Proof. This is a straightforward generalization of [20, Corollary 3.12]. □

Lemma 3.3.3. *Suppose that a is countable and transitive. Then for all Lp-good a -premise M , there exists an Lp-good a -premouse N such that $M \triangleleft N$.*

Proof. Since M projects to 1 and is sound, there exists a surjection $f : J_1(a) \twoheadrightarrow M$ which is definable over M . Now, if look at $J_1(M)$, organized as an a -premouse, we see that it is obtained as the rudimentary closer of $M \cup \{M\}$. However, the rudimentary functions can be listed recursively, so the facts that $V_\omega \cup \{f\} \subseteq J_1(a)$ allow us to define a surjection

$$g : J_1(a) \twoheadrightarrow J_1(M)$$

over $J_1(M)$. This means that $J_1(M)$ projects to 1 and is consequently Lp-good. □

Supremum of all Lp-good a -mice is denoted by $\text{Lp}(a)$. This object is itself an a -mouse and it looks like the power set of a . Since sets appearing in $\text{Lp}(a)$ are not too complicated, i.e. they come from mice that have strategies in $L(\mathbb{R})$, this object can be understood as a *lower part* of the full powerset of a (hence the abbreviation Lp).

Definition 3.3.4. Suppose that a is countable and transitive. Then $\text{Lp}(a)$ is the unique a -premouse satisfying:

- a. for all Lp-good a -premise M , we have that $M \trianglelefteq \text{Lp}(a)$;
- b. no proper initial segment of $\text{Lp}(a)$ satisfies the previous close. □

Lemma 3.3.5. *Suppose that a is countable and transitive. Then the following holds:*

- a. $k(\text{Lp}(a)) = 0$,
- b. $\text{Lp}(a)$ is countable.

Proof.

- 1° If it were the case that $k(\text{Lp}(a)) > 0$, then the a -premouse obtained from $\text{Lp}(a)$ by decreasing $k(\text{Lp}(a))$ by 1 would also satisfy condition a of Definition 3.3.4, while being a strict initial segment of $\text{Lp}(a)$. This shows that a must hold.
- 2° To establish b, let us assume otherwise. Then the set C of all $\alpha \in (\text{rank}(a), \omega_1)$ such that there exists an Lp-good a -premouse M satisfying $\hat{\delta}(M) = \alpha$ is cofinal in ω_1 .
- 3° For all $\alpha \in C$, an M witnessing this fact is unique and we denote it by M_α .
- 4° Let $(f_\alpha : \alpha \in C), g$ be as follows:

- 1. for all $\alpha \in C$, $f_\alpha : a \cup \{a\} \twoheadrightarrow M_\alpha$ is given by the soundness,

2. $g : \omega \rightarrow a \cup \{a\}$ is an arbitrary enumeration,
 3. for all $\alpha \in C$, $e_\alpha := \{(m, n) : f_\alpha(g(m)) \in f_\alpha(g(n))\} \subseteq \omega^2$.
- 5° The sequence $(e_\alpha : \alpha \in C)$ is injective and it belongs to $L(\mathbb{R})$. This contradicts $L(\mathbb{R}) \models \text{AD}$.

□

We said that $\text{Lp}(a)$ should be understood as a lower part of the powerset of a . It turns out that if a is countable, transitive, and self-wellorderable¹⁰, it is in fact the case that

$$\text{Lp}(a) = H(|a|^+)^{\mathcal{M}_\omega(a)} = H(|a|^+)^{\text{HOD}^{L(\mathbb{R})}(a \cup \{a\})}.$$

This is the content of [7, Theorem 6.4], which we reproduce here.

Theorem 3.3.6. *Suppose that*

1. a is countable and transitive,
2. $b \subseteq a$.

Then the following are equivalent.

- a. b is definable over $(L(\mathbb{R}), \in)$ from parameters in $\text{Ord} \cup a \cup \{a\}$.
- b. $b \in \text{Lp}(a)$.
- c. $b \in \mathcal{M}_\omega(a)$.

□

Corollary 3.3.7. *Suppose that a is countable, transitive, and self-wellorderable. Then it holds that*

- a. $\text{Lp}(a) = H(|a|^+)^{\mathcal{M}_\omega}$,
- b. $\text{Lp}(a) \models \text{ZFC}^-$,
- c. $\text{Lp}(a)$ is ω -sound.

□

We are ready to introduce the notion of a suitable premouse. Here and later, we will need reorganize premice into premice over their initial segments (when this is possible).

Notation 3.3.8. Suppose that M is a premouse and δ is a strong cutpoint of M . Then we denote by M/δ the canonical reorganization of M into an $(M \parallel \delta)$ -premouse. □

¹⁰A transitive set a is self-wellorderable iff $J_1(a) \models$ “there exists a wellordering on a ”.

Definition 3.3.9. Suppose that M is a premouse. Then M is **suitable** iff there exists $\delta < \omega_1$ such that

- a. $M \models \text{ZFC}^- + \text{“}\delta \text{ is the largest cardinal”} + \text{“}\delta \text{ is Woodin”}$,
- b. δ is a strong cutpoint of M ,
- c. for all $\eta \leq \delta$, $\text{Lp}(M \parallel \eta) \subseteq M$,
- d. for all $\eta \leq \delta$, if η is a strong cutpoint of M , then $\text{Lp}(M \parallel \eta) = (M \parallel \eta^{+M})/\eta$,
- e. for all $\eta < \delta$, $\text{Lp}(M \parallel \eta) \models \text{“}\eta \text{ is not Woodin”}$. □

We observe that the part of the definition is for δ to be countable. This has as an immediate consequence the fact that all suitable premisses are countable.

Lemma 3.3.10. *Suppose that M is suitable. Then M is countable.*

Proof. This follows from Lemma 3.3.5. and the fact that there exists $\delta < \omega_1$ such that $M/\delta = \text{Lp}(M \parallel \delta)$. □

3.4 Super-suitable Premices

A super-suitable premouse is obtained from an iterate of \mathcal{M}_ω by cutting it at the successor of its least Woodin. We make precise in next few definitions. Here and later, we shall introduce several variations on the notion of an iterate. What we call here simply “an iterate” is elsewhere called (more clumsily) “a nondropping iterate”. Since we will not have the need to talk about dropping iterates, we omit this additional qualifier. Similarly, since we will not have the need to talk about uncountable iterates, we omit the qualifier “countable” as well and incorporate the countability in the definitions.

Definition 3.4.1. Suppose that

- 1. M, N are countable premisses,
- 2. Σ_M is an $(\omega_1 + 1)$ -iteration strategy for M .

Then N is a **normal Σ_M -iterate** of M iff there exists a countable normal tree \mathcal{T} on M according to Σ_M whose last model is N and whose main branch does not drop. □

Definition 3.4.2. Suppose that

- 1. M, N are countable premisses,
- 2. Σ_M is an $(\omega_1, \omega_1 + 1)$ -iteration strategy for M .

Then N is a Σ_M -iterate of M iff there exists a countable stack of countable normal trees on M according to Σ_M whose last model is N and whose main branch does not drop. \square

Definition 3.4.3. Suppose that

1. $\tau := \delta(\mathcal{M}_\omega)^{+\mathcal{M}_\omega}$,
2. $\mathcal{M} := \mathcal{M}_\omega \parallel \tau$,
3. $\Sigma_{\mathcal{M}}$ is the iteration strategy for \mathcal{M} obtained from the unique $(\omega_1, \omega_1 + 1)$ -iteration strategy for \mathcal{M}_ω ,
4. M, N are countable premice.

Then we define the following.

- a. M is **super-suitable** iff M is a $\Sigma_{\mathcal{M}}$ -iterate of \mathcal{M} .
- b. If M is super-suitable, then Σ_M is the $(\omega_1, \omega_1 + 1)$ -iteration strategy for M induced by $\Sigma_{\mathcal{M}}$. \square

Since the supremum of Woodin cardinals of \mathcal{M}_ω is countable and since all iterates are countable by our choice of the definition, we have that all super-suitable mice are countable. Furthermore, since all super-suitable mice are elementarily equivalent to the mouse \mathcal{M} of the previous definition, they are all tame. We highlight this in the following lemma.

Lemma 3.4.4. *All super-suitable premice are countable and tame.* \square

When we consider super-suitable mice, we will only consider them together with their canonical strategies. Thus, we can simplify the terminology of iterates a bit.

Notation 3.4.5. Suppose that M is super-suitable and that N is a countable premouse. Then we define the following.

- a. N is a **normal iterate** of M iff N is a normal Σ_M -iterate of M .
- b. N is an **iterate** of M iff N is a Σ_M -iterate of M . \square

As we have already pointed it out, the suitability is an approximation to the super-suitability, so the following proposition is to be expected.

Proposition 3.4.6. *Suppose that M is a super-suitable premouse. Then M is suitable.*

Proof. See the paragraph immediately below Definition 6.8 of [7]. \square

If M is super-suitable, then M is in particular tame. This has for a consequence that $\delta(\mathcal{T})$ is a strong cutpoint in $\mathcal{Q}(\mathcal{T}, b)$ whenever \mathcal{T} is a normal tree on M , b a cofinal well-founded branch through \mathcal{T} , and $\mathcal{Q}(\mathcal{T}, b)$ exists. In particular, the $\mathcal{M}(\mathcal{T})$ -premouse

$$\mathcal{Q}(\mathcal{T}, b)/\delta(\mathcal{T})$$

is defined. If $\mathcal{Q}(\mathcal{T}, b)$ is iterable, then the canonical strategy Σ_M must pick the branch b for the tree \mathcal{T} , which is the content of the following proposition.

Proposition 3.4.7. *Suppose that*

1. M is a super-suitable premouse,
2. \mathcal{T} is a countable normal tree on M of limit length according to Σ_M ,
3. b is a cofinal wellfounded branch through \mathcal{T} ,
4. $\mathcal{Q}(\mathcal{T}, b)$ exists and is $(\omega_1 + 1)$ -iterable.

Then $\mathcal{Q}(\mathcal{T}, b)/\delta(\mathcal{T}) \trianglelefteq \text{Lp}(\mathcal{M}(\mathcal{T}))$ and $\Sigma_M(\mathcal{T}) = b$.

Proof.

- 1° By Proposition 3.2.14, we have that $\mathcal{Q}(\mathcal{T}, b)$ has an $(\omega_1 + 1)$ -iteration strategy in $L(\mathbb{R})$. In particular, $\mathcal{Q}(\mathcal{T}, b)$ is Lp-good and consequently,

$$\mathcal{Q}(\mathcal{T}, b)/\delta(\mathcal{T}) \trianglelefteq \text{Lp}(\mathcal{M}(\mathcal{T})).$$

- 2° Let us now verify that $\Sigma_M(\mathcal{T}) = b$. We assume towards contradiction that the branch $c := \Sigma_M(\mathcal{T})$ is distinct from b .

- 3° The Q-structure $\mathcal{Q}(\mathcal{T}, c)$ does not exist, for otherwise it would be $(\omega_1 + 1)$ -iterable, while there can be only one branch with an iterable Q-structure. Hence, we get that c does not drop and $\mathcal{M}_c^{\mathcal{T}} \models \text{“}\delta(\mathcal{T}) \text{ is Woodin”}$.

- 4° Since c does not drop, we have that $\mathcal{M}_c^{\mathcal{T}}$ is super-suitable.

- 5° Putting together 3° and 4°, we get that $\mathcal{M}_c^{\mathcal{T}}$ is suitable and that $\delta(\mathcal{T})$ is its unique Woodin. This means that

$$\mathcal{M}_c^{\mathcal{T}}/\delta(\mathcal{T}) = \text{Lp}(\mathcal{M}_c^{\mathcal{T}} \parallel \delta(\mathcal{T})) = \text{Lp}(\mathcal{M}(\mathcal{T})).$$

- 6° Adding to this the conclusion of 1°, we have that

$$\mathcal{Q}(\mathcal{T}, b) \trianglelefteq \text{Lp}(\mathcal{M}(\mathcal{T})) = \mathcal{M}_c^{\mathcal{T}}.$$

- 7° Since $\delta(\mathcal{T})$ is Woodin in $\mathcal{M}_c^{\mathcal{T}}$, it follows that $\mathcal{Q}(\mathcal{T}, b) = \mathcal{M}_c^{\mathcal{T}}$.

- 8° Now, this would mean that $\mathcal{Q}(\mathcal{T}, c)$ does exist (and is equal to $\mathcal{Q}(\mathcal{T}, b)$), which is in contradiction with 3°.

□

3.5 Short tree Iterability

The model $L(\mathbb{R})$ does not contain an ω_1 -iteration strategy for \mathcal{M}_ω . The reason for this is that otherwise one could easily construct, using the genericity iterations, \mathbb{R}^\sharp inside $L(\mathbb{R})$. A consequence of this fact is that $L(\mathbb{R})$ cannot check if a given countable normal tree on the suitable initial segment of \mathcal{M}_ω is according to the strategy for \mathcal{M}_ω . This also means that one cannot expect to be able to verify inside $L(\mathbb{R})$ whether a premouse is super-suitable and for this reason, we introduced the notion of suitability. Now, given a super-suitable premouse M , we can verify inside $L(\mathbb{R})$ that it is suitable, but we cannot know the strategy Σ_M . What we do have is a partial strategy, one which knows how to continue the tree as long as the tree is *short* and breaks down once the tree stops being short (such trees are called *maximal*). This partial strategy is called *the short tree strategy* and denoted by Σ_M^{st} .

Definition 3.5.1. Suppose that

1. M is suitable,
2. \mathcal{T} is a countable normal tree on M of limit length.

The predicates “ \mathcal{T} is short”, “ \mathcal{T} is maximal”, “ $\mathcal{T} \in \text{dom}(\Sigma_M^{\text{st}})$ ” and the value $\Sigma_M^{\text{st}}(\mathcal{T})$ are defined by recursion on $\text{lh}(\mathcal{T})$, as follows. If \mathcal{T} is according to Σ_M^{st} , then

- a. \mathcal{T} is short iff $\mathcal{Q}(\mathcal{T})$ exists,
- b. \mathcal{T} is maximal iff it is not short,
- c. $\mathcal{T} \in \text{dom}(\Sigma_M^{\text{st}})$ iff \mathcal{T} is short and there exists a cofinal wellfounded branch b through \mathcal{T} such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$,
- d. if $\mathcal{T} \in \text{dom}(\Sigma_M^{\text{st}})$, the $\Sigma_M^{\text{st}}(\mathcal{T})$ is the unique cofinal wellfounded branch b through \mathcal{T} satisfying $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. \square

To explain the previous definition, at successor stages, the tree is built by picking and applying an extender, so the strategy is concerned by limit stages only. If the tree is built according to the short tree strategy and if we reach a limit stage, several things can occur. One possibility is that the tree is maximal, in which case the strategy breaks down. In particular, no extension of such a tree will be according to the short tree strategy. The other possibility is that the tree is short. In this case, we would like to say that $\Sigma_M^{\text{st}}(\mathcal{T})$ is defined, but the fact that $\mathcal{Q}(\mathcal{T})$ exists does not guarantee that there is a branch whose \mathcal{Q} -structure is exactly $\mathcal{Q}(\mathcal{T})$. This last scenario is a pathology and we would like to exclude it. Assuming that this pathology does not occur, we then know that the branch b whose \mathcal{Q} -structure is equal to $\mathcal{Q}(\mathcal{T})$ is unique, so we can set $\Sigma_M^{\text{st}}(\mathcal{T}) := b$. The pathology just mentioned is dealt with by introducing the notion of short tree iterability. This notion also excludes another pathology, that of the possibility that an ultrapower in the tree may produce an illfounded model, and it additionally guarantees that the iterates of M stay suitable (which is sometimes called *being fullness preserving*).

Definition 3.5.2. Suppose that M is suitable. Then M is **short tree iterable** iff for all countable trees \mathcal{T} on M according to Σ_M^{st} , we have that

- a. if \mathcal{T} has a last model N , then
 - i. \mathcal{T} can be normally extended by any extender on the N -sequence without producing an illfounded model,
 - ii. if the branch M -to- N of \mathcal{T} does not drop, then N is suitable,
- b. if \mathcal{T} is short, then $\mathcal{T} \in \text{dom}(\Sigma_M^{\text{st}})$,
- c. if \mathcal{T} is maximal, then there exists a nondropping cofinal wellfounded branch b through \mathcal{T} such that $\mathcal{M}_b^{\mathcal{T}}$ is suitable. □

We have already seen that a super-suitable premouse is suitable. In this sense, the suitability is an approximation to the super-suitability. We want to see next that Σ_M^{st} is an approximation to Σ_M whenever M is super-suitable.

Proposition 3.5.3. *Suppose that M is super-suitable. Then $\Sigma_M^{\text{st}} \subseteq \Sigma_M$.*

Proof.

1° By induction on $\text{lh}(\mathcal{T})$, we show that for all $\mathcal{T} \in \text{dom}(\Sigma_M^{\text{st}})$, if

$$\forall \eta < \text{lh}(\mathcal{T}) \text{ limit, } \mathcal{T} \upharpoonright \eta \in \text{dom}(\Sigma_M),$$

then $\mathcal{T} \in \text{dom}(\Sigma_M)$ and $\Sigma_M(\mathcal{T}) = \Sigma_M^{\text{st}}(\mathcal{T})$.

2° Since \mathcal{T} is according to Σ_M , we immediately have that $\mathcal{T} \in \text{dom}(\Sigma_M)$.

3° Let us denote by b the branch $\Sigma_M^{\text{st}}(\mathcal{T})$. By the definition of Σ_M^{st} , we have that $\mathcal{Q}(\mathcal{T})$ exists and satisfies $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. Thus, $\mathcal{Q}(\mathcal{T}, b) = \mathcal{Q}(\mathcal{T})$ exists and is $(\omega_1 + 1)$ -iterable.

4° By Proposition 3.4.7, we must have that $\Sigma_M(\mathcal{T}) = b$, as required. □

Super-suitable mice are not only suitable, but also iterable. Since we are interested in $L(\mathbb{R})$ -approximations, we should verify the short tree iterability of these mice.

Proposition 3.5.4. *Suppose that M is super-suitable. Then M is short tree iterable.*

Proof.

1° Let \mathcal{T} be a countable normal tree on M according to Σ_M^{st} . By Proposition 3.5.3, \mathcal{T} is also according to Σ_M .

2° What needs to be verified depends on the type of the tree \mathcal{T} , as can be seen from Definition 3.5.2. We have three cases.

3° **Case.** \mathcal{T} is of successor length.

Proof. First, we need to verify that when \mathcal{T} is extended by one extender, we do not get an illfounded model. However, this is obvious since \mathcal{T} is according to Σ_M and Σ_M is an $(\omega_1, \omega_1 + 1)$ -iteration strategy for M . The second thing to verify is that if the main branch of \mathcal{T} does not drop, then the last model of \mathcal{T} is suitable. For this, observe that if the main branch of \mathcal{T} does not drop, then the last model of \mathcal{T} is an iterate of a super-suitable premouse M , so it must itself be super-suitable. \square

4° **Case.** \mathcal{T} is short.

Proof.

- 1' Being short means that $\mathcal{Q}(\mathcal{T})$ exists, so we only need to verify that there exists a cofinal wellfounded branch b through \mathcal{T} such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ works. We claim that $b := \Sigma_M(\mathcal{T})$ works.
- 2' Let us assume towards contradiction that b does not drop. Then $\mathcal{M}_b^{\mathcal{T}}$ is suitable and $\delta(\mathcal{T})$ is its Woodin.
- 3' By definition, the premouse $\mathcal{Q}(\mathcal{T})$ is $(\omega_1 + 1)$ -iterable and $\delta(\mathcal{T})$ is its strong cutpoint. This means that $\mathcal{Q}(\mathcal{T})/\delta(\mathcal{T})$ is Lp-good and consequently,

$$\mathcal{Q}(\mathcal{T})/\delta(\mathcal{T}) \trianglelefteq \text{Lp}(\mathcal{M}(\mathcal{T})).$$

4' However, we have by suitability that

$$\mathcal{M}_b^{\mathcal{T}}/\delta(\mathcal{T}) = \text{Lp}(\mathcal{M}_b^{\mathcal{T}} \parallel \delta(\mathcal{T})) = \text{Lp}(\mathcal{M}(\mathcal{T})),$$

so we conclude that $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

- 5' If it were the case that $o(\mathcal{Q}(\mathcal{T})) < o(\mathcal{M}_b^{\mathcal{T}})$, it would hold that $\delta(\mathcal{T})$ is not Woodin in $\mathcal{M}_b^{\mathcal{T}}$. This being absurd, we conclude that the J-structures underlying premice $\mathcal{Q}(\mathcal{T})$ and $\mathcal{M}_b^{\mathcal{T}}$ are equal.
- 6' However, this is a contradiction since $\rho(\mathcal{Q}(\mathcal{T})) \leq \delta(\mathcal{T})$ while

$$\rho_\omega(\mathcal{M}_b^{\mathcal{T}}) = \hat{\delta}(\mathcal{M}_b^{\mathcal{T}}).$$

7' The conclusion is then that b drops, which means that $\mathcal{Q}(\mathcal{T}, b)$ exists.

8' By the uniqueness of the Q-structure above $\mathcal{M}(\mathcal{T})$ (cf. Proposition 3.2.7), it follows that $\mathcal{Q}(\mathcal{T}, b) = \mathcal{Q}(\mathcal{T})$ and consequently, $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, as required. \square

5° **Case.** \mathcal{T} is maximal.

Proof.

- 1' In this case, we need to verify that there exists a nondropping cofinal well-founded branch b through \mathcal{T} such that $\mathcal{M}_b^{\mathcal{T}}$ is suitable. We claim that $b := \Sigma_M(\mathcal{T})$ works.
- 2' If b dropped, then $\mathcal{Q}(\mathcal{T}, b)$ would exist and be $(\omega_1 + 1)$ -iterable. In other words, $\mathcal{Q}(\mathcal{T})$ would exist and be equal to $\mathcal{Q}(\mathcal{T}, b)$, which is not the case.
- 3' Thus, b does not drop.
- 4' It follows that $\mathcal{M}_b^{\mathcal{T}}$ is an iterate of a super-suitable premouse M , as witnessed by $\mathcal{T} \cap b$, so $\mathcal{M}_b^{\mathcal{T}}$ is super-suitable.
- 5' In particular, $\mathcal{M}_b^{\mathcal{T}}$ is suitable, as required. □

6° The three cases above cover all possibilities, so the verification is concluded. □

Hence, the right $L(\mathbb{R})$ -approximations for super-suitable mice are suitable, short tree iterable premice. The notion of a normal iterate for them is not in general defined, so we introduce a more general notion, which we shall call a *normal pseudo-iterate*.

Definition 3.5.5. The relation $M \xrightarrow{\mathcal{T}}^{\text{st}} N$ holds iff

- a. M is suitable and short tree iterable,
- b. \mathcal{T} is a tree on M according to Σ_M^{st} ,
- c. if \mathcal{T} has a last model, then N is the last model of \mathcal{T} and the M -to- N branch of \mathcal{T} does not drop,
- d. if \mathcal{T} does not have a last model, then \mathcal{T} is maximal and there exists a nondropping cofinal wellfounded branch b through \mathcal{T} such that $\mathcal{M}_b^{\mathcal{T}}$ is suitable and $\mathcal{M}_b^{\mathcal{T}} = N$. □

This relation is absolute between $L(\mathbb{R})$ and V . In the part c, one has the behavior that one would expect from a normal iterate. It is the part d that is unusual inasmuch that we do not know what is the branch leading to the final model. Let us look closer to this case.

Lemma 3.5.6. *Suppose that $M \xrightarrow{\mathcal{T}}^{\text{st}} N$. Then N is suitable and if \mathcal{T} is maximal, then $\delta(\mathcal{T})$ is the Woodin of N and $N \parallel \delta(\mathcal{T}) = \mathcal{M}(\mathcal{T})$.*

Proof. This is immediate from the definition since it subsumes the short tree iterability of M . □

What could happen is that there could be many branches that are “good enough”, so we do not force ourselves to choose, but we know that they all lead to the same model. In other words, by Lemma 3.5.6, we have that

$$N/\delta(\mathcal{T}) = \text{Lp}(\mathcal{M}(\mathcal{T})),$$

which means that N depends only on \mathcal{T} , but not on b . That there are indeed such branches follows from the definition of the short tree iterability, but (assuming that M is super-suitable) we do not know which one of them is chosen by Σ_M . We observe just below that in the case that M is super-suitable, the relation

$$M \longrightarrow_{\mathcal{T}}^{\text{st}} N$$

simply means that N is a normal iterate of M via \mathcal{T} .

Lemma 3.5.7. *Suppose that*

1. M is super-suitable,
2. N is a countable premouse,
3. \mathcal{T} is a countable normal tree on M ,
4. for all limit $\gamma < \text{lh}(\mathcal{T})$, $\mathcal{Q}(\mathcal{T} \upharpoonright \gamma)$ exists.

Then $M \longrightarrow_{\mathcal{T}}^{\text{st}} N$ holds if and only if the following holds:

- a. \mathcal{T} is according to Σ_M ,
- b. if \mathcal{T} has a last model, then $\mathcal{T} \upharpoonright (\text{lh}(\mathcal{T}) - 1)$ is not maximal¹¹, N is the last model of \mathcal{T} , and the M -to- N branch of \mathcal{T} does not drop,
- c. if \mathcal{T} does not have a last model, then for $b := \Sigma_M(\mathcal{T})$, we have that b does not drop and $N = \mathcal{M}_b^{\mathcal{T}}$.

Proof.

Implication (\Rightarrow)

- 1° The part a follows from Proposition 3.5.3, while the part b follows from the part a.
- 2° We want to verify the part c. In this case, the tree \mathcal{T} is maximal.
- 3° If b dropped, then $\mathcal{Q}(\mathcal{T}, b)$ would exist and be $(\omega_1 + 1)$ -iterable, which would mean that $\mathcal{Q}(\mathcal{T})$ exists and is equal to $\mathcal{Q}(\mathcal{T}, b)$. This being contradictory, we conclude that b does not drop.

¹¹We do not say “short” here because we $\mathcal{T} \upharpoonright (\text{lh}(\mathcal{T}) - 1)$ might not be of limit length.

4° It follows that $\mathcal{M}_b^{\mathcal{T}}$ is super-suitable and that $\delta(\mathcal{T})$ is its Woodin. In particular,

$$\mathcal{M}_b^{\mathcal{T}}/\delta(\mathcal{T}) = \text{Lp}(\mathcal{M}(\mathcal{T})) = N/\delta(\mathcal{T}),$$

where the second equality follows from Lemma 3.5.6.

5° This suffices for the conclusion that $N = \mathcal{M}_b^{\mathcal{T}}$.

Implication (\Leftarrow)

6° The assumptions yield that if \mathcal{T} has a last model, then $\mathcal{T} \upharpoonright (\text{lh}(\mathcal{T}) - 1)$ is not maximal.

7° We also know that there cannot exist ξ such that $\xi + 1 < \text{lh}(\mathcal{T})$ and $\mathcal{T} \upharpoonright \xi$ is maximal. The reason for this is that $\delta(\mathcal{T}) = \delta(\mathcal{M}_\xi^{\mathcal{T}})$ is a strong cutpoint of $\mathcal{M}_\xi^{\mathcal{T}}$, so any cofinal branch through \mathcal{T} must contain ξ and must drop at the successor of ξ .

8° These two fact together with Proposition 3.5.3 imply that \mathcal{T} is according to Σ_M^{st} . The rest is now a routine verification. □

Definition 3.5.8. Suppose that M is suitable and N is a countable premouse. Then N is a **normal pseudo-iterate** of M iff there exists \mathcal{T} such that $M \rightarrow_{\mathcal{T}}^{\text{st}} N$. □

Proposition 3.5.9. *Suppose that*

1. M is super-suitable,
2. N is a countable premouse.

Then the following are equivalent.

- a. N is a normal pseudo-iterate of M .
- b. N is a normal iterate of M .

Proof. This follows from Lemma 3.5.7. □

It turns out that the short tree iterability suffices for the comparison.

Proposition 3.5.10. *Suppose that M, N are suitable and short tree iterable. Then there exists P which is a normal pseudo-iterate of both M and N .*

Proof.

1° Let T be the tree of a $(\Sigma_1^2)^{L(\mathbb{R})}$ -scale on a universal $(\Sigma_1^2)^{L(\mathbb{R})}$ -set and let

$$\mu := \max\{\delta(M), \delta(N)\}^{+L[T, M, N]}.$$

We want to show that $\mu < \omega_1$, for once we do this, the conclusion follows from [7, Theorem 6.14].

2° Let x be a real coding M and N . It suffices to show that $\omega_1^{L[T, x]} < \omega_1$.

3° The theorem at the bottom of the page 77 of [8] implies that

$$L[T] \parallel \delta_1^2 = \text{HOD} \parallel \delta_1^2,$$

while [6, Corollary 10.3.7] implies that $\text{HOD}_x = \text{HOD}[x]$.

4° We can now compute that

$$L[T, x] \parallel \delta_1^2 = (L[T] \parallel \delta_1^2)[x] = (\text{HOD} \parallel \delta_1^2)[x] = \text{HOD}[x] \parallel \delta_1^2 = \text{HOD}_x \parallel \delta_1^2.$$

5° By [7, Lemma 6.34], ω_1 is the least measurable of HOD. Exactly the same arguments yields the same conclusion for HOD_x .

6° Thus, it follows that

$$\omega_1^{\text{HOD}_x} < (\text{“the first measurable”})^{\text{HOD}_x} = \omega_1,$$

as required. □

3.6 HOD as a Direct Limit of Mice

In this section, we describe $\text{HOD} \parallel \Theta$ as the direct limit of a certain directed family of mice. This is just a slight reformulation of the well-known results of [7].

Definition 3.6.1. Suppose that M, N are super-suitable. Then we define the following.

- a. $\mathcal{I}(M)$ is the set of all normal iterates of M .
- b. $M \leq_{\mathcal{I}} N$ iff N is a normal iterate of M . □

We note that if the main branch of a normal tree on a super-suitable mouse does not drop, then the tree is based below the Woodin. The reason why extenders above the Woodin cannot be used is that once such an extender is used, all later extenders need to be above the Woodin, while any such extender necessarily leads to a drop. Instead of restricting ourselves to normal iterates, we could have looked at all iterates (via stacks of normal trees). However, by results of [21], stacks of trees can be normalized, so nothing would be gained from this extension and we are still in the scenario of [20, Section 8] and [7, Section 6]. In particular, the structure $(\mathcal{I}(M), \leq_{\mathcal{I}})$ is a countably directed partial order.

Lemma 3.6.2. *Suppose that M is super-suitable. Then $(\mathcal{I}(M), \leq_{\mathcal{I}})$ is a countably directed partial order.* \square

This partial order will index our directed family. To every pair (M, N) satisfying $M \leq_{\mathcal{I}} N$, we need to assign an arrow $M \rightarrow N$, which is provided by the next lemma.

Lemma 3.6.3. *Suppose that*

1. M is super-suitable,
2. N is a normal iterate of M .

Then there exists a unique normal tree \mathcal{T} on M according to Σ_M with the last model N . This tree is countable and its main branch does not drop.

Proof. Such a tree \mathcal{T} exists by the fact that N is a normal iterate of M . The uniqueness follows from the fact that any such tree must be the first coordinate of the coiteration of (M, Σ_M) and (N, Σ_N) . \square

Definition 3.6.4. Suppose that

1. M is super-suitable,
2. N is a normal iterate of M .

Then the mapping $\pi_{M,N}$ is defined to be the mapping

$$\pi^{\mathcal{T}} : M \longrightarrow N$$

where \mathcal{T} is the normal tree on M according to Σ_M whose last model is N . \square

We are ready to introduce the directed system.

Definition 3.6.5. Suppose that M is super-suitable. Then we define $\vec{\mathcal{I}}(M)$ to be the system

$$(P, \pi_{P,Q} : P, Q \in \mathcal{I}(M), P \leq_{\mathcal{I}} Q).$$

\square

Proposition 3.6.6. *Suppose that M is super-suitable. Then $\vec{\mathcal{I}}(M)$ is a directed system with a wellfounded direct limit.*

Proof. The commutativity follows from the normalization (cf. [21, Theorem 1.1]), while wellfoundedness follows from the fact that $(\mathcal{I}(M), \leq_{\mathcal{I}})$ is countably directed. \square

By the results of Section 3.5, and in particular Proposition 3.5.10, if M and N are two super-suitable mice, then there exists a super-suitable P which is a normal iterate of both. We get that

$$\vec{\mathcal{I}}(M) \upharpoonright P = \vec{\mathcal{I}}(N) \upharpoonright P = \vec{\mathcal{I}}(P).$$

This leads to the following consequence.

Proposition 3.6.7. *Suppose that M, N are super-suitable. Then the direct limits of the systems $\vec{\mathcal{I}}(M)$ and $\vec{\mathcal{I}}(N)$ are equal and for all $P \in \mathcal{I}(M) \cap \mathcal{I}(N)$,*

$$\pi_{P,\infty}^{\vec{\mathcal{I}}(M)} = \pi_{P,\infty}^{\vec{\mathcal{I}}(N)}.$$

□

Definition 3.6.8. Let us denote $\mathcal{M} := \mathcal{M}_\omega \parallel \delta(\mathcal{M}_\omega)^{+\mathcal{M}_\omega}$. We define objects $\mathcal{M}_\infty, \pi_{P,\infty}$ (for all $P \in \mathcal{I}(\mathcal{M})$) as follows:

$$(\mathcal{M}_\infty, \pi_{P,\infty} : P \in \mathcal{I}(\mathcal{M}))$$

is the direct limit of the system $\vec{\mathcal{I}}(\mathcal{M})$.

□

We can now state the characterization of $\text{HOD} \parallel \Theta$ in the style of [7] that we shall use.

Theorem 3.6.9. *It holds that $\delta(\mathcal{M}_\infty) = \Theta$ and $\mathcal{M}_\infty \parallel \Theta = \text{HOD} \parallel \Theta$.*

□

3.7 HOD as a Normal Iterate

In order to get from a super-suitable mouse to the mouse $\text{HOD} \parallel \Theta$, we did the direct limit construction. In that limit construct, one moves from one mouse to another via a normal tree, so we see that, after collapsing everything relevant to be countable, we get a stack of normal tree leading from a super-suitable mouse to \mathcal{M}_∞ . This stack again can be normalized, leading to a single normal tree on the super-suitable mouse, ending with \mathcal{M}_∞ . This tree will not be countable, but it will be according to the canonical extension of the canonical strategy for the super-suitable mouse. We will dissect this normal tree in order to come up with a desired local definition of $\text{HOD} \parallel \Theta$.

Proposition 3.7.1. *Suppose that M is super-suitable. Then there exists a unique normal tree \mathcal{T} on M according¹² to Σ_M whose last model is \mathcal{M}_∞ . Moreover, the main branch of \mathcal{T} does not drop, $\pi^\mathcal{T} = \pi_{M,\infty}$, and $\text{lh}(\mathcal{T}) = \Theta + 1$.*

Proof. The existence of \mathcal{T} follows from [21, Corollary 1.2]. The uniqueness follows from the fact that, when comparing M to \mathcal{M}_∞ using Σ_M , all disagreements are on the M side and the comparison tree is exactly \mathcal{T} . This characterization also implies that $\text{lh}(\mathcal{T}) \leq \Theta + 1$, while the regularity¹³ of Θ in $L(\mathbb{R})$ implies the equality. The rest of the proposition follows from [21, Theorem 1.1].

□

¹²A priori, $\text{lh}(\mathcal{T}) \leq \mathfrak{c}^+ + 1$, so $\Gamma := \Sigma_M$ needs to be extended. This is done in the standard way: for $\mathcal{U} \subseteq H(\mathfrak{c}^+)$, pick any g which is V -generic for $\text{Col}(\omega, \mathcal{P}(\mathbb{R}))$ and define $\Gamma(\mathcal{U}) := \Gamma_g(\mathcal{U})$.

¹³Cf. [24, Lemma 2.19]

Notation 3.7.2. Suppose that M is super-suitable. Then \mathcal{T}_M denotes the unique normal tree on M according to Σ_M with the last model \mathcal{M}_∞ and b_M denotes the main branch of \mathcal{T}_M . \square

For initial segments of \mathcal{T}_M , the Q-structures exist, so $L(\mathbb{R})$ knows how to pick the right branches of those trees. This means that the proper initial segments of \mathcal{T}_M are in $L(\mathbb{R})$. On the other hand, once we reach $\mathcal{T}_M \upharpoonright \Theta$, we reach the Woodin cardinal and we need the strategy to tell us which branch to pick, the strategy which is not in $L(\mathbb{R})$. In fact, the final branch is not in $L(\mathbb{R})$, for otherwise we would have the embedding

$$\pi^{\mathcal{T}_M} : M \longrightarrow \mathcal{M}_\infty$$

which is continuous at the Woodin and whose image would thus witness that

$$\text{cof}^{L(\mathbb{R})}(\Theta) = \omega$$

(whereas Θ is regular in $L(\mathbb{R})$).

Proposition 3.7.3. *Suppose that M is super-suitable. Then every proper initial segment of \mathcal{T}_M belongs to $L(\mathbb{R})$.*

Proof.

- 1° We will show that extenders and branches of \mathcal{T}_M , except possibly the last branch, are picked definably over $L(\mathbb{R})$, which suffices for the conclusion.
- 2° For all ξ satisfying $\xi + 1 < \text{lh}(\mathcal{T}_M)$, the extender $E_\xi^{\mathcal{T}_M}$ is exactly the first extender on the $\mathcal{M}_\xi^{\mathcal{T}_M}$ -sequence which is not on the $\text{HOD} \parallel \Theta$ -sequence, hence chosen definably.
- 3° Let ξ be a limit ordinal satisfying $\xi + 1 < \text{lh}(\mathcal{T}_M)$. We want to show that the branch $[0, \xi)_{\mathcal{T}_M}$ is picked definably over $L(\mathbb{R})$.
- 4° **Claim.** $\mathcal{Q}(\mathcal{T}_M \upharpoonright \xi, [0, \xi)_{\mathcal{T}_M})$ exists and it is an initial segment of $\text{HOD} \parallel \Theta$.

Proof.

- 1' Let $\delta := \delta(\mathcal{T}_M \upharpoonright \xi) = \bigcup_{i < \xi} \text{lh}(E_i^{\mathcal{T}_M})$. Since $\xi + 1 < \text{lh}(\mathcal{T}_M)$, we must have that $\delta < \Theta$.
- 2' If $[0, \xi)_{\mathcal{T}_M}$ drops, then $\mathcal{Q}(\mathcal{T}_M \upharpoonright \xi, [0, \xi)_{\mathcal{T}_M})$ exists.
- 3' If $[0, \xi)_{\mathcal{T}_M}$ does not drop, then $\delta < \delta(\mathcal{M}_\xi^{\mathcal{T}_M})$, for otherwise $\mathcal{T}_M \upharpoonright \xi$ could not have a strict normal extension whose main branch does not drop. In particular, $\mathcal{Q}(\mathcal{T}_M \upharpoonright \xi, [0, \xi)_{\mathcal{T}_M})$ exists in this case as well.
- 4' Let $Q := \mathcal{Q}(\mathcal{T}_M \upharpoonright \xi, [0, \xi)_{\mathcal{T}_M}) \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}_M}$. We want to show that $Q \trianglelefteq \text{HOD} \parallel \Theta$.
- 5' Since \mathcal{T}_M is normal, $\text{lh}(E_\xi^{\mathcal{T}_M}) \geq \delta$.

- 6' Recall that for all $i < \xi$, $\text{lh}(E_i^{\mathcal{T}_M})$ is a cardinal of $\mathcal{M}_{i+1}^{\mathcal{T}_M}$, which means that δ is a limit cardinal of $\mathcal{M}_\xi^{\mathcal{T}_M}$.
- 7' Consequently, there are no extenders in $\mathcal{M}_\xi^{\mathcal{T}_M}$ indexed at δ and it must be the case that $\text{lh}(E_\xi^{\mathcal{T}_M}) > \delta$.
- 8' Thus, $\mathcal{P}(\delta) \cap \mathcal{M}_\xi^{\mathcal{T}_M} = \mathcal{P}(\delta) \cap \text{HOD}$.
- 9' Since δ is not Woodin¹⁴ in HOD, there exists $\zeta < \delta^{+\text{HOD}}$ such that

$$\text{HOD} \parallel (\zeta + 1) \models \text{“}\delta \text{ is not Woodin”}.$$

- 10' Note now that $\text{HOD} \parallel (\zeta + 1) \triangleleft \mathcal{M}_\xi^{\mathcal{T}_M}$, so $Q = \text{HOD} \parallel \zeta$.

□

- 5° For all cofinal wellfounded branches b through $\mathcal{T}_M \upharpoonright \xi$, $(M, \mathcal{T}_M \upharpoonright \xi, b)$ is not an anomaly. This simply follows from the fact that M is sound.
- 6° If b is a cofinal wellfounded branch through $\mathcal{T} \upharpoonright \xi$ different from $[0, \xi)_{\mathcal{T}_M}$ which has a Q-structure, then that Q-structure is not equal to

$$\mathcal{Q}(\mathcal{T}_M \upharpoonright \xi, [0, \xi)_{\mathcal{T}_M}).$$

In particular, $\mathcal{Q}(\mathcal{T}_M \upharpoonright \xi, b) \not\equiv \text{HOD} \parallel \Theta$.

- 7° We conclude that $[0, \xi)_{\mathcal{T}_M}$ is exactly the unique cofinal wellfounded branch through $\mathcal{T}_M \upharpoonright \xi$ which has a Q-structure which is an initial segment of $\text{HOD} \parallel \Theta$. This shows that $[0, \xi)_{\mathcal{T}_M}$ is picked definably over $L(\mathbb{R})$.

□

We now start working towards establishing local definability of HOD. We will work at some fixed HOD-cardinal η .

Section Assumptions 3.7.4. We fix a HOD-cardinal η satisfying that $\delta_1^2 \leq \eta < \Theta$. □

We want to show that $\text{HOD} \parallel \eta^{+\text{HOD}}$ is definable over some local structure at η . We would like to say that it is definable over $H(\eta^+)^{L(\mathbb{R})}$, but this is not good enough since Choice fails. We propose two alternatives, the first one being the structure \mathcal{A}_η defined just below and the second one being \mathcal{B}_η defined in Notation 3.9.5. The structure \mathcal{B}_η is arguably more canonical, but establishing the definability over it is considerably harder.

Notation 3.7.5. We fix the following notation.

- a. \mathcal{H}_η is the premouse $\text{HOD} \parallel \eta$,

¹⁴This is because $\delta < \Theta$, the models HOD and \mathcal{M}_∞ agree below Θ , and Θ is the only Woodin of \mathcal{M}_∞ .

- b. \mathcal{Q}_η is the Q-structure $\mathcal{Q}(\text{HOD} \parallel \Theta, \eta)$,
- c. κ_η is the least $\kappa > o(\mathcal{Q}_\eta)$ such that $L_\kappa(\mathbb{R}, \mathcal{Q}_\eta) \models \text{ZF}^-$,
- d. \mathcal{A}_η is the structure $(L_{\kappa_\eta}(\mathbb{R}, \mathcal{Q}_\eta), \in, \mathbb{R}, \mathcal{Q}_\eta)$, where \mathbb{R} and \mathcal{Q}_η are treated as constants. □

Lemma 3.7.6. *The ordinal κ_η is well defined and is strictly less than Θ .*

Proof.

- 1° We work in $L(\mathbb{R})$ and we denote by θ the cardinal Θ^{+10} .
- 2° Let us first verify that $L_\theta(\mathbb{R})$ satisfies ZF^- . What is nontrivial is to see that this model satisfies Collection, so we concentrate on that.
- 3° Let $x \in L_\theta(\mathbb{R})$ and $R \subseteq L_\theta(\mathbb{R})^2$ be arbitrary. We want to find $y \in L_\theta(\mathbb{R})$ such that for all $u \in x \cap \text{dom}(R)$, $R[u] \cap y \neq \emptyset$.
- 4° For all $u \in x$, let $g(u)$ be the least $\eta < \theta$ such that $R[u] \cap L_\eta(\mathbb{R}) \neq \emptyset$. We have defined a function $g : x \rightarrow \theta$.
- 5° **Claim.** g is not cofinal.

Proof.

- 1' Let us assume otherwise.
- 2' There exist $\xi < \theta$ and a surjection $f : \xi \times \mathbb{R} \rightarrow x$.
- 3' Let $h : \omega \rightarrow \mathbb{R}$ code a generic for $\text{Col}(\omega, \mathbb{R})$ and let us work in $L(\mathbb{R})[h] = L[h]$. We have that θ remains a successor.
- 4' Since Choice holds, we get that θ regular.
- 5' The mapping $g \circ f \circ (\text{id}_\xi \times h) : \xi \times \omega \rightarrow \theta$ is cofinal. This contradicts the regularity of θ . □

- 6° Let $\zeta := \sup(\text{ran}(g)) < \theta$ and let $y := L_\zeta(\mathbb{R})$. It is immediate that y is as required in 2°.
- 7° We have established that $L_\theta(\mathbb{R}) \models \text{ZF}^-$. Since \mathcal{Q}_η is coded by a set of reals and since $\theta > \Theta$, we get that

$$L_\theta(\mathbb{R}, \mathcal{Q}_\eta) = L_\theta(\mathbb{R}) \models \text{ZF}^-.$$

- 8° Since there exists a surjection $\theta \times \mathbb{R} \rightarrow L_\theta(\mathbb{R})$, there exists $X \prec L_\theta(\mathbb{R}, \mathcal{Q}_\eta)$ satisfying that $\mathbb{R} \cup \mathcal{Q}_\eta \cup \{\mathbb{R}, \mathcal{Q}_\eta\} \subseteq X$ and satisfying that there exists a surjection $\mathbb{R} \rightarrow X$.

9° The transitive collapse of X is of the form $L_\kappa(\mathbb{R}, \mathcal{Q}_\eta)$ with $\kappa < \Theta$. This suffices for the conclusion. □

Showing that $\text{HOD}\|\eta^{+\text{HOD}}$ is definable over \mathcal{A}_η amounts to showing that $\mathcal{P}(\eta) \cap \text{HOD}$ is a subset of and definable over \mathcal{A}_η . We now work towards giving a characterization of this set which we will be able to approximate in \mathcal{A}_η .

Definition 3.7.7. Suppose that M is super-suitable. Then we define the following.

- a. γ_M is the least i such that $\mathcal{M}_i^{\mathcal{T}_M}\|\eta = \text{HOD}\|\eta$.
- b. \mathcal{U}_M is the tree $\mathcal{T}_M \upharpoonright (\gamma_M + 1)$.
- c. \mathcal{P}_M is the premouse $\mathcal{M}_{\gamma_M}^{\mathcal{T}_M}$. □

Lemma 3.7.8. Suppose that M is super-suitable. Then it holds that

- a. $\text{lh}(\mathcal{U}_M) \leq \eta + 1$,
- b. $\mathcal{U}_M \in L(\mathbb{R})$,
- c. $|\mathcal{P}_M|^{L(\mathbb{R})} \leq |\eta|^{L(\mathbb{R})} < \Theta$.

Proof. Part a follows from the fact that \mathcal{U}_M the tree arising from comparing (M, Σ_M) to $\text{HOD}\|\eta$, part b follows from Proposition 3.7.3, while part c follows from the previous two. □

Definition 3.7.9. Suppose that M is super-suitable. Then

- a. M is η -exact iff $\eta \in \text{ran}(\pi_{M, \infty})$,
- b. if M is η -exact, then η_M denotes the preimage of η by $\pi_{M, \infty}$. □

The characterization of $\mathcal{P}(\eta) \cap \text{HOD}$ that we announced is given below. Namely, this powerset is the same as the powerset $\mathcal{P}(\eta) \cap \mathcal{P}_M$ where M is any η -exact super-suitable mouse.

Proposition 3.7.10. Suppose that M is super-suitable and η -exact. Then it holds that

- a. $\gamma_M \in b_M$,
- b. $\text{crit}(\pi_{\gamma_M, \text{lh}(\mathcal{T}_M)-1}^{\mathcal{T}_M}) > \eta$,
- c. $\eta \in \text{ran}(\pi_{0, \gamma_M}^{\mathcal{T}_M})$,

$$d. \mathcal{P}(\eta) \cap \mathcal{P}_M = \mathcal{P}(\eta) \cap \text{HOD}.$$

Proof.

1° Let us assume towards contradiction that a fails. Then there exist $\xi, \zeta \in b_M$ such that $\gamma_M \in (\xi, \zeta)$ and ξ is a \mathcal{T}_M -predecessor of ζ .

2° Note that $\zeta = \bar{\zeta} + 1$ for some $\bar{\zeta} \geq \gamma_M$ and

$$\mathcal{M}_{\bar{\zeta}}^{\mathcal{T}_M} = \text{Ult}(\mathcal{M}_{\xi}^{\mathcal{T}_M}, E_{\bar{\zeta}}^{\mathcal{T}_M}).$$

3° We have that

$$\text{crit}(E_{\bar{\zeta}}^{\mathcal{T}_M}) < \text{lh}(E_{\xi}^{\mathcal{T}_M}) \leq \eta \leq \text{lh}(E_{\bar{\zeta}}^{\mathcal{T}_M}),$$

where the first inequality follows from the fact that \mathcal{T}_M is normal, the second one follows by definition of γ_M , and the third one follows from the fact $\mathcal{M}_{\bar{\zeta}}^{\mathcal{T}_M} \parallel \eta = \mathcal{M}_{\gamma_M}^{\mathcal{T}_M} \parallel \eta = \mathcal{M}_{\infty} \parallel \eta$.

4° The previous two points then imply that $\eta \notin \text{ran}(\pi_{\xi, \zeta}^{\mathcal{T}_M})$, which contradicts the fact that $\eta \in \text{ran}(\pi^{\mathcal{T}_M})$.

5° Let us now verify part b. Let E be the extender used at γ_M along b_M . We have that $\text{lh}(E) \geq \eta$.

6° The case $\text{crit}(E) \leq \eta$ would imply $\eta \notin \text{ran}(\pi_{\gamma_M, \text{lh}(\mathcal{T}_M)-1}^{\mathcal{T}_M})$, which is not possible since $\eta \in \text{ran}(\pi^{\mathcal{T}_M})$.

7° Thus, $\text{crit}(\pi_{\gamma_M, \text{lh}(\mathcal{T}_M)-1}^{\mathcal{T}_M}) = \text{crit}(E) > \eta$, which establishes part b.

8° Parts c and d now easily follow.

□

3.8 Local Definition of HOD I

We will now establish the definability of $\text{HOD} \parallel \eta^{+\text{HOD}}$ over \mathcal{A}_η . This is done by approximating the characterization of Proposition 3.7.10 inside \mathcal{A}_η . The first step is to expand the notions of the previous section to a larger domain of premisses, in order to make them definable inside the structure that we are interested in.

Definition 3.8.1. Suppose that M is a countable premouse. Then we define the following.

- a. The tree \mathcal{U}_M on M is the unique normal tree \mathcal{U} on M of the maximal possible length satisfying that

- i. for all i such that $i + 1 < \text{lh}(\mathcal{U})$, the least disagreement between $\mathcal{M}_i^{\mathcal{U}}$ and \mathcal{H}_η is on the $\mathcal{M}_i^{\mathcal{U}}$ -side and $E_i^{\mathcal{U}}$ is that disagreement,
 - ii. for all limit $i < \text{lh}(\mathcal{U})$, we have that the Q-structure $\mathcal{Q}(\mathcal{U} \upharpoonright i, [0, i)_{\mathcal{U}})$ exists and is an initial segment of \mathcal{Q}_η .
- b. We say that \mathcal{U}_M **converges** iff there exists N such that
- i. N is the last model of \mathcal{U}_M ,
 - ii. the M -to- N branch of \mathcal{U}_M does not drop,
 - iii. either $N \trianglelefteq \mathcal{H}_\eta$ or $\mathcal{H}_\eta \trianglelefteq N$.
- c. If \mathcal{U}_M converges, we denote by \mathcal{P}_M the last model of \mathcal{U}_M and we denote by σ_M the mapping
- $$\pi^{\mathcal{U}_M} : M \longrightarrow \mathcal{P}_M.$$
- d. The premouse M is **η -exact** iff \mathcal{U}_M converges, $\mathcal{P}_M \triangleright \mathcal{H}_\eta$, and $\eta \in \text{ran}(\sigma_M)$.
- e. If M is η -exact, we denote η_M the ordinal $\sigma_M^{-1}(\eta)$ and by $\gamma_M \text{lh}(\mathcal{U}_M) - 1$. □

The tree \mathcal{U}_M is built by comparing M to \mathcal{H}_η , using the strategy given by Q-structures at limit stages. We want that \mathcal{H}_η does not move in this comparison, so we set up the definition in such a way that the construction breaks down if \mathcal{H}_η has to move (or if the strategy is not strong enough). The notion of convergence is simply ensuring that breakdown does not happen. If M is super-suitable and η -exact in the sense of Definition 3.7.9, then it is η -exact in this new sense (cf. Proposition 3.7.10) and the objects η_M , \mathcal{U}_M , γ_M , and \mathcal{P}_M correspond to those introduced in Definition 3.7.9 and Definition 3.7.7. In that case, the embedding σ_M is exactly the embedding

$$\pi_{0, \gamma_M}^{\mathcal{T}_M} = \pi^{\mathcal{U}_M} : M \longrightarrow \mathcal{P}_M.$$

Recall that we want to locally approximate the notion of an η -exact super-suitable mouse. We do so as follows.

Definition 3.8.2. Suppose that M is a countable premouse. Then M is **η -good** iff

- a. M is suitable,
- b. M is short tree iterable,
- c. all normal pseudo-iterates N of M are η -exact and satisfy that

$$\mathcal{P}(\eta) \cap \mathcal{P}_N = \mathcal{P}(\eta) \cap \mathcal{P}_M.$$

□

Proposition 3.8.3. *Suppose that M is super-suitable and η -exact. Then M is η -good.*

Proof.

- 1° We already know that super-suitable premice are suitable and short tree iterable, so it remains to verify condition c of Definition 3.8.2.
- 2° Let N be a normal pseudo-iterate of M . We want to show that N is η -exact and that $\mathcal{P}(\eta) \cap \mathcal{P}_N = \mathcal{P}(\eta) \cap \mathcal{P}_M$.
- 3° Let \mathcal{T} be the tree satisfying $M \xrightarrow{\text{st}}_{\mathcal{T}} N$. By Lemma 3.5.7, we have that \mathcal{T} is according to Σ_M and N is either the last model of \mathcal{T} or $N = \mathcal{M}_{\Sigma_M(\mathcal{T})}^{\mathcal{T}}$. In addition, the main branch of \mathcal{T} or the branch $\Sigma_M(\mathcal{T})$ does not drop.
- 4° This implies that N is super-suitable. Since $\eta \in \text{ran}(\pi_{M,\infty})$ and $\pi_{M,\infty} = \pi_{N,\infty} \circ \pi_{M,N}$, we get that $\eta \in \text{ran}(\pi_{N,\infty})$, which is another way of saying that N is η -exact.
- 5° By Proposition 3.7.10, we have that

$$\mathcal{P}(\eta) \cap \mathcal{P}_N = \mathcal{P}(\eta) \cap \text{HOD} = \mathcal{P}(\eta) \cap \mathcal{P}_M,$$

which concludes the verification. □

We are finally ready to state our local characterization of $\mathcal{P}(\eta) \cap \text{HOD}$.

Proposition 3.8.4. *Suppose that $A \subseteq \eta$. Then $A \in \text{HOD}$ if and only if there exists an η -good M such that $A \in \mathcal{P}_M$.¹⁵*

Proof.

- 1° For the implication (\Rightarrow), we can pick a super-suitable M satisfying $\eta \in \text{ran}(\pi_{M,\infty})$ and reach the conclusion by referencing Propositions 3.8.3, 3.5.4, and 3.7.10. We now focus on the implication (\Leftarrow).
- 2° Let M be η -good such that $A \in \mathcal{P}_M$. We want to show that $A \in \text{HOD}$.
- 3° There exists a super-suitable N such that $\eta \in \text{ran}(\pi_{N,\infty})$.
- 4° By Proposition 3.5.10, there exists a suitable P which is a normal pseudo-iterate of both M and N .
- 5° Since M is η -good, we have that

$$\mathcal{P}(\eta) \cap \mathcal{P}_M = \mathcal{P}(\eta) \cap \mathcal{P}_P.$$

- 6° Since N is super-suitable and $\eta \in \text{ran}(\pi_{N,\infty})$, we have that P is super-suitable as well and $\eta \in \text{ran}(\pi_{P,\infty})$.

