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Paraconsistent and fuzzy modal logics for reasoning about uncertainty

Daniil Kozhemiachenko

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THÈSE présentée par :

Daniil KOZHEMIACHENKO

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**Paraconsistent and fuzzy modal logics
for reasoning about uncertainty**

THÈSE DIRIGÉE PAR :

M. NGUYEN Benjamin

Professeur des Universités, INSA Centre Val de Loire (Directeur), France

Mme. FRITTELLA Sabine

Maîtresse de Conférences, INSA Centre Val de Loire (Encadrante), France

RAPPORTEURS :

M. BALBIANI Philippe

Directeur de recherche, Institut de recherche en informatique de Toulouse, France

M. GODO LACASA Lluís

Professeur, Artificial Intelligence Research Institute, CSIC, Espagne

JURY :

M. BALBIANI Philippe (président)

Directeur de recherche, Institut de recherche en informatique de Toulouse, France

Mme. BIENVENU Meghyn

Directrice de recherche, Université de Bordeaux, France

Mme. FRITTELLA Sabine

Maîtresse de Conférences, INSA Centre Val de Loire, France

M. GODO LACASA Lluís

Professeur, Artificial Intelligence Research Institute, CSIC, Espagne

Mme. HANIKOVÁ Zuzana

Chercheuse, Institute of Computer Science, Czech Academy of Sciences, Tchèque

M. NGUYEN Benjamin

Professeur des Universités, INSA Centre Val de Loire, France

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Daniil Kozhemiachenko

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Abstract

This dissertation is devoted to the study of fuzzy modal logics that formalise (paraconsistent) reasoning about uncertainty. The understanding of ‘uncertain information (data)’ here includes any combination of the following three characteristics. First, the information can be graded, i.e., the statement is equipped with a *truth degree* rather than a *truth value*. Second, the information can be incomplete. Third, the information can be contradictory.

All the logics in question can be divided into two kinds. First, the more ‘traditional’ modal logics defined on $[0, 1]$ -valued Kripke models (possibly, with fuzzy accessibility relations) whose language includes modal operators $\Box\phi$ and $\Diamond\phi$ interpreted as, respectively, infima and suprema of ϕ ’s values in the accessible states.

The second kind of logics contains so-called ‘two-layered’ logics. In this framework, the language is divided into three parts: the *inner layer* \mathcal{L}_i , the *outer layer* \mathcal{L}_o and the *non-nesting modality* \mathbf{M} . The idea is to use \mathcal{L}_i to *describe events*, interpret \mathbf{M} as a *measure on the set of events* (e.g., as a probability function, belief function, plausibility, etc.) corresponding to the degree of the agent’s (un)certainty in a given event, and then *reason about* this (un)certainty in \mathcal{L}_o . A *frame* in a two-layered logic is, thus, a set with a measure defined thereon.

These two kinds of logics correspond to two ways of interpreting uncertainty. In the less formal one that is closer to the intuitive understanding of constructions such as ‘I believe’, ‘I am certain that’, etc., we will be using the logics with the Kripke-frame semantics. In the more formal case where the degree of one’s certainty or belief in ϕ is assumed to behave as a concrete uncertainty measure, we will use the two-layered logics.

The logics studied in the manuscript can be also divided into ‘qualitative’ and ‘quantitative’ ones depending on the operations the agent is supposed to be able to carry out with their degree of certainty in ϕ . In the qualitative case, the agent is only supposed to be capable of comparison of their degrees of certainty in different statements (e.g., ‘I am more certain that it is going to snow today than that there is going to be a hailstorm’) or state their complete certainty or disbelief therein (e.g., ‘I am completely sure that it is not going to rain’). In other words, the agent does not know the exact numerical value of their certainty. In contrast to that, in the quantitative case, the agent is supposed to know this value, whence, they are able to conduct some basic arithmetic operations with them: e.g., addition or subtraction.

The logics formalising quantitative reasoning will thus be based on the Łukasiewicz logic and its expansions as it can express the arithmetic operations. The logics for the quantitative reasoning, in their turn, will use Gödel logic as its propositional fragment. We will mainly focus on providing the axiomatisations for the logics formalising reasoning about uncertainty, establishing their complexity evaluations and devising decision procedures as well as on investigating their semantical properties. Among those, we will be mostly concerned with the correspondence between formulas and the classes of frames they define, and faithful translations and embeddings of the logics into one another.

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1. Introduction

People believe in many things, and one of the logician’s tasks is to attempt the formalisation of these beliefs. In doing so, they need to choose a framework wherein this formalisation shall occur. And while the use of the classical logic is a well-established tradition in the representation of beliefs or knowledge, as well as in the reasoning about uncertainty, it is (as the title of the dissertation suggests) not going to be followed here. Why?

The intuitions about beliefs and uncertainty tell us (among other things) the following.

Desideratum 1. Given two statements ϕ and χ , one can be more certain in ϕ than in χ but still, neither believe in ϕ *completely* nor consider χ completely impossible.

Desideratum 2. Given two trusted sources, one can still prefer one source to the other.

Desideratum 3. One can believe in a contradiction but still not believe in something else.

Desideratum 4. Given two statements, it is possible that one cannot always compare their degrees of certainty in them (if, e.g., these statements have no common content).

Desideratum 5. If the (dis)belief is supposed to be built only upon the available evidence and the accounts provided by sources and if there is no evidence at all concerning ϕ , then one cannot consider ‘I believe in ϕ ’ (as well as ‘I believe in $\phi \vee \neg\phi$ ’¹) true nor false.

Unfortunately, none of these desiderata can be straightforwardly formalised using classical² logic. Indeed, every statement is either true or false from the classical standpoint, thus one cannot have truth degrees.³ Likewise, if we represent the sources as states in a classical Kripke frame and use sRt to stand for ‘ s trusts t ’, there will be no degrees of trust. Thus, Desiderata 1 and 2 call for the use of *fuzzy logics*, i.e., the logics where the values of formulas range over $[0, 1]$. In particular, if we represent sources as states in a Kripke frame, the degree of trust one source assigns to another can be represented via a fuzzy accessibility relation (we discuss this in further detail in Chapters 5–7).

Desiderata 3 and 4 show the need to employ *paraconsistent logics*, i.e., those where the *explosion principle* — $p, \neg p \models q$ — fails. Desideratum 3 just says that a modalisation of explosion should fail. To understand why Desideratum 4 is connected to paraconsistent logics, recall, that it is customary⁴ to consider truth and falsity in the paraconsistent logics to be independent. The entailment can be then understood as the preservation of truth from the premise to the conclusion and the preservation of falsity from the conclusion to the premise: if the premise is true, so must be the conclusion (or, in the fuzzy setting, the conclusion is *at least as true* as the premise); if the conclusion is false, so must be the premise (the premise must be *at least as false* as the conclusion). In this sense, the values of ϕ and χ are incomparable if, e.g., ϕ is *both more true and more false than* χ .

¹Recall that in *paracomplete* logics, the law of excluded middle is not valid, whence an agent reasoning according to them, does not have to believe in an instance of LEM. Moreover, in this setting, believing in $\phi \vee \neg\phi$ means that the agent *has evidence that ϕ does not behave in a paracomplete manner*.

²The term ‘classical logic’ can be sometimes vague. Here, we interpret it as ‘a logic that expands the classical propositional logic’. I.e., \mathbf{K} is a (minimal normal modal) classical logic but the intuitionistic propositional logic is not.

³We refer the reader to [43] for a more detailed discussion of truth degrees.

⁴This approach can be traced to Belnap [15] (reprinted in [16]). Even earlier, in [110], there are independent *positive* and *negative* realisations of arithmetical predicates.

1. Introduction

The [fifth](#) desideratum indicates the need to use *paracomplete* logics, i.e., those where the law of excluded middle does not hold. In this text, most (propositional fragments of) logics are built over the Belnap–Dunn logic BD which is both paraconsistent and paracomplete.⁵

In the remainder of the chapter, we will give the broader context to the dissertation and discuss related work. Namely, we provide a short survey of how the reasoning about uncertainty and belief can be (and has been) formalised using fuzzy and (or) paraconsistent modal logics.

1.1. Reasoning about uncertainty

Over the course of this dissertation, we will construe ‘(un)certainty’ in one of the two following senses. The first one is an intuitive understanding of natural-language phrases such as ‘I am certain that it is raining outside now’, ‘I think that the rain tomorrow is more likely than the hailstorm’, etc. The second one is a more formal interpretation where the value of such statements is computed via an *uncertainty measure*: a probability, a belief function, plausibility, capacity, etc. Both these approaches in the classical framework are well established (cf., e.g., [86] for an introduction and overview).

Formally, these two readings of uncertainty correspond to two types of logics that we cover in the dissertation. The first one is the modal logics on $[0, 1]$ -valued Kripke frames, possibly, with $[0, 1]$ -valued accessibility relations⁶, and (in the case of paraconsistent logics) with two independent valuations standing for the support of truth and support of falsity. The usual modal formulas $\Box\phi$ and $\Diamond\phi$ will be evaluated as, respectively, infima and suprema of ϕ ’s values across the accessible states (cf., e.g., [123] for some simple examples of many-valued modal logics where modalities are interpreted as infima and suprema or minima and maxima). The interpretation of $\Box\phi$ or $\Diamond\phi$ is then taken as ‘the agent believes that ϕ ’⁷ or ‘the agent is certain that ϕ ’.

The second kind of logics that corresponds to the ‘belief-as-measure’ approach is the *two-layered logics*. The main idea is to separate the description of events from the reasoning about these events on the syntactic level. Namely, the language is split into three parts: the *inner language* \mathcal{L}_i that describes events, the *measure modality* \mathbf{M} defined on the sample space of events, and the *outer language* \mathcal{L}_o where the reasoning about events is formalised. In this text, we use a very simple $\{\neg, \wedge, \vee\}$ language (mostly, equipped with BD semantics) to describe events. In particular, we do not use implication (unless it can be defined using other connectives) as conditional statements do not correspond to event descriptions. The choice of the outer language and its semantics will depend on the scenarios we wish to formalise. In general, however, the outer logic is a fuzzy logic. This ties into the existing tradition of using fuzzy logics for reasoning about vagueness [143], beliefs [73, 144], and uncertainty [55].

In this manuscript, we are dealing with two approaches to the belief as a measure. The first one is *quantitative*, i.e., we assume that given a statement such as ‘it is going to be windy today’, the agent can give a numerical value to their certainty and say something like ‘I am 73% certain that it is going to be windy today’ or ‘I think that the rain today is twice more likely than snow’. The belief, then, can be more precisely described via a probability measure, belief function, plausibility, etc. For this approach, we choose the Łukasiewicz logic $\mathbf{\mathbb{L}}$ and its expansions on the outer layer since it can reason with arithmetic functions on $[0, 1]$.

The second approach is *qualitative*. Here, given two statements, the agent can only say that one is more or less likely than the other, that they have the same likelihood, that they are completely certain in one or both, or (in case of paraconsistent reasoning) say that their likelihoods are incomparable. This approach is formalised via preference relations on sample space

⁵Logics that are both paraconsistent and paracomplete are sometimes called *paradefinite* or *paranormal* [4].

⁶We will thus have two kinds of frames: *crisp*, where the accessibility relation is $\{0, 1\}$ -valued; and *fuzzy* with $[0, 1]$ -valued relations.

⁷It is clear that both \Box and \Diamond can be construed as belief modality, their difference being that to *reject* the \Box -belief, one counterexample suffices, and to *accept* the \Diamond -belief, the agent needs one supporting example.

(total preorders on the powerset of the sample space) of events which are then characterised by their measure counterparts. More formally, given a set of events W , a measure μ agrees with a preference relation \preceq iff

$$\forall X, Y \subseteq W : X \preceq Y \Leftrightarrow \mu(X) \leq \mu(Y)$$

For the qualitative reasoning, we will use expansions of Gödel logic since it can express order relations but not arithmetic functions.

As regards the formal aspect, we will focus on providing the axiomatisations of the logics formalising the mentioned ways to reason about uncertainty, studying their expressivity and model-theoretic properties, and establishing decidability and complexity evaluations. The last point is an important direction of research in the classical reasoning about uncertainty (cf., e.g., [87, 8, 13] for the complexity results regarding reasoning using ‘traditional’ modal logics and [61] for the complexity of logics with modalities interpreted as measures), especially, in its connection with knowledge representation and reasoning. We extend this direction to the non-classical reasoning about uncertainty.

1.2. Fuzzy modal logics

As we have already discussed, it makes sense to have modal statements with different truth degrees: there are obligations of different strengths and convictions more or less dear to us. Thus, there are deontic, doxastic, epistemic fuzzy logics (cf., e.g., [50] and [46]).

Different (propositional) fuzzy logics have different expressive capacities. One can roughly divide them into three classes: the ones that can express (truncated) addition and subtraction; the ones that can express order on $[0, 1]$; those that can do neither of those. The most well-known examples are, respectively, Łukasiewicz, Gödel, and Product logic (cf., e.g., [83] or [106] for a detailed presentation of these logics).

When formalising natural-language modal statements, it is reasonable to expect that an agent can compare them (e.g., ‘I think that the rain today is more likely than a tornado’). On the other hand, it is rare to see somebody who says ‘I am 67% certain that Paula’s dog is a golden retriever’ while ‘I think that Paula’s dog is rather a golden retriever than a dachshund’ is a completely natural sentence. Thus, the logics of the second kind seem to be the most reasonable choice.

(Propositional) Gödel logic \mathbf{G} can be thought of as a logic of comparative truth since the value of a formula depends not on the values but rather on the order of the variables. Thus, it is well-suited to the formalisation of modal statements. The expansion of \mathbf{G} with \Box and \Diamond (\mathbf{KG}) with semantics on $[0, 1]$ -valued frames with fuzzy accessibility relations was first introduced in [39] and since then well studied along with its axiomatic extensions corresponding to the axiomatic extensions of \mathbf{K} . In particular, axiomatisations of both \Box and \Diamond fragments and the axiomatisations of bi-modal fuzzy [40] and crisp [130] logics are known. Moreover, both fuzzy and crisp \mathbf{KG} , and its monomodal fragments are \mathbf{PSpace} -complete [104, 105, 37, 131, 38], likewise, a Gödel counterpart of $\mathbf{S4}$ is also \mathbf{PSpace} -complete [51]. Furthermore, there are applications of Gödel modal logics to reasoning about uncertainty. In particular, a Gödel $\mathbf{K45}$ counterpart is known to be complete w.r.t. frames where \Box and \Diamond are interpreted as non-normalised necessity and possibility measures⁸ on a Kripke frame [129].

Bi-Gödel (symmetric Gödel in [76]) logic \mathbf{biG} expands \mathbf{G} with \prec (co-implication which is interpreted as ‘excludes’) or Baaz’ Delta operator Δ [9] (interpreted ‘it is true that’). This allows expressing *strict* order. Thus, modal expansions of \mathbf{biG} can formalise statements such as ‘I think that Paula’s dog is rather a golden retriever than a dachshund’ given above where ‘rather’ is construed as ‘strictly more confident’. \mathbf{KbiG} (the expansion of \mathbf{biG} with \Box and \Diamond) was introduced in [20] and given an axiomatisation in [21]. Additionally, a temporal expansion of bi-Gödel logic was introduced in [2]. The satisfiability and validity of both logics are also in \mathbf{PSpace} .

⁸Normalised possibility measures correspond to the Gödelian $\mathbf{KD45}$.

1. Introduction

	is true when	is false when
$\neg\phi$	ϕ is false	ϕ is true
$\phi_1 \wedge \phi_2$	ϕ_1 and ϕ_2 are true	ϕ_1 is false or ϕ_2 is false
$\phi_1 \vee \phi_2$	ϕ_1 is true or ϕ_2 is true	ϕ_1 and ϕ_2 are false

Table 1.1.: Truth and falsity conditions of BD formulas

Just as classical description logics are notational variants of (global) classical modal logics, so Gödel description logics are the counterparts of Gödel modal logics. Their fuzziness allows for the expression of vague and uncertain data which is not possible in classical ontologies. The decidability and expressivity of Gödel description logics is well studied [30, 31, 33, 32, 29] and the complexity usually coincides their classical counterparts (cf., e.g., [8]). This shows another (this time, practical) advantage of Gödel description logics over Łukasiewicz ones since the latter ones are not decidable unless they don't use the Łukasiewicz t-norm-based conjunction [34, 41, 95, 148]. In fact, global Łukasiewicz modal logic is not even axiomatisable [149].

1.3. Paraconsistent modal logics

As we pointed out in the [beginning of this chapter](#), paraconsistent modal logics can formalise statements expressing belief or certainty more intuitively than the classical ones. Paraconsistent logics (mostly expanding BD, the Priest's Logic of Paradox LP [121], and related systems) also found their use in knowledge representation since they can straightforwardly formalise non-trivial reasoning and querying over contradictory ontologies. Paraconsistent description logics have been attracting much attention. In particular, the paraconsistent counterparts of \mathcal{ALC} [113, 114, 155] as well as much more expressive systems [101] were proposed and studied; inconsistency-tolerant versions of the Web Ontology Language (OWL) were developed [100, 99, 98, 102]; there has also been an investigation into querying over inconsistent ontologies [157].

We have also remarked that it is customary to treat the truth and falsity of formulas independently in the paraconsistent setting. Formally, this means that we can consider Kripke frames not with one but two valuations following [122, 150, 72, 137, 115, 116, 52]. The valuations are, as expected, interpreted as independent supports of truth and falsity (or positive and negative supports). The idea follows Belnap's and Dunn's 'useful four-valued logic' [56, 15, 14, 16] (alias, BD or FDE — 'first-degree entailment').

The truth and falsity conditions for negation, conjunction, and disjunction in BD are intuitive and can be summarised as shown in Table 1.1. The modal logics that we will be considering are built upon biG, thus we also need to come up with the falsity conditions of \rightarrow and \leftarrow . In this dissertation, we are focussing on the expansion of G that defines $\neg(\phi \rightarrow \phi') \leftrightarrow (\neg\phi' \leftarrow \neg\phi)$ and $\neg(\phi \leftarrow \phi') \leftrightarrow (\neg\phi' \rightarrow \neg\phi)$ after I_4C_4 ⁹ from [151] when dealing with the propositional fragment of modal logics with Kripke frame semantics. We henceforth call this logic $G_{(\rightarrow, \leftarrow)}^2$.

$G_{(\rightarrow, \leftarrow)}^2$ has some nice properties. First, all $G_{(\rightarrow, \leftarrow)}^2$ connectives have their duals. Second, in contrast to G, *it is not the case that either $p \rightarrow q$ or $q \rightarrow p$ has designated value under any valuation*. This means that not all statements are comparable. Indeed, when reasoning about beliefs, it is safe to assume that they are not always comparable: people do not have to believe that a thunderstorm is going to happen today *more (or less) than* they believe that their cousin twice removed has two dogs.

An expected next step after introducing separate valuations of formulas' truth and falsity is to introduce separate accessibility relations as it is done in [137, 52]: one relation is used to determine whether a modal formula *is true at w*, and the other whether it is *false at w*.

⁹The logic was introduced independently by different authors [151, 97], and further studied in [111]. It is the propositional fragment of Moisil's modal logic [107]. We are grateful to Heinrich Wansing who pointed this out to us.

We finish the section with a quick summary of the modalities we are going to consider. Namely, we differentiate between two kinds of paraconsistent modalities. $\Box\phi$ whose negative support is defined as the *supremum* of negative supports of ϕ across the accessible states (and its dual \Diamond), and $\blacksquare\phi$ ($\blacklozenge\phi$, respectively) where the negative support is the *infimum* (supremum) of negative supports of ϕ ¹⁰ in the accessible states. We will study these modalities both on fuzzy and crisp frames and both on mono- and bi-relational frames.

1.4. Two-layered modal logics

In Section 1.1, we said that we will be using two-layered logics to reason about uncertainty when it is construed in terms of measures. Two-layered logics are less expressive than the logics that do allow for the nesting of modalities. Even though this restriction can be seen as too strong, it is actually justifiable.

Indeed, an obvious alternative to two-layered logics would be those where M can nest. There are multiple examples of such systems. For instance, an expansion of an epistemic logic with conditional probabilities is proposed in [47]; qualitative counterparts of probability measures are axiomatised in [69] and [48, 49] using a binary modality \lesssim interpreted as the preference relation. Note, however, that nested modalities are difficult to interpret in the natural language, and people rarely say something like *it is probable that p and that q is probable too*. While Mp can be understood as ‘ p is probable’, ‘the agent believes that p is the case’, etc. depending on M , and its value can be straightforwardly derived from the measure of the subset of the sample space where p is true, the interpretation of formulas such as $M(p \wedge Mq)$ that corresponds to the italicised phrase from the previous sentence is considerably less intuitive.¹¹

From the formal side, the decision procedures for such logics are not straightforward and cannot be used to obtain a sharp complexity evaluation (e.g., filtration is used in [47] to establish the decidability of CKL, while Gärdenfors [69] enumerates preference orderings on canonical models of formulas).

On the other hand, the decision procedures for the two-layered logics are usually intuitive and can often be adapted from those for their outer logics [61, 60, 85]. In addition, it is often the case that the decidability of a two-layered logic is not harder than that of its outer layer. The (outer layer) formulas are also straightforward to interpret since they are just propositional combinations of modal atoms (formulas of the form $M\phi$ where ϕ is an inner-layer formula).

Usually, two-layered logics are formalising classical reasoning about uncertainty (there is, however, a *de facto* two-layered logic formalising intuitionistic probabilistic reasoning [89]). In the qualitative case, the axioms for the preference order can be simply translated into the corresponding modal formulas. In the quantitative case, there are two options. The first and simpler one is to just use arithmetic operations on the outer level as done in [61, 60]. A ‘logically puristic’ alternative is to use a fuzzy logic that can express the required operations. Usually, (expansions of) Łukasiewicz or Product logic (or a combination of these two) is used [84, 73, 66, 44] since for a two-layered probabilistic logic to be complete, it has to express the (finite) additivity property of the probability measures and belief functions. Recently [11], these two approaches were shown to be equivalent to one another via mutual faithful translations. In this manuscript, we will be working with the ‘puristic’ two-layered logics whose outer layer is an expansion of Łukasiewicz logic (when dealing with quantitative uncertainty) or of Gödel logic (for qualitative uncertainty). This is mostly because the completeness proofs of such two-layered logics can be reduced to the completeness proofs of the outer-layer logics expanded with additional axioms governing the uncertainty measure.

Again, when dealing with paraconsistent expansions of \mathfrak{L} and \mathfrak{G} , we have to come up with falsity conditions for the implications. The first option was presented in the previous section —

¹⁰We will provide a more detailed motivation for such modalities in Chapter 7.

¹¹This is even more the case with nested \lesssim : $(p \lesssim q) \lesssim (r \lesssim s)$ is not a natural sentence to utter either.

1. Introduction

the dualisation via the co-implication for \mathbf{G} and defining $\neg(\phi \rightarrow \chi) \leftrightarrow (\neg\chi \ominus \neg\phi)$ for a paraconsistent expansion of \mathbf{L} . This results in the congruential implication that can define order. The second option is to use a more intuitive understanding ‘ $\phi \rightarrow \chi$ is false when ϕ is true but χ is false’. The idea behind the second implication comes from Nelson’s interpretation of its falsity condition [110]. Another benefit of Nelson’s implication is that (in contrast to the strong or congruential implication \rightarrow) it allows talking about support of truth and support of falsity *separately*.

1.5. Structure of the dissertation

The remainder of the dissertation is structured as follows. In Part I, we present some preliminaries on the Belnap–Dunn logic (Chapter 2) and propositional fragments of our modal logics, namely, paraconsistent expansions of Łukasiewicz and Gödel logics (Chapters 3 and 4, respectively). We define their semantics, provide complete axiomatisations, devise decision procedures based on constraint tableaux, establish complexity evaluations, and investigate some instructive semantical properties. Chapter 3 is based on [19] (Sections 3.2 and 3.3) and [26] (Sections 3.1.1 and 3.1.2), although some proofs are conducted in a slightly different manner. Chapter 4 is based on [19] (Sections 4.2 and 4.3.1) and [24] (Section 4.1). The results in Section 4.3.2, however, have never been published.

The main body of the manuscript is divided into two parts. Part II is dedicated to the modal logics with Kripke frame-based semantics and Part III to the two-layered logics.

In Chapter 5, we present a modal expansion of \mathbf{biG} on crisp frames denoted \mathbf{KbiG} and construct a strongly complete Hilbert-style calculus for it. We then show that \mathbf{KbiG} is decidable and explore its expressivity and correspondence theory. In particular, we show how the addition of Δ or \prec affects the expressivity of \mathbf{KbiG} in comparison to \mathbf{KG} and study the classes of formulas that define the same classes of frames in \mathbf{K} and \mathbf{KbiG} . This chapter is based on [21].

In Chapter 6, we construct \mathbf{KG}^{2c} — a paraconsistent expansion of \mathbf{KbiG} . We show that \mathbf{KG}^{2c} validity is reducible to \mathbf{KbiG} validity and use this fact to devise a complete axiomatisation of \mathbf{KG}^{2c} and obtain the decidability result. We also construct a simple tableaux calculus for the \mathbf{KG}^{2c} over finitely branching frames and prove an analogue of Glivenko’s theorem. The contents of the chapter have first appeared in [21] (Section 6.2 and the counterpart of the Glivenko’s theorem in Section 6.3) and [20] (Section 6.1 and the tableaux in Section 6.3).

In Chapter 7, we consider paraconsistent Gödel modal logics on bi-relational and fuzzy frames. We consider both the logics with \Box and \Diamond as well as with \blacksquare and \blacklozenge . We investigate their semantical properties and show that the modalities in the pair are not interdefinable. Moreover, we show that $\mathbf{KG}^{2\pm}$ (the logic with \Box and \Diamond) does not extend fuzzy \mathbf{KbiG} and that $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ (the logic with \blacksquare and \blacklozenge) is non-normal but regular. We investigate the definability of different classes of frames. In particular, we show that just as in \mathbf{KbiG} and \mathbf{KG}^{2c} , both fuzzy and crisp finitely branching frames are definable. For both logics over finitely branching frames, we create tableaux systems and use them to prove decidability and provide complexity evaluations. The chapter is built upon [23, 22] (Sections 7.1 and 7.2, respectively).

In Chapter 8, we present two-layered logics $\mathbf{Pr}_{\Delta}^{\mathbf{L}^2}$ and $\mathbf{4Pr}^{\mathbf{L}\Delta}$ that formalise reasoning with the ‘non-standard’¹² and four-valued probabilities proposed in [92], respectively. We provide their axiomatisations and prove (weak) completeness theorems. We also show that these logics can be faithfully embedded into one another, provide decision procedures for them based on constraint tableaux and establish complexity evaluations. Section 8.2 is based upon [26] while Sections 8.3 and 8.4 on [25].

Chapter 9 is dedicated to two-layered logics formalising (both classical and paraconsistent) qualitative reasoning about uncertainty. We present logics \mathbf{QG} , \mathbf{MCB} , and \mathbf{NMCB} that are based

¹²In this manuscript, however, we will call them ‘ \pm -probabilities’ since all non-classical probability theories are not ‘standard’.

on bi-Gödel logic and its paraconsistent expansions $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ and $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$. We construct strongly complete calculi for these logics and study their correspondence theory. The results of the chapter are published in [24].

Finally, in [Conclusion](#), we summarise the results obtained in the dissertation and provide a roadmap for future research.

Part I.

Propositional fragments

2. Preliminaries

In this part of the dissertation, we provide some necessary logical preliminaries to the results discussed afterwards. More precisely, we introduce \mathfrak{L}^2 and \mathbf{G}^2 expansions¹³ of Łukasiewicz and Gödel logics whose semantics is defined in terms of *two* valuations on $[0, 1]$ (Chapters 3 and 4) connected with a De Morgan negation \neg . These valuations — v_1 and v_2 — can be interpreted as support of truth and support of falsity, respectively. We also present the BD logic to which the present chapter is mostly devoted. First, we are going to use BD as the inner layer of some logics in Part III; second, one can interpret \mathfrak{L}^2 and \mathbf{G}^2 as hybrids between \mathfrak{L} and BD on the one hand, and \mathbf{G} and BD on the other.

Remark 2.1 (Logics). Henceforth, we will be using the term ‘logic’ in two senses:

- to designate the set of tautologies or theorems if *we do not provide a strongly complete axiomatisation*;
- to designate the ‘set–formula’ entailment relation, otherwise.

Convention 2.1 (Calculi, validities, and entailments). Given a logic \mathbf{L} , we use

- $\mathcal{L}_{\mathbf{L}}$ to designate its¹⁴ language;
- $\models_{\mathbf{L}}$ for the entailment relation of \mathbf{L} , and $\mathbf{L} \models \phi$ to designate that ϕ is \mathbf{L} -valid;
- \mathcal{HL} for its Hilbert-style axiomatisation and write $\Gamma \vdash_{\mathcal{HL}} \phi$ and $\mathcal{HL} \vdash \chi$ to designate that ϕ is derivable from Γ in \mathcal{HL} and χ is provable without assumptions;
- $\mathcal{T}(\mathbf{L})$ for its tableaux calculus.

There are several equivalent semantics for BD (cf. [118] for the examples). Here, we provide two semantics: the truth-table semantics and the set semantics (or frame semantics) which is a slight generalisation of Dunn’s relational semantics from [56]. We fix a countable set of propositional variables \mathbf{Prop} and define the language of BD via the following grammar.

$$\mathcal{L}_{\mathbf{BD}} \ni \phi := p \in \mathbf{Prop} \mid \neg\phi \mid (\phi \wedge \psi) \mid (\phi \vee \psi)$$

Convention 2.2. Henceforth, we use $\mathbf{Prop}(\phi)$ to stand for the set of variables occurring in ϕ and $\mathbf{Prop}[\Gamma]$ to stand for the set of variables occurring in the set of formulas Γ .

The set of literals is defined as $\mathbf{Lit} = \mathbf{Prop} \cup \{\neg p : p \in \mathbf{Prop}\}$. $\mathbf{Lit}(\phi)$ and $\mathbf{Lit}[\Gamma]$ stand, respectively, for the set of literals occurring in ϕ and Γ . Note that if a variable occurs in ϕ under \neg only, it is counted only once in $\mathbf{Lit}(\phi)$. E.g.,

$$\mathbf{Lit}(p \vee (\neg p \wedge q)) = \{\neg p, p, q\} \qquad \mathbf{Lit}(p \vee (\neg p \wedge \neg q)) = \{\neg p, p, \neg q\}$$

The sets of all subformulas of a given formula ϕ or set of formulas Γ are denoted with $\mathbf{Sf}(\phi)$ and $\mathbf{Sf}[\Gamma]$, respectively.

Definition 2.1 (BD: truth-table semantics). A 4-valuation is a map $v_4 : \mathbf{Prop} \rightarrow \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$ that is extended to complex formulas according to Table 2.1. A sequent $\phi \vdash \chi$ is *valid* iff it holds

¹³As we mentioned in Chapter 1, we introduce two \mathfrak{L}^2 ’s and two \mathbf{G}^2 ’s — with a congruential and with Nelson’s implications.

¹⁴Note that sometimes several logics use the same language. In this case, we will only use the first designation.

2. Preliminaries

\neg		\wedge	T	B	N	F	\vee	T	B	N	F
T	F	T	T	B	N	F	T	T	T	T	T
B	B	B	B	B	F	F	B	T	B	T	B
N	N	N	N	F	N	F	N	T	T	N	N
F	T	F	F	F	F	F	F	T	B	N	F

Table 2.1.: Truth-table semantics of BD.

$w \models^+ p$	iff	$w \in v^+(p)$	$w \models^- p$	iff	$w \in v^-(p)$
$w \models^+ \neg\phi$	iff	$w \models^- \phi$	$w \models^- \neg\phi$	iff	$w \models^+ \phi$
$w \models^+ \phi \wedge \phi'$	iff	$w \models^+ \phi$ and $w \models^+ \phi'$	$w \models^- \phi \wedge \phi'$	iff	$w \models^- \phi$ or $w \models^- \phi'$
$w \models^+ \phi \vee \phi'$	iff	$w \models^+ \phi$ or $w \models^+ \phi'$	$w \models^- \phi \vee \phi'$	iff	$w \models^- \phi$ and $w \models^- \phi'$

Table 2.2.: Truth and falsity conditions of \mathcal{L}_{BD} formulas on sets.

that

$$\forall v_4 : v_4(\phi) \in \{\mathbf{T}, \mathbf{B}\} \Rightarrow v_4(\chi) \in \{\mathbf{T}, \mathbf{B}\}$$

Definition 2.2 (BD: frame semantics). Let $\phi, \phi' \in \mathcal{L}_{\text{BD}}$. For a model $\mathfrak{M} = \langle W, v^+, v^- \rangle$ with $W \neq \emptyset$ and $v^+, v^- : \text{Prop} \rightarrow 2^W$, we define notions of $w \models^+ \phi$ and $w \models^- \phi$ for $w \in W$ as in Table 2.2. We define the *positive* and *negative interpretations* of ϕ as follows:

$$|\phi|^+ = \{w \in W \mid w \models^+ \phi\} \quad |\phi|^- = \{w \in W \mid w \models^- \phi\} \quad (2.1)$$

We say that a sequent $\phi \vdash \chi$ is *satisfied on* \mathfrak{M} (denoted, $\mathfrak{M} \models [\phi \vdash \chi]$) iff $|\phi|^+ \subseteq |\chi|^+$ and $|\chi|^- \subseteq |\phi|^-$. $\phi \vdash \chi$ is *valid* iff it is satisfied on every model. In this case, we say that ϕ *entails* χ and write $\phi \models_{\text{BD}} \chi$.

Convention 2.3. In what follows, when presenting BD-models, we will use the shorthands shown in Table 2.3 to denote the values of variables in states.

Remark 2.2. From Definition 2.1, it is clear that BD is decidable (in fact, its validity is in **coNP** [146, 5]). Furthermore, we will not be using BD by itself *to reason* about anything, its only use will be to describe events about which we reason in outer-layer logics. This is why, even though there exists a multitude of complete calculi (cf., e.g., [68, 57, 139, 124, 140, 138]), when we need to incorporate BD-valid sequents into other calculi, we will compress them into one axiom similar to what is sometimes done when presenting axiomatisations of modal logics.

Example 2.1. Let us clarify which information corresponds to which truth value using the following example. Assume that we read an announcement about a dog being lost by its owner.

A female golden retriever was lost on the 5th of October. The last time I saw her on the 7th of October, she had a wide blue collar made of leather. Any finder is kindly requested to call +33625153633.

notation	meaning
$w : p^+$	$w \models^+ p$ and $w \not\models^- p$
$w : p^-$	$w \not\models^+ p$ and $w \models^- p$
$w : p^\pm$	$w \models^+ p$ and $w \models^- p$
$w : \not p$	$w \not\models^+ p$ and $w \not\models^- p$

Table 2.3.: Notation in the models.

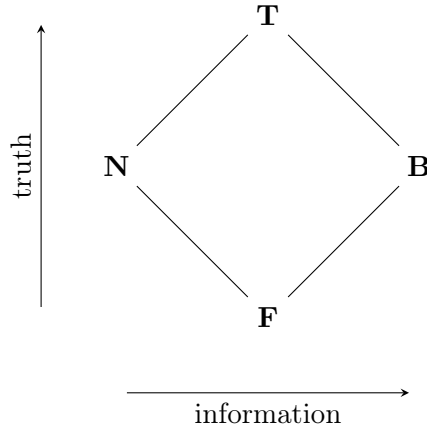


Figure 2.1.: Bi-lattice 4 with two orders: the truth order (upwards) and the information order (left-to-right).

It is clear that it is *true only* that the dog lost was a golden retriever, and *false only* that it was male. However, it is *both true and false* that the owner lost her on the 5th of October: *the announcement contradicts itself* saying that the owner saw the dog for the last time *two days after* the loss. Furthermore, since there is *no information* regarding the location where the dog was lost (or seen last), a statement such as ‘the dog was lost near the city theatre’ would be *neither true nor false*.

The semantics given in Definition 2.2 is suitable when we use BD to describe events since it will be possible to define a measure on W and then reason about measures corresponding to the interpretations of the formulas in the given model.

Definition 2.1 shows a connection between BD and bi-lattices. Indeed, the set of BD truth values is interpreted in [15] as the four information states¹⁵ an agent¹⁶ can have regarding a statement ϕ .

- **T** — ‘only told that ϕ is true’.
- **F** — ‘only told that ϕ is false’.
- **B** — ‘told both that ϕ is true and false’.
- **N** — ‘neither told that ϕ is true nor that it is false’.

The \mathcal{L}_{BD} connectives correspond to the operations w.r.t. the truth (upwards) order on the following bi-lattice (Fig. 2.1). The idea to treat the reasoning about uncertain (and possibly, contradictory) information was later studied further in [70] (where bi-lattices were first comprehensively studied and applied to the reasoning in the AI context) and then developed in the context of bi-lattice logics [127, 90].

In the dissertation, we are going to use BD as the inner-layer logic to describe events (Part III). This is for two reasons. First, as we have mentioned above, the semantics of BD allows for the representation of all kinds of information an agent may have regarding a given event ϕ (namely, that ϕ takes place, that ϕ does not take place, contradictory information, and no information at all — cf. Example 2.1). In addition, a given theory can be simultaneously incomplete and inconsistent (e.g., $\{p, \neg p\}$ is inconsistent and is also incomplete if the language includes q).

¹⁵This is not the only interpretation of the truth values of BD, however. E.g., in [53, 54] **B** and **N** are interpreted in a doxastic manner as ‘contradiction’ and ‘ignorance’ (which leads to some counter-intuitive consequences).

This interpretation, however, was extensively criticised in [152], in particular, for a confusion between degrees of belief and information states.

¹⁶In *How a computer should think* cited above, it was a *computer* (whence, the title of the paper) or a database.

2. Preliminaries

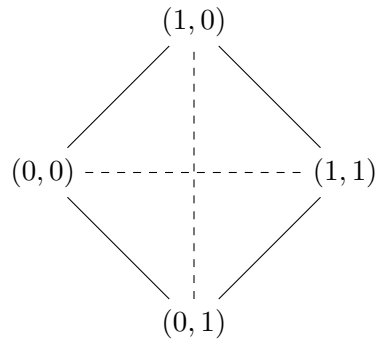


Figure 2.2.: $[0, 1]^{\times}$ with the ‘classical’ and ‘confused’ diagonals.

Second, \mathcal{L}_{BD} contains all the necessary connectives to represent the natural-language sentences one uses to describe events (‘it did not rain yesterday’, ‘Paula has a dog and a cat’, ‘Paula or Quinn came to the birthday party last Saturday’, etc.). Indeed, conditional statements do not usually express events in the natural language, whence the lack of an implication is not an issue.

To model the expansions of \mathbf{L} and \mathbf{G} with an additional paraconsistent negation \neg described in Chapters 3 and 4, we can use a continuous expansion of $\mathbf{4}$ ¹⁷, the bi-lattice $[0, 1]^{\times} = [0, 1] \times [0, 1]^{\text{op}}$ ¹⁸ (Fig. 2.2). Note also that if the first coordinate is interpreted as support of truth and the second support of falsity, then the line from $(1, 0)$ to $(0, 1)$ represents classical information (since supports of truth and falsity sum up to 1), and the horizontal line represents the complete confusion (since the support of truth is equal to the support of falsity).

Convention 2.4 (Notation in $[0, 1]^{\times}$). In what follows, we use the following notational conventions when dealing with $[0, 1]^{\times}$.

- $\leq_{[0,1]^{\times}}$ stands for the upwards order.
- Given $(x, y) \in [0, 1]^{\times}$, we set $(x, y)^{\uparrow} = \{(x', y') : (x, y) \leq_{[0,1]^{\times}} (x', y')\} = \{(x', y') : x \leq x' \text{ and } y \geq y'\}$.

¹⁷An expansion of \mathbf{L} with \neg was proposed in [27] (denoted $\mathbf{L}(\neg)$ there). However, the idea of a paraconsistent expansion of \mathbf{L} (actually, the Rational Pavelka Logic) where a formula ϕ is associated with a pair of numbers (evidence couple) corresponding to the evidence *for* and *against* ϕ had been already devised in [145]. The *values* of the formulas, however, are given as 2×2 matrices. The author is grateful to Lluís Godo Lacasa for the reference.

¹⁸In the context of Nelson’s paraconsistent logics such product construction has been called a twisted product of algebras [147] or a twist structure [112, Chapter 8]. Note, moreover, that $\mathbf{4}$ can itself be represented as a twisted product of two Boolean algebras over $\{0, 1\}$.

3. Paraconsistent expansions of Łukasiewicz logic

To make the text more self-contained, we begin with \mathfrak{L}_Δ , the expansion of \mathfrak{L} with the Baaz' Delta operator.

Definition 3.1. The standard \mathfrak{L}_Δ -algebra is a tuple $\langle [0, 1], \sim_\mathfrak{L}, \Delta_\mathfrak{L}, \wedge_\mathfrak{L}, \vee_\mathfrak{L}, \rightarrow_\mathfrak{L}, \odot_\mathfrak{L}, \oplus_\mathfrak{L}, \ominus_\mathfrak{L} \rangle$ with the operations are defined as follows.

$$\sim_\mathfrak{L} a := 1 - a \qquad \Delta_\mathfrak{L} a := \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{lll} a \wedge_\mathfrak{L} b := \min(a, b) & a \vee_\mathfrak{L} b := \max(a, b) & a \rightarrow_\mathfrak{L} b := \min(1, 1 - a + b) \\ a \odot_\mathfrak{L} b := \max(0, a + b - 1) & a \oplus_\mathfrak{L} b := \min(1, a + b) & a \ominus_\mathfrak{L} b := \max(0, a - b) \end{array}$$

Definition 3.2 (Łukasiewicz logic with Δ). The language of \mathfrak{L}_Δ is given via the following grammar

$$\mathcal{L}_{\mathfrak{L}_\Delta} \ni \phi := p \in \mathbf{Prop} \mid \sim \phi \mid \Delta \phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\phi \odot \phi) \mid (\phi \oplus \phi) \mid (\phi \ominus \phi)$$

We will also write $\phi \leftrightarrow \chi$ as a shorthand for $(\phi \rightarrow \chi) \odot (\chi \rightarrow \phi)$ and use $\mathcal{L}_\mathfrak{L}$ to denote the Δ -free fragment of $\mathcal{L}_{\mathfrak{L}_\Delta}$.

An \mathfrak{L}_Δ valuation is a map $v : \mathbf{Prop} \rightarrow [0, 1]$ that is extended to the complex formulas as expected: $v(\phi \circ \chi) = v(\phi) \circ_\mathfrak{L} v(\chi)$.

ϕ is \mathfrak{L}_Δ -valid iff $v(\phi) = 1$ for every v . Γ entails χ ($\Gamma \models_{\mathfrak{L}_\Delta} \chi$) in \mathfrak{L}_Δ iff there is no valuation v s.t. $v(\phi) = 1$ for every $\phi \in \Gamma$ but $v(\chi) \neq 1$.

Remark 3.1. Note that \sim and \rightarrow can be used to define all other connectives and constants as follows.

$$\begin{array}{lll} \phi \vee \chi := (\phi \rightarrow \chi) \rightarrow \chi & \phi \wedge \chi := \sim(\sim\phi \vee \sim\chi) & \phi \oplus \chi := \sim\phi \rightarrow \chi \\ \phi \odot \chi := \sim(\phi \rightarrow \sim\chi) & \phi \ominus \chi := \phi \odot \sim\chi & \mathbf{1} := p \rightarrow p \\ \mathbf{0} := \sim\mathbf{1} \end{array}$$

Remark 3.2. Observe from the definition of \leftrightarrow (Definitions 3.1 and 3.2) that $v(\phi_1 \leftrightarrow \phi_2) = 1 - |v(\phi_1) - v(\phi_2)|$. Moreover, $v(\psi_1 \leftrightarrow \psi_2)$ is the *complement to the distance* between $v(\psi_1)$ and $v(\psi_2)$ on $[0, 1]$.

Convention 3.1. Given a set of formulas Γ and a valuation v , we use $v[\Gamma] = x$ to denote ‘ $\inf\{v(\phi) : \phi \in \Gamma\} = x$ ’.

We construct $\mathcal{H}\mathfrak{L}_\Delta$ the Hilbert-style calculus for \mathfrak{L}_Δ by adding Δ axioms and rules from [9], [83, Definition 2.4.5], or [36, Chapter I,2.2.1] to the Hilbert-style calculus for \mathfrak{L} from [106, §6.2].

Definition 3.3 ($\mathcal{H}\mathfrak{L}$ — the Hilbert-style calculus for \mathfrak{L}). The calculus contains the following axioms and rules.

w: $\phi \rightarrow (\chi \rightarrow \phi)$.

3. Paraconsistent expansions of Łukasiewicz logic

sf: $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi))$.

waj: $((\phi \rightarrow \chi) \rightarrow \chi) \rightarrow ((\chi \rightarrow \phi) \rightarrow \phi)$.

co: $(\sim\chi \rightarrow \sim\phi) \rightarrow (\phi \rightarrow \chi)$.

MP: $\frac{\phi \quad \phi \rightarrow \chi}{\chi}$.

Definition 3.4 (Δ axioms).

$\Delta 1$: $\Delta\phi \vee \sim\Delta\phi$.

$\Delta 2$: $\Delta\phi \rightarrow \phi$.

$\Delta 3$: $\Delta\phi \rightarrow \Delta\Delta\phi$.

$\Delta 4$: $\Delta(\phi \vee \chi) \rightarrow \Delta\phi \vee \Delta\chi$.

$\Delta 5$: $\Delta(\phi \rightarrow \chi) \rightarrow (\Delta\phi \rightarrow \Delta\chi)$.

$\Delta \text{ nec}$: $\frac{\phi}{\Delta\phi}$.

We also recall the following property of Δ .

Proposition 3.1 (Δ deduction theorem). *Let $\Gamma \subseteq \mathcal{L}_{\mathfrak{k}_\Delta}$ be finite. Then*

$$\Gamma, \phi \vdash_{\mathcal{H}\mathfrak{k}_\Delta} \chi \text{ iff } \Gamma \vdash_{\mathcal{H}\mathfrak{k}_\Delta} \Delta\phi \rightarrow \chi$$

Łukasiewicz logic is known to lack compactness [83, Remark 3.2.14], whence, $\mathcal{H}\mathfrak{k}_\Delta$ is only *finitely strongly complete*.

Proposition 3.2 (Finite strong completeness of $\mathcal{H}\mathfrak{k}_\Delta$). *Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathfrak{k}_\Delta}$ be finite. Then*

$$\Gamma \models_{\mathfrak{k}_\Delta} \phi \text{ iff } \Gamma \vdash_{\mathcal{H}\mathfrak{k}_\Delta} \phi$$

3.1. Semantics and axiomatisation

In this section, we are going to define two expansions of \mathfrak{k} with \neg : $\mathfrak{k}_{(\Delta, \rightarrow)}^2$ and $\mathfrak{k}_{(\rightarrow)}^2$ that we collectively denote with \mathfrak{k}^2 .

Definition 3.5 ($\mathfrak{k}_{(\Delta, \rightarrow)}^2$: language and semantics). The language is constructed using the following grammar.

$$\mathcal{L}_{\mathfrak{k}_{(\Delta, \rightarrow)}^2} \ni \phi := p \in \mathbf{Prop} \mid \neg\phi \mid \sim\phi \mid \Delta\phi \mid (\phi \rightarrow \phi)$$

The semantics is given by *two* $\mathfrak{k}_{(\Delta, \rightarrow)}^2$ valuations v_1 (support of truth) and v_2 (support of falsity) $v_1, v_2 : \mathbf{Prop} \rightarrow [0, 1]$ that are extended as follows (other connectives can be introduced as in Definition 3.2).

$$\begin{array}{ll} v_1(\neg\phi) = v_2(\phi) & v_2(\neg\phi) = v_1(\phi) \\ v_1(\sim\phi) = \sim_{\mathfrak{k}} v_1(\phi) & v_2(\sim\phi) = \sim_{\mathfrak{k}} v_2(\phi) \\ v_1(\Delta\phi) = \Delta_{\mathfrak{k}} v_1(\phi) & v_2(\Delta\phi) = \sim_{\mathfrak{k}} \Delta_{\mathfrak{k}} \sim_{\mathfrak{k}} v_2(\phi) \\ v_1(\phi \rightarrow \chi) = v_1(\phi) \rightarrow_{\mathfrak{k}} v_1(\chi) & v_2(\phi \rightarrow \chi) = v_2(\chi) \ominus_{\mathfrak{k}} v_2(\phi) \end{array}$$

We say that ϕ is \mathfrak{k}_{Δ}^2 -*valid* iff for every v_1 and v_2 , it holds that $v_1(\phi) = 1$ and $v_2(\phi) = 0$. Γ *entails* χ ($\Gamma \models_{\mathfrak{k}_{(\Delta, \rightarrow)}^2} \chi$) iff there are no v_1 and v_2 s.t. $v_1(\phi) = 1$ and $v_2(\phi) = 0$ for every $\phi \in \Gamma$ but $v_1(\chi) \neq 1$ or $v_2(\chi) \neq 0$.

Definition 3.6 ($\mathfrak{L}_{(\rightarrow)}^2$: language and semantics). The language is constructed via the following grammar.

$$\mathcal{L}_{\mathfrak{L}_{(\rightarrow)}^2} \ni \phi := p \mid \sim\phi \mid \neg\phi \mid (\phi \wedge \psi) \mid (\phi \rightarrow \psi)$$

The support of truth and support of falsity conditions are given by the following extensions of $v_1, v_2 : \mathbf{Prop} \rightarrow [0, 1]$ ($\mathfrak{L}_{(\rightarrow)}^2$ valuations) to the complex formulas.

$$\begin{array}{ll} v_1(\neg\phi) = v_2(\phi) & v_2(\neg\phi) = v_1(\phi) \\ v_1(\sim\phi) = \sim_{\mathfrak{L}} v_1(\phi) & v_2(\sim\phi) = v_1(\phi) \\ v_1(\phi \wedge \chi) = v_1(\phi) \wedge_{\mathfrak{L}} v_1(\chi) & v_2(\phi \wedge \chi) = v_2(\phi) \vee_{\mathfrak{L}} v_2(\chi) \\ v_1(\phi \rightarrow \chi) = v_1(\phi) \rightarrow_{\mathfrak{L}} v_1(\chi) & v_2(\phi \rightarrow \chi) = v_1(\phi) \odot_{\mathfrak{L}} v_2(\chi) \end{array}$$

Other connectives can be introduced as follows.

$$\begin{array}{ll} \phi \odot \chi := \sim(\phi \rightarrow \sim\chi) & \phi \Rightarrow \chi := (\phi \rightarrow \chi) \wedge (\neg\chi \rightarrow \neg\phi) \\ \phi \leftrightarrow \chi := (\phi \rightarrow \chi) \wedge (\chi \rightarrow \phi) & \phi \Leftrightarrow \chi := (\phi \leftrightarrow \chi) \wedge (\neg\phi \leftrightarrow \neg\chi) \\ \phi \oplus \chi := \sim\phi \rightarrow \chi & \phi \ominus \chi := \sim(\phi \rightarrow \chi) \\ \phi \vee \chi := \neg(\neg\phi \wedge \neg\chi) & \end{array}$$

We say that ϕ is $\mathfrak{L}_{(\rightarrow)}^2$ -valid iff $v_1(\phi) = 1$ for every v_1 . Γ entails χ ($\Gamma \models_{\mathfrak{L}_{(\rightarrow)}^2} \chi$) iff there is no v_1 s.t. $v_1(\phi) = 1$ for every $\phi \in \Gamma$ and $v_1(\chi) \neq 1$.

Remark 3.3 (Some notes about \mathfrak{L}^2 semantics). We note, first of all, that it is possible to define semantical conditions of \mathfrak{L}^2 connectives using one valuation $v = (v_1, v_2)$ on $[0, 1]^{\times}$ (this is why, we will further use $v(\phi) = (x, y)$ as a shorthand for ‘ $v_1(\phi) = x$ and $v_2(\phi) = y$ ’). This way \wedge and \vee will be the meet and join of $[0, 1]^{\times}$, and \neg will be the symmetry over the horizontal line (recall Fig. 2.2). The Łukasiewicz negation \sim will be the symmetry over $(0.5, 0.5)$ in $\mathfrak{L}_{(\Delta, \rightarrow)}^2$; in $\mathfrak{L}_{(\rightarrow)}^2$, however, $v(\sim\phi)$ is always on the classical line.

Moreover, there is an important distinction between \rightarrow and \Rightarrow . Namely, \Rightarrow is a weak implication in the sense that if $\phi \rightarrow \chi$ is designated, then we only know that $v_1(\phi) \leq v_1(\chi)$ (and, accordingly, $\phi \leftrightarrow \chi$ is designated iff $v_1(\phi) = v_1(\chi)$). This aligns with the Nelsonian interpretation of the implication whose positive support is defined intuitionistically (i.e., there must be a recursive function that transforms every realisation of ϕ into a realisation of χ) and negative support classically (ϕ should be realised positively and χ negatively).

This means that \Rightarrow preserves $(1, 1)$: $v(p \Rightarrow p) = (1, 1)$ if $v(p) = (1, 1)$. On the other hand, \rightarrow is strong (or congruential on $[0, 1]^{\times}$) since $v(\phi \rightarrow \chi) = (1, 0)$ iff $v_1(\phi) \leq v_1(\chi)$ and $v_2(\phi) \geq v_2(\chi)$ (i.e., iff $v(\phi) \leq_{[0, 1]^{\times}} v(\chi)$). This is why, we define the *congruential Nelsonian implication* \Rightarrow : $v_1(\phi \Rightarrow \chi) = 1$ iff $v(\phi) \leq_{[0, 1]^{\times}} v(\chi)$.

Remark 3.4 (Paraconsistency of \mathfrak{L}^2). We are calling \mathfrak{L}^2 's *paraconsistent counterparts* of \mathfrak{L} . What we mean here is that neither $p \wedge \neg p \rightarrow q$ nor $p \wedge \neg p \Rightarrow q$ is \mathfrak{L}^2 -valid. On the other hand, only the entailment of $\mathfrak{L}_{(\rightarrow)}^2$ is not explosive: $p \wedge \neg p \models_{\mathfrak{L}_{(\Delta, \rightarrow)}^2} q$ since there is no valuation s.t. $v(p \wedge \neg p) = (1, 0)$. At the end of Section 3 (cf. Remark 3.9), we will briefly discuss how to make $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ entailment paraconsistent and how to axiomatise it.

One can notice, however, that the Łukasiewicz negation \sim is itself paraconsistent in the above sense. Indeed, $p \wedge \sim p \rightarrow q$ is not \mathfrak{L} -valid (although it is the case that $p \wedge \sim p \models_{\mathfrak{L}} q$). Why did we then introduce another negation to \mathfrak{L}^2 's? First of all, in our approach, we assume that the agents reason only with the information provided by their sources. In this sense, if all sources give a contradictory account regarding p (i.e., claim that p is both true and false), the agent should consider p both true and false; and if there is no account on whether q is true, then the agent considers it neither true nor false. I.e., the agent cannot infer that q is true or false if they don't have any information about q 's truth or falsity. This situation, however, cannot be modelled in \mathfrak{L} since $(p \wedge \sim p) \rightarrow (q \vee \sim q)$ is \mathfrak{L} -valid.

3. Paraconsistent expansions of Łukasiewicz logic

The following statement shows that \mathfrak{L}^2 's extend \mathfrak{L} as one would expect.

Proposition 3.3.

1. Let $\phi \in \mathcal{L}_{\mathfrak{L}^2_{(\Delta, \rightarrow)}}$. Then, ϕ is $\mathfrak{L}^2_{(\Delta, \rightarrow)}$ -valid iff $v_1(\phi) = 1$ for every v_1 .
2. Let $\phi \in \mathcal{L}_{\mathfrak{L}}$. Then, ϕ is \mathfrak{L} -valid iff ϕ is $\mathfrak{L}^2_{(\rightarrow)}$ -valid.

Proof. Since $\mathfrak{L}^2_{(\rightarrow)}$ validity only takes into account v_1 and since ϕ is \neg -free, 2. follows immediately from the fact that v_1 semantical conditions coincide with \mathfrak{L} semantical conditions. Let us now tackle 1.

For 1., we proceed as follows. Let $v = (v_1, v_2)$ be a couple of $\mathfrak{L}^2_{(\Delta, \rightarrow)}$ valuations. We define $v^*(p) = (1 - v_2(p), 1 - v_1(p))$. It now suffices to prove that we have $v^*(\phi) = (1 - v_2(\phi), 1 - v_1(\phi))$ for every $\phi \in \mathfrak{L}_{\Delta}$. Indeed, if ϕ is not $\mathfrak{L}^2_{(\Delta, \rightarrow)}$ -valid, then either $v_1(\phi) \neq 1$ or $v_2(\phi) \neq 0$ for some v_1 and v_2 . In the first case, we have the result immediately since ϕ is \neg -free, in the second case, we apply the statement.

Notice that $v(\phi) = (x', y')$ iff $v(\neg\sim\phi) = (1 - y', 1 - x')$ and $v(\psi_1 \leftrightarrow \psi_2) = (1, 0)$ iff $v(\psi_1) = v(\psi_2)$. Moreover, it is easy to establish that

$$\begin{aligned} v(\neg\sim\neg\psi \leftrightarrow \neg\neg\sim\psi) &= (1, 0) \\ v(\neg\sim\sim\psi \leftrightarrow \sim\neg\sim\psi) &= (1, 0) \\ v(\neg\sim\Delta\psi \leftrightarrow \Delta\neg\sim\psi) &= (1, 0) \\ v(\neg\sim(\psi_1 \rightarrow \psi_2) \leftrightarrow (\neg\sim\psi_1 \rightarrow \neg\sim\psi_2)) &= (1, 0) \end{aligned}$$

We have thus shown that $\neg\sim$ can be pushed to the variables preserving the equivalence. Since $v(\neg\sim p) = v^*(p)$, the result follows. \square

Remark 3.5. The composition $\neg\sim$ (or, equivalently, $\sim\neg$) of negations can be thought of as a symmetry *across the vertical (classical) line*. It can be understood as an analogue of conflation on **4** (cf., e.g., [65, 117] for further details) — an involutive negation w.r.t. informational (left-to-right) order.

Observe that

$$v(\Delta\phi) = \begin{cases} (1, 0) & \text{if } v(\phi) = (1, 0) \\ (0, 1) & \text{otherwise} \end{cases}$$

does not hold in $\mathfrak{L}^2_{(\Delta, \rightarrow)}$. Indeed, if $v(\phi) = (0, 0)$, then $v(\Delta\phi) = (0, 0)$; further, $v(\phi) = v(\Delta\phi)$ if $v(\phi) \in \{(1, 0); (1, 1); (0, 0); (0, 1)\}$. However, we can define Δ^\top as follows

$$\Delta^\top\phi := \Delta\phi \wedge \Delta\sim\neg\phi \tag{3.1}$$

It is clear that

$$v(\Delta^\top\phi) := \begin{cases} (1, 0) & \text{if } v(\phi) = (1, 0) \\ (0, 1) & \text{otherwise} \end{cases} \tag{3.2}$$

Proposition 3.4. Let $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathfrak{L}^2_{(\Delta, \rightarrow)}}$ and $\Gamma^{\Delta^\top} = \{\Delta^\top\phi : \phi \in \Gamma\}$. Then

1. $\Gamma \models_{\mathfrak{L}^2_{(\Delta, \rightarrow)}} \chi$ iff $\Gamma^{\Delta^\top} \models_{\mathfrak{L}^2_{(\Delta, \rightarrow)}} \chi$;
2. $\Gamma^{\Delta^\top} \models_{\mathfrak{L}^2_{(\Delta, \rightarrow)}} \chi$ iff there is no v_1 s.t. $v_1(\Delta^\top\phi) = 1$ for every $\Delta^\top\phi \in \Gamma^{\Delta^\top}$ and $v_1(\chi) \neq 1$.

Proof. 1. follows immediately from (3.2) and Definition 3.5. Let us now consider 2.

Let $\Gamma^{\Delta^\top} \not\vdash_{\mathfrak{L}_{(\Delta, \rightarrow)}^2} \chi$. Then, there is a valuation v s.t. $v[\Gamma^{\Delta^\top}] = (1, 0)$ but $v(\chi) \neq (1, 0)$. If $v_1(\chi) \neq 1$, we are done. If $v_2(\chi) \neq 0$, we apply Proposition 3.3 and have that $v_1^*(\chi) \neq 1$ but $v_1^*[\Gamma^{\Delta^\top}] = 1$, as required. For the converse, let there be v_1 s.t. $v_1(\chi) \neq 1$ but $v_1[\Gamma^{\Delta^\top}] = 1$. By (3.2), we have that $v_2[\Gamma^{\Delta^\top}] = 0$. Thus, $v[\Gamma^{\Delta^\top}] = (1, 0)$ but $v(\chi) \neq (1, 0)$, as required. \square

We are now ready to provide the Hilbert-style axiomatisations and establish their (weak) completeness.

3.1.1. Axiomatisation of $\mathfrak{L}_{(\Delta, \rightarrow)}^2$

Definition 3.7 ($\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2$ — Hilbert-style calculus for $\mathfrak{L}_{(\Delta, \rightarrow)}^2$). The calculus expands $\mathcal{H}\mathfrak{L}_\Delta$ (cf. Definitions 3.3 and 3.4) with the following axioms and rules.

$$\neg\neg: \neg\neg\phi \leftrightarrow \phi.$$

$$\neg\sim: \neg\sim\phi \leftrightarrow \sim\neg\phi.$$

$$\sim\neg\rightarrow: (\sim\neg\phi \rightarrow \sim\neg\chi) \leftrightarrow \sim\neg(\phi \rightarrow \chi).$$

$$\neg\Delta: \neg\Delta\phi \leftrightarrow \sim\Delta\sim\neg\phi.$$

$$\text{conf: } \frac{\phi}{\sim\neg\phi}.$$

To prove the completeness of $\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2$, we will reduce the proofs therein to the $\mathcal{H}\mathfrak{L}_\Delta$ proofs. To do that, we first observe that we can push \neg 's to variables using $\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2$ and thus obtain negation normal forms (NNF's) w.r.t. \neg .

Lemma 3.1. *Let $\phi \in \mathcal{L}_{\mathfrak{L}_{(\Delta, \rightarrow)}^2}$. Then there exists ϕ^\neg where all occurrences of \neg are applied to variables only s.t.*

$$1. v(\phi) = v(\phi^\neg) \text{ for every } v;$$

$$2. \mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2 \vdash \phi \leftrightarrow \phi^\neg.$$

Proof. We begin with 1. To obtain ϕ^\neg , we take ϕ and apply the following reductions to each subformula $\neg\psi$ where $\psi \notin \text{Prop}$.

$$\neg\neg\psi \Rightarrow \psi; \quad \neg\sim\psi \Rightarrow \sim\neg\psi; \quad \neg\Delta\psi \Rightarrow \sim\Delta\sim\neg\psi; \quad \neg(\psi \rightarrow \psi') \Rightarrow \sim(\sim\neg\psi \rightarrow \sim\neg\psi')$$

It is clear that \Rightarrow defined above preserves the value of any formula. Thus, 1. is proven.

To prove 2., it suffices to show that this transformation is provably equivalent. The first three cases are instances of $\neg\neg$, $\neg\sim$, and $\neg\Delta$ axioms. To prove the last one, we proceed as follows.

$$\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2 \vdash \neg(\psi \rightarrow \psi') \leftrightarrow \sim\sim\neg(\psi \rightarrow \psi') \quad (\text{provable in } \mathcal{H}\mathfrak{L}_\Delta)$$

$$\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2 \vdash \sim\sim\neg(\psi \rightarrow \psi') \leftrightarrow \sim(\sim\neg\psi \rightarrow \sim\neg\psi') \quad (\text{using } \sim\neg\rightarrow)$$

$$\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2 \vdash \neg(\psi \rightarrow \psi') \leftrightarrow \sim(\sim\neg\psi \rightarrow \sim\neg\psi')$$

\square

Lemma 3.2. *For every finite $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathfrak{L}_{(\Delta, \rightarrow)}^2}$, it holds that*

$$\Gamma \vdash_{\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2} \chi \text{ iff } \Gamma^{\Delta^\top} \vdash_{\mathcal{H}\mathfrak{L}_{(\Delta, \rightarrow)}^2} \chi$$

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Proof. Immediately since having ϕ , one obtains $\Delta^\top \phi$ by applications of **conf**, Δ **nec**, and the definition of \wedge . Conversely, having $\Delta^\top \phi$, one can derive ϕ using the definition of \wedge (Remark 3.1) and $\Delta 2$ (Definition 3.4). \square

Definition 3.8. Let $\phi \in \mathcal{L}_{\mathbf{k}_{(\Delta, \rightarrow)}^2}$ be in \neg negation normal form. We define ϕ^* to be the result of the replacement of each $\neg p$ occurring in ϕ with a new variable p^* .

Lemma 3.3. For every $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathbf{k}_{(\Delta, \rightarrow)}^2}$, it holds that

$$\Gamma^{\Delta^\top} \models_{\mathbf{k}_{(\Delta, \rightarrow)}^2} \chi \text{ iff } \left(\left(\Gamma^{\Delta^\top} \right)^\neg \right)^* \models_{\mathbf{k}_\Delta} (\chi^\neg)^*$$

where $\left(\left(\Gamma^{\Delta^\top} \right)^\neg \right)^* = \left\{ \left(\left(\phi^{\Delta^\top} \right)^\neg \right)^* : \phi \in \Gamma \right\}$.

Proof. We reason by contraposition. Assume that $\Gamma^{\Delta^\top} \not\models_{\mathbf{k}_{(\Delta, \rightarrow)}^2} \chi$. By Proposition 3.4, we can assume w.l.o.g. that $v_1 \left[\Gamma^{\Delta^\top} \right] = 1$ but $v_1(\chi) \neq 1$. Now, transform all formulas into \neg NNF's. Clearly, $v_1 \left[\left(\Gamma^{\Delta^\top} \right)^\neg \right] = 1$ and $v_1(\chi^\neg) \neq 1$. It remains to obtain the falsifying valuation for $\mathcal{L}_{\mathbf{k}_\Delta}$ formulas. To do this, we let $v(p) = v_1(p)$ and $v(p^*) = v_1(\neg p)$ for every literal. Since v_1 semantical conditions and semantical conditions in \mathbf{k}_Δ coincide, the result follows.

The converse direction follows by Definitions 3.2 and 3.5 since the semantical conditions of the support of the truth of $\mathbf{k}_{(\Delta, \rightarrow)}^2$ -formulas coincide with those of \mathbf{k}_Δ -formulas. \square

Theorem 3.1 ($\mathcal{H}\mathbf{k}_{(\Delta, \rightarrow)}^2$ completeness). Let $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathbf{k}_{(\Delta, \rightarrow)}^2}$ be finite. Then

$$\Gamma \models_{\mathbf{k}_{(\Delta, \rightarrow)}^2} \chi \text{ iff } \Gamma \vdash_{\mathcal{H}\mathbf{k}_{(\Delta, \rightarrow)}^2} \chi$$

Proof. For the soundness part, we need to check that the axioms are valid and rules preserve the designated value. The proper \mathbf{k}_Δ axioms and rules are valid by Propositions 3.3 and 3.4. The validity of the axioms (except for $\sim \neg \rightarrow$) governing \neg is shown in Lemma 3.1.

We show that $\sim \neg \rightarrow$ is valid here. Take any v ; we have:

$$\begin{aligned} v(\sim \neg(\phi \rightarrow \chi)) &= (v_1(\sim \neg(\phi \rightarrow \chi)), v_2(\sim \neg(\phi \rightarrow \chi))) \\ &= (\sim_{\mathbf{k}} v_1(\neg(\phi \rightarrow \chi)), \sim_{\mathbf{k}} v_2(\neg(\phi \rightarrow \chi))) \\ &= (\sim_{\mathbf{k}} v_2(\phi \rightarrow \chi), \sim_{\mathbf{k}} v_1(\phi \rightarrow \chi)) \\ &= ((v_2(\chi) \rightarrow_{\mathbf{k}} v_2(\phi)), \sim_{\mathbf{k}}(v_1(\phi) \rightarrow_{\mathbf{k}} v_1(\chi))) \end{aligned}$$

and

$$\begin{aligned} v(\sim \neg \phi \rightarrow \sim \neg \chi) &= (v_1(\sim \neg \phi \rightarrow \sim \neg \chi), v_2(\sim \neg \phi \rightarrow \sim \neg \chi)) \\ &= (\sim_{\mathbf{k}} v_2(\phi) \rightarrow_{\mathbf{k}} \sim_{\mathbf{k}} v_2(\chi), v_1(\phi) \odot_{\mathbf{k}} \sim_{\mathbf{k}} v_1(\chi)) \\ &= ((v_2(\chi) \rightarrow_{\mathbf{k}} v_2(\phi)), \sim_{\mathbf{k}}(v_1(\phi) \rightarrow_{\mathbf{k}} v_1(\chi))) \end{aligned}$$

Finally, it is clear that if $v(\phi) = (1, 0)$, then $v(\sim \neg \phi) = (1, 0)$.

For completeness, we reason as follows. Let $\Gamma \models_{\mathbf{k}_{(\Delta, \rightarrow)}^2} \chi$. This is equivalent to $\Gamma^{\Delta^\top} \models_{\mathbf{k}_{(\Delta, \rightarrow)}^2} \chi$ since $\models_{\mathbf{k}_{(\Delta, \rightarrow)}^2}$ is defined via the preservation of $(1, 0)$ (recall Definition 3.5) and Δ^\top is $(1, 0)$ -detecting.

From here, by Lemma 3.3, we have that $\left(\left(\Gamma^{\Delta^\top} \right)^\neg \right)^* \models_{\mathbf{k}_\Delta} (\chi^\neg)^*$. Hence, $\left(\left(\Gamma^{\Delta^\top} \right)^\neg \right)^* \vdash_{\mathcal{H}\mathbf{k}_\Delta} (\chi^\neg)^*$ by Proposition 3.2. It remains to take this $\mathcal{H}\mathbf{k}_\Delta$ derivation and apply Lemmas 3.1 and 3.2 to recover the derivation of χ from Γ . \square

3.1.2. Axiomatisation of $\mathfrak{L}_{(\rightarrow)}^2$

Definition 3.9 ($\mathcal{H}\mathfrak{L}_{(\rightarrow)}^2$ — Hilbert-style calculus for $\mathfrak{L}_{(\rightarrow)}^2$). We replace all occurrences of \rightarrow with \rightarrow 's and expand $\mathcal{H}\mathfrak{L}$ (cf. Definition 3.3) with the following axioms.

$$\neg\neg: \neg\neg\phi \Leftrightarrow \phi.$$

$$\neg\rightarrow: \neg(\phi \rightarrow \chi) \Leftrightarrow (\phi \odot \neg\chi).$$

$$\neg\sim: \neg\sim\phi \Leftrightarrow \phi.$$

$$\text{DeM: } \neg(\phi \vee \chi) \Leftrightarrow (\neg\phi \wedge \neg\chi); \neg(\phi \wedge \chi) \Leftrightarrow (\neg\phi \vee \neg\chi).$$

$$\odot \Leftrightarrow: (\phi \odot \chi) \Leftrightarrow \sim(\phi \rightarrow \sim\chi).$$

$$\wedge \Leftrightarrow: (\phi \wedge \chi) \Leftrightarrow ((\phi \rightarrow \chi) \odot \phi).$$

$$\vee \Leftrightarrow: ((\phi \rightarrow \chi) \rightarrow \chi) \Leftrightarrow \phi \vee \chi.$$

Remark 3.6. Observe that even though all $\mathcal{H}\mathfrak{L}$ theorems are inherited by $\mathcal{H}\mathfrak{L}_{(\rightarrow)}^2$, we cannot always use \Rightarrow where \rightarrow was originally in place because \rightarrow , in a sense, ‘forgets’ the support of falsity. In particular, it is easy to see that $\neg(\phi \wedge \chi) \Leftrightarrow \neg((\phi \rightarrow \chi) \odot \phi)$ is not $\mathfrak{L}_{(\rightarrow)}^2$ -valid. Furthermore, we need to add $\wedge \Leftrightarrow$ since in contrast to \mathfrak{L} , \wedge is no longer a definable connective, whence, we cannot really use $\mathcal{H}\mathfrak{L}$ -derivability to obtain it.

Again, to prove the completeness of $\mathcal{H}\mathfrak{L}_{(\rightarrow)}^2$, we reduce its proofs to proofs in $\mathcal{H}\mathfrak{L}$. This time, however, we use the weak equivalence (\Leftrightarrow) since $\mathfrak{L}_{(\rightarrow)}^2$ validity depends only on v_1 .

The next statement can be proven in the same way as Lemma 3.1.

Lemma 3.4. *Let $\phi \in \mathcal{L}_{\mathfrak{L}_{(\rightarrow)}^2}$. Then there exists ϕ^\neg where all occurrences of \neg are applied to variables only s.t.*

$$1. v_1(\phi) = v_1(\phi^\neg) \text{ for every } v;$$

$$2. \mathcal{H}\mathfrak{L}_{(\rightarrow)}^2 \vdash \phi \Leftrightarrow \phi^\neg.$$

Proof. To prove 1., we apply the following reductions to every subformula of ϕ .

$$\begin{array}{ll} \neg\neg\psi \Rightarrow \psi & \neg(\psi \rightarrow \psi') \Rightarrow \sim(\psi \rightarrow \sim\neg\psi') \\ \neg\sim\psi \Rightarrow \psi & \neg(\psi \wedge \psi') \Rightarrow ((\neg\psi \rightarrow \neg\psi') \rightarrow \neg\psi') \end{array}$$

Note that these reductions preserve the values of v_1 . To prove 2., we observe that these reductions are provable in $\mathcal{H}\mathfrak{L}_{(\rightarrow)}^2$. \square

We can now use the negation normal forms from the previous lemma and apply Definition 3.8 to them.

Lemma 3.5. *Let $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathfrak{L}_{(\rightarrow)}^2}$. Then, it holds that*

$$\Gamma \models_{\mathfrak{L}_{(\rightarrow)}^2} \chi \text{ iff } (\Gamma^\neg)^* \models_{\mathfrak{L}} (\chi^\neg)^*$$

Proof. Assume that $\Gamma \not\models_{\mathfrak{L}_{(\rightarrow)}^2} \chi$. Then, there is v_1 s.t. $v_1[\Gamma] = 1$ but $v_1(\chi) \neq 1$. From Lemma 3.4, it is clear that $v_1[\Gamma^\neg] = 1$ and $v_1(\chi^\neg) \neq 1$. We obtain the falsifying valuation for \mathfrak{L} as follows. For ‘old’ p 's, we set $v(p) = v_1(p)$; for the new p^* 's, we set $v(p^*) = v_1(\neg p)$. Since the semantics of \mathfrak{L} coincides with v_1 conditions of $\mathfrak{L}_{(\rightarrow)}^2$, it follows that $v[(\Gamma^\neg)^*] = 1$ but $v((\chi^\neg)^*) \neq 1$, as required. The converse direction can be proved in the same manner. \square

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Theorem 3.2. *Let $\Gamma \cup \{\chi\} \subseteq \mathcal{L}_{\mathbf{k}^2(\rightarrow)}$ be finite. Then*

$$\Gamma \models_{\mathbf{k}^2(\rightarrow)} \chi \text{ iff } \Gamma \vdash_{\mathcal{H}\mathbf{k}^2(\rightarrow)} \chi$$

Proof. The soundness part can be established by a routine check of the axioms. For completeness, we proceed as follows. Let $\Gamma \models_{\mathbf{k}^2(\rightarrow)} \chi$. By Lemma 3.5, it is equivalent to $(\Gamma^\neg)^* \models_{\mathbf{k}} (\chi^\neg)^*$. Thus, by the completeness of $\mathcal{H}\mathbf{k}$, we have that $(\Gamma^\neg)^* \vdash_{\mathcal{H}\mathbf{k}} (\chi^\neg)^*$. We take this $\mathcal{H}\mathbf{k}$ proof and replace p^* 's with $\neg p$'s and then use Lemma 3.4 to transform the negation normal forms back into the original formulas. \square

3.2. Tableaux and complexity

In this section, we present a complete tableaux calculus for \mathbf{k}^2 . To do this, we expand the calculus presented in [77, 78, 79, 81, 80, 82] with additional rules for the new connectives. The main idea behind the constraint tableaux is to label every formula not with one value as is usually done in the ‘traditional’ tableaux calculi but with a set of values. In the case of \mathbf{k}^2 (and \mathbf{G}^2 , cf. Section 4.2), we need two sorts of labels: corresponding to v_1 and v_2 . This idea comes from the tableaux for BD presented in [1]. Thus, just as \mathbf{k}^2 and \mathbf{G}^2 are hybrids of \mathbf{k} and \mathbf{G} with BD, so are their proof systems.

The notion of the *branch closure* is then reinterpreted accordingly. While in a traditional tableau, a branch is closed when it contains a formula *with two different labels* (i.e., a formula that is asserted to have two different values), in a constraint tableau, branches are closed when there is a formula labelled *with two disjoint sets of values*.

Let us now give a general definition of a constraint tableaux that we will then adapt to different logics.

Definition 3.10 (Constraint tableaux). Let **Label** be a set of labels and \mathcal{L} a set of formulas. A *constraint* is one of these three expressions:

- *labelled formulas* of the form $L : \phi$ with $L \in \mathbf{Label}$ and $\phi \in \mathcal{L}$;
- *numerical constraints* of the form $c \leq d$ or $c < d$ with c and d ranging over $[0, 1]$;
- *formulaic constraints* of the form $L : \phi \leq L' : \phi'$ or $L : \phi < L' : \phi'$ with $L, L' \in \mathbf{Label}$ and $\phi, \phi' \in \mathcal{L}$.

A *constraint tableau* is a downward branching tree each branch of which is a non-empty set of constraints. Each branch \mathcal{B} can be extended by applications of a given set of rules. \mathcal{B} is *complete* iff for every premise of a rule occurring thereon, one¹⁹ of the conclusions is also on \mathcal{B} .

As expected, in labelled formulas, L is some *set of values*. Thus, the intended interpretation of $L : \phi$ is ‘ ϕ has some value from L ’. In formulaic constraints, L and L' are *components of ϕ 's valuation*. Hence, the intended interpretation of $L : \phi \leq L' : \phi'$ is ‘the component of ϕ 's valuation denoted by L is less or equal to the component of ϕ' 's valuation denoted by L' ’.

Let us now present the constraint tableaux for \mathbf{k}^2 .

Definition 3.11 (Constraint tableau for \mathbf{k}^2 — $\mathcal{T}(\mathbf{k}^2)$). Branches contain *labelled formulas* of the form $\phi \leq_1 i$, $\phi \leq_2 i$, $\phi \geq_1 i$, or $\phi \geq_2 i$, and *numerical constraints* of the form $i \leq j$ with $i, j \in [0, 1]$. We call *atomic labelled formulas* labelled formulas where $\phi \in \mathbf{Prop}$.

Each branch can be extended by an application of one of the rules in Figure 3.1 where $i, j \in [0, 1]$. Let i 's be in $[0, 1]$ and x 's be variables ranging over the real interval $[0, 1]$. We define the

¹⁹Note that branching rules have *two* conclusions.

$$\begin{array}{cccc}
 \sim \leq_1 \frac{\sim \phi \leq_1 i}{\phi \geq_1 1 - i} & \sim \leq_2 \frac{\sim \phi \leq_2 i}{\phi \geq_2 1 - i} & \sim \geq_1 \frac{\sim \phi \geq_1 i}{\phi \leq_1 1 - i} & \sim \geq_2 \frac{\sim \phi \geq_2 i}{\phi \leq_2 1 - i} \\
 \Delta \geq_1 \frac{\Delta \phi \geq_1 i}{i \leq 0 \mid \begin{array}{l} \phi \geq_1 j \\ j \geq 1 \end{array}} & \Delta \leq_1 \frac{\Delta \phi \leq_1 i}{i \geq 1 \mid \begin{array}{l} \phi \leq_1 j \\ j < 1 \end{array}} & \Delta \leq_2 \frac{\Delta \phi \leq_2 i}{i \geq 1 \mid \begin{array}{l} \phi \leq_2 j \\ j \leq 0 \end{array}} & \Delta \geq_2 \frac{\Delta \phi \geq_2 i}{i \leq 0 \mid \begin{array}{l} \phi \geq_2 j \\ j > 0 \end{array}} \\
 \neg \leq_1 \frac{\neg \phi \leq_1 i}{\phi \leq_2 i} & \neg \leq_2 \frac{\neg \phi \leq_2 i}{\phi \leq_1 i} & \neg \geq_1 \frac{\neg \phi \geq_1 i}{\phi \geq_2 i} & \neg \geq_2 \frac{\neg \phi \geq_2 i}{\phi \geq_1 i} \\
 \rightarrow \leq_1 \frac{\phi_1 \rightarrow \phi_2 \leq_1 i}{i \geq 1 \mid \begin{array}{l} \phi_1 \geq_1 1 - i + j \\ \phi_2 \leq_1 j \\ j \leq i \end{array}} & \rightarrow \leq_2 \frac{\phi_1 \rightarrow \phi_2 \leq_2 i}{\phi_1 \geq_2 j \\ \phi_2 \leq_2 i + j} & \rightarrow \geq_1 \frac{\phi_1 \rightarrow \phi_2 \geq_1 i}{\phi_1 \leq_1 1 - i + j \\ \phi_2 \geq_1 j} & \rightarrow \geq_2 \frac{\phi_1 \rightarrow \phi_2 \geq_2 i}{i \leq 0 \mid \begin{array}{l} \phi_1 \leq_2 j \\ \phi_2 \geq_2 i + j \\ j \leq 1 - i \end{array}} \\
 \rightarrow \leq_1 \frac{\phi_1 \rightarrow \phi_2 \leq_1 i}{i \geq 1 \mid \begin{array}{l} \phi_1 \geq_1 1 - i + j \\ \phi_2 \leq_1 j \\ j \leq i \end{array}} & \rightarrow \leq_2 \frac{\phi_1 \rightarrow \phi_2 \leq_2 i}{\phi_1 \leq_2 i + j \\ \phi_2 \leq_2 1 - j} & \rightarrow \geq_1 \frac{\phi_1 \rightarrow \phi_2 \geq_1 i}{\phi_1 \leq_1 1 - i + j \\ \phi_2 \geq_1 j} & \rightarrow \geq_2 \frac{\phi_1 \rightarrow \phi_2 \geq_2 i}{i \leq 0 \mid \begin{array}{l} \phi_1 \geq_2 i + j \\ \phi_2 \geq_2 1 - j \\ j \leq 1 - i \end{array}} \\
 \wedge \leq_1 \frac{\phi_1 \wedge \phi_2 \leq_1 i}{\phi_1 \leq_1 i \mid \phi_2 \leq_1 i} & \wedge \leq_2 \frac{\phi_1 \wedge \phi_2 \leq_2 i}{\phi_1 \leq_2 i \\ \phi_2 \leq_2 i} & \wedge \geq_1 \frac{\phi_1 \wedge \phi_2 \geq_1 i}{\phi_1 \geq_1 i \\ \phi_2 \geq_1 i} & \wedge \geq_2 \frac{\phi_1 \wedge \phi_2 \geq_2 i}{\phi_1 \geq_2 i \mid \phi_2 \geq_2 i}
 \end{array}$$

 Figure 3.1.: Rules of $\mathcal{T}(\mathfrak{t}^2)$. Vertical bars denote the splitting of the branch.

translation τ from labelled formulas to linear inequalities as follows:

$$\tau(\phi \leq_1 i) = x_\phi^L \leq i; \quad \tau(\phi \geq_1 i) = x_\phi^L \geq i; \quad \tau(\phi \leq_2 i) = x_\phi^R \leq i; \quad \tau(\phi \geq_2 i) = x_\phi^R \geq i$$

Let $*$ $\in \{\leq_1, \geq_1, \leq_2, \geq_2\}$. A tableau branch

$$\mathcal{B} = \{\phi_1 * i_1, \dots, \phi_m * i_m, k_1 \leq l_1, \dots, k_q \leq l_q\}$$

is *closed* if the system of inequalities

$$\tau(\phi_1 * i_1), \dots, \tau(\phi_m * i_m), k_1 \leq l_1, \dots, k_q \leq l_q$$

does not have solutions. Otherwise, \mathcal{B} is *open*.

