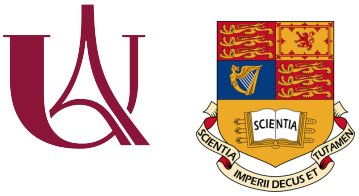
Pricing and hedging of transition risk in Credit Portfolio

Lionel Sopgoui Mbeukam



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Pricing and hedging of climate transition risk in Credit Portfolio

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed: Lionel SOPGOUI

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Notations

- $\mathbb{N}$ is the set of non-negative integers, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and $\mathbb{Z}$ is the set of integers.
- $\mathbb{R}^d$ is the d-dimensional Euclidean space, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^* := (0, \infty)$.
- $\mathbb{R}^{n \times d}$ is the set of real-valued $n \times d$ matrices ($\mathbb{R}^{n \times 1} = \mathbb{R}^n$), $\mathbf{I}_n$ is the identity $n \times n$ matrix.
- x^i denotes the i-th component of the vector $x \in \mathbb{R}^d$. For all $A := (A^{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$, we denote by $A^\top := (A^{ji})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ the transpose matrix.
- $\bigotimes$ is the Kronecker product while $\odot$ is the Hadamard product.
- For a given finite set S, we define $\#S$ as its cardinal.
- For any $x, y \in \mathbb{R}^d$, we denote the scalar product $x^\top y$, the Euclidean norm $|x| := \sqrt{x^\top x}$ and for a matrix $M \in \mathbb{R}^{d \times d}$, we denote $|M| := \sup_{a \in \mathbb{R}^d, |a| \leq 1} |Ma|$.
- If $v = \begin{bmatrix} v_1 \\ \vdots \\ v_I \end{bmatrix} \in \mathbb{R}^I$, then $\mathrm{Diag}(v) = \begin{bmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & v_I \end{bmatrix}$.
- We note $\mathbf{1} := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^I$.
- $(\Omega, \mathcal{H}, \mathbb{P})$ is a complete probability space.
- For $p \in [1, \infty]$, E is a finite dimensional Euclidean vector space and for a σ-field $\mathcal{H}$, $\mathcal{L}^p(\mathcal{H}, E)$, denotes the set of $\mathcal{H}$-measurable random variable X with values in E such that $\|X\|_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}} < \infty$ for $p < \infty$ and for $p = \infty$, $\|X\|_\infty := \mathrm{esssup}_{\omega \in \Omega} |X(\omega)| < \infty$.
- For a filtration $\mathbb{G}$, $p \in [1, +\infty]$ and $I \in \mathbb{N}^*$, $\mathscr{L}_+^p(\mathbb{G}, (0, \infty)^I)$ is the set of discrete-time processes that are $\mathbb{G}$-adapted valued in $(0, \infty)^I$ and which satisfy

$$\|X_t\|_p < \infty \text{ for all } t \in \mathbb{N}.$$

- If X and Y are two random variables $\mathbb{R}^d$-valued, for $x \in \mathbb{R}^d$, we note $Y|X = x$ the conditional distribution of Y given $X = x$, and $Y|\mathcal{F}$ the conditional distribution of Y given the filtration $\mathcal{F}$.
- If $f : \mathbb{R} \to \mathbb{R}, t \mapsto f(t)$ is a differentiable function, we note $\dot{f}$ its first derivative.
- If $f : \mathbb{R} \to \mathbb{R}, t \mapsto f(t)$ is a function, we note for $v = \begin{bmatrix} v_1 \\ \vdots \\ v_I \end{bmatrix} \in \mathbb{R}^d$, $f(v) = \begin{bmatrix} f(v_1) \\ \vdots \\ f(v_d) \end{bmatrix}$.

Résumé [1]

L'objectif de cette thèse est d'étudier l'impact de la transition climatique sur les portefeuilles de crédit. Il s'agit précisément de calculer les mesures de risques d'un portefeuille de crédits. Ces derniers peuvent être garantis ou non garantis. Ils sont contractés par des entreprises appartenant à une économie organisée en secteurs, dirigée par une productivité dynamique et stochastique (un vecteur autorégressif en temps discret ou un processus Ornstein-Uhlenbeck multidimensionnel en temps continu), et soumise à la transition climatique modélisée à l'aide d'un processus dynamique et déterministe représentant le prix du carbone. La principale nouveauté de notre approche est que nous proposons une méthodologie de bout en bout, partant d'un scénario de transition modélisation par le prix du carbone, jusqu'à l'impact sur différentes mesures de risque de crédit. Elle est divisée en deux parties.

La première partie est divisée en deux chapitres, et le temps est supposé discret. Dans le premier chapitre, nous modélisons l'environnement économique dans lequel opèrent les entreprises d'un portefeuille de crédits supposés tous non collatéralisés. Nous considérons une entreprise représentative pour chaque secteur et un ménage représentatif pour l'ensemble de l'économie. De plus, les émissions de gaz à effet de serre (GES) émises par les entreprises lorsqu'elles produisent des biens et/ou des services et lorsqu'elles consomment des entrées intermédiaires, ainsi que par les ménages lorsqu'ils consomment les biens et/ou les services, sont facturées à l'aide du prix du carbone. En transformant le problème d'optimisation des profits des entreprises en une séquence de problèmes déterministes et en résolvant le problème d'optimisation des ménages par une méthode variationnelle, nous obtenons l'impact du prix du carbone sur la dynamique des variables macroéconomiques sectorielles telles que les émissions de GES, la production, la consommation, les entrées intermédiaires, etc. Dans le deuxième chapitre, nous utilisons la méthode *Discounted Cash Flow* (DCF) pour déterminer la valeur des firmes opérant dans l'économie décrite précédemment. Nous utilisons ensuite cette valeur dans un modèle structurel de crédit. Cela nous permet de calculer différentes mesures de risques d'un portefeuille de crédits : la probabilité de défaut de chaque entreprise, ainsi que les pertes attendues et

[1]Ce travail a été financé par de BPCE S.A. dans le cadre d'une thèse CIFRE. Les opinions ici exprimées sont celles des auteurs et ne sont pas destinées à représenter les opinions ou les positions officielles de BPCE S.A.

inattendues. Ces dernières dépendront en particulier du prix du carbone puisqu'elles dérivent d'entreprises appartenant à une économie qui en est soumise.

Dans la deuxième partie, nous travaillons en temps continu. Le problème est le même, mais nous supposons cette fois que les prêts peuvent être garantis. Les mesures de risques dépendent par conséquent de la valeur des garanties au moment du défaut. Ces dernières appartenant à la même économie que les entreprises prêteuses, elles subissent également la transition climatique. Nous modélisons deux exemples de garanties: l'une mobilière (un actif financier) et l'autre matérielle (un bien immobilier). Lorsque la garantie est un actif financier, nous déterminons sa valeur par la méthode DCF puisqu'il génère un flux de trésorerie. En revanche, quand il s'agit d'un bien immobilier, nous montrons que sa valeur dépend la différence entre un processus d'Ornstein-Uhlenbeck exponentiel et des coûts de rénovation (pour rendre le logement efficace énergétiquement) ainsi que le surplus de coûts d'énergie (avant rénovation). Nous calculons enfin, comme dans la première partie, différentes mesures de risques avec en plus la Loss Given Default.

Les différents modèles sont calibrés et appliqués sur les données de l'économie Française ainsi que sur les données financières réelles des portefeuilles du Groupe BPCE (pour un usage interne) ou fictives (pour publications).

Mots clé— Risque de crédit, Risque climatique, Collatéral, Modélisation stochastique, Risque de transition, Prix du carbone, Évaluation d'entreprise, Mesures de risque

Abstract [2]

The objective of this thesis is to quantify the impact of climate transition on credit portfolio. It is specifically about calculating the risk measures of a portfolio of loans. These can be secured and/or unsecured. They are contracted by firms belonging to an economy divided in sectors, driven by a dynamic and stochastic productivity (modelled as a vector autoregressive process or a multidimensional Ornstein-Uhlenbeck process), and subject to climate transition modeled using a dynamic and deterministic process representing the carbon price. The main contribution of our approach is that we propose an end-to-end methodology, starting from a transition scenario modeled by the carbon price, to the impact on different credit risk measures. Our work is divided into two parts.

The first part is divided into two chapters, and time is assumed to be discrete. In the first chapter, we model the economic environment in which the companies of a portfolio of credits, all assumed unsecured, operate. We consider a representative firm for each sector and a representative household for the entire economy. In addition, greenhouse gas (GHG) emissions generated by companies when they produce goods and/or services and when they consume intermediate inputs, as well as by households when they consume goods and/or services, are charged using the carbon price. By deriving the firms' problem into a sequence of deterministic problems and solving the households' problem by a variational method, we obtain the impact of the carbon price on the dynamics of sectoral macroeconomic variables such as GHG emissions, production, consumption, intermediary inputs, etc. In the second chapter, we use the Discounted Cash Flow (DCF) method to determine the value of firms operating in the previously described economy. We then use this value into a structural credit model. This allows us to calculate different risk measures of a credit portfolio: the probability of default of each firm, as well as the expected and unexpected losses. These will particularly depend on the carbon price since they derive from companies belonging to an economy that is subject to it.

In the second part, we work in continuous time. The problem is the same, but we

[2]This research are funded by a CIFRE grant from BPCE S.A. The opinions expressed in this research are those of the authors and are not meant to represent the opinions or official positions of BPCE S.A.

assume now that the loans can be secured. The risk measures, especially the losses, therefore depend on the value of the guarantees at the default time. These guarantees, because they belong to the same economy as the lending companies, also undergo the climate transition. We model two examples of guarantees: one intangible (a financial asset) and the other tangible (a property). When the guarantee is a *financial asset*, we determine its value by the DCF method since it generates a stream of cash flows. On the other hand, when it is a *building*, we show that its value depends on the difference between an exponential Ornstein Uhlenbeck process and renovation costs (to make the housing energy efficient) as well as the surplus of energy costs (before renovation). We finally calculate, as in the first part, different risk measures with in addition the Loss Given Default.

The different models are calibrated and applied to data from the French economy as well as to real financial data from the BPCE's portfolios or to fictitious data (for publications).

Key words— Credit risk, Climate risk, Collateral, Stochastic modelling, Transition risk, Carbon price, Firm valuation, Risk measures

Chapter 1

Résumé détaillé

Le but de ce manuscrit est de construire des modèles mathématiques permettant quantifier et de projeter les mesures de risques d'un portefeuille de crédit dans un contexte de transition climatique. Nous considérons un portefeuille de crédits d'entreprises appartenant chacune à une unique économie. Nous supposons que l'économie est fermée, dirigée par une productivité dynamique et stochastique, et soumise à la transition climatique modélisée par un prix de carbone dynamique et déterministe.

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1.1 Contexte

1.1.1 Un défi de notre temps

Il est maintenant évident que les changements climatiques ont et auront des effets indésirables sur l'environnement, sur le réchauffement de la planète, et sur les sociétés humaines. La consommation des hydrocarbures émettent du dioxyde de carbone, l'agriculture et l'élévage produisent quant à eux du méthane notamment à cause des engrais ou des excréments d'animaux, etc. Le dioxyde de carbone comme le méthane sont deux exemples de gaz à effet de serre (GES), qui, lorsqu'ils s'accumulent dans l'atmosphère, perturbent la force radiative, qui elle assure l'équilibre thermique à la surface de la terre et dans les océans. Depuis quelques décennies, de nombreux sommets internationaux ont été organisés afin de faire face à ce nouveau risque: le Protocole de Kyoto en 1997, l'Accord de Copenhague en 2009, l'Accord de Paris sur le climat en 2015, etc. Ces rencontres ont permis de prendre enfin la mesure de l'enjeu et d'opérer une transition vers une économie bas carbone.

Le risque climatique a deux composantes. Il est composé d'une part du risque physique, qui est dû au changement des variables climatiques (comme l'augmentation de la température et la fonte des glaces) ou à la survenue des événements météorologiques extrêmes (tels des sécheresse ou des typhons) qui peuvent endommager les infrastructures et mettre les populations en danger. D'autre part, il est composé du risque de transition qui vient de la nécessité de la transition vers une économie bas carbone, entrainant la mise en oeuvre des mesures réglementaires, des changements technologiques potentiels, et de l'évolution des préférences des consommateurs. Ces évolutions auront des nombreux impacts dans tous les secteurs et à toutes les échelles de l'économie. Nous nous intéressons dans cette thèse au secteur financier et bancaire.

Les crises financières précédentes ont montré l'existence d'une relation forte entre la sphère financière et la sphère réelle. C'est pourquoi les régulateurs encouragent désormais les acteurs à prendre en compte le risque de transition dans leurs modèles. Les institutions financières et de régulation telles la Banque de France (voir Devulder and Lisack [2020], Allen et al. [2020]), ont commencé à mener des recherches pour comprendre et quantifier l'impact du risque climatique sur les institutions financières (parmi lesquelles les banques commerciales, les assureurs, et les gestionnaires d'actifs). On constate par exemple que le risque de transition peut modifier les trois principales composantes du risque de crédit que sont *les cash flows de l'emprunteur, la valeur de ses actifs, et la valeur de ses garanties.* En effet, la transition climatique implique la réduction de la demande des produits avec une empreinte carbone élevée, l'augmentation des coûts de recherche et de production de produits bas-carbone, la réévaluation des actifs des entreprises ainsi que ceux des ménages, et l'augmentation des actifs irrécupérables.

1.1.2 Des questions émergentes

Dans nos économies modernes, ce sont les banques commerciales qui jouent le principal rôle pour le financement des agents économiques (aussi bien les entreprises et les ménages, que les Etats). Elles le font entre autres en accordant des crédits à ces derniers. Tous les agents, en particulier les entreprises, peuvent faire *défaut* pour de multiples raisons. Pour assurer la viabililité de son business model, satisfaire les exigences réglementaires, voire la stabilité de l'économie (particulièrement pour les institutions systémiques), une banque se doit d'avoir des provisions – calculées à partir de la perte attendue, c'est-à-dire la perte moyenne anticipée sur une période de temps définie. Les pertes attendues représentent un coût d'exploitation et doivent généralement être absorbées par une partie du résultat opérationnel – ainsi que du capital économique et réglementaire – qui peut être décrit comme une protection contre les pertes futures inattendues (c'est-à-dire les pertes réelles potentielles moins les pertes attendues) à un niveau de confiance donné. Tant les provisions que le capital économique dépendent de la distribution des pertes des portefeuilles de crédits. Par ailleurs, nous avons souligné la nécessité de la transition vers une économie bas-carbone ainsi les effets possibles de cette transition sur les entreprises, c'est-à dire les emprunteurs. Il y a donc un lien entre le risque de crédit et le risque de transition climatique.

Trois questions principales émergent:

1. **Comment modéliser la transition climatique?** Le secteur financier considère généralement trois principaux drivers au risque de transition: les changements dans les préférences des consommateurs, les changements technologiques, et les changements politiques. Pour ces derniers, la déclinaison la plus connue est le prix du carbone. Cela revient à dire que les émissions de GES des agents économiques sont facturées. C'est celle que nous allons adopter ici.

2. **Comment modéliser le *défaut* d'un empruteur et le lier à la transition climatique ?** On peut déclencher le *défaut* d'une entreprise si celle-ci fait face à une crise d'illiquidité c'est-à-dire que ses flux de trésorerie ne suffisent pas à satisfaire les demandes de paiement des créanciers (intérêts et remboursements nets) comme prévu dans le contrat, ou une crise d'insolvabilité c'est-à-dire que la valeur de marché de sa dette dépasse la valeur de marché de l'entreprise (on parle aussi de *surendettement*). A partir de cette définition, considérons deux exemples pour montrer que la transition climatique, et notamment le prix du carbone, peut affecter le *surendettement*: (1) les émissions de GES de l'entreprise sont facturées, sa trésorerie peut diminuer et donc sa distance au *défaut* est également réduite, (2) les actifs d'une entreprise émettent tellement de GES que le coût potentiel de ces émissions affectent leur valeur, cela impacte la valeur de marché de l'entreprise et donc sa distance au *défaut*, voire sa date de *défaut*.

3. **Comment modéliser la perte en cas de *défaut* et la lier à la transition climatique ?** Quand un emprunteur fait *défaut*, la banque peut perdre une partie de l'emprunt qui est la différence entre la montant de l'exposition au moment du *défaut* et de la valeur liquidative de l'entreprise et de ses garanties. Nous avons vu que la valeur marché de l'entreprise peut être affectée par des charges financières supplémentaires du fait de ses émissions de GES. Il en va de même pour ses garanties. Si par exemple, ces dernières sont des biens immobiliers, et des passoires énergétiques, elles subiront directement une dépréciation et/ou nécessiteront des rénovations.

1.1.3 Un problème à formaliser

Nous considérons un espace de probabilité complet $(\Omega, \mathcal{H}, \mathbb{P})$ et $\mathbb{K} \in \{\mathbb{N}, \mathbb{R}\}$. Soit N un entier naturel non nul. Soient trois processus stochastiques Θ d-dimensionnel ainsi $\mathfrak{b}$ et $\overline{\mathfrak{b}}$ N-dimensionnels, définis sur l'espace probabilisé $(\Omega, \mathcal{H}, \mathbb{P})$, et qui représenteront respectivement la productivité de l'économie, le bruit de la valeur de l'emprunteur, et le bruit du collatéral. On considère aussi un processus déterministe n-dimensionnel δ. On note en outre les filtrations $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{K}}$, $\mathbb{G}^{\mathfrak{b}} := \left(\mathcal{G}_t^{\mathfrak{b}}\right)_{t\in\mathbb{K}}$ et $\mathbb{G}^{\overline{\mathfrak{b}}} := \left(\mathcal{G}_t^{\overline{\mathfrak{b}}}\right)_{t\in\mathbb{K}}$, telles que $t \in \mathbb{K}$, $\mathcal{G}_t := \sigma\left(\{\Theta_s : s \in [0,t] \cap \mathbb{K}\}\right)$, $\mathcal{G}_t^{\mathfrak{b}} := \sigma\left(\{\Theta_s, \mathfrak{b}_s : s \in [0,t] \cap \mathbb{K}\}\right)$, et $\mathcal{G}_t^{\overline{\mathfrak{b}}} := \sigma\left(\left\{\Theta_s, \overline{\mathfrak{b}}_s : s \in [0,t] \cap \mathbb{K}\right\}\right)$.

Soient $(A_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ et $(U_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ deux séquences dépendantes de δ et respectivement $\mathbb{G}^{\overline{\mathfrak{b}}}$ et $\mathbb{G}^{\mathfrak{b}}$-mesurables. Nous définissons un portefeuille composé de N crédits contractés par des entreprises appartenant toutes à une même économie, telle que pour tous $t \in \mathbb{K}$ et $n \in \{1, \dots, N\}$, $A_t^n > 0$ représente l'exposition nette de l'emprunteur n à la date t et U_t^n représente la perte par exposition à l'emprunteur n à la date t. La *perte du portefeuille* à la date t notée L_t^N s'écrit

$$L_t^N := \sum_{n=1}^{N} A_t^n U_t^n. \tag{1.1.1}$$

Une première étape en risque de crédit consiste à regrouper les entreprises en sous-groupes homogènes basés par exemple sur leur industrie, leur géographie, leur taille, ou leur note de crédit. Cependant, étant donné que nous nous intéressons ici au risque de transition climatique, il serait mieux de classer les entreprises par intensité carbone : les entreprises ayant des intensités carbone proches appartiennent à un même secteur/groupe homogène.

Nous supposons donc qu'il existe $I \in \mathbb{N}^*$ ($I \leq N$) secteurs d'émissions de GES homogènes dans l'économie. Puisque nous disposons rarement des émissions/intensités carbone individuelles, nous supposons que chaque entreprise a la même intensité carbone que son secteur d'activité. Cela revient à regrouper les "secteurs d'activités" en I "secteurs d'émission de carbone". À partir de maintenant, les secteurs doivent être interprétés comme des secteurs d'émission carbone, ce qui nous permet de construire des sous-portefeuilles homogènes.