¹⁵Cf. [17, Lemma 3.5].

7° By Proposition 3.7.10, we have that

$$\mathcal{P}(\eta) \cap \mathcal{P}_P = \mathcal{P}(\eta) \cap \text{HOD}.$$

8° By 5° and 7°, we have that $A \in \text{HOD}$.

□

To see that this characterization definable in \mathcal{A}_η , the main ingredient is to observe that the function Lp is definable over \mathcal{A}_η . Recall that Lp is defined on countable transitive sets and returns such sets as values. Since these sets are coded by reals, the function Lp is indeed a (partial) function in the structure \mathcal{A}_η . In other words, it really makes sense to ask whether now this function is definable over that structure.

Lemma 3.8.5. *The function Lp is lightface definable over \mathcal{A}_η .*

Proof. Definition 3.2.13 implies that Σ_1 -formulas with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$ are absolute between \mathcal{A}_η and $L(\mathbb{R})$, from which it easily follows that the definition of Lp is absolute between \mathcal{A}_η and $L(\mathbb{R})$. □

Once we have Lp in \mathcal{A}_η , we can run the standard definitions of the suitability, short tree iterability, and η -goodness inside this structure and obtain their definability as well.

Corollary 3.8.6. *The following sets are contained in \mathcal{A}_η and definable over it:*

a. $\{M : M \text{ is suitable}\}$

b. $\{M : M \text{ is short tree iterable}\},$

c. $\{M : M \text{ is } \eta\text{-good}\}.$ □

Putting this corollary together with Proposition 3.8.4, we get the conclusion.

Corollary 3.8.7. *The premouse $\text{HOD} \parallel \eta^{+\text{HOD}}$ is contained in $L_{\kappa_\eta}(\mathbb{R}, \mathcal{Q}_\eta)$ and definable over it from the parameter \mathcal{Q}_η .* □

As an immediate consequence, we get an upper bound on $\eta^{+\text{HOD}}$.

Corollary 3.8.8. $\eta^{+\text{HOD}} < \kappa_\eta.$ □

3.9 Local Definition of HOD II

In the previous section, we established local definability of $\text{HOD}\|\eta^{\text{HOD}}$ from \mathcal{Q}_η . We now want to obtain a more canonical structure to define $\text{HOD}\|\eta^{\text{HOD}}$ over it, a structure that does not refer to \mathcal{Q}_η . Hence, the work done in this section has for its goal locally defining \mathcal{Q}_η from some other objects (cf. the structure \mathcal{B}_η of Notation 3.9.5). Most of this work will be done internally in $L(\mathbb{R})$, so we position ourselves accordingly.

Section Assumptions 3.9.1. In this section, we work in $L(\mathbb{R})$, unless otherwise stated. \square

We remind the reader that a supercompactness measure on $[X]^\omega$ is just a normal fine ultrafilter on that set. Since AD holds (in $L(\mathbb{R})$), the sets that can be wellordered in the ordertype $< \Theta$ come with unique supercompactness measures.

Theorem 3.9.2. *For all X satisfying $|X| < \Theta$, there exists a unique supercompactness measure on $[X]^\omega$.*

Proof. See the first theorem on page 32 of [25]. \square

Definition 3.9.3. Suppose that $|X| < \Theta$. Then μ_X denotes the unique supercompactness measure on $[X]^\omega$. \square

Since these measures are unique, they must project to each other.

Lemma 3.9.4. *Suppose that*

1. $|X|, |Y| < \Theta$,
2. $X \subseteq Y$,
3. $D \in \mu_Y$.

Then $\{\sigma \cap X : \sigma \in D\} \in \mu_X$.

Proof. We have that

$$\nu := \{\{\sigma \cap X : \sigma \in E\} : E \in \mu_Y\}$$

is a supercompactness measure on $[X]^\omega$. By the uniqueness of supercompactness measures (Theorem 3.9.2), we get that $\nu = \mu_X$. This now means that

$$\{\sigma \cap X : \sigma \in D\} \in \nu = \mu_X.$$

\square

We are now able to define the new structure over which we will define $\text{HOD}\|\eta^{\text{HOD}}$. This structure has the measure μ_η as one of its predicates.

Notation 3.9.5. We use the following notation.

1. χ_η denotes the second of ordinals $\chi > \eta$ satisfying that

$$L_{\chi_\eta}(\mathcal{H}_\eta^\omega)[\mu_\eta] \models \text{ZF}^- + \text{“}\mathcal{P}([\eta]^\omega) \text{ exists”}.$$

2. \mathcal{B}_η denotes the structure

$$(L_{\chi_\eta}(\mathcal{H}_\eta^\omega)[\mu_\eta], \in, \mu_\eta, \mathcal{H}_\eta),$$

where (the restriction of) \in is a binary predicate, (the restriction of) μ_η is a unary predicate, and \mathcal{H}_η is a constant. \square

Lemma 3.9.6. *The ordinal χ_η is well defined and strictly less than Θ .*

Proof. This is verified similarly to Lemma 3.7.6. \square

The goal is to define \mathcal{Q}_η over \mathcal{B}_η . Let M be an arbitrary η -exact super-suitable mouse. This mouse belongs to \mathcal{B}_η and it is not hard to see that the tree $\mathcal{U}_M \upharpoonright \gamma_M$ also belongs to \mathcal{B}_η . What will require some work is to see that the branch b_M of this tree belongs to \mathcal{B}_η as well. Once we know this, we will get that \mathcal{Q}_η is definable over \mathcal{B}_η as the \mathcal{Q} -structure corresponding the branch b_M of the tree $\mathcal{U}_M \upharpoonright \gamma_M$. Our approach to establishing that b_M belongs to \mathcal{B}_η is to represent it as a certain ultraproduct and then argue that that ultraproduct can be computed inside \mathcal{B}_η . Now, in the absence of Choice, we do not have full Łoś’s Theorem, but we will be able to salvage a part of it which will suffice for our purposes.

Notation 3.9.7. Suppose that

1. $W \ni \eta$ is a rudimentarily closed transitive set,
2. $\mathcal{W} := (W, \in, \dots)$ is a first-order structure in a countable language with a lightface definable wellordering,
3. $\sigma \subseteq W$.

Then we denote by

- a. $\text{Hull}^{\mathcal{W}}(\sigma)$ the Skolem hull of σ inside \mathcal{W} , understood both as a set and as a substructure,
- b. $\text{cHull}^{\mathcal{W}}(\sigma)$ the transitive collapse of $\text{Hull}^{\mathcal{W}}(\sigma)$,
- c. $\pi_\sigma^{\mathcal{W}}$ the anticollapse associated to $\text{Hull}^{\mathcal{W}}(\sigma)$, where the superscript is omitted whenever it is clear from the context. \square

These hulls are well defined because we assumed that \mathcal{W} defines a wellordering. We can now get the following form of Łoś’s Theorem.

Lemma 3.9.8. *Suppose that*

1. $W \ni \eta$ is a rudimentarily closed transitive set,
2. $\mathcal{W} := (W, \in, \dots)$ is a first-order structure in a countable language with a lightface definable wellordering,
3. $\mathcal{W}^* := \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}}(\sigma) / \mu_\eta$,
4. $\phi(x)$ is a formula in the language of the structure \mathcal{W} ,
5. $f \in \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}}(\sigma)$.

Then

$$\mathcal{W}^* \models \phi[[f]] \iff \{\sigma \in [\eta]^\omega : \text{cHull}^{\mathcal{W}}(\sigma) \models \phi[f(\sigma)]\} \in \mu_\eta.$$

Proof.

1° This is shown by induction on the complexity of $\phi(\bar{x})$, as in the usual proof of Łoś's Theorem. We verify here the nontrivial step.

2° Suppose that

$$\{\sigma \in [\eta]^\omega : \text{cHull}^{\mathcal{W}}(\sigma) \models \exists y, \phi(f(\sigma), y)\} \in \mu_\eta.$$

We want to show that $\mathcal{W}^* \models \exists y, \phi([f], y)$.

3° Let

$$C := \{\sigma \in [\eta]^\omega : \text{cHull}^{\mathcal{W}}(\sigma) \models \exists y, \phi(f(\sigma), y)\}.$$

There exists a mapping $C \ni \sigma \mapsto (t_\sigma, \alpha_\sigma)$ such that for all $\sigma \in C$, t_σ is a Skolem term, $\alpha_\sigma \in \sigma$, and

$$\text{cHull}^{\mathcal{W}}(\sigma) \models \phi(f(\sigma), t_\sigma(\alpha_\sigma)).$$

4° Since μ_η is countably closed, we may assume w.l.o.g. that there exists a Skolem term t such that for all $\sigma \in C$, $t_\sigma = t$.

5° By normality of μ_η , we may assume w.l.o.g. that there exists $\alpha < \eta$ such that for all $\sigma \in C$, $\alpha_\sigma = \alpha$. In particular, for all $\sigma \in C$,

$$\text{cHull}^{\mathcal{W}}(\sigma) \models \phi[f(\sigma), t(\alpha)].$$

6° Let $g \in \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}}(\sigma)$ be defined by

$$\forall \sigma \in [\eta]^\omega, g(\sigma) := \pi_\sigma^{-1}(t(\alpha)).$$

By the inductive hypothesis, we have that $\mathcal{W}^* \models \phi[[f], [g]]$.

□

We can use the above lemma to show that in fact \mathcal{W}^* embeds canonically into \mathcal{W} . This in particular shows that the ultraproduct \mathcal{W}^* is wellfounded.

Lemma 3.9.9. *Suppose that*

1. $W \ni \eta$ is a rudimentarily closed transitive set,
2. $\mathcal{W} := (W, \in, \dots)$ is a first-order structure in a countable language with a lightface definable wellordering,
3. $\mathcal{W}^* := \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}}(\sigma) / \mu_\eta$.

Then \mathcal{W}^* is wellfounded and there exists a unique embedding

$$j : \mathcal{W}^* \xrightarrow[\Sigma_\omega]{} \mathcal{W}$$

satisfying that for all $\alpha < \eta$,

$$j([\text{otp}(\alpha \cap \sigma) : \sigma \in [\eta]^\omega]) = \alpha.$$

Moreover, $\text{ran}(j) = \text{Hull}^{\mathcal{W}}(\eta)$.

Proof.

- 1° The wellfoundedness follows from the existence of j .
- 2° Let $f \in \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}}(\sigma)$ be arbitrary. By applying the countable completeness and the normality of μ_η as in the proof of Lemma 3.9.8, we see that there exist $C \in \mu_\eta$, a Skolem term t , and an ordinal $\alpha < \eta$ such that for all $\sigma \in C$, $f(\sigma) = \pi_\sigma^{-1}(t(\alpha))$. We denote by (t_f, α_f) the minimal such (t, α) .
- 3° For all $f \in \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}}(\sigma)$, we define $j([f]) := t_f(\alpha_f)$. By Lemma 3.9.8, it follows that j is well defined and

$$j([\text{otp}(\alpha \cap \sigma) : \sigma \in [\eta]^\omega]) = \alpha.$$

- 4° The “moreover” part is now easily verified.

□

We are now ready to do the main work, which involves showing that \mathcal{U}_M belongs to \mathcal{B}_η , where M is η -exact and super-suitable. This breaks down into first observing that $\mathcal{U}_M \upharpoonright \gamma_M$ belongs to \mathcal{B}_η and then proving that $b_M = [0, \gamma_M)_{\mathcal{U}_M}$ also belongs to \mathcal{B}_η .

Lemma 3.9.10. *Suppose that M is an η -exact super-suitable premouse. Then $\mathcal{U}_M \upharpoonright \gamma_M$ belongs to \mathcal{B}_η .*

Proof. The tree $\mathcal{U}_M \upharpoonright \gamma_M$ is obtained by comparing M to \mathcal{H}_η , where at limit stages, we pick the branches whose Q-structures are initial segments of \mathcal{H}_η . In this comparison, \mathcal{H}_η does not move. This description relativizes correctly to \mathcal{B}_η , yielding the conclusion. \square

The previous proof does not go through when we need to pick the final branch of \mathcal{U}_M . The reason for this is that this branch has for its Q-structure the structure \mathcal{Q}_η , which we are actually trying to obtain. We thus need a more complicated argument, which we give below.

Proposition 3.9.11. *Suppose that M is an η -exact super-suitable premouse. Then the branch b_M belongs to \mathcal{B}_η .*

Proof.

1° Let x_M be a real coding M and let $\theta > \eta$ be the least ordinal satisfying that $W := L_\theta(x_M, \mathcal{H}_\eta, \mathcal{U}_M \upharpoonright \gamma_M) \models \text{ZFC}^-$. We have that $\theta < \chi_\eta$ and that W belongs to \mathcal{B}_η .

2° We use the following notation:

1. $U := \mathcal{U}_M \upharpoonright \gamma_M$,
2. $\mathcal{W} := (W, \in, x_M, \mathcal{H}_\eta, U)$,
3. for all $\sigma \in [\eta]^\omega$, $U_\sigma := (\pi_\sigma^\mathcal{W})^{-1}(U)$.

3° Let B consist of all pairs (σ, B_σ) where

- a. $\sigma \in [\eta]^\omega$,
- b. U_σ is a short tree on M ,
- c. $B_\sigma = \Sigma_M^{\text{st}}(U_\sigma)$.

We have that B is a function and belongs to \mathcal{B}_η (cf. Proposition 3.2.14).

4° Let

1. $\lambda \in (\theta, \Theta)$ be such that $W^* := L_\lambda(\mathcal{W}, \mathcal{Q}_\eta) \models \text{ZFC}^-$,
2. $\mathcal{W}^* := (W^*, \in, \mathcal{W}, \mathcal{Q}_\eta)$,
3. for all $\sigma \in [\eta]^\omega$, $\text{Hull}_+^{\mathcal{W}}(\sigma) := \text{Hull}^{\mathcal{W}^*}(\sigma) \cap W$,
4. for all $\sigma \in [\eta]^\omega$, $\text{cHull}_+^{\mathcal{W}}(\sigma)$ be the transitive collapse of $\text{Hull}_+^{\mathcal{W}}(\sigma)$,
5. for all $\sigma \in [\eta]^\omega$, $k_\sigma : \text{cHull}_+^{\mathcal{W}}(\sigma) \xrightarrow{\Sigma_\omega} \text{cHull}_+^{\mathcal{W}}(\sigma)$ be such that the diagram

$$\begin{array}{ccc}
 \text{Hull}^{\mathcal{W}}(\sigma) & \xrightarrow{\subseteq} & \text{Hull}_+^{\mathcal{W}}(\sigma) \\
 \pi_\sigma^{\mathcal{W}} \uparrow & & \pi_\sigma^{\mathcal{W}^*} \uparrow \\
 \text{cHull}^{\mathcal{W}}(\sigma) & \xrightarrow{k_\sigma} & \text{cHull}_+^{\mathcal{W}}(\sigma)
 \end{array}$$

commutes.

Note that $b_M \in W^*$ and it is definable over \mathcal{W}^* (with no additional parameters) as the unique cofinal wellfounded branch through U having \mathcal{Q}_η for its \mathcal{Q} -structure.

5° **Claim.** There exists $S \in \mu_\eta$ such that for all $\sigma \in S$,

- a. $\sigma \in \text{dom}(B)$,
- b. $\text{crit}(k_\sigma) > (\pi_\sigma^{\mathcal{W}})^{-1}(\eta)$,
- c. $B_\sigma \in \text{cHull}^{\mathcal{W}^*}(\sigma)$.

Proof.

1' Let $C^* := \{X^* \in [W^*]^\omega : X^* \prec \mathcal{W}^*\} \in \mu_{W^*}$. By Proposition 3.9.4, we have that

$$S := \{X^* \cap \eta : X^* \in C^*\} \in \mu_\eta.$$

2' We want to show that S is as required. To that end, fix an arbitrary $\sigma \in S$.

3' Let $X^* \in C^*$ be such that $\sigma = X^* \cap \eta$. We have that

$$\text{Hull}^{\mathcal{W}}(\sigma) \cap \eta = \text{Hull}^{\mathcal{W}^*}(\sigma) \cap \eta = \sigma,$$

and in particular, $k_\sigma \upharpoonright (\pi_\sigma^{\mathcal{W}})^{-1}(\eta)$ is the identity.

4' In addition, η is definable over both \mathcal{W} and \mathcal{W}^* . This suffices for b.

5' Note that $U \in \text{Hull}^{\mathcal{W}^*}(\sigma)$ and that U is coded by a subset of η . Since $\text{crit}(k_\sigma) > (\pi_\sigma^{\mathcal{W}})^{-1}(\eta)$, we conclude that $(\pi_\sigma^{\mathcal{W}^*})^{-1}(U) = U_\sigma$.

6' We have that $\mathcal{Q}_\eta \in \text{Hull}^{\mathcal{W}^*}(\sigma)$, so we can denote $Q := (\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$. By elementarity, $\text{cHull}^{\mathcal{W}^*}(\sigma) \models "Q \text{ is the } \mathcal{Q}\text{-structure of the tree } U_\sigma"$.

7' Being the case that Q embeds into \mathcal{Q}_η , it is $(\omega_1 + 1)$ -iterable (in V and in $L(\mathbb{R})$; cf. Proposition 3.2.14). This means that Q is the true \mathcal{Q} -structure of the tree U_σ .

8' In particular, U_σ is short and $\sigma \in \text{dom}(B)$. This establishes a.

9' Now, B_σ is the unique cofinal wellfounded branch through U_σ which has Q as its \mathcal{Q} -structure, which means that

$$B_\sigma = (\pi_\sigma^{\mathcal{W}^*})^{-1}([0, \gamma_M]_{\mathcal{U}_M}) \in \text{cHull}^{\mathcal{W}^*}(\sigma),$$

as required in c.

□

6° Let

$$b := \{\alpha < \eta : \forall_{\mu_\eta}^* \sigma, \text{otp}(\alpha \cap \sigma) \in B_\sigma\} \subseteq \eta.$$

By 3°, b belongs to \mathcal{B}_η .

7° **Claim.** $b = b_M$

Proof.

1' Let

$$j : \prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}^*}(\sigma)/\mu_\eta \xrightarrow{\Sigma_\omega} \mathcal{W}^*$$

be the unique embedding satisfying $j \upharpoonright \eta = \text{id}_\eta$ (cf. Lemma 3.9.9). Since η is definable over \mathcal{W}^* , we also have that $j(\eta) = \eta$.

2' By 4° and 1', we have that

- a. $j^{-1}(\mathcal{Q}_\eta) = \mathcal{Q}_\eta$,
- b. b_M is definable over \mathcal{W}^* , $b_M \in \text{ran}(j)$, and $j^{-1}(b_M) = b_M$,
- c. $\prod_{\sigma \in [\eta]^\omega} \text{cHull}^{\mathcal{W}^*}(\sigma)/\mu_\eta \models$ “ b_M is the unique cofinal wellfounded branch through U having \mathcal{Q}_η for its Q-structure”.

3' By Lemma 3.9.8, if we fix some f satisfying $[f] = b_M$, we get that for μ_η -almost all σ , it holds that $\text{cHull}^{\mathcal{W}^*}(\sigma) \models$ “ $f(\sigma)$ is the unique cofinal wellfounded branch through $(\pi_\sigma^{\mathcal{W}^*})^{-1}(U)$ having $(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$ for its Q-structure”.

4' By Claim 5° and the step 3', there exists $S \in \mu_\eta$ such that for all $\sigma \in S$,

- a. $\sigma \in \text{dom}(B)$,
- b. $\text{crit}(k_\sigma) > (\pi_\sigma^{\mathcal{W}})^{-1}(\eta)$,
- c. $B_\sigma \in \text{cHull}^{\mathcal{W}^*}(\sigma)$,
- d. $f(\sigma)$ is the unique cofinal wellfounded branch through

$$(\pi_\sigma^{\mathcal{W}^*})^{-1}(U)$$

having $(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$ for its Q-structure.

5' Thus, for all $\sigma \in S$, we have that

- a. U_σ is short,
- b. $(\pi_\sigma^{\mathcal{W}^*})^{-1}(U) = U_\sigma$,
- c. $(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$ is $(\omega_1 + 1)$ -iterable (for this, we reason as in the step 5°7').

6' Putting 4' and 5' together, we get that for all $\sigma \in S$,

- a. U_σ is short,
- b. $(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$ is the Q-structure of the tree U_σ ,

c. $f(\sigma)$ is the unique cofinal wellfounded branch through U_σ having

$$(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$$

for its Q-structure.

Hence, for all $\sigma \in S$, $f(\sigma) = B_\sigma$.

7' Now, we can compute as follows:

$$b_M = \{\alpha < \eta : \alpha \in [f]\} \tag{3.1}$$

$$= \{\alpha < \eta : \forall_{\mu_\eta}^* \sigma, \text{otp}(\alpha \cap \sigma) \in f(\sigma)\} \tag{3.2}$$

$$= \{\alpha < \eta : \forall_{\mu_\eta}^* \sigma, \text{otp}(\alpha \cap \sigma) \in B_\sigma\} \tag{3.3}$$

$$= b. \tag{3.4}$$

□

8° By the point 6° and Claim 7°, we conclude that b_M belongs to \mathcal{B}_η .

□

Thanks to the previous proposition, we can obtain \mathcal{Q}_η as a definable element of \mathcal{B}_η , which then implies that the structure \mathcal{A}_η is a definable element of the structure \mathcal{B}_η .

Corollary 3.9.12. \mathcal{A}_η belongs to and is definable over \mathcal{B}_η with no additional parameters.

Proof. Let M be an arbitrary η -exact super-suitable premouse. Since \mathcal{U}_M belongs to \mathcal{B}_η , so does $\mathcal{Q}_\eta = \mathcal{Q}(\mathcal{U}_M \upharpoonright \gamma_M, b_M)$. It remains to show that \mathcal{Q}_η is lightface definable over \mathcal{B}_η . To that end, notice that \mathcal{Q}_η is the Q-structure above \mathcal{H}_η (in the sense of Definition 3.2.8). Since \mathcal{B}_η has the strategies for the hulls of this structure (cf. Proposition 3.2.14), this characterization relativizes correctly to \mathcal{B}_η . □

We now get the main local definability result.

Corollary 3.9.13. The premouse $\text{HOD} \parallel \eta^{+\text{HOD}}$ is definable over the structure

$$(L_{\chi_\eta}(\mathcal{H}_\eta^\omega)[\mu_\eta], \in, \mu_\eta, \mathcal{H}_\eta^\omega)$$

with no additional parameters.¹⁶

□

We of course get one more upper bound for $\eta^{+\text{HOD}}$.

Corollary 3.9.14. $\eta^{+\text{HOD}} < \chi_\eta$.

□

¹⁶Here, \mathcal{H}_η^ω is treated as a constant and \in and μ_η as predicates (and they are naturally assumed to be restricted to the domain of the structure).

Chapter 4

Building Models of Determinacy from Below

4.1 Introduction

4.1.1 Statement of the Result

It was known through the work of Sargsyan that if there exists an inner model containing all reals and satisfying $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$, then there exists the smallest such model.¹ However, this was an existence result rather than a construction, so it was left open what this model looks like. In this chapter we will describe an L -like construction that can produce this model, and somewhat larger models.

The minimal model of $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ satisfies $V = L(\mathcal{P}(\mathbb{R}))$, so we will focus only on such models. Each such model \mathcal{M} is completely determined by the collection of its sets of reals. More precisely, if Δ denotes $\mathcal{P}(\mathbb{R}) \cap \mathcal{M}$, then $\mathcal{M} = L(\Delta)$. So the problem is to construct the powerset of reals of the minimal model of $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$.

Let us call $\Delta \subseteq \mathcal{P}(\mathbb{R})$ a *determinacy pointclass* if

$$\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Delta) = \Delta$$

and $L(\Delta) \models \text{AD}^+$. For any set X , we denote by μ_X the club filter on $[X]^\omega$. Now, given a determinacy pointclass Δ and an ordinal κ , we let $\mathcal{O}(\Delta, \kappa)$ be the pointclass

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega, \kappa^\omega)[\mu_\Delta, \mu_\kappa].$$

For any determinacy pointclass Δ , if there exists κ such that

$$\Delta \subsetneq \mathcal{O}(\Delta, \kappa),$$

¹It was shown in [11] that if there are two divergent inner models containing all reals and satisfying AD^+ , then there exists an inner model containing all reals and satisfying $\text{AD}_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$ contained in both.

we let κ_Δ be the smallest such κ , otherwise κ_Δ is undefined. Finally, the pointclass $\mathcal{O}(\Delta)$ is defined if and only if κ_Δ is defined, and it is set to be $\mathcal{O}(\Delta, \kappa_\Delta)$.

The sequence

$$(\mathcal{A}_\alpha : \alpha < \omega)$$

of determinacy pointclasses is defined recursively by iterating the operator \mathcal{O} as long as possible. If $L(\mathbb{R})$ does not satisfy AD^+ , the sequence is empty, otherwise \mathcal{A}_0 is set to be the powerset of reals of $L(\mathbb{R})$. If \mathcal{A}_α is defined, then $\mathcal{A}_{\alpha+1}$ is set to be $\mathcal{O}(\mathcal{A}_\alpha)$, if the latter is defined and a determinacy pointclass. The recursion terminates if the last condition is not met. At the limit stages we take the constructive closure of the union of the previous pointclasses. Again, the construction terminates if the obtained pointclass is not a determinacy pointclass. Our main theorem is then the following.

Theorem 4.1.1. *Suppose \mathcal{M} is an inner model containing all reals and satisfying $\text{AD}^+ + \text{AD}_{\mathbb{R}} +$ “there is no least branch hod pair with a measurable limit of Woodin cardinals”. Then there exists $\alpha < \omega$ such that $\mathcal{P}(\mathbb{R}) \cap \mathcal{M} = \mathcal{A}_\alpha$. \square*

Corollary 4.1.2. *If there exists an inner model containing all reals and satisfying $\text{AD}_{\mathbb{R}} +$ “ Θ is regular”, then there exists $\alpha < \omega$ such that $L(\mathcal{A}_\alpha)$ is the minimal model of $\text{AD}_{\mathbb{R}} +$ “ Θ is regular”.*

Proof. The point is that in the minimal model of $\text{AD}_{\mathbb{R}} +$ “ Θ is regular”, there is no sharp for the minimal hod mouse with an inaccessible limit of Woodin cardinals. To see this, let V be this minimal model and suppose M is such a sharp. Let λ denote the inaccessible limit of Woodin cardinals in it. First, the top measure of M can be iterated Ord times to produce an inner model M^* . Second, M^* is ω_1 -iterable below λ by essentially the same strategy as M . We can then go to the forcing extension by $\text{Col}(\omega, \mathbb{R})$ and make all reals of V generic over Woodin cardinals of a countable iterate M^{**} of M^* . The derived model of M^{**} at λ is then of the form $L(\mathbb{R}^V, \Delta)$, where Δ is its powerset of reals, and it satisfies $\text{AD}_{\mathbb{R}} +$ “ Θ is regular” by [26, Theorem 1.3]. It is easy to see that in fact Δ is in V and is a surjective image of the reals there, which contradicts the minimality assumption on V . \square

We will now try to motivate why the operator \mathcal{O} looks the way it does. The idea behind it is that we want a way to go from any determinacy pointclass to a strictly bigger one.

If there exists an inner model containing all reals and satisfying AD^+ , then $L(\mathbb{R})$ is the least such model. This minimal model is obtained by a well-known recursive construction:

$$\begin{aligned} J_1(\mathbb{R}) &= V_{\omega+1} \\ J_{\alpha+1}(\mathbb{R}) &= \text{rud}(J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\}) \\ J_\gamma(\mathbb{R}) &= \bigcup_{\alpha < \gamma} J_\alpha(\mathbb{R}) \quad (\gamma \text{ limit}) \\ L(\mathbb{R}) &= \bigcup_{\alpha \in \text{Ord}} J_\alpha(\mathbb{R}). \end{aligned}$$

In other words, we start with the reals and iterate the operator J_1 . From time to time a new set of reals is added to $J_\alpha(\mathbb{R})$ and by the time we reach $\alpha = \Theta^{L(\mathbb{R})}$, all sets of reals of $L(\mathbb{R})$ have been added. This results in the pointclass \mathcal{A}_0 of all sets of reals of $L(\mathbb{R})$, the \subseteq -minimal determinacy pointclass.

A strictly bigger determinacy pointclass cannot be obtained by the simple L -construction: an oracle must be added to this purely recursive approach.² It turns out that if there is a determinacy pointclass strictly larger than \mathcal{A}_0 , then

$$\mathcal{P}(\mathbb{R}) \cap L(\mathcal{A}_0^\omega)$$

is such a pointclass. More generally, the same is true if \mathcal{A}_0 is replaced by any determinacy pointclass Δ such that $\Theta^{L(\Delta)}$ has countable cofinality.³ So the oracle we use in this case is the sequences of sets in the pointclass. On the other hand, if $\Theta^{L(\Delta)}$ has uncountable cofinality, then the pointclass

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega)$$

is just Δ .⁴ To continue, we need to distinguish two cases, depending on whether $L(\Delta) \models \text{AD}_{\mathbb{R}}$.

An $\text{AD}_{\mathbb{R}}$ pointclass is a determinacy pointclass whose constructible closure satisfies $\text{AD}_{\mathbb{R}}$. Suppose there exists an $\text{AD}_{\mathbb{R}}$ pointclass strictly larger than Δ . In the first case, if Δ is not an $\text{AD}_{\mathbb{R}}$ pointclass, we have that

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta)[\mu_\Delta]$$

is a determinacy pointclass strictly larger than Δ . On the other hand, in the second case, where Δ is an $\text{AD}_{\mathbb{R}}$ pointclass, the answer to how to get a larger determinacy pointclass is more subtle and only partially known.

We would like to say that there exists κ such that the powerset of reals of $L(\Delta, \kappa^\omega)[\mu_\kappa]$ is strictly larger than Δ . However, this can fail in general.⁵ What we have been able to show is that there is such a κ if Δ is not too complicated.⁶ Furthermore, for the smallest κ as above, we know that $L(\Delta, \kappa^\omega)[\mu_\kappa]$ satisfies AD^+ . So in this last case,

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta, \kappa^\omega)[\mu_\kappa]$$

is a determinacy pointclass strictly larger than Δ .

²If there exists a model of AD^+ which extends $L(\mathbb{R})$ and has a larger Θ , then by [6, Theorem 10.1.8] this model contains \mathbb{R}^\sharp . In particular, whatever operation is used to get to this model must be able to produce the sharp of the reals.

³The equality $\text{cof}(\Theta^{L(\mathbb{R})}) = \omega$ holds since it is witnessed by the direct limit embedding coming from the HOD analysis for $L(\mathbb{R})$.

⁴This follows from the fact that if \mathcal{M} is an inner model containing all reals and satisfying AD^+ , then for all $\alpha < \Theta^{\mathcal{M}}$, $\alpha^\omega \subseteq \mathcal{M}$.

⁵For example, it fails if there is a proper class of Woodin limit of Woodin cardinals and if Δ is the powerset of reals of the Chang-plus model. We give more details below.

⁶The upper bound on the complexity of the pointclass is given in terms of strategies for hod mice. The exact statement is given below.

4.1.2 Outline of the Proof

Before starting the proof of the Main Theorem, we include a section summarizing some facts regarding HOD mice and HOD analysis (Section 4.2). Near the end of Section 4.3, we start working towards the proof of the Main Theorem. The introductory part of that section recalls and systemizes some facts concerning the club filter on the collection of countable subsets of a set. More precisely, we describe the notion of quasi-club filter in the context independent of the Axiom of Choice, which reduces the usual club filter if the Choice holds.

Let now \mathcal{M} be an inner model containing all reals and satisfying $\text{AD}^+ + \text{AD}_{\mathbb{R}} +$ “there is no least branch hod pair with a measurable limit of Woodin cardinals”. We want to show that there exists $\alpha < u$ such that

$$\mathcal{P}(\mathbb{R}) \cap \mathcal{M} = \mathcal{I}_\alpha.$$

The proof comes down to showing that the construction does not break down too soon and that a long enough initial segment of the construction is absolute between V and \mathcal{M} . The point here is that the sequence of pointclasses that the construction produces is strictly increasing, so as long as it stays in \mathcal{M} and does not break down, it will eventually reach the powerset of reals in \mathcal{M} .

The absoluteness of the construction comes down to the absoluteness of the operator

$$\mathcal{O} : \Delta \mapsto \mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega, \kappa_\Delta^\omega)[\mu_\Delta, \mu_{\kappa_\Delta}].$$

As long as we know that $\Theta^{L(\Delta)}$ and κ_Δ are strictly smaller than $\Theta^{\mathcal{M}}$, this absoluteness follows from the fact that \mathcal{M} contains all the ω -sequences which are bounded in Θ and from the fact that quasi-club filters are sufficiently absolute between \mathcal{M} and V (the latter is established in Section 4.3). Thus, the hard part is to show that κ_Δ is well-defined and strictly smaller than $\Theta^{\mathcal{M}}$.

If $\text{cof}(\Theta^{L(\Delta)}) = \omega$, then the joint of any ω -sequence which is Wadge-cofinal in Δ is a set of reals which is not in Δ . This shows that $L(\Delta^\omega)$ contains a new set of reals, so $\kappa_\Delta = 0$. If $\text{cof}(\Theta^{L(\Delta)}) > \omega$, we have to consider two subcases. First, if Δ is not an $\text{AD}_{\mathbb{R}}$ pointclass, then $L(\Delta)[\mu_\Delta]$ contains a new set of reals. If this were not the case, then $L(\Delta)[\mu_\Delta]$ would satisfy $\text{AD}^+ + \text{DC} + \neg\text{AD}_{\mathbb{R}} +$ “ ω_1 is $\mathcal{P}(\mathbb{R})$ -supercompact”, which will be shown inconsistent in Section 4.3. This means that in the first subcase of the second case, we also have $\kappa_\Delta = 0$.

Thus, the remaining subcase is when Δ is an $\text{AD}_{\mathbb{R}}$ pointclass. We place ourselves in \mathcal{M} and argue as follows. Let (K, Ψ) be a mouse pair which is not in Δ and is minimal in the mouse order with respect to such mouse pairs. In Section 4.4, we show that K has a largest cardinal and is coded by its powerset. Let $\mathcal{M}_\infty(K, \Psi)$ denote the direct limit of all non-dropping iterates of (K, Ψ) and let κ be the height of this direct limit. We will show that $\kappa_\Delta \leq \kappa$ by establishing that $L(\Delta, \kappa^\omega)[\mu_\kappa]$ contains a set of reals which is not in Δ .

This step is broken into two parts. Let S be the set of all ω -sequences of elements of $\mathcal{M}_\infty(K, \Psi)$. In Section 4.5, we show that $L(\Delta, S)$ contains the non-dropping part of Ψ , which is essentially a set of reals not in Δ . This further reduces our goal to showing that $S \in L(\Delta, \kappa^\omega)[\mu_\kappa]$ and this is done in Section 4.6. In Section 4.7, we put everything together and formally state and prove the main theorem.

4.1.3 History

Woodin has constructed⁷ in an unpublished⁸ work the minimal model of $\text{AD}_{\mathbb{R}} + \text{DC}$. Steel extended this work in [27, Theorem 3.1] and reached the minimal model of $\text{AD}_{\mathbb{R}} + \text{DC} + “\omega_1$ is $\mathcal{P}(\mathbb{R})$ -supercompact”.

This latter model does not satisfy $V = L(\mathcal{P}(\mathbb{R}))$, but its powerset of reals is presumably strictly larger than the powerset of reals in the former model.⁹ The construction that Steel proposes roughly corresponds to the first two oracles of our construction: if Δ is the current determinacy pointclass, then if $\text{cof}(\Theta^{L(\Delta)}) = \omega$, the next pointclass is

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega),$$

otherwise, the next pointclass is

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta)[\mu_\Delta],$$

but only in the case that it is strictly bigger than Δ . When this step fails to produce a bigger pointclass or produces a non-determined set, the construction terminates. In the cited paper, Steel argues that the construction does not terminate too soon. The pointclass that Steel’s construction reaches is strictly smaller than the powerset of reals of the minimal model of $\text{AD}_{\mathbb{R}} + “\Theta$ is regular”.¹⁰

4.1.4 Future Prospects

Our construction produces the powersets of the reals of all $\text{AD}_{\mathbb{R}}$ models below a strategy for a hod mouse with a measurable limit of Woodin cardinals. We believe that this is not the ultimate extent of the method and that the construction reaches even larger $\text{AD}_{\mathbb{R}}$ models. However, our construction can reach at most the Chang-plus model

$$\text{CM}^+ := L(\text{Ord}^\omega)[(\mu_{\alpha^\omega} : \alpha \in \text{Ord})].$$

Namely, if there exists a proper class of Woodin limit of Woodin cardinals, the Chang-plus model satisfies $\text{AD}_{\mathbb{R}} + \text{DC} + “\omega_1$ is supercompact”.¹¹ Moreover, the proof of the supercompactness of ω_1 (due to Woodin) shows that it is witnessed by the restrictions of the club filters μ_X (for all X). Letting Δ be the powerset of reals in the Chang-plus model, this is enough to conclude that for all κ ,

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega, \kappa^\omega)[\mu_\Delta, \mu_\kappa] = \Delta.$$

In other words, if our construction can reach Δ , it cannot go beyond it. This means that a new kind of oracle is required at this stage. Let us observe the following lower bound on the consistency of the determinacy beyond the Chang-plus model.

⁷in the sense of that we described in Subsection 4.1.1

⁸but see [27, Section 3]

⁹We could not locate a proof of this claim in the literature.

¹⁰To see this, work in $\text{AD}_{\mathbb{R}} + “\Theta$ is regular”.

It turns out that, letting $\Delta := \Delta_{\theta_{\omega_1}}$, the model $L(\Delta)[\mu_\Delta]$ satisfies $\text{AD}_{\mathbb{R}} + \text{DC} + “\omega_1$ is $\mathcal{P}(\mathbb{R})$ -supercompact” and its powerset of reals is just Δ . Once we verify the latter, then the former follows by the results of [28]. The verification is based on the observation that every set of reals in $L(\Delta)[\mu_\Delta]$ is ordinal definable from a member of Δ and thus, in Δ .

¹¹It follows from [14, Theorem 1.3] that it satisfies $\text{AD}^+ + “\omega_1$ is supercompact”, while the rest follows by results of [13].

Corollary 4.1.3. *Suppose that CM^+ satisfies $\text{AD}^+ + \text{“}\omega_1 \text{ is supercompact”}$ and that there exists an inner model \mathcal{M} such that*

$$\mathcal{P}(\mathbb{R}) \cap \text{CM}^+ \not\subseteq \mathcal{M} \models \text{AD}^+ + \text{AD}_{\mathbb{R}}.$$

Then in \mathcal{M} , there exists a least branch hod pair with a measurable limit of Woodin cardinals.

Proof. If there is such pair in CM^+ , we are done. Otherwise, the Main Theorem implies that there exists $\alpha < u$ such that

$$\mathcal{P}(\mathbb{R}) \cap \text{CM}^+ = \mathcal{I}_\alpha.$$

By the observations given above, our construction cannot continue further, so $u = \alpha + 1$. Applying the Main Theorem this time to $\mathcal{P}(\mathbb{R}) \cap \mathcal{M}$ shows that there must exist in \mathcal{M} a least branch hod pair with a measurable limit of Woodin cardinals. \square

Another missing point of our construction are the powersets of the reals of determinacy models satisfying $\text{AD}^+ + \neg\text{AD}_{\mathbb{R}}$. Looking at our argument, one thing that we lack is the HOD analysis of these models in terms of least branch hod mice. However, the conceptual difficulty lies in the usage of club filters. Under $\text{AD}_{\mathbb{R}}$, they are ultrafilters on any set which is a surjective image of the reals, but this may fail in the absence of $\text{AD}_{\mathbb{R}}$.

Let us also say a few words on possible applications. An important question that we would like to address is, working in some extension of ZFC,¹² show that there are models of ZFC with a given large cardinal.¹³ One approach is to first exhibit an inner model containing all reals and satisfying AD^+ and then show that its HOD, or a rank initial segment thereof, has the target large cardinal. Our method shows what this model must be, so what remains to show is that the construction lasts long enough as to ensure that HOD of the corresponding model is complex enough.¹⁴

4.1.5 Some Technical Remarks

For the sake of precision, let us mentioned that by an *inner model* we mean a transitive, proper class model satisfying ZF. We will be mostly interested in inner models containing all reals and satisfying AD^+ , which we might sometimes call *determinacy models*.¹⁵ Every such determinacy model contains $L(\mathbb{R})$. If $M \subseteq N$ are determinacy models, then $\mathcal{P}(\mathbb{R})^M \not\subseteq \mathcal{P}(\mathbb{R})^N$ if and only if $\Theta^M < \Theta^N$. Also, if N is a determinacy model and $M \subseteq N$ is an inner model containing all reals, then M is a determinacy model as well.¹⁶

¹²for example, ZFC + PFA

¹³In the PFA case, that can be a Woodin limit of Woodin. Of course, this would require that our method is extended further.

¹⁴In this scenario, the study of HOD would probably require the HOD analysis, which would require showing HPC in bigger and bigger determinacy models.

¹⁵We expect that the reader is familiar with the basic theory of AD^+ and $\text{AD}_{\mathbb{R}}$, but as a general reference we recommend [6].

¹⁶This follows from [6, Theorem 8.22]. The way this theorem is stated, it would require that M is a definable subclass of N , but this issue can be avoid by assuming that both M and N satisfy $V = L(\mathcal{P}(\mathbb{R}))$. The reason why we can add this assumption is that an inner model P satisfies AD^+ if and only if $L(\mathcal{P}(\mathbb{R})^P)$ does so.

The construction that is the central topic of this chapter concerns sufficiently closed pointclasses. For our purposes, a pointclass is sufficiently closed if it is the collection of all sets of reals of some inner model containing all reals. If $\Delta_0 \subseteq \Delta_1$ are closed pointclasses and $L(\Delta_1)$ satisfies AD^+ , then $L(\Delta_0)$ satisfies AD^+ as well and it holds that $\Delta_0 \subsetneq \Delta_1$ if and only if

$$\Theta^{L(\Delta_0)} < \Theta^{L(\Delta_1)}.$$

Notation 4.1.4. A subset Δ of $\mathcal{P}(\mathbb{R})$ is said to be a **closed pointclass** iff

$$\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Delta) = \Delta.$$

If Δ is a closed pointclass, then we denote by Θ_Δ the ordinal $\Theta^{L(\Delta)}$. □

Let us now fix some relatively standard notation in the AD context that we will use through the chapter.

Notation 4.1.5. Let us assume AD.

- a. We denote by Ω and θ_α (for $\alpha \leq \Omega$) the length and the members of the Solovay sequence¹⁷, respectively.¹⁸
- b. For $A \subseteq \mathbb{R}$, we denote by $w(A)$ the Wadge rank of A .
- c. For $\alpha \leq \Theta$, Δ_α denotes the set of all $A \subseteq \mathbb{R}$ satisfying $w(A) < \alpha$. □

For the fine structure, hod mice, and HOD analysis, our main sources are [12] and [29]. We will try to follow their notation as much as possible. One point where we diverge is the notation for the pull-back strategy.

Notation 4.1.6. Let M be a premouse, let (N, T) be a mouse pair, and let $\pi : M \rightarrow N$ be nearly elementary. Then we denote by π^*T the pull-back strategy of T along π . □

Another detail that needs to be adjusted concerns cutpoints.

Definition 4.1.7. Let M be a premouse and let $\eta \leq o(M)$. Then we say that

- a. η is a **cutpoint** of M iff for all $\xi < \eta$, $o^M(\xi) \leq \eta$,
- b. η is a **strong cutpoint** of M iff for all $\xi < \eta$, $o^M(\xi) < \eta$,
- c. η is a **clean cutpoint** of M iff for all M -extenders E , if $\text{len}(E) > o(M)$, then $\text{crit}(E) > o(M)$. □

Cutpoints and strong cutpoints appear in [29, Definition 2.16], while the clean cutpoints are new. The following relation is the reason why we introduce them.

¹⁷Cf. the beginning of [6, Section 6.3].

¹⁸So in particular, $\theta_\Omega = \Theta$.

Definition 4.1.8. Let M and N be premice. Then $M \trianglelefteq^* N$ iff $M \trianglelefteq N$ and $o(M)$ is a clean cutpoint of N . \square

This relation corresponds to the strong initial segment relation from [29, Definition 4.1], but the way the relation is defined there excludes the possibility that $M \trianglelefteq^* N$ whenever M is extender-active. We believe this to be a typo and that our formulation captures the point.

4.2 HOD Mice and HOD Analysis

We try to follow [12] regarding the fine structure as much as possible. To avoid cumbersome wording, we will use the following convention.

Notation 4.2.1. When we say “premouse” and “mouse pair”, we mean the one of the least branch hod type. The scope of a mouse pair is assumed to be H_{ω_1} , unless stated otherwise. \square

Notation 4.2.2. Assume AD^+ . Let (M, Σ) be a mouse pair. We denote by $(M, \Sigma) \parallel \xi$ the mouse pair

$$(M \parallel \xi, \Sigma_{M \parallel \xi}),$$

whenever $\xi \leq \hat{o}(M)$. The pairs $(M, \Sigma) \parallel \xi$ and $(M, \Sigma) \parallel (\xi, n)$ are defined similarly (for $(\xi, n) \leq l(M)$). \square

We consider that every notion defined for premice is inherited by mouse pairs via their premouse component, unless explicitly stated otherwise. For example, a mouse pair (M, Σ) is strongly stable iff M is strongly stable, where the definition of strong stability of premice reads as follows.

Notation 4.2.3. A premouse M is **strongly stable** iff the $r\Sigma_{k(M)}$ -cofinality of $\rho_{k(M)}(M)$ is not measurable by the M -sequence.

The comparison theorem that we work with is proved for strongly stable mouse pairs, cf. [12, Theorem 9.3.6]. This shows the importance of strongly stable mouse pairs. In particular, given a mouse pair (M, Σ) which is not strongly stable, we will often have to replace it with one that is so. The construction that we most often use in this chapter comes from [12, Page 111, last paragraph] and [29, Page 43, penultimate paragraph] and it goes as follows. Since M is not strongly stable, we necessarily have that $k := k(M) > 0$. We denote by \bar{M} the premouse

$$\mathfrak{C}(M^-)$$

and by $\bar{\Sigma}$ the pullback of Σ by the anti-core embedding. Then $(\bar{M}, \bar{\Sigma})$ is essentially a mouse pair, except that \bar{M} may fail to be pfs. The only thing that can fail is in fact that its k^{th} projectum is measurable. In other words, pair $(\bar{M}, \bar{\Sigma})$ is either equal to (M, Σ) or $(M, \Sigma)^-$ is the ultrapower of $(\bar{M}, \bar{\Sigma})^-$ by an order zero measure on $\rho_k(\bar{M})$. Since M is assumed not

to be strongly stable, there is an order zero measure D on the $r\Sigma_k$ -cofinality of $\rho_k(M)$. Let (M^*, Σ^*) be the ultrapower of $(\bar{M}, \bar{\Sigma})$ via D . Then (M^*, Σ^*) is a strongly stable mouse pair.

One drawback of the above construction is that if \bar{M} is not equal to M , then $(M, \Sigma)^-$ is an ultrapower of $(\bar{M}, \bar{\Sigma})^-$, but it is not clear that we can reconstruct the full strategy Σ from $\bar{\Sigma}$. This prevents us from showing that when passing from Σ to Σ^* we do not drop in Wadge rank. However, there is another way to make a mouse pair strongly stable and it is based on [30, Section 3].¹⁹ Since we are assuming that M is not strongly stable, we can take the order zero measure D on η_k^M and produce the ultrapower $\text{Ult}(M, D)$. An issue is that the premouse $\text{Ult}(M, D)$ might not be k -sound, i.e. it might be a pfs premouse of type 2. If that happens, we have to make sense of iterations of such non-sound structures in order to talk about the mouse pair $(\text{Ult}(M, D), \Sigma_{(D)})$. This is done in [30, Lemma 3.49]. What we care about is here is the k^{th} core M^* of $\text{Ult}(M, D)$, together with the pullback strategy Σ^* of $\Sigma_{(D)}$ along the anti-core map. This is a strongly stable mouse pair (of type 1). The cited lemma shows that $w(\Sigma_{(D)}) \geq w(\Sigma^*)$, while the reverse inequality follows is witnessed simply by the anti-core map. Since strategies Σ and $\Sigma_{(D)}$ are an $r\Sigma_k$ -ultrapower away from each other, they also have the same Wadge-rank, finally ensuring the conclusion that $w(\Sigma^*) = w(\Sigma)$.

One way in which this construction turns out to be useful is with respect to HPC. We invite the reader to consult [29, Definition 1.7.1] and the paragraph that follows if they are not familiar with this principle. As stated there, assuming AD^+ , principle HPC is equivalent to the strategies of mouse pairs being Wadge-cofinal in Suslin co-Suslin sets of reals. What the above construction brings to the table is that we can make sure that these mouse pairs are strongly stable, which gives the following lemma.

Lemma 4.2.4. *Suppose that AD^+ holds. Then HPC holds if and only if for all Suslin co-Suslin sets of reals A , there exists a strongly stable mouse pair (M, Σ) such that $w(\Sigma) \geq w(A)$.²⁰ \square*

The principle HPC is conjectured to follow from AD^+ and the assumption that there are no mouse pairs with long extenders.²¹ If we (significantly) strengthen this additional requirement, then HPC is known to hold. The anti-large-cardinal assumption that will suffice for our purposes reads as follows.

Notation 4.2.5. The statement **NMLW** (“no measurable limit of Woodins”) asserts that “there does not exist a mouse pair (M, Σ) such that M has a measurable cardinal which is a limit of Woodin cardinals”. \square

The important theorem which will often be implicitly used is the following.

Theorem 4.2.6. $\text{ZF} + \text{AD}^+ + \text{NMLW} \vdash \text{HPC}$

Proof. Working in $\text{ZF} + \text{AD}^+ + \text{NMLW}$, we know by [31, Theorem 10.2.1] that the *Mouse Capturing* holds. It was shown in [32] that the *Mouse Capturing* is equivalent to the principle LEC defined in [12, Definition 1.7.1]. By [12, Definition 10.4.3], LEC implies HPC. \square

¹⁹We thank John Steel for drawing our attention to this.

²⁰This was stated without an explanation in the paragraph following [29, Definition 0.6].

²¹Cf. [12, Conjecture 1.7.6].

The importance of HPC lies in the fact that it allows *HOD analysis*, which will be the main technical tool in the chapter. Our starting point for that method is [29] and we will summarize here the aspects of the paper that will be important for us.

Definition 4.2.7. Assume AD^+ . Let Δ be a selfdual pointclass and let (M, Σ) be a mouse pair. Then (M, Σ) is a **generator** for Δ iff

- a. (M, Σ) is strongly stable,
- b. Σ is not in Δ ,
- c. for all mouse pairs $(N, \text{T}) <^* (P, \Sigma)$, we have that T is in Δ . □

We impose the strong stability on the generators from the beginning.²² The reason for this is that strongly stable mouse pairs are pre-well-ordered by the mouse order.²³ Hence, a generator for Δ is the least mouse pair outside of Δ and all the generators for Δ are mouse-equivalent. When generating pointclasses, essentially only Suslin co-Suslin sets matter.

Lemma 4.2.8. *Let Δ be a closed pointclass and let Δ' be the pointclass of Suslin co-Suslin sets in $L(\Delta)$. Then (M, Σ) is a generator for Δ if and only if it is a generator for Δ' .*

Proof. This follows from the fact that the iteration strategies are Suslin co-Suslin. □

The mouse limits of generators for non-limit Solovay pointclasses are known full in the following sense.

Lemma 4.2.9. *Suppose that*

- 1. $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{HPC}$,
- 2. $\alpha < \Omega$ is not a limit,
- 3. (M, Σ) is a generator for Δ_{θ_α} ,
- 4. (N, T) is a strongly stable mouse pair,
- 5. $(M, \Sigma) \leq^* (N, \text{T})$.

Then $\mathcal{M}_\infty(M, \Sigma) \trianglelefteq^* \mathcal{M}_\infty(N, \text{T})$.

Proof. This is [29, Corollary 4.5]. □

This fullness allows us to stack all such mouse limits.

²²In [29, Definition 3.3], the author did not require the strong stability, which we believe to be a non-essential omission.

²³Cf. [12, Corollary 9.3.7.b].

Definition 4.2.10. Assume $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{HPC}$. We define \mathcal{H} to be the least branch premouse of the least possible length satisfying that for all $\xi < \Omega$, for all (M, Σ) which are generators for $\Delta_{\theta_{\xi+1}}$, we have that $\mathcal{M}_{\infty}(M, \Sigma) \sqsubseteq^* \mathcal{H}$. \square

It turns out that this stack is exactly the HOD of the determinacy model.

Theorem 4.2.11. Assume $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{HPC}$. The domain of \mathcal{H} is equal to $\text{HOD} \parallel \Theta$.

Proof. This is [29, Theorem 4.7]. \square

Notation 4.2.12. Let Γ be a closed pointclass such that $L(\Gamma) \models \text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{HPC}$. Then we denote by \mathcal{H}_{Γ} the premouse $\mathcal{H}^{L(\Gamma)}$. \square

We will also need some additional properties of generators of the non-limit Solovay pointclasses. Before we state them, let us recall some notation.

Definition 4.2.13. Suppose that M is a premouse. Then we define the following.

- a. The ordinal τ^M is the supremum of all κ^{+M} where $\kappa < \rho^-(M)$ and there exists $\eta \in (\kappa^{+M}, o(M)]$ such that $\kappa = \text{crit}(E_{\eta}^M)$.
- b. If τ^M is not a strong cutpoint of M , then the ordinal β^M the least ordinal $\xi < o(M)$ satisfying that $o(\xi)^M \geq \tau^M$.²⁴ \square

Definition 4.2.14. Let (M, Σ) be a strongly stable mouse pair. Then we define the following.

- a. The ordinal $\tau_{\infty}(M, \Sigma)$ is defined as $\pi_{(M, \Sigma), \infty}(\tau^M)$.
- b. If τ^M is not a strong cutpoint of M , then the ordinal $\beta_{\infty}(M, \Sigma)$ is defined as

$$\pi_{(M, \Sigma), \infty}(\beta^M).$$

\square

The properties that we want now read as follows.

Theorem 4.2.15. Suppose that

1. $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{HPC}$,
2. $\alpha < \Omega$ is not a limit,
3. (M, Σ) is a generator for $\Delta_{\theta_{\alpha}}$,

²⁴We remind the reader that $o(\xi)^M$ denotes the ordinal $\sup\{\eta \leq o(M) : \text{crit}(E_{\eta}^M) = \xi\}$.

4. Δ is the pointclass of Suslin co-Suslin sets of $L(\Delta_{\theta_\alpha})$.

Then it holds that

- a. $o(M) = \tau^M$ and $k(M) = 0$,
- b. M is passive,
- c. the domain of $\mathcal{M}_\infty(M, \Sigma)$ is $\text{HOD} \parallel \theta_\alpha$,
- d. $\beta_\infty(M, \Sigma) = o(\Delta)$ and $\tau_\infty(M, \Sigma) = \theta_\alpha$.

Proof. These facts are listed in the first paragraph of [29, Page 38], except b. To see b, fix a generator (N, T) for $\Delta_{\theta_{\alpha+1}}$. We have that $o(\mathcal{M}_\infty(N, T)) = \theta_{\alpha+1}$. By Lemma 4.2.9, we get that $\mathcal{M}_\infty(M, \Sigma) \triangleleft \mathcal{M}_\infty(N, T)$. Since $\theta_\alpha = o(\mathcal{M}_\infty(M, \Sigma))$ is a cardinal in V , it is so in $\mathcal{M}_\infty(N, T)$ as well. This means that in $\mathcal{M}_\infty(N, T)$, neither a branch nor an extender are indexed at $o(\mathcal{M}_\infty(M, \Sigma))$, which shows that $\mathcal{M}_\infty(M, \Sigma)$ is passive. \square

4.3 Club Filter

The goal of this section is to describe the notion of club filter in the choiceless context. This notion will generalize the usual notion of club filter under the Axiom of Choice. The definition that we will work with essentially comes from [28].