A tableau is *closed* if all its branches are closed. ϕ has a $\mathcal{T}(\mathfrak{t}_\Delta^2)$ *proof* if the tableau beginning with $\{\phi \leq_1 c, c < 1\}$ is closed.

Remark 3.7. Note that Proposition 3.3 allows us to check only the support of truth when establishing the validity of $\mathcal{L}_{\mathfrak{t}_{(\Delta, \rightarrow)}^2}$ formulas.

Remark 3.8 (How to interpret the rules of $\mathcal{T}(\mathfrak{t}^2)$?). Consider for instance the rule $\rightarrow \leq_2$. Its meaning is: $v_2(\phi_1 \rightarrow \phi_2) \leq i$ iff there is $j \in [0, 1]$ s.t. $v_2(\phi_1) \geq j$ and $v_2(\phi_2) \leq i + j$. While rule $\wedge \leq_1$ means $v_1(\phi_1 \wedge \phi_2) \leq i$ iff either $v_1(\phi_1) \leq i$ or $v_1(\phi_2) \leq i$.

Convention 3.2 (Premises and conclusions). Given a tableaux rule, we call the entries to which it is applied *premises* and the entries that are added to the branch *conclusion*. If the rule splits the branch, we say that it has several conclusions (according to the number of branches into which the original branch is split).

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For example, in the following instance of $\rightarrow_{\leq 1}$, the premise is red, the first conclusion is green, and the second conclusion is blue.

$$\frac{p \rightarrow q \leq_1 i}{\begin{array}{l|l} & p \geq_1 1 - i + j \\ i \geq 1 & q \leq_1 j \\ & j \leq i \end{array}}$$

To prove completeness and soundness, we need the following definitions.

Definition 3.12 (Satisfying valuation of a branch). Let v be a valuation and $k \in \{1, 2\}$. v satisfies a labelled formula $\phi \leq_k i$ (respectively, $\phi \geq_k i$) iff $v_k(\phi) \leq i$ (respectively, $v_k(\phi) \geq i$). v satisfies a branch \mathcal{B} iff v satisfies any labelled formula in \mathcal{B} . A branch \mathcal{B} is satisfiable iff there is a valuation which satisfies it.

Theorem 3.3 (Completeness of $\mathcal{T}(\mathfrak{L}^2)$). ϕ is \mathfrak{L}^2 -valid iff there is a $\mathcal{T}(\mathfrak{L}^2)$ proof for it.

Proof. The soundness follows from the fact that no closed branch is realisable and that if a premise of the rule is realisable, then all labelled formulas are satisfied in at least one of the conclusions.

We consider the case of $\rightarrow_{\leq 1}$. Let $v_1(\phi_1 \rightarrow \phi_2) \leq i$. We show that either $v_1(\phi_1) \geq 1 - i + j$ and $v_1(\phi_2) \leq j$ for $j \leq i$ or $i \geq 1$. We have two cases: either $i = 1$ or $i < 1$. In the first case, we have that $v_1(\phi_1 \rightarrow \phi_2)$ is arbitrary.

In the second case, $\min(1, 1 - v_1(\phi_1) + v_1(\phi_2)) \leq i < 1$ implies that $1 - v_1(\phi_1) + v_1(\phi_2) \leq i$. Hence, $1 - i + v_1(\phi_2) \leq v_1(\phi_1)$, which means that there is $j \in [0, 1]$ s.t. $v_1(\phi_2) \leq j$ and $1 - i + j \leq v_1(\phi_1)$. Hence, there is $j \in [0, 1]$ s.t. $v_1(\phi_2) \leq j$ and $1 - i + j \leq v_1(\phi_1)$. Notice that we necessarily have $j \leq i$ otherwise we would get $1 < v_1(\phi_1)$. Hence, the conclusion of the rule holds. Furthermore, $v_1(\phi_1 \rightarrow \phi_2) = \min(1, 1 - v_1(\phi_1) + v_1(\phi_2)) = 1 - v_1(\phi_1) + v_1(\phi_2)$. From here, it follows that $v_1(\phi_2) \leq i$ and that $v_1(\phi_1) \geq 1 - i + v_1(\phi_2)$. Thus, we choose some $j \leq i$ and $\phi_2 \leq_1 j$ is satisfied, as required.

To show completeness, we proceed by contraposition. We need to check that complete open branches are satisfiable.

Assume that \mathcal{B} is a complete open branch. We construct the satisfying valuation as follows. Let $*$ $\in \{\leq_1, \geq_1, \leq_2, \geq_2\}$ and p_1, \dots, p_m be the propositional variables appearing in the atomic labelled formulas in \mathcal{B} . Let $\{p_1 * i_1, \dots, p_m * i_n\}$ and $\{k_1 \leq l_1, \dots, k_q \leq l_q\}$ be the sets of all atomic labelled formulas and all numerical constraints in \mathcal{B} . Notice that one variable might appear in many atomic labelled formulas, hence we might have $m \neq n$. Since \mathcal{B} is complete and open, the following system of linear inequalities over the set of variables $\{x_{p_1}^L, x_{p_1}^R, \dots, x_{p_m}^L, x_{p_m}^R\}$ must have at least one solution under the constraints listed:

$$\tau(p_1 * i_1), \dots, \tau(p_m * i_n), k_1 \leq l_1, \dots, k_q \leq l_q.$$

Let $c = (c_1^L, c_1^R, \dots, c_m^L, c_m^R)$ be a solution to the above system of inequalities s.t. c_j^L (respectively, c_j^R) is the value of $x_{p_j}^L$ (respectively, $x_{p_j}^R$). Define the valuation v as follows: $v(p_j) = (c_j^L, c_j^R)$.

It remains to show by induction on ϕ that all labelled formulas present at \mathcal{B} are satisfied by v . The basis case of variables holds by the construction of v . We consider only the most instructive case of $\phi_1 \rightarrow \phi_2 \geq_2 i$ as the other ones are straightforward.

Assume that $\phi_1 \rightarrow \phi_2 \geq_2 i \in \mathcal{B}$. Then, by completeness of \mathcal{B} , either $i \leq 0 \in \mathcal{B}$, in which case, $\phi_1 \rightarrow \phi_2 \geq_2 i$ is trivially satisfied, or $\phi_1 \leq_2 j, \phi_2 \geq_2 i + j \in \mathcal{B}$. Furthermore, by the induction hypothesis, v satisfies $\phi_1 \leq_2 j$ and $\phi_2 \geq_2 i + j$, and we also have that $j \leq 1 - i$. Now, to show that v satisfies $\phi_1 \rightarrow \phi_2 \geq_2 i$, recall from semantics that $v_2(\phi_1 \rightarrow \phi_2) = \max(0, v_2(\phi_2) - v_2(\phi_1))$.

Now, we have $\max(0, v_2(\phi_2) - v_2(\phi_1)) \geq \max(0, i + j - j) = \max(0, i) = i$, as desired.

The cases of other connectives can be tackled similarly. \square

The following proof of \mathfrak{L}^2 NP-completeness uses the reduction of Łukasiewicz formulas to bounded mixed-integer problems (bMIPs) as given in [77, 78, 79]. To make the text self-contained, we state the required definitions and results here.

Definition 3.13 (Mixed-integer problem). Let $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{Z}^m$ be variables, A and B be integer matrices and h an integer vector, and $f(\mathbf{x}, \mathbf{y})$ be a $k + m$ -place linear function.

1. A general MIP is to find \mathbf{x} and \mathbf{y} s.t. $f(\mathbf{x}, \mathbf{y}) = \min\{f(\mathbf{x}, \mathbf{y}) : A\mathbf{x} + B\mathbf{y} \geq h\}$.
2. A bounded MIP (bMIP) is to find all solutions that belong to $[0, 1]$.

Proposition 3.5. *Bounded MIP is NP-complete.*

Theorem 3.4. *Satisfiability in \mathfrak{L}^2 's is NP-complete while their validities are coNP-complete.*

Proof. Let $|\phi|$ be the number of symbols in ϕ . Observe, from the proof of Theorem 3.3, that each tableau branch gives rise to two bounded mixed-integer programming problems (bMIP) — each of the length $O(\rho(|\phi|))$ for some polynomial ρ . Recall that bMIP is NP-complete (Proposition 3.5). Thus we can non-deterministically guess an open branch and then solve its two bMIPs (one arising from inequalities with \leq_1 , and the other from those with \leq_2). This yields the NP- and coNP-membership for satisfiability and validity, respectively.

The hardness follows from Proposition 3.3 since both \mathfrak{L}^2 's extend \mathfrak{L} which is known to be NP-complete (w.r.t. satisfiability; coNP-complete w.r.t. validity). \square

3.3. Semantical properties

In Definitions 3.5 and 3.6 we set the designated values to be either $(1, 0)$ (for $\mathfrak{L}_{(\Delta, \rightarrow)}^2$) or $(1, 1)^\uparrow$ (for $\mathfrak{L}_{(\rightarrow)}^2$ ²⁰). In this section, we are investigating the logics arising from altering the set of designated values that we will take to be a filter on $[0, 1]^\times$ (prime filters extending $(1, 1)^\uparrow$ in case of $\mathfrak{L}_{(\rightarrow)}^2$). Let us first adapt the notion of validity.

Definition 3.14. Let $D \subseteq [0, 1]^\times$ and \mathcal{L} be some language equipped with semantics. We say that $\phi \in \mathcal{L}$ is D -valid iff $v(\phi) \in D$ for every v .

First, we observe that in $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ some filters can be reduced to others.

Proposition 3.6. *For every $\phi \in \mathcal{L}_{\mathfrak{L}_{(\Delta, \rightarrow)}^2}$, the following holds.*

- Let $y \geq 1 - x$. Then ϕ is $(x, y)^\uparrow$ -valid iff ϕ is $(x, 1 - x)^\uparrow$ -valid.
- Let $y < 1 - x$. Then ϕ is $(x, y)^\uparrow$ -valid in iff ϕ is $(1 - y, y)^\uparrow$ -valid.

Proof. Analogous to Proposition 3.3. \square

Convention 3.3. We use $\mathfrak{L}_{(\Delta, \rightarrow)}^2(x, y)^\uparrow$ and $\mathfrak{L}_{(\rightarrow)}^2(x, y)^\uparrow$ to designate the sets $\mathcal{L}_{\mathfrak{L}_{(\Delta, \rightarrow)}^2}$ and, respectively, $\mathcal{L}_{\mathfrak{L}_{(\rightarrow)}^2}(x, y)^\uparrow$ -valid formulas.

Let us now show the expected result that modus ponens is sound only in the sets of designated values chosen in Definitions 3.5 and 3.6. To do this, we construct a family of \mathfrak{L} formulas that are not \mathfrak{L} -valid but can never have value 0.

²⁰It does not make much sense to consider filters that do not include $(1, 1)$ since many theorems of $\mathcal{H}\mathfrak{L}_{(\rightarrow)}^2$ (e.g., $p \rightarrow p$) will be invalidated on such filters.

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Lemma 3.6. *Consider the following family of formulas with $n \geq 2$.*

$$F_n := \bigvee_{1 \leq i < j \leq n+1} (p_i \leftrightarrow p_j)$$

It holds that $v(F_n) \geq \frac{n-1}{n}$ for every v .

Proof. Note from Remark 3.2 that $v(F_n)$ is the maximal complement to the distance between any two points out of n on $[0, 1]$. The lower bound on $v(F_n)$ is produced when we place points on $[0, 1]$ with equal intervals between them. The result follows. \square

Theorem 3.5. *For $1 \geq x > 0$ the following holds.*

1. *Let $\mathbf{k}_{(\Delta, \rightarrow)}^2(x, y)^\uparrow \neq \mathbf{k}_{(\Delta, \rightarrow)}^2(x, y)^\uparrow$, then $\mathbf{k}_{(\Delta, \rightarrow)}^2(x, y)^\uparrow$ is not closed under modus ponens.*
2. *Let $\mathbf{k}_{(\rightarrow)}^2(x, 1)^\uparrow \neq \mathbf{k}_{(\rightarrow)}^2(x, 1)^\uparrow$, then $\mathbf{k}_{(\rightarrow)}^2(x, 1)^\uparrow$ is not closed under modus ponens.*

Proof. We begin with 1. First of all, we have by Definition 3.5 that $\mathbf{k}_{(\Delta, \rightarrow)}^2 = \mathbf{k}_{(\Delta, \rightarrow)}^2(1, 0)^\uparrow$. Second, by Proposition 3.6, it suffices to consider only $(x, 1-x)^\uparrow$'s since $\mathbf{k}_{(\Delta, \rightarrow)}^2(x, y)^\uparrow$ coincides with $\mathbf{k}_{(\Delta, \rightarrow)}^2(x, 1-x)^\uparrow$ or $\mathbf{k}_{(\Delta, \rightarrow)}^2(1-y, y)^\uparrow$.

Now, since for any ψ and ψ' , $\psi \rightarrow (\psi' \rightarrow (\psi \odot \psi'))$ is \mathbf{k} -valid (whence, $\mathbf{k}_{(\Delta, \rightarrow)}^2$ -valid, by Proposition 3.3), it is enough to find $(x, 1-x)^\uparrow$ -valid formulas ψ and ψ' s.t. $\psi \odot \psi'$ is not $(x, 1-x)^\uparrow$ -valid. Recall F_n 's from Lemma 3.6. It is clear that $v_2(F_n)$ is the minimal distance between any two points out of n on $[0, 1]$ since $v_2(\psi_1 \leftrightarrow \psi_2)$ is the distance between $v_2(\psi_1)$ and $v_2(\psi_2)$. Thus, $v_2(F_n) \leq \frac{1}{n}$.

Let now $(1, 0)^\uparrow \subsetneq (x, 1-x)^\uparrow$. It is clear that either (1) F_2 is $(x, 1-x)^\uparrow$ -valid or (2) there is some $(x, 1-x)^\uparrow$ -valid F_k s.t. F_{k-1} is not $(x, 1-x)^\uparrow$ -valid. Indeed, assume that F_2 is not $(x, 1-x)^\uparrow$ -valid. Then, $(x, 1-x)^\uparrow \subsetneq (\frac{1}{2}, \frac{1}{2})^\uparrow$. But then there is $k \in \mathbb{N}$ s.t. $\frac{k-2}{k-1} < x \leq \frac{k-1}{k}$ and thus, F_k is $(x, 1-x)^\uparrow$ -valid but F_{k-1} is not, as required.

Now recall that

$$v_1(F_2 \odot F_2) = \max(0, v_1(F_2) + v_1(F_2) - 1) \quad v_2(F_2 \odot F_2) = \min(1, v_2(F_2) + v_2(F_2))$$

and observe that $v(F_2 \odot F_2) = (0, 1)$ when $v(F_2) = (\frac{1}{2}, \frac{1}{2})$ (i.e., $F_2 \odot F_2$ is not valid in any $(x, 1-x)^\uparrow$). This tackles (1). For (2), we notice that $v(F_k \odot F_k) = (\frac{k-2}{k}, \frac{2}{k})$ when $v(F_k) = (\frac{k-1}{k}, \frac{1}{k})$. We can see that $v(F_{k-1}) \geq (\frac{k-2}{k-1}, \frac{1}{k-1})$ for any v . Since F_{k-1} is not $(x, 1-x)^\uparrow$ -valid by (2) and there is a valuation v s.t. $v(F_k \odot F_k) <_{[0,1]^\times} v(F_{k-1})$, it follows that $F_k \odot F_k$ is not $(x, 1-x)^\uparrow$ -valid either, as required.

The second part can be dealt with in a similar manner. Again, it suffices to consider F_n 's. This time, however, we will substitute \leftrightarrow 's with \leftrightarrow 's. Since we care only for the support of truth in $\mathbf{k}_{(\rightarrow)}^2$, the result follows. \square

Remark 3.9 (Prime filters as the sets of designated values in $\mathbf{k}_{(\Delta, \rightarrow)}^2$). Changing the set of designated values in $\mathbf{k}_{(\Delta, \rightarrow)}^2$ to the filter $(1, 1)^\uparrow$ does not change the set of valid formulas as follows from Proposition 3.4. Define now the entailment relation corresponding to each filter $(x, y)^\uparrow$:

$$\phi \models_{(x, y)^\uparrow} \chi \text{ iff } \forall v : v(\phi) \in (x, y)^\uparrow \Rightarrow v(\chi) \in (x, y)^\uparrow$$

and note that it is paraconsistent for $(1, 1)^\uparrow$: $p \wedge \neg p \not\models_{(1, 1)^\uparrow} q$.

$\models_{(1, 1)^\uparrow}$ can be axiomatised by making the conflation rule **conf** (recall Definition 3.3) applicable to theorems only. Its $\{\wedge, \vee, \neg\}$ fragment coincides with BD, while the $\{\wedge, \vee, \neg\}$ fragment of

$\mathfrak{L}_{(\Delta, \rightarrow)}^2$ coincides with ETL²¹ from [120]. We will not use this logic here, however, it can be proved complete in a similar manner as $\mathfrak{L}_{(\Delta, \rightarrow)}^2$.

We finish the section by showing the correspondence between first-degree fragments of \mathfrak{L}^2 on the one hand and BD and ETL on the other.

Definition 3.15. Let ϕ and χ be over $\{\neg, \wedge, \vee\}$. We define \models_{ETL} on $\mathbf{4}$ as follows:

$$\phi \models_{\text{ETL}} \chi \text{ iff } \forall v : v(\phi) = \mathbf{T} \Rightarrow v(\chi) = \mathbf{T}$$

Lemma 3.7. For any v on $[0, 1]^\times$, define $v_{\mathbf{4}}$ on $\mathbf{4}$ as follows.

- If $v(p) \in \{(1, 0), (1, 1), (0, 0), (0, 1)\}$, then $v_{\mathbf{4}}(p) \in \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$, respectively.
- Otherwise, $v_{\mathbf{4}}$ is defined in the following manner.

$$v_{\mathbf{4}}(p) = \begin{cases} \mathbf{T} & \text{if } v_1(p) = 1 \text{ and } v_2(p) \neq 1 \\ \mathbf{F} & \text{if } v_1(p) \neq 1 \text{ and } v_2(p) = 1 \\ \mathbf{N} & \text{otherwise} \end{cases}$$

Then for any ϕ over $\{\neg, \wedge, \vee\}$, it holds that

$$\begin{aligned} v_1(\phi) = 1 & \text{ iff } v_{\mathbf{4}}(\phi) \in \{\mathbf{T}, \mathbf{B}\} \\ v_2(\phi) = 1 & \text{ iff } v_{\mathbf{4}}(\phi) \in \{\mathbf{F}, \mathbf{B}\} \end{aligned}$$

Proof. We proceed by induction on ϕ .

For the basis case of $\phi = p$, the statement holds by construction.

$$\boxed{\phi = \neg\psi}$$

$$\begin{aligned} v_1(\neg\psi) = 1 & \text{ iff } v_2(\psi) = 1 \\ & \text{ iff } v_{\mathbf{4}}(\psi) \in \{\mathbf{F}, \mathbf{B}\} && \text{(by IH)} \\ & \text{ iff } v_{\mathbf{4}}(\neg\psi) \in \{\mathbf{T}, \mathbf{B}\} \end{aligned}$$

$$\begin{aligned} v_2(\neg\psi) = 1 & \text{ iff } v_1(\psi) = 1 \\ & \text{ iff } v_{\mathbf{4}}(\psi) \in \{\mathbf{T}, \mathbf{B}\} && \text{(by IH)} \\ & \text{ iff } v_{\mathbf{4}}(\neg\psi) \in \{\mathbf{F}, \mathbf{B}\} \end{aligned}$$

$$\boxed{\phi = \psi \wedge \psi'}$$

$$\begin{aligned} v_1(\psi \wedge \psi') = 1 & \text{ iff } v_1(\psi) = 1 \text{ and } v_1(\psi') = 1 \\ & \text{ iff } v_{\mathbf{4}}(\psi) \in \{\mathbf{T}, \mathbf{B}\} \text{ and } v_{\mathbf{4}}(\psi') \in \{\mathbf{T}, \mathbf{B}\} && \text{(by IH)} \\ & \text{ iff } v_{\mathbf{4}}(\psi \wedge \psi') \in \{\mathbf{T}, \mathbf{B}\} \end{aligned}$$

$$\begin{aligned} v_2(\psi \wedge \psi') = 1 & \text{ iff } v_2(\psi) = 1 \text{ or } v_2(\psi') = 1 \\ & \text{ iff } v_{\mathbf{4}}(\psi) \in \{\mathbf{F}, \mathbf{B}\} \text{ or } v_{\mathbf{4}}(\psi') \in \{\mathbf{F}, \mathbf{B}\} && \text{(by IH)} \\ & \text{ iff } v_{\mathbf{4}}(\psi \wedge \psi') \in \{\mathbf{F}, \mathbf{B}\} \end{aligned}$$

$$\boxed{\phi = \psi \vee \psi'}$$
 is obtained dually. □

²¹The difference between \models_{BD} and \models_{ETL} is that the latter is defined via preservation of \mathbf{T} (cf. Definition 3.15), whence the name.

3. Paraconsistent expansions of Łukasiewicz logic

Proposition 3.7. *Let ϕ and χ be over $\{\neg, \wedge, \vee\}$. Then the following equivalences hold.*

$$\begin{aligned} \phi \models_{\mathbf{L}_{(\Delta, \rightarrow)}^2} \chi & \text{ iff } \phi \models_{\text{ETL}} \chi \\ \phi \models_{(1,1)^\uparrow} \chi & \text{ iff } \phi \models_{\text{BD}} \chi \end{aligned}$$

Proof. Observe that since $\mathbf{4}$ is a sublattice of $[0, 1]^\boxtimes$, the left-to-right direction is obvious. Indeed, if $\phi \not\models_{\text{ETL}} \chi$, then there is some valuation v s.t. $v(\phi) = (1, 0)$ and $v(\chi) \neq (1, 0)$, i.e., $\phi \not\models_{\mathbf{L}_{(\Delta, \rightarrow)}^2} \chi$. Now, consider the converse.

Let $\phi \not\models_{\mathbf{L}_{(\Delta, \rightarrow)}^2} \chi$. Then, there is a valuation v s.t. $v(\phi) = (1, 0)$ and $v(\chi) \neq (1, 0)$. Furthermore, by Proposition 3.4, we have w.l.o.g. that $v_1(\chi) \neq 1$. Then, by Lemma 3.7, we have that $v_4(\phi) = \mathbf{T}$ and $v_4(\chi) \neq \mathbf{T}$. Thus, $\phi \not\models_{\text{ETL}} \chi$, as desired.

Now let $\phi \not\models_{(1,1)^\uparrow} \chi$. Then, there is a valuation v s.t. $v_1(\phi) = 1$ and $v_1(\chi) \neq 1$. Then, by Lemma 3.7, $v_4(\phi) \in \{\mathbf{T}, \mathbf{B}\}$ but $v_4(\chi) \notin \{\mathbf{T}, \mathbf{B}\}$. Thus, $\phi \not\models_{\text{BD}} \chi$, as desired. \square

4. Paraconsistent expansions of Gödel logic

This chapter is structured in the same way as Chapter 3. First, we recall Gödel logic and its expansions with Δ ($\mathbf{G}\Delta$) or \prec (\mathbf{biG}). Then, we introduce two paraconsistent expansions of \mathbf{biG} , provide their strongly complete axiomatisations, establish complexity evaluations, and, finally, discuss some semantical properties.

Gödel logic is an infinite-valued propositional logic, with its standard algebraic semantics being based on the full $[0, 1]$ interval, where 1 is the designated value. The truth values are (densely) ordered, and, together with the semantics of Gödel implication, this makes Gödel logic suitable for formalising comparisons. Gödel logic is one of the three basic t-norm-based fuzzy logics, and it is also closely related to intuitionistic logic: it is the logic of linearly ordered Heyting algebras and can also be characterised as the logic of linearly ordered intuitionistic Kripke structures, and axiomatized by extending the intuitionistic logic with the axiom of prelinearity $(p \rightarrow q) \vee (q \rightarrow p)$. A more detailed exposition of Gödel logics can be found e.g., in [10].

We are going to formulate Gödel logic expanded with a co-implication connective \prec and refer to it as \mathbf{biG} (bi-Gödel logic or symmetric Gödel logic in the terminology of [76]), as it can naturally be obtained by extending the bi-intuitionistic logic with the axioms of prelinearity. Note that instead of \prec , one could add the Baaz delta operator Δ and obtain $\mathbf{G}\Delta$ that is expressively equivalent to \mathbf{biG} (cf. Remark 4.1). In the following definition of bi-Gödel algebras, we leave both Δ and \prec as it will facilitate the formalisation of comparative belief statements.

Definition 4.1. The bi-Gödel algebra $[0, 1]_{\mathbf{G}} = \langle [0, 1], 0, 1, \wedge_{\mathbf{G}}, \vee_{\mathbf{G}}, \rightarrow_{\mathbf{G}}, \prec, \sim_{\mathbf{G}}, \Delta_{\mathbf{G}} \rangle$ is defined as follows: for all $a, b \in [0, 1]$, $\wedge_{\mathbf{G}}$ and $\vee_{\mathbf{G}}$ are given by $a \wedge_{\mathbf{G}} b := \min(a, b)$, $a \vee_{\mathbf{G}} b := \max(a, b)$. The remaining operations are defined below:

$$a \rightarrow_{\mathbf{G}} b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{else} \end{cases} \quad a \prec_{\mathbf{G}} b = \begin{cases} 0 & \text{if } a \leq b \\ a & \text{else} \end{cases} \quad \sim_{\mathbf{G}} a = \begin{cases} 0 & \text{if } a > 0 \\ 1 & \text{else} \end{cases} \quad \Delta_{\mathbf{G}} a = \begin{cases} 0 & \text{if } a < 1 \\ 1 & \text{else} \end{cases}$$

Definition 4.2 (Language and semantics of \mathbf{biG}). We set \mathbf{Prop} to be a countable set of propositional variables and consider the following languages $\mathcal{L}_{\mathbf{biG}}$ and $\mathcal{L}_{\mathbf{G}\Delta}$.

$$\begin{aligned} \mathcal{L}_{\mathbf{biG}} \ni \phi &:= p \in \mathbf{Prop} \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\phi \prec \phi) \\ \mathcal{L}_{\mathbf{biG}} \ni \phi &:= p \in \mathbf{Prop} \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \Delta \phi \mid \sim \phi \end{aligned}$$

We let $v : \mathbf{Prop} \rightarrow [0, 1]$. Using bi-Gödel operations from Definition 4.1, a \mathbf{biG} valuation v is extended to complex formulas in the expected manner: $v(\phi \circ \phi') = v(\phi) \circ_{\mathbf{G}} v(\phi')$. We say that ϕ is *valid* iff $v(\phi) = 1$ for any v . Furthermore, we define the entailment:

$$\Gamma \models_{\mathbf{biG}} \chi \text{ iff } \inf\{v(\phi) : \phi \in \Gamma\} \leq v(\chi) \text{ for any } v$$

Convention 4.1 (Notational conventions). We will further use the following shorthands. In $\mathcal{L}_{\mathbf{biG}}$:

$$\top := p \rightarrow p \quad \perp := p \prec p \quad \sim \phi := \phi \rightarrow \perp$$

and in $\mathcal{L}_{\mathbf{G}\Delta}$:

$$\top := p \rightarrow p \quad \perp := \sim \top$$

Note that

$$v(\top) = 1 \quad v(\perp) = 0$$

Finally, we write $v[\Gamma] = x$ iff $\inf\{v(\phi) : \phi \in \Gamma\} = x$.

4. Paraconsistent expansions of Gödel logic

Remark 4.1. Observe that \prec and Δ are interdefinable:

$$\Delta\phi := \top \prec (\top \prec \phi) \qquad \phi \prec \phi' := \phi \wedge \sim\Delta(\phi \rightarrow \phi')$$

Indeed, $\phi \rightarrow \phi'$ is true (has value 1) iff the value of ϕ is less or equal to that of ϕ' . In simpler words, for an implication to be true, the value cannot *decrease* from the antecedent to the consequent. On the other hand, if $\phi \rightarrow \phi'$ is not true, then we can safely assume that its truth degree is not smaller than that of ϕ' . In Parts II and III, we will use \prec (and treat Δ as a derived connective if we need it) when dealing with *two-layered* logics based on **biG** and its expansions since there is no straightforward Nelsonian counterpart of Δ . When dealing with ‘traditional’ modal logics based on **biG**, we will use Δ as a primitive connective and not consider \prec since Δ behaves better with \Box and \Diamond .

Let us now present the Hilbert-style calculus for **biG** that we dub $\mathcal{H}\mathbf{biG}$.

Definition 4.3 ($\mathcal{H}\mathbf{biG}$). The calculus has the following axioms and rules.

$$\text{Int}\rightarrow: (\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)).$$

$$\text{Int}\vee: \phi \rightarrow (\phi \vee \chi); \chi \rightarrow (\phi \vee \chi); (\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow ((\phi \vee \chi) \rightarrow \psi)).$$

$$\text{Int}\wedge: (\phi \wedge \chi) \rightarrow \phi; (\phi \wedge \chi) \rightarrow \chi; (\phi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\chi \wedge \psi))).$$

$$\text{IntRes}: (\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \wedge \chi) \rightarrow \psi); ((\phi \wedge \chi) \rightarrow \psi) \rightarrow (\phi \rightarrow (\chi \rightarrow \psi)).$$

$$\text{Int}\sim: (\phi \rightarrow \chi) \rightarrow (\sim\chi \rightarrow \sim\phi).$$

$$\text{HB}\prec: (\phi \prec \chi) \rightarrow (\top \prec (\phi \rightarrow \chi)); \sim(\phi \prec \chi) \rightarrow (\phi \rightarrow \chi).$$

$$\text{HB}\vee: \phi \rightarrow (\chi \vee (\phi \prec \chi)); ((\phi \prec \chi) \prec \psi) \rightarrow (\phi \prec (\chi \vee \psi)).$$

$$\text{prel}: (\phi \rightarrow \chi) \vee (\chi \rightarrow \phi); \top \prec ((\phi \prec \chi) \wedge (\chi \prec \phi)).$$

$$\text{MP}: \frac{\phi \quad \phi \rightarrow \chi}{\chi}.$$

$$\text{HBnec}: \frac{\mathcal{H}\mathbf{biG} \vdash \phi}{\mathcal{H}\mathbf{biG} \vdash \sim(\top \prec \phi)}.$$

Observe that in the definition above, the first five groups of axioms formalise intuitionistic logic, adding the next two axioms produces the axiomatisation of Heyting–Brouwer (bi-Intuitionistic in the terminology of [74]²² or HB) logic [125, 126]. Finally, axiom **prel** stands for the linearity conditions for \rightarrow and \prec .

Remark 4.2 ($\mathcal{H}\mathbf{G}\Delta$). The axiomatisation of $\mathbf{G}\Delta$ (Gödel logic with Baaz delta) $\mathcal{H}\mathbf{G}\Delta$ can be easily obtained from Definition 4.3. Instead of the axioms and rules with \prec , one should add the axioms and rules for Δ ²³ from Definition 3.4 and $\sim\phi \rightarrow (\phi \rightarrow \chi)$ (or $\sim\sim\Delta\phi \rightarrow \Delta\phi$). As expected, the Δ deduction theorem (Proposition 3.1) will also hold for $\mathcal{H}\mathbf{G}\Delta$ even for an infinite Γ *but the resulting calculus will be complete only w.r.t. the entailment defined as 1-preservation*. For the completeness w.r.t. entailment as order on $[0, 1]$, $\Delta\mathbf{nec}$ can be applied *only to theorems*.

Remark 4.3 (Entailment in Gödel logics). Recall from Definition 3.2 that the entailment in the Łukasiewicz logic was defined via the preservation of the designated values (1 in the case of \mathbf{L}_Δ and $(1, 0)$ and $(1, 1)^\uparrow$ in the cases of $\mathbf{L}_{(\Delta, \rightarrow)}^2$ and $\mathbf{L}_{(\rightarrow)}^2$, respectively). In the bi-Gödel logic and its expansions, however, we are defining entailments via the preservation of the order on the underlying algebra.

²²The symbol \prec is also due to [74].

²³Note that $\Delta\mathbf{nec}$ should, in this case, be applied only to theorems. In addition, it is clear that $\mathbf{biG} \models \Delta\phi \leftrightarrow \sim(\top \prec \phi)$, i.e., \mathbf{HBnec} is just a notational variant of $\Delta\mathbf{nec}$.

This is done for the following reasons. The biG entailment defined as the order on $[0, 1]$ corresponds to the local entailment on the linearly ordered Kripke²⁴ frames. On the other hand, there is (to the best of the author's knowledge) no Kripke semantic for \mathfrak{L} in the traditional²⁵ sense. Moreover, since \mathfrak{L} and its expansions are not compact, and hence, cannot have axiomatisations complete w.r.t. countable theories, we are mostly interested in the expansions of \mathfrak{L} as in sets of theorems or valid formulas (recall Remark 2.1). For this purpose, it suffices to consider entailments defined via the preservation of the designated values.

Before proceeding to the paraconsistent expansions, we state completeness theorems for $\mathcal{H}\text{biG}$ and $\mathcal{H}\text{G}\Delta$.

Proposition 4.1. *Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\text{biG}}$ and $\Xi \cup \{\chi\} \subseteq \mathcal{L}_{\text{G}\Delta}$. Then*

$$\Gamma \models_{\text{biG}} \phi \text{ iff } \Gamma \vdash_{\mathcal{H}\text{biG}} \phi \qquad \Xi \models_{\text{biG}} \chi \text{ iff } \Xi \vdash_{\mathcal{H}\text{G}\Delta} \chi$$

4.1. Semantics and axiomatisation

Just as in the paraconsistent expansions of \mathfrak{L} , the main difference in the expansions of biG is going to be in the falsity conditions of (co)implications. We again choose two of them: (1) an intuitive or Nelsonian option, ‘implication is false when its antecedent is true and consequent is false’ (i.e., the way we disprove classical implication) (\rightarrow); and (2) via co-implication (\rightarrow) that produces a self-dual logic. The falsity conditions of co-implications (\prec and \dashv) are dual to those of implications (\rightarrow and \dashv).

Definition 4.4 (Language and semantics of G^2). We fix a countable set Prop of propositional letters and consider the following language:

$$\phi := p \in \text{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\phi \prec \phi) \mid \sim\phi \mid \Delta\phi \mid (\phi \dashv \phi) \mid (\phi \dashv\vdash \phi)$$

Let $v_1, v_2 : \text{Prop} \rightarrow [0, 1]$. We extend the G^2 -valuation as follows.

$$\begin{array}{ll} v_1(\neg\phi) = v_2(\phi) & v_2(\neg\phi) = v_1(\phi) \\ v_1(\sim\phi) = \sim_{\text{G}}v_1(\phi) & v_2(\sim\phi) = 1 \prec_{\text{G}}v_2(\phi) \\ v_1(\phi \wedge \phi') = v_1(\phi) \wedge_{\text{G}}v_1(\phi') & v_2(\phi \wedge \phi') = v_2(\phi) \vee_{\text{G}}v_2(\phi') \\ v_1(\phi \vee \phi') = v_1(\phi) \vee_{\text{G}}v_1(\phi') & v_2(\phi \vee \phi') = v_2(\phi) \wedge_{\text{G}}v_2(\phi') \\ v_1(\phi \rightarrow \phi') = v_1(\phi) \rightarrow_{\text{G}}v_1(\phi') & v_2(\phi \rightarrow \phi') = v_2(\phi') \prec_{\text{G}}v_2(\phi) \\ v_1(\phi \prec \phi') = v_1(\phi) \prec_{\text{G}}v_1(\phi') & v_2(\phi \prec \phi') = v_2(\phi') \rightarrow_{\text{G}}v_2(\phi) \\ v_1(\Delta\phi) = \Delta_{\text{G}}v_1(\phi) & v_2(\Delta\phi) = \sim_{\text{G}}\sim_{\text{G}}v_2(\phi) \\ v_1(\phi \dashv\vdash \phi') = v_1(\phi) \rightarrow_{\text{G}}v_1(\phi') & v_2(\phi \dashv\vdash \phi') = v_1(\phi) \wedge_{\text{G}}v_2(\phi') \\ v_1(\phi \dashv\vdash \phi') = v_1(\phi) \prec_{\text{G}}v_1(\phi') & v_2(\phi \dashv\vdash \phi') = v_2(\phi) \vee_{\text{G}}v_1(\phi') \end{array}$$

We will consider two separate logics: $\text{G}^2_{(\rightarrow, \prec)}$ and $\text{G}^2_{(\dashv, \dashv\vdash)}$ and their respective languages $\mathcal{L}_{\text{G}^2_{(\rightarrow, \prec)}}$ and $\mathcal{L}_{\text{G}^2_{(\dashv, \dashv\vdash)}}$ which have only one set of (co-)implications²⁶ indicated in the brackets.

²⁴HB is strongly complete w.r.t. partially (pre-)ordered Kripke frames [125]. Adding prel makes it strongly complete w.r.t. linearly (pre-)ordered Kripke frames. We show the correspondence between the local entailment on Kripke frames and the upwards order on $[0, 1]^{\mathfrak{K}}$ for the case of paraconsistent expansions of biG in Section 4.1 — cf. Lemmas 4.1 and 4.2. The proof for biG can be recovered if one considers \neg -free formulas.

²⁵Nominally, [108] does propose a Kripke semantics for \mathfrak{L} . However, it is, essentially, a translation of the conditions on the many-valued algebras into the relation-semantic framework.

²⁶In what follows, we will treat Δ as a definable connective of $\mathcal{L}_{\text{G}^2_{(\rightarrow, \prec)}}$ and also assume that Δ is not present in $\mathcal{L}_{\text{G}^2_{(\dashv, \dashv\vdash)}}$.

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Convention 4.2. In what follows, we will use several shorthands (p is a fixed fresh variable).

$$\begin{array}{ll} \top_{\mathbf{1}} := p \rightarrow p & \top_{\mathbf{N}} := p \rightarrow p \\ \perp_{\mathbf{0}} := p \prec p & \perp_{\mathbf{N}} := p \multimap p \\ \sim_{\mathbf{0}} p := p \rightarrow \perp_{\mathbf{0}} & \sim_{\mathbf{N}} p := p \rightarrow \perp_{\mathbf{N}} \end{array}$$

When there is no risk of confusion, we will drop the subscripts.

We define entailments in \mathbf{G}^2 as follows.

Definition 4.5 (\mathbf{G}^2 entailments). Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$ and $\Delta \cup \{\chi\} \subseteq \mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$. We define:

$$\begin{array}{l} \Gamma \models_{\mathbf{G}^2(\rightarrow, \prec)} \phi \text{ iff } \forall v_1, v_2 : \inf\{v_1(\psi) : \psi \in \Gamma\} \leq v_1(\phi) \text{ and } \sup\{v_2(\psi) : \psi \in \Gamma\} \geq v_2(\phi) \\ \Theta \models_{\mathbf{G}^2(\rightarrow, \multimap)} \chi \text{ iff } \forall v_1 : \inf\{v_1(\psi) : \psi \in \Theta\} \leq v_1(\chi) \end{array}$$

Note that both these entailments are paraconsistent.

Proposition 4.2.

1. Let $\phi, \neg\phi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$ be non-valid. Then there is $\chi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$ s.t. $\phi, \neg\phi \not\models_{\mathbf{G}^2(\rightarrow, \prec)} \chi$.
2. Let $\phi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$ and let there be v_1 and v_2 s.t. $v_1(\phi) = 1$ and $v_2(\phi) = 1$. Then there is $\chi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$ s.t. $\phi, \neg\phi \not\models_{\mathbf{G}^2(\rightarrow, \multimap)} \chi$.

Proof. We prove the first statement. The second can be obtained in the same way. Let $q \notin \text{Prop}(\phi)$ and set $\chi = q$. Now let v be the valuation s.t. $v(\phi) \notin \{(1, 0), (0, 1)\}$ and $v(q) = (0, 1)$. v refutes the entailment. \square

Remark 4.4. Note that $\mathbf{0}$ and $\mathbf{1}$ s.t. $v(\mathbf{1}) = (1, 0)$ and $v(\mathbf{0}) = (0, 1)$ are definable in $\mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$ and $\mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$.

$$\begin{array}{ll} \mathbf{0} := \top_{\mathbf{N}} \multimap \top_{\mathbf{N}} & \mathbf{0} := p \prec p \\ \mathbf{1} := \mathbf{0} \rightarrow \mathbf{0} & \mathbf{1} := p \rightarrow p \end{array}$$

Note that the definitions of $\mathbf{1}$ and $\mathbf{0}$ in $\mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$ coincide with $\top_{\mathbf{1}}$ and $\perp_{\mathbf{0}}$ which is not the case for $\mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$.

It is also straightforward to verify that $p \rightarrow p$ is not a constant. Thus, the standard definition of Δ cannot be transferred to $\mathbf{G}^2(\rightarrow, \multimap)$ which prevents its intuitive axiomatisation. On the other hand, \multimap obeys the expected De Morgan law that is dual to that of \rightarrow (cf. Definition 4.6).

Note also that just as checking v_1 to establish validity was enough in $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ (Propositions 3.3 and 3.4), so it is enough in $\mathbf{G}^2(\rightarrow, \prec)$.

Proposition 4.3. Let $\phi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$. For any $v(p) = (x, y)$, let $v^*(p) = (1 - y, 1 - x)$. Then $v(\phi) = (x, y)$ iff $v^*(\phi) = (1 - y, 1 - x)$.

Proof. We show that, for all $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \{1, 2\}$ s.t. $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x}' \neq \mathbf{y}'$, we have

$$v_{\mathbf{x}}(\chi) \geq v_{\mathbf{x}'}(\chi') \Leftrightarrow v_{\mathbf{y}'}^*(\chi') \geq v_{\mathbf{y}}^*(\chi) \quad (4.1)$$

We proceed by induction on the number of unary and binary connectives in both χ and χ' . The only non-trivial case is that of \rightarrow . For \rightarrow , we have

$$\begin{array}{l} v_1(\psi_1 \rightarrow \psi_2) < v_{\mathbf{x}}(\chi) \text{ iff } v_1(\psi_1) > v_1(\psi_2) \text{ and } v_1(\psi_2) < v_{\mathbf{x}}(\chi) \\ \text{iff } v_2^*(\psi_1) < v_2^*(\psi_2) \text{ and } v_2^*(\psi_2) > v_{\mathbf{y}}^*(\chi) \end{array} \quad (\text{by IH})$$

$$\text{iff } v_2^*(\psi_1 \rightarrow \psi_2) > v_{\mathbf{y}}^*(\chi)$$

and

$$\begin{aligned} v_1(\psi_1 \rightarrow \psi_2) = v_{\mathbf{x}}(\chi) &\Leftrightarrow v_1(\psi_1) > v_1(\psi_2) = v_{\mathbf{x}}(\chi) \text{ or } \begin{bmatrix} v_1(\psi_1) \leq v_1(\psi_2) \\ \text{and} \\ v_{\mathbf{x}}(\chi) = 1 \end{bmatrix} \\ &\text{iff } v_2^*(\psi_1) < v_2^*(\psi_2) = v_{\mathbf{y}}^*(\chi) \text{ or } \begin{bmatrix} v_2^*(\psi_1) \geq v_2^*(\psi_2) \\ \text{and} \\ v_{\mathbf{y}}^*(\chi) = 0 \end{bmatrix} && \text{(by IH)} \\ &\text{iff } v_2^*(\psi_1 \rightarrow \psi_2) = v_{\mathbf{y}}^*(\chi) \end{aligned}$$

The $v_2(\psi_1 \rightarrow \psi_2)$ case can be tackled similarly. Now we can prove the statement by induction on ϕ . The basis cases of variables and constants hold by the construction of v^* .

We only present the case of \rightarrow . We consider two cases: $(x, y) \neq (1, 0)$ and $(x, y) = (1, 0)$. In the first and second cases, we have

$$\begin{aligned} v(\psi_1 \rightarrow \psi_2) = (x, y) &\Leftrightarrow v_1(\psi_1) > v_1(\psi_2) = x \text{ and } v_2(\psi_1) < v_2(\psi_2) = y \\ &\text{iff } v_2^*(\psi_1) < v_2^*(\psi_2) = 1 - x \text{ and } v_1^*(\psi_1) > v_1^*(\psi_2) = 1 - y && \text{(by IH and (4.1))} \\ &\text{iff } v^*(\psi_1 \rightarrow \psi_2) = (1 - y, 1 - x) \end{aligned}$$

and

$$\begin{aligned} v(\psi_1 \rightarrow \psi_2) = (1, 0) &\text{ iff } v_1(\psi_1) \leq v_1(\psi_2) \text{ and } v_2(\psi_1) \geq v_2(\psi_2) \\ &\text{ iff } v_2^*(\psi_1) \geq v_2^*(\psi_2) \text{ and } v_1^*(\psi_1) \leq v_1^*(\psi_2) && \text{(by IH and (4.1))} \\ &\text{ iff } v^*(\psi_1 \rightarrow \psi_2) = (1, 0). \end{aligned}$$

□

Corollary 4.1. $\Gamma \models_{\mathbf{G}_{(\rightarrow, <) }^2} \chi$ iff $\inf\{v_1(\phi) : \phi \in \Gamma\} \leq v_1(\chi)$ for every v_1 .

Proof. The ‘only if’ part follows directly from Definition 4.5. We consider the ‘if’ part. Let $\Gamma \not\models_{\mathbf{G}_{(\rightarrow, <) }^2} \chi$. We show that there is a v_1 s.t. $\inf\{v_1(\phi) : \phi \in \Gamma\} > v_1(\chi)$. Assume w.l.o.g. that there is a \mathbf{v}_2 s.t. $y = \sup\{v_2(\phi) : \phi \in \Gamma\} < v_2(\chi) = y'$. But then, by Proposition 4.3, we have that $1 - y = \inf\{v_1^*(\phi) : \phi \in \Gamma\} > v_1^*(\chi) = 1 - y'$, as required. □

In the remainder of this subsection, we are going to present Hilbert calculi for $\mathbf{G}_{(\rightarrow, \ominus)}^2$ and $\mathbf{G}_{(\rightarrow, <) }^2$ and prove their completeness. The calculi are straightforward expansions of $\mathcal{H}\text{biG}$ with De Morgan axioms.

Definition 4.6 ($\mathcal{H}\mathbf{G}_{(\rightarrow, <) }^2$ and $\mathcal{H}\mathbf{G}_{(\rightarrow, \ominus)}^2$). To obtain $\mathcal{H}\mathbf{G}_{(\rightarrow, <) }^2$, we add the following axiom schemas (below, $\phi \leftrightarrow \chi$ is a shorthand for $(\phi \rightarrow \chi) \wedge (\chi \rightarrow \phi)$).

$$\text{neg: } \neg\neg\phi \leftrightarrow \phi.$$

$$\text{DeM}\wedge: \neg(\phi \wedge \chi) \leftrightarrow (\neg\phi \vee \neg\chi).$$

$$\text{DeM}\vee: \neg(\phi \vee \chi) \leftrightarrow (\neg\phi \wedge \neg\chi).$$

$$\text{DeM}\rightarrow: \neg(\phi \rightarrow \chi) \leftrightarrow (\neg\chi \prec \neg\phi).$$

$$\text{DeM}\prec: \neg(\phi \prec \chi) \leftrightarrow (\neg\chi \rightarrow \neg\phi).$$

For $\mathcal{H}\mathbf{G}_{(\rightarrow, \ominus)}^2$, we replace \rightarrow with \rightarrow , \prec with \ominus , and \sim with $\sim_{\mathbb{N}}$ in $\mathcal{H}\text{biG}$. We also add neg and De Morgan laws for \wedge and \vee (with \leftrightarrow instead of \leftrightarrow). Finally, the De Morgan laws for (co-)implication are as follows.

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$$\text{DeM}\rightarrow: \neg(\phi \rightarrow \chi) \leftrightarrow (\phi \wedge \neg\chi).$$

$$\text{DeM}\multimap: \neg(\phi \multimap \chi) \leftrightarrow (\neg\phi \vee \chi).$$

Note that $\mathcal{HG}_{(\rightarrow, \prec)}^2$ and $\mathcal{HG}_{(\rightarrow, \multimap)}^2$ extend, respectively, $\mathsf{I}_4\mathsf{C}_4$ and $\mathsf{I}_1\mathsf{C}_1$ (Nelson's logic with co-implication) from [151] with prelinearity axioms.

Remark 4.5 (Δ De Morgan axioms). In Part II, we will mostly need the $\mathsf{G}_{(\rightarrow, \prec)}^2$ with Δ and \sim (treated as a primitive connective) instead of \prec . To obtain its axiomatisation, we replace $\text{DeM}\rightarrow$ and $\text{DeM}\prec$ with the following axioms to $\mathcal{HG}\Delta$ (recall Remark 4.5).

$$\text{DeM}\rightarrow: \neg(\phi \rightarrow \chi) \leftrightarrow (\neg\chi \wedge \sim\Delta(\neg\chi \rightarrow \neg\phi)).$$

$$\text{DeM}\Delta: \neg\Delta\phi \leftrightarrow \sim\sim\neg\phi.$$

$$\text{DeM}\sim: \neg\sim\phi \leftrightarrow \sim\Delta\neg\phi.$$

In the remainder of the section, we prove the strong completeness of $\mathcal{HG}_{(\rightarrow, \prec)}^2$ and $\mathcal{HG}_{(\rightarrow, \multimap)}^2$. Just as with the calculi for L^2 , the idea is to reduce the proofs to the $\mathcal{H}\text{biG}$ -proofs. Unfortunately, to the best of our knowledge, there are no proofs of *algebraic* completeness of biG . Thus, the proof will go via the equivalence between the algebraic semantics in Definition 4.4 and its semantics on *linearly ordered* bi-intuitionistic Kripke frames²⁷ with two valuations shown in [151].

We begin with the frame semantics for G^2 .

Definition 4.7 (G^2 -frames). A G^2 -frame is a tuple $\mathbb{F} = \langle W, \preceq \rangle$ with $W \neq \emptyset$ and \preceq being a reflexive linear (total) order on W .

Definition 4.8 (Models and semantics). A G^2 -model on the frame \mathbb{F} is a tuple $\mathfrak{M} = \langle \mathbb{F}, v^+, v^- \rangle$ with $v^+, v^- : \text{Prop} \rightarrow 2^W$ (positive and negative valuations) s.t. if $s \in v^+(p)$ and $s \preceq s'$, then $s' \in v^+(p)$ (and likewise for v^-). Using these maps, the positive and negative support of formulas at state $s \in W$ is defined as follows.

$$\begin{aligned} \mathfrak{M}, s \models^+ p & \text{ iff } s \in v^+(p) \\ \mathfrak{M}, s \models^- p & \text{ iff } s \in v^-(p) \\ \mathfrak{M}, s \models^+ \neg\phi & \text{ iff } \mathfrak{M}, s \models^- \phi \\ \mathfrak{M}, s \models^- \neg\phi & \text{ iff } \mathfrak{M}, s \models^+ \phi \\ \mathfrak{M}, s \models^+ \phi_1 \wedge \phi_2 & \text{ iff } \mathfrak{M}, s \models^+ \phi_1 \text{ and } \mathfrak{M}, s \models^+ \phi_2 \\ \mathfrak{M}, s \models^- \phi_1 \wedge \phi_2 & \text{ iff } \mathfrak{M}, s \models^- \phi_1 \text{ or } \mathfrak{M}, s \models^- \phi_2 \\ \mathfrak{M}, s \models^+ \phi_1 \vee \phi_2 & \text{ iff } \mathfrak{M}, s \models^+ \phi_1 \text{ or } \mathfrak{M}, s \models^+ \phi_2 \\ \mathfrak{M}, s \models^- \phi_1 \vee \phi_2 & \text{ iff } \mathfrak{M}, s \models^- \phi_1 \text{ and } \mathfrak{M}, s \models^- \phi_2 \\ \mathfrak{M}, s \models^+ \phi_1 \rightarrow \phi_2 & \text{ iff } \forall s' \succeq s : \mathfrak{M}, s' \models^+ \phi_1 \Rightarrow \mathfrak{M}, s' \models^+ \phi_2 \\ \mathfrak{M}, s \models^- \phi_1 \rightarrow \phi_2 & \text{ iff } \exists s' \preceq s : \mathfrak{M}, s' \models^- \phi_1 \ \& \ \mathfrak{M}, s' \models^- \phi_2 \\ \mathfrak{M}, s \models^+ \phi_1 \prec \phi_2 & \text{ iff } \exists s' \preceq s : \mathfrak{M}, s' \models^+ \phi_1 \ \& \ \mathfrak{M}, s' \not\models^+ \phi_2 \\ \mathfrak{M}, s \models^- \phi_1 \prec \phi_2 & \text{ iff } \forall s' \preceq s : \mathfrak{M}, s' \not\models^- \phi_1 \Rightarrow \mathfrak{M}, s' \not\models^- \phi_2 \\ \mathfrak{M}, s \models^+ \phi_1 \multimap \phi_2 & \text{ iff } \forall s' \succeq s : \mathfrak{M}, s' \models^+ \phi_1 \Rightarrow \mathfrak{M}, s' \models^+ \phi_2 \\ \mathfrak{M}, s \models^- \phi_1 \multimap \phi_2 & \text{ iff } \mathfrak{M}, s \models^+ \phi_1 \ \& \ \mathfrak{M}, s \models^- \phi_2 \\ \mathfrak{M}, s \models^+ \phi_1 \multimap \phi_2 & \text{ iff } \exists s' \preceq s : \mathfrak{M}, s' \models^+ \phi_1 \ \& \ \mathfrak{M}, s' \not\models^+ \phi_2 \\ \mathfrak{M}, s \models^- \phi_1 \multimap \phi_2 & \text{ iff } \mathfrak{M}, s \models^- \phi_1 \text{ or } \mathfrak{M}, s \models^+ \phi_2 \end{aligned}$$

Observe that in the definition above \models^+ conditions coincide with the Kripke semantics of Heyting–Brouwer logic [125]. Note, in particular, that the \models^+ conditions of co-implications (\prec and \multimap) use the converse order on the frame. This condition is dual to that of \rightarrow and \rightarrow and is in line with the usual interpretation of intuitionistic Kripke semantics.

²⁷Recall that linearity axioms correspond to the linearity conditions on the frames.

Namely, if ϕ is *true at w* , it is considered constructively proven (whence, it can never become not-true²⁸). The implication is a transformation of *every* proof of the antecedent into that of the succedent. This is why, the succedent should be true in every accessible state where the antecedent is true. Co-implication $\phi \prec \chi$ ²⁹, on the other hand, means that the ϕ excludes χ . This means that the ϕ must be true in a preceding state (and hence, in the current one) where χ was not true.

Definition 4.9 (Entailment in Kripke models). Γ (locally) entails ϕ in \mathbf{G}^2 — denoted $\Gamma \models \phi$ — iff for any \mathfrak{M} and $s \in \mathfrak{M}$ it holds that

$$\text{if } \mathfrak{M}, s \models^+ [\Gamma] \text{ then } \mathfrak{M}, s \models^+ \phi$$

Theorem 4.1 ($\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2$ completeness). *Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathbf{G}_{(\rightarrow, \prec)}^2}$. Then $\Gamma \models \phi$ iff $\Gamma \models_{\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2} \phi$.*

Proof. Recall first, that $\mathcal{H}\mathbf{biG}$ is strongly complete w.r.t. linear frames since **prel** defines prelinear frames (Definition 4.3) and other axioms along with modus ponens and necessitation axiomatise the bi-intuitionistic logic which is complete w.r.t. partially ordered frames. We use De Morgan laws to transform every formula into its negation normal form. Now let ϕ^* be a negation normal form of ϕ where each literal $\neg p$ is substituted with a fresh variable p^* . We know from [151, Lemma 12] that ϕ is falsified on a $\mathbf{G}_{(\rightarrow, \prec)}^2$ model iff ϕ^* is falsified on an HB model *over the same frame*. Thus, $\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2$ is complete w.r.t. the class of linearly ordered frames. \square

The completeness result for $\mathcal{H}\mathbf{G}_{(\rightarrow, \dashv)}^2$ can be proved in the same manner.

Theorem 4.2 ($\mathcal{H}\mathbf{G}_{(\rightarrow, \dashv)}^2$ completeness). *Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathbf{G}_{(\rightarrow, \dashv)}^2}$. Then $\Gamma \models \phi$ iff there is a derivation of ϕ from Γ in $\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2$ s.t. *nec* is applied only to $\mathcal{H}\mathbf{G}_{(\rightarrow, \dashv)}^2$ theorems.*

It now remains to show that every pair of \mathbf{G}^2 -valuations v_1 and v_2 on $[0, 1]$ can be faithfully transformed into valuations on some linear model and vice versa.

Definition 4.10 (Model counterpart of a \mathbf{G}^2 -valuation). Let \mathbf{v} be a \mathbf{G}^2 -valuation on $[0, 1]^{\mathbb{Q}}$. A model $\mathfrak{M}_{\mathbf{v}} = \langle \mathbb{Q}, \leq_{\mathbf{v}}, v_{\mathbf{v}}^+, v_{\mathbf{v}}^- \rangle$ is a *counterpart* of \mathbf{v} if for its valuations $v_{\mathbf{v}}^+$ and $v_{\mathbf{v}}^-$ it holds that:

$$\begin{aligned} v_{\mathbf{v}}^+(p) = \mathbb{Q} &\text{ iff } \mathbf{v}_1(p) = 1 & v_{\mathbf{v}}^-(p) = \mathbb{Q} &\text{ iff } \mathbf{v}_2(p) = 1 \\ v_{\mathbf{v}}^+(p) = \emptyset &\text{ iff } \mathbf{v}_1(p) = 0 & v_{\mathbf{v}}^-(p) = \emptyset &\text{ iff } \mathbf{v}_2(p) = 0 \\ v_{\mathbf{v}}^+(p) \subseteq v_{\mathbf{v}}^+(q) &\text{ iff } \mathbf{v}_1(p) \leq \mathbf{v}_1(q) & v_{\mathbf{v}}^-(p) \subseteq v_{\mathbf{v}}^-(q) &\text{ iff } \mathbf{v}_2(p) \leq \mathbf{v}_2(q) \\ v_{\mathbf{v}}^-(p) \subseteq v_{\mathbf{v}}^-(q) &\text{ iff } \mathbf{v}_2(p) \leq \mathbf{v}_1(q) & v_{\mathbf{v}}^+(p) \subseteq v_{\mathbf{v}}^+(q) &\text{ iff } \mathbf{v}_1(p) \leq \mathbf{v}_2(q) \end{aligned}$$

Lemma 4.1. *Let $\phi, \phi' \in \mathcal{L}_{\mathbf{G}_{(\rightarrow, \prec)}^2} \cup \mathcal{L}_{\mathbf{G}_{(\rightarrow, \dashv)}^2}$, \mathbf{v} be a \mathbf{G}^2 -valuation on $[0, 1]^{\mathbb{Q}}$, and $\mathfrak{M}_{\mathbf{v}}$ be a counterpart of \mathbf{v} . Then it holds that*

$$\begin{aligned} \mathbf{v}_1(\phi) \leq \mathbf{v}_1(\phi') &\text{ iff } v_{\mathbf{v}}^+(\phi) \subseteq v_{\mathbf{v}}^+(\phi') \\ \mathbf{v}_2(\phi) \leq \mathbf{v}_2(\phi') &\text{ iff } v_{\mathbf{v}}^-(\phi) \subseteq v_{\mathbf{v}}^-(\phi') \\ \mathbf{v}_1(\phi) \leq \mathbf{v}_2(\phi') &\text{ iff } v_{\mathbf{v}}^+(\phi) \subseteq v_{\mathbf{v}}^-(\phi') \\ \mathbf{v}_2(\phi) \leq \mathbf{v}_1(\phi') &\text{ iff } v_{\mathbf{v}}^-(\phi) \subseteq v_{\mathbf{v}}^+(\phi') \end{aligned}$$

Proof. We proceed by induction on ϕ . Then the basis cases of variables and constants hold by construction.

²⁸This is reflected in the persistence property: if $w \preceq w'$ and ϕ is true at w , it is also true at w' .

²⁹We mention only \prec for brevity.

4. Paraconsistent expansions of Gödel logic

For the induction steps, we consider only the case of $\phi = \psi \rightarrow \psi'$. The other ones are straightforward or can be obtained in a similar manner. In the following, we let $v_{\mathbf{v}}^{\circ}$ to stand for the counterpart of \mathbf{v}_j .

$$\begin{aligned} \mathbf{v}_1(\psi \rightarrow \psi') \leq \mathbf{v}_j(\phi') &\text{ iff } \begin{bmatrix} \mathbf{v}_1(\psi) \leq \mathbf{v}_1(\psi') \\ \text{and} \\ \mathbf{v}_j(\phi') \geq \mathbf{v}_2(\mathbf{0}) \end{bmatrix} \text{ or } \begin{bmatrix} \mathbf{v}_1(\psi) > \mathbf{v}_1(\psi') \\ \text{and} \\ \mathbf{v}_1(\psi') \leq \mathbf{v}_j(\phi') \end{bmatrix} \\ &\text{ iff } \begin{bmatrix} v_{\mathbf{v}}^+(\psi) \subseteq v_{\mathbf{v}}^+(\psi') \\ \text{and} \\ v_{\mathbf{v}}^{\circ}(\phi') \supseteq v_{\mathbf{v}}^-(\mathbf{0}) \end{bmatrix} \text{ or } \begin{bmatrix} v_{\mathbf{v}}^+(\psi) \supseteq v_{\mathbf{v}}^+(\psi') \\ \text{and} \\ v_{\mathbf{v}}^+(\psi') \subseteq v_{\mathbf{v}}^{\circ}(\phi') \end{bmatrix} & \text{(by IH)} \\ &\text{ iff } v_{\mathbf{v}}^+(\psi \rightarrow \psi') \subseteq v_{\mathbf{v}}^{\circ}(\phi') \end{aligned}$$

$$\begin{aligned} \mathbf{v}_2(\psi \rightarrow \psi') \leq \mathbf{v}_j(\phi') &\text{ iff } \mathbf{v}_1(\psi) \leq \mathbf{v}_j(\phi') \text{ or } \mathbf{v}_2(\psi') \leq \mathbf{v}_j(\phi') \\ &\text{ iff } v_{\mathbf{v}}^+(\psi) \subseteq v_{\mathbf{v}}^{\circ}(\phi') \text{ or } v_{\mathbf{v}}^-(\psi') \subseteq v_{\mathbf{v}}^{\circ}(\phi') & \text{(by IH)} \\ &\text{ iff } v_{\mathbf{v}}^-(\psi \rightarrow \psi') \subseteq v_{\mathbf{v}}^{\circ}(\phi') \end{aligned}$$

□

Definition 4.11 (Algebraic counterparts). Let $\mathfrak{M} = \langle W, \preceq, v^+, v^- \rangle$ be a \mathbf{G}^2 -model. We say that algebraic valuations $v_1^{\mathfrak{M}}$ and $v_2^{\mathfrak{M}}$ are *counterparts* of \mathfrak{M} if it holds that:

$$\begin{aligned} v_1^{\mathfrak{M}}(p) = 1 &\text{ iff } v^+(p) = W & v_2^{\mathfrak{M}}(p) = 1 &\text{ iff } v^-(p) = W \\ v_1^{\mathfrak{M}}(p) = 0 &\text{ iff } v^+(p) = \emptyset & v_2^{\mathfrak{M}}(p) = 0 &\text{ iff } v^-(p) = \emptyset \\ v_1^{\mathfrak{M}}(p) \leq v_1^{\mathfrak{M}}(q) &\text{ iff } v^+(p) \subseteq v^+(q) & v_2^{\mathfrak{M}}(p) \leq v_2^{\mathfrak{M}}(q) &\text{ iff } v^-(p) \subseteq v^-(q) \\ v_1^{\mathfrak{M}}(p) \leq v_2^{\mathfrak{M}}(q) &\text{ iff } v^+(p) \subseteq v^-(q) & v_2^{\mathfrak{M}}(p) \leq v_1^{\mathfrak{M}}(q) &\text{ iff } v^-(p) \subseteq v^+(q) \end{aligned}$$

Lemma 4.2. Let $\phi, \phi' \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)} \cup \mathcal{L}_{\mathbf{G}^2(\rightarrow, \dashv)}$. Then, for any \mathbf{G}^2 -model $\mathfrak{M} = \langle \mathbb{F}, v^+, v^- \rangle$ and any valuations $v_1^{\mathfrak{M}}$ and $v_2^{\mathfrak{M}}$ that are counterparts of \mathfrak{M} , it holds that:

$$\begin{aligned} v^+(\phi) \subseteq v^+(\phi') &\text{ iff } v_1^{\mathfrak{M}}(\phi) \leq v_1^{\mathfrak{M}}(\phi') \\ v^-(\phi) \subseteq v^-(\phi') &\text{ iff } v_2^{\mathfrak{M}}(\phi) \leq v_2^{\mathfrak{M}}(\phi') \\ v^+(\phi) \subseteq v^-(\phi') &\text{ iff } v_1^{\mathfrak{M}}(\phi) \leq v_2^{\mathfrak{M}}(\phi') \\ v^-(\phi) \subseteq v^+(\phi') &\text{ iff } v_2^{\mathfrak{M}}(\phi) \leq v_1^{\mathfrak{M}}(\phi') \end{aligned}$$

Proof. Analogously to Lemma 4.1. □

We can now finally prove the algebraic completeness.

Theorem 4.3. $\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2$ and $\mathcal{H}\mathbf{G}_{(\rightarrow, \dashv)}^2$ are strongly complete:

$$\Gamma \models_{\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2} \phi \text{ iff } \Gamma \models_{\mathbf{G}_{(\rightarrow, \prec)}^2} \phi \qquad \Gamma \models_{\mathcal{H}\mathbf{G}_{(\rightarrow, \dashv)}^2} \phi \text{ iff } \Gamma \models_{\mathbf{G}_{(\rightarrow, \dashv)}^2} \phi$$

Proof. The soundness part can be easily proved by a routine check of axioms and rules. Now, assume that $\Gamma \not\models_{\mathbf{G}_{(\rightarrow, \prec)}^2} \phi$. Then, by Corollary 4.1, there is a \mathbf{G}^2 -valuation s.t. $v_1[\Gamma] > v_1(\phi)$. Hence, by Lemma 4.1, there is a model \mathfrak{M} and $w \in \mathfrak{M}$ s.t. $\mathfrak{M}, w \models^+ [\Gamma]$ but $\mathfrak{M}, w \not\models^+ \phi$. Thus, by Theorem 4.1, we obtain that ϕ is not $\mathcal{H}\mathbf{G}_{(\rightarrow, \prec)}^2$ derivable from Γ .

The case of $\mathcal{H}\mathbf{G}_{(\rightarrow, \dashv)}^2$ can be tackled in a similar manner. □

Remark 4.6. Note from the proofs that in Lemmas 4.1 and 4.2, we can mix languages: not only can it happen that $\phi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \prec)}$ and $\phi' \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \dashv)}$ (or vice versa) but formulas can contain both kinds of (co-)implications.

4.2. Tableaux and complexity

In this section, we construct a constraint tableaux calculus for G^2 that we will expand with modal rules in Part II. We are adapting the idea³⁰ of constraint tableaux for \mathbf{L} . Now, however, we will be using not only numbers as constraints but also other formulas too. Our calculus is, thus, a middle ground between constraint tableaux and relational sequent calculi (cf., e.g., [106] for a relational sequent calculus for G). An alternative would be, for instance, to expand the decomposition calculus from [6, 7]. However, we argue that the constraint tableau approach is more streamlined since we do not need additional rules for the implication based on the type of the antecedent or succedent: all our rules are going to be of the form $\phi \circ \chi \lesssim \psi$ and $\phi \circ \chi \gtrsim \psi$ where $\lesssim \in \{\leq, <\}$, \circ is a language connective and ϕ , χ , and ψ are *arbitrary*.

Definition 4.12 (Constraint tableaux for $G^2 - \mathcal{T}(G^2)$). Let $\lesssim \in \{<, \leq\}$ and $\gtrsim \in \{<, \leq\}$. Branches contain:

- *formulaic constraints* of the form $\mathbf{x} : \phi \lesssim \mathbf{x}' : \phi'$ with $\mathbf{x}, \mathbf{x}' \in \{1, 2\}$;
- *numerical constraints* of the form $c \lesssim c'$ with $c, c' \in \{1, 0\}$ ³¹;
- *labelled formulas* of the form $\mathbf{x} : \phi * c$ with $* \in \{\lesssim, \gtrsim\}$ with $\mathbf{x} \in \{1, 2\}$.

We abbreviate all these types of entries with $\mathfrak{X} \lesssim \mathfrak{X}'$. Each branch can be extended by an application of one of the rules in Figure 4.1 where $\mathbf{c} \neq \mathbf{c}'$, $c \neq c'$, $\mathbf{c}, \mathbf{c}' \in \{0, 1\}$ and $c, c' \in \{0, 1\}$.

A tableau's branch \mathcal{B} is *closed* (and *open* otherwise) iff at least one of the following conditions applies:

- the transitive closure of \mathcal{B} under \lesssim contains $\mathfrak{X} < \mathfrak{X}$;
- $0 \geq 1 \in \mathcal{B}$, or $\mathfrak{X} > 1 \in \mathcal{B}$, or $\mathfrak{X} < 0 \in \mathcal{B}$.

A tableau is *closed* iff all its branches are closed. We say that there is a *tableau proof* of ϕ iff there is a closed tableau starting from $1 : \phi < 1$.

Remark 4.7 (Interpretation of constraints). Formulaic constraint $\mathbf{x} : \phi \leq \mathbf{x}' : \phi'$ encodes the fact that $v_{\mathbf{x}}(\phi) \leq v_{\mathbf{x}'}(\phi')$, similarly labelled formula $\mathbf{x} : \phi \leq c$ encodes the fact that $v_{\mathbf{x}}(\phi) \leq c$.

Definition 4.13 (Satisfying valuation of a branch). Let $\mathbf{x}, \mathbf{x}' \in \{1, 2\}$. Branch \mathcal{B} is *satisfied* by a valuation v iff

- $v_{\mathbf{x}}(\phi) \leq v_{\mathbf{x}'}(\phi')$ for any $\mathbf{x} : \phi \leq \mathbf{x}' : \phi' \in \mathcal{B}$ and
- $v_{\mathbf{x}}(\phi) \leq c$ for any $\mathbf{x} : \phi \leq c \in \mathcal{B}$ s.t. $c \in \{0, 1\}$.

Theorem 4.4 (Soundness and completeness of $\mathcal{T}(G^2)$). ϕ is G^2 -valid iff it has a $\mathcal{T}(G^2)$ proof.

Proof. For soundness, we check that if the premise of the rule is satisfied, then so is at least one of its conclusions. We consider $\rightarrow_2 \gtrsim$ as an example. Indeed, assume that $2 : \phi_1 \rightarrow \phi_2 \gtrsim \mathfrak{X}$ is satisfied and w.l.o.g. that $\mathfrak{X} = 1 : \psi$. Then, we have

$$\begin{aligned} v_2(\phi_1 \rightarrow \phi_2) \geq v_1(\psi) & \quad \text{iff} \quad \min(v_1(\phi_1), v_2(\phi_2)) \geq v_1(\psi) \\ & \quad \text{iff} \quad v_1(\phi_1) \geq v_1(\psi) \text{ and } v_2(\phi_2) \geq v_1(\psi) \end{aligned}$$

And thus $1 : \phi_1 \geq 1 : \psi$ and $2 : \phi_2 \geq 1 : \psi$ are both satisfied. Since no valuation can satisfy a closed branch, the result follows.

³⁰A similar decision procedure for Rational Pavelka logic was proposed in [96].

³¹Note that here, in contrast to the tableaux for \mathbf{L}^2 , c and c' are numbers. In fact, (cf. Fig. 4.1), 0 and 1 can appear as entries in the tableau.

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$$\begin{array}{c}
\neg_1 \lesssim \frac{1:\neg\phi \lesssim \mathfrak{X}}{2:\phi \lesssim \mathfrak{X}} \quad \neg_2 \lesssim \frac{2:\neg\phi \lesssim \mathfrak{X}}{1:\phi \lesssim \mathfrak{X}} \quad \neg_1 \gtrsim \frac{1:\neg\phi \gtrsim \mathfrak{X}}{2:\phi \gtrsim \mathfrak{X}} \quad \neg_2 \gtrsim \frac{2:\neg\phi \gtrsim \mathfrak{X}}{1:\phi \gtrsim \mathfrak{X}} \\
\\
\wedge_1 \gtrsim \frac{1:\phi \wedge \phi' \gtrsim \mathfrak{X}}{1:\phi \gtrsim \mathfrak{X} \quad 1:\phi' \gtrsim \mathfrak{X}} \quad \wedge_2 \lesssim \frac{2:\phi \wedge \phi' \lesssim \mathfrak{X}}{2:\phi \lesssim \mathfrak{X} \quad 2:\phi' \lesssim \mathfrak{X}} \quad \wedge_1 \lesssim \frac{1:\phi \wedge \phi' \lesssim \mathfrak{X}}{1:\phi \lesssim \mathfrak{X} \mid 1:\phi' \lesssim \mathfrak{X}} \quad \wedge_2 \gtrsim \frac{2:\phi \wedge \phi' \gtrsim \mathfrak{X}}{2:\phi \gtrsim \mathfrak{X} \mid 2:\phi' \gtrsim \mathfrak{X}} \\
\\
\vee_1 \lesssim \frac{1:\phi \vee \phi' \lesssim \mathfrak{X}}{1:\phi \lesssim \mathfrak{X} \quad 1:\phi' \lesssim \mathfrak{X}} \quad \vee_2 \gtrsim \frac{2:\phi \vee \phi' \gtrsim \mathfrak{X}}{2:\phi \gtrsim \mathfrak{X} \quad 2:\phi' \gtrsim \mathfrak{X}} \quad \vee_1 \gtrsim \frac{1:\phi \vee \phi' \gtrsim \mathfrak{X}}{1:\phi \gtrsim \mathfrak{X} \mid 1:\phi' \gtrsim \mathfrak{X}} \quad \vee_2 \lesssim \frac{2:\phi \vee \phi' \lesssim \mathfrak{X}}{2:\phi \lesssim \mathfrak{X} \mid 2:\phi' \lesssim \mathfrak{X}} \\
\\
\Delta_1 \leq \frac{1:\Delta\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid 1:\phi < 1} \quad \Delta_1 \geq \frac{1:\Delta\phi \geq \mathfrak{X}}{\mathfrak{X} \leq 0 \mid 1:\phi \geq 1} \quad \Delta_2 \leq \frac{2:\Delta\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid 2:\phi \leq 0} \quad \Delta_2 \geq \frac{2:\Delta\phi \geq \mathfrak{X}}{\mathfrak{X} \leq 0 \mid 2:\phi > 0} \\
\\
\rightarrow_1 \leq \frac{1:\phi \rightarrow \phi' \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \frac{1:\phi' \leq \mathfrak{X}}{1:\phi > 1:\phi'}} \quad \rightarrow_1 \gtrsim \frac{1:\phi \rightarrow \phi' \gtrsim \mathfrak{X}}{1:\phi \leq 1:\phi' \mid \frac{1:\phi' \gtrsim \mathfrak{X}}{1:\phi > 1:\phi'}} \quad \rightarrow_1 < \frac{1:\phi \rightarrow \phi' < \mathfrak{X}}{1:\phi' < \mathfrak{X} \mid 1:\phi > 1:\phi'} \\
\\
\rightarrow_2 \lesssim \frac{2:\phi \rightarrow \phi' \lesssim \mathfrak{X}}{2:\phi' \leq 2:\phi \mid \frac{2:\phi' \lesssim \mathfrak{X}}{0 \lesssim \mathfrak{X}}} \quad \rightarrow_2 \gtrsim \frac{2:\phi \rightarrow \phi' \gtrsim \mathfrak{X}}{\mathfrak{X} \leq 0 \mid \frac{2:\phi' \gtrsim \mathfrak{X}}{2:\phi' > 2:\phi}} \quad \rightarrow_2 > \frac{2:\phi \rightarrow \phi' > \mathfrak{X}}{2:\phi' > \mathfrak{X} \mid 2:\phi' > 2:\phi} \\
\\
\leftarrow_1 \lesssim \frac{1:\phi < \phi' \lesssim \mathfrak{X}}{1:\phi \leq 1:\phi' \mid \frac{1:\phi \lesssim \mathfrak{X}}{0 \lesssim \mathfrak{X}}} \quad \leftarrow_1 > \frac{1:\phi < \phi' > \mathfrak{X}}{1:\phi > \mathfrak{X} \mid 1:\phi > 1:\phi'} \quad \leftarrow_1 \geq \frac{1:\phi < \phi' \geq \mathfrak{X}}{\mathfrak{X} \leq 0 \mid \frac{1:\phi \geq \mathfrak{X}}{1:\phi > 1:\phi'}} \\
\\
\leftarrow_2 \gtrsim \frac{2:\phi < \phi' \gtrsim \mathfrak{X}}{2:\phi \gtrsim \mathfrak{X} \mid \frac{2:\phi' \leq 2:\phi}{1 \gtrsim \mathfrak{X}}} \quad \leftarrow_2 \leq \frac{1:\phi < \phi' \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \frac{2:\phi \leq \mathfrak{X}}{2:\phi' > 2:\phi}} \quad \leftarrow_2 < \frac{2:\phi < \phi' < \mathfrak{X}}{2:\phi < \mathfrak{X} \mid 2:\phi < 2:\phi'} \\
\\
\rightarrow_1 \leq \frac{1:\phi \rightarrow \phi' \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \frac{1:\phi' \leq \mathfrak{X}}{1:\phi > 1:\phi'}} \quad \rightarrow_1 \gtrsim \frac{1:\phi \rightarrow \phi' \gtrsim \mathfrak{X}}{1:\phi \leq 1:\phi' \mid \frac{1:\phi' \gtrsim \mathfrak{X}}{1:\phi > 1:\phi'}} \quad \rightarrow_2 \lesssim \frac{2:\phi \rightarrow \phi' \lesssim \mathfrak{X}}{1:\phi \lesssim \mathfrak{X} \mid 2:\phi' \lesssim \mathfrak{X}} \\
\\
\rightarrow_1 < \frac{1:\phi \rightarrow \phi' < \mathfrak{X}}{1:\phi' < \mathfrak{X} \mid 1:\phi > 1:\phi'} \quad \rightarrow_2 \gtrsim \frac{2:\phi \rightarrow \phi' \gtrsim \mathfrak{X}}{1:\phi \gtrsim \mathfrak{X} \mid 2:\phi' \gtrsim \mathfrak{X}} \quad \rightarrow_1 > \frac{1:\phi \rightarrow \phi' > \mathfrak{X}}{1:\phi > \mathfrak{X} \mid 1:\phi > 1:\phi'} \quad \rightarrow_2 \lesssim \frac{2:\phi \rightarrow \phi' \lesssim \mathfrak{X}}{2:\phi \lesssim \mathfrak{X} \mid 1:\phi' \lesssim \mathfrak{X}} \\
\\
\rightarrow_1 \geq \frac{1:\phi \rightarrow \phi' \geq \mathfrak{X}}{\mathfrak{X} \leq 0 \mid \frac{1:\phi \geq \mathfrak{X}}{1:\phi > 1:\phi'}} \quad \rightarrow_1 \lesssim \frac{1:\phi \rightarrow \phi' \lesssim \mathfrak{X}}{1:\phi \leq 1:\phi' \mid \frac{1:\phi \lesssim \mathfrak{X}}{0 \lesssim \mathfrak{X}}} \quad \rightarrow_2 \gtrsim \frac{2:\phi \rightarrow \phi' \gtrsim \mathfrak{X}}{2:\phi \gtrsim \mathfrak{X} \mid 1:\phi' \gtrsim \mathfrak{X}}
\end{array}$$

Figure 4.1.: Rules of $\mathcal{T}(\mathbf{G}^2)$. Vertical bars denote branching; $c \neq c'$, $c, c' \in \{0, 1\}$.