Definition 1.1.1 (Sous-portefeuilles)**.** Nous divisons notre portefeuille en I sous-portefeuilles disjoints $g_1, \dots, g_I$ de sorte que chaque sous-portefeuille représente une seule classe de risque et que les entreprises de chacun d'eux appartiennent à un seul secteur d'émission de carbone. A partir de maintenant, nous notons $\mathcal{I}$ de cardinal $I \in \mathbb{N}^*$ l'ensemble des secteurs (et des sous-portefeuilles). Nous posons également $n_i := \min\{n \in \{1, \dots, N\} \text{ tel que } n \in g_i\}$ pour tout $i \in \mathcal{I}$. Par conséquent, l'entreprise n_i est un représentant du groupe i.

Le but de ce travail est de décrire les mesures de risque du portefeuille (et des pertes des sous-portefeuilles $(g_i)_{i\in\mathcal{I}}$) c'est à dire par exemple pour $t, T \in \mathbb{K}$,

$$\mathrm{EL}_t^{N,T} := \mathbb{E}\left[L_{t+T}^N - L_t^N \middle| \mathcal{G}_t\right],$$

appelée *perte espérée* et pour $\alpha \in [0,1]$,

$$\mathrm{UL}_{t,\alpha}^{N,T} := \mathrm{VaR}_{t,\delta}^{\alpha,N,T} - \mathrm{EL}_t^{N,T}, \quad \text{avec} \quad 1-\alpha = \mathbb{P}\left[L_{t+T}^N - L_t^N \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right],$$

appelée *perte inespérée* et définie à l'aide de la Value at Risk (VaR), si les entreprises appartiennent à une économie fermée (en d'autres termes, sans importation ni exportation) drivée par une productivité décrite par Θ et soumise à une transition climatique décrite par δ. Cela revient précisément à quantifier la distorsion dans le temps des mesures du risque de crédit créée par l'introduction d'un prix du carbone.

Nous nous proposons de résoudre ce problème en deux parties: une partie discrète (qui se prête mieux à la modélisation économique et à la modélisation de la perte sans collatéral) et une partie continue (qui se prête mieux à la modélisation de la perte avec collatéral). Dans la première partie, nous supposons que les prêts sont tous non collatéralisés tandis que Θ et δ sont des processus à temps discret (c'est-à-dire $\mathbb{K} = \mathbb{N}$ et $(A_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ est déterministe) et dans la deuxième partie, certains prêts peuvent être collatéralisés tandis que Θ et δ sont des processus à temps continu (c'est-à-dire $\mathbb{K} = \mathbb{R}_+$ et $(A_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ est stochastique).

Dans le chapitre 3, nous proposons une modélisation du cadre dans lequel opèrent les entreprises du portefeuille. Ce cadre est décrit par un modèle économique dynamique, stochastique, et multisectoriel dans lequel les émissions directes et indirectes de GES des entreprises ainsi que les émissions directes de GES des ménages sont facturées. Nous choisissons une entreprise représentative dans chaque secteur et un ménage représentatif pour l'ensemble de l'économie. Dans le chapitre 4, nous supposons que chaque entreprise appartient à un secteur et que ses flux de trésorerie sont une proportion de ses ventes. Ces dernières étant une proportion de la production sectorielle, nous obtenons la dynamique des flux de trésorerie que nous utilisons pour modéliser la valeur des entreprises dans un environnement où les émissions de GES sont facturées. Enfin, à partir d'un modèle structurel dans lequel une entreprise fait *défaut* si elle est surendettée c'est-à-dire si sa valeur devient inférieure à sa dette, nous calculons la probabilité de *défaut* de chaque entreprise ainsi que la perte (et les statistiques associées) du portefeuille (et des sous portefeuilles) déformée par le prix du carbone.

Dans le chapitre 5, après avoir étendu les trois modèles de la partie I en temps continu, nous proposons une définition de la perte d'un portefeuille de crédit en toute date. Précisement, lorsqu'une contrepartie fait *défaut*, s'il n y a pas de garantie, la banque récupère une proportion déterministe de son exposition à la date du *défaut*. S'il y a une garantie, la banque perd une proportion du montant de son exposition restant après la liquidation de la garantie. Dans le contexte de la transition climatique, nous détaillons ensuite le cas où la garantie est un actif financier (elle est dite intangible) suivie du cas où la garantie est un bien immobilier (elle est dite tangible). Nous définissons enfin des mesures de risques de crédit que sont la probabilité de *défaut*, la perte attendue, et la perte inattendue. Dans chaque partie et chaque chapitre, des simulations sont réalisées sur des données réelles ou fictives pour différents scénarios de transition (c'est à dire pour différentes trajectoires de prix de carbone).

1.2 Revue de la littérature

Les premiers stress tests climatiques ont eu lieu en France entre 2019 et 2021. Neuf groupes bancaires et quinze groupes d'assurance se sont impliqués sur la base du volontariat. Comme indiqué dans ACPR [2023], ils ont permis une première quantification de l'impact du risque climatique sur les institutions financières. La méthodologie utilisée pour réaliser ces stress tests est principalement inspirée de Allen et al. [2020]. Dans ce dernier, les auteurs évaluent l'impact du risque de transition climatique sur les portefeuilles de crédit en trois étapes: (1) des modèles climato-économiques (voir Luderer et al. [2015]) qui permettent d'obtenir la trajectoire du produit intérieur brut (PIB) et du prix du carbone selon les scénarios de transition climatique (par exemple, moins de 2°C en 2050), (2) des modèles macroéconomiques et multisectoriels (voir Bertram et al. [2021] et Devulder and Lisack [2020]) qui permettent d'obtenir des variables macroéconomiques en fonction d'une taxe carbone, et enfin (3) des modèles financiers pour calculer des probabilités de défauts. Dans ce travail, nous allons non seulement nous inspirer de cette méthodologie mais aussi l'étendre.

1.2.1 Des modèles climato-économiques

Il s'agit de modèles qui combinent les processus climatiques, macro-économiques, et les rétroactions entre le climat et l'économie dans un cadre de modélisation unique. Comme indiqué par Farmer et al. [2015], il existe trois classes de tels modèles: les modèles d'évaluation intégrée (IAM), les modèles d'équilibre général dynamique et stochastique (DSGE), et les modèles basés sur des agents (ABM). Quelque soit leur type, il s'agit de modèles d'optimisation ou d'évaluation qui visent à approcher la transition par le prix du carbone (ou la taxe carbone) aussi bien exogène qu'endogène. Par exemple, Nordhaus [1993], Reis and Augusto [2013], et Golosov et al. [2014] proposent des modèles dont le but est de déterminer une trajectoire endogène et optimale du prix de carbone à suivre pour (ne pas) atteindre un objectif de

température ou d'émisions de GES. En revanche, Golosov et al. [2014], McKibbin and Wilcoxen [2013] et Devulder and Lisack [2020] proposent quant à eux une modélisation multisectorielle mais la taxe carbone est exogène pour le dernier modèle, même s'il est statique et déterministe. D'autres auteurs se concentrent sur des secteurs plus spécifiques. C'est le cas pour Golosov et al. [2014] qui traitent spécifiquement des secteurs énergétiques, ou de Ter Steege and Vogel [2021] pour le secteur immobilier. Ces derniers travaillent en temps discret et supposent que la différence de prix par mètre carré entre deux propriétés batiments, devrait être uniquement expliquée par la somme de la valeur actualisée de la différence de coût énergétique attendue.

1.2.2 Des modèles de risque de crédit

La littérature sur les modèles de risque de crédit est très vaste et date à peu près de Merton [1974]. Le but reste de calculer la perte du portefeuille introduite en (1.1.1) (ou à minima certaines de ses statistiques). La formulation la plus utilisée (notamment parce qu'encouragée par les régulateurs) est proposée par Basel Committee on Banking Supervision [2017] et qui est essentiellement basée sur les travaux de Merton [1974], Vasicek [2002]. Elle consiste à réécrire pour tout $t \in \mathbb{N}$, la perte L_t^N définie en (1.1.1) comme

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot D_t^n,$$

où pour chaque contrepartie $n \in \{1, \dots, N\}$,

- EAD_t^n (ou *exposure at default*) est l'encours de la dette au moment du *défaut*
- LGD_t^n (ou *loss given default*) est pourcentage l'EAD qui est perdu en cas de *défaut*,
- D_t^n est l'évènement de *défaut* ($D_t^n = 1$ s'il y a *défaut* en t et $D_t^n = 0$ sinon).

Modélisation de EAD Taplin et al. [2007] propose comme modélisation suivante de EAD_t^n:

$$\mathrm{EAD}_t^n(\tau^n) = B^n(t) + CCF_t^n(\tau^n)(L^n(t) - B^n(t)),$$

où $B^n(t)$ représente le solde restant (ou le montant actuellement tiré) en t, $L^n(t)$ est la limite de crédit actuellement non utilisée de la facilité de crédit et $CCF_t^n(\tau^n)$ est le facteur de conversion de crédit. Cela signifie que l'exposition au moment du *défaut* τ^n a deux composantes: l'exposition actuelle $B^n(t)$ qui est déterministe, et l'exposition future $CCF_t^n(\tau^n)(L^n(t) - B^n(t))$ qui est aléatoire.

Modélisation de LGD La deuxième variable LGD_t^n est obtenue à l'aide d'un ensemble de flux de trésorerie estimés, qui sont le résultat d'un processus de redressement et de recouvrement, et qui sont correctement actualisés à une date de

défaut. Bastos [2010], Roncalli [2020] proposent une modélisation économique où LGD est une fonction (linéaire ou non linéaire) de nombreux facteurs qui peuvent être externes à l'émetteur, spécifiques à l'émetteur, ou spécifiques à l'émission de la dette. On peut écrire

$$\mathrm{LGD}_t^n = g^n(X_t^1, \ldots, X_t^m),$$

où $(X_t^1, \ldots, X_t^m)$ est une séquence de facteurs de risques et g^n est une function de $\mathbb{R}^m$ dans $[0, 1]$ qui peut être obtenues par régression logistique, arbres de régression, ou réseaux de neurones. Chalupka and Kopecsni [2008] et Roncalli [2020][Page 193] proposent quant à eux une modélisation stochastique, où LGD_t^n suit une certaine distribution (paramétrique ou non paramétrique). Par exemple

$$\mathrm{LGD}_t^n \sim \mathfrak{B}(\alpha^n, \beta^n),$$

où $\mathfrak{B}(\alpha^n, \beta^n)$ est la loi beta de paramètres α^n et β^n.

Modélisation de l'évènelment de défaut D Pour ce qui est de D_t^n, même s'il existe une modélisation dite "en forme réduite" basée sur les fonctions de survie, les modélisations les plus utilisées sont dites "structurelles". Les plus populaires sont ainsi celle de Merton [1974] où $D_t^n = \mathbf{1}_{A^n(t) \leq B^n}$, et celle de Black and Cox [1976] où $D_t^n = \mathbf{1}_{\inf_{0 \leq s \leq t} A^n(s) \leq B^n}$, où A^n est la dynamique des actifs de l'emprunteur n qui suit un mouvement Brownien géométrique tandis que $B \geq 0$ est une barrière donnée. Basel Committee on Banking Supervision [2017] utilise cependant l'extension proposée par Merton [1974], Vasicek [2002] où la dynamique des actifs est donné par $A^n(t) = \sqrt{\rho} X_t + \sqrt{1 - \rho^2} \epsilon_n$ avec $X_t \sim \mathcal{N}(0, 1)$ qui est un facteur de risque systématique, tandis que $\epsilon_n \sim \mathcal{N}(0, 1)$ est un facteur de risque idiosynchratique, et $\rho \in [-1, 1]$.

La difficulté avec la plupart de ces modèles est qu'ils ne sont pas toujours microfondés: dans le modèle de Vasicek [2002] par exemple, les facteurs systémiques et idiosynchratiques étant gaussiens centrés réduits, l'intégration de l'environnement économique ne peut se faire qu'indirectement à travers l'estimation de ρ. Cela rend délicate l'intégration du risque de transition climatique, dont on aimerait mesurer le plus finement possible la propagation dans un portefeuille de crédit. C'est pourquoi le première étape de notre travail consiste à modéliser l'environnement qui subit la transition et dans lequel vivent les entreprises de notre portefeuille.

1.2.3 Des liens entre le risque climatique et le risque de crédit

Avec les accords de Paris en 2015, on a constaté de nombreux projets de recherche et une littérature croissante sur le sujet. Cartellier [2022] discute des méthodologies et des approches utilisées par les banques et les chercheurs dans les tests de stress climatique. Battiston and Monasterolo [2019] traitent de l'évaluation du risque de transition dans les portefeuilles d'obligations souveraines tandis que Allen et al. [2020] se concentrent sur l'évaluation du crédit aux entreprises. Garnier [2021] et Gaudemet

et al. [2022] proposent deux modèles. Le premier, appelé CERM (Climate Extended Risk Model), est un modèle basé sur celui de Merton avec un facteur systémique gaussien multidimensionnel, où le risque de transition est diffusé vers le risque de crédit par les facteurs de charge définis comme les corrélations entre les facteurs de risque systématiques et les actifs. Le second introduit un modèle climato-économique pour calibrer le modèle du premier. D'autres travaux, tels que Bourgey et al. [2021] ou Bouchet and Le Guenedal [2020], prennent en compte la structure économique et capitalistique de l'entreprise dans la mesure du risque carbone. En particulier, Bourgey et al. [2021] dérive la valeur de l'entreprise en utilisant la méthodologie des flux de trésorerie actualisés sur des flux de trésorerie qui sont affectés par la politique de transition de l'entreprise, tandis que Bouchet and Le Guenedal [2020] affecte directement la valeur de l'entreprise par un choc dépendant du ratio entre le coût du carbone et l'EBITDA.

De plus, Le Guenedal and Tankov [2022b] utilisent un modèle structurel pour évaluer les obligations émises par une entreprise soumise au risque de transition climatique et, en particulier, prennent en compte l'incertitude du scénario de transition. Enfin, Livieri et al. [2023] utilisent un modèle de risque de crédit Jump-Diffusion où les sauts vers le bas décrivent les politiques vertes prises par les entreprises, pour évaluer les obligations à coupons et les Credit Default Swaps.

1.3 Contribution principales

La démarche méthodologique utilisée dans ce travail s'inspire de Allen et al. [2020]. Notre objectif est de développer une méthodologie de bout en bout, partant des scénarios de transition climatique jusqu'aux mesures de risques du portefeuille de crédit.

1.3.1 Modélisation d'un cadre économique prenant compte du coût des émissions de GES

Nous considérons une économie fermée composée de I secteurs (on note $\mathcal{I}$ l'ensemble des secteurs). Le but dans ce chapitre est de décrire la dynamique des variables macroéconomiques par secteur que sont la production des entreprises, la consommation des ménages, l'offre de travail, ainsi que les entrées intermédiaires.

C'est pour cette raison que l'on s'inspire principalement de Devulder and Lisack [2020], de Golosov et al. [2014], de Galí [2015], ainsi que de Miranda-Pinto and Young [2019]. Notre modèle est à l'intersection de ces travaux-là, parce que nous souhaitons par la suite l'appliquer à la modélisation d'un portefeuille de crédit. Nous avons ainsi besoin de certaines caractéristiques dont la principale est que les variables macroéconomiques soient stochastiques avec des distributions paramétriques. Nos principales contributions dans le chapitre 3 sont les suivantes. D'une part, les productivités des secteurs sont stochastiques et corrélées. D'autre part, que les émissions de GES émises par les entreprises lorsqu'elles produisent des biens et/ou des

services et lorsqu'elles consomment des entrées intermédiaires, ainsi que celles générées par les ménages lorsqu'elles consomment les biens et/ou les services, sont facturées à l'aide prix du carbone déterministe et exogène.

Le cadre proposé suppose une entreprise représentative dans chaque secteur qui maximise ses profits en choisissant, à chaque instant et pour une productivité donnée, les quantités de travail et d'entrées intermédiaires, pour produire un unique bien/service. On peut donc assimiler l'entreprise représentative à son secteur, et au bien/service qu'il produit. Un ménage représentatif résout quant à lui un problème d'optimisation dynamique pour décider comment répartir ses dépenses de consommation entre les différents biens/services et les heures travaillées et entre les différents secteurs. Introduisons les deux hypothèses suivantes:

Standing Assumption 1.3.1. Considérons les processus Θ et $\mathcal{A}$ dans $\mathbb{R}^I$ et qui evoluent comme suit

$$\begin{cases} \mathcal{A}_t &= \mathcal{A}_{t-1} + \Theta_t, \\ \Theta_t &= \mu + \Gamma\Theta_{t-1} + \varepsilon\mathcal{E}_t, \end{cases} \quad \text{pour tout } t \in \mathbb{N}^*,$$

où les constantes $\mu, \mathcal{A}_0 \in \mathbb{R}^I$ et où la matrice $\Gamma \in \mathbb{R}^{I\times I}$ est telle que les valeurs absolues de ses valeurs propres sont strictement inférieures à 1, $0 < \varepsilon \leq 1$ est un paramètre de bruit que l'on suppose fixé. De plus, les bruits $(\mathcal{E}_t)_{t\in\mathbb{Z}}$ sont indépendants et identiquement distribués (iid) avec pour tout $t \in \mathbb{Z}$, $\mathcal{E}_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$ tel que $\Sigma \in \mathbb{R}^{I\times I}$. Par ailleurs, nous avons, $\Theta_0 \sim \mathcal{N}(\overline{\mu}, \varepsilon^2\overline{\Sigma})$ tel que $\overline{\mu} := (\mathbf{I}_I - \Gamma)^{-1}\mu$, et $\text{vec}(\overline{\Sigma}) := (\mathbf{I}_{I\times I} - \Gamma \bigotimes \Gamma)^{-1}\text{vec}(\Sigma)$, avec
$M \in \mathbb{R}^{d\times d}$, $\text{vec}(M) := [M^{11}, \ldots, M^{d1}, M^{21}, \ldots, M^{d2}, \ldots, M^{1d}, \ldots, M^{dd}]^\top$. Enfin, les processus $(\mathcal{E}_t)_{t\in\mathbb{N}}$ et la variable aléatoire Θ_0 sont en outre independants.

Pour tout $i \in \mathcal{I}$, les processus Θ^i et $\mathcal{A}^i$ jouent un rôle majeur dans notre modèle puisque, la productivité totale des facteurs du secteur i est définie comme suit

$$A_t^i := \exp(\mathcal{A}_t^i),$$

de sorte que Θ^i est la croissance logarithmique de la productivité et $\mathcal{A}^i$ est la croissance logarithmique cumulative de la productivité.

Standing Assumption 1.3.2. Nous introduisons aussi les intensités carbones τ, κ, ζ (définies comme la quantité de GES en tonnes émis pour chaque euro de production/consommation) et le prix du carbone δ qui sont des processus déterministes tels que:

1. Etant donné $0 \leq t_\circ < t_\star$. Le prix du carbone δ satisfait pour tout $t \in \mathbb{N}$,
   - si $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in \mathbb{R}_+$, c'est-à-dire qu'il est constant avant le début de la transition;
   - si $t \in (t_\circ, t_\star)$, $\delta_t \in \mathbb{R}_+$, c'est-à-dire qu'il peut être dynamique pendant la transition;

- for $t \geq t_\star$, $\delta_t = \delta_{t_\star} \in \mathbb{R}_+$, c'est-à-dire qu'il redevient constant après la transition.

Ainsi, la transition commence en $t_\circ$ avec un prix du carbone constant et se termine $t_\star$ avec un prix du carbone constant.

2. Les processus déterministes τ, ζ, et κ appartiennent respectivement à $\mathbb{R}_+^I$, $\mathbb{R}_+^{I\times I}$, et $\mathbb{R}_+^I$, et représentent respectivement les intensités de carbone sur la production des entreprises, sur la consommation intermédiaire des entreprises, et sur la consommation des ménages, et satisfaisant pour tout $t \in \mathbb{N}$ et pour tout $\mathfrak{y} \in \{\tau^1, \ldots, \tau^I, \zeta^{11}, \zeta^{12}, \ldots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \ldots, \kappa^I\}$,

$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t)}{\theta_{\mathfrak{y}}}\right) & \text{si } t \leq t_\star \\ \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t_\star)}{\theta_{\mathfrak{y}}}\right) & \text{sinon,} \end{cases}$$

avec $\mathfrak{y}_0, -g_{\mathfrak{y},0}, \theta_{\mathfrak{y}} > 0$. Pour tout $t \in \mathbb{N}$, nous appelons $\mathfrak{y}_t\delta_t$ le *taux de coût des émissions* à l'instant t.