Section Assumptions 4.3.1. Suppose that

- 1. ZF holds,
- 2. S is an infinite set,
- 3. DC_S holds. \square

Under these assumptions we know that the set $[S]^\omega$ of all countable subsets of S is non-empty. We want to talk about filters on $[S]^\omega$, one instance of which is the club filter. Below are some properties which are usually associated with the club filter.

Definition 4.3.2. Let \mathcal{F} be a filter on $[S]^\omega$.

- a. \mathcal{F} is said to be **fine** iff for all $\tau \in [S]^\omega$,

$$\{\sigma \in [S]^\omega : \tau \subseteq \sigma\} \in \mathcal{F}.$$

b. \mathcal{F} is said to be **normal** iff for all A which are \mathcal{F} positive, for all

$$F : A \longrightarrow [S]^\omega$$

satisfying that

$$\forall \sigma \in A, \emptyset \subset F(\sigma) \subseteq \sigma,$$

there exists $a \in S$ such that

$$\{\sigma \in A : a \in F(\sigma)\}$$

is \mathcal{F} -positive.

c. \mathcal{F} is said to have the **diagonal intersection property** iff for all families $(A_a : a \in S)$ of sets in \mathcal{F} , the set

$$\Delta_{a \in S} A_a := \{\sigma \in [S]^\omega : \forall a \in \sigma, \sigma \in A_a\}$$

belongs to \mathcal{F} . □

The notion of normality is adjusted to the choiceless context. This adjustment is appropriate, since we have the following standard equivalence.

Proposition 4.3.3. *Let \mathcal{F} be a filter on $[S]^\omega$. Then \mathcal{F} is normal iff \mathcal{F} has the diagonal intersection property.*

Proof. By the proof of [28, Lemma 4.5]. □

We also have that normal fine filters are ω_1 -complete.

Lemma 4.3.4. *Suppose that \mathcal{F} is a normal, fine filter on $[S]^\omega$. Then \mathcal{F} is ω_1 -complete.*

Proof. Let $(A_n : n < \omega) \in \mathcal{F}^\omega$ be arbitrary. There exists an injection $(a_n : n < \omega) \in S^\omega$. We define $(B_a : a \in S) \in \mathcal{F}^S$ as follows: for $a \in S$,

$$B_a := \begin{cases} A_n, & n < \omega, a = a_n \\ S, & \text{otherwise.} \end{cases}$$

Since \mathcal{F} is fine, we have that

$$C := \{\sigma \in [S]^\omega : \{a_n\}_{n < \omega} \subseteq \sigma\} \in \mathcal{F}.$$

Note that

$$\bigcap_{n < \omega} A_n \cap C = \Delta_{a \in S} B_a \cap C.$$

Since \mathcal{F} is normal, it follows that $\bigcap_{n < \omega} A_n \in \mathcal{F}$. □

Following the terminology [33], we introduce the so-called *club game*. This game is what will allow us to characterize the club filter.

Definition 4.3.5. Let $A \subseteq [S]^\omega$. The game $\mathfrak{D}^{\text{club}}(S, A)$ is a length ω two player game of the form

$$\begin{array}{c|ccc} \text{I} & a_0 & a_1 & \cdots \\ \hline \text{II} & b_0 & b_1 & \cdots \end{array}$$

where for all $n < \omega$, $a_n, b_n \in S$. Player II wins iff $\{a_n, b_n : n < \omega\} \in A$. □

We will introduce now what is supposed to be the club filter, but we will avoid using the word “club” unless the Axiom of Choice holds. The reason for this is that different characterizations of being a club are not necessarily equivalent in the choiceless context. In informal discussions, we might use the term “quasi-club filter”.

Definition 4.3.6. We define μ_S to consist of all $A \subseteq [S]^\omega$ for which Player II has a winning quasi-strategy in $\mathfrak{D}^{\text{club}}(S, A)$.

The set μ_S is a filter and it will have some of the usual properties of the club filter.

Proposition 4.3.7. μ_S is a fine filter.

Proof. To see that μ_S is a filter, the only non-trivial part is to verify the closure for intersections. This is argued similarly to the countable completeness of [28, Lemma 4.1], but we work with quasi-strategies instead of strategies. Our case is simpler insomuch that we are only considering intersections of two sets. Let us explain why μ_S is fine. Let $\sigma \in [S]^\omega$ and let

$$A := \{\sigma' \in [S]^\omega : \sigma \subseteq \sigma'\}.$$

We need to give a winning quasi-strategy for Player II in $\mathfrak{D}^{\text{club}}(S, A)$. If we fix an enumeration $(a_n : n < \omega)$ of σ , then Player II can win $\mathfrak{D}^{\text{club}}(S, A)$ by playing a_n 's, regardless of the moves of Player I. □

The normality seems to require more than just the assumptions that we are working under. One way to ensure it is to assume more choice.

Proposition 4.3.8. Suppose that $\text{DC}_{\mathcal{P}(S^{<\omega})}$ holds. Then filter μ_S is normal.

Proof. Suppose that $(A_a : a \in S)$ is a family of sets in μ_S . Implicit in the proof of [28, Lemma 4.6] is a quasi-strategy τ for Player II in the game in the game

$$\mathfrak{D}^{\text{club}}(S, \Delta_{a \in S} A_a).$$

Note that we do not pick a family $(\tau_a : a \in S)$ of winning quasi-strategy for Player II in the games

$$(\mathfrak{D}^{\text{club}}(S, A_a) : a \in A)$$

as it is done there, but we play a tree of auxiliary games according to all possible such quasi-strategies. In the end, if f is a play according to τ , we can use $\text{DC}_{\mathcal{P}(S^{<\omega})}$ to pick a sequences of $(\tau_a : a \in \text{ran}(f))$ of winning quasi-strategies for Player II in the games

$$(\mathfrak{D}^{\text{club}}(S, A_a) : a \in \text{ran}(f))$$

which ensure that for all $a \in \text{ran}(f)$, $\text{ran}(f) \in A_a$. \square

We will soon observe another natural way to obtain the normality. In the choiceless context that we care about the most, the quasi-club filter will be both ultra and normal. We call such filters *supercompactness measures*.

Definition 4.3.9. A **supercompactness measure** on $[S]^\omega$ is a normal, fine ultrafilter on $[S]^\omega$. \square

It follows from Lemma 4.3.4 that every supercompactness measure on $[S]^\omega$ is ω_1 -complete. We will now show that every supercompactness measure must extend the quasi-club filter.

Proposition 4.3.10. *Suppose that μ is a supercompactness measure on $[S]^\omega$. Then $\mu_S \subseteq \mu$.*

Proof. This follows from the proof of [34, Theorem 1.2]. \square

One immediate consequence is that the existence of a supercompactness measure implies the normality of the quasi-club filter. Another consequence is that if the quasi-club filter itself is a supercompactness measure, then it is the only supercompactness measure. This situation naturally occurs in the determinacy context (see Proposition 4.3.12 below).

Lemma 4.3.11. *Suppose that $\text{AD}_{\mathbb{R}}$ holds and that \mathbb{R} surjects onto S . Then in every game on S , one of the players has a winning quasi-strategy.*

Proof.

1° Let $A \subseteq S$ be arbitrary. We want to show that one of the players in the game $\mathfrak{D}(S, A)$ on S with A as the payoff set for Player II has a winning quasi-strategy.

2° Let $\pi : \mathbb{R} \twoheadrightarrow S$ be a surjection and let

$$B := \{y \in \mathbb{R}^\omega : \pi \circ y \in A\}.$$

By $\text{AD}_{\mathbb{R}}$, one of the players in the game on \mathbb{R} with B as the payoff set for Player II has a winning strategy. We can assume without loss of generality that it is Player I who has a winning strategy τ .

- 3° Regarding τ as a tree, let $\sigma := \{\pi \circ y : y \in \tau\}$. Then σ is a quasi-strategy for Player I in $\mathfrak{D}(S, A)$. We want to show that it is winning.
- 4° Let f be a play according to σ . By $\text{DC}_{\mathbb{R}}$, there exists g according to τ such that $f = \pi \circ g$.
- 5° Since g is according to τ , we have that $g \in B^{\mathfrak{C}}$. This implies $f \in A^{\mathfrak{C}}$, as required.

□

Proposition 4.3.12. *Suppose that $\text{AD}_{\mathbb{R}}$ holds and that \mathbb{R} surjects onto S . Then μ_S is a supercompactness measure. Furthermore, it is the only supercompactness measure on $[S]^{\omega}$.²⁵*

Proof. By Lemma 4.3.11, either Player I or Player II has a winning quasi-strategy in

$$\mathfrak{D}^{\text{club}}(S, A).$$

If Player II has a winning quasi-strategy, then $A \in \mu_S$ by the definition. On the other hand, if Player I has a winning quasi-strategy in $\mathfrak{D}^{\text{club}}(S, A)$, then we can easily show that Player II has a winning quasi-strategy in $\mathfrak{D}^{\text{club}}(S, A^{\mathfrak{C}})$, which means that $A^{\mathfrak{C}} \in \mu_S$. Thus, μ_S is a fine ultrafilter on $[S]^{\omega}$. To see that it is also normal, we can run the proof of [28, Lemma 4.6]. On the other hand, the uniqueness follows from 4.3.10. □

In the case that S is a mouse limit, there is an important collection of quasi-clubs that we will have to be aware of in the proof of Theorem 4.6.9.

Proposition 4.3.13. *Suppose that*

1. AD^+ holds,
2. (M, Σ) is a mouse pair,
3. $S := \mathcal{M}_{\infty}(M, \Sigma)$.²⁶

Then the set $\{\text{ran}(\pi_{(N, T), \infty}) : (N, T) \text{ is a non-dropping iterate of } (M, \Sigma)\}$ belongs to μ_S .

Proof. Let us denote by A the said set. A winning quasi-strategy for Player II in $\mathfrak{D}^{\text{club}}(S, A)$ consists in the following. Let $(M_{-1}, \Sigma_{-1}) := (M, \Sigma)$. As the game progresses and Player I plays some $a_n \in S$ (for $n < \omega$), Player II produces a non-dropping iterate (M_n, Σ_n) of (M_{n-1}, Σ_{n-1}) so that $a_n \in \text{ran}(\pi_{(M_n, \Sigma_n), \infty})$. Player II then makes sure to list each element of $\text{ran}(\pi_{(M_n, \Sigma_n), \infty})$ in one of his future moves. The set produced in the end is equal to

$$\bigcup_{n < \omega} \text{ran}(\pi_{(M_n, \Sigma_n), \infty}),$$

²⁵[34, Theorem 1.5] proves that there exists a unique supercompactness measure on $[S]^{\omega}$, but we need a more precise statement.

²⁶Note that in this case DC_S holds because S is wellordered, so adding this assumption is not necessary.

which is easily seen to belong to A . One thing to keep in mind is that we actually never have to pick an iterate (M_n, Σ_n) (or its enumeration in order type ω) since we only need to exhibit a quasi-strategy. \square

Another fact that we will use in the proof of Theorem 4.6.9 is that quasi-club filters project onto each other.

Lemma 4.3.14. *Suppose that*

1. $\bar{S} \subseteq S$ is infinite,²⁷
2. A belongs to μ_S .

Then the set $\{\sigma \cap \bar{S} : \sigma \in A\}$ belongs to $\mu_{\bar{S}}$.

Proof. Let u be an arbitrary element of \bar{S} and let A' consist of all $\sigma \in A$ satisfying that $u \in \sigma$. Since μ_S is fine, it follows that $A' \in \mu_S$. Let us consider an arbitrary winning quasi-strategy for Player II in $\mathcal{D}^{\text{club}}(S, A')$. We transform this quasi-strategy into a quasi-strategy for Player II in

$$\mathcal{D}^{\text{club}}(\bar{S}, \{\sigma \cap \bar{S} : \sigma \in A'\})$$

by asserting that Player II should follow it faithfully as long as it prescribes an element of \bar{S} and otherwise, he should play u instead. This new quasi-strategy is easily seen to be winning, thus witnessing that

$$\{\sigma \cap \bar{S} : \sigma \in A'\} \in \mu_{\bar{S}}.$$

The conclusion now readily follows from the fact that $A \supseteq A'$. \square

Let us state for the record that under the Axiom of Choice, the quasi-club filter is simply the club filter.

Proposition 4.3.15. *Suppose that AC holds. Then μ_S is equal to the club filter on $[S]^\omega$.*

Proof. AC allows us to work with strategies instead of quasi-strategies. The conclusion then follows from [33, Proposition 6.19]. \square

From this point on, we work towards proving the Main Theorem. The following proposition has for a consequence that the construction described in the introduction is sufficiently absolute.

Proposition 4.3.16. *Suppose that*

1. M is an inner model,

²⁷Observe that DC_S implies $\text{DC}_{\bar{S}}$ since every tree on \bar{S} is also a tree on S .

2. $S \in M$,
3. $S^\omega \subseteq M$.

Then $\mu_S^M \subseteq \mu_S$. In particular, if μ_S^M is an ultrafilter in M , then $\mu_S^M = \mu_S \cap M$.

Proof. Suppose that $A \in \mu_S^M$. Then there exists τ such that $M \models \text{“}\tau \text{ is a winning quasi-strategy for Player II in } \mathcal{D}^{\text{club}}(S, A)\text{”}$. Since $S^\omega \subseteq M$, we see that τ is a winning quasi-strategy for Player II in $\mathcal{D}^{\text{club}}(S, A)$ (in V). This shows that $A \in \mu_S$. \square

In the introduction, we said that in the case that Δ is a determinacy pointclass which is not an $\text{AD}_{\mathbb{R}}$ pointclass and for which $\text{cof}(\Theta^{L(\Delta)}) > \omega$, the model $L(\Delta)[\mu_\Delta]$ contains a set of reals which is not in Δ . This is the content of the corollary below, which states in the counter-positive.

Proposition 4.3.17. *Suppose that AD^+ and DC hold and that there exists a supercompactness measure on $[\mathcal{P}(\mathbb{R})]^\omega$. Then $\text{AD}_{\mathbb{R}}$ holds.²⁸*

Proof. Since we are assuming AD^+ , to get $\text{AD}_{\mathbb{R}}$ it suffices to show that every Suslin set is co-Suslin.²⁹ Let $\lambda < \Theta$ and let T be a tree on $\omega \times \lambda$. We want to show that $p[T]^\mathbb{C}$ is Suslin. We can run the proof of [13, Theorem 2] to get that T is weakly homogeneously Suslin. Since DC holds, we can construct a Martin-Solovay tree for T and show that it projects onto $p[T]^\mathbb{C}$. This tree witnesses that $p[T]^\mathbb{C}$ is Suslin.

We briefly explain why the cited proof goes through. Its examination shows that the authors are using that DC holds, that there exists a supercompactness measure on³⁰

$$\left[\bigcup_{n < \omega} (\mathcal{P}(\lambda^n) \cup \text{MEAS}_n^{\omega_1, \lambda}) \right]^\omega,$$

and that there exists a supercompactness measure on $[\mathcal{P}(\lambda)]^\omega$. In our case, DC holds by the assumption. To get the required supercompactness measures, observe that $\lambda < \Theta$ means that \mathbb{R} surjects to λ , so by Coding Lemma, \mathbb{R} surjects onto $\mathcal{P}(\lambda)$. This suffices to get the second supercompactness measure. To get the first one, we need to show that there is a surjection

$$\mathcal{P}(\mathbb{R}) \twoheadrightarrow \bigcup_{n < \omega} (\mathcal{P}(\lambda^n) \cup \text{MEAS}_n^{\omega_1, \lambda}).$$

Since DC holds, it suffices to show that for each $n < \omega$, there exists a surjection

$$\mathcal{P}(\mathbb{R}) \twoheadrightarrow \mathcal{P}(\lambda^n) \cup \text{MEAS}_n^{\omega_1, \lambda}.$$

²⁸[35, Theorem 2.2.1] shows that the conclusion does not follow if instead of assuming that ω_1 is $\mathcal{P}(\mathbb{R})$ -supercompact, we just assume that it is \mathbb{R} -supercompact.

²⁹Cf. [6, Theorem 0.3].

³⁰Following the cited opus, $\text{MEAS}_n^{\omega_1, \lambda}$ denotes the set of all countably complete measures on λ^n .

Since there is a canonical bijection $\lambda \leftrightarrow \lambda^n$ for $n > 1$, we may assume that $n = 1$. In that case,

$$\mathcal{P}(\lambda^n) \cup \text{MEAS}_n^{\omega_1, \lambda} \subseteq \mathcal{P}(\lambda) \cup \mathcal{P}(\mathcal{P}(\lambda)),$$

so the required surjection is obtained by the fact that \mathbb{R} surjects onto $\mathcal{P}(\lambda)$. \square

Corollary 4.3.18. *Suppose that AD^+ holds, that $\mu_{\mathcal{P}(\mathbb{R})}$ is a supercompactness measure, and that $V = L(\mathcal{P}(\mathbb{R}))[\mu_{\mathcal{P}(\mathbb{R})}]$. Then $V \models \text{AD}_{\mathbb{R}} + \text{DC}$.*

Proof. Note that ω_1 is $\mathcal{P}(\mathbb{R})$ -supercompact. The proof of [13, Theorem 1] then shows that $\text{DC}_{\mathcal{P}(\mathbb{R})}$ holds. Since $V = L(\mathcal{P}(\mathbb{R}))[\mu_{\mathcal{P}(\mathbb{R})}]$, we conclude that full DC holds. Proposition 4.3.17 shows that $\text{AD}_{\mathbb{R}}$ holds. \square

4.4 Description of a Generator

Section Assumptions 4.4.1. Suppose that

1. $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+ + \text{HPC}$,
2. there does not exist a mouse pair with a superstrong cardinal,
3. Γ is a closed pointclass such that $L(\Gamma) \models \text{AD}_{\mathbb{R}}$,
4. $\omega < \text{cof}(\Theta_{\Gamma}) \leq \Theta_{\Gamma} < \Theta$,
5. (K, Ψ) is a generator for Γ . \square

The goal of this section is establish how the generator K for Γ looks like. For the final description, we will have to strengthen the anti-large-cardinal assumption to the non-existence of a mouse pair with a measurable limit of Woodin cardinals. In that case, K has the largest cardinal, which is a limit of Woodin cardinals, and its fine structural degree is 0. To start arguing, we will require the following technical notion from [29].

Definition 4.4.2. Suppose that $(M, \Sigma) \leq^* (N, \text{T})$ are strongly stable mouse pairs. Then $(\mathcal{T}, \mathcal{U})$ is a **minimal comparison** of (M, Σ) with (N, T) iff

- a. \mathcal{T} is a normal tree on (M, Σ) with the last pair (M', Σ') ,
- b. \mathcal{U} is a normal tree on (N, T) with the last pair (N', T') ,
- c. the main branch of \mathcal{T} does not drop,
- d. $(M', \Sigma') \leq^* (N', \text{T}')$,
- e. for all $\alpha < \text{len}(\mathcal{U})$, $\text{crit}(E_{\alpha}^{\mathcal{U}}) \leq o(M')$. \square

Lemma 4.4.3. *Suppose that $(M, \Sigma) \leq^* (N, \mathbb{T})$ are strongly stable mouse pairs. Then there exists a minimal comparison of (M, Σ) with (N, \mathbb{T}) . \square*

We will also need a fullness lemma for generators of Solovay pointclasses.

Lemma 4.4.4. *Let (M, Σ) be a generator for some $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$, where $\alpha < \Omega^{L(\Gamma)}$. Then for all strong cutpoint cardinals κ of M , letting $\bar{M} := M|_{\kappa^{+M}}$, for all mouse pairs $(N, \mathbb{T}) \in L(\Gamma)$ satisfying that $(N, \mathbb{T}) \geq (\bar{M}, \Sigma_{\bar{M}})$, it holds that that $\rho(N) > \beta^M$.*

Proof. This is [29, Lemma 4.3] relativized to $L(\Gamma)$. \square

Furthermore, we have to observe that every mouse pair weaker than the generator (K, Ψ) of Γ is in Γ . This is just the definition of a generator in the case that the former is strongly stable, but otherwise, an argument is required.

Lemma 4.4.5. *Suppose that (M, Σ) is a dropping iterate of (K, Ψ) . Then (M, Σ) belongs to $L(\Gamma)$.*

Proof.

- 1° Let us assume otherwise. By the definition of a generator, this can only occur if M is not strongly stable.
- 2° Let \bar{M} be the strong $k(M)$ -core of M and let $\bar{\Sigma}$ be the pullback of Σ along the anti-core embedding.
- 3° Let D be the order zero measure on $\eta_{k(M)}^M$ ³¹ and let

$$(N, \mathbb{T}) := \text{Ult}((\bar{M}, \bar{\Sigma}), D).$$

Premouse N is strongly stable.

- 4° Since $(N, \mathbb{T}) \notin L(\Gamma)$, it follows that $(K, \Psi) \leq^* (N, \mathbb{T})$. Thus, there exist a stack t on (N, \mathbb{T}) and a nearly elementary embedding

$$\sigma : (K, \Psi) \longrightarrow (\mathcal{M}_\infty(t), \mathbb{T}_t).$$

- 5° Let \bar{s} be the stack $(D) \frown t$ on $(\bar{M}, \bar{\Sigma})$, let s be the lift of \bar{s} via the anti-core embedding $\bar{M} \longrightarrow M$, and let

$$\pi : \mathcal{M}_\infty(t) \longrightarrow P \sqsubseteq \mathcal{M}_\infty(s)$$

be the embedding from lifting. We now have a nearly elementary embedding

$$\pi \circ \sigma : (K, \Psi) \longrightarrow (P, \Sigma_{s,P}).$$

³¹We remind the reader that $\eta_{k(M)}^M$ denotes the $r\Sigma_{k(M)}^M$ -cofinality of $\rho_{k(M)}^M$.

6° On the other hand, $(P, \Sigma_{s,P})$ is a dropping iterate of (K, Ψ) : we first have a dropping stack from (K, Ψ) to (M, Σ) and then the stack $s^\frown(P)$. This contradicts Dodd-Jensen Lemma³².

□

Having done these technical preparations, we are ready to make a very important point. Namely, if (M, Σ) is a generator for a non-limit Solovay pointclass of $L(\Gamma)$, then (K, Ψ) is of course stronger than (M, Σ) , but it turns out that in a minimal comparison witnessing this, neither tree drops.

Proposition 4.4.6. *Suppose that*

1. $\alpha < \Omega^{L(\Gamma)}$,
2. (M, Σ) is a generator for some $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$,
3. $(\mathcal{T}, \mathcal{U})$ is a minimal comparison of (M, Σ) with (K, Ψ) .

*Then the main branch of \mathcal{U} does not drop.*³³

Proof.

1° Let us assume otherwise and let (M', Σ') and (K', Ψ') be the last pairs of \mathcal{T} and \mathcal{U} , respectively. We have that $(M', \Sigma') \leq^* (K', \Psi')$ and that $\Psi' \in L(\Gamma)$ (cf. Lemma 4.4.5).

2° **Claim.** For all (N, T) for which

$$(M', \Sigma') \leq (N, T) \leq (K', \Psi'),$$

for all $\eta < o(M')$, for all $r \in N$, we have that the set

$$\text{Hull}_{k(N)+1}^N(\eta \cup \{r\})$$

is not cofinal below $o(M')$.³⁴

Proof.

1' Let us assume otherwise and let N be the counter-example with the least possible $l(N)$.

³²Cf. [12, Theorem 9.3.4].

³³The proof of this proposition follows closely the proof of [29, Lemma 4.4], but unfortunately, we cannot directly use that lemma.

³⁴We are essentially claiming that $o(M')$ is $r\Sigma_{k(N)}$ -regular in N .

2' By the minimality of N , we get that $o(M')$ is $r\Sigma_{k(N)}^N$ -regular in N . (Note that the case $N = M'$ is covered by Lemma 4.2.15.)

3' This regularity together with the fact that $(M', \Sigma') \leq^* (N, T)$ implies that

- a. $\mathcal{M}_\infty(M', \Sigma') \leq^* \mathcal{M}_\infty(N, T)$,
- b. $\pi_{(N, T), \infty} \upharpoonright M' = \pi_{(M', \Sigma'), \infty}$,
- c. $\pi_{(N, T), \infty}(o(M')) = o(\mathcal{M}_\infty(M', \Sigma'))$.

4' Let $\eta < o(M')$ and $r \in N$ be such that the set

$$\text{Hull}_{k(N)+1}^N(\eta \cup \{r\})$$

is cofinal below $o(M')$ and let

$$(\eta_\infty, r_\infty) := \pi_{(N, T), \infty}((\eta, r)).$$

We have that the set

$$\text{Hull}_{k(N)+1}^{\mathcal{M}_\infty(N, T)}(\eta_\infty \cup \{r_\infty\})$$

is cofinal below $o(\mathcal{M}_\infty(M', \Sigma'))$.³⁵

5' Since $T \in \Gamma$, we get that $\mathcal{M}_\infty(N, T) \in \mathcal{H}_\Gamma$. Thus,

$$\text{cof}^{\mathcal{H}_\Gamma}(o(\mathcal{M}_\infty(M', \Sigma'))) \leq \eta_\infty < o(\mathcal{M}_\infty(M', \Sigma')).$$

6' However, Lemma 4.2.15 shows that $o(\mathcal{M}_\infty(M', \Sigma')) = \theta_{\alpha+1}$, so it is Woodin in \mathcal{H}_Γ . This is a contradiction. □

3° **Claim.** There exists a limit ordinal γ such that $\text{len}(\mathcal{U}) = \gamma + 1$.

Proof.

1' Let us assume otherwise. Then there are ξ and η such that $\text{len}(\mathcal{U}) = \eta + 2$ and ξ is the \mathcal{U} -predecessor of $\eta + 1$.

2' Recall that

$$K' = \mathcal{M}_{\eta+1}^{\mathcal{U}} = \text{Ult}(\mathcal{M}_{\eta+1}^{*\mathcal{U}}, E_\eta^{\mathcal{U}}),$$

where $\mathcal{M}_{\eta+1}^{*\mathcal{U}}$ is the longest initial segment to which $E_\eta^{\mathcal{U}}$ can be applied.

3' Since the main branch of \mathcal{U} drops, it follows that K' is sound above $\lambda(E_\eta^{\mathcal{U}})$.

4' In other words, the equality

$$K' = \text{Hull}_{k(K')+1}^{K'}(\lambda(E_\eta^{\mathcal{U}}) \cup \{p(K')\})$$

holds. Since Claim 2° excludes this possibility in the case that $\lambda(E_\eta^{\mathcal{U}}) < o(M')$, we conclude that

$$\lambda(E_\eta^{\mathcal{U}}) \geq o(M').$$

³⁵The fact that we have enough elementarity to push this statement about the cofinality from N to the mouse limit follows from the pfs version of [36, Corollary 4.3].

5' Let us denote by κ the critical point of $E_\eta^{\mathcal{U}}$. The minimality of $(\mathcal{T}, \mathcal{U})$ implies that $\kappa \leq o(M')$.

6' Thus, we know so far that

$$\kappa \leq o(M') \leq \lambda(E_\eta^{\mathcal{U}}) < \text{len}(E_\eta^{\mathcal{U}}).$$

The next thing we verify is that the first inequality is strict.

7' **Subclaim.** $\kappa < o(M')$

Proof. Let us assume otherwise, i.e. that $\kappa = o(M')$. Theorem 4.2.15 implies that

$$o^{M'}(\beta^{M'}) = \tau^{M'} = o(M') = \kappa.$$

Since $M' \triangleleft K'$ and $E_\eta^{\mathcal{U}}$ is on the K' -sequence and overlaps $o(M')$, it follows that

$$o^{K'}(\beta^{M'}) > \kappa,$$

but this contradicts the fact that $M' \leq^* K'$, i.e. that $o(M')$ is a cutpoint of K' . \square

8' Thus, $\kappa < o(M') < \text{len}(E_\eta^{\mathcal{U}})$. This in particular means that κ is a cardinal in M' .

9' **Subclaim.** κ is a strong cutpoint in M' .

Proof. Let us assume otherwise, i.e. that for some $\alpha < \kappa$, we have that $o^{M'}(\alpha) \geq \kappa$. This implies that

$$o^{K'}(\alpha) \geq \lambda(E_\eta^{\mathcal{U}}) > o(M'),$$

which contradicts $M' \leq^* K'$. \square

10' Subclaim 9' ensures that $\kappa \leq \beta^{M'}$.

11' **Subclaim.** $\rho(K') \leq \kappa$

Proof.

1'' If $\eta + 1 \in D^{\mathcal{U}}$, then $\rho(K') = \rho(\mathcal{M}_{\eta+1}^{\mathcal{U}*}) \leq \kappa$ by definition.

2'' Otherwise, if $\eta + 1 \notin D^{\mathcal{U}}$, we have that

$$[0, \xi]_{\mathcal{U}} \cap D^{\mathcal{U}} \neq \emptyset.$$

Let $\alpha + 1$ be the largest element of this set.

3'' It follows that

$$\rho(K') = \rho(\mathcal{M}_{\alpha+1}^{\mathcal{U}*}) \leq \text{crit}(E_{\alpha+1}^{\mathcal{U}}) < \lambda(E_{\alpha+1}^{\mathcal{U}}) \leq \text{crit}(E_{\eta+1}^{\mathcal{U}}) = \kappa.$$

4'' In either case, we get that $\rho(K') \leq \kappa$. \square

12' Thus, $\rho(K') \leq \kappa \leq \beta^{M'}$, which contradicts Lemma 4.4.4. (One should recall the first line of the proof where we assumed that (K', Ψ') is a dropping iterate of (K, Ψ) , which meant that $\Psi' \in L(\Gamma)$.)

□

4° We have just argued that $\text{len}(\mathcal{U}) = \gamma + 1$ for some limit γ . Let us now pick α on the main branch of \mathcal{U} so that there are no drops in the interval $[\alpha, \gamma)_{\mathcal{U}}$ and so that there is an $i_{\alpha, \gamma}^{\mathcal{U}}$ -preimage δ of $o(M')$.

5° Let F be the extender applied at α along $[0, \gamma)_{\mathcal{U}}$. We have that $\delta \geq \text{crit}(F)$.

6° Let α^* be the successor of α in $[0, \gamma)_{\mathcal{U}}$, let $\delta^* := i_{\alpha, \alpha^*}^{\mathcal{U}}(\delta)$, let η be the supremum of generators of F , and let $W := \mathcal{M}_{\alpha^*}^{\mathcal{U}}$. We have that $\eta < \lambda_F \leq \delta^*$ ³⁶ and that

$$W = \text{Hull}_{k(W)+1}^W(\eta \cup \{p(W)\}).$$

7° In particular, $\text{Hull}_{k(W)+1}^W(\eta \cup \{p(W)\})$ is cofinal below δ^* .

8° By the elementarity³⁷ and the continuity³⁸ at δ^* of the embedding $i_{\alpha^*, \gamma}^{\mathcal{U}} : W \rightarrow K'$, it follows that

$$\text{Hull}_{k(K')+1}^{K'}(i_{\alpha^*, \gamma}^{\mathcal{U}}(\eta) \cup \{p(K')\})$$

is cofinal below $i_{\alpha^*, \gamma}^{\mathcal{U}}(\delta^*) = o(M')$, which is in contradiction with Claim 2°.

□

What we have just proven essentially says that we may assume that a generator (M, Σ) for a non-limit Solovay pointclass of $L(\Gamma)$ is an initial segment of (K, Ψ) . The next point that we want to make is that K does not project strictly across the height of M .

Proposition 4.4.7. *Suppose that*

1. $\alpha < \Omega^{L(\Gamma)}$,
2. (M, Σ) is a generator for some $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$,
3. $(\mathcal{T}, \mathcal{U})$ is a minimal comparison of (M, Σ) with (K, Ψ) .

Then $o(\mathcal{M}_{\infty}^{\mathcal{T}}) \leq \rho^{-}(\mathcal{M}_{\infty}^{\mathcal{U}})$.³⁹

³⁶We are using here the assumption that there are no mouse pairs with superstrong cardinals, which we made at the beginning of the section.

³⁷The level of the elementarity that we need again follows from the pfs version of [36, Corollary 4.3].

³⁸We are using here that $o(M')$ is not measurable and is $r_{\Sigma_{k(K')}}$ -regular in K' .

³⁹The proof is similar to the proof of Claim 1 of the proof of [29, Lemma 4.4].

Proof.

1° Let us assume otherwise and let (M', Σ') and (K', Ψ') be the final models of \mathcal{T} and \mathcal{U} , respectively. The situation is that

- a. neither \mathcal{T} nor \mathcal{U} drops,
- b. $(M', \Sigma') \leq^* (K', \Psi')$,
- c. $\rho^-(K') < o(M')$.

2° Let us denote by (P, Υ) the shortest initial segment of (K', Ψ') for which there exist $\eta < o(M')$ and $r \in P$ so that

$$\text{Hull}_{k(P)+1}^P(\eta \cup \{r\})$$

is cofinal in $o(M')$.⁴⁰

3° Observe that

- a. $(M', \Sigma') \trianglelefteq (P, \Upsilon) \triangleleft (K', \Psi')$,
- b. $o(M') \leq \rho^-(P)$.

4° **Case I.** P is strongly stable.

Proof.

1' This assumption implies that $\mathcal{M}_\infty(P, \Upsilon)$ belongs to HOD of $L(\Gamma)$,⁴¹ i.e.

$$\mathcal{M}_\infty(P, \Upsilon) \in \mathcal{H}_\Gamma.$$

2' Considering that $o(M')$ is a clean cutpoint of P and is $r\Sigma_{k(P)}^P$ -regular, we get that

$$\mathcal{M}_\infty(M', \Sigma') \trianglelefteq \mathcal{M}_\infty(P, \Upsilon)$$

and that

$$\pi_{(P, \Upsilon), \infty}(o(M')) = o(\mathcal{M}_\infty(M', \Sigma')) = \theta_{\alpha+1}^{L(\Gamma)}.$$

3' For all $\eta < o(M')$ and $r \in P$ for which

$$\text{Hull}_{k(P)+1}^P(\eta \cup \{r\})$$

is cofinal in $o(M')$, we have that

$$\text{Hull}_{k(P)+1}^{\mathcal{M}_\infty(P, \Upsilon)}(\pi_{(P, \Upsilon), \infty}(\eta) \cup \{\pi_{(P, \Upsilon), \infty}(r)\})$$

is cofinal in $\theta_{\alpha+1}^{L(\Gamma)}$.

⁴⁰We are essentially picking the least initial segment that singularizes $o(M')$ at the next degree of the fine structure. Such P exists since this singularization occurs for $(K')^-$.

⁴¹Cf. [29, Proposition 2.1].

4' However, $\mathcal{M}_\infty(P, \Upsilon) \in \mathcal{H}_\Gamma$, so it follows that $\theta_{\alpha+1}^{L(\Gamma)}$ is not regular in \mathcal{H}_Γ , which of course absurd. □

5° **Case II.** P is not strongly stable.

Proof.

1' Let \bar{P} be the $k(P)^{\text{th}}$ strong core of P and let $\bar{\Upsilon}$ be the pullback strategy of Υ along the anti-core embedding. The fact that $o(M') \leq \rho^-(P)$ yields that $(M', \Sigma') \trianglelefteq (\bar{P}, \bar{\Upsilon})$.

2' Let D be the order zero measure on $\eta_{k(P)}^P$ on the \bar{P} -sequence and let

$$(P^*, \Upsilon^*) := (\text{Ult}(\bar{P}, D), \bar{\Upsilon}_D).$$

3' Since $o(M')$ is a clean cutpoint of P , it follows that either D is on the M' -sequence or that the critical point of D is strictly above $o(M')$. This means that in the respective cases, either⁴²

$$i_D^{\bar{P}}(M') = \text{Ult}(M', D) \text{ or } i_D^{\bar{P}}(M') = M'.$$

4' Again in the corresponding cases, let us denote by (M^*, Σ^*) either the pair $(\text{Ult}(M', D), \Sigma'_D)$ or the pair (M', Σ') . The situation now is as follows:

- a. (P^*, Υ^*) is strongly stable,
- b. $(M^*, \Sigma^*) \trianglelefteq^* (P^*, \Upsilon^*)$,
- c. (M^*, Σ^*) is a generator for $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$,
- d. $o(M^*)$ is $r\Sigma_{k(P^*)}^{P^*}$ -regular,
- e. there exist $\eta < o(M^*)$ and $r \in P^*$ such that

$$\text{Hull}_{k(P^*)+1}^{P^*}(\eta \cup \{r\})$$

is cofinal in $o(M^*)$.

5' This suffices to run the argument of Case I and obtain a contradiction in the same way. □

6° We showed that both cases lead to a contradiction, which concludes the proof. □

⁴²For the first case, one should recall that $o(M')$ is $r\Sigma_{k(P)}^{\bar{P}}$ -regular.

We can now conclude that the direct limit associated to (K, Ψ) extends \mathcal{H}_Γ and does not project strictly across its height.

Corollary 4.4.8. *It holds that $\mathcal{H}_\Gamma \trianglelefteq \mathcal{M}_\infty(K, \Psi)$ and that*

$$\rho^-(\mathcal{M}_\infty(K, \Psi)) \geq \Theta_\Gamma.$$

Proof.

- 1° Let $\alpha < \Omega^{L(\Gamma)}$ be arbitrary. We want to show that $\mathcal{H}_\Gamma \parallel \theta_{\alpha+1}^{L(\Gamma)} \trianglelefteq \mathcal{M}_\infty(K, \Psi)$ and that $\rho^-(\mathcal{M}_\infty(K, \Psi)) > \theta_{\alpha+1}^{L(\Gamma)}$.
- 2° Let (M, Σ) be a generator for $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$ and let (N, T) be a generator for $\Delta_{\theta_{\alpha+2}}^{L(\Gamma)}$. Without loss of generality, we may assume that $(M, \Sigma) \triangleleft^* (N, T)$.
- 3° By [29, Lemma 4.4], we have that $o(M) \leq \rho^-(N)$ and that $o(M)$ is $r\Sigma_{k(N)}$ -regular in M .
- 4° Let $(\mathcal{T}, \mathcal{U})$ be a minimal comparison of (N, T) with (K, Ψ) , let (N', T') and (K', Ψ') (resp.) be the last pairs of these trees, let $M' := i^{\mathcal{T}}(M)$, and let $\Sigma' := T'_{M'}$. Lines 2° and 3° imply that (M', Σ') is a pair on the main branch of \mathcal{T} and that \mathcal{T} up to that pair can be seen as a normal tree on (M, Σ) .
- 5° Propositions 4.4.6 and 4.4.7 imply that \mathcal{U} does not drop,

$$(M', \Sigma') \triangleleft^* (N', T') \trianglelefteq^* (K', \Psi'),$$

and $\rho^-(K') \geq o(N') > o(M')$.

- 6° Lines 3° and 5° imply that $o(M')$ is $r\Sigma_{k(K')}$ -regular in K' . It follows that

$$\mathcal{M}_\infty(M', \Sigma') \trianglelefteq^* \mathcal{M}_\infty(K', \Psi')$$

and $\rho^-(\mathcal{M}_\infty(K', \Psi')) > o(\mathcal{M}_\infty(M', \Sigma'))$.

- 7° In other words,

$$\mathcal{H}_\Gamma \parallel \theta_{\alpha+1}^{L(\Gamma)} = \mathcal{M}_\infty(M', \Sigma') \trianglelefteq \mathcal{M}_\infty(K, \Psi)$$

and $\rho^-(\mathcal{M}_\infty(K, \Psi)) > \theta_{\alpha+1}^{L(\Gamma)}$, as required.

□

We are finally ready to state the description of K that we require. As pointed out in the beginning, it is necessary for us to strengthen the anti-large-cardinal hypothesis.

Proposition 4.4.9. *Assume NMLW.⁴³ Then $k(K) = 0$ and that Θ_Γ is the largest cardinal of $\mathcal{M}_\infty(K, \Psi)$.*

⁴³That is, we are assuming that there is no mouse pair with a measurable limit of Woodin cardinals.

Proof.

1° It is not possible that $\rho^-(\mathcal{M}_\infty(K, \Psi)) = \Theta_\Gamma$. For otherwise, the fact that K is strongly stable would imply that $\pi_{(K, \Psi), \infty} \upharpoonright \rho^-(K)$ is cofinal in Θ_Γ , contradicting the fact that $\text{cof}(\Theta_\Gamma) > \omega$.

2° Thus, $\rho^-(\mathcal{M}_\infty(K, \Psi)) > \Theta_\Gamma$.

3° Let us assume towards contradiction that either $k(K) > 0$ or Θ_Γ is not the largest cardinal of $\mathcal{M}_\infty(K, \Psi)$.

4° Let (K', Ψ') be a non-dropping iterate of (K, Ψ) such that for some $\delta < o(K')$,

$$\pi_{(K', \Psi'), \infty}(\delta) = \Theta_\Gamma,$$

let $P := K' \upharpoonright \delta^{+K'}$, and let $\Upsilon := \Psi'_P$. Our assumptions ensure that $(P, \Upsilon) \triangleleft (K', \Psi')$ and that $\rho^-(K') \geq o(P)$. (Note that we might have $\delta^{+K'} = o(K')$.)

5° **Claim.** (P, Υ) is a generator for Γ .

Proof.

1' It suffices to show that for all $\alpha < \Omega^{L(\Gamma)}$ and for all generators (M, Σ) for $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$, it holds that $(M, \Sigma) \leq^* (P, \Upsilon)$.

2' Let $(\mathcal{T}, \mathcal{U})$ be a minimal comparison of (M, Σ) with (K', Ψ') , with the last pairs (M'', Σ'') and (K'', Ψ'') , and let

$$Q := i^{\mathcal{U}}(P) = K'' \upharpoonright i^{\mathcal{U}}(\delta^{+K'}).$$

3' Since K is strongly stable and since δ is not measurable⁴⁴ in K , it follows that (Q, Ψ''_Q) is a non-dropping iterate of (P, Υ) .

4' By Corollary 4.4.8, it follows that $(M'', \Sigma'') \trianglelefteq (Q, \Psi''_Q)$, so we get that $(M, \Sigma) \leq^* (P, \Upsilon)$. □

6° Claim 5° contradicts the fact that $(P, \Upsilon) <^* (K, \Psi)$. □

Corollary 4.4.10. *Assume NMLW. Then some ordinal is mapped to Θ_Γ by the direct limit embedding $\pi_{(K, \Psi), \infty}$.* □

⁴⁴This is the reason why we had to strengthen our anti-large-cardinal assumption for this proposition.

4.5 Realizability Strategy

Section Assumptions 4.5.1. We are assuming that

1. $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+$,
2. there does not exist a mouse pair with a measurable limit of Woodin cardinals,
3. Γ is a closed pointclass such that $L(\Gamma) \models \text{AD}_{\mathbb{R}}$ and

$$\omega < \text{cof}(\Theta_{\Gamma}) \leq \Theta_{\Gamma} < \Theta,$$

4. (K, Ψ) is a generator for Γ . □

Note that these assumptions together with Theorem 4.2.6 imply that HPC holds. This fact will be used tacitly from now on.

Notation 4.5.2. We denote by K_{∞} the domain of $\mathcal{M}_{\infty}(K, \Psi)$. □

The goal of this section is to show that the model $L(\Gamma, K_{\infty}^{\omega})$ has a set of reals which is not in Γ . This set of reals is essentially the fragment of the strategy Ψ acting on the non-dropping⁴⁵ normal trees. In order to see that the model $L(\Gamma, K_{\infty}^{\omega})$ can reconstruct this fragment, we will have to give a sufficiently absolute description of it. The idea how to proceed comes from [11, Section 6.3]. Roughly, we have enough information in Γ to iterate initial segments of K , while for picking a non-dropping branch through a tree, we can ask whether the model associated to a branch embeds into K_{∞} in a commutative manner (i.e. whether the branch is “realizable”). In the later case, the point is that all the embeddings belong to K_{∞}^{ω} .

Notation 4.5.3. Let M be a premouse which is elementarily equivalent⁴⁶ to K . Then the ordinals λ^M and δ_{α}^M (for $\alpha \leq \lambda^M$) are defined in such a way that the sequence $(\delta_{\alpha}^M : \alpha \leq \lambda^M)$ is a continuous, strictly increasing enumeration of cutpoint Woodin cardinals and their limits in M . We also set δ^M to be $\delta_{\lambda^M}^M$ and δ_{-1}^M to be 0. □

By the results of Section 4.4, λ^M is a limit ordinal, δ^M is the largest cardinal of M , and $k(M) = 0$. Furthermore, under the direct limit embedding

$$\pi_{(K, \Psi), \infty} : K \longrightarrow \mathcal{M}_{\infty}(K, \Psi),$$

the ordinal δ^K is mapped to Θ_{Γ} , $\mathcal{M}_{\infty}(K, \Psi) \parallel \Theta_{\Gamma} = \mathcal{H}_{\Gamma}$, and the ordinals δ_{α}^K (for $\alpha < \lambda^M$) are mapped to members of the Solovay sequence of $L(\Gamma)$.

The sequence $(\delta_{\alpha}^K : \alpha \leq \lambda^M)$ naturally divides K and its iterates into windows. While using the extenders between two successive members of this sequence, we only need to

⁴⁵Recall that the non-dropping trees are exactly those which (1) have the final model and (2) the branch leading to the final model does not drop.

⁴⁶We mean that $k(M) = k(K)$ and that these premice have the same $r_{\Sigma_{k(K)}}$ -theory.

know the iteration strategy for the initial segment up to the larger member. This is what we think of as a window. The fragment of the iteration strategy for a single window is contained in Γ , but the complexity increases when changing a window. In that case, we pick the correct branch of the tree by asking whether it can be realized into K_∞ . This is an intuitive explanation why the information in $\Gamma \cup K_\infty^\omega$ is enough to reconstruct the fragment of Ψ concerning non-dropping trees.

We will now work on formalizing the intuition just given. The first step is to make sure that we can choose canonically the strategy for any single window. This will be enough for the part of the previous description concerning iterations that stay inside one window.

Lemma 4.5.4. *Suppose that*

1. M is a countable premouse elementarily equivalent to K ,
2. $\sigma : \lfloor M \rfloor \xrightarrow[\Sigma_1]{\text{cof}} K_\infty$,⁴⁷
3. $\mathcal{H}_\Gamma \in \text{ran}(\sigma)$,
4. $\xi < \lambda^M$.

Then there exists at most one strategy Υ such that $(M \parallel \delta_{\xi+1}^M, \Upsilon)$ is a generator for $\Delta_{\sigma(\delta_{\xi+1}^M)}^{L(\Gamma)}$ and $\sigma \upharpoonright (M \parallel \delta_{\xi+1}^M) = \pi_{(M \parallel \delta_{\xi+1}^M, \Upsilon), \infty}$.

Proof. This is immediate from [12, Proposition 2.1(b)]. □

Definition 4.5.5. Suppose that

1. M is a countable premouse elementarily equivalent to K ,
2. $\sigma : \lfloor M \rfloor \xrightarrow[\Sigma_1]{\text{cof}} K_\infty$,
3. $\mathcal{H}_\Gamma \in \text{ran}(\sigma)$.

We say that σ is **Γ -certified** iff for all $\xi < \lambda^M$, there exists a strategy Υ such that

$$(M \parallel \delta_{\xi+1}^M, \Upsilon)$$

is a generator for $\Delta_{\sigma(\delta_{\xi+1}^M)}^{L(\Gamma)}$ and $\sigma \upharpoonright (M \parallel \delta_{\xi+1}^M) = \pi_{(M \parallel \delta_{\xi+1}^M, \Upsilon), \infty}$. □

If the premouse M of the previous definition is a non-dropping iterate of (K, Ψ) , it will be Γ -certified. The previous lemma and definition together mean that a Γ -certified premouse M has a unique strategy with the relevant properties for each initial segment $M \parallel \delta_{\xi+1}^M$.

⁴⁷We remind the reader that $\lfloor M \rfloor$ denotes the domain of the premouse M .

Definition 4.5.6. Suppose that

1. M is a countable premouse elementarily equivalent to K ,
2. $\sigma : \lfloor M \rfloor \longrightarrow K_\infty$ is Γ -certified,
3. δ is a cutpoint Woodin cardinal of M .

Then we denote by Υ_δ^σ the unique strategy Υ such that $(M \parallel \delta, \Upsilon)$ is a generator for $\Delta_{\sigma(\delta)}^{L(\Gamma)}$ and $\sigma \upharpoonright (M \parallel \delta) = \pi_{(M \parallel \delta, \Upsilon), \infty}$. \square

Observe that the model $L(\Gamma, \sigma)$ can correctly identify each Υ_δ^σ , which means that we can simply use this strategy to iterate below δ . This was the case of iterations within a single window. We will now proceed to study the case when windows change. If we have built a normal iteration tree with a final model M and then picked an extender E outside of the current window, this means that the branch leading to M does not drop. Let us distinguish nodes where a change of the window can occur.

Definition 4.5.7. Suppose that \mathcal{T} is a countable normal tree on K . Then for $\alpha < \text{len}(\mathcal{T})$, we say that α a **terminal node** of \mathcal{T} iff the branch $[0, \alpha]_{\mathcal{T}}$ does not drop. We denote by $\text{Ter}(\mathcal{T})$ the set of all terminal nodes of \mathcal{T} . \square

If we have built a normal tree whose last node is terminal, then the tree can easily be continued in a way that changes the window. In particular, in order to go forward, we might require an iteration strategy for a larger initial segment. These strategies are derived from the realization into K_∞ , so we want to make sure to record the realizations of the terminal nodes as we build an iteration.

Definition 4.5.8. Suppose that \mathcal{T} is a countable normal tree on K . Then a $\pi_{(K, \Psi), \infty}$ -**realization** of \mathcal{T} is a sequence

$$\vec{\sigma} = (\sigma_\alpha : \alpha \in \text{Ter}(\mathcal{T})),$$

such that for all $\alpha, \beta \in \text{Ter}(\mathcal{T})$, for all $\gamma \in (\alpha, \text{len}(\mathcal{T}))$, it holds that

- a. $\sigma_\alpha : \lfloor \mathcal{M}_\alpha^{\mathcal{T}} \rfloor \longrightarrow K_\infty$ is Γ -certified,
- b. if $\alpha = 0$, then $\sigma_\alpha = \pi_{(K, \Psi), \infty}$,
- c. if $\alpha \leq_{\mathcal{T}} \beta$, then the diagram

$$\begin{array}{ccc} & K_\infty & \\ & \uparrow \sigma_\alpha & \swarrow \sigma_\beta \\ \lfloor \mathcal{M}_\alpha^{\mathcal{T}} \rfloor & \xrightarrow{i_{\alpha, \beta}^{\mathcal{T}}} & \lfloor \mathcal{M}_\beta^{\mathcal{T}} \rfloor \end{array}$$

commutes,

- d. if $\alpha + 1 < \text{len}(\mathcal{T})$, then $\text{len}(E_\alpha^\mathcal{T}) < \delta^{\mathcal{M}_\alpha^\mathcal{T}}$,
- e. for all $\xi < \lambda^{\mathcal{M}_\alpha^\mathcal{T}}$, letting $\delta := \delta_{\xi+1}(\mathcal{M}_\alpha^\mathcal{T})$, if $\mathcal{T} \upharpoonright [\alpha, \gamma]$ is a tree on $\mathcal{M}_\alpha^\mathcal{T} \parallel \delta$, then
 - i. $\mathcal{T} \upharpoonright [\alpha, \gamma]$ is according to $\Upsilon_\delta^{\sigma_\alpha}$, and
 - ii. if γ is a terminal node of \mathcal{T} , then, letting $\delta^* := i_{\alpha, \gamma}^\mathcal{T}(\delta)$, the last pair of $\mathcal{T} \upharpoonright [\alpha, \gamma]$, seen as a tree on $(\mathcal{M}_\alpha^\mathcal{T} \parallel \delta, \Upsilon_\delta^{\sigma_\alpha})$, is the pair $(\mathcal{M}_\gamma^\mathcal{T} \parallel \delta^*, \Upsilon_{\delta^*}^{\sigma_\gamma})$. \square

Definition 4.5.9. Suppose that \mathcal{T} is a countable normal tree on K . Then \mathcal{T} is $\pi_{(K, \Psi), \infty}$ -realizable iff there exists a $\pi_{(K, \Psi), \infty}$ -realization of \mathcal{T} . \square

We will be able to identify the restriction of Ψ to the non-dropping trees by characterizing it as the strategy that produces $\pi_{(K, \Psi), \infty}$ -realizable trees. The crucial point is that this characterization is absolute between V and $L(\Gamma, K_\infty^\omega)$.⁴⁸

Lemma 4.5.10. *Suppose that*

1. \mathcal{T} is a countable, normal tree on K ,
2. $\vec{\sigma}$ is a $\pi_{(K, \Psi), \infty}$ -realization of \mathcal{T} .

Then

- a. \mathcal{T} is according to Ψ , and
- b. for all $\alpha \in \text{Ter}(\mathcal{T})$, we have that $\sigma_\alpha = \pi_{(\mathcal{M}_\alpha^\mathcal{T}, \Psi_{\mathcal{T} \upharpoonright [0, \alpha]})}$.

Proof.

1° We show this by induction on the length of \mathcal{T} . To that end, let us suppose that

- a. \mathcal{T} is of the length $\gamma + 1$, where γ is a limit ordinal,
- b. that $\mathcal{T} \upharpoonright \gamma$ is according to Ψ ,
- c. for all $\alpha \in \text{Ter}(\mathcal{T}) \cap \gamma$, we have that $\sigma_\alpha = \pi_{(\mathcal{M}_\alpha^\mathcal{T}, \Psi_{\mathcal{T} \upharpoonright [0, \alpha]})}$.

We want to show that $[0, \gamma]_\mathcal{T} = \Psi(\mathcal{T} \upharpoonright \gamma)$ and that if $\gamma \in \text{Ter}(\mathcal{T})$, then

$$\sigma_\gamma = \pi_{(\mathcal{M}_\gamma^\mathcal{T}, \Psi_\mathcal{T})}$$

2° The set $W \subseteq \text{Ter}(\mathcal{T}) \cap \gamma$ and the sequence $(\delta_\alpha : \alpha \in W)$ are defined recursively so that for all $\alpha < \gamma$:

⁴⁸Note that the model $L(\Gamma, K_\infty^\omega)$ does not need to compute the embedding $\pi_{(K, \Psi), \infty}$ since it can use it as a parameter.

- a. if $\alpha \in W$, then $\delta_\alpha = \delta_{\xi+1}^{\mathcal{M}_\alpha^T}$, where $\xi < \lambda^{\mathcal{M}_\alpha^T}$ is the least such that $\text{len}(E_\alpha^T) < \delta_{\xi+1}^{\mathcal{M}_\alpha^T}$,
- b. $\alpha \in W$ if and only if for all $\beta \in W \cap \alpha$, $\mathcal{T} \upharpoonright [\beta, \alpha + 1]$ is not a tree on $\mathcal{M}_\beta^T \parallel \delta_\beta$.

Observe that $0 \in W$ and that W is closed in γ .

3° **Case I.** W is cofinal in γ .

Proof.