For completeness, we show that every complete open branch \mathcal{B} is satisfiable. We construct the satisfying valuation as follows. If $\mathbf{x} : p \geq 1 \in \mathcal{B}$, we set $v_{\mathbf{x}}(p) = 1$. If $\mathbf{x} : p \leq 0 \in \mathcal{B}$, we set $v_{\mathbf{x}}(p) = 0$. To set the values of the remaining variables q_1, \dots, q_n , we proceed as follows. Denote \mathcal{B}^+ the transitive closure of \mathcal{B} under \lesssim and let

$$[\mathbf{x} : q_i] = \left\{ \mathbf{x}' : q_j \mid \begin{array}{l} (\mathbf{x} : q_i \leq \mathbf{x}' : q_j \in \mathcal{B}^+ \text{ or } \mathbf{x} : q_i \geq \mathbf{x}' : q_j \in \mathcal{B}^+) \\ \text{and} \\ \mathbf{x} : q_i < \mathbf{x}' : q_j \notin \mathcal{B}^+ \text{ and } \mathbf{x} : q_i > \mathbf{x}' : q_j \notin \mathcal{B}^+ \end{array} \right\}$$

It is clear that there are at most $2n$ $[\mathbf{x} : q_i]$'s since the only possible loop in \mathcal{B}^+ is $\mathbf{x} : r \leq \dots \leq \mathbf{x} : r$, but in such a loop all elements belong to $[\mathbf{x} : r]$. We put $[\mathbf{x} : q_i] \preceq [\mathbf{x}' : q_j]$ iff there are $\mathbf{x} : r \in [\mathbf{x} : q_i]$ and $\mathbf{x}' : r' \in [\mathbf{x}' : q_j]$ s.t. $\mathbf{x} : r \leq \mathbf{x}' : r' \in \mathcal{B}^+$.

We now set the valuation of these variables as follows

$$v_{\mathbf{x}}(q_i) = \frac{|\{[\mathbf{x}' : q_j] \mid [\mathbf{x}' : q_j] \preceq [\mathbf{x} : q_i]\}|}{2n} \quad (4.2)$$

Thus, all constraints containing only variables are satisfied.

It remains to show that all other constraints are satisfied. For that, we prove that if at least one conclusion of the rule is satisfied, then so is the premise. We consider only the case of $\rightarrow_2 \lesssim$. Let $1 : \phi_1 \lesssim \mathfrak{X}$ be satisfied. W.l.o.g., assume that $\mathfrak{X} = 2 : \psi$ and $\lesssim = <$. Thus, $v_1(\phi_1) < v_2(\psi)$. Recall that $v_2(\phi_1 \rightarrow \phi_2) = \min(v_1(\phi_1), v_2(\phi_2))$. Hence, $v_2(\phi_1 \rightarrow \phi_2) < v_2(\psi)$, and $2 : \phi_1 \rightarrow \phi_2 < 2 : \psi$ is satisfied as desired. By the same reasoning, we have that if $2 : \phi_2 \lesssim \mathfrak{X}$ is satisfied, then so is $2 : \phi_1 \rightarrow \phi_2 \lesssim \mathfrak{X}$.

The cases of other rules can be shown in the same fashion. \square

Theorem 4.5. *Satisfiability for $G_{(\rightarrow, <)}^2$ and $G_{(\rightarrow, \rightarrow)}^2$ is NP-complete. Validity is coNP-complete for both G^2 's.*

Proof. It follows from the proof of Theorem 4.4 that the satisfiability of $G_{(\rightarrow, <)}^2$ and $G_{(\rightarrow, \rightarrow)}^2$ is in NP: we obtain the valuation from (4.2), and it takes polynomial time to check that it indeed satisfies the formula.

The NP-hardness follows since G^2 's are conservative extensions of G whose satisfiability and validity are NP- and coNP-complete respectively. \square

4.3. Semantical properties

Recall from Lemma 3.6 that changing the set of designated values in \mathfrak{L}^2 resulted in new formulas becoming valid. In this section, we will see that this is not the case in G^2 . Moreover, we will show that there are only finitely many G^2 entailments generated by filters on $[0, 1]^{\mathfrak{M}}$.

4.3.1. Validity in filters on $[0, 1]^{\mathfrak{M}}$

First, we show that the values \mathcal{L}_{biG} - and $\mathcal{L}_{G_{(\rightarrow, <)}}^2$ -formulas range over 0, 1, and the values of their variables. (The same property is known for the Gödel logic [80, Section 9.1].)

Lemma 4.3. *For every $\phi \in \mathcal{L}_{\text{biG}}$, every valuation v s.t. $v(\phi) < 1$, and every $0 < x \leq 1$, we have that*

1. $v(\phi) \in \{v(p) : p \in \text{Prop}(\phi)\} \cup \{0\}$, and
2. there exists a valuation v' s.t. $v'(\phi) \leq x$.

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Proof. We begin with 1. and proceed by induction. Since $v(\phi) \in \{0, 1\} \cup \{v(p) : p \in \mathbf{Prop}(\phi)\}$ for Gödel formulas [82, Theorem 2], we only need to consider $\phi = \phi_1 \prec \phi_2$.

If $v(\phi_1 \prec \phi_2) < 1$, then $v(\phi_1) < 1$ and $v(\phi_1 \prec \phi_2) = v(\phi_1)$ or $v(\phi_1 \prec \phi_2) = 0$ and $v(\phi_1) \leq v(\phi_2)$. By the induction hypothesis, there exist $q_1 \in \mathbf{Prop}(\phi_1)$ and $q_2 \in \mathbf{Prop}(\phi_2)$ s.t. one of the following holds.

- i. $v(q_1) = v(\phi_1)$ and $v(q_2) = v(\phi_2)$.
- ii. $v(\phi_1) = 0$ and $v(\phi_2) = 0$.
- iii. $v(q_1) = v(\phi_1)$ and $v(\phi_2) = 0$.
- iv. $v(\phi_1) = 0$ and $v(q_2) = v(\phi_2)$.

In every case i.–iv., it follows that $v(\phi_1 \prec \phi_2) \in \{v(q_1), v(q_2), 0\}$, as required.

For 2., we let $v(\phi) < 1$, $n > 0$ and $\frac{1}{n} \leq x$. We construct $v'(\phi)$ as follows: $v'(p) = \frac{v(p)}{n}$. We can prove by induction on ϕ that $v'(\phi) < 1$. Hence, by the previous item, we have that

$$v'(\phi) \in \left\{ \frac{v(p)}{n} : p \in \mathbf{Prop}(\phi) \right\} \cup \{0\}$$

whence, $v'(\phi) \leq x$ as required. \square

Lemma 4.4.

1. Let $\phi \in \mathcal{L}_{\mathbf{G}^2_{(\rightarrow, \prec)}}$. Then for every valuation v s.t. $v_1(\phi) \neq 1$ and for every $0 < x \leq 1$, we have that
 - a) $v_1(\phi) \in \{v_1(p) : p \in \mathbf{Prop}(\phi)\} \cup \{v_2(p) : p \in \mathbf{Prop}(\phi)\} \cup \{0\}$ and
 - b) there exists a valuation v' s.t. $v'_1(\phi) \leq x$.
2. Let $\phi \in \mathcal{L}_{\mathbf{G}^2_{(\rightarrow, \rightarrow\circ)}}$. Then for every valuation v s.t. $v_1(\phi) \neq 1$ and for every $0 < x \leq 1$, we have that
 - a) $v_1(\phi) \in \{v_1(p) : p \in \mathbf{Prop}(\phi)\} \cup \{v_2(p) : p \in \mathbf{Prop}(\phi)\} \cup \{0\}$ and
 - b) there exists a valuation v' s.t. $v'_1(\phi) \leq x$.

Proof. We prove the first part, as the second one can be established in the same way. First, we can assume that ϕ is in \neg negation normal form (we have all the necessary De Morgan laws).³²

For (a), we let v be a valuation s.t. $v_1(\phi) \neq 1$. Since ϕ is in negation normal form, it can be perceived as a formula $\phi^\neg \in \mathcal{L}_{\mathbf{biG}}$ but over literals instead of propositional variables. Now let \mathbf{v} be the new valuation over the set of literals defined as follows: $\mathbf{v}(p) = v_1(p)$ and $\mathbf{v}(\neg p) = v_1(\neg p) = v_2(p)$.

By applying Lemma 4.3 to ϕ^\neg and \mathbf{v} , we get that

$$\mathbf{v}(\phi^\neg) \in \{\mathbf{v}(l) : l \in \mathbf{Lit}(\phi^\neg)\} \cup \{0\} = (\{v_1(p) : p \in \mathbf{Prop}(\phi)\} \cup \{v_2(p) : p \in \mathbf{Prop}(\phi)\} \cup \{0\})$$

as required.

To prove (b), we notice that by Lemma 4.3, there exists a \mathbf{biG} valuation \mathbf{v}' on the set of literals s.t. $\mathbf{v}'(\phi) \leq x$. Let now v' be defined as follows: $v'(p) = (\mathbf{v}'(p), \mathbf{v}'(\neg p))$ for every $p \in \mathbf{Prop}$. We get that $v'_1(\phi) \leq x$, as required. \square

Theorem 4.6.

1. Let $\phi \in \mathbf{G}^2_{(\rightarrow, \prec)}$ be s.t. $v(\phi) \geq_{[0,1]^\times} (x, y)$ for any v and some fixed $(x, y) \neq (0, 1)$. Then ϕ is $\mathbf{G}^2_{(\rightarrow, \prec)}$ -valid.

³²Note that this also works in $\mathbf{G}^2_{(\rightarrow, \rightarrow\circ)}$ because we only care about v_1 .

2. Let $\phi \in \mathbf{G}_{(\rightarrow, \rightarrow)}^2$ be s.t. $v_1(\phi) \geq x$ for any v and some fixed $x > 0$. Then ϕ is $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$ -valid.

Proof. Again, we prove only 1. since 2. can be shown similarly. We assume w.l.o.g. that ϕ is in negation normal form.

Observe that (x, x) points are not affected by \neg . Now recall that by Proposition 4.3, if $v(\phi) \neq (1, 0)$, there is a v' s.t. $v'_1(\phi) \neq 1$. Furthermore, notice that if $\{(0, 0), (1, 1)\} \subseteq (x, y)^\uparrow$ then $(x, y)^\uparrow = (0, 1)^\uparrow$ because $(x, y)^\uparrow$ is a filter. Hence, we have that $\{(0, 0), (1, 1)\} \not\subseteq (x, y)^\uparrow$, which implies that

$$\begin{aligned} \exists(z, z) \forall(x', y') \in (x, y)^\uparrow : (z, z) \neq (1, 1) \text{ and } z \geq y' \\ \text{or} \\ \exists(z, z) \forall(x', y') \in (x, y)^\uparrow : (z, z) \neq (0, 0) \text{ and } z \leq x' \end{aligned}$$

By Proposition 4.3, we know that $v(\phi) = (0, 0)$ iff $v^*(\phi) = (1, 1)$, whence we can state w.l.o.g. that $(0, 0) \notin (x, y)^\uparrow$ and that $v(\phi) \neq (0, 0)$ for every v . Thus, there is a (z, z) s.t. $(z, z) \neq (0, 0)$ and for any $(x', y') \in (x, y)^\uparrow$, we have $z \leq x'$.

Assume for contradiction, that $v'(\phi) \neq (1, 0)$. There are two cases.

Case 1: $v'_1(\phi) \neq 1$. Since $x > 0$, by Lemma 4.4, we have that there exists a valuation v' s.t. $v'_1(\phi) \leq \frac{x}{2} < x$. Hence, $v'(\phi) \notin (x, y)^\uparrow$, which contradicts the fact that ϕ is $\mathbf{G}_{(x, y)}^2$ -valid.

Case 2: $v'_1(\phi) = 1$. Then $v'_2(\phi) \neq 0$ and $v'_1^*(\phi) = 1 - v'_2(\phi) \neq 1$ (see Proposition 4.3 for definition of v'^*). We proceed as in the previous case. \square

4.3.2. Entailments over filters on $[0, 1]^\times$

We show that there is only a finite set of entailments arising from the sets of designated values defined as point-generated filters on $[0, 1]^\times$. First, we define several (non-exclusive) classes of filters on $[0, 1]^\times$.

Definition 4.14 (Classes of filters). We are going to discern between the following classes of filters $(x, y)^\uparrow$.

- *Nonparaconsistent* — where $x > y$.
- *Paraconsistent* — where $x \leq y$.
- *Non-prime* — where $y < 1$ and $x > 0$.
- *Prime* — where either $y = 1$, or $x = 0$.

Given a filter $(x, y)^\uparrow$, we denote with $\models_{(x, y)^\uparrow}$ the \mathbf{G}^2 -entailment generated by it.

Henceforth, we will be considering *only prime filters* when dealing with $\mathcal{L}_{\mathbf{G}^2(\rightarrow, \rightarrow)}$ formulas since, for example, $(p \wedge q) \rightarrow q$ is not valid on any filter not including $(1, 1)$ (and every filter including $(1, 1)$ is prime).

Lemma 4.5. For any v , let \bar{v} to be a valuation s.t. $v_i(p) \leq v_j(p')$ iff $\bar{v}_i(p) \leq \bar{v}_j(p')$ ($i, j \in \{1, 2\}$). Then, for any $\phi, \phi' \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \rightarrow)} \cup \mathcal{L}_{\mathbf{G}^2(\rightarrow, \rightarrow)}$, it holds that $v_i(\phi) \leq v_j(\phi')$ iff $\bar{v}_i(\phi) \leq \bar{v}_j(\phi')$.

Proof. We assume w.l.o.g. that ϕ and ϕ' are in NNF. The cases of variables hold by construction.

$$\boxed{\phi = \neg p}$$

$$\begin{aligned} v_i(\neg p) \leq v_k(\phi') \text{ iff } v_j(p) \leq v_k(\phi') & \quad (i \neq j; i, j, k \in \{1, 2\}) \\ \text{iff } \bar{v}_j(p) \leq \bar{v}_k(\phi') & \quad (\text{by IH}) \\ \text{iff } \bar{v}_i(\neg p) \leq \bar{v}_k(\phi') & \end{aligned}$$

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$$\boxed{\phi = \psi \rightarrow \psi'}$$

$$\begin{aligned} v_1(\psi \rightarrow \psi') \leq v_j(\phi') \text{ iff } & \begin{bmatrix} v_1(\psi') \leq v_j(\phi') \\ \text{and} \\ v_1(\psi) > v_1(\psi') \end{bmatrix} \text{ or } \begin{bmatrix} 1 = v_j(\phi') \\ \text{and} \\ v_1(\psi) \leq v_1(\psi') \end{bmatrix} \\ \text{iff } & \begin{bmatrix} \bar{v}_1(\psi') \leq \bar{v}_j(\phi') \\ \text{and} \\ \bar{v}_1(\psi) > \bar{v}_1(\psi') \end{bmatrix} \text{ or } \begin{bmatrix} \bar{v}_1(\mathbf{1}) = \bar{v}_j(\phi') \\ \text{and} \\ \bar{v}_1(\psi) \leq \bar{v}_1(\psi') \end{bmatrix} \quad (\text{by IH}) \\ \text{iff } & \bar{v}_1(\psi \rightarrow \psi') \leq \bar{v}_j(\phi') \end{aligned}$$

$$\begin{aligned} v_2(\psi \rightarrow \psi') \leq v_j(\phi') \text{ iff } & v_2(\psi') \leq v_j(\phi') \text{ or } v_2(\psi') \leq v_2(\psi) \\ \text{iff } & \bar{v}_2(\psi') \leq \bar{v}_j(\phi') \text{ or } \bar{v}_2(\psi') \leq \bar{v}_2(\psi) \quad (\text{by IH}) \\ \text{iff } & \bar{v}_2(\psi \rightarrow \psi') \leq \bar{v}_j(\phi') \end{aligned}$$

$$\boxed{\phi = \psi \prec \psi'} \text{ Dually to } \phi = \psi \rightarrow \psi'.$$

The cases of \wedge , \vee , \rightarrow , and \rightarrow are straightforward. \square

Proposition 4.4. *Let $(x, y)^\dagger$ be paraconsistent and not prime. Then, there exist a satisfiable formula τ s.t. $\tau \wedge \neg\tau \models_{(x,y)^\dagger} \mathbf{0}$ and $\models_{(x,y)^\dagger}$ is not closed under*

$$\frac{\phi \models \psi \quad \chi \models \psi}{\phi \vee \chi \models \psi}$$

Proof. Consider $\tau(p) := (p \wedge \neg p) \rightarrow \mathbf{0}$. It is evident that

$$\forall v : v(\tau(p)) \in \{(1, 0), (1, 1), (0, 0), (0, 1)\}$$

Since $(x, y)^\dagger$ is not prime, $(0, 0), (1, 1) \notin (x, y)^\dagger$. Thus, $\tau(p) \wedge \neg\tau(p) \models_{(x,y)^\dagger} \mathbf{0}$ because $v(\tau(p) \wedge \neg\tau(p)) \notin (x, y)^\dagger$ for any v . Define now $v'(p) = (1, 1)$ and $v'(q) = (0, 0)$. Observe that $v'(\tau(p)) = (0, 0)$ and $v'(\tau(q)) = (1, 1)$. But then,

$$v'((\tau(p) \wedge \neg\tau(p)) \vee (\tau(q) \wedge \neg\tau(q))) = (1, 0)$$

whence

$$(\tau(p) \wedge \neg\tau(p)) \vee (\tau(q) \wedge \neg\tau(q)) \not\models_{(x,y)^\dagger} \mathbf{0}$$

as desired. \square

The next statement establishes the *lower bound on the number of \mathbf{G}^2 -entailments over $\mathcal{L}_{\mathbf{G}^2_{(\rightarrow, \prec)}}$ formulas.*

Proposition 4.5. *Let $(x, y)^\dagger$ be a non-paraconsistent filter, let $(x', y')^\dagger$ be a paraconsistent and non-prime filter, and $0 < z < 1$. Then $\models_{(1,0)^\dagger}$, $\models_{(1,1)^\dagger}$, $\models_{(x,y)^\dagger}$, $\models_{(\frac{1}{2}, \frac{1}{2})^\dagger}$, $\models_{(z,1)^\dagger}$, and $\models_{(x',y')^\dagger}$ (recall Remark 3.9) are all pairwise distinct.*

Proof. The following statements are easy to establish by a routine check.

- $p \wedge \neg p \models q$ is valid only for $(x, y)^\dagger$ and $(1, 0)^\dagger$.
- $(p \prec q) \wedge q \models r$ is valid only for $(1, 0)^\dagger$ and $(1, 1)^\dagger$.
- $(p \prec \neg p) \wedge \neg p \models q$ is valid only on $(1, 0)^\dagger$, $(1, 1)^\dagger$, and $(\frac{1}{2}, \frac{1}{2})^\dagger$.
- $\tau(p) \wedge \neg\tau(p) \models q$ is valid only for non-prime filters.

□

Let us now prove that the bound is exact.

Definition 4.15 (\Downarrow -equivalence). We say that $(x, y)^\uparrow$ is \Downarrow -reducible to $(x', y')^\uparrow$ iff for any valuation v and formula ϕ , there is a valuation v^\Downarrow s.t. $v(\phi) \in (x, y)^\uparrow$ iff $v^\Downarrow(\phi) \in (x', y')^\uparrow$.

Two filters $(x, y)^\uparrow$ and $(x', y')^\uparrow$ are \Downarrow -equivalent iff they are \Downarrow -reducible to one another.

It is evident that if $(x, y)^\uparrow$ and $(x', y')^\uparrow$ are \Downarrow -equivalent, $\models_{(x, y)^\uparrow}$ coincides with $\models_{(x', y')^\uparrow}$.

The next lemma gives a sufficient condition for the reduction of $\models_{(x, y)^\uparrow}$ to $\models_{(x', y')^\uparrow}$.

Lemma 4.6. Let $(x, y), (x', y') \neq (1, 0)$ and let $(x, y)^\uparrow$ be a filter. And further, for any v and any filter $(x', y')^\uparrow$, let v^\Downarrow be a valuation s.t.: for all $p, p' \in \text{Prop}$,

$$\begin{aligned} v_1(p) \geq x & \text{ iff } v_1^\Downarrow(p) \geq x' \\ v_1(\neg p) \geq x & \text{ iff } v_1^\Downarrow(\neg p) \geq x' \\ v_2(p) \leq y & \text{ iff } v_2^\Downarrow(p) \leq y' \\ v_2(\neg p) \leq y & \text{ iff } v_2^\Downarrow(\neg p) \leq y' \\ v_i(p) \leq v_j(p') & \text{ iff } v_i^\Downarrow(p) \leq v_j^\Downarrow(p') \quad \text{for all } i, j \in \{1, 2\} \end{aligned}$$

Then, for any $\phi, \phi' \in \mathcal{L}_{\mathcal{G}_{(\rightarrow, \prec)}^2}$, it holds that

$$\begin{aligned} v_1(\phi) \geq x & \text{ iff } v_1^\Downarrow(\phi) \geq x' \\ v_1(\neg \phi) \geq x & \text{ iff } v_1^\Downarrow(\neg \phi) \geq x' \\ v_2(\phi) \leq y & \text{ iff } v_2^\Downarrow(\phi) \leq y' \\ v_2(\neg \phi) \leq y & \text{ iff } v_2^\Downarrow(\neg \phi) \leq y' \\ v_i(\phi) \leq v_j(\phi') & \text{ iff } v_i^\Downarrow(\phi) \leq v_j^\Downarrow(\phi') \quad \text{for all } i, j \in \{1, 2\} \end{aligned}$$

Proof. We proceed by induction on ϕ . In addition, observe that the last clause holds by Lemma 4.5. Now, assume, w.l.o.g. that ϕ is in NNF. The basis cases of literals hold by the construction of v^\Downarrow . The only non-trivial case is that of $\phi = \psi \rightarrow \psi'$ ($\phi = \psi \prec \psi'$ is obtained dually). Thus, it suffices to consider only the first and the third clauses.

$$\begin{aligned} v_1(\psi \rightarrow \psi') \geq x & \text{ iff } v_1(\psi) \leq v_1(\psi') \text{ or } v_1(\psi') \geq x \\ & \text{ iff } v_1^\Downarrow(\psi) \leq v_1^\Downarrow(\psi') \text{ or } v_1^\Downarrow(\psi') \geq x' & \text{(by IH and Lemma 4.5)} \\ & \text{ iff } v_1^\Downarrow(\psi \rightarrow \psi') \geq x' \end{aligned}$$

$$\begin{aligned} v_2(\psi \rightarrow \psi') \leq y & \text{ iff } v_2(\psi') \leq v_2(\psi) \text{ or } v_2(\psi') \leq y \\ & \text{ iff } v_2^\Downarrow(\psi') \leq v_2^\Downarrow(\psi) \text{ or } v_2^\Downarrow(\psi') \leq y' & \text{(by IH and Lemma 4.5)} \\ & \text{ iff } v_2^\Downarrow(\psi \rightarrow \psi') \leq y' \end{aligned}$$

□

Lemma 4.7. Define v^\uparrow as follows.

$$v_i^\uparrow(p) = \begin{cases} 1 & \text{iff } v_i(p) \geq x \\ v_i(p) & \text{otherwise} \end{cases} \quad (i \in \{1, 2\})$$

Then, for any $\phi, \phi' \in \mathcal{L}_{\mathcal{G}_{(\rightarrow, \prec)}^2}$ the following holds.

$$\begin{aligned} v_i(\phi) \geq x & \text{ iff } v_i^\uparrow(\phi) = 1 \\ v_i(\phi) < x & \text{ iff } v_i^\uparrow(\phi) < 1 \\ v_i(\phi) \leq v_j(\phi') & \text{ then } v_i^\uparrow(\phi) \leq v_j^\uparrow(\phi') \end{aligned}$$

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Proof. Again, let w.l.o.g. ϕ and ϕ' be in NNF. Then, the cases of variables, literals and constants hold by construction. The cases of \wedge and \vee are straightforward. We consider the implicative case ($\phi = \psi \prec \psi'$ is obtained dually).

$$\begin{aligned} v_1(\psi \rightarrow \psi') \geq x &\text{ iff } v_1(\psi) \leq v_1(\psi') \text{ or } v_1(\psi') \geq x \\ &\text{ then } v_1^\uparrow(\psi) \leq v_1(\psi') \text{ or } v_1^\uparrow(\psi') = 1 && \text{(by IH)} \\ &\text{ then } v_1^\uparrow(\psi \rightarrow \psi') = 1 \end{aligned}$$

$$\begin{aligned} v_1(\psi \rightarrow \psi') < x &\text{ iff } v_1(\psi) > v_1(\psi') < x \\ &\text{ then } v_1^\uparrow(\psi) > v_1(\psi') > v_1^\uparrow(\psi') < 1 && \text{(by IH)} \\ &\text{ then } v_1^\uparrow(\psi \rightarrow \psi') < 1 \end{aligned}$$

$$\begin{aligned} v_1(\psi \rightarrow \psi') \leq v_j(\phi') &\text{ iff } \left[\begin{array}{c} v_1(\psi) > v_1(\psi') \\ \text{and} \\ v_1(\psi') \leq v_j(\phi') \end{array} \right] \text{ or } \left[\begin{array}{c} v_1(\psi) \leq v_1(\psi') \\ \text{and} \\ 1 \leq v_j(\phi') \end{array} \right] \\ &\text{ then } \left[\begin{array}{c} v_1^\uparrow(\psi) > v_1^\uparrow(\psi') \\ \text{and} \\ v_1^\uparrow(\psi') \leq v_j^\uparrow(\phi') \end{array} \right] \text{ or } \left[\begin{array}{c} v_1^\uparrow(\psi) \leq v_1^\uparrow(\psi') \\ \text{and} \\ 1 \leq v_j^\uparrow(\phi') \end{array} \right] && \text{(by IH)} \\ &\text{ then } v_1^\uparrow(\psi \rightarrow \psi') \leq v_j^\uparrow(\phi') \end{aligned}$$

$$\begin{aligned} v_2(\psi \rightarrow \psi') \geq x &\text{ iff } v_2(\psi') \geq x \text{ and } v_2(\psi') > v_2(\psi) && \text{(since } x > 0) \\ &\text{ then } v_2^\uparrow(\psi') = 1 \text{ and } v_2^\uparrow(\psi') > v_2^\uparrow(\psi) && \text{(by IH)} \\ &\text{ then } v_2^\uparrow(\psi \rightarrow \psi') = 1 \end{aligned}$$

$$\begin{aligned} v_2(\psi \rightarrow \psi') < x &\text{ iff } v_2(\psi') < x \text{ or } v_2(\psi') \leq v_2(\psi) \\ &\text{ then } v_2^\uparrow(\psi') < 1 \text{ or } v_2^\uparrow(\psi') \leq v_2^\uparrow(\psi) && \text{(by IH)} \\ &\text{ then } v_2^\uparrow(\psi \rightarrow \psi') < 1 \end{aligned}$$

$$\begin{aligned} v_2(\psi \rightarrow \psi') \leq v_j(\phi') &\text{ iff } v_2(\psi') \leq x \text{ or } v_2(\psi') \leq v_2(\psi) \\ &\text{ then } v_2^\uparrow(\psi') \leq v_j^\uparrow(\phi') \text{ or } v_2^\uparrow(\psi') \leq v_2^\uparrow(\psi) && \text{(by IH)} \\ &\text{ then } v_2^\uparrow(\psi \rightarrow \psi') \leq v_j^\uparrow(\phi') \end{aligned}$$

□

Theorem 4.7. *The following filters are \Downarrow -equivalent.*

1. Any nonparaconsistent filters $(x, y)^\uparrow$ and $(x', y')^\uparrow$ for $x, x' \neq 1$ and $y, y' \neq 0$.
2. Any non-prime paraconsistent filters $(x, y)^\uparrow$ and $(x', y')^\uparrow$.
3. Any filters $(z, z)^\uparrow$ and $(z', z')^\uparrow$.
4. Any prime filters $(x, 1)^\uparrow$ and $(x', 1)^\uparrow$ for $x, x' < 1$ or $(0, y)^\uparrow$ and $(0, y')^\uparrow$ for $y, y' > 0$.
5. Any filters $(1, y)^\uparrow$ and $(1, y')$ for $y, y' < 1$ or $(x, 0)^\uparrow$ and $(x', 0)^\uparrow$ for $x, x' > 0$.

6. $(1, 1)^\uparrow$ and $(0, 0)^\uparrow$.

Proof of Theorem 4.7.1. Let $(x_0, y_0)^\uparrow$ and $(x_1, y_1)^\uparrow$ be two nonparaconsistent filters. We have three cases.

1. $(x_0, y_0)^\uparrow \subseteq (x_1, y_1)^\uparrow$
2. $(x_0, y_0)^\uparrow \supseteq (x_1, y_1)^\uparrow$
3. $(x_0, y_0)^\uparrow \not\subseteq (x_1, y_1)^\uparrow$ and $(x_0, y_0)^\uparrow \not\supseteq (x_1, y_1)^\uparrow$

Case 1. Let v be a \mathbb{G}^2 valuation. We construct v^\uparrow s.t.

- (1.) $v_1(p) \geq x_1$ iff $v_1^\uparrow(p) \geq x_0$
- (2.) $v_1(\neg p) \geq x_1$ iff $v_1^\uparrow(\neg p) \geq x_0$
- (3.) $v_2(p) \leq y_1$ iff $v_2^\uparrow(p) \leq y_0$
- (4.) $v_2(\neg p) \leq y_1$ iff $v_2^\uparrow(\neg p) \leq y_0$
- (5.) $v_i(p) \leq v_j(p')$ iff $v_i^\uparrow(p) \leq v_j^\uparrow(p')$ for all $i, j \in \{1, 2\}$

We define it as follows.

$$v_i^\uparrow(p) = \begin{cases} v_i(p) & \text{iff } x_1 > v_i(p) > y_1 \\ x_0 & \text{iff } v_i(p) = x_1 \\ y_0 & \text{iff } v_i(p) = y_1 \\ \frac{1-x_0}{1-x_1} \cdot (v_i(p) - x_1) + x_0 & \text{iff } v_i(p) > x_1 \\ \frac{y_0}{y_1} \cdot v_i(p) & \text{iff } v_i(p) < y_1 \end{cases}$$

(1.) and (3.) hold by construction.

For (2.) we have

$$\begin{aligned} v_1(\neg p) \geq x_1 & \text{ iff } v_2(p) \geq x_1 \\ & \text{ iff } v_2^\uparrow(p) = x_0 \text{ or } v_2^\uparrow(p) = \underbrace{\frac{1-x_0}{1-x_1} \cdot (v_2(p) - x_1) + x_0}_{=X} \\ & \text{ iff } v_2^\uparrow(p) \geq x_0 & \text{ (since } X \geq x_0) \\ & \text{ iff } v_1^\uparrow(\neg p) \geq x_0 \end{aligned}$$

For (4.) we have

$$\begin{aligned} v_2(\neg p) \leq y_1 & \text{ iff } v_1(p) \leq y_1 \\ & \text{ iff } v_1^\uparrow(p) = y_0 \text{ or } v_1^\uparrow(p) = \underbrace{\frac{y_0}{y_1} \cdot v_1(p)}_{=Y} \\ & \text{ iff } v_1^\uparrow(p) \leq y_0 & \text{ (since } Y \leq y_0) \\ & \text{ iff } v_2^\uparrow(\neg p) \leq y_0 \end{aligned}$$

Finally, (5.) holds since $x_1 > y_1$ since \uparrow increases v_i for the values $> x_1$, and decreases v_i for the values $< y_1$.

4. Paraconsistent expansions of Gödel logic

Now, for the reduction in the other direction, we construct v^{\Downarrow} s.t.

$$\begin{aligned} v_1(p) \geq x_0 & \text{ iff } v_1^{\Downarrow}(p) \geq x_1 \\ v_1(\neg p) \geq x_0 & \text{ iff } v_1^{\Downarrow}(\neg p) \geq x_1 \\ v_2(p) \leq y_0 & \text{ iff } v_2^{\Downarrow}(p) \leq y_1 \\ v_2(\neg p) \leq y_0 & \text{ iff } v_2^{\Downarrow}(\neg p) \leq y_1 \\ v_i(p) \leq v_j(p') & \text{ iff } v_i^{\Downarrow}(p) \leq v_j^{\Downarrow}(p') \text{ for all } i, j \in \{1, 2\} \end{aligned}$$

We define $v_i^{\Downarrow}(p)$ as follows.

$$v_i^{\Downarrow}(p) = \begin{cases} v_i(p) & \text{iff } v_i(p) > x_0 \text{ or } v_i(p) < y_0 \\ x_1 & \text{iff } v_i(p) = x_0 \\ \frac{x_1 - y_1}{x_0 - y_0} \cdot (v_i(p) - y_0) + y_1 & \text{iff } y_0 > v_i(p) > x_0 \\ y_1 & \text{iff } v_i(p) = y_0 \end{cases}$$

Again, observe that the only transformation by v^{\Downarrow} is monotone w.r.t. z , thus (5.) is satisfied. (1.)–(4.) can be proved in the same fashion as in the previous part.

Case 2. can be shown in the same fashion.

Case 3. Assume w.l.o.g. that $x_0 \leq x_1$ and $y_0 \leq y_1$. Furthermore, since the filters are non-paraconsistent, $x_0 > y_0$ and $x_1 > y_1$. Thus we get $x_1 \geq x_0 > y_0$ and $x_1 > y_1 \geq y_0$ whence

$$(a) \ x_1 \geq x_0 > y_1 \geq y_0 \text{ or } (b) \ x_1 > y_1 \geq x_0 > y_0$$

Now observe that in both cases we have that $(x_1, y_1)^\uparrow \supseteq (x_1, y_0)^\uparrow \subseteq (x_0, y_0)^\uparrow$. Thus, we use **1.** to first obtain equivalence between $(x_1, y_1)^\uparrow$ and $(x_1, y_0)^\uparrow$, and then between $(x_1, y_0)^\uparrow$ and $(x_0, y_0)^\uparrow$. \square

Proof of Theorem 4.7.2. Let $(x_0, y_0)^\uparrow$ and $(x_1, y_1)^\uparrow$ be two non-prime paraconsistent filters. We show how to \Downarrow -reduce them to one another. We have three cases.

1. $(x_0, y_0)^\uparrow \subseteq (x_1, y_1)^\uparrow$
2. $(x_0, y_0)^\uparrow \supseteq (x_1, y_1)^\uparrow$
3. $(x_0, y_0)^\uparrow \not\subseteq (x_1, y_1)^\uparrow$ and $(x_0, y_0)^\uparrow \not\supseteq (x_1, y_1)^\uparrow$

Again, observe, that the second and third cases will be reduced to the first one.

Case 1. Since $(x_0, y_0)^\uparrow \subseteq (x_1, y_1)^\uparrow$ are both paraconsistent and not of the form $(z, z)^\uparrow$, we have that $x_0 < y_0$ and $x_1 < y_1$. Furthermore, $x_1 \leq x_0$ and $y_0 \leq y_1$. Thus, we have $x_1 \leq x_0 < y_0 \leq y_1$.

First, we reduce $(x_1, y_1)^\uparrow$ to $(x_0, y_0)^\uparrow$. Define v_i^{\Downarrow} as follows.

$$v_i^{\Downarrow}(p) = \begin{cases} v_i(p) & \text{iff } v_i(p) < x_1 \text{ or } v_i(p) > y_1 \\ x_0 & \text{iff } v_i(p) = x_1 \\ y_0 & \text{iff } v_i(p) = y_1 \\ \frac{y_0 - x_0}{y_1 - x_1} \cdot (v_i(p) - y_1) + y_0 & \text{iff } y_1 > v_i(p) > x_1 \end{cases}$$

Observe that this definition is the same as in 4.7.1.**1.** where we defined reduction from one *nonparaconsistent* filter to another which was contained in it up to renaming of variables.

The other direction is obtained by the following definition of v_i^{\Uparrow} .

$$v_i^{\Uparrow}(p) = \begin{cases} v_i(p) & \text{iff } y_0 > v_i(p) > x_0 \\ x_1 & \text{iff } v_i(p) = x_0 \\ y_1 & \text{iff } v_i(p) = y_0 \\ \frac{x_1}{x_0} \cdot v_i(p) & \text{iff } v_i(p) < x_0 \\ \frac{1 - y_1}{1 - y_0} \cdot (v_i(p) - y_0) + y_1 & \text{iff } v_i > y_0 \end{cases}$$

Again, it is obtained via renaming of variables from the definition of v^\updownarrow in 4.7.1. \square

Case 2. Can be shown in the same manner.

Case 3. Assume, w.l.o.g. that $x_0 \leq x_1$ and $y_0 \leq y_1$. Then, we have $(x_0, y_0)^\uparrow \subseteq (x_0, y_1)^\uparrow \supseteq (x_1, y_1)^\uparrow$. Thus, we again reduce to \square \square

Proof of Theorem 4.7.3. Let $(z_0, z_0)^\uparrow$ and $(z_1, z_1)^\uparrow$ be two filters. We show that $(z_0, z_0)^\uparrow$ is \updownarrow -equivalent to $(z_1, z_1)^\uparrow$. Assume, w.l.o.g. that $z_0 > z_1$.

The reduction from $(z_0, z_0)^\uparrow$ to $(z_1, z_1)^\uparrow$ is obtained as follows.

$$v_i^\updownarrow(p) = \begin{cases} z_1 & \text{iff } v_i(p) = z_0 \\ v_i(p) & \text{iff } v_i(p) > z_0 \\ \frac{z_1}{z_0} \cdot v_i(p) & \text{iff } v_i(p) < z_0 \end{cases}$$

We now check the conditions from Lemma 4.6. In particular, we need to show that the following holds.

- (1.) $v_1(p) \geq z_0$ iff $v_1^\updownarrow(p) \geq z_1$
- (2.) $v_1(\neg p) \geq z_0$ iff $v_1^\updownarrow(\neg p) \geq z_1$
- (3.) $v_2(p) \leq z_0$ iff $v_2^\updownarrow(p) \leq z_1$
- (4.) $v_2(\neg p) \leq z_0$ iff $v_2^\updownarrow(\neg p) \leq z_1$
- (5.) $v_i(p) \leq v_j(p')$ iff $v_i^\updownarrow(p) \leq v_j^\updownarrow(p')$ for all $i, j \in \{1, 2\}$

(1.) and (3.) follow straightforwardly from the construction of v^\updownarrow 's. For (2.), we have

$$\begin{aligned} v_1(\neg p) \geq z_0 & \text{ iff } v_2(p) \geq z_0 \\ & \text{ iff } v_2^\updownarrow(p) = z_1 \text{ or } v_2^\updownarrow(p) = v_2(p) \geq z_0 && \text{(by construction)} \\ & \text{ iff } v_2^\updownarrow(p) \geq z_1 && \text{(since } z_0 > z_1) \\ & \text{ iff } v_1^\updownarrow(\neg p) \geq z_1 \end{aligned}$$

(4.) can be shown in the same fashion.

For (5.), observe that the only transformation v^\updownarrow 's do is monotone w.r.t. z .

The reduction from $(z_1, z_1)^\uparrow$ to $(z_0, z_0)^\uparrow$ is obtained as follows.

$$v_i^\updownarrow(p) = \begin{cases} z_0 & \text{iff } v_i(p) = z_1 \\ v_i(p) & \text{iff } v_i(p) < z_1 \\ \frac{1-z_0}{1-z_1} \cdot (v_i(p) - z_1) + z_0 & \text{iff } v_i(p) > z_1 \end{cases}$$

The soundness of the conditions of Lemma 4.6 can be obtained similarly. \square

Proof of Theorem 4.7.4. We show \updownarrow -equivalence of all prime filters of the form $(x, 1)^\uparrow$ and $(0, y)^\uparrow$ with $x < 1$ and $y > 0$. Observe that by Proposition 4.3, we can consider only the filters of the form $(x_0, 1)^\uparrow$ and $(x_1, 1)^\uparrow$. Assume, w.l.o.g. that $x_0 > x_1$.

The reduction from $(x_0, 1)^\uparrow$ to $(x_1, 1)^\uparrow$ is obtained as follows.

$$v_i^\updownarrow(p) = \begin{cases} x_1 & \text{iff } v_i(p) = x_0 \\ v_i(p) & \text{iff } v_i(p) > x_0 \\ \frac{x_1}{x_0} \cdot v_i(p) & \text{iff } v_i(p) < x_0 \end{cases}$$

The other direction is obtained as follows.

$$v_i^\updownarrow(p) = \begin{cases} x_0 & \text{iff } v_i(p) = x_1 \\ v_i(p) & \text{iff } v_i(p) < x_1 \\ \frac{1-x_0}{1-x_1} \cdot (v_i(p) - x_1) + x_0 & \text{iff } v_i(p) > x_1 \end{cases}$$

4. Paraconsistent expansions of Gödel logic

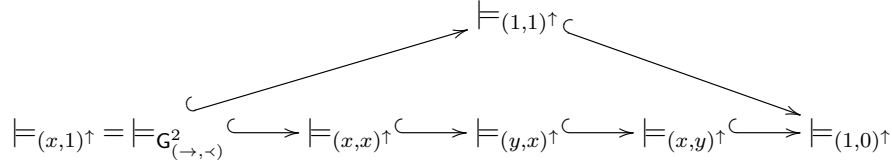


Figure 4.2.: Only canonical examples are shown. $1 > x > y > 0$.

Observe that these definitions are obtained from those in the previous case by renaming of the variables. \square

Proof of Theorem 4.7.5. We show that all nonparaconsistent filters of the form $(x, 0)^\uparrow$ or $(1, y)^\uparrow$ for $x < 1$ and $y > 0$ are pairwise \Downarrow -equivalent.

Again, by Proposition 4.3, we can consider only filters of the form $(x_0, 0)^\uparrow$ and $(x_1, 0)^\uparrow$. Assume, w.l.o.g. that $x_0 > x_1$.

First of all, assume that $\Gamma \not\models_{(1,0)^\uparrow} \phi$. Then, there is a valuation on $[0, 1] \times [0, 1]$ \mathbf{v} , s.t. $\mathbf{v}[\Gamma] = (1, 0)$ but $\mathbf{v}(\phi) \neq (1, 0)$. If $\mathbf{v}(\phi) \notin (x', 0)^\uparrow$ for some given x' , we are good. Otherwise, let $\mathbf{v}(\phi) = (x'', 0)$ with $x'' \geq x'$, we consider \mathbf{v}^* . In this case, we have that $\mathbf{v}^*[\Gamma] = (1, 0)$ and $\mathbf{v}^*(\phi) = (0, 1 - x'')$ as desired.

Thus, we have that $\Gamma \models_{(x,0)^\uparrow} \phi$ entails $\Gamma \models_{(1,0)^\uparrow} \phi$ for any x .

For the other direction, we define the reduction as in Lemma 4.7. It is now easy to check by induction on ϕ that $v(\phi) \in (x, 0)^\uparrow$ iff $v^\uparrow(\phi) = (1, 0)$.

We consider only the most instructive case of $\phi = \psi \prec \psi'$.

$$\begin{aligned} v(\psi \prec \psi') \in (x, 0)^\uparrow & \text{ then } v_1(\psi) > v_1(\psi') \text{ and } v_1(\psi) \geq x \text{ and } v_2(\psi') \leq v_2(\psi) \\ & \text{ then } v_1^\uparrow(\psi) > v_1^\uparrow(\psi') \text{ and } v_1^\uparrow(\psi) = 1 \text{ and } v_2^\uparrow(\psi') \leq v_2^\uparrow(\psi) \quad (\text{Lemma 4.7}) \\ & \text{ then } v^\uparrow(\psi \prec \psi') = (1, 0) \end{aligned}$$

$$\begin{aligned} v(\psi \prec \psi') \notin (x, 0)^\uparrow & \text{ then } v_1(\psi) < x \text{ or } v_1(\psi) \leq v_1(\psi') \text{ or } v_2(\psi') \geq v_2(\psi) \\ & \text{ then } v_1^\uparrow(\psi) < 1 \text{ or } v_1^\uparrow(\psi) \leq v_1^\uparrow(\psi') \text{ or } v_2^\uparrow(\psi') \geq v_2^\uparrow(\psi) \quad (\text{Lemma 4.7}) \\ & \text{ then } v^\uparrow(\psi \prec \psi') \neq (1, 0) \end{aligned}$$

\square

Proof of Theorem 4.7.6. Immediately from Proposition 4.3. \square

Corollary 4.2. For formulas in $\mathcal{L}_{\mathcal{G}_{(\rightarrow, \prec)}^2}$, there are only six entailments on $[0, 1]^\boxtimes$ generated by $(x, y)^\uparrow$ filters whose inclusion hierarchy is shown in Fig. 4.2.

The next theorem can be obtained in the same manner as Theorem 4.7.

Theorem 4.8. There are only two entailments over $\mathcal{L}_{\mathcal{G}_{(\rightarrow, \rightarrow)}^2}$ generated by non-trivial prime filters on $[0, 1]^\boxtimes$ extending $(1, 1)^\uparrow$: namely, $\models_{(x,1)^\uparrow} = \models_{\mathcal{G}_{(\rightarrow, \rightarrow)}^2}$ ($x > 0$) and $\models_{(1,1)^\uparrow}$.

Note, however, that since \rightarrow is not definable from other $\mathcal{L}_{\mathcal{G}_{(\rightarrow, \rightarrow)}^2}$ connectives, its removal from the language makes all prime filters on $[0, 1]^\boxtimes$ that extend $(1, 1)^\uparrow$ indistinguishable.

Lemma 4.8. For any valuation v and prime filter $(x, 1)^\uparrow$ define v^\Downarrow as follows.

$$\begin{aligned} v_i(p) = 1 & \quad \text{iff} \quad v_i^\Downarrow(p) = 1 \\ v_i(p) \neq 1 & \quad \text{iff} \quad v_i^\Downarrow(p) = v_i(p) \cdot x \end{aligned}$$

Then, for any \multimap -free $\phi, \phi' \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$, it holds that

$$\begin{aligned} v_i(\phi) = 1 & \quad \text{iff} \quad v_i^{\downarrow}(\phi) = 1 \\ v_i(\phi) \neq 1 & \quad \text{iff} \quad v_i^{\downarrow}(\phi) < x \\ v_i(\phi) \leq v_j(\phi') & \quad \text{iff} \quad v_i^{\downarrow}(\phi) \leq v_j^{\downarrow}(\phi') \quad \text{for all } i, j \in \{1, 2\} \end{aligned}$$

Proof. Analogously to Lemma 4.6. □

Lemma 4.9. For any valuation v and a prime filter $(x, 1)^\uparrow$ define v^\uparrow as follows.

$$\begin{aligned} v(p) \in (x, 1)^\uparrow & \quad \text{iff} \quad v^\uparrow(p) = (1, v_2(p)) \\ v(\neg p) \in (x, 1)^\uparrow & \quad \text{iff} \quad v^\uparrow(p) = (v_2(p), 1) \\ v(p), v(\neg p) \notin (x, 1)^\uparrow & \quad \text{iff} \quad v^\uparrow(p) = v(p) \end{aligned}$$

Then, for any \multimap -free $\phi \in \mathcal{L}_{\mathbf{G}^2(\rightarrow, \multimap)}$

$$\begin{aligned} v(\phi) \in (x, 1)^\uparrow & \quad \text{iff} \quad v^\uparrow(\phi) = (1, v_2(\phi)) \\ v(\neg\phi) \in (x, 1)^\uparrow & \quad \text{iff} \quad v^\uparrow(\phi) = (v_2(\phi), 1) \\ v(\phi), v(\neg\phi) \notin (x, 1)^\uparrow & \quad \text{iff} \quad v^\uparrow(\phi) = v(\phi) \end{aligned}$$

Proof. First, observe that if $v(p) \leq v(p')$, then $v^\uparrow(p) \leq v^\uparrow(p')$. Thus, the condition of Lemma 4.5 holds. We now proceed by induction on ϕ . We assume w.l.o.g. that ϕ is in NNF. Now, the cases of literals hold by construction.

$$\boxed{\phi = \psi \wedge \psi'}$$

$$\begin{aligned} v(\psi \wedge \psi') \in (x, 1)^\uparrow & \quad \text{iff } v(\psi) \in (x, 1)^\uparrow \text{ and } v(\psi') \in (x, 1)^\uparrow \\ & \quad \text{iff } v^\uparrow(\psi) = (1, v_2(\psi)) \text{ and } v^\uparrow(\psi') = (1, v_2(\psi')) \quad (\text{by IH}) \\ & \quad \text{iff } v^\uparrow(\psi \wedge \psi') = (1, v_2(\psi \wedge \psi')) \end{aligned}$$

$$\begin{aligned} v(\neg(\psi \wedge \psi')) \in (x, 1)^\uparrow & \quad \text{iff } v(\neg\psi) \in (x, 1)^\uparrow \text{ or } v(\neg\psi') \in (x, 1)^\uparrow \\ & \quad \text{iff } v^\uparrow(\psi) = (v_2(\psi), 1) \text{ or } v^\uparrow(\psi') = (v_2(\psi'), 1) \quad (\text{by IH}) \\ & \quad \text{iff } v^\uparrow(\psi \wedge \psi') = (v_2(\psi \wedge \psi'), 1) \end{aligned}$$

$$\begin{aligned} v(\neg(\psi \wedge \psi')), v(\psi \wedge \psi') \notin (x, 1)^\uparrow & \quad \text{iff} \quad \left[\begin{array}{c} v(\neg\psi) \notin (x, 1)^\uparrow \\ \text{and} \\ v(\psi) \notin (x, 1)^\uparrow \end{array} \right] \text{ or } \left[\begin{array}{c} v(\neg\psi') \notin (x, 1)^\uparrow \\ \text{and} \\ v(\psi') \notin (x, 1)^\uparrow \end{array} \right] \\ & \quad \text{iff } v^\uparrow(\psi) = v(\psi) \text{ or } v^\uparrow(\psi') = v(\psi') \quad (\text{by IH}) \\ & \quad \text{iff } v^\uparrow(\psi \wedge \psi') = v(\psi \wedge \psi') \end{aligned}$$

$\boxed{\phi = \psi \vee \psi'}$ is obtained dually.

$$\boxed{\phi = \psi \rightarrow \psi'}$$

$$\begin{aligned} v(\psi \rightarrow \psi') \in (x, 1)^\uparrow & \quad \text{iff } v_1(\psi) \leq v_1(\psi') \text{ or } v(\psi') \in (x, 1)^\uparrow \\ & \quad \text{then } v_1^\uparrow(\psi) \leq v_1^\uparrow(\psi') \text{ or } v^\uparrow(\psi') = (1, v_2(\psi'))^\uparrow \quad (\text{IH, Lemma 4.5}) \\ & \quad \text{then } v^\uparrow(\psi \rightarrow \psi') = v(1, v_2(\psi \rightarrow \psi')) \end{aligned}$$

$$\begin{aligned} v(\neg(\psi \rightarrow \psi')) \in (x, 1)^\uparrow & \quad \text{iff } v(\psi), v(\neg\psi') \in (x, 1)^\uparrow \\ & \quad \text{iff } v^\uparrow(\psi) = (1, v_2(\psi)) \text{ and } v^\uparrow(\psi') = (v_2(\psi'), 1) \quad (\text{by IH}) \end{aligned}$$

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$$\text{iff } v^\uparrow(\psi \wedge \neg\psi') = (v_2(\psi \wedge \neg\psi'), 1)$$

$$\text{iff } v^\uparrow(\psi \rightarrow \psi') = (1, v_2(\psi \wedge \neg\psi'))$$

$$\begin{aligned} & \left[\begin{array}{c} v(\neg(\psi \rightarrow \psi')) \notin (x, 1)^\uparrow \\ \text{and} \\ v(\psi \rightarrow \psi') \notin (x, 1)^\uparrow \end{array} \right] \text{ iff } v(\psi \wedge \neg\psi') \notin (x, 1)^\uparrow \text{ and } v(\psi \rightarrow \psi') \notin (x, 1)^\uparrow \\ & \text{iff } \left[\begin{array}{c} v_1(\psi) > v_1(\psi') \\ \text{and} \\ v(\psi') \notin (x, 1)^\uparrow \\ \text{and} \\ v(\psi) \notin (x, 1)^\uparrow \end{array} \right] \text{ or } \left[\begin{array}{c} v_1(\psi) > v_1(\psi') \\ \text{and} \\ v(\psi') \notin (x, 1)^\uparrow \\ \text{and} \\ v(\neg\psi') \notin (x, 1)^\uparrow \end{array} \right] \\ & \text{then } \left[\begin{array}{c} v_1^\uparrow(\psi) > v_1^\uparrow(\psi') \\ \text{and} \\ v_1^\uparrow(\psi') = v_1(\psi') \\ \text{and} \\ v_1^\uparrow(\psi) = v_1(\psi) \end{array} \right] \text{ or } \left[\begin{array}{c} v_1^\uparrow(\psi) > v_1^\uparrow(\psi') \\ \text{and} \\ v_1^\uparrow(\psi') = v_1(\psi') \\ \text{and} \\ v_1^\uparrow(\neg\psi') = v_1(\neg\psi') \end{array} \right] \quad (\text{IH, Lemma 4.5}) \\ & \text{then } v^\uparrow(\psi \rightarrow \psi') = v^\uparrow(\psi \rightarrow \psi') \end{aligned}$$

□

Proposition 4.6. *Let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathbf{G}^2(\rightarrow, \rightarrow)}$ be \rightarrow -free. Then $\Gamma \models_{(x,1)^\uparrow} \phi$ iff $\Gamma \models_{(1,1)^\uparrow} \phi$.*

Proof. From Lemmas 4.8 and 4.9. □

Finally, a statement similar to Proposition 3.7 holds for \mathbf{G}^2 too.

Proposition 4.7. *Let ϕ and χ be over $\{\neg, \wedge, \vee\}$, and $\Rightarrow \in \{\rightarrow, \rightarrow\}$. Then, the following equivalences hold:*

$$\underbrace{\mathbf{G}_{(\rightarrow, \rightarrow)}^2 \models \phi \rightarrow \chi}_1 \text{ iff } \underbrace{\mathbf{G}_{(\rightarrow, \leftarrow)}^2 \models \phi \rightarrow \chi}_2 \text{ iff } \underbrace{\phi \models_{(1,1)^\uparrow} \chi}_3 \text{ iff } \underbrace{\phi \models_{\mathbf{G}_{(\rightarrow, \leftarrow)}^2} \chi}_4 \text{ iff } \underbrace{\phi \models_{\text{BD}} \chi}_5$$

and

$$\underbrace{\phi \models_{(1,0)} \chi}_6 \text{ iff } \underbrace{\phi \models_{\text{ETL}} \chi}_7$$

Proof. First, we have $2 \Leftrightarrow 4$ by Definition 4.4. Second, observe from Definitions 3.5, 3.6, and 4.4 that the semantics of \neg , \wedge , and \vee are the same in \mathbf{L}^2 's and \mathbf{G}^2 's. Thus, $1 \Leftrightarrow 2$ follows immediately by Proposition 4.3, while $6 \Leftrightarrow 7$ and $3 \Leftrightarrow 5$ can be proved in the same manner as in Proposition 4.7. Finally, $1 \Leftrightarrow 3$ follows from Proposition 4.6. □

Part II.

Modal logics on $[0, 1]$ -valued Kripke frames

5. Crisp bi-Gödel modal logic

In Part II, we will be dealing with modal expansions of biG and $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$. We are choosing $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ (denoted here with \mathbf{G}_{Δ}^2 or just \mathbf{G}^2) instead of $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ for the propositional fragment of the paraconsistent modal logics since there is Δ in the former which will allow for more straightforward formalisations of statements expressing comparisons of beliefs. We will also be referring to the crisp and fuzzy bimodal³³ Gödel modal logics provided in [40] and [130]. The following picture (Fig. 5.1) summarises the logics and their relations.

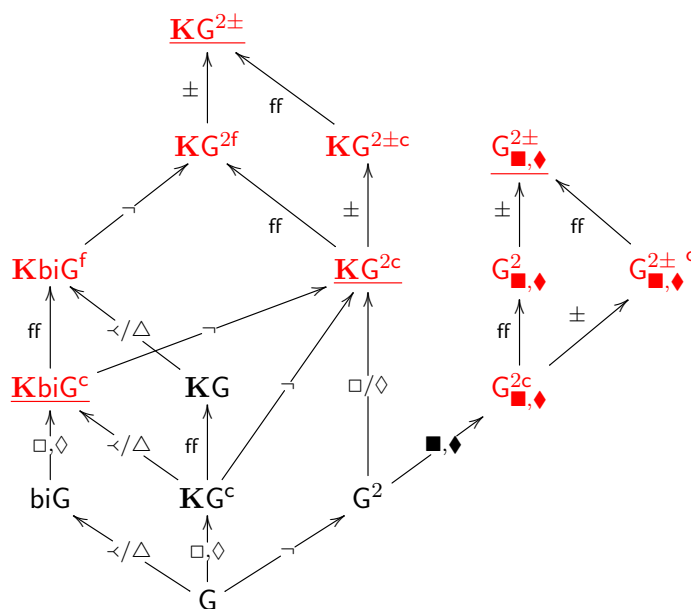


Figure 5.1.: Logics in Part II. ff stands for ‘permitting fuzzy frames’; \pm for ‘permitting birelational frames’. Subscripts on arrows denote language expansions. $/$ stands for ‘or’ and comma for ‘and’. The new modal logics are highlighted in red. Among those, we are mostly focussing on the underlined ones.

We will be mostly considering the semantical properties of these modal logics, investigating their expressivity, and establishing complexity evaluations of their sets of formulas valid over *finitely branching frames*. For this, we are going to construct modal expansions of $\mathcal{T}(\mathbf{G}^2)$ (recall Definition 4.12). In addition, we will also axiomatise \mathbf{KbiG}^c and \mathbf{KG}^{2c} (in Chapters 5 and 6, respectively) and show their PSpace -completeness utilising a technique from [38].

³³Note that in Gödel modal logics, \Box and \Diamond are not interdefinable

5.1. Language and semantics

5.1.1. Frames and models

Let us now provide the semantics of **KbiG** (both fuzzy and crisp). The language $\mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ is defined via the following grammar.

$$\mathcal{L}_{\mathbf{G}\Delta,\square,\diamond} \ni \phi := p \in \mathbf{Prop} \mid \sim\phi \mid \Delta\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \square\phi \mid \diamond\phi$$

In this part, we will use **1** and **0** to stand as shorthands for, respectively, $p \rightarrow p$ and $p \prec p$ (cf. Remark 4.4).

Definition 5.1 (Frames).

- A *fuzzy frame* is a tuple $\mathfrak{F} = \langle W, R \rangle$ with $W \neq \emptyset$ and $R : W \times W \rightarrow [0, 1]$.
- A *crisp frame* is a tuple $\mathfrak{F} = \langle W, R \rangle$ with $W \neq \emptyset$ and $R \subseteq W \times W$.

Definition 5.2 (**KbiG** models). A **KbiG** model is a tuple $\mathfrak{M} = \langle W, R, v \rangle$ with $\langle W, R \rangle$ being a (crisp or fuzzy) frame, and $v : \mathbf{Prop} \times W \rightarrow [0, 1]$. v (a **KbiG**-valuation) is extended on complex $\mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ formulas according to Definition 4.2 in the propositional cases:

$$v(\phi \circ \phi', w) = v(\phi, w) \circ_{\mathbf{G}} v(\phi', w) \quad (\circ \in \{\sim, \Delta, \wedge, \vee, \rightarrow\})$$

The interpretation of modal formulas on *fuzzy* frames is as follows:

$$v(\square\phi, w) = \inf_{w' \in W} \{wRw' \rightarrow_{\mathbf{G}} v(\phi, w')\} \quad v(\diamond\phi, w) = \sup_{w' \in W} \{wRw' \wedge_{\mathbf{G}} v(\phi, w')\}$$

On *crisp* frames, the interpretation is simpler (here, $\inf(\emptyset) = 1$ and $\sup(\emptyset) = 0$):

$$v(\square\phi, w) = \inf\{v(\phi, w') : wRw'\} \quad v(\diamond\phi, w) = \sup\{v(\phi, w') : wRw'\}$$

We say that $\phi \in \mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ is **KbiG** valid on frame \mathfrak{F} (denote, $\mathfrak{F} \models_{\mathbf{KbiG}} \phi$) iff for any $w \in \mathfrak{F}$, it holds that $v(\phi, w) = 1$ for any model \mathfrak{M} on \mathfrak{F} . Γ entails χ (on \mathfrak{F}), denoted $\Gamma \models_{\mathbf{KbiG}} \chi$ ($\Gamma \models_{\mathfrak{F}}^{\mathbf{KbiG}} \chi$), iff for every model \mathfrak{M} (on \mathfrak{F}) and every $w \in \mathfrak{M}$, it holds that

$$\inf\{v(\phi, w) : \phi \in \Gamma\} \leq v(\chi, w)$$

Definition 5.3 (Frame definability). Given a class of frames \mathbb{F} , we say that $\Sigma \subseteq \mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ defines \mathbb{F} when it holds that $\mathfrak{F} \models_{\mathbf{KbiG}} \Sigma$ iff $\mathfrak{F} \in \mathbb{F}$.

Convention 5.1. For each frame \mathfrak{F} and each $w \in \mathfrak{F}$, we denote

$$\begin{aligned} R(w) &= \{w' : wRw' = 1\} && \text{(for fuzzy frames)} \\ R^+(w) &= \{w' : wRw' > 0\} && \text{(for fuzzy frames)} \\ R(w) &= \{w' : wRw'\} && \text{(for crisp frames)} \end{aligned}$$

And set that $R^+(w) = R(w)$ for crisp frames.

Convention 5.2. In what follows, we use \mathbf{KbiG}^c to stand for the set of all $\mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ formulas valid on all *crisp* frames and \mathbf{KbiG}^f to stand for the set of all $\mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ formulas valid on all *fuzzy* frames. **KbiG** without superscripts stands for both logics.

KG and \mathbf{KG}^c stand for, respectively, the sets of all Δ -free $\mathcal{L}_{\mathbf{G}\Delta,\square,\diamond}$ formulas valid on all frames and all *crisp* frames. \mathcal{HKG} and \mathcal{HKG}^c denote their Hilbert-style axiomatisations from [40] and [130].

$$w_1 : p = \frac{1}{2} \longleftarrow w_0 : p = 1 \longrightarrow w_2 : p = \frac{2}{3}$$

Figure 5.2.: All variables have the same values in all states exemplified by p ; the relation is crisp.

In the remainder of the section, we are going to give an informal interpretation of our language. In particular, we will show how one can formalise statements expressing *comparative belief*.

On the *crisp* frames, the states can represent *sources of information* that claim some statements to be true. The degree of certainty of a source w in the statement ϕ is represented by the value $v(\phi, w)$. $\Box\phi$ and $\Diamond\phi$ are then two aggregations of information regarding ϕ : the ‘pessimistic’ (since one counterexample is enough to reject $\Box\phi$) and the ‘optimistic’ (it suffices that one source confirms ϕ to accept $\Diamond\phi$) ones.

In *fuzzy* frames, the value of tRt' can be thought of as the degree of trust t has in t' . Then, $\Diamond\phi$ represents the search for evidence from trusted sources that supports ϕ : $v(\Diamond\phi, t) > 0$ iff there is t' s.t. $tRt' > 0$ and $v(\phi, t') > 0$, i.e., there must be a source t' to which t has positive degree of trust and that has at least some certainty in ϕ . On the other hand, if no source is trusted by t (i.e., $tRu = 0$ for all u), then $v(\Diamond\phi, t) = 0$. Likewise, $\Box\chi$ can be construed as the search of a rejection of χ given by trusted sources: $v(\Box\chi, t) < 1$ iff there is a source t' that gives to χ less certainty than t gives trust to t' . In other words, if t trusts no sources or if all sources have at least as high confidence in χ as t has in them, then t fails to find a trustworthy enough rejection.

5.1.2. Δ and its expressivity

Let us first see how Δ allows us to express the statements of *comparative belief*. For instance, consider the following sentence

Example 5.1.

weather: *Paula considers a rain happening today strictly more likely than a hailstorm.*

To formalise this statement, one needs a formula that is true iff the value of $\Box r$ (Paula believes it is going to rain today) is strictly greater than that of $\Box s$ (Paula believes that a hailstorm is going to happen today). Paula also does not state that she *believes completely* in the rain, nor does she exclude the possibility of a hailstorm. Hence, $\Box r \wedge \Box \sim s$ does not suit the purpose. In fact, there is no Gödel formula $\phi(p, q)$ s.t.

$$v(\phi) = 1 \text{ iff } v(p) > v(q)$$

On the other hand, it is easy to see that

$$v(\sim\Delta(\Box r \rightarrow \Box s), w) = 1 \text{ iff } v(\Box r, w) > v(\Box s, w)$$

and thus is a suitable formalisation of **weather**.

Note also that just as in **KG**, \Box and \Diamond are not interdefinable in **KbiG^c**.

Proposition 5.1. \Box and \Diamond are not interdefinable in **KbiG^c**.

Proof. Consider the frame on Fig. 5.2. It is clear that $v(\Box p, w_0) = \frac{1}{2}$ and $v(\Diamond p, w_0) = \frac{2}{3}$. It remains to show that there is no \Box -free χ s.t. $v(\chi, w_0) = \frac{1}{2}$ and no \Diamond -free ψ s.t. $v(\psi, w_0) = \frac{2}{3}$.

To do this, we show by induction that for every \Box -free χ , $v(\chi, w_0) \in \{0, \frac{2}{3}, 1\} = X$ and for every \Diamond -free ψ , $v(\psi, w_0) \in \{0, \frac{1}{2}, 1\} = Y$. Note, first of all, that X and Y are closed under $G\Delta$ operations. Note, moreover, that the following holds for every $\tau \in \mathcal{L}_{G\Delta, \Box, \Diamond}$.

$$v(\tau, w_1) = 1 \text{ iff } v(\tau, w_2) = 1$$

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$$\begin{aligned} v(\tau, w_1) &= \frac{1}{2} \text{ iff } v(\tau, w_2) = \frac{2}{3} \\ v(\tau, w_1) &= 0 \text{ iff } v(\tau, w_2) = 0 \end{aligned} \tag{5.1}$$

We proceed by induction on χ and ψ . The basis cases of variables hold by the construction of the model. The cases of propositional connectives hold because X and Y are closed under $\mathbf{G}\Delta$ operations. Now let $\chi = \diamond\chi'$. It follows immediately from (5.1) that $v(\diamond\chi') \in X$. The case of $\psi = \square\psi'$ can be proven in the same manner.

Observe that we have found a counterexample using a finitely branching crisp model. Since such models are a subclass of *all* \mathbf{KbiG}^c models, it follows that \square and \diamond are not interdefinable in \mathbf{KbiG}^c . \square

As we have just seen, the addition of Δ allows us to formalise the statements we were not able to treat without it. On a more formal side, however, Δ makes both \square and \diamond fragments³⁴ of \mathbf{KbiG} more expressive. Namely, \diamond fragment of \mathbf{KG} has finite model property while crisp and fuzzy \square fragments coincide [39]. We show that neither of these is the case in \mathbf{KbiG}^c .

Proposition 5.2.

1. $\mathfrak{F} \models_{\mathbf{KbiG}} \Delta\square p \rightarrow \square\Delta p$ iff \mathfrak{F} is crisp.
2. There are only infinite countermodels of $\Delta\diamond p \rightarrow \diamond\Delta p$.

Proof. We begin with 1. Assume that \mathfrak{F} is crisp, and let v be a valuation thereon s.t. $v(\Delta\square p, w) = 1$. Then, $v(\square p, w) = 1$. But \mathfrak{F} is crisp, whence, $v(p, w') = 1$ and thus, $v(\Delta p, w') = 1$ for every accessible w' . Thus, $v(\square\Delta p, w) = 1$, as required. For the converse, assume that \mathfrak{F} is fuzzy and that w.l.o.g. $wRw' = \frac{1}{2}$. We refute $\Delta\square p \rightarrow \square\Delta p$ at w as follows. Set $v(p, w') = \frac{2}{3}$ and $v(p, w'') = 1$ in all other states. It is clear that $v(\Delta\square p, w) = 1$ but $v(\square\Delta p, w) = 0$ for we have $v(\Delta p, w') = 0$.

For 2, we proceed as follows. Let \mathfrak{M} be a finite model and let $v(\Delta\diamond p, w) = 1$. Then, there is $w' \in R(w)$ s.t. $v(p, w') = 1$, whence $v(\diamond\Delta p, w) = 1$. For the converse, assume that $v(\Delta\diamond p, w) = 1$ and $v(\diamond\Delta p, w) < 1$. We define an infinite fuzzy³⁵ countermodel as follows.

- $W = \{w\} \cup \{w_i : i \in \mathbb{N} \text{ and } i \geq 1\}$.
- $wRw_i = \frac{i}{i+1}$; $uRu' = 0$ for every $u \neq w$ and $u \neq w_i$.
- $v(p, w_i) = \frac{i}{i+1}$.

It is clear that this model is infinite and that $v(\Delta\diamond p \rightarrow \diamond\Delta p, w) = 0$. \square

Remark 5.1. Note that it is also easy to show that $\diamond\Delta p \rightarrow \Delta\diamond p$ defines crisp frames but, of course, one can define crisp frames in the \diamond fragment without Δ : $\sim\sim\diamond p \rightarrow \diamond\sim\sim p$ [39].

The following proposition shows that Δ adds expressivity to the bi-modal Gödel logic as well.

Proposition 5.3. There are \mathbf{KbiG} -definable classes of fuzzy frames that are not \mathbf{KG} -definable

Proof. Consider $\tau = \sim\Delta\diamond\mathbf{1} \wedge \sim\square\mathbf{0}$. It is clear that

$$\mathfrak{F} \models_{\mathbf{KbiG}} \tau \text{ iff } \underbrace{\forall u \in \mathfrak{F} : \sup\{uRu' : u' \in \mathfrak{F}\}}_{\Psi} < 1 \tag{5.2}$$

Denote the class of frames satisfying Ψ in (5.2) with \mathbb{S} . Observe now $\mathfrak{F} \in \mathbb{S}$ iff $v(\diamond\mathbf{1} \vee \square\mathbf{0}, w) < 1$ in every $w \in \mathfrak{F}$ since it is always the case that $v(\square\mathbf{0}, w), v(\sim\Delta\diamond\mathbf{1}, w) \in \{0, 1\}$ and since $v(\sim\Delta\diamond\mathbf{1}, w) = 1$ iff $v(\diamond\mathbf{1}, w) < 1$. On the other hand, $\diamond\mathbf{1}$ can take any value from $[0, 1]$ on a fuzzy frame. But there is no Gödel formula that is true iff $v(\diamond\mathbf{1} \vee \square\mathbf{0}, w) < 1$ because $\diamond\mathbf{1} \vee \square\mathbf{0}$ can have any value from 0 to 1; thus to say that it has a value less than 1, one needs Δ or \prec . Thus, \mathbb{S} is not \mathbf{KG} -definable. \square

³⁴Note that \square and \diamond are not interdefinable in \mathbf{KG} [130, Corollary 6.2], nor in \mathbf{KbiG}^c , as we have just shown.

³⁵Recall from [39] that the crisp \diamond fragment of \mathbf{KG} lacks FMP.

5.2. Axiomatisation

We are now finally ready to formulate Hilbert-style calculus for \mathbf{KbiG}^c and prove its completeness. Our completeness proof follows the approach of [40] and [130]. Note, however, that we cannot completely copy the original proof from [130] because it employs that the entailment in Gödel logic can be *equivalently* defined either as preservation of the order on $[0, 1]$ or as preservation of 1 as the designated value. This is, of course, false in \mathbf{biG} since p entails Δp if 1 is the designated value but, of course, $p \not\vdash_{\mathbf{biG}} \Delta p$ (recall Definition 4.2).

We begin with the calculus for \mathbf{KbiG}^c which we dub \mathcal{HKbiG}^c .

Definition 5.4 (\mathcal{HKbiG}^c — Hilbert-style calculus for \mathbf{KbiG}^c). The calculus has the following axiom schemas and rules.

biG: All substitution instances of $\mathcal{HG}\Delta$ theorems and rules.

0: $\sim\Diamond 0$

K: $\Box(\phi \rightarrow \chi) \rightarrow (\Box\phi \rightarrow \Box\chi)$; $\Diamond(\phi \vee \chi) \rightarrow (\Diamond\phi \vee \Diamond\chi)$

FS: $\Diamond(\phi \rightarrow \chi) \rightarrow (\Box\phi \rightarrow \Diamond\chi)$; $(\Diamond\phi \rightarrow \Box\chi) \rightarrow \Box(\phi \rightarrow \chi)$

$\sim\Delta\Diamond$: $\sim\Delta(\Diamond\phi \rightarrow \Diamond\chi) \rightarrow \Diamond\sim\Delta(\phi \rightarrow \chi)$

Cr: $\Box(\phi \vee \chi) \rightarrow (\Box\phi \vee \Box\chi)$; $\Delta\Box\phi \rightarrow \Box\Delta\phi$

nec: $\frac{\vdash \phi}{\vdash \Box\phi}$; $\frac{\vdash \phi \rightarrow \chi}{\vdash \Diamond\phi \rightarrow \Diamond\chi}$

As one sees from the definition above, we have added two modal axioms to the Hilbert-style calculus \mathcal{HKG}^c (\mathcal{GK}^c , in the notation of [130]) that axiomatises \mathbf{KG}^c . $\sim\Delta\Diamond$ says that if the supremum of ϕ is strictly greater than supremum of χ , then there must be a state where the value of ϕ is greater than that of χ . The second axiom is the definition of crisp frames without \Diamond but with Δ . It is also easy to see that the deduction theorem holds for \mathcal{HKbiG}^c :

$$\Gamma, \phi \vdash_{\mathcal{HKbiG}^c} \chi \text{ iff } \Gamma \vdash_{\mathcal{HKbiG}^c} \phi \rightarrow \chi \quad (5.3)$$

In what follows, we denote the set of \mathcal{HKbiG}^c theorems (i.e., formulas provable without assumptions) with $\text{Th}(\mathcal{HKbiG}^c)$. Observe that $\text{Th}(\mathcal{HKG}^c) \subseteq \text{Th}(\mathcal{HKbiG}^c)$. In particular, $\mathbf{P} := \Box(\phi \rightarrow \chi) \rightarrow (\Diamond\phi \rightarrow \Diamond\chi)$ is provable and

$$M_{\Box} : \frac{\Gamma \vdash \phi}{\Box\Gamma \vdash \Box\phi}$$

is admissible. Furthermore, another rule is admissible in \mathcal{HKG} .

Lemma 5.1. *The following rule is admissible in \mathcal{HKG} .*

$$\frac{\mathcal{HKG} \vdash \sim\phi \vee \phi}{\mathcal{HKG} \vdash \sim\Box\phi \vee \Box\phi}$$

Proof. Since \mathcal{HKG} is complete w.r.t. all frames, it suffices to show that if $\sim\phi \vee \phi$ is \mathbf{KG} -valid, then so is $\sim\Box\phi \vee \Box\phi$. We proceed by contraposition. Assume that $v(\sim\Box\phi \vee \Box\phi, w) < 1$ for some frame $\mathfrak{F} = \langle W, R \rangle$ and some $w \in W$. Then, it follows that $v(\sim\Box\phi, w) < 1$ (whence, $v(\Box\phi, w) > 0$) and $v(\Box\phi, w) < 1$. Thus,

$$\inf_{w' \in W} \{wRw' \rightarrow v(\phi, w')\} \in (0, 1)$$

Hence, there exists $w' \in W$ s.t. $v(\phi, w') \in (0, 1)$ and thus, $v(\phi \vee \sim\phi, w') < 1$. \square

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Using this, we obtain the following statement.

Proposition 5.4.

1. The Barcan's formula $\Box\Delta\phi \rightarrow \Delta\Box\phi$ is provable in \mathcal{HKbiG}^c without using **Cr**.
2. The $\Diamond\Delta$ definition of crispness $\Diamond\Delta\phi \rightarrow \Delta\Diamond\phi$ is provable in \mathcal{HKbiG}^c .

Proof. We begin with 1. First, the following rule is admissible in \mathcal{HKG} by Lemma 5.1.

$$\frac{\mathcal{HKG} \vdash \sim\phi \vee \phi}{\mathcal{HKG} \vdash \sim\Box\phi \vee \Box\phi}$$

Furthermore, the following rule is admissible in $\mathcal{HG}\Delta$:

$$\frac{\mathcal{HG}\Delta \vdash \sim\phi \vee \phi \quad \mathcal{HG}\Delta \vdash \phi \rightarrow \chi}{\mathcal{HG}\Delta \vdash \phi \rightarrow \Delta\chi}$$

Thus, since \mathcal{HKbiG}^c is conservative over \mathcal{HKG}^c , we can prove Barcan's formula as follows.

1. $\sim\Delta\phi \vee \Delta\phi$ — a theorem of $\mathcal{HG}\Delta$
2. $\sim\Box\Delta\phi \vee \Box\Delta\phi$ — from 1
3. $\Delta\phi \rightarrow \phi$ — a theorem of $\mathcal{HG}\Delta$
4. $\Box\Delta\phi \rightarrow \Box\phi$ — from 3, **nec**, and **K**
5. $\Box\Delta\phi \rightarrow \Delta\Box\phi$ — from 2 and 4

To prove the $\Diamond\Delta$ definition of the crispness, we proceed as follows.

1. $\sim\Delta\phi \vee \Delta\phi$ — a theorem of $\mathcal{HG}\Delta$
2. $\Box(\sim\Delta\phi \vee \Delta\phi)$ — from 1 using **nec**
3. $\Box\sim\Delta\phi \vee \Diamond\Delta\phi$ — from 2 using **Cr**
4. $\sim\Diamond\Delta\phi \vee \Diamond\Delta\phi$ — from 3 since $\mathcal{HKG} \vdash \Box\sim\phi \rightarrow \sim\Diamond\phi$
5. $\Delta\phi \rightarrow \phi$ — a theorem of $\mathcal{HG}\Delta$
6. $\Diamond\Delta\phi \rightarrow \Diamond\phi$ — from 5 using **K**
7. $\Diamond\Delta\phi \rightarrow \Delta\Diamond\phi$ — from 4 and 6

□

Furthermore, just as in the case of **KG**, the modal rules of \mathcal{HKbiG}^c are restricted to theorems. Thus, we can reduce the proofs in \mathcal{HKbiG}^c to the $\mathcal{HG}\Delta$ derivations from $\text{Th}(\mathcal{HKbiG}^c)$.

Proposition 5.5. For any $\Gamma \cup \{\phi, \chi\} \subseteq \mathcal{L}_{\mathcal{G}\Delta, \Box, \Diamond}$, it holds that

$$\Gamma \vdash_{\mathcal{HKbiG}^c} \phi \text{ iff } \Gamma, \text{Th}(\mathcal{HKbiG}^c) \vdash_{\mathcal{HG}\Delta} \phi$$

We are now ready to prove the completeness theorem. Our proof is a modification of the completeness theorem for crisp Gödel modal logic in [130].

Convention 5.3. For any $\phi \in \mathcal{L}_{\mathcal{G}\Delta, \Box, \Diamond}$, we denote with $\text{Sf}^{0,1}(\phi)$ the set containing all its subformulas and the constants **1** and **0**.

For every $\tau \in \mathcal{L}_{\mathcal{G}\Delta, \Box, \Diamond}$ s.t. $\mathcal{HKbiG}^c \not\vdash \tau$, we are building a canonical model \mathfrak{M}^τ that refutes it.

Definition 5.5 (Canonical model for τ). We define $\mathfrak{M}^\tau = \langle W^\tau, R^\tau, v^\tau \rangle$ as follows.

- W^τ is the set of all $\mathbf{G}\Delta$ homomorphisms $u : \mathcal{L}_{\mathbf{G}\Delta, \square, \diamond} \rightarrow [0, 1]_{\text{biG}}$ ³⁶ s.t. all theorems of \mathcal{HKbiG}^c are evaluated at 1.
- $uR^\tau u'$ iff $u(\square\psi) \leq u'(\psi)$ and $u'(\psi) \leq u(\diamond\psi)$ for all $\psi \in \text{Sf}^{0,1}(\tau)$.
- $v^\tau(p, u) = u(p)$.

Following [130], we introduce the following notation.

Convention 5.4. Let $u \in W^\tau$, $\alpha \in [0, 1]$, $\heartsuit \in \{\square, \diamond\}$, and $\Upsilon \in \{<, \leq, >, \geq, =\}$. We set

$$\heartsuit_u^{\Upsilon\alpha} := \{\chi \in \text{Sf}^{0,1}(\tau) : u(\heartsuit\chi) \Upsilon \alpha\} \quad * \square_u^{\leq 1} := \{\psi : u(\square\psi) = 1\}$$

Observe that $\heartsuit_u^{\Upsilon\alpha}$ is always finite, whence $\bigwedge \heartsuit_u^{\Upsilon\alpha}, \bigvee \heartsuit_u^{\Upsilon\alpha} \in \mathcal{L}_{\mathbf{G}\Delta, \square, \diamond}$. Furthermore, if $\heartsuit_u^{\Upsilon\alpha} = \emptyset$, we set $\bigwedge \emptyset = \mathbf{1}$ and $\bigvee \emptyset = \mathbf{0}$.

The following two statements are the analogues of Lemma 4.1 and Remark 4.2 from [130] and can be proven in exactly the same manner.

Proposition 5.6. Let $\phi \in \square_u^{\leq \alpha}$ with $\alpha < 1$ and set

$$\delta := \left(\bigwedge \square_u^{\leq \alpha} \rightarrow \phi \right) \rightarrow \phi$$

Then $u(\square\delta) > \alpha$.

Proposition 5.7. For any biG homomorphism v s.t. $v(\delta) = 1$ and $v(\phi) < 1$, it holds that $v(\phi) < v(\chi)$ for any $\chi \in \square_u^{\leq \alpha}$.

We are now ready to prove the analogue of [130, Proposition 4.3]. Note, however, that we cannot exactly follow the original proof step by step as it uses the fact that the propositional entailment in Gödel logic can be *equivalently* defined either via preservation of the order on $[0, 1]$ or via preservation of 1. This, of course, is not the case in $\mathbf{G}\Delta$ as we have noted above.