3. Pour tout $i \in \mathcal{I}$ et pour tout $t \in \mathbb{N}$,

$$\delta_t \max_{i\in\mathcal{I}} \tau_0^i < 1.$$

Introduisons aussi la filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{N}}$ with $\mathcal{G}_0 := \sigma(\Theta_0)$ et pour tout $t \geq 1$, $\mathcal{G}_t := \sigma\left(\{\Theta_0, \mathcal{E}_s : s \in (0,t] \cap \mathbb{N}^*\}\right)$. Pour tout $i \in \mathcal{I}$, considérons les processus $\mathbb{G}$-mesurables et positifs suivants:

- Y^i la production du secteur i,
- P^i le prix du bien/service i,
- N^i la demande de travail dans le secteur i,
- H^i l'offre de travail dans le secteur i,
- W^i le salaire dans le secteur i,
- C^i la consommation des biens/services du secteur i par les ménages,
- pour tout $j \in \mathcal{I}$, Z^{ji} la consommation par le secteur i des entrées intermédiaires produites par le secteur j.

On considère aussi le processus réel positif et déterministe r représentant le taux d'intérêt.

Pour chaque secteur i, l'entreprise représentative maximise ses profits en résolvant le problème suivant:

$$\max_{N_t^i, Z_t^{1i}, \ldots, Z_t^{Ii}} \left\{ P_t^i Y_t^i - W_t^i N_t^i - \sum_{j \in \mathcal{I}} Z_t^{ji} P_t^j - \left(\tau_t^i P_t^i Y_t^i \delta_t + \sum_{j \in \mathcal{I}} \zeta_t^{ji} Z_t^{ji} P_t^j \delta_t \right) \right\}, \quad (1.3.1)$$

sous contrainte de la fonction de production

$$Y_t^i = A_t^i (N_t^i)^{\psi^i} \prod_{j \in \mathcal{I}} (Z_t^{ji})^{\lambda^{ji}},$$

et des rendements d'échelle constants

$$\psi^i + \sum_{j \in \mathcal{I}} \boldsymbol{\lambda}^{ji} = 1, \qquad \text{pour tout } i \in \mathcal{I}.$$

Notons que Y_t^i représente la production en volume du secteur i, $P_t^i Y_t^i$ ses revenus bruts en euros, $\tau_t^i P_t^i Y_t^i$ ses émissions de GES en tonnes, générées lors de la production, donc $\tau_t^i P_t^i Y_t^i \delta_t$ le coût des émissions de GES du secteur.

Etant donné la fonction d'utilité $U : (0, \infty)^2 \to \mathbb{R}$ définie comme suit: pour $\varphi \geq 0$, on a $U(x, y) := \frac{x^{1-\sigma}}{1-\sigma} - \frac{y^{1+\varphi}}{1+\varphi}$ si $\sigma \in [0, 1) \cup (1, +\infty)$ et $U(x, y) := \log(x) - \frac{y^{1+\varphi}}{1+\varphi}$, si $\sigma = 1$. Le ménage représentatif à durée de vie infinie cherche à résoudre le problème suivant :

$$\max_{(C_i^i, H_t^i)_{t \in \mathbb{N}, i \in \mathcal{I}}} \quad \sum_{i \in \mathcal{I}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(C_i^t, H_i^t) \right], \quad (1.3.2)$$

sachant que

$$\mathbb{E}\left[\sum_{i \in \mathcal{I}} \sum_{t=0}^{\infty} \beta^t \left| U(C_i^t, H_i^t) \right| \right] < \infty,$$

et le processus de richesse Q est tel que pour tout $t \geq 0$

$$Q_t := (1 + r_{t-1}) Q_{t-1} + \sum_{i \in \mathcal{I}} W_t^i H_t^i - \sum_{i \in \mathcal{I}} P_t^i C_t^i - \sum_{i \in \mathcal{I}} \kappa_t^i P_t^i C_t^i \delta_t,$$

avec la convention $Q_{-1} = 0$, $r_{-1} = 0$, et $\lim_{T \uparrow \infty} \mathbb{E}[Q_T | \mathcal{G}_t] \geq 0$.

Pour chaque $i \in \mathcal{I}$, $P_t^i C_t^i$ représente la consommation du ménage dans le secteur i et $\kappa_t^i P_t^i C_t^i \delta_t$ le coût payé en raison de ses émissions lorsqu'il consomme les biens/services i, $W_t^i H_t^i$ est le revenu du travail du ménage offert par le secteur i, $(1 + r_{t-1}) Q_{t-1}$ le revenu du capital du ménage.

Nous obtenons les conditions de premier ordre du problème des entreprises (1.3.1) en remarquant qu'à chaque pas de temps et pour chaque réalisation de la productivité, le problème devient statique et déterministe. Nous déterminons les conditions de premier ordre du problème des ménages (1.3.2) en utilisant une approche variationnelle. Lorsque les problèmes d'optimisation (1.3.1) et (1.3.2) sont résolus, que les marchés des

biens/services et du travail s'équilibrent, nous montrons que l'ensemble des variables macroéconomiques (N, H, Z, C, Y) ne dépendent que de la production $(Y_t^i)_{t\in\mathbb{N},i\in\mathcal{I}}$ et de la consommation $(C_t^i)_{t\in\mathbb{N},i\in\mathcal{I}}$, qui elles, vérifient à chaque instant, un système de $2I$ équations non linéaires à $2I$ inconnues. On ne sait pas résoudre analytiquement ce système, mais lorsque la fonction d'utilité est logarithmique en la consommation (c'est-à-dire $\sigma = 1$), et que d'autres conditions structurelles sont satisfaites, nous obtenons une trajectoire unique pour la production et la consommation, et donc pour toutes les autres variables. Chaque variable dépendant de la productivité, du prix du carbone, des intensités carbone, et des paramètres structurels du modèle (φ, $(\psi^i)_{i\in\mathcal{I}}$ et $(\boldsymbol{\lambda}^{ji})_{i,j\in\mathcal{I}}$). Nous remarquons en outre que du fait du choix de fonctions de production et d'utilité séparables, les processus prix et salaire n'interviennent pas à la fin.

Nous définissons et déterminons en outre

- la croissance logarithmique des variables entre deux instants (exemple en Figure 1.1).
- Les sensibilités de la croissance des variables au prix du carbone. Ce qui permet de répondre aux types de questions suivantes: *quelle est l'impact sur toutes les variables si dans toute l'économie, on ne facture que les émissions de GES venant de la production des entreprises du secteur* 1*?* La principale difficulté ici vient du fait que les coefficients de l'inverse de la matrice des élasticités $(\boldsymbol{\lambda}^{ji})_{i,j\in\mathcal{I}}$ ne sont pas connus.
- La loi de la croissance logarithmique des variables macroéconomiques qui est obtenue en utilisant la gaussiannité et la stationnarité de la croissance logarithmique.
- l'évolution de la contribution de chaque secteur dans la production/consommation ainsi que l'évolution des émissions de GES.

Nous calibrons et estimons les paramètres en utilisant des données historiques de l'économie française venant de INSEE [2023] et le prix du carbone de NGFS [2022]. Pour une transition de 10 ans commençant en 2021 et se terminant en 2030, nous réalisons enfin les simulations en considérant 4 secteurs et 4 scénarios de transition.

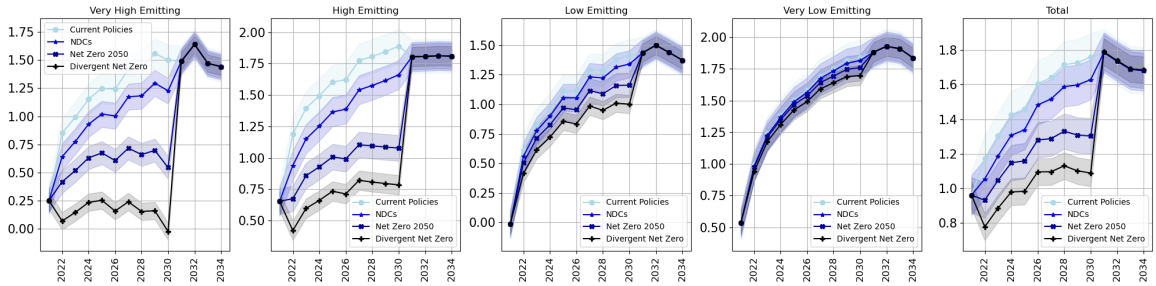


Figure 1.1: Moyenne et intervalle de confiance à 95% de la croissance annuelle de la production

1.3.2 Modèles de valorisation des firmes et modèle de risque de crédit

Nous nous intéressons dans le chapitre 4 au calcul de la valeur de chaque firme $n \in \{1, \dots, N\}$, de la probabilité de défaut, et de la déformation de la perte sans collatéral (et de ses statistiques) en présence d'un prix du carbone dans l'économie que nous avons modélisée précédemment. Dans le but de définir la perte, introduisons les hypothèses suivantes.

Assumption 1.3.3. Considérons un portefeuille de $N \in \mathbb{N}^*$ crédits. Pour tout $1 \leq n \leq N$,

(1) L'entreprise n peut émettre deux catégories de titres : des actions et des obligations;

(2) $(\mathrm{EAD}_t^n)_{t\in\mathbb{N}^*}$ est un processus déterministe de $\mathbb{R}_*^+$, représentant l'encours de la dette au moment du défaut;

(3) $(\mathrm{LGD}_t^n)_{t\in\mathbb{N}^*}$ est un processus déterministe de $(0, 1]$, représentant le pourcentage attendu de l'EAD qui est perdu en cas de défaut du débiteur;

(4) la barrière de *surendettement* $\mathcal{D}^n \in \mathbb{R}^+$ est un scalaire déterministe que nous utiliserons pour définir les conditions dans lesquelles un emprunteur est considéré comme étant en *défaut*.

(5) la valeur de l'entreprise n, notée $\mathcal{V}_t^n$, au temps t est supposée être un actif négociable.

On peut désormais définir à tout instant $t \in \mathbb{N}$, la perte potentielle d'un portefeuille composé de N prêts par

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbf{1}_{\{V_t^n \leq \mathcal{D}^n\}}. \tag{1.3.3}$$

Dans cette équation, l'évènement $\{\mathcal{V}_t^n \leq \mathcal{D}^n\}$ correspond à la définition du *défaut* dans les modèles structurels (voir Merton [1974] et Black and Cox [1976]), et précisément au *risque d'insolvabilité* qui survient lorsque la valeur de marché de la dette $\mathcal{D}^n$ dépasse la valeur de marché de l'entreprise $\mathcal{V}_t^n$. Notons que pour simplifier, on a supposé ici la valeur $\mathcal{D}^n$ de la dette fixe.

Les processus LGD et EAD étant supposés déterministes de même que $\mathcal{D}^n$, le prix du carbone ne peut se transmettre qu'à partir de $\mathcal{V}^n$.

Valeur de la firme

Il existe de nombreux modèles d'évaluation de la valeur d'une entreprise: les méthodes de rendement (évaleur la capacité à générer des profits dans le futur), les méthodes comparatives (comparer à une autre entreprise de profil similaire qui a été vendue

récemment), et les méthodes patrimoniales (évaluer la valeur des actifs de l'entreprise). La littérature en risque de crédit reste parfois floue sur le type de méthode utilisée, et ne précise toujours pas si on évalue les actifs, la valeur marché, ou une autre grandeur. Néanmoins, mathématiquement, les modèles les plus utilisées supposent que la valeur de la firme suit un brownien géométrique (KMV [Merton Merton [1974]], CreditGrades [Black and Cox Black and Cox [1976]]) ou qu'elle dépend de facteurs de risque systémiques communs et d'un facteur de risque idiosyncrasique (Bale II [Vasicek Vasicek [2002]]). Etant donné que nous souhaitons analyser l'impact du prix du carbone sur la valeur d'une entreprise, il est plus intéressant d'utiliser un modèle microfondé: une méthode de rendement s'y prête mieux.

Pour tout $n \in \{1, \dots, N\}$ et pout tout $t \in \mathbb{N}$, la valeur de l'entreprise n, notée V_t^n, est l'espérance conditionnelle de la somme actualisée de tous ses cash flows futurs $(F_{t+s}^n)_{s\in\mathbb{N}}$. Pour déterminer la dynamique des cash flows, on remarque d'une part que, pour chaque entreprise n appartient à un unique secteur (d'émissions) $i \in \mathcal{I}$, tel que $n \in g_i$. Ainsi, la production/le revenu de l'entreprise à la date t est une fraction de la production Y_t^i du secteur i à la date t. Aussi, on suppose qu'à chaque date t, le cash flow de l'entreprise est une fraction de son revenu. En écrivant la croissance logarithmique des cash flows de l'entreprise sous forme vectorielle, nous avons donc l'hypothèse suivante.

Assumption 1.3.4. Pour tout $n \in \{1, \dots, N\}$, le processus de croissance des flux de trésorerie de l'entreprise n à valeur $\mathbb{R}^N$, noté $(\omega_t^n)_{t\in\mathbb{N}^*}$ est linéaire par rapport à des facteurs économiques (précisément la croissance logarithmique de la production), mathématiquement, nous avons pour tous $t \in \mathbb{N}$,

$$\omega_t^n = \tilde{\mathfrak{a}}^{n\cdot}\Delta_t^Y + \mathfrak{b}_t^n = \mathfrak{a}^{n\cdot}\left(\Theta_t + \mathfrak{v}(\mathfrak{d}_t) - \mathfrak{v}(\mathfrak{d}_{t-1})\right) + \mathfrak{b}_t^n,$$

avec $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$, $\mathfrak{a}^{n\cdot} := \tilde{\mathfrak{a}}^{n\cdot}(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}$, et où le bruit idiosyncratique $(\mathfrak{b}_t)_{t\in\mathbb{N}} := (\mathfrak{b}_t^n)_{t\in\mathbb{N},1\le n\le N}$ est i.i.d. avec pour loi $\mathcal{N}(0, \mathrm{diag}(\sigma_{\mathfrak{b}^n}^2))$ telle $\sigma_{\mathfrak{b}^n} > 0$ pour $n \in \{1, \dots, N\}$. De plus, $(\Delta_t^Y)_{t\in\mathbb{N}^*}$ et $(\mathfrak{b}_t)_{t\in\mathbb{N}}$ sont independants.

Par ailleurs, en définissant la filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}}$ par $\mathcal{F}_t = \sigma\left(\mathcal{G}_t \cup \sigma\left\{\mathfrak{b}_s : s \in [0,t] \cap \mathbb{N}\right\}\right)$ pour tout $t \in \mathbb{N}$, on note $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$. La valeur de la firme s'écrit donc

$$V_t^n := \mathbb{E}_t\left[\sum_{s=0}^{+\infty} e^{-rs} F_{t+s}^n\right].$$

Nous commençons par vérifier si cette valeur est bien définie dans le théorème 4.1.4. Puis, nous devrions la calculer. Cependant, trouver une expression simple de V n'est pas évidente à cause de la structure autoregressive de la productivité Θ (voir Hypothèse 1.3.1). Pour avancer, on peut utiliser une méthode numérique comme des réseaux de neurones Hammad et al. [2022] ou des méthodes par itérations de Picard Berinde and Takens [2007]. Cependant, à partir du paramètre de bruit ε introduit lors de la définition de $(\mathcal{A}, \Theta)$ dans l'hypothèse 1.3.1, nous pouvons introduire un proxy $\mathcal{V}$

à la valeur de la firme qu'on définit comme suit

$$\mathcal{V}_t^n := F_t^n \left(1 + \sum_{s=1}^{+\infty} e^{-rs} \mathbb{E}_t \left[\exp\left(\mathfrak{a}^{n\cdot} \left(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t) + s\overline{\mu} \right) + \sum_{u=1}^{s} \mathfrak{b}_{t+u}^n \right) \right] \right), \quad (1.3.4)$$

et qui provient du développement asymptotique à l'ordre 1 de $\frac{V_t^n}{F_t^n}$. Ainsi, pour tout $n \in \{1, \dots, N\}$, sous les conditions de la proposition 4.1.7, pour tout $t \in \mathbb{N}$, V_t^n converge vers $\mathcal{V}_t^n$ quand ε tend vers 0. Par la suite, quand nous parlerons de la valeur de la firme, il s'agira de $\mathcal{V}$ et non de V. Par conséquent, nous en obtenons une expression simple (voir Lemme 4.1.6) et nous savons déterminer ses lois conditionnelles. En particulier, pour tout $t, T \in \mathbb{N}$,

$$\mathcal{V}_{t+T}^n | \mathcal{G}_t \sim \mathcal{LN}\left(\log(F_0^n) + \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t), \mathcal{L}^n(t, T) \right),$$

où $\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)$ et $\mathcal{L}^n(t, T)$ sont donnés en (4.1.22) et (4.1.23) respectivement.

Nous obtenons déjà à cette étape la déformation de la valeur d'une entreprise qui baigne dans l'économie décrite dans le chapitre 3. Il en résulte que (1) plus la croissance des cash flows d'une entreprise est dépendante d'un secteur polluant, plus sa valeur baisse, (2) plus la transition se durcit (prix du carbone élevé), plus la valeur des entreprises baisse, et (3) la distribution de la valeur de la firme se déforme et translate en présence du prix du carbone.

Probabilité de *surendettement*, pertes attendue et inattendue

Sachant que nous avons désormais la valeur de la firme (1.3.4), et que LGD et EAD sont déterministes, nous pouvons revenir à la perte du portefeuille L^N (1.3.3). Avec le proxy de la valeur de la firme $\mathcal{V}$, elle s'écrit désormais:

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}}.$$

Dans cette expression, la productivité de l'économie $(\mathcal{A}, \Theta)$ et le bruit des cash flows $(\mathfrak{b}_t^n)_{t\in\mathbb{N}, 1\leq n\leq N}$ sont les deux facteurs d'aléa: le premier est dit *systémique* et le second dit *idiosyncratique*. Cependant, pour $t \in \mathbb{N}$ donné, les variables aléatoires $(\mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}})_{n\in\{1,\dots,N\}}$ sont des lois de Bernouilli non indépendantes et non identiquement distribuées. Décrire analytiquement la loi de L_t^N s'avère donc ardu. Mais en ajoutant une hypothèse de non concentration dans le portefeuille, on peut utiliser un résultat introduit par Gordy [2003b]. On obtient le théorème suivant.

Theorem 1.3.5. *Si les hypothèses introduites en 1.3.3 sont satisfaites et que pour tout $t \in \mathbb{N}$, la famille $(\mathrm{EAD}_t^n)_{n=1,\dots,N}$ est une séquence de constantes positives telles que*

1. $\sum_{n\geq 1} \mathrm{EAD}_t^n = +\infty$;

2. il existe $\upsilon > 0$ *tel que* $\frac{\mathrm{EAD}_t^n}{\sum_{n=1}^N \mathrm{EAD}_t^n} = \mathcal{O}(N^{-(\frac{1}{2}+\upsilon)})$*, quand* N *tend vers l'infini.*

Alors, $L_t^N - \mathrm{L}_t^{\mathbb{G},N}$ *converge vers zéro presque sûrement lorsque* N *tend vers l'infini, avec*

$$\mathrm{L}_t^{\mathbb{G},N} := \mathbb{E}\left[L_t^N \middle| \mathcal{G}_t\right] = \sum_{n=1}^N \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbb{P}\left[\mathcal{V}_t^n \leq \mathcal{D}^n | \mathcal{G}_t\right]. \tag{1.3.5}$$

Cela signifie qu'à toute date, lorsque le portefeuille est bien diversifié, on peut approcher, presque sûrement, la perte potentielle L_t^N par la perte espérée conditionnellement aux facteurs systémiques $\mathrm{L}_t^{\mathbb{G},N}$. Dans la suite, on remplace L_t^N par $\mathrm{L}_t^{\mathbb{G},N}$. Bien que la distribution de la perte du portefeuille contienne plus d'informations, nous allons nous concentrer sur quelques unes de ses statistiques. Par conséquent, étant donnés les dates t et T, considérons les mesures de risques ci-dessous.