- 1' In this case, there is a unique cofinal branch through \mathcal{T} : the one obtain by closing W downwards for $\leq_{\mathcal{T}}$. This means that necessarily $[0, \gamma)_{\mathcal{T}} = \Psi(\mathcal{T} \upharpoonright \gamma)$ and that this branch does not drop.
- 2' It remains to verify that $\sigma_\gamma = \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}$.
- 3' Let $x \in \mathcal{M}_\gamma^T$ be arbitrary. There exists $\xi \in W$ and $\bar{x} \in \mathcal{M}_\xi^T$ such that $i_{\xi, \gamma}^T(\bar{x}) = x$.
- 4' We then have that

$$\sigma_\gamma(x) = \sigma_\gamma(i_{\xi, \gamma}^T(\bar{x})) \quad (4.1)$$

$$= \sigma_\xi(\bar{x}) \quad (4.2)$$

$$= \pi_{(\mathcal{M}_\xi^T, \Psi_{\mathcal{T} \upharpoonright [0, \xi]}, \infty)}(\bar{x}) \quad (4.3)$$

$$= \pi_{(\mathcal{M}_\xi^T, \Psi_{\mathcal{T}}), \infty}(i_{\xi, \gamma}^T(\bar{x})) \quad (4.4)$$

$$= \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(x), \quad (4.5)$$

where

(4.2) follows from property c of Definition 4.5.8 of a $\pi_{(K, \Psi), \infty}$ -realization,

(4.3) follows by the inductive hypothesis,

(4.4) follows from the fact that $\mathcal{T} \upharpoonright [\xi, \gamma]$ is a normal tree on

$$(\mathcal{M}_\xi^T, \Psi_{\mathcal{T} \upharpoonright [0, \xi]}).$$

□

4° **Case II.** W is bounded in γ .

Proof.

- 1' Since W is closed in γ , there exists the largest element α of W .
- 2' We have that $\mathcal{T} \upharpoonright [\alpha, \gamma]$ is a tree on $(\mathcal{M}_\alpha^T \parallel \delta_\alpha, \Upsilon_{\delta_\alpha}^{\sigma_\alpha})$.
- 3' Since $\sigma_\alpha = \pi_{(\mathcal{M}_\alpha^T, \Psi_{\mathcal{T} \upharpoonright [0, \alpha]}, \infty)}$, we have that $\Upsilon_{\delta_\alpha}^{\sigma_\alpha} = \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta_\alpha}$.⁴⁹

⁴⁹In other words, we are saying that the pullback strategy for $\mathcal{M}_\alpha^T \parallel \delta_\alpha$ along σ_α is exactly the same as the tail strategy of Ψ obtained by first iterating along the tree $\mathcal{T} \upharpoonright [0, \alpha]$ and then dropping to $\mathcal{M}_\alpha^T \parallel \delta_\alpha$.

4' By the property e of Definition 4.5.8, it holds that $\mathcal{T} \upharpoonright [\alpha, \gamma]$ is a tree on

$$(\mathcal{M}_\alpha^T \parallel \delta_\alpha, \Upsilon_{\delta_\alpha}^{\sigma_\alpha}).$$

5' Adding to this the equality of strategies in 3', we conclude that

$$[0, \gamma]_{\mathcal{T}} = \Upsilon_{\delta_\alpha}^{\sigma_\alpha}(\mathcal{T} \upharpoonright [\alpha, \gamma]) = \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta_\alpha}(\mathcal{T} \upharpoonright [\alpha, \gamma]).$$

6' By the internal lift consistency of Ψ , we get that

$$[0, \gamma]_{\mathcal{T}} = \Psi_{\mathcal{T} \upharpoonright [0, \alpha]}(\mathcal{T} \upharpoonright [\alpha, \gamma]) = \Psi(\mathcal{T} \upharpoonright \gamma).$$

7' It remains to show that if $\gamma \in \text{Ter}(\mathcal{T})$, then $\sigma_\gamma = \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}$.

8' Let $\delta^* := i_{\alpha, \gamma}^T(\delta_\alpha)$. We will use the following claim.

9' **Claim.** $\pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty} \upharpoonright (\mathcal{M}_\gamma^T \parallel \delta^*) = \pi_{(\mathcal{M}_\gamma^T \parallel \delta^*, \Upsilon_{\delta^*}^{\sigma_\gamma}), \infty}$

Proof. Property e of Definition 4.5.8 implies that $(\mathcal{M}_\gamma^T \parallel \delta^*, \Upsilon_{\delta^*}^{\sigma_\gamma})$ is an iterate of $(\mathcal{M}_\alpha^T \parallel \delta_\alpha, \Upsilon_{\delta_\alpha}^{\sigma_\alpha})$ via $\mathcal{T} \upharpoonright [\alpha, \gamma]$. Adding to this 4', we obtain that $\Upsilon_{\delta^*}^{\sigma_\gamma} = \Psi_{\mathcal{T}, \mathcal{M}_\gamma^T \parallel \delta^*}$. Thus,

$$\pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty} \upharpoonright (\mathcal{M}_\gamma^T \parallel \delta^*) = \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}) \parallel \delta^*, \infty} = \pi_{(\mathcal{M}_\gamma^T \parallel \delta^*, \Upsilon_{\delta^*}^{\sigma_\gamma}), \infty}.$$

□

10' Let $x \in \mathcal{M}_\gamma^T$ be arbitrary. Since $\mathcal{T} \upharpoonright [\alpha, \gamma]$ can be seen as a tree on $(\mathcal{M}_\alpha^T \parallel \delta_\alpha, \Upsilon_{\delta_\alpha}^{\sigma_\alpha})$, we have that there exists a function $f \in \mathcal{M}_\alpha^T$ and some $z \in \mathcal{M}_\gamma^T \parallel \delta^*$ such that

$$x = i_{\alpha, \gamma}^T(f)(z).$$

11' We now have that

$$\sigma_\gamma(x) = \sigma_\gamma(i_{\alpha, \gamma}^T(f)(z)) \tag{4.6}$$

$$= \sigma_\gamma(i_{\alpha, \gamma}^T(f))(\sigma_\gamma(z)) \tag{4.7}$$

$$= \sigma_\alpha(f)(\sigma_\gamma(z)) \tag{4.8}$$

$$= \pi_{(\mathcal{M}_\alpha^T, \Psi \upharpoonright [0, \alpha]), \infty}(f)(\sigma_\gamma(z)) \tag{4.9}$$

$$= \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(i_{\alpha, \gamma}^T(f))(\sigma_\gamma(z)) \tag{4.10}$$

$$= \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(i_{\alpha, \gamma}^T(f))(\pi_{(\mathcal{M}_\gamma^T \parallel \delta^*, \Upsilon_{\delta^*}^{\sigma_\gamma}), \infty}(z)) \tag{4.11}$$

$$= \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(i_{\alpha, \gamma}^T(f))(\pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(z)) \tag{4.12}$$

$$= \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(i_{\alpha, \gamma}^T(f)(z)) \tag{4.13}$$

$$= \pi_{(\mathcal{M}_\gamma^T, \Psi_{\mathcal{T}}), \infty}(x), \tag{4.14}$$

where

(4.8) follows from property c of Definition 4.5.8 of a $\pi_{(K, \Psi), \infty}$ -realization,

(4.9) follows by the inductive hypothesis,

(4.10) follows from the fact that $\mathcal{T} \upharpoonright [\alpha, \gamma]$ can be seen as a tree on $(\mathcal{M}_\alpha^T, \Psi_{\mathcal{T} \upharpoonright [0, \alpha]})$,

(4.11) follows from the property a of Definition 4.5.8 of a $\pi_{(K, \Psi), \infty}$ -realization,

(4.12) follows from Claim 9'.

12' This concludes the verification of Case II. □

5° The verification of the two cases concludes the proof of the lemma. □

Corollary 4.5.11. *Let \mathcal{T} be a countable, non-dropping, normal tree on K . Then \mathcal{T} is according to Ψ if and only if it is $\pi_{(K, \Psi), \infty}$ -realizable.*

Proof. The implication (\Leftarrow) follows from the lemma. For the other direction, let \mathcal{T} be according to Ψ . We claim that

$$\vec{\sigma} := (\pi_{(\mathcal{M}_\alpha^T, \Psi_{\mathcal{T} \upharpoonright [0, \alpha]})} : \alpha \in \text{Ter}(\mathcal{T}))$$

is a $\pi_{(K, \Psi), \infty}$ -realization of \mathcal{T} . Our assumption on the extenders of K ensures that for all $\alpha \in \text{Ter}(\mathcal{T})$ such that $\alpha + 1 < \text{len}(\mathcal{T})$, we have that $\text{len}(E_\alpha^T) < \delta^{\mathcal{M}_\alpha^T}$. This shows the property d of Definition 4.5.8, while properties b and c are obvious. The rest will follow once we verify the following. For all $\alpha \in \text{Ter}(\mathcal{T})$ and for all $\xi < \lambda^{\mathcal{M}_\alpha^T}$, letting $\delta := \delta_{\xi+1}(\mathcal{M}_\alpha^T)$, the pair

$$(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta})$$

is a generator for $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$ and

$$\sigma \upharpoonright (\mathcal{M}_\alpha^T \parallel \delta) = \pi_{(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta})}.$$

The verification of this is given below.

1° Let us assume otherwise. Then, letting (M, Σ) be a generator for $\Delta_{\theta_{\alpha+1}}^{L(\Gamma)}$, we have that either

$$(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta}) <^* (M, \Sigma) \text{ or } (M, \Sigma) <^* (\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta}).$$

2° Let $\theta := \pi_{(\mathcal{M}_\alpha^T, \Psi), \infty}(\delta)$. Then θ is a successor member of the Solovay sequence of $L(\Gamma)$ and

$$\mathcal{M}_\infty(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta}) = \mathcal{M}_\infty(K, \Psi) \parallel \theta = \mathcal{H}_\Gamma \parallel \theta = \mathcal{M}_\infty(M, \Sigma).$$

3° If it was the case that $(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta}) <^* (M, \Sigma)$, we would have that

$$(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta}) \in \Delta_\theta^{L(\Gamma)}$$

and consequently,

$$o(\mathcal{M}_\infty(\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta})) < \theta.$$

This is clearly a contradiction.

4° Thus, the only remaining possibility is that

$$(M, \Sigma) <^* (\mathcal{M}_\alpha^T \parallel \delta, \Psi_{\mathcal{T} \upharpoonright [0, \alpha], \mathcal{M}_\alpha^T \parallel \delta}).$$

5° Let $(N, \mathbb{T}) \triangleleft^* (P, \Upsilon)$ be the result of a minimal comparison witnessing the above inequality. It follows from [29, Lemma 4.4] that neither of the trees in this comparison drop, that $o(N) \leq \rho^-(P)$, and that $o(N)$ is $r\Sigma_{k(P)}^P$ -regular.

6° This suffices for the conclusion that $\mathcal{M}_\infty(N, \mathbb{T}) \triangleleft \mathcal{M}_\infty(P, \Upsilon)$, which again contradicts 2°.

□

The fact just established, together with the absoluteness of the $\pi_{(K, \Psi), \infty}$ -realizability, gives us the absoluteness of the relevant fragment of Ψ .

Corollary 4.5.12. *The restriction of Ψ to non-dropping normal trees belongs to $L(\Gamma, K_\infty^\omega)$.*

□

This fragment of Ψ is coded by a set of reals which cannot belong to Γ : the reason for this is the ability to define from it a surjection $\mathbb{R} \twoheadrightarrow \mathcal{H}_\Gamma$. The surjection in question is just the one coming from the direct limit construction for $\mathcal{M}_\infty(K, \Psi)$ and it witnesses that Θ_Γ is not the “real” Θ .

Corollary 4.5.13. *There exists a set of reals in $L(\Gamma, K_\infty^\omega)$ which is not in Γ .*

Proof.

1° Let \mathcal{F} consist of all non-dropping normal trees on (K, Ψ) . The mapping

$$\mathcal{F} \longrightarrow \mathcal{F}(K, \Psi) : \mathcal{T} \mapsto (\mathcal{M}_\infty^T, \Psi_{\mathcal{T}})$$

is a bijection.⁵⁰

2° Corollary 4.5.12 implies that $\mathcal{F} \in L(\Gamma, K_\infty^\omega)$.

⁵⁰We remind the reader that $\mathcal{F}(K, \Psi)$ denotes the set of all of non-dropping iterates of (K, Ψ) .

3° For $\mathcal{T}, \mathcal{T}' \in \mathcal{F}$, we define $\mathcal{T} \leq_{\mathcal{F}} \mathcal{T}'$ iff there exists a non-dropping $\pi_{(\mathcal{M}^{\mathcal{T}}, \Psi_{\mathcal{T}}), \infty}$ -realizable tree \mathcal{U} on $\mathcal{M}_{\infty}^{\mathcal{T}}$ such that the stack $\mathcal{T} \sim \mathcal{U}$ normalizes to \mathcal{T}' .

4° Note that for all $\mathcal{T} \in \mathcal{F}$, $\mathcal{M}_{\infty}(\mathcal{M}^{\mathcal{T}}, \Psi_{\mathcal{T}}) = K_{\infty}$ and consequently,

$$L(\Gamma, K_{\infty}^{\omega})$$

can correctly identify $\pi_{(\mathcal{M}^{\mathcal{T}}, \Psi_{\mathcal{T}}), \infty}$ -realizable trees. In particular, $\leq_{\mathcal{F}} \in L(\Gamma, K_{\infty}^{\omega})$.

5° Observe that for all $\mathcal{T}, \mathcal{T}' \in \mathcal{F}$, we have that $\mathcal{T} \leq_{\mathcal{F}} \mathcal{T}'$ if and only if $(\mathcal{M}_{\infty}^{\mathcal{T}'}, \Psi_{\mathcal{T}'})$ is a non-dropping iterate of $(\mathcal{M}_{\infty}^{\mathcal{T}}, \Psi_{\mathcal{T}})$. Furthermore, if \mathcal{U} witnesses that $\mathcal{T} \leq_{\mathcal{F}} \mathcal{T}'$, then \mathcal{U} is equal to the unique normal tree on $(\mathcal{M}_{\infty}^{\mathcal{T}}, \Psi_{\mathcal{T}})$ with the last pair $(\mathcal{M}_{\infty}^{\mathcal{T}'}, \Psi_{\mathcal{T}'})$.

6° This means that for $\mathcal{T} \leq_{\mathcal{F}} \mathcal{T}'$, we can define

$$\pi_{\mathcal{T}, \mathcal{T}'} := \pi^{\mathcal{U}} : \mathcal{M}_{\infty}^{\mathcal{T}} \longrightarrow \mathcal{M}_{\infty}^{\mathcal{T}'},$$

where \mathcal{U} is the unique tree witnessing that $\mathcal{T} \leq_{\mathcal{F}} \mathcal{T}'$.

7° In this way we obtain a family of embeddings $(\pi_{\mathcal{T}, \mathcal{T}'} : \mathcal{T} \leq_{\mathcal{F}} \mathcal{T}')$ which belongs to $L(\Gamma, K_{\infty}^{\omega})$.

8° On the other hand, for all $\mathcal{T} \leq_{\mathcal{F}} \mathcal{T}'$,

$$\pi_{\mathcal{T}, \mathcal{T}'} = \pi_{(\mathcal{M}_{\infty}^{\mathcal{T}}, \Psi_{\mathcal{T}}), (\mathcal{M}_{\infty}^{\mathcal{T}'}, \Psi_{\mathcal{T}'})}.$$

9° Thus, the family

$$(\mathcal{M}_{\infty}^{\mathcal{T}}, \pi_{\mathcal{T}, \mathcal{T}'} : \mathcal{T}, \mathcal{T}' \in \mathcal{F}, \mathcal{T} \leq_{\mathcal{F}} \mathcal{T}')$$

belongs to $L(\Gamma, K_{\infty}^{\omega})$ and forms a directed system which is equivalent to the standard directed system leading to $\mathcal{M}_{\infty}(K, \Psi)$.

10° Now, for each $x \in \mathbb{R}$ which codes a pair (\mathcal{T}, α) , where $\mathcal{T} \in \mathcal{F}$ and $\alpha < \delta^{\mathcal{M}_{\infty}^{\mathcal{T}}}$, we can map x to

$$\pi_{\mathcal{T}, \infty}(x) \in K_{\infty}.$$

This defines a surjection onto Θ_{Γ} inside $L(\Gamma, K_{\infty}^{\omega})$, which concludes the proof. □

4.6 Adding Sequences and Measures

Section Assumptions 4.6.1. We are assuming that

1. $V = L(\mathcal{P}(\mathbb{R})) \models \text{AD}^+ + \text{AD}_{\mathbb{R}}$,

2. there does not exist a mouse pair with a measurable limit of Woodin cardinals,
3. Γ is a closed pointclass such that $L(\Gamma) \models \text{AD}_{\mathbb{R}}$ and

$$\omega < \text{cof}(\Theta_{\Gamma}) \leq \Theta_{\Gamma} < \Theta.$$

□

The goal of this section is to show that for some $\kappa < \Theta$, the model

$$L(\Gamma, \kappa^{\omega})[\mu_{\kappa}]$$

contains a set of reals which is not in Γ .⁵¹ By the results of the previous section, it will suffice to find $\kappa < \Theta$ so that K_{∞}^{ω} belongs to $L(\Gamma, \kappa^{\omega})[\mu_{\kappa}]$, where K_{∞} comes from a generator (K, Ψ) for Γ . It will turn out that we can take κ to be the ordinal height of K_{∞} .

The first step is to show that K_{∞} belongs to $L(\Gamma, \kappa^{\omega})[\mu_{\kappa}]$. The argument that we have in mind is to give a sufficiently absolute description of this set. We will represent K_{∞} as a μ_{κ} -ultraproduct of the transitive collapses of the hulls of X inside $\mathcal{M}_{\infty}(K, \Psi)$, where X varies over the countable subsets of κ . The point here is that this family of collapsed hulls is equal almost everywhere to a family inside $L(\Gamma, \kappa^{\omega})[\mu_{\kappa}]$. More concretely, almost every collapsed hull of $\mathcal{M}_{\infty}(K, \Psi)$ is a non-dropping iterate of (K, Ψ) and its part below its largest cardinal is just the corresponding hull of \mathcal{H}_{Γ} . This means that the only remaining issue is to describe the parts of the iterates of (K, Ψ) above their largest cardinals in a way that relativizes to $L(\Gamma, \kappa^{\omega})[\mu_{\kappa}]$.

Thus, the first problem that we will address is, given a generator for Γ , compute the domain of its premouse component from its part up to its largest cardinal inside $L(\Gamma)$. Denoting by (K, Ψ) the generator, we would like to say that K can be obtained by stacking the premouse components of all mouse pairs that belong to $L(\Gamma)$, extend $(K, \Psi)|\delta^K$, and project to δ^K . This corresponds to the notion of the *lower part operator*, but it seems that this scenario does not fully work out in the least branch hierarchy. For now, let us distinguish the pairs that we want to stack.

Definition 4.6.2. Let (K, Ψ) be a generator for Γ and let (N, T) be a mouse pair in $L(\Gamma)$. We say that (N, T) **anticipates** (K, Ψ) iff $(K, \Psi)|\delta^K \leq^* (N, T)$ and $\rho^-(N) = \rho_{\omega}(N) = \delta^K$. □

We would like to say that if (N, T) and (P, Υ) both anticipate (K, Ψ) , then one of the premice N and P is an initial segment of the other and the both are initial segments of K , but this does not seem to be true. Nevertheless, we will be able to show that either N and P have the same domain or one of them is an *element* of the other. We will also establish that both of N and P must be elements of K . Thus, the premouse components of the anticipating pairs all align with each other and with K as well, but with respect the membership relation only. Furthermore, we will be able to argue that the domain of K is

⁵¹We remind the reader that μ_{κ} is the quasi-club filter on $[\kappa]^{\omega}$ introduced in Definition 4.3.6. This filter is an ultrafilter by Proposition 4.3.12.

the union of the definable powersets all these premisses, therefor obtaining a lower-part-like characterization of the *set underlining* K . This will suffice for the purposes of this chapter. What follows below, excepting the final theorem of the section, is verifying what we have just elaborated. We start from a useful technical observation.

Lemma 4.6.3. *Suppose that*

1. (K, Ψ) is a generator for Γ ,
2. (N, T) anticipates (K, Ψ) .

Then $\bar{\mathfrak{C}}_{k(N)}(N) = N$.⁵²

Proof. Otherwise, $\rho^-(N)$ would be a measurable limit of Woodin cardinals in $\bar{\mathfrak{C}}_{k(N)}(N)$. This is not possible since we are assuming that there is no mouse pair with a measurable limit of Woodin cardinals. \square

If a pair (N, T) anticipates (K, Ψ) , then N and K agree up to δ^K , but might be incomparable above it. It turns out that we can produce a non-dropping tree on their common part below δ^K which aligns them. More precisely, when this tree is lifted to N and K , the final model corresponding to N is an initial segment of the final model corresponding to K .

Lemma 4.6.4. *Suppose that*

1. (K, Ψ) is a generator for Γ ,
2. (N, T) anticipates (K, Ψ) .

Then there exists a non-dropping, normal tree \mathcal{T} on $(K, \Psi)|\delta^K$ such that, when regarded as a tree on (N, T) and (K, Ψ) , it has the last models N^* and K^* (resp.) satisfying that $N^* \triangleleft K^*$.⁵³

Proof.

- 1^o If N is not strongly stable, then, by Lemma 4.6.3, letting D be the order zero measure on $\eta_{k(N)}^N$, we have that

$$(\text{Ult}(N, D), T_{(D)})$$

is a strongly stable mouse pair.

⁵²In other words, N is strongly sound.

⁵³Since N projects to δ^K , the lifting of a tree on $K|\delta^K$ simply means applying exactly the same extenders. The similar fact is true for K due to δ^K being its largest cardinal.

- 2° These facts, together with the comparison for strongly stable mouse pairs and the full normalization⁵⁴, imply that there are a normal tree \mathcal{U} on (N, \mathbb{T}) with the last pair (N^*, \mathbb{T}^*) and a normal tree \mathcal{V} on (K, Ψ) with the last pair (K', Ψ') such that the main branch of \mathcal{U} does not drop and $(N^*, \mathbb{T}^*) \triangleleft^* (K', \Psi')$.
- 3° There exists the largest ordinal $\gamma_{\mathcal{U}}$ such that $\mathcal{U} \upharpoonright \gamma_{\mathcal{U}}$ can be regarded as a tree on $K \upharpoonright \delta^K$. There exists the analogous ordinal $\gamma_{\mathcal{V}}$ with respect to \mathcal{V} .
- 4° We have that $\gamma_{\mathcal{U}}$ and $\gamma_{\mathcal{V}}$ are on the main branches of \mathcal{U} and \mathcal{V} (resp.).
- 5° Let $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{V}}$ be the trees $\mathcal{U} \upharpoonright [0, \gamma_{\mathcal{U}}]$ and $\mathcal{V} \upharpoonright [0, \gamma_{\mathcal{V}}]$ (resp.) regarded as trees on

$$(K, \Psi) \upharpoonright \delta^K.$$

These are non-dropping, normal trees on $(K, \Psi) \upharpoonright \delta^K$.

- 6° All extenders $E_{\xi}^{\mathcal{U}}$ for $\xi \geq \gamma_{\mathcal{U}}$ have the critical point strictly larger than $o(\mathcal{M}_{\gamma_{\mathcal{U}}}^{\mathcal{T}_{\mathcal{U}}})$ and all extenders $E_{\xi}^{\mathcal{V}}$ for $\xi \geq \gamma_{\mathcal{V}}$ have the critical point strictly larger than $o(\mathcal{M}_{\gamma_{\mathcal{V}}}^{\mathcal{T}_{\mathcal{V}}})$. In both cases, the said height is a limit of Woodin cardinals and consequently, a clean cutpoint in $\mathcal{M}_{\gamma_{\mathcal{U}}}^{\mathcal{U}}$ (resp. $\mathcal{M}_{\gamma_{\mathcal{V}}}^{\mathcal{V}}$).
- 7° Now, since $(N^*, \mathbb{T}^*) \triangleleft (K', \Psi')$, we have that one of the final pairs of $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{V}}$ is an initial segment of the other. Since these two mouse pairs are equivalent, they must be equal.
- 8° This has as a consequence that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{V}}$. Letting \mathcal{T} denote this tree and (M^*, Σ^*) be its last pair, we have that $\mathcal{T} \trianglelefteq \mathcal{U}, \mathcal{V}$ and

$$(M^*, \Sigma^*) \trianglelefteq (N^*, \mathbb{T}^*), (K', \Psi').$$

- 9° Since \mathcal{U} is non-dropping, we must have that in fact $\mathcal{U} = \mathcal{T} \trianglelefteq \mathcal{V}$.
- 10° Let us denote by K^* the model $\mathcal{M}_{\gamma_{\mathcal{V}}}^{\mathcal{V}}$, i.e. the last model of \mathcal{T} when considered on $(K, \Psi) \upharpoonright \delta^K$. The fact that all extenders $E_{\xi}^{\mathcal{V}}$ for $\xi \geq \gamma_{\mathcal{V}}$ have the critical point strictly larger than $o(M^*)$ implies that $\mathcal{P}(o(M^*)) \cap K' \subseteq K^*$.
- 11° Since $\rho^-(N^*) = o(M^*)$, we can conclude that $N^* \triangleleft K^*$. This shows that the tree \mathcal{T} is as required.

□

Even though we do not seem to be able to pull this alignment along \mathcal{T} back to the starting preimage N and K , we are able to pull back the alignment with respect to the membership. We will prove this after stating another useful technical observation.

Lemma 4.6.5. *Suppose that*

⁵⁴Our reference for the full normalization is [37].

1. (K, Ψ) is a generator for Γ ,
2. (N, T) anticipates (K, Ψ) ,
3. \mathcal{T} is a non-dropping normal tree on $K \parallel \delta^K$,
4. \mathcal{U} and \mathcal{V} are the tree \mathcal{T} seen as a tree on (N, T) and (K, Ψ) , respectively.

Then for all $\alpha \leq \beta$ on the main branch of \mathcal{T} ,

- a. $\mathcal{M}_\alpha^\mathcal{T}$ is an initial segment of both $\mathcal{M}_\alpha^\mathcal{U}$ and $\mathcal{M}_\alpha^\mathcal{V}$,
- b. $\rho^-(\mathcal{M}_\alpha^\mathcal{U}) = o(\mathcal{M}_\alpha^\mathcal{T})$,
- c. $i_{\alpha,\beta}^\mathcal{T} = i_{\alpha,\beta}^\mathcal{U} \upharpoonright \mathcal{M}_\alpha^\mathcal{T} = i_{\alpha,\beta}^\mathcal{V} \upharpoonright \mathcal{M}_\alpha^\mathcal{T}$. □

Proposition 4.6.6. *Suppose that*

1. (K, Ψ) is a generator for Γ ,
2. (N, T) anticipates (K, Ψ) .

Then $N \in K$

Proof.

- 1° By Lemma 4.6.4, there exists a non-dropping, normal tree \mathcal{T} on $(K, \Psi) \parallel \delta^K$ such that, when lifted to a tree \mathcal{U} on (N, T) with final model \tilde{N} and a tree \mathcal{V} on (K, Ψ) with the final model \tilde{K} , we have $\tilde{N} \triangleleft \tilde{K}$. In particular, $\tilde{N} \in \tilde{K}$.
- 2° Let ζ on the main branch of \mathcal{T} be the least such that

$$\mathcal{M}_\zeta^\mathcal{U} \in \mathcal{M}_\zeta^\mathcal{V}.$$

We will be done if we show that $\zeta = 0$.

- 3° Let $N^* := \mathcal{M}_\zeta^\mathcal{U}$ and let $K^* := \mathcal{M}_\zeta^\mathcal{V}$.
- 4° **Claim.** ζ is not a successor.

Proof.

- 1' Let us assume otherwise, i.e. that there exists γ such that $\zeta = \gamma + 1$.
- 2' Let γ' be the \mathcal{T} -predecessor of $\gamma + 1$, let $N' := \mathcal{M}_{\gamma'}^\mathcal{U}$, let $K' := \mathcal{M}_{\gamma'}^\mathcal{V}$, and let $E := E_{\gamma'}^\mathcal{T}$. We have that $N' \notin K'$ and that

$$\text{Ult}(N', E) = N^* \in K^* = \text{Ult}(K', E).$$

3' We will reach a contradiction by showing that $N' \in K'$. To that end, it suffices to show that $\text{Th}_{k(N')}^{N'}(\rho^-(N') \cup \{p_{k(N)}(N')\}) \in K'$.

4' Let ϕ be a formula, let $s \in [\rho^-(N')]^{<\omega}$, and let

$$N' \models \phi[s, p_{k(N)}(N')].$$

5' It follows by the elementarity that

$$N^* \models \phi[i_E^{N'}(s), p_{k(N)}(N^*)].$$

6' There exist an $r\Sigma_{k(K')}^{K'}$ -function $f : \kappa \rightarrow K'$ and $\alpha < \lambda_E$ such that $N^* = i_E^{K'}(f)(\alpha)$.

7' Lemma 4.6.5 implies that $i_E^{N'}(s) = i_E^{K'}(s)$. Adding to this 5', we conclude that

$$i_E^{K'}(f)(\alpha) \models \phi[i_E^{K'}(s), p_{k(N)}(i_E^{K'}(f)(\alpha))].$$

8' It follows that for E_α -almost every $\xi < \kappa_E$,

$$f(\xi) \models \phi[s, p_{k(N)}(f(\xi))].$$

9' Thus,

$$\text{Th}_{k(N')}^{N'}(\rho^-(N') \cup \{p_{k(N)}(N')\})$$

consists of all those ϕ such that for E_α -almost every $\xi < \kappa_E$,

$$\phi \in \text{Th}_{k(N')}^{f(\xi)}(\rho^-(N') \cup \{p_{k(N)}(f(\xi))\}).$$

10' Since E is close to K' , we get that

$$\text{Th}_{k(N')}^{N'}(\rho^-(N') \cup \{p_{k(N)}(N')\}) \in K'$$

and consequently, that $N' \in K'$, which is a contradiction. □

5° **Claim.** ζ is not a limit.

Proof.

1' Let us assume otherwise.

2' Since $N^* \in K^*$, there exist $\gamma \in [0, \zeta)_\mathcal{T}$ and $N'' \in \mathcal{M}_\gamma^\mathcal{V}$ such that

$$i_{\gamma, \zeta}^\mathcal{V}(N'') = N^*.$$

3' Let $N' := \mathcal{M}_\gamma^\mathcal{U}$ and $K' := \mathcal{M}_\gamma^\mathcal{V}$. We will reach a contradiction by showing that $N' \in K'$.

4' Let $\rho := \rho^-(N') = o(\mathcal{M}_\gamma^T)$. Since $N'' \in K'$, it suffices to show that

$$N' = \text{cHull}_{k(N)}^{N''}(\rho \cup \{p_{k(N)}(N'')\}),$$

which in turn comes down to showing that

$$\text{Th}_{k(N)}^{N'}(\rho \cup \{p_{k(N)}(N')\}) = \text{Th}_{k(N)}^{N''}(\rho \cup \{p_{k(N)}(N'')\}).$$

5' Suppose that ϕ is a formula, $s \in [\rho]^{<\omega}$, and

$$N' \models \phi[s, p_{k(N)}(N')].$$

6' By the elementarity of $i_{\gamma, \zeta}^U$, it follows that

$$N^* \models \phi[i_{\gamma, \zeta}^U(s), p_{k(N)}(N^*)].$$

7' Lemma 4.6.5 implies that $i_{\gamma, \zeta}^U(s) = i_{\gamma, \zeta}^V(s)$. Adding to this line 2', we conclude that

$$i_{\gamma, \zeta}^V(N'') \models \phi[i_{\gamma, \zeta}^V(s), p_{k(N)}(i_{\gamma, \zeta}^V(N''))].$$

8' It follows by the elementarity of $i_{\gamma, \zeta}^V$ that

$$N'' \models \phi[s, p_{k(N)}(N'')].$$

9' Since ϕ and s were arbitrary, we conclude that

$$\text{Th}_{k(N)}^{N'}(\rho \cup \{p_{k(N)}(N')\}) = \text{Th}_{k(N)}^{N''}(\rho \cup \{p_{k(N)}(N'')\}),$$

which in turn yields $N' \in K'$, as explained in line 4'. We have reached a contradiction. □

6° Thus, the only remaining possibility is that $\zeta = 0$, which means that

$$N = N^* \in K^* = K. \quad \square$$

This shows that the anticipating pairs align with the generator. Essentially the same arguments show that they also align among themselves.

Proposition 4.6.7. *Suppose that*

1. (K, Ψ) is a generator for Γ ,
2. (N, \mathbb{T}) and (P, Υ) anticipate (K, Ψ) .

Then either $\hat{N} \in \hat{P}$, or $\hat{N} = \hat{P}$, or $\hat{P} \in \hat{N}$.⁵⁵

Proof. Essentially the same proof as that of Lemma 4.6.4 shows that there exists a non-dropping normal tree on $(K, \Psi)|\delta^K$ such that, when lifted to the tree on (N, T) with the final model N^* and to the tree on (P, Υ) with the final model P^* , it holds that either $N^* \leq^* P^*$ or $P^* \leq^* N^*$. If $o(N^*) = o(P^*)$, then it follows that $\hat{N} = \hat{P}$. Hence, we may assume without loss of generality that $o(N^*) < o(P^*)$, i.e. that $N^* \in P^*$. We can now run the proof of Proposition 4.6.6 to establish that in fact $N \in P$. \square

As a corollary, we obtain a lower-part-like characterization of the generator.

Corollary 4.6.8. *Let (K, Ψ) be a generator for Γ . Then the domain of K is equal to the union of all $J_1(N)$, where (N, T) anticipates (K, Ψ) .*

Proof.

- 1° If (N, T) anticipates (K, Ψ) , then it holds by Proposition 4.6.6 that $J_1(N) \subseteq K$. This verifies one inclusion.
- 2° For the other inclusion, it suffices to show that for all $A \in \mathcal{P}(\delta^K) \cap K$, there exists a mouse pair (N, T) which anticipates (K, Ψ) so that $A \in J_1(N)$.⁵⁶
- 3° Since $A \in \mathcal{P}(\delta^K) \cap K$, there exists $\alpha < \hat{o}(K)$ such that $\rho_\omega(K|\alpha) = \delta^K$ and $A \in J_1(K|\alpha)$.
- 4° Let $k < \omega$ be such that $\rho_k(K|\alpha) = \delta^K$ and let $(N, T) := (K, \Psi)|\alpha$. It follows from Lemma 4.4.5 that $(N, T) \in L(\Gamma)$, so (N, T) anticipates (K, Ψ) .
- 5° Since $A \in J_1(N)$, this verifies the other inclusion.

\square

We are finally ready to state the main result of the section.

Theorem 4.6.9. *Let (K, Ψ) be a generator for Γ and let κ be the height of $\mathcal{M}_\infty(K, \Psi)$. Then $L(\Gamma, \kappa^\omega)[\mu_\kappa]$ contains a set of reals which is not in Γ .*

Proof.

- 1° Let us assume otherwise and let $\mathcal{M} := L(\Gamma, \kappa^\omega)[\mu_\kappa]$. We have that

$$\text{a. } \mathcal{P}(\mathbb{R}) \cap \mathcal{M} = \Gamma,$$

⁵⁵Recall that \hat{N} denotes the bare premouse underlining N , i.e. the structure obtained from N by disregarding the soundness degree $k(N)$.

⁵⁶The reason why this is sufficient is that δ^K is the largest cardinal of K .

- b. $\kappa^\omega \subseteq \mathcal{M}$,
- c. $\mu_{\kappa}^{\mathcal{M}} = \mu_{\kappa} \cap \mathcal{M}$ is a supercompactness measure in \mathcal{M} .

2° For $X \in [\kappa]^\omega$, let

$$\sigma_X : \mathcal{K}_X \longrightarrow \mathcal{M}_\infty(K, \Psi)$$

be the anti-collapse of $\text{Hull}_1^{\mathcal{M}_\infty(K, \Psi)}(X)$, where we are assuming the pullback premouse structure on \mathcal{K}_X .

3° **Claim.** $\prod_{\mu_{\kappa}} \mathcal{K}_X = \mathcal{M}_\infty(K, \Psi)$

Proof.

1' For all $X \in [\kappa]^\omega$ and for all $x \in \mathcal{K}_X$, there exists the least pair $(t_{X,x}, p_{X,x})$, with $t_{X,x}$ a Skolem term and $p_{X,x} \in [X]^{<\omega}$, such that

$$\sigma_X(x) = t_{X,x}(p_{X,x}).$$

2' For all $f \in \prod_{X \in [\kappa]^\omega} \mathcal{K}_X$, there exists a unique pair (t_f, p_f) such that for μ_{κ} -almost every X ,

$$(t_{X,f(X)}, p_{X,f(X)}) = (t_f, p_f).$$

(This follows from the fact that μ_{κ} is a supercompactness measure.)

3' Let \mathcal{K} be the ultraproduct $\prod_{\mu_{\kappa} \cap \mathcal{M}} \mathcal{K}_X$, as computed inside \mathcal{M} . The mapping

$$j : \mathcal{K} \longrightarrow \mathcal{M}_\infty(K, \Psi) : [f] \mapsto t_f(p_f)$$

is well defined and elementary.

4' We want to see that j is a surjection. Let $y \in \mathcal{M}_\infty(K, \Psi)$ be arbitrary and let us find $f \in \prod_{X \in [\kappa]^\omega} \mathcal{K}_X \cap \mathcal{M}$ such that $j([f]) = y$.

5' For all $X \in [\kappa]^\omega$ satisfying that $y \in \text{Hull}_1^{\mathcal{M}_\infty(K, \Psi)}(X)$, we let $f(X) := \sigma_X^{-1}(y)$. Function f is defined μ_{κ} -almost everywhere.

6' It follows now that $j([f]) = y$. This shows that j is surjective.

7' Thus, j is the identity, which suffices for the conclusion of the claim. □

4° For $X \in [\kappa]^\omega$, let us denote by

- a. $\pi_X : \mathcal{H}_X \longrightarrow \mathcal{H}_\Gamma$ the anti-collapse of $\text{Hull}_1^{\mathcal{H}_\Gamma}(X \cap \Theta_\Gamma)$,
- b. Ξ_X the π_X -pullback strategy on \mathcal{H}_X ,⁵⁷

⁵⁷Note that $\sup \pi_X[o(\mathcal{H}_X)] < \Theta_\Gamma$, so this pullback is a well-defined, complete strategy.

c. K_X the union of all $J_1(N)$, where (N, T) is a mouse pair satisfying that

$$(\mathcal{H}_X, \Xi_X) \triangleleft^* (N, T)$$

and $\rho^-(N) = \rho_\omega(N) = o(\mathcal{H}_X)$.⁵⁸

Observe that $(K_X : X \in [\kappa]^\omega) \in \mathcal{M}$.

5° **Claim.** For μ_κ -almost every X , the set K_X is the domain of the premouse \mathcal{K}_X .

Proof.

1' Let A consist of the sets $\text{ran}(\pi_{(K', \Psi'), \infty})$, where (K', Ψ') is an arbitrary non-dropping iterate of (K, Ψ) . Proposition 4.3.13 implies that $A \in \mu_{K_\infty}$.

2' It follows from Lemma 4.3.14 that the set

$$B := \{Y \cap \kappa : Y \in A\}$$

belongs to μ_κ .

3' The fact that every element of a premouse is Σ_1 -definable from a finite sequence of ordinals yields that for all $X \in B$, the set $\text{Hull}_1^{\mathcal{M}_\infty(K, \Psi)}(X)$ is the unique $Y \in A$ such that $X = Y \cap \kappa$.

4' This shows that for all $X \in B$, it (\mathcal{K}_X, Ψ_X) is a non-dropping iterate of (K, Ψ) .

5' Reasoning similarly to the line 3', we conclude that for all $X \in B$,

$$\text{Hull}_1^{\mathcal{M}_\infty(K, \Psi)}(X) \cap \mathcal{H}_\Gamma = \text{Hull}_1^{\mathcal{H}_\Gamma}(X \cap \Theta_\Gamma).$$

6' The last two lines mean that for all $X \in B$, the mouse pair

$$(\mathcal{K}_X, \Psi_X) \upharpoonright \delta^{\mathcal{K}_X}$$

is equal to (\mathcal{H}_X, Ξ_X) and that the domain of \mathcal{K}_X is equal to the union of $J_1(N)$, where (N, T) is an arbitrary mouse pair anticipating (K, Ψ) . For the second conclusion, we used Corollary 4.6.8.

7' What has been just said is simply another way of saying that for all $X \in B$, the domain of \mathcal{K}_X is equal to K_X . Since B belongs to μ_κ , the claim is proven. □

6° Let

$$j : \left(\prod_{\mu_\kappa} K_X \right)^{\mathcal{M}} \longrightarrow \prod_{\mu_\kappa} [\mathcal{K}_X]$$

be the canonical embedding. This embedding is Σ_0 -elementary.⁵⁹

⁵⁸Observe that this is how we express “ (N, T) anticipates (K, Ψ) ” inside \mathcal{M} .

⁵⁹This comes down to the fact that j commutes with rudimentary functions.

7° **Claim.** j is surjective.

Proof.

- 1' Let $y \in \prod_{\mu_\kappa} [\mathcal{K}_X]$ be arbitrary and let us show that $y \in \text{ran}(j)$.
- 2' For μ_κ -almost every X , the value $\sigma_X^{-1}(y)$ is a well-defined element of K_X .
- 3' For such an X , we define $\hat{N}_{X,y}$ to be the \in -least bare premouse \hat{N} coming from a mouse pair (N, T) anticipating (K, Ψ) and satisfying that $\sigma_X^{-1}(y) \in J_1(\hat{N})$ (cf. Proposition 4.6.7).
- 4' We want to show that the family $(\hat{N}_{X,y})_X$ is almost everywhere equal to some family in \mathcal{M} .
- 5' Let $\alpha := o(\prod_{\mu_\kappa} \hat{N}_{X,y}) < o(\mathcal{M}_\infty(K, \Psi))$. We have that α is represented in the ultraproduct $\prod_{\mu_\kappa} [\mathcal{K}_X]$ by the function

$$h : [\kappa]^\omega \longrightarrow \text{Ord} : X \mapsto \text{otp}(\alpha \cap X).$$

- 6' Observe that $h \in \mathcal{M}$ and that for μ_κ -almost every X , $\hat{N}_{X,y}$ the \in -least bare premouse coming from a mouse pair (N, T) which anticipates (K, Ψ) and satisfies that $o(\hat{N}) = h(X)$. This shows that the function $X \mapsto \hat{N}_{X,y}$ is equal μ_κ -almost everywhere to some function in \mathcal{M} .
- 7' Since μ_κ is a supercompactness measure, there exist a Skolem term t_y and a $p_y \in [\alpha]^{<\omega}$ such that for μ_κ -almost every X ,

$$\sigma_X^{-1}(y) = t_y^{\hat{N}_{X,y}}(\sigma_X^{-1}(p_y)).$$

- 8' For μ_κ -almost every X ,

$$\sigma_X^{-1}(p_y) = \{\sigma_X^{-1}(\xi) : \xi \in p_y\} = \{\text{otp}(\xi \cap X) : \xi \in p_y\}.$$

Thus, the function $(\sigma_X^{-1}(p_y))_X$ belongs to \mathcal{M} up to a negligible permutation.

- 9' The previous three points show that the function $(\sigma_X^{-1}(y))_X$ belongs to \mathcal{M} up to a negligible permutation. The corresponding equivalence class represents an element in $(\prod_{\mu_\kappa} K_X)^{\mathcal{M}}$ whose image via j is equal to y . This concludes the verification that j is surjective. □

8° Since j is Σ_0 -elementary and surjective, it is the identity. Thus,

$$K_\infty := [\mathcal{M}_\infty(K, \Psi)] = \prod_{\mu_\kappa} [\mathcal{K}_X] = \left(\prod_{\mu_\kappa} K_X \right)^{\mathcal{M}} \in \mathcal{M}.$$

9° **Claim.** $\mathcal{M} \models |K_\infty| \leq \kappa$

Proof. We work in \mathcal{M} . By the proof of [38, Lemma 1.17], for every J-structure M , there exists a surjection $f : [o(M)]^{<\omega} \rightarrow M$ which is Σ_1^M , uniformly in M . Adding to this Proposition 4.6.7, we get that for μ_κ -almost every X , there exists a surjection

$$f_X : [o(K_X)]^{<\omega} \rightarrow K_X,$$

which is uniformly definable in X . By passing to the ultraproduct, we construct the surjection

$$f : [o(K_\infty)]^{<\omega} \rightarrow K_\infty.$$

This suffices for the conclusion. □

10° Since $\kappa^\omega \subseteq \mathcal{M}$, the last claim implies that $K_\infty^\omega \subseteq \mathcal{M}$. However, this means that $L(\Gamma, K_\infty^\omega) \subseteq \mathcal{M}$, which together with Proposition 4.5.13 shows that $\mathcal{P}(\mathbb{R})^\mathcal{M}$ is strictly larger than Γ . This is a contradiction with 1°. □

4.7 Synthesis

We are now going to formulate precisely our construction of pointclasses and show that it reaches every model of $\text{AD}^+ + \text{AD}_\mathbb{R}$ in which there is no mouse pair with a measurable limit of Woodin cardinals.

Section Assumptions 4.7.1. We are assuming that $V \models \text{ZFC}$. □

Let us remind the reader that the filter μ_X has been defined in Definition 4.3.6 for all sets X . Since we are assuming AC, this filter is simply the club filter on $[X]^\omega$. The construction to be defined can at most reach the minimal *Chang type* pointclass.

Definition 4.7.2. Let Δ be a closed pointclass. Then Δ is said to be of **Chang type** iff for all κ , $\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega, \kappa^\omega)[\mu_\Delta, \mu_\kappa] = \Delta$. □

Our construction of pointclasses is essentially obtained by iterating the operator *Next* to produce the next pointclass. The definition of the operator that we propose cannot be realized if the current pointclass is of Chang type, so we will immediately exclude those pointclasses.

Definition 4.7.3. Suppose that Δ is a closed pointclass which is not of Chang type. Let κ be the least ordinal such that

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega, \kappa^\omega)[\mu_\Delta, \mu_\kappa] \supset \Delta.$$

We define **Next**(Δ) to be $\mathcal{P}(\mathbb{R}) \cap L(\Delta^\omega, \kappa^\omega)[\mu_\Delta, \mu_\kappa]$. □

When trying to reach by our construction the collection of all sets of reals of a certain determinacy model, it suffices to verify that a closed pointclass belonging to the model is not of Chang type and that the operator Next produces a pointclass that still belongs to the model in question. The point is that Next can be applied to any non-Chang type pointclass and that it produces a strictly bigger pointclass by the definition. This means that as long as the next pointclass is still inside the model, a simple recursive construction will relativize to the determinacy model and eventually reach all sets of reals.

Proposition 4.7.4. *Suppose that \mathcal{M} is an inner model containing all reals and satisfying $\text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{NMLW}$. Let $\Delta \in \mathcal{M}$ be a closed pointclass satisfying that $\Theta_{\Delta} < \Theta^{\mathcal{M}}$. Then Δ is not of Chang type and $\text{Next}(\Delta) \in \mathcal{M}$.*

Proof. Recall that for $\alpha < \Theta^{\mathcal{M}}$, we have that $\alpha^{\omega} \in \mathcal{M}$. This implies that also $\Delta^{\omega} \in \mathcal{M}$. Adding to this Proposition 4.3.16, we conclude that for all $\kappa < \Theta^{\mathcal{M}}$,

$$L(\Delta^{\omega}, \kappa^{\omega})[\mu_{\Delta}, \mu_{\kappa}] = L(\Delta^{\omega}, \kappa^{\omega})[\mu_{\Delta}, \mu_{\kappa}]^{\mathcal{M}}.$$

Hence, we will be done if we show that there exists $\kappa < \Theta^{\mathcal{M}}$ such that

$$\mathcal{P}(\mathbb{R}) \cap L(\Delta^{\omega}, \kappa^{\omega})[\mu_{\Delta}, \mu_{\kappa}] \supset \Delta.$$

If $\text{cof}(\Theta_{\Delta}) = \omega$, then there exists a Wadge-cofinal sequence $(A_n : n < \omega)$ in Δ . The joint of this sequence is a set of reals in $L(\Delta^{\omega})$, which is not in Δ . Hence, it remains to consider the case when $\text{cof}(\Theta_{\Delta}) > \omega$. By Propositions 4.3.16 and 4.3.12, ω_1 is Δ -supercompact in

$$L(\Delta)[\mu_{\Delta}] = L(\Delta)[\mu_{\Delta}]^{\mathcal{M}}.$$

If $L(\Delta) \models \neg \text{AD}_{\mathbb{R}}$, then Corollary 4.3.18 implies that

$$\Delta \subset \mathcal{P}(\mathbb{R}) \cap L(\Delta)[\mu_{\Delta}].$$

This means that the remaining case to verify is when $L(\Delta) \models \text{AD}_{\mathbb{R}}$. In this case, Theorem 4.6.9 shows that there exists $\kappa < \Theta^{\mathcal{M}}$ such that

$$\Delta \subset \mathcal{P}(\mathbb{R}) \cap L(\Delta, \kappa^{\omega})[\mu_{\kappa}].$$

□

Let us now formulate the construction. The first pointclass in the construction is the collection of sets of reals in $L(\mathbb{R})$, provided that determinacy holds there. If the determinacy fails in $L(\mathbb{R})$, then the construction terminates immediately and is empty. The construction also terminates if the current pointclass is of Chang type. Otherwise, we apply the operator Next, which produces a bigger pointclass, but we take the bigger pointclass as a part of the construction only if it is a determinacy pointclass. This means that the construction terminates as well if we ever reach a pointclass with a non-determined set. At limit stages, we take the minimal closed pointclass extending all the previous ones, again terminating if a non-determined set is produced.

Definition 4.7.5. We define

$$(\mathcal{A}_\alpha : \alpha < \omega)$$

to be the longest sequence of closed pointclasses satisfying that for all $\alpha < \omega$,

- a. $L(\mathcal{A}_\alpha) \models \text{AD}^+$,
- b. if $\alpha = 0$, then $\mathcal{A}_\alpha = \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$,
- c. if α is a successor, then $\mathcal{A}_{\alpha-1}$ is not of Chang type and $\mathcal{A}_\alpha = \text{Next}(\mathcal{A}_{\alpha-1})$,
- d. if α is a limit, then, letting $\mathcal{A}_{<\alpha} := \bigcup_{\xi < \alpha} \mathcal{A}_\xi$, it holds that

$$\mathcal{A}_\alpha = \mathcal{P}(\mathbb{R}) \cap L(\mathcal{A}_{<\alpha}).$$

□

The main result of the chapter now reads as follows.

Theorem 4.7.6. *Suppose that \mathcal{M} is an inner model containing all reals and satisfying $\text{AD}^+ + \text{AD}_{\mathbb{R}} + \text{NMLW}$. Then there exists $\alpha < \omega$ such that*

$$\mathcal{P}(\mathbb{R}) \cap \mathcal{M} = \mathcal{A}_\alpha.$$

Proof. This follows immediately from Proposition 4.7.4. □

Chapter 5

Forcing and Infinitary Logic

5.1 Introduction

One of the fundamental problems in logic is to characterize when a given formula has a model and what that model is. The most important instance of this problem is the well-known *Entscheidungsproblem*, which asks whether a first-order theory is satisfiable. In this chapter, we will be interested in certain forms of the problem which concern infinitary propositional formulas. In this case, the most basic form of a model-existence property is, of course, the *satisfiability*.

Definition 5.1.1. Let ϕ be an infinitary propositional formula. Then ϕ is **satisfiable** iff there exists a poset \mathbb{P} such that in $V^{\mathbb{P}}$, there exists a model¹ for ϕ . \square

A superficial inspection of this definition might cause doubts as to whether it is natural to look at models in all generic extensions. These doubts are quickly eliminated once we realize that this notion is equivalent to the property of “not proving the contradiction”. To fully appreciate this last comment, one has to recall the fact that the infinitary propositional logic comes equipped with a natural notion of a proof system, one which is a relatively straightforward generalization of the proof system for the standard (finitary) propositional logic.² This explains the central role in Logic of this notion of satisfiability. The starting point of our investigation is the question of what happens with this property if we restrict the class of posets which are to be considered for adding a model to the formula.

Definition 5.1.2. Let ϕ be an infinitary propositional formula. Then ϕ is **proper-satisfiable** (resp. **semiproper-satisfiable**, **ssp-satisfiable**³) iff there exists a proper (resp. semiproper, ssp) poset \mathbb{P} such that in $V^{\mathbb{P}}$, there exists a model for ϕ . \square

For each of these three notions, we will give a game-theoretic characterization. The motivation for such pursuit is the fact that the notion of satisfiability has, in addition to the

¹A model for ϕ is an evaluation for its propositional letters which assigns the value 1 to ϕ .

²See, for example, [39, Section 1].

³The abbreviation “ssp” stands for “stationary set preserving”.

proof-theoretic characterization, an important game theoretic one. The latter is given in terms of the *Model Existence Game*⁴.

Definition 5.1.3. Suppose that ϕ is an infinitary propositional formula in negation normal form⁵. Then the game $\mathfrak{D}^{\text{meg}}(\phi)$ is defined as the length ω two player game of the form

$$\begin{array}{c|ccc} \text{I} & Q_0 & Q_1 & \cdots \\ \hline \text{II} & w_0 & w_1 & \cdots \end{array}$$

where $w_{-1} := \{\phi\}$ and for all $n < \omega$, the following is satisfied.

- a. Player I must ensure that either
 - i. $Q_n = \psi$ for some $\psi \in w_{n-1}$ of the form $\bigvee_{i \in I} \theta_i$, or
 - ii. $Q_n = (\psi, i)$ for some $\psi \in w_{n-1}$ of the form $\bigwedge_{i \in I} \theta_i$ and some $i \in I$.
- b. Player II must ensure that if $Q_n = \psi$ for some $\psi \equiv \bigvee_{i \in I} \theta_i$, then for some $i \in I$, $\theta_i \in w_n$.
- c. Player II must ensure that if $Q_n = (\psi, i)$ for some $\psi \equiv \bigwedge_{i \in I} \theta_i$, then $\theta_i \in w_n$,
- d. Player II must ensure that w_n does not contain any propositional letter together with its negation.

The infinite plays with no rules broken are won by Player II. □

The characterization of satisfiability that we want to imitate reads as follows.

Theorem 5.1.4. *Let ϕ be an infinitary propositional formula in negation normal form. Then ϕ is satisfiable if and only if Player II has a winning strategy in $\mathfrak{D}^{\text{meg}}(\phi)$.* □⁶

We will describe versions of the Model Existence Game that correspond to the notions of proper-, semi-proper, and ssp-satisfiability. They are denoted by $\mathfrak{D}_\kappa^{\text{p}}(o)$, $\mathfrak{D}_\kappa^{\text{sp}}(o)$, and $\mathfrak{D}_\kappa^{\text{ssp}}(o)$ (respectively), and depend implicitly on ϕ and explicitly on a beth-fixed point κ . We prove the following.⁷

Theorem 5.1.5. *Let ϕ be an infinitary propositional formula in negation normal form. Suppose there exists a proper class of inaccessible cardinals. Then the following are equivalent.*

⁴See [33, Section 8.3]. The game presented there greatly simplifies when we are only interested in propositional formulas.

⁵A formula is a **negation normal form** iff all the negations are immediately in front of propositional letters. Every formula is canonically equivalent to a formula in negation normal form. This form is obtained simply by pushing the negations “all the way inside” using De Morgan’s laws.

⁶This follows by the proof of [33, Theorem 8.12]. The point is that once the set of all subformulas of ϕ is made countable, we are in the scenario of the quoted theorem. Since the game is closed for Player II, it is generically absolute whether or not Player II wins.

⁷See Corollaries 5.4.10, 5.6.9, and 5.7.8.

- a. ϕ is proper- (resp. semiproper-, ssp-) satisfiable.
- b. For some inaccessible κ , Player II has a winning strategy in $\mathcal{D}_\kappa^p(o)$ (resp. $\mathcal{D}_\kappa^{\text{SP}}(o), \mathcal{D}_\kappa^{\text{SSP}}(o)$). □

This theorem essentially characterizes when a model to a Σ_1 formula can be added by forcing of the given kind. Let us mention some examples. There is a formula whose models (in all generic extensions) naturally code surjections $\omega \rightarrow \omega_1^V$. This is an example of a satisfiable formula which is not ssp-satisfiable. On the other hand, we will see a formula whose models naturally code a cofinal map $\omega \rightarrow \omega_2^V$.⁸ This is an example of an ssp-satisfiable formula which is not proper-satisfiable.⁹ An example of a proper-satisfiable formula drawn from the literature is given in Section 5.5.¹⁰

In addition to simply answering the question of whether a model for a given formula can be forced by a poset of a certain kind, our method provides such a poset whenever the answer is positive. The poset thus obtained is made up of conditions consisting of working parts and *side-conditions*. This is exactly the form of proper posets that dominated the literature on applications of PFA.¹¹ Thus, our result can also be seen as an abstract unification theorem for those seemingly ad hoc constructions.