Proposition 5.8. For any $\alpha < 1$ and $\phi \in \square_u^{\leq \alpha}$, there exists a propositional homomorphism $h : \mathcal{L}_{\mathbf{G}\Delta, \square, \diamond} \rightarrow [0, 1]_{\text{biG}}$, s.t.:

C1: $h(\chi) = 1$ for any $\chi \in \text{Th}(\mathcal{HKbiG}^c)$;

C2: $h(\psi) = 1$ for every $\psi \in * \square_u^{\leq 1}$;

C3: $h(\rho) < 1$ for every $\rho \in \diamond_u^{\leq 1}$;

C4: $h(\phi) < h(\sigma)$ for every $\sigma \in \square_u^{\leq \alpha}$.

Proof. Recall that for any ϕ_1, ϕ_2 , and ϕ_3 , it holds that

$$\mathcal{HG}\Delta \vdash (((\phi_1 \rightarrow \phi_2) \rightarrow \phi_2) \wedge (\phi_2 \rightarrow \phi_3)) \vee (((\phi_1 \rightarrow \phi_2) \rightarrow \phi_2) \rightarrow (\phi_3 \rightarrow \phi_2))$$

We replace ϕ_1 with $\bigwedge \square_u^{\leq \alpha}$, ϕ_2 with ϕ , and use δ from Proposition 5.6 which gives us that

$$\mathcal{HKbiG}^c \vdash \left(\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{\leq 1} \right) \right) \vee \left(\delta \rightarrow \left(\bigvee \diamond_u^{\leq 1} \rightarrow \phi \right) \right)$$

Since

$$\frac{\mathcal{HG}\Delta \vdash (\varphi \wedge \varphi') \vee \eta}{\mathcal{HG}\Delta \vdash (\Delta\varphi \wedge \varphi') \vee \eta}$$

³⁶I.e., modal formulas $\square\chi$ and $\diamond\chi$ are treated as atoms in this setting.

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is admissible in $\mathcal{H}\mathbf{G}\Delta$, we have

$$\mathcal{H}\mathbf{KbiG}^c \vdash \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \vee \left(\delta \rightarrow \left(\bigvee \diamond_u^{<1} \rightarrow \phi \right) \right)$$

Now, we use the commutativity of \vee , apply **nec**, and then **Cr** to obtain that

$$\mathcal{H}\mathbf{KbiG}^c \vdash \diamond \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \vee \square \left(\delta \rightarrow \left(\bigvee \diamond_u^{<1} \rightarrow \phi \right) \right)$$

Since $u \in W^\tau$, we have that one of the following holds:

(A) $u \left(\diamond \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \right) = 1$ or

(B) $u \left(\square \left(\delta \rightarrow \left(\bigvee \diamond_u^{<1} \rightarrow \phi \right) \right) \right) = 1$.

We prove the statement in both cases.

Assume that (A) holds. We show that

$$\text{Th}(\mathcal{H}\mathbf{KbiG}^c), \Delta(*\square_u^{=1}), \Delta\delta \not\models_{\text{biG}} \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \rightarrow \bigvee \diamond_u^{<1} \quad (5.4)$$

We reason for the contradiction. Note that $\mathcal{H}\mathbf{G}\Delta$ is strongly complete w.r.t. biG , and that $\mathcal{H}\mathbf{G}\Delta \vdash \Delta\delta \rightarrow \Delta\Delta\delta$. Thus, applying Proposition 5.5, we obtain

$$\Delta(*\square_u^{=1}) \vdash_{\mathcal{H}\mathbf{KbiG}^c} \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \rightarrow \bigvee \diamond_u^{<1}$$

We apply \mathbf{M}_\square , \mathbf{P} , and \mathbf{K} and get

$$\square\Delta(*\square_u^{=1}) \vdash_{\mathcal{H}\mathbf{KbiG}^c} \diamond \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \rightarrow \bigvee \diamond \diamond_u^{<1}$$

Now, since $\Delta\square\phi \rightarrow \square\Delta\phi$ is an axiom scheme **Cr**, we have that

$$\Delta\square(*\square_u^{=1}) \vdash_{\mathcal{H}\mathbf{KbiG}^c} \diamond \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \rightarrow \bigvee \diamond \diamond_u^{<1}$$

We apply Proposition 5.5 again which gives us that

$$\text{Th}(\mathcal{H}\mathbf{KbiG}^c), \Delta\square(*\square_u^{=1}) \not\models_{\text{biG}} \diamond \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \rightarrow \bigvee \diamond \diamond_u^{<1}$$

However, we can show that u refutes this entailment. Indeed, observe that $u(\text{Th}(\mathcal{H}\mathbf{KbiG}^c)) = \{1\}$ since $u \in W^\tau$. Moreover, since $u(\square*\square_u^{=1}) = \{1\}$ by definition, we have that $u(\Delta\square*\square_u^{=1}) = 1$ as well. Finally, (A) gives us that $u \left(\diamond \left(\Delta\delta \wedge \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \right) \right) = 1$ but $u \left(\bigvee \diamond \diamond_u^{<1} \right) < 1$ by definition.

Thus, since the premises of (5.4) are either theorems of $\mathcal{H}\mathbf{KbiG}^c$ or formulas whose main connective is Δ , there is a homomorphism h that sends the premises of (5.4) to 1 and the conclusion to a lesser value. We show that h satisfies the conditions of the statement. Indeed, **C1** is obtained immediately since $\text{Th}(\mathcal{H}\mathbf{KbiG}^c)$ is closed under Δ . To see that **C2** holds, we note that $h(\Delta(*\square_u^{=1})) = \{1\}$, whence $h(*\square_u^{=1}) = \{1\}$ are evaluated at 1 too.

Since h refutes the conclusion of (5.4), we have that $h \left(\bigvee \diamond \diamond_u^{<1} \right) < h \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right)$. Hence, h satisfies **C3**. Finally, $h(\Delta\delta) = 1$ entails that $h(\delta) = 1$. But one can see that $h(\phi) \leq h \left(\bigvee \diamond_u^{<1} \right)$, whence **C4** also holds w.r.t. h by Proposition 5.7.

We consider (B). We assume that

$$\text{Th}(\mathcal{H}\mathbf{KbiG}^c), \Delta(*\square_u^{=1}), \delta, \delta \rightarrow \left(\bigvee \diamond_u^{<1} \rightarrow \phi \right) \models_{\text{biG}} \phi \quad (5.5)$$

and reason for contradiction. For this, we apply Proposition 5.5, strong completeness of $\mathcal{H}\mathbf{G}\Delta$ w.r.t. biG , and \mathbf{M}_\square to obtain

$$\text{Th}(\mathcal{H}\mathbf{KbiG}^c), \square\Delta(*\square_u^{=1}), \square\delta, \square \left(\delta \rightarrow \left(\bigvee \diamond_u^{<1} \rightarrow \phi \right) \right) \models_{\text{biG}} \square\phi$$

Now, we apply **Cr** which gives us

$$\text{Th}(\mathcal{HKbiG}^c), \Delta \square(*\square_u^{-1}), \square \delta, \square \left(\delta \rightarrow \left(\bigvee \diamond_u^{<1} \rightarrow \phi \right) \right) \models_{\text{biG}} \square \phi$$

Again, we can refute this entailment with u . Since $u \in W^\tau$, $u(\text{Th}(\mathcal{HKbiG}^c)) = \{1\}$. Furthermore, $u(\Delta \square(*\square_u^{-1})) = 1$ since $u(\square(*\square_u^{-1})) = 1$ by definition of u , and $u(\square \delta) > \alpha$ by Proposition 5.6, and $u(\square(\delta \rightarrow (\bigvee \diamond_u^{<1} \rightarrow \phi))) = 1$ by assumption (B). On the contrary, $u(\square \phi) = \alpha < 1$.

Thus, there exists a homomorphism h that evaluates the premises of (5.5) at 1^{37} and ϕ at a lesser value.

It remains to show that h satisfies **C1–C4**. Indeed, **C1** and **C2** hold because the premises of (5.5) are sent to 1. Furthermore, by the same reason, we have that $h(\bigvee \diamond_u^{<1}) < 1$ (**C3**). Finally, since $h(\phi) < h(\delta) = 1$, we obtain **C4** via an application of Proposition 5.7. \square

Remark 5.2. Let us return to the proof. Observe that it is crucial to use $\Delta \delta$ and $\Delta(*\square_u^{-1})$ and not δ and $*\square_u^{-1}$ in the premises of (5.4). Indeed, if we use the Δ -less versions, then

$$\text{Th}(\mathcal{HKbiG}), *\square_u^{-1}, \delta \not\models_{\text{biG}} \left(\phi \rightarrow \bigvee \diamond_u^{<1} \right) \rightarrow \bigvee \diamond_u^{<1}$$

does not guarantee the existence of h s.t. $h(\delta) = 1$ (which is necessary to establish **C4**) and $h(\psi) = 1$ (**C2**) for every $\psi \in *\square_u^{-1}$.

Next, we prove the counterpart of Proposition 4.7 in [130]. Again, we will not be able to mimic it step-by-step since biG-entailment is not equivalent to the preservation of 1. Thus, we need a stronger version of [130, Lemma 4.6]. Our next proposition serves exactly this goal.

Proposition 5.9. *Let $\phi \in \diamond_u^{=\alpha}$ for some $\alpha > 0$ and set*

$$\delta' := \left(\phi \rightarrow \bigvee \diamond_u^{<\alpha} \right) \rightarrow \bigvee \diamond_u^{<\alpha}$$

Then $u(\diamond \Delta \delta') = 1$.

Proof. Note that $u(\diamond \phi) > u(\bigvee \diamond \diamond_u^{<\alpha})$ by definition. Thus we have $u(\diamond \phi \rightarrow \bigvee \diamond \diamond_u^{<\alpha}) < 1$ by **K**, whence $u(\sim \Delta(\diamond \phi \rightarrow \bigvee \diamond \diamond_u^{<\alpha})) = 1$. Now, we use $\sim \Delta \diamond$ axiom to obtain that $u(\diamond \sim \Delta(\phi \rightarrow \bigvee \diamond_u^{<\alpha})) = 1$. From here, since $\mathcal{HG}\Delta \vdash \sim \Delta(\phi \rightarrow \bigvee \diamond_u^{<\alpha}) \rightarrow \Delta((\phi \rightarrow \bigvee \diamond_u^{<\alpha}) \rightarrow \bigvee \diamond_u^{<\alpha})$, we obtain $u(\diamond \Delta \delta') = 1$ by an application of **nec** and **K**, as required. \square

We are now ready to prove the \diamond counterpart of Proposition 5.8.

Proposition 5.10. *For any $\alpha > 0$ and $\phi \in \diamond_u^{=\alpha}$, there exists a propositional homomorphism $h : \mathcal{L}_{\mathcal{G}\Delta, \square, \diamond} \rightarrow [0, 1]_{\text{biG}}$, s.t.:*

C1: $h(\chi) = 1$ for any $\chi \in \text{Th}(\mathcal{HKbiG}^c)$;

C2: $h(\psi) = 1$ for every $\psi \in *\square_u^{-1}$;

C3: $h(\rho) < 1$ for every $\rho \in \diamond_u^{<1}$;

C4': $h(\phi) > h(\sigma)$ for every $\sigma \in \diamond_u^{<\alpha}$.

Proof. We assume

$$\text{Th}(\mathcal{HKbiG}^c), \Delta(*\square_u^{-1}), \Delta \delta' \models_{\text{biG}} \bigvee \diamond_u^{<1} \tag{5.6}$$

and reason for contradiction. Again, we use the completeness of $\mathcal{HG}\Delta$, Proposition 5.5, **K**, **M** $_{\square}$, **P**, and **Cr** to arrive at

$$\text{Th}(\mathcal{HKbiG}^c), \Delta \square(*\square_u^{-1}) \models_{\text{biG}} \diamond \Delta \delta' \rightarrow \bigvee \diamond \diamond_u^{<1}$$

³⁷Note that theorems are closed under Δ and that $h(\delta \rightarrow (\bigvee \diamond_u^{<1} \rightarrow \phi)), h(\delta) \in \{1, h(\phi)\}$.

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It is easy to see that u refutes this entailment: $u(\text{Th}(\mathcal{HKbiG}^c)) = 1$ since $u \in W^\tau$, $u(\Box^* \Box_u^=1) = 1$ by definition, whence $u(\Delta \Box^* \Box_u^=1) = 1$, and $u(\Diamond \Delta \delta') = 1$ by Proposition 5.9 but $u(\bigvee \Diamond \Diamond_u^{<1}) < 1$ by definition.

Thus, there exists a homomorphism h that sends the premises of (5.6) to 1³⁸ and its conclusion to a lesser value. Hence, h satisfies **C1** and **C2**. Furthermore, $h(\bigvee \Diamond_u^{<1}) < 1$, and thus, **C3** is satisfied. Finally, since $h(\Delta \delta') = 1$, we have that $h(\delta') = 1$, whence **C4'** is satisfied as well. \square

Remark 5.3. Again, the considerations of Remark 5.2 apply for Proposition 5.8 as well. Since we need $h(\delta') = 1$ to establish **C4'**, we cannot use δ' by itself in the premise of (5.6) as the failure of the entailment does not (in general) guarantee that the premises are evaluated at 1.

Remark 5.4. It is clear that since $h(\Box 1) = 1$, every homomorphism h satisfying the conditions of Proposition 5.8 satisfies

C4.1: $h(\phi) < 1$.

Furthermore, from $h(\Diamond 0) = 0$, it follows that for every h that satisfies the conditions of Proposition 5.10, it holds that

C4'.1: $h(\phi) > 0$.

Finally, if **C1**–**C3** are true for h , then the following properties hold for all $\theta, \theta' \in \mathcal{L}_{G\Delta, \Box, \Diamond}$.

C2.a: If $u(\Diamond \theta) \leq u(\Box \theta')$, then $h(\theta) \leq h(\theta')$ since $\theta \rightarrow \theta' \in * \Box_u^=1$ using **FS**.

C2.b: If $\theta \in \text{Sf}^{0,1}(\tau)$ and $u(\Diamond \theta) < u(\Box \theta')$, then $h(\theta) < h(\theta')$. For

$$\mathcal{HKbiG}^c \vdash ((\Box \theta' \rightarrow \Diamond \theta) \rightarrow \Diamond \theta) \rightarrow (\Box((\theta' \rightarrow \theta) \rightarrow \theta) \vee \Diamond \theta)$$

$u(\Diamond \theta) < 1$, and $u(\Box \theta' \rightarrow \Diamond \theta) \rightarrow \Diamond \theta = 1$ imply that $(\theta' \rightarrow \theta) \rightarrow \theta \in * \Box_u^=1$ and **C3** implies $h(\theta) < 1$.

C2.c: If $u(\Box \theta) > 0$, then $h(\theta) > 0$.

C2.d: If $u(\Diamond \theta) = 0$, then $h(\theta) = 0$.

We can now establish the next statement which is analogous to propositions 4.5 and 4.8 in [130] using Propositions 5.8 and 5.10 as well as Remark 5.4. The proof is exactly the same as in the original version.

Proposition 5.11.

1. For any $\phi \in \Box_u^=\alpha$, $\alpha < 1$, and $\varepsilon > 0$ there is $u' \in W^\tau$ s.t. $uR^\tau u'$ and $u'(\phi) \in [\alpha, \alpha + \varepsilon]$.
2. For any $\phi \in \Diamond_u^=\alpha$, $\alpha > 0$, and $\varepsilon > 0$ there is $u' \in W^\tau$ s.t. $uR^\tau u'$ and $u'(\phi) \in [\alpha - \varepsilon, \alpha]$.

The truth lemma can be established using Proposition 5.11 that guarantees that for every value α of $\Box \phi$ or $\Diamond \phi$, one can find an accessible state where the value of ϕ is arbitrarily close to α . Thus, $\Box \phi$ will be indeed evaluated as the infimum and $\Diamond \phi$ as the supremum of ϕ 's values in the accessible states. Again, the proof can be conducted in the same manner as in [130].

Proposition 5.12 (Truth lemma). For any $\phi \in \text{Sf}^{0,1}(\tau)$, it holds that $v^\tau(\phi, u) = u(\phi)$.

Now, weak completeness will follow from the truth lemma and the validity of axioms and rules.

Theorem 5.1. \mathcal{HKbiG}^c is weakly complete: for any $\phi \in \mathcal{L}_{G\Delta, \Box, \Diamond}$, it holds that $\mathbf{KbiG}^c \models \phi$ iff $\mathcal{HKbiG}^c \vdash \phi$.

³⁸Again, observe that $\text{Th}(\mathcal{HKbiG}^c)$ is closed under Δ and all other premises have Δ as their main connective.

The strong completeness is a bit more complicated.

Theorem 5.2. \mathcal{HKbiG}^c is strongly complete: for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathcal{G}\Delta, \square, \diamond}$, it holds that $\Gamma \models_{\mathbf{KbiG}^c} \phi$ iff $\Gamma \vdash_{\mathcal{HKbiG}^c} \phi$.

Proof. The proof follows [130, Corollary 4.12]. The only two differences are that we need to account for Δ and that the \mathbf{KbiG}^c entailment $\Gamma \models_{\mathbf{KbiG}^c} \chi$ is defined via the order on $[0, 1]$. That is, if the entailment is refuted by v , then $\inf\{v(\phi, w) : \phi \in \Gamma\} > v(\chi, w)$ for some $w \in \mathfrak{F}$. This, in turn, is equivalent to

$$\exists d \in (0, 1] \forall \phi \in \Gamma : v(\phi, w) \geq d \text{ but } v(\chi, w) < d$$

Now let $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathcal{G}\Delta, \square, \diamond}$ and $\Gamma \not\models_{\mathcal{HKbiG}^c} \phi$. We consider the classical first order theory Γ^* whose signature contains two unary predicates W and P , one binary predicate $<$, binary functions \circ and \mathfrak{s} , unary function \blacktriangle , constants $0, 1, c, d$, and a function symbol f_θ for each $\theta \in \mathcal{L}_{\mathcal{G}\Delta, \square, \diamond}$. Intuitively, $W(x)$ stands for ‘ x is a state’; $P(x)$ for ‘ x is a number’; $<$ is going to be the order on numbers. \circ is used to define the value of the Gödel implication; \blacktriangle is the counterpart of Δ ; \mathfrak{s} is the relation between states.

We can now axiomatise the \mathbf{KbiG} semantics in the classical first-order logic as follows.

- $\forall x \sim(W(x) \wedge P(x))$
- $\forall x(W(x) \vee \sim W(x))$
- $P(d)$
- ‘ $\langle P, < \rangle$ is a strict linear order s.t. $0 < d \leq 1$, 0 and 1 are the minimum and the maximum of $\langle P, < \rangle$ ’.
- $\forall x \forall y ((W(x) \wedge W(y)) \rightarrow (\mathfrak{s}(x, y) = 1 \vee \mathfrak{s}(x, y) = 0))$
- $\forall x \forall y ((P(x) \wedge P(y)) \rightarrow ((x \leq y \wedge x \circ y = 1) \vee (x > y \wedge x \circ y = y)))$
- $\forall x(P(x) \rightarrow ((x = 1 \wedge \blacktriangle(x) = 1) \vee (x < 1 \wedge \blacktriangle(x) = 0)))$
- For each $\theta, \theta' \in \mathcal{L}_{\mathcal{G}\Delta, \square, \diamond}$, we add the following formulas.
 - $\forall x(W(x) \rightarrow P(f_\theta(x)))$
 - $\forall x(W(x) \rightarrow f_{\sim\theta}(x) = (f_\theta(x) \circ 0))$
 - $\forall x(W(x) \rightarrow f_{\Delta\theta}(x) = \blacktriangle(f_\theta(x)))$
 - $\forall x(W(x) \rightarrow f_{\theta \wedge \theta'}(x) = \min\{f_\theta(x), f_{\theta'}(x)\})$
 - $\forall x(W(x) \rightarrow f_{\theta \vee \theta'}(x) = \max\{f_\theta(x), f_{\theta'}(x)\})$
 - $\forall x(W(x) \rightarrow f_{\theta \rightarrow \theta'}(x) = f_\theta(x) \circ f_{\theta'}(x))$
 - $\forall x(W(x) \rightarrow f_{\square\theta}(x) = \inf_y \{\mathfrak{s}(x, y) \circ f_\theta(y)\})$
 - $\forall x(W(x) \rightarrow f_{\diamond\theta}(x) = \sup_y \{\min\{\mathfrak{s}(x, y), f_\theta(y)\}\})$
- For each $\gamma \in \Gamma$, we add $f_\gamma(c) \geq d$.
- We also add $W(c) \wedge (f_\phi(c) < d)$.

The rest of the proof is identical to that in [130]. For each finite subset Γ^- of Γ^* , we let $\mathcal{L}_{\mathcal{G}\Delta, \square, \diamond}^- = \{\theta : f_\theta \text{ occurs in } \Gamma^-\}$. Since $\mathcal{L}_{\mathcal{G}\Delta, \square, \diamond}^- \cap \Gamma \not\models_{\mathcal{HKbiG}^c} \phi$ by assumption, Theorem 5.1

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and (5.3) entail that there is a crisp pointed model $\langle \mathfrak{M}, c \rangle$ with $\mathfrak{M} = \langle W, \mathbf{s}^{\Gamma^-}, e^{\Gamma^-} \rangle$ being such that $e^{\Gamma^-}(\phi, c) < d$ and $e^{\Gamma^-}(\theta, c) \geq d$ for every $\theta \in \Gamma \cap \Gamma^-$. Thus, the following structure

$$\langle W \sqcup [0, 1], W, [0, 1], <, 0, 1, c, d, \circ, \blacktriangle, \mathbf{s}^{\Gamma^-}, \{f_\theta\}_{\theta \in \mathcal{L}_{\mathbb{G}\Delta, \square, \diamond}} \rangle$$

is a model of Γ^- . Now, by compactness and the downward Löwenheim–Skolem theorem, Γ^* has a countable model

$$\mathfrak{M}^* = \langle B, W, P, <, 0, 1, c, d, \circ, \blacktriangle, \mathbf{s}, \{f_\theta\}_{\theta \in \mathcal{L}_{\mathbb{G}\Delta, \square, \diamond}} \rangle$$

Now, we can embed $\langle P, < \rangle$ into $\langle \mathbb{Q} \cap [0, 1], < \rangle$ preserving 0 and 1 as well as all infima and suprema. Hence, we may w.l.o.g. assume that \mathbf{s} is crisp and the ranges of f_θ 's are contained in $[0, 1]$. Then, it is straightforward to verify that $\mathfrak{M} = \langle W, S, e \rangle$, where $e(\theta, w) = f_\theta(w)$ for all $w \in W$ and $\theta \in \mathcal{L}_{\mathbb{G}\Delta, \square, \diamond}$, is a crisp \mathbf{KbiG}^c model with a distinguished world c such that $v[\Gamma, c] \geq d$ and $v(\phi, c) < d$ for some $0 < d \leq 1$. Hence, $\inf\{v(\gamma, c) : \gamma \in \Gamma\} > v(\phi, c)$, and thus, $\Gamma \not\models_{\mathbf{KbiG}} \phi$. \square

5.3. Model-theoretic properties

This section is mostly done employing \mathbf{KG} only (i.e., we are considering Δ -free formulas unless explicitly stated otherwise) and is devoted to two questions. First, we explore which *classical* definitions of classes of frames can be transferred to \mathbf{KG}^c . Second, we establish the class of frames where Glivenko's theorem holds.

5.3.1. Frame definability

Note first of all, that every class of frames definable in \mathbf{K} is definable in \mathbf{KG} (cf. [104, 105, 37, 38] for details). Namely, if ϕ^∇ is the result of replacing every p occurring in ϕ with $\sim\sim p$, then $\mathfrak{F} \models_{\mathbf{K}} \phi$ iff $\mathfrak{F} \models_{\mathbf{KG}} \phi^\nabla$ (for the mono-modal fragments of \mathbf{KG}). For the bi-modal \mathbf{KG} , the embedding is given by placing $\sim\sim$ in front of every subformula of ϕ .

It is easy to establish that \mathbf{KG} is *more expressive* than \mathbf{K} . Namely, it can define some frame properties undefinable in \mathbf{K} . We show that finitely branching frames (i.e., those where $R^+(w)$ is finite for every w) are definable.

Proposition 5.13. *A (crisp or fuzzy) frame \mathfrak{F} is finitely branching iff $\mathfrak{F} \models_{\mathbf{KbiG}} \sim\sim\square(p \vee \sim p)$.*

Proof. We show only the fuzzy case as the crisp one can be proven in a similar manner.

Assume that \mathfrak{F} is finitely branching. Then, clearly, $v(\square(p \vee \sim p), w) = \min\{R(w, w') \rightarrow_{\mathbb{G}} v(p \vee \sim p, w') : w' \in W\} > 0$. Hence, $v(\sim\sim\square(p \vee \sim p), w) = 1$.

Now let \mathfrak{F} be infinitely branching, let $X \subseteq R^+(w)$ be countable and w.l.o.g. $R(w, w_i) \geq R(w, w_j)$ iff $i < j$ for every $w_i, w_j \in X$. We define $v(p, w_1) = \frac{R(w, w_1)}{2}$ and

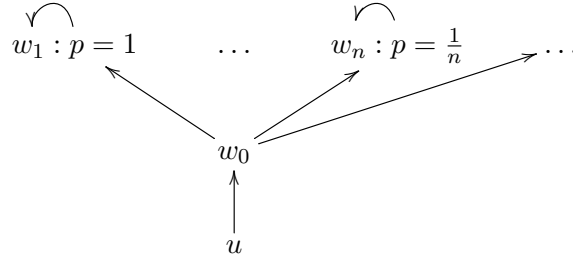
$$v(p, w_{i+1}) = \begin{cases} \frac{v(p, w_i)}{2} & \text{iff } v(p, w_i) \leq R(w, w_{i+1}) \\ \frac{R(w, w_{i+1})}{2} & \text{otherwise} \end{cases}$$

It is clear that $v(\sim p, w_i) = 0$ and that $v(p \vee \sim p, w_i) = v(p, w_i)$ for every $w_i \in R(w)$.

Observe that $\inf\{R(w, w') \rightarrow_{\mathbb{G}} v(p, w') : w' \in W\} = 0$. Thus, $v(\square(p \vee \sim p), w) = 0$, and thus $v(\sim\sim\square(p \vee \sim p), w) = 0$, as required. \square

Observe, however, that some formulas defining useful classes of frames do not require any translation at all. For example [130], the following formulas define the same classes of frames both in \mathbf{K} and \mathbf{KG}^c .

$$\begin{array}{lll} \square p \rightarrow p & p \rightarrow \diamond p & \text{(reflexivity)} \\ \square p \rightarrow \square \square p & \diamond \diamond p \rightarrow \diamond p & \text{(transitivity)} \end{array}$$

Figure 5.3.: $v(\Box\sim\sim p \rightarrow \sim\sim\Box p, w_0) = 0$ and $v(\phi, u) = 0$.

$$\begin{array}{lll}
 p \rightarrow \Box\Diamond p & \Diamond\Box p \rightarrow p & \text{(symmetry)} \\
 \Diamond p \rightarrow \Box\Diamond p & \Diamond\Box p \rightarrow \Box p & \text{(Euclideaness)} \\
 \Diamond\mathbf{1} & & \text{(seriality)}
 \end{array}$$

One should remember, though, that since \Box and \Diamond are not interdefinable in \mathbf{KG}^c [130, Lemma 6.1], nor in \mathbf{KbiG}^c (Proposition 5.1), one needs *both* formulas to define a class of frames in the bi-modal languages.

A natural question now is whether *every* classical definition of a class of frames \mathbb{F} defines \mathbb{F} in \mathbf{KG}^c . Evidently, the answer is negative. For consider $\Diamond(p \vee \sim p)$. Even though it defines serial frames in classical modal logic, it does not do so in the Gödel modal logic. In fact, $\Diamond(p \vee \sim p)$ can be refuted *on every frame*: indeed, it just suffices to set $v(p, w) = \frac{1}{2}$ for every state.

One could also think that every ϕ that classically defines \mathbb{F} , defines it in \mathbf{KG}^c as long as ϕ^- (ϕ with all modalities removed) is a \mathbf{G} -tautology. This turns out to be false too. For consider $\phi = \Diamond(\Box\sim\sim p \rightarrow \sim\sim\Box p)$. Clearly, ϕ^- is a \mathbf{G} -tautology. Classically, ϕ defines serial frames. However, it is not valid on the (serial) frame in Fig. 5.3.

A question thus arises: which classes of formulas are *transferrable*, i.e., define the same frames in \mathbf{K} and \mathbf{KG}^c . In this section, we establish several such classes.

Definition 5.6 (Transferrable formulas). A Δ -free $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \Box, \Diamond}$ is called *transferrable* iff for any crisp frame \mathfrak{F} and $w \in \mathfrak{F}$, it holds that $\mathfrak{F}, w \models_{\mathbf{K}} \phi$ iff $\mathfrak{F}, w \models_{\mathbf{KbiG}} \phi$.

Proposition 5.14. *Every closed formula (i.e., built only from constants $\mathbf{0}$ and $\mathbf{1}$) ϕ is transferrable.*

Proof. Immediately since closed formulas on crisp frames have values in $\{0, 1\}$. \square

Theorem 5.3. *Let ϕ, ϕ' , and ψ be transferrable. Let further, $\text{Prop}(\phi) \cap \text{Prop}(\psi) = \emptyset$. Then, $\phi \wedge \phi'$, $\phi \vee \psi$, and $\Box\phi$ are transferrable.*

Proof. The case of $\phi \wedge \phi'$ is straightforward, so we will only consider $\phi \vee \psi$ and $\Box\phi$.

$$\boxed{\phi \vee \psi}$$

$$\begin{aligned}
 \mathfrak{F}, w \not\models_{\mathbf{KbiG}} \phi \vee \psi & \text{ iff } \mathfrak{F}, w \not\models_{\mathbf{KbiG}} \phi \text{ and } \mathfrak{F}, w \not\models_{\mathbf{KbiG}} \psi \\
 & \text{ iff } \mathfrak{F}, w \not\models_{\mathbf{K}} \phi \text{ and } \mathfrak{F}, w \not\models_{\mathbf{K}} \psi & \text{(by assumption)} \\
 & \text{ iff } \mathfrak{F}, w \not\models_{\mathbf{K}} \phi \vee \psi
 \end{aligned}$$

$$\boxed{\Box\phi}$$

$$\begin{aligned}
 \mathfrak{F}, w \not\models_{\mathbf{KbiG}} \Box\phi & \text{ iff } \exists w' : wRw' \text{ and } \mathfrak{F}, w' \not\models_{\mathbf{KbiG}} \phi \\
 & \text{ iff } \exists w' : wRw' \text{ and } \mathfrak{F}, w' \not\models_{\mathbf{K}} \phi & \text{(by assumption)} \\
 & \text{ iff } \mathfrak{F}, w \not\models_{\mathbf{K}} \Box\phi
 \end{aligned}$$

\square

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Note that in the above proof, we cannot use arbitrary disjunctions. Indeed, it does not follow from the classical validity of $\phi \vee \psi$ that ϕ or ψ is valid unless ϕ and ψ do not have common variables. Moreover, it is crucial that we define transfer on *pointed frames* since it is false that

$$\mathfrak{F} \models_{\mathbf{K}} \phi \vee \psi \text{ iff } \mathfrak{F} \models_{\mathbf{K}} \phi \text{ or } \mathfrak{F} \models_{\mathbf{K}} \psi \quad (\mathbf{Prop}(\phi) \cap \mathbf{Prop}(\psi) = \emptyset)$$

The easiest counterexample is $\diamond(p \vee \sim p) \vee \Box(q \wedge \sim q)$ which is valid in \mathbf{K} , although $\diamond(p \vee \sim p)$ and $\Box(q \wedge \sim q)$ are not valid in \mathbf{K} .

To establish further transfer results, we will need the notions of *positive* and *monotone* formulas.

Definition 5.7.

- $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \Box, \Diamond}$ is called *monotone* iff it does not contain \rightarrow , \sim , and Δ .
- A *monotone* formula is called *positive* iff it does not contain $\mathbf{1}$ and $\mathbf{0}$.

Lemma 5.2. *Let ϕ and ϕ' be monotone. Let further, $v(\phi, w) > v(\phi', w') = x'$. Define*

$$v^{\text{cl}}(p, u) = \begin{cases} 1 & \text{iff } v(p, u) > x' \\ 0 & \text{iff otherwise} \end{cases} \quad (\text{vCL})$$

Then $v^{\text{cl}}(\phi, w) = 1$ and $v^{\text{cl}}(\phi', w') = 0$.

Proof. We proceed by induction on the total number of connectives in ϕ and ϕ' . The basis case of ϕ and ϕ' being variables or constants is straightforward. The cases of propositional connectives are easy as well.

For $v(\diamond\phi, w) > v(\phi', w')$, we proceed as follows.

$$\begin{aligned} v(\diamond\phi, w) > v(\phi', w') & \text{ iff } \sup\{v(\phi, u) : wRu\} > v(\phi', w') \\ & \text{ iff } \exists u : wRu \text{ and } v(\phi, u) > v(\phi', w') \\ & \text{ iff } \exists u : wRu \text{ and } v^{\text{cl}}(\phi, u) = 1 \text{ and } v^{\text{cl}}(\phi', w') = 0 \quad (\text{by IH}) \\ & \text{ iff } v^{\text{cl}}(\diamond\phi, w) = 1 \text{ and } v^{\text{cl}}(\phi', w') = 0 \end{aligned}$$

Other cases of modalities can be tackled in a similar manner. □

Theorem 5.4. *Let ϕ and ϕ' be monotone. Then $\phi \rightarrow \phi'$ is transferrable.*

Proof. Immediately from Lemma 5.2. □

The final transfer result we are going to discuss in this section is that Sahlqvist formulas are transferrable. We recall the definition from [28].

Definition 5.8. A *Sahlqvist implication* (SI) is a formula $\phi \rightarrow \chi$ with

$$\text{SI} \ni \phi := l \in \mathbf{Prop} \cup \{\sim p : p \in \mathbf{Prop}\} \cup \underbrace{\{\Box \dots \Box p : p \in \mathbf{Prop}, k \in \mathbb{N}\}}_{k \text{ times}} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \diamond\phi$$

and χ being *positive*. *Sahlqvist formulas* (SF) are obtained using the following grammar:

$$\text{SF} \ni \psi, \psi' := \tau \in \text{SI} \mid \psi \wedge \psi \mid \psi \vee \psi' \quad (\mathbf{Prop}(\psi) \cap \mathbf{Prop}(\psi') = \emptyset) \mid \Box\psi$$

Theorem 5.5. *Sahlqvist formulas are transferrable.*

Proof. By Theorem 5.3, it suffices to prove the statement only for Sahlqvist implications.

Let $\phi \rightarrow \chi \in \text{SI}$. Assume that $v(\phi, w) > x'$ and $v(\chi, w') = x' \neq 1$.

We show by induction on the total number of connectives that

$$\begin{aligned} v(\phi, u) > x' &\Rightarrow v^{\text{cl}}(\phi, u) = 1 && (\phi \text{ as in definition 5.8}) \\ v(\chi, u') \leq x' &\Rightarrow v^{\text{cl}}(\chi, u') = 0 && (\chi \text{ is positive}) \end{aligned}$$

The basis case of variables and constants is straightforward.

$$\boxed{v(\sim p, w) = 1, v(\chi, w') < 1}$$

$$\begin{aligned} v(\sim p, w) = 1 \text{ and } v(\chi, w') < 1 &\Rightarrow v(p, w) = 0 \text{ and } v(\chi, w') = x' < 1 \\ &\Rightarrow v^{\text{cl}}(p, w) = 0 \text{ and } v^{\text{cl}}(\chi, w') = 0 \quad (\text{by IH since } p \text{ is positive}) \\ &\Rightarrow v^{\text{cl}}(\sim p, w) = 1 \text{ and } v^{\text{cl}}(\chi, w') = 0 \end{aligned}$$

The cases of propositional connectives as well as $\phi = \underbrace{\Box \dots \Box}_k p$ are easy as well.

$$\boxed{v(\Diamond \phi', w) > x', v(\chi, w') = x' < 1}$$

$$\begin{aligned} v(\Diamond \phi', w) > x' \text{ and } v(\chi, w') = x' < 1 &\Rightarrow \sup\{v(\phi', u) : wRu\} > x' \text{ and } v(\chi, w') = x' \\ &\Rightarrow \exists u : wRu \text{ and } v(\phi', u) > x' \text{ and } v(\chi, w') = x' \\ &\Rightarrow \exists u : wRu \text{ and } v^{\text{cl}}(\phi', u) = 1 \text{ and } v^{\text{cl}}(\chi, w') = 0 \quad (\text{by IH}) \\ &\Rightarrow v^{\text{cl}}(\Diamond \phi', w) = 1 \text{ and } v^{\text{cl}}(\chi, w') = 0 \end{aligned}$$

□

Note that the two classes of transferrable formulas in Theorems 5.4 and 5.5 do not coincide as there are Sahlqvist implications that are not monotone and there are implications of monotone formulas that are not Sahlqvist. Note, furthermore, that the above theorems *do not* characterise the class of transferrable formulas completely: for example, we can show that the Gödel-Löb formula $\Box(\Box p \rightarrow p) \rightarrow \Box p$ is transferrable, even though it is neither monotone, nor Sahlqvist, nor obtained from transferrable formulas via Theorem 5.3.

Proposition 5.15. *Let $\mathfrak{F} = \langle W, R \rangle$ be a crisp frame. Then, $\mathfrak{F}, w \models_{\mathbf{KbiG}} \Box(\Box p \rightarrow p) \rightarrow \Box p$ iff R is transitive and does not contain an infinite chain $wRw_0Rw_1Rw_2R\dots$ originating from w (i.e., conversely well-founded).*

Proof. Since \mathbf{KbiG}^c valuations preserve classical values, we only prove the ‘if’ direction. Let $\mathfrak{F} = \langle W, R \rangle$ be a crisp frame s.t. R is transitive and conversely well-founded. We let

$$v(\Box(\Box p \rightarrow p), w) = x > 0$$

for some $w \in \mathfrak{F}$ and v on \mathfrak{F} . Then, for every $w' \in R(w)$, it holds either $v(p, w) \geq x$ or $v(\Box p, w') \leq v(p, w')$.

Recall that R does not have infinite chains beginning from w . Thus, $v(p, w'') \geq x$ for every $w'' \in R(w)$ s.t. $R(w'') = \emptyset$ because $v(\Box p, w'') = 1$ for every such w'' . Denote the set of these states with W_0 .

In general, for every $n \in \mathbb{N}$, we define W_{-n} to be the set of all $t \in R(w)$ s.t. the longest R -sequence originating from t has n members.

It is clear that $w \in W_{-(k+1)}$ for some $k \in \mathbb{N}$ and that $R(w) = \bigcup_{i=0}^k W_{-i}$. We show by induction on k that $v(p, u) \geq x$ for every $u \in \bigcup_{i=0}^k W_{-i-1}$. The basis case is already shown. Assume that the

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statement holds for some l . We show it for $l+1$ and reason for a contradiction. Let $u' \in W_{-(l+1)}$ and $v(p, u') < x$. But then, since R is transitive and irreflexive, we have $v(\Box p, u') \geq x$ by the induction hypothesis. Hence, $v(\Box p \rightarrow p, u') < x$ and further, $v(\Box(\Box p \rightarrow p), w) < x$, contrary to the assumption.

Thus, $v(p, u) \geq x$ for every $u \in R(w)$. But then $v(\Box p, w) \geq x$, as required. \square

5.3.2. Glivenko's theorem

In this section, we study the fragments of \mathbf{KbiG}^c that admit Glivenko's theorem [71] that we present in its semantical form.

Theorem 5.6. *ϕ is a classical propositional tautology iff $\sim\sim\phi$ is a (super-)intuitionistically valid propositional formula.*

Glivenko's theorem in non-intermediate propositional logics is well studied (cf., e.g. [119] and the literature referred to therein). It is also known [91] that the theorem holds for the \exists fragment of the first-order intuitionistic logic. Furthermore, versions of Glivenko's theorem for modal intuitionistic logics are studied in [18].

Considering \mathbf{KbiG}^c and \mathbf{KbiG}^f , we, first, notice that the unrestricted version of Glivenko's theorem (unsurprisingly) fails: $\sim\sim\Box(p \vee \sim p)$ is not \mathbf{KbiG}^c valid (as we have seen in Proposition 5.13, it defines finitely branching frames).

In what follows, we will show that Glivenko's theorem holds in all finitely branching frames, and that, conversely, if Glivenko's theorem holds for a logic of a class of frames \mathbb{F} , then \mathbb{F} does not contain infinitely branching frames. For this, we require some preliminary definitions and statements.

Definition 5.9 (Logic of \mathbb{F}). Let \mathbb{F} be a class of frames. The \mathbf{KbiG}^c logic of \mathbb{F} is the set $L \subseteq \mathcal{L}_{G\Delta, \Box, \Diamond}$ s.t. $\mathfrak{F} \models_{\mathbf{KbiG}} L$ for every $\mathfrak{F} \in \mathbb{F}$.

Definition 5.10. For any model $\mathfrak{M} = \langle W, R, v \rangle$, define a model $\mathfrak{M}^{cl} = \langle W, R^{cl}, v^{cl} \rangle$ s.t.

$$wR^{cl}w' = \begin{cases} 1 & \text{iff } wRw' \neq 0 \\ 0 & \text{iff } wRw' = 0 \end{cases} \quad v^{cl}(p, w) = \begin{cases} 1 & \text{iff } v(p, w) \neq 0 \\ 0 & \text{iff } v(p, w) = 0 \end{cases}$$

For any frame $\mathfrak{F} = \langle W, R \rangle$, we set $\mathfrak{F}^{cl} = \langle W, R^{cl} \rangle$.

Lemma 5.3. *Let $\phi \in \mathcal{L}_{G\Delta, \Box, \Diamond}$ be Δ -free. Then for any finitely branching frame \mathfrak{F} and for any v on \mathfrak{F} , it holds that*

$$v^{cl}(\phi, w) = \begin{cases} 1 & \text{iff } v(\phi, w) \neq 0 \\ 0 & \text{iff } v(\phi, w) = 0 \end{cases} \quad (5.7)$$

Proof. We prove by induction. The cases when $\phi = p$ or $\phi = \mathbf{0}$ are trivial.

$$\boxed{\phi = \psi \wedge \psi'}$$

$$\begin{aligned} v(\psi \wedge \psi', w) = 0 & \text{ iff } v(\psi, w) = 0 \text{ or } v(\psi', w) = 0 \\ & \text{ iff } v^{cl}(\psi, w) = 0 \text{ or } v^{cl}(\psi', w) = 0 & \text{(by IH)} \\ & \text{ iff } v^{cl}(\psi \wedge \psi', w) = 0 \end{aligned}$$

$$\boxed{\phi = \psi \vee \psi'}$$
 is dual.

$$\boxed{\phi = \psi \rightarrow \psi'}$$

$$v(\psi \rightarrow \psi', w) = 0 \text{ iff } v(\psi, w) \neq 0 \text{ and } v(\psi', w) = 0$$

$$\begin{aligned} & \text{iff } v^{\text{cl}}(\psi, w) = 1 \text{ or } v^{\text{cl}}(\psi', w) = 0 && \text{(by IH)} \\ & \text{iff } v^{\text{cl}}(\psi \rightarrow \psi', w) = 0 \end{aligned}$$

$$\begin{aligned} v(\psi \wedge \psi', w) \neq 0 & \text{ iff } v(\psi', w) \neq 0 \\ & \text{ iff } v^{\text{cl}}(\psi', w) = 1 && \text{(by IH)} \\ & \text{ iff } v^{\text{cl}}(\psi \wedge \psi', w) = 1 \end{aligned}$$

$$\boxed{\phi = \Box\psi}$$

$$\begin{aligned} v(\Box\psi, w) = 0 & \text{ iff } \exists w' : wRw' > 0 \text{ and } v(\psi, w') = 0 \\ & \text{ iff } \exists w' : wRw' = 1 \text{ and } v^{\text{cl}}(\psi, w') = 0 && \text{(by IH)} \\ & \text{ iff } v^{\text{cl}}(\Box\psi, w) = 0 && \text{(by finite branching)} \end{aligned}$$

$$\boxed{\phi = \Diamond\psi}$$

$$\begin{aligned} v(\Diamond\psi, w) \neq 0 & \text{ iff } \exists w' : wRw' > 0 \wedge v(\psi, w') \neq 0 \\ & \text{ iff } \exists w' : wRw' = 1 \Rightarrow v^{\text{cl}}(\psi, w') = 1 && \text{(by IH)} \\ & \text{ iff } v^{\text{cl}}(\Diamond\psi, w) = 1 \end{aligned}$$

□

The following unsurprising statement is immediate.

Proposition 5.16. *Let $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \Box, \Diamond}$ be $\{\Box, \Delta\}$ -free. Then*

1. ϕ is \mathbf{K} valid iff $\sim\sim\phi$ is \mathbf{KbiG}^c valid iff $\sim\sim\phi$ is \mathbf{KbiG}^f valid;
2. $\mathfrak{F} \models_{\mathbf{K}} \phi$ iff $\mathfrak{F} \models_{\mathbf{KbiG}} \phi$ for every crisp \mathfrak{F} .

Proof. Note that in the proof of Lemma 5.3, we use the finite branching only in the \Box case but ϕ is \Box -free. □

Theorem 5.7.

1. Let $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \Box, \Diamond}$ be Δ -free. Then it is \mathbf{K} -valid iff $\sim\sim\phi$ is \mathbf{KbiG}^c -valid (\mathbf{KbiG}^f) on all finitely branching frames.
2. Let \mathbb{F} be a class of (fuzzy or crisp) frames and let \mathbf{L} be the \mathbf{KbiG} logic of \mathbb{F} . Then, $\{\sim\sim\phi : \phi \text{ is } \mathbb{F} \models_{\mathbf{K}} \phi\} \subseteq \mathbf{L}$ implies that every $\mathfrak{F} \in \mathbb{F}$ is finitely branching.

Proof. We begin with 1. Clearly, if ϕ is not valid in \mathbf{K} , there is a finite branching frame where it is invalidated by a classical valuation. But classical valuations are preserved in \mathbf{KbiG}^c .

For the converse, let $\sim\sim\phi$ be not \mathbf{KbiG} -valid on some finitely branching frame \mathfrak{F} . Then, there exist $w \in \mathfrak{F}$ and v on \mathfrak{F} s.t. $v(\sim\sim\phi, w) \neq 1$. But then, $v(\phi, w) = 0$. Hence, by Lemma 5.3, we have a classical valuation v^{cl} on \mathfrak{F}^{cl} s.t. $v^{\text{cl}}(\phi, w) = 0$. The result follows.

Consider 2. We reason by contraposition. Assume that \mathbb{F} contains some infinitely branching frame \mathfrak{F} . But then $\mathfrak{F} \not\models_{\mathbf{KbiG}} \sim\sim\Box(p \vee \sim p)$. Thus, $\{\sim\sim\phi : \phi \text{ is } \mathbf{K} \text{ valid on } \mathbb{F}\} \not\subseteq \mathbf{L}$ as required. □

$$w : \longrightarrow w' : p = \frac{1}{2}$$

Figure 5.4.: $T(w) = \{0, 1\}$, $T(w')$ can be arbitrary.

5.4. Decidability and complexity

In this section, we establish that, as expected, the satisfiability and validity³⁹ of \mathbf{KbiG}^c are PSpace-complete. We apply the approach proposed in [37, 38].

The next definition is a straightforward adaptation of [37] to \mathbf{KbiG}^c .

Definition 5.11 (F-models of \mathbf{KbiG}^c). An F-model is a tuple $\mathfrak{M} = \langle W, R, T, v \rangle$ with $\langle W, R, v \rangle$ being a \mathbf{KbiG}^c model and $T : W \rightarrow \mathcal{P}_{<\omega}([0, 1])$ be s.t. $\{0, 1\} \subseteq T(w)$ for all $w \in W$. v is extended to the complex formulas as in \mathbf{KbiG}^c in the cases of propositional connectives, and in the modal cases, as follows.

$$\begin{aligned} v(\Box\phi, w) &= \max\{x \in T(w) : x \leq \inf\{v(\phi, w') : wRw'\}\} \\ v(\Diamond\phi, w) &= \min\{x \in T(w) : x \geq \sup\{v(\phi, w') : wRw'\}\} \end{aligned}$$

Example 5.2 (A finite F-model). Recall that there are no finite \mathbf{KbiG}^c countermodels for $\phi = \Delta\Diamond p \rightarrow \Diamond\Delta p$. It is, however, easy to provide a finite F-model of ϕ (cf. Fig. 5.4). Indeed, it is clear that $v(\phi, w) = 0$. One sees that $v(p, w') = \frac{1}{2}$, whence $\inf\{v(p, w') : wRw'\} = \frac{1}{2}$ as well. But then the minimal $T(w)$ that is at least as great as $\frac{1}{2}$ is 1. Thus, $v(\Delta\Diamond p, w) = 1$. On the other hand, $v(\Delta p, w') = 0$, whence, $v(\Diamond\Delta p, w) = 0$.

The next lemma is a straightforward extension of [37, Theorem 1] to \mathbf{KbiG}^c . The proof is essentially the same since we add only Δ to the language.

Lemma 5.4. ϕ is \mathbf{KbiG}^c valid iff ϕ is true in all F-models iff ϕ is true in all F-models whose depth is $O(|\phi|)$ s.t. $|W| \leq (|\phi| + 2)^{|\phi|}$ and $|T(w)| \leq |\phi| + 2$ for all $w \in W$.

It is now clear that \mathbf{KbiG} are decidable. To establish their complexity, we can utilise the algorithm described in [38]. The algorithm will work for \mathbf{KbiG} since its only difference from \mathbf{KG}^c is Δ which is an extensional connective. Another alternative would be to expand the tableaux calculus for \mathbf{KG}^c from [131] with the rules for Δ and use it to construct the decision procedure. The following statement is now immediate.

Theorem 5.8. *The satisfiability of \mathbf{KbiG}^c is PSpace-complete.*

³⁹Satisfiability and falsifiability (non-validity) are reducible to each other using Δ : ϕ is satisfiable (falsifiable) iff $\sim\Delta\phi$ is falsifiable (satisfiable).

6. Paraconsistent crisp Gödel modal logic

In [the introduction](#), we put forth five desiderata for a logic that can intuitively formalise statements about belief. So far, we have only considered **KbiG** that satisfies only [the first](#) and (in the fuzzy version) [the second](#) ones. In this chapter, we are studying **KG^{2c}** (recall [Fig. 5.1](#)) that additionally satisfies [Desiderata 3, 4, and 5](#).

6.1. Language and semantics

The language $\mathcal{L}_{\mathbf{G}_{\Delta, \square, \diamond}^2}$ is defined via the following grammar.

$$\mathcal{L}_{\mathbf{G}_{\Delta, \square, \diamond}^2} \ni \phi := p \in \mathbf{Prop} \mid \neg\phi \mid \sim\phi \mid \Delta\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid \square\phi \mid \diamond\phi$$

Definition 6.1 (**KG^{2c}**-models). A **KG^{2c}**-model is a tuple $\mathfrak{M} = \langle W, R, v_1, v_2 \rangle$ with $\langle W, R \rangle$ being a *crisp* frame (recall [Definition 5.1](#)), and **KG^{2c}**-valuations $v_1, v_2 : \mathbf{Prop} \times W \rightarrow [0, 1]$. The valuations which we interpret as support of truth and support of falsity, respectively, are extended on complex formulas as expected.

Namely, the propositional connectives are defined state-wise according to [Definition 4.4](#). The modalities are defined as follows.

$$\begin{aligned} v_1(\square\phi, w) &= \inf\{v_1(\phi, w') : wRw'\} & v_2(\square\phi, w) &= \sup\{v_2(\phi, w') : wRw'\} \\ v_1(\diamond\phi, w) &= \sup\{v_1(\phi, w') : wRw'\} & v_2(\diamond\phi, w) &= \inf\{v_2(\phi, w') : wRw'\} \end{aligned}$$

We say that $\phi \in \mathcal{L}_{\mathbf{G}_{\Delta, \square, \diamond}^2}$ is **KG^{2c}** *valid on frame* \mathfrak{F} ($\mathfrak{F} \models_{\mathbf{KG}^{2c}} \phi$) iff for any $w \in \mathfrak{F}$, it holds that $v_1(\phi, w) = 1$ and $v_2(\phi, w) = 0$ for any model \mathfrak{M} on \mathfrak{F} . Γ *entails* χ (on \mathfrak{F}), denoted $\Gamma \models_{\mathbf{KG}^{2c}} \chi$ ($\Gamma \models_{\mathbf{KG}^{2c}}^{\mathfrak{F}} \chi$), iff for every model \mathfrak{M} (on \mathfrak{F}) and every $w \in \mathfrak{M}$, it holds that

$$\inf\{v_1(\phi, w) : \phi \in \Gamma\} \leq v_1(\chi, w) \text{ and } \sup\{v_2(\phi, w) : \phi \in \Gamma\} \geq v_2(\chi, w).$$

We begin with an evident observation that **KG^{2c}** permits NNF's.

Proposition 6.1. *For every $\phi \in \mathcal{L}_{\mathbf{G}_{\Delta, \square, \diamond}^2}$ there exists its negation normal form ϕ^\neg s.t. $v(\phi, w) = v(\phi^\neg, w)$ for every valuation v and every state w .*

Proof. We proceed by induction on ϕ . The propositional cases are straightforward since De Morgan laws for $\sim, \wedge, \vee, \rightarrow, \prec$, and Δ (recall [Definition 4.6](#) and [Remark 4.5](#)) are **G²**-valid and $v(\neg\neg\phi, w) = v(\phi, w)$. The case of modal connectives is also simple since $v(\neg\square\phi, w) = v(\diamond\neg\phi, w)$ and $v(\neg\diamond\phi, w) = v(\square\neg\phi, w)$. \square

Note, furthermore, that \square and \diamond are interdefinable in **KG^{2c}**.

$$\square\phi \leftrightarrow \neg\diamond\neg\phi \qquad \diamond\phi \leftrightarrow \neg\square\neg\phi$$

Moreover, the following statements show that we can reduce **KG^{2c}** validity to **KbiG^c** validity using NNF's.

Proposition 6.2. $\mathfrak{F} \models_{\mathbf{KG}^{2c}} \phi$ iff for any model \mathfrak{M} on \mathfrak{F} and any $w \in \mathfrak{F}$, $v_1(\phi, w) = 1$.

Proof. The ‘only if’ direction is evident from the definition of validity. We show the ‘only if’ part. It suffices to show that the following statement holds for any ϕ and $w \in \mathfrak{F}$:

6. Paraconsistent crisp Gödel modal logic

for any $v(p, w) = (x, y)$, let $v^*(p, w) = (1 - y, 1 - x)$. Then $v(\phi, w) = (x, y)$ iff $v^*(\phi, w) = (1 - y, 1 - x)$.

We proceed by induction on ϕ . The proof of propositional cases is identical to the one in Proposition 4.3. We consider only the case of $\phi = \Box\psi$ since \Box and \Diamond are interdefinable.

Let $v(\Box\psi, w) = (x, y)$. Then $\inf\{v_1(\psi, w') : wRw'\} = x$, and $\sup\{v_2(\psi, w') : wRw'\} = y$. Now, we apply the induction hypothesis to ψ , and thus if $v(\psi, s) = (x', y')$, then $v^*(\psi, s) = (1 - y', 1 - x')$ for any $s \in R(w)$. But then $\inf\{v_1^*(\psi, w') : wRw'\} = 1 - y$, and $\sup\{v_2^*(\psi, w') : wRw'\} = 1 - x$ as required.

Now, assume that $v_1(\phi, w) = 1$ for any v_1 and w . We can show that $v_2(\phi, w) = 0$ for any w and v_2 . Assume for contradiction that $v_2(\phi, w) = y > 0$ but $v_1(\phi, w) = 1$. Then, $v^*(\phi) = (1 - y, 1 - 1) = (1 - y, 0)$. But since $y > 0$, $v^*(\phi) \neq (1, 0)$. \square

Proposition 6.3. *Let $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \Box, \Diamond}$. Then, $\mathfrak{F} \models_{\mathbf{KbiG}} \phi$ iff $\mathfrak{F} \models_{\mathbf{KG}^{2c}} \phi$, for any crisp \mathfrak{F} .*

Proof. The ‘only if’ direction is straightforward since the semantic conditions of v_1 in \mathbf{KG}^{2c} -models and v in \mathbf{KbiG} models coincide for \neg -free formulas. The ‘only if’ direction follows from Proposition 6.2: if ϕ is \mathbf{KbiG} -valid on \mathfrak{F} , then $v(\phi, w) = 1$ for any $w \in \mathfrak{F}$ and any v on \mathfrak{F} . But then, $v_1(\phi, w) = 1$ for any $w \in \mathfrak{F}$. Hence, $\mathfrak{F} \models_{\mathbf{KG}^{2c}} \phi$. \square

Proposition 6.4. *Let \mathfrak{F} be a crisp frame and $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \Box, \Diamond}$. Then $\mathfrak{F} \models_{\mathbf{KG}^{2c}} \phi$ iff $\mathfrak{F} \models_{\mathbf{KbiG}} (\phi^\neg)^*$ (recall Definition 3.8).*

Proof. By Proposition 6.1, we have that $v(\phi, w) = v(\phi^\neg, w)$. By Proposition 6.2, we have that $\mathfrak{F} \models_{\mathbf{KG}^{2c}} \phi^\neg$ iff in every \mathbf{KG}^{2c} -model \mathfrak{M} on \mathfrak{F} and every $w \in \mathfrak{M}$ it holds that $v_1(\phi^\neg, w) = 1$. It remains to construct a \mathbf{KbiG} model \mathfrak{M}^* on the same frame where $v_1(\phi^\neg, u) = v^*((\phi^\neg)^*, u)$ for every ϕ^\neg and u .

For any $u \in W$, define the valuation as follows:

$$v^*(p, u) = v_1(p, u) \qquad v^+(p^*, u) = v_2(p, u)$$

It now suffices to show that $v^*((\phi^\neg)^*, u) = v_1(\phi^\neg, u)$ for any ϕ^\neg and w . We proceed by induction on ϕ . The basis cases of literals are straightforward as well as those of the propositional connectives. Thus, we consider the case of $\phi^\neg = \Box\phi'$.

$$\begin{aligned} v_1(\Box\phi', u) &= \inf\{v_1(\phi', u') : uRu'\} \\ &= \inf\{v^+(\phi'^*, u') : uRu'\} && \text{(by IH)} \\ &= v^+(\Box\phi'^*, u) \end{aligned}$$

The case of $\phi = \Diamond\phi'$ can be considered in the same manner. \square

The above propositions also have the expected corollary.

Corollary 6.1. *Let \mathbb{F} be a class of crisp frames. Then \mathbb{F} is \mathbf{KbiG} -definable iff it is \mathbf{KG}^{2c} -definable. Moreover, satisfiability and validity in \mathbf{KG}^{2c} are PSpace-complete.*

Proof. Immediately from Propositions 6.3 and 6.4. \square

We finish the section by discussing the desiderata from the introduction. First of all, it is clear that since we assume the accessibility relation to be crisp, Desideratum 2 is not satisfied.⁴⁰ Second, neither \Box nor \Diamond ⁴¹ is trivialised by contradictions: in contrast to \mathbf{K} , $\Box(p \wedge \neg p) \rightarrow \Box q$ is not \mathbf{KG}^{2c} valid, and neither is $\Diamond(p \wedge \neg p) \rightarrow \Diamond q$. Intuitively, this means that one can have contradictory but non-trivial beliefs, whence, Desideratum 3 is fulfilled. Third, $\Box(p \vee \neg p)$ and $\Diamond(p \vee \neg p)$ are not valid either which corresponds to the fifth desideratum. The next examples explain how \mathbf{KG}^{2c} fulfills Desiderata 1 and 4.

⁴⁰We will deal with it in Chapter 7.

⁴¹ \Box and \Diamond can be then viewed as two simple aggregation strategies: a pessimistic one (the infimum of positive support and the supremum of the negative support), and an optimistic one (the dual strategy), respectively.

Example 6.1. Let us recall Example 5.1. We need to formalise the following statement.

weather: *Paula considers a rain happening today strictly more likely than a hailstorm.*

Again, we need a formula that is true (i.e., has value $(1, 0)$) iff $v(\Box r, w) > v(\Box s, w)$. This time, however, we cannot just use $\sim\Delta(\Box r \rightarrow \Box s)$ since the valuation has *two* components and the truth of the implication depends on both of them. Moreover, $v(\Delta\phi, w) \in \{(1, 0), (1, 1), (0, 0), (0, 1)\}$, and we would like to have a $(1, 0)$ -detecting connective.

We define

$$\Delta^\top\phi := \Delta\phi \wedge \neg\sim\Delta\phi \quad (6.1)$$

One can see that

$$v(\Delta^\top\phi, w) = \begin{cases} (1, 0) & \text{if } v(\phi, w) = (1, 0) \\ (0, 1) & \text{otherwise} \end{cases}$$

and that $\Delta^\top(p \rightarrow q) \vee \Delta^\top(q \rightarrow p)$ is not \mathbf{G}^2 valid while $\Delta(p \rightarrow q) \vee \Delta(q \rightarrow p)$ is \mathbf{biG} valid. Now, to formalise **weather** in a \mathbf{KG}^{2c} setting, we use the following formula.

$$\psi := \Delta^\top(\Box s \rightarrow \Box r) \wedge \sim\Delta^\top(\Box r \rightarrow \Box s)$$

One can check, that, indeed

$$v(\psi, w) = \begin{cases} (1, 0) & \text{iff } v(\Box s, w) <_{[0,1]^\times} v(\Box r, w) \\ (0, 1) & \text{otherwise} \end{cases}$$

As we have seen, Δ^\top can formalise statements where one event is considered to be *more or less likely* than the other. In \mathbf{KG}^{2c} , we can also write down formulas corresponding to situations where the agents cannot or refuse to compare their certainty in different events.

Example 6.2.

weather and dog: Paula thinks that the likelihood of a hailstorm happening today and that of her cousin's spouse having a dachshund is incomparable.

Again, we use \Box to denote 'Paula finds it likely' or 'Paula believes that'. Since the likelihoods are incomparable, it means that $v(\Box s \rightarrow \Box d, w) \neq (1, 0)$ and $v(\Box d \rightarrow \Box s, w) \neq (1, 0)$. Thus, we can formalise this sentence as follows.

$$\sim\Delta^\top(\Box s \rightarrow \Box d) \wedge \sim\Delta^\top(\Box d \rightarrow \Box s)$$

Remark 6.1. Note that the Gödel logic is *paracomplete* (i.e., $p \vee \sim p$ is not \mathbf{G} -valid). Thus, it should be possible to represent statements like 'the agent has no information about p ' already in \mathbf{KG} and its extensions. Indeed, it is possible [129] to interpret \Box and \Diamond in **45** and **D45** extensions of \mathbf{KG} as a necessity and possibility measure on a set of states, respectively. In this approach, $\sim\Box p \wedge \Diamond p$ can be read as 'nothing is known about p '⁴² (since the necessity measure is 0 and the possibility is 1). Moreover, $\Diamond\mathbf{1}$ is not valid (in **KG45**), hence $\Box\mathbf{0}$ is satisfiable which might be seen as a representation of an inconsistent belief.

This approach, however, has some problems. Although it is the case that the interpretation of \Box and \Diamond via necessity and possibility measures is natural, we maintain that 'nothing is known about p ' is not the same as ' x has no information about p at all' which is the meaning behind $v(\underbrace{\Box p \wedge \Diamond p \wedge \Delta^\top(\Box p \leftrightarrow \Diamond p)}_{\text{noInfo}}, x) = (0, 0)$. Indeed, for $\sim\Box p \wedge \Diamond p$ to be true, there must be states

⁴²The author is grateful to Lluís Godo Lacasa for mentioning this.

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where the value of p is positive. I.e., there is some information about it, although, it is not conclusive and does not allow to form a belief. On the other hand, for $v(\text{noInfo}, x) = (0, 0)$ to hold, there has to be no state $y \in R(x)$ where $v_1(p, y) > 0$ or $v_2(p, y) > 0$.

Moreover, even though $\Box \mathbf{0}$ can be satisfiable, the contradictions in \mathbf{G} and \mathbf{G}^2 are treated differently. Namely, in the framework of \mathbf{KG} (just as in the case of \mathbf{K} mentioned above), if one believes in one contradiction, one also believes in every other contradiction as well: $\Box(p \wedge \sim p) \rightarrow \Box(q \wedge \sim q)$ ⁴³ is \mathbf{KG} -valid. On the other hand, \neg allows to treat contradictory beliefs in a non-trivial manner since $\Box(p \wedge \neg p) \rightarrow \Box q$ is not \mathbf{KG}^{2c} -valid.

6.2. Axiomatisation

In the previous section, we reduced the \mathbf{KG}^{2c} -validity to \mathbf{KbiG} -validity (Proposition 6.4). We will use this to construct a strongly complete calculus for \mathbf{KG}^{2c} .

Definition 6.2 (\mathcal{HKG}^{2c} — Hilbert-style calculus for \mathbf{KG}^{2c}). \mathcal{HKG}^{2c} contains the following axioms and rules.

A0: All instances of \mathcal{HKbiG}^c rules and axioms in $\mathcal{L}_{\mathbf{G}^2, \Box, \Diamond}$ language from Definition 5.4.

neg: $\neg\neg\phi \leftrightarrow \phi$

DeM \wedge : $\neg(\phi \wedge \chi) \leftrightarrow (\neg\phi \vee \neg\chi)$

DeM \vee : $\neg(\phi \vee \chi) \leftrightarrow (\neg\phi \wedge \neg\chi)$

DeM \rightarrow : $\neg(\phi \rightarrow \chi) \leftrightarrow (\neg\chi \wedge \sim\Delta(\neg\chi \rightarrow \neg\phi))$

DeM Δ : $\neg\Delta\phi \leftrightarrow \sim\sim\neg\phi$

DeM \sim : $\neg\sim\phi \leftrightarrow \sim\Delta\neg\phi$

DeM $\Box\Diamond$: $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

The following completeness theorem is a straightforward corollary of Theorem 5.1 and Proposition 6.4 since every $\phi \in \mathcal{L}_{\mathbf{G}^2, \Box, \Diamond}$ can be transformed into its NNF using the axioms of \mathcal{HKG}^{2c} .

Theorem 6.1. \mathcal{HKG}^{2c} is strongly complete: for any $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathbf{G}^2, \Box, \Diamond}$, it holds that $\Gamma \models_{\mathbf{KG}^{2c}} \phi$ iff $\Gamma \vdash_{\mathcal{HKG}^{2c}} \phi$.

It is easy to check that \neg contraposition $\frac{\phi \rightarrow \chi}{\neg\chi \rightarrow \neg\phi}$ is admissible in $\mathcal{HG}_{(\rightarrow, \neg)}^2$. Moreover, by the completeness of \mathcal{HKG}^{2c} , it is also admissible there. The next proposition shows that if contraposition is allowed as a rule applied to theorems, some \mathcal{HKbiG} axioms become redundant.

Proposition 6.5. The following \mathcal{HKbiG}^c axioms are redundant in \mathcal{HKG}^{2c} with contraposition:

- $\sim\Diamond\mathbf{0}$;
- $\Diamond(\phi \vee \chi) \rightarrow (\Diamond\phi \vee \Diamond\chi)$;
- $\sim\Delta(\Diamond\phi \rightarrow \Diamond\chi) \rightarrow \Diamond\sim\Delta(\phi \rightarrow \chi)$;
- $\Delta\Box\phi \rightarrow \Box\Delta\phi$.

⁴³Or, in fact, in *anything*: $\Box(p \wedge \sim p) \rightarrow \Box q$ is \mathbf{KG} -valid as well.

Proof.

$$\boxed{\sim\Diamond\mathbf{0}}$$

Since \sim can be defined via \rightarrow , we need to prove $\Diamond\mathbf{0} \rightarrow \mathbf{0}$. By contraposition, we can prove $\neg\mathbf{0} \rightarrow \neg\Diamond\mathbf{0}$. By De Morgan laws, we transform this into $\neg\mathbf{0} \rightarrow \Box\neg\mathbf{0}$. Now recall that $\mathcal{H}\mathbf{G}_{(\rightarrow, \neg)}^2 \vdash \neg\mathbf{0}$, whence $\mathcal{H}\mathbf{K}\mathbf{G}^{2c} \vdash \neg\mathbf{0} \rightarrow \Box\neg\mathbf{0}$.

$$\boxed{\Diamond(\phi \vee \chi) \rightarrow (\Diamond\phi \vee \Diamond\chi)}$$

Note again that we can prove $\neg(\Diamond\phi \vee \Diamond\chi) \rightarrow \neg\Diamond(\phi \vee \chi)$ instead. But this is equivalent to $\Box(\neg\phi \wedge \neg\chi) \rightarrow (\Box\neg\phi \wedge \Box\neg\chi)$ which is provable in $\mathbf{K}\mathbf{G}$.

$$\boxed{\sim\Delta(\Diamond\phi \rightarrow \Diamond\chi) \rightarrow \Diamond\sim\Delta(\phi \rightarrow \chi)}$$

Observe that

$$\Box(\sim\neg\chi \vee \Delta(\neg\chi \rightarrow \neg\phi)) \rightarrow (\Diamond\sim\neg\chi \vee \Delta(\Box\neg\chi \rightarrow \Box\neg\phi))$$

can be proven via an application of \vee -commutativity to \mathbf{Cr} , the Barcan's formula, and \mathbf{K} .

From here, since, $\mathcal{H}\mathbf{K}\mathbf{G}^c \vdash \Diamond\sim\neg\chi \rightarrow \sim\Box\neg\chi$, we obtain

$$\Box(\sim\neg\chi \vee \Delta(\neg\chi \rightarrow \neg\phi)) \rightarrow (\sim\Box\neg\chi \vee \Delta(\Box\neg\chi \rightarrow \Box\neg\phi))$$

Now, applying $\sim\sim\Delta\psi \leftrightarrow \Delta\psi$, we obtain

$$\Box(\sim\neg\chi \vee \sim\sim\Delta(\neg\chi \rightarrow \neg\phi)) \rightarrow (\sim\Box\neg\chi \vee \sim\sim\Delta(\Box\neg\chi \rightarrow \Box\neg\phi))$$

We use the De Morgan law for \sim — $\sim(\psi \wedge \psi') \leftrightarrow (\sim\psi \vee \sim\psi')$ to get

$$\Box\sim(\neg\chi \wedge \sim\Delta(\neg\chi \rightarrow \neg\phi)) \rightarrow \sim(\Box\neg\chi \wedge \sim\Delta(\Box\neg\chi \rightarrow \Box\neg\phi))$$

At this point, we apply the De Morgan laws for \rightarrow and Δ and \Diamond and \Box which give us

$$\Box\sim\neg(\phi \rightarrow \chi) \rightarrow \sim\neg(\Diamond\phi \rightarrow \Diamond\chi)$$

Observe that $\mathcal{H}\mathbf{G}\Delta \vdash \sim\psi \leftrightarrow \sim\sim\sim\psi$ and $\mathcal{H}\mathbf{G}\Delta \vdash \Delta\sim\psi \leftrightarrow \sim\psi$ since $\sim\psi \leftrightarrow \sim\sim\sim\psi$ and $\Delta\sim\psi \leftrightarrow \sim\psi$ are $\mathbf{G}\Delta$ -valid and $\mathcal{H}\mathbf{G}\Delta$ is complete (Remark 4.2). Thus, we have

$$\Box\sim\Delta\sim\sim\neg(\phi \rightarrow \chi) \rightarrow \sim\Delta\sim\sim\neg(\Diamond\phi \rightarrow \Diamond\chi)$$

The application of $\text{DeM}\Delta$ gives us

$$\Box\sim\Delta\neg\Delta(\phi \rightarrow \chi) \rightarrow \sim\Delta\neg\Delta(\Diamond\phi \rightarrow \Diamond\chi)$$

We can now apply $\text{DeM}\sim$ to obtain

$$\Box\neg\sim\Delta(\phi \rightarrow \chi)\neg \rightarrow \neg\sim\Delta(\Diamond\phi \rightarrow \Diamond\chi)$$

Finally, we use the \neg contraposition neg , and $\text{DeM}\Box\Diamond$ to get

$$\sim\Delta(\Diamond\phi \rightarrow \Diamond\chi) \rightarrow \Diamond\sim\Delta(\phi \rightarrow \chi)$$

$$\boxed{\Delta\Box\phi \rightarrow \Box\Delta\phi}$$

At last, we can see that $\Delta\Box\phi \rightarrow \Box\Delta\phi$ can be transformed via contraposition and De Morgan laws into $\Diamond\sim\sim\neg\phi \rightarrow \sim\sim\Diamond\neg\phi$ which is provable in $\mathcal{H}\mathbf{K}\mathbf{G}^c$. \square

6.3. $\mathbf{KG}_{\text{fb}}^{2c}$ — \mathbf{KG}^{2c} over finitely branching frames

Up to this point, we have mostly considered logics of all crisp frames. One should remember, however, that \mathbf{KG} and \mathbf{KG}^c (and hence, \mathbf{KbiG} and \mathbf{KG}^{2c}) lack the finite model property [39]. In fact, in contrast to \mathbf{K} , finitely branching frames are definable (Proposition 5.13). Moreover, classical epistemic and doxastic logics are usually complete w.r.t. finite frames (cf. [87, 62] for details). It is reasonable since for practical reasoning, agents cannot consider infinitely many alternatives. In our case, however, if we wish to use \mathbf{KbiG} and \mathbf{KG}^{2c} for knowledge representation, we need to *impose* finite branching explicitly.

Furthermore, allowing for infinitely branching frames in \mathbf{KbiG} or \mathbf{KG}^{2c} leads to counter-intuitive consequences. In particular, it is possible that $v(\Box\phi, w) = (0, 1)$ even though there are no $w', w'' \in R(w)$ (with $R(w) \neq \emptyset$) s.t. $v_1(\phi, w') = 0$ or $v_2(\phi, w'') = 1$. In other words, no source decisively falsifies ϕ , furthermore, all sources have some evidence *for* ϕ , and yet we somehow believe that ϕ is completely false and untrue. Dually, it is possible that $v(\Diamond\phi, w) = (1, 0)$ although there are no $w', w'' \in R(w)$ s.t. $v_1(\phi, w') = 1$ or $v_2(\phi, w'') = 0$. Even though \Diamond is an ‘optimistic’ aggregation, it should not ignore the fact that *all* sources have some evidence *against* ϕ but *none* supports it completely.

Of course, this situation is impossible if we consider only finitely branching frames for infima and suprema will become minima and maxima. There, all values of modal formulas will be *witnessed* by some accessible states in the following sense. For $\heartsuit \in \{\Box, \Diamond\}$, $i \in \{1, 2\}$, if $v_i(\heartsuit\phi, w) = x$, then there is $w' \in R(w)$ s.t. $v_i(\phi, w') = x$. Intuitively speaking, finitely branching frames represent the situation when our degree of certainty in some statement is based uniquely on the data given by the sources.

Convention 6.1. We will further use $\mathbf{KbiG}_{\text{fb}}^c$ and $\mathbf{KG}_{\text{fb}}^{2c}$ to denote the sets of all $\mathcal{L}_{G\Delta, \Box, \Diamond}$ and $\mathcal{L}_{G^2, \Box, \Diamond}$ formulas valid on finitely branching crisp frames.

In this section, we are studying $\mathbf{KG}_{\text{fb}}^{2c}$. In particular, we show that a special counterpart of Glivenko’s theorem⁴⁴ holds. Namely, instead of $\sim\sim$, we can add $\neg\sim$ on top of classical formulas. Furthermore, we construct a simple constraint tableaux calculus for $\mathbf{KG}_{\text{fb}}^{2c}$ (and hence, for $\mathbf{KbiG}_{\text{fb}}^c$) and use it to establish PSpace-completeness.

Theorem 6.2.

1. Let $\phi \in \mathcal{L}_{G\Delta, \Box, \Diamond}$ be Δ -free. Then it is \mathbf{K} -valid iff $\neg\sim\phi$ is \mathbf{KG}^{2c} -valid on all finitely branching crisp frames.
2. Let \mathbb{F} be a class of crisp frames, and let \mathbf{L} be the \mathbf{KG}^{2c} logic of \mathbb{F} . Then, $\{\neg\sim\phi : \mathbb{F} \models_{\mathbf{K}} \phi\} \subseteq \mathbf{L}$ implies that every $\mathfrak{F} \in \mathbb{F}$ is finitely branching.

Proof. Consider 1. It is clear that no classically valid ϕ can have $v_1(\phi, w) = 0$, nor $v_2(\phi, w) = 1$. Otherwise, by Lemma 5.3 and Proposition 6.2, there is a classical valuation v^{cl} s.t. $v^{\text{cl}}(\phi, w) = (0, 1)$. Thus, $v(\neg\sim\phi, w) = (1, 0)$, as required. The converse direction holds since the classical valuations are preserved.

For 2, assume that \mathbb{F} contains an infinitely branching frame \mathfrak{F} . Let now $R(w)$ be infinite for some $w \in \mathfrak{F}$ and $\{w_i : i \geq 1, i \in \mathbb{N}\} \subseteq R(w)$. We set $v(p, w_i) = \left(\frac{1}{i+1}, 1 - \frac{1}{i}\right)$. It is easy to see that $v(\Box(p \vee \sim p), w) = (0, 1)$, whence $v(\neg\sim\Box(p \vee \sim p), w) = (0, 1)$, and thus, $\{\neg\sim\phi : \mathbb{F} \models_{\mathbf{K}} \phi\} \not\subseteq \mathbf{L}$. \square

Before constructing the tableaux system for $\mathbf{KG}_{\text{fb}}^{2c}$ and $\mathbf{KbiG}_{\text{fb}}^c$, let us first remark that the finite model property is not entirely for granted. In the classical and super-intuitionistic logics, it is often established via the filtration technique. It is interesting to note that several expected ways of defining filtration (cf. [42, 28] for more details thereon) fail.

⁴⁴The Glivenko’s theorem itself holds for $\mathbf{KG}_{\text{fb}}^{2c}$ by Proposition 6.3.

$$\mathfrak{M} : w_1 \longrightarrow w_2 \longrightarrow \dots \longrightarrow w_n \longrightarrow \dots$$

 Figure 6.1.: $v(p, w_n) = \frac{1}{n+1}$

Let $\Sigma \subseteq \mathcal{L}_{\mathbf{G}\Delta, \square, \diamond}$ be closed under subformulas. If we want to have filtration for $\mathbf{KbiG}_{\text{fb}}^c$, there are three intuitive ways to define \sim_Σ on the carrier of a model that is supposed to relate states satisfying the same formulas.

1. $w \sim_\Sigma^1 w'$ iff $v(\phi, w) = v(\phi, w')$ for all $\phi \in \Sigma$.
2. $w \sim_\Sigma^2 w'$ iff $v(\phi, w) = 1 \Leftrightarrow v(\phi, w') = 1$ for all $\phi \in \Sigma$.
3. $w \sim_\Sigma^3 w'$ iff $v(\phi, w) \leq v(\phi', w) \Leftrightarrow v(\phi, w') \leq v(\phi', w')$ for all $\phi, \phi' \in \Sigma \cup \{\mathbf{0}, \mathbf{1}\}$.

Consider the model on Fig. 6.1 and two formulas:

$$\phi^\leq := \sim\sim(p \rightarrow \diamond p) \qquad \phi^\gt := \sim\Delta(p \rightarrow \diamond p)$$

Now let Σ be the set of all subformulas of $\phi^\leq \wedge \phi^\gt$.

First of all, it is clear that $v(\phi^\leq \wedge \phi^\gt, w) = 1$ for any $w \in \mathfrak{M}$. Observe now that all states in \mathfrak{M} are *distinct* w.r.t. \sim_Σ^1 . Thus, the first way of constructing the carrier of the new model does not give the FMP.

As regards \sim_Σ^2 and \sim_Σ^3 , one can check that for any $w, w' \in \mathfrak{M}$, it holds that $w \sim_\Sigma^2 w'$ and $w \sim_\Sigma^3 w'$. So, if we construct a filtration of \mathfrak{M} using equivalence classes of either of these two relations, the carrier of the resulting model is going to be finite. Even more so, it is going to be a singleton.

However, we can show that there is *no finite model* $\mathfrak{N} = \langle U, S, e \rangle$ s.t.

$$\forall s \in \mathfrak{N} : e(\phi^\leq \wedge \phi^\gt, s) = 1.$$

Indeed, $e(\phi^\leq, t) = 1$ iff $e(p, t') > 0$ for some $t' \in S(t)$, while $e(\phi^\gt, t) = 1$ iff $e(p, t) > e(p, t')$ for any $t' \in S(t)$. Now, if U is finite, we have two options: either (1) there is $u \in U$ s.t. $R(u) = \emptyset$, or (2) U contains a finite S -cycle.

For (1), note that $v(\diamond p, u) = 0$, and we have two options: if $e(p, u) = 0$, then $e(\phi^\gt, u) = 0$; if, on the other hand, $e(p, u) > 0$, then $e(\phi^\leq, u) = 0$. For (2), assume w.l.o.g. that the S -cycle looks as follows: $u_0 S u_1 S u_2 \dots S u_n S u_0$.

If $e(p, u_0) = 0$, $e(\phi^\gt, u_0) = 0$, so $e(p, u_0) > 0$. Furthermore, $e(p, u_i) > e(p, u_{i+1})$. Otherwise, again, $e(\phi^\gt, u_i) = 0$. But then we have $e(\phi^\gt, u_i) = 0$.

But this means that \sim_Σ^2 and \sim_Σ^3 do not preserve the truth of formulas from w to $[w]_\Sigma$, i.e., neither of these two relations can be used to define filtration. Thus, to explicitly prove the finite model property and establish complexity evaluations for $\mathbf{KbiG}_{\text{fb}}^c$ and $\mathbf{KG}_{\text{fb}}^{2c}$, we will provide a tableaux calculus. It will also serve as a decision procedure for the satisfiability and validity of formulas.