- La probabilité de *surendettement* de l'entité $n \in \{1, \ldots, N\}$ entre t et $t+T$ calculé à l'instant t, on note

$$\mathrm{PD}_{t,T,\mathfrak{d}}^n := \mathbb{P}\left(\mathcal{V}_{t+T}^n \leq \mathcal{D}^n | \mathcal{G}_t\right).$$

- La perte attendue du portefeuille (et des sous-portefeuilles g_i, $i \in \mathcal{I}$) sur la période $[t, t+T]$ et calculée en t, on note

$$\mathrm{EL}_{t,\mathfrak{d}}^{N,T} := \mathbb{E}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \middle| \mathcal{G}_t\right]. \tag{1.3.6}$$

- La perte inattendue du portefeuille (et des sous-portefeuilles g_i, $i \in \mathcal{I}$) sur la période $[t, t+T]$ et calculée en t, on note pour $\alpha \in (0,1)$,

$$\mathrm{UL}_{t,\alpha,\mathfrak{d}}^{N,T} := \mathrm{VaR}_t^{\alpha,N,T} - \mathrm{EL}_{t,\mathfrak{d}}^{N,T}, \quad \text{avec} \quad 1-\alpha = \mathbb{P}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]. \tag{1.3.7}$$

Le proxy de la valeur de la firme $\mathcal{V}$ introduit en (1.3.4) ainsi que le proxy de la perte introduit en (1.3.5) nous permettent enfin d'obtenir des expressions simples de $(\mathrm{PD}_{t,T,\mathfrak{d}}^n)_{n\in\{1,\ldots,N\}}$, $\mathrm{EL}_{t,\mathfrak{d}}^{N,T}$, et $\mathrm{UL}_{t,\alpha,\mathfrak{d}}^{N,T}$, à partir notamment de la fonction de répartition de la Gaussienne centrée réduite Φ en (4.2.5), en (4.2.6) et en (4.2.8) notamment. La présence de $\mathfrak{d}$ appelé taux du coût des émissions (vecteur sans dimension qui est le produit du prix du carbone et des intensités carbone) et définit pour tout $t \in \mathbb{N}$ par

$$\mathfrak{d}_t := (\tau_t \delta_t, \zeta_t \delta_t, \kappa_t \delta_t),$$

nous permet d'insister sur la dépendance de nos mesures de risques en les variables de la transition climatique.

En considérant une transition de 10 ans commençant en 2021 et se terminant en 2030 et en utilisant la méthode de Monte Carlo pour calculer les mesures de risques, nous obtenons que le prix du carbone dans l'économie

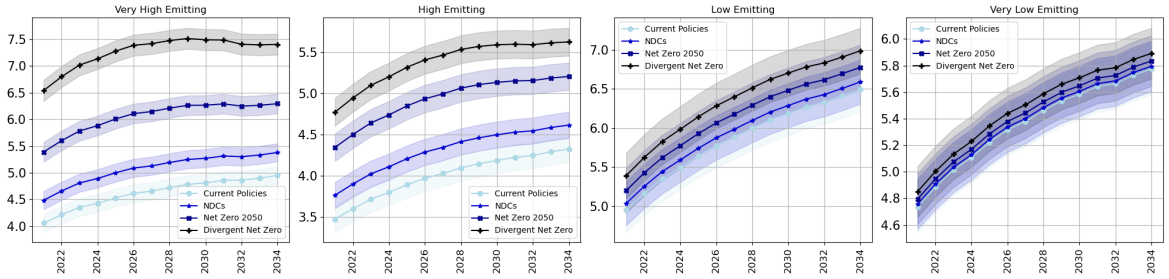


Figure 1.2: Probabilité de *défaut* annuelle par scénario et par sous-portefeuille

1. augmente la probabilité de défaut des entreprises et que cette hausse est accentuée quand le scénario de transition se durcit et/ou quand l'entreprise appartient à un secteur très polluant (voir Figure 1.2);

2. augmente les frais bancaires (matérialisés par le niveau des provisions calculées à partir de la perte attendue EL) facturés aux clients ou supportés par les revenus d'exploitation des banques;

3. et réduit la solvabilité des banques (traduite par le capital économique calculé à partir de la perte inattendue UL).

Dans ce chapitre, nous avons supposé que la LGD était déterministe, constante, et indépendante du prix du carbone. C'est une hypothèse assez forte car la LGD (et donc la perte du portefeuille) dépend des actifs des emprunteurs et de leurs garanties, qui eux peuvent être directement affectés par la transition climatique. Nous revenons dans la troisième partie sur cette hypothèse.

1.3.3 Impact de la transition climatique sur les pertes du portefeuille de crédit avec des garanties stochastiques

Dans les portefeuilles de crédit d'une banque, il peut exister des prêts garantis (encore appelés) et les prêts non garantis. Quand une contrepartie est mise *surendettement*, la banque commence par liquider les garanties de la contrepartie défaillante si elle en a, et si cela ne couvre pas son exposition à la date du *surendettement*, elle entreprend des procédures de recouvrement par d'autres moyens (liquidation des actifs, procédures judiciaires, etc.) Une garantie, encore appelée collatéral, peut être constituée d'actifs corporels (bâtiments, équipements professionnels, stocks de marchandises, etc.) ou incorporels (dépôts en espèces, obligations publiques, titres financiers, etc.) Dans ce chapitre, nous considérons une garantie corporelle – un bien immobilier – et une garantie incorporelle – actif financier (actions ou obligations d'entreprises). Nous avons vu dans le chapitre 4 que la valeur d'une entreprise peut-être affectée par la transition climatique et précisement par le prix du carbone, ainsi si la valeur des actions ou des obligations en seront aussi affectées. De la même manière, le prix des bâtiments

commerciaux ou résidentiels sera affecté par la transition, par exemple par le biais de la performance (ou efficacité) énergétique.

Par ailleurs, il se trouve qu'une modélisation en temps continu correspond mieux au marché immobilier. Aussi, il est plus intéressant pour une banque de suivre l'évolution de son portefeuille à des fréquences plus resserrées. C'est la raison pour laquelle, avant de modéliser la perte avec collatéral, nous commençons par étendre les résultats précédents en temps continu. La croissance logarithmique de la productivité suit désormais un processus Ornstein-Uhlenbeck multidimensionnel en lieu et place d'un vecteur autoregressif.

Standing Assumption 1.3.6. Nous définissons les processus $\mathcal{Z}$ et $\mathcal{A}$ à valeurs $\mathbb{R}^I$ qui évoluent comme suit

$$\begin{cases} \mathrm{d}\mathcal{Z}_t &= -\Gamma\mathcal{Z}_t\mathrm{d}t + \Sigma\mathrm{d}B_t^{\mathcal{Z}} \\ \mathrm{d}\mathcal{A}_t &= (\mu + \varsigma\mathcal{Z}_t)\,\mathrm{d}t \end{cases} \quad \text{pour tout } t \in \mathbb{R}^*,$$

où $(B_t^{\mathcal{Z}})_{t\mathbb{R}^*}$ est un mouvement brownien à I dimensions, et où les constantes $\mu, \mathcal{A}_0 \in \mathbb{R}^I$, les matrices $\Gamma, \Sigma \in \mathbb{R}^{I\times I}$, $\mathcal{Z}_0 \sim \mathcal{N}\left(0, \Sigma\Sigma^\top\right)$ et $0 < \varsigma \leq 1$ est un paramètre d'intensité du bruit qui est fixé : il sera utilisé plus tard pour obtenir une approximation intéressante de la valeur de l'entreprise. De plus, Σ est une matrice définie positive et $-\Gamma$ est une matrice de Hurwitz, c'est-à-dire que ses valeurs propres ont des parties réelles strictement négatives.

Nous introduisons également la filtration suivante $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{R}^*}$ avec $\mathcal{G}_0 := \sigma(\Theta_0)$ et pour tout $t > 0$, $\mathcal{G}_t := \sigma\left(\left\{\Theta_0, B_s^{\mathcal{Z}} : s \leq t\right\}\right)$. Le prix du carbone quant à lui évolue commme suit.

Standing Assumption 1.3.7. Introduisons aussi le prix du carbone δ et les intensités carbones τ, κ, ζ qui sont des processus telles que:

1. Etant donné $0 \leq t_\circ < t_\star$. Le prix du carbone δ satisfait
   - si $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in \mathbb{R}_+$, c'est-à-dire qu'il est constant;
   - si $t \in (t_\circ, t_\star)$, $\delta_t \in \mathbb{R}_+$, c'est-à-dire qu'il peut évoluer autour d'une position déterministe et dynamique donnée;
   - si $t \geq t_\star$, $\delta_t = \delta_{t_\star} \in \mathbb{R}_+$, c'est-à-dire qu'il est constant.
2. Les processus déterministes τ, ζ, et κ appartiennent respectivement à $\mathbb{R}_+^I$, $\mathbb{R}_+^{I\times I}$, et $\mathbb{R}_+^I$, et représentent respectivement les intensités de carbone sur la production des entreprises, sur la consommation intermédiaire des entreprises, et sur la consommation des ménages, et satisfaisant pour tout $t \in \mathbb{R}_+$ et pour tout $\mathfrak{y} \in \{\tau^1, \ldots, \tau^I, \zeta^{11}, \zeta^{12}, \ldots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \ldots, \kappa^I\}$,

$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t)}{\theta_{\mathfrak{y}}}\right) & \text{si } t \leq t_\star \\ \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t_\star)}{\theta_{\mathfrak{y}}}\right) & \text{sinon,} \end{cases}$$

avec $\mathfrak{y}_0, -g_{\mathfrak{y},0}, \theta_{\mathfrak{y}} > 0$. Pour tout $t \in \mathbb{R}_+$, on appelle $\mathfrak{y}_t\delta_t$ le *taux de coût des émissions* à l'instant t.

3. Pour tout $i \in \mathcal{I}$ et pour tout $t \in \mathbb{R}_+$,

$$\delta_t \max_{i\in\mathcal{I}} \tau_0^i < 1.$$

On note de la même manière

$$\mathfrak{d}_t := (\tau_t\delta_t, \zeta_t\delta_t, \kappa_t\delta_t).$$

A partir des hypothèses 1.3.6 et 1.3.7, on reécrit le modèle multisectorielle en temps continu. Nous le résolvons dans l'annexe C.2 en nous inspirant des méthodes utilisées dans le chapitre 3. On obtient ainsi les processus $\mathbb{G}$-mesurables dans $\mathbb{R}^I$, Y, C, N, et Z qui répresentent respectivement la production, la consommation, le travail, et les entrées intermédiaires. Ces grandeurs dépendent des processus $\mathfrak{d}$, $\mathcal{A}$, et des paramètres du modèles (élasticités et fonction d'utilité notamment). Pour préciser la dépendance en $\mathfrak{d}$ et $\mathcal{A}$, nous écrivons par exemple pour tout $t \in \mathbb{R}_+$, $Y_t(\mathfrak{d}_t, \mathcal{A}_t) \in \mathbb{R}^d$ la production à la date t.

De la même manière, on écrit la valeur de toute firme $n \in \{1, \dots, N\}$ à toute date $t \in \mathbb{R}_+$,

$$V^n_{t,\mathfrak{d}} := \mathbb{E}_t\left[\int_t^{+\infty} e^{-rs} F^n_{s,\mathfrak{d}} \mathrm{d}s\right], \tag{1.3.8}$$

avec la dynamique des cash flows

$$\mathrm{d}\log F^n_{t,\mathfrak{d}} = \mathfrak{a}^{n\cdot}\mathrm{d}\log\left(Y_t(\mathfrak{d}_t, \mathcal{A}_t)\right) + \sigma_n \mathrm{d}\mathcal{W}^n_t,$$

pour $\mathfrak{a}^{n\cdot} \in \mathbb{R}^I$ et $(\mathcal{W}^n_t)_{t\in\mathbb{R}_+}$ es un $\mathbb{R}^N$-mouvement brownien avec $\sigma_n > 0$. De plus, $B^{\mathcal{Z}}$ et $\mathcal{W}^n$ sont indépendants.

Les conditions d'existence et de convergence doivent légèrement être adaptées parce que les paramètres de l'Ornstein-Uhlenbeck (en continu) et du vecteur autoregressif (en discret) ne sont pas les mêmes. On introduit ensuite un proxy de la valeur de la firme $\mathcal{V}$ en nous inspirant du modèle discret et en utilisant le paramètre ς introduit dans l'hypothèse 1.3.6.

Soit $n \in \{1, \dots, N\}$ et $t \in \mathbb{R}_+$. On note le processus réel positif C la valeur de la garantie (qui est nulle si le prêt est non garanti). Rappelons qu'il y a *surendettement* à la date t si $\{\mathcal{V}^n_t \leq \mathcal{D}^n\}$. Lorsqu'il y a *surendettement*, la banque liquide la contrepartie dans un délai de $a > 0$ et reçoit $(1-k)e^{-ra}C^n_{t+a}$. Si les liquidations ne couvrent pas la totalité de la dette, c'est-à-dire $\mathrm{EAD}^n_t \geq (1-k)e^{-ra}C^n_{t+a}$, la banque déploie d'autres actions pour récupérer une fraction supplémentaire notée $\gamma \in [0,1]$. Elle obtient donc en plus $\gamma(\mathrm{EAD}^n_t - (1-k)e^{-ra}C^n_{t+a})_+$. La perte du portefeuille s'écrit donc

$$L^N_t := \sum_{n=1}^N L_{n,t},$$

où $L_{n,t}$ est la perte liée à la contrepartie n à la date t qui s'écrit

$$L_{n,t} := (1-\gamma)(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n)_+ \cdot \mathbf{1}_{\{\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{B}^n\}}.$$

Avec les mêmes hypothèses que celles du théorème 1.3.5 (déclinées en temps continu), on introduit la perte espérée conditionnellement aux facteurs systémiques $\mathrm{L}_t^{\mathbb{G},N}$ comme un proxy de la perte potentielle L_t^N lorsque le portefeuille n'est pas concentré. On a le théorème suivant:

Theorem 1.3.8 (et définition)**.** *Pour tout $t \in \mathbb{R}_+$, définissons*

$$\mathrm{L}_t^{\mathbb{G},N} := \sum_{n=1}^{N} L_{n,t}^{\mathbb{G}} \quad \textit{où} \quad L_{n,t}^{\mathbb{G}} = \mathbb{E}\left[L_{n,t}|\mathcal{G}_t\right] = \mathrm{EAD}_t^n \cdot \mathrm{LGD}_{t,\mathfrak{d}}^n \cdot \mathrm{PD}_{t,\mathfrak{d}}^n,$$

avec

$$\begin{aligned}
\mathrm{PD}_{t,\mathfrak{d}}^n &:= \mathbb{P}\left(\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{B}^n|\mathcal{G}_t\right), \\
\mathrm{LGD}_{t,\mathfrak{d}}^n &:= (1-\gamma)\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{C_{t+a}^n}{\mathrm{EAD}_t^n}\right)_+ \middle| F_{t,\mathfrak{d}}^n < \mathcal{D}_t^n, \mathcal{G}_t\right].
\end{aligned}$$

Alors, $L_t^N - \mathrm{L}_t^{\mathbb{G},N}$ converge vers zéro presque sûrement lorsque N tend vers l'infini, pour tout $t \in \mathbb{R}_+$.

On obtient ainsi dans la proposition 5.2.9 une expression de la probabilité de *surendettement* $\mathrm{PD}_{t,\mathfrak{d}}^n$ en fonction de Φ de la même manière que dans la partie discrète. Nous nous concentrons sur la LGD, et en particulier, sur la dynamique du collatéral pour la contrepartie n.

S'il n y a pas de collatéral

On a $C_{t+a}^n = 0$, et donc directement $\mathrm{LGD}_{t,\mathfrak{d}}^n = 1-\gamma$.

Si le collatéral est un actif financier

Si le collatéral est un actif financier comme les actions d'une entreprise, il en représente donc une part notée $\alpha^n \in (0,1]$. Comme tout investissement, il doit générer un flux de trésorerie de sorte qu'à chaque instant, nous puissions calculer sa valeur en utilisant le modèle d'actualisation des flux de trésorerie introduit en (1.3.8). Par conséquent, la valeur du collatéral ici s'obtient de la même manière la valeur de toute firme $n \in \{1,\ldots,N\}$ à toute date $t \in \mathbb{R}_+$,

$$C_{t,\mathfrak{d}}^n := \alpha^n \mathbb{E}_t\left[\int_{s=t}^{+\infty} e^{-rs}\overline{F}_{s,\mathfrak{d}}^n \mathrm{d}s\right], \tag{1.3.10}$$

avec

$$\mathrm{d}\log\overline{F}_{t,\mathfrak{d}}^n = \overline{\mathfrak{a}}^n \cdot \mathrm{d}\log\left(Y_t(\mathfrak{d}_t, \mathcal{A}_t)\right) + \overline{\sigma}_n \mathrm{d}\overline{\mathcal{W}}_t^n, \tag{1.3.11}$$

pour $\overline{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ et où $(\overline{\mathcal{W}}_t^n)_{t\in\mathbb{R}_+}$ est un mouvement brownien dans $\mathbb{R}^N$ avec $\overline{\sigma}_n > 0$. De plus, $B^{\mathcal{Z}}$ (bruit de la productivité), $\overline{\mathcal{W}}^n$ (bruit des garanties) et $\mathcal{W}^n$ (bruit des débiteurs) sont indépendants. Partant du proxy $\mathcal{C}^n_{\cdot,\mathfrak{d}}$ (en s'inspirant une fois de plus du modèle discret et en utilisant le paramètre ς introduit dans l'hypothèse 1.3.6), nous avons le théorème suivant:

Theorem 1.3.9. *La perte en cas de surendettement de l'emprunteur n à l'instant $t \in \mathbb{R}_+$, lorsque la garantie est l'actif financier décrit par* (1.3.11) *et* (1.3.10)*, conditionnellement à $\mathcal{G}_t$; est*

$$\mathrm{LGD}^n_{t,\mathfrak{d}} = (1-\gamma)\left[\Phi\left(\frac{w_t^n}{\overline{\sigma}_{\mathfrak{b}_n}\sqrt{t}}\right) - \exp\left(-w_t^n + \frac{1}{2}t\overline{\sigma}_n^2\right)\Phi\left(\frac{w_t^n}{\overline{\sigma}_n\sqrt{t}} - \overline{\sigma}_n\sqrt{t}\right)\right],$$

avec

$$w_t^n := \log\left(\frac{\mathrm{EAD}_t^n}{(1-k)e^{-ra}}\right) - \mathbb{E}[\mathcal{C}^n_{t,\mathfrak{d}}|\mathcal{G}_t].$$

Si le collatéral est bien immobilier

Les bâtiments résidentiels ou commerciaux sont l'un des plus gros émetteurs de GES. La directive sur la performance énergétique des bâtiments (DPEB), introduite en 2002 par la Commission Européenne et révisée en 2010 et ultérieurement, est un instrument clé pour augmenter la performance énergétique des bâtiments dans l'UE. Comme le prix du carbone, c'est un moyen de mettre en oeuvre la transition climatique. Il consiste par exemple à classer les bâtiments en fonction de leur efficacité énergétique (EE) en utilisant des lettres de A à G (où A est le plus efficace et G le moins). Des études récentes montrent que la performance énergétique d'un logement commence à avoir une importance similaire que le prix ou la localisation. C'est ce que nous modélisons dans cette partie.

Pour le logement qui sert de garantie à la contrepartie $n \in \{1,\dots,N\}$, nous supposons que :

1. en l'absence de transition climatique, le prix du logement suit un processus d'Ornstein-Uhlenbeck exponentiel ;

2. chaque logement consomme une certaine quantité d'énergie par mètre carré, qui est utilisée pour déterminer son efficacité énergétique, notée α^n et exprimée en kilowattheure par mètre carré et par an $(kWh/m^2/an)$;

3. en conséquence, le prix du logement est déprécié (ou apprécié) par la somme actualisée des coûts énergétiques futurs ;

4. une fois qu'un certain niveau d'efficacité énergétique $\alpha^\star$ est atteint, le marché est insensible à ce facteur ;

5. le prix de l'énergie est une fonction déterministe à deux variables $\mathfrak{f}$, la première variable est le prix du carbone et la deuxième est la source d'énergie,

6. chaque logement pourrait être vendu à une date aléatoire $T_n > 0$ (par exemple, suivant une loi exponentielle avec paramètre $\lambda_n > 0$),

7. pendant la vie du bien, il peut subir des rénovations qui déplacent l'efficacité énergétique de α^n à $\alpha^\star$, et dont le coût par mètre carré est une fonction $\mathfrak{c}$ de son efficacité énergétique,

8. la date de rénovation $\mathfrak{t}_n$ d'un logement est inconnue, mais doit être optimisée.