Finally, in a recent breakthrough of D. Asperó and R. Schindler,¹² a new sophisticated construction of ssp posets was presented. We observe that the concrete aim towards which their poset was built can be expressed by an infinitary propositional formula. One is then confronted by the question of whether that formula is ssp-satisfiable. Once this is verified, our method provides an ssp poset forcing a model for that formula.¹³ Thus, the final question that we want to answer is how to verify the ssp-satisfiability for the formulas of “Asperó-Schindler type”.¹⁴

We resolve this final question by providing an abstract condition on infinitary propositional formulas that we call the *AS-goodness* and show that it implies ssp-satisfiability. This condition is stated in terms of generic elementary embeddings and we will verify that the concrete formula relevant for the Asperó-Schindler result satisfies it. This verification is also an explanation of why we expect this condition to hold in other applications of their method¹⁵.

⁸See Example 5.2.8.

⁹The fact that the formula is ssp-satisfiable follows from the fact that the Namba forcing is always ssp. We prove it explicitly in Example 5.8.9, but by assuming more than ZFC. On the other hand, it is a matter of routine to verify that the cofinality of ω_2^V cannot be changed to ω by a proper poset, which explains why the formula is not proper-satisfiable.

¹⁰The formula is given implicitly, since we work with the notion of a *problem*. This notion and its exact relationship to infinitary propositional formulas is described in Section 5.2.

¹¹See, for example, [40] or [41].

¹²[16]

¹³Let us note here that the Asperó-Schindler poset is somewhat analogous to the posets made up of pairs of working parts and side conditions, but it did not really fit into this paradigm, while our poset is exactly of this form.

¹⁴For example, in addition to the construction of [16] itself, its variation plays an important role in the argument of [42].

¹⁵Such as, for example, the one that appears in [42].

Let us give a brief outline of the chapter.

Sec. 5.2 We define a simply combinatorial notion of a *problem* and explain why it is same whether we work with infinitary propositional formulas or with problems. We will then, for the rest of the chapter, mostly work with problems, which is essentially just a matter of simplifying some proofs.

Sec. 5.3 We recall the characterization of genericity and semigenericity which will play the main role in designing our game-theoretic characterizations. Additionally, we introduce the notion of a virtual model and some basic notation around it.

Sec. 5.4 This is where we present the game-theoretic characterization of proper satisfiability.

Sec. 5.5 Here we give an example a formula whose proper satisfiability has been (implicitly) considered in the literature. We illustrate how to apply our method in this case.

Sec. 5.6 This is where we present the variation of our method that concerns the semi-proper satisfiability.

Sec. 5.7 Here we give the variation of the method for the ssp satisfiability.

Sec. 5.8 In this section, we prove the ssp-satisfiability criterion based on generic embeddings.

Sec. 5.9 Finally, we illustrate that criterion on the example of a formula coding the main task of [16].

5.2 The Notion of a Problem

Given an infinitary propositional formula, we can think of it as a problem. A solution to this problem is a model for the formula, i.e. an evaluation that assigns the value 1 to the formula. The following poset is a natural attempt to force a model of a formula.

Definition 5.2.1. Let ϕ be a satisfiable infinitary propositional formula in negation normal form. Then the poset $\mathbb{H}(\phi)$ consists of all finite satisfiable sets of subformulas of ϕ that contain ϕ , ordered by the reverse inclusion. \square

Forcing with this poset adds a model of the formula, but it will usually collapse ω_1 . If we want to force a solution with, say, a proper forcing, then we must not produce generics for $\mathbb{H}(\phi)$. Luckily, we do not actually need full generics.

Definition 5.2.2. Let ϕ be a satisfiable infinitary propositional formula in negation normal form. Then the set $\mathcal{D}(\phi)$ of dense subsets of $\mathbb{H}(\phi)$ is defined to be

$$\{D_\psi, D_{\theta,i}\}_{\psi,\theta,i},$$

where

- a. for ψ a subformula of ϕ with $\psi \equiv \bigvee_{i \in I} \phi_i$,

$$D_\psi := \{w \in \mathbb{H}(\phi) : \psi \in w \implies \exists i \in I, \psi_i \in w\},$$

- b. for θ a subformula of ϕ with $\theta \equiv \bigwedge_{i \in I} \theta_i$ and for $i \in I$,

$$D_{\theta,i} := \{w \in \mathbb{H}(\phi) : \theta \in w \implies \theta_i \in w\}.$$

□

It turns out that adding a filter for $\mathbb{H}(\phi)$ which is just $\mathcal{D}(\phi)$ -generic¹⁶ suffices.

Proposition 5.2.3. *Let ϕ be a satisfiable infinitary propositional formula in negation normal form. Then there exists a $\mathcal{D}(\phi)$ -generic filter for $\mathbb{H}(\phi)$ if and only if there exists a model for ϕ .*

Proof. If we are given a $\mathcal{D}(\phi)$ -generic filter g for $\mathbb{H}(\phi)$, we can set

$$\mu_g(l) := \begin{cases} 1, & \exists w \in g, l \in w \\ 0, & \text{otherwise,} \end{cases}$$

for all propositional letters l appearing in ϕ . It then follows by the induction on the complexity of a subformula that $\mu_g \models \phi$. On the other hand, if we are given a model $\mu \models \phi$, we define

$$g_\mu := \{w \in \mathbb{H}(\phi) : \mu \models w\}$$

and routinely verify that g_μ is a $\mathcal{D}(\phi)$ -generic filter for $\mathbb{H}(\phi)$. □

Thus, we are lead to the following generalization of the notion of a problem given by a propositional formula.

Definition 5.2.4. A pair $(\mathbb{H}, \mathcal{D})$ is a **problem** iff

- \mathbb{H} is a poset¹⁷,
- for all $p, q \in \mathbb{H}$, if $p \parallel q$, then $\inf\{p, q\}$ exists (and it will be denoted by pq),
- \mathcal{D} is a collection of pre-dense subsets of \mathbb{H} . □

A solution for a problem then becomes simply a \mathcal{D} -generic filter for \mathbb{H} . The proper satisfiability then naturally becomes the property of being able to force a solution by a proper forcing.

Definition 5.2.5. A problem $(\mathbb{H}, \mathcal{D})$ is **proper-satisfiable** (resp. **semiproper-satisfiable**, **ssp-satisfiable**) iff there exists a proper (resp. semiproper, ssp) poset \mathbb{Q} such that in $V^{\mathbb{Q}}$, there exists a \mathcal{D} -generic filter for \mathbb{H} . □

¹⁶In other words, a filter that intersects all the sets in $\mathcal{D}(\phi)$, but not necessarily all dense sets in V .

¹⁷For the sake of precision, we will mean by “poset” a partial order with the distinguished maximum.

Lemma 5.2.6. *Let ϕ be a satisfiable infinitary propositional formula in negation normal form. Then ϕ is proper- (resp. semiproper-, ssp-) satisfiable if and only if $(\mathbb{H}(\phi), \mathcal{D}(\phi))$ is so. \square*

Not all problems come from an infinitary propositional formula, but the most complex ones do. The possibilities of the expressing power of propositional formulas will be illustrated in Section 5.9. The apparent increased generality that problems introduce is just that, apparent. We explain further in the following example.

Example 5.2.7. Let $(\mathbb{H}, \mathcal{D})$ be a problem. We want to find a propositional formula χ such that there is a canonical correspondence between \mathcal{D} -generic filters for \mathbb{H} and models for χ (in all generic extensions). We will use the set

$$\{g_w : w \in \mathbb{H}\}$$

of propositional letters to build χ . The idea is that the evaluation for g_w answers “yes” or “no” to the question of whether w belongs to the corresponding filter. The formula χ is then the conjunction of the following formulas:

- a. $g_{1_{\mathbb{H}}}$,
- b. $\bigwedge_{w \leq w'} (g_w \implies g_{w'})$,
- c. $\bigwedge_{w, w' \in \mathbb{H}} (g_w \wedge g_{w'} \implies g_{ww'})$,
- d. $\bigwedge_{D \in \mathcal{D}} \bigvee_{w \in D} g_w$.

Now, in any generic extension, if g is a \mathcal{D} -generic filter for \mathbb{H} , then

$$\{g_w : w \in \mathbb{H}\} \longrightarrow 2 : g_w \mapsto \begin{cases} 1, & w \in g \\ 0, & w \notin g \end{cases}$$

is a model for χ . On the other hand, if μ is a model for χ , then

$$\{w \in \mathbb{H} : \mu(g_w) = 1\}$$

is a \mathcal{D} -generic filter for \mathbb{H} . Furthermore, these two correspondences are mutual inverses. \square

The next example illustrates that sometimes it might be easier to work with the abstract notion of a problem than to pass through a propositional formula.

Example 5.2.8. We explain how to code the “Namba problem”. More precisely, we want to force a cofinal function $\omega \longrightarrow \omega_2^V$ by an ssp poset. Such a function is a subset of $\omega \times \omega_2$, so we can build a natural propositional formula ϕ with the propositional letters

$$\{f_{n,\alpha} : n < \omega, \alpha < \omega_2\}$$

such that for every evaluation ν in every generic extension, $\nu \models \phi$ if and only if the set

$$\{(n, \alpha) : \nu(f_{n,\alpha}) = 1\}$$

is a cofinal function $\omega \rightarrow \omega_2^V$. The Namba problem then becomes the question whether ϕ is ssp-satisfiable, or equivalently, whether $(\mathbb{H}(\phi), \mathcal{D}(\phi))$ is ssp-satisfiable.

On the other hand, the generality that we get from the notion of a problem allows us to code the Namba problem more directly, i.e. without passing through a propositional formula. We then set

$$\mathbb{H} := \{w \subseteq \omega \times \omega_2 : w \text{ is a finite function}\},$$

ordered by the reverse inclusion, and we set $\mathcal{D} := \{D_n, C_\alpha : n < \omega, \alpha, \omega_2\}$, where

$$D_n := \{w \in \mathbb{H} : n \in \text{dom}(w)\},$$

$$C_\alpha := \{w \in \mathbb{H} : \exists m \in \text{dom}(w), w(m) > \alpha\}.$$

Then a filter F for \mathbb{H} (in a generic extension) is \mathcal{D} -generic if and only if

$$\bigcup F : \omega \xrightarrow{\text{cof}} \omega_2^V.$$

□

5.3 Some Simple Remarks

In our arguments, we will define a class of preconditions, and then we will define a way to pick among them those that should be conditions in our final posets. The preconditions will simply be pairs of a working part together with a chain of side conditions. The criterion for picking the conditions is based on games. The games are designed to ensure that the final poset adds a correct object. However, they must also ensure that a condition of the poset is (semi)generic for a model that appears as a side condition. To do this, we use the following slight modification of the standard (semi)genericity criterion.

Lemma 5.3.1. *Suppose that*

1. \mathbb{Q} is a poset,
2. $\theta \gg \text{rank}(\mathbb{Q})$ is regular,
3. $M \prec (H_\theta, \in, \mathbb{Q})$ is countable,
4. $p \in \mathbb{Q}$.

Then the following are equivalent.

- a. p is generic (resp. semigeneric) for (M, \mathbb{Q}) .
- b. For all $q \leq p$, for all $E \in M$ which contain q , there exists $r \in E \cap \mathbb{Q}$ such that $r \in M$ (resp. $\delta(\text{Hull}(M, r)) = \delta(M)$)¹⁸ and $r \parallel q$.

Proof. The modification here is that E is not required to be (pre)dense. In order to apply the standard lemma, we just need to replace E by $E \cup E^\perp$, where

$$E^\perp := \{r \in \mathbb{Q} : \forall q' \in E, r \perp q'\}.$$

□

The side conditions in our posets will be *virtual models*. They were first defined by Veličković when studying alternative ways to iterate semiproper posets, see for example [43]. In the cited paper, one can also find a detailed elaboration on how to work with virtual models. We review here only the basics. A prototype of a virtual model is an elementary submodel of some H_θ , with θ regular, but there is more to the notion.

Definition 5.3.2. A set M is a **virtual model** iff $M \prec \widehat{M} \models \text{ZFC}^-$.¹⁹

□

One of the fundamental operations for virtual models is the hull operation.

Definition 5.3.3. Let M be a virtual model and X an arbitrary set. Then $\text{Hull}(M, X)$ is defined as the set of all $f(x)$ where f is a function in M and x is an element of $X \cap \text{dom}(f)$.

□

$\text{Hull}(M, X)$ is elementary in \widehat{M} and it contains $X \cap \widehat{M}$. In fact, it is the smallest virtual model with respect to these two properties. The most important virtual models which are not elementary submodels of some H_θ originate from an application of the following operation.

Definition 5.3.4. Suppose that M is a virtual model and that λ is an infinite ordinal. We define $M \downarrow \lambda$ to be the image of M under the transitive collapse of $\text{Hull}(M, V_\lambda)$.

□

The anti-collapse of a virtual model is essentially a partial extender. The following parameter measures the strength of that extender.

Definition 5.3.5. Suppose that M is a virtual model. Then λ_M is defined as the largest λ such that $V_\lambda \in M$, when it exists.

□

We will mostly talk about virtual models which belong to \mathcal{C}_λ , for a beth-fixed point λ . These collections are defined just below. Since we will refer a lot to beth-fixed points, it helps to introduce the following notation.

¹⁸We write $\delta(M)$ or δ_M for the intersection of M with ω_1 . We use the same notation for all *virtual models* (cf. Definition 5.3.2).

¹⁹ \widehat{M} denotes the transitive closure of M .

Notation 5.3.6. \beth_{fix} denotes the class of all beth-fixed points. \square

Definition 5.3.7. Suppose that $\lambda \in \beth_{\text{fix}}$. Then \mathcal{C}_λ denotes the set of all virtual model M satisfying that

a. $\lambda_M = \lambda$,

b. $\widehat{M} = \text{Hull}(M, V_\lambda)$. \square

Definition 5.3.8. The class \mathcal{C} is defined as $\bigcup_{\lambda \in \beth_{\text{fix}}} \mathcal{C}_\lambda$. \square

If $M \in \mathcal{C}$ and if \mathbb{Q} is a poset that belongs to $M \cap V_{\lambda_M}$, we can make sense of the phrase “ \mathbb{Q} is (semi)proper for M ”. One way to do so is to just run the usual definition, but with respect to this more general object M .²⁰ It turns out that this notion is equivalent to saying that \mathbb{Q} is (semi)proper for $M \cap H_\theta$ for any and all regular $\theta \in (2^{2^\kappa}, \lambda_M) \cap M$. However, $M \cap H_\theta \prec H_\theta$, so we do not diverge much from the usual notion of (semi)properness.

When using virtual models as side conditions, we will require them to be \in -chains. In fact, we will require a bit more, as explained in the following definition.

Definition 5.3.9. Suppose that $\mathcal{M} \subseteq \mathcal{C}$. Then \mathcal{M} is a **vm-chain** iff for all $M, N \in \mathcal{M}$,

a. if $\delta_M = \delta_N$, then $M = N$,

b. if $\delta_M < \delta_N$, then $M \in N$ and $\lambda_M < \lambda_N$. \square

When passing from a virtual model M to its projection $M \downarrow \lambda$, we lose information. In a sense, M then certifies that $M \downarrow \lambda$ is “consistent” with more information.²¹ This is not true for all models $N \in \mathcal{C}_\lambda$ and even if it is true for some such N , a virtual model M such that $M \downarrow \lambda = N$ is not necessarily unique. We will use the following notion in order to talk about this possibility of increasing the information content of a virtual model.

Definition 5.3.10. Suppose that

1. M and N are virtual models,
2. $M \in \mathcal{C}$,
3. $\pi : M \longrightarrow N$.

Then π is a **lifting** iff, letting

$$\rho : \widehat{N \downarrow \lambda_M} \xrightarrow{\cong} \text{Hull}(N, V_{\lambda_M}) \prec \widehat{N}$$

be the anti-collapse, we have that $M = N \downarrow \lambda_M$ and $\pi = \rho \upharpoonright M$. \square

²⁰The point is that M is not necessarily an elementary submodel of some H_θ .

²¹Our intuition is that it is more complicated to embed a model into a transitive structure of a larger rank.

5.4 Proper Satisfiability

Section Assumptions 5.4.1. We fix a problem $(\mathbb{H}, \mathcal{D})$. □

We want to characterize when this problem is proper-satisfiable. In this case, we will also give a poset that adds a \mathcal{D} -generic filter for \mathbb{H} . We start by defining a class-sized poset of preconditions. The required will be a (set-sized) suborder of it.

Definition 5.4.2.

- a. The class $\mathbb{P}^{\mathbb{P}}$ consists of all $p = (w_p, \mathcal{M}_p)$ such that $w_p \in \mathbb{H}$ and \mathcal{M}_p is a finite vm-chain.
- b. The order $\leq^{\mathbb{P}}$ on vm-chains is the reverse inclusion \supseteq .
- c. The order $\leq^{\mathbb{P}}$ on $\mathbb{P}^{\mathbb{P}}$ is defined by asserting that $p \leq q$ iff $w_p \leq_{\mathbb{H}} w_q$ and $\mathcal{M}_p \leq^{\mathbb{P}} \mathcal{M}_q$. □

The criterion for deciding which preconditions should be taken as conditions in the final poset is based on a game. We describe this game in the following definition. Player I plays by asking certain questions and Player II plays by answering those questions. Preconditions will be conditions if Player II (the “good” player) can always play the corresponding game for ω rounds, i.e. without failing to produce an answer of the required form. There are three types of questions that Player I can ask. The first ensures that the final poset actually forces a \mathcal{D} -generic filter for \mathbb{H} . The other two types of questions ensure that the final poset is proper. They are based on the genericity criterion given in Lemma 5.3.1.

Definition 5.4.3. Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$ and $p \in \mathbb{P}^{\mathbb{P}} \cap V_{\kappa}$. Then the game $\mathfrak{D}_{\kappa}^{\mathbb{P}}(p)$ is defined as the length ω two Player game of the form

$$\begin{array}{c|cccc} \text{I} & & Q_0 & Q_1 & \cdots \\ \hline \text{II} & p_{-1} & p_0 & p_1 & \cdots \end{array}$$

where $p_{-1} \in \mathbb{P}^{\mathbb{P}} \cap V_{\kappa}$ is such that $p_{-1} \leq p$ and for all $n < \omega$, the following is satisfied.

- a. Player I must ensure that either
 - i. $Q_n = D$ for some $D \in \mathcal{D}$, or
 - ii. $Q_n = M$ for some countable $M \prec H((2^{\kappa})^+)$, or
 - iii. $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$.
- b. Player II must ensure that $p_n \in \mathbb{P}^{\mathbb{P}} \cap V_{\kappa}$ and $p_n \leq p_{n-1}$.
- c. Player II must ensure that if $Q_n = D$ for some $D \in \mathcal{D}$, then there exists $w \in D$ such that $w_{p_n} \leq w$.

- d. Player I must ensure that if $Q_n = M$ for some countable $M \prec H((2^\kappa)^+)$, then $p_{n-1}, \kappa \in M$.
- e. Player II must ensure that if $Q_n = M$ for some countable $M \prec H((2^\kappa)^+)$, then there exists $\lambda \in \mathfrak{J}_{\text{fix}} \cap \kappa$ such that $\kappa \cap \text{Hull}(M, V_\lambda) \subseteq \lambda$ and $M \downarrow \lambda \in \mathcal{M}_{p_n}$.
- f. Player I must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$, then there exist $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

- g. Player II must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$, then there exists $q \in E$ such that $q \in M$ and $p_n \leq q$.

The infinite plays with no rules broken are won by Player II. □

We will be interested in those p for which Player II has a winning strategy in $\mathfrak{D}_\kappa^p(p)$. Something that we can observe right now, and which will be useful throughout, is that if σ is a winning strategy for Player II in $\mathfrak{D}_\kappa^p(p)$, then any move q that this strategy prescribes for Player II has the same property: Player II has a winning strategy in $\mathfrak{D}_\kappa^p(q)$. To see this, one can simply use the tail strategy.

Definition 5.4.4. Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$. We let \mathbb{C}_κ^p consist of all $p \in \mathbb{P}^p \cap V_\kappa$ such that Player II wins $\mathfrak{D}_\kappa^p(p)$. □

The set \mathbb{C}_κ^p might be empty. If it is not empty, then it inherits the ordering from \mathbb{P}^p which makes it into a poset. It also has the natural unit element.

Notation 5.4.5. We denote by o the pair $(1_{\mathbb{H}}, \emptyset)$. □

It turns out that \mathbb{C}_κ^p is an initial segment of (\mathbb{P}^p, \leq^p) . By this, we simply mean that it is upwards closed. In particular, if \mathbb{C}_κ^p contains any element whatsoever, then it contains o . On the other hand, if \mathbb{C}_κ^p contains o , then it also contains $\text{ran}(\sigma)$ for any winning strategy σ for Player II in $\mathfrak{D}_\kappa^p(o)$.

Proposition 5.4.6. *Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$. Then the following holds.*

- a. \mathbb{C}_κ^p is an initial segment of (\mathbb{P}^p, \leq^p) .
- b. If \mathbb{C}_κ^p is non-empty, then $o \in \mathbb{C}_\kappa^p$ and $(\mathbb{C}_\kappa^p, \leq^p, o)$ is a poset.
- c. If \mathbb{C}_κ^p is non-empty, then, letting g be a V -generic for \mathbb{C}_κ^p , we have that the set

$$\{w_p : p \in g\}$$

is a \mathcal{D} -generic filter for \mathbb{H} .

Proof. The fact that \mathbb{C}_κ^p is an initial segment of (\mathbb{P}^p, \leq^p) easily follows from the fact that we allowed Player II to arbitrarily strengthen condition p at the very beginning of game $\mathcal{D}_\kappa^p(p)$. Part b is then immediate. Let us now verify part c.

1° Suppose that $\mathbb{C}_\kappa^p \neq \emptyset$.

2° We have that for all $p \in \mathbb{C}_\kappa^p$, for all $w \in \mathbb{H}$ satisfying that $w \geq w_p$, it holds that $(w, \emptyset) \in \mathbb{C}_\kappa^p$ and $(w, \emptyset) \geq p$. This readily implies that

$$\mathbb{C}_\kappa^p \Vdash \text{“}\{w_p : p \in g\} \text{ is a filter in } \mathbb{H}\text{”}.$$

3° It remains to verify genericity. Let $D \in \mathcal{D}$ be arbitrary. We want to show that

$$\mathbb{C}_\kappa^p \Vdash \exists p \in \dot{g}, w_p \in D.$$

4° To this end, it suffices to show that the set

$$E := \{p \in \mathbb{C}_\kappa^p : \exists w \in D, w_p \leq w\}$$

is dense in \mathbb{C}_κ^p .

5° Let $p \in \mathbb{C}_\kappa^p$ be arbitrary. We want to find $q \in E$ such that $q \leq p$.

6° Since $p \in \mathbb{C}_\kappa^p$, there exists a winning strategy σ for Player II in $\mathcal{D}_\kappa^p(p)$.

7° Let us consider the following partial play of $\mathcal{D}_\kappa^p(p)$ according to σ .

$$\begin{array}{c|cc} \text{I} & & D \\ \hline \text{II} & p_{-1} & p_0 \end{array}$$

8° Since p_{-1} and p_0 appear in a play according to a winning strategy for Player II, we have that $p_{-1}, p_0 \in \mathbb{C}_\kappa^p$.

9° The rules of the game imply that

$$p_0 \leq p_{-1} \leq p$$

and that there exist $w \in D$ such that $w_{p_0} \leq w$.

10° This imply that $p_0 \in E$, so $q := p_0$ is as required. □

Proposition 5.4.7. *Suppose that $\kappa \in \beth_{\text{fix}}$ and that \mathbb{C}_κ^p is non-empty. Then \mathbb{C}_κ^p is a proper poset.*

Proof.

1° For $p \in \mathbb{C}_\kappa^p$, let σ_p be a winning strategy for Player II in the game $\mathfrak{D}_\kappa^p(p)$.

2° Let

$$M \prec (H((2^\kappa)^+), \in, \kappa, \mathbb{H}, \mathcal{D}, (\sigma_p : p \in \mathbb{C}_\kappa^p))$$

be an arbitrary countable virtual model. We want to show that \mathbb{C}_κ^p is proper for M .

3° Let $p \in \mathbb{C}_\kappa^p \cap M$ be arbitrary. We need to find $p_0 \leq p$ which is generic for (M, \mathbb{C}_κ^p) .

4° Consider the following partial play of $\mathfrak{D}_\kappa^p(p)$ according to σ_p .

$$\begin{array}{c|cc} \text{I} & & M \downarrow \kappa \\ \hline \text{II} & p_{-1} & p_0 \end{array}$$

Since $p_{-1} = \sigma_p(\emptyset)$ and $p \in M$, we get that $p_{-1} \in M \downarrow \kappa$. In other words, Player I did not break the rules.

5° Consequently, it follows that $p_0 \leq p_{-1} \leq p$ and that there exists $\lambda \in \mathfrak{J}_{\text{fix}} \cap \kappa$ such that

$$\kappa \cap \text{Hull}(P, V_\lambda) \subseteq \lambda$$

and $M \downarrow \lambda \in \mathcal{M}_{p_0}$.

6° We want to show that p_0 is as required. This comes down to the following. Let $q \leq p_0$ and let $E \in M$ be an open dense subset of \mathbb{C}_κ^p containing q . We need to show that there exist $r \in E$ such that $r \in M$ and $r \parallel q$.

7° Let

$$\pi : \widehat{M \downarrow \lambda} \xrightarrow{\cong} \text{Hull}(M, V_\lambda) \prec H_\theta$$

be the anti-collapse. We have that $\pi^{-1}(\kappa) = \lambda$ and $\pi^{-1}(E) = E \cap V_\lambda$.

8° Let us consider the following partial play of $\mathfrak{D}_\kappa^p(q)$ according to σ_q .

$$\begin{array}{c|cc} \text{I} & & (M \downarrow \lambda, E \cap V_\lambda) \\ \hline \text{II} & q_{-1} & q_0 \end{array}$$

Player I did not break the rules since

- a. $M \downarrow \lambda \in \mathcal{M}_q \subseteq \mathcal{M}_{q_{-1}}$,
- b. $E \cap V_\lambda = \pi^{-1}(E) \in \pi^{-1}[M] = M \downarrow \lambda$,
- c. $\pi \upharpoonright (M \downarrow \lambda) : M \downarrow \lambda \rightarrow M$ is a lifting,
- d. $q_{-1} \in E = \pi(E \cap V_\lambda)$ (E is open and $q_{-1} \leq q$).

9° The rules then imply that $q_0 \leq q$ and that there exists $r \in E \cap V_\lambda$ such that $q_0 \leq r$ and $r \in M \downarrow \lambda$.

10° Thus, $r \in E$, $r \parallel q$ (as witnessed by q_0), and $r \in (M \downarrow \lambda) \cap V_\lambda \subseteq M$.

□

Thus, if Player II has a winning strategy in $\mathcal{D}_\kappa^p(o)$, then \mathbb{C}_κ^p is a proper poset which adds a \mathcal{D} -generic filter for \mathbb{H} . It turns out that a converse is true as well. To show it, we need the following lemma.

Lemma 5.4.8. *The following are equivalent.*

- a. $(\mathbb{H}, \mathcal{D})$ is proper- (resp. semiproper-, ssp-) satisfiable.
- b. There exists a proper (resp. semiproper, ssp), complete boolean algebra \mathbb{B} and a mapping $i : \mathbb{H} \rightarrow \mathbb{B}$ satisfying that
 - i. $i(1) = 1$,
 - ii. for all $p, q \in \mathbb{H}$, if $p \leq q$, then $i(p) \leq i(q)$,
 - iii. for all $p, q \in \mathbb{H}$, if $p \perp q$, then $i(p)i(q) = 0$,
 - iv. for all $p, q \in \mathbb{H}$, if $p \parallel q$, then $i(pq) = i(p)i(q)$,
 - v. for all $D \in \mathcal{D}$, $\sum i[D] = 1$.

Proof. We explain the proper case, the other two being analogous. Let us first show that (a \Rightarrow b). Since $(\mathbb{H}, \mathcal{D})$ is proper-satisfiable, there exists a proper poset \mathbb{Q} which adds a \mathcal{D} -generic filter for \mathbb{H} . Let \mathbb{B} be the complete boolean algebra obtained as the completion of \mathbb{Q} and let \dot{F} be a \mathbb{B} -name satisfying that

$$\Vdash_{\mathbb{B}} \text{“}\dot{F} \text{ is } \mathcal{D}\text{-generic for } \mathbb{H}\text{.”}$$

Then $i : \mathbb{H} \rightarrow \mathbb{B} : w \mapsto \|w \in \dot{F}\|$ is as required. To see that (b \Rightarrow a), fix $i : \mathbb{H} \rightarrow \mathbb{B}$ as in b. Then for g which is V -generic for \mathbb{B} , we have that $i^{-1}[g]$ is \mathcal{D} -generic for \mathbb{H} . □

Theorem 5.4.9. *Suppose that*

1. *there exists a proper class of inaccessible cardinals,*
2. $(\mathbb{H}, \mathcal{D})$ *is proper-satisfiable.*

Then for some inaccessible κ , Player II has a winning strategy in $\mathcal{D}_\kappa^p(o)$.

Proof.

- 1 $^\circ$ By Lemma 5.4.8, there exists a proper, complete boolean algebra \mathbb{B} and a mapping $i : \mathbb{H} \rightarrow \mathbb{B}$ satisfying that

- a. $i(1) = 1$,

- b. for all $p, q \in \mathbb{H}$, if $p \leq q$, then $i(p) \leq i(q)$,
- c. for all $p, q \in \mathbb{H}$, if $p \perp q$, then $i(p)i(q) = 0$,
- d. for all $p, q \in \mathbb{H}$, if $p \parallel q$, then $i(pq) = i(p)i(q)$,
- e. for all $D \in \mathcal{D}$, $\sum i[D] = 1$.

2° Let κ be an inaccessible cardinal satisfying that $\text{rank}(\mathbb{H}), \text{rank}(\mathbb{B}) < \kappa$ and let us show that Player II has a winning strategy in $\mathcal{D}_\kappa^{\mathbb{P}}(o)$.

3° Let us assume otherwise. Since the game is closed for Player II, it follows that Player I has a winning strategy σ . We will reach a contradiction by showing that Player II can defeat σ .

4° For $p \in \mathbb{P}^{\mathbb{P}}$ and $b \in \mathbb{B}$, let $\Psi(p, b)$ be the conjunction of the following statements:

- a. for all $M \in \mathcal{M}_p$, $\mathbb{H}, \mathbb{B}, i \in M$,
- b. for all $M \in \mathcal{M}_p$, $\lambda_M > \text{rank}(\mathbb{B})$,
- c. $0 < b \leq i(w_p)$,
- d. for all $M \in \mathcal{M}_p$, b is generic for (M, \mathbb{B}) .

We will show that Player II can play by maintaining that for all $n \in [-1, \omega)$, there exists $b \in \mathbb{B}$ such that $\Psi(p_n, b)$ holds.

5° Suppose first that $n = -1$. Let

$$M \prec (H((2^\kappa)^+), \in, \kappa, \mathbb{H}, \mathbb{B}, i)$$

be countable, let $\lambda := \sup(\kappa \cap M)$, and let $p_{-1} := (\emptyset, \{M \downarrow \lambda\})$. We see that conditions 4°a and 4°b are met.

6° Since \mathbb{B} is proper and $1_{\mathbb{B}} \in M$, there exists $b \in \mathbb{B}$ such that

$$0 < b \leq 1 = i(w_{p_{-1}})$$

and such that b is generic for (M, \mathbb{B}) .

7° It is now easily seen that $\Psi(p_{-1}, b)$ holds.

8° Let us now inductively consider the case n assuming that there exists $b \in \mathbb{B}$ such that $\Psi(p_{n-1}, b)$ holds. Let Player I play a move Q_n and let us show how Player II can answer in such a way as to ensure that there exists $c \in \mathbb{B}$ such that $\Psi(p_n, c)$ holds.

9° **Case I.** $Q_n = D$ for some $D \in \mathcal{D}$.

Proof.

1' Since $\sum i[D] = 1$, there exists $w \in D$ such that $i(w)b > 0$.

2' Let $p_n := (w_{p_{n-1}}w, \mathcal{M}_{p_{n-1}})$ and let $c := i(w)b$. We have that

$$0 < c \leq i(w_{p_n})$$

and $\mathcal{M}_{p_n} = \mathcal{M}_{p_{n-1}}$.

3' Since also $c \leq b$, it follows easily that $\Psi(p_n, c)$ holds.

□

10° **Case II.** $Q_n = M$ for some countable $M \prec H((2^\kappa)^+)$.

Proof.

1' We have that $p_{n-1}, \kappa \in M$.

2' We need to find $p_n \in \mathbb{P}^{\mathbb{P}}$, $\lambda \in \mathfrak{J}_{\text{fix}} \cap \kappa$, and $c \in \mathbb{B}$ such that

- a. $p_n \leq p_{n-1}$,
- b. $\kappa \cap \text{Hull}(M, V_\lambda) \subseteq \lambda$,
- c. $M \downarrow \lambda \in \mathcal{M}_{p_n}$,
- d. $\Psi(p_n, c)$ holds.

3' Let $F : \kappa \longrightarrow \kappa$ be defined by setting

$$F(\xi) := \sup(\kappa \cap \text{Hull}(M, V_\xi)) < \kappa$$

for $\xi < \kappa$. There exists $\lambda \in \mathfrak{J}_{\text{fix}} \cap \kappa$ such that $F[\lambda] \subseteq \lambda$.

4' Let $p_n := (w_{p_{n-1}}, \mathcal{M}_{p_{n-1}} \cup \{M \downarrow \lambda\}) \in \mathbb{P}^{\mathbb{P}}$. We have that $p_n \leq p_{n-1}$ and $M \downarrow \lambda \in \mathcal{M}_{p_n}$. It remains to find $c \in \mathbb{B}$ such that $\Psi(p_n, c)$.

5' By elementarity, there exists $b_M \in \mathbb{B} \cap M$ such that $\Psi(p_{n-1}, b_M)$ holds.

6' Since $\mathcal{M}_{p_{n-1}} \subseteq \mathcal{M}_{p_n} \in M$, we have that $\mathbb{B} \in M$. This means that \mathbb{B} is proper for M .

7' Hence, there exists $c \in \mathbb{B}$ such that $0 < c \leq b_M$ and such that c is generic for (M, \mathbb{B}) . It follows that $\Psi(p_n, c)$ holds, as required.

□

11° **Case III.** $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$.

Proof.

1' We have that there exist $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

- 2' We need to find $p_n \leq p_{n-1}$, $q \in E$, and $c \in \mathbb{B}$ such that
- a. $q \in M$,
 - b. $p_n \leq q$,
 - c. $\Psi(p_n, c)$ holds.
- 3' Let $\mathcal{N} := \mathcal{M}_{p_{n-1}} \cap V_{\lambda_M} \in M$ and let F consist of all $d \in \mathbb{B}$ such that there exists $q \in E \cap \mathbb{P}^{\mathbb{P}}$ satisfying that
- a. for all $P \in \mathcal{M}_q, \mathbb{H}, \mathbb{B}, i \in P$,
 - b. for all $P \in \mathcal{M}_q, \lambda_P > \text{rank}(\mathbb{B})$,
 - c. $0 < d \leq i(w_q)$,
 - d. for all $P \in \mathcal{M}_q, d$ is generic for (P, \mathbb{B}) ,
 - e. $\mathcal{N} \subseteq \mathcal{M}_q$.
- We have that F is an open subset of $\mathbb{B} - \{0\}$ and $F \in M$.
- 4' Let $F^\perp := \{a \in \mathbb{B} - \{0\} : \forall d \in F, ad = 0\}$. We have that $F \cup F^\perp$ is dense in \mathbb{B} and belongs to M .
- 5' Since b is generic for (M, \mathbb{B}) , there exists $d \in F \cup F^\perp$ such that $bd > 0$ and $d \in M$.
- 6' Since $p_{n-1} \in \pi(E)$ and $\Psi(p_{n-1}, b)$ holds, it follows that $b \in \pi(F)$.
- 7' Since $F \subseteq \mathbb{B}$, we have that $F \in M \cap V_{\lambda_M}$. This means that $\pi(F) = F$ and consequently, $b \in F$.
- 8' Since $bd > 0$ and $b \in F$, we have that $d \notin F^\perp$, which means that $d \in F$.
- 9' Since $d \in F \cap M$, there exists $q \in E \cap \mathbb{P}^{\mathbb{P}} \cap M$ such that
- a. for all $P \in \mathcal{M}_q, \mathbb{H}, \mathbb{B}, i \in P$,
 - b. for all $P \in \mathcal{M}_q, \lambda_P > \text{rank}(\mathbb{B})$,
 - c. $0 < d \leq i(w_q)$,
 - d. for all $P \in \mathcal{M}_q, d$ is generic for (P, \mathbb{B}) ,
 - e. $\mathcal{M}_{p_{n-1}} \cap V_{\lambda_M} \subseteq \mathcal{M}_q$.
- 10' Let $p_n := (w_{p_{n-1}}w_q, \mathcal{M}_{p_{n-1}} \cup \mathcal{M}_q)$. We have that $p_n \in \mathbb{P}^{\mathbb{P}}$. (Note that $0 < bd \leq i(w_{p_{n-1}})i(w_q)$, so $w_{p_{n-1}} \parallel w_q$.)
- 11' We also have that $p_n \leq p_{n-1}$ and $p_n \leq q$, while $q \in E \cap M$.
- 12' Let $c := bd > 0$. We get that $\Psi(p_n, c)$ holds, concluding the argument. □

12° The three cases show together that Player II can survive ω moves against σ . This contradicts the fact that σ is winning for Player I. □

Hence, we obtain a game-theoretic characterization of proper satisfiability.

Corollary 5.4.10. *Suppose that there is a proper class of inaccessible cardinals. Then the following are equivalent.*

- a. $(\mathbb{H}, \mathcal{D})$ is proper-satisfiable.
- b. For some inaccessible κ , Player II has a winning strategy in $\mathfrak{D}_\kappa^{\mathbb{P}}(o)$. □

If $(\mathbb{H}, \mathcal{D})$ is proper-satisfiable, then for any sufficiently large inaccessible κ , the poset $\mathbb{C}_\kappa^{\mathbb{P}}$ is a proper poset adding a \mathcal{D} -generic filter for \mathbb{H} . It turns out that this is the maximal poset in a collection of posets which are methods identify and which have the the same property.

Proposition 5.4.11. *Suppose that*

- 1. $\lambda \in \mathfrak{J}_{\text{fix}}$ is strictly larger than $\text{rank}(\mathbb{H})$,
- 2. \mathbb{C} is a poset contained in $\mathbb{P}^{\mathbb{P}} \cap V_\lambda$,
- 3. for all $p \in \mathbb{C}$, there exists a winning strategy σ for Player II in $\mathfrak{D}_\lambda^{\mathbb{P}}(p)$ such that $\text{ran}(\sigma) \subseteq \mathbb{C}$.²²

Then \mathbb{C} is a proper poset that adds a \mathcal{D} -generic filter for \mathbb{H} . Moreover, \mathbb{C} is contained in $\mathbb{C}_\lambda^{\mathbb{P}}$

Proof. These assumptions are sufficient to run the proofs of Propositions 5.4.6.c and 5.4.7. □

There are also minimal ones among the posets that this proposition describes. They are obtained from by taking the range of a single winning strategy for Player II in $\mathfrak{D}_\lambda^{\mathbb{P}}(o)$. We will have the analogous situation in the semiproper and stationary set preserving case. In Section 5.9, we will actually have to work with a smaller poset in the corresponding collection of posets in the stationary set preserving case.

5.5 An Example for the Proper Satisfiability

In this section, we will illustrate the method described in Section 5.4 using a concrete example. First, let us fix our assumptions.

Section Assumptions 5.5.1. Suppose that

- 1. $X \subseteq \mathbb{R}$ is uncountable,

²²In other words, the moves that σ prescribes for Player II are all in \mathbb{C} .

2. $\Gamma : [X]^2 \longrightarrow 2$ is an open coloring,
3. κ is an inaccessible,
4. $<_{V_\kappa}$ is a wellordering of V_κ . □

The goal is to show that there exists a proper poset \mathbb{Q} such that in $V^\mathbb{Q}$, there exists a sequence $(X_n : n < \omega)$ such that $X = \bigsqcup_{n < \omega} X_n$ and such that for all $n < \omega$, $\Gamma \upharpoonright [X_n]^2 = \text{const.}$ ²³ This goal can be naturally reformulated as a problem (in our formal sense).

Definition 5.5.2.

- a. Let \mathbb{H} consist of finite partial functions $w \subseteq X \times \mathbb{Z}$ such that for all $n \in \mathbb{Z}$,

$$\Gamma \upharpoonright [w^{-1}(n)]^2 = \text{const.}$$

The order on \mathbb{H} is the reverse inclusion.

- b. For $x \in X$, let D_x be $\{w \in \mathbb{H} : x \in \text{dom}(w)\}$.
- c. Let \mathcal{D} be the set $\{D_x : x \in X\}$. □

Partial functions $w \in \mathbb{H}$ are approximations of the final partition. We chose to index that partial by \mathbb{Z} in order to simplify notation in arguments below, but of course, any countable set works equally well. The dense sets inside \mathcal{D} are such as to ensure that the \mathcal{D} -generic filter for \mathbb{H} gives a total partition of X . The following lemma is a simply observation that the problem is what is designed to be.

Lemma 5.5.3. *Let \mathbb{Q} be an arbitrary poset. Then in $V^\mathbb{Q}$, the following are equivalent:*

- a. *there exists a sequence $(X_n : n < \omega)$ such that $X = \bigsqcup_{n < \omega} X_n$ and such that for all $n < \omega$, $\Gamma \upharpoonright [X_n]^2 = \text{const.}$;*
- b. *there exists a \mathcal{D} -generic filter for \mathbb{H} .* □

This means that our goal reduces to showing that Player II has a winning strategy in $\mathcal{D}_\kappa^p(o)$, where the game is defined with respect to the problem $(\mathbb{H}, \mathcal{D})$. Our approach for describing this strategy is to define a set of good preconditions in such a way as to be able to show that Player II can win by simply maintaining that all of their moves are good. The hardest part of this definition is to establish in which way the working part w_p of a good precondition p has to respect its side condition part \mathcal{M}_p . To that end, we first work towards describing a coloring of X “over” M given by Γ , for a countable $M \prec H(\mathfrak{c}^+)$.

Notation 5.5.4. We denote by $<_{\mathcal{H}}$ the wellordering $<_{V_\kappa} \upharpoonright H(\mathfrak{c}^+)$ of $H(\mathfrak{c}^+)$ and we denote by \mathcal{H} the first-order structure

$$(H(\mathfrak{c}^+), \in, <_{\mathcal{H}}, X, \Gamma).$$

□

²³This was previously known by the results of [44].

The following lemma is a slight generalization of the well-known fact that if $M \prec \mathcal{H}$ is countable and $\phi(x)$ is a formula with parameters in M and with a realization outside of M , then $\phi(x)$ has uncountably many realizations.

Lemma 5.5.5. *Suppose that*

1. $M \prec \mathcal{H}$ is countable,
2. $a \subseteq X$ is finite such that $a \cap M = \emptyset$,
3. $\phi(x) \in \text{tp}^{\mathcal{H}}(a/M)$.²⁴

Then for all countable $S \subseteq X$, there exists a finite $b \subseteq X - S$ such that $\mathcal{H} \models \phi[b]$.

Proof. For all countable $S \subseteq X$ satisfying that $S \in M$, it holds that $M \models \phi[a] \wedge a \cap S = \emptyset$. Since $M \prec \mathcal{H}$, it follows that

$$\mathcal{H} \models \forall S \in [X]^\omega, \exists x \in [X - S]^{<\omega}, \phi(x).$$

This suffices for the conclusion. □

We can strengthen the above lemma in a way that better takes into the account the coloring Γ . Namely, if $a = \{a_i : i < n\}$, we can find fixed colors $(z_i : i < n)$ such that for all formulas ϕ and all countable $S \subseteq X$ as above, the corresponding tuple $b = \{b_i : i < n\}$ satisfies additionally that the color $\Gamma(a_i, b_i)$ is exactly z_i for all $i < n$.

Lemma 5.5.6. *Suppose that*

1. $M \prec \mathcal{H}$ is countable,
2. $\sigma \subseteq X - M$ is finite.

Then there exists $\varepsilon : \sigma \rightarrow 2$ such that for all $\phi(x) \in \text{tp}^{\mathcal{H}}(\sigma/M)$, for all $S \in [X]^\omega$, there exists $f : \sigma \rightarrow X - (S \cup \sigma)$ such that

- a. $\mathcal{H} \models \phi[\text{ran}(f)]$,
- b. for all $a \in \sigma$, $\Gamma(a, f(a)) = \varepsilon(a)$.

Proof.

- 1° Let us assume otherwise. Then for all $\varepsilon : \sigma \rightarrow 2$, there exist $\phi_\varepsilon(x) \in \text{tp}^{\mathcal{H}}(\sigma/M)$ and $S_\varepsilon \in [X]^\omega$ such that for all $f : \sigma \rightarrow X - (S_\varepsilon \cup \sigma)$, if $\mathcal{H} \models \phi_\varepsilon[\text{ran}(f)]$, then for some $a \in \sigma$, $\Gamma(a, f(a)) \neq \varepsilon(a)$.

²⁴We use the notation “ $\text{tp}^{\mathcal{H}}(a/M)$ ” to denote the type of the object a in the structure \mathcal{H} with respect to parameters in M .

2° Let $\phi^\wedge(x) := \bigwedge_{z:\sigma \rightarrow 2} \phi_z(x)$ and let $S := \bigcup_{z:\sigma \rightarrow 2} S_z \cup \sigma$. It follows that for all $f : \sigma \rightarrow X - S$, if $\mathcal{H} \models \phi^\wedge[\text{ran}(f)]$, then

$$\forall z : \sigma \rightarrow 2, \exists a \in \sigma, \Gamma(a, f(a)) \neq z(a).$$

3° Lemma 5.5.5 implies that there exists $f : \sigma \rightarrow X - S$ such that $\mathcal{H} \models \phi^\wedge[\text{ran}(f)]$.

4° Let $z : \sigma \rightarrow 2 : a \mapsto \Gamma(a, f(a))$. The existence of this z contradicts 2°.

□

Note that in this lemma, the coloring obtained depends on σ , so let us call it z_σ . If we have another finite set $\tau \subseteq X - M$, we would like the colorings z_σ and z_τ to be coherent, i.e. that $z_\sigma \upharpoonright (\sigma \cap \tau) = z_\tau \upharpoonright (\sigma \cap \tau)$. A priori, there is no reason for this to hold, but the following lemma shows that we can *pick* the colorings in such a way as to ensure it.

Lemma 5.5.7. *There exists a function $M \rightarrow \Gamma_M$ which is defined on all countable $M \prec \mathcal{H}$ and which satisfies that for all countable $M \prec \mathcal{H}$,*

a. $\Gamma_M : X - M \rightarrow 2$,

b. *for all finite $\sigma \subseteq X - M$, for all $\phi(x) \in \text{tp}^{\mathcal{H}}(\sigma/M)$, for all $S \in [X]^\omega$, there exists $f : \sigma \rightarrow X - (S \cup \sigma)$ such that $\mathcal{H} \models \phi[\text{ran}(f)]$ and such that for all $a \in \sigma$, $\Gamma(a, f(a)) = \Gamma_M(a)$.*

Proof.

1° For $M \prec \mathcal{H}$ countable and for $\sigma \subseteq X - M$ finite, let $\tilde{\Gamma}(\sigma/M)$ be defined as the $<_{\mathcal{H}}$ -least $z : \sigma \rightarrow 2$ such that for all $\phi(x) \in \text{tp}^{\mathcal{H}}(\sigma/M)$, for all $S \in [X]^\omega$, there exists $f : \sigma \rightarrow X - (S \cup \sigma)$ satisfying that $\mathcal{H} \models \phi[\text{ran}(f)]$ and that for all $a \in \sigma$, $\Gamma(a, f(a)) = z(a)$. This is possible by Lemma 5.5.6.

2° Let \mathcal{U} be an ultrafilter on $[X]^{<\omega}$ satisfying that for all $\sigma \in [X]^{<\omega}$,

$$\{\tau \in [X]^{<\omega} : \sigma \subseteq \tau\} \in \mathcal{U}.$$

3° For a countable $M \prec \mathcal{H}$ and for $\sigma \subseteq X - M$ finite, we define $\Gamma(\sigma/M)$ to be that $z : \sigma \rightarrow 2$ which satisfies that for \mathcal{U} -almost all τ , it holds that $\sigma \subseteq \tau$ and $\tilde{\Gamma}(\tau/M) \upharpoonright \sigma = z$.

4° Observe that for all countable $M \prec \mathcal{H}$ and for all finite $\sigma \subseteq \tau \subseteq X - M$, it holds that $\Gamma(\tau/M) \upharpoonright \sigma = \Gamma(\sigma/M)$.

5° For a countable $M \prec \mathcal{H}$, we let

$$\Gamma_M := \bigcup_{\sigma \in [X-M]^{<\omega}} \Gamma(\sigma/M).$$

It is easily seen that $M \mapsto \Gamma_M$ is as required. □

Notation 5.5.8.

- a. Let us denote by $M \mapsto \Gamma_M$ the $<_{V_\kappa}$ -least mapping whose existence is established by Lemma 5.5.7.
- b. For $M \in \mathcal{C}$, we denote by Γ_M the function $\Gamma_{M \cap \mathcal{H}}$.
- c. We denote by r_2 the mapping which assigns to each element of \mathbb{Z} its remainder modulo 2. □

We are now ready to define the set of *good* preconditions.

Definition 5.5.9. We define a subset **Good** of $\mathbb{P}^p \cap V_\kappa$ by asserting that $p \in \text{Good}$ iff

- a. $\mathcal{M}_p \neq \emptyset$,
- b. for all $M \in \mathcal{M}_p$, it holds that $(\Gamma_P : P \prec \mathcal{H} \text{ countable}) \in M$,
- c. for all $a \in \text{dom}(w_p)$, it is true that $w_p(a) < 0$ if and only if a belongs to the \in -least element of \mathcal{M}_p ,
- d. for all $n \in \mathbb{Z}_{\geq 0}$, for all distinct $a, b \in w_p^{-1}(n)$, it holds that $\Gamma(a, b) = r_2(n)$,
- e. for all $M, N \in \mathcal{M}_p$, if $M \in N$ and $\mathcal{M}_p \cap (N - M) = \emptyset$, then letting $\sigma := \text{dom}(w_p) \cap (N - M)$, it holds that $\Gamma_M(\sigma) = (r_2 \circ w_p) \upharpoonright \sigma$,
- f. $\text{dom}(w_p) \subseteq \bigcup \mathcal{M}_p$. □

Recall that the goal is to show that Player II can win $\mathfrak{D}_\kappa^p(o)$ by maintaining that for all $n \in [-1, \omega)$, $p_n \in \text{Good}$. The next three lemmas exactly address the three types of moves that Player I can making and show that Player II can keep answering with good preconditions.

Lemma 5.5.10. *Suppose that $p \in \text{Good}$ and that $a \in X$. Then there exists $q \in \text{Good}$ such that $q \leq p$ and $a \in \text{dom}(w_q)$.*

Proof.

1° If $a \in \text{dom}(w_p)$, we can take $q := p$.

2° Hence, let us assume that $a \notin \text{dom}(w_p)$.

3° If a is in the \in -least model of \mathcal{M}_p , then let

- a. $n \in \mathbb{Z}_{<0} - \text{ran}(w_p)$,
- b. $w_q := w_p \cup \{(a, n)\}$,
- c. $\mathcal{M}_q := \mathcal{M}_p$,
- d. $q := (w_q, \mathcal{M}_q)$.

It is now easily seen that $q \in \text{Good}$, $q \leq p$, and $a \in \text{dom}(w_q)$.

4° Hence, let us assume that a is not in the \in -least model of \mathcal{M}_p .

5° If $a \in \bigcup \mathcal{M}_p$, let

- a. $\mathcal{M}_q := \mathcal{M}_p$,
- b. $M \in N \in \mathcal{M}_p$ be such that $\mathcal{M}_p \cap (N - M) = \emptyset$ and $a \in N - M$,
- c. $n \in \omega - \text{ran}(w_p)$ be such that $r_2(n) = \Gamma_M(\{a\})$,
- d. $w_q := w_p \cup \{(a, n)\}$,
- e. $q := (w_q, \mathcal{M}_q)$.

It is now easily seen that $q \in \text{Good}$, $q \leq p$, and $a \in \text{dom}(w_q)$.

6° If $a \notin \bigcup \mathcal{M}_p$, let

- a. $N \in \mathcal{C}_{<\kappa}$ be such that $\mathcal{H}, (P \mapsto \Gamma_P), p \in N$,
- b. $\mathcal{M}_q := \mathcal{M}_p \cup \{N\}$,
- c. M be the \in -largest element of \mathcal{M}_p ,
- d. $n \in \omega - \text{ran}(w_p)$ be such that $r_2(n) = \Gamma_M(\{a\})$,
- e. $w_q := w_p \cup \{(a, n)\}$,
- f. $q := (w_q, \mathcal{M}_q)$.

It is now easily seen that $q \in \text{Good}$, $q \leq p$, and $a \in \text{dom}(w_q)$.

7° The four cases conclude the proof.

□

Lemma 5.5.11. *Suppose that $p \in \text{Good}$ and that $M \prec H((2^\kappa)^+)$ is countable satisfying that $p, \kappa \in M$. Then there exist $q \in \text{Good}$ and $\lambda \in \beth_{\text{fix}} \cap \kappa$ such that $q \leq p$, $\kappa \cap \text{Hull}(M, V_\lambda) \subseteq \lambda$, and $M \downarrow \lambda \in \mathcal{M}_q$.*

Proof. It is easily seen that we can take $\mathcal{M}_q := \mathcal{M}_p \cup \{M \downarrow \lambda\}$ and $q := (w_p, \mathcal{M}_q)$, for an appropriate λ . \square

Lemma 5.5.12. *Suppose that*

1. $p \in \text{Good}$,
2. $N \in \mathcal{M}_p$,
3. $\pi : N \longrightarrow N^* \prec H((2^\kappa)^+)$ is a lifting,
4. $E \in N$ is such that $p \in \pi(E)$.

Then there exist $q \in E \cap N$ and $p' \in \text{Good}$ such that $p' \leq p, q$.

Proof.

1° Let $n := |w_p|$ and let $(a_i : i < n)$ be the $<_{\mathcal{H}}$ -increasing enumeration of $\text{dom}(w_p)$. Since Γ is an open coloring, there exists a sequence $(t_i : i < n) \in (\omega^{<\omega})^n$ such that

- a. for all $i < n$, $t_i \triangleleft a_i$,
- b. for all $i < j < n$, t_i and t_j are not initial segments of each other,
- c. for all $i < j < n$, for all $b, c \in X$, if $t_i \triangleleft b$ and $t_j \triangleleft c$, then $\Gamma(b, c) = \Gamma(a_i, a_j)$.

2° Let $k \leq n$ be such that $\{a_i : i < k\} = \text{dom}(w_p) \cap N$.

3° Since $p \in \pi(E) \in N^*$, we may assume without loss of generality that for all $q \in E$, it holds that

- a. $q \in \text{Good}$,
- b. $|\text{dom}(w_q)| = n$,
- c. $\mathcal{M}_p \cap N \subseteq \mathcal{M}_q$,
- d. the first k elements of $\text{dom}(w_q)$ are exactly a_0, \dots, a_{k-1} ,²⁵
- e. for all $i < n$, letting b be the i^{th} element of $\text{dom}(w_q)$, it holds that $w_q(b) = w_p(a_i)$ and $t_i \triangleleft b$.