Our tableaux system $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2c})$ is a straightforward modal expansion of constraint tableaux for $\mathcal{T}(\mathbf{G}^2)$ from Section 4.2. In a sense, one can see it as a hybrid between the following systems: tableaux for BD from [1] because we use two types of labels, one for each valuation; relational calculi for mono-modal fragments of \mathbf{KG} [104, 105]; labelled sequent calculi for (classical) modal logics presented in, e.g. [109] because we use two types of entries: constraints of the form $w : i : \phi \leq w' : j : \phi'$ ($i, j \in \{1, 2\}$) and relational terms $w R w'$.

Definition 6.3 ($\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2c})$). We fix a set of state-labels \mathbb{W} and let $\lesssim \in \{<, \leq\}$ and $\gtrsim \in \{>, \geq\}$. Let further $w \in \mathbb{W}$, $\mathbf{x} \in \{1, 2\}$, $\phi \in \mathcal{L}_{\mathbf{G}\Delta, \square, \diamond}$, and $c \in \{0, 1\}$. A structure is either $w : \mathbf{x} : \phi$ or c . We denote the set of structures with Str .

We define a *constraint tableau* as a downward branching tree whose branches are sets containing the following types of entries:

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$$\begin{array}{c}
\frac{w:1:\Box\phi \gtrsim \mathfrak{X}}{\Box_1 \gtrsim \frac{wRw'}{w':1:\phi \gtrsim \mathfrak{X}}} \quad \Box_1 \leq \frac{w:1:\Box\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \frac{wRw''}{w'':1:\phi \leq \mathfrak{X}}} \quad \Box_1 < \frac{w:1:\Box\phi < \mathfrak{X}}{wRw''} \\
\frac{w:1:\Diamond\phi \gtrsim \mathfrak{X}}{\Diamond_1 \gtrsim \frac{wRw''}{w'':1:\phi \gtrsim \mathfrak{X}}} \quad \frac{w:1:\Diamond\phi \lesssim \mathfrak{X}}{\Diamond_1 \lesssim \frac{wRw'}{w':1:\phi \lesssim \mathfrak{X}}} \quad \Box_2 \gtrsim \frac{w:2:\Box\phi \gtrsim \mathfrak{X}}{w'':2:\phi \gtrsim \mathfrak{X}} \quad \Box_2 \lesssim \frac{w:2:\Box\phi \lesssim \mathfrak{X}}{wRw'} \\
\frac{w:2:\Diamond\phi \gtrsim \mathfrak{X}}{\Diamond_2 \gtrsim \frac{wRw'}{w':2:\phi \gtrsim \mathfrak{X}}} \quad \Diamond_2 \leq \frac{w:2:\Diamond\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \frac{wRw''}{w'':2:\phi \leq \mathfrak{X}}} \quad \Diamond_2 < \frac{w:2:\Diamond\phi < \mathfrak{X}}{wRw''}
\end{array}$$

Figure 6.2.: Modal rules of $\mathcal{T}(\mathbf{KG}_{fb}^{2c})$. w'' is fresh on the branch.

entry	interpretation
$\frac{w:1:\phi \leq w':2:\phi'}{w:2:\phi \leq c}$	$\begin{array}{l} v_1(\phi, w) \leq v_2(\phi', w') \\ v_2(\phi, w) \leq c \text{ with } c \in \{0, 1\} \end{array}$

Table 6.1.: Interpretations of $\mathcal{T}(\mathbf{KG}_{fb}^{2c})$ entries.

- *relational constraints* of the form wRw' with $w, w' \in W$;
- *structural constraints* of the form $\mathfrak{X} \lesssim \mathfrak{X}'$ with $\mathfrak{X}, \mathfrak{X}' \in \text{Str}$.

Each branch can be extended by an application of a rule⁴⁵ from Fig. 4.1 or Fig. 6.2.

A tableau's branch \mathcal{B} is *closed* iff one of the following conditions applies:

- the transitive closure of \mathcal{B} under \lesssim contains $\mathfrak{X} < \mathfrak{X}$;
- $0 \geq 1 \in \mathcal{B}$, or $\mathfrak{X} > 1 \in \mathcal{B}$, or $\mathfrak{X} < 0 \in \mathcal{B}$.

A tableau is *closed* iff all its branches are closed. We say that there is a *tableau proof* of ϕ iff there is a closed tableau starting from the constraint $w:1:\phi < 1$.

An open branch \mathcal{B} is *complete* iff the following condition is met.

- * *If all premises of a rule occur on \mathcal{B} , then one of its conclusions⁴⁶ occurs on \mathcal{B} .*

Remark 6.2. Note that due to Proposition 6.2, we need to check only one valuation of ϕ to verify its validity.

Convention 6.2 (Interpretation of constraints). Table 6.1 gives the interpretations of structural constraints on the example of \leq .

As one can see from Fig. 4.1 and Fig. 6.2, the rules follow the semantical conditions from Definition 6.1. For example, to apply $\Box_1 <$ to $w:1:\Box\phi < \mathfrak{X}$, we introduce a new state w'' that is seen by w . Since we work in a finite branching model, w'' can witness the value of $\Box\phi$. Thus, we add $w'':1:\phi < \mathfrak{X}$.

We also provide an example of how our tableaux work. On Fig. 6.3, one can see a successful proof on the left and a failed proof on the right.

⁴⁵If $\mathfrak{X} < 1$ and $\mathfrak{X} < \mathfrak{X}'$ (or $0 < \mathfrak{X}'$ and $\mathfrak{X} < \mathfrak{X}'$) occur on \mathcal{B} , then the rules are applied only to $\mathfrak{X} < \mathfrak{X}'$.

⁴⁶Note that branching rules have *two* conclusions.

$$\begin{array}{ll}
 w_0:1:\Delta\Box p \rightarrow \Box\Delta p < 1 & w_0:1:\Box p \rightarrow \Box\Box p < 1 \\
 w_0:1:\Delta\Box p > w_0:1:\Box\Delta p & w_0:1:\Box\Box p < 1 \\
 w_0:1:\Box\Delta p < 1 & w_0:1:\Box p > w_0:1:\Box\Box p \\
 w_0:1:\Box p \geq 1 & w_0Rw_1 \\
 w_0Rw_1 & w_0:1:\Box p > w_1:1:\Box p \\
 w_1:1:\Delta p < 1 & w_1:1:p > w_1:1:\Box p \\
 w_1:1:p < 1 & w_1Rw_2 \\
 w_1:1:p \geq 1 & w_1:1:p > w_2:1:p \\
 \times & \odot
 \end{array}$$

Figure 6.3.: \times indicates closed branches; \odot indicates complete open branches.

Definition 6.4 (Branch realisation). We say that a model $\mathfrak{M} = \langle W, R, v_1, v_2 \rangle$ with $W = \{w : w \text{ occurs on } \mathcal{B}\}$ and $R = \{\langle w, w' \rangle : wRw' \in \mathcal{B}\}$ realises a branch \mathcal{B} of a tree iff the following conditions are met.

- $v_{\mathbf{x}}(\phi, w) \leq v_{\mathbf{x}'}(\phi', w')$ for any $w : \mathbf{x} : \phi \leq w' : \mathbf{x}' : \phi' \in \mathcal{B}$ with $\mathbf{x}, \mathbf{x}' \in \{1, 2\}$.
- $v_{\mathbf{x}}(\phi, w) \leq c$ for any $w : \mathbf{x} : \phi \leq c \in \mathcal{B}$ with $c \in \{0, 1\}$.

Theorem 6.3 (Completeness). ϕ is $\mathbf{KG}_{\text{fb}}^{2c}$ valid iff it has a $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2c})$ proof.

Proof. For soundness, we check that if the premise of the rule is realised, then so is at least one of its conclusions. We consider the case $\Box_1 \leq$. Assume that $w : 1 : \Box\phi \leq \mathfrak{X}$ is realised and assume w.l.o.g. that $\mathfrak{X} = u : 2 : \psi$. Thus, $v_1(\Box\phi, w) \leq v_2(\psi, u)$. Then, since the model is finitely branching, there is an accessible state w'' s.t. $v_1(\phi, w'') \leq v_2(\psi, u)$. Thus, $w'' : 1 : \phi \leq \mathfrak{X}$ is realised too.

As no closed branch is realisable, the result follows.

For completeness, we show that every complete open branch \mathcal{B} is realisable. We construct the model as follows. We let $W = \{w : w \text{ occurs in } \mathcal{B}\}$, and set $R = \{\langle w, w' \rangle : wRw' \in \mathcal{B}\}$. Now, it remains to construct suitable valuations.

For $i \in \{1, 2\}$, if $w : i : p \geq 1 \in \mathcal{B}$, we set $v_i(p, w) = 1$. If $w : i : p \leq 0 \in \mathcal{B}$, we set $v_i(p, w) = 0$. To set the values of the remaining variables q_1, \dots, q_n , we proceed as follows. Denote \mathcal{B}^+ the transitive closure of \mathcal{B} under \lesssim and let

$$[w : \mathbf{x} : q_i] = \left\{ w' : \mathbf{x}' : q_j \left| \begin{array}{l} w : \mathbf{x} : q_i \leq w' : \mathbf{x}' : q_j \in \mathcal{B}^+ \text{ and } w : \mathbf{x} : q_i < w' : \mathbf{x}' : q_j \notin \mathcal{B}^+ \\ \text{or} \\ w : \mathbf{x} : q_i \geq w' : \mathbf{x}' : q_j \in \mathcal{B}^+ \text{ and } w : \mathbf{x} : q_i > w' : \mathbf{x}' : q_j \notin \mathcal{B}^+ \end{array} \right. \right\}$$

It is clear that there are at most $2 \cdot n \cdot |W|$ $[w : \mathbf{x} : q_i]$'s since the only possible loop in \mathcal{B}^+ is $w_{i_1} : \mathbf{x} : r \leq \dots \leq w_{i_1} : \mathbf{x} : r$, but in such a loop all elements belong to $[w_{i_1} : \mathbf{x} : r]$. We put $[w : \mathbf{x} : q_i] \prec [w' : \mathbf{x}' : q_j]$ iff there are $w_k : \mathbf{x} : r \in [w : \mathbf{x} : q_i]$ and $w'_k : \mathbf{x}' : r' \in [w' : \mathbf{x}' : q_j]$ s.t. $w_k : \mathbf{x} : r < w'_k : \mathbf{x}' : r' \in \mathcal{B}^+$.

We now set the valuation of these variables as follows

$$v_{\mathbf{x}}(q_i, w) = \frac{|\{[w' : \mathbf{x}' : q'] \mid [w' : \mathbf{x}' : q'] \prec [w : \mathbf{x} : q_i]\}|}{2 \cdot n \cdot |W|}$$

Note that if $s \in \text{Prop}(\phi)$ for some ϕ but \mathcal{B}^+ contains no inequality with it, the above definition ensures that s is going to be evaluated at 0. Thus, all constraints containing only variables are satisfied.

It remains to show that all other constraints are satisfied. For that, we prove that if at least one conclusion of the rule is satisfied, then so is the premise. The propositional cases

6. Paraconsistent crisp Gödel modal logic

are straightforward and can be tackled in the same manner as in Theorem 4.4. We consider only the case of $\diamond_2 \succcurlyeq$. Assume w.l.o.g. that $\succcurlyeq = \geq$ and $\mathfrak{X} = u : 1 : \psi$. Since \mathcal{B} is complete, if $w : 2 : \diamond \phi \geq u : 1 : \psi \in \mathcal{B}$, then for any w' s.t. $wRw' \in \mathcal{B}$, we have $w' : 2 : \phi \geq u : 1 : \psi \in \mathcal{B}$, and all of them are realised by \mathfrak{M} . But then $w : 2 : \diamond \phi \geq u : 1 : \psi$ is realised too, as required. \square

We can now use tableaux to obtain decidability results for $\mathbf{KbiG}_{\text{fb}}^{\text{c}}$ and $\mathbf{KG}_{\text{fb}}^{2\text{c}}$.

Theorem 6.4.

1. Let $\phi \in \mathcal{L}_{\mathcal{G}_{\Delta, \square, \diamond}^2}$ be not $\mathbf{KG}_{\text{fb}}^{2\text{c}}$ valid, and let $|\phi|$ denote the number of symbols in it. Then there is a model \mathfrak{M} of the size $O(|\phi|^{|\phi|})$ and depth $O(|\phi|)$ and $w \in \mathfrak{M}$ s.t. $v_1(\phi, w) \neq 1$.
2. $\mathbf{KG}_{\text{fb}}^{2\text{c}}$ validity and satisfiability are PSpace-complete.

Proof. We begin with 1. By Theorem 6.3, if ϕ is not $\mathbf{KG}_{\text{fb}}^{2\text{c}}$ valid, we can build a falsifying model using tableaux. It is also clear from the rules on Fig. 6.2 that the depth of the constructed model is bounded from above by the maximal number of nested modalities in ϕ . The width of the model is bounded by the maximal number of modalities on the same level of nesting. The sharpness of the bound is obtained using the embedding of \mathbf{K} into $\mathbf{KG}_{\text{fb}}^{2\text{c}}$ since \mathbf{K} is complete w.r.t. finitely branching models and it is possible to force trees of exponential size in \mathbf{K} (cf., e.g. [28, §6.7]) whose depth is bounded from above by the number of modalities in the formula. The embedding also entails PSpace-hardness. It remains to tackle membership.

First, observe from the proof of Theorem 6.3 that $\phi(p_1, \dots, p_n)$ is satisfiable (falsifiable) on $\mathfrak{M} = \langle W, R, v_1, v_2 \rangle$ iff there are v_1 and v_2 that give variables values from

$$\mathbb{V} = \left\{ 0, \frac{1}{2 \cdot n \cdot |W|}, \dots, \frac{2 \cdot n \cdot |W| - 1}{2 \cdot n \cdot |W|}, 1 \right\} \quad (6.2)$$

under which ϕ is satisfied (falsified).

As we mentioned, $|W|$ is bounded from above by k^{k+1} with k being the number of modalities in ϕ . Therefore, we replace structural constraints with labelled formulas of the form $w : i : \phi = v$ ($v \in \mathbb{V}$) avoiding comparisons of values of formulas in different states. As expected, we close the branch if it contains $w : i : \psi = v$ and $w : i : \psi = v'$ for $v \neq v'$.

Now we replace the rules with new ones that work with labelled formulas instead of structural constraints. Below, we give as an example new rules for \rightarrow ⁴⁷ and \diamond ⁴⁸ (with $|\mathbb{V}| = m + 1$):

$$\frac{w : 1 : \phi \rightarrow \phi' = 1}{w : 1 : \phi = 0 \left| \begin{array}{c} w : 1 : \phi = \frac{1}{m+1} \\ w : 1 : \phi' = \frac{1}{m+1} \end{array} \right| \left| \begin{array}{c} w : 1 : \phi = \frac{1}{m+1} \\ w : 1 : \phi' = \frac{2}{m+1} \end{array} \right| \dots \left| \begin{array}{c} w : 1 : \phi = \frac{m-1}{m+1} \\ w : 1 : \phi' = \frac{m}{m+1} \end{array} \right| w : 1 : \phi' = 1}$$

$$\frac{w : 1 : \phi \rightarrow \phi' = \frac{r}{m+1}}{w : 1 : \phi = \frac{r+1}{m+1} \left| \dots \left| \begin{array}{c} w : 1 : \phi = \frac{1}{m+1} \\ w : 1 : \phi' = \frac{r}{m+1} \end{array} \right.} \quad \frac{w : 1 : \diamond \phi = \frac{r}{m+1}}{wRw''} \quad \frac{w : 1 : \diamond \phi = \frac{r}{m+1}; wRw'}{w' : 1 : \phi = 0 \mid \dots \mid w' : 1 : \phi = \frac{r}{m+1}}$$

$$w'' : 1 : \phi = \frac{r}{m+1}$$

We now show how to build a satisfying model for ϕ using polynomial space. We begin with $w_0 : 1 : \phi = 1$ and start applying propositional rules (first, those that do not require branching). If we implement a branching rule, we pick one branch and work only with it: either until the branch is closed, in which case we pick another one; until no more rules are applicable (then, the

⁴⁷For the implication, we will need two rules. First, for the case $v(\phi \rightarrow \phi', w) = 1$. Here, we split the branch as follows: we check for $v(\phi, w) = 0$, for $v(\phi', w) = 1$, and for every $v, v' \in \mathbb{V}$ s.t. $v = v(\phi, w) \leq v(\phi', w) = v'$. Second, if $v(\phi \rightarrow \phi', w) = v' < 1$, we check every $v, v' \in \mathbb{V}$ s.t. $v = v(\phi, w) > v(\phi', w) = v'$.

⁴⁸Intuitively, for a value $1 > v > 0$ of $\diamond \phi$ at w , we add a new state that witnesses v , and for a state on the branch, we guess a value smaller or equal to v . Other modal rules can be rewritten similarly.

model is constructed); or until we need to apply a modal rule to proceed. At this stage, we need to store only the subformulas of ϕ with labels denoting their value at w_0 .

Now we guess a modal formula (say, $w_0 : 2 : \Box\chi = \frac{1}{m+1}$) whose decomposition requires an introduction of a new state (w_1) and apply this rule. Then we apply all modal rules that use $w_0\mathbf{R}w_1$ as a premise (again, if those require branching, we guess only one branch) and start from the beginning with the propositional rules. If we reach a contradiction, the branch is closed. Again, the only new entries to store are subformulas of ϕ (now, with fewer modalities), their values at w_1 , and a relational term $w_0\mathbf{R}w_1$. Since the depth of the model is $O(|\phi|)$ and since we work with modal formulas one by one, we need to store subformulas of ϕ with their values $O(|\phi|)$ times, so, we need only $O(|\phi|^2)$ space.

Finally, if no rule is applicable and there is no contradiction, we mark $w_0 : 2 : \Box\chi = \frac{1}{m+1}$ as ‘safe’. Now we *delete all entries of the tableau below it* and pick another unmarked modal formula that requires an introduction of a new state. Dealing with these one by one allows us to construct the model branch by branch. But since the length of each branch of the model is bounded by $O(|\phi|)$ and since we delete *branches of the model* once they are shown to contain no contradictions, we need only polynomial space. \square

7. Modal logics on bi-relational frames

This is the last chapter of Part II. Here, we present two logics that fulfil all five desiderata outlined in [the introduction](#). In fact, we go one step further since we consider the logics on frames with *two* fuzzy relations — R^+ and R^- — standing for degrees of trust in confirmations and denials given by a source. This separation makes sense when, for example, an agent u dealing with a test t that gives more false positives than false negatives (in which case, $uR^+t < uR^-t$); or when u has a source s which is known to be extremely sceptical (whence, $uR^+s > uR^-s$), etc. Formally, the idea of frames with separate accessibility relations for positive and negative support in paraconsistent logics can be traced to [137] (and see [52] for other examples). This is the next expected step after having two valuations on a frame.

We are considering two types of modalities. The normal \Box and \Diamond that we already had in \mathbf{KG}^{2c} and the ‘informational’ \blacksquare and \blacklozenge . Let us quickly explain the naming. Consider Fig. 7.1 and Definition 6.1. It is clear that $\Box\phi$ can be understood as an infinitary \wedge (the conjunction w.r.t. upwards or truth order) on $[0, 1]^\times$ across the accessible states. Likewise, \Diamond is an infinitary \vee on $[0, 1]^\times$ across the accessible states. But $[0, 1]^\times$ is a *bi-lattice*, whence it has \sqcap and \sqcup (the conjunction and disjunction w.r.t. rightwards or information order). \blacksquare and \blacklozenge will be then defined as infinitary \sqcap and \sqcup across the accessible states.

We have already mentioned in Chapter 6 that \Box and \Diamond can represent ‘pessimistic’ and ‘optimistic’ aggregations of evidence. Their informational counterparts represent ‘sceptical’ (or ‘cautious’) and ‘credulous’ (or ‘gullible’) aggregations. These aggregations were first analysed in [27]. There, however, they were described in a two-layered framework which prohibits the nesting of modalities. Furthermore, BD that lacks implication was chosen as the propositional fragment of both inner and outer layers. In Section 7.2, we extend that approach to the Kripke semantics to incorporate possible references between the sources and the sources’ ability to give modalised statements. Furthermore, we use \mathbf{G}^2 as the propositional fragment.

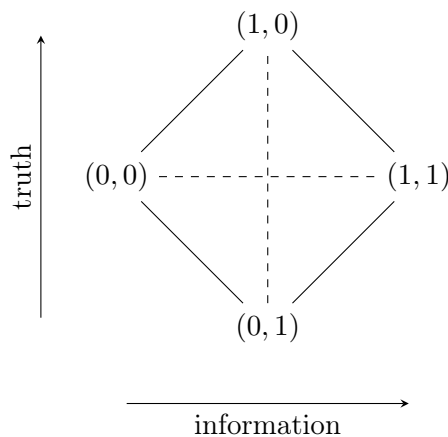


Figure 7.1.: $[0, 1]^\times$: the truth order goes upwards and the information order goes rightwards.

7.1. Paraconsistent Gödel logic with normal modalities

We begin with \mathbf{KG}^{2c} on fuzzy bi-relational frames. We dub the logic $\mathbf{KG}^{2\pm}$ (recall Fig. 5.1). We also retain $\mathcal{L}_{\mathcal{G}_{\Delta, \square, \diamond}^2}$ as our language but we need to change the notion of a frame and the semantics of modalities.

Definition 7.1. A *bi-relational frame* is a tuple $\mathfrak{F} = \langle W, R^+, R^- \rangle$ with $W \neq \emptyset$ and

1. $R^+, R^- : W \times W \rightarrow \{0, 1\}$ if \mathfrak{F} is *crisp*⁴⁹;
2. $R^+, R^- : W \times W \rightarrow [0, 1]$ if \mathfrak{F} is *fuzzy*.

If $R^+ = R^-$, we call the frame *mono-relational*.

Definition 7.2 (Semantics of $\mathbf{KG}^{2\pm}$). A $\mathbf{KG}^{2\pm}$ -*model* is a tuple $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ with $\langle W, R^+, R^- \rangle$ being a crisp or fuzzy frame and $\mathbf{KG}^{2\pm}$ valuations $v_1, v_2 : \mathbf{Prop} \rightarrow [0, 1]$. The semantics of propositional connectives is the same as in Definition 6.1 and the semantics of modalities are as follows.

$$\begin{aligned} v_1(\Box\phi, w) &= \inf_{w' \in W} \{wR^+w' \rightarrow_{\mathcal{G}} v_1(\phi, w')\} & v_2(\Box\phi, w) &= \sup_{w' \in W} \{wR^-w' \wedge_{\mathcal{G}} v_2(\phi, w')\} \\ v_1(\Diamond\phi, w) &= \sup_{w' \in W} \{wR^+w' \wedge_{\mathcal{G}} v_1(\phi, w')\} & v_2(\Diamond\phi, w) &= \inf_{w' \in W} \{wR^-w' \rightarrow_{\mathcal{G}} v_2(\phi, w')\} \end{aligned}$$

We say that ϕ is *v_1 -valid on \mathfrak{F}* ($\mathfrak{F} \models^+ \phi$) iff for every model \mathfrak{M} on \mathfrak{F} and every $w \in \mathfrak{M}$, it holds that $v_1(\phi, w) = 1$. ϕ is *v_2 -valid on \mathfrak{F}* ($\mathfrak{F} \models^- \phi$) iff for every model \mathfrak{M} on \mathfrak{F} and every $w \in \mathfrak{M}$, it holds that $v_2(\phi, w) = 0$. ϕ is *strongly valid on \mathfrak{F}* ($\mathfrak{F} \models \phi$) iff it is v_1 and v_2 -valid.

Convention 7.1. In this chapter, if S is a *fuzzy relation* on W (i.e., $S : W \times W \rightarrow [0, 1]$), we set $S(w) = \{\langle w, w' \rangle : w, w' \in W \text{ and } wSw' > 0\}$.

Remark 7.1. Note that the semantical conditions for the support of truth in $\mathbf{KG}^{2\pm}$ coincide with the semantics of \mathbf{KbiG}^f (recall Definition 5.2). \mathbf{KG}^{2c} can be retrieved as expected: we should just stipulate that $R^+ = R^-$ are crisp.

Note also that we do not give the definition of $\mathbf{KG}^{2\pm}$ entailment, although, it can be given in the same manner as for \mathbf{KG}^{2c} (recall Definition 6.1). This is because we are not going to provide axiomatisation of $\mathbf{KG}^{2\pm}$. Instead, we will mostly consider its semantical and computational properties.

7.1.1. \mathbf{KbiG} , \mathbf{KG}^{2c} , and $\mathbf{KG}^{2\pm}$

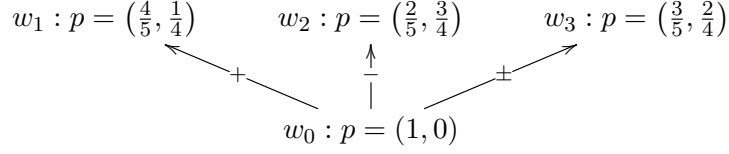
Definition 7.2 gives a reason to believe that $\mathbf{KG}^{2\pm}$ is in a sense intermediate between \mathbf{KbiG}^c and \mathbf{KG}^{2c} . In this section, we investigate the following questions.

1. \Box and \Diamond are not interdefinable in \mathbf{KbiG} (Proposition 5.1) but $\neg\Box\neg p \leftrightarrow \Diamond p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ are \mathbf{KG}^{2c} valid. Are \Box and \Diamond interdefinable in $\mathbf{KG}^{2\pm}$?
2. \mathbf{KG}^{2c} extends \mathbf{KbiG}^c and is conservative w.r.t. \neg -free formulas (Proposition 6.3). Does $\mathbf{KG}^{2\pm}$ (on mono- or bi-relational frames) extend \mathbf{KbiG}^f ? Does crisp $\mathbf{KG}^{2\pm}$ on bi-relational frames extend \mathbf{KbiG}^c ?

We first show that \Box and \Diamond are, in fact, not interdefinable in $\mathbf{KG}^{2\pm}$.

Theorem 7.1. \Box and \Diamond are not interdefinable.

Proof. Denote with \mathcal{L}_{\Box} and \mathcal{L}_{\Diamond} the \Diamond -free and \Box -free fragments of $\mathcal{L}_{\mathcal{G}_{\Delta, \square, \diamond}^2}$, respectively. To prove the statement, it suffices to find a pointed model $\langle \mathfrak{M}, w \rangle$ s.t. there is no \mathcal{L}_{\Diamond} formula that has the same value at w as $\Box p$ and vice versa.


 Figure 7.2.: All variables have the same values in all states exemplified by p .

Consider the model on Fig. 7.2. We have $v(\Box p, w_0) = \left(\frac{3}{5}, \frac{3}{4}\right)$ and $v(\Diamond p, w_0) = \left(\frac{4}{5}, \frac{2}{4}\right)$.

It is easy to check that $v(\phi, t) \in \{v(p, t), v(\neg p, t), (1, 0), (0, 1)\}$ for every $\phi \in \mathcal{L}_{\mathbf{G}_{\Delta, \Box, \Diamond}^2}$ over one variable on the single-point irreflexive frame with a state t . Thus, for every $\chi \in \mathcal{L}_{\Box}$ and every $\psi \in \mathcal{L}_{\Diamond}$ it holds that

$$\begin{aligned}
 v(\Box \chi, w_0) &\in \left\{ (0; 1), \left(\frac{3}{5}; \frac{3}{4}\right), \left(\frac{1}{4}; \frac{3}{5}\right), \left(\frac{3}{4}; \frac{3}{5}\right), \left(\frac{3}{5}; \frac{1}{4}\right), (1; 0) \right\} = X \\
 v(\Diamond \psi, w_0) &\in \left\{ (0; 1), \left(\frac{4}{5}; \frac{2}{4}\right), \left(\frac{2}{4}; \frac{2}{5}\right), \left(\frac{2}{4}; \frac{4}{5}\right), \left(\frac{2}{5}; \frac{2}{4}\right), (1; 0) \right\} = Y
 \end{aligned}$$

Now, let X^c and Y^c be the closures of X and Y under propositional operations. It is clear that $\left(\frac{3}{5}; \frac{3}{4}\right) \notin Y^c$ and $\left(\frac{4}{5}; \frac{2}{4}\right) \notin X^c$. It is also easy to verify by induction that for all $\chi' \in \mathcal{L}_{\Box}$ and $\psi' \in \mathcal{L}_{\Diamond}$, it holds that $v(\chi', w_0) \in X^c$ and $v(\psi', w_0) \in Y^c$. The result now follows. \square

The next statement gives the negative answer to the first half of the second question.

Theorem 7.2. *Fuzzy $\mathbf{KG}^{2\pm}$ does not extend \mathbf{KbiG}^f .*

Proof. Recall that $\Diamond \sim \sim p \rightarrow \sim \sim \Diamond p$ is a theorem of fuzzy Gödel modal logic and that \mathbf{KbiG}^f extends fuzzy \mathbf{KG} . Thus, $\Diamond \sim \sim p \rightarrow \sim \sim \Diamond p$ is \mathbf{KbiG}^f valid. Consider the model below.

$$w \xrightarrow{R^+ = R^- = \frac{1}{2}} w' : p = \left(1, \frac{2}{3}\right)$$

It is clear that $v_2(\sim \sim \Diamond p, w) = 1$ but $v_2(\Diamond \sim \sim p, w) = 0$. Thus, $v_2(\Diamond \sim \sim p \rightarrow \sim \sim \Diamond p, w) = 1$, i.e., $\Diamond \sim \sim p \rightarrow \sim \sim \Diamond p$ is not valid in $\mathbf{KG}^{2\pm}$. \square

Note that we used a *mono-relational* fuzzy frame in the proof of the above theorem. It remains to consider the *crisp $\mathbf{KG}^{2\pm}$ over bi-relational frames*. In the remainder of the section, we show that it *does extend \mathbf{KbiG}^c* . The next lemma is a straightforward generalisation of Proposition 6.2.

Lemma 7.1. *Let $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ be a crisp $\mathbf{KG}^{2\pm}$ -model. We define*

$$\mathfrak{M}^* = \langle W, (R^+)^*, (R^-)^*, v_1^*, v_2^* \rangle$$

to be as follows: $(R^+)^ = R^-$, $(R^-)^* = R^+$, $v_1^*(p, w) = 1 - v_2(p, w)$, and $v_2^*(p, w) = 1 - v_1(p, w)$.*

Then, $v(\phi, w) = (x, y)$ iff $v^(\phi, w) = (1 - y, 1 - x)$.*

Proof. We proceed by induction on ϕ . The basis case of propositional variables holds by the construction of \mathfrak{M}^* . The cases of propositional connectives hold by Proposition 4.3. We consider the case of $\phi = \Box \psi$.

Let $v(\Box \psi, w) = (x, y)$. Then $\inf\{v_1(\psi, w') : wR^+w'\} = x$, and $\sup\{v_2(\psi, w') : wR^-w'\} = y$. Now, we apply the induction hypothesis to ψ , and thus if $v(\psi, s) = (x', y')$, then $v_1^*(\psi, s) = 1 - y'$ and $v_2^*(\psi, s) = 1 - x'$ for any $s \in R^+(w) = (R^-)^*(w)$ and $s' \in R^-(w) = (R^+)^*(w)$. But then $\inf\{v_1^*(\psi, w') : w(R^+)^*w'\} = 1 - y$, and $\sup\{v_2^*(\psi, w') : w(R^-)^*w'\} = 1 - x$, as required. \square

⁴⁹Equivalently, R^+ and R^- in crisp frames are *relations* on W .

Theorem 7.3. *Let ϕ be a \neg -free formula. Then, ϕ is \mathbf{KbiG}^c -valid iff it is crisp $\mathbf{KG}^{2\pm}$ -valid.*

Proof. It is clear that if ϕ is not \mathbf{KbiG} valid, then it is not $\mathbf{KG}^{2\pm}$ valid either (recall Remark 7.1). For the converse, it follows from Lemma 7.1 that if $v_2(\phi, w) > 0$ for some frame $\mathfrak{F} = \langle W, R^+, R^- \rangle$, $w \in \mathfrak{F}$ and v_2 on \mathfrak{F} , then $v_1^*(\phi, w) < 1$. But ϕ does not contain \neg and thus its interpretation depends only on v_2 and R^- , whence v_1^* is a \mathbf{KbiG} valuation on $\langle W, R^- \rangle$. Thus, ϕ is not \mathbf{KbiG}^c valid either. \square

Note that Lemma 7.1 implies that in order to check crisp $\mathbf{KG}^{2\pm}$ validity of ϕ , it suffices to check whether it is always the case that $v_1(\phi, w) = 1$ in crisp models. On the other hand, this reduction, evidently, does not hold for fuzzy $\mathbf{KG}^{2\pm}$ due to Theorem 7.2.

7.1.2. Correspondence theory and frame definability

In this section, we investigate the modal (un)definability of frame classes in crisp and fuzzy $\mathbf{KG}^{2\pm}$. We begin with corollaries of Lemma 7.1 that concern the definability of crisp frames.

Definition 7.3. Let \mathbb{K} be a class of crisp frames. A first- or second-order formula $F(R)$ containing a free predicate letter R defines \mathbb{K} iff for every $\mathfrak{F} = \langle W, R \rangle$, it holds that $\mathfrak{F} \in \mathbb{K}$ iff $\mathfrak{F} \models F(R)$.

1. The $+$ -counterpart of \mathbb{K} is the class \mathbb{K}^+ of frames $\mathfrak{F} = \langle W, R^+, R^- \rangle$ s.t. $\langle W, R^+ \rangle \models F(R^+)$.
2. The $-$ -counterpart of \mathbb{K} is the class \mathbb{K}^- of frames $\mathfrak{F} = \langle W, R^+, R^- \rangle$ s.t. $\langle W, R^- \rangle \models F(R^-)$.
3. The \pm -counterpart of \mathbb{K} is the class $\mathbb{K}^\pm = \mathbb{K}^+ \cap \mathbb{K}^-$.

Corollary 7.1. *Let ϕ be a \neg -free formula that defines a class of frames $\mathfrak{F} = \langle W, R \rangle$, \mathbb{K} in \mathbf{KbiG}^c and let \mathbb{K}^\pm be the \pm -counterpart of \mathbb{K} . Then ϕ defines \mathbb{K}^\pm in $\mathbf{KG}^{2\pm c}$.*

Proof. Assume that ϕ does not define \mathbb{K}^\pm . Then, either (1) there is $\mathfrak{F} \notin \mathbb{K}^\pm$ s.t. $\mathfrak{F} \models \phi$ or (2) $\mathfrak{H} \not\models \phi$ for some $\mathfrak{H} \in \mathbb{K}^\pm$. Since ϕ defines \mathbb{K} in \mathbf{KbiG} , it is clear that $\mathfrak{F}, \mathfrak{H} \in \mathbb{K}^+$. Thus, we need to reason for contradiction in the case when $\mathfrak{F} \notin \mathbb{K}^-$ or $\mathfrak{H} \notin \mathbb{K}^-$. We prove only (1) as (2) can be tackled in a dual manner.

Observe that $v_2(\phi, w) = 0$ for every $w \in \mathfrak{F}$ and v_2 defined on $\mathfrak{F} = \langle W, R^+, R^- \rangle$. But then, by Lemma 7.1, we have that $v_1^*(\phi, w) = 1$ for every $w \in \mathfrak{F}$ and v_1^* defined on \mathfrak{F} . Thus, since for every v_1^* there is v_2 from which it could be obtained, ϕ is \mathbf{KbiG} -valid on a frame $\langle W, R^- \rangle$ where R^- is not definable via $F(R^-)$. Hence, ϕ does not define \mathbb{K} in \mathbf{KbiG} either. A contradiction. \square

A natural question now is whether it is possible to have definitions of classes of frames that are only $+$ -counterparts (or $-$ -counterparts) of \mathbf{KbiG} -definable frame classes. E.g., a class of frames whose R^+ is reflexive but R^- is not necessarily so. The next statement provides a negative answer.

Corollary 7.2. *Let $F(R^+)$ and $F(R^-)$ be two first- or second-order formulas defining relations R^+ and R^- . Then, the class \mathbb{K} of crisp frames $\mathfrak{F} = \langle W, R^+, R^- \rangle$ with only R^+ being definable by $F(R^+)$ (resp., only R^- being definable by $F(R^-)$) is not definable in $\mathbf{KG}^{2\pm}$.*

Proof. We reason for contradiction. Assume that ϕ defines \mathbb{K} , and let $\mathfrak{F} \in \mathbb{K}$ with $\mathfrak{F} = \langle W, R^+, R^- \rangle$ s.t. $F(R^-)$ does not hold of \mathfrak{F} . Now denote $\mathfrak{F}^* = \langle W, R^-, R^+ \rangle$. Clearly, $\mathfrak{F}^* \notin \mathbb{K}$. However, by Lemma 7.1, we have that $\mathfrak{F}^* \models \phi$, i.e., ϕ does not define \mathbb{K} . A contradiction. \square

As of now, we have discussed the definability of *different classes of crisp frames*. $\Box(p \vee q) \rightarrow (\Box p \vee \Diamond q)$ defines crisp frames in \mathbf{KG} [130] and $\Delta\Box p \rightarrow \Box\Delta p$ in \mathbf{KbiG} (Proposition 5.4)⁵⁰, however, $\mathbf{KG}^{2\pm}$ does not extend \mathbf{KbiG} (nor \mathbf{KG}), whence the definability of the *class of all crisp frames* is not immediate.

⁵⁰It makes sense to speak of definability of crisp frames in \mathbf{KG} and \mathbf{KbiG} separately since $\Delta\Box p \rightarrow \Box\Delta p$ is essential in the completeness proof.

Theorem 7.4. Let $\mathfrak{F} = \langle W, R^+, R^- \rangle$.

1. R^+ is crisp iff $\mathfrak{F} \models \Delta \Box p \rightarrow \Box \Delta p$.
2. R^- is crisp iff $\mathfrak{F} \models \Diamond \sim \sim p \rightarrow \sim \sim \Diamond p$.

Proof. Note, first of all, that $v_i(\Delta \phi, w), v_i(\sim \sim \phi, w) \in \{0, 1\}$ for every ϕ and $i \in \{1, 2\}$. Now let R^+ be crisp. We have

$$\begin{aligned} v_1(\Delta \Box p, w) = 1 & \text{ then } v_1(\Box p, w) = 1 \\ & \text{ then } \inf\{v_1(p, w') : wR^+w'\} = 1 & (R^+ \text{ is crisp}) \\ & \text{ then } \inf\{v_1(\Delta p, w') : wR^+w'\} = 1 \\ & \text{ then } v_1(\Box \Delta p, w) = 1 \end{aligned}$$

$$\begin{aligned} v_2(\Delta \Box p, w) = 0 & \text{ then } v_2(\Box p, w) = 0 \\ & \text{ then } \sup_{w' \in W} \{wR^-w' \wedge_{\mathbf{G}} v_2(p, w')\} = 0 \\ & \text{ then } \sup_{w' \in W} \{wR^-w' \wedge_{\mathbf{G}} v_2(\Delta p, w')\} = 0 \\ & \text{ then } v_2(\Box \Delta p, w) = 0 \end{aligned}$$

For the converse, let $wR^+w' = x$ with $0 < x < 1$. We set $v(p, w') = (x, 0)$ and $v(p, w'') = (1, 0)$ elsewhere. It is clear that $v(\Delta \Box p, w) = (1, 0)$ but $v(\Box \Delta p, w) = (0, 0)$. Thus, $v(\Delta \Box p \rightarrow \Box \Delta p, w) \neq (1, 0)$, as required.

The case of R^- is considered dually. For crisp R^- , we have

$$\begin{aligned} v_1(\Diamond \sim \sim p, w) = 1 & \text{ then } \sup_{w' \in W} \{wR^-w' \wedge_{\mathbf{G}} v_1(\sim \sim p, w')\} = 1 \\ & \text{ then } \sup_{w' \in W} \{wR^-w' \wedge_{\mathbf{G}} v_1(p, w')\} > 0 \\ & \text{ then } v_1(\Diamond p, w) > 0 \\ & \text{ then } v_1(\sim \sim \Diamond p, w) = 1 \end{aligned}$$

$$\begin{aligned} v_2(\Diamond \sim \sim p, w) = 0 & \text{ then } \inf\{v_2(\sim \sim p, w') : wR^-w'\} = 0 & (R^- \text{ is crisp}) \\ & \text{ then } \inf\{v_2(p, w') : wR^-w'\} < 1 \\ & \text{ then } v_2(\Diamond p, w) < 1 \\ & \text{ then } v_2(\sim \sim \Diamond p, w) = 0 \end{aligned}$$

For the converse, let $wR^-w' = y$ with $y \in (0, 1)$. We set $v(p, w') = (1, y)$ and $v(p, w'') = (1, 0)$ elsewhere. It is clear that $v(\Diamond \sim \sim p, w) = (1, 0)$ but $v(\sim \sim \Diamond p, w) = (1, 1)$. Thus, $v(\Diamond \sim \sim p \rightarrow \sim \sim \Diamond p, w) \neq (1, 0)$, as required. \square

The above statement highlights an important contrast between crisp and fuzzy bi-relational frames: while it is impossible to define R^+ and R^- separately in crisp frames, we can define a class of frames where only R^+ (or only R^-) is crisp. It is now instructive to ask whether we can define some relations *between* R^+ and R^- . In particular, we show that

1. frames where $wR^+w' = 0$ or $wR^-w' = 0$ for all w and w' , are not definable;
2. mono-relational frames (both crisp and fuzzy) are definable.

Definition 7.4. Let $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ be a model. We define its *splitting* to be $\mathfrak{M}^s = \langle W^s, (R^+)^s, (R^-)^s, v_1^s, v_2^s \rangle$ with

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- $W^s = \{\ulcorner wSww' \urcorner : wSw' > 0, S \in \{R^+, R^-\}\} \cup \{\ulcorner \emptyset Su \urcorner : \neg \exists t tSu > 0, S \in \{R^+, R^-\}\}$;
- $\ulcorner uSu' \urcorner S^s \ulcorner wSw' \urcorner = u'Sw'$ with $S \in R^+, R^-$;
- for every $\ulcorner wSw' \urcorner$ and $i \in \{1, 2\}$, $v_i^s(p, \ulcorner wSw' \urcorner) = v_i(p, w')$.

We will further denote

$$\llbracket w \rrbracket = \{\ulcorner \emptyset Sw \urcorner, \ulcorner uSw \urcorner : S \in \{R^+, R^-\}\}$$

It is clear that there are no $u, u' \in W^s$ s.t. $u(R^+)^s u', u(R^-)^s u' > 0$. We will further call such models *split models* since we separate R^+ from R^- .

The next statement is easy to prove.

Lemma 7.2. *Let $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ be a model and \mathfrak{M}^s be its splitting. Then $v_i(\phi, w) = v_i^s(\phi, w^s)$ for every $\phi \in \mathcal{L}_{\Delta, \square, \diamond}^2$ and $w^s \in \llbracket w \rrbracket$.*

Proof. We proceed by induction. The basis case of propositional variables holds by the construction of \mathfrak{M}^s . The cases of propositional connectives are straightforward. We consider the case of $\phi = \square \chi$ (the \diamond case can be considered dually).

Let $\ulcorner uR^+w \urcorner \in \llbracket w \rrbracket$ be arbitrary. We have

$$\begin{aligned} v_1^s(\square \chi, \ulcorner uR^+w \urcorner) &= \inf_{\ulcorner wR^+w' \urcorner \in W^s} \{\ulcorner uR^+w \urcorner (R^+)^s \ulcorner wR^+w' \urcorner \rightarrow_{\mathbf{G}} v_1^s(\chi, \ulcorner wR^+w' \urcorner)\} \\ &= \inf_{w' \in W} \{wR^+w' \rightarrow_{\mathbf{G}} v_1(\chi, w')\} && \text{(by IH)} \\ &= v_1(\square \chi, w) \end{aligned}$$

$$\begin{aligned} v_2^s(\square \chi, \ulcorner uR^+w \urcorner) &= \sup_{\ulcorner wR^-w' \urcorner \in W^s} \{\ulcorner uR^-w \urcorner (R^-)^s \ulcorner wR^-w' \urcorner \wedge_{\mathbf{G}} v_2^s(\chi, \ulcorner wR^-w' \urcorner)\} \\ &= \sup_{w' \in W} \{wR^-w' \wedge_{\mathbf{G}} v_2(\chi, w')\} && \text{(by IH)} \\ &= v_2(\square \chi, w) \end{aligned}$$

Note that we could apply the induction hypothesis because $\ulcorner uR^+w \urcorner \in \llbracket w \rrbracket$, $\ulcorner wR^+w' \urcorner \in \llbracket w' \rrbracket$, and the values of wR^+w' (resp., wR^-w') are the values of $\ulcorner uR^+w \urcorner (R^+)^s \ulcorner wR^+w' \urcorner$ (resp., $\ulcorner uR^-w \urcorner (R^-)^s \ulcorner wR^-w' \urcorner$). The result follows. \square

The following corollary is now immediate.

Corollary 7.3. *The class of (crisp or fuzzy) frames $\mathfrak{F} = \langle W, R^+, R^- \rangle$ s.t. for every $w, w' \in W$, $wR^+w' = 0$ or $wR^-w' = 0$ is not definable in $\mathbf{KG}^{2\pm}$.*

Proof. Indeed, let \mathbb{P} denote this class of frames. We show that the sets of formulas

$$\mathbf{L}(\mathbb{P}) = \{\phi : \mathbb{P} \models_{\mathbf{KG}^{2\pm}} \phi\} \quad \text{and} \quad \mathbf{KG}^{2\pm} = \{\chi : \chi \text{ is } \mathbf{KG}^{2\pm}\text{-valid}\}$$

coincide. It is clear that $\mathbf{KG}^{2\pm} \subseteq \mathbf{L}(\mathbb{P})$. Now, assume that χ is *not* $\mathbf{KG}^{2\pm}$ -valid and, namely, that there is a frame $\mathfrak{F} \not\models \chi$ s.t. $\mathfrak{F} \notin \mathbb{P}$. Then, there is a model $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ on \mathfrak{F} and $w \in W$ s.t. $v(\chi, w) \neq (1, 0)$. But by Lemma 7.2, we have that $v^s(\chi, w^s) \neq (1, 0)$ where v^s is a valuation on $\mathfrak{M}^s = \langle W^s, (R^+)^s, (R^-)^s, v_1^s, v_2^s \rangle$ (the splitting of \mathfrak{M}). Thus, $\mathbb{P} \not\models_{\mathbf{KG}^{2\pm}} \chi$. \square

Let us now prove the definability of mono-relational frames.

Theorem 7.5. *Let $\mathfrak{F} = \langle W, R^+, R^- \rangle$. Then $\mathfrak{F} \models \square p \leftrightarrow \neg \diamond \neg p$ iff $R^+ = R^-$.*

Proof. Let $R^+ = R^-$, we have

$$\begin{aligned} v_1(\neg\Diamond\neg p, w) &= v_2(\Diamond\neg p, w) \\ &= \inf_{w' \in W} \{wRw' \rightarrow_{\mathcal{G}} v_2(\neg p, w')\} \\ &= \inf_{w' \in W} \{wRw' \rightarrow_{\mathcal{G}} v_1(p, w')\} \\ &= v_1(\Box p, w) \end{aligned}$$

v_2 can be tackled similarly.

Now let $R^+ \neq R^-$, i.e., $wR^+w' = x$ and $wR^-w' = y$ for some $w, w' \in \mathfrak{F}$, and assume w.l.o.g. that $x > y$. We define the values of p as follows: $v(p, w'') = (1, 0)$ for all $w'' \neq w'$ and $v(p, w') = (x, y)$. It is clear that $v(\Box p, w) = (1, 0)$ but $v(\neg\Diamond\neg p, w) = (y, x) \neq (1, 0)$, as required. \square

In the remaining part of Section 7.1, we will be considering (fuzzy) $\mathbf{KG}^{2\pm}$ over *finitely branching frames*, i.e., frames $\langle W, R^+, R^- \rangle$ where $|R^+(w)|, |R^-(w)| < \aleph_0$ for every $w \in W$. We will denote this logic $\mathbf{KG}_{\text{fb}}^{2\pm}$. As we have discussed in Section 6.3, this is a natural restriction if we want to use the logic to formalise natural-language statements about beliefs.

We finish the current section by showing that fuzzy and crisp⁵¹ finitely branching frames are definable. Note, however, that now we need *two formulas*.

Theorem 7.6. \mathfrak{F} is finitely branching iff $\mathfrak{F} \models \sim\sim\Box(p \vee \sim p)$ and $\mathfrak{F} \models \mathbf{1} \prec \Diamond\neg(p \vee \sim p)$.

Proof. From left to right, we observe that $v_1(p \vee \sim p, w) > 0$ and $v_2(p \vee \sim p, w) < 1$ for every $w \in \mathfrak{F}$. Since \mathfrak{F} is finitely branching, $\inf_{w' \in W} \{wSw' \rightarrow_{\mathcal{G}} v_1(p \vee \sim p, w')\} > 0$ and $\sup_{w' \in W} \{wSw' \wedge_{\mathcal{G}} v_2(p \vee \sim p, w')\} < 1$ for $S \in \{R^+, R^-\}$. Thus, $v_1(\Box(p \vee \sim p), w) > 0$ and $v_2(\Box(p \vee \sim p), w) < 1$, whence, $v(\sim\sim\Box(p \vee \sim p), w) = (1, 0)$. Likewise, $v_1(\Diamond\neg(p \vee \sim p), w) < 1$ and $v_2(\Diamond\neg(p \vee \sim p), w) > 0$, whence $v(\mathbf{1} \prec \Diamond\neg(p \vee \sim p), w) = (1, 0)$.

For the converse, we proceed by contraposition and assume that \mathfrak{F} is not finitely branching. We have two cases: (1) $|R^+(w)| \geq \aleph_0$ or (2) $|R^-(w)| \geq \aleph_0$ for some $w \in \mathfrak{F}$. In the first case, we let $X \subseteq R^+(w)$ be countably infinite and define the value of p as follows: $v(p, w'') = (1, 0)$ for every $w'' \notin X$ and $v(p, w_i) = (wR^+w' \cdot \frac{1}{i}, 0)$ for every $w_i \in X$. It is clear that $\inf_{w' \in W} \{wR^+w' \rightarrow_{\mathcal{G}} v_1(p \vee \sim p, w')\} = 0$, whence $v_1(\sim\sim\Box(p \vee \sim p)) = 0$ as required.

In the second case, $Y \subseteq R^-(w)$ be countable and define the value of p as follows: $v(p, w'') = (1, 0)$ for every $w'' \notin Y$ and $v(p, w_i) = (wR^-w' \cdot \frac{1}{i}, 0)$. It is clear that $\inf_{w' \in W} \{wR^-w' \rightarrow_{\mathcal{G}} v_2(\neg(p \vee \sim p), w')\} = 0$, whence $v_2(\Diamond\neg(p \vee \sim p)) = 0$ and $v_2(\mathbf{1} \prec \Diamond\neg(p \vee \sim p)) = 1$ as required. \square

7.1.3. Constraint tableaux

In this section, we present constraint tableaux for $\mathbf{KG}_{\text{fb}}^{2\pm}$. The present calculus which we call $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ is an easy adaptation of $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\epsilon})$ from Definition 6.3.

Definition 7.5 ($\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ — the tableaux calculus for $\mathbf{KG}_{\text{fb}}^{2\pm}$). We fix a set of state-labels W and let $\lesssim \in \{<, \leq\}$ and $\gtrsim \in \{>, \geq\}$. Let further $w \in W$, $\mathbf{x} \in \{1, 2\}$, $\phi \in \mathcal{L}_{\mathbf{G}_{\Delta, \Box, \Diamond}^2}$, and $c \in \{0, 1\}$. A *structure* is either $w : \mathbf{x} : \phi, c, wR^+w'$, or wR^-w' . We denote the set of structures with Str . Structures of the form $w : \mathbf{x} : p, wR^+w'$, and wR^-w' are called *atomic* (denoted AStr).

We define a *constraint tableau* as a downward branching tree whose branches are sets containing constraints $\mathfrak{X} \lesssim \mathfrak{X}'$ ($\mathfrak{X}, \mathfrak{X}' \in \text{Str}$). Each branch can be extended by an application of a rule⁵² from Fig. 4.1 or Fig. 7.3.

⁵¹In fact, the definability of *crisp* finitely branching frames follows from Corollary 7.1 since $\sim\sim\Box(p \vee \sim p)$ defines finitely branching frames in \mathbf{KbiG} (Proposition 5.13).

⁵²If $\mathfrak{X} < 1$, $\mathfrak{X} < \mathfrak{X}' \in \mathcal{B}$ or $0 < \mathfrak{X}', \mathfrak{X} < \mathfrak{X}' \in \mathcal{B}$, the rules are applied only to $\mathfrak{X} < \mathfrak{X}'$.

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$$\begin{array}{c}
\Box_1 \gtrsim \frac{w:1:\Box\phi \gtrsim \mathfrak{X}}{w':1:\phi \gtrsim \mathfrak{X} \mid \begin{array}{l} wR^+w' \leq w':1:\phi \\ 1 \gtrsim \mathfrak{X} \end{array}} \quad \Box_1 \leq \frac{w:1:\Box\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \begin{array}{l} wR^+w'' > w'':1:\phi \\ w'':1:\phi \leq \mathfrak{X} \end{array}} \quad \Box_1 < \frac{w:1:\Box\phi < \mathfrak{X}}{wR^+w'' > w'':1:\phi \\ w'':1:\phi < \mathfrak{X}} \\
\\
\Diamond_1 \gtrsim \frac{w:1:\Diamond\phi \gtrsim \mathfrak{X}}{wR^+w'' \gtrsim \mathfrak{X} \\ w'':1:\phi \gtrsim \mathfrak{X}} \quad \Diamond_1 \lesssim \frac{w:1:\Diamond\phi \lesssim \mathfrak{X}}{w':1:\phi \lesssim \mathfrak{X} \mid wR^+w' \lesssim \mathfrak{X}} \quad \Box_2 \gtrsim \frac{w:2:\Box\phi \gtrsim \mathfrak{X}}{wR^-w'' \gtrsim \mathfrak{X} \\ w'':2:\phi \gtrsim \mathfrak{X}} \quad \Box_2 \lesssim \frac{w:2:\Box\phi \lesssim \mathfrak{X}}{w':2:\phi \lesssim \mathfrak{X} \mid wR^-w' \lesssim \mathfrak{X}} \\
\\
\Diamond_2 \gtrsim \frac{w:2:\Diamond\phi \gtrsim \mathfrak{X}}{w':2:\phi \gtrsim \mathfrak{X} \mid \begin{array}{l} wR^-w' \leq w':2:\phi \\ 1 \gtrsim \mathfrak{X} \end{array}} \quad \Diamond_2 \leq \frac{w:2:\Diamond\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid \begin{array}{l} wR^-w'' > w'':2:\phi \\ w'':2:\phi \leq \mathfrak{X} \end{array}} \quad \Diamond_2 < \frac{w:2:\Diamond\phi < \mathfrak{X}}{wR^-w'' > w'':2:\phi \\ w'':2:\phi < \mathfrak{X}}
\end{array}$$

Figure 7.3.: Bars denote branching, $i, j \in \{1, 2\}$, $i \neq j$, wR^+w' and wR^-w' occur on the branch, w'' is fresh on the branch.

entry	interpretation
$w:1:\phi \leq w':2:\phi'$	$v_1(\phi, w) \leq v_2(\phi', w')$
$w:2:\phi \leq c$	$v_2(\phi, w) \leq c$ with $c \in \{0, 1\}$
$wR^-w' \leq w':2:\phi$	$wR^-w' \leq v_2(\phi, w')$

Table 7.1.: Interpretations of $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ entries.

The notions of *closed*, *open*, and *complete branches* are the same as in Definition 6.3. We also say that there is a *tableau proof* of ϕ iff there are closed tableaux starting from $w:1:\phi < 1$ and $w:2:\phi > 0$.

Before proceeding to the completeness proof, let us explain how $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ works. First, we summarise the meanings of tableaux entries in Table 7.1. Note that we only add the interpretation of relational constraints to Table 6.1.

We are also adapting the notion of branch realisation from $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ (recall Definition 6.4).

Definition 7.6 (Branch realisation). A model $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ realises a branch \mathcal{B} of a tableau iff $W = \{w : w \text{ occurs on } \mathcal{B}\}$ and there is a function $\text{rl} : \text{Str} \rightarrow [0, 1]$ s.t. for every $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Y}', \mathfrak{Z}, \mathfrak{Z}' \in \text{Str}$ with $\mathfrak{X} = w : \mathbf{x} : \phi$, $\mathfrak{Y} = w_i R^+ w_j$, and $\mathfrak{Y}' = w'_i R^- w'_j$ the following holds ($\mathbf{x} \in \{1, 2\}$, $c \in \{0, 1\}$).

- If $\mathfrak{Z} \lesssim \mathfrak{Z}' \in \mathcal{B}$, then $\text{rl}(\mathfrak{Z}) \lesssim \text{rl}(\mathfrak{Z}')$.
- $\text{rl}(\mathfrak{X}) = v_{\mathbf{x}}(\phi, w)$, $\text{rl}(c) = c$, $\text{rl}(\mathfrak{Y}) = w_i R^+ w_j$, $\text{rl}(\mathfrak{Y}') = w'_i R^- w'_j$

Let us now provide an example of a failed proof with a complete open branch and construct a model realising it. The proof (Fig. 7.4) goes as follows: first, we apply all the possible propositional rules, then the modal rules that introduce new states, and then the modal rules using the newly introduced states. We repeat the process until we decompose all structures into atoms.

We then extract a model from the complete open branch s.t. $v_1(\Box p \rightarrow \Box \neg \Diamond p, w_0) < 1$. We use w 's on the branch as the carrier and assign the values of variables and relations so that they correspond to \lesssim .

The following completeness proof is a straightforward adaptation of that of Theorem 6.3.

Theorem 7.7 ($\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ completeness). ϕ is $\mathbf{KG}_{\text{fb}}^{2\pm}$ valid iff there is a tableau proof of ϕ .

Proof. We consider only the most important cases.

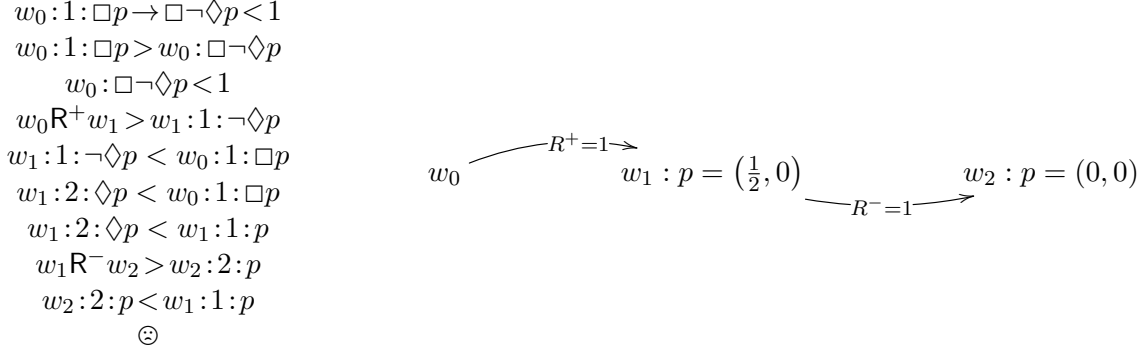


Figure 7.4.: A failed $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ proof (\odot denotes that the branch is complete and open) and its corresponding realising branch.

For soundness, we prove that if the premise of the rule is realised, then so is at least one of its conclusions. Note that since we work with finitely branching frames, infima and suprema from Definition 7.2 become maxima and minima. Since propositional rules are exactly the same as in $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2c})$, we consider only the most interesting cases of modal rules. We tackle $\Box_1 \succsim$ (cf. Definition 7.5) and show that if $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ realises the premise of the rule, it also realises one of its conclusions.

Assume w.l.o.g. that $\mathfrak{X} = w'' : 2 : \psi$, and let \mathfrak{M} realise $w : 1 : \Box \phi \geq w'' : 2 : \psi$. Now, since R^+ is finitely branching, we have $\min_{w' \in W} \{wR^+w' \rightarrow_{\mathcal{G}} v_1(\phi, w')\} \geq v_2(\psi, w'')$, whence at each $w' \in W$ s.t. $wR^+w' > 0$ ⁵³, either $v_1(\phi, w') \geq v_2(\psi, w'')$ or $wR^+w' \leq v_2(\phi, w')$. Thus, at least one conclusion of the rule is satisfied.

Other rules can be dealt with similarly. Since closed branches are not realisable, the result follows.

To prove completeness, we show that a realising model can be built for every complete open branch \mathcal{B} . First, we set $W = \{w : w \text{ occurs in } \mathcal{B}\}$. Denote the set of atomic structures appearing on \mathcal{B} with $\text{AStr}(\mathcal{B})$ and let \mathcal{B}^+ be the transitive closure of \mathcal{B} under \lesssim . Now, we assign values. For $i \in \{1, 2\}$, if $w : i : p \geq 1 \in \mathcal{B}$, we set $v_i(p, w) = 1$. If $w : i : p \leq 0 \in \mathcal{B}$, we set $v_i(p, w) = 0$. If $wSw' < \mathfrak{X} \notin \mathcal{B}^+$, we set $wSw' = 1$. If $w : i : p$ or wSw' with $S \in \{R^+, R^-\}$ does not occur on \mathcal{B} , we set $v_i(p, w) = 0$ and $wSw' = 0$.

For each $\text{str} \in \text{AStr}$, we now set

$$[\text{str}] = \left\{ \text{str}' \mid \begin{array}{l} \text{str} \leq \text{str}' \in \mathcal{B}^+ \text{ and } \text{str} < \text{str}' \notin \mathcal{B}^+ \\ \text{or} \\ \text{str} \geq \text{str}' \in \mathcal{B}^+ \text{ and } \text{str} > \text{str}' \notin \mathcal{B}^+ \end{array} \right\}$$

Denote the number of $[\text{str}]$'s with $\#\text{str}$. Since the only possible loop in \mathcal{B}^+ is $\text{str} \leq \text{str}' \leq \dots \leq \text{str}$ where all elements belong to $[\text{str}]$, it is clear that $\#\text{str} \leq 2 \cdot |\text{AStr}(\mathcal{B})| \cdot |W|$. Put $[\text{str}] \prec [\text{str}']$ iff there are $\text{str}_i \in [\text{str}]$ and $\text{str}_j \in [\text{str}']$ s.t. $\text{str}_i < \text{str}_j \in \mathcal{B}^+$. We now set the valuation of these structures as follows:

$$\text{str} = \frac{|\{[\text{str}'] : [\text{str}'] \prec [\text{str}]\}|}{\#\text{str}}$$

It is clear that constraints containing only atomic structures and constants are now satisfied. To show that all other constraints are satisfied, we prove that if at least one conclusion of the rule is satisfied, then so is the premise. The proof is done by considering the cases of rules. We consider only the case of $\Box_1 \succsim$ and assume w.l.o.g. that $\mathfrak{X} = w'' : 2 : \psi$.

⁵³Recall that if $uR^+u' \notin \mathcal{B}$, we set $uR^+u' = 0$.

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For $\Box_1 \succsim$, assume that for every w' s.t. wR^+w' is on the branch, either $w':1:\phi \geq w':2:\psi$ or $wR^+w' \leq w':1:\phi$ is realisable. Thus, by the inductive hypothesis, for every $w' \in R^+(w)$, it holds that $v_1(\phi, w') \geq v_2(\psi, w')$ or $wR^+w' \leq v_1(\phi, w')$. Hence, $v_1(\Box\phi, w) \geq v_2(\psi, w')$ and $w:1:\Box\phi \geq w':2:\psi$ is realised. \square

We can now use $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ to obtain the expected finite model property and upper bound on the size of satisfying (or falsifying) models.

Corollary 7.4. *Let $\phi \in \mathcal{L}_{\mathcal{G}_{\Delta, \Box, \Diamond}^2}$ be not $\mathbf{KG}_{\text{fb}}^{2\pm}$ valid, and let k be the number of modalities in it. Then there is a model \mathfrak{M} of the size $\leq k^{k+1}$ and depth $\leq k$ and $w \in \mathfrak{M}$ s.t. $v_1(\phi, w) \neq 1$ or $v_2(\phi, w) \neq 0$.*

Proof. By theorem 7.7, if ϕ is not $\mathbf{KG}_{\text{fb}}^{2\pm}$ valid, we can build a falsifying model using tableaux. It is also clear from the rules in Definition 7.5 that the depth of the constructed model is bounded from above by the maximal number of nested modalities in ϕ . The width of the model is bounded by the maximal number of modalities on the same level of nesting. \square

Now, using the upper bound on the size of the model, we can reduce the satisfiability in $\mathbf{KG}_{\text{fb}}^{2\pm}$ to the satisfiability on the models where the values of subformulas and relations are over some finite bi-Gödel algebra. This allows us to avoid comparisons of formulas in different states, whence, we can build the satisfying model branch by branch. We adapt the algorithm from [20].

Theorem 7.8. *$\mathbf{KG}_{\text{fb}}^{2\pm}$ validity and satisfiability are PSpace complete.*

Proof. For the membership, observe from the proof of Theorem 7.7 that ϕ is satisfiable (falsifiable) on $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ iff all variables, wR^+w' 's, and wR^-w' 's have values from $\mathbf{V} = \left\{ 0, \frac{1}{\#\text{str}}, \dots, \frac{\#\text{str}-1}{\#\text{str}}, 1 \right\}$ under which ϕ is satisfied (falsified). Since satisfiability and falsifiability are reducible to each other and since PSpace = coPSpace, we prove the PSpace-membership of the $\mathbf{KG}_{\text{fb}}^{2\pm}$ -satisfiability.

Since $\#\text{str}$ is bounded from above, we can now replace constraints with labelled formulas and relational structures of the form $w:i:\phi = \mathbf{v}$ or $wSw' = \mathbf{v}$ ($\mathbf{v} \in \mathbf{V}$) avoiding comparisons of values of formulas in different states. We close the branch if it contains $w:i:\psi = \mathbf{v}$ and $w:i:\psi = \mathbf{v}'$ or $wSw' = \mathbf{v}$ and $wSw' = \mathbf{v}'$ for $\mathbf{v} \neq \mathbf{v}'$.

Now we replace the $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ rules with ones that work with labelled structures. Below, we give as an example the rules⁵⁴ that replace $\Diamond_1 \prec$.

$$\frac{w:1:\Diamond\phi = \frac{r}{\#\text{str}} \quad \left(wR^+w'' = 1 \mid wR^+w'' = \frac{r}{\#\text{str}} \mid \dots \mid wR^+w'' = \frac{r}{\#\text{str}} \right)}{w:1:\phi = \frac{r}{\#\text{str}} \mid w:1:\phi = 1 \mid \dots \mid w:1:\phi = \frac{r}{\#\text{str}}} \quad \frac{w:1:\Diamond\phi = \frac{r}{\#\text{str}}; \left(wR^+w' = \mathbf{v} \text{ for } \mathbf{v} \geq \frac{r}{\#\text{str}} \right) \text{ is on the branch}}{w':1:\phi = \frac{r}{\#\text{str}} \mid \dots \mid w':1:\phi = 0}$$

Note that the rules of such form prevent us from comparing values of formulas in different states.

We can now build a satisfying model for ϕ using polynomial space. We begin with $w_0:1:\phi = 1$ (the algorithm for $w_0:1:\phi = 0$ is the same) and start applying propositional rules (first, those that do not require branching). If we implement a branching rule, we pick one branch and work only with it: either until the branch is closed, in which case we pick another one; or until no more rules are applicable (then, the model is constructed); or until we need to apply a modal rule to proceed. At this stage, we need to store only the subformulas of ϕ with labels denoting their value at w_0 .

⁵⁴For a value $\mathbf{v} > 0$ of $\Diamond\phi$ at w , we add a new state that witnesses \mathbf{v} , and for a state on the branch, we guess a value not greater than \mathbf{v} . Other modal rules can be rewritten similarly.

Now we guess a modal formula (say, $w_0 : 1 : \diamond\chi = \frac{1}{\#str}$) whose decomposition requires an introduction of a new state (w_1) and apply this rule. Then we apply all modal rules whose implementation requires that $w_0R^+w_1$ occur on the branch (again, if those require branching, we guess only one branch) and start from the beginning with the propositional rules. If we reach a contradiction, the branch is closed. Again, the only new entries to store are subformulas of ϕ (now, with fewer modalities), their values at w_1 , and a relational term $w_0R^+w_1$ with its value. Since the depth of the model is $O(|\phi|)$ and since we work with modal formulas one by one, we need to store subformulas of ϕ with their values $O(|\phi|)$ times, so, we need only $O(|\phi|^2)$ space.

Finally, if no rule is applicable and there is no contradiction, we mark $w_0 : 2 : \diamond\chi = \frac{1}{\#str}$ as ‘safe’. Now we *delete all entries of the tableau below it* and pick another unmarked modal formula that requires an introduction of a new state. Dealing with them one by one allows us to construct the model branch by branch. But since the length of each branch of the model is bounded by $O(|\phi|)$ and since we delete *branches of the model* once they are shown to contain no contradictions, we need only polynomial space.

To establish PSpace-hardness, we provide a reduction from \mathbf{K} validity. For ϕ over $\{\mathbf{0}, \wedge, \vee, \rightarrow, \square\}$, we construct $\phi^!$ as follows. First, we replace every variable p with $\Delta p \wedge \neg \sim \Delta p$ and then put Δ in front of every \square . It is clear that $|\phi^!| = O(|\phi|)$. Moreover, it is easy to check by induction that $v(\phi^!, w) \in \{(1, 0), (0, 1)\}$ for any valuation v on any frame.

It remains to show that $\mathbf{K} \models \phi$ iff $\mathbf{KG}_{fb}^{2\pm} \models \phi^!$. It is clear that $\mathbf{KG}_{fb}^{2\pm} \not\models \phi^!$ when $\mathbf{K} \not\models \phi$ since classical values are preserved by $\mathcal{L}_{G_{\Delta, \square, \diamond}^{2\pm}}$ connectives. For the converse, let $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ be a $\mathbf{KG}_{fb}^{2\pm}$ model s.t. $v(\phi^!, w) = (0, 1)$ for some $w \in W$. We construct a *classical* model $\mathfrak{M}^! = \langle W, R^!, v^! \rangle$ as follows: $R^! = R^+ \cup R^-$; $w \in v^!(p)$ iff $v(p, w) = (1, 0)$. We check by induction that $v(\phi^!, w) = (1, 0)$ iff $\mathfrak{M}^!, w \models \phi$. The basis case of $\phi^! = \Delta p \wedge \neg \sim \Delta p$ and $\phi = p$ holds by construction of $\mathfrak{M}^!$. The cases of propositional connectives can be proven directly from the induction hypothesis. Finally, if $\phi^! = \Delta \square(\psi^!)$ and $\phi = \square\psi$, we have that $v(\Delta \square(\psi^!), w) = (1, 0)$ iff $v(\psi^!, w') = (1, 0)$ for every $w' \in (R^+ \cup R^-)$, which, by the induction hypothesis, is equivalent to $\mathfrak{M}, w' \models \psi$ for every $w' \in R^!(w)$, and thus $\mathfrak{M}, w \models \square\psi$. \square

7.2. Paraconsistent Gödel logic with informational modalities

We finish Part II with a discussion of $G_{\blacksquare, \blacklozenge}^{2\pm}$. Let us begin with the language and its semantics.

Definition 7.7 (Language and semantics of $G_{\blacksquare, \blacklozenge}^{2\pm}$). The language $\mathcal{L}_{G_{\Delta, \blacksquare, \blacklozenge}^{2\pm}}$ expands $\mathcal{L}_{G_{(\rightarrow, <) }^{2\pm}}$ (recall Definition 4.4) with two modalities: \blacksquare and \blacklozenge .

A $G_{\blacksquare, \blacklozenge}^{2\pm}$ -model is a tuple $\mathfrak{M} = \langle W, R^+, R^-, v_1, v_2 \rangle$ with $\langle W, R^+, R^- \rangle$ being a bi-relational frame (recall Definition 7.1) and $G_{\blacksquare, \blacklozenge}^{2\pm}$ valuations $v_1, v_2 : \text{Prop} \rightarrow [0, 1]$. The semantics of propositional connectives is as in Definition 6.1. The semantics of modalities is given below.

$$\begin{aligned} v_1(\blacksquare\phi, w) &= \inf_{w' \in W} \{wR^+w' \rightarrow_G v_1(\phi, w')\} & v_2(\blacksquare\phi, w) &= \inf_{w' \in W} \{wR^-w' \rightarrow_G v_2(\phi, w')\} \\ v_1(\blacklozenge\phi, w) &= \sup_{w' \in W} \{wR^+w' \wedge_G v_1(\phi, w')\} & v_2(\blacklozenge\phi, w) &= \sup_{w' \in W} \{wR^-w' \wedge_G v_2(\phi, w')\} \end{aligned}$$

We say that $\phi \in \mathcal{L}_{G_{\Delta, \blacksquare, \blacklozenge}^{2\pm}}$ is *v_1 -valid on \mathfrak{F}* ($\mathfrak{F} \models_{G_{\blacksquare, \blacklozenge}^{2\pm}}^+ \phi$) iff for every model \mathfrak{M} on \mathfrak{F} and every $w \in \mathfrak{M}$, it holds that $v_1(\phi, w) = 1$. ϕ is *v_2 -valid on \mathfrak{F}* ($\mathfrak{F} \models_{G_{\blacksquare, \blacklozenge}^{2\pm}}^- \phi$) iff for every model \mathfrak{M} on \mathfrak{F} and every $w \in \mathfrak{M}$, it holds that $v_2(\phi, w) = 0$. ϕ is *strongly valid on \mathfrak{F}* ($\mathfrak{F} \models_{G_{\blacksquare, \blacklozenge}^{2\pm}} \phi$) iff it is v_1 and v_2 -valid.

ϕ is v_1 (resp., v_2 , strongly) $G_{\blacksquare, \blacklozenge}^{2\pm}$ valid iff it is v_1 (resp., v_2 , strongly) valid on every frame. We will further use $G_{\blacksquare, \blacklozenge}^{2\pm}$ to designate the set of all $\mathcal{L}_{G_{\Delta, \blacksquare, \blacklozenge}^{2\pm}}$ formulas *strongly valid* on every frame.

Observe in the definition above that the semantical conditions governing the support of the truth of $G_{\blacksquare, \blacklozenge}^{2\pm}$ connectives (except for \neg) coincide with the semantics of \mathbf{KbiG} (recall Definition 5.2). The following example clarifies the semantics of \blacksquare and \blacklozenge .

7. Modal logics on bi-relational frames

$$f_1 : \begin{array}{l} s = (0.5, 0.5) \\ d = (0.7, 0.3) \end{array} \xleftarrow{(0.8, 0.9)} t \xrightarrow{(0.7, 0.2)} f_2 : \begin{array}{l} s = (1, 0.4) \\ d = (0, 0) \end{array}$$

Figure 7.5.: (x, y) over the arrows stands for $wR^+w' = x, wR^-w' = y$. R^+ (resp., R^-) is interpreted as the tourist's threshold of trust in positive (negative) statements by the friends.

$$w_1 : \begin{array}{l} p = (1, 0) \\ q = (0, 1) \end{array} \xleftarrow{\quad} w_0 \xrightarrow{\quad} w_2 : \begin{array}{l} p = (0, 1) \\ q = (1, 0) \end{array}$$

Figure 7.6.: All arrows are crisp; $R^+ = R^-$.

Example 7.1. A tourist (t) wants to go to a restaurant and asks their two friends (f_1 and f_2) to describe their impressions regarding the politeness of the staff (s) and the quality of the desserts (d). Of course, the friends' opinions are not always internally consistent, nor is it always the case that one or the other even noticed whether the staff was polite or was eating desserts. Furthermore, t trusts their friends to different degrees when it comes to their positive and negative opinions. The situation is depicted in Fig. 7.5.

The first friend says that half of the staff was really nice but the other half is unwelcoming and rude and that the desserts (except for the tiramisu and soufflé) are tasty. The second friend, unfortunately, did not have the desserts at all. Furthermore, even though, they praised the staff, they also said that the manager was quite obnoxious.

The tourist now makes up their mind. If they are sceptical w.r.t. s and d , they look for *trusted rejections*⁵⁵ of both positive and negative supports of s and d . Thus t uses the values of R^+ and R^- as thresholds above which the information provided by the source does not count as a trusted enough rejection. In our case, we have $v(\blacksquare s, t) = (0.5, 0.5)$ and $v(\blacksquare d, t) = (0, 0)$. On the other hand, if t is credulous, they look for *trusted confirmations of both positive and negative supports* and use R^+ and R^- as thresholds up to which they accept the information provided by the source. Thus, we have $v(\blacklozenge s, t) = (0.7, 0.2)$ and $v(\blacklozenge d, t) = (0.7, 0.3)$.

Note as well that just as in \mathbf{KG}^{2c} , \blacksquare and \blacklozenge are not trivialised by contradictions: $\blacklozenge(p \wedge \neg p) \rightarrow \blacklozenge q$ and $\blacksquare(p \wedge \neg p) \rightarrow \blacksquare q$ are not valid.

Recall that [at the beginning of the chapter](#), we mentioned that \blacksquare and \blacklozenge can be interpreted as infinitary generalisations of \sqcap and \sqcup . From here, it is expected that \blacksquare and \blacklozenge are not normal in the following sense.

Proposition 7.1. *None of the following formulas is strongly valid.*

1. $\blacksquare \mathbf{1}, \blacklozenge \mathbf{0} \leftrightarrow \mathbf{0}$.
2. $\blacksquare(p \wedge q) \leftrightarrow (\blacksquare p \wedge \blacksquare q), \blacklozenge(p \vee q) \leftrightarrow (\blacklozenge p \vee \blacklozenge q)$.

Proof. We begin with 1. It is easy to see that if $R^+(w), R^-(w) = \emptyset$, then $v(\blacksquare \mathbf{1}, w) = (1, 1)$ and $v(\blacklozenge \mathbf{0}, w) = (0, 0)$. To prove 2., we provide the following counter-model (cf. Fig. 7.6). It is clear that the following holds.

$$\begin{array}{lll} v(\blacksquare(p \wedge q), w_0) = (0, 1) & v(\blacksquare p, w_0) = (0, 0) & v(\blacksquare q, w_0) = (0, 0) \\ v(\blacklozenge(p \vee q), w_0) = (1, 0) & v(\blacklozenge p, w_0) = (1, 1) & v(\blacklozenge q, w_0) = (1, 1) \end{array} \quad (7.1)$$

⁵⁵We differentiate between a *rejection* which we treat as *lack of support* and a *denial, disproof, refutation, counterexample*, etc. which we interpret as the *negative support*.

$$w_1 : p = \left(\frac{2}{3}, \frac{1}{2}\right) \longleftarrow w_0 : p = (1, 0) \longrightarrow w_2 : p = \left(\frac{1}{3}, \frac{1}{4}\right)$$

Figure 7.7.: In each state, all variables share the same value. We only indicate the value of p .
 $R^+ = R^-$ is crisp, $v(\blacksquare p, w_0) = \left(\frac{1}{3}, \frac{1}{4}\right)$, $v(\blacklozenge p, w_0) = \left(\frac{2}{3}, \frac{1}{2}\right)$.

From (7.1), it is immediate that $v(\blacksquare(p \wedge q) \leftrightarrow (\blacksquare p \wedge \blacksquare q), w_0) \neq (1, 0)$ and $v(\blacklozenge(p \vee q) \leftrightarrow (\blacklozenge p \vee \blacklozenge q), w_0) \neq (1, 0)$. \square

Observe that in both cases, we used crisp mono-relational frames⁵⁶ to refute the normality of \blacksquare and \blacklozenge . Note also that we refuted the formulas by providing suitable models with *positive support of falsity* while it would be, of course, impossible to provide models where their support of truth is less than 1. On the other hand, both modalities are *regular*.

Proposition 7.2. *Let $\phi \rightarrow \phi'$ and $\chi \rightarrow \chi'$ be strongly valid. Then $\blacksquare\phi \rightarrow \blacksquare\phi'$ and $\blacklozenge\chi \rightarrow \blacklozenge\chi'$ are strongly valid too.*

Proof. We prove only the \blacksquare case. Let $\blacksquare\phi \rightarrow \blacksquare\phi'$ be *not strongly valid* in some frame \mathfrak{F} . Then, there is a $w \in \mathfrak{F}$ as well as v_1 and v_2 thereover s.t. $v(\blacksquare\phi \rightarrow \blacksquare\phi', w) \neq (1, 0)$. Since v_1 conditions of \blacksquare coincide with the \mathbf{KbiG}^f semantics of \square (and since \square is obviously regular in \mathbf{KbiG}^f), it suffices to check the case when $v_2(\blacksquare\phi \rightarrow \blacksquare\phi', w) > 0$.

We have that

$$\begin{aligned} v_2(\blacksquare\phi \rightarrow \blacksquare\phi', w) > 0 &\text{ iff } v_2(\blacksquare\phi, w) < v_2(\blacksquare\phi', w) \\ &\text{ iff } \inf_{w' \in W} \{wR^-w' \rightarrow_{\mathbf{G}} v_2(\phi)\} < \inf_{w' \in W} \{wR^-w' \rightarrow_{\mathbf{G}} v_2(\phi')\} \\ &\text{ then } \exists w' \in R^-(w) : v_2(\phi, w') < v_2(\phi', w') \\ &\text{ then } v_2(\phi \rightarrow \phi', w') > 0 \end{aligned}$$

The regularity of \blacklozenge can be tackled similarly. \square

Before proceeding to discuss the frame definability, we establish the expected result that \blacklozenge and \blacksquare are *not interdefinable*.

Theorem 7.9. *\blacksquare and \blacklozenge are not interdefinable.*

Proof. Denote with $\mathcal{L}_{\blacksquare}$ and $\mathcal{L}_{\blacklozenge}$ the \blacklozenge - and \blacksquare -free fragments of $\mathcal{L}_{\mathbf{G}_{\Delta}^2, \blacksquare, \blacklozenge}$. We build a pointed model $\langle \mathfrak{M}, w \rangle$ s.t. there is no \blacklozenge -free formula that has the same value at w as $\blacksquare p$ (and vice versa). Consider Fig. 7.7.