9. après les rénovations, le prix du bâtiment devient insensible aux coûts énergétiques.

En remarquant qu'un bien immobilier est un actif appartenant à l'économie décrite par 1.3.6 et 1.3.7, nous avons l'hypothèse suivante.

Assumption 1.3.10. On se place en risque neutre.

1. *Prix du logement sans transition climatique:* la valeur marchande, du bâtiment indexé par n à $t \geq 0$, est donnée par

$$C_t^n := R_n C_0^n e^{K_t},$$

où

$$\begin{aligned} \mathrm{d}K_t &= \left(\frac{d\Psi}{dt} - \overline{\lambda}\overline{\sigma} + \nu(\Psi_t - K_t) \right) \mathrm{d}t + \overline{\sigma}\mathrm{d}\overline{B}_t, \\ \mathrm{d}\overline{B}_t &= \rho^\top \mathrm{d}B_t^{\mathcal{Z}} + \sqrt{1 - \|\rho\|^2}\mathrm{d}\overline{\mathcal{W}}_t, \end{aligned}$$

où $(\overline{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ est un mouvement brownien standard indépendant de $B^{\mathcal{Z}}$ tandis que C_0^n, r, R_n, $\overline{\sigma}$, $\overline{\lambda} > 0$, $\rho \in \mathbb{R}_+^I$, et $\Psi \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$.

2. Un batiment est un investissement comme un autre. Sa possession entraine par conséquent la génération d'un flux de trésorerie de revenus/dividendes noté $(D_t^n)_{t\geq 0}$ qui est continu et adapté à $\mathbb{U}$. Par conséquent, pour tout $t \geq 0$, une autre façon d'écrire le prix C_t^n de l'immeuble est la suivante :

$$C_t^n = R_n \mathbb{E}\left[\int_0^{+\infty} e^{-\bar{r}s} D_{t+s}^n \, ds \middle| \mathcal{U}_t \right],$$

avec $\bar{r} > 0$ et $\mathbb{U} := (\mathcal{U}_t)_{t\in\mathbb{R}^*}$ la filtration telle que pour tout $t \geq 0$, $\mathcal{U}_t := \sigma\left(\left\{\overline{W}_s, B_s^{\mathcal{Z}} : s \leq t\right\}\right)$

3. *Prix du logement avec transition climatique*: la valeur marchande du bâtiment servant de garantie à l'entreprise n à $t \geq 0$, étant donné la séquence de prix du carbone δ, est représentée par

$$\mathcal{C}_{t,\delta}^n := R_n \operatorname*{ess\,sup}_{\theta\geq t} \mathbb{E}\left[\begin{array}{c} \int_t^\theta \left[D_s^n - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)}\mathrm{d}u - \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\theta-t)} \\ + \int_\theta^{+\infty} e^{-\bar{r}(s-t)} D_{t+s}^n \mathrm{d}s \end{array} \middle| \mathcal{U}_t \right],$$

où

- $\mathfrak{c}$ est une fonction continue de $(\mathbb{R}_+)^2$ à $\mathbb{R}_+^*$,
- pour chaque source d'énergie $\mathfrak{p}$, $\mathfrak{f}(.,\mathfrak{p})$ est une fonction continue de $\mathbb{R}_+$ à $\mathbb{R}_+^*$,
- α^n, $\alpha^\star > 0$ avec $\alpha^n > \alpha^\star$,
- la filtration $\mathbb{U} := (\mathcal{U}_t)_{t\in\mathbb{R}^*}$ avec pour $t \geq 0$, $\mathcal{U}_t := \sigma\left(\left\{\overline{B}_s : s \leq t\right\}\right)$.

Dans l'hypothèse ci-dessus, le terme $[D_u^n - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})]\, e^{-\bar{r}(u-t)}$ est le revenu actualisé à t du bien avant la date optimale de rénovation $\mathfrak{t}_n$ tandis que le terme $D_u^n e^{-\bar{r}(u-t)}$ est le revenu actualisé à t du bien après $\mathfrak{t}_n$. En outre, le terme $\mathfrak{c}(\alpha^n, \alpha^\star)e^{-r(\mathfrak{t}_n - t)}$ représente les coûts actualisés des rénovations qui sont effectuées à la date $\mathfrak{t}_n$. Nous notons enfin que pour tout $t \geq 0$, $C_t^n - \mathcal{C}_{t,\delta}^n \leq R_n\mathfrak{c}(\alpha^n, \alpha^\star)$, c'est-à-dire que l'agent peut toujours rénover la maison immédiatement, mais cela peut ne pas être optimal.

Par suite, sous certaines hypothèses précisées dans le Théorème 5.3.10, on peut écrire la valeur marchande du bâtiment comme

$$\mathcal{C}_{t,\delta}^n = C_t^n - R_n X_{t,\delta}^n.$$

Nous montrons que $X_{t,\delta}^n$ est le choc de transition climatique sur le prix du bâtiment. Il dépend notamment de la date optimale de rénovation, des coûts de rénovation, du prix du carbone (via le prix de l'énergie), et de l'efficacité énergétique du bâtiment.

On utilise enfin cette expression pour obtenir la LGD pour un collatéral immobilier par le théorème suivant

Theorem 1.3.11. *Etant donné le processus du prix du carbone δ. La perte en cas de surendettement du débiteur n à la date $t \in \mathbb{R}_+$, lorsque son collatéral est le bien immobilier décrit en 1.3.10, conditionnellement à $\mathcal{G}_t$, est de*

$$\mathrm{LGD}_{t,\alpha^n,\delta}^n = (1-\gamma)\left[\left(1 + (1-k)e^{-ra}\frac{R_n X_t^n}{\mathrm{EAD}_t^n}\right)\Phi\left(\frac{w_t^n}{\sqrt{v_{t,t}^n}}\right) - \exp\left(-w_t^n + \frac{1}{2}v_{t,t}^n\right)\Phi\left(\frac{w_t^n}{\sqrt{v_{t,t}^n}} - \sqrt{v_{t,t}^n}\right)\right],$$

avec

$$w_t^n := \log\left(\frac{\mathrm{EAD}_t^n}{(1-k)R_n e^{-ra}} + X_t^n\right) - \log\left(C_0^n\right) - \mathbb{E}[Y_t|\mathcal{G}_t].$$

On note $\mathrm{LGD}_{t,\alpha^n,\delta}^n$ (avec $\alpha^n\delta$ en indice) pour présicer la dépendance de la LGD en l'efficacité énergétique du batiment α^n et du prix du carbone δ.

Pour un portefeuille composée de N crédits soit non garantis, soit garantis par des actifs financiers, soit garantis par des actifs immobiliers, on écrit, pour tout t, T, la perte attendue (respectivement inattendue) entre t et $t+T$, notée $\mathrm{EL}_{t,\mathfrak{d}}^{N,T}$ introduite en (1.3.6) (respectivement $\mathrm{UL}_{t,\alpha,\mathfrak{d}}^{N,T}$ introduite en (1.3.7)). Il suffit pour cela de remarquer que la loi de $\mathrm{L}_t^{\mathbb{G},N}|\mathcal{G}_t$ est une fonction de la loi de $\mathcal{A}_{t+T}|\mathcal{G}_t$.

Enfin, de la même manière que dans le chapitre 4, nous considérons 4 scénarios de transition climatique de 2021 à 2030. Nous considérons aussi 4 sous-portefeuilles représentant chacun un secteur de l'économie française divisée en 4 secteurs. Nous

supposons en outre que dans les sous-portefeuilles, certains prêts sont non collatéralisés, tandis d'autres sont garantis soit par des actifs financiers, soit par des biens immobiliers. Nous calculons donc les différentes mesures de risques associées: PD, LGD, EL, et UL. Nous obtenons, en plus des résultats déjà obtenues dans le chapitre 4, que

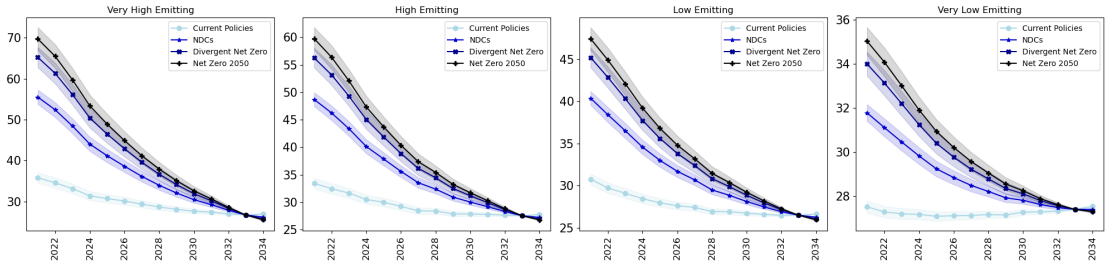


Figure 1.3: LGD avec un bien immobilier en collatéral

1. la présence de garanties réduit LGD, EL, et UL;

2. lorsque le scénario de transition se durcit, l'impact de la présence du collatéral sur LGD, EL, et UL s'atténue parce que la valeur liquidative de la garantie baisse (exemple en Figure 1.3);

3. pour le cas de l'immobilier, la date optimale de rénovation décroit avec un scénario de transition qui se durcit.

4. Lorsque la date optimale de rénovation est atteinte, la baisse du prix de bien immobilier s'arrête, voire s'inverse selon son efficacité énergétique.

Tous ces modèles sont à chaque fois calibrés sur des données historiques et appliqués sur des portefeuilles réels de la banque BPCE.

Ce travail a des implications très pratiques pour la gestion des risques de la banque BPCE. Précisément, la *perte attendue* étant couverte par les provisions provenant des frais facturés aux clients, une augmentation de la *perte attendue* à cause du prix du carbone implique une augmentation des frais de crédit pour les clients et des coûts opérationnels pour la banque. Dans le même temps, la *perte inattendue* est couverte par le capital économique et réglementaire de la banque, l'une des mesures de sa solvabilité, qui est financée par les actionnaires. Une augmentation de la *perte inattendue* signifierait donc soit une baisse de la solvabilité, soit une augmentation du capital. Par conséquent, pour réduire son exposition à la transition climatique (en diminuant les *pertes attendues et inattendues*), la banque peut décider d'y être moins exposée, c'est-à-dire d'accorder moins de crédit aux secteurs polluants et d'accepter des garanties moins polluantes.

Chapter 2

Introduction

The aim of this work is to construct mathematical models to quantify and project credit portfolio's risk measures. We consider a portfolio of corporate credits, each belonging to a unique economy. We assume that the economy is closed, driven by dynamic and stochastic productivity, and subject to climate transition modeled by a dynamic and deterministic carbon price.

Contents

2.1 Context

2.1.1 A Challenge of Our Time

It is now clear that climate change has and will have adverse effects on the environment, on global warming, and on human societies. The consumption of hydrocarbons emits carbon dioxide, while agriculture and livestock produce methane, notably due to fertilizers or animal excrement, etc. Carbon dioxide and methane are two examples of greenhouse gases (GHGs), which, when they accumulate in the atmosphere, disrupt the radiative force, which ensures thermal equilibrium at the surface of the earth and in the oceans. Over the past few decades, numerous international summits have been organized to address this new risk: the Kyoto Protocol in 1997, the Copenhagen Agreement in 2009, the Paris Climate Agreement in 2015, etc. These meetings have finally allowed us to take the measure of the challenge and to initiate a transition to a low-carbon economy.

Climate risk has two components. The first one is physical risk, which is due to changes in climate variables (such as rising temperatures and melting ice) or the occurrence of extreme weather events (such as droughts or typhoons) that can damage infrastructure and endanger populations. The second one is transition risk which comes from the necessity of transitioning to a low-carbon economy, leading to the implementation of regulatory measures, potential technological changes, and the evolution of consumer preferences. These developments will have numerous impacts in all sectors, at all scales, and on all the agents of the economy. We focus in this thesis on the financial and banking sector.

Previous financial crises have shown the existence of a strong relationship between the financial sphere and the real one. This is why regulators now encourage the institutions to take into account transition risk in their models. Financial and regulatory authorities such as the Bank of France (see Devulder and Lisack [2020], Allen et al. [2020]), have begun to conduct research to understand and quantify the impact of climate risk on financial institutions (including commercial banks, insurers, and asset managers). For example, it is observed that transition risk can modify the three main components of credit risk, which are *the borrower's cash flows, the value of his assets, and the value of his guarantees.* Indeed, the climate transition implies the reduction in demand for products with a high carbon footprint, the increase in research and production costs of low-carbon products, the revaluation of corporate and household assets, and the increase in stranded assets.

2.1.2 Emerging Questions

In our modern economies, Commercial banks play the main role in financing economic agents (both businesses and households, as well as states). They do this, among

other things, by granting loans to them. All agents, especially businesses, can *default* for multiple reasons. To ensure the viability of its business model, meet regulatory requirements, or even the stability of the economy (especially for systemic institutions), a bank must have provisions – calculated from the expected loss, that is, the average anticipated loss over a defined period of time. Expected losses represent an operating cost and must generally be absorbed by a part of the operational result – as well as economic and regulatory capital – which can be described as protection against future unexpected losses (that is, potential real losses minus expected losses) at a given confidence level. Both provisions and economic capital depend on the distribution of losses from credit portfolios. Moreover, we have highlighted the need for transition to a low-carbon economy and the possible effects of this transition on businesses, i.e., borrowers. There is therefore a link between credit risk and climate transition risk.

Three main questions emerge:

1. **How to model the climate transition?** The financial sector generally considers three main types of drivers for the transition risk: changes in consumer preferences, technological changes, and political changes. For the latter, the most well-known variant is the carbon price. This means that the GHG emissions of economic agents are charged. This is the one we will adopt here.

2. **How to model the *default* of a borrower and link it to the climate transition?** A bank can trigger the *default* of a firm if the latter faces a liquidity crisis, i.e., its cash flows are not sufficient to fulfill the creditor's payment demands (interest and net repayment) as planned in the contract, or a insolvency crisis, i.e., the market value of its debt exceeds the market value of the company (This is known as *over-indebtedness*). From this definition, let us consider two examples to show that the climate transition, and notably the carbon price, can affect the *over-indebtedness*: (1) the company's GHG emissions are charged, its cash flow can decrease and therefore its distance to *default* also decreases, (2) the assets of a company emit so much GHG that the potential costs of these emissions affect their value, this impacts the market value of the company and therefore its distance to *default*, or even its *default* date.

3. **How to model the loss in case of *default* and link it to the climate transition?** when the bank activates the default of a borrower, the former can lose a part of the loan. This loss is the difference between the amount of exposure at the time of *default* and the liquidation value of the company and its guarantees. We have seen that the market value of the company can be affected by the additional charge due to its GHG emissions. The same applies to its guarantees. If, for example, these are real estate assets, and energy sieves, they will directly depreciate.

2.1.3 A Problem to Formalize

We consider a complete probability space $(\Omega, \mathcal{H}, \mathbb{P})$ and $\mathbb{K} \in \{\mathbb{N}, \mathbb{R}\}$. Let N be a non-zero natural integer. Let Θ, $\mathfrak{b}$, and $\overline{\mathfrak{b}}$ be three stochastic processes, defined on the probability space $(\Omega, \mathcal{H}, \mathbb{P})$, with Θ being d-dimensional whereas $\mathfrak{b}$ and $\overline{\mathfrak{b}}$ being N-dimensional. They will respectively represent the productivity of the economy, the noise of the borrower's value, and the noise of the collateral. We also consider a deterministic n-dimensional process δ. We further denote the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{K}}$, $\mathbb{G}^{\mathfrak{b}} := \left(\mathcal{G}_t^{\mathfrak{b}}\right)_{t\in\mathbb{K}}$ and the filtration $\mathbb{G}^{\overline{\mathfrak{b}}} := \left(\mathcal{G}_t^{\overline{\mathfrak{b}}}\right)_{t\in\mathbb{K}}$, such that for $t \in \mathbb{K}$, $\mathcal{G}_t := \sigma\left(\{\Theta_s : s \in [0,t] \cap \mathbb{K}\}\right)$, $\mathcal{G}_t^{\mathfrak{b}} := \sigma\left(\{\Theta_s, \mathfrak{b}_s : s \in [0,t] \cap \mathbb{K}\}\right)$, and $\mathcal{G}_t^{\overline{\mathfrak{b}}} := \sigma\left(\left\{\Theta_s, \overline{\mathfrak{b}}_s : s \in [0,t] \cap \mathbb{K}\right\}\right)$.

Let $(A_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ and $(U_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ be two sequences dependent on δ and respectively $\mathbb{G}^{\overline{\mathfrak{b}}}$ and $\mathbb{G}^{\mathfrak{b}}$-measurable. We define a bank's portfolio composed of N loans contracted by firms all belonging to the same economy, such that for all $t \in \mathbb{K}$ and $n \in \{1, \dots, N\}$, $A_t^n > 0$ represents the exposure of the bank to borrower n at date t and U_t^n represents the loss per exposure to borrower n at date t. The *portfolio loss* at date t, denoted L_t^N, is

$$L_t^N := \sum_{n=1}^{N} A_t^n U_t^n.$$

In credit risk assessment, one of the first steps is to create homogeneous sub-portfolios of firms. As we are dealing here with climate transition risk, we would like to classify firms by carbon intensity: firms with similar carbon intensities belong to a same homogeneous sub-portfolio. It should be noted that in the absence of a climate transition, firms are traditionally clustered in terms of industry, geography, size, and credit rating, for example.

We therefore assume that there exist $I \in \mathbb{N}^*$ ($I \leq N$) sectors of GHG emissions in the economy assumed homogeneous. The set of sectors is denoted by $\mathcal{I}$. Because we rarely have individual carbon emissions/intensities, we assume that each company has the same carbon intensity as its sector of activity. This amounts to grouping the "activity sectors" into I "carbon emission sectors". From now on, sectors must be interpreted as carbon emission sectors, which allows us to construct homogeneous sub-portfolios.

Definition 2.1.1 (Sub-portfolios)**.** We divide our portfolio into I disjoint sub-portfolios $g_1, \dots, g_I$ so that each sub-portfolio represents a single risk class and that the companies in each of them belong to a single carbon emission sector. From now on, we denote $\mathcal{I}$ of cardinal $I \in \mathbb{N}^*$ the set of sectors (and sub-portfolios). We also set, for all $i \in \mathcal{I}$, $n_i := \min\{n \in \{1, \dots, N\} \text{ such that } n \in g_i\}$. Therefore, the company n_i is a representative of the group i.

The aim of this work is to calculate the risk measures of the portfolio (and of the sub-portfolios $(g_i)_{i\in\mathcal{I}}$) i.e., for example, for $t, T \in \mathbb{K}$,

$$\mathrm{EL}_t^{N,T} := \mathbb{E}\left[L_{t+T}^N - L_t^N \middle| \mathcal{G}_t\right],$$

called *expected loss* and for $\alpha \in [0,1]$,

$$\mathrm{UL}_{t,\alpha}^{N,T} := \mathrm{VaR}_{t,\delta}^{\alpha,N,T} - \mathrm{EL}_t^{N,T}, \quad \text{with} \quad 1-\alpha = \mathbb{P}\left[L_{t+T}^N - L_t^N \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right],$$

called *unexpected loss*, if the companies belong to a closed economy (in other words, without import or export) driven by a productivity described by Θ and subject to a climate transition described by δ. This amounts precisely to quantifying the distortion in time of credit risk measures created by the introduction of a carbon price.