4° **Claim.** There exists $q \in E \cap N$ such that for all $i \in [k, n)$, letting b be the i^{th} element of $\text{dom}(w_q)$, it holds that $\Gamma(a_i, b) = r_2(w_p(a_i))$.

Proof.

²⁵We mean the first k in the order $<_{\mathcal{H}}$.

- 1' Let $\bar{E} \in N$ be the set of all n -tuples that increasingly enumerate $\text{dom}(w_q)$ for some $q \in E$. Since $N \prec \hat{N}$, it suffices to show that there exists $\sigma \in \bar{E}$ such that for all $i \in [k, n)$, it holds that $\Gamma(a_i, \sigma_i) = r_2(w_p(a_i))$. We may assume without loss of generality that $k < n$, for otherwise there is nothing to show.
- 2' Let $(M_i : i \leq l)$ be the \in -increasing enumeration of $\mathcal{M}_p - N$ and let $(n_i : i \leq l)$ be the increasing sequence of natural numbers such that for all $i \leq l - 1$,

$$\text{dom}(w_p) \cap (M_{i+1} - M_i) = \{a_j : n_i \leq j < n_{i+1}\}.$$

Since $k > 0$, we must have that $l > 0$.

- 3' Let $\phi_l(x_k, \dots, x_{n-1})$ be the formula in the language of \mathcal{H} , with parameters in $N \cap H(\mathfrak{c}^+)$, asserting that

$$a_0 \widehat{\ } a_1 \widehat{\ } \dots \widehat{\ } a_{k-1} \widehat{\ } x_k \widehat{\ } \dots \widehat{\ } x_{n-1} \in \bar{E}.$$

Observe that $\mathcal{H} \models \phi_l[a_k, \dots, a_{n-1}]$.

- 4' By a reverse recursion on $i < l$, we define sequences

$$(r_j : n_i \leq j < n_{i+1}), (s_j : n_i \leq j < n_{i+1}) \in (\omega^{<\omega})^{n_{i+1}-n_i}$$

and a formula $\phi_i(x_k, \dots, x_{n_{i-1}})$. We maintain as the recursive hypothesis that

$$\mathcal{H} \models \phi_{i+1}[a_k, \dots, a_{n_{i+1}-1}].$$

- 5' Since $p \in \text{Good}$, we have that for all $j \in [n_i, n_{i+1})$, $\Gamma_{M_i}(\{a_j\}) = r_2(w_p(a_j))$.
- 6' By the definition of Γ_{M_i} and the fact that

$$\mathcal{H} \models \phi_{i+1}[a_k, \dots, a_{n_{i+1}-1}],$$

we have that there exist $b_{n_i}, \dots, b_{n_{i+1}-1} \in X - M_{i+1}$ such that

$$\mathcal{H} \models \phi_{i+1}[a_k, \dots, a_{n_{i-1}}, b_{n_i}, \dots, b_{n_{i+1}-1}]$$

and such that for all $j \in [n_i, n_{i+1})$,

$$\Gamma(a_j, b_j) = \Gamma_{M_i}(\{a_j\}) = r_2(w_p(a_j)).$$

- 7' For $j \in [n_i, n_{i+1})$, let $r_j \triangleleft a_j$ and $s_j \triangleleft b_j$ be finite sequences of natural numbers which are not initial segments of each other and which satisfy that for all $c, d \in X$, if $r_i \triangleleft c$ and $s_i \triangleleft d$, then

$$\Gamma(c, d) = \Gamma(a_j, b_j) = r_2(w_p(a_j)).$$

(We are using here that the coloring is open.)

8' Let $\phi_i(x_k, \dots, x_{n_i-1})$ be the formula asserting that

$$\begin{aligned} & \exists y_{n_i}, \dots, y_{n_{i+1}-1}, \bigwedge_{n_i \leq j < n_{i+1}} s_j \triangleleft y_j \wedge \\ & \wedge \phi_{i+1}(x_k, \dots, x_{n_i-1}, y_{n_i}, \dots, y_{n_{i+1}-1}). \end{aligned}$$

This concludes the reverse recursion.

9' Observe that $\mathcal{H} \models \phi_0$. By recursion on $i < l$, we can witness the existential quantifiers in ϕ_i so that we obtain $\sigma_{n_i}, \dots, \sigma_{n_{i+1}-1} \in X \cap N$ satisfying that $\mathcal{H} \models \phi_i[\sigma_k, \dots, \sigma_{n_{i+1}-1}]$.

10' Let $\sigma := (a_0, \dots, a_{k-1}, \sigma_k, \dots, \sigma_{n-1})$. Line 9' yields that $\sigma \in \bar{E}$ and that for all $j \in [k, n)$, $s_j \triangleleft \sigma_j$.

11' For all $j \in [k, n)$, the fact that $r_j \triangleleft a_j$ and $s_j \triangleleft \sigma_j$ ensures that $\Gamma(a_j, \sigma_j) = r_2(w_p(a_j))$ (cf. line 7'). This shows that σ is as required in 1'. □

5° Let q be as provided by Claim 4° and let

$$p' := (w_p \cup w_q, \mathcal{M}_p \cup \mathcal{M}_q).$$

We will show that q and p' are as required.

6° We already now that $q \in E \cap N$ and that $p' \leq p, q$, so the only thing to verify is that $p' \in \text{Good}$. For this, in turn, the only non-trivial point is that for all $n \in \mathbb{Z}_{\geq 0}$, for all distinct $b, c \in w_{p'}^{-1}(n)$, it holds that $\Gamma(b, c) = r_2(n)$. Let us show that this is indeed the case.

7° This is clear if $b, c \in w_p^{-1}(n)$ or $b, c \in w_q^{-1}(n)$, so let us assume that $b \in w_p^{-1}(n) - N$ and $c \in w_q^{-1}(n) - N$.

8° Let $i, j \in [k, n)$ be such that b is the i^{th} element of $\text{dom}(w_p)$ and c is the j^{th} element of $\text{dom}(w_q)$. If $i \neq j$, we have that

$$\Gamma(b, c) = \Gamma(a_i, a_j) = r_2(n)$$

because $t_i \triangleleft b$ and $t_j \triangleleft c$ (which gives the first equality) and

$$w_p(a_i) = w_p(b) = n = w_q(c) = w_p(a_j)$$

(which gives the seconde equality).

9° If, on the other hand, $i = j$, then we have that

$$\Gamma(b, c) = \Gamma(a_i, c) = r_2(w_p(a_i)) = r_2(n)$$

because of the way we chose q (cf. Claim 4°).

10° This shows that q and p' are as required. □

Putting together the three lemma just proved, we get the desired conclusion.

Corollary 5.5.13. *Player II has a winning strategy in $\mathfrak{D}_\kappa^p(o)$.*

Proof. One can easily find $M \in \mathcal{C}_{<\kappa}$ such that

$$p_{-1} := (\emptyset, \{M\}) \in \text{Good}.$$

Lemmas 5.5.10, 5.5.11, and 5.5.12 then show that Player II can survive ω moves by maintaining that for all $n < \omega$, $p_n \in \text{Good}$. This describes a winning strategy for Player II in $\mathfrak{D}_\kappa^p(o)$. □

As observed in the beginning, this exactly solves the problem that we posed.

Corollary 5.5.14. *There exists a proper poset \mathbb{Q} such that in $V^{\mathbb{Q}}$, there exists a sequence*

$$(X_n : n < \omega)$$

such that $X = \bigsqcup_{n < \omega} X_n$ and for all $n < \omega$, $\Gamma \upharpoonright [X_n]^2 = \text{const}$.

Proof. This follows by Corollary 5.5.13 and Lemma 5.5.3. □

5.6 Semiproper Satisfiability²⁶

Section Assumptions 5.6.1. We fix a problem $(\mathbb{H}, \mathcal{D})$. □

In this section, we want to characterize the semiproper satisfiability of $(\mathbb{H}, \mathcal{D})$ in a way analogous to what we did for the proper satisfiability. We will closely follow the outline of Section 5.4. We start by introducing the preconditions. They are the same as in the proper case, except that we order them differently.

Definition 5.6.2.

- a. The class \mathbb{P}^{SP} consists of all $p = (w_p, \mathcal{M}_p)$ such that $w_p \in \mathbb{H}$ and \mathcal{M}_p is a finite vm-chain.
- b. The ordered \leq^{SP} on vm-chains is defined by asserting that $\mathcal{M} \leq^{\text{SP}} \mathcal{N}$ holds iff for all $N \in \mathcal{N}$, there exists $M \in \mathcal{M}$ such that

²⁶A version of this section was done jointly with Ben De Bondt. He pursued some of these ideas in a different direction in his thesis [45].

- i. $\delta_M = \delta_N$,
 - ii. $\lambda_M = \lambda_N$,
 - iii. there exists $X \subseteq V_{\lambda_N}$ such that $M = \text{Hull}(N, X)$.
- c. The order \leq^{SP} on \mathbb{P}^{SP} is defined by asserting that $p \leq q$ holds iff $w_p \leq_{\mathbb{H}} w_q$ and $\mathcal{M}_p \leq^{\text{SP}} \mathcal{M}_q$. \square

The game in this case differs from the one in the proper case only in that Player II is required to do less when answering questions of the third kind.

Definition 5.6.3. Suppose that $\kappa \in \beth_{\text{fix}}$ and $p \in \mathbb{P}^{\text{SP}} \cap V_{\kappa}$. Then the game $\mathcal{G}_{\kappa}^{\text{SP}}(p)$ is defined as the length ω two Player game of the form

I	Q_0	Q_1	\cdots
II	p_{-1}	p_0	$p_1 \quad \cdots$

where $p_{-1} \in \mathbb{P}^{\text{SP}} \cap V_{\kappa}$ is such that $p_{-1} \leq p$, and for all $n < \omega$, the following is satisfied.

- a. Player I must ensure that either
 - i. $Q_n = D$ for some $D \in \mathcal{D}$, or
 - ii. $Q_n = M$ for some countable $M \prec H((2^{\kappa})^+)$, or
 - iii. $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$.
- b. Player II must ensure that $p_n \in \mathbb{P}^{\text{SP}} \cap V_{\kappa}$ and $p_n \leq p_{n-1}$.
- c. Player II must ensure that if $Q_n = D$ for some $D \in \mathcal{D}$, then there exists $w \in D$ such that $w_{p_n} \leq w$.
- d. Player I must ensure that if $Q_n = M$ for some countable $M \prec H((2^{\kappa})^+)$, then $p_{n-1}, \kappa \in M$.
- e. Player II must ensure that if $Q_n = M$ for some countable $M \prec H((2^{\kappa})^+)$, then there exists $\lambda \in \beth_{\text{fix}} \cap \kappa$ such that $\kappa \cap \text{Hull}(M, V_{\lambda}) \subseteq \lambda$ and $M \downarrow \lambda \in \mathcal{M}_{p_n}$.
- f. Player I must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exist $M^* \prec (H((2^{\kappa})^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

- g. Player II must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exists $q \in E$ such that $\delta(\text{Hull}(M, q)) = \delta(M)$ and $p_n \leq q$.

The infinite plays with no rules broken are won by Player II. \square

Definition 5.6.4. Suppose that $\kappa \in \beth_{\text{fix}}$. We let $\mathbb{C}_\kappa^{\text{SP}}$ consist of all $p \in \mathbb{P}^{\text{SP}} \cap V_\kappa$ such that Player II wins $\mathcal{D}_\kappa^{\text{SP}}(p)$. \square

Proposition 5.6.5. Suppose that $\kappa \in \beth_{\text{fix}}$. Then the following holds.

- a. The set $\mathbb{C}_\kappa^{\text{SP}}$ is an initial segment of $(\mathbb{P}^{\text{SP}}, \leq^{\text{SP}}, o)$.
- b. If $\mathbb{C}_\kappa^{\text{SP}}$ is non-empty, then $o \in \mathbb{C}_\kappa^{\text{SP}}$ and $(\mathbb{C}_\kappa^{\text{SP}}, \leq^{\text{SP}}, o)$ is a poset.
- c. If $\mathbb{C}_\kappa^{\text{SP}}$ is non-empty, then, letting g be a V -generic for $\mathbb{C}_\kappa^{\text{SP}}$, we have that the set

$$\{w_p : p \in g\}$$

is a \mathcal{D} -generic filter for \mathbb{H} .

Proof. This is shown analogously to Proposition 5.4.6. \square

Proposition 5.6.6. Suppose that $\kappa \in \beth_{\text{fix}}$ and that $\mathbb{C}_\kappa^{\text{SP}}$ is non-empty. Then $\mathbb{C}_\kappa^{\text{SP}}$ is semiproper.

Proof.

1° For $p \in \mathbb{C}_\kappa^{\text{SP}}$, let σ_p be a winning strategy for Player II in the game $\mathcal{D}_\kappa^{\text{SP}}(p)$.

2° Let

$$M \prec (H((2^\kappa)^+), \in, \kappa, \mathbb{H}, \mathcal{D}, (\sigma_p : p \in \mathbb{C}_\kappa^{\text{SP}}))$$

be an arbitrary countable virtual model. We want to show that $\mathbb{C}_\kappa^{\text{SP}}$ is semiproper for M .

3° Let $p \in \mathbb{C}_\kappa^{\text{SP}} \cap M$ be arbitrary. We need to find $p_0 \leq p$ which is semigeneric for $(M, \mathbb{C}_\kappa^{\text{SP}})$.

4° Consider the following partial play of $\mathcal{D}_\kappa^{\text{SP}}(p)$ according to σ_p .

$$\begin{array}{c|cc} \text{I} & & M \\ \hline \text{II} & p_{-1} & p_0 \end{array}$$

Since $p_{-1} = \sigma_p(\emptyset)$ and $p \in M$, we get that $p_{-1} \in M$. Thus, Player I did not break the rules.

5° Consequently, it follows that $p_0 \leq p_{-1} \leq p$ and that there exists $\lambda \in \beth_{\text{fix}} \cap \kappa$ such that

$$\kappa \cap \text{Hull}(M, V_\lambda) \subseteq \lambda$$

and such that $M \downarrow \lambda \in \mathcal{M}_{p_0}$.

6° We want to show that p_0 is as required. This comes down to the following. Let $q \leq p_0$ and let $E \in M$ be an open dense subset of $\mathbb{C}_\kappa^{\text{sp}}$ containing q . We need to show that there exist $r \in E$ such that $\delta(\text{Hull}(M, r)) = \delta(M)$ and $r \parallel q$.

7° Let

$$\pi : \widehat{M \downarrow \lambda} \xrightarrow{\simeq} \text{Hull}(M, V_\lambda) \prec H_\theta$$

be the anti-collapse. We have that $\pi^{-1}(\kappa) = \lambda$ and $\pi^{-1}(E) = E \cap V_\lambda$.

8° Let us consider the following partial play of $\mathcal{D}_\kappa^{\text{sp}}(q)$ according to σ_q .

$$\begin{array}{c|cc} \text{I} & & (N, E \cap V_\lambda) \\ \hline \text{II} & q_{-1} & q_0 \end{array}$$

Here, N is the unique virtual model in $\mathcal{M}_{q_{-1}}$ such that $\delta_N = \delta_M$. (It exists since $M \downarrow \lambda \in \mathcal{M}_{p_0}$ and $q_{-1} \leq p_0$.)

9° Player I did not break the rules since

- a. $N \in \mathcal{M}_{q_{-1}}$,
- b. $E \cap V_\lambda = \pi^{-1}(E) \in M \downarrow \lambda \subseteq N$,
- c. $\pi \upharpoonright N : N \longrightarrow \text{Hull}(M, X) \prec H((2^\kappa)^+)$ is a lifting, where $X \subseteq V_\lambda$ is such that $N = \text{Hull}(M \downarrow \lambda, X)$,
- d. $q_{-1} \in E = \pi(E \cap V_\lambda)$ (because E is open and $q_{-1} \leq q$).

10° The rules then imply that $q_0 \leq q$ and that there exists $r \in E \cap V_\lambda$ such that

$$\delta(\text{Hull}(N, r)) = \delta(N)$$

and $q_0 \leq r$.

11° Thus, $r \in E$, $r \parallel q$ (as witnessed by q_0), and $\delta(\text{Hull}(M, r)) = \delta(M)$. (For the last point, note that

$$\begin{aligned} \delta(M) &\leq \delta(\text{Hull}(M, r)) = \delta(\text{Hull}(M \downarrow \lambda, r)) \leq \\ &\leq \delta(\text{Hull}(N, r)) = \delta(N) = \delta(M). \end{aligned}$$

□

We now proceed to showing that the semiproper satisfiability implies that Player II has a winning strategy in $\mathcal{D}_\kappa^{\text{sp}}(o)$, for some κ . We will use the following simple lemma.

Lemma 5.6.7. *Suppose that*

1. \mathbb{Q} is a poset,
2. $\theta \gg \text{rank}(\mathbb{Q})$ is regular,

3. $M \prec (H_\theta, \in, \mathbb{Q})$ is countable,
4. p is semigeneric for (M, \mathbb{Q}) ,
5. $p \in E \in M$.

Then there exist $r \in E \cap \mathbb{Q}$ and $s \leq p, r$ such that $s \Vdash \check{r} \in \check{M}(\dot{g})$. □

Theorem 5.6.8. *Suppose that*

1. *there exists a proper class of inaccessible cardinals,*
2. $(\mathbb{H}, \mathcal{D})$ *is semiproper-satisfiable.*

Then for some inaccessible κ , Player II has a winning strategy in $\mathfrak{D}_\kappa^{\text{sp}}(o)$.

Proof.

- 1° By Lemma 5.4.8, there exists a semiproper, complete boolean algebra \mathbb{B} and a mapping $i : \mathbb{H} \rightarrow \mathbb{B}$ satisfying that
 - a. $i(1) = 1$,
 - b. for all $p, q \in \mathbb{H}$, if $p \leq q$, then $i(p) \leq i(q)$,
 - c. for all $p, q \in \mathbb{H}$, if $p \perp q$, then $i(p)i(q) = 0$,
 - d. for all $p, q \in \mathbb{H}$, if $p \parallel q$, then $i(pq) = i(p)i(q)$,
 - e. for all $D \in \mathcal{D}$, $\sum i[D] = 1$.
- 2° Let κ be an inaccessible cardinal satisfying that $\text{rank}(\mathbb{H}), \text{rank}(\mathbb{B}) < \kappa$ and let us show that $o \in \mathbb{C}_\kappa^{\text{sp}}$. We need to show that Player II has a winning strategy in $\mathfrak{D}_\kappa^{\text{sp}}(o)$.
- 3° Let us assume otherwise. Since the game is closed for Player II, it follows that Player I has a winning strategy σ . We will reach a contradiction by showing that Player II can defeat σ .
- 4° For $p \in \mathbb{P}^{\text{sp}}$ and $b \in \mathbb{B}$, let $\Psi(p, b)$ be the conjunction of the following statements:
 - a. for all $M \in \mathcal{M}_p$, $\mathbb{H}, \mathbb{B}, i \in M$,
 - b. for all $M \in \mathcal{M}_p$, $\lambda_M > \text{rank}(\mathbb{B})$,
 - c. $0 < b \leq i(w_p)$,
 - d. for all $M \in \mathcal{M}_p$, b is semigeneric for (M, \mathbb{B}) .

We will show that Player II can play by maintaining that for all $n \in [-1, \omega)$, there exists $b \in \mathbb{B}$ such that $\Psi(p_n, b)$ holds.

5° Suppose first that $n = -1$. Let

$$M \prec (H((2^\kappa)^+), \in, \kappa, \mathbb{H}, \mathbb{B}, i)$$

be countable, let $\lambda := \sup(\kappa \cap M)$, and let $p_{-1} := (\emptyset, \{M \downarrow \lambda\})$. We see that conditions 4°a and 4°b are met.

6° Since \mathbb{B} is semiproper and $1_{\mathbb{B}} \in M$, there exists $b \in \mathbb{B}$ such that

$$0 < b \leq 1 = i(w_{p_{-1}})$$

and such that b is semigeneric for (M, \mathbb{B}) .

7° It is now easily seen that $\Psi(p_{-1}, b)$ holds.

8° Let us now inductively consider the case n assuming that there exists $b \in \mathbb{B}$ such that $\Psi(p_{n-1}, b)$ holds. Let Player I play a move Q_n and let us show how Player II can answer in such a way as to ensure that there exists $c \in \mathbb{B}$ such that $\Psi(p_n, c)$ holds.

9° **Case I.** $Q_n = D$ for some $D \in \mathcal{D}$.

Proof. This is shown like Case I of the proof of Theorem 5.4.9. □

10° **Case II.** $Q_n = M$ for some countable $M \prec H((2^\kappa)^+)$.

Proof. This is shown like Case II of the proof of Theorem 5.4.9. □

11° **Case III.** $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$.

Proof.

1' We have that there exist $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

2' We need to find $p_n \leq p_{n-1}$, $q \in E$, and $c \in \mathbb{B}$ such that

- a. $\delta(\text{Hull}(M, q)) = \delta(M)$,
- b. $p_n \leq q$,
- c. $\Psi(p_n, c)$ holds.

3' Let $\mathcal{N} := \mathcal{M}_{p_{n-1}} \cap V_{\lambda_M} \in M$ and let F consist of all $d \in \mathbb{B}$ such that there exists $q \in E \cap \mathbb{P}^{\text{sp}}$ satisfying that

- a. for all $N \in \mathcal{M}_q, \mathbb{H}, \mathbb{B}, i \in N$,
- b. for all $N \in \mathcal{M}_q, \lambda_N > \text{rank}(\mathbb{B})$,

- c. $0 < d \leq i(w_q)$,
- d. for all $N \in \mathcal{M}_q$, d is semigeneric for (N, \mathbb{B}) ,
- e. \mathcal{N} is an initial segment of (\mathcal{M}_q, \in) .

We have that $F \in M$.

4' **Claim.** There exists $d \in F$ and $c \in \mathbb{B}$ such that

- a. for all $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$, $\delta(\text{Hull}(N, d)) = \delta(N)$,
- b. $0 < c \leq bd$,
- c. for all $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$, c is semigeneric for $(\text{Hull}(N, d), \mathbb{B})$.

Proof.

- 1'' Since $p_{n-1} \in \pi(E)$ and $\Psi(p_{n-1}, b)$ holds, it follows that $b \in \pi(F)$.
- 2'' Since $F \subseteq \mathbb{B}$, we have that $F \in M \cap V_{\lambda_M}$. This means that $\pi(F) = F$ and consequently, $b \in F$.
- 3'' By Lemma 5.6.7, there exists $d \in F$ and $c \in \mathbb{B}$ such that

$$0 < c \leq bd$$

and $c \Vdash d \in M(\dot{g})$.

- 4'' Let $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$ be arbitrary. We have to verify that $\delta(\text{Hull}(N, d)) = \delta(N)$ and c is semigeneric for $(\text{Hull}(N, d), \mathbb{B})$.
- 5'' Since $c \leq b$ and b is semigeneric for N , we have that

$$c \Vdash \delta(N(\dot{g})) = \delta(N).$$

- 6'' Since $c \Vdash d \in M(\dot{g})$, we have that $c \Vdash d \in N(\dot{g})$
- 7'' Consequently, $c \Vdash N \prec \text{Hull}(N, d) \prec N(\dot{g})$.
- 8'' Lines 5'' and 7'' imply that $\delta(\text{Hull}(N, d)) = \delta(N)$.
- 9'' Line 7'' also implies that $c \Vdash \text{Hull}(N, d)(\dot{g}) = N(\dot{g})$.

- 10'' By referencing line 5'' once again, we get that c is semigeneric for

$$(\text{Hull}(N, d), \mathbb{B}).$$

□

5' Since $d \in F \cap \text{Hull}(M, d)$, there exists $q \in E \cap \mathbb{P}^{\text{SP}} \cap \text{Hull}(M, d)$ such that

- a. for all $N \in \mathcal{M}_q$, $\mathbb{H}, \mathbb{B}, i \in N$,
- b. for all $N \in \mathcal{M}_q$, $\lambda_N > \text{rank}(\mathbb{B})$,
- c. $0 < d \leq i(w_q)$,
- d. for all $N \in \mathcal{M}_q$, d is semigeneric for (N, \mathbb{B}) ,
- e. $\mathcal{M}_{p_{n-1}} \cap V_{\lambda_M}$ is an initial segment of (\mathcal{M}_q, \in) .

6' Let

- a. $w_{p_n} := w_{p_{n-1}} w_q$,
- b. $\mathcal{M}_{p_n} := \mathcal{M}_q \cup \{\text{Hull}(N, d) : N \in \mathcal{M}_{p_{n-1}}, \delta_N \geq \delta_M\}$,
- c. $p_n := (w_{p_n}, \mathcal{M}_{p_n})$.

7' **Claim.** p_n, q , and c are as required in 2'

Proof.

- 1'' Since $0 < c \leq bd \leq i(w_{p_{n-1}})i(w_q)$, it follows that $w_{p_{n-1}} \parallel w_q$. This means that w_{p_n} is a well defined element of \mathbb{H} .
- 2'' Since $\mathcal{M}_q \in \text{Hull}(M, d)$, we can easily verify that \mathcal{M}_{p_n} is a vm-chain.
- 3'' This means that $p_n \in \mathbb{P}^{\text{SP}}$.
- 4'' The fact that $p_n \leq p_{n-1}$ is self-evident, modulo the observation that for all $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$, we have that

$$\delta(\text{Hull}(N, d)) = \delta(N) \text{ and } \lambda(\text{Hull}(N, d)) = \lambda(N).$$

- 5'' q was picked so that $q \in E$, while the fact that $q \in \text{Hull}(M, d)$ easily implies that

$$\delta(\text{Hull}(M, q)) = \delta(M).$$

- 6'' The fact that $p_n \leq q$ is also self-evident.
- 7'' It remains to verify that $\Psi(p_n, c)$ holds, i.e. that
 - a. for all $N \in \mathcal{M}_{p_n}, \mathbb{H}, \mathbb{B}, i \in N$,
 - b. for all $N \in \mathcal{M}_{p_n}, \lambda_N > \text{rank}(\mathbb{B})$,
 - c. $0 < c \leq i(w_{p_n})$,
 - d. for all $N \in \mathcal{M}_{p_n}, c$ is semigeneric for (N, \mathbb{B}) .

Parts a and b are immediate, while for c, we have

$$c \leq bd \leq i(w_{p_{n-1}})i(w_q) = i(w_{p_n}).$$

- 8'' Let us explain part d. By the choice of q , we have that d is semigeneric for (N, \mathbb{B}) for all $N \in \mathcal{M}_q$.
- 9'' By the choice of c , we know that it is semigeneric for

$$(\text{Hull}(N, d), \mathbb{B}),$$

for all $N \in \mathcal{M}_q - \mathcal{N}$.

- 10'' Since $c \leq d$, the last two line imply that c is generic for (N, \mathbb{B}) for all $N \in \mathcal{M}_{p_n}$.

□

8' The claim concludes verification of Case III.

□

12° The three cases show together that Player II can survive ω moves against σ . This contradicts the fact that σ is winning for Player I. □

Corollary 5.6.9. *Suppose that there is a proper class of inaccessible cardinals. Then the following are equivalent.*

a. $(\mathbb{H}, \mathcal{D})$ is semiproper-satisfiable.

b. For some inaccessible κ , Player II has a winning strategy in $\mathfrak{D}_\kappa^{\text{sp}}(o)$. □

5.7 Stationary Set Preserving Satisfiability

Section Assumptions 5.7.1. We fix a problem $(\mathbb{H}, \mathcal{D})$. □

In this section, we want to characterize the ssp satisfiability of $(\mathbb{H}, \mathcal{D})$ in a way analogous to what we did for proper and semiproper satisfiability. We follow closely the outline of Sections 5.4 and 5.6. In this case, the preconditions and their ordering are exactly the same as in the semiproper case. The game $\mathfrak{D}_\kappa^{\text{ssp}}(p)$ that we define below differs from the game $\mathfrak{D}_\kappa^{\text{sp}}(p)$ in the questions of the second type that Player I can ask. While a semiproper poset is semiproper for club many models, this is not true for an ssp poset. What we have is that an ssp poset is semiproper for projectively stationary many models (cf. [46, Lemma 4.8]). The change in the game is such that it reflects this fact.

Definition 5.7.2. Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$ and $p \in \mathbb{P}^{\text{sp}} \cap V_\kappa$. Then the game $\mathfrak{D}_\kappa^{\text{ssp}}(p)$ is defined as the length ω two Player game of the form

$$\begin{array}{c|ccc} \text{I} & Q_0 & Q_1 & \cdots \\ \hline \text{II} & p_{-1} & p_0 & p_1 \cdots \end{array}$$

where $p_{-1} \in \mathbb{P}^{\text{sp}} \cap V_\kappa$ is such that $p_{-1} \leq p$ and for all $n < \omega$, the following is satisfied.

a. Player I must ensure that either

i. $Q_n = D$ for some $D \in \mathcal{D}$, or

ii. $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary,

iii. $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$.

b. Player II must ensure that $p_n \in \mathbb{P}^{\text{sp}} \cap V_\kappa$ and $p_n \leq p_{n-1}$.

c. Player II must ensure that if $Q_n = D$ for some $D \in \mathcal{D}$, then there exists $w \in D$ such that $w_{p_n} \leq w$.

- d. Player II must ensure that if $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary, then there exist

$$M \prec (H((2^\kappa)^+), \in, \kappa, p_{n-1}, U)$$

and $\lambda \in \mathfrak{J}_{\text{fix}} \cap \kappa$ such that

$$\kappa \cap \text{Hull}(M, V_\lambda) \subseteq \lambda,$$

$M \downarrow \lambda \in \mathcal{M}_{p_n}$, and $\delta_M \in S$.

- e. Player I must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exist $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

- f. Player II must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exists $q \in E$ such that $\delta(\text{Hull}(M, q)) = \delta(M)$ and $p_n \leq q$.

The infinite plays with no rules broken are won by Player II. □

Definition 5.7.3. Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$. We let $\mathbb{C}_\kappa^{\text{SSP}}$ consist of all $p \in \mathbb{P}^{\text{SP}} \cap V_\kappa$ such that Player II wins $\mathfrak{D}_\kappa^{\text{SSP}}(p)$. □

Proposition 5.7.4. Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$. Then the following holds.

- a. The set $\mathbb{C}_\kappa^{\text{SSP}}$ is an initial segment of $(\mathbb{P}^{\text{SP}}, \leq^{\text{SP}}, o)$.
- b. If $\mathbb{C}_\kappa^{\text{SSP}}$ is non-empty, then $o \in \mathbb{C}_\kappa^{\text{SSP}}$ and $(\mathbb{C}_\kappa^{\text{SSP}}, \leq^{\text{SP}}, o)$ is a poset.
- c. If $\mathbb{C}_\kappa^{\text{SSP}}$ is non-empty, then, letting g be a V -generic for $\mathbb{C}_\kappa^{\text{SSP}}$, we have that the set

$$\{w_p : p \in g\}$$

is a \mathcal{D} -generic filter for \mathbb{H} .

Proof. This is verified as in the proof of Proposition 5.4.6. □

Proposition 5.7.5. Suppose that $\kappa \in \mathfrak{J}_{\text{fix}}$ and that $\mathbb{C}_\kappa^{\text{SSP}}$ is non-empty. Then $\mathbb{C}_\kappa^{\text{SSP}}$ is stationary set preserving.

Proof.

- 1° Let us assume otherwise. Then there exist $S \subseteq \omega_1$ which is stationary, $p \in \mathbb{C}_\kappa^{\text{SSP}}$, and \dot{C} which is a canonical name for a subset of ω_1 such that $p \Vdash \dot{C}$ is a club and $\check{S} \cap \dot{C} = \emptyset$.

2° For all $q \in \mathbb{C}_\kappa^{\text{SSP}}$, let σ_q be a winning strategy for Player II in the game $\mathcal{D}_\kappa^{\text{SSP}}(p)$.

3° Consider the following partial play of $\mathcal{D}_\kappa^{\text{SSP}}(p)$ according to σ_p .

$$\begin{array}{c|cc} \text{I} & & (U, S) \\ \hline \text{II} & p_{-1} & p_0 \end{array}$$

Here, $U \subseteq H((2^\kappa)^+)$ is some canonical coding of the tuple

$$(\mathbb{H}, \mathcal{D}, (\sigma_p : p \in \mathbb{C}_\kappa^{\text{SSP}}), \dot{C}).$$

4° The rules then imply that $p_0 \leq p$ and that for some countable

$$M \prec (H((2^\kappa)^+), \in, \kappa, p_{-1}, \mathbb{H}, \mathcal{D}, (\sigma_p : p \in \mathbb{C}_\kappa^{\text{SSP}}), \dot{C})$$

and for some $\lambda \in \mathfrak{J}_{\text{fix}} \cap \kappa$, we have that

$$\kappa \cap \text{Hull}(M, V_\lambda) \subseteq \lambda,$$

$M \downarrow \lambda \in \mathcal{M}_{p_0}$, and $\delta_M \in S$.

5° **Claim.** p_0 is semigeneric for $(M, \mathbb{C}_\kappa^{\text{SSP}})$.

Proof.

1' We need to verify the following. Let $q \in \mathbb{C}_\kappa^{\text{SSP}}$ be such that $q \leq p_0$ and let E be a dense open subset of $\mathbb{C}_\kappa^{\text{SSP}}$ such that $q \in E \in M$. We need to find $r \in E$ such that $\delta(\text{Hull}(M, r)) = \delta(M)$ and $r \parallel q$.

2' Let us consider the following partial play of $\mathcal{D}_\kappa^{\text{SSP}}(q)$ according to σ_q .

$$\begin{array}{c|cc} \text{I} & & (N, E \cap V_\lambda) \\ \hline \text{II} & q_{-1} & q_0 \end{array}$$

Here, N is the unique virtual model in $\mathcal{M}_{q_{-1}}$ such that $\delta_N = \delta_M$. (It exists since $M \downarrow \lambda \in \mathcal{M}_{p_0}$ and $q_{-1} \leq p_0$.)

3' Let

$$\pi : \widehat{M \downarrow \lambda} \xrightarrow{\cong} \text{Hull}(M, V_\lambda) \prec H_\theta$$

be the anti-collapse. We have that $\pi^{-1}(\kappa) = \lambda$ and $\pi^{-1}(E) = E \cap V_\lambda$.

4' Player I did not break the rules since

- a. $N \in \mathcal{M}_{q_{-1}}$,
- b. $E \cap V_\lambda = \pi^{-1}(E) \in M \downarrow \lambda \subseteq N$,
- c. $\pi \upharpoonright N : N \longrightarrow \text{Hull}(M, X) \prec H((2^\kappa)^+)$ is a lifting, where $X \subseteq V_\lambda$ is such that $N = \text{Hull}(M \downarrow \lambda, X)$,
- d. $q_{-1} \in E = \pi(E \cap V_\lambda)$ (because E is open and $q_{-1} \leq q$).

5' The rules then imply that $q_0 \leq q$ and that there exists $r \in E \cap V_\lambda$ such that $\delta(\text{Hull}(N, r)) = \delta(N)$ and $q_0 \leq r$.

6' Thus, $r \in E, r \parallel q$ (as witnessed by q_0), and $\delta(\text{Hull}(M, r)) = \delta(M)$. (For the last point, note that

$$\begin{aligned} \delta(M) &\leq \delta(\text{Hull}(M, r)) = \delta(\text{Hull}(M \downarrow \lambda, r)) \leq \\ &\leq \delta(\text{Hull}(N, r)) = \delta(N) = \delta(M). \end{aligned}$$

□

6° Now, let g be a V -generic filter for $\mathbb{C}_\kappa^{\text{ssp}}$ that contains p_0 . Since p_0 is semigeneric for $(M, \mathbb{C}_\kappa^{\text{ssp}})$, we have that

$$\delta_{M[g]} = \delta_M.$$

7° Since $\dot{C} \in M$ and \dot{C}_g is a club in ω_1^V , we have that

$$\delta_M = \delta_{M[g]} \in \dot{C}_g.$$

8° Since $\delta_M \in S$, we get that $\dot{C}_g \cap S \neq \emptyset$, which contradicts line 1° and the fact that $p_0 \leq p$.

□

We now turn towards showing that the ssp satisfiability implies the existence of a winning strategy for Player II in $\mathfrak{D}_\kappa^{\text{ssp}}(o)$, for some κ . We will need the following lemma.

Lemma 5.7.6. *Suppose that*

1. \mathbb{Q} is a poset,
2. $\theta \gg \text{rank}(\mathbb{Q})$ is regular,
3. \mathbb{Q} is stationary set preserving,
4. $S \subseteq \omega_1$ is stationary,
5. U is an arbitrary subset of H_θ .

Then there exists a countable $M \prec (H_\theta, \in, U)$ such that $\delta_M \in S$ and \mathbb{Q} is semiproper for M .

Proof. See [46, Lemma 4.8].

□

Theorem 5.7.7. *Suppose that*

1. *there exists a proper class of inaccessible cardinals,*

2. $(\mathbb{H}, \mathcal{D})$ is *ssp-satisfiable*.

Then for some inaccessible κ , Player II has a winning strategy in $\mathcal{D}_\kappa^{\text{ssp}}(o)$.

Proof.

1° By Lemma 5.4.8, there exists an ssp, complete boolean algebra \mathbb{B} and a mapping $i : \mathbb{H} \rightarrow \mathbb{B}$ satisfying that

- a. for all $w \in \mathbb{H}$, $i(w) > 0$,
- b. $i(1) = 1$,
- c. for all $p, q \in \mathbb{H}$, if $p \leq q$, then $i(p) \leq i(q)$,
- d. for all $p, q \in \mathbb{H}$, if $p \perp q$, then $i(p)i(q) = 0$,
- e. for all $p, q \in \mathbb{H}$, if $p \parallel q$, then $i(pq) = i(p)i(q)$,
- f. for all $D \in \mathcal{D}$, $\sum i[D] = 1$.

2° Let κ be an inaccessible cardinal satisfying that $\text{rank}(\mathbb{H}), \text{rank}(\mathbb{B}) < \kappa$ and let us show that $o \in \mathbb{C}_\kappa^{\text{ssp}}$. We need to show that Player II has a winning strategy in $\mathcal{D}_\kappa^{\text{ssp}}(o)$.

3° Let us assume otherwise. Since the game is closed for Player II, it follows that Player I has a winning strategy σ . We will reach a contradiction by showing that Player II can defeat σ .

4° For $p \in \mathbb{P}^{\text{ssp}}$ and $b \in \mathbb{B}$, let $\Psi(p, b)$ be the conjunction of the following statements:

- a. for all $M \in \mathcal{M}_p$, $\mathbb{H}, \mathbb{B}, i \in M$,
- b. for all $M \in \mathcal{M}_p$, $\lambda_M > \text{rank}(\mathbb{B})$,
- c. $0 < b \leq i(w_p)$,
- d. for all $M \in \mathcal{M}_p$, b is semigeneric for (M, \mathbb{B}) .

We will show that Player II can play by maintaining that for all $n \in [-1, \omega)$, there exists $b \in \mathbb{B}$ such that $\Psi(p_n, b)$ holds.

5° Suppose first that $n = -1$. By Lemma 5.7.6, there exists a countable

$$M \prec (H((2^\kappa)^+), \in, \kappa, \mathbb{H}, \mathbb{B}, i)$$

such that \mathbb{B} is semiproper for M .

6° Let $\lambda := \sup(\kappa \cap M)$ and let $p_{-1} := (\emptyset, \{M \downarrow \lambda\})$. We see that conditions 4°a and 4°b are met.

7° Since \mathbb{B} is semiproper for M and $1_{\mathbb{B}} \in M$, there exists $b \in \mathbb{B}$ such that

$$0 < b \leq 1 = i(w_{p_{-1}})$$

and such that b is semigeneric for (M, \mathbb{B}) . It is easily seen that $\Psi(p_{-1}, b)$ holds.

8° Let us now inductively consider the case n assuming that there exists $b \in \mathbb{B}$ such that $\Psi(p_{n-1}, b)$ holds. Let Player I make a move Q_n and let us show how Player II can answer in such a way as to ensure that there exists $c \in \mathbb{B}$ such that $\Psi(p_n, c)$ holds.

9° **Case I.** $Q_n = D$ for some $D \in \mathcal{D}$.

Proof. This is verified in the same way as Case I of the proof of Theorem 5.4.9. \square

10° **Case II.** $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary.

Proof.

1' By Lemma 5.7.6, there exists

$$M \prec (H((2^\kappa)^+), \in, \kappa, p_{n-1}, U)$$

such that $\delta_M \in S$ and \mathbb{B} is semiproper for M .

2' Let $F : \kappa \rightarrow \kappa$ be defined by setting

$$F(\xi) := \sup(\kappa \cap \text{Hull}(M, V_\xi)) < \kappa$$

for $\xi < \kappa$. There exists $\lambda \in \beth_{\text{fix}} \cap \kappa$ such that $F[\lambda] \subseteq \lambda$.

3' Let $p_n := (w_{p_{n-1}}, \mathcal{M}_{p_{n-1}} \cup \{M \downarrow \lambda\}) \in \mathbb{P}^{\text{sp}}$. We have that $p_n \leq p_{n-1}$ and $M \downarrow \lambda \in \mathcal{M}_{p_n}$. It remains to find $c \in \mathbb{B}$ such that $\Psi(p_n, c)$.

4' By elementarity, there exists $b_M \in \mathbb{B} \cap M$ such that $\Psi(p_{n-1}, b_M)$ holds.

5' Since \mathbb{B} is semiproper for M , there exists $c \in \mathbb{B}$ such that $0 < c \leq b_M$ and such that c is semigeneric for (M, \mathbb{B}) . It follows that $\Psi(p_n, c)$ holds, as required. \square

11° **Case III.** $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$.

Proof. This is shown like Case III of the proof of Theorem 5.6.8. \square

12° The three cases show together that Player II can survive ω moves against σ . This contradicts the fact that σ is winning for Player I.

□

Corollary 5.7.8. *Suppose that there is a proper class of inaccessible cardinals. Then the following are equivalent.*

a. $(\mathbb{H}, \mathcal{D})$ is ssp-satisfiable.

b. For some inaccessible κ , Player II has a winning strategy in $\mathfrak{D}_\kappa^{\text{ssp}}(o)$.

□

Proposition 5.7.9. *Suppose that*

1. $\lambda \in \mathfrak{J}_{\text{fix}}$ is strictly larger than $\text{rank}(\mathbb{H})$,

2. \mathbb{C} is a poset contained in $\mathbb{P}^{\text{ssp}} \cap V_\lambda$,

3. for all $p \in \mathbb{C}$, there exists a winning strategy σ for Player II in $\mathfrak{D}_\lambda^{\text{ssp}}(p)$ such that $\text{ran}(\sigma) \subseteq \mathbb{C}$.

Then \mathbb{C} is a stationary set preserving poset that adds a \mathcal{D} -generic filter for \mathbb{H} .

Proof. These assumptions are sufficient to run the proofs of Propositions 5.7.4.c and 5.7.5.

□

5.8 AS Goodness

Section Assumptions 5.8.1. We fix a problem $(\mathbb{H}, \mathcal{D})$.

□

We want to give a sufficient condition for this problem to be ssp-satisfiable. This is the content of the following definition.

Definition 5.8.2. Suppose that $\kappa > \text{rank}(\mathbb{H})$ is inaccessible. Then $(\mathbb{H}, \mathcal{D})$ is **AS-good** at κ iff for all stationary $S \subseteq \omega_1$, it holds in $V^{\text{Col}(\omega, < \kappa)}$ that for all \mathcal{D} -generic filters F for \mathbb{H} , there exists an elementary embedding

$$j : V \longrightarrow M$$

such that $\text{crit}(j) = \omega_1^V \in j(S)$ and there exists a $j(\mathcal{D})$ -generic filter F^* for $j(\mathbb{H})$ such that $j[F] \subseteq F^*$.

□

In Section 5.9, we will illustrate how this criterion can be used in practice. We now proceed to show that it implies the ssp satisfiability. More precisely, we will show how Player II can win $\mathfrak{D}_\kappa^{\text{ssp}}(o)$. In the next definition, we recursively define a family of posets

$$(\mathbb{C}_\lambda^{\text{BV}} : \lambda \in \mathfrak{J}_{\text{fix}}, \lambda > \text{rank}(\mathbb{H}))$$

which is closely related to the posets $\mathbb{C}_\lambda^{\text{SSP}}$. The conditions in these posets are chosen based on a modification of the game $\mathfrak{D}_\kappa^{\text{SSP}}(p)$: Player I is not allowed any more to play the moves $Q_n = (U, S)$, while in the moves $Q_n = (M, E)$, he needs to meet a slightly different requirement.

Definition 5.8.3. Suppose that

1. $\eta \in \mathfrak{J}_{\text{fix}}$ is strictly larger than $\text{rank}(\mathbb{H})$,
2. for all $\xi \in \mathfrak{J}_{\text{fix}} \cap \eta$, the poset $\mathbb{C}_\xi^{\text{BV}}$ has been defined,
3. for all $\xi \in \mathfrak{J}_{\text{fix}} \cap \eta$, poset $\mathbb{C}_\xi^{\text{BV}}$ is an initial segment of $\mathbb{P}^{\text{SSP}} \cap V_\xi$.

For $p \in \mathbb{P}^{\text{SSP}} \cap V_\eta$, the game $\mathfrak{D}_\eta^{\text{BV}}(p)$ is defined as the length ω two Player game of the form

$$\begin{array}{c|cccc} \text{I} & & Q_0 & Q_1 & \cdots \\ \hline \text{II} & p_{-1} & p_0 & p_1 & \cdots \end{array}$$

where $p_{-1} \in \mathbb{P}^{\text{SSP}} \cap V_\eta$ is such that $p_{-1} \leq p$ and for all $n < \omega$, the following is satisfied.

- a. Player I must ensure that either
 - i. $Q_n = D$ for some $D \in \mathcal{D}$, or
 - ii. $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$.
- b. Player II must ensure that $p_n \in \mathbb{P}^{\text{SSP}} \cap V_\eta$ and $p_n \leq p_{n-1}$.
- c. Player II must ensure that if $Q_n = D$ for some $D \in \mathcal{D}$, then there exists $w \in D$ such that $w_{p_n} \leq w$.
- d. Player I must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then E is a dense subset $\mathbb{C}_{\lambda_M}^{\text{BV}}$.
- e. Player II must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exists $q \in E$ such that $\delta(\text{Hull}(M, q)) = \delta(M)$ and $p_n \leq q$.

The infinite plays with no rules broken are won by Player II. The set $\mathbb{C}_\eta^{\text{BV}}$ is defined to consist of all $p \in \mathbb{P}^{\text{SSP}} \cap V_\eta$ such that Player II has a winning strategy in $\mathfrak{D}_\eta^{\text{BV}}(p)$.²⁷ \square

Note that this game is local: there is no essential difference between $\mathfrak{D}_\eta^{\text{BV}}(p)$ and $\mathfrak{D}_{\eta'}^{\text{BV}}(p)$, as long as $\eta, \eta' > \text{rank}(p)$. We will show that Player II can win $\mathfrak{D}_\kappa^{\text{SSP}}(o)$ by maintaining that for each of his moves p_n , he can win the (auxiliary) game $\mathfrak{D}_\kappa^{\text{BV}}(p_n)$. The following lemma observes some basic facts about the posets $\mathbb{C}_\eta^{\text{BV}}$, which are standard by now.

²⁷This definition is essentially due to De Bondt and Veličković.

Lemma 5.8.4. *Suppose that $\eta \in \beth_{\text{fix}}$ is strictly larger than $\text{rank}(\mathbb{H})$. The following holds.*

- a. *The poset $\mathbb{C}_\eta^{\text{BV}}$ is well-defined.*
- b. *For all $w \in \mathbb{H}$, $(w, \emptyset) \in \mathbb{C}_\eta^{\text{BV}}$.*
- c. *The poset $\mathbb{C}_\eta^{\text{BV}}$ is an initial segment of $\mathbb{P}^{\text{SP}} \cap V_\eta$.*
- d. *For all $\zeta \in \beth_{\text{fix}}$ which are larger than η , $\mathbb{C}_\zeta^{\text{BV}} \cap V_\eta = \mathbb{C}_\eta^{\text{BV}}$.*
- e. *If g is V -generic for $\mathbb{C}_\eta^{\text{BV}}$, then $\{w_p : p \in g\}$ is a \mathcal{D} -generic filter for \mathbb{H} . □*

As we said, Player II will win $\mathfrak{D}_\kappa^{\text{SSP}}(o)$ by maintaining that he can win $\mathfrak{D}_\kappa^{\text{BV}}(p_n)$ for all $n < \omega$. The following proposition shows how he can answer a question of the form $Q_n = (U, S)$ of Player I in $\mathfrak{D}_\kappa^{\text{SSP}}(o)$. The proof uses the AS goodness in a crucial way.

Proposition 5.8.5. *Suppose that*

1. *$\kappa > \text{rank}(\mathbb{H})$ is inaccessible,*
2. *$(\mathbb{H}, \mathcal{D})$ is AS-good at κ ,*
3. *$p \in \mathbb{C}_\kappa^{\text{BV}}$,*
4. *$U \subseteq H((2^\kappa)^+)$,*
5. *$S \subseteq \omega_1$ is stationary.*

Then there exists a countable

$$M \prec (H((2^\kappa)^+), \in, U, \kappa, p)$$

with $\delta_M \in S$, there exists $\lambda \in \beth_{\text{fix}} \cap \kappa$ with $\text{Hull}(M, V_\lambda) \cap \kappa \subseteq \lambda$, and there exists $q \in \mathbb{C}_\kappa^{\text{BV}}$ such that $q \leq p$ and $M \downarrow \lambda \in \mathcal{M}_q$.

Proof.

- 1° Let $\mathcal{H} := (H((2^\kappa)^+), \in, U)$. Up to modifying U , we may assume that the structure \mathcal{H} has a lightface definable wellordering and that $\kappa, p, \mathbb{H}, \mathcal{D}$ are lightface definable singletons over \mathcal{H} .
- 2° Let $\lambda < \kappa$ be such that there exists $H \prec \mathcal{H}$ satisfying that $\kappa \cap H = \lambda$, let $g \in V[h]$ be V -generic for $\mathbb{C}_\kappa^{\text{BV}} \cap V_\lambda = \mathbb{C}_\lambda^{\text{BV}}$ containing p , and let h be V -generic for $\text{Col}(\omega, < \kappa)$ such that $g \in V[h]$. We work in $V[h]$.
- 3° Let $F := \{w_p : p \in g\}$. By Lemma 5.8.4, F is a \mathcal{D} -generic filter for \mathbb{H} .

4° Since $(\mathbb{H}, \mathcal{D})$ is AS-good at κ (in V), there exist (in $V[h]$) an elementary embedding $\tau : V \longrightarrow W$ with

$$\text{crit}(\tau) = \omega_1^V \in \tau(S)$$

and a $\tau(\mathcal{D})$ -generic filter $F^* \subseteq \tau(\mathbb{H})$ such that $\tau[F] \subseteq F^*$.

5° Let $N := \text{Hull}^{\tau(\mathcal{H})}(\omega_1^V \cup \{\tau(p)\}) \prec \tau(\mathcal{H})$. It holds that

- a. $N \in W$,
- b. $|N|^W = \omega$,
- c. $N \prec \tau[\mathcal{H}]$,
- d. $\delta_N = \omega_1^V \in \tau(S)$.

6° **Claim.** In W , there exists $q \in \tau(\mathbb{C}_\kappa^{\text{BV}})$ such that $q \leq \tau(p)$ and $N \downarrow \tau(\lambda) \in \mathcal{M}_q$.

Proof.

1' We will verify that one can take

$$q := (\tau(w_p), \tau(\mathcal{M}_p) \cup \{N \downarrow \tau(\lambda)\}).$$

To that end, it suffices to show that $q \in \tau(\mathbb{C}_\kappa^{\text{BV}})$.

2' Let us assume otherwise. Then there exists a winning strategy σ for Player I in $\mathfrak{D}_\kappa^{\text{BV}}(q)^W$. We will reach a contradiction by defeating this strategy in $V[h]$. (We use in this step that the game $\mathfrak{D}_\kappa^{\text{BV}}(q)^W$ is closed for Player II.)

3' Let Player I play according to σ and let us show that Player II can play by maintaining that for all $n \in [-1, \omega)$, there exist r, X, P such that

- a. $r \in g$,
- b. $w_{p_n} \in F^*$ and $w_{p_n} \leq \tau(w_r)$,
- c. X is a finite subset of $\tau[V_\lambda]$,
- d. $P = \text{Hull}^{\tau(\mathcal{H})}(\omega_1^V \cup X)$,
- e. $\mathcal{M}_{p_n} = \tau(\mathcal{M}_r) \cup \{P \downarrow \tau(\lambda)\}$.

4' The above conditions are satisfied for $n = -1$ and $p_{-1} = q$. Let us consider the case $n \geq 0$ and the move Q_n of Player I. We let r, X, P be as above, but for the index $n - 1$.

5' **Case I.** $Q_n = D$ for some $D \in \tau(\mathcal{D})$.

Proof. Since F^* is $\tau(\mathcal{D})$ -generic, there exists $w \in F^* \cap D$. Let $w_{p_n} := w_{p_{n-1}} w \in F^*$ and let $\mathcal{M}_{p_n} := \mathcal{M}_{p_{n-1}}$. It is easily seen that $p_n := (w_{p_n}, \mathcal{M}_{p_n})$ is as required. \square

6' **Case IIa.** $Q_n = (P \downarrow \tau(\lambda), E)$ for some $E \in P \downarrow \tau(\lambda)$.

Proof. Since $P \subseteq \text{im}(\tau)$, it follows that $P \downarrow \tau(\lambda) \subseteq \text{ran}(\tau)$. Hence, there is $\bar{E} \in V$ such that $\tau(\bar{E}) = E$. By elementarity, \bar{E} is dense in $\mathbb{C}_\lambda^{\text{BV}}$ (in V). Since g is V -generic, there exists $s \in \bar{E} \cap g$ such that $s \leq r$. Observe that $\tau(s) \in \tau(\bar{E}) = E$ and that

$$P' := \text{Hull}^{\tau(\mathcal{H})}(\omega_1^V \cup X \cup \{\tau(s)\})$$

satisfies that $\delta(P') = \delta(P) = \omega_1^V$. Thus, we may set

$$p_n := (\tau(w_s)w_{p_{n-1}}, \tau(\mathcal{M}_s) \cup \{P' \downarrow \tau(\lambda)\}) \in \tau(\mathbb{P}^{\text{SP}} \cap V_\kappa).$$

□

7' **Case IIb.** $Q_n = (M, E)$ for some $M \in \tau(\mathcal{M}_r)$ and some $E \in M$.

Proof. Observe that the preimages $\bar{M} := \tau^{-1}(M) \in \mathcal{M}_r$ and $\bar{E} := \tau^{-1}(E) \in \bar{M}$ are defined. By elementarity of τ , \bar{E} is dense in $\mathbb{C}_{\lambda_{\bar{M}}}^{\text{BV}}$ (in V). We will verify below that the set

$$\{s \leq r : \exists t \in \bar{E}, s \leq t, \delta(\text{Hull}(\bar{M}, t)) = \delta(\bar{M})\} \quad (5.1)$$

is dense below r (in $\mathbb{C}_\lambda^{\text{BV}}$, in V). Once we know this, the genericity of g yields $s \in g$ and $t \in \bar{E}$ such that $s \leq r, t$ and $\delta(\text{Hull}(\bar{M}, t)) = \delta(\bar{M})$. For $P' := \text{Hull}^{\tau(\mathcal{H})}(\omega_1^V \cup X \cup \{\tau(s)\})$, we have that $\delta_{P'} = \omega_1^V = P$. Thus, we may set

$$p_n := (\tau(w_s)w_{p_{n-1}}, \tau(\mathcal{M}_s) \cup \{P' \downarrow \tau(\lambda)\}) \in \tau(\mathbb{P}^{\text{SP}}).$$

Let us now verify that the set (5.1) is dense. Let $r' \in \mathbb{C}_\lambda^{\text{BV}}$ be an arbitrary satisfying that $r' \leq r$ and let us find $s \leq r'$ which belongs to the set (5.1). Then there exists $\bar{M}' \in \mathcal{M}_{r'}$ such that $\delta(\bar{M}') = \delta(\bar{M})$, $\lambda_{\bar{M}'} = \lambda_{\bar{M}}$, and $\bar{M} \subseteq \bar{M}'$. Using the winning strategy for Player II in $\mathfrak{D}_\lambda^{\text{BV}}(r')$, we obtain $s \in \mathbb{C}_\lambda^{\text{BV}}$ such that $s \leq r'$ and such that there exists $t \in \bar{E}$ satisfying $s \leq t$ and $\delta(\text{Hull}(\bar{M}', t)) = \delta(\bar{M}')$. We have that

$$\delta(\bar{M}) \leq \delta(\text{Hull}(\bar{M}, t)) \leq \delta(\text{Hull}(\bar{M}', t)) = \delta(\bar{M}') = \delta(\bar{M}),$$

which shows that all of the above values are equal. Hence, $s \leq r'$ is as required. □

8' Cases I, IIa, and IIb show that Player II can survive ω steps against the strategy σ , which is a contradiction. □

7° The conclusion of the proposition now follows from Claim 6°, by the elementarity of τ . □

We are now ready to show how Player II can win $\mathfrak{D}_{\kappa}^{\text{ssp}}(o)$, provided that $(\mathbb{H}, \mathcal{D})$ is AS-good at κ .