One can check by induction that if $\phi \in \mathcal{L}_{\mathbf{G}_{\Delta}^2, \blacksquare, \blacklozenge}$, then

$$\begin{aligned} v(\phi, w_1) &\in \left\{ (0; 1), \left(\frac{1}{2}; \frac{2}{3}\right), \left(\frac{2}{3}; \frac{1}{2}\right), (0; 0), (1; 1), (1; 0) \right\} \\ v(\phi, w_2) &\in \left\{ (0; 1), \left(\frac{1}{4}; \frac{1}{3}\right), \left(\frac{1}{3}; \frac{1}{4}\right), (0; 0), (1; 1), (1; 0) \right\} \end{aligned}$$

Moreover, on the single-point irreflexive frame whose only state is u , it holds for every $\phi(p) \in \mathcal{L}_{\mathbf{G}_{\Delta}^2, \blacksquare, \blacklozenge}$, $v(\phi, u) \in \{v(p, u), v(\neg p, u), (1, 0), (1, 1), (0, 0), (0, 1)\}$.

Thus, for every \blacklozenge -free χ and every \blacksquare -free ψ it holds that

$$v(\blacksquare\chi, w_0) \in \left\{ (0; 1), \left(\frac{1}{3}; \frac{1}{4}\right), \left(\frac{1}{4}; \frac{1}{3}\right), (0; 0), (1; 1), (1; 0) \right\} = X$$

⁵⁶In fact, we even used BD values, not the fuzzy ones.

7. Modal logics on bi-relational frames

$$v(\diamond\psi, w_0) \in \left\{ (0; 1), \left(\frac{1}{2}; \frac{2}{3}\right), \left(\frac{2}{3}; \frac{1}{2}\right), (0; 0), (1; 1), (1; 0) \right\} = Y$$

Since X and Y are closed w.r.t. propositional operations, it is now easy to check by induction that for every $\chi' \in \mathcal{L}_{\blacksquare}$ and $\psi' \in \mathcal{L}_{\blacklozenge}$, $v(\chi', w_0) \in X$ and $v(\psi', w_0) \in Y$. \square

7.2.1. Frame definability

In this section, we explore some classes of frames that can be defined in $\mathcal{L}_{\mathbb{G}_{\Delta}^2, \blacksquare, \blacklozenge}$. However, since \blacksquare and \blacklozenge are non-normal and since we have two independent relations on frames, we expand the traditional notion of modal definability. Moreover, in contrast to a traditional approach to non-normal modal logics and their Kripke semantics (cf., e.g. [88]), we do not postulate ‘non-normal worlds’⁵⁷ in our frames. Thus, we have definability w.r.t. each kind of validity outlined in Definition 7.7.

Definition 7.8.

1. ϕ *positively defines* a class of frames \mathbb{F} iff for every \mathfrak{F} , it holds that $\mathfrak{F} \models^+ \phi$ iff $\mathfrak{F} \in \mathbb{F}$.
2. ϕ *negatively defines* a class of frames \mathbb{F} iff for every \mathfrak{F} , it holds that $\mathfrak{F} \models^- \phi$ iff $\mathfrak{F} \in \mathbb{F}$.
3. ϕ (*strongly*) *defines* a class of frames \mathbb{F} iff for every \mathfrak{F} , it holds that $\mathfrak{F} \in \mathbb{F}$ iff $\mathfrak{F} \models \phi$.

With the help of the above definition, we can show that every class of frames definable in \mathbf{KbiG} is *positively definable* in $\mathbb{G}_{\blacksquare, \blacklozenge}^{2\pm}$.

Definition 7.9. Let $\mathfrak{F} = \langle W, S \rangle$ be a (fuzzy or crisp) frame.

1. An R^+ -*counterpart* of \mathfrak{F} is any bi-relational frame $\mathfrak{F}^+ = \langle W, S, R^- \rangle$.
2. An R^- -*counterpart* of \mathfrak{F} is any bi-relational frame $\mathfrak{F}^- = \langle W, R^+, S \rangle$.

Convention 7.2. Let $\phi \in \mathcal{L}_{\mathbb{G}_{\Delta, \square, \diamond}}$.

1. We denote with $\phi^{+\bullet}$ the formula obtained from ϕ by replacing all \square 's and \diamond 's with \blacksquare 's and \blacklozenge 's.
2. We denote with $\phi^{-\bullet}$ the formula obtained from ϕ by replacing all \square 's and \diamond 's with $\neg\blacksquare\neg$'s and $\neg\blacklozenge\neg$'s.

Theorem 7.10. Let $\mathfrak{F} = \langle W, S \rangle$ and let \mathfrak{F}^+ and \mathfrak{F}^- be its R^+ and R^- counterparts. Then, for any $\phi \in \mathcal{L}_{\mathbb{G}_{\Delta, \square, \diamond}}$, it holds that

$$\mathfrak{F} \models_{\mathbf{KbiG}} \phi \quad \text{iff} \quad \mathfrak{F}^+ \models_{\mathbb{G}_{\blacksquare, \blacklozenge}^{2\pm}}^+ \phi^{+\bullet} \quad \text{iff} \quad \mathfrak{F}^- \models_{\mathbb{G}_{\blacksquare, \blacklozenge}^{2\pm}}^+ \phi^{-\bullet}$$

Proof. Since the semantics of \mathbf{KbiG} connectives is identical to v_1 conditions of Definition 7.7, we only prove that $\mathfrak{F} \models \phi$ iff $\mathfrak{F}^- \models^+ \phi^{-\bullet}$. It suffices to prove by induction the following statement.

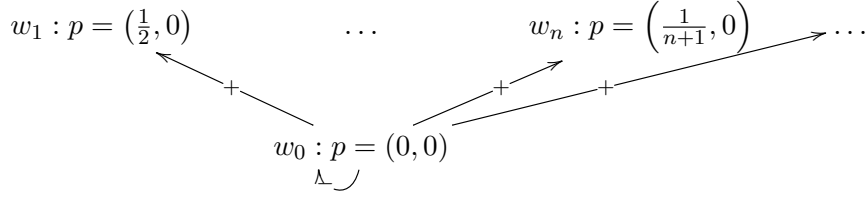
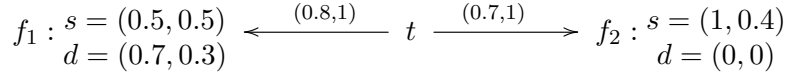
Let \mathbf{v} be a \mathbf{KbiG} valuation on \mathfrak{F} , $\mathbf{v}(p, w) = v_1(p, w)$ for every $w \in \mathfrak{F}$, and v_2 be arbitrary. Then

$$\mathbf{v}(\phi, w) = v_1(\phi^{-\bullet}, w) \text{ for every } \phi.$$

The case of $\phi = p$ holds by Convention 7.2, the cases of propositional connectives are straightforward. Consider $\phi = \square\chi$. We have that $\phi^{-\bullet} = \neg\blacksquare\neg(\chi^{-\bullet})$ and thus

$$\begin{aligned} v_1(\neg\blacksquare\neg(\chi^{-\bullet}), w) &= v_2(\blacksquare\neg(\chi^{-\bullet}), w) \\ &= \inf_{w' \in W} \{wSw' \rightarrow_{\mathbb{G}} v_2(\neg(\chi^{-\bullet}))\} \end{aligned}$$

⁵⁷Given a frame $\mathfrak{F} = \langle W, R \rangle$, $n \in W$ is *non-normal* iff $n \models \diamond\phi$ and $n \not\models \square\phi$ for every ϕ . The validity and entailment are then defined w.r.t. *normal* worlds.


 Figure 7.8.: R^+ and R^- are crisp, $v(\sim\blacksquare(p \vee \sim p), w_0) = (1, 0)$.

 Figure 7.9.: s stands for ‘staff is polite’; d for ‘desserts are good’.

$$\begin{aligned}
 &= \inf_{w' \in W} \{wSw' \rightarrow_{\mathbf{G}} v_1(\chi^{-\bullet})\} \\
 &= \inf_{w' \in W} \{wSw' \rightarrow_{\mathbf{G}} \mathbf{v}(\chi)\} && \text{(by IH)} \\
 &= \mathbf{v}(\Box\chi, w)
 \end{aligned}$$

□

The above theorem allows us to *positively* define in $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ all classes of frames that are definable in \mathbf{KbiG} . In particular, all \mathbf{K} -definable frames are positively definable. Moreover, it follows that $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ (as \mathbf{KG} and \mathbf{KbiG}) lacks the finite model property: $\sim\Box(p \vee \sim p)$ is false on every finite frame, and thus, $\sim\blacksquare(p \vee \sim p)$ is too. On the other hand, there are infinite models satisfying $\sim\blacksquare(p \vee \sim p)$ as Fig. 7.8.

Furthermore, Theorem 7.10 gives us a degree of flexibility. For example, one can check that $\neg\blacksquare\neg(p \vee q) \rightarrow (\neg\blacksquare\neg p \vee \neg\blacklozenge\neg q)$ positively defines frames with crisp R^- but not necessarily crisp R^+ . This models a situation when an agent *completely (dis)believes* in denials given by their sources while may have some degree of trust between 0 and 1 when the sources assert something. Let us return to Example 7.1.

Example 7.2. Assume that the tourist *completely trusts* the negative (but not positive) opinions of their friends. Thus, instead of Fig. 7.5, we have the model on Fig. 7.9.

The new values for the cautious and credulous aggregation are as follows: $v(\blacksquare s, t) = (0.5, 0.4)$, $v(\blacksquare d, t) = (0, 0)$, $v(\blacklozenge s, t) = (0.7, 0.5)$, and $v(\blacklozenge d, t) = (0.7, 0.3)$.

Furthermore, the agent can trust the sources to the same degree no matter whether they confirm or deny statements. This can be modelled with *mono-relational* frames where $R^+ = R^-$. We show that they are *strongly definable*.

Theorem 7.11. \mathfrak{F} is *mono-relational* iff $\mathfrak{F} \models \blacksquare\neg p \leftrightarrow \neg\blacksquare p$ and $\mathfrak{F} \models \blacklozenge\neg p \leftrightarrow \neg\blacklozenge p$.

Proof. Let \mathfrak{F} be mono-relational and $R^+ = R^- = R$. Now observe that

$$\begin{aligned}
 v_i(\blacksquare\neg p, w) &= \inf_{w' \in W} \{wRw' \rightarrow_{\mathbf{G}} v_i(\neg p, w')\} && (i \in \{1, 2\}) \\
 &= \inf_{w' \in W} \{wRw' \rightarrow_{\mathbf{G}} v_j(p, w')\} && (i \neq j) \\
 &= v_j(\blacksquare p, w) \\
 &= v_i(\neg\blacksquare p, w)
 \end{aligned}$$

$$\begin{array}{c}
 \blacksquare_i \gtrsim \frac{w:i:\blacksquare\phi \gtrsim \mathfrak{X}}{w':i:\phi \gtrsim \mathfrak{X} \mid wSw' \leq w':i:\phi} \quad \blacksquare_i \leq \frac{w:i:\blacksquare\phi \leq \mathfrak{X}}{\mathfrak{X} \geq 1 \mid wSw'' > w'':i:\phi} \quad \blacksquare_i < \frac{w:i:\blacksquare\phi < \mathfrak{X}}{wSw'' > w'':i:\phi} \\
 \frac{1 \gtrsim \mathfrak{X}}{\phantom{w':i:\phi \gtrsim \mathfrak{X}}} \quad \frac{w'':i:\phi \leq \mathfrak{X}}{\phantom{w'':i:\phi < \mathfrak{X}}} \\
 \\
 \blacklozenge_i \gtrsim \frac{w:i:\blacklozenge\phi \gtrsim \mathfrak{X}}{wSw'' \gtrsim \mathfrak{X}} \quad \blacklozenge_i \lesssim \frac{w:i:\blacklozenge\phi \lesssim \mathfrak{X}}{w':i:\phi \lesssim \mathfrak{X} \mid wSw' \lesssim \mathfrak{X}} \quad \left[\begin{array}{l} w'' \text{ is fresh on the branch} \\ \text{if } i=1, \text{ then } S=R^+ \\ \text{if } i=2, \text{ then } S=R^- \\ \text{in } \blacksquare_i \gtrsim, \blacklozenge_i \lesssim \text{ } wSw' \text{ is on the branch} \end{array} \right] \\
 \frac{w'':i:\phi \gtrsim \mathfrak{X}}{\phantom{wSw'' \gtrsim \mathfrak{X}}}
 \end{array}$$

 Figure 7.10.: Modal rules of $\mathcal{T}(\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm})$; bars denote branching, $i, j \in \{1, 2\}$, $i \neq j$.

For the converse, let $R^+ \neq R^-$ and, in particular, $wR^+w' = x$ and $wR^-w' = y$. Assume w.l.o.g. that $x > y$. We set the valuation of p : $v(p, w') = (x, y)$ and for every $w'' \neq w'$, we have $v(p, w'') = (1, 1)$. It is clear that $v(\neg\blacksquare p, w) = (1, 1)$. On the other hand, $v(\neg p, w') = (y, x)$, whence $v_1(\blacksquare\neg p) \neq 1$.

The case of \blacklozenge can be tackled in a dual manner. \square

In the remainder of the section, we will be concerned with $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ — $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ over finitely branching (both fuzzy and crisp) frames. Recall that since both fuzzy and crisp finitely branching frames can be defined in \mathbf{KG} with $\sim\sim\Box(p \vee \sim p)$ (cf. Proposition 5.13). Thus, by Theorem 7.10, frames with finitely R^+ are positively definable via $\sim\sim\blacksquare(p \vee \sim p)$ and those with finitely branching R^- via $\sim\sim\neg\blacksquare\neg(p \vee \sim p)$.

7.2.2. Constraint tableaux

Let us now construct a sound and complete constraint tableaux system $\mathcal{T}(\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm})$ for $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$. The calculus builds upon $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2c})$ in the same manner as $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ (cf. Definition 7.5). The only difference between the two is the modal rules.

Definition 7.10 ($\mathcal{T}(\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm})$ — tableaux rules for $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$). A *constraint tableau* is a downward branching tree whose branches are sets containing constraints $\mathfrak{X} \lesssim \mathfrak{X}'$ ($\mathfrak{X}, \mathfrak{X}' \in \text{Str}$). Each branch can be extended by an application of a rule⁵⁸ from Fig. 4.1 or Fig. 7.10. The notions of closed, open, and complete branches are the same as in Definition 6.3. We say that there is a $\mathcal{T}(\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm})$ *proof* of ϕ iff there are closed tableaux starting from $w:1:\phi < 1$ and $w:2:\phi > 0$.

The notion of branch realisation (Definition 7.6), as well as the interpretations of tableaux entries (Table 7.1), are the same as in $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$.

Let us now give an example of a failed tableau proof and extracted counter-model (Fig. 7.11). The proof goes as follows: first, we apply all the possible propositional rules, then the modal rules that introduce new states, and then those that use the states already on the branch. We repeat the process until all structures are decomposed into atomic ones. We can now extract a model from the complete open branch marked with \ominus s.t. $v_2(\neg\blacksquare p \rightarrow \blacksquare\neg p, w_0) > 0$. We use w 's that occur thereon as the carrier and assign the values of variables and relations so that they correspond to \lesssim .

Completeness can be proved in the same manner as Theorem 7.7.

Theorem 7.12 ($\mathcal{T}(\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm})$ completeness). ϕ is v_1 -valid (v_2 -valid) in $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ iff there is a closed tableau beginning with $w:1:\phi < 1$ ($w:2:\phi > 0$).

⁵⁸If $\mathfrak{X} < 1, \mathfrak{X} < \mathfrak{X}' \in \mathcal{B}$ or $0 < \mathfrak{X}', \mathfrak{X} < \mathfrak{X}' \in \mathcal{B}$, the rules are applied only to $\mathfrak{X} < \mathfrak{X}'$.

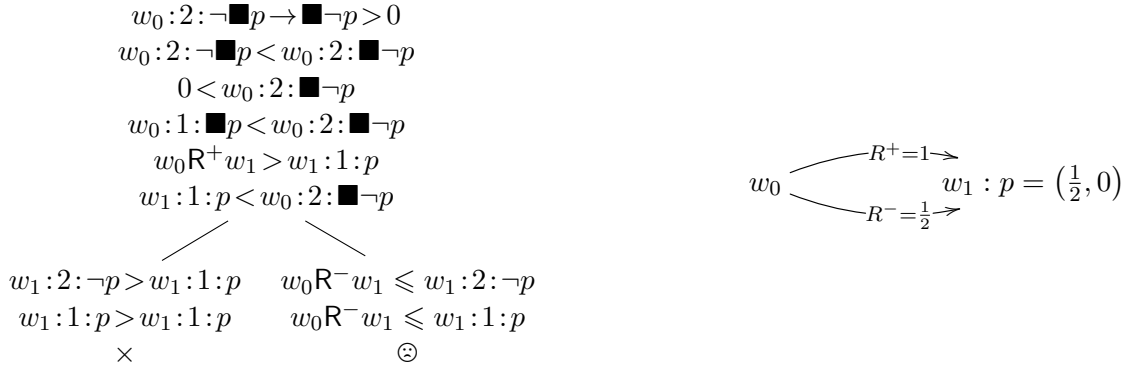


Figure 7.11.: A failed tableau proof and a countermodel. The complete open branch is marked with \odot .

The following statement follows immediately from Theorem 7.12.

Corollary 7.5. *Let $\phi \in \mathcal{L}_{\mathbb{G}_{\Delta}^2, \blacksquare, \blacklozenge}$ be not $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ valid, and let k be the number of modalities in it. Then there is a model \mathfrak{M} of the size $\leq k^{k+1}$ and depth $\leq k$ and $w \in \mathfrak{M}$ s.t. $v_1(\phi, w) \neq 1$ or $v_2(\phi, w) \neq 0$.*

We finish the chapter by establishing the PSpace-completeness of $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$. To tackle the PSpace-hardness of *strong validity*, we introduce an additional constant \mathbf{B} to $\mathcal{L}_{\mathbb{G}_{\Delta}^2, \blacksquare, \blacklozenge}$. The semantics is as expected: $v(\mathbf{B}, w) = (1, 1)$. Note that the dual constant \mathbf{N} s.t. $v(\mathbf{N}, w) = (0, 0)$ is definable via \mathbf{B} as $\mathbf{N} := \sim \mathbf{B}$.

Let us now use $\mathbb{G}_{\blacksquare, \blacklozenge}^{2\pm}(\mathbf{B})$ to denote the expansion of $\mathbb{G}_{\blacksquare, \blacklozenge}^{2\pm}$ with \mathbf{B} . The following statement is immediate.

Proposition 7.3.

1. $\phi \in \mathcal{L}_{\mathbb{G}_{\Delta}^2, \blacksquare, \blacklozenge}$ is v_1 -valid on \mathfrak{F} iff $\mathbf{B} \rightarrow \phi$ is strongly valid on \mathfrak{F} .
2. $\phi \in \mathcal{L}_{\mathbb{G}_{\Delta}^2, \blacksquare, \blacklozenge}$ is v_2 -valid on \mathfrak{F} iff $\mathbf{N} \rightarrow \phi$ is strongly valid on \mathfrak{F} .

It is also clear that adding the following rules to the tableaux calculus in Definition 7.10 will make it complete w.r.t. $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}(\mathbf{B})$.

$$\frac{w : i : \mathbf{B} \lesssim \mathfrak{X}}{w : i : 1 \lesssim \mathfrak{X}} \qquad \frac{w : i : \mathbf{B} \gtrsim \mathfrak{X}}{w : i : 1 \gtrsim \mathfrak{X}} \qquad (i \in \{1, 2\})$$

Theorem 7.13.

1. $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}(\mathbf{B})$ validity and satisfiability are PSpace complete.
2. v_1 - and v_2 -validities in $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ are PSpace-complete

Proof. The proof of the membership part is the same as in Theorem 7.8. For hardness, we reduce the \mathbf{KG} validity of $\{\mathbf{0}, \wedge, \vee, \rightarrow, \diamond\}$ formulas to strong validity as well as to v_1 - and v_2 -validities. Recall that the \diamond fragment of \mathbf{KG} has the finite model property [39, Theorem 7.1] and is PSpace-complete [105, Theorem 5.9].

Since the semantics of \mathbf{KG} is the same as \mathbf{KbiG} (cf. Definition 5.2) and coincides with the v_1 -conditions of $\mathbb{G}_{\blacksquare, \blacklozenge}^{2\pm}$ (recall Definition 7.7), it is immediate by Theorem 7.10 that ϕ over $\{\mathbf{0}, \wedge, \vee, \rightarrow, \diamond\}$ is \mathbf{KG} -valid iff $\phi^{+\bullet}$ is v_1 -valid. This also gives us the reduction to $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}(\mathbf{B})$ strong validity using Proposition 7.3: ϕ is \mathbf{KG} -valid iff $\mathbf{B} \rightarrow \phi^{+\bullet}$ is strongly $\mathbb{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ -valid.

7. Modal logics on bi-relational frames

For v_2 -validity, we proceed as follows. We let ϕ be over $\{\mathbf{0}, \wedge, \vee, \rightarrow, \diamond\}$ and inductively define ϕ^∂ :

$$\begin{aligned}
 p^\partial &= p \\
 (\chi \circ \psi)^\partial &= \chi^\partial \bullet \psi^\partial && (\circ, \bullet \in \{\wedge, \vee\}, \circ \neq \bullet) \\
 (\chi \rightarrow \psi)^\partial &= \psi^\partial \prec \chi^\partial \\
 (\diamond \chi)^\partial &= \blacklozenge(\chi^\partial)
 \end{aligned}$$

It is clear that ϕ is **KG**-valid on a given frame $\mathfrak{F} = \langle W, S \rangle$ iff $v_2(\phi^\partial, w) = 1$ for every valuation v_2 and every state w in the R^- -counterpart \mathfrak{F}^- of \mathfrak{F} . Hence, ϕ is **KG**-valid iff $\mathbf{1} \prec \phi^\partial$ is v_2 -valid. \square

Part III.

Two-layered modal logics

8. Logics for quantitative reasoning

In Part III, we are going to consider two-layered logics that formalise classical and paraconsistent reasoning about uncertainty measures. Syntactically, this is reflected in them using CPL or, respectively, BD to describe events. Semantically, however, all two-layered logics that we consider are interpreted over the models of the form $\mathcal{M} = \langle W, v, \mu, e \rangle$ where $W \neq \emptyset$ is a set of states, v is the *inner valuation* that determines which *inner-language formulas* are true at which state, μ is some *uncertainty measure on W* , and e is the *outer valuation* determining the *truth degree of the outer-language formulas*.

Convention 8.1. We will use two kinds of formulas: the inner- and the outer-layer ones (or just inner and outer formulas). To make the differentiation between them simpler, we use Greek letters from the end of the alphabet (ϕ, χ, ψ , etc.) to designate the first kind and the letters from the beginning of the alphabet (α, β, γ , ...) for the second kind. Furthermore, we use v (with indices) to stand for the valuations of inner-layer formulas and e (with indices) for the outer-layer formulas. When axiomatising our logics, we will use the terms ‘probabilistic axioms’, ‘uncertainty axioms’ etc. or ‘modal axioms’ to stand for the axioms that formalise the behaviour of the uncertainty measure.

In this chapter, we focus on logics that assume that an agent can precisely determine their certainty in a given event. This implies that they can conduct some basic arithmetic operations with it. Moreover, we will deal with paraconsistent counterparts of probability measures that satisfy weaker forms of the additivity condition. Therefore, we will use expansions of \mathfrak{L} on the outer layer.

8.1. Paraconsistent theories of uncertainty

Before proceeding any further, let us define and discuss the most important notion of these two chapters, namely, *uncertainty measure*.

Definition 8.1 (Measure on a set). For a $W \neq \emptyset$, an *uncertainty measure on W* is a monotone w.r.t. \subseteq map $\mu : 2^W \rightarrow [0, 1]$ s.t. $\mu(W) > \mu(\emptyset)$.

Remark 8.1 (Capacities and their generalisations). Usually [75, 156], the most general measure on a set is taken to be a *capacity*, i.e., a measure s.t. $\mu(W) = 1$ and $\mu(\emptyset) = 0$. This condition is usually called *normalisation*.⁵⁹ The difference between normalised and non-normalised measures is related to the difference between closed and open-world assumptions and to the difference between normal and non-normal modal logics. In other words, an agent may not necessarily believe that they have access to the whole sample space. Thus, even though, a tautology ϕ is true in all states accessible by an agent, they are still not convinced therein to assign 1 as ϕ ’s degree of certainty.

Historically, there have been several approaches to the paraconsistent theories of probability and uncertainty. In [17], the reasoning with possibility and necessity functions is formalised using da Costa’s logic C_1 from [45]. In [35, 128], a probability theory based on the logic of formal inconsistency (LFI) which is an expansion of BD with an implication \rightarrow and a *consistency operator* \circ is developed.

⁵⁹ $\mu(W) > \mu(\emptyset)$ is usually called ‘non-triviality’.

To the best of our knowledge, the earliest formalisation of probability theory in terms of BD was provided in [103]. Another formalisation is given in [58]. Related results [59, 67] present a formalisation of paraconsistent reasoning in $\mathfrak{L}^{\leq 60}$. We, however, will use the probability axioms as they were given in [92]. This is for several reasons.

First, the conditional statements do not correspond to event descriptions (and thus, the presence of an implication not reducible to \neg , \wedge , and \vee is not required), whence it suffices to use BD for this purpose. This is why, LFI is too expressive for our purposes. Moreover, the law of excluded middle is valid in C_1 which means that we cannot reason about incomplete information. On the other hand, BD is *paradefinite*.

Second, the probability of $\phi \wedge \chi$ in [58] can be computed directly from the probabilities of ϕ and χ . Furthermore, the probability is interpreted not as a ‘real probability’ of an event but rather as an agent’s degree of certainty in the event. While this is a reasonable assumption in the classical case, one can argue (cf. [53] for further details) that if the available information is contradictory or incomplete, the agent’s certainty is, in fact, not compositional.

Third, \mathfrak{L}^{\leq} and $RPL^{\leq 61}$, unfortunately, cannot represent the contexts where the agent has *no information at all* or *only contradictory information* about p . Indeed (recall Remark 3.4), even though \mathfrak{L}^{\leq} and RPL^{\leq} are not *explosive* w.r.t. \sim negation, they are *safe* (i.e., $p \wedge \sim p \rightarrow q \vee \sim q$ is valid), and, moreover, the value of $p \wedge \sim p$ can never be 1 (i.e., contradictions can never be true) while the value of $q \vee \sim q$ is never 0 (the instances of the LEM are never false). This means that the usual Łukasiewicz negation is not well suited to model contradictory or incomplete information, and hence, we need to introduce the Belnapian negation \neg that allows $p \wedge \neg p$ to be true and $q \vee \neg q$ to be false.

Finally, the probabilistic axioms in [103] and [92] are very close, with their sole distinction being that Mares postulates (axiom Pr 1) that the probability of the whole sample set is equal to 1 (and, accordingly, the probability of the empty set is 0). Note, however, that there are no BD-valid formulas, nor the formulas that are always false. Thus, Pr 1 does not have an immediate analogue in the language of BD. This is why, we assume the probability measures defined in terms of BD to be *non-normalised* by default. This is also related to the idea first proposed in [141, 142] where the positive mass of the empty set⁶² was used to account for the contradictory evidence.

In this chapter, we present two logics, that formalise two equivalent approaches to the probability in Belnap–Dunn framework outlined in [92]. The first one, $\text{Pr}_{\Delta}^{\mathfrak{L}^2} = \langle \text{BD}, \{\text{Pr}\}, \mathfrak{L}_{(\Delta, \rightarrow)}^2 \rangle$, formalises the reasoning with the \pm -probabilities⁶³ that give each $\phi \in \mathcal{L}_{\text{BD}}$ two independent assignments: one for ϕ itself and the other for its negation. The second one — $4\text{Pr}^{\mathfrak{L}^{\Delta}} = \langle \text{BD}, \{\text{Bl}, \text{Db}, \text{Cf}, \text{Uc}\}, \mathfrak{L}_{\Delta} \rangle$ — axiomatises the 4-probabilities⁶⁴: here, every $\phi \in \mathcal{L}_{\text{BD}}$ has four assignments that correspond to the Belnapian values. Our goal is to obtain their Hilbert-style axiomatisations, construct embeddings of one into the other, and establish complexity evaluations.

8.2. Logic of \pm -probabilities

We begin with the notion of \pm -probabilities on a BD model (recall Definition 2.2). We adapt the definition from [92].

⁶⁰Łukasiewicz where the entailment is defined via the preservation of order on $[0, 1]$.

⁶¹Rational Pavelka Logic — the expansion of \mathfrak{L} with constants corresponding to rational numbers between 0 and 1 — where the entailment is defined as the order on $[0, 1]$.

⁶²Mass function on W is a map $\mathfrak{m} : 2^W \rightarrow [0, 1]$ s.t. $\sum_{X \subseteq W} \mathfrak{m}(X) = 1$.

⁶³Originally, they were called ‘non-standard probabilities’. This is, however, too broad of a term since none of the probability theories described in the previous paragraphs is ‘standard’.

⁶⁴Originally, ‘four-valued probabilities’.

$$w_1 : p^+ \quad w_2 : p^\pm \quad w'_1 : p^+ \quad w'_2 : p^\pm \quad w'_3 : \text{?}$$

Figure 8.1.: The values of all variables coincide with the values of p state-wise. $\mu(X) = \frac{1}{2}$ for every $X \subseteq W$; $\mu'(\emptyset) = \mu'(\{w'_1\}) = 0$, $\mu'(W') = 1$, $\mu'(X') = \frac{1}{2}$ otherwise.

Definition 8.2 (BD models with \pm -probabilities). A BD model with a \pm -probability is a tuple $\mathfrak{M}_\mu = \langle \mathfrak{M}, \mu \rangle$ with \mathfrak{M} being a BD model and $\mu : 2^W \rightarrow [0, 1]$ satisfying:

\pm mon: if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$ (monotonicity);

\pm neg: $\mu(|\phi|^-) = \mu(|\neg\phi|^+)$ (negative extension);

\pm ex: $\mu(|\phi \vee \chi|^+) = \mu(|\phi|^+) + \mu(|\chi|^+) - \mu(|\phi \wedge \chi|^+)$ (import-export).

Remark 8.2. It is easy to see that a \pm -probability *does not have to be a measure*. Indeed, for every $c \in [0, 1]$ and every BD model $\langle W, v^+, v^- \rangle$, it is easy to check that the uniform assignment $\forall X \subseteq W : \mu(X) = c$ is a \pm -probability.

Note, furthermore, that the general import-export condition

$$\mu(X \cup Y) = \mu(X) + \mu(Y) - \mu(X \cap Y) \quad (\text{IE})$$

does not hold in general either. For consider $W = \{u_1, u_2\}$ with $\mu(\{u_1\}) = \mu(\{u_2\}) = \frac{1}{3}$, $\mu(W) = 1$, and $\mu(\emptyset) = 0$ and let, further, $v^+(p) = v^-(p) = \emptyset$ for all $p \in \text{Prop}$. One can see that μ is monotone and that $\mu(|\phi|^+) = \mu(|\phi|^-) = 0$ for every $\phi \in \mathcal{L}_{\text{BD}}$ (whence, μ is a \pm -probability). On the other hand, it is clear that $\mu(W) \neq \mu(\{u_1\}) + \mu(\{u_2\}) - \mu(\emptyset)$.

This, however, is not a problem since for every BD model with a \pm -probability $\langle W, v^+, v^-, \mu \rangle$, there exists a BD model $\langle W', v'^+, v'^-, \pi \rangle$ with a *classical* probability measure π s.t. $\pi(|\phi|^+) = \mu(|\phi|^+)$ [92, Theorem 4]. Namely, consider Fig. 8.1 and set $W = \{w_1, w_2\}$ and $W' = \{w'_1, w'_2, w'_3\}$. It is clear that for every $\phi \in \mathcal{L}_{\text{BD}}$, $\mu(|\phi|^+) = \pi(|\phi|^+)$.

Thus, we will further assume w.l.o.g. that μ is a *classical probability measure* on W .

Remark 8.3. Note that both \pm - and 4-probabilities (cf. Definition 8.5) were defined in [92] over (finite) Lindenbaum algebras of BD. In this text, we opt for their presentation as maps defined on powersets since it is a more convenient setting for the semantics of two-layered logics. These two approaches are equivalent by [92, Theorems 4–5].

Our next step is to expand BD models with \pm -probabilities so that they become a model for their two-layered logic $\text{Pr}_\Delta^{\mathbf{k}^2}$.

Definition 8.3 ($\text{Pr}_\Delta^{\mathbf{k}^2}$: language and semantics). The language of $\text{Pr}_\Delta^{\mathbf{k}^2}$ is given by the following grammar

$$\mathcal{L}_{\text{Pr}_\Delta^{\mathbf{k}^2}} \ni \alpha := \text{Pr}\phi \mid \sim\alpha \mid \neg\alpha \mid \Delta\alpha \mid (\alpha \rightarrow \alpha) \quad (\phi \in \mathcal{L}_{\text{BD}})$$

A $\text{Pr}_\Delta^{\mathbf{k}^2}$ -model is a tuple $\mathbb{M} = \langle \mathfrak{M}, \mu, e_1, e_2 \rangle$ with $\langle \mathfrak{M}, \mu \rangle$ being a BD model with \pm -probability and $e_1, e_2 : \mathcal{L}_{\text{Pr}_\Delta^{\mathbf{k}^2}} \rightarrow [0, 1]$ s.t. $e_1(\text{Pr}\phi) = \mu(|\phi|^+)$, $e_2(\text{Pr}\phi) = \mu(|\phi|^-)$, and the values of complex formulas being computed following Definition 3.5.⁶⁵ We say that α is $\text{Pr}_\Delta^{\mathbf{k}^2}$ -*valid* iff $e(\alpha) = (1, 0)$ in every model; Ξ *entails* α in $\text{Pr}_\Delta^{\mathbf{k}^2}$ ($\Xi \models_{\text{Pr}_\Delta^{\mathbf{k}^2}} \alpha$) iff there is no model s.t. $e(\xi) = (1, 0)$ for every $\xi \in \Xi$ (i.e., $e[\Xi] = (1, 0)$ and $e(\alpha) \neq (1, 0)$).

⁶⁵We will say that e_1 and e_2 are *induced* by μ .

8. Logics for quantitative reasoning

To axiomatise $\text{Pr}_{\Delta}^{\mathbf{k}^2}$, we now need to translate the conditions in Definition 8.2 into $\mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$ formulas. We dub the calculus $\mathcal{HPr}_{\Delta}^{\mathbf{k}^2}$.

Definition 8.4 ($\mathcal{HPr}_{\Delta}^{\mathbf{k}^2}$ — a Hilbert-style calculus for $\text{Pr}_{\Delta}^{\mathbf{k}^2}$). The calculus has the following axioms and rules.

$\mathbf{k}_{(\Delta, \rightarrow)}^2$: $\mathbf{k}_{(\Delta, \rightarrow)}^2$ -valid formulas and $\mathcal{H}\mathbf{k}_{(\Delta, \rightarrow)}^2$ rules instantiated in $\mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$;

$\pm\text{mon}$: $\text{Pr}\phi \rightarrow \text{Pr}\chi$ with $\phi \models_{\text{BD}} \chi$;

$\pm\text{neg}$: $\text{Pr}\neg\phi \leftrightarrow \neg\text{Pr}\phi$;

$\pm\text{ex}$: $\text{Pr}(\phi \vee \chi) \leftrightarrow ((\text{Pr}\phi \ominus \text{Pr}(\phi \wedge \chi)) \oplus \text{Pr}\chi)$.

Remark 8.4. Note that the direct translation of the import-export axiom $\text{Pr}(\phi \vee \chi) \leftrightarrow (\text{Pr}\phi \oplus \text{Pr}\chi) \ominus \text{Pr}(\phi \wedge \chi)$ is not valid since it is possible that $\mu(|\phi|^+) + \mu(|\chi|^+) > 1$. In this case, however, $e_1(\text{Pr}\phi \oplus \text{Pr}\chi) = 1$. To account for this, it is necessary to conduct the truncated subtraction (\ominus) first and only then add the measure of the second disjunct.

Let us now prove the (weak) completeness of $\mathcal{HPr}_{\Delta}^{\mathbf{k}^2}$.

Theorem 8.1 ($\mathcal{HPr}_{\Delta}^{\mathbf{k}^2}$ completeness). *Let $\Xi \cup \{\alpha\} \subseteq \mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$ be finite. Then*

$$\Xi \models_{\text{Pr}_{\Delta}^{\mathbf{k}^2}} \alpha \text{ iff } \Xi \vdash_{\mathcal{HPr}_{\Delta}^{\mathbf{k}^2}} \alpha$$

Proof. We begin with the soundness part. Since every outer valuation on a $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ -model is, in fact, a $\mathbf{k}_{(\Delta, \rightarrow)}^2$ valuation, it is clear that $\mathbf{k}_{(\Delta, \rightarrow)}^2$ -valid formulas and rules are also $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ -valid. For the **mon** axioms, observe that if $\phi \models_{\text{BD}} \chi$, then $\mu(|\phi|^+) \leq \mu(|\chi|^+)$ and $\mu(|\chi|^-) \leq \mu(|\phi|^-)$ ⁶⁶ in every model. Thus, $\text{Pr}\phi \rightarrow \text{Pr}\chi$ will also be valid. For **neg**, observe that

$$\begin{aligned} e(\neg\text{Pr}\phi) &= (e_2(\text{Pr}\phi), e_1(\text{Pr}\phi)) \\ &= (\mu(|\phi|^-), \mu(|\phi|^+)) \\ &= (\mu(|\neg\phi|^+), \mu(|\neg\phi|^-)) \\ &= e(\text{Pr}\neg\phi) \end{aligned}$$

Finally, for **ex**, we have that

$$\begin{aligned} e_1(\text{Pr}(\phi \vee \chi)) &= \mu(|\phi \vee \chi|^+) \\ &= \mu(|\phi|^+) + \mu(|\chi|^+) - \mu(|\phi \wedge \chi|^+) \\ &= (e_1(\text{Pr}\phi) - e_1(\text{Pr}(\phi \wedge \chi))) + e_1(\text{Pr}\chi) \\ &= e_1(\text{Pr}\phi \ominus \text{Pr}(\phi \wedge \chi)) + e_1(\text{Pr}\chi) && \text{(since } \mu(|\phi|^+) \geq \mu(|\phi \wedge \chi|^+) \text{)} \\ &= e_1((\text{Pr}\phi \ominus \text{Pr}(\phi \wedge \chi)) \oplus \text{Pr}\chi) && \text{(since } \mu(|\phi \vee \chi|^+) \leq 1 \text{)} \end{aligned}$$

and

$$\begin{aligned} e_2((\text{Pr}\phi \ominus \text{Pr}(\phi \wedge \chi)) \oplus \text{Pr}\chi) &= (e_2(\text{Pr}(\phi \wedge \chi)) \rightarrow_{\mathbf{k}} e_2(\text{Pr}\phi)) \odot_{\mathbf{k}} e_2(\text{Pr}\chi) \\ &= (1 - \mu(|\phi \wedge \chi|^-) + \mu(|\phi|^-)) \odot_{\mathbf{k}} \mu(|\chi|^-) \\ & && \text{(since } \mu(|\phi|^-) \leq \mu(|\phi \wedge \chi|^-) \text{)} \\ &= \mu(|\neg\phi|^+) + \mu(|\neg\chi|^+) - \mu(|\neg\phi \vee \neg\chi|^+) \\ &= \mu(|\neg\phi \wedge \neg\chi|^+) \end{aligned}$$

⁶⁶Recall from [57] that BD admits contraposition by \neg .

$$\begin{aligned} &= \mu(|\phi \vee \chi|^-) \\ &= e_2(\text{Pr}(\phi \vee \chi)) \end{aligned}$$

For completeness, we reason by contraposition. Assume that $\Xi \not\vdash_{\mathcal{H}\text{Pr}_\Delta^{\mathbf{k}^2}} \alpha$. Observe that $\mathcal{H}\text{Pr}_\Delta^{\mathbf{k}^2}$ is an extension of $\mathcal{H}\mathbf{k}_{(\Delta, \rightarrow)}^2$, with additional axioms and let Ξ^* denote Ξ expanded with all instances of probabilistic axioms constructed from all pairwise non-equivalent \mathcal{L}_{BD} -formulas over $\text{Prop}[\Xi \cup \{\alpha\}]$. It is clear that Ξ^* is also finite because BD is tabular. Moreover, $\Xi^* \not\vdash_{\mathcal{H}\text{Pr}_\Delta^{\mathbf{k}^2}} \alpha$ either.

It remains to construct a suitable $\text{Pr}_\Delta^{\mathbf{k}^2}$ -model \mathbb{M} that refutes $\Xi^* \not\vdash_{\mathcal{H}\text{Pr}_\Delta^{\mathbf{k}^2}} \alpha$. We set $W = 2^{\text{Lit}[\Xi^* \cup \{\alpha\}]}$, $w \in v^+(p)$ iff $p \in w$, and $w \in v^-(p)$ iff $\neg p \in w$. v^+ and v^- are then extended to all \mathcal{L}_{BD} formulas as usual. Since $\mathcal{H}\mathbf{k}_{(\Delta, \rightarrow)}^2$ is complete (recall Theorem 3.1), we have a $\mathbf{k}_{(\Delta, \rightarrow)}^2$ valuation e s.t. $e[\Xi^*] = (1, 0)$ and $e(\alpha) \neq (1, 0)$. It remains to define μ . For $\text{Pr}\phi \in \text{Sf}[\Xi^* \cup \{\alpha\}]$, we set $\mu(|\phi|^+) = e_1(\text{Pr}\phi)$ and $\mu(|\phi|^-) = e_2(\text{Pr}\phi)$. From here, we have that every monotone w.r.t. \subseteq from 2^W to $[0, 1]$ that extends μ is, actually, a \pm -probability because all constraints in Definition 8.2 are satisfied. \square

We finish the section with a remark on the expressivity of $\text{Pr}_\Delta^{\mathbf{k}^2}$.

Remark 8.5. Just as in \mathbf{KG}^{2c} (recall Examples 6.1 and 6.2), we can express that the probability of one statement is higher, lower, or incomparable to that of the other using Δ^\top from (3.1) since the $\text{Pr}_\Delta^{\mathbf{k}^2}$ semantic of \rightarrow conforms to the upwards order on $[0, 1]^\times$. However, in contrast to \mathbf{KG}^{2c} , we can also stipulate that a formula α has a classical value, i.e., that $e_1(\alpha) = 1 - e_2(\alpha)$ and, consequently, $e_2(\alpha) = 1 - e_1(\alpha)$.

For this, we recall that

$$\begin{aligned} e(\sim\neg\alpha) &= (1 - e_1(\neg\alpha), 1 - e_2(\neg\alpha)) \\ &= (1 - e_2(\alpha), 1 - e_1(\alpha)) \end{aligned}$$

and that $e(\beta \leftrightarrow \beta') = (1, 0)$ iff $e(\beta) = e(\beta')$.

$$e(\Delta^\top(\alpha \leftrightarrow \sim\neg\alpha)) = \begin{cases} (1, 0) & \text{if } e_1(\alpha) = 1 - e_2(\alpha) \\ (0, 1) & \text{otherwise} \end{cases}$$

In particular, we can call ϕ a *classical event* when $\mu(|\phi|^+) = 1 - \mu(|\phi|^-)$ (and hence, $e(\text{Pr}\phi \leftrightarrow \sim\neg\text{Pr}\phi) = (1, 0)$). Now note that

$$\text{Pr}_\Delta^{\mathbf{k}^2} \models \Delta^\top \sim \text{Pr}(\phi \wedge \neg\phi) \leftrightarrow \Delta^\top(\text{Pr}\phi \leftrightarrow \sim\neg\text{Pr}\phi)$$

This provides an expected characterisation of classical events in $\text{Pr}_\Delta^{\mathbf{k}^2}$: ϕ has classical probability assignment iff $\mu(|\phi \wedge \neg\phi|^+) = 0$ and $\mu(|\phi \wedge \neg\phi|^-) = 1$.

8.3. Logic of 4-probabilities

In this section, we deal with the logic of 4-probabilities defined on BD models from Definition 2.2. First of all, to facilitate the presentation, we introduce four new extensions of a formula following [92].

Convention 8.2. Let $\mathfrak{M} = \langle W, v^+, v^- \rangle$ be a BD model, $\phi \in \mathcal{L}_{\text{BD}}$. We set

$$\begin{aligned} |\phi|^b &= |\phi|^+ \setminus |\phi|^- & |\phi|^d &= |\phi|^- \setminus |\phi|^+ \\ |\phi|^c &= |\phi|^+ \cap |\phi|^- & |\phi|^u &= W \setminus (|\phi|^+ \cup |\phi|^-) \end{aligned}$$

8. Logics for quantitative reasoning

We call these extensions, respectively, *pure belief*, *pure disbelief*, *conflict*, and *uncertainty in ϕ* , following [92]. Note that they correspond to the subsets of W where ϕ has one of the Belnapian values (recall Definition 2.1).

We can now define $\mathbf{4}$ -probabilities over BD models.

Definition 8.5 (BD models with $\mathbf{4}$ -probabilities). A BD model with a $\mathbf{4}$ -probability is a tuple $\mathfrak{M}_4 = \langle \mathfrak{M}, \mu_4 \rangle$ with \mathfrak{M} being a BD model and $\mu_4 : 2^W \rightarrow [0, 1]$ satisfying:

4part: $\mu_4(|\phi|^b) + \mu_4(|\phi|^d) + \mu_4(|\phi|^u) + \mu_4(|\phi|^c) = 1$ (partition);

4neg: $\mu_4(|\neg\phi|^b) = \mu_4(|\phi|^d)$, $\mu_4(|\neg\phi|^c) = \mu_4(|\phi|^c)$ (negation);

4contr: $\mu_4(|\phi \wedge \neg\phi|^b) = 0$, $\mu_4(|\phi \wedge \neg\phi|^c) = \mu_4(|\phi|^c)$ (contradiction);

4mon: if $\mathfrak{M} \models [\phi \vdash \chi]$, then $\mu_4(|\phi|^b) + \mu_4(|\phi|^c) \leq \mu_4(|\chi|^b) + \mu_4(|\chi|^c)$ (monotonicity);

4ex: $\mu_4(|\phi|^b) + \mu_4(|\phi|^c) + \mu_4(|\psi|^b) + \mu_4(|\psi|^c) = \mu_4(|\phi \wedge \psi|^b) + \mu_4(|\phi \wedge \psi|^c) + \mu_4(|\phi \vee \psi|^b) + \mu_4(|\phi \vee \psi|^c)$ (import-export).

Remark 8.6. Notice again that μ_4 is not necessarily a measure on 2^W according to Definition 8.1. Indeed, it is not necessarily monotone w.r.t. \subseteq since not every subset of a model is represented by an extension of an \mathcal{L}_{BD} formula. Again, it is not a problem since for every BD model with $\mathbf{4}$ -probability $\langle W, v^+, v^-, \mu_4 \rangle$, there exist a BD model $\langle W', v'^+, v'^-, \pi \rangle$ with a *classical* probability measure π s.t. $\pi(|\phi|^x) = \mu_4(|\phi|^x)$ for $x \in \{b, d, c, u\}$ [92, Theorem 5].

Let us now present $\mathbf{4Pr}^{\mathbf{k}\Delta}$ — the logic of $\mathbf{4}$ -probabilities.

Definition 8.6 ($\mathbf{4Pr}^{\mathbf{k}\Delta}$: language and semantics). The language of $\mathbf{4Pr}^{\mathbf{k}\Delta}$ is constructed by the following grammar:

$$\mathcal{L}_{\mathbf{4Pr}^{\mathbf{k}\Delta}} \ni \alpha := \text{Bl}\phi \mid \text{Db}\phi \mid \text{Cf}\phi \mid \text{Uc}\phi \mid \sim\alpha \mid \Delta\alpha \mid (\alpha \rightarrow \alpha) \quad (\phi \in \mathcal{L}_{\text{BD}})$$

A $\mathbf{4Pr}^{\mathbf{k}\Delta}$ -model is a tuple $\mathbb{M} = \langle \mathfrak{M}, \mu_4, e \rangle$ with $\langle \mathfrak{M}, \mu_4 \rangle$ being a BD model with $\mathbf{4}$ -probability and e a map from the set of modal atoms to $[0, 1]$ s.t. $e(\text{Bl}\phi) = \mu_4(|\phi|^b)$, $e(\text{Db}\phi) = \mu_4(|\phi|^d)$, $e(\text{Cf}\phi) = \mu_4(|\phi|^c)$, $e(\text{Uc}\phi) = \mu_4(|\phi|^u)$, and the values of complex formulas⁶⁷ are computed via Definition 3.2. We say that α is $\mathbf{4Pr}^{\mathbf{k}\Delta}$ *valid* iff $e(\alpha) = 1$ in every model. A set of formulas Γ *entails* α ($\Gamma \models_{\mathbf{4Pr}^{\mathbf{k}\Delta}} \alpha$) iff there is no \mathbb{M} s.t. $e(\gamma) = 1$ for every $\gamma \in \Gamma$ but $e(\alpha) \neq 1$.

To make the semantics clearer, we provide the following example.

Example 8.1. Consider the following BD model.

$$w_0 : p^\pm, \not{q} \quad w_1 : p^-, q^-$$

Let μ_4 be defined as follows: $\mu_4(\{w_0\}) = \frac{2}{3}$, $\mu_4(\{w_1\}) = \frac{1}{3}$, $\mu_4(W) = 1$, $\mu_4(\emptyset) = 0$. It is easy to check that μ satisfies the conditions of Definition 8.5. Now let e_4 be the $\mathbf{k}\Delta$ *valuation* induced by μ_4 .

Consider two BD formulas: $p \vee q$ and p . We have $e_4(\text{Bl}(p \vee q)) = \frac{2}{3}$, $e_4(\text{Db}(p \vee q)) = \frac{1}{3}$, $e_4(\text{Cf}p) = \frac{2}{3}$, $e_4(\text{Cf}(p \vee q)) = \frac{2}{3}$, $e_4(\text{Uc}(p \vee q)) = 0$, $e_4(\text{Bl}p) = \frac{2}{3}$, $e_4(\text{Uc}p) = 0$, $e_4(\text{Cf}p) = \frac{2}{3}$, and $e_4(\text{Db}p) = \frac{1}{3}$.

Let us now proceed to the axiomatisation of $\mathbf{4Pr}^{\mathbf{k}\Delta}$. Since its outer layer expands $\mathbf{k}\Delta$, we will need to encode the conditions on μ_4 therein. The axiomatisation will consist of two types of axioms: those that axiomatise $\mathbf{k}\Delta$ and modal axioms that encode the conditions from Definition 8.5. For the sake of brevity, we will compress the axiomatisation of $\mathbf{k}\Delta$ into one axiom that allows us to use $\mathbf{k}\Delta$ theorems without proof just as we did for $\text{Pr}^{\mathbf{k}\Delta}$ (recall Definition 8.4).

⁶⁷We will say that e is *induced* by μ_4 .

Definition 8.7 ($\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}$ — Hilbert-style calculus for $4\text{Pr}^{\mathbf{k}\Delta}$). The calculus $\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}$ consists of the following axioms and rules.

\mathbf{k}_{Δ} : \mathbf{k}_{Δ} -valid formulas and sound rules instantiated in $\mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$.

4equiv: $X\phi \leftrightarrow X\chi$ for every $\phi, \chi \in \mathcal{L}_{\text{BD}}$ s.t. $\phi \dashv\vdash \chi$ is BD-valid and $X \in \{\text{Bl}, \text{Db}, \text{Cf}, \text{Uc}\}$ (equivalence axioms).

4contr: $\sim\text{Bl}(\phi \wedge \neg\phi)$; $\text{Cf}\phi \leftrightarrow \text{Cf}(\phi \wedge \neg\phi)$ (contradiction axioms).

4neg: $\text{Bl}\neg\phi \leftrightarrow \text{Db}\phi$; $\text{Cf}\neg\phi \leftrightarrow \text{Cf}\phi$ (negation axioms).

4mon: $(\text{Bl}\phi \oplus \text{Cf}\phi) \rightarrow (\text{Bl}\chi \oplus \text{Cf}\chi)$ for every $\phi, \chi \in \mathcal{L}_{\text{BD}}$ s.t. $\phi \vdash \chi$ is BD-valid (monotonicity axioms).

4part1: $\text{Bl}\phi \oplus \text{Db}\phi \oplus \text{Cf}\phi \oplus \text{Uc}\phi$ (partition axioms).

4part2: $((X_1\phi \oplus X_2\phi \oplus X_3\phi \oplus X_4\phi) \ominus X_4\phi) \leftrightarrow (X_1\phi \oplus X_2\phi \oplus X_3\phi)$ with $X_i \neq X_j$, $X_i \in \{\text{Bl}, \text{Db}, \text{Cf}, \text{Uc}\}$.

4ex: $(\text{Bl}(\phi \vee \chi) \oplus \text{Cf}(\phi \vee \chi)) \leftrightarrow ((\text{Bl}\phi \oplus \text{Cf}\phi) \ominus (\text{Bl}(\phi \wedge \chi) \oplus \text{Cf}(\phi \wedge \chi))) \oplus (\text{Bl}\chi \oplus \text{Cf}\chi)$ (import-export axioms).

The axioms above are simple translations of properties from Definition 8.5. We split **4part** in two axioms to ensure that the values of $\text{Bl}\phi$, $\text{Db}\phi$, $\text{Cf}\phi$, and $\text{Uc}\phi$ sum up exactly to 1. The completeness proof is essentially the same as that of $\mathcal{H}\text{Pr}_{\Delta}^{\mathbf{k}^2}$ (recall Theorem 8.1).

Theorem 8.2. *Let $\Xi \subseteq \mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$ be finite. Then $\Xi \models_{4\text{Pr}^{\mathbf{k}\Delta}} \alpha$ iff $\Xi \vdash_{\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}} \alpha$.*

Proof. Soundness can be established by the routine check of the axioms' validity. Thus, we prove completeness. We reason by contraposition. Assume that $\Xi \not\models_{\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}} \alpha$. Now, observe that $\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}$ proofs are, actually, \mathbf{k}_{Δ} proofs with additional probabilistic axioms. Let Ξ^* stand for Ξ extended with probabilistic axioms built over all pairwise non-equivalent \mathcal{L}_{BD} formulas constructed from $\text{Prop}[\Xi \cup \{\alpha\}]$. Clearly, $\Xi^* \not\models_{\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}} \alpha$ either. Moreover, Ξ^* is finite as well since BD is tabular (and whence, there exist only finitely many pairwise non-equivalent \mathcal{L}_{BD} formulas over a finite set of variables). Now, by the weak completeness of \mathbf{k}_{Δ} (Proposition 3.2), there exists an \mathbf{k}_{Δ} valuation e s.t. $e[\Xi^*] = 1$ and $e(\alpha) \neq 1$.

It remains to construct a $4\text{Pr}^{\mathbf{k}\Delta}$ -model \mathbb{M} falsifying $\Xi^* \models_{4\text{Pr}^{\mathbf{k}\Delta}} \alpha$ using e . We proceed as follows. First, we set $W = 2^{\text{Lit}[\Xi^* \cup \{\alpha\}]}$, and for every $w \in W$ define $w \in v^+(p)$ iff $p \in w$ and $w \in v^-(p)$ iff $\neg p \in w$. We extend the valuations to $\phi \in \mathcal{L}_{\text{BD}}$ in the usual manner. Then, for $X\phi \in \text{Sf}[\Xi^* \cup \{\alpha\}]$, we set $\mu_4(|\phi|^X) = e(X\phi)$ according to modality X .

It remains to extend μ_4 to the whole 2^W . Observe, however, that any map from 2^W to $[0, 1]$ that extends μ_4 is, in fact, a 4 -probability. Indeed, all requirements from Definition 8.5 concern *only the extensions of formulas*. But the model is finite, BD is tabular, and Ξ^* contains all the necessary instances of probabilistic axioms and $e[\Xi^*] = 1$, whence all constraints on the formulas are satisfied. \square

Remark 8.7. Observe that we could have used a *classical probability measure* in the proof of Theorem 8.2 because of [92, Theorem 5]. This, however, would require us to show that the extensions of formulas form a subalgebra of 2^W . On the other hand, it is simpler to use 4 -probabilities instead of classical probabilities since we can immediately extend them to the full powerset from the extensions of formulas by Definition 8.5.

8.4. Comparing $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ and $4\text{Pr}^{\mathbf{k}\Delta}$

In this section, we show the expected result that just as their underlying probabilities, $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ and $4\text{Pr}^{\mathbf{k}\Delta}$ are equivalent. In particular, we show that they are equally expressive and that they have the same complexity.

8.4.1. Expressivity

At first glance, $4\text{Pr}^{\mathbf{k}\Delta}$ gives a more fine-grained view on a BD model than $\text{Pr}^{\mathbf{k}\Delta^2}$ since it can evaluate each extension of a given $\phi \in \mathcal{L}_{\text{BD}}$, while $\text{Pr}^{\mathbf{k}\Delta^2}$ always considers $|\phi|^+$ and $|\phi|^-$ together. In this section, we show how to faithfully translate these logics into one another.

Recall from Proposition 3.4 that $\mathbf{k}_{(\Delta, \rightarrow)}^2$ validity and entailment can be established only using the support of the truth. We can show that a similar result holds for $\text{Pr}^{\mathbf{k}\Delta^2}$.

Lemma 8.1. *Let $\alpha \in \mathcal{L}_{\text{Pr}^{\mathbf{k}\Delta^2}}$. Then, α is $\text{Pr}^{\mathbf{k}\Delta^2}$ valid iff $e_1(\alpha) = 1$ in every $\text{Pr}^{\mathbf{k}\Delta^2}$ -model.*

Proof. Let $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2 \rangle$ be a $\text{Pr}^{\mathbf{k}\Delta^2}$ -model s.t. $e_2(\alpha) \neq 0$. We construct a model $\mathbb{M}^* = \langle W, (v^*)^+, (v^*)^-, \mu, e_1^*, e_2^* \rangle$ where $e_1^*(\alpha) \neq 1$. To do this, we define new BD valuations $(v^*)^+$ and $(v^*)^-$ on W as follows.

$$\begin{aligned} \text{if } w \in v^+(p) \text{ and } w \notin v^-(p) \text{ then } w \in (v^*)^+(p) \text{ and } w \notin (v^*)^-(p) \\ \text{if } w \in v^+(p) \text{ and } v^-(p) \text{ then } w \notin (v^*)^+(p) \text{ and } (v^*)^-(p) \\ \text{if } w \notin v^+(p) \text{ and } v^-(p) \text{ then } w \in (v^*)^+(p) \text{ and } (v^*)^-(p) \\ \text{if } w \notin v^+(p) \text{ and } w \in v^-(p) \text{ then } w \notin (v^*)^+(p) \text{ and } w \in (v^*)^-(p) \end{aligned}$$

It can be easily checked by induction on $\phi \in \mathcal{L}_{\text{BD}}$ that

$$|\phi|_{\mathbb{M}}^+ = W \setminus |\phi|_{\mathbb{M}^*}^- \quad |\phi|_{\mathbb{M}}^- = W \setminus |\phi|_{\mathbb{M}^*}^+$$

Now, since we can w.l.o.g. assume that μ is a (classical) probability measure on W (recall Remark 8.2), we have that

$$e^*(\text{Pr}\phi) = (1 - \mu(|\phi|^-), 1 - \mu(|\phi|^+)) = (1 - e_2(\text{Pr}\phi), 1 - e_1(\text{Pr}\phi))$$

Observe that if $e(\alpha) = (x, y)$, then $e(\neg\alpha) = (1 - y, 1 - x)$. Furthermore, by Proposition 3.4, the following formulas are valid.

$$\begin{aligned} \neg\sim\neg\alpha &\leftrightarrow \neg\neg\sim\alpha & \neg\sim\sim\alpha &\leftrightarrow \sim\neg\sim\alpha \\ \neg\sim\Delta\alpha &\leftrightarrow \Delta\neg\sim\alpha & \neg\sim(\alpha \rightarrow \alpha') &\leftrightarrow \neg\sim\alpha \rightarrow \neg\sim\alpha' \end{aligned}$$

Hence, $e^*(\alpha) = (1 - e_2(\alpha), 1 - e_1(\alpha))$ for every $\alpha \in \mathcal{L}_{\text{Pr}^{\mathbf{k}\Delta^2}}$. The result follows. \square

The next statement is also easy to obtain.

Lemma 8.2. *Let $\alpha \in \mathcal{L}_{\text{Pr}^{\mathbf{k}\Delta^2}}$. Then, there exists α^+ s.t. all \neg 's occur inside modal atoms and $\text{Pr}^{\mathbf{k}\Delta^2} \models \alpha \leftrightarrow \alpha^+$.*

Proof. First, recall that $\mathbf{k}_{(\Delta, \rightarrow)}^2$ admits \neg NNF's (Lemma 3.1). Since $\text{Pr}^{\mathbf{k}\Delta^2}$ extends $\mathbf{k}_{(\Delta, \rightarrow)}^2$, we can transform α in such a way that \neg 's occur close to modal atoms. Finally, to push \neg 's inside modal atoms, we recall that $\text{Pr}^{\mathbf{k}\Delta^2} \models \neg\text{Pr}\phi \leftrightarrow \text{Pr}\neg\phi$. This gives us the desired α^+ . \square

Convention 8.3. We will say that $\alpha \in \mathcal{L}_{\text{Pr}^{\mathbf{k}\Delta^2}}$ is *outer- \neg -free* when \neg 's appear only inside modal atoms.

Let us now define the embeddings between $\mathcal{L}_{\text{Pr}^{\mathbf{k}\Delta^2}}$ and $\mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$. Lemma 8.2 allows us to consider only outer- \neg -free formulas.

Definition 8.8. For an outer- \neg -free $\alpha \in \mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$, we define $\alpha^{\mathbf{4}} \in \mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$ as follows.

$$\begin{aligned} (\text{Pr}\phi)^{\mathbf{4}} &= \text{Bl}\phi \oplus \text{Cf}\phi \\ (\heartsuit\alpha)^{\mathbf{4}} &= \heartsuit\alpha^{\mathbf{4}} \quad (\heartsuit \in \{\Delta, \sim\}) \\ (\alpha \rightarrow \alpha')^{\mathbf{4}} &= \alpha^{\mathbf{4}} \rightarrow \alpha'^{\mathbf{4}} \end{aligned}$$

Let $\beta \in \mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$. We define β^{\pm} as follows.

$$\begin{aligned} (\text{Bl}\phi)^{\pm} &= \text{Pr}\phi \ominus \text{Pr}(\phi \wedge \neg\phi) \\ (\text{Cf}\phi)^{\pm} &= \text{Pr}(\phi \wedge \neg\phi) \\ (\text{Uc}\phi)^{\pm} &= \sim\text{Pr}(\phi \vee \neg\phi) \\ (\text{Db}\phi)^{\pm} &= \text{Pr}\neg\phi \ominus \text{Pr}(\phi \wedge \neg\phi) \\ (\heartsuit\beta)^{\pm} &= \heartsuit\beta^{\pm} \quad (\heartsuit \in \{\Delta, \sim\}) \\ (\beta \rightarrow \beta')^{\pm} &= \beta^{\pm} \rightarrow \beta'^{\pm} \end{aligned}$$

Theorem 8.3. $\alpha \in \mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$ is $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ valid iff $(\alpha^+)^{\mathbf{4}}$ is $4\text{Pr}^{\mathbf{k}\Delta}$ valid.

Proof. Recall that because of [92, Theorem 5] (Remark 8.6), we have that for each BD model with $\mathbf{4}$ -probability $\mathfrak{M}_{\mathbf{4}} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e \rangle$, there is a BD model with \pm -probability $\mathfrak{M}^{\pm} = \langle W', v'^+, v'^-, \mu^{\pm}, e_1, e_2 \rangle$ s.t.

$$\forall \phi \in \mathcal{L}_{\text{BD}} : \mu^{\pm}(|\phi|^+) = \mu_{\mathbf{4}}(|\phi|^+) = \mu_{\mathbf{4}}(|\phi|^{\text{b}} \cup |\phi|^{\text{c}}) = \mu_{\mathbf{4}}(|\phi|^{\text{b}}) + \mu_{\mathbf{4}}(|\phi|^{\text{c}})$$

We now let w.l.o.g. $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2 \rangle$ be a BD model with \pm -probability where μ is a classical probability measure and let $e(\alpha) = (x, y)$. We show that in the BD model $\mathbb{M}_{\mathbf{4}} = \langle W, v^+, v^-, \mu, e_1 \rangle$ with *four-probability* μ , $e_1((\alpha^+)^{\mathbf{4}}) = x$. This is sufficient to prove the result. Indeed, by Lemma 8.1, it suffices to verify that $e_1(\alpha) = 1$ for every e_1 , to establish the validity of $\alpha \in \mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$.

We proceed by induction on α^+ (recall that $\alpha \leftrightarrow \alpha^+$ is $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ valid by Lemma 8.2). If $\alpha = \text{Pr}\phi$, then $e_1(\text{Pr}\phi) = \mu(|\phi|^+) = \mu(|\phi|^{\text{b}} \cup |\phi|^{\text{c}})$. But $|\phi|^{\text{b}}$ and $|\phi|^{\text{c}}$ are disjoint, whence $\mu(|\phi|^{\text{b}} \cup |\phi|^{\text{c}}) = \mu(|\phi|^{\text{b}}) + \mu(|\phi|^{\text{c}})$, and since $\mu(|\phi|^{\text{b}}) + \mu(|\phi|^{\text{c}}) \leq 1$, we have that $e_1(\text{Bl}\phi \oplus \text{Cf}\phi) = \mu(|\phi|^{\text{b}}) + \mu(|\phi|^{\text{c}}) = e_1(\text{Pr}\phi)$, as required.

The induction steps are straightforward since the semantic conditions of support of truth in $\mathbf{k}_{(\Delta, \rightarrow)}^2$ coincide with the semantics of \mathbf{k}_{Δ} (cf. Definitions 3.5 and 3.2). \square

Theorem 8.4. $\beta \in \mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$ is $\mathcal{L}_{4\text{Pr}^{\mathbf{k}\Delta}}$ valid iff β^{\pm} is $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ valid.

Proof. Again, by [92, Theorem 5] (Remark 8.6), we have that for each BD model with \pm -probability $\mathfrak{M}^{\pm} = \langle W', v'^+, v'^-, \mu^{\pm}, e_1, e_2 \rangle$, there is a BD model with $\mathbf{4}$ -probability $\mathfrak{M}_{\mathbf{4}} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e \rangle$ s.t.

$$\forall \phi \in \mathcal{L}_{\text{BD}} \forall x \in \{\mathbf{b}, \mathbf{d}, \mathbf{c}, \mathbf{u}\} : \mu_{\mathbf{4}}(|\phi|^x) = \mu^{\pm}(|\phi|^x)$$

Assume now, w.l.o.g. that $\mathbb{M} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e \rangle$ is a BD model with a $\mathbf{4}$ -probability where $\mu_{\mathbf{4}}$ is a classical probability measure and $e(\beta) = x$. We define a BD model with \pm -probability $\mathbb{M}^{\pm} = \langle W, v^+, v^-, \mu_{\mathbf{4}}, e_1, e_2 \rangle$ and show that $e_1(\beta^{\pm}) = x$. Again, it is sufficient for us by Lemma 8.1.

We proceed by induction on β . If $\beta = \text{Bl}\phi$, then $e(\text{Bl}\phi) = \mu_{\mathbf{4}}(|\phi|^{\text{b}})$. Now observe that $\mu_{\mathbf{4}}(|\phi|^+) = \mu(|\phi|^{\text{b}} \cup |\phi|^{\text{c}}) = \mu_{\mathbf{4}}(|\phi|^{\text{b}}) + \mu_{\mathbf{4}}(|\phi|^{\text{c}})$ since $|\phi|^{\text{b}}$ and $|\phi|^{\text{c}}$ are disjoint. But $\mu_{\mathbf{4}}(|\phi|^+) = e_1(\text{Pr}\phi)$ and $\mu_{\mathbf{4}}(|\phi|^{\text{c}}) = \mu_{\mathbf{4}}(|\phi \wedge \neg\phi|^+)$ since $|\phi \wedge \neg\phi|^+ = |\phi|^{\text{c}}$. Thus, $\mu_{\mathbf{4}}(|\phi|^{\text{b}}) = e_1(\text{Pr}\phi \ominus \text{Pr}(\phi \wedge \neg\phi))$ as required.

Other basis cases of $\text{Cf}\phi$, $\text{Uc}\phi$, and $\text{Db}\phi$ can be tackled in a similar manner. The induction steps are straightforward since the support of truth in $\mathbf{k}_{(\Delta, \rightarrow)}^2$ coincides with semantical conditions in \mathbf{k}_{Δ} . \square

Remark 8.8. Theorems 8.3 and 8.4 mean, in a sense, that $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ and $4\text{Pr}^{\mathbf{k}\Delta}$ can be treated as syntactic variants of one another. Conceptually, however, they are somewhat different. Namely, $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ assigns *two independent measures* to each formula ϕ corresponding to the likelihoods of ϕ itself and $\neg\phi$. On the other hand, $4\text{Pr}^{\mathbf{k}\Delta}$ treats the extensions ϕ as a separation of the underlying sample set into four parts whose measures must add up to 1.

Remark 8.9. Before proceeding to establish the complexity of $4\text{Pr}^{\mathbf{k}\Delta}$ and $\text{Pr}_{\Delta}^{\mathbf{k}^2}$, let us quickly recall the desiderata stated in the introduction. Since $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ and $4\text{Pr}^{\mathbf{k}\Delta}$ are equally expressive, we will discuss only $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ as it has only one modality. From Remark 8.5, we know that we can compare the degrees to which the agent is certain in given statements and likewise state that these degrees are incomparable. Thus, desiderata 1 and 4 are satisfied. Moreover, in line with desiderata 3 and 5, $\text{Pr}(p \wedge \neg p) \rightarrow \text{Pr}q$ and $\text{Pr}p \rightarrow \text{Pr}(q \vee \neg q)$ are not valid.

Finally, to represent different sources, one can consider BD models with *several measures* and, accordingly, expand the language with other modalities corresponding to these new measures. Then, we can state, for example, that one source (s_1) considers ϕ to be more likely than the other source does, i.e., the value of $\text{Pr}_{s_1}\phi$ is *smaller than* the value of $\text{Pr}_{s_2}\phi$. This can be formalised as follows.

$$\Delta^{\top}(\text{Pr}_{s_1}\phi \rightarrow \text{Pr}_{s_2}\phi) \wedge \sim\Delta^{\top}(\text{Pr}_{s_2}\phi \rightarrow \text{Pr}_{s_1}\phi)$$

Unfortunately, there seems to be no direct way of representing the *degrees of trust* the agent assigns to s_1 and s_2 using only modalities interpreted as measures in the Łukasiewicz setting. A traditional way (cf. [136, p.252]) of accounting for the degree of trust in a given source is to multiply the value a mass function gives to $X \subseteq W$ by some $x \in [0, 1]$. Thus, to model this approach, one would need a combination of Rational Pavelka and Product logics. Another option⁶⁸ would be to redefine $e_1(\text{Pr}_s\phi) = (\text{tr}_s \odot_{\mathbf{k}} \mu(|\phi|^+))$ and $e_2(\text{Pr}\phi) = (\text{tr}_s \odot_{\mathbf{k}} \mu(|\phi|^-))$ (with $\text{tr}_s \in [0, 1]$ standing for the trust in source s). It is unclear, however, whether this new logic is going to be an extension of $\text{Pr}_{\Delta}^{\mathbf{k}^2}$.

It is possible, though, to make different modalities stand for different measures (e.g., Pr_{s_1} can be a \pm -probability while Pr_{s_2} only a belief function⁶⁹). This represents the *different ways of aggregating the data* the agent can have.

8.4.2. Complexity

In the proof of Theorem 8.2, we reduced $\mathcal{H}4\text{Pr}^{\mathbf{k}\Delta}$ proofs to $\mathcal{H}\mathbf{k}_{\Delta}$ proofs. We know that validity and finitary entailment of \mathbf{k}_{Δ} are coNP-complete (since \mathbf{k} is coNP-complete and Δ has truth-functional semantics). Likewise, $\mathcal{H}\text{Pr}_{\Delta}^{\mathbf{k}^2}$ proofs are also reducible to $\mathcal{H}\mathbf{k}_{(\Delta, \rightarrow)}^2$ proofs (cf. Theorem 8.1) from substitution instances of $\mathcal{H}\text{Pr}_{\Delta}^{\mathbf{k}^2}$ axioms. Thus, it is clear that the validity and satisfiability of $4\text{Pr}^{\mathbf{k}\Delta}$ and $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ are coNP-hard and NP-hard, respectively.

In this section, we provide a simple decision procedure for $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ and $4\text{Pr}^{\mathbf{k}\Delta}$ and show that their satisfiability and validity are NP- and coNP-complete, respectively. Namely, we adapt constraint tableaux for \mathbf{k}^2 (cf. Definition 3.11). We then adapt the NP-completeness proof $\text{FP}(\mathbf{k})$ from [85] to establish our result.

First, we show that we can *completely remove* \neg 's from $\mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$ formulas while preserving their satisfiability.

Lemma 8.3. *For any outer- \neg -free $\alpha \in \mathcal{L}_{\text{Pr}_{\Delta}^{\mathbf{k}^2}}$, there exists α^* where \neg does not occur at all s.t. α is $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ -satisfiable iff α^* is $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ -satisfiable.*

⁶⁸The author would like to thank Lluís Godo Lacasa for this idea.

⁶⁹We do not provide two-layered axiomatisations of belief functions in \mathbf{k}^2 in this text. An interested reader may find them in [26].

Proof. We construct α^* as follows. First, we take every modal atom $\text{Pr}\phi$ and replace ϕ with its \neg NNF. Second, we replace every literal $\neg p$ occurring in $\text{Pr}(\text{NNF}(\phi))$ with p^* . Outer-layer connectives remain the same. Observe, that this transformation increases the number of symbols in α at most linearly. It remains to check that satisfiability is preserved.

Let $e(\alpha) = (1, 0)$ at some $\text{Pr}_{\Delta}^{\mathfrak{k}^2}$ -model. By Lemma 8.1, this is equivalent to $e_1(\alpha) = 1$ in some $\mathbb{M} = \langle W, v^+, v^-, \mu, e_1, e_2 \rangle$. Now, let $\mathbb{M}^* = \langle W, v^{*+}, v^{*-}, \mu, e_1^*, e_2^* \rangle$ and, in particular, $v^-(p) = v^{*+}(p^*)$. It suffices to show that $e_1(\alpha) = e_1^*(\alpha^*)$.

We proceed by induction on α . Let $\alpha = \text{Pr}\phi$ for some $\phi \in \mathcal{L}_{\text{BD}}$. We have that $e_1(\alpha) = \mu(|\phi|^+) = \mu(|\text{NNF}(\phi)|^+)$. We check that $|\text{NNF}(\phi)|^+ = |\text{NNF}(\phi)^*|^+$ by induction on $\text{NNF}(\phi)$. The basis cases of variables and literals hold by the construction of \mathbb{M}^* . The cases of \wedge and \vee hold by the induction hypothesis. It follows now that $|\text{NNF}(\phi)|^+ = |\text{NNF}(\phi)^*|^+$ and thus, $e_1(\text{Pr}\phi) = e_1^*(\text{Pr}(\text{NNF}(\phi)^*))$. The cases of $\mathfrak{k}_{(\Delta, \rightarrow)}^2$ connectives can be proven by a straightforward application of the induction hypothesis. The result follows. \square

We can now apply this lemma to adapt the proof of the NP-completeness of $\text{FP}(\mathfrak{k})$.

Theorem 8.5. *Satisfiability of $\text{Pr}_{\Delta}^{\mathfrak{k}^2}$ and $4\text{Pr}^{\mathfrak{k}\Delta}$ is NP-complete.*

Proof. Recall that $\text{Pr}_{\Delta}^{\mathfrak{k}^2}$ and $4\text{Pr}^{\mathfrak{k}\Delta}$ can be linearly embedded into one another (Theorems 8.3 and 8.4). Thus, it remains to provide a non-deterministic polynomial algorithm for one of these logics. We choose $\text{Pr}_{\Delta}^{\mathfrak{k}^2}$ since it has only one modality.

Let $\alpha \in \mathcal{L}_{\text{Pr}_{\Delta}^{\mathfrak{k}^2}}$. We can w.l.o.g. assume that \neg occurs only in modal atoms and that in every modal atom $\text{Pr}\phi_i$, ϕ_i is in negation normal form. Now, transform α in α^* . By Lemma 8.3, this preserves satisfiability. Let us now construct a satisfying valuation for α^* .

First, we replace every modal atom $\text{Pr}\phi_i$ with a fresh variable q_{ϕ_i} . Denote the new formula $(\alpha^*)^-$. It is clear that the size of $(\alpha^*)^-$ — $|(\alpha^*)^-|$ — is at most linearly greater than $|\alpha|$. We construct a tableau beginning with $\{(\alpha^*)^- \geq_1 c, c \geq 1\}$. Every branch gives us an instance of a bMIP (recall Definition 3.13) equivalent to the \mathfrak{k} -satisfiability of $(\alpha^*)^-$: $(\alpha^*)^-$ is satisfiable iff at least one instance of a bMIP has a solution. Now, write z_i for the values of q_{ϕ_i} 's in $(\alpha^*)^-$. Our instance of the bMIP also has additional variables x_j ranging over $[0, 1]$ and equalities $k = 1$ and $k' = 0$ obtained from entries $k \geq 1$ and $k' \leq 0$. It is clear that both the number of (in)equalities l_1 and the number of variables l_2 in the bMIP are linear w.r.t. $|(\alpha^*)^-|$. Denote this instance MIP(1).

We need to show that z_i 's are coherent as probabilities of ϕ_i 's (here, $i \leq n$ indexes the modal atoms of $(\alpha^*)^-$). We introduce 2^n variables u_v indexed by n -letter words over $\{0, 1\}$ and denoting whether the variables of ϕ_i 's are true under v^+ .⁷⁰ We let $a_{i,v} = 1$ when ϕ_i is true under v^+ and $a_{i,v} = 0$ otherwise. Now add new equalities to MIP(1), namely, $\sum_v u_v = 1$ and $\sum_v (a_{i,v} \cdot u_v) = z_i$ and denote them with MIP(2 exp). The new problem — $\text{MIP}(1) \cup \text{MIP}(2 \text{ exp})$ — has a non-negative solution iff $(\alpha^*)^-$ is $\text{Pr}_{\Delta}^{\mathfrak{k}^2}$ -satisfiable. Furthermore, although there are $l_2 + 2^n + n$ variables in $\text{MIP}(1) \cup \text{MIP}(2 \text{ exp})$, it has no more than $l_1 + n + 1$ (in)equalities. Thus by [61, Lemma 2.5], it has a non-negative solution with at most $l_1 + n + 1$ non-zero entries. We guess a list L of at most $l_1 + n + 1$ words v (its size is $n \cdot (l_1 + n + 1)$). We can now compute the values of $a_{i,v}$'s for $i \leq n$ and $v \in L$ and obtain a new MIP which we denote MIP(2 poly): $\sum_{v \in L} u_v = 1$ and $\sum_{v \in L} (a_{i,v} \cdot u_v) = z_i$. It is clear that $\text{MIP}(1) \cup \text{MIP}(2 \text{ poly})$ is of polynomial size w.r.t. $|\alpha|$ and has a non-negative solution iff the corresponding branch of the tableau is open. Thus, we can solve it in non-deterministic polynomial time as required. \square

⁷⁰Note that \neg does not occur in $(\alpha^*)^-$ and thus we care only about e_1 and v^+ . Furthermore, while n is the number of ϕ_i 's, we can add extra modal atoms or variables to make it also the number of variables.

9. Logics for qualitative reasoning

In the previous chapter, we presented two logics that axiomatised the reasoning with probabilities defined over Belnap–Dunn logic. In this chapter, we tackle the logics for the qualitative reasoning about uncertainty measures: both classical and paraconsistent. This is because, in contrast to the classical *quantitative reasoning* that has been axiomatised using Łukasiewicz or Product logic (or a combination of these two) [84, 73, 44, 11] on the outer layer, there seem to be no previous results on the two-layered formalisations of qualitative reasoning about uncertainty.

To the best of our knowledge, qualitative reasoning about uncertainty has been formalised⁷¹ using logics with nesting modalities starting from [69] where the logic of qualitative probabilities QP was axiomatised (the axiomatisation, however, was infinite). A similar approach was used in the qualitative logic of possibility provided in [63] where the authors extend classical propositional logic with axioms of qualitative possibility formulated with \lesssim . The logic is then translated into the quantitative possibility logic and shown to preserve validity between quantitative and qualitative models.

Another approach to the logic of qualitative possibility is presented in [86, Section 2.9]. There, Halpern introduces a qualitative notion of relative likelihood and shows that every order based on a possibility measure is in fact a relative likelihood. In other words, his axiomatization of qualitative possibility gives sufficient conditions for the existence of a compatible possibility measure, he does not discuss the question if these conditions are also necessary.

Recently [48, 49] a new axiomatisation of qualitative probability that addresses the infinitude of Gärdenfors’ axiomatisation was proposed. Namely, an additional connective \oplus that defines qualitative probability on sequences of formulas is introduced.

$$\bigoplus_{i=1}^n \phi_i \lesssim \bigoplus_{i=1}^n \psi_i \text{ iff } \sum_{i=1}^n \mathbf{p}(\|\phi_i\|) \leq \sum_{i=1}^n \mathbf{p}(\|\psi_i\|) \quad (\mathbf{p} \text{ is a probability measure})$$

This allows for a finite axiomatisation in contrast to the Gärdenfors’ QP of the logic as \oplus can be used to express additivity. On the other hand, \oplus makes the logic hybrid rather than purely qualitative. Moreover, the existence of a quantitative measure is assumed rather than derived from the qualitative relation as it is traditionally.

Yet another treatment of qualitative probabilities inspired by [133] is presented in [12]. The authors devise a sequence of finitely axiomatised calculi that approximate qualitative reasoning with probability measures.