We propose to solve this problem in two parts: a discrete part (which is better suited to economic modeling and the modeling of loss without collateral) and a continuous part (which is better suited to the modeling of loss with collateral). In the first part, we assume that the loans are all unsecured while Θ and δ are discrete time processes (i.e., $\mathbb{K} = \mathbb{N}$ and $(A_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ is deterministic) and in the second part, some loans can be collateralized while Θ and δ are continuous time processes (i.e., $\mathbb{K} = \mathbb{R}_+$ and $(A_t^n)_{t\in\mathbb{K}, n\in\{1,\dots,N\}}$ is stochastic).

In Chapter 3, we propose a modeling of the framework in which the companies in the portfolio operate. This framework is described by a dynamic, stochastic, and multisectoral economic model in which the direct and indirect GHG emissions of companies as well as the direct GHG emissions of households are charged. We choose a representative company in each sector and a representative household for the entire economy. In Chapter 4, we assume that each company belongs to a sector and that its cash flows are a proportion of its sales. Since the latter are a proportion of sectoral production, we obtain the dynamics of cash flows that we use to model the value of companies in an economic environment where GHG emissions are charged. Finally, from a structural model in which a company defaults if it is *over-indebted*, i.e., if its value becomes less than its debt, we calculate the probability of default of each company as well as the loss (and the associated statistics) of the portfolio (and of the sub-portfolios) distorted by the carbon price. In Chapter 5, after extending the three models of Part I into continuous time, we propose a definition of the loss of a credit portfolio at any date. Specifically, when a borrower defaults, if there is no guarantee, the bank recovers a deterministic and constant proportion of its exposure at the default date. If there is a guarantee, the bank loses a proportion of the amount of its exposure remaining after the liquidation of the guarantee. In the context of the climate transition, we then detail the case where the guarantee is a *financial asset* (it is said to be intangible) followed by the case where the guarantee is a *real estate* asset (it is said to be tangible). We finally define credit risk measures such as the probability of default, the expected loss, and the unexpected loss. In each part and each chapter, simulations are carried out on real or simulated data for different transition scenarios (i.e., for different carbon price trajectories) and on different portfolio.

2.2 Literature Review

The first climate stress tests took place in France between 2019 and 2021. Nine banking groups and fifteen insurance groups were involved on a voluntary basis. As indicated in ACPR [2023] [1], they allowed a first quantification of the impact of climate risk on financial institutions. The methodology used to carry out these stress tests is mainly inspired by Allen et al. [2020]. In the latter, the authors assess the impact of climate transition risk on credit portfolios in three steps: (1) climato-economic models (see Luderer et al. [2015]) that allow obtaining the trajectory of the gross domestic product (GDP) and the carbon price according to climate transition scenarios (for example, less than 2 °C in 2050), (2) macroeconomic and multisectoral models (see Bertram et al. [2021] and Devulder and Lisack [2020]) that allow obtaining macroeconomic variables as a function of a carbon tax, and finally (3) financial models to calculate probabilities of default. In this work, we will not only be inspired but also extend this methodology.

2.2.1 Climato-Economic Models

These are models that combine climate processes (GHG emissions, temperature, etc.), macroeconomic processes (output, consumption, etc.), and feedbacks between climate and economy (damages, abatement, etc.) in a single modeling framework. As indicated by Farmer et al. [2015], there are three classes of such models: Integrated Assessment Models (IAM), Dynamic Stochastic General Equilibrium (DSGE) models, and Agent Based Models (ABM). Regardless of their type, these are optimization or evaluation models that aim to approach the transition by the carbon price (or carbon tax) both exogenous and endogenous. For example, Nordhaus [1993], Reis and Augusto [2013], and Golosov et al. [2014] propose models to find an endogenous and optimal trajectory of the carbon price to follow to (not) reach a temperature or GHG emissions target. On the other hand, Golosov et al. [2014], McKibbin and Wilcoxen [2013] and Devulder and Lisack [2020] propose a multisectoral modeling but the carbon tax is exogenous for the last model, even if it is static and deterministic. Other authors focus on more specific sectors. This is the case for Golosov et al. [2014] who specifically deal with energy sectors, or Ter Steege and Vogel [2021], who, in discrete time, give the price difference per square meter between two building properties should be solely explained by the sum of the discounted value of the expected energy cost difference.

2.2.2 Credit Risk Models

The literature on credit risk models is quite vast and dates back to about Merton [1974]. The goal remains to calculate the portfolio loss introduced in (1.1.1) (or at least some of its statistics). The most used formulation (especially because it is encouraged by regulators) is proposed by Basel Committee on Banking Supervision [2017] and

[1] Autorité de Contrôle Prudentiel et de Résolution - the financial regulatory authority which supervises and regulates the French financial firms.

is essentially based on the work of Merton [1974] and Vasicek [2002]. It consists in rewriting for all $t \in \mathbb{N}$, the loss L_t^N defined in (1.1.1) as

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot D_t^n,$$

where for each counterparty $n \in \{1, \ldots, N\}$,

- EAD_t^n is the outstanding debt at the time of *default*
- LGD_t^n is the percentage of the EAD that is lost in case of *default*,
- D_t^n is the *default* event ($D_t^n = 1$ if there is *default* at t and $D_t^n = 0$ otherwise).

Modeling of EAD Taplin et al. [2007] proposes the following modeling of EAD_t^n:

$$\mathrm{EAD}_t^n(\tau^n) = B^n(t) + CCF_t^n(\tau^n)(L^n(t) - B^n(t)),$$

where $B^n(t)$ represents the remaining balance (or the amount currently drawn) at t, $L^n(t)$ is the currently unused credit limit of the credit facility and $CCF_t^n(\tau^n)$ is the credit conversion factor. This means that the exposure at the time of *default* τ^n has two components: the current exposure $B^n(t)$ which is deterministic, and the future exposure $CCF_t^n(\tau^n)(L^n(t) - B^n(t))$ which is random.

Modeling of LGD The second variable LGD_t^n is derived from a set of estimated cash flows, which are the result of the recovery process, and which are correctly discounted at a *default* date. Bastos [2010] and Roncalli [2020] propose an economic modeling where LGD is a function (linear or non-linear) of many factors that can be external to the issuer, specific to the issuer or specific to the debt issue. We can then write

$$\mathrm{LGD}_t^n = g^n(X_t^1, \ldots, X_t^m),$$

where $(X_t^1, \ldots, X_t^m)$ is a sequence of risk factors and g^n is a function from $\mathbb{R}^m$ to $[0, 1]$ which can be obtained by logistic regression, regression trees, or neural networks. Chalupka and Kopecsni [2008] and Roncalli [2020][Page 193] propose a stochastic modeling, where LGD_t^n follows a certain distribution (parametric or non-parametric). For example

$$\mathrm{LGD}_t^n \sim \mathfrak{B}(\alpha^n, \beta^n),$$

where $\mathfrak{B}(\alpha^n, \beta^n)$ is the beta law of parameters α^n and β^n.

Modeling of the default event D As for D_t^n, even if there is a "reduced form" modeling based on survival functions, the most used models are "structural". The most popular are that of Merton [1974] where $D_t^n = \mathbf{1}_{A^n(t) \leq B^n}$, and that of Black and Cox [1976] where $D_t^n = \mathbf{1}_{\inf_{0 \leq s \leq t} A^n(s) \leq B^n}$, where A^n is the dynamics of the borrower n's assets which follows a geometric Brownian motion while $B \geq 0$ is a given barrier. Basel Committee on Banking Supervision [2017] however uses the extension of the Merton [1974] model proposed by Vasicek [2002] where the dynamics of the assets is given by $A^n(t) = \sqrt{\rho} X_t + \sqrt{1-\rho^2}\epsilon_n$ with $X_t \sim \mathcal{N}(0,1)$ is the systematic risk factor, $\epsilon_n \sim \mathcal{N}(0,1)$ is the idiosyncratic risk factor, and $\rho \in [-1,1]$.

The difficulty with most of these models is that they are not always microfounded: in the model of Vasicek [2002] for example, the systemic and idiosyncratic factors being centered reduced Gaussians, the integration of the economic environment can only be done indirectly through the estimation of ρ. This makes it difficult to integrate the risk of climate transition, which we would like to measure the diffusion in a credit portfolio. This is why the first step of our work is to model the environment that is undergoing the transition and in which the firms in the portfolio live.

2.2.3 Links Between Climate Risk and Credit Risk

With the Paris agreements in 2015, numerous research projects and a growing literature on the subject have been observed. Cartellier [2022] discusses the methodologies and approaches used by banks and researchers in climate stress tests. Battiston and Monasterolo [2019] deal with the assessment of transition risk in sovereign bond portfolios while Allen et al. [2020] focus on the assessment of corporate credit. Garnier [2021] and Gaudemet et al. [2022] propose two models. The first, called CERM (Climate Extended Risk Model), is a model based on the Vasicek-Merton one with a multidimensional Gaussian systemic factor, where the transition risk is diffused to the credit risk by the load factors defined as the correlations between the systematic risk factors and the assets. The second introduces a climato-economic model to calibrate the model of the first. Other works, such as Bourgey et al. [2021] or Bouchet and Le Guenedal [2020], take into account the economic and capital structure of the company in the measurement of carbon risk. In particular, Bourgey et al. [2021] derive the value of the company using the discounted cash flow methodology on cash flows that are affected by the company's optimal transition policy, while Bouchet and Le Guenedal [2020] directly affect the value of the company by a shock dependent on the ratio between the cost of carbon and EBITDA (i.e. Earnings Before Interest, Taxes, Depreciation, and Amortization).

In addition, Le Guenedal and Tankov [2022b] use a structural model to assess the bonds issued by a company subject to climate transition risk and, in particular, take into account the uncertainty of the transition scenario. Finally, Livieri et al. [2023] use a Jump-Diffusion credit risk model where the downward jumps describe the green policies taken by companies, to assess default coupon bonds and Credit Default Swaps.

2.3 Main Contributions

The methodological approach used in this work is inspired by Allen et al. [2020]. Our goal is to develop an end-to-end methodology, starting from climate transition scenarios to portfolio risk measures.

2.3.1 Modeling of an economic framework taking into account the GHG emissions costs

We consider a closed economy composed of I sectors (recall that $\mathcal{I}$ is the set of sectors). The aim here is to describe the dynamics of macroeconomic variables by sector, which are the production of companies, consumption of households, labor supply, and intermediate inputs.

For this reason, we are mainly inspired by Devulder and Lisack [2020], Golosov et al. [2014], Galí [2015], and Miranda-Pinto and Young [2019]. Our setup is at the intersection of these works. The extensions made are motivated by the goal to apply it later on to model the credit portfolio's risk measures. We thus need certain characteristics, the main one being that the macroeconomic variables are stochastic with parametric distributions. Our main contributions in Chapter 3 are as follows. On the one hand, the productivities of the sectors are stochastic and correlated. On the other hand, the GHG emissions emitted by companies when they produce goods and/or services and when they consume intermediate inputs, as well as by households when they consume goods and/or services, are charged at a deterministic and exogenous carbon price.

The proposed framework assumes a representative company in each sector that maximizes its profits by choosing, at each instant and for a given productivity, the quantities of labor and intermediate inputs, to produce a unique good/service. We can therefore assimilate the representative company to its sector, and to the good/service it produces. A representative household, on the other hand, solves a dynamic optimization problem to decide how to distribute its consumption expenditures among the different goods/services and hours worked and among the different sectors. Let us introduce the following two assumptions:

Standing Assumption 2.3.1. Consider the processes Θ and $\mathcal{A}$ in $\mathbb{R}^I$ and which evolve as follows

$$\begin{cases} \mathcal{A}_t &= \mathcal{A}_{t-1} + \Theta_t, \\ \Theta_t &= \mu + \Gamma\Theta_{t-1} + \varepsilon\mathcal{E}_t, \end{cases} \qquad \text{for all } t \in \mathbb{N}^*,$$

where the constants $\mu, \mathcal{A}_0 \in \mathbb{R}^I$ and where the matrix $\Gamma \in \mathbb{R}^{I\times I}$ is such that the absolute values of its eigenvalues are strictly less than 1, $0 < \varepsilon \leq 1$ is a noise parameter that we assume fixed. In addition, the noises $(\mathcal{E}_t)_{t\in\mathbb{Z}}$ are independent and identically distributed (iid) with for all $t \in \mathbb{Z}$, $\mathcal{E}_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$ such that $\Sigma \in \mathbb{R}^{I\times I}$. Moreover, we have, $\Theta_0 \sim \mathcal{N}(\overline{\mu}, \varepsilon^2\overline{\Sigma})$ such that $\overline{\mu} := (\mathbf{I}_I - \Gamma)^{-1}\mu$, and $\mathrm{vec}(\overline{\Sigma}) := (\mathbf{I}_{I\times I} - \Gamma \bigotimes \Gamma)^{-1}\mathrm{vec}(\Sigma)$, with

$M \in \mathbb{R}^{d\times d}$, $\mathrm{vec}(M) := [M^{11}, \dots, M^{d1}, M^{21}, \dots, M^{d2}, \dots, M^{1d}, \dots, M^{dd}]^\top$. Finally, the processes $(\mathcal{E}_t)_{t\in\mathbb{N}}$ and the random variable Θ_0 are also independent.

For all $i \in \mathcal{I}$, the processes Θ^i and $\mathcal{A}^i$ play a major role in our model since, the total factor productivity of sector i is defined as follows

$$A_t^i := \exp\left(\mathcal{A}_t^i\right),$$

so that Θ^i is the logarithmic growth of productivity and $\mathcal{A}^i$ is the cumulative logarithmic growth of productivity.

Standing Assumption 2.3.2. We also introduce the carbon intensities τ, κ, ζ (defined as the quantity of GHG in tons emitted for each euro of production/consumption) and the carbon price δ which are deterministic processes such that:

1. Given $0 \le t_\circ < t_\star$. The carbon price δ satisfies for all $t \in \mathbb{N}$,

   - if $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in \mathbb{R}_+$, i.e. it is constant before the start of the transition;
   - if $t \in (t_\circ, t_\star)$, $\delta_t \in \mathbb{R}_+$, i.e. it can be dynamic during the transition;
   - for $t \ge t_\star$, $\delta_t = \delta_{t_\star} \in \mathbb{R}_+$, i.e. it becomes constant again after the transition.

   Thus, the transition begins at $t_\circ$ with a constant carbon price and ends at $t_\star$ with a constant carbon price.

2. The deterministic processes τ, ζ, and κ belong respectively to $\mathbb{R}_+^I$, $\mathbb{R}_+^{I\times I}$, and $\mathbb{R}_+^I$, and respectively represent the carbon intensities on the production of companies, on the intermediate consumption of companies, and on the consumption of households, and satisfying for all $t \in \mathbb{N}$ and for all $\mathfrak{y} \in \{\tau^1, \dots, \tau^I, \zeta^{11}, \zeta^{12}, \dots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \dots, \kappa^I\}$,

$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t)}{\theta_{\mathfrak{y}}}\right) & \text{if } t \le t_\star \\ \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t_\star)}{\theta_{\mathfrak{y}}}\right) & \text{otherwise,} \end{cases}$$

   with $\mathfrak{y}_0, g_{\mathfrak{y},0}, \theta_{\mathfrak{y}} > 0$. For all $t \in \mathbb{N}$, we call $\mathfrak{y}_t\delta_t$ the *rate of emission cost* at time t.

3. For all $i \in \mathcal{I}$ and for all $t \in \mathbb{N}$,

$$\delta_t \max_{i\in\mathcal{I}} \tau_0^i < 1.$$

Let us introduce the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{N}}$ with $\mathcal{G}_0 := \sigma(\Theta_0)$ and for all $t \ge 1$, $\mathcal{G}_t := \sigma\left(\{\Theta_0, \mathcal{E}_s : s \in (0,t] \cap \mathbb{N}^*\}\right)$. For all $i \in \mathcal{I}$, let us consider the following $\mathbb{G}$-measurable and positive processes:

- Y^i the production of sector i,

- P^i the price of good/service i,
- N^i the labor demand in sector i,
- H^i the labor supply in sector i,
- W^i the wage in sector i,
- C^i the consumption of goods/services of sector i by households,
- for all $j \in \mathcal{I}$, Z^{ji} the consumption by sector i of intermediate inputs produced by sector j.

We also consider a deterministic real process r representing the interest rate.

For each sector i, the representative firm maximizes its profits by solving the following problem:

$$\max_{N_t^i, Z_t^{1i}, \dots, Z_t^{Ii}} \left\{ P_t^i Y_t^i - W_t^i N_t^i - \sum_{j \in \mathcal{I}} Z_t^{ji} P_t^j - \left(\tau_t^i P_t^i Y_t^i \delta_t + \sum_{j \in \mathcal{I}} \zeta_t^{ji} Z_t^{ji} P_t^j \delta_t \right) \right\},$$

under the constraint of the production function

$$Y_t^i = A_t^i (N_t^i)^{\psi^i} \prod_{j \in \mathcal{I}} (Z_t^{ji})^{\lambda^{ji}},$$

and constant returns to scale

$$\psi^i + \sum_{j \in \mathcal{I}} \boldsymbol{\lambda}^{ji} = 1, \qquad \text{for all } i \in \mathcal{I}.$$

Note that Y_t^i represents the volume production of sector i, $P_t^i Y_t^i$ its gross revenues in euros, $\tau_t^i P_t^i Y_t^i$ its GHG emissions in tons, emitted during production process, so $\tau_t^i P_t^i Y_t^i \delta_t$ the cost of the sector's GHG emissions.

Given the following utility function $U : (0, \infty)^2 \to \mathbb{R}$ given, for $\varphi \geq 0$, by $U(x, y) := \frac{x^{1-\sigma}}{1-\sigma} - \frac{y^{1+\varphi}}{1+\varphi}$ if $\sigma \in [0, 1) \cup (1, +\infty)$ and by $U(x, y) := \log(x) - \frac{y^{1+\varphi}}{1+\varphi}$, if $\sigma = 1$. The representative infinitely-lived household seeks to solve the following problem:

$$\max_{(C_t^i, H_t^i)_{t \in \mathbb{N}, i \in \mathcal{I}}} \quad \sum_{i \in \mathcal{I}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(C_t^i, H_t^i) \right],$$

in order to determine its consumption and working time, knowing that

$$\mathbb{E}\left[\sum_{i \in \mathcal{I}} \sum_{t=0}^{\infty} \beta^t \left| U(C_t^i, H_t^i) \right| \right] < \infty,$$

and the wealth process Q is such that for all $t \geq 0$

$$Q_t := (1 + r_{t-1}) Q_{t-1} + \sum_{i \in \mathcal{I}} W_t^i H_t^i - \sum_{i \in \mathcal{I}} P_t^i C_t^i - \sum_{i \in \mathcal{I}} \kappa_t^i P_t^i C_t^i \delta_t,$$

with the convention $Q_{-1} = 0$, $r_{-1} = 0$, and $\lim_{T\uparrow\infty} \mathbb{E}[Q_T|\mathcal{G}_t] \geq 0$.

For each $i \in \mathcal{I}$, $P_t^i C_t^i$ represents the household's consumption in sector i and $\kappa_t^i P_t^i C_t^i \delta_t$ the cost paid due to its emissions when it consumes the goods/services i, $W_t^i H_t^i$ is the household's labor income offered by sector i, $(1+r_{t-1})Q_{t-1}$ the household's capital income.

We obtain the first order conditions of the firm's problem (1.3.1) by noting that at each time step and for each realization of productivity, the problem becomes static and deterministic. We determine the first order conditions of the household's problem (1.3.2) using a variational approach. When the optimization problems (1.3.1) and (1.3.2) are solved, that the goods/services and labor markets balance, we show that the set of macroeconomic variables (N, H, Z, C, Y) only depend on the production $(Y_t^i)_{t\in\mathbb{N},i\in\mathcal{I}}$ and consumption $(Y_t^i)_{t\in\mathbb{N},i\in\mathcal{I}}$, which themselves, satisfy at each instant, a system of $2I$ nonlinear equations with $2I$ unknowns. We cannot solve this system analytically, but when the utility function is logarithmic in consumption (that is, $\sigma = 1$), and other structural conditions are satisfied, we obtain a unique trajectory for production and consumption, and therefore for all other variables. Each variable depending on productivity, carbon price, carbon intensities, and structural parameters of the model (φ, $(\psi^i)_{i\in\mathcal{I}}$ and $(\boldsymbol{\lambda}^{ji})_{i,j\in\mathcal{I}}$). We also note that because of the choice of separable production and utility functions, the price and wage processes do not intervene in the end.