Theorem 5.8.6. *Suppose that*

1. $\kappa > \text{rank}(\mathbb{H})$ is inaccessible,
2. $(\mathbb{H}, \mathcal{D})$ is AS-good at κ .

Then Player II has a winning strategy in $\mathfrak{D}_{\kappa}^{\text{ssp}}(o)$.

Proof.

- 1° We show that Player II can play by maintaining that for all $n \in [-1, \omega)$, $p_n \in \mathbb{C}_{\kappa}^{\text{BV}}$.
- 2° For $n = -1$, Player II can simply play $p_{-1} := o$.
- 3° Let us consider the move Q_n of Player I.
- 4° **Case I.** $Q_n = D$ for some $D \in \mathcal{D}$.

Proof. Let us consider a partial play

$$\begin{array}{c|cc} \text{I} & & D \\ \hline \text{II} & q_{-1} & q_0 \end{array}$$

of $\mathfrak{D}_{\kappa}^{\text{BV}}(p_{n-1})$ according to some winning strategy for Player II. The rules imply that for some $w \in D$, $w_{q_0} \leq w$. They also imply that $q_0 \leq q_{-1} \leq p_{n-1}$. This is enough to conclude that Player II can play $p_n := q_0$ in the play of $\mathfrak{D}_{\kappa}^{\text{ssp}}(o)$. \square

- 5° **Case II.** $Q_n = (U, S)$ for some $U \subseteq H((2^{\kappa})^+)$ and for some $S \subseteq \omega_1$ which is stationary.

Proof. By Proposition 5.8.5, there exist a countable

$$M \prec (H((2^{\kappa})^+), \in, U, \kappa, p_{n-1})$$

with $\delta_M \in S$, an ordinal $\lambda \in \beth_{\text{fix}} \cap \kappa$ with $\text{Hull}(M, V_{\lambda}) \cap \kappa \subseteq \lambda$, and a condition $p_n \in \mathbb{C}_{\kappa}^{\text{BV}}$ such that $p_n \leq p_{n-1}$ and $M \downarrow \lambda \in \mathcal{M}_{p_n}$. We see now that Player II can respond by playing p_n . \square

- 6° **Case III.** $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$.

Proof.

1' We have that there exists $M^* \prec H((2^\kappa)^+)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

2' Let

- a. $F := \mathbb{C}_{\lambda_M}^{\text{BV}} \cap E$,
- b. $F^\perp := \{q \in \mathbb{C}_{\lambda_M}^{\text{BV}} : \forall r \in F, q \perp_{\mathbb{C}_{\lambda_M}^{\text{BV}}} r\}$,
- c. $G := F \cup F^\perp$.

We have that

- a. $F, F^\perp, G \in M$,
- b. G is dense in $\mathbb{C}_{\lambda_M}^{\text{BV}}$,
- c. $p_{n-1} \in \pi(F)$.

3' Let us consider a partial play

$$\begin{array}{c|cc} \text{I} & & (M, G) \\ \hline \text{II} & q_{-1} & q_0 \end{array}$$

of $\mathfrak{D}_\kappa^{\text{BV}}(p_{n-1})$ according to some winning strategy for Player II.

4' The rules imply that there exists $r \in G$ such that $\delta(\text{Hull}(M, r)) = \delta(M)$ and that $q_0 \leq r, q_{-1}$.

5' Note that $r \in V_{\lambda_M}$. This means that $r = \pi(r) \in \pi(F) \cup \pi(F^\perp)$.

6' Since $q_0 \leq r$ and $q_0 \leq q_{-1} \leq p_{n-1}$, we see that $r \parallel p_{n-1}$ in the sense of $\mathbb{C}_{\lambda_M}^{\text{BV}}$. However, this also means that $r \parallel p_{n-1}$ in the sense of $\mathbb{C}_\kappa^{\text{BV}}$.

7' Since $p_{n-1} \in \pi(F)$, we get that $r \notin \pi(F^\perp)$. This means that $r \in \pi(F)$ and consequently, $r \in F$.

8' Since $F \subseteq E$, we get that

- a. $r \in E$ and $\delta(\text{Hull}(M, r)) = \delta(M)$,
- b. $q_0 \leq p_{n-1}, r$,
- c. $q_0 \in \mathbb{C}_\kappa^{\text{BV}}$.

In other words, Player II can respond in the play of $\mathfrak{D}_\kappa^{\text{SSP}}(o)$ by playing $p_n := q_0$.

□

7° The verification of three cases shows that Player II can always play ω many rounds of $\mathfrak{D}_\kappa^{\text{SSP}}(o)$ without breaking the rules. This concludes the argument since all such plays are won by Player II.

□

Corollary 5.8.7. *Suppose that*

1. $\kappa > \text{rank}(\mathbb{H})$ is inaccessible,
2. $(\mathbb{H}, \mathcal{D})$ is AS-good at κ .

Then $\mathbb{C}_\kappa^{\text{ssp}}$ is a stationary set preserving poset that adds a \mathcal{D} -generic filter for \mathbb{H} . □

Our arguments actually show that $\mathbb{C}_\kappa^{\text{BV}} \subseteq \mathbb{C}_\kappa^{\text{ssp}}$ and that this set is closed for some winning strategy for Player II in $\mathfrak{D}_\kappa^{\text{ssp}}(-)$. The point is that the winning strategy for Player II in $\mathfrak{D}_\kappa^{\text{ssp}}(-)$ is simply “play so as to remain inside $\mathbb{C}_\kappa^{\text{BV}}$ ”. This ensures that $\mathbb{C}_\kappa^{\text{BV}}$ is yet another ssp poset adding a \mathcal{D} -generic filter for \mathbb{H} . We will have to come back to this poset in Section 5.9, so we observe the following corollary as well.

Corollary 5.8.8. *Suppose that*

1. $\kappa > \text{rank}(\mathbb{H})$ is inaccessible,
2. $(\mathbb{H}, \mathcal{D})$ is AS-good at κ .

Then $\mathbb{C}_\kappa^{\text{BV}}$ is an initial segment of $\mathbb{C}_\kappa^{\text{ssp}}$, it is stationary set preserving, and it adds a \mathcal{D} -generic filter for \mathbb{H} .

Proof. The proof of Theorem 5.8.6 actually shows that for all $p \in \mathbb{C}_\kappa^{\text{BV}}$, Player II has a winning strategy σ in $\mathfrak{D}_\kappa^{\text{ssp}}(p)$ such that $\text{ran}(\sigma) \subseteq \mathbb{C}_\kappa^{\text{BV}}$. This shows that $\mathbb{C}_\kappa^{\text{BV}} \subseteq \mathbb{C}_\kappa^{\text{ssp}}$ and that $\mathbb{C}_\kappa^{\text{BV}}$ is a stationary set preserving poset adding a \mathcal{D} -generic filter for \mathbb{H} (Lemma 5.7.9). We observed in Lemma 5.8.4 that $\mathbb{C}_\kappa^{\text{BV}}$ is an initial segment of $\mathbb{P}^{\text{ssp}} \cap V_\kappa$, whence the final conclusion. □

Example 5.8.9. Suppose that NS_{ω_1} is saturated and that κ is an inaccessible cardinal. We want to show that there exists a stationary set preserving poset adding a cofinal function $f : \omega \rightarrow \omega_2^V$. This comes down to showing that the Namba problem $(\mathbb{H}, \mathcal{D})$ of Example 5.2.8 is ssp-satisfiable. A sufficient condition for the ssp satisfiability is the AS goodness at κ and this problem satisfies that. More precisely, let $S \subseteq \omega_1$ be stationary, let g be V -generic for $\text{Col}(\omega, < \kappa)$, and let F be a \mathcal{D} -generic filter for \mathbb{H} . We want to find

$$\tau : V \longrightarrow W$$

satisfying that $\text{crit}(\tau) = \omega_1^V \in \tau(S)$ and a $\tau(\mathcal{D})$ -generic filter F^* for $\tau(\mathbb{H})$ satisfying that $\tau[F] \subseteq F^*$.

Let G be a V -generic filter for $(\text{NS}_{\omega_1}^+)^V$ containing S and let

$$\tau : V \longrightarrow W$$

be the corresponding generic embedding. Observe that $f := \bigcup F$ is a cofinal function $\omega \rightarrow \omega_2^V$. Since τ is continuous at ω_2^V , it follows that $\tau \circ f$ is a cofinal function $\omega \rightarrow \omega_2^W$. Hence,

$$F^* := \{w \in \tau(\mathbb{H}) : w \subseteq \tau \circ f\}$$

is a $\tau(\mathcal{D})$ -generic filter for $\tau(\mathbb{H})$. This shows that τ and F^* are as required.

We have shown that the Namba problem is AS-good at κ . This means that there exists a stationary set preserving poset adding a solution to it. This solution is exactly a cofinal function $\omega \rightarrow \omega_2^V$. Furthermore, we know that $\mathbb{C}_\kappa^{\text{ssp}}$ and $\mathbb{C}_\kappa^{\text{BV}}$ are stationary set preserving posets adding such a function. \square

5.9 An Example for the AS Goodness

In this section, we show how our methods can be used to give another proof of the main lemma of [16]. Before we precisely state what we are doing, we need a following auxiliary notion.

Definition 5.9.1. Suppose that θ is a cardinal and $\Gamma \subseteq \mathcal{P}(\mathbb{R})$. Then Γ is θ -well behaved iff

- a. for all $A \in \Gamma$, A is θ -universally Baire,
- b. for all $A \in \Gamma$, for all formulas $\phi(x, \dot{A})$ projective in \dot{A} , we have

$$B := \{x \in \mathbb{R} : \phi(x, A)\} \in \Gamma$$

$$\text{and } V^{\text{Col}(\omega, \theta)} \models B_{\dot{G}} = \{x \in \mathbb{R}_{\dot{G}} : \phi(x, A_{\dot{G}})\}. \quad \square$$

Section Assumptions 5.9.2. Suppose that

1. NS_{ω_1} is saturated,
2. $2^{\omega_1} = \omega_2$,
3. MA_{ω_1} holds,
4. D is dense in \mathbb{P}_{\max} ,
5. for all θ , there exists θ -well behaved Γ such that D is coded by a set in Γ ,
6. $A \subseteq \omega_1$ satisfies that $\omega_1^{L[A]} = \omega_1$,
7. κ is inaccessible. \square

Our goal is to construct an ssp poset \mathbb{Q} such that in $V^{\mathbb{Q}}$, there exist generic iterations \mathcal{I} and \mathcal{J} of \mathbb{P}_{\max} conditions of length $\omega_1 + 1$ such that

- a. $M_0^{\mathcal{J}} \in D_{\dot{G}}$,
- b. $\mathcal{I} \upharpoonright_{\omega_1}^{M_0^{\mathcal{J}}}$ witnesses that $M_0^{\mathcal{J}} < M_0^{\mathcal{I}}$,
- c. $\pi^{\mathcal{J}}(\mathcal{I} \upharpoonright_{\omega_1}^{M_0^{\mathcal{J}}}) = \mathcal{I}$,
- d. $M_{\omega_1}^{\mathcal{I}} = (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V$,
- e. $\text{NS}_{\omega_1} \cap \mathcal{P}(\omega_1)^{M_{\omega_1}^{\mathcal{J}}} = I^{M_{\omega_1}^{\mathcal{J}}}$.

We will start by forgetting for a moment about the requirement e and recasting the rest of the problem as adding a model to an infinitary propositional formula by an ssp forcing. It will turn out that we can get e for free.

Notation 5.9.3. Suppose $\mathcal{M} = (\omega, \dots)$ is a structure in a recursively enumerable language. Then $\vartheta(\mathcal{M})$ denotes the truth function of the model $(\mathcal{M}, m : m < \omega)$, i.e. $\vartheta(\mathcal{M}) : \omega \rightarrow 2$ is defined in such a way that for all $n < \omega$, for ψ denoting the formula in the language of

$$(\mathcal{M}, m : m < \omega)$$

with $\ulcorner \psi \urcorner = n$, it holds that

$$\vartheta(\mathcal{M})(n) = 1 \iff (\mathcal{M}, m : m < \omega) \models \psi.$$

□

Notation 5.9.4.

- a. If $E \subseteq X \times Y$ and $x \in X$, we denote $E_x := \{y \in Y : (x, y) \in E\}$.
- b. If $a : X \rightarrow Y$ and $x \in X$, then $a_x := a(x)$.

□

It is not clear that we can express the fact that something is a \mathbb{P}_{\max} condition using propositional logic. Our workaround is to fix a tree that projects to codes of \mathbb{P}_{\max} conditions and just make sure that a code of the object that we are forcing belongs to the projection of that tree. Similarly, we will not express directly that one of the conditions that we are forcing belongs to the reinterpretation of the set D , but rather that its code belongs to the projection of the corresponding tree.

Section Assumptions 5.9.5. Let T, U be ω_2 -absolutely complemented trees on $\omega \times \omega_2$ such that

- a. $p[T]$ is the set of all $\vartheta(\omega, E, I, a)$ for which there exists $p \in \mathbb{P}_{\max}$ satisfying

$$(\omega, E, I, a) \simeq p,$$

- b. $p[U]$ is the set of all $\vartheta(\omega, E, I, a)$ for which there exists $p \in D$ satisfying $(\omega, E, I, a) \simeq p$.

□

Having made the necessary compromises, we are ready to restate the goal in a way that can easily be coded by an infinitary propositional formula.

Goal 5.9.6. Construct an ssp poset \mathbb{Q} such that in $V^{\mathbb{Q}}$, there exist $E, F, I, J, a, b, u, v, \pi, \sigma, \mathcal{A}, \mathcal{H}$ such that

- a. $E, F \subseteq \omega_1 \times (\omega \times \omega)$,
- b. $I, J \subseteq \omega_1 \times \omega$,
- c. $a, b : \omega_1 \longrightarrow \omega$,
- d. $u, v : \omega \longrightarrow \omega_2$,
- e. for all $n < \omega$, $(\vartheta(\omega, E_0, I_0, a_0) \upharpoonright n, u \upharpoonright n) \in T$,
- f. for all $n < \omega$, $(\vartheta(\omega, F_0, J_0, b_0) \upharpoonright n, v \upharpoonright n) \in U$,
- g. $\pi, \sigma \subseteq \{(\alpha, \beta) : \alpha \leq \beta < \omega_1\} \times (\omega \times \omega)$,
- h. for all $\alpha \leq \beta < \omega_1$, $\pi_{\alpha, \beta} : (\omega, E_\alpha, I_\alpha, a_\alpha) \xrightarrow{\Sigma_\omega} (\omega, E_\beta, I_\beta, a_\beta)$,
- i. for all $\alpha \leq \beta \leq \gamma < \omega_1$, $\pi_{\beta, \gamma} \circ \pi_{\alpha, \beta} = \pi_{\alpha, \gamma}$,
- j. for all $\alpha < \omega_1$, there exists $k < \omega$ such that
 - i. $(\omega, E_{\alpha+1}) \models \dot{k} < \dot{\omega}_1$,
 - ii. for all $\iota < \omega$, there exists $y < \omega$ such that $(\omega, E_\alpha) \models \dot{y} : \dot{\omega}_1 \longrightarrow \dot{V}$ and such that for $u := \pi_{\alpha, \alpha+1}(y)$, $(\omega, E_{\alpha+1}) \models \dot{u}(k) = \dot{i}$,
 - iii. for $g_\alpha := \{w < \omega : (\omega, E_\alpha, I_\alpha) \models \dot{w} \in I_\alpha^+, k E_{\alpha+1} \pi_{\alpha, \alpha+1}(w)\}$, we have that $(\omega, E_\alpha, I_\alpha, g_\alpha) \models \text{“}g_\alpha \text{ is generic for } I_\alpha^+ \text{”}$,
- k. for all limit $\gamma < \omega_1$, for all $n < \omega$, there exist $\alpha < \gamma$ and $m < \omega$ such that $\pi_{\alpha, \gamma}(m) = n$,
- l. for all $\alpha \leq \beta < \omega_1$, $\sigma_{\alpha, \beta} : (\omega, F_\alpha, J_\alpha, b_\alpha) \xrightarrow{\Sigma_\omega} (\omega, F_\beta, J_\beta, b_\beta)$,
- m. for all $\alpha \leq \beta \leq \gamma < \omega_1$, $\sigma_{\beta, \gamma} \circ \sigma_{\alpha, \beta} = \sigma_{\alpha, \gamma}$,
- n. for all $\alpha < \omega_1$, there exists $k < \omega$ such that
 - i. $(\omega, F_{\alpha+1}) \models \dot{k} < \dot{\omega}_1$,
 - ii. for all $\iota < \omega$, there exists $y < \omega$ such that $(\omega, F_\alpha) \models \dot{y} : \dot{\omega}_1 \longrightarrow \dot{V}$ and such that for $u := \sigma_{\alpha, \alpha+1}(y)$, $(\omega, F_{\alpha+1}) \models \dot{u}(k) = \dot{i}$,
 - iii. for $h_\alpha := \{w < \omega : (\omega, F_\alpha, J_\alpha) \models \dot{w} \in J_\alpha^+, k F_{\alpha+1} \sigma_{\alpha, \alpha+1}(w)\}$, we have that $(\omega, F_\alpha, J_\alpha, h_\alpha) \models \text{“}h_\alpha \text{ is generic for } J_\alpha^+ \text{”}$,
- o. for all limit $\gamma < \omega_1$, for all $n < \omega$, there exist $\alpha < \gamma$ and $m < \omega$ such that $\sigma_{\alpha, \gamma}(m) = n$,

- p. $\mathcal{N} \subseteq \omega_1 \times (\omega \times \omega)$,
- q. for all $\alpha < \omega_1$, $\mathcal{N}_\alpha : (\omega, E_\alpha, I_\alpha, a_\alpha) \xrightarrow{\Sigma_0} (\omega, F_\alpha, J_\alpha, b_\alpha)$,
- r. for all $\alpha < \omega_1$, for all $\iota < \omega$, for all $y < \omega$, if $y \in F_\alpha \cap \mathcal{N}_\alpha(\iota)$, then for some $u < \omega$, $\mathcal{N}_\alpha(u) = y$,
- s. for all $\alpha \leq \beta < \omega_1$, $\mathcal{N}_\beta \circ \pi_{\alpha,\beta} = \sigma_{\alpha,\beta} \circ \mathcal{N}_\alpha$,
- t. $\mathcal{H} \subseteq \omega_1 \times (\omega \times H_{\omega_2}^V)$,
- u. for all $\alpha < \omega_1$, $\mathcal{H}_\alpha : (\omega, E_\alpha, I_\alpha, a_\alpha) \xrightarrow{\Sigma_\omega} (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V$,
- v. for all $\alpha \leq \beta < \omega_1$, $\mathcal{H}_\alpha = \mathcal{H}_\beta \circ \pi_{\alpha,\beta}$,
- w. for all $x \in H_{\omega_2}^V$, there exist $\alpha < \omega_1$ and $\iota < \omega$ such that $\mathcal{H}_\alpha(\iota) = x$. \square

Note that if $|\mathbb{Q}| > \omega_2$, then every element of $p[T]$ in $V^{\mathbb{Q}}$ will be of the form $\vartheta(\mathcal{M})$ for some \mathcal{M} isomorphic to a \mathbb{P}_{\max} -condition, but the converse might not hold, and similarly for U . However, this suffices for our purposes. Here are some examples how to convert the requirements of Goal 5.9.6 into a propositional formula.

- a. Let $\dot{E}_{\alpha,m,n}, \dot{F}_{\alpha,m,n}, \dot{I}_{\alpha,m}, \dot{J}_{\alpha,m}, \dot{a}_{\alpha,m}, \dot{b}_{\alpha,m}, \dot{u}_{m,\xi}, \dot{v}_{m,\xi}, \dot{\pi}_{\alpha,\beta,m,n}, \dot{\sigma}_{\alpha,\beta,m,n}, \dot{\mathcal{N}}_{\alpha,m,n}, \dot{\mathcal{H}}_{\alpha,m,x}$ be propositional letters, for all $\alpha \leq \beta < \omega_1, \gamma < \omega_1, m, n < \omega, \xi < \omega_2, x \in H_{\omega_2}$.
- b. If ψ is the language of some $(\omega, E_\alpha, I_\alpha, a_\alpha)$, we can express

$$(\omega, E_\alpha, I_\alpha, a_\alpha) \models \psi$$

as a propositional formula by induction on complexity of ψ . Similarly for $(\omega, F_\alpha, J_\alpha, b_\alpha)$.

- c. We can now express requirement e as

$$\bigwedge_{n < \omega} \bigvee_{\substack{s \in T, \\ |s|=n}} \bigwedge_{i < n} \theta_{s,i}$$

where

- i. for $s(i) = (1, \xi)$ and for ψ_i being the formula with code i , $\theta_{s,i}$ is defined as the propositional formula asserting

$$“(\omega, E_\alpha, I_\alpha, a_\alpha) \models \psi_i” \wedge \dot{u}_{i,\xi},$$

- ii. for $s(i) = (0, \xi)$ and for ψ_i being the formula with code i , $\theta_{s,i}$ is defined as the propositional formula asserting

$$“(\omega, E_\alpha, I_\alpha, a_\alpha) \models \neg \psi_i” \wedge \dot{u}_{i,\xi}.$$

Similarly for the requirement f.

One can proceed like this to code all of the requirements of Goal 5.9.6 and obtain a formula which will have ω_2 propositional letters and whose every conjunction and disjunction will have the length at most ω_2 .

Definition 5.9.7. We denote by ϕ_{AS} the negation normal form of the formula sketched just above. \square

The poset \mathbb{Q} required in Goal 5.9.6 will be the poset \mathbb{C}_κ^{BV} of Definition 5.8.3, defined with respect to the problem $(\mathbb{H}(\phi_{AS}), \mathcal{D}(\phi_{AS}))$. The several things need to be verified, starting from the fact that the said problem is well-defined.

Lemma 5.9.8. ϕ_{AS} is satisfiable.

Proof. This is shown as [16, Lemma 3.6]. \square

The following lemma simply states that the formula ϕ_{AS} codes that which it is designed to code.

Lemma 5.9.9. Suppose that $\mu \models \phi_{AS}$ belongs to an outer model. Then in the outer model, there exist unique generic iterations \mathcal{I} and \mathcal{J} of \mathbb{P}_{\max} conditions of length $\omega_1^V + 1$ such that

- a. $\mathcal{I} \upharpoonright \omega_1^{M_0^{\mathcal{J}}}$ witnesses that $M_0^{\mathcal{J}} < M_0^{\mathcal{I}}$,
- b. $\pi^{\mathcal{J}}(\mathcal{I} \upharpoonright \omega_1^{M_0^{\mathcal{J}}}) = \mathcal{I}$,
- c. $M_{\omega_1}^{\mathcal{I}} = (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V$,
- d. $M_0^{\mathcal{J}} \in D_g$,
- e. for all $\alpha < \omega_1^V$, $(\omega, E_\alpha, I_\alpha, a_\alpha) \simeq M_\alpha^{\mathcal{I}}$ and $(\omega, F_\alpha, J_\alpha, b_\alpha) \simeq M_\alpha^{\mathcal{J}}$,
- f. for all $\alpha \leq \beta < \omega_1^V$, the diagrams

$$\begin{array}{ccc} (\omega, E_\alpha, I_\alpha, a_\alpha) & \xrightarrow{\pi_{\alpha,\beta}} & (\omega, E_\beta, I_\beta, a_\beta) \\ \downarrow \simeq & & \downarrow \simeq \\ M_\alpha^{\mathcal{I}} & \xrightarrow{\pi_{\alpha,\beta}^{\mathcal{I}}} & M_\beta^{\mathcal{I}} \end{array}$$

$$\begin{array}{ccc} (\omega, E_\alpha, I_\alpha, a_\alpha) & \xrightarrow{\mathfrak{H}_\alpha} & (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V \\ \downarrow \simeq & & \downarrow = \\ M_\alpha^{\mathcal{I}} & \xrightarrow{\pi_{\alpha,\omega_1^V}^{\mathcal{I}}} & (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V \end{array}$$

$$\begin{array}{ccc}
 (\omega, F_\alpha, J_\alpha, b_\alpha) & \xrightarrow{\sigma_{\alpha,\beta}} & (\omega, F_\beta, J_\beta, b_\beta) \\
 \downarrow \simeq & & \downarrow \simeq \\
 M_\alpha^{\mathcal{J}} & \xrightarrow{\pi_{\alpha,\beta}^{\mathcal{J}}} & M_\beta^{\mathcal{J}}
 \end{array}$$

commute. □

Notation 5.9.10. In the case of the above proposition, we denote by \mathcal{I}^μ and \mathcal{J}^μ the generic iterations \mathcal{I} and \mathcal{J} , respectively. We also denote

$$E^\mu := \{(\alpha, m, n) \in \omega_1^V \times (\omega \times \omega) : \mu(\dot{E}_{\alpha,m,n}) = 1\},$$

$$F^\mu := \{(\alpha, m, n) \in \omega_1^V \times (\omega \times \omega) : \mu(\dot{F}_{\alpha,m,n}) = 1\},$$

$$I^\mu := \{(\alpha, m) \in \omega_1^V \times \omega : \mu(\dot{I}_{\alpha,m}) = 1\},$$

and so on. □

We now want to show that ϕ_{AS} is AS-good at κ . By this, we simply mean that the problem $(\mathbb{H}(\phi_{AS}), \mathcal{D}(\phi_{AS}))$ is AS-good at κ . After unraveling the definitions, it turns out that we need to verify that for all stationary $S \subseteq \omega_1$, it holds in $V^{\text{Col}(\omega, < \kappa)}$ that for all $\mu \models \phi$, there exist elementary $\tau : V \rightarrow W$ and $\hat{\mu} \models \tau(\phi)$ such that $\text{crit}(\tau) = \omega_1^V \in \tau(S)$ and for all subformulas ψ of ϕ , $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

Proposition 5.9.11. ϕ_{AS} is AS-good at κ .

Proof.

1° Let h be V -generic for $\text{Col}(\omega, < \kappa)$ and let us work in $V[h]$. We fix an arbitrary valuation $\mu \models \phi$ and an arbitrary stationary $S \subseteq \omega_1^V$ and we are going to produce elementary $\tau : V \rightarrow W$ and $\hat{\mu} \models \tau(\phi)$ such that $\text{crit}(\tau) = \omega_1^V \in \tau(S)$ and for all subformulas ψ of ϕ , $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

2° Let $\hat{\mathcal{J}}$ be a generic iteration of length $\omega_1 + 1$ extending \mathcal{J}^μ and satisfying $S \in g_{\omega_1^V}^{\hat{\mathcal{J}}}$ and let $\hat{\mathcal{I}} := \pi_{\omega_1^V, \omega_1}^{\hat{\mathcal{J}}}(\mathcal{I})$. Since

$$M_{\omega_1^V}^{\mathcal{I}} = (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V,$$

there exists a unique generic iteration \mathcal{K} of V of length $\omega_1 + 1$ such that for all $\alpha < \omega_1$, $g_\alpha^{\mathcal{K}} = g_{\omega_1^V + \alpha}^{\hat{\mathcal{I}}}$.

3° We let $\tau := \pi^{\mathcal{K}} : V \rightarrow M_{\omega_1^V}^{\mathcal{K}} =: W$.

4° For all $\alpha < \omega_1$, let $e_\alpha : \omega \rightarrow M_\alpha^{\hat{\mathcal{I}}}$ be a bijection such that if $\alpha < \omega_1^V$, then e_α is the transitive collapse of (ω, E_α) . We define for all $\alpha < \omega_1$,

$$\hat{E}_\alpha := \{(m, n) : e_\alpha(m) \in e_\alpha(n)\}.$$

In the case $\alpha < \omega_1^V$, we get the old $\hat{E}_\alpha = E_\alpha$.

5° In the similar fashion, we extended other sets from Goal 5.9.6, producing the “hat versions” of them. Then the valuation $\hat{\mu}$ is defined according to these “hat versions”.

6° **Claim.** For ψ a subformula of ϕ , $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

Proof. This requires a tedious verification. Suppose for example that ψ comes from the part w. of Goal 5.9.6. Then ψ might of the formula

$$\bigwedge_{x \in H_{\omega_2}^V} \bigvee_{\alpha < \omega_1^V} \bigvee_{\iota < \omega} \mathcal{H}C_{\alpha, \iota, x}.$$

In this case, $\mu(\psi) = 1$, $\tau(\psi)$ is the formula

$$\bigwedge_{x \in H_{\omega_2}^W} \bigvee_{\alpha < \omega_1} \bigvee_{\iota < \omega} \mathcal{H}C_{\alpha, \iota, x},$$

and we need to verify that $\tau(\psi)$ is evaluated as true by $\hat{\mu}$. However, this is trivial, since

$$(H_{\omega_2}^W, \mathcal{H}C_{\alpha} : \alpha < \omega_1)$$

is the direct limit

$$(M_{\alpha}^{\hat{\mathcal{I}}}, \pi_{\alpha, \beta}^{\hat{\mathcal{I}}} : \alpha \leq \beta < \omega_1).$$

We leave it to the reader to convince herself of the rest. □

7° Thus, τ and $\hat{\mu}$ are as required. □

We now know that both $\mathbb{C}_{\kappa}^{\text{SSP}}$ and $\mathbb{C}_{\kappa}^{\text{BV}}$ are stationary set preserving and add a model for ϕ_{AS} . Let g be a V -generic for one of them and let $\mu_g := \mu$, $\mathcal{I} := \mathcal{I}^{\mu}$, and $\mathcal{J} := \mathcal{J}^{\mu}$. The \mathbb{P}_{max} -iterations \mathcal{I} and \mathcal{J} are like what we wanted, except the thing that we forgot about. Namely, recall that we did not code into ϕ_{AS} the requirement that

$$\text{NS}_{\omega_1} \cap \mathcal{P}(\omega_1)^{M_{\omega_1}^{\mathcal{J}}} = I^{M_{\omega_1}^{\mathcal{J}}}.$$

It turns out that if we force with the smaller of the posets, i.e. $\mathbb{C}_{\kappa}^{\text{BV}}$, then we get this property for “free”.

Proposition 5.9.12. *It holds in $V^{\mathbb{C}_{\kappa}^{\text{BV}}}$ that*

$$\text{NS}_{\omega_1} \cap \mathcal{P}(\omega_1)^{M_{\omega_1}^{\mathcal{J}^{\mu_g}}} = I^{M_{\omega_1}^{\mathcal{J}^{\mu_g}}}.$$

Proof.

1° Let us assume otherwise. Since (\supseteq) clearly holds, the other inclusion must fail. More precisely, there exist $p \in \mathbb{C}_{\kappa}^{\text{BV}}$, $\dot{C} \in V^{\mathbb{C}_{\kappa}^{\text{BV}}}$, $\alpha < \omega_1$, and $m < \omega$ such that p forces that

- a. $(\omega, F_\alpha^{\mu_{\dot{g}}}) \models \text{“}\dot{m} \text{ is a subset of } \omega_1 \text{ which is not in } J_\alpha^{\mu_{\dot{g}}}\text{”}$,
- b. \dot{C} is a club in ω_1^V ,
- c. the image of m in the final iterate of $\mathcal{J}^{\mu_{\dot{g}}}$ has the empty intersection with $\dot{C} \subseteq V_\kappa$.

2° Let

1. $\theta := (2^\kappa)^+$,
2. R be a wellordering of H_θ ,
3. $\mathcal{H} := (H_\theta, \in, R, \kappa, \dot{C}, T, U, A)$,
4. $\lambda < \kappa$ be such that there exists $H \prec \mathcal{H}$ satisfying that $p \in H$ and $\lambda = \kappa \cap H$,
5. g be V -generic for $\mathbb{C}_\lambda^{\text{BV}}$ containing p ,
6. h be V -generic for $\text{Col}(\omega, < \kappa)$ such that $g \in V[h]$,
7. $\mu := \mu_g \models \phi$.

We work in $V[h]$.

3° Let S_0 be the image of m under the transitive collapse of (ω, F_α^μ) and let $S := \pi_{\alpha, \omega_1^V}^{\mathcal{J}^\mu}(S_0)$.

4° Let $\hat{\mathcal{L}}, \hat{\mathcal{J}}, \mathcal{K}, \tau, W$, and $\hat{\mu}$ be obtained as in the proof of Proposition 5.9.11. (The reason why we cannot simply quote this proposition is because now S is not necessarily in V .) We have that

- a. $\tau : V \longrightarrow W$ elementarily,
- b. $\text{crit}(\tau) = \omega_1^V$,
- c. $\omega_1^V \in \pi_{\omega_1^V, \omega_1}^{\hat{\mathcal{J}}}(S) = \pi_{\alpha, \omega_1}^{\hat{\mathcal{J}}}(S_0)$,
- d. for all subformulas ψ of ϕ , $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

5° Let

1. \check{h}_δ be the formula coding the assertion that “ $\omega_1^{M_\delta^{\hat{\mathcal{J}}}} = \delta$ and this ordinal belongs to the image of the set coded by m under the α -to- $(\delta + 1)$ iteration embedding”,
2. $N := \text{Hull}^{\tau(\mathcal{H})}(\omega_1^V \cup \{\tau(p)\})$,
3. $q := (\tau(w_p) \cup \{\check{h}_{\omega_1^V}\}, \tau(\mathcal{M}_p) \cup \{N \downarrow \tau(\lambda)\}) \in \tau(\mathbb{P}^{\text{SP}})$.

6° **Claim.** $q \in \tau(\mathbb{C}_\kappa^{\text{BV}})$

Proof. This is shown like Claim 6° of the proof of Proposition 5.8.5. □

7° By elementarity, it holds in V that there exist $r \in \mathbb{P}_\lambda$, $\delta < \omega_1$, and $P \prec \mathcal{H}$ such that

- a. $r = (w_p \cup \{\check{h}_\delta\}, \mathcal{M}_p \cup \{P \downarrow \lambda\}) \in \mathbb{P}_\lambda$,
- b. $p \in P$,
- c. $\delta_P = \delta > \alpha$.

8° **Claim.** $r \Vdash_{\mathbb{C}_\kappa^{\text{BV}}}^V$ “ δ belongs to both the image of the set coded by m along the \mathcal{J} -iteration embedding and to the club \dot{C} ”.

Proof.

- 1' Since $\check{h}_\delta \in w_r$, it is immediate that $r \Vdash$ “ δ belongs to the image of the set coded by m along the \mathcal{J} -iteration embedding”. It remains to verify that $r \Vdash \delta \in \dot{C}$.
- 2' The proof of Claim 5° of Proposition 5.7.5 shows that r is semigeneric for $(P, \mathbb{C}_\kappa^{\text{BV}})$ (in V). The point here is that $P \downarrow \lambda \in \mathcal{M}_r$, so the rules of the game $\mathfrak{D}_\kappa^{\text{BV}}(r)$ suffice to show the semigenericity.
- 3' Let g' be an arbitrary V -generic for $\mathbb{C}_\kappa^{\text{BV}}$ containing r . We have that $\delta(P[g']) = \delta(P) = \delta$.
- 4' Since $\dot{C}^{g'} \in P[g']$ is a club in ω_1 , we have that $\delta \in C$, as required.

□

9° The previous claim is in contradiction with 1°c.

□

Finally, we arrive at the conclusion.

Corollary 5.9.13. *The poset $\mathbb{C}_\kappa^{\text{BV}}$ is stationary set preserving and there exist in $V^{\mathbb{C}_\kappa^{\text{BV}}}$ generic iterations \mathcal{I} and \mathcal{J} of \mathbb{P}_{\max} conditions of length $\omega_1 + 1$ such that*

- a. $M_0^{\mathcal{J}} \in D_{\dot{G}}$,
- b. $\mathcal{I} \upharpoonright \omega_1^{M_0^{\mathcal{J}}}$ witnesses that $M_0^{\mathcal{J}} < M_0^{\mathcal{I}}$,
- c. $\pi^{\mathcal{J}}(\mathcal{I} \upharpoonright \omega_1^{M_0^{\mathcal{J}}}) = \mathcal{I}$,
- d. $M_{\omega_1}^{\mathcal{I}} = (H_{\omega_2}, \in, \text{NS}_{\omega_1}, A)^V$,
- e. $\text{NS}_{\omega_1} \cap \mathcal{P}(\omega_1)^{M_{\omega_1}^{\mathcal{J}}} = I^{M_{\omega_1}^{\mathcal{J}}}$.

□

Chapter 6

Generic Absoluteness and Maximality

6.1 Introduction

It follows from results of [15] and [16] that Martin's Maximum⁺⁺ implies¹ the following generic-maximality principle.

Definition 6.1.1. The sentence $(\dagger)_{\text{UB}}$ asserts that for every universally Baire set A and every Π_2 sentence ϕ in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$, if

$$\exists \mathbb{P}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_g) \models \phi),$$

then $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi$. □

More precisely, under large cardinal assumptions, $(\dagger)_{\text{UB}}$ is equivalent to the following bounded form of Martin's Maximum⁺⁺.

Definition 6.1.2. The sentence $\text{UB} - \text{BMM}^{++}$ asserts that for every universally Baire set A , every Σ_1 formula $\phi(v_a)$ in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$ and every $a \in H_{\omega_2}$, if

$$\exists \mathbb{P} \text{ ssp}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_g) \models \phi[a]),$$

then $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi[a]$.² □

Theorem 6.1.3. *Suppose there exists a proper class of Woodin cardinals. Then the following holds.*

a. $\text{UB} - \text{BMM}^{++}$ implies $(\dagger)_{\text{UB}}$.

b. *If there exists a supercompact cardinal, then $(\dagger)_{\text{UB}}$ implies $\text{UB} - \text{BMM}^{++}$.*

¹assuming proper class of Woodin cardinals in the background; cf. Theorem 6.1.3.

²This principle follows from MM^{++} - see Lemma 6.2.10. The origin of the name of the principle will be clear from the proof (explanation) of this fact.

Proof. In Section 6.2 we put together explicitly facts from [15] and [16] to obtain this conclusion; see Corollary 6.2.13. \square

A corollary of the theorem is that MM^{++} implies $(\dagger)_{\text{UB}}$. We would like to obtain a similar result for just MM . However, an argument for a proof of generic maximality under MM has to be at least somewhat different: the argument of Section 6.2 goes through a form of the axiom $(*)$, while it is known that MM does not imply $(*)$.³ We will be interested in the following versions of $(\dagger)_{\text{UB}}$ and $\text{UB} - \text{BMM}^{++}$.

Definition 6.1.4. The sentence $(\dagger)_{\text{UB}}^-$ asserts that for every universally Baire set A and every Π_2 sentence ϕ in the language $\{R_{\in}^2, c_{\omega_1}, R_A^1\}$, if

$$\exists \mathbb{P}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \omega_1, A_{\dot{g}}) \models \phi),$$

then $(H(\omega_2), \in, \omega_1, A) \models \phi$. \square

Definition 6.1.5. The sentence $\text{UB} - \text{BMM}$ asserts that for every universally Baire set A , every Σ_1 formula $\phi(v_a)$ in the language $\{R_{\in}^2, R_A^1\}$ and every $a \in H_{\omega_2}$, if

$$\exists \mathbb{P} \text{ ssp}, V^{\mathbb{P}} \models ((H(\omega_2), \in, A_{\dot{g}}) \models \phi[a]),$$

then $(H(\omega_2), \in, A) \models \phi[a]$. \square

By unraveling the proofs of lemmas used for the results of [16] and [15] used in Section 6.2, we obtain the following result. We include its proof here as an example for application of the AS-goodness condition for ssp-forcibility.

Proposition 6.1.6. *Suppose there exists a proper class of Woodin cardinals. Then $\text{UB} - \text{BMM}$ implies $(\dagger)_{\text{UB}}^-$.*

Proof. See Corollary 6.4.5. \square

As a corollary, we infer that MM implies $(\dagger)_{\text{UB}}^-$.^{4,5} We conjecture that the converse of Proposition 6.1.6 also holds (assuming existence of a supercompact cardinal).

Conjecture 6.1.7. *Suppose there exists a proper class of Woodin cardinals and a supercompact cardinal. Then $(\dagger)_{\text{UB}}^-$ implies $\text{UB} - \text{BMM}$.* \square

Once again, what is missing when compared with Theorem 6.1.3 is a version of the axiom $(*)$. We propose such a statement in Definition 6.5.2. Section 6.5 contains a brief discussion on how to approach Conjecture 6.1.7.

While developing the argument for Proposition 6.1.6, we observed that the following generic maximality simply follows from large cardinals. It readily implies generic absoluteness for the mentioned formulas - see Corollary 6.6.4.

³See [47].

⁴The point is that MM implies $\text{UB} - \text{BMM}$; cf. [15, Theorem 10.124].

⁵A corollary of $(\dagger)_{\text{UB}}^-$ is that every Π_2 sentence over H_{ω_2} that can be forced is true. The fact that this instance of $(\dagger)_{\text{UB}}^-$ follows from MM was previously known through [48, Theorem 3.1].

Theorem 6.1.8. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *ϕ is a Σ_1 sentence in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$,*
4. *there exists a poset that forces that $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi$.*

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$.

Proof. See Proposition 6.6.1. □

One curiosity about this theorem is that, assuming CH, one can try to add \mathbb{R} as a parameter in ϕ . This form of generic maximality is a strengthening of generic Σ_1^2 -maximality, which is known to hold assuming a proper class of measurable Woodin cardinals and CH.⁶ Hence, it makes sense to conjecture the following.⁷

Conjecture 6.1.9. *Suppose*

1. *there exists a proper class of measurable Woodin cardinals,*
2. *CH holds,*
3. *A is a universally Baire set,*
4. *$\phi(v_A)$ is a Σ_1 formula in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1\}$,*
5. *there exists a poset that forces that CH and $(H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A_{\dot{g}}]$.*

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A]$. □

We do not know if the conjecture is true. However, if the predicate for NS_{ω_1} is omitted, then the proof of Woodin's theorem on Generic Σ_1^2 -absoluteness condition on CH gives the following.

Theorem 6.1.10. *Suppose*

1. *there exists a proper class of measurable Woodin cardinals,*
2. *CH holds,*
3. *A is a universally Baire set,*

⁶This fact is due to Woodin; see [5] for a proof. We have given our interpretation of this proof in [49].

⁷The difference with Theorem 6.1.8 is that the universally Baire set is now added as a parameter.

4. $\phi(v_A)$ is a Σ_1 formula in the language of set theory,
5. there exists a poset that forces that $H_{\omega_2} \models \phi[A_g]$.

Then $H_{\omega_2} \models \phi[A]$.

Proof. See the proof of Theorem 6.7.8 for details. □

Finally, we propose in Section 6.6 an approach to Conjecture 6.1.9 based on the proof of Conjecture 6.1.9.

6.2 Proof of Theorem 6.1.3

Notation 6.2.1. We denote by dec the canonical surjection $\mathbb{R} \twoheadrightarrow H_{\omega_1}$. □

Definition 6.2.2. A set $A \subseteq H_{\omega_1}$ is **universally Baire in codes** iff $\text{dec}^{-1}[A]$ is a universally Baire set of reals. □

Definition 6.2.3. Let $A \subseteq H_{\omega_1}$ be universally Baire in codes and let g be a V -generic. Then A_g is defined to be $\text{dec}[(\text{dec}^{-1}[A])_g]$.

Definition 6.2.4. Suppose $a \subseteq \omega_1$. We denote by g_a the set of all $p \in \mathbb{P}_{\max}$ for which there exists a generic iteration \mathcal{I} satisfying that

$$i^{\mathcal{I}}(I^p) = \text{NS}_{\omega_1} \cap \mathcal{P}_{\infty}^{\mathcal{I}} \text{ and } i^{\mathcal{I}}(a^p) = a.$$

□

Definition 6.2.5. The sentence $(*)_{\text{UB}}$ asserts that for all $a \subseteq \omega_1$ such that $\omega_1^{L[a]} = \omega_1$ and for all dense subsets D of \mathbb{P}_{\max} that are universally Baire in codes, it holds that $g_a \cap D \neq \emptyset$. □

Lemma 6.2.6. *Suppose there exists a proper class of Woodin cardinals. Then $\text{UB} - \text{BMM}^{++}$ holds if and only if $(*)_{\text{UB}}$ holds.*

Proof. This follows from [16, Theorem 2.16]. □

Lemma 6.2.7. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *$(*)_{\text{UB}}$ holds,*
3. *$a \subseteq \omega_1$ satisfies $\omega_1^{L[a]} = \omega_1$,*
4. *A is a universally Baire set.*

Then g_a is an $L(\mathbb{R}, A)[g_a]$ -generic filter for \mathbb{P}_{\max} and $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R}, A)[g_a]$.

Proof.

1° **Claim.** For all $p, q \in g_a$, $p \parallel q$.

Proof.

1' Let G be a V -generic such that in $V[G]$,

- a. $\text{NS}_{\omega_1} \cap V = \text{NS}_{\omega_1}^V$,
- b. NS_{ω_1} is saturated.

2' Let κ be an inaccessible cardinal in $V[G]$ and let h be $V[G]$ -generic for $\text{Col}(\omega, \kappa)$. We work in $V[G][h]$.

3' Let $r := (H_\kappa, \in, \text{NS}_{\omega_1}, a)^{V[G]}$. Notice that $r \in \mathbb{P}_{\max}$ and $r \leq p, q$.

4' This shows that $p \parallel q$. By absoluteness, $V \models p \parallel q$.

□

2° Since every set of reals in $L(\mathbb{R}, A)$ is universally Baire, g_a intersects every dense subset of \mathbb{P}_{\max} which is in $L(\mathbb{R}, A)$. This, together with Claim 1°, implies that g_a is an $L(\mathbb{R}, A)$ -generic filter for \mathbb{P}_{\max} .

3° It remains to verify that for a given $b \subseteq \omega_1$, it holds that $b \in L(\mathbb{R})[g_a]$.

4° Let $c := \{2\alpha : \alpha \in a\} \cup \{2\alpha + 1 : \alpha \in b\}$. By $(*)_{\text{UB}}$, $g_c \neq \emptyset$.

5° Let

- a. $p \in g_c$,
- b. $a' := \{\alpha : 2\alpha \in a^p\}$,
- c. $b' := \{\alpha : 2\alpha + 1 \in a^p\}$,
- d. $q := (M^p, \in, I^p, a')$.

Then $q \in g_a$ and $i^{\mathcal{I}_{q, g_a}}(b') = b$.

6° This shows that $b \in L(\mathbb{R})[g_a]$, as required.

□

Lemma 6.2.8. *Suppose*

1. *there exists a proper class of Woodin cardinals,*

2. A is a universally Baire set of reals,
3. ϕ is a Π_2 sentence in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$,
4. there exists a poset that forces “ $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_j) \models \phi$ ”.

Then it holds in $L(\mathbb{R}, A)^{\mathbb{P}_{\max}}$ that $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$.

Proof. This follows by the proof of [50, Theorem 7.3]. □

Proposition 6.2.9. *Suppose there exists a proper class of Woodin cardinals and $\text{UB} - \text{BMM}^{++}$. Then $(\dagger)_{\text{UB}}$ holds.*

Proof.

- 1° Let A be a universally Baire set and let ϕ be a Π_2 sentence in the language

$$\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}.$$

Suppose that

$$\exists \mathbb{P}, \Vdash_{\mathbb{P}} ((H(\omega_2), \in, \text{NS}_{\omega_1}, A_j) \models \phi).$$

We want to show that $(H(\omega_2), \in, \text{NS}_{\omega_1}, A) \models \phi$.

- 2° By Lemma 6.2.8, $L(\mathbb{R}, A)^{\mathbb{P}_{\max}} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi)$.
- 3° By Lemma 6.2.6, there exists an $L(\mathbb{R}, A)$ -generic filter g for \mathbb{P}_{\max} such that $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R}, A)[g]$.
- 4° This means that the formula “ $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$ ” holds in $L(\mathbb{R}, A)[g]$ and consequently, in V .

□

Lemma 6.2.10. *Suppose MM^{++} holds. Then $\text{UB} - \text{BMM}^{++}$ holds.*

Proof. The principle $A - \text{BMM}^{++}$ is defined in [15, Definition 10.123]. As observed in [15, Theorem 10.124], MM^{++} implies that for all $A \in \text{UB}$, $A - \text{BMM}^{++}$ holds. On the other hand, this consequent is equivalent to our principle $\text{UB} - \text{BMM}^{++}$: $A - \text{BMM}^{++}$ is equivalent to the instance of $\text{UB} - \text{BMM}^{++}$ that corresponds to A by the methods of [51]. □

Proposition 6.2.11. *Suppose*

1. there exists a proper class of Woodin cardinals,
2. there exists a supercompact cardinal,

3. $(\dagger)_{\text{UB}}$ holds.

Then $(*)_{\text{UB}}$ holds.

Proof.

1° Let D be a dense subset of \mathbb{P}_{max} which is universally Baire in codes. We want to show that For every $c \subseteq \omega_1$ satisfying $\omega_1^{L[c]}$, $g_c \cap D \neq \emptyset$.

2° Let D^* be a universally Baire set that codes D . There exists a Π_2 sentence ψ in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_{D^*}^1\}$ that is equivalent to the statement “for all $c \subseteq \omega_1$ satisfying $\omega_1^{L[c]}$, there exists $p \in D_{\dot{g}}^* \cap g_c$ ” over the structure

$$(H_{\omega_2}, \in, \text{NS}_{\omega_1}, D_{\dot{g}}^*),$$

in every forcing extension.

3° Let \mathbb{P} be a poset that forces MM^{++} . By Lemmas 6.2.6 and 6.2.10, $V^{\mathbb{P}} \models (*_{D_{\dot{g}}^*})$, which means in $V^{\mathbb{P}}$,

$$(H_{\omega_2}, \in, \text{NS}_{\omega_1}, D_{\dot{g}}^*) \models \psi.$$

4° By the assumption 3, $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, D^*) \models \psi$. This shows that for every $c \subseteq \omega_1$ satisfying $\omega_1^{L[c]}$, $g_c \cap D \neq \emptyset$, as required. □

Proposition 6.2.12. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *there exists a supercompact cardinal,*
3. $(\dagger)_{\text{UB}}$ holds.

Then $\text{UB} - \text{BMM}^{++}$ holds.

Proof. This follows from Proposition 6.2.11 and Lemma 6.2.6. □

Corollary 6.2.13. *Suppose there exists a proper class of Woodin cardinals. Then the following holds.*

- a. $\text{UB} - \text{BMM}^{++}$ implies $(\dagger)_{\text{UB}}$.
- b. *If there exists a supercompact cardinal, then $(\dagger)_{\text{UB}}$ implies $\text{UB} - \text{BMM}^{++}$.*

Proof. This follows from Propositions 6.2.9 and 6.2.12. □

6.3 Pre-conditions

Definition 6.3.1. A **pre-condition** is a structure (M, \in, I) satisfying that

- a. M is a countable transitive model of ZFC,
- b. I is a normal ideal on ω_1 in M ,
- c. (M, I) is generically iterable. □

Notation 6.3.2. We denote by \mathbb{P}_{\max}^* the set of all pre-conditions. For $p \in \mathbb{P}_{\max}^*$, we denote by M^p and I^p its first and third coordinate, respectively. □

Notation 6.3.3. Suppose p be a pre-condition and \mathcal{I} is a (generic) iteration of p . Then

- a. for $\alpha < \text{len}(\mathcal{I})$, $\mathcal{P}_\alpha^\mathcal{I}$ denotes the α^{th} structure in the iteration \mathcal{I} ,
- b. for $\alpha \leq \beta < \text{len}(\mathcal{I})$, $i_{\alpha,\beta}^\mathcal{I}$ denotes the embedding $\mathcal{P}_\alpha^\mathcal{I} \longrightarrow \mathcal{P}_\beta^\mathcal{I}$ given by \mathcal{I} ,
- c. $\mathcal{P}_\infty^\mathcal{I}$ denotes the the direct limit of the iteration \mathcal{I} ,
- d. for $\alpha < \text{len}(\mathcal{I})$, $i_{\alpha,\infty}^\mathcal{I}$ denotes the direct limit embedding $\mathcal{P}_\alpha^\mathcal{I} \longrightarrow \mathcal{P}_\infty^\mathcal{I}$ given by \mathcal{I} ,
- e. $i^\mathcal{I}$ denotes $i_{0,\infty}^\mathcal{I}$. □

Definition 6.3.4. The relation $\mathcal{I} : p > q$ holds iff

- a. p and q are pre-conditions,
- b. $p \in H_{\omega_1}^q$,
- c. $\mathcal{I} \in q$,
- d. \mathcal{I} is a generic iteration of p ,
- e. $i^\mathcal{I}(I^p) = I^q \cap \mathcal{P}_\infty^\mathcal{I}$. □

Lemma 6.3.5. *Suppose*

1. J is a normal ideal on ω_1 ,
2. p is a pre-condition.

Then there exists an iteration \mathcal{I} of p such that $i^\mathcal{I}(I^p) = J \cap \mathcal{P}_\infty^\mathcal{I}$.

Proof. This result comes from [15], but we learned it from [50]. See [52, Proposition 2-X] for our presentation of the proof. □

Definition 6.3.6. Let p be a pre-condition and let A be a set of reals. Then p is *A-iterable* iff $A \cap p \in p$ and for every countable generic iteration \mathcal{I} of p , $i^{\mathcal{I}}(A \cap p) = A \cap \mathcal{P}_{\infty}^{\mathcal{I}}$. \square

Lemma 6.3.7. *Suppose*

1. δ is a limit of Woodin cardinals,
2. A is a $<\delta$ -universally Baire set,
3. $\theta > \delta$ is inaccessible,
4. M is countable and transitive,
5. $\pi : (M, \in, \bar{\delta}, \bar{A}) \longrightarrow (V_{\theta}, \in, \delta, A)$ is elementary,
6. g is $<\bar{\delta}$ -generic over M ,
7. I is a precipitous ideal on ω_1 in $M[g]$.

Then $(M[g], I)$ is an *A-iterable pre-condition*.

Proof. It is our understanding that this comes from [15], but we learned it from [50]. See [53, Proposition 2.10] for our presentation of the proof. \square

Lemma 6.3.8. *Suppose*

1. δ is a beth-fixed point,
2. A is a $<\delta$ -universally Baire set of reals,
3. $\theta > \delta$ is inaccessible,
4. M is transitive,
5. $\pi : (M, \in, \bar{\delta}, \bar{A}) \longrightarrow (H_{\theta}, \in, \delta, A)$ is elementary,
6. g is $<\bar{\delta}$ -generic over M .

Then $\bar{A}_g = A \cap M[g]$.