We aim to close these gaps. First, we provide a two-layered logic built over biG that axiomatises reasoning with uncertainty measures (recall Definition 8.1) and then show how to extend it with new axioms corresponding to the qualitative counterparts of stronger measures: in particular, capacities, belief functions, and probabilities. Second, we construct logics that formalise *paraconsistent* qualitative reasoning with uncertainty measures.

⁷¹Note that the standard approach to the formalisation of probabilistic reasoning via expansions of Łukasiewicz logic [84, 85, 73, 66, 44, 11, 67] can also be considered *qualitative* in the sense that the comparison of degrees of certainty is, of course, expressible. However, the arithmetic operations *are definable in \mathcal{L}* which means that we are presupposing that the agent knows exactly how certain they are in a given statement. In this text, however, we do not assume that for the qualitative reasoning and construe ‘qualitative’ as ‘only qualitative’.

9.1. Qualitative characterisations of uncertainty measures

A measure μ on a set W gives rise to a total preorder \preceq on 2^W that is defined as follows: $X \preceq Y$ iff $\mu(X) \leq \mu(Y)$. In this case, μ is said to *agree* with \preceq . If W is the sample space, this preorder can be interpreted as a preference relation between events. Namely, $X \preceq Y$ stands for ‘the agent finds X at most as likely as Y ’.

Historically, the study of the qualitative counterparts of uncertainty measures began with the qualitative probability that was undertaken in [64]. The complete axiomatisation, however, was presented more than twenty years after [94]. The qualitative counterparts of capacities and belief functions were studied as well, albeit, relatively recently (see e.g., [154, 153]). We summarise the results considering various uncertainty measures in the following theorem.

Definition 9.1. Let $W \neq \emptyset$, μ be an uncertainty measure on W , and \preceq be a linear preorder on 2^W . μ is a counterpart of \preceq iff

$$\forall X, Y \subseteq W : X \preceq Y \Leftrightarrow \mu(X) \leq \mu(Y)$$

Theorem 9.1 (Qualitative characterisations of uncertainty measures). *Let $W \neq \emptyset$ and let further \preceq be a linear preorder on 2^W . Consider the following conditions on \preceq for all $X, Y, Z \subseteq W$.*

Q1 $\emptyset \preceq X \preceq W$.

Q2 $\emptyset \prec W$.

Q3 If $X \subseteq Y$, then $X \preceq Y$.

PM If $X \subsetneq Y$, $X \prec Y$, and $Y \cap Z = \emptyset$, then $X \cup Z \prec Y \cup Z$.

KPS_m For any $m \in \mathbb{N}$ and all $X_i, Y_i \subseteq W$ ($i \in \{0, \dots, m\}$), it holds that if $X_j \preceq Y_j$ for all $j < m$ and if any $w \in W$ belongs to as many X_i 's as Y_i 's, then $X_m \succcurlyeq Y_m$.

Then, it holds that

1. the counterparts of \preceq are uncertainty measures and capacities iff \preceq satisfies **Q1–Q3**;
2. the counterparts of \preceq are belief functions iff \preceq satisfies **Q1–Q3** and **PM**;
3. the counterparts of \preceq are probability measures iff \preceq satisfies **Q1–Q3** and **KPS_m**.

The list in Theorem 9.1 is compiled from different sources, whence its obvious redundancy: **Q3** entails **Q1**, and moreover, **Q1**, **Q2**, and **KPS_m** entail **Q3**. We decided to leave the redundant conditions to make the presentation more uniform. Note, furthermore, that the qualitative characterisations do not distinguish between normalised and non-normalised measures.

The original conditions **PM** (‘partial monotonicity’, in the terminology of [154, 153]) and **KPS_m** (Kraft–Pratt–Seidenberg conditions) can be reformulated in terms of measures. For any uncertainty measure μ , its qualitative counterpart is an ordering corresponding to a belief function iff μ **PM** holds and an ordering corresponding to a probability measure iff μ **KPS_m** holds.

μ **PM**: If $\mu(X) < \mu(Y)$, $X \subsetneq Y$, and $Y \cap Z = \emptyset$, then $\mu(X \cup Z) < \mu(Y \cup Z)$.

μ **KPS_m**: For any $m \in \mathbb{N}$ and all $X_i, Y_i \subseteq W$ ($i \in \{0, \dots, m\}$), it holds that if $\mu(X_j) \leq \mu(Y_j)$ for all $j < m$ and if any $w \in W$ belongs to as many X_i 's as Y_i 's, then $\mu(X_m) \geq \mu(Y_m)$.

In what follows, we present two-layered logics that formalise qualitative reasoning with relative likelihood orderings corresponding to uncertainty measures, capacities, belief functions and probabilities.

9.2. Classical qualitative two-layered logics

In this section, we formulate two-layered modal logic $\text{QG} = \langle \text{CPL}, \{\text{B}\}, \text{biG} \rangle$ ⁷² and also study its extensions. Here, we represent the events with the classical propositional logic but reason with the beliefs concerning these events using biG — Gödel logic with co-implication.

One of the most important distinctions of our approach from the ones discussed above is that we use *unary* belief modalities $\text{B}\phi$ whose values are understood as truth degrees of the statement ‘agent believes that ϕ ’ while traditionally, a *binary* modality \lesssim (‘at least as likely as’) is used. At first glance, the use of a binary modality seems to be more justified and straightforward when one deals with qualitative or comparative contexts. However, as one can see from Theorem 9.1, \lesssim cannot distinguish between normalised measures and non-normalised ones. In a sense, binary modalities cannot express statements such as ‘the agent has a positive belief in p ’. Later (cf. Remark 9.3), we will also see some formulas expressing some natural properties of measures that cannot be stated using \lesssim .

Moreover, we claim that our approach is purely qualitative in contrast to those of Delgrande–Renne–Sack in the following sense. First, we do not *a priori* assume that the measure μ on the frame is in fact a belief function, a probability, etc. but merely that the order \preceq corresponding to μ conforms to the needed conditions from Theorem 9.1. Second, our language does not express the quantitative axioms of any uncertainty measure in contrast to that of [48, 49]. In this, we follow the approach of Gärdenfors [69]: we begin with frames whereon a measure is defined and then axiomatise the qualitative conditions corresponding to this measure.

9.2.1. QG — the minimal qualitative logic

Let us now introduce QG in a formal manner. We are first building our qualitative framework correspondingly to generic uncertainty measures. Then, we will add conditions characterising qualitative counterparts of stronger measures which are usually discussed in the literature: in particular, capacities, belief functions, and probability measures.

Definition 9.2 (Language and semantics of QG). We define the following grammar.

$$\mathcal{L}_{\text{QG}} \ni \alpha := \text{B}\phi \mid \sim\alpha \mid (\alpha \wedge \alpha') \mid (\alpha \vee \alpha') \mid (\alpha \rightarrow \alpha) \mid (\alpha \prec \alpha') \quad (\phi \in \mathcal{L}_{\text{CPL}})$$

An *uncertainty frame*⁷³ is a tuple $\mathbb{F} = \langle W, \mu \rangle$. Here, $W \neq \emptyset$, and μ is an uncertainty measure on W . A *QG-model* is a tuple $\mathfrak{M} = \langle W, v, \mu, e \rangle$ with $\langle W, \mu \rangle$ being a frame and $v : \text{Prop} \rightarrow 2^W$. The truth of a formula in a given state ($\mathfrak{M}, x \models \phi$ for $x \in W$ and $\phi \in \mathcal{L}_{\text{CPL}}$) is defined as follows.

- $\mathfrak{M}, x \models p$ iff $x \in v(p)$.
- $\mathfrak{M}, x \models \sim\phi$ iff $\mathfrak{M}, x \not\models \phi$.
- $\mathfrak{M}, x \models \phi \wedge \phi'$ iff $\mathfrak{M}, x \models \phi$ and $\mathfrak{M}, x \models \phi'$.

e is a bi-Gödel valuation (cf. Definition 4.2) s.t. $e(\text{B}\phi) = \mu(\|\phi\|)$ with $\|\phi\| = \{x : \mathfrak{M}, x \models \phi\}$.

QG entailment is defined in the same manner as that of biG :

$$\Xi \models_{\text{QG}} \alpha \text{ iff } \inf\{e(\gamma) : \gamma \in \Xi\} \leq e(\alpha) \text{ for any } e \text{ induced by an uncertainty measure}$$

Finally, $\alpha \in \mathcal{L}_{\text{QG}}$ is valid on \mathbb{F} ($\mathbb{F} \models \alpha$) iff $e(\alpha) = 1$ for any e and v defined on \mathbb{F} .

⁷²We will use a minimalistic $\{\sim, \wedge\}$ language of the classical propositional logic. Other connectives: \supset (implication), \vee (disjunction), \equiv (equivalence), etc. can be introduced in an expected manner.

⁷³In this chapter we will use ‘frame’ as a shorthand for ‘uncertainty frame’.

9. Logics for qualitative reasoning

Remark 9.1. One can notice that \mathbf{B} is *non-compositional* in the sense that neither $\mathbf{B}(t \wedge r) \leftrightarrow (\mathbf{B}t \wedge \mathbf{B}r)$, nor $\mathbf{B}(t \vee r) \leftrightarrow (\mathbf{B}t \vee \mathbf{B}r)$ are \mathbf{QG} valid⁷⁴. However, it can be argued [53] that belief should not be compositional. In fact, all standard uncertainty measures are non-compositional, so it is expected that belief based on these measures is non-compositional too. It is even more true for the case when the truth value of a given belief statement is graded.

Indeed, let t stand for ‘the temperature in the cellar is 26°C’ and r for ‘it is raining outside right now’. The agent is in the cellar right now but there is no thermometer and no windows either. The agent does not feel very cold or hot, so t seems reasonable (say, $v(\mathbf{B}t) = 0.7$); half an hour ago it was cloudy and wet, so the rain is not at all excluded (say, $v(\mathbf{B}r) = 0.5$). However, t and r are not entirely independent, thus, it is hardly possible to precisely determine the degree of certainty in either $\mathbf{B}(t \vee r)$ or $\mathbf{B}(t \wedge r)$.

It is clear that we can formalise the statements of comparative belief in \mathbf{QG} similarly to how we were doing this in \mathbf{KbiG} (recall Example 5.1).

Example 9.1 (Comparing certainty in \mathbf{QG}). Assume that two people, Paula and Quinn, come to you and say that a recently found stray dog belongs to them (and not to the other person). Thus, we have two events: $p \wedge \sim q$ (the dog belongs to Paula but not to Quinn) and $\sim p \wedge q$ (vice versa). Now, assume further that you trust Paula more than you trust Quinn for some reason. Thus, the following statement should be true

dog: I am more certain that the dog belongs to Paula than to Quinn.

We formalise it as follows⁷⁵

$$\sim \Delta(\mathbf{B}(p \wedge \sim q) \rightarrow \mathbf{B}(\sim p \wedge q)) \quad (9.1)$$

Recall [80, Theorem 35] that $\phi(p_1, \dots, p_n) \in \mathcal{L}_{\mathbf{biG}}$ is valid iff it is valid for all valuations that range over $\left\{0, \frac{1}{n+1}, \dots, \frac{n}{n+1}, 1\right\}$. We can use this fact to provide a natural language interpretation of formulas avoiding reference to numerical values altogether.

Since there are two $\mathcal{L}_{\mathbf{QG}}$ atoms in (9.1) — $\mathbf{B}(p \wedge \sim q)$ and $\mathbf{B}(\sim p \wedge q)$ — we need four (numerical) values corresponding to the ordered set: $\left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. We can associate them with the following subjective values: *certainly false, unlikely, likely, certainly true*.

Now, we can have the following assignment: I find it *likely* that the dog belongs to Paula and not Quinn (thus, $e(\mathbf{B}(p \wedge \sim q)) = \text{‘likely’}$) and *unlikely* that the dog belongs to Quinn, not Paula (i.e., $e(\mathbf{B}(\sim p \wedge q)) = \text{‘unlikely’}$). Hence, we conclude that I find that the dog belongs to Paula rather than to Quinn i.e., $\sim \Delta(\mathbf{B}(p \wedge \sim q) \rightarrow \mathbf{B}(\sim p \wedge q))$, *certainly true*.

We are now ready to present the calculus for \mathbf{QG} and prove its completeness.

Definition 9.3 ($\mathcal{H}\mathbf{QG}$). The calculus $\mathcal{H}\mathbf{QG}$ has the following axioms and rules.

nontriv: $\sim \Delta(\mathbf{B}\phi \rightarrow \mathbf{B}\chi)$ for any ϕ and χ s.t. $\text{CPL} \models \phi$ and $\text{CPL} \models \sim \chi$ (nontriviality axioms).

reg: $\mathbf{B}\phi \rightarrow \mathbf{B}\phi'$ with $\text{CPL} \models \phi \supset \phi'$ (regularity axioms).

biG: instantiations of $\mathcal{H}\mathbf{biG}$ axioms and rules with $\mathcal{L}_{\mathbf{QG}}$ formulas.

Remark 9.2. In Definition 9.3, **nontriv** formalises the non-triviality condition on measures since $\|\phi\| = W$ and $\|\chi\| = \emptyset$ when ϕ is a tautology and χ is not classically satisfiable. **reg**⁷⁶ captures the monotonicity condition.

Note, however, that \mathbf{K} is not valid: $\mathbf{B}(p \supset \perp) \rightarrow (\mathbf{B}p \rightarrow \mathbf{B}\perp)$ is easy to disprove. Indeed, consider the model in Fig. 9.1. \mathbf{K} is, however, valid *on single-point models*. Moreover, $\Delta(\mathbf{B}\perp \rightarrow \mathbf{B}\phi)$ and $\Delta(\mathbf{B}\phi \rightarrow \mathbf{B}\top)$ which correspond to **Q1** (Theorem 9.1) are provable in $\mathcal{H}\mathbf{biG}$ from **reg**.

⁷⁴In fact, it is easy to see that there is no general definition of $\mathbf{B}\phi(p_1, \dots, p_n)$ via $\mathbf{B}p_i$ ’s.

⁷⁵We choose this formalisation instead of a simpler $\sim \Delta(\mathbf{B}p \rightarrow \mathbf{B}q)$ because the agent does not consider the event $p \wedge q$ (the dog belongs to both Paula and Quinn) which is not excluded in $\sim \Delta(\mathbf{B}p \rightarrow \mathbf{B}q)$.

⁷⁶The name comes from the notion of *regular modalities*: \mathbf{M} is regular iff the validity of $\sigma \rightarrow \tau$ entails the validity of $\mathbf{M}\sigma \rightarrow \mathbf{M}\tau$.

$$w : p \qquad w' : \sim p$$

Figure 9.1.: $\mu(\{w, w'\}) = 1$, $\mu(\{w\}) = \mu(\{w'\}) = \frac{1}{2}$, $\mu(\emptyset) = 0$.

The completeness result can be proved in a way similar to Theorems 8.1 and 8.2.

Theorem 9.2 (Completeness of $\mathcal{H}\text{QG}$). *Let $\Xi \cup \{\alpha\} \subseteq \mathcal{L}_{\text{QG}}$. Then*

$$\Xi \models_{\text{QG}} \alpha \text{ iff } \Xi \vdash_{\mathcal{H}\text{QG}} \alpha$$

Proof. As regards the soundness part, we just need to check that the axioms are valid. It is clear that if ϕ is a propositional tautology, then $w \models \phi$ for any $w \in W$, and if χ is a contradictory formula, then $w \not\models \chi$ for any $w \in W$. But then, it is clear that $\mu(\|\phi\|) > \mu(\|\chi\|)$, whence $\sim\Delta(\text{B}\phi \rightarrow \text{B}\chi)$ is valid. Furthermore, if $\phi \supset \phi'$ is classically valid, then $v(\phi) \subseteq v(\phi')$ in any model, whence, $\text{B}\phi \rightarrow \text{B}\phi'$ is valid as well.

For completeness, we reason by contraposition. Assume that $\Xi \not\vdash_{\mathcal{H}\text{QG}} \alpha$. We can now extend Ξ with the set of all formulas of the form $\sim(\top \prec \xi)$ with ξ being a modal axiom composed from the subformulas of Ξ and α . Such an extension is possible via applications of the necessitation rule **HBnec** (Definition 4.3) to the modal axioms. Denote the resulting set Ξ^* . It is clear that $\Xi^* \not\vdash_{\mathcal{H}\text{QG}} \alpha$ and that, moreover, by the completeness theorem for bi-Gödel logic (Proposition 4.1), there is a valuation e s.t. $e[\Xi^*] > e(\alpha)$.

It remains to construct a model falsifying the entailment. The biG valuation is already given. We proceed as follows. First, we set $W = 2^{\text{Prop}(\Xi^* \cup \{\alpha\})}$. Then, we define $w \in v(p)$ iff $p \in w$ for any $w \in W$ and extend it to $\|\cdot\|$ in a usual fashion. Finally, for any $\text{B}\phi \in \text{Sf}[\Xi^* \cup \{\alpha\}]$, we set $\mu(\|\phi\|) = e(\text{B}\phi)$. For other $X \subseteq W$, we set $\mu(X) = \sup\{\mu(\|\phi\|) : \phi \in \text{Sf}[\Psi^* \cup \{\alpha\}], \|\phi\| \subseteq X\}$. It remains to check that μ thus defined satisfies Definition 9.2.

To show that μ is monotone, let $X \subseteq X'$. If there exist $\phi, \phi' \in \mathcal{L}_{\text{CPL}}$ s.t. $\|\phi\| = X$ and $\|\phi'\| = X'$, it is clear from the construction of W that $\phi \supset \phi'$ is a classical tautology, whence, $\text{B}\phi \rightarrow \text{B}\phi'$ is an axiom and $\sim(\top \prec (\text{B}\phi \rightarrow \text{B}\phi')) \in \Xi^*$, and thus $\mu(\|\phi\|) \leq \mu(\|\phi'\|)$, as required. Otherwise, recall that

$$\begin{aligned} \mu(X) &= \sup\{\mu(\|\phi\|) : \phi \in \text{Sf}[\Psi^* \cup \{\alpha\}], \|\phi\| \subseteq X\} \\ \mu(X') &= \sup\{\mu(\|\phi'\|) : \phi' \in \text{Sf}[\Psi^* \cup \{\alpha\}], \|\phi'\| \subseteq X'\} \end{aligned}$$

whence, clearly, $\mu(X) \leq \mu(X')$, as required. Finally, since $\sim\Delta(\text{B}\top \rightarrow \text{B}\perp)$ is an instance of the **nontriv** axiom, we have that $\mu(W) > \mu(\emptyset)$. \square

9.2.2. Correspondence theory for weak uncertainty measures

It is clear that **QG** is the logic of all uncertainty frames. However, **QG** does not validate some statements regarding measures one usually expects to hold in the classical case. For example, from the classical point of view and if we subscribe to the closed-world assumption, we *know for certain* that the event ‘it rained in Paris on 28.03.2021 or it did not’ (formally, $r \vee \sim r$) occurred. However, $\text{B}(r \vee \sim r)$ is *not valid* in **QG**⁷⁷ since it can be that $\mu(\|r \vee \sim r\|) < 1$.

Moreover, if belief is represented as a generic uncertainty measure or capacity, it is still possible for two incompatible (or even complementary) events to have measure 1 *at the same time*. Moreover, it is also possible that $\mu(Y \cup Y') > \mu(Y)$ even if $\mu(Y') = 0$. In other words, it is possible that the agent’s certainty in $\phi \vee \phi'$ is strictly higher than that in ϕ even if they are completely certain that ϕ' is not the case.

In this section, we will show how to axiomatise these and other conditions.

⁷⁷This shows that **QG** can distinguish between capacities and generic uncertainty measures: **BT** is valid on a frame if its uncertainty measure is a capacity.

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Convention 9.1. We introduce the following naming conventions for several formulas.

1compl: $\Delta Bp \leftrightarrow \sim B\sim p$

disj+: $(\sim B(p \wedge p') \wedge \sim \sim Bp \wedge \sim \sim Bp') \rightarrow (\sim \Delta(B(p \vee p') \rightarrow Bp) \wedge \sim \Delta(B(p \vee p') \rightarrow Bp'))$

disj0: $\sim Bp \rightarrow \Delta(Bp' \leftrightarrow B(p \vee p'))$

cap: $B\top \wedge \sim B\perp$

In the list above, **1compl** states that the agent completely believes in p iff they completely disbelieve in $\sim p$ (the classical negation of p). **disj+** stipulates that if the beliefs in two incompatible events are positive, then the belief in their disjunction should be strictly greater than either of those. **disj0** states that if the agent completely disbelieves in p , then the degree of their belief in $p \vee p'$ should be equal to the belief in p' . Finally, **cap** stands for the capacity condition on the measure.

The next theorem states that the formulas from [Convention 9.1](#) indeed define their corresponding properties.

Theorem 9.3. *Let $\mathbb{F} = \langle W, \mu \rangle$ be an uncertainty frame. Then the following equivalences hold*

$$\mathbb{F} \models \mathbf{1compl} \text{ iff } \mu(X) = 1 \Leftrightarrow \mu(W \setminus X) = 0 \quad (\text{I})$$

$$\mathbb{F} \models \mathbf{disj+} \text{ iff if } \mu(Y \cap Y') = 0 \text{ and } \mu(Y), \mu(Y') > 0 \text{ then } \mu(Y \cup Y') > \mu(Y), \mu(Y') \quad (\text{II})$$

$$\mathbb{F} \models \mathbf{disj0} \text{ iff } \mu(Y) = 0 \Rightarrow \mu(Y \cup Y') = \mu(Y') \quad (\text{III})$$

$$\mathbb{F} \models \mathbf{cap} \text{ iff } \mu \text{ is a capacity} \quad (\text{IV})$$

Proof. We consider III, the other cases can be tackled in a similar manner.

Indeed, let $\mu(Y) = 0$ but $\mu(Y \cup Y') \neq \mu(Y')$ for some $Y, Y' \subseteq W$. Now let $v(p) = Y$ and $v(p') = Y'$. Then, it is clear that $e(Bp) = 0$ but $e(Bp') \neq e(B(p \vee p'))$. Hence, $e(\mathbf{disj0}) \neq 1$, as required.

For the converse, we assume that $e(\mathbf{disj0}) \neq 1$. But then, $e(\mathbf{disj0}) = 0$ since **disj0** is composed of Δ -formulas and \sim -formulas. Thus $e(\sim Bp) = 1$ but $e(\Delta(Bp' \leftrightarrow B(p \vee p')) = 0$ (i.e., $\mu(\|p\| \cup \|p'\|) \neq \mu(\|p'\|)$), whence $\mu(\|p\|) = 0$ but $\mu(\|p\| \cup \|p'\|) \neq \mu(\|p'\|)$, as required. \square

Now, if we want to formalise qualitative counterparts of belief functions, we need to transform $\mu\mathbf{PM}$ into an axiom. However, there is no two-layered formula that can formalise $X \subsetneq Y$. Thus, we have to introduce a new *axiom schema*.

$$\begin{aligned} & \sim \Delta(B\chi \rightarrow B\phi) \rightarrow \sim \Delta(B(\chi \vee \psi) \rightarrow B(\phi \vee \psi)) \\ & \text{with } \text{CPL} \models \phi \supset \chi, \text{CPL} \models \sim(\chi \wedge \psi), \text{ and } \text{CPL} \not\models \chi \supset \phi \end{aligned} \quad (\text{QBel})$$

Theorem 9.4. *Let $\mathbb{F} = \langle W, \mu \rangle$ be an uncertainty frame. Then μ satisfies $\mu\mathbf{PM}$ iff $\mathbb{F} \models \text{QBel}$.*

Proof. Let $\mu\mathbf{PM}$ hold, and let further ϕ, χ , and ψ be as in [\(QBel\)](#). Thus, $\|\phi\| \subseteq \|\chi\|$ and $\|\chi\| \cap \|\psi\| = \emptyset$. Note also that both $\sim \Delta(B\chi \rightarrow B\phi)$ and $\sim \Delta(B(\chi \vee \psi) \rightarrow B(\phi \vee \psi))$ can have values only in $\{0, 1\}$.

Now, if $e(\sim \Delta(B\chi \rightarrow B\phi)) = 1$, then $\mu(\|\phi\|) < \mu(\|\chi\|)$ and, in fact, $\|\phi\| \subsetneq \|\chi\|$. But then $\mu(\|\phi\| \cup \|\psi\|) < \mu(\|\chi\| \cup \|\psi\|)$, whence $\mu(\|\phi \vee \psi\|) < \mu(\|\chi \vee \psi\|)$, and thus $e(\sim \Delta(B(\chi \vee \psi) \rightarrow B(\phi \vee \psi))) = 1$, as well.

For the converse, let $\mu\mathbf{PM}$ fail for \mathbb{F} , and let, in particular, $\mu(X) < \mu(Y)$, $X \subsetneq Y$, and $Y \cap Z = \emptyset$, but $\mu(X \cup Z) \geq \mu(Y \cup Z)$. We show how to falsify [QBel](#).

We let $\|p\| = X$, $\|p \vee q\| = Y$, and $\|\sim q \wedge r\| = Z$. Now, it is easy to see that

$$e(\sim \Delta(B(p \vee q) \rightarrow Bp) \rightarrow \sim \Delta(B((p \vee q) \vee (\sim q \wedge r)) \rightarrow B(p \vee (\sim q \wedge r)))) = 0$$

as required. \square

9.2.3. Logics of qualitative probabilities

The main objective of this section is to provide a two-layered axiomatisation of qualitative probabilities. To do this, we need to transform $\mu\mathbf{KPS}_m$ into axioms. This, however, is not straightforward. This is why, we will take a detour through the classical logic of qualitative probability QP introduced in [69]. The language of QP is given by the following grammar.

$$\mathcal{L}_{\text{QP}} \ni \phi := p \in \mathbf{Prop} \mid \sim\phi \mid (\phi \wedge \psi) \mid (\phi \lesssim \psi).$$

The semantics uses probabilistic frames and models built upon them.

Definition 9.4 (Frame semantics of QP [69]). A *Gärdenfors probabilistic frame* is a tuple $\mathbb{F} = \langle U, \{\mathbf{P}_x\}_{x \in U} \rangle$ with $U \neq \emptyset$ and $\{\mathbf{P}_x\}_{x \in U}$ being a family of probability measures on 2^U . A *Gärdenfors model* is a tuple $\mathfrak{M} = \langle U, \{\mathbf{P}_x\}_{x \in U}, v \rangle$ with $\langle U, \{\mathbf{P}_x\}_{x \in U} \rangle$ being a frame and $v : \mathbf{Prop} \rightarrow 2^U$ being a valuation that is extended to a satisfaction relation $\|\cdot\|$ as follows:

- $\|p\| = v(p)$;
- $\|\sim\phi\| = U \setminus \|\phi\|$;
- $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$;
- $\|\phi \lesssim \psi\| = \{x \in U \mid \mathbf{P}_x(\|\phi\|) \leq \mathbf{P}_x(\|\psi\|)\}$.

For any model \mathfrak{M} , we say that ϕ is *true in \mathfrak{M}* ($\mathfrak{M} \models \phi$) iff $\|\phi\| = U$. Furthermore, ϕ is *valid in \mathbb{F}* ($\mathbb{F} \models \phi$) iff ϕ is true in every model on \mathbb{F} .

Furthermore, we can define an additional notion of satisfaction in a state.

Definition 9.5 (Pointed model semantics). Let \mathfrak{M} be a Gärdenfors model and $x \in \mathfrak{M}$. We define $\mathfrak{M}, x \models \phi$ (ϕ is true at x) as follows.

- $\mathfrak{M}, x \models p$ iff $x \in v(p)$.
- $\mathfrak{M}, x \models \sim\phi$ iff $\mathfrak{M}, x \not\models \phi$.
- $\mathfrak{M}, x \models \phi \wedge \phi'$ iff $\mathfrak{M}, x \models \phi$ and $\mathfrak{M}, x \models \phi'$.
- $\mathfrak{M}, x \models \phi \lesssim \phi'$ iff $\mathbf{P}_x(\|\phi\|) \leq \mathbf{P}_x(\|\phi'\|)$.

In the remainder of the paper, we will call a tuple $\langle \mathfrak{M}, x \rangle$ a *pointed model*.

Other connectives and modalities can be introduced in an expected fashion:

$$\begin{aligned} \phi \vee \phi' &:= \sim(\sim\phi \wedge \sim\phi') & \phi \supset \phi' &:= \sim\phi \vee \phi' & \phi \equiv \phi' &:= (\phi \supset \phi') \wedge (\phi' \supset \phi) \\ \phi \approx \phi' &:= (\phi \lesssim \phi') \wedge (\phi' \lesssim \phi) & \top &:= p \supset p & \phi < \phi' &:= (\phi \lesssim \phi') \wedge \sim(\phi' \lesssim \phi) \end{aligned}$$

Let us now recall the axiomatisation of QP. For this, we borrow the E-notation from [135] and [69]. This will help us express the Kraft–Pratt–Seidenberg conditions in a more concise manner.

Convention 9.2 (E-notation). Consider \mathcal{L}_{QP} formulas ϕ_1, \dots, ϕ_n and χ_1, \dots, χ_n . Let further, $\phi^\circ \in \{\phi, \sim\phi\}$ and $\chi^\circ \in \{\chi, \sim\chi\}$. We introduce a new operator **E** and write

$$\phi_1, \dots, \phi_n \mathbf{E} \chi_1, \dots, \chi_n$$

to designate that necessarily the same number of ϕ_i° 's as of χ_j° 's are actually of the form $\sim\phi_i$ and $\sim\chi_j$, respectively. For example

$$p_1, p_2 \mathbf{E} q_1, q_2 := ((p_1 \wedge p_2 \wedge q_1 \wedge q_2) \vee (\sim p_1 \wedge p_2 \wedge \sim q_1 \wedge q_2))$$

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$$\begin{aligned} & \vee (\sim p_1 \wedge p_2 \wedge q_1 \wedge \sim q_2) \vee (p_1 \wedge \sim p_2 \wedge q_1 \wedge \sim q_2) \\ & \vee (p_1 \wedge \sim p_2 \wedge \sim q_1 \wedge q_2) \vee (\sim p_1 \wedge \sim p_2 \wedge \sim q_1 \wedge \sim q_2) \approx \top \end{aligned}$$

More formally, we let $M = \{1, \dots, m\}$, $K, L \subseteq M$ and set

$$\phi_1, \dots, \phi_m \mathbf{E} \chi_1, \dots, \chi_m := \left(\bigvee_{\substack{i=0 \\ |K|=i \\ |L|=i}}^m \bigvee \left(\bigwedge_{k \in K} \sim \phi_k \wedge \bigwedge_{k' \in M \setminus K} \phi_{k'} \wedge \bigwedge_{l \in L} \sim \chi_l \wedge \bigwedge_{l' \in M \setminus L} \chi_{l'} \right) \right) \approx \top$$

The axiomatisation of QP which we call \mathcal{HQP} expands the classical propositional rules with new axioms and rules concerning \lesssim .

Definition 9.6 (\mathcal{HQP} — Hilbert-style calculus for QP). The calculus contains the following axioms and rules.

(PC): All propositional tautologies.

(A0): $((\phi_1 \equiv \phi_2) \approx \top) \wedge ((\psi_1 \equiv \psi_2) \approx \top) \supset ((\phi_1 \lesssim \psi_1) \equiv (\phi_2 \lesssim \psi_2))$.

(A1): $\perp \lesssim \phi$.

(A2): $(\phi \lesssim \psi) \vee (\psi \lesssim \phi)$.

(A3): $\perp < \top$.

(A4)_m: $\left((\phi_1, \dots, \phi_m \mathbf{E} \psi_1, \dots, \psi_m) \wedge \bigwedge_{i=1}^{m-1} (\phi_i \lesssim \psi_i) \right) \supset (\psi_m \lesssim \phi_m)$

The rules are modus ponens and necessitation:

$$\text{MP} : \frac{\phi \supset \chi \quad \phi}{\chi} \qquad \text{nec} : \frac{\vdash \phi}{\vdash \phi \approx \top}$$

Let us consider the modal axioms. (A0) allows for substitutions of ‘believably equivalent’ formulas. Other axioms correspond to the conditions on the preorders we cited in Theorem 9.1. In particular, (A1) corresponds to Q1; (A2) is the linearity condition on \preccurlyeq ; (A3) corresponds to Q2. Finally, the family of axioms (A4)_m corresponds to the \mathbf{KPS}_m . As expected [69, P.179], the expansion of the classical propositional logic with these axioms is complete w.r.t. all Gärdenfors’ probabilistic frames.

Note that \mathcal{L}_{QP} does allow for the nesting of \lesssim while \mathcal{L}_{QG} prohibits the nesting of B. However, a specific fragment of \mathcal{L}_{QP} can be embedded into \mathcal{L}_{QG} .

Definition 9.7 (Embedding of simple inequality formulas). We define *simple inequality formulas* (SIF’s) using the following grammar ($\chi, \chi' \in \mathcal{L}_{CPL}$):

$$\text{SIF} \ni \phi := \chi \lesssim \chi' \mid \sim \phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \supset \phi)$$

We define a translation Δ of SIF’s into \mathcal{L}_{QG} as follows.

$$\begin{aligned} (\chi \lesssim \chi')^\Delta &= \Delta(\mathbf{B}\chi \rightarrow \mathbf{B}\chi') \\ (\sim \phi)^\Delta &= \sim \phi^\Delta \\ (\phi \circ \phi')^\Delta &= \phi^\Delta \circ \phi'^\Delta \\ (\phi \supset \phi')^\Delta &= \phi^\Delta \rightarrow \phi'^\Delta \end{aligned} \qquad (\circ \in \{\wedge, \vee\})$$

Remark 9.3. It is instructive to observe that not all statements about comparing beliefs can be represented as SIF's and their translations into \mathcal{L}_{QG} . Indeed, cap and disj+ are *not* translations of SIF's. In fact, $\sim\sim\text{B}p$ stipulates that the agent's belief in p is *positive*. In QP, it can only be expressed as $p > \perp$. However, as we have already mentioned, QP cannot distinguish between normalised and non-normalised measures. Thus, one could demand that P_x 's be not probability measures but any uncertainty measures satisfying Kraft–Pratt–Seidenberg conditions. This means that $p > \perp$ is *stronger than* $\sim\sim\text{B}p$ for the latter is *compatible with* $\Delta(\text{B}p \leftrightarrow \text{B}\perp)$.

In what follows, we will say that a QG-model $\mathfrak{M} = \langle W, v, \mu, e \rangle$ is a QPG model if μ satisfies μKPS_m . In other words, in a QPG model, the order on 2^W induced by μ is a qualitative counterpart of a probability measure.

We can now establish that the translation in Definition 9.7 is indeed faithful. We do this by showing how to transform a given pointed Gärdenfors model $\langle \mathfrak{M}, x \rangle$ (recall Definition 9.5) into a QPG model that satisfies exactly the translations of SIF's that $\langle \mathfrak{M}, x \rangle$ satisfies. And conversely, how to provide a Gärdenfors model using a given QPG model preserving all satisfied SIF's.

Definition 9.8 (G-counterparts). Let $\mathfrak{M} = \langle U, \{\text{P}_x\}_{x \in U}, v \rangle$ be a Gärdenfors model. A *G-counterpart* of a pointed model $\langle \mathfrak{M}, x \rangle$ is the QPG model $\mathfrak{M}_{\text{QPG}} = \langle U, v, \text{P}_x, e \rangle$.

Remark 9.4. Note that a Δ -less translation of $\chi \lesssim \chi'$ as $\text{B}\chi \rightarrow \text{B}\chi'$ does not preserve truth. Indeed, let $\text{P}_x(\|p\|) = 0.7$, $\text{P}_x(\|q\|) = 0.6$, $\text{P}_x(\|r\|) = 0.5$, $\text{P}_x(\|s\|) = 0.4$. Then $(p \lesssim q) \supset (r \lesssim s)$ is *true* at x but $e((\text{B}p \rightarrow \text{B}q) \rightarrow (\text{B}r \rightarrow \text{B}s)) = 0.4$.

Lemma 9.1. *Let $\langle \mathfrak{M}, x \rangle$ be a pointed Gärdenfors model and $\mathfrak{M}_{\text{QPG}}$ its G-counterpart, then $\mathfrak{M}, x \models \phi$ iff $e(\phi^\Delta) = 1$ for any $\phi \in \text{SIF}$.*

Proof. First, it is clear that for any classical formula χ , it holds that $\|\chi\| = v(\chi)$ since \mathfrak{M} and \mathfrak{M}_{G} have the same valuation.

We proceed by induction on ϕ . First, let $\phi := (\chi \lesssim \chi')$.

$$\begin{aligned} \mathfrak{M}, x \models \chi \lesssim \chi' &\text{ iff } \text{P}_x(\|\chi\|) \leq \text{P}_x(\|\chi'\|) \\ &\text{ iff } \text{P}_x(v(\chi)) \leq \text{P}_x(v(\chi')) && (\|\chi\| = v(\chi)) \\ &\text{ iff } e(\text{B}\chi \rightarrow \text{B}\chi') = 1 \\ &\text{ iff } e(\Delta(\text{B}\chi \rightarrow \text{B}\chi')) = 1 \end{aligned}$$

For the inductive step, we consider $\phi := \phi_1 \wedge \phi_2$ and $\phi := \sim\phi'$.

$$\begin{aligned} \mathfrak{M}, x \models \phi_1 \wedge \phi_2 &\text{ iff } \mathfrak{M}, x \models \phi_1 \text{ and } \mathfrak{M}, x \models \phi_2 \\ &\text{ iff } e(\phi_1^\Delta) \text{ and } e(\phi_2^\Delta) = 1 && (\text{by IH}) \\ &\text{ iff } e((\phi_1 \wedge \phi_2)^\Delta) = 1 \end{aligned}$$

$$\begin{aligned} \mathfrak{M}, x \models \sim\phi' &\text{ iff } \mathfrak{M}, x \not\models \phi' \\ &\text{ iff } e(\phi'^\Delta) \neq 1 && (\text{by IH}) \\ &\text{ iff } e(\phi'^\Delta) = 0 && (\phi'^\Delta \text{ is a Boolean combination of } \Delta\text{-formulas}) \\ &\text{ iff } e((\sim\phi')^\Delta) = 1 \end{aligned}$$

□

Definition 9.9 (QP-counterparts). Let $\mathfrak{M} = \langle W, v, \mu, e \rangle$ be a QPG model. Its *QP-counterpart* is any QP pointed model $\langle \mathfrak{M}_{\text{G}}, w \rangle$ with $w \in W$ s.t. $\mathfrak{M}_{\text{G}} = \langle W, v, \{\pi_{\mu_x}\}_{x \in W} \rangle$ and π_{μ_x} is a probability measure s.t. $\mu(X) \leq \mu(Y)$ iff $\pi_{\mu_x}(X) \leq \pi_{\mu_x}(Y)$ for all $X, Y \subseteq W$.

Proposition 9.1. *For any QPG model, there exists its QP-counterpart.*

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Proof. Note that μ conforms to Kraft–Pratt–Seidenberg conditions [94, 134]. Thus, there is a probability measure on the same set that preserves all orders from μ . \square

Note that we do not demand QP-counterparts to be unique as we are able to prove the next statement regardless.

Lemma 9.2. *Let $\mathfrak{M} = \langle W, v, \mu, e \rangle$ be a QPG model and $\langle \mathfrak{M}_{\mathbf{G}}, w \rangle$ one of its counterparts. Then, $e(\phi^{\Delta}) = 1$ iff $\mathfrak{M}_{\mathbf{G}}, w \models \phi$ for any $\phi \in \text{SIF}$.*

Proof. Let $e(\phi^{\Delta}(\mathbf{s}_1^{\Delta}, \dots, \mathbf{s}_n^{\Delta})) = 1$ with $\mathbf{s}_i = \chi_i \lesssim \chi'_i$ and $\mathbf{s}_i^{\Delta} = \Delta(\mathbf{B}\chi_i \rightarrow \mathbf{B}\chi'_i)$. Since the measure on the QP-counterpart preserves all order relations from \mathfrak{M} , it is clear that $\mathfrak{M}, w \models \mathbf{s}_i$ iff $e(\mathbf{s}_i) = 1$ for all $i \leq n$. But then we have that $e(\phi^{\Delta}) = 1$ iff $\mathfrak{M}_{\mathbf{G}}, w \models \phi$ since $e(\mathbf{s}_i^{\Delta}) \in \{0, 1\}$ and Gödel connectives behave classically on values 0 and 1. \square

Theorem 9.5. *Let $\phi \in \text{SIF}$. Then ϕ is QP valid iff ϕ^{Δ} is QPG valid.*

Proof. Immediately from Lemmas 9.1 and 9.2. \square

Now, observe that if we instantiate ϕ_i 's and ψ_i 's in $(\mathbf{A4})_{\mathbf{m}}$ with propositional formulas, these formulas are going to be SIF's. This means that to obtain the axiomatisation of the logic complete w.r.t. QPG frames, we only need to translate $(\mathbf{A4})_{\mathbf{m}}$ into \mathcal{L}_{QG} .

Convention 9.3 (E-notation for QPG). Consider \mathcal{L}_{CPL} -formulas ϕ_1, \dots, ϕ_n and χ_1, \dots, χ_n . Let further, $\phi^{\circ} \in \{\phi, \sim\phi\}$ and $\chi^{\circ} \in \{\chi, \sim\chi\}$. We introduce operator $\mathbf{E}_{\mathbf{G}}$ and write

$$\phi_1, \dots, \phi_n \mathbf{E}_{\mathbf{G}} \chi_1, \dots, \chi_n$$

to designate that necessarily the same number of ϕ_i° 's as of χ_j° 's are actually of the form $\sim\phi_i$ and $\sim\chi_j$, respectively.

More formally, we let $M = \{1, \dots, m\}$, $K, L \subseteq M$, and set

$$\phi_1, \dots, \phi_m \mathbf{E}_{\mathbf{G}} \chi_1, \dots, \chi_m := \Delta \left(\mathbf{B} \left(\bigvee_{i=0}^m \bigvee_{\substack{|K|=i \\ |L|=i}} \left(\bigwedge_{k \in K} \sim\phi_k \wedge \bigwedge_{k' \in M \setminus K} \phi_{k'} \wedge \bigwedge_{l \in L} \sim\chi_l \wedge \bigwedge_{l' \in M \setminus L} \chi_{l'} \right) \right) \leftrightarrow \mathbf{B}\top \right)$$

$\mathbf{E}_{\mathbf{G}}$ has the same intended meaning as \mathbf{E} . Namely, that the measure of

$$\bigvee_{i=0}^m \bigvee_{\substack{|K|=i \\ |L|=i}} \left(\bigwedge_{k \in K} \sim\phi_k \wedge \bigwedge_{k' \in M \setminus K} \phi_{k'} \wedge \bigwedge_{l \in L} \sim\chi_l \wedge \bigwedge_{l' \in M \setminus L} \chi_{l'} \right)$$

is equal to the measure of \top .

Finally, we define

$$\text{KPS}_m: \left((\phi_1, \dots, \phi_m \mathbf{E}_{\mathbf{G}} \chi_1, \dots, \chi_m) \wedge \bigwedge_{i=1}^{m-1} \Delta(\mathbf{B}\phi_i \rightarrow \mathbf{B}\chi_i) \right) \rightarrow \Delta(\mathbf{B}\chi_m \rightarrow \mathbf{B}\phi_m).$$

In what follows, we use $\mathcal{H}\text{QPG}$ to designate the extension of $\mathcal{H}\text{QG}$ with KPS_m axioms for every $m > 0$.

The next statements are straightforward corollaries from Theorems 9.3 and 9.5.

Theorem 9.6. *Let $\text{KPS} = \{\text{KPS}_m : m \in \mathbb{N}\}$ and $\mathbb{F} = \langle W, \mu \rangle$. Then $\mathbb{F} \models \text{KPS}$ iff μ satisfies Kraft–Pratt–Seidenberg conditions.*

Convention 9.4. Let \mathcal{H} be a Hilbert-style calculus and Φ be a scheme of formulas. We denote with $\mathcal{H} \otimes \Phi$ the calculus obtained by adding Φ to \mathcal{H} as an *axiom scheme*. We also say that \mathcal{H} is the logic of a class \mathbb{K} of frames iff

$$\Xi \vdash_{\mathcal{H}} \alpha \text{ iff } \Xi \models_{\mathbb{K}} \alpha$$

Theorem 9.7.

1. $\mathcal{H}\text{QG} \otimes \text{1compl}$ is the logic of the frames satisfying condition (I) from Theorem 9.3.
2. $\mathcal{H}\text{QG} \otimes \text{disj+}$ is the logic of the frames satisfying condition (II) from Theorem 9.3.
3. $\mathcal{H}\text{QG} \otimes \text{disj0}$ is the logic of the frames satisfying condition (III) from Theorem 9.3.
4. $\mathcal{H}\text{QG} \otimes \text{cap}$ is the logic of the frames whose measure is a capacity.
5. $\mathcal{H}\text{QG} \otimes \text{QBel}$ is the logic of the frames whose measure satisfies μPM .
6. $\mathcal{H}\text{QPG}$ is the logic of QPG frames.

Proof. All the proofs can be conducted in a similar manner. This is why, we provide only the most instructive case — that of $\mathcal{H}\text{QPG}$. Soundness follows immediately from Theorem 9.6. For completeness, we reason by contraposition.

Assume that $\Xi \not\vdash_{\mathcal{H}\text{QPG}} \alpha$. We extend Ξ with all formulas of the form $\sim(\top \prec \xi)$ with ξ being an instance of a modal axiom composed from $\text{Sf}[\Xi \cup \{\alpha\}]$ and denote the resulting set Ξ^* . Since all such formulas are theorems in $\mathcal{H}\text{QPG}$, it is clear that $\Xi^* \not\vdash_{\mathcal{H}\text{QPG}} \alpha$. But then, by the strong completeness of biG , we have that there is a biG valuation e s.t. $e[\Xi^*] > e(\alpha)$. Furthermore, it is clear that $e(\sim(\top \prec \xi)) = 1$ (whence, $e(\xi) = 1$) for every ξ .

We now need to construct the falsifying model. The outer valuation (e) is already given. Now, we set $W = 2^{\text{Prop}(\Xi^* \cup \{\alpha\})}$. Then, we define $w \in v(p)$ iff $p \in w$ for any $w \in W$ and extend it to $\|\cdot\|$ in a usual fashion. Finally, for any $\text{B}\phi \in \text{Sf}[\Xi^* \cup \{\alpha\}]$, we set $\mu(\phi) = e(\text{B}\phi)$. It is clear that μ is defined on a subalgebra of W over set union, intersection and complement and that it satisfies **Q1–Q3** and **KPS_m** from Theorem 9.1. But then, there is a (possibly non-normalised) probability measure \mathbf{p}_μ on this subalgebra that agrees with μ on the order. Thus, we can extend \mathbf{p}_μ to the entire W , and clearly, the extended measure will satisfy **Q1–Q3** and **KPS_m** because it is a probability measure. \square

9.3. Paraconsistent qualitative two-layered logics

As we have already discussed, an agent does not need to be able to assign a number to their certainty in a given statement. Furthermore, the evidence regarding a given statement may be contradictory or incomplete, whence if we want to compare our certainty in different statements, we need to treat evidence in favour and evidence against independently. Consider, e.g., the following situation: all sources give contradictory information regarding ϕ but no information regarding χ , both our certainty that ϕ is true and our certainty in its falsity should be greater than our certainty in either truth or falsity of χ .

These characteristics of evidence can be illustrated in the context of court proceedings. Indeed, the evidence in court has the features listed above: rarely, one can reliably measure one's certainty in any given piece thereof, instead, the court tries to establish whose claims are more compelling; the evidence presented by witnesses can be incomplete or inconsistent; in any court proceeding there are two parties, and the non-contradictory evidence can be treated as favouring one of them (or irrelevant to the process).

This is why we analyse these contexts using paraconsistent Gödel logics $\mathbf{G}_{(\rightarrow, \prec)}^2$ and $\mathbf{G}_{(\rightarrow, \dashv)}^2$ introduced in Chapter 4 on the outer layer (while **BD** is the inner-layer logic). Since Gödel logic can be seen as a logic of comparative truth, its paraconsistent expansion with a De Morgan negation \neg can be seen as a logic of comparative truth and falsity.

9.3.1. Language and semantics

In this section, we provide two logics of monotone comparative belief based on G^2 : $MCB = \langle BD, \{C\}, G^2_{(\rightarrow, <)} \rangle$ (stands for ‘monotone comparative belief’) and the Nelson-like MCB designated $NMCB = \langle BD, \{C\}, G^2_{(\rightarrow, \neg\circ)} \rangle$. We will treat each atomic modal formula $C\phi$ (read ‘the agent is certain in the truth of ϕ ’) as a piece of evidence. The first coordinate supports one party, and the second the other. Pieces of evidence can be combined in different fashions: we can compare our certainty therein using (co-)implication; choose the more or less certain one with \vee and \wedge , etc. Gödelian negation represents the countering of a given statement — we show that it is absurd. Finally, \neg is the operator that swaps the support of truth and the support of falsity. But in the context of a court session, if a statement is used as an argument for one party, then its negation is actually an argument for the other. Thus, we posit that $\neg C\phi$ is equivalent to $C\neg\phi$.

Furthermore, there is a difference between criminal and civil proceedings (as well as arbitrations). Namely, during a criminal proceeding, a defendant is pronounced innocent as long as they were able to present conclusive evidence in their favour or counter the evidence of the prosecution. Furthermore, contradictions are usually interpreted in favour of the defendant. Thus, the two parties in the proceeding are not equal in this respect. On the other hand, both parties in a civil court (say, two relatives settle an inheritance dispute in court) or an arbitration present evidence in their own favour, after which the court determines whose evidence was more compelling.

This difference can be formalised if we recall two G^2 logics and their entailments. $G^2_{(\rightarrow, <)}$ takes into account *both* coordinates of a given valuation which makes it closer to the reasoning demonstrated in a civil process. On the other hand, $G^2_{(\rightarrow, \neg\circ)}$ takes into account only the first coordinate. Thus, we can associate each coordinate to a party of the process (for $G^2_{(\rightarrow, \neg\circ)}$, the first coordinate stands for the defence, and the second one for the prosecution).

Now, if $\Gamma \cup \{\phi\}$ is a set of statements concerning some evidence, then entailment relations can be interpreted as preservation of the degree of certainty from premises to the conclusion. We will call C the modality of monotone comparative belief. Here, ‘monotone’ means that C conforms to the underlying BD entailment in the sense that if $\phi \vdash \chi$ is valid in BD , $C\phi$ implies $C\chi$ in the outer-layer logic. ‘Comparative’ relates to the fact that we use Gödel logic which can be thought of as a logic of comparative truth.

Definition 9.10 (Languages of MCB and NMCB). The languages of MCB and NMCB (\mathcal{L}_{MCB} and \mathcal{L}_{NMCB} , respectively) are defined via the following grammars.

$$\mathcal{L}_{MCB} : \alpha := C\phi \mid \neg\alpha \mid \alpha \circ \alpha \quad (\circ \in \{\wedge, \vee, \rightarrow, <\}, \phi \in \mathcal{L}_{BD})$$

$$\mathcal{L}_{NMCB} : \alpha := C\phi \mid \neg\alpha \mid \alpha \circ \alpha \quad (\circ \in \{\wedge, \vee, \rightarrow, \neg\circ\}, \phi \in \mathcal{L}_{BD})$$

Definition 9.11 (Semantics of MCB and NMCB). An MCB-model (NMCB-model) is a tuple $\mathcal{M} = \langle W, v^+, v^-, \pi, e_1, e_2 \rangle$ with $\langle W, v^+, v^- \rangle$ being a BD model (cf. Definition 2.2), $\pi : 2^W \rightarrow [0, 1]$ being an uncertainty measure (recall Definition 8.1).

Semantic conditions of atomic \mathcal{L}_{MCB} and \mathcal{L}_{NMCB} formulas are as follows.

$$\begin{aligned} e_1(C\phi) &= \pi(\{w : w \models^+ \phi\}) = \pi(|\phi|^+) \\ e_2(C\phi) &= \pi(\{w : w \models^- \phi\}) = \pi(|\phi|^-) \end{aligned}$$

Values of complex formulas are computed according to Definition 4.4.

For a frame $\mathbb{F} = \langle W, \pi \rangle$ on an MCB (NMCB) model \mathcal{M} , we say that $\alpha \in \mathcal{L}_{MCB}$ ($\beta \in \mathcal{L}_{NMCB}$) is valid on \mathbb{F} ($\mathbb{F} \models \alpha$ and $\mathbb{F} \models \beta$, respectively) iff $e(\alpha) = (1, 0)$ ($e_1(\beta) = 1$) for every e_1 and e_2 on \mathbb{F} . Finally, for $\Psi \cup \{\alpha\} \subseteq \mathcal{L}_{MCB}$ and $\Omega \cup \{\beta\} \subseteq \mathcal{L}_{NMCB}$, we define the same entailment relations as in Definition 4.5.

Convention 9.5. In what follows, we will use \Rightarrow and \Leftrightarrow — congruential versions of \rightarrow and \leftrightarrow — defined as in Definition 3.6.

One can check that

$$\begin{aligned} e_1(\alpha \Rightarrow \beta) &= 1 \text{ iff } e_1(\alpha) \leq e_1(\beta) \text{ and } e_2(\alpha) \geq e_2(\beta) \\ e_1(\alpha \Leftrightarrow \beta) &= 1 \text{ iff } e_1(\alpha) = e_1(\beta) \text{ and } e_2(\alpha) = e_2(\beta) \end{aligned}$$

As expected (and as was the case in \mathbf{KG}^{2c} , $\mathbf{KG}^{2\pm}$, and $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$) there is a difference between MCB and NMCB on one hand and QG on the other hand. Namely, in QPG an agent can compare their beliefs in any two given statements. This, however, is not the case in MCB and NMCB.

To see this, we recall Δ^\top from Example 6.1 (cf. equation (6.1)) and define a new connective Δ^N (recall notation introduced in Convention 4.2).

$$\Delta^N \beta := \sim_N(\top_N \multimap \beta)$$

One can check that

$$e_1(\Delta^N \beta) = \begin{cases} 1 & \text{iff } e_1(\beta) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, it is easy to see that in contrast to $\Delta(\alpha \rightarrow \alpha') \vee \Delta(\alpha' \rightarrow \alpha)$ that is QG-valid, neither $\Delta^\top(\alpha \rightarrow \alpha') \vee \Delta^\top(\alpha' \rightarrow \alpha)$ nor $\Delta^N(\alpha \Rightarrow \alpha') \vee \Delta^N(\alpha' \Rightarrow \alpha)$ are valid in MCB and NMCB, respectively.

Intuitively, this failure of comparability is justified. First, α and α' can be irrelevant to one another. Indeed, we cannot always answer conclusively what we consider more likely: that it will rain tomorrow or that we will find our lost dog. Second, even if the events are related, we are not necessarily able to compare our confidence in them when the evidence is of different nature.

Recall the situation of Paula and Quinn claiming that the dog is theirs from Example 9.1. Assume now that Paula shows a photo of her with the dog on the leash and Quinn shows the (same or at least very similar) leash. Neither piece of evidence is conclusive and, without further investigation, it might not be clear whether one is stronger than the other.

Third, if the events are described *classically* (as done in QP, QG, and QPG), then all contradictory events have measure 0 (or the least possible positive measure). However, if an agent tries to align their beliefs with what they are told by their sources, this is not necessarily the case. Indeed, if I do not have any information at all regarding p , then $\pi(|p|^+) = 0$ and $\pi(|p|^-) = 0$, whence $e(\mathbf{C}(p \wedge \neg p)) = (0, 0)$. On the other hand, if I have somewhat reliable sources claiming that q is true and some others (less trusted ones) that it is false, then I can posit that $\pi(|q|^+) = 0.5$ and $\pi(|q|^-) = 0.3$ ⁷⁸, whence $e(\mathbf{C}(q \wedge \neg q)) = (0.3, 0.5)$. But then my certainty in $p \wedge \neg p$ is incomparable to that in $q \wedge \neg q$. Finally, if I know for certain that r is true (i.e., $\pi(|r|^+) = 1$ and $\pi(|r|^-) = 0$), then $e(\mathbf{C}(r \wedge \neg r)) = (0, 1)$. Thus my certainty in $r \wedge \neg r$ is strictly below that in both $p \wedge \neg p$ and $q \wedge \neg q$.

9.3.2. Axiomatisation

Let us now introduce Hilbert-style calculi for MCB and NMCB.

Definition 9.12 ($\mathcal{H}\text{MCB}$). The calculus consists of the following axioms and rules ($\phi, \chi \in \mathcal{L}_{\text{BD}}$ and $\alpha, \beta \in \mathcal{L}_{\text{MCB}}$).

$\mathcal{H}\text{MCB}_{\text{BD}}$: $\mathbf{C}\phi \rightarrow \mathbf{C}\chi$ for any $\phi, \chi \in \mathcal{L}_{\text{BD}}$ s.t. $\phi \vdash \chi$ is BD valid.

$\mathcal{H}\text{MCB}_{\neg}$: $\mathbf{C}\neg\phi \leftrightarrow \neg\mathbf{C}\phi$.

⁷⁸Recall, that both in biG and \mathbf{G}^2 , the exact numbers assigned to our certainty in a given event are of little importance. What matters is that (in this case) I have some information that suggests that q is true and some that it is false.

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$\mathcal{G}_{(\rightarrow, \leftarrow)}^2$: all theorems and rules of $\mathcal{HG}_{(\rightarrow, \leftarrow)}^2$ instantiated with MCB formulas.

Definition 9.13 ($\mathcal{H}\text{NMCB}$). The calculus consists of the following axioms and rules (below, $\phi, \chi \in \mathcal{L}_{\text{BD}}$ and $\alpha, \beta \in \mathcal{L}_{\text{NMCB}}$).

$\mathcal{H}\text{NMCB}_{\text{BD}}$: $\mathbf{C}\phi \Rightarrow \mathbf{C}\chi$ for any $\phi, \chi \in \mathcal{L}_{\text{BD}}$ s.t. $\phi \vdash \chi$ is BD valid.

$\mathcal{H}\text{NMCB}_{\neg}$: $\mathbf{C}\neg\phi \Leftrightarrow \neg\mathbf{C}\phi$.

$\mathcal{G}_{(\rightarrow, \rightarrow)}^2$: all theorems and rules of $\mathcal{HG}_{(\rightarrow, \rightarrow)}^2$ instantiated with NMCB formulas.

As expected, $\mathcal{H}\text{MCB}_{\text{BD}}$ and $\mathcal{H}\text{NMCB}_{\text{BD}}$ correspond to the monotonicity of π while $\mathcal{H}\text{MCB}_{\neg}$ and $\mathcal{H}\text{NMCB}_{\neg}$ establish the connection between the support of truth and support of falsity of a given $\phi \in \mathcal{L}_{\text{BD}}$.

We finish the section by establishing strong completeness results.

Theorem 9.8 (Completeness of $\mathcal{H}\text{MCB}$ and $\mathcal{H}\text{NMCB}$). *For any $\Psi \cup \{\alpha\} \subseteq \mathcal{L}_{\text{MCB}}$ and $\Omega \cup \{\beta\} \subseteq \mathcal{L}_{\text{NMCB}}$, it holds that*

$$\Psi \vdash_{\mathcal{H}\text{MCB}} \alpha \text{ iff } \Psi \models_{\text{MCB}} \alpha \qquad \Omega \vdash_{\mathcal{H}\text{NMCB}} \beta \text{ iff } \Omega \models_{\text{NMCB}} \beta$$

Proof. We show only the case of $\mathcal{H}\text{MCB}$ since $\mathcal{H}\text{NMCB}$ can be proved similarly.

For the soundness part, it suffices to establish validity of $\mathcal{H}\text{MCB}_{\text{BD}}$ and $\mathcal{H}\text{MCB}_{\neg}$. Indeed, if $\phi \vdash \chi$ is BD valid, then $|\phi|^+ \subseteq |\chi|^+$ and $|\chi|^- \subseteq |\phi|^-$ for any v^+ and v^- . Hence, $\pi(|\phi|^+) \leq \pi(|\chi|^+)$ and $\pi(|\phi|^-) \geq \pi(|\chi|^-)$. Thus, $e(\mathbf{C}\phi \rightarrow \mathbf{C}\chi) = (1, 0)$, as required.

Likewise, $e_1(\mathbf{C}\neg\phi) = \pi(|\neg\phi|^+) = \pi(|\phi|^-)$ and $e_2(\mathbf{C}\neg\phi) = \pi(|\neg\phi|^-) = \pi(|\phi|^+)$, while $e_1(\neg\mathbf{C}\phi) = e_2(\mathbf{C}\phi) = \pi(|\phi|^-)$ and $e_2(\neg\mathbf{C}\phi) = e_1(\mathbf{C}\phi) = \pi(|\phi|^+)$.

For the completeness part, we reason by contraposition. An $\mathcal{H}\text{MCB}$ prime theory is $\Pi \subseteq \mathcal{L}_{\text{MCB}}$ s.t. $\Pi \vdash_{\mathcal{H}\text{MCB}} \gamma$ iff $\gamma \in \Pi$ and for any $\gamma \vee \gamma' \in \Pi$, $\gamma \in \Pi$ or $\gamma' \in \Pi$.

Assume now, that α cannot be inferred from Ψ . We construct a model refuting $\Psi \models_{\text{MCB}} \alpha$. Assume an enumeration of all MCB formulas. We let $\Psi = \Psi_0$ and define

$$\Psi_{n+1} = \begin{cases} \Psi_n \cup \{\alpha_n\} & \text{iff } \Psi_n, \alpha_n \not\vdash_{\mathcal{H}\text{MCB}} \alpha \\ \Psi_n & \text{otherwise} \end{cases}$$

We now define $\Psi^* = \bigcup_{n < \omega} \Psi_n$. It is clear that Ψ^* is a maximal prime theory that does not contain α , whence $\Psi^* \not\vdash_{\mathcal{H}\text{MCB}} \alpha$. But observe that all formulas are actually $\mathcal{L}_{\mathcal{G}_{(\rightarrow, \leftarrow)}^2}$ formulas with $\mathbf{C}\phi$'s instead of variables. Thus, by Theorem 4.3, there is a \mathcal{G}^2 valuation e s.t. $e_1[\Psi^*] > e_1(\alpha)$ or $e_2[\Psi^*] < e_2(\alpha)$. It is also clear that $\Delta^\top \xi \in \Psi^*$ for every ξ being an instance of a modal axiom since

$$\frac{\mathcal{HG}_{(\rightarrow, \leftarrow)}^2 \vdash \xi}{\mathcal{HG}_{(\rightarrow, \leftarrow)}^2 \vdash \Delta^\top \xi}$$

is admissible in $\mathcal{HG}_{(\rightarrow, \leftarrow)}^2$ and $\mathcal{H}\text{MCB}$ extends $\mathcal{HG}_{(\rightarrow, \leftarrow)}^2$. Thus, e evaluates all modal axioms with $(1, 0)$.

It remains to define π and v^\pm . We set $W = 2^{\text{Lit}[\Psi^* \cup \{\alpha\}]}$. Then for any $w \in W$, we let $w \in v^+(p)$ iff $p \in w$ and $w \in v^-(p)$ iff $\neg p \in w$. And finally, for any $\mathbf{C}\phi \in \mathbf{Sf}[\Psi^* \cup \{\alpha\}]$, we set $\pi(|\phi|) = e_1(\mathbf{C}\phi)$. For other $X \subseteq W$, we set $\pi(X) = \sup\{\pi(|\phi|^+) : \phi \in \mathbf{Sf}[\Psi^* \cup \{\alpha\}], |\phi|^+ \subseteq X\}$ and $\pi(W) = 1$ and $\pi(\emptyset) = 0$. It is straightforward to check that π thus defined conforms to Definition 9.11.

The result follows. \square

9.3.3. Extensions

The logics of monotone comparative belief provided in the previous subsection were, in a sense, minimal. It is thus instructive to consider their extensions with additional axioms corresponding to additional conditions imposed on π .

First, observe that since **BD** lacks tautologies and universally false formulas, **cap** does not have any analogues in **MCB**, nor in **NMCB**. In fact, one can prove Theorem 9.8 even without requiring that $\pi(W) > \pi(\emptyset)$ ⁷⁹. This shows that the truth and falsity of **MCB** and **NMCB** formulas depend only on the order relations between different uncertainty measures of different events, not on the values of these measures. However, in contrast to **QP**, it is not problematic: if one describes events using **BD**, there is no event whose uncertainty measure one could know *a priori*⁸⁰ just by its description. This, again, is in line with that **MCB** and **NMCB** formalise reasoning with uncertainty when the agent tries to build their beliefs using only the information provided by their sources as we discussed in Section 9.3.1. Furthermore, since \neg is not related to the set-theoretic complement, **KPS_m** axioms cannot be meaningfully translated either.

Still, we may assert that two events ϕ and ϕ' are incompatible if $\pi(|\phi \wedge \phi'|^+) = 0$ and $\pi(|\phi \wedge \phi'|^-) = 1$. Indeed, this statement corresponds to $\sim\mathbf{C}(\phi \wedge \phi')$ having value $(1, 0)$ or to the formula $\Delta^\top \sim\mathbf{C}(\phi \wedge \phi')$. To express incompatibility in **NMCB**, we define the following connective:

$$\Delta^{\text{IN}}\alpha := \Delta^{\text{N}}(\mathbf{1} \Rightarrow \alpha)$$

Now Δ^{IN} can be used to express that the agent is completely certain in ϕ as follows: $\Delta^{\text{IN}}\mathbf{C}\phi$. It is clear that

$$e_1(\Delta^{\text{IN}}\alpha) = \begin{cases} 1 & \text{iff } e(\alpha) = (1, 0) \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\Delta^{\text{IN}}\sim\mathbf{N}\mathbf{C}(\phi \wedge \phi')$ corresponds to the incompatibility of ϕ and ϕ' in **NMCB**.

The next statement establishes some correspondence results for **MCB** and **NMCB**.

Convention 9.6. We introduce the following naming conventions.

$$\text{disj}+\neg: (\Delta^\top \sim\mathbf{C}(p \wedge p') \wedge \sim\Delta^\top \sim\mathbf{C}p \wedge \sim\Delta^\top \sim\mathbf{C}p') \rightarrow (\sim\Delta^\top (\mathbf{C}(p \vee p') \rightarrow \mathbf{C}p) \wedge \sim\Delta^\top (\mathbf{C}(p \vee p') \rightarrow \mathbf{C}p'))$$

$$\text{disj}0^\neg: \Delta^\top \sim\mathbf{C}p \rightarrow \Delta^\top (\mathbf{C}p' \leftrightarrow \mathbf{C}(p \vee p'))$$

$$\text{disj}+\text{N}: (\Delta^{\text{N}}\sim\mathbf{N}\mathbf{C}(p \wedge p') \wedge \sim\mathbf{N}\sim\mathbf{N}\mathbf{C}p \wedge \sim\mathbf{N}\sim\mathbf{N}\mathbf{C}p') \rightarrow (\sim\mathbf{N}\Delta^{\text{N}}(\mathbf{C}(p \vee p') \rightarrow \mathbf{C}p) \wedge \sim\mathbf{N}\Delta^{\text{N}}(\mathbf{C}(p \vee p') \rightarrow \mathbf{C}p'))$$

$$\text{disj}0^{\text{N}}: \Delta^{\text{N}}\sim\mathbf{N}\mathbf{C}p \rightarrow \Delta^{\text{N}}(\mathbf{C}p' \leftrightarrow \mathbf{C}(p \vee p'))$$

Theorem 9.9. *Let $\mathbb{F} = \langle W, \pi \rangle$ be a frame. Then the following statements hold.*

$$\mathbb{F} \models \text{disj}+\neg \text{ iff } \left[\begin{array}{l} \pi(X \cap X') = 0 \text{ and } \pi(Y \cup Y') = 1 \\ \text{and} \\ \min(\pi(X), \pi(X')) > 0 \\ \text{and} \\ \max(\pi(Y), \pi(Y')) < 1 \end{array} \right] \Rightarrow \left[\begin{array}{l} \pi(X \cup X') > \max(\pi(X), \pi(X')) \\ \text{or} \\ \pi(Y \cap Y') < \min(\pi(Y), \pi(Y')) \end{array} \right] \quad (\text{I})$$

$$\mathbb{F} \models \text{disj}0^\neg \text{ iff } \left[\begin{array}{l} \pi(X) = 0 \\ \text{and} \\ \pi(Y) = 1 \end{array} \right] \Rightarrow \left[\begin{array}{l} \pi(X \cup X') = \pi(X') \\ \text{and} \\ \pi(Y \cap Y') = \pi(Y') \end{array} \right] \quad (\text{II})$$

⁷⁹Note, however, that in this case, π is not going to be a measure but just a \subseteq -monotone map from 2^W to $[0, 1]$.

⁸⁰One can, however, *express* in **MCB** and **NMCB** that the agent is completely certain in a given statement using Δ^\top and Δ^{N} .

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$$\mathbb{F} \models \text{disj}^{\text{N}} \text{ iff } \left[\begin{array}{c} Y \cap Y' = \emptyset \\ \text{and} \\ \min(\pi(Y), \pi(Y')) > 0 \end{array} \right] \Rightarrow \pi(Y \cup Y') > \max(\pi(Y), \pi(Y')) \quad (\text{III})$$

$$\mathbb{F} \models \text{disj}^{\text{N}} \text{ iff } \pi(Y) = 0 \Rightarrow \pi(Y \cup Y') = \pi(Y') \quad (\text{IV})$$

Proof. Analogously to Theorem 9.3. □

The formulas from the previous theorem are paraconsistent analogues of those from Theorem 9.3. Note, first of all, that we do not translate 1compl. It tells that if an agent is completely certain in some statement, then they should completely disbelieve its negation. In a paraconsistent setting, however, it might be the case that all sources provide contradictory information about p , whence this principle is not justified.

Moreover, there is a considerable difference in the expressivity of MCB and NMCB. The former takes into account both support of truth and support of falsity, while the latter only support of truth. It means that the properties of the uncertainty measures that can be axiomatised using MCB are considerably weaker than those axiomatisable in QPG or NMCB because every outer-layer formula corresponds not to one but two subsets of the carrier.

Finally, in Section 9.3.1, we discussed that MCB and NMCB can express both comparability and incomparability of beliefs. It is clear from Definition 9.11 that $\mathbb{L} \not\models \mathbb{C}(p \wedge \neg p) \rightsquigarrow \mathbb{C}q$ and $\mathbb{L} \not\models \mathbb{C}p \rightsquigarrow \mathbb{C}(q \vee \neg q)$ for $\mathbb{L} \in \{\text{MCB}, \text{NMCB}\}$ and $\rightsquigarrow \in \{\rightarrow, \rightarrow, \Rightarrow\}$. This corresponds to Desiderata 1 and 3–5 from the introduction. To satisfy Desideratum 2, we can use the idea outlined in Remark 8.9 and employ multimodal two-layered logics expanding MCB and NMCB where different modalities have different properties (e.g., the ones given in Theorem 9.9).

END OF PART III.

10. Conclusion

Let us recapitulate the main results of the manuscript and provide the roadmap for future research.

10.1. Summary

For the last time, we return to the desiderata we put forth in [the introduction](#). In this dissertation, we were aiming to provide and study logics that conform to them. In Chapter 7, we presented paraconsistent logics with Kripke-frame semantics that satisfy all five desiderata and in Chapters 8 and 9, we presented the idea on how to expand the two-layered logics with additional modalities in such a way that all the desiderata are satisfied.

We are now turning to a more detailed summary of the results. In Chapter 3, we presented two paraconsistent expansions of $\mathfrak{L} — \mathfrak{L}_{(\Delta, \rightarrow)}^2$ and $\mathfrak{L}_{(\rightarrow)}^2$. We provided their Hilbert-style axiomatisations and proved⁸¹ their completeness (Theorems 3.1 and 3.2). Furthermore, we constructed a unified tableaux calculus $\mathcal{T}(\mathfrak{L}^2)$ for both \mathfrak{L}^2 's, established its completeness (Theorem 3.3), and used it to prove NP-completeness of $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ and $\mathfrak{L}_{(\rightarrow)}^2$ (Theorem 3.4). We have also explored semantical properties of $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ and $\mathfrak{L}_{(\rightarrow)}^2$ and shown that adding new axioms to $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ and $\mathfrak{L}_{(\rightarrow)}^2$ makes modus ponens unsound (Theorem 3.5).

In Chapter 4, we constructed paraconsistent expansions of $\mathbf{biG} — \mathbf{G}_{(\rightarrow, \leftarrow)}^2$ and $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$. We constructed strongly complete Hilbert and tableaux calculi for \mathbf{G}^2 (Theorems 4.3 and 4.4). We proved the completeness of the Hilbert-style calculi by establishing mutual translations between $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$ - and $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ -valuations on $[0, 1]$ and bi-valued linear Kripke models (as defined in [151]) for $\mathbf{I}_1\mathbf{C}_1$ and $\mathbf{I}_4\mathbf{C}_4$, respectively (Theorems 4.1 and 4.2). We established that in contrast to \mathfrak{L}^2 's the set of valid \mathbf{G}^2 -formulas remains the same as long as the filter of designated values on $[0, 1]^\mathbb{K}$ extends $(1, 0)^\uparrow$ or $(1, 1)^\uparrow$ (for $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ and $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$, respectively — Theorem 4.6). Moreover, we proved that there are only six entailment relations over $\mathcal{L}_{\mathbf{G}_{(\rightarrow, \leftarrow)}^2}$ generated by filters on $[0, 1]^\mathbb{K}$ and only two entailment relations on $\mathcal{L}_{\mathbf{G}_{(\rightarrow, \rightarrow)}^2}$ generated by prime filters on $[0, 1]^\mathbb{K}$ of the form $(x, 1)^\uparrow$ (Theorems 4.7 and 4.8 and Corollary 4.2).

In Chapter 5, we presented $\mathbf{KbiG} —$ an expansion of $\mathbf{G}\Delta$ with \Box and \Diamond and proved the strong completeness of its crisp axiomatisation (Theorem 5.2) adapting the method from [130]. We studied the model theory of \mathbf{KbiG} . In particular, we established several classes of formulas transferrable from \mathbf{K} (Theorems 5.4 and 5.5) and characterised frames whereon Glivenko's theorem holds (Theorem 5.7). Moreover, by an adaptation of the \mathbf{PSpace} -completeness proof of \mathbf{KG}^c from [38], we established \mathbf{PSpace} -completeness of \mathbf{KbiG}^c (Theorem 5.8).

Chapter 6 was dedicated to $\mathbf{KG}^{2c} —$ a paraconsistent expansion of \mathbf{KbiG}^c . Using the reduction of \mathbf{KG}^{2c} validity to \mathbf{KbiG} validity, we proved that their expressivities coincide (Corollary 6.1) and obtained a complete axiomatisation of \mathbf{KG}^{2c} (Theorem 6.1). In addition, we studied $\mathbf{KG}_{\text{fb}}^{2c} —$ \mathbf{KG}^{2c} over finitely branching frames. Namely, we have characterised crisp frames forcing a paraconsistent counterpart of Glivenko's theorem (Theorem 6.2) and provided a simple constraint tableaux calculus for $\mathbf{KG}_{\text{fb}}^{2c}$ that we used to establish its finite model property and \mathbf{PSpace} -completeness (Theorems 6.3 and 6.4).

⁸¹Note that originally the completeness of the Δ -less fragment of $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ was established in [27]. The weak completeness proofs of $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ and $\mathfrak{L}_{(\rightarrow)}^2$ were first given in [26]. The proofs in this manuscript are simplified versions of those in [26].

In Chapter 7, we provided further generalisations of \mathbf{KbiG} and \mathbf{KG}^{2c} . We constructed two logics on bi-relational fuzzy frames — $\mathbf{KG}^{2\pm}$ and $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ — that expand \mathbf{G}^2 with normal (\square and \diamond) and informational (\blacksquare and \blacklozenge) modalities, respectively. We showed that neither normal nor informational modalities are interdefinable in the bi-relational setting (Theorems 7.1 and 7.9). In addition, we proved that mono-relational frames are definable in both languages (Theorems 7.5 and 7.11). We also established that crisp $\mathbf{KG}^{2\pm}$ extends (and its validity is actually reducible to the validity in) crisp \mathbf{KbiG} (Theorem 7.3) while fuzzy $\mathbf{KG}^{2\pm}$ does not (Theorem 7.2). Furthermore, we demonstrated that in $\mathbf{KG}^{2\pm}$ a class of crisp bi-relational frames is definable only if both its relations are definable by the same formula in \mathbf{KbiG} (Corollaries 7.1 and 7.2). On the other hand, it is possible to define both crisp and fuzzy bi-relational frames with different relations in $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ (Theorem 7.10). Lastly, we have proved the definability of finitely branching frames in $\mathbf{KG}^{2\pm}$ and $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ (Theorem 7.6 and as an immediate corollary to Theorem 7.10) and constructed complete tableaux calculi for $\mathbf{KG}_{\text{fb}}^{2\pm}$ and $\mathbf{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ (Theorems 6.3 and 7.12). We then utilised $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2\pm})$ and $\mathcal{T}(\mathbf{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm})$ to create decision procedures and establish PSpace-completeness of $\mathbf{KG}_{\text{fb}}^{2\pm}$ and $\mathbf{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ (Theorems 7.8 and 7.13).

In Chapter 8, we considered two-layered logics $\text{Pr}_{\Delta}^{\mathbf{k}^2}$ and $4\text{Pr}^{\mathbf{k}^2}$ based on $\mathbf{k}_{(\Delta, \rightarrow)}^2$ and \mathbf{k}_{Δ} and formalising quantitative reasoning about \pm - and 4 -probabilities, respectively. We constructed their Hilbert-style axiomatisations (Theorems 8.1⁸² and 8.2) as well as faithful embeddings into one another (Theorems 8.3 and 8.4). We then utilised $\mathcal{T}(\mathbf{k}^2)$ and these embeddings to show that both logics are NP-complete (Theorem 8.5).

In Chapter 9, we examined the logics for qualitative reasoning about uncertainty measures: both classical (QG and its extensions) and paraconsistent (MCB and NMCB). We built their Hilbert-style axiomatisations and proved their completeness (Theorems 9.2 and 9.8) and established correspondence between formulas and properties of measures that they encode (Theorems 9.3, 9.4, 9.7, and 9.9).

10.2. Open questions and future research

The work done leaves some important questions to solve. Since the dissertation was divided into three parts, we will split this discussion according to them.

10.2.1. Propositional fragments

Let us recall Corollary 4.2 once again. Say that ϕ is *globally true* in a \mathbf{G}^2 model \mathfrak{M} iff $w \models^+ \phi$ for every $w \in \mathfrak{M}$ and *globally designated* in \mathfrak{M} iff, in addition, $w \not\models^- \phi$ for every $w \in \mathfrak{M}$.

Problem 10.1 (Filters on $[0, 1]^{\mathfrak{M}}$ and entailment relations on \mathbf{G}^2 Kripke models). It is clear from Definition 4.10 that the following holds for every $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathbf{G}^2_{(\rightarrow, \leftarrow)}}$.

- $\Gamma \models_{(1,0)\uparrow} \phi$ iff for every \mathbf{G}^2 model \mathfrak{M} where Γ is globally designated, ϕ is globally designated too.
- $\Gamma \models_{(1,1)\uparrow} \phi$ iff for every \mathbf{G}^2 model \mathfrak{M} where Γ is globally true, ϕ is globally true too.

This leaves out $(x, x)\uparrow$, $(x, y)\uparrow$, and $(y, x)\uparrow$ entailments. Is it possible to define their counterparts on \mathbf{G}^2 Kripke models?

Problem 10.2 (Axiomatisation of \mathbf{G}^2 entailments). In Proposition 4.5, we give first-degree consecutions that separate different filter entailments. If we add them to $\mathcal{HG}_{(\rightarrow, \leftarrow)}^2$ as rules, will it produce complete axiomatisations?

⁸²The proof borrowed from [26].

10.2.2. Fuzzy modal logics

Problem 10.3 (Fuzzy \mathbf{KbiG}). We were mostly considering *crisp* \mathbf{KbiG} that was obtained from \mathbf{KG}^c by adding Δ axioms from Definition 3.4, one additional crispness axiom, and one additional axiom governing the relation between Δ and \diamond . If we remove both crispness axioms, will we obtain a complete axiomatisation of \mathbf{KbiG}^f ?

Problem 10.4 (Axiomatisation and decidability of $\mathbf{KG}^{2\pm}$ and $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$). The axiomatisation of \mathbf{KG}^{2c} and the proof of its \mathbf{PSpace} -completeness (Theorem 6.1 and Corollary 6.1) utilised \neg NNF's. Neither $\mathbf{KG}^{2\pm}$, nor $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ admit NNF's. In fact, they do not even extend \mathbf{KbiG}^f , whence there is no immediate reduction to \mathbf{KbiG} validity.

The conjecture that both these logics are still \mathbf{PSpace} -complete seems reasonable. It is unclear, however, how they can be axiomatised and what their relation to \mathbf{KbiG} is. Neither is it clear whether the approach from [38] will help in establishing the complexity evaluation.

Problem 10.5 (Global and non-standard \mathbf{G}^2 modalities and description logics). Gödel description logics (and thus, Gödel logics with global modalities) are well studied (cf. [32] for a summary of the foundational results). It thus makes sense to introduce global modalities in $\mathbf{KG}^{2\pm}$. Moreover, to the best of our knowledge, non-standard modalities (such as contingency, accident, etc.) are not being studied in the DL framework nor in the context of paraconsistent and fuzzy modal logics in general. In fact, it seems that the only paper on paraconsistent, although, not fuzzy logics with non-standard modalities are [3] and [93] (the latter of which was co-written by the author of the present manuscript). It makes sense, then, to introduce not only global \square and \diamond but also global \blacksquare and \blacklozenge .

10.2.3. Two-layered logics

Problem 10.6 (Four-valued belief functions). In [26], paraconsistent counterparts of belief functions defined over De Morgan algebras are considered. Ideologically, they are close to \pm -probabilities as to each statement ϕ they assign two independent values: the belief in ϕ itself and in $\neg\phi$. In addition, the (quantitative) reasoning with them is formalised using a two-layered logic expanding $\mathfrak{L}_{(\Delta, \rightarrow)}^2$. It is thus reasonable to continue the direction of research outlined in [92] and propose a four-valued counterpart of these belief functions as well as provide its two-layered axiomatisation.