We further define and determine

- the logarithmic growth of variables between two instants (example in Figure 2.1).
- The sensitivities of the growth of variables to the carbon price. This allows us to answer the following types of questions: *what is the impact on all variables if in the whole economy, we only charge GHG emissions coming from the production of companies in sector* 1*?* The main difficulty here comes from the fact that the coefficients of the inverse of the elasticity matrix $(\boldsymbol{\lambda}^{ji})_{i,j\in\mathcal{I}}$ are not known.
- The law of the logarithmic growth of macroeconomic variables that we obtain by using the Gaussianity and stationarity of logarithmic growth.
- the evolution of the contribution of each sector in production/consumption as well as the evolution of GHG emissions.

We calibrate and estimate the parameters based on the historical data from the French economy given by INSEE [2023] and the carbon price given by NGFS [2022]. For a 10-year transition starting in 2021 and ending in 2030, we finally carry out simulations by considering 4 sectors and 4 transition scenarios.

2.3.2 Discrete-time firm valuation and credit risk models

We are interested in Chapter 4 in calculating the value of each firm $n \in \{1, \ldots, N\}$, the probability of default, and the distortion of the loss (and its statistics) of the portfolio

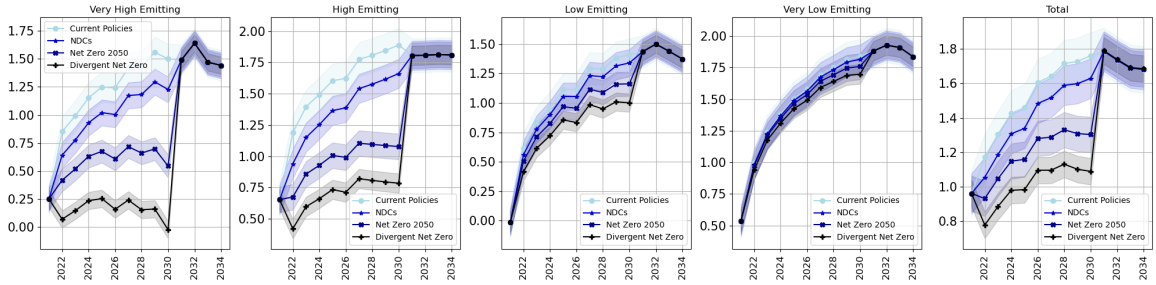


Figure 2.1: Mean and 95% confidence interval of annual production growth

in the presence of a carbon price in the economy that we have previously modeled. In order to define the loss, let us introduce the following assumptions.

Assumption 2.3.3. Consider a portfolio of $N \in \mathbb{N}^*$ credits. For all $1 \leq n \leq N$,

(1) Firm n has issued two categories of securities: shares and bonds;

(2) $(\mathrm{EAD}_t^n)_{t\in\mathbb{N}^*}$ is a deterministic process of $\mathbb{R}_*^+$, representing the outstanding debt at the time of default;

(3) $(\mathrm{LGD}_t^n)_{t\in\mathbb{N}^*}$ is a deterministic process of $(0,1]$, representing the expected percentage of the EAD that is lost in case of debtor default;

(4) the *over-indebtedness* barrier $\mathcal{D}^n \in \mathbb{R}^+$ is a deterministic scalar that we will use to define the conditions under which a borrower is considered to be in *over-indebtedness*.

(5) the value of firm n, denoted $\mathcal{V}_t^n$, at time t is assumed to be a tradable asset.

We can now define at any time $t \in \mathbb{N}$, the potential loss of a portfolio composed of N loans by

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbf{1}_{\{V_t^n \leq \mathcal{D}^n\}}. \tag{2.3.1}$$

In this equation, the event $\{\mathcal{V}_t^n \leq \mathcal{D}^n\}$ corresponds to the definition of *over-indebtedness* in structural models Merton [1974], Black and Cox [1976], and precisely to the risk of insolvency which occurs when the market value of the debt $\mathcal{D}^n$ exceeds the market value of the firm $\mathcal{V}_t^n$. Note that for simplicity, we have assumed here the value of the debt to be fixed.

The processes LGD and EAD being assumed deterministic as well as $\mathcal{D}^n$, the carbon price can only be transmitted from $\mathcal{V}^n$.

Firm Value

There are many methods/models for valuing a company: yield methods (evaluate the ability to generate profits in the future), comparative methods (compare to another

similar profile company that has recently been sold), and asset-based methods (evaluate the value of the company's assets). The literature on credit risk is sometimes vague about the type of method used, and does not always specify whether we are valuing assets, market value, or another quantity. However, mathematically, the most used models assume that the value of the firm follows a geometric Brownian motion (KMV [Merton Merton [1974]], CreditGrades [Black and Cox Black and Cox [1976]]) or that it depends on common systemic risk factors and an idiosyncratic risk factor (Bale II [Vasicek Vasicek [2002]]). Given that we want to analyze the impact of the carbon price on the value of a company, it is more interesting to use a microfounded model: a yield method is better suited.

For all $n \in \{1, \dots, N\}$ and for all $t \in \mathbb{N}$, the value of the company n, denoted V_t^n, is the conditional expectation of the discounted sum of all its future cash flows $(F_{t+s}^n)_{s\in\mathbb{N}}$. To determine the dynamics of cash flows, we note on the one hand that, for each company n belongs to a unique sector (of emissions) $i \in \mathcal{I}$, such that $n \in g_i$. Thus, the production/revenue of the company at date t is a fraction of the production Y_t^i of sector i at date t. Also, we assume that the cash flow of the company at date t is a fraction of the revenue of the company at date t. By writing the logarithmic growth of the company's cash flows in vector form, we therefore have the following assumption:

Assumption 2.3.4. For all $n \in \{1, \dots, N\}$, the process of growth of the cash flows of the company n with value $\mathbb{R}^N$, denoted $(\omega_t^n)_{t\in\mathbb{N}^*}$ is linear with respect to economic factors (precisely the logarithmic growth of production), mathematically, we have for all $t \in \mathbb{N}$,

$$\omega_t^n = \tilde{\mathfrak{a}}^{n\cdot}\Delta_t^Y + \mathfrak{b}_t^n = \mathfrak{a}^{n\cdot}\left(\Theta_t + \mathfrak{v}(\mathfrak{d}_t) - \mathfrak{v}(\mathfrak{d}_{t-1})\right) + \mathfrak{b}_t^n,$$

with $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$, $\mathfrak{a}^{n\cdot} := \tilde{\mathfrak{a}}^{n\cdot}(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}$, and where the idiosyncratic noise $(\mathfrak{b}_t)_{t\in\mathbb{N}} := (\mathfrak{b}_t^n)_{t\in\mathbb{N},1\le n\le N}$ is i.i.d. with law $\mathcal{N}(0, \mathrm{diag}(\sigma^2_{\mathfrak{b}^n}))$ such $\sigma_{\mathfrak{b}^n} > 0$ for $n \in \{1, \dots, N\}$. Moreover, $(\Delta_t^Y)_{t\in\mathbb{N}^*}$ and $(\mathfrak{b}_t)_{t\in\mathbb{N}}$ are independent.

Furthermore, by defining the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}}$ by $\mathcal{F}_t = \sigma\left(\mathcal{G}_t \cup \sigma\left\{\mathfrak{b}_s : s \in [0,t] \cap \mathbb{N}\right\}\right)$ for all $t \in \mathbb{N}$, we denote $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$. The firm value is therefore written

$$V_t^n := \mathbb{E}_t\left[\sum_{s=0}^{+\infty} e^{-rs} F_{t+s}^n\right].$$

We start by checking if this value is well defined in theorem 4.1.4. Then, we should calculate it. However, finding a simple expression of V is not obvious because of the autoregressive structure of productivity Θ (see Assumption 1.3.1). To move forward, we can use a numerical method like neural networks Hammad et al. [2022] or Picard iteration methods Berinde and Takens [2007]. However, from the noise parameter ε introduced when defining $(\mathcal{A}, \Theta)$ in assumption 1.3.1, we can introduce a proxy $\mathcal{V}$ to the firm value that we define as follows

$$\mathcal{V}_t^n := F_t^n\left(1 + \sum_{s=1}^{+\infty} e^{-rs}\mathbb{E}_t\left[\exp\left(\mathfrak{a}^{n\cdot}\left(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t) + s\overline{\mu}\right) + \sum_{u=1}^{s}\mathfrak{b}_{t+u}^n\right)\right]\right), \quad (2.3.2)$$

and which comes from the asymptotic development of $\frac{V_t^n}{F_t^n}$ to order 1. Thus, for all $n \in \{1, \dots, N\}$, under the conditions of proposition 4.1.7, for all $t \in \mathbb{N}$, V_t^n converges towards $\mathcal{V}_t^n$ when ε tends to 0. Subsequently, the terminology "firm value" will be used within a context that will allow the reader to understand if the term refers to $\mathcal{V}$ or V. Therefore, we obtain a simple expression (see Lemma 4.1.6) and we know how to determine its conditional laws. In particular, for all $t, T \in \mathbb{N}$,

$$\mathcal{V}_{t+T}^n | \mathcal{G}_t \sim \mathcal{LN}\left(\log(F_0^n) + \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t), \mathcal{L}^n(t, T)\right),$$

where $\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)$ and $\mathcal{L}^n(t, T)$ are given in (4.1.22) and (4.1.23) respectively.

We already obtain at this stage the deformation of the value of a company which belongs in the economy described in Chapter 3. It follows that (1) the more the growth of a company's cash flows is dependent on a polluting sector, the more its value decreases, (2) the harder the transition (high carbon price), the more the value of companies decreases, and (3) the distribution of the firm value deforms and translates in the presence of the carbon price.

Probability of default, expected and unexpected losses

Given that we now have the value of the firm (2.3.2), and that LGD and EAD are deterministic, we can return to the portfolio loss L^N (2.3.1). With the proxy of the firm value $\mathcal{V}$, it is now written as:

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}}.$$

In this expression, the productivity of the economy $(\mathcal{A}, \Theta)$ and the noise of cash flows $(\mathfrak{b}_t^n)_{t \in \mathbb{N}, 1 \leq n \leq N}$ are the two factors of randomness: the first is said to be *systemic* and the second is said to be *idiosyncratic*. However, for a given $t \in \mathbb{N}$, the random variables $(\mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}})_{n \in \{1, \dots, N\}}$ are non-independent and non-identically distributed Bernoulli laws. Therefore, describing analytically the law of L_t^N is difficult. But by adding a non-concentration assumption in the portfolio, we can use a result introduced by Gordy [2003b]. We obtain the following theorem.

Theorem 2.3.5. *If the assumptions introduced in 2.3.3 are satisfied and that for all $t \in \mathbb{N}$, the family $(\mathrm{EAD}_t^n)_{n=1,\dots,N}$ is a sequence of positive constants such that*

1. $\sum_{n \geq 1} \mathrm{EAD}_t^n = +\infty$;
2. *there exists $\upsilon > 0$ such that $\frac{\mathrm{EAD}_t^n}{\sum_{n=1}^N \mathrm{EAD}_t^n} = \mathcal{O}(N^{-(\frac{1}{2}+\upsilon)})$, when N tends to infinity.*

Then, $L_t^N - \mathrm{L}_t^{\mathbb{G},N}$ converges to zero almost surely when N tends to infinity, with

$$\mathrm{L}_t^{\mathbb{G},N} := \mathbb{E}\left[L_t^N \middle| \mathcal{G}_t\right] = \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbb{P}\left[\mathcal{V}_t^n \leq \mathcal{D}^n | \mathcal{G}_t\right]. \tag{2.3.3}$$

This means that at any date, when the portfolio is well diversified, we can approach, almost surely, the potential loss L_t^N by the expected loss conditionally on systemic factors $\mathrm{L}_t^{\mathbb{G},N}$. In the following, we replace L_t^N by $\mathrm{L}_t^{\mathbb{G},N}$. Although the distribution of the portfolio loss contains more information, we will focus on some of its statistics. Therefore, given the dates t and T, let us consider the risk measures below.

- The probability of default of the entity $n \in \{1, \dots, N\}$ between t and $t+T$ calculated at time t, we note

$$\mathrm{PD}_{t,T,\mathfrak{d}}^n := \mathbb{P}\left(\mathcal{V}_{t+T}^n \leq \mathcal{D}^n | \mathcal{G}_t\right).$$

- The expected loss of the portfolio (and sub-portfolios g_i, $i \in \mathcal{I}$) calculated at t and over the period $[t, t+T]$, we note

$$\mathrm{EL}_{t,\mathfrak{d}}^{N,T} := \mathbb{E}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \middle| \mathcal{G}_t\right]. \tag{2.3.4}$$

- The unexpected loss of the portfolio (and sub-portfolios g_i, $i \in \mathcal{I}$) calculated at t and over the period $[t, t+T]$, we note for $\alpha \in (0,1)$,

$$\mathrm{UL}_{t,\alpha,\mathfrak{d}}^{N,T} := \mathrm{VaR}_t^{\alpha,N,T} - \mathrm{EL}_{t,\mathfrak{d}}^{N,T}, \quad \text{where} \quad 1-\alpha = \mathbb{P}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} - \mathrm{L}_t^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]. \tag{2.3.5}$$

The proxy of the firm value $\mathcal{V}$ introduced in (2.3.2) and the proxy of the loss introduced in (2.3.3) allow us to obtain simple expressions of $(\mathrm{PD}_{t,T,\mathfrak{d}}^n)_{n\in\{1,\dots,N\}}$, $\mathrm{EL}_{t,\mathfrak{d}}^{N,T}$, and $\mathrm{UL}_{t,\alpha,\mathfrak{d}}^{N,T}$, in particular from the cumulative distribution function of the standard Gaussian Φ in (4.2.5), e, (4.2.6) and in (4.2.8) in particular. The presence of $\mathfrak{d}$ called rate of the cost of emissions (dimensionless vector which is the product of the carbon price and carbon intensities) and defined for all $t \in \mathbb{N}$ by

$$\mathfrak{d}_t := (\tau_t \delta_t, \zeta_t \delta_t, \kappa_t \delta_t),$$

allows us to insist on the dependence of our risk measures on the variables of the climate transition.

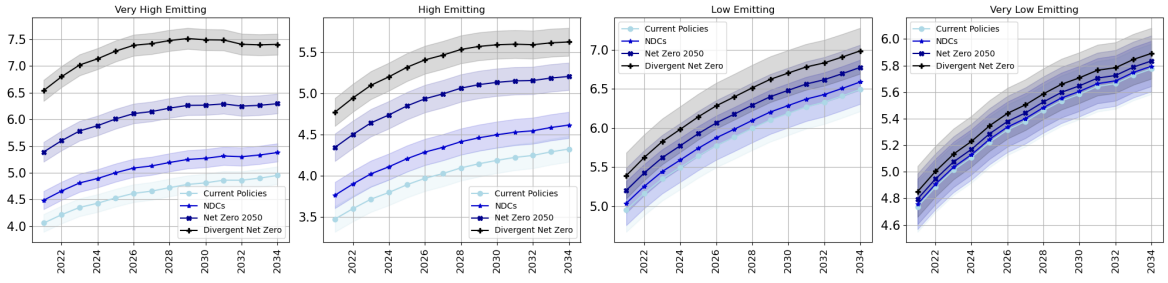


Figure 2.2: Annual probability of *default* by scenario and by sub-portfolio

Considering a 10-year transition starting in 2021 and ending in 2030 and using the Monte Carlo method to calculate risk measures, we find that the carbon price in the economy

1. increases the probability of default of companies and this increase is accentuated when the transition scenario hardens and/or when the company belongs to a very polluting sector (see Figure 2.2);

2. increases bank charges (materialized by the level of provisions calculated from the expected loss EL) charged to customers or by the operating income of banks;

3. and reduces the solvency of banks (translated by the economic capital calculated from the unexpected loss UL).

In this part, we assumed that the LGD was deterministic, constant, and independent of the carbon price. This is a strong assumption because the LGD (and therefore the portfolio loss) depends on assets and guarantees, which can be directly affected by the climate transition. We remove this assumption in the next part.

2.3.3 Impact of climate transition on credit portfolio's loss with stochastic collateral

In a bank's credit portfolio, there may exist secured loans (or collateralized loans) and unsecured loans. When a counterparty is put into *over-indebtedness*, the bank begins by liquidating the guarantees, if it has any, of the defaulting counterparty, and if this does not cover its exposure at the date of *over-indebtedness*, it undertakes recovery procedures by other means (liquidation of assets, legal proceedings, etc.) A guarantee, also called collateral, can be made up of tangible assets (buildings, professional equipment, stocks of goods, etc.) or intangible assets (cash deposits, public bonds, financial securities, etc.) In this chapter, we consider a tangible guarantee – a real estate property – and an intangible guarantee – financial asset (company shares or bonds). We obtained in the previous result (see (2.3.2)) that the value of a company can be affected by the climate transition and precisely by the carbon price, so the value of its shares or bonds will also be affected. Similarly, the price of commercial or residential buildings will be affected by the transition, for example through energy performance (or efficiency).

Moreover, it turns out that a continuous time modeling corresponds better to the real estate market. Also, it is more interesting for a bank to follow the evolution of its portfolio at more frequent frequencies. Hence, before modeling the loss with collateral, we start by extending the previous results in continuous time. The logarithmic growth of productivity now follows a multidimensional Ornstein-Uhlenbeck process instead of an autoregressive vector.

Standing Assumption 2.3.6. We define the processes $\mathcal{Z}$ and $\mathcal{A}$ with values $\mathbb{R}^I$ that evolve as follows

$$\begin{cases} \mathrm{d}\mathcal{Z}_t &= -\Gamma\mathcal{Z}_t\mathrm{d}t + \Sigma\mathrm{d}B_t^{\mathcal{Z}} \\ \mathrm{d}\mathcal{A}_t &= (\mu + \varsigma\mathcal{Z}_t)\,\mathrm{d}t \end{cases} \quad \text{for all } t \in \mathbb{R}^*,$$

where $(B_t^{\mathcal{Z}})_{t\mathbb{R}^*}$ is a Brownian motion in I dimensions, and where the constants $\mu, \mathcal{A}_0 \in \mathbb{R}^I$, the matrices $\Gamma, \Sigma \in \mathbb{R}^{I\times I}$, $\mathcal{Z}_0 \sim \mathcal{N}\left(0, \Sigma\Sigma^\top\right)$ and $0 < \varsigma \leq 1$ is a noise intensity parameter that is fixed: it will be used later to obtain an interesting approximation of the value of the company. In addition, Σ is a positive definite matrix and $-\Gamma$ is a Hurwitz matrix, i.e., its eigenvalues have strictly negative real parts.

We also introduce the following filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{R}^*}$ with $\mathcal{G}_0 := \sigma(\Theta_0)$ and for all $t > 0$, $\mathcal{G}_t := \sigma\left(\left\{\Theta_0, B_s^{\mathcal{Z}} : s \leq t\right\}\right)$. The carbon price evolves as follows.