Proof. This is probably folklore, but see [53, Lemma 1.6] for our presentation of the proof. \square

Lemma 6.3.9. *Suppose δ is a Woodin cardinal. Then there exists a semi-proper poset that forces “ NS_{ω_1} is saturated”.*

Proof. This is an unpublished result of Shelah. For a new proof of this fact, see [43, Proposition [3.3.6]]. \square

Proposition 6.3.10. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *$\phi(x)$ is a Σ_1 formula in the language $\{R_{\in}^2, c_{\omega_1}, R_A^1\}$,*
4. *there exists a poset that forces “ $(H_{\omega_2}, \in, \omega_1, A_g) \models \forall x, \phi(x)$ ”,*
5. *p is a pre-condition.*

Then there are \mathcal{I} and q such that

- a. $\mathcal{I} : p > q$,
- b. q is A -iterable,
- c. $(H_{\omega_2}^q, \in, \omega_1^q, A \cap q) \models \forall x \in i^{\mathcal{I}}(H_{\omega_2}^p), \phi(x)$.

Proof.

1° Let \mathbb{P} be a poset such that $(H_{\omega_2}, \in, \omega_1, A_g) \models \forall x, \phi(x)$ holds in $V^{\mathbb{P}}$, let $\delta > \text{rank}(\mathbb{P})$ be a limit of Woodin cardinals, and let $\theta > \delta$ be an inaccessible cardinal.

2° Let $\pi : (M, \in, p, \bar{\mathbb{P}}, \bar{\delta}, \bar{A}) \rightarrow (V_{\theta}, \in, p, \mathbb{P}, \delta, A)$ be a countable elementary embedding, where M is transitive, and let g be M -generic for $\bar{\mathbb{P}}$. Observe that in $M[g]$, $(H_{\omega_2}, \in, \omega_1, \bar{A}_g) \models \forall x, \phi(x)$.

3° By Lemma 6.3.5, there exists in $M[g]$ a generic iteration \mathcal{I} of p such that $I^{\mathcal{I}}_{\infty} = \text{NS}_{\omega_1}^{M[g]} \cap \mathcal{P}_{\infty}^{\mathcal{I}}$. Observe that

$$(H_{\omega_2}^{M[g]}, \in, \omega_1^{M[g]}, \bar{A}_g) \models \forall x \in i^{\mathcal{I}}(H_{\omega_2}^p), \phi(x).$$

4° By Lemma 6.3.9, there exists a $< \bar{\delta}$ -generic h over $M[g]$ such that

- a. $\text{NS}_{\omega_1}^{M[g][h]} \cap M[g] = \text{NS}_{\omega_1}^{M[g]}$,
- b. NS_{ω_1} is saturated in $M[g][h]$.

5° Let q be the triple $(M[g][h], \in, \text{NS}_{\omega_1}^{M[g][h]})$. We want to show that q is as required.

6° By Lemma 6.3.7, q is an A -iterable pre-condition.

7° It is immediate that $\mathcal{I} : p > q$.

8° By upward absoluteness applied to 2°, we have that

$$(H_{\omega_2}^q, \in, \omega_1, \bar{A}_{g*h}) \models \forall x \in i^{\mathcal{I}}(H_{\omega_2}^p), \phi(x).$$

Since, by Lemma 6.3.8, $\bar{A}_{g*h} = A \cap q$, this suffices for the conclusion.

□

6.4 Proof of Proposition 6.1.6

Definition 6.4.1. Suppose

1. θ is an inaccessible cardinal,
2. $\phi \in V_\theta$ is an infinitary propositional formula in negation normal form.

Then ϕ is **AS-good at θ** iff ϕ is satisfiable and for all stationary sets $S \subseteq \omega_1$, it holds in $V^{\text{Col}(\omega, < \theta)}$ that for every model μ of ϕ , there exists a well-founded elementary embedding $\tau : V \longrightarrow M$ and a model $\hat{\mu}$ for $\tau(\phi)$ such that

- a. $\text{crit}(\tau) = \omega_1^V \in \tau(S)$,
- b. for every subformula ψ of ϕ , $\mu(\psi) = \hat{\mu}(\tau(\psi))$. □

Lemma 6.4.2. *Suppose*

1. θ is an inaccessible cardinal,
2. $\phi \in V_\theta$ is an infinitary propositional formula in negation normal form,
3. ϕ is AS-good at θ .

Then there exists an ssp poset $\mathbb{P} \subseteq V_\theta$ such that in $V^\mathbb{P}$, there is a model for ϕ .

Proof. This follows from Corollary 5.8.7. □

Proposition 6.4.3. *Suppose*

1. there exists a proper class of Woodin cardinals,
2. $D \subseteq \{(i, p, q) : \exists \mathcal{I} : p > q, i = i^\mathcal{I}\}$ is universally Baire in codes,
3. for all $p \in \mathbb{P}_{\max}^*$, there are q and i such that $(i, p, q) \in D$.

Then there exists an ssp poset \mathbb{P} such that in $V^\mathbb{P}$, there are $p, q, \mathcal{I}, \mathcal{J}$ satisfying that

- a. $(i^\mathcal{I}, p, q) \in D_{\dot{g}}$,
- b. \mathcal{J} is a generic iteration of q of length $\omega_1 + 1$,
- c. $H(\omega_2)^{\mathcal{P}_\infty^{i^\mathcal{J}(\mathcal{I})}} = H(\omega_2)^V$.

Proof.

1° Since it can be forced by an ssp poset, we may assume without loss of generality that NS_{ω_1} is saturated.

2° For $n < \omega$, let ψ_n be the formula in the language

$$\{R_{\mathbb{E}}^2, c_I, c_p, c_i, c_m : m < \omega\}$$

whose Gödel number is n . Let D^* consist of all reals x for which there are a triple $(i, p, q) \in D$ and surjection $a : \omega \rightarrow M^q$ such that the set

$$\{\psi_n : x(n) = 0\}$$

is the theory of the structure

$$(M^q, \in, I^q, p, i, a_m : m < \omega).$$

3° Let κ be an inaccessible cardinal and let T_{D^*} be a κ -absolutely complemented tree on $\omega \times \kappa$ such that $p[T_{D^*}] = D^*$.⁸

4° Let Φ be the negation normal form of the canonical infinitary propositional formula whose models (in all generic extensions) code⁹

- a sequence

$$((\omega, E_\alpha^q, I_\alpha^q, p_\alpha, i_\alpha, m)_{m < \omega} : \alpha < \omega_1^V)$$

of structures in the language $\{R_{\mathbb{E}}^2, c_I, c_p, c_i, c_m : m < \omega\}$,

- a sequence $u : \omega \rightarrow \kappa$,
- a family $(\pi_{\alpha, \beta} : \alpha \leq \beta < \omega_1^V)$,
- a sequence $(\mathcal{H}c_\alpha : \alpha < \omega_1^V)$

such that

- a. letting $t : \omega \rightarrow \{0, 1\}$ be the function defined by asserting that for all $n < \omega$,

$$t(n) = 1 \iff (\omega, E_0^q, I_0^q, p_0, i_0, m)_{m < \omega} \models \psi_n,$$

it holds that $(t, u) \in T_{D^*}$,

- b. for all $\alpha \leq \beta < \omega_1^V$,

$$\pi_{\alpha, \beta} : (\omega, E_\alpha^q, I_\alpha^q, p_\alpha, i_\alpha) \xrightarrow{\Sigma_\omega} (\omega, E_\beta^q, I_\beta^q, p_\beta, i_\beta),$$

- c. the family $((\omega, E_\alpha^q, I_\alpha^q), \pi_{\alpha, \beta} : \alpha \leq \beta < \omega_1^V)$ is a generic iteration of a pre-condition, modulo the transitive collapse,

⁸We gave a presentation of the proof of this fact in [4, Proposition 1.3-IV]; we learned the fact from [3].

⁹For more details on how to get Φ , see Section 6.8.

d. for all $\alpha < \omega_1^V$, letting

- $\mathcal{U}_\alpha < \omega$ be such that $(\omega, E_\alpha^q, I_\alpha^q) \models \text{“}\mathcal{U}_\alpha \text{ is the transitive closure of } \text{ran}(i_\alpha)\text{”}$,
- y_α be the set of all $m < \omega$ satisfying $m E_\alpha^q \mathcal{U}_\alpha$,

it holds that

$$\mathcal{H}_\alpha : (y_\alpha, E_\alpha^q \upharpoonright y_\alpha, I_\alpha^q \upharpoonright y_\alpha) \xrightarrow{\Sigma_\omega} (V_\kappa, \in, \text{NS}_{\omega_1}^V)$$

and for all $\beta \in [\alpha, \omega_1^V)$, the diagram

$$\begin{array}{ccc} y_\alpha & \xrightarrow{\pi_{\alpha,\beta} \upharpoonright y_\alpha} & y_\beta \\ & \searrow \mathcal{H}_\alpha & \downarrow \mathcal{H}_\beta \\ & & V_\kappa \end{array}$$

commutes.

5° Let $\theta > \kappa$ be an inaccessible cardinal.

6° **Claim.** Φ is satisfiable.

Proof.

- 1' Let G be V -generic for $\text{Col}(\omega, \kappa)$. We work in $V[G]$.
- 2' Notice that $p := (V_\kappa, \in, \text{NS}_{\omega_1}^V)$ is a pre-condition.
- 3' By the assumptions on D , there are i and q such that $(i, p, q) \in D_G$.
- 4' Let \mathcal{I} be the generic iteration given by i , let \mathcal{J} be an arbitrary generic iteration of q of length $\omega_1 + 1$, and let $\hat{\mathcal{I}}$ be the generic iteration of $i^{\hat{\mathcal{J}}}(\mathcal{I})$.
- 5' Since $H(\omega_2)^{\mathcal{P}_\infty^{\hat{\mathcal{I}}}} = H(\omega_2)^V$ and since NS_{ω_1} is saturated in V , there exists a unique generic iteration \mathcal{I}_V of V satisfying that for all $\alpha < \theta$, $g_\alpha^{\mathcal{I}_V} = g_{\text{len}(\mathcal{I})+\alpha}^{\hat{\mathcal{I}}}$.
- 6' Let $\tau := i^{\mathcal{I}_V}$ and $M := \mathcal{P}_\infty^{\mathcal{I}_V}$. So, $\tau : V \longrightarrow M$.
- 7' Let $t : \omega \longrightarrow \{0, 1\}$ be the function defined by asserting that for all $n < \omega$,

$$t(n) = 1 \iff (\omega, E_0^q, I_0^q, p_0, i_0, m)_{m < \omega} \models \psi_n.$$

Since $(i, p, q) \in D_G$, it follows that $t \in p[T_{D^*}]$.

- 8' Let $u : \omega \longrightarrow \kappa$ be such that $(t, u) \in T_{D^*}$ and let $v := \tau \circ u$.
- 9' Let $\hat{\mu}$ be obtained from $\hat{\mathcal{I}}, \hat{\mathcal{J}}, v$ in the manner analogous to what is described in steps 7' and 8' of the proof of Claim 7°. We have that $\hat{\mu} \models \tau(\Phi)$.
- 10' By Σ_1^1 -absoluteness, $\tau(\Phi)$ has a model in $M^{\text{Col}(\omega, \kappa)}$. This means that $\tau(\Phi)$ is satisfiable in M and consequently, Φ is satisfiable in V .

□

7° **Claim.** Φ is AS-good at θ .

Proof.

- 1' Let S be a stationary subset of ω_1 and let G be V -generic for $\text{Col}(\omega, < \theta)$. We work in $V[G]$.
- 2' Let μ be an arbitrary model for Φ . We want to find an elementary embedding $\tau : V \rightarrow M$ and a model $\hat{\mu}$ for $\tau(\Phi)$ such that
 - a. $\text{crit}(\tau) = \omega_1^V \in \tau(S)$,
 - b. for every subformula ϕ of Φ , $\mu(\phi) = \hat{\mu}(\tau(\phi))$.
- 3' Let the following be the canonical decoding of μ :

- a sequence

$$(((\omega, E_\alpha^q, I_\alpha^q, p_\alpha^*, i_\alpha, m)_{m < \omega} : \alpha < \omega_1^V)$$

of structures in the language $\{R_{\in}^2, c_I, c_p, c_i, c_m : m < \omega\}$,

- a sequence $u : \omega \rightarrow \kappa$,
- a family $(\pi_{\alpha, \beta} : \alpha \leq \beta < \omega_1^V)$,
- a sequence $(\mathcal{H}_\alpha : \alpha < \omega_1^V)$,

where

- a. letting $t : \omega \rightarrow \{0, 1\}$ be the function defined by asserting that for all $n < \omega$,

$$t(n) = 1 \iff (\omega, E_0^q, I_0^q, p_0^*, i_0, m)_{m < \omega} \models \psi_n,$$

it holds that $(t, u) \in T_{D^*}$,

- b. for all $\alpha \leq \beta < \omega_1^V$,

$$\pi_{\alpha, \beta} : (\omega, E_\alpha^q, I_\alpha^q, p_\alpha^*, i_\alpha) \xrightarrow{\Sigma_\omega} (\omega, E_\beta^q, I_\beta^q, p_\beta^*, i_\beta),$$

- c. the family $((\omega, E_\alpha^q, I_\alpha^q), \pi_{\alpha, \beta} : \alpha \leq \beta < \omega_1^V)$ is a generic iteration of a pre-condition, modulo the transitive collapse,

- d. for all $\alpha < \omega_1^V$, letting

- $\iota_\alpha < \omega$ be such that $(\omega, E_\alpha^q, I_\alpha^q) \models \iota_\alpha$ is the transitive closure of $\text{ran}(i_\alpha)$,
- y_α be the set of all $m < \omega$ satisfying $m E_\alpha^q \iota_\alpha$,

it holds that

$$\mathcal{H}_\alpha : (y_\alpha, E_\alpha^q \upharpoonright y_\alpha, I_\alpha^q \upharpoonright y_\alpha) \xrightarrow{\Sigma_\omega} (V_\kappa, \in, \text{NS}_{\omega_1^V}^V)$$

and for all $\beta \in [\alpha, \omega_1^V)$, the diagram

$$\begin{array}{ccc} y_\alpha & \xrightarrow{\pi_{\alpha, \beta} \upharpoonright y_\alpha} & y_\beta \\ & \searrow \mathcal{H}_\alpha & \downarrow \mathcal{H}_\beta \\ & & V_\kappa \end{array}$$

commutes.

4' Let

a. \mathcal{J} be the generic iteration given by the transitive collapse of

$$((\omega, E_\alpha^q, I_\alpha^q), \pi_{\alpha,\beta} : \alpha \leq \beta < \omega_1^V),$$

b. p and i be the images of p_0 and i_0 (resp.) under the transitive collapse of (ω, E_0^q) ,

c. \mathcal{I} be the generic iteration of p given by i ,

d. $\hat{\mathcal{J}}$ be a generic iteration of length θ which end extends \mathcal{J} and satisfies that $S \in g_{\text{len}(\mathcal{J})}^{\hat{\mathcal{J}}}$,

e. $\hat{\mathcal{I}}$ be the generic iteration $i^{\hat{\mathcal{J}}}(\mathcal{I})$.

5' Since $H(\omega_2)^{\mathcal{P}_\infty^{\mathcal{I}}} = H(\omega_2)^V$ and since NS_{ω_1} is saturated in V , there exists a unique generic iteration \mathcal{I}_V of V satisfying that for all $\alpha < \theta$, $g_\alpha^{\mathcal{I}_V} = g_{\text{len}(\mathcal{I})+\alpha}^{\hat{\mathcal{I}}}$.

6' Let $\tau : i^{\mathcal{I}_V} : V \longrightarrow \mathcal{P}_\infty^{\mathcal{I}_V}$. We have that $\text{crit}(\tau) = \omega_1^V \in \tau(S)$.

7' For $\alpha \in [\text{len}(\mathcal{J}), \theta)$,

a. let $a_\alpha : \omega \longrightarrow \mathcal{P}_\alpha^{\hat{\mathcal{J}}}$ be a surjection,

b. let $(E_\alpha^q, I_\alpha^q, p_\alpha, i_\alpha)$ be the unique 4-tuple satisfying that

$$(\omega, E_\alpha^q, I_\alpha^q, p_\alpha, i_\alpha, m : m < \omega) \simeq (\mathcal{P}_\alpha^{\hat{\mathcal{J}}}, p, i_{0,\alpha}^{\hat{\mathcal{J}}}(i), a_m : m < \omega),$$

c. let $u^* := \tau \circ u$,

d. for $\xi \leq \alpha$, let

$$\pi_{\xi,\alpha} : (\omega, E_\xi^q, I_\xi^q, p_\xi^*, i_\xi) \xrightarrow{\Sigma_\omega} (\omega, E_\alpha^q, I_\alpha^q, p_\alpha^*, i_\alpha)$$

be obtained from $i_{\xi,\alpha}^{\hat{\mathcal{J}}}$ via the anti-collapse from b of its domain and codomain,

e. let \mathcal{H}_α be obtained from $i_{\alpha,\infty}^{\hat{\mathcal{J}}}$, restricted to the final model of $i_{0,\alpha}^{\hat{\mathcal{J}}}(i)$, via the anti-collapse of $\mathcal{P}_\alpha^{\hat{\mathcal{J}}}$ from b.

8' Let $\hat{\mu}$ canonically code the following information:

- the sequence

$$(((\omega, E_\alpha^q, I_\alpha^q, p_\alpha^*, i_\alpha, m)_{m < \omega} : \alpha < \theta),$$

- the sequence $u^* : \omega \longrightarrow \tau(\kappa)$,
- the family $(\pi_{\alpha,\beta} : \alpha \leq \beta < \theta)$,
- the sequence $(\mathcal{H}_\alpha : \alpha < \theta)$.

It is a matter of routine to verify that for every subformula ϕ of Φ , $\hat{\mu}(\tau(\phi)) = \mu(\phi)$.

□

8° By Claim 7° and Lemma 6.4.2, there exists an ssp poset \mathbb{P} that adds a model for Φ . It is easily seen that \mathbb{P} is as required. □

Proposition 6.4.4. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *$\phi(x)$ is a Σ_1 formula in the language $\{R_{\check{c}}, c_{\omega_1}, R_A^1\}$,*
4. *there exists a poset that forces “ $(H_{\omega_2}, \in, \omega_1, A_j) \models \forall x, \phi(x)$ ”.*

Then there exists an ssp poset \mathbb{Q} such that it holds in $V^{\mathbb{Q}}$ that $H_{\omega_2}^V \in H_{\omega_2}$ and $(H_{\omega_2}, \in, \omega_1, A_{\check{H}}) \models \forall x \in H_{\omega_2}^V, \phi(x)$.

Proof.

1° Let D consist of all $(i^{\mathcal{I}}, p, q)$ where $\mathcal{I} : p > q$, q is A -iterable, and

$$(H_{\omega_2}^q, \in, \omega_1^q, A \cap q) \models \forall x \in i^{\mathcal{I}}(H_{\omega_2}^p), \phi(x).$$

Since D is first-order-definable over (H_{ω_1}, \in, A) , it is universally Baire codes.

2° Proposition 6.3.10 implies that for all $p \in \mathbb{P}_{\max}^*$, there exist q and \mathcal{I} such that $(i^{\mathcal{I}}, p, q) \in D$.

3° By Proposition 6.4.3, there exists an ssp poset \mathbb{Q} such that $V^{\mathbb{Q}}$, there are $p, q, \mathcal{I}, \mathcal{J}$ satisfying that

- a. $(i^{\mathcal{I}}, p, q) \in D_{\check{H}}$,
- b. \mathcal{J} is a generic iteration of q of length $\omega_1 + 1$,
- c. $H(\omega_2)^{\mathcal{P}_{\infty}^{i^{\mathcal{J}}(x)}} = H(\omega_2)^V$.

4° We want to show that \mathbb{Q} is as required. Let H be V -generic for \mathbb{Q} and let $p, q, \mathcal{I}, \mathcal{J}$ be as above. We work in $V[H]$.

5° Since $(i^{\mathcal{I}}, p, q) \in D_H$, we conclude that

- a. $\mathcal{I} : p > q$,
- b. q is A_H -iterable,
- c. $(H_{\omega_2}^q, \in, \omega_1^q, A_H \cap q) \models \forall x \in i^{\mathcal{I}}(H_{\omega_2}^p), \phi(x)$.

6° By elementarity,

$$(H_{\omega_2}^{\mathcal{P}^{\mathcal{J}}}, \in, \omega_1, i^{\mathcal{J}}(A_H \cap q)) \models \forall x \in i^{\mathcal{J}}(i^{\mathcal{I}}(H_{\omega_2}^{\mathcal{P}})), \phi(x).$$

7° By upward absoluteness, we get that

$$(H_{\omega_2}, \in, \omega_1, i^{\mathcal{J}}(A_H \cap q)) \models \forall x \in i^{\mathcal{J}}(i^{\mathcal{I}}(H_{\omega_2}^{\mathcal{P}})), \phi(x).$$

8° Since q is A_H -iterable, we have that $i^{\mathcal{J}}(A_H \cap q) = A_H$.

9° We also have that

$$i^{\mathcal{J}}(i^{\mathcal{I}}(H_{\omega_2}^{\mathcal{P}})) = i^{i^{\mathcal{J}}(\mathcal{I})}(H_{\omega_2}^{\mathcal{P}}) = H_{\omega_2}^{\mathcal{P}^{i^{\mathcal{J}}(\mathcal{I})}} = H_{\omega_2}^{\mathcal{V}}.$$

10° Thus, $(H_{\omega_2}, \in, \omega_1, A_H) \models \forall x \in H_{\omega_2}^{\mathcal{V}}, \phi(x)$, as required. □

Corollary 6.4.5. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *UB – BMM holds.*

Then $(\dagger)_{\text{UB}}^-$ holds.

Proof.

1° Let $\phi(x)$ be a Σ_1 formula in the language $\{R_{\in}^2, c_{\omega_1}, R_A^1\}$ such that

$$\exists \mathbb{P}, V^{\mathbb{P}} \models ((H_{\omega_2}, \in, \omega_1, A_{\dot{g}}) \models \forall x, \phi(x)).$$

We want to show that

$$(H_{\omega_2}, \in, \omega_1, A) \models \forall x, \phi(x).$$

2° Let $a \in H_{\omega_2}$ be arbitrary and let us show that

$$(H_{\omega_2}, \in, \omega_1, A) \models \phi[a].$$

3° By Proposition 6.4.4, there exists an ssp poset \mathbb{P} such that

$$V^{\mathbb{P}} \models ((H_{\omega_2}, \in, \omega_1, A_{\dot{g}}) \models \phi[a]).$$

4° Since UB – BMM holds, we get that

$$(H_{\omega_2}, \in, \omega_1, A) \models \phi[a],$$

as required. □

6.5 Towards Conjecture 6.1.7

Definition 6.5.1. A subset D of H_{ω_1} is called a **densoid** iff

- a. every element of D is a triple (i, p, q) , where p and q are pre-conditions and for some $\mathcal{I} : p > q, i = i^{\mathcal{I}}$;
- b. for every precondition p , there exist i and q such that $(i, p, q) \in D$. □

Definition 6.5.2. The sentence $(*)_{\text{UB}}^-$ asserts that for all $a \in H_{\omega_2}$ and for all densoids D which are universally Baire in codes, there exist p, q, i, \mathcal{J} such that

- a. $(i, p, q) \in D$,
- b. \mathcal{J} is a generic iteration of q of length $\omega_1 + 1$,
- c. $a \in \text{ran}(i^{\mathcal{J}}(i))$. □

Proposition 6.5.3. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *UB – BMM holds.*

Then $()_{\text{UB}}^-$ holds.*

Proof. This follows from Proposition 6.4.3. □

Conjecture 6.5.4. *Suppose there exists a proper class of Woodin cardinals. Then the following is true.*

- a. $(*)_{\text{UB}}^-$ *implies* UB – BMM.
- b. *If there exists a supercompact cardinal, then $(\dagger)_{\text{UB}}^-$ implies $(*)_{\text{UB}}^-$.* □

6.6 Towards Conjecture 6.1.9

Proposition 6.6.1. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *ϕ is a Σ_1 sentence in the language $\{R_{\in}^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$,*

4. there exists a poset that forces that $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi$.

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$.

Proof.

- 1° Let \mathbb{P} be a poset that forces that $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi$, let $\delta > \text{rank}(\mathbb{P})$ be a limit of Woodin cardinals, and let $\theta > \delta$ be an inaccessible cardinal.
- 2° Let $\pi : (M, \in, p, \bar{\mathbb{P}}, \bar{\delta}, \bar{A}) \longrightarrow (V_\theta, \in, p, \mathbb{P}, \delta, A)$ be a countable elementary embedding, where M is transitive, and let g be M -generic for $\bar{\mathbb{P}}$. Observe that in $M[g]$, $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, \bar{A}_g) \models \phi$.
- 3° By Lemma 6.3.9, there exists a $< \bar{\delta}$ -generic h over $M[g]$ such that
 - a. $\text{NS}_{\omega_1}^{M[g][h]} \cap M[g] = \text{NS}_{\omega_1}^{M[g]}$,
 - b. NS_{ω_1} is saturated in $M[g][h]$.
- 4° Let q be the triple $(M[g][h], \in, \text{NS}_{\omega_1}^{M[g][h]})$. By Lemma 6.3.7, q is an A -iterable pre-condition.
- 5° By Lemma 6.3.5, there exists a generic iteration \mathcal{I} of q such that $i^{\mathcal{I}}(I^q) = \text{NS}_{\omega_1} \cap \mathcal{P}_\infty^{\mathcal{I}}$. Observe that by A -iterability, $i^{\mathcal{I}}(A \cap q) = A \cap \mathcal{P}_\infty^{\mathcal{I}}$.
- 6° It follows that $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$, as required.

□

This generic maximality implies corresponding generic absoluteness. To show it, we will need to recall Woodin's theorem that Ω -satisfiability of Σ_2 formulas is generically absolute. We need a slight generalization of it which uses universally Baire sets as parameters; this generalization follows from the proof of said theorem.

Lemma 6.6.2. *Suppose*

1. there exists a proper class of Woodin cardinals,
2. A is a universally Baire set,
3. G is a V -generic,
4. M is an inner model containing all reals (in $V[G]$),
5. $j : V \longrightarrow M$ is an elementary embedding.

Then $j(A) = A_G$.

Proof.

- 1° Let \mathbb{P} be the underlining poset for G and let T, S be \mathbb{P} -absolutely complementing trees in V satisfying $p[T]^V = A$. Then $j(A) = p[j(T)]^M$ and $A_G = p[T]$.
- 2° Since $\mathbb{R} \subseteq M$, it follows that $j(A) = p[j(T)]$.
- 3° Note that $p[T] \subseteq p[j(T)]$ and $p[S] \subseteq p[j(S)]$. Since T and S project to complements and since $p[j(T)] \cap p[j(S)] = \emptyset$ by elementarity of j and absoluteness of wellfoundedness, it follows that $p[T] = p[j(T)]$.
- 4° This means that $A_G = j(A)$, as required.

□

Proposition 6.6.3. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *$\phi(v_A)$ is a formula,*
4. *there are a poset \mathbb{X} such that $\Vdash_{\mathbb{X}}$ “for some inaccessible κ , $R_\kappa \models \phi[A_{\dot{G}_\kappa}]$ ”,*
5. *\mathbb{P} is an arbitrary poset.*

Then there exist a complete super-poset \mathbb{Y} of \mathbb{P} such that $\Vdash_{\mathbb{Y}}$ “for some inaccessible λ , $R_\lambda \models \phi[A_{\dot{G}_\lambda}]$ ”¹⁰

Proof.

- 1° Let \mathbb{X} be as in the assumption 4 and let δ be a Woodin cardinal satisfying that $\mathbb{X}, \mathbb{P} \in V_\delta$ and $\Vdash_{\mathbb{X}}$ “for some inaccessible $\kappa < \delta$, $R_\kappa \models \phi[A_{\dot{G}_\kappa}]$ ”. Since \mathbb{P} completely embeds into \mathbb{P}_δ , we will be done we show find a complete super-poset \mathbb{Y} of \mathbb{P}_δ such that $\Vdash_{\mathbb{Y}}$ “for some inaccessible λ , $R_\lambda \models \phi[A_{\dot{G}_\lambda}]$ ”.
- 2° Let $\dot{\mathbb{Z}}$ be the canonical \mathbb{P}_δ -name for the poset $j_{\dot{G}_{\mathbb{P}_\delta}}^V(\mathbb{X})$. We want to show that we can take $\mathbb{Y} := \mathbb{P}_\delta * \dot{\mathbb{Z}}$.
- 3° Let G be V -generic for \mathbb{P}_δ and let us work in $V[G]$.
- 4° Let $j := j_G^V : V \rightarrow \text{Ult}(V, G)$ and let $\mathbb{Z} := \dot{\mathbb{Z}}^G = j(\mathbb{X})$. By elementarity, $\text{Ult}(V, G)^\mathbb{Z} \models$ “for some inaccessible $\kappa < j(\delta)$, $R_\kappa \models \phi[j(A)_{\dot{G}_\kappa}]$ ”.

¹⁰Cf. [54, Theorem 2.2].

- 5° Let k be a $V[G]$ -generic filter for \mathbb{Z} and let us work in $V[G][k]$.
- 6° Let $\kappa < j(\delta)$ be inaccessible in $\text{Ult}(V, G)[k]$ such that $R_\kappa^{\text{Ult}(V, G)[k]} \models \phi[j(A)_k]$. Since $j(\delta) = \delta$ and $R_\delta^{\text{Ult}(V, G)} = R_\delta^{V[G]}$, we conclude that
- a. $\kappa < \delta$,
 - b. κ is inaccessible,
 - c. $R_\kappa \models \phi[j(A)_k]$.
- 7° By Lemma 6.6.2 applied in $V[G]$, $j(A) = A_G$. Thus, $R_\kappa \models \phi[A_{G*k}]$, as required. □

Corollary 6.6.4. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *A is a universally Baire set,*
3. *ϕ is a Σ_1 sentence in the language $\{R_\infty^2, R_{\text{NS}_{\omega_1}}^1, R_A^1\}$,*
4. *\mathbb{P} is an arbitrary poset.*

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$ if and only if $V^\mathbb{P} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi)$.

Proof. The implication (\Leftarrow) follows from Proposition 6.6.1, so let us argue for the other one. Let us assume that $(H_{\omega_2}, \in, \text{NS}_{\omega_1}, A) \models \phi$. This fact and Proposition 6.6.3 imply that in $V^\mathbb{P}$, there exists a poset \mathbb{Q} that forces

$$(H_{\omega_2}, \in, \text{NS}_{\omega_1}, (A_{\dot{G}_\mathbb{P}})_{\dot{G}_\mathbb{Q}}) \models \phi.$$

By Proposition 6.6.1 applied in $V^\mathbb{P}$, we conclude that

$$V^\mathbb{P} \models ((H_{\omega_2}, \in, \text{NS}_{\omega_1}, A_{\dot{g}}) \models \phi).$$

□

Conjecture 6.6.5. *Suppose*

1. *there exists a proper class of measurable Woodin cardinals,*
2. *CH holds,*
3. *A is a universally Baire set,*
4. *$\phi(v_A)$ is a Σ_1 formula in the language $\{R_\infty^2, R_{\text{NS}_{\omega_1}}^1\}$,*

5. *there exists a poset that forces that CH and $(H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A_{\dot{g}}]$.*

Then $(H_{\omega_2}, \in, \text{NS}_{\omega_1}) \models \phi[A]$. □

Theorem 6.7.8 shows that the conjecture is true if we omit the predicate for NS_{ω_1} . It is not clear at the moment how to modify the proof of Theorem 6.7.8 to be able to add NS_{ω_1} . We think that a different approach, based on the proof of Proposition 6.6.1, is more promising. There, we first obtained an A -iterable pre-condition $q = (N, \in, \text{NS}_{\omega_1}^N)$ such that

$$(H_{\omega_2}^N, \in, \text{NS}_{\omega_1}^N, A \cap N) \models \phi[A \cap N].$$

We then produced a generic iteration $i : q \rightarrow q^*$ such that $i(\text{NS}_{\omega_1}^N) = \text{NS}_{\omega_1} \cap q^*$. This sufficed for the conclusion there, but for Conjecture 6.6.5, more is need.

What would suffice for this conjecture is obtain an embedding $i : q \rightarrow q^*$ such that

- a. $i(\text{NS}_{\omega_1}^N) = \text{NS}_{\omega_1} \cap q^*$,
- b. $i(A \cap N) = A \cap q^*$,
- c. $\mathbb{R} \subseteq q^*$.

The first two points are as before, but the third one is new. It is not immediately impossible, since we are assuming CH. Together with the second point, it would ensure that $A = A \cap q^* \in q^*$ and consequently, the conclusion of the conjecture. So, the question is how to ensure $\mathbb{R} \subseteq q^*$.

Our idea is to produce a generic iteration as for Proposition 6.6.1, but with cofinally many extra steps. If we have a model X at some stage, the extra step consists in picking a real x and producing an embedding $k : X \rightarrow X^*$ with critical point $> \omega_1^X$. It is not clear whether this is possible, but the following fact is encouraging (even though it is stated under the assumption that NS_{ω_1} is saturated).

Proposition 6.6.6 (De Bondt, Veličković). *Suppose*

- 1. NS_{ω_1} is saturated,
- 2. θ is large enough regular.

Then for projectively stationary many countable $M \prec H_\theta$, for all $x \in \mathbb{R}$, there exist N and f such that

- a. $M \prec N \prec H_\theta$,
- b. N is countable and $\delta_N = \delta_M$,
- c. $f \in N$ is a function and $f(\delta_N) = x$.

Proof.

1° Let $S \subseteq \omega_1$ be stationary, let $U \subseteq H_\theta$, and let $\mathcal{H} := (H_\theta, \in, U)$. We may assume without loss of generality that there exists a lightface definable wellordering of \mathcal{H} .

2° We want to show that there exists a countable $M \prec \mathcal{H}$ such that

- a. $\delta_M \in S$,
- b. for all $x \in \mathbb{R}$, there exists $f : \omega_1 \rightarrow H_\theta$ such that $\delta(\text{Hull}(M, f)) = \delta(M)$ and $f(\delta_M) = x$.

3° Let g be V -generic for $\text{NS}_{\omega_1}^+$ containing S and let us work in $V[g]$.

4° Let

$$j := j_g^V : V \rightarrow \text{Ult}(V, g) =: W$$

be the generic ultrapower. We will be done if we show that it holds in W that there exists a countable $M \prec j(\mathcal{H})$ such that

- a. $\delta_M \in j(S)$,
- b. for all $x \in \mathbb{R}$, there exists $f : \omega_1 \rightarrow H_{j(\theta)}$ such that $\delta(\text{Hull}(M, f)) = \delta(M)$ and $f(\delta_M) = x$.

5° Let $M := \text{Hull}^{j(\mathcal{H})}(\omega_1^V) \prec j[\mathcal{H}] \prec j(\mathcal{H})$. Observe that $M \in W$, that $|M|^W = \omega$, and that $\delta_M = \omega_1^V \in j(S)$.

6° We will be done if we show that in W , for a given real x , there exists $f : \omega_1 \rightarrow H_{j(\theta)}$ such that $\delta(\text{Hull}(M, f)) = \delta(M)$ and $f(\delta_M) = x$.

7° Since $x \in W$, there exists $f \in V$ such that $f : \omega_1^V \rightarrow \mathbb{R}^V$ and $j(f)(\omega_1^V) = x$.

8° Since $M \prec \text{Hull}(M, j(f)) \prec j[H_\theta^V]$, it follows that $\delta(M) = \delta(\text{Hull}(M, j(f))) = \delta(j[H_\theta^V])$.

9° Since $j(f)(\delta_M) = j(f)(\omega_1^V) = x$, we seen that $j(f)$ is as required.

□

6.7 Proof of Theorem 6.1.10

The proof is essentially the same as the proof of Woodin's theorem on the equivalence of Generic Σ_1^2 -maximality and CH, see [5, Theorem 3.2.1]. We broke Woodin's proof into lemmas in [49]; we explain below how to modify these lemmas to show Theorem 6.1.10.

Definition 6.7.1. Suppose

1. δ is Woodin,
2. $\theta > \delta$ is regular,
3. $M \prec (H_\theta, \in, \delta)$.

Then M is said to be \mathbb{Q}_δ -self-generic iff for all $D \in M$ which are predense in \mathbb{Q}_δ , there exists $S \in D \cap M$ such that $M \cap (\cup S) \in S$. \square

Definition 6.7.2. Suppose

1. N is a transitive model of ZFC^- ,
2. ω_1 is an element of N ,
3. C is a subset of ω_1 .

Then C is an N -slow club iff

- a. every $\alpha \in C$ is Woodin in N ,
- b. C is a club,
- c. for all limit points γ of C , for all $E \in N$ which are club subsets of γ , there exists $\rho < \gamma$ such that $C \cap (\rho, \gamma) \subseteq E$. \square

Notation 6.7.3. For a set X , $\text{tc}(X)$ denotes its transitive collapse. \square

Lemma 6.7.4. Suppose κ is a measurable Woodin cardinal. Then for stationary many $M \prec H(\kappa^+)$,

- a. $\text{tc}(\kappa \cap M) = \omega_1$,
- b. M is \mathbb{Q}_κ -self-generic,
- c. there exists a $\text{tc}(M)$ -slow club.

Proof. See [5, Proof of Theorem 3.2.1, the claim]. \square

Proposition 6.7.5. Suppose

1. κ is a measurable Woodin cardinal,
2. $\delta > \kappa$ is Woodin,
3. $\theta \in (\kappa, \delta)$ is inaccessible.

Then there are stationary many $M \prec (H_\theta, \in, \kappa)$ satisfying that

- a. $\text{tc}(\kappa \cap M) = \omega_1$,
- b. there exists a $\text{tc}(M)$ -slow club,
- c. every real is $< \omega_1^V$ -generic over $\text{tc}(M)$.

Proof.

1° We denote by \mathcal{U} the set of all $M \prec (H_\theta, \in, \kappa)$ such that

- a. $\text{tc}(\kappa \cap M) = \omega_1$,
- b. M is \mathbb{Q}_κ -self-generic,
- c. there exists a $\text{tc}(M)$ -slow club.

Lemma 6.7.4 implies that \mathcal{U} is stationary in $\mathcal{P}(V_\theta)$.

2° Let G be a V -generic filter for \mathbb{P}_δ containing \mathcal{U} and let

$$j := j_G^V : V \longrightarrow \text{Ult}(V, G).$$

We have that $j[V_\theta] = [(M : M \subseteq V_\theta)^V]_G \in \text{Ult}(V, G)$ and

$$\text{Ult}(V, G) \models j[V_\theta] \prec (H_{j(\theta)}, \in, j(\kappa)).$$

3° By Łoś's Theorem, it holds in $\text{Ult}(V, G)$ that

- a. $\text{tc}(j(\kappa) \cap j[V_\theta]) = \omega_1$,
- b. $j[V_\theta]$ is $\mathbb{Q}_{j(\kappa)}$ -self-generic,
- c. there exists a $\text{tc}(j[V_\theta])$ -slow club.

4° **Claim.** It holds in $\text{Ult}(V, G)$ that every real is $< \omega_1$ -generic over $\text{tc}(j[V_\theta])$.

Proof. It follows from 3° that $G \cap \mathbb{Q}_\kappa^V$ is V -generic for \mathbb{Q}_κ^V and that

$$k : \text{Ult}(V, G \cap \mathbb{Q}_\kappa^V) \longrightarrow \text{Ult}(V, G) : [f]_{G \cap \mathbb{Q}_\kappa^V} \mapsto [f]_G$$

is well-defined, elementary, and has the critical point $> \kappa = \omega_1^{\text{Ult}(V, G \cap \mathbb{Q}_\kappa^V)}$. This means that all reals of $V[G]$ are contained in $\text{Ult}(V, G \cap \mathbb{Q}_\kappa^V)$ and consequently, in $V_\theta[G \cap \mathbb{Q}_\kappa^V]$. The properties of the \mathbb{Q} -tower ensure now that every real is $< \kappa$ -generic over V_θ , as required. \square

5° Since $j[V_\theta] = [(M : M \subseteq V_\theta)^V]_G$, Łoś's Theorem now implies that there exists $S \in \mathcal{P}(V_\theta) \cap G$ such that for all $M \in S$, it holds in V that

- a. $\text{tc}(\kappa \cap M) = \omega_1$,
- b. there exists a $\text{tc}(M)$ -slow club,
- c. every real is $<\omega_1^V$ -generic over $\text{tc}(M)$.

This finishes the proof because S is necessarily stationary in V .

□

Lemma 6.7.6. *Suppose*

1. N is a transitive model of ZFC,
2. ω_1^V is a Woodin cardinal in N ,
3. there exists an N -slow club,
4. every real is $<\omega_1^V$ -generic over N ,
5. h is a $<\omega_1^V$ -generic filter over N .

Then there exists an $N[h]$ -slow club and every real is $<\omega_1^V$ -generic filter over $N[h]$

Proof. See the argument that starts at the last paragraph of [5, Page 95].

□

Lemma 6.7.7. *Suppose*

1. P is a transitive model of ZFC,
2. ω_1^V is a Woodin cardinal in P ,
3. there exists an P -slow club,
4. every real is $<\omega_1^V$ -generic over P .

Then there exists an P -generic filter H for $\mathbb{Q}_{\omega_1^V}^P$ such that $\mathbb{R} \subseteq P[H]$.

Proof. See the argument that starts at the last paragraph of [5, Page 95].

□

Theorem 6.7.8. *Suppose*

1. there exists a proper class of measurable Woodin cardinals,
2. CH holds,
3. A is a universally Baire set,

4. $\phi(v_A)$ is a Σ_1 formula in the language of set theory,

5. there exists a poset that forces that $H_{\omega_2} \models \phi[A_{\dot{g}}]$.

Then $H_{\omega_2} \models \phi[A]$.

Proof.

1° Let

- a. \mathbb{P} be a poset that forces “ $H(\omega_2) \models \phi[A_{\dot{g}}]$ ”,
- b. $\kappa > \text{rank}(\mathbb{P})$ be a measurable Woodin cardinal,
- c. $\delta > \kappa$ be Woodin,
- d. (T, S) be κ^+ -absolutely complementing trees on $\omega \times \kappa^+$ satisfying that $p[T] = A$,
- e. $\theta \in (\kappa, \delta)$ be an inaccessible cardinal.

2° By Lemma 6.7.5, there exists $M \prec (H_\theta, \in, \kappa, T, S, \mathbb{P})$ such that

- a. $\text{tc}(\kappa \cap M) = \omega_1$,
- b. there exists a $\text{tc}(M)$ -slow club,
- c. every real is $< \omega_1^V$ -generic over $\text{tc}(M)$.

3° Let N be the transitive collapse of M . For $x \in N$, let \bar{x} denote the image of x by this transitive collapse.

4° Let h be N -generic for $\bar{\mathbb{P}}$. Note that $N[h] \models (H(\omega_2) \models \phi[\bar{A}_h])$.

5° By Lemma 6.7.6, there exists an $N[h]$ -slow club and every real is $< \omega_1$ -generic over $N[h]$. Thus, Lemma 6.7.7 implies the existence of a $N[h]$ -generic filter H for $\mathbb{Q}_{\omega_1^V}^{N[H]}$ satisfying that $\mathbb{R} \subseteq N[h][H]$.

6° The elementarity of

$$j := j_H^{N[h]} : N[h] \longrightarrow \text{Ult}(N[h], H) =: Q$$

implies that

$$Q \models (H(\omega_2) \models \phi[j(\bar{A}_h)]).$$

7° Since ϕ is Σ_1 and since $H(\omega_2)^Q$ is a transitive set contained in $H(\omega_2)$, we will be done if we show that $j(\bar{A}_h) = A$.

8° The facts that

$$p[j(\bar{T})]^Q = j(\bar{A}_h) = \mathbb{R}^Q - p[j(\bar{S})] \text{ and } \mathbb{R}^Q = \mathbb{R}^{N[h][H]} = \mathbb{R}$$

yield that

$$j(\bar{A}_h) = p[j(\bar{T})].$$

9° **Claim.** $p[j(\bar{T})] = p[T]$

Proof.

- 1' Let $x \in p[T]$ and let us show that $x \in p[j(\bar{T})]$.
- 2' Since x is $<\omega_1^V$ -generic over N , we have that $N[x]$ is a forcing extension of N via a poset in $N \upharpoonright \omega_1^V = N \upharpoonright \bar{\kappa}$.
- 3' Since (\bar{T}, \bar{S}) are $\bar{\kappa}$ -absolutely complemented in N , it follows that either $x \in p[\bar{T}]$ or $x \in p[\bar{S}]$.
- 4' Since $p[\bar{S}] \subseteq p[S]$, it is not the case that $x \in p[\bar{S}]$. Hence, $x \in p[\bar{T}]$.
- 5' Taking into the account the fact that $p[\bar{T}] \subseteq p[j(\bar{T})]$, we get that $x \in [j(\bar{T})]$. This shows that $p[T] \subseteq p[j(\bar{T})]$.
- 6' We can analogously show that $p[S] \subseteq p[j(\bar{S})]$.
- 7' Since $p[T] = \mathbb{R} - p[S]$, it follows that $p[T] = p[j(\bar{T})]$ and $p[S] = p[j(\bar{S})]$.

□

10° It now follows that $j(\bar{A}_h) = p[j(\bar{T})] = p[T] = A$, as required.

□

Theorem 6.7.9. *Suppose*

1. *there exists a proper class of Woodin cardinals,*
2. *CH holds,*
3. *A is a universally Baire set,*
4. *$\phi(v_A)$ is a Σ_1 sentence in the language of set theory,*
5. *\mathbb{P} is an arbitrary poset that forces CH.*

Then $(H_{\omega_2}, \in) \models \phi[A]$ if and only if $V^{\mathbb{P}} \models ((H_{\omega_2}, \in) \models \phi[A_{\dot{g}}])$.

Proof. The implication (\Leftarrow) follows immediately from Theorem 6.7.8. For the other implication, note that if $(H_{\omega_2}, \in) \models \phi[A]$, then Proposition 6.6.3 that in $V^{\mathbb{P}}$, there exists a poset \mathbb{Q} that forces

$$(H_{\omega_2}, \in) \models \phi[(A_{\dot{G}_{\mathbb{P}}})_{\dot{G}_{\mathbb{Q}}}]$$

The conclusion then follows by again applying Theorem 6.7.8.

□

6.8 Appendix: Coding with Propositional Formulas

Section Assumptions 6.8.1. Suppose

1. there exists a proper class of Woodin cardinals,
2. D is a densoid which is universally Baire in codes,
3. κ is inaccessible. □

Notation 6.8.2. For $n < \omega$, let ψ_n be the formula in the language

$$\{R_{\in}^2, c_I, c_p, c_i, c_m : m < \omega\}$$

whose Gödel number is n . □

Notation 6.8.3. Let D^* consist of all reals x for which there are a triple $(i, p, q) \in D$ and surjection $a : \omega \rightarrow M^q$ such that the set

$$\{\psi_n : x(n) = 0\}$$

is the theory of the structure

$$(M^q, \in, I^q, p, i, a_m : m < \omega).$$

□

Section Assumptions 6.8.4. Let T_{D^*} be a κ -absolutely complemented tree on $\omega \times \kappa$ such that $p[T_{D^*}] = D^*$. □

Our goal is to describe an infinitary propositional formula Φ whose models (in all generic extensions) code

- a sequence

$$(((\omega, E_\alpha^q, I_\alpha^q, p_\alpha, i_\alpha, m)_{m < \omega} : \alpha < \omega_1^V)$$

of structures in the language $\{R_{\in}^2, c_I, c_p, c_i, c_m : m < \omega\}$,

- a sequence $u : \omega \rightarrow \kappa$,
- a family $(\pi_{\alpha, \beta} : \alpha \leq \beta < \omega_1^V)$,
- a sequence $(\mathcal{H}_\alpha : \alpha < \omega_1^V)$

such that

- a. letting $t : \omega \rightarrow \{0, 1\}$ be the function defined by asserting that for all $n < \omega$,

$$t(n) = 1 \iff (\omega, E_0^q, I_0^q, p_0, i_0, m)_{m < \omega} \models \psi_n,$$

it holds that $(t, u) \in T_{D^*}$,

b. for all $\alpha \leq \beta < \omega_1^V$,

$$\pi_{\alpha,\beta} : (\omega, E_\alpha^q, I_\alpha^q, p_\alpha, i_\alpha) \xrightarrow{\Sigma_\omega} (\omega, E_\beta^q, I_\beta^q, p_\beta, i_\beta),$$

c. the family $((\omega, E_\alpha^q, I_\alpha^q), \pi_{\alpha,\beta} : \alpha \leq \beta < \omega_1^V)$ is a generic iteration of a pre-condition, modulo the transitive collapse,

d. for all $\alpha < \omega_1^V$, letting

- $\iota_\alpha < \omega$ be such that $(\omega, E_\alpha^q, I_\alpha^q) \models \iota_\alpha$ is the transitive closure of $\text{ran}(i_\alpha)$,
- y_α be the set of all $m < \omega$ satisfying $m E_\alpha^q \iota_\alpha$,

it holds that

$$\mathcal{H}c_\alpha : (y_\alpha, E_\alpha^q \upharpoonright y_\alpha, I_\alpha^q \upharpoonright y_\alpha) \xrightarrow{\Sigma_\omega} (V_\kappa, \in, \text{NS}_{\omega_1^V}^V)$$

and for all $\beta \in [\alpha, \omega_1^V)$, the diagram

$$\begin{array}{ccc} y_\alpha & \xrightarrow{\pi_{\alpha,\beta} \upharpoonright y_\alpha} & y_\beta \\ & \searrow \mathcal{H}c_\alpha & \downarrow \mathcal{H}c_\beta \\ & & V_\kappa \end{array}$$

commutes.

We will try to convince the reader that this can indeed be expressed by an infinitary propositional formulas. To give more details about what exactly should be expressed, let us introduce some notation.

Notation 6.8.5.

- a. If $E \subseteq X \times Y$ and $x \in X$, we denote $E_x := \{y \in Y : (x, y) \in E\}$.
- b. If $a : X \rightarrow Y$ and $x \in X$, then $a_x := a(x)$. □

Notation 6.8.6. Suppose $\mathcal{M} = (\omega, \dots)$ is a structure in the language $\{R_\infty^2, c_I, c_p, c_i\}$. Then $\vartheta(\mathcal{M})$ denotes the truth function of the model $(\mathcal{M}, m : m < \omega)$, i.e. $\vartheta(\mathcal{M}) : \omega \rightarrow 2$ is defined in such a way that for all $n < \omega$, for ψ denoting the formula in the language of

$$(\mathcal{M}, m : m < \omega)$$

with $\ulcorner \psi \urcorner = n$, it holds that

$$\vartheta(\mathcal{M})(n) = 1 \iff (\mathcal{M}, m : m < \omega) \models \psi.$$

□

Now, we need to construct a formula Φ whose models code $E, I, p, i, u, \pi, \mathcal{H}c$ such that

- a. $E \subseteq \omega_1 \times (\omega \times \omega)$,
- b. $I, p, i : \omega_1 \longrightarrow \omega$,
- c. $u : \omega \longrightarrow \kappa$,
- d. for all $n < \omega$, $(\vartheta(\omega, E_0, I_0, p_0, i_0) \upharpoonright n, u \upharpoonright n) \in T_{D^*}$,
- e. $\pi \subseteq \{(\alpha, \beta) : \alpha \leq \beta < \omega_1\} \times (\omega \times \omega)$,
- f. for all $\alpha \leq \beta < \omega_1$, $\pi_{\alpha, \beta} : (\omega, E_\alpha, I_\alpha, p_\alpha, i_\alpha) \xrightarrow{\Sigma_\omega} (\omega, E_\beta, I_\beta, p_\beta, i_\beta)$,
- g. for all $\alpha \leq \beta \leq \gamma < \omega_1$, $\pi_{\beta, \gamma} \circ \pi_{\alpha, \beta} = \pi_{\alpha, \gamma}$,
- h. for all $\alpha < \omega_1$, there exists $k < \omega$ such that
- i. $(\omega, E_{\alpha+1}) \models \dot{k} < \dot{\omega}_1$,
 - ii. for all $\mathfrak{u} < \omega$, there exists $y < \omega$ such that $(\omega, E_\alpha) \models \dot{y} : \dot{\omega}_1 \longrightarrow \dot{V}$ and such that for $u := \pi_{\alpha, \alpha+1}(y)$, $(\omega, E_{\alpha+1}) \models \dot{u}(k) = \dot{i}$,
 - iii. for $g_\alpha := \{w < \omega : (\omega, E_\alpha, I_\alpha) \models \dot{w} \in I_\alpha^+, k E_{\alpha+1} \pi_{\alpha, \alpha+1}(w)\}$, we have that $(\omega, E_\alpha, I_\alpha, g_\alpha) \models \text{“}g_\alpha \text{ is generic for } I_\alpha^+\text{”}$,
- i. for all limit $\gamma < \omega_1$, for all $n < \omega$, there exist $\alpha < \gamma$ and $m < \omega$ such that $\pi_{\alpha, \gamma}(m) = n$,
- j. $\mathcal{H} \subseteq \omega_1 \times (\omega \times V_\kappa)$,
- k. for all $\alpha < \omega_1^V$, letting
- $\mathfrak{u}_\alpha < \omega$ be such that $(\omega, E_\alpha) \models \text{“}\mathfrak{u}_\alpha \text{ is the transitive closure of } \text{ran}(i_\alpha)\text{”}$,
 - y_α be the set of all $m < \omega$ satisfying $(\omega, E_\alpha) \models m \in \mathfrak{u}_\alpha$,
 - z_α be the set of all $m < \omega$ satisfying that $(\omega, E_\alpha) \models m \in I_\alpha \cap \mathfrak{u}_\alpha$,

it holds that

- i. for all $\alpha < \omega_1$, $\mathcal{H}_\alpha : (y_\alpha, E_\alpha, z_\alpha) \xrightarrow{\Sigma_\omega} (V_\kappa, \in, \text{NS}_{\omega_1}^V)$,
- ii. for all $\alpha \leq \beta < \omega_1$, $\mathcal{H}_\alpha = \mathcal{H}_\beta \circ \pi_{\alpha, \beta} \upharpoonright y_\alpha$,
- iii. for all $x \in V_\kappa$, there exist $\alpha < \omega_1$ and $\mathfrak{u} < \omega$ such that $\mathcal{H}_\alpha(\mathfrak{u}) = x$. □

Here are some examples how to convert these requirements into a propositional formula.

- Let $\dot{E}_{\alpha, m, n}, \dot{I}_{\alpha, m}, \dot{p}_{\alpha, m}, \dot{i}_{\alpha, m}, \dot{u}_{m, \xi}, \dot{\pi}_{\alpha, \beta, m, n}, \dot{\mathcal{H}}_{\alpha, m, x}$ be propositional letters, for all $\alpha \leq \beta < \omega_1$, $\gamma < \omega_1$, $m, n < \omega$, $\xi < \omega_2$, $x \in V_\kappa$.
- If ψ is the language $\{R_{\in}^2, c_I, c_p, c_i\}$, we can express

$$(\omega, E_\alpha, I_\alpha, p_\alpha, i_\alpha) \models \psi$$

as a propositional formula by induction on complexity of ψ .

-
- We can now express requirement d as

$$\bigwedge_{n < \omega} \bigvee_{\substack{s \in T, \\ |s|=n}} \bigwedge_{i < n} \theta_{s,i}$$

where

- for $s(i) = (1, \xi)$ and for ψ_i being the formula with code i , $\theta_{s,i}$ is defined as the propositional formula asserting

$$“(\omega, E_\alpha, I_\alpha, p_\alpha, i_\alpha) \models \psi_i” \wedge \dot{u}_{i,\xi},$$

- for $s(i) = (0, \xi)$ and for ψ_i being the formula with code i , $\theta_{s,i}$ is defined as the propositional formula asserting

$$“(\omega, E_\alpha, I_\alpha, p_\alpha, i_\alpha) \models \neg\psi_i” \wedge \dot{u}_{i,\xi}.$$

One can proceed like this to code all of the requirements of listed above and obtain a formula which will have κ propositional letters and whose every conjunction and disjunction will have the length at most κ . If Φ is the formula just described and μ its model, then one obtains E, I , and so on, in the obvious way:

$$E^\mu := \{(\alpha, m, n) \in \omega_1^V \times (\omega \times \omega) : \mu(\dot{E}_{\alpha,m,n}) = 1\},$$

$$I^\mu := \{(\alpha, m) \in \omega_1^V \times \omega : \mu(\dot{I}_{\alpha,m}) = 1\},$$

and so on.

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