Problem 10.7 (Qualitative \pm -probabilities and belief functions). Qualitative counterparts of most important *classical* uncertainty measures (capacities, belief functions, and probabilities) are well-known [94, 154, 153]. On the other hand, it is open whether qualitative characterisations of their BD-counterparts can be established.

One of the immediate technical difficulties is that the conditions for qualitative belief functions and probabilities in Theorem 9.1 use set-theoretic complements and non-intersecting sets which are not expressible in \mathcal{L}_{BD} . It might be possible to circumvent this by using a (weakly) functionally complete expansion of BD, for instance, $\text{BD}\Delta$ [132] (cf. [124] for a study of its algebraic semantics) on the inner layer.

Bibliography

- [1] M.D' Agostino. *Investigations into the complexity of some propositional calculi*. Oxford University Computing Laboratory, Oxford, 1990.
- [2] J.P. Aguilera, M. Diéguez, D. Fernández-Duque, and B. McLean. Time and Gödel: Fuzzy Temporal Reasoning in PSPACE. In *Logic, Language, Information, and Computation*, volume 13368 of *Lecture notes in computer science*, pages 18–35. Springer International Publishing, Cham, 2022.
- [3] H. Antunes, W. Carnielli, A. Kapsner, and A. Rodrigues. Kripke-style models for logics of evidence and truth. *Axioms*, 9(3):100, August 2020.
- [4] O. Arieli and A. Avron. Four-valued parafinite logics. *Studia Logica*, 105(6):1087–1122, December 2017.
- [5] O. Arieli and M. Denecker. Reducing preferential paraconsistent reasoning to classical entailment. *Journal of Logic and Computation*, 13(4):557–580, August 2003.
- [6] A. Avron. A tableau system for Gödel–Dummett logic based on a hypersequent calculus. In R. Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2000*, volume 1847 of *Lecture notes in computer science*, pages 98–111. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.
- [7] A. Avron and B. Konikowska. Decomposition proof systems for Gödel–Dummett logics. *Studia Logica*, 69(2):197–219, 2001.
- [8] F. Baader, D. Calvanese, D.L. McGuinness, D. Nardi, and P.-F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation and Applications*. Cambridge University Press, 2010.
- [9] M. Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. In *Gödel'96: Logical foundations of mathematics, computer science and physics—Kurt Gödel's legacy, Brno, Czech Republic, August 1996, proceedings*, pages 23–33. Association for Symbolic Logic, 1996.
- [10] M. Baaz and N. Preining. Gödel–Dummett Logics. In P. Cintula, P. Hájek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic*, volume 38 of *Studies in Logic*. College Publications, 2011.
- [11] P. Baldi, P. Cintula, and C. Noguera. Classical and Fuzzy Two-Layered Modal Logics for Uncertainty: Translations and Proof-Theory. *International Journal of Computational Intelligence Systems*, 13:988–1001, 2020.
- [12] P. Baldi and H. Hosni. Logical approximations of qualitative probability. In A. Cano, J. de Bock, En. Miranda, and S. Moral, editors, *Proceedings of the Twelfth International Symposium on Imprecise Probability: Theories and Applications*, volume 147 of *Proceedings of Machine Learning Research*, pages 12–21. PMLR, 06–09 Jul 2021.

- [13] B. Bednarczyk, E. Kieroński, and P. Witkowski. On the complexity of graded modal logics with converse. In F. Calimeri, N. Leone, and M. Manna, editors, *Logics in Artificial Intelligence. JELIA 2019*, volume 11468 of *Lecture notes in computer science*, pages 642–658. Springer International Publishing, Cham, 2019.
- [14] N.D. Belnap. A Useful Four-Valued Logic. In J.M. Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 5–37, Dordrecht, 1977. Springer Netherlands.
- [15] N.D. Belnap. How a computer should think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*. Oriel Press, 1977.
- [16] N.D. Belnap. How a computer should think. In H. Omori and H. Wansing, editors, *New Essays on Belnap-Dunn Logic*, volume 418 of *Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science)*, pages 35–53. Springer, Cham, 2019.
- [17] P. Besnard and J. Lang. Possibility and necessity functions over non-classical logics. In *Uncertainty Proceedings 1994*, pages 69–76. Elsevier, 1994.
- [18] G. Bezhanishvili. Glivenko type theorems for intuitionistic modal logics. *Studia Logica*, 67(1):89–109, 2001.
- [19] M. Bílková, S. Frittella, and D. Kozhemiachenko. Constraint Tableaux for Two-Dimensional Fuzzy Logics. In A. Das and S. Negri, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 12842 of *Lecture Notes in Computer Science*, pages 20–37. Springer International Publishing, 2021.
- [20] M. Bílková, S. Frittella, and D. Kozhemiachenko. Paraconsistent Gödel Modal Logic. In *Automated Reasoning*, volume 13385 of *Lecture notes in computer science*, pages 429–448. Springer International Publishing, Cham, 2022.
- [21] M. Bílková, S. Frittella, and D. Kozhemiachenko. Crisp Bi-Gödel modal logic and its paraconsistent expansion. *Logic Journal of the IGPL*, September 2023.
- [22] M. Bílková, S. Frittella, and D. Kozhemiachenko. Non-standard modalities in paraconsistent Gödel logic. In S. Gaggl, M.V. Martinez, and M. Ortiz, editors, *Logics in Artificial Intelligence*, volume 14281 of *Lecture Notes in Artificial Intelligence*, pages 420–436. Springer, 2023.
- [23] M. Bílková, S. Frittella, and D. Kozhemiachenko. Paraconsistent Gödel modal logic on bi-relational frames. [arXiv:2303.14164](https://arxiv.org/abs/2303.14164) (unpublished manuscript), 2023.
- [24] M. Bílková, S. Frittella, D. Kozhemiachenko, and O. Majer. Qualitative reasoning in a two-layered framework. *International Journal of Approximate Reasoning*, 154:84–108, March 2023.
- [25] M. Bílková, S. Frittella, D. Kozhemiachenko, and O. Majer. Two-layered logics for paraconsistent probabilities. In H.H. Hansen, A. Scedrov, and R.J. de Queiroz, editors, *Logic, Language, Information, and Computation. WoLLIC 2023*, volume 13923 of *Lecture notes in computer science*, pages 101–117. Springer Nature Switzerland, Cham, 2023.
- [26] M. Bílková, S. Frittella, D. Kozhemiachenko, O. Majer, and S. Nazari. Reasoning with belief functions over Belnap–Dunn logic. *Annals of Pure and Applied Logic*, page 103338, July 2023.
- [27] M. Bílková, S. Frittella, O. Majer, and S. Nazari. Belief Based on Inconsistent Information. In M.A. Martins and I. Sedlár, editors, *Dynamic Logic. New Trends and Applications*, pages 68–86, Cham, 2020. Springer International Publishing.

- [28] P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic*, volume 53 of *Cambridge tracts in theoretical computer science*. Cambridge University Press, 4. print. with corr. edition, 2010.
- [29] F. Bobillo, M. Cerami, F. Esteva, Á. García-Cerdaña, R. Peñaloza, and U. Straccia. Introduction to Mathematical Fuzzy Logic. In P. Cintula, C.G. Fermüller, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic*, volume 58 of *Studies in logic*, pages 1105–1182. College Publications, 2015.
- [30] F. Bobillo, M. Delgado, J. Gómez-Romero, and U. Straccia. Fuzzy description logics under Gödel semantics. *International Journal of Approximate Reasoning*, 50(3):494–514, March 2009.
- [31] F. Bobillo, M. Delgado, J. Gómez-Romero, and U. Straccia. Joining Gödel and Zadeh fuzzy logics in fuzzy description logics. *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems*, 20(04):475–508, August 2012.
- [32] S. Borgwardt. *Fuzzy Description Logics with General Concept Inclusions*. PhD thesis, Technische Universität Dresden, 2014.
- [33] S. Borgwardt, F. Distel, and R. Peñaloza. Decidable Gödel description logics without the finitely-valued model property. In *Fourteenth International Conference on the Principles of Knowledge Representation and Reasoning*, 2014.
- [34] S. Borgwardt and R. Peñaloza. Undecidability of fuzzy description logics. In *Thirteenth International Conference on the Principles of Knowledge Representation and Reasoning*, pages 232–242, 2012.
- [35] J. Bueno-Soler and W. Carnielli. Paraconsistent probabilities: Consistency, contradictions and Bayes’ theorem. *Entropy (Basel)*, 18(9):325, September 2016.
- [36] L. Běhounek, P. Cintula, and P. Hájek. Introduction to Mathematical Fuzzy Logic. In P. Cintula, P. Hájek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic*, volume 37 of *Studies in logic*, pages 1–102. College Publications, 2011.
- [37] X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger. A finite model property for Gödel modal logics. In L. Libkin, U. Kohlenbach, and R. de Queiroz, editors, *Logic, Language, Information, and Computation. WoLLIC 2013*, volume 8071 of *Lecture Notes in Computer Science*, pages 226–237. Springer, 2013.
- [38] X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger. Decidability of order-based modal logics. *Journal of Computer and System Sciences*, 88:53–74, September 2017.
- [39] X. Caicedo and R.O. Rodríguez. Standard Gödel modal logics. *Studia Logica*, 94(2):189–214, 2010.
- [40] X. Caicedo and R.O. Rodríguez. Bi-modal Gödel logic over $[0,1]$ -valued Kripke frames. *Journal of Logic and Computation*, 25(1):37–55, 2015.
- [41] M. Cerami and U. Straccia. On the (un)decidability of fuzzy description logics under Łukasiewicz t-norm. *Information Sciences*, 227:1–21, April 2013.
- [42] A. Chagrov and M. Zakharyashev. *Modal Logic*. Clarendon Press, 1997.
- [43] P. Cintula, C.G. Fermüller, and C. Noguera. Fuzzy Logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2021 edition, 2021.

- [44] P. Cintula and C. Noguera. Modal logics of uncertainty with two-layer syntax: A general completeness theorem. In *International Workshop on Logic, Language, Information, and Computation*, pages 124–136. Springer, 2014.
- [45] N.C.A. da Costa. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, 15(4):497–510, 1974.
- [46] M. Daňková and L. Běhounek. Fuzzy neighborhood semantics for multi-agent probabilistic reasoning in games. In M.-J. Lesot, S. Vieira, M.Z. Reformat, J.P. Carvalho, A. Wilbik, B. Bouchon-Meunier, and R.R. Yager, editors, *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 680–693, Cham, 2020. Springer International Publishing.
- [47] Š. Dautović, D. Doder, and Z. Ognjanović. An epistemic probabilistic logic with conditional probabilities. In *Logics in Artificial Intelligence. JELIA 2021*, volume 12678 of *Lecture notes in computer science*, pages 279–293. Springer International Publishing, Cham, 2021.
- [48] J. Delgrande and B. Renne. The logic of qualitative probability. In *Twenty-Fourth International Joint Conference on Artificial Intelligence*, pages 2904–2910. AAAI Press, 2015.
- [49] J.P. Delgrande, B. Renne, and J. Sack. The logic of qualitative probability. *Artificial Intelligence*, 275:457–486, October 2019.
- [50] P. Dellunde and L. Godo. Introducing grades in deontic logics. In R. van der Meyden and L. van der Torre, editors, *Deontic Logic in Computer Science*, pages 248–262, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.
- [51] M. Diéguez and D. Fernández-Duque. Decidability for **S4** Gödel Modal Logics. In *Computational Intelligence and Mathematics for Tackling Complex Problems 4*, volume 1040 of *Studies in computational intelligence*, pages 1–7. Springer International Publishing, Cham, 2023.
- [52] S. Drobyshevich. A general framework for FDE-based modal logics. *Studia Logica*, 108(6):1281–1306, December 2020.
- [53] D. Dubois. On ignorance and contradiction considered as truth-values. *Logic Journal of the IGPL*, 16(2):195–216, 2008.
- [54] D. Dubois. Author’s response to Wansing and Belnap’s Generalized truth-values. *Logic Journal of the IGPL*, 18(6):936–940, December 2010.
- [55] D. Dubois, H. Prade, and S. Schockaert. Generalized possibilistic logic: Foundations and applications to qualitative reasoning about uncertainty. *Artificial Intelligence*, 252:139–174, November 2017.
- [56] J.M. Dunn. Intuitive semantics for first-degree entailments and ‘coupled trees’. *Philosophical Studies*, 29(3):149–168, 1976.
- [57] J.M. Dunn. Partiality and Its Dual. *Studia Logica*, 66(1):5–40, Oct 2000.
- [58] J.M. Dunn. Contradictory information: Too much of a good thing. *Journal of Philosophical Logic*, 39:425–452, 2010.
- [59] R. Ertola, F. Esteva, T. Flaminio, L. Godo, and C. Noguera. Paraconsistency properties in degree-preserving fuzzy logics. *Soft Computing*, 19(3):531–546, 2015.
- [60] R. Fagin and J.Y. Halpern. Uncertainty, belief, and probability. *Computational Intelligence*, 7(3):160–173, 1991.

- [61] R. Fagin, J.Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and computation*, 87(1–2):78–128, 1990.
- [62] R. Fagin, J.Y. Halpern, Y. Moses, and M.Y. Vardi. *Reasoning About Knowledge*. MIT Press, Cambridge, MA, USA, 2003.
- [63] L. Fariñas del Cerro and A. Herzig. A modal analysis of possibility theory. In Ph. Jordan and J. Kelemen, editors, *Fundamentals of Artificial Intelligence Research*, volume 535 of *Lecture notes in computer science*, pages 11–18. Springer Berlin Heidelberg, Berlin, Heidelberg, 1991.
- [64] B. de Finetti. La prévision: ses lois logiques, ses sources subjectives. *Annales de l'institut Henri Poincaré*, 7(1):1–68, 1937.
- [65] M. Fitting. Kleene’s three-valued logics and their children. *Fundamenta informaticae*, 20(1, 2, 3):113–131, 1994.
- [66] T. Flaminio, L. Godo, and E. Marchioni. Reasoning about uncertainty of fuzzy events: an overview. In *Understanding Vagueness: Logical, Philosophical, and Linguistic Perspectives*, volume 36 of *Studies in Logic*, pages 367–400. College Publications London, 2011.
- [67] T. Flaminio, L. Godo, and S. Ugolini. An Approach to Inconsistency-Tolerant Reasoning About Probability Based on Łukasiewicz Logic. In *International Conference on Scalable Uncertainty Management*, pages 124–138. Springer, 2022.
- [68] J.M. Font. Belnap’s Four-Valued Logic and De Morgan Lattices. *Logic Journal of the IGPL*, 5:1–29, 05 1997.
- [69] P. Gärdenfors. Qualitative probability as an intensional logic. *Journal of Philosophical Logic*, pages 171–185, 1975.
- [70] M.L. Ginsberg. Multivalued logics: A uniform approach to reasoning in AI. *Computer Intelligence*, 4:256–316, 1988.
- [71] V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bulletins de la classe des sciences*, 15(5):183–188, 1929.
- [72] L. Goble. Paraconsistent modal logic. *Logique et Analyse*, 49(193):3–29, 2006.
- [73] L. Godo, P. Hájek, and F. Esteva. A fuzzy modal logic for belief functions. *Fundamenta Informaticae*, 57(2–4):127–146, 2003.
- [74] R. Goré. Dual Intuitionistic Logic Revisited. In R. Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods. TABLEAUX 2000*, volume 1847 of *Lecture notes in computer science*, pages 252–267. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.
- [75] M. Grabisch. Belief functions on lattices. *International Journal of Intelligent Systems*, 24(1):76–95, January 2009.
- [76] R. Grigolia, T. Kiseliova, and V. Odisharia. Free and Projective Bimodal Symmetric Gödel Algebras. *Studia Logica*, 104(1):115–143, 2016.
- [77] R. Hähnle. A new translation from deduction into integer programming. In *International Conference on Artificial Intelligence and Symbolic Mathematical Computing*, pages 262–275. Springer, 1992.

Bibliography

- [78] R. Hähnle. Many-valued logic and mixed integer programming. *Annals of Mathematics and Artificial Intelligence*, 12(3-4):231–263, 1994.
- [79] R. Hähnle. Tableaux for many-valued logics. In M. D’Agostino, D.M. Gabbay, R. Hähnle, and J. Posegga, editors, *Handbook of Tableau Methods*, pages 529–580. Springer-Science+Business Media, B.V., 1999.
- [80] R. Hähnle. Advanced many-valued logics. In D.M. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, volume 2nd, pages 297–395, Dordrecht, 2001. Springer Netherlands.
- [81] R. Hähnle. Tableaux and related methods. In J.A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume 2, pages 100–178. Elsevier and MIT Press, 2001.
- [82] R. Hähnle. Complexity of many-valued logics. In M. Fitting and E. Orłowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, pages 211–233, Heidelberg, 2003. Springer.
- [83] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Springer, Dordrecht, 1998.
- [84] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. In P. Besnard and S. Hanks, editors, *IJAI’95: Proceedings of the Eleventh Annual Conference on Uncertainty in Artificial Intelligence, Montreal, Quebec, Canada, August 18–20, 1995*, pages 237–244. Morgan Kaufmann, 1995.
- [85] P. Hájek and S. Tulipani. Complexity of fuzzy probability logics. *Fundamenta Informaticae*, 45(3):207–213, 2001.
- [86] J.Y. Halpern. *Reasoning about Uncertainty*. The MIT Press, 2nd edition, 2017.
- [87] J.Y. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54(3):319–379, April 1992.
- [88] G.E. Hughes and M.J. Cresswell. *A new introduction to modal logic*. Routledge, London New York, 1968.
- [89] A. Ilić-Stepić, M. Knežević, and Z. Ognjanović. Intuitionistic propositional probability logic. *Mathematical Logic Quarterly*, 68(4):479–495, November 2022.
- [90] R. Jansana and U. Rivieccio. Residuated bilattices. *Soft Computing*, 16(3):493–504, 2012.
- [91] S.C. Kleene. *Introduction to metamathematics*. North-Holland, Amsterdam, 1952.
- [92] D. Klein, O. Majer, and S. Rafiee Rad. Probabilities with gaps and gluts. *Journal of Philosophical Logic*, 50(5):1107–1141, October 2021.
- [93] D. Kozhemiachenko and L. Vashentseva. Non-contingency in a paraconsistent setting. *Logic Journal of the IGPL*, January 2023.
- [94] C.H. Kraft, J.W. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, 30(2):408–419, 1959.
- [95] A. Kułacka, D. Pattinson, and L. Schröder. Syntactic Labelled Tableaux for Łukasiewicz Fuzzy ALC. In *Twenty-Third International Joint Conference on Artificial Intelligence*. AAAI Press, 2013.

- [96] L. di Lascio and A. Gisolfi. Graded tableaux for rational Pavelka logic. *International journal of intelligent systems*, 20(12):1273–1285, 2005.
- [97] H. Leitgeb. Hype: A system of hyperintensional logic (with an application to semantic paradoxes). *Journal of Philosophical Logic*, 48(2):305–405, 2019.
- [98] Y. Ma and P. Hitzler. Paraconsistent reasoning for OWL 2. In *Web Reasoning and Rule Systems*, volume 5837 of *Lecture notes in computer science*, pages 197–211. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.
- [99] Y. Ma, P. Hitzler, and Z. Lin. Algorithms for paraconsistent reasoning with OWL. In E. Franconi, M. Kifer, and W. May, editors, *The Semantic Web: Research and Applications. ESWC 2007*, volume 4519 of *Lecture notes in computer science*, pages 399–413. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.
- [100] Y. Ma, Z. Lin, and Z. Lin. Inferring with inconsistent OWL DL ontology: A multi-valued logic approach. In T. Grust, H. Höpfner, A. Illarramendi, S. Jablonski, M. Mesiti, S. Müller, P.-L. Patranjan, K.-U. Sattler, M. Spiliopoulou, and J. Wijsen, editors, *Current Trends in Database Technology – EDBT 2006*, volume 4254 of *Lecture notes in computer science*, pages 535–553. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.
- [101] F. Maier. Extending Paraconsistent *SRQLQ*. In *Web Reasoning and Rule Systems*, volume 5837 of *Lecture notes in computer science*, pages 118–132. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.
- [102] F. Maier, Y. Ma, and P. Hitzler. Paraconsistent OWL and related logics. *Semantic Web*, 4(4):395–427, 2013.
- [103] E.D. Mares. Paraconsistent probability theory and paraconsistent bayesianism. *Logique et analyse*, 160:375–384, 1997.
- [104] G. Metcalfe and N. Olivetti. Proof systems for a Gödel modal logic. In *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, pages 265–279. Springer, 2009.
- [105] G. Metcalfe and N. Olivetti. Towards a Proof Theory of Gödel Modal Logics. *Logical Methods in Computer Science*, 7, 2011.
- [106] G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics*, volume 36 of *Applied Logic Series*. Springer, 2008.
- [107] G.M. Moisil. Logique modale. *Disquisitiones mathematicae et physicae*, 2:3–98, 1942.
- [108] F. Montagna and L. Sacchetti. Kripke-style semantics for many-valued logics. *Mathematical Logic Quarterly: Mathematical Logic Quarterly*, 49(6):629–641, 2003.
- [109] S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5-6):507–544, October 2005.
- [110] D. Nelson. Constructible falsity. *The Journal of Symbolic Logic*, 14(1):16–26, 1949.
- [111] S. Odintsov and H. Wansing. Routley star and hyperintensionality. *Journal of Philosophical Logic*, 50:33–56, 2021.
- [112] S.P. Odintsov. *Constructive negations and paraconsistency*, volume 26 of *Trends in logic*. Springer, 2008.

Bibliography

- [113] S.P. Odintsov and H. Wansing. Inconsistency-tolerant description logic: Motivation and basic systems. In *50 Years of Studia Logica*, volume 21 of *Trends in Logic*, pages 301–335. Springer Netherlands, Dordrecht, 2003.
- [114] S.P. Odintsov and H. Wansing. Inconsistency-tolerant description logic. Part II: A tableau algorithm for \mathcal{CALC}^C . *Journal of Applied Logic*, 6(3):343–360, September 2008.
- [115] S.P. Odintsov and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20(3):279–301, 2010.
- [116] S.P. Odintsov and H. Wansing. Disentangling FDE-Based Paraconsistent Modal Logics. *Studia Logica*, 105(6):1221–1254, 2017.
- [117] H. Omori and K. Sano. Generalizing Functional Completeness in Belnap–Dunn Logic. *Studia Logica*, 103(5):883–917, Oct 2015.
- [118] H. Omori and H. Wansing. 40 years of FDE: An Introductory Overview. *Studia Logica*, 105(6):1021–1049, Dec 2017.
- [119] H. Ono. Glivenko theorems revisited. *Annals of Pure and Applied Logic*, 161(2):246–250, November 2009.
- [120] A. Pietz and U. Rivieccio. Nothing but the truth. *Journal of Philosophical Logic*, 42(1):125–135, Feb 2013.
- [121] G. Priest. The Logic of Paradox. *Journal of Philosophical Logic*, 8(1):219–241, 1979.
- [122] G. Priest. *An Introduction to Non-Classical Logic. From If to Is*. Cambridge University Press, 2nd edition, 2008.
- [123] G. Priest. Many-valued modal logics: a simple approach. *The Review of Symbolic Logic*, 1(2):190–203, 2008.
- [124] A. Přenosil. *Reasoning with Inconsistent Information*. PhD thesis, Charles University, Prague, 2018.
- [125] C. Rauszer. Applications of Kripke Models to Heyting-Brouwer Logic. *Studia Logica*, 36(1/2):61–71, 1977.
- [126] C. Rauszer. *An algebraic and Kripke-style approach to a certain extension of intuitionistic logic*. dissertation, Institute of Mathematics, Polish Academy of Sciences, 1980.
- [127] U. Rivieccio. *An Algebraic Study of Bilattice-based Logics*. PhD thesis, University of Barcelona — University of Genoa, 2010.
- [128] A. Rodrigues, J. Bueno-Soler, and W. Carnielli. Measuring evidence: a probabilistic approach to an extension of Belnap–Dunn logic. *Synthese*, 198(S22):5451–5480, October 2021.
- [129] R.O. Rodriguez, O.F. Tuyt, F. Esteva, and L. Godo. Simplified Kripke semantics for K45-like Gödel modal logics and its axiomatic extensions. *Studia Logica*, 110(4):1081–1114, August 2022.
- [130] R.O. Rodriguez and A. Vidal. Axiomatization of Crisp Gödel Modal Logic. *Studia Logica*, 109:367–395, 2021.
- [131] J. Rogger. *Decidability of Order-Based Modal Logics*. PhD thesis, University of Bern, 2016.

- [132] K. Sano and H. Omori. An expansion of first-order Belnap–Dunn logic. *Logic Journal of IGPL*, 22(3):458–481, 2014.
- [133] L.J. Savage. *The foundations of statistics*. Dover, 2nd edition, 1972.
- [134] D. Scott. Measurement structures and linear inequalities. *Journal of mathematical psychology*, 1(2):233–247, 1964.
- [135] K. Segerberg. Qualitative probability in a modal setting. In *Studies in Logic and the Foundations of Mathematics*, volume 63, pages 341–352. Elsevier, 1971.
- [136] G. Shafer. *A mathematical theory of evidence*. Princeton university press, 1976.
- [137] E.Yu. Sherkhonov. Modal operators over constructive logic. *Journal of Logic and Computation*, 18(6):815–829, October 2008.
- [138] Y. Shramko. Hilbert-style axiomatization of first-degree entailment and a family of its extensions. *Annals of Pure and Applied Logic*, 172(9):103011, October 2021.
- [139] Y. Shramko, D. Zaitsev, and A. Belikov. First-Degree Entailment and its Relatives. *Studia Logica*, 105(6):1291–1317, Dec 2017.
- [140] Y. Shramko, D. Zaitsev, and A. Belikov. The FMLA-FMLA axiomatizations of the exactly true and non-falsity logics and some of their cousins. *Journal of Philosophical Logic*, 48(5):787–808, October 2019.
- [141] P. Smets. Belief functions. In P. Smets, Mamdani A., D. Dubois, and H. Prade, editors, *Non-Standard Logics for Automated Reasoning*, pages 253–286. Academic Press, 1988.
- [142] P. Smets. The nature of the unnormalized beliefs encountered in the transferable belief model. In D. Dubois and M.P. Wellman, editors, *UAI '92: Proceedings of the Eighth Annual Conference on Uncertainty in Artificial Intelligence, Stanford University, Stanford, CA, USA, July 17–19, 1992*, pages 292–297. Morgan Kaufmann, 1992.
- [143] N.J.J. Smith. *Vagueness and Degrees of Truth*. Oxford University Press, USA, 2008.
- [144] F. Touazi, C. Cayrol, and D. Dubois. Possibilistic reasoning with partially ordered beliefs. *Journal of Applied Logic*, 13(4):770–798, December 2015.
- [145] E. Turunen, M. Öztürk, and A. Tsoukiás. Paraconsistent semantics for Pavelka style fuzzy sentential logic. *Fuzzy Sets And Systems*, 161(14):1926–1940, July 2010.
- [146] A. Urquhart. The complexity of decision procedures in relevance logic. In A. Urquhart, J.M. Dunn, and A. Gupta, editors, *Truth or Consequences, Essays in Honor of Nuel Belnap*, pages 61–76. Kluwer Academic Publishers Dordrecht, 1990.
- [147] D. Vakarelov. Notes on N-lattices and constructive logic with strong negation. *Studia logica*, 36(1–2):109–125, 1977.
- [148] A. Vidal. On transitive modal many-valued logics. *Fuzzy Sets and Systems*, 407:97–114, March 2021.
- [149] A. Vidal. Undecidability and non-axiomatizability of modal many-valued logics. *The Journal of Symbolic Logic*, 87(4):1576–1605, December 2022.
- [150] H. Wansing. Connexive modal logic. In R. Schmidt, I. Pratt-Hartmann, M. Reynolds, and H. Wansing, editors, *AiML-2004: Advances in Modal Logic*, volume 5 of *Advances in modal logic*, pages 387–399, 2005.

Bibliography

- [151] H. Wansing. Constructive negation, implication, and co-implication. *Journal of Applied Non-Classical Logics*, 18(2–3):341–364, 2008.
- [152] H. Wansing and N. Belnap. Generalized truth values.: A reply to Dubois. *Logic Journal of the IGPL*, 18(6):921–935, December 2010.
- [153] S.K.M. Wong, Y. Yao, and P. Bollmann. Characterization of comparative belief structures. *International journal of man-machine studies*, 37(1):123–133, 1992.
- [154] S.K.M. Wong, Y.Y. Yao, P. Bollmann, and H.C. Bürger. Axiomatization of qualitative belief structure. *IEEE Transactions on Systems, Man, and Cybernetics*, 21(4):726–734, 1991.
- [155] X. Zhang, Z. Lin, and K. Wang. Towards a paradoxical description logic for the semantic web. In S. Link and H. Prade, editors, *Foundations of Information and Knowledge Systems*, volume 5956 of *Lecture notes in computer science*, pages 306–325. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.
- [156] C. Zhou. Belief functions on distributive lattices. *Artificial Intelligence*, 201:1–31, 2013.
- [157] L. Zhou, H. Huang, G. Qi, Y. Ma, Z. Huang, and Y. Qu. Paraconsistent query answering over DL-Lite ontologies. *Web Intelligence and Agent Systems: An International Journal*, 10(1):19–31, 2012.

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Part IV.

Appendices

A. Synopsis (en français)

Ce manuscrit est dédiée à l'étude des logiques modales floues qui formalisent le raisonnement (paraconsistant) sur l'incertitude. Ici, l'interprétation d'«information (données) incertaine(s)» inclut toute combinaison des trois propriétés suivantes. Premièrement, l'information peut être quantifiée, i.e., la proposition est associée à un *degré de vérité* plutôt qu'une *valeur de vérité*. Deuxièmement, l'information peut être incomplète. Troisièmement, l'information peut être contradictoire.

Toutes les logiques que nous allons étudier se divisent en deux groupes. Les logiques modales plus «traditionnelles» dont la sémantique est construite sur des modèles de Kripke où les formules (et parfois, même des relations d'accessibilité) prennent des valeurs de $[0, 1]$ constituent le premier. Le second groupe contient des logiques dites «bi-stratifiées». Dans ces logiques, le langage est partagé en trois : la strate intérieure désignée par \mathcal{L}_i ; \mathcal{L}_o (la strate extérieure) ; et la modalité non-nichante M . On utilise \mathcal{L}_i pour *décrire les événements* et interprète M comme une mesure sur l'univers (e.g., une mesure de probabilité, fonction de croyance, fonction de plausibilité, etc.) correspondante au degré de (in)certitude de l'agent dans une proposition donnée. Le raisonnement sur cette (in)certitude est conduit dans \mathcal{L}_o . Les cadres dans des logiques bi-stratifiées sont, alors, des ensembles munis de mesures.

Chacun de ces deux genres de logiques correspond à l'une des façons d'interpréter l'incertitude. Dans le cas moins formel et plus proche à l'intuition concernant des phrases telles que «je crois que», «je suis certain(e) que», etc., nous utiliserons les logiques avec la sémantique de Kripke. Dans le cas plus formel où l'on assume que le degré de certitude se comporte comme une mesure d'incertitude concrète, nous utiliserons les logiques bi-stratifiées.

Les logiques que nous étudions se divisent aussi en logiques «qualitatives» et «quantitatives» selon ce que l'agent peut faire avec son degré de certitude en ϕ . Dans le cas qualitatif, l'agent n'est capable que de comparer ces degrés concernant des propositions données (comme, par exemple, dans «j'ai une plus grande certitude qu'il neigera aujourd'hui plutôt que ce soit de la grêle») ou exprimer sa complète certitude ou incrédulité («je suis complètement sûr(e) qu'il fera beau aujourd'hui»). C'est-à-dire, l'agent(e) ne connaît pas la valeur exacte de sa certitude. Au contraire, dans le cas quantitatif, on suppose que l'agent(e) connaît ces valeurs et alors peut conduire des opérations arithmétiques avec elles : e.g., les additionner ou soustraire.

Ainsi, les logiques qui formalisent le raisonnement quantitatif seront bâties sur la logique de Łukasiewicz et ses extensions puisqu'elle est capable d'exprimer les opérations arithmétiques. Les logiques pour le raisonnement qualitatif, à leur tour, utiliseront la logique de Gödel pour ses fragments propositionnels. Notre objectif premier sera de construire les axiomatisations, de déterminer leur complexités et rechercher leurs propriétés sémantiques. Parmi celles-ci, nous nous intéresserons principalement à la correspondance entre les formules et les classes de cadres qu'elles définissent ainsi que des traductions entre elles qui préservent leur validité.

B. Introduction (en français)

On croit en beaucoup de choses et l'une des tâches d'un(e) logicien(ne) est de tenter de formaliser ces croyances. Pour cela on a besoin de choisir un environnement dans lequel on puisse construire une formalisation. Bien que l'utilisation de la logique classique soit un moyen bien établi dans la représentation de connaissances et croyances et dans le raisonnement sur l'incertitude, elle ne sera pas (comme l'indique le titre du manuscrit) utilisée ici. Pourquoi ?

Les intuitions que nous souhaitons prendre en compte par rapport aux croyances et incertitudes peuvent s'exprimer (entre autres) par les desiderata suivants.

Desideratum 1. Étant données deux propositions ϕ et χ , on peut être plus certain de ϕ que de χ mais néanmoins ni croire *complètement* ϕ , ni considérer χ comme absolument impossible.

Desideratum 2. Étant données deux sources fiables, on peut préférer l'une à l'autre.

Desideratum 3. On peut croire en une contradiction et toutefois ne pas croire pas en une autre proposition.

Desideratum 4. Étant données deux propositions, il est possible que l'on ne puisse pas toujours comparer leur degré d'incertitude (si, par exemple, ces propositions n'ont aucun contenu en commun).

Desideratum 5. Si l'on suppose que les croyances sont basées sur l'évidence disponible et les témoignages donnés par des sources et si, en plus, il n'y a pas d'évidence du tout concernant ϕ , on n'est pas, alors, capable de poser «je crois ϕ » (ou même «je crois $\phi \vee \neg\phi$ »¹) est vraie ou fausse.

Malheureusement, aucun de ces desiderata ne peut être facilement formalisé dans le cadre de la logique classique.² En effet, toute proposition est soit vraie, soit fausse d'un point de vue classique, alors il n'y a pas de degrés de vérité.³ Aussi, si on représente les sources par les états dans un modèle de Kripke classique et interprète *sRt* comme «*s* fait confiance à *t*», il n'y aura pas de degrés de confiance. Ainsi, les Desiderata 1 and 2 impliquent la nécessité d'utiliser des *logiques floues*, c'est-à-dire, des logiques où les formules prennent des valeurs de $[0, 1]$.

Les Desiderata 3 et 4 nécessitent l'utilisation des *logiques paraconsistantes*, i.e., telles que le principe de l'explosion ($p, \neg p \models q$) n'est pas valide. En fait, le Desideratum 3 dit que la modalisation de l'explosion échoue. Pour comprendre la liaison entre le Desideratum 4 et les logiques paraconsistantes, il faut se rappeler qu'il est d'usage⁴ de traiter la vérité et la fausseté de propositions comme indépendantes. Le conséquence peut alors être définie comme la préservation de la vérité des prémisses à la conclusion, et la préservation de fausseté de la conclusion aux prémisses : si la prémisse est vraie, la conclusion l'est aussi (ou dans le cas flou, la conclusion est *au moins aussi vraie* que la prémisse) ; si la conclusion est fausse, la prémisse l'est aussi (la prémisse est au moins aussi fausse que la conclusion). Dans cette interprétation, les valeurs de ϕ et celles de χ sont incomparables si, par exemple, ϕ est *en même temps plus fausse et plus vraie* que χ .

¹À noter que le principe du tiers exclu n'est pas valide dans les logiques *paracomplètes*. Par conséquent, si le raisonnement de l'agent(e) en suit une, iel n'est pas obligé(e) de croire en une instance donnée du principe du tiers exclu.

²Le terme «logique classique» est parfois imprécis. Ici, nous désignons avec ce terme tout «logique qui étend la logique classique propositionnelle». Donc, (par exemple) **K** est une logique classique ou encore la logique minimale normale modale. Par contre, la logique intuitionniste ne l'est pas.

³Nous référons les lecteur(trices) à [43] pour une discussion détaillée sur les degrés de vérité.

⁴Cette approche peut être tracée jusqu'à Belnap [15] (réimprimé dans [16]). Même plus tôt, dans [110] il y a des réalisations positives et négatives indépendantes de prédicats arithmétiques.

B. Introduction (en français)

Le **cinquième** desideratum indique la nécessité d'utiliser des logiques *paracomplètes*, c'est-à-dire celles où le principe du tiers exclu ne vaut pas. Dans cette thèse, la plupart de logiques (dont les fragments propositionnels) sont bâties sur la logique de Belnap et Dunn BD qui est en même temps paraconsistante et paracomplète.⁵

Dans la partie restante de l'introduction, nous présenterons le contexte plus général de la thèse et discuterons l'état de l'art. Nous donnerons une revue des moyens de formalisation de raisonnement sur l'incertitude et les croyances dans le cadre des logiques modales floues et/ou paraconsistantes.

B.1. Raisonnement sur l'incertitude

Dans le manuscrit, nous considérons le terme «(in)certitude» dans l'un des deux sens suivants. La première interprétation vient de la compréhension intuitive de phrases utilisées dans le discours courant telles que «je suis certain(e) qu'il pleut dehors maintenant», «je pense que la pluie demain est plus probable que la grêle», etc. La seconde interprétation est plus formelle. Dans ce cas, on calcule des valeurs de telles assertions par une *mesure d'incertitude* : une probabilité, fonction de croyance, plausibilité, capacité, etc. Toutes les deux approches sont bien établies dans la logique classique (cf., e.g., [86] pour une introduction et revue).

D'un point de vue formel, ces deux lectures d'incertitude correspondent à deux genres de logiques que nous couvrirons dans la thèse. Le premier correspond aux logiques modales définies sur les cadres de Kripke évalués sur $[0, 1]$, possiblement avec des relations d'accessibilité qui sont aussi évaluées sur $[0, 1]$, et dans le cas de logiques paraconsistantes, avec deux valuations indépendantes interprétées comme le soutien de la vérité et soutien de la fausseté. Les formules modales habituelles $\Box\phi$ et $\Diamond\phi$ seront évaluées comme des infima et suprema de valeurs de ϕ dans les états accessibles (on trouve dans [123] des exemples simples des logiques modales plurivalentes où des modalités sont définies par des infima et suprema ou maxima et minima). $\Box\phi$ ou $\Diamond\phi$ sont interprétées comme «l'agent(e) croit que ϕ est vraie»⁶ ou «l'agent(e) est certain(e) de ϕ ».

Le deuxième type de logiques correspond à l'approche dit «croyance comme mesure». Il comprend des logiques *bi-stratifiées*. L'idée principale est de séparer la description d'événements du raisonnement de ces événements sur le niveau syntactique. Pour cela le langage est ordonné en trois parties : le *langage intérieure* \mathcal{L}_i pour décrire des événements, la *modalité mesure* M définie sur l'univers des événements et le *langage extérieure* \mathcal{L}_o qui formalise le raisonnement sur les événements. Ici, nous utilisons un langage très simple $\{\neg, \wedge, \vee\}$ (qui sera dans la plupart de cas muni de la sémantique de BD) pour décrire des événements. En particulier, nous n'utiliserons pas l'implication (sauf si elle est définissable par des autres conjonctifs) puisque les propositions conditionnelles ne correspondent pas aux descriptions des événements. Le choix de langage extérieur et dont la sémantique sera faite selon des scénarii que nous formaliserons. En général, les logiques extérieures seront floues. Ceci est lié à la tradition existante de son utilisation dans la formalisation de raisonnements sur l'imprécision [143], les [73, 144] et l'incertitude [55].

De la même manière qu'il y a deux types d'incertitudes en général, il y a aussi deux genres de mesures. Le premier est compris de mesures quantitatives, où l'on assume que l'agent(e) peut donner une valeur numérique à chaque proposition comme «il y aura du vent aujourd'hui» et la préciser comme, par exemple, «je suis 73% certain(e) qu'il y aura du vent aujourd'hui» ou «je suis deux fois plus certain(e) qu'il pleuvra aujourd'hui qu'il neigera». La croyance peut après être décrite plus précisément avec une mesure de probabilité, une fonction de croyance, plausibilité, etc. Dans cette approche, nous choisissons la logique de Łukasiewicz \mathbb{L} et ses extensions pour formaliser le raisonnement dans la strate extérieure parce que \mathbb{L} est capable d'exprimer les

⁵De telles logiques sont parfois désignées *paradéfinies* or *paranormales* [4].

⁶Évidemment, \Box ainsi que \Diamond peut être interprété comme une modalité de croyance. La différence entre eux étant que l'on *rejette* la \Box -croyance s'il y a un contre-exemple, mais on *accepte* la \Diamond -croyance s'il y a un exemple soutenant.

opérations arithmétiques sur $[0, 1]$.

La seconde approche est *qualitative*. Étant données deux propositions, l'agent(e) ne peut que déterminer la plus ou moins probable des deux, postuler qu'elles ont la même probabilité, que l'on est complètement sur(e) de l'une ou l'autre, ou si l'on raisonne de manière paraconsistante, poser que les probabilités de deux propositions ne sont pas comparables. Cette approche est formalisée par des relations de préférence sur les univers d'événements (i.e., des preordres totaux sur l'ensemble des parties de l'univers) caractérisés par ses analogues mesures. Plus formellement, étant donné un ensemble d'événements W , une mesure μ s'accorde avec une relation de préférence \preceq si et seulement si

$$\forall X, Y \subseteq W : X \preceq Y \Leftrightarrow \mu(X) \leq \mu(Y)$$

Pour le raisonnement qualitatif nous utiliserons les extensions de la logique de Gödel parce qu'elles peuvent exprimer les relations d'ordre mais pas les fonctions arithmétiques.

En ce qui concerne le côté formel, nous viserons principalement à construire les axiomatisations des logiques formalisant les moyens mentionnés de raisonner sur l'incertitude, à étudier dont l'expressivité et les propriétés sémantiques et à établir ainsi la décidabilité et complexité. Le dernier point est une direction de recherche importante dans le raisonnement classique (on peut trouver les résultats qui concernent la complexité du raisonnement avec des logiques modales dans, e.g., [87, 8, 13] et la complexité des logiques probabilistes dans [61]), notamment en lien avec la représentation des connaissances. Nous poursuivons avec l'utilisation du raisonnement non-classique pour l'incertitude.

B.2. Logiques modales floues

Comme nous l'avons déjà discuté, il est raisonnable d'avoir des propositions modales qui peuvent avoir plusieurs degrés de vérité. En effet, il y a des obligations plus ou moins importantes et des convictions plus ou moins fortes. Ainsi, il existe des logiques déontiques, doxastiques, épistémiques floues (cf., e.g., [50] et [46]).

Différentes logiques floues (propositionnelles) possèdent différentes capacités par rapport à l'expressivité. On peut grossièrement les partager en trois catégories : celles qui peuvent exprimer l'addition et la soustraction (tronquées) ; celles qui peuvent exprimer l'ordre sur $[0, 1]$; celles qui n'en sont pas capables. Les exemples les mieux connus sont les logiques de Łukasiewicz, de Gödel et la logique du produit (cf., e.g., [83] ou [106] pour dont la présentation détaillée).

Lorsqu'on formalise des propositions modales qui se produisent dans la langue naturelle, on attend à ce que l'agent(e) puisse les comparer (par exemple, «je pense que la pluie est plus probable aujourd'hui que l'ouragan»). Par ailleurs, il est rare de voir quelqu'un(e) qui dirait «je suis sûr(e) à 67% que le chien de Paula est un Golden retriever» bien que «je pense que le chien de Paula est un Golden retriever plutôt qu'un Teckel» est une phrase absolument naturelle. Ainsi, les logiques de second genre semblent être le choix le plus raisonnable pour cela.

La logique (propositionnelle) de Gödel \mathbf{G} peut être traitée comme une logique de vérité comparative puisque les valeurs des formules ne dépendent que de l'ordre de valeurs des variables. Ainsi, elle est bien adaptée à la formalisation de propositions modales. L'augmentation de \mathbf{G} avec \square et \diamond (\mathbf{KG}) munie de la sémantique sur les cadres évalués sur $[0, 1]$ et ayant des relations d'acceptabilité floues fut pour la première fois introduite dans [39] et fut elle-même (ainsi que ses extensions axiomatiques correspondant à celles de \mathbf{K}) bien étudiée depuis lors. En particulier, les axiomatisations de fragments \square et \diamond ainsi que celles de logiques bi-modales floues [40] et fraîches [130] sont connues. Par exemple \mathbf{KG} est floue et fraîche et ses fragments mono-modaux sont \mathbf{PSpace} -complets [104, 105, 37, 131, 38]. L'analogue Gödel de $\mathbf{S4}$ est aussi \mathbf{PSpace} -complet [51]. Par ailleurs, il existe des applications de logiques modales de Gödel au raisonnement sur l'incertitude. Par exemple, l'analogue Gödel de $\mathbf{K45}$ est complet par rapport aux cadres où \square et \diamond

B. Introduction (en français)

	est vraie quand	est fausse quand
$\neg\phi$	ϕ est fausse	ϕ est vraie
$\phi_1 \wedge \phi_2$	ϕ_1 et ϕ_2 sont vraies	ϕ_1 est fausse ou ϕ_2 est fausse
$\phi_1 \vee \phi_2$	ϕ_1 est vraie ou ϕ_2 est vraie	ϕ_1 et ϕ_2 sont fausses

TABLE B.1. : Les conditions de vérité et de fausseté des formules BD.

sont interprétés comme mesures de nécessité et possibilité non-normalisées⁷ sur un cadre de Kripke [129].

La logique bi-Gödelienne (Gödelienne symétrique dans [76]) **biG** étend **G** avec \prec (la co-implication, interprétée comme ‘exclut’) ou l’opérateur Delta de Baaz [9] écrit Δ (interprété ‘il est vraie que’). Cela permet d’exprimer l’ordre strict. Ainsi, les extensions modales de **biG** sont capables de formaliser des phrases telles que «je pense que le chien de Paula est un Golden retriever plutôt qu’un Teckel» (comme dessus) où «plutôt que» est traité comme «strictement plus confiant(e)». **KbiG** (l’augmentation de **biG** avec \Box et \Diamond) fut introduite dans [20] et permet du calcul de Hilbert dans [21]. En outre, une augmentation temporelle de **biG** fut proposée dans [2]. La satisfaisabilité et la validité des deux logiques appartiennent à **PSpace**.

De la même manière que les logiques de description classiques sont des variants notationnels de logiques modales (globales) classiques, les logiques de description Gödel sont les analogues des logiques modales Gödel. En effet, le flou leur permet d’exprimer les données vagues et incertaines, ce qui est au-delà des capacités des ontologies classiques. La décidabilité et l’expressivité de logiques de description Gödel sont bien étudiés [30, 31, 33, 32, 29]. De plus, leur complexité est souvent la même que dans les cas classiques (cf., e.g., [8]). Cela montre un avantage (cette fois, pratique) de DLs Gödel par comparaison avec des Łukasiewicz puisque ces dernières ne sont pas décidables sauf si elles n’utilisent pas la conjonction définie par la t-norme Łukasiewicz [34, 41, 95, 148]. En fait, la logique globale modale de Łukasiewicz n’est même pas axiomatisable [149].

B.3. Logiques modales paraconsistantes

Comme nous le remarquâmes déjà dans le début de ce chapitre, les logiques modales paraconsistantes peuvent formaliser des propositions qui expriment les croyances ou certitudes d’une manière plus intuitive que les logiques classiques. Les logiques paraconsistantes (principalement celles qui augmentent BD, la Logique de Paradox LP de Priest [121], ainsi que les systèmes liés) trouvèrent aussi leurs applications dans la représentation de connaissances puisqu’elles peuvent facilement formaliser le raisonnement et les requêtes sur les ontologies contradictoires. Les logiques paraconsistantes de description attirent aussi beaucoup d’attention. En particulier, les analogues paraconsistants d’**ALC** [113, 114, 155] et des systèmes beaucoup plus expressifs [101] furent proposés et étudiés. Des versions tolérantes aux contradictions de Web Ontology Language (OWL) furent développées [100, 99, 98, 102]. Il y eut aussi des recherches sur les requêtes sur des ontologies inconsistantes [157].

Nous observâmes aussi qu’il est de coutume de traiter la vérité et la fausseté de formules comme indépendantes dans le contexte des logiques paraconsistantes. Formellement, cela veut dire que l’on considère les cadres de Kripke munis de deux valuations et pas d’une seule, selon [122, 150, 72, 137, 115, 116, 52]. Ces valuations sont interprétées, comme attendu, comme des soutiens indépendants de vérité et de fausseté ou comme des soutiens positifs et négatifs. L’idée suit «la logique utile» de Belnap et Dunn [56, 15, 14, 16] (alias, BD ou FDE — «first-degree entailment (conséquence de premier degré)»).

Les conditions gérant la vérité et la fausseté des négations, conjonctions, disjonctions dans BD sont intuitives et peuvent être résumées dans la Table B.1. Les logiques modales que nous

⁷Les mesures normalisées correspondent à l’analogue Gödel de **KD45**.

considérerons sont bâties sur \mathbf{biG} . Nous aurons ainsi besoin de concevoir les conditions de fausseté de \rightarrow et \prec . Dans ce manuscrit, nous nous concentrons principalement sur l'extension de \mathbf{G} qui définit $\neg(\phi \rightarrow \phi') \leftrightarrow (\neg\phi' \prec \neg\phi)$ et $\neg(\phi \prec \phi') \leftrightarrow (\neg\phi' \rightarrow \neg\phi)$ suivant $\mathbf{l}_4\mathbf{C}_4$ ⁸ de [151] quand il s'agit des fragments propositionnels de logiques modales avec la sémantique de Kripke. Nous désignons désormais cette logique avec $\mathbf{G}_{(\rightarrow, \prec)}^2$.

$\mathbf{G}_{(\rightarrow, \prec)}^2$ a de bonnes propriétés. D'abord, tous les connecteurs possèdent des duals. Deuxièmement, contrairement à \mathbf{G} , *il n'est pas vrai que $p \rightarrow q$ ou $q \rightarrow p$ prend une valeur désignée sous toute valuation*. Cela implique l'existence de propositions non-comparables, ce qu'il est prudent d'assumer si l'on raisonne sur des croyances. On n'est pas obligé de croire qu'il va y avoir un orage aujourd'hui *plus ou moins que* l'on croit que sa deuxième cousine ait deux chiens.

Un pas suivant attendu après l'introduction de deux valuations différentes pour la vérité et la fausseté est d'introduire deux relations d'accessibilité comme fait dans [137, 52]. La première relation déterminera si une formule modale *est vraie dans w* et l'autre si elle est *fausse dans w* .

Nous concluons la section par un bref résumé des modalités paraconsistantes que nous considérons. Nous différencions entre deux types. $\Box\phi$ dont le soutien négatif est défini comme le *supremum* de soutiens négatifs de ϕ dans les états accessibles (et \Diamond — le dual de \Box) est la première modalité. Le deuxième couple est $\blacksquare\phi$ et $\blacklozenge\phi$. Ici le soutien négatif est l'infimum de soutiens négatifs de ϕ ⁹ (respectivement le supremum dans le cas de \blacklozenge) dans les états accessibles. Nous étudierons ces modalités sur les cadres flous et frais et dans l'environnement mono et bi-relationnel.

B.4. Logiques modales bi-stratifiées

Dans la Section B.1 nous mentionnâmes que nous utiliserons des logiques bi-stratifiées pour le raisonnement sur l'incertitude interprétée par des mesures. Les logiques bi-stratifiées sont moins expressives que celles permettant la nidification de modalités. Bien que cette restriction puisse sembler trop forte, elle est, en fait, justifiable.

Une alternative évidente aux logiques bi-stratifiées seraient les logiques qui permettent la nidification de \mathbf{M} . Il y a beaucoup d'exemples de ces systèmes comme une augmentation de la logique épistémique avec des probabilités conditionnelles proposée dans [47]. D'ailleurs, les analogues qualitatifs des mesures de probabilité sont axiomatisés dans [69] puis dans [48, 49] en utilisant une modalité binaire \lesssim interprétée comme la relation de préférence. Il faut cependant noter que les modalités nichantes sont difficiles à interpréter dans la langue naturelle puisque l'on dit rarement quelque chose comme *il est probable que p et que q est probable aussi*. On peut comprendre $\mathbf{M}p$ comme « p est probable» ou «l'agent(e) croit que p est vraie», etc. selon \mathbf{M} . La valeur de $\mathbf{M}p$ est facilement dérivée de la mesure de sous-ensemble de l'univers où p est vraie. En revanche, l'interprétation de formules comme $\mathbf{M}(p \wedge \mathbf{M}q)$ qui correspond au fragment en italique de phrase précédente est considérablement moins intuitive.¹⁰

Du côté formel, les procédures de décision pour telles logiques ne sont pas faciles et ne peuvent alors pas être utilisées pour obtenir une évaluation exacte de complexité (par exemple, la filtration fut utilisée dans [47] pour démontrer la décidabilité de CKL ; Gärdenfors [69] dénombre les ordonnances de préférence sur les modèles canoniques de formules).

D'autre part, les procédures de décision pour les logiques bi-stratifiées sont d'habitude assez intuitives et peuvent souvent être adaptées de celles pour les strates extérieures [61, 60, 85]. En plus, la décidabilité d'une logique bi-stratifiée n'est souvent pas plus difficile à calculer que celle

⁸Cette logique fut indépendamment introduite par plusieurs auteurs [151, 97] puis étudiée dans [111]. Elle constitue le fragment propositionnel de la logique modale de Moisil [107]. Nous remercions Heinrich Wansing qui nous l'indiqua.

⁹Nous donnons une motivation plus détaillée dans le Chapitre 7.

¹⁰C'est encore plus le cas par rapport au \lesssim nichant : $(p \lesssim q) \lesssim (r \lesssim s)$ n'est pas une phrase naturelle à prononcer non plus.

de sa strate extérieure. Les formules extérieures sont aussi faciles à interpréter puisqu’elles ne sont que des combinaisons propositionnelles d’atomes modaux (i.e., de formules de forme $M\phi$, où ϕ est une formule interne).

D’habitude, des logiques bi-stratifiées formalisent le raisonnement classique sur l’incertitude (il y a quand même une logique *de facto* bi-stratifiée qui formalise le raisonnement probabiliste intuitionniste [89]). Dans le cas du raisonnement qualitatif, les axiomes gérants l’ordonnance de préférences peuvent être traduits dans les formules modales correspondantes. Dans le cas quantitatif, il y a deux options. La première et la plus simple est d’utiliser les opérations arithmétiques dans la strate extérieure comme dans [61, 60]. Une alternative «logiquement puriste» est d’utiliser une logique floue qui peut exprimer les opérations requises. Pour cela, on utilise les logiques de produit ou de Łukasiewicz dont la combinaison et ses extensions [84, 73, 66, 44] puisque pour que la logique bi-stratifiée soit complète elle doit être capable d’exprimer l’additivité (finie) de mesures de probabilité et de fonctions de croyances. Récemment [11] ont prouvé que les deux approches sont équivalentes dans le sens où il y a des traductions qui préservent la validité de formules. Dans ce texte, nous travaillerons avec des logiques bi-stratifiées «puristes» dont la strate extérieure est une augmentation de la logique de Łukasiewicz (quand il s’agira d’incertitude quantitative) ou de la logique de Gödel (dans le cas qualitatif). La principale raison est la possibilité de réduction des preuves de complétude de telles logiques aux preuves de complétude de logiques extérieures étendues avec des axiomes gérants les mesures d’incertitude.

Enfin, en traitant les extensions paraconsistantes de \mathbf{L} et \mathbf{G} , nous devons concevoir les conditions de fausseté des implications. La première option fut présentée dans la section précédente : dans \mathbf{G} , on dualise l’implication par la co-implication et dans \mathbf{L} , on définit $\neg(\phi \rightarrow \chi) \leftrightarrow (\neg\chi \ominus \neg\phi)$. Cela produit une implication congruentielle capable de définir l’ordre. La deuxième option est d’utiliser une interprétation plus intuitive : « $\phi \rightarrow \chi$ est fausse quand ϕ est vraie mais χ est fausse». L’idée d’une telle implication vient de l’interprétation de la condition de fausseté par Nelson [110]. Un autre avantage de l’implication de Nelson est la possibilité de *séparer* le soutien de vérité du soutien de la fausseté avec elle qui contraste de l’implication forte ou congruentielle (\rightarrow).

B.5. Structure du manuscrit

La partie restante du manuscrit est structurée comme suit. La Partie I couvre des préliminaires par rapport à la logique de Belnap–Dunn (Chapitre 2) et les fragments propositionnels des logiques modales : les augmentations paraconsistantes des logiques de Łukasiewicz et de Gödel (Chapitres 3 et 4). Nous définissons ainsi les sémantiques, construisons les axiomatisations complètes, créons les procédures de décision en utilisant des tableaux de contraintes, établissons les complexités, et prouvons plusieurs propriétés importantes. Le Chapitre 3 est basé sur [19] (Sections 3.2 et 3.3) et [26] (Sections 3.1.1 et 3.1.2). Certains preuves ont été étayées par rapport aux versions publiées. Le Chapitre 4 est basé sur [19] (Sections 4.2 et 4.3.1) et [24] (Section 4.1). Les résultats présentés dans la Section 4.3.2 n’ont jamais été publiés pour le moment.

La partie principale du manuscrit est divisée en deux parties. La Partie II est dédiée aux logiques modales avec la sémantique de Kripke et la Partie III aux logiques bi-stratifiées.

Dans le Chapitre 5, nous présentons \mathbf{KbiG} — une augmentation modale de \mathbf{biG} sur des cadres frais et construisons un calcul de Hilbert fortement complet. Puis, nous prouvons que \mathbf{KbiG} est décidable et explorons son expressivité et la théorie de la correspondance. En particulier, nous montrons comment l’addition de Δ ou \prec change l’expressivité de \mathbf{KbiG} en comparaison avec \mathbf{KG} et étudions les classes de formules qui définissent les mêmes classes de cadres dans \mathbf{K} et \mathbf{KbiG} . Ce chapitre est basé sur [21].

Dans le Chapitre 6, nous bâtissons \mathbf{KG}^{2c} — une augmentation paraconsistante de \mathbf{KbiG} . Nous montrons que la validité de \mathbf{KG}^{2c} est réductible à celle de \mathbf{KbiG} et utilisons ce fait pour concevoir l’axiomatisation complète de \mathbf{KG}^{2c} et obtenir ainsi la décidabilité. Nous construisons aussi un

calcul de tableaux simple pour \mathbf{KG}^{2c} sur les cadres au branchements fini et obtenons un analogue du théorème de Glivenko. Le contenu du chapitre fut plubié pour la première fois dans [21] (Section 6.2 l’analogue du théorème de Glivenko dans la Section 6.3) et [20] (Section 6.1 et le tableau de la Section 6.3).

Le Chapitre 7 aborde les logiques modales paraconsistantes de Gödel sur des cadres flous munis de deux relations. Nous considérons les logiques avec \square et \diamond ainsi que celles avec \blacksquare et \blacklozenge . Nous examinons ainsi leurs propriétés sémantiques et prouvons que les modalités dans chaque couple ne sont pas interdéfinissables. En plus, nous établissons que $\mathbf{KG}^{2\pm}$ (la logique avec \square et \diamond) n’étend pas \mathbf{KbiG} floue et que $\mathbf{G}_{\blacksquare, \blacklozenge}^{2\pm}$ (la logique avec \blacksquare et \blacklozenge) est non-normale mais régulière. Nous examinons aussi la définissabilité des classes des cadres et montrons, entre autres, que les cadres aux branchements fini flous et frais sont définissables dans toutes les deux logiques. Nous construisons pour elles les calculs tableaux et les utilisons pour obtenir les résultats à propos de la décidabilité et des évaluations de la complexité. Le chapitre est basé sur [23, 22] (Sections 7.1 et 7.2, respectivement).

Le Chapitre 8 est dédié aux logiques bi-stratifiées $\mathbf{Pr}_{\Delta}^{\pm 2}$ et $\mathbf{4Pr}^{\pm \Delta}$ qui formalisent le raisonnement avec des \pm -probabilités et des probabilités à quatre valeurs proposees dans [92].¹¹ Nous construisons ainsi les axiomatisations et prouvons les complétudes faibles. Nous démontrons aussi qu’il y a des traductions fidèles entre $\mathbf{Pr}_{\Delta}^{\pm 2}$ et $\mathbf{4Pr}^{\pm \Delta}$, concevons des procédures de décision par des tableaux aux contraintes et obtenons les évaluations des complexités. Le contenu de la Section 8.2 est emprunté de [26] et celui des Sections 8.3 et 8.4 de [25].

Le Chapitre 9 présente les logiques bi-stratifiées qui formalisent le raisonnement qualitatif (classique ainsi que paraconsistant) sur l’incertitude. Nous présentons les logiques \mathbf{QG} , \mathbf{MCB} , et \mathbf{NMCB} qui étendent la logique bi-Gödelienne et leurs extensions paraconsistantes $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ et $\mathbf{G}_{(\rightarrow, \multimap)}^2$. Nous bâtissons des calculs fortement complets pour ces trois logiques et etudions ainsi la théorie de la correspondance. Les résultats du chapitre sont publiés dans [24].

La conclusion résume les résultats obtenus dans la dissertation et dessine un plan de recherche futur.

¹¹Les \pm -probabilités furent nommées «des probabilités non-standard» dans [92]. Nous utilisons une désignation plus spécifique.

C. Conclusion (en français)

Nous récapitulons les résultats principaux de la dissertation et indiquons des pistes de recherches futures.

C.1. Résumé des chapitres

Nous revenons aux desiderata proposés dans l'introduction. Dans ce manuscrit nous visâmes à proposer et à étudier des logiques qui les respectent. Le Chapitre 7 présente des logiques munies de la sémantique de Kripke qui satisfont les cinq desiderata. Dans les Chapitres 8 et 9 nous proposons également les moyens avec lesquels on peut étendre les logiques bi-stratifiées avec de nouvelles modalités qui satisfont tous les desiderata également.

Nous présentons ici le résumé détaillé des résultats obtenus dans le manuscrit. Dans le Chapitre 3, deux extensions paraconsistantes de $\mathfrak{L} \text{ — } \mathfrak{L}_{(\Delta, \rightarrow)}^2$ et $\mathfrak{L}_{(\rightarrow)}^2 \text{ — }$ furent présentées. Nous construisîmes leurs axiomatisations sous forme de calculs de Hilbert et prouvâmes¹² sur la complétude (Théorèmes 3.1 et 3.2). En outre, nous bâtîmes $\mathcal{T}(\mathfrak{L}^2)$ un calcul de tableaux unifié pour les deux logiques et démontrâmes sa complétude (Theorem 3.3), qui fut utilisé pour obtenir la NP-complétude de \mathfrak{L}^2 's (Théorème 3.4). Nous explorâmes aussi les propriétés sémantiques de \mathfrak{L}^2 's et montrâmes que l'addition de nouveaux axiomes à $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ et $\mathfrak{L}_{(\rightarrow)}^2$ rend modus ponens invalide (Théorème 3.5).

Dans le Chapitre 4, nous conçûmes $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ et $\mathbf{G}_{(\rightarrow, \rightarrow)}^2 \text{ — }$ des extensions paraconsistantes de \mathbf{biG} . Pour elles, des calculs de Hilbert et tableaux fortement complètes furent proposés (Theorems 4.3 et 4.4). La complétude des calculs de Hilbert fut prouvée par la construction de traductions mutuelles entre des $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$ - et $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ -valuations sur $[0, 1]$ et des modèles de Kripke linéaires de $\mathbf{I}_1\mathbf{C}_1$ et $\mathbf{I}_4\mathbf{C}_4$ utilisant deux valuations comme défini dans [151] (Théorèmes 4.1 et 4.2). Nous établîmes que contrairement aux \mathfrak{L}^2 's, l'ensemble de formules \mathbf{G}^2 -valides reste le même tant que le filtre sur $[0, 1]^{\mathfrak{K}}$ correspondant aux valeurs désignées étend $(1, 0)^\uparrow$ ou $(1, 1)^\uparrow$ (pour $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ et $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$, respectivement — Théorème 4.6). De plus, il n'y a que six relations de conséquence correspondant à la sémantique de $\mathbf{G}_{(\rightarrow, \leftarrow)}^2$ générées par des filtres sur $[0, 1]^{\mathfrak{K}}$ et seulement deux relations qui correspondent à la sémantique de $\mathbf{G}_{(\rightarrow, \rightarrow)}^2$ générée par des filtres premiers de forme $(x, 1)^\uparrow$ (Théorèmes 4.7 et 4.8 et Corollaire 4.2).

Le Chapitre 5 présente \mathbf{KbiG} — une augmentation de $\mathbf{G}\Delta$ avec \square et \diamond . Nous prouvâmes la complétude forte dont le fragment frais (Théorème 5.2) en empruntant la méthode de [130]. Nous étudiâmes aussi la théorie des modèles de \mathbf{KbiG} : nous établîmes, notamment, plusieurs classes de formules transférables de \mathbf{K} (Théorèmes 5.4 et 5.5) et caractérisâmes les cadres qui permettent le théorème de Glivenko (Théorème 5.7). De plus, nous adaptâmes l'approche à la preuve de PSpace-complétude de \mathbf{KG}^c de [38] et l'utilisâmes pour prouver le même résultat à propos de \mathbf{KbiG}^c (Théorème 5.8).

Le Chapitre 6 fut dédié à \mathbf{KG}^{2c} — une augmentation paraconsistante de \mathbf{KbiG}^c . Nous réduisîmes la validité de \mathbf{KG}^{2c} à celle de \mathbf{KbiG} et utilisâmes cela pour prouver que leurs capacités expressives sont identiques (Corollaire 6.1) et obtenir l'axiomatisation complète de \mathbf{KG}^{2c} (Théorème 6.1). Par ailleurs, nous étudiâmes $\mathbf{KG}_{\text{fb}}^{2c} \text{ — } \mathbf{KG}^{2c}$ sur des cadres au branchement fini : nous

¹²À noter, que la complétude de $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ sans Δ a été obtenue dans [27]. De plus, les preuves de complétude faible de $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ et $\mathfrak{L}_{(\rightarrow)}^2$ furent en premier données dans [26]. Les preuves présentées dans ce texte sont des versions simplifiées de celles de [27].

C. Conclusion (en français)

caractérisâmes les cadres qui valident un analogue paracosistant de théorème de Glivenko (Théorème 6.2) et construisîmes un simple calcul de tableaux pour $\mathbf{KG}_{\text{fb}}^{2c}$ qui fut utilisé pour prouver dont l'approximation aux modèles finis et la PSpace-complétude (Théorèmes 6.3 et 6.4).

Nous généralisâmes davantage \mathbf{KbiG} et \mathbf{KG}^{2c} dans le Chapitre 7. Là, nous créâmes deux logiques qui augmentent G^2 avec des modalités normales (\square et \diamond) et informationnelles (\blacksquare et \blacklozenge) — $\mathbf{KG}^{2\pm}$ et $G_{\blacksquare, \blacklozenge}^{2\pm}$ dont les sémantiques sont définies sur des cadres à deux relations. Nous démontrâmes que dans aucun couple les modalités ne sont définissable l'une vers l'autre si le cadre est muni de deux relations (Théorèmes 7.1 et 7.9). Nous prouvâmes aussi que les cadres mono-relationnels sont définissables dans les deux langages (Théorèmes 7.5 et 7.11) et que $\mathbf{KG}^{2\pm}$ frais étend \mathbf{KbiG} frais et, en fait, la validité de $\mathbf{KG}^{2\pm}$ frais peut être réduite à celle de \mathbf{KbiG} (Théorème 7.3). En revanche, cela ne vaut pas à propos de $\mathbf{KG}^{2\pm}$ flou (Théorème 7.2). Par ailleurs, nous démontrâmes que les classes de cadres frais sont $\mathbf{KG}^{2\pm}$ -définissables si et seulement si toutes les deux dont les relations sont \mathbf{KbiG} -définissables par la même formule (Corollaires 7.1 et 7.2). Par contre, il est possible de définir des cadres frais ou flous munis des relations différentes dans $G_{\blacksquare, \blacklozenge}^{2\pm}$ (Théorème 7.10). Nous montrâmes aussi la définissabilité des cadres au branchement fini dans $\mathbf{KG}^{2\pm}$ et $G_{\blacksquare, \blacklozenge}^{2\pm}$ (Théorème 7.6 et comme un corollaire immédiat du Théorème 7.10) et construisîmes des calculs tableaux complets pour $\mathbf{KG}_{\text{fb}}^{2\pm}$ et $G_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ (Théorèmes 6.3 et 7.12). Nous les utilisâmes pour construire des procédures de décision et prouver la PSpace-complétude de $\mathbf{KG}_{\text{fb}}^{2\pm}$ et $G_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ (Théorèmes 7.8 et 7.13).

Le Chapitre 8 couvrit deux logiques bi-stratifiées basées sur $\mathfrak{L}_{(\Delta, \rightarrow)}^2$ et $\mathfrak{L}_{\Delta} - \text{Pr}_{\Delta}^{\mathfrak{L}^2}$ et $4\text{Pr}^{\mathfrak{L}^{\Delta}} -$ qui formalisent le raisonnement quantitatif sur \pm - et 4-probabilités. Nous fournîmes dont les axiomatisations Hilbert (Théorèmes 8.1¹³ et 8.2) ainsi que les traductions bilatérales (Théorèmes 8.3 et 8.4) qui furent puis utilisées pour prouver la NP-complétude de $\text{Pr}_{\Delta}^{\mathfrak{L}^2}$ et $4\text{Pr}^{\mathfrak{L}^{\Delta}}$ (Théorème 8.5).

Dans le Chapitre 9, nous examinâmes les logiques pour le raisonnement qualitatif sur les mesures d'incertitude : les classiques (QG et ses extensions axiomatiques) and paracosistent (MCB et NMCB). Nous bâtîmes les calculs de Hilbert et prouvâmes leur complétude (Théorèmes 9.2 et 9.8) et établîmes la correspondance entre les formules et les propriétés de mesures codées par ces formules (Théorèmes 9.3, 9.4, 9.7, et 9.9).

C.2. Questions ouvertes et pistes de recherches futures

Notre travail laisse plusieurs questions importantes à résoudre. Puisque le manuscrit est divisée en trois parties, la discussion aussi est divisée selon ces parties.

C.2.1. Fragments propositionnels

Rappelons-nous du Corollaire 4.2. Disons que ϕ est *globalement vraie* dans un modèle \mathfrak{M} de G^2 si et seulement si $w \models^+ \phi$ vaut pour tout $w \in \mathfrak{M}$ et *globalement désignée* si, en plus, $w \not\models^- \phi$ vaut pour tout $w \in \mathfrak{M}$.

Problème 1 (Filtres sur $[0, 1]^{\aleph}$ et relations de conséquence sur modèles Kripke de G^2). Il est clair de Définition 4.10 que les propositions suivantes sont vraies pour chaque $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{G^2(\rightarrow, \leftarrow)}$.

- $\Gamma \models_{(1,0)\uparrow} \phi$ si et seulement si ϕ est globalement désignée dans tout modèle \mathfrak{M} de G^2 où Γ est globalement désigné.
- $\Gamma \models_{(1,1)\uparrow} \phi$ si et seulement si ϕ est globalement vraie dans tout modèle \mathfrak{M} de G^2 où Γ est globalement vrai.

Or, la question est comme suit : est-il possible de définir les analogues de conséquences générées par $(x, x)^\uparrow$, $(x, y)^\uparrow$, et $(y, x)^\uparrow$ sur des modèles Kripke de G^2 ?

¹³La preuve fut empruntée de [26].

Problème 2 (Axiomatisation de conséquences G^2). Dans la Proposition 4.5, nous établimes les conséquences de premier degré qui séparent les relations de conséquence définies par des filtres. Si l'on les ajoute à $\mathcal{HG}_{(\rightarrow, \leftarrow)}^2$ comme des règles, produira-t-il des axiomatisations complètes?

C.2.2. Logiques modales floues

Problème 3 (**KbiG** floue). Nous considérons davantage **KbiG***fraiche* obtenue de **KG**^c par l'ajout d'axiomes géranes Δ de Définition 3.4, un axiome additionnel de fraîcheur et un axiome géran la relation entre Δ et \Diamond . Si l'on enlève tous les deux axiomes de fraîcheur, obtiendra-t-on un calcul complète pour **KbiG**^f?

Problème 4 (Axiomatisation et décidabilité de **KG**^{2±} et $G_{\blacksquare, \blacklozenge}^{2\pm}$). L'axiomatisation de **KG**^{2c} et la preuve de sa **PSpace**-complétude (Théorème 6.1 et Corollaire 6.1) étaient simples parce qu'il était possible d'utiliser les formes normales négatives avec \neg . En revanche, ni **KG**^{2±}, ni $G_{\blacksquare, \blacklozenge}^{2\pm}$ ne les admettent. En fait, elles n'étendent pas **KbiG**^f, d'où il n'y a pas de réduction immédiate vers la validité **KbiG**.

Il semble raisonnable que les deux logiques soient aussi **PSpace**-complètes. Il est, cependant, difficile à deviner comment on peut les axiomatiser et quelle est leur relation avec **KbiG**. Il n'est pas clair non plus si la méthode de [38] sera utile pour obtenir la complexité de ces logiques.

Problème 5 (Modalités G^2 globales et non-standard et logiques de description). Les logiques de description Gödel (et, par conséquence, les logiques de Gödel avec des modalités globales) sont bien étudiées (cf. [32] pour un résumé des résultats). Il est ainsi raisonnable d'introduire des modalités globales dans **KG**^{2±}. En plus, au mieux de nos connaissances, des modalités non-standard (telles que la contingence, l'accident, etc.) ne sont pas étudiées dans le cadre de DLs ni dans le contexte des logiques paraconsistantes et modales en général. En fait, il semble qu'il n'y a que deux articles sur les logiques paraconsistantes (bien que fraîches) avec des modalités non-standard : [3] et [93] (dont le deuxième fut coécrit par l'auteur de ce manuscrit). L'introduction des \blacksquare and \blacklozenge globaux en plus des \square and \diamond globaux a aussi du sens.

C.2.3. Logiques bi-stratifiées

Problème 6 (Fonctions de croyance à quatre valeurs). Nous considérâmes des analogues paraconsistants des fonctions de croyance définies sur des algèbres de De Morgan dans [26]. Ces fonctions sont proches des \pm -probabilités puisqu'elles donnent deux valeurs indépendantes à chaque formule ϕ : la croyance en ϕ elle-même et sa négation $\neg\phi$. Par ailleurs, le raisonnement (quantitatif) est formalisé dans une logique bi-stratifiée qui augmente $\mathfrak{L}_{(\Delta, \rightarrow)}^2$. Cela donne du sens de continuer les recherches commencées dans [92] et de proposer un analogue de ces fonctions de croyance fourni de quatre valeurs et construire leur axiomatisation bi-stratifiée.

Problème 7 (\pm -probabilités et fonctions de croyance qualitatives). Les analogues qualitatifs de mesures classiques les plus importantes (capacités, fonctions de croyance et probabilités) sont bien connus [94, 154, 153]. En revanche, la tâche de construire leurs analogues en BD reste ouvert.

L'une des difficultés techniques les plus évidentes est l'utilisation des compléments et ensembles avec des intersections vides dans les définitions des analogues qualitatifs des fonctions de croyances et probabilités dans le Théorème 9.1. Il peut être possible de les circonvenir en se reposant sur une augmentation (faiblement) fonctionnement complète de BD sur la strate intérieure comme, par exemple, $BD\Delta$ [132] (cf. [124] pour l'étude de sa sémantique algébrique).

D. Author's contributions

The dissertation is based on several works co-written by its author over the course of their studies and is mostly a continuation of the project proposed in [27]. The papers whose results were used in the writing of this text are listed in chronological order w.r.t. submission with the author's contributions relevant to the dissertation's content detailed for each item. The study of paraconsistent and modal expansions of bi-Gödel logic was mainly undertaken due to the author of this manuscript; most technical results and motivation were also provided by them.

- [19] in collaboration with Sabine Frittella and Marta Bílková, the author constructed tableau calculi designated here with $\mathcal{T}(\mathfrak{L}^2)$ ¹⁴ and $\mathcal{T}(\mathfrak{G}^2)$; they also proved semantical properties of \mathfrak{L}^2 and \mathfrak{G}^2 discussed in Sections 3.3 and 4.3.1.
- [20] in collaboration with Sabine Frittella and Marta Bílková, the author constructed the tableau calculus designated here with $\mathcal{T}(\mathbf{KG}_{\text{fb}}^{2c})$ and established its semantical properties outlined in Section 6.1.
- [26] the author contributed equally to the completeness proof of $\mathcal{HPr}_{\Delta}^{\mathfrak{L}^2}$ with Marta Bílková.
- [24] in collaboration with Sabine Frittella, Marta Bílková, and Ondrej Majer, the author constructed the calculi axiomatising QG, MCB, NMCB, and their extensions.
- [21] in collaboration with Sabine Frittella and Marta Bílková, the author constructed the Hilbert-style axiomatisations for \mathbf{KbiG} and \mathbf{KG}^{2c} , established their PSpace-completeness, and proved semantical properties discussed in Sections 5.3 and 6.3.
- [25] in collaboration with Sabine Frittella, Marta Bílková, and Ondrej Majer, the author devised the calculus axiomatising $4Pr^{\mathfrak{L}^{\Delta}}$, constructed the embeddings between $4Pr^{\mathfrak{L}^{\Delta}}$ and $Pr_{\Delta}^{\mathfrak{L}^2}$, and proved the NP-completeness of these logics.
- [23] in collaboration with Sabine Frittella and Marta Bílková, the author devised the tableau calculus for $\mathbf{KG}_{\text{fb}}^{2\pm}$ and proved the model-theoretic properties discussed in Sections 7.1.1 and 7.1.2.
- [22] in collaboration with Sabine Frittella and Marta Bílková, the author devised the tableau calculus for $\mathbf{G}_{\blacksquare, \blacklozenge, \text{fb}}^{2\pm}$ and proved its model-theoretic properties.

¹⁴More precisely, only the Δ -free fragment of $\mathcal{T}(\mathfrak{L}^2)$ was provided in [19]; Δ was added in [25].

Daniil KOZHEMIACHENKO

Logiques modales paraconsistantes et floues pour le raisonnement sur l'incertitude

Ce manuscrit est dédié à l'étude des logiques modales floues qui formalisent le raisonnement (paraconsistant) sur l'incertitude. Ici, l'interprétation d'«information (données) incertaine(s)» inclut toute combinaison des trois propriétés suivantes. Premièrement, l'information peut être quantifiée, i.e., la proposition est associée à un *degré de vérité* plutôt qu'une *valeur de vérité*. Deuxièmement, l'information peut être incomplète. Troisièmement, l'information peut être contradictoire.

Toutes les logiques étudiées se divisent en deux groupes. Les plus «traditionnelles» dont la sémantique est construite sur des modèles de Kripke où les formules (et parfois, même des relations d'accessibilité) prennent des valeurs dans $[0, 1]$ constituent le premier groupe. Le second groupe contient des logiques dites «bi-stratifiées». Ici, le langage est composé de trois parties : la strate intérieure désignée par \mathcal{L}_i ; \mathcal{L}_o (la strate extérieure); et la modalité non-nichante M . On utilise \mathcal{L}_i pour *décrire les événements*. On interprète M comme une mesure sur l'univers (e.g., une mesure de probabilité, fonction de croyance, fonction de plausibilité, etc.) correspondante au degré de (in)certitude de l'agent dans une proposition donnée. Le raisonnement sur cette (in)certitude est conduit dans \mathcal{L}_o . Les cadres dans des logiques bi-stratifiées sont alors des ensembles munis de mesures.

Chacun de ces deux genres de logiques correspond à l'une des façons d'interpréter l'incertitude. Dans le cas moins formel, nous utiliserons les logiques avec la sémantique de Kripke. Dans le cas plus formel où l'on assume que le degré de certitude se comporte comme une mesure d'incertitude concrète, nous utiliserons les logiques bi-stratifiées.

Mots clés : logiques modales floues, logiques modales paraconsistantes, logiques bi-stratifiées, complexité.

Paraconsistent and fuzzy modal logics for reasoning about uncertainty

This dissertation is devoted to the study of fuzzy modal logics that formalise (paraconsistent) reasoning about uncertainty. The understanding of 'uncertain information (data)' here includes any combination of the following three characteristics. First, the information can be graded, i.e., the statement is equipped with a *truth degree* rather than a *truth value*. Second, the information can be incomplete. Third, the information can be contradictory.

All the logics in question can be divided into two kinds. First, the more 'traditional' modal logics defined on $[0, 1]$ -valued Kripke models (possibly, with fuzzy accessibility relations) whose language includes modal operators $\Box\phi$ and $\Diamond\phi$ interpreted as, respectively, infima and suprema of ϕ 's values in the accessible states.

The second kind of logics contains so-called 'two-layered' logics. In this framework, the language is divided into three parts: the *inner layer* \mathcal{L}_i , the *outer layer* \mathcal{L}_o , and the *non-nesting modality* M . The idea is to use \mathcal{L}_i to *describe events*, interpret M as a *measure on the set of events* (e.g., as a probability function, belief function, plausibility, etc.) corresponding to the degree of the agent's (un)certainty in a given event, and then *reason about this (un)certainty* in \mathcal{L}_o . A *frame* in a two-layered logic is, thus, a set with a measure defined thereon.

These two kinds of logics correspond to two ways of interpreting uncertainty. In the less formal one, we will be using the logics with the Kripke-frame semantics. In the more formal case where the degree of one's certainty or belief in ϕ is assumed to behave as a concrete uncertainty measure, we will use the two-layered logics.

Keywords: paraconsistent modal logics, fuzzy modal logics, two-layered logics, complexity.