Standing Assumption 2.3.7. We also introduce the carbon price δ and the carbon intensities τ, κ, ζ which are processes such that:

1. Given $0 \leq t_\circ < t_\star$. The carbon price δ satisfies
   - if $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in \mathbb{R}_+$, that is, it is constant;
   - if $t \in (t_\circ, t_\star)$, $\delta_t \in \mathbb{R}_+$, that is, it can evolve around a given deterministic and dynamic position;
   - for $t \geq t_\star$, $\delta_t = \delta_{t_\star} \in \mathbb{R}_+$, that is, it is constant.
2. The deterministic processes τ, ζ, and κ belong respectively to $\mathbb{R}_+^I$, $\mathbb{R}_+^{I\times I}$, and $\mathbb{R}_+^I$, and represent respectively the carbon intensities on the production of companies, on the intermediate consumption of companies, and on the consumption of households, and satisfying for all $t \in \mathbb{R}_+$ and for all $\mathfrak{y} \in \{\tau^1, \ldots, \tau^I, \zeta^{11}, \zeta^{12}, \ldots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \ldots, \kappa^I\}$,
$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t)}{\theta_{\mathfrak{y}}}\right) & \text{if } t \leq t_\star \\ \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_{\mathfrak{y}}t_\star)}{\theta_{\mathfrak{y}}}\right) & \text{otherwise,} \end{cases}$$
with $\mathfrak{y}_0, g_{\mathfrak{y},0}, \theta_{\mathfrak{y}} > 0$. For all $t \in \mathbb{N}$, we call $\mathfrak{y}_t\delta_t$ the *rate of cost of emissions* at time t.
3. For all $i \in \mathcal{I}$ and for all $t \in \mathbb{R}_+$,
$$\delta_t \max_{i\in\mathcal{I}} \tau_0^i < 1.$$
We note in the same way
$$\mathfrak{d}_t := (\tau_t\delta_t, \zeta_t\delta_t, \kappa_t\delta_t).$$

2.3.4 Impact of the Climate Transition on Credit Portfolio Losses with Stochastic Guarantees, in Continuous Time

Starting from assumptions 1.3.6 and 1.3.7, we rewrite the multisectoral model in continuous time. We solve it in appendix C.2 by drawing inspiration from the methods used in chapter 3. We thus obtain the $\mathbb{G}$-measurable processes in $\mathbb{R}^I$, Y, C, N, and

Z which represent respectively the production, consumption, labor, and intermediate inputs. These quantities depend on the processes $\mathfrak{d}$, $\mathcal{A}$, and the parameters of the models (elasticities and utility function in particular). To specify the dependence on $\mathfrak{d}$ and $\mathcal{A}$, we write for example for all $t \in \mathbb{R}_+$, $Y_t(\mathfrak{d}_t, \mathcal{A}_t) \in \mathbb{R}^d$ the production at date t.

Similarly, we write the value of any firm $n \in \{1, \dots, N\}$ at any date $t \in \mathbb{R}_+$,

$$V^n_{t,\mathfrak{d}} := \mathbb{E}_t \left[\int_t^{+\infty} e^{-rs} F^n_{s,\mathfrak{d}} \mathrm{d}s \right],$$

with

$$\mathrm{d} \log F^n_{t,\mathfrak{d}} = \mathfrak{a}^{n\cdot} \mathrm{d} \log \left(Y_t(\mathfrak{d}_t, \mathcal{A}_t) \right) + \sigma_n \mathrm{d}\mathcal{W}^n_t,$$

for $\mathfrak{a}^{n\cdot} \in \mathbb{R}^I$ and $(\mathcal{W}^n_t)_{t\in\mathbb{R}_+}$ is a $\mathbb{R}^N$-Brownian motion with $\sigma_n > 0$. In addition, $B^{\mathcal{Z}}$ and $\mathcal{W}^n$ are independent.

The conditions of existence and convergence must be slightly adapted because the parameters of the Ornstein-Uhlenbeck (in continuous) and the autoregressive vector (in discrete) are not the same. We then introduce a proxy of the firm value $\mathcal{V}$ by drawing inspiration from the discrete model and using the parameter ς introduced in assumption 1.3.6.

Let $n \in \{1, \dots, N\}$ and $t \in \mathbb{R}_+$. We denote the positive real process C the value of the guarantee (which is null if the loan is unsecured). Recall that there is *over-indebtedness* at date t if $\{\mathcal{V}^n_t \leq \mathcal{D}^n\}$. When there is *over-indebtedness*, the bank liquidates the counterparty within a delay of $a > 0$ and receives $(1-k)e^{-ra}C^n_{t+a}$. If the liquidations do not cover the entire debt, that is $\mathrm{EAD}^n_t \geq (1-k)e^{-ra}C^n_{t+a}$, the bank deploys other actions to recover an additional fraction denoted $\gamma \in [0,1]$. It therefore obtains in addition $\gamma(\mathrm{EAD}^n_t - (1-k)e^{-ra}C^n_{t+a})_+$. The portfolio loss is therefore written

$$L^N_t := \sum_{n=1}^N L_{n,t},$$

where $L_{n,t}$ is the loss related to the counterparty n at date t which is written

$$L_{n,t} := (1-\gamma)(\mathrm{EAD}^n_t - (1-k)e^{-ra}C^n_{t+a})_+ \cdot \mathbf{1}_{\{\mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{B}^n\}}.$$

With the same assumptions as those of theorem 2.3.5 (declined in continuous time), we introduce the expected loss conditional on systemic factors $\mathrm{L}^{\mathbb{G},N}_t$ as a proxy for the potential loss L^N_t when the portfolio is not concentrated. We have the following theorem:

Theorem 2.3.8 (and definition)**.** *For all $t \in \mathbb{R}_+$, we define*

$$\mathrm{L}^{\mathbb{G},N}_t := \sum_{n=1}^N L^{\mathbb{G}}_{n,t} \quad \textit{where} \quad L^{\mathbb{G}}_{n,t} = \mathbb{E}\left[L_{n,t} | \mathcal{G}_t\right] = \mathrm{EAD}^n_t \cdot \mathrm{LGD}^n_{t,\mathfrak{d}} \cdot \mathrm{PD}^n_{t,\mathfrak{d}},$$

with

$$\begin{aligned} \mathrm{PD}^n_{t,\mathfrak{d}} &:= \mathbb{P}\left(\mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{B}^n | \mathcal{G}_t\right), \\ \mathrm{LGD}^n_{t,\mathfrak{d}} &:= (1-\gamma)\mathbb{E}\left[\left(1 - (1-k)e^{-ra} \frac{C^n_{t+a}}{\mathrm{EAD}^n_t} \right)_+ \middle| F^n_{t,\mathfrak{d}} < \mathcal{D}^n_t, \mathcal{G}_t \right]. \end{aligned}$$

Then, $L_t^N - \mathrm{L}_t^{\mathbb{G},N}$ *converges to zero almost surely when* N *tends to infinity, for all* $t \in \mathbb{R}_+$.

We thus obtain in proposition 5.2.9 an expression of the probability of *default* $\mathrm{PD}_{t,\mathfrak{d}}^n$ as a function of Φ in the same way as in the discrete part. We focus on the LGD, and in particular, on the dynamics of the collateral for the firm n

If there is no collateral

We have $C_{t+a}^n = 0$, and therefore directly $\mathrm{LGD}_{t,\mathfrak{d}}^n = 1 - \gamma$.

If the Collateral is a Financial Asset

If the collateral is a financial asset such as a company's shares, it therefore represents a share denoted by $\alpha^n \in (0, 1]$. Like any investment, it must generate a cash flow so that at each instant, we can calculate its value using the discounted cash flow model introduced in (1.3.8). Therefore, the value of the collateral here is obtained in the same way as the value of any firm $n \in \{1, \ldots, N\}$ at any date $t \in \mathbb{R}_+$,

$$C_{t,\mathfrak{d}}^n := \alpha^n \mathbb{E}_t \left[\int_{s=t}^{+\infty} e^{-rs} \overline{F}_{s,\mathfrak{d}}^n \mathrm{d}s \right], \tag{2.3.7}$$

with

$$\mathrm{d} \log \overline{F}_{t,\mathfrak{d}}^n = \overline{\mathfrak{a}}^{n\cdot} \mathrm{d} \log \left(Y_t(\mathfrak{d}_t, \mathcal{A}_t) \right) + \overline{\sigma}_n \mathrm{d} \overline{\mathcal{W}}_t^n, \tag{2.3.8}$$

for $\overline{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ and where $(\overline{\mathcal{W}}_t^n)_{t \in \mathbb{R}_+}$ is a Brownian motion in $\mathbb{R}^N$ with $\overline{\sigma}_n > 0$. In addition, $B^{\mathcal{Z}}$ (noise of productivity), $\overline{\mathcal{W}}^n$ (noise of guarantees) and $\mathcal{W}^n$ (noise of debtors) are independent. Starting from the proxy $\mathcal{C}_{\cdot,\mathfrak{d}}^n$ (drawing inspiration once again from the discrete model and using the parameter ς introduced in assumption 1.3.6), we have the following theorem:

Theorem 2.3.9. *The Loss Given Default of the borrower* n *in over-indebtedness at time* $t \in \mathbb{R}_+$*, when the collateral is the financial asset described in* (2.3.8) *and* (2.3.7)*, conditionally on* $\mathcal{G}_t$ *is*

$$\mathrm{LGD}_{t,\mathfrak{d}}^n = (1 - \gamma) \left[\Phi \left(\frac{w_t^n}{\overline{\sigma}_{\mathfrak{b}_n} \sqrt{t}} \right) - \exp \left(-w_t^n + \frac{1}{2} t \overline{\sigma}_n^2 \right) \Phi \left(\frac{w_t^n}{\overline{\sigma}_n \sqrt{t}} - \overline{\sigma}_n \sqrt{t} \right) \right],$$

with

$$w_t^n := \log \left(\frac{\mathrm{EAD}_t^n}{(1-k)e^{-ra}} \right) - \mathbb{E}[\mathcal{C}_{t,\mathfrak{d}}^n | \mathcal{G}_t].$$

If the Collateral is Real Estate

Residential or commercial buildings are one of the largest emitters of GHGs. The Energy Performance of Buildings Directive (EPBD), introduced in 2002 by the European Commission and revised in 2010 and subsequently, is a key instrument for

increasing the energy performance of buildings in the EU. Like the carbon price, it is a means of implementing the climate transition. It consists, for example, of classifying buildings according to their energy efficiency (EE) using letters from A to G (where A is the most efficient and G the least). Recent studies show that the energy performance of a dwelling is beginning to have a similar importance to that of price or location. This is what we model.

For the housing that serves as collateral to the counterparty $n \in \{1, \dots, N\}$, we assume that:

1. in the absence of climate transition, the price of housing follows an exponential Ornstein-Uhlenbeck process;

2. each dwelling consumes a certain amount of energy per square meter, which is used to determine its energy efficiency, denoted α^n and expressed in kilowatts per square meter (KWh/m^2);

3. consequently, the price of housing is depreciated (or appreciated) by the discounted sum of future energy costs;

4. once a certain level of energy efficiency $\alpha^\star$ is reached, the market is insensitive to this factor;

5. the price of energy is a deterministic function of two variables $\mathfrak{f}$, the first variable is the carbon price and the second is the energy source,

6. each dwelling could be sold at a random date $T_n > 0$ (for example, following an exponential law with parameter $\lambda_n > 0$),

7. during the life of the property, it can undergo renovations that move the energy efficiency from α^n to $\alpha^\star$, and whose cost per square meter is a function $\mathfrak{c}$ of its energy efficiency,

8. the renovation date $\mathfrak{t}_n$ of a dwelling is unknown, but must be optimized.

9. after renovations, the price of the building becomes insensitive to energy costs.

Noting that real estate is an asset belonging to the economy described by 2.3.6 and 2.3.7, we have the following hypothesis.

Assumption 2.3.10. We place ourselves in a risk-neutral setting.

1. *Price of housing without climate transition:* the market value of the building indexed by n at $t \geq 0$ is given by

$$C_t^n := R_n C_0^n e^{K_t},$$

where

$$\begin{aligned} \mathrm{d}K_t &= \left(\frac{d\Psi}{dt} - \overline{\lambda}\overline{\sigma} + \nu(\Psi_t - K_t)\right)\mathrm{d}t + \overline{\sigma}\mathrm{d}\overline{B}_t, \\ \mathrm{d}\overline{B}_t &= \rho^\top \mathrm{d}B_t^{\mathcal{Z}} + \sqrt{1 - \|\rho\|^2}\mathrm{d}\overline{\mathcal{W}}_t, \end{aligned}$$

Where $(\overline{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion independent of $B^{\mathcal{Z}}$ while C_0^n, r, R_n, $\overline{\sigma}$, $\overline{\lambda} > 0$, $\rho \in \mathbb{R}_+^I$, and $\Psi \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$.

2. A building is an investment like any other. Its possession therefore generates a cash flow of income/dividends denoted $(D_t^n)_{t\geq 0}$ which is continuous and adapted to $\mathbb{U}$. Therefore, for all $t \geq 0$, another way to write the price C_t^n of the building is as follows :

$$C_t^n = R_n \mathbb{E}\left[\int_0^{+\infty} e^{-\bar{r}s} D_{t+s}^n \, ds \middle| \mathcal{U}_t\right],$$

with $\bar{r} > 0$ and $\mathbb{U} := (\mathcal{U}_t)_{t\in\mathbb{R}^*}$ the filtration such that for all $t \geq 0$, $\mathcal{U}_t := \sigma\left(\left\{\overline{W}_s, B_s^{\mathcal{Z}} : s \leq t\right\}\right)$

3. *Housing price with climate transition*: the market value of the building serving as collateral to company n at $t \geq 0$, given the carbon price sequence δ, is represented by

$$\mathcal{C}_{t,\delta}^n := R_n \operatorname*{ess\,sup}_{\theta\geq t} \mathbb{E}\left[\begin{array}{c} \int_t^\theta \left[D_s^n - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)}\mathrm{d}u - \mathfrak{c}(\alpha^n, \alpha^\star)e^{-\bar{r}(\theta-t)} \\ + \int_\theta^{+\infty} e^{-\bar{r}(s-t)} D_{t+s}^n \mathrm{d}s \end{array} \middle| \mathcal{U}_t\right],$$

where

- $\mathfrak{c}$ is a continuous function from $(\mathbb{R}_+)^2$ to $\mathbb{R}_+^*$,
- for each energy source $\mathfrak{p}$, $\mathfrak{f}(., \mathfrak{p})$ is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+^*$,
- α^n, $\alpha^\star > 0$ with $\alpha^n > \alpha^\star$,
- the filtration $\mathbb{U} := (\mathcal{U}_t)_{t\in\mathbb{R}^*}$ with for $t \geq 0$, $\mathcal{U}_t := \sigma\left(\left\{\overline{B}_s : s \leq t\right\}\right)$.

In the above assumption, the term $\left[D_u^n - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)}$ is the income at t of the property before the optimal renovation date $\mathfrak{t}_n$ while the term $D_u^n e^{-\bar{r}(u-t)}$ is the income at t of the property after $\mathfrak{t}_n$. In addition, the term $\mathfrak{c}(\alpha^n, \alpha^\star)e^{-r(\mathfrak{t}_n - t)}$ represents the discounted renovation costs that are carried out at time $\mathfrak{t}_n$. We finally note that for all $t \geq 0$, $C_t^n - \mathcal{C}_{t,\delta}^n \leq R_n\mathfrak{c}(\alpha^n, \alpha^\star)$, that is, the agent can always renovate the house immediately, but this may not be optimal.

As a result, under certain assumptions specified in Theorem 5.3.10, we can write the market value of the building as

$$\mathcal{C}_{t,\delta}^n = C_t^n - R_n X_{t,\delta}^n.$$

We show that $X^n_{t,\delta}$ is the climate transition shock on the building price. It depends especially on the optimal renovation date, on the renovation costs, on the carbon price (through the energy price), and on the energy efficiency of the building.

We finally use this expression to obtain the LGD for a real estate collateral by the following theorem

Theorem 2.3.11. *Given the carbon price process δ. The Loss Given Default of debtor n at date $t \in \mathbb{R}_+$, when its collateral is the property described in 2.3.10, conditionally on $\mathcal{G}_t$, is*

$$\mathrm{LGD}^n_{t,\alpha^n,\delta} = (1-\gamma)\left[\left(1+(1-k)e^{-ra}\frac{R_n X^n_t}{\mathrm{EAD}^n_t}\right)\Phi\left(\frac{w^n_t}{\sqrt{v^n_{t,t}}}\right) - \exp\left(-w^n_t + \frac{1}{2}v^n_{t,t}\right)\Phi\left(\frac{w^n_t}{\sqrt{v^n_{t,t}}} - \sqrt{v^n_{t,t}}\right)\right],$$

with

$$w^n_t := \log\left(\frac{\mathrm{EAD}^n_t}{(1-k)R_n e^{-ra}} + X^n_t\right) - \log\left(C^n_0\right) - \mathbb{E}[Y_t|\mathcal{G}_t].$$

We note $\mathrm{LGD}^n_{t,\alpha^n,\delta}$ (with $\alpha^n\delta$ in index) to specify the dependence of the LGD on the energy efficiency of the building α^n and the carbon price δ.

For a portfolio composed of N loans either unsecured, or secured by financial assets, or real estate shares, we write, for all t, T, the expected loss (respectively unexpected) between t and $t+T$, denoted $\mathrm{EL}^{N,T}_{t,\mathfrak{d}}$ introduced in (2.3.4) (respectively $\mathrm{UL}^{N,T}_{t,\alpha,\mathfrak{d}}$ introduced in (2.3.5)). It is enough to notice that the law of $\mathrm{L}^{\mathbb{G},N}_t|\mathcal{G}_t$ is a function of the law of $\mathcal{A}_{t+T}|\mathcal{G}_t$.

Finally, in the same way as in chapter 4, we consider 4 climate transition scenarios from 2021 to 2030. We also consider 4 sub-portfolios each representing a sector of the French economy divided into 4 sectors. We further assume that in the sub-portfolios, some loans are uncollateralized, while others are guaranteed either by financial assets or by real estate. We therefore calculate the various associated risk measures: PD, LGD, EL, and UL. We obtain, in addition to the results already obtained in chapter 4,

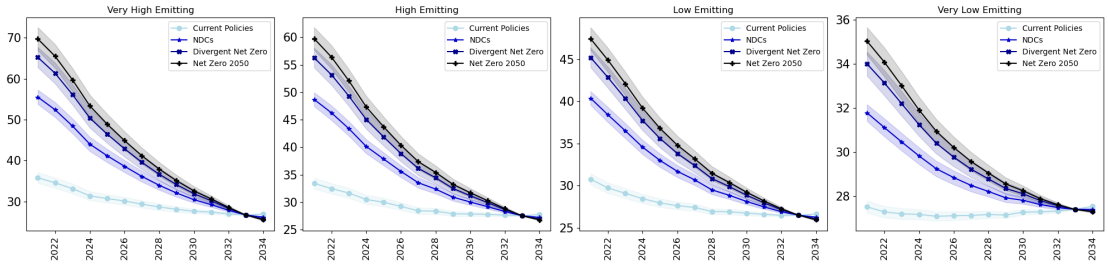


Figure 2.3: LGD with a real estate collateral

that

1. the presence of guarantees reduces LGD, EL, and UL;

2. when the transition scenario hardens, the impact of the presence of the collateral on LGD, EL, and UL is attenuated because the liquidation value of the guarantee decreases (example in Figure 2.3);

3. for the case of real estate, the optimal renovation date decreases when the transition scenario hardens.

4. When the optimal renovation date is reached, the decrease in the price of real estate stops, or even reverses depending on its energy efficiency.

All these models are each time calibrated on historical data and applied on real portfolios of the BPCE bank.

This work has very practical implications for BPCE's risk management. Precisely, EL being covered by the provisions coming from the fees charged to clients, an increase in EL due to the carbon price implies an increase in credit fees for clients and in operating costs for the bank. At the same time, UL is covered by the bank's economic and regulatory capital, one of the measures of its solvency, which comes from the shareholders. An increase in UL would therefore mean either a fall in solvency or an increase in capital. Finally, to reduce its exposure to the climate transition (by lowering EL and UL), the bank can decide to be less exposed, i.e. to grant less credit to polluting sectors and accept less polluting guarantees.

Part I

Loss without collateral in discrete time

Chapter 3

A Multisectoral Model with Carbon emissions costs

This chapter is based on the paper Bouveret et al. [2023][Section 2] under revisions.

We consider a closed economy with I various sectors subject to GHG emissions. In this chapter, our main goal is to derive the dynamics of macroeconomic variables (output, consumption, labour, and intermediary inputs) per sector. We thus use a discrete-time model with infinite time horizon. The main point here is that firms provoke GHG emissions when they consume intermediary inputs from other sectors and emit GHG when they produce the output. Likewise, households emit GHG when they consume. All these emissions are charged using a dynamic and deterministic carbon price. This will allow us in particular to describe the transition process towards a decarbonized economy.

Contents

We build a stochastic and multisectoral model where we introduce greenhouse gases (GHG) emissions costs which are the product of carbon prices, provided by the NGFS transition scenarios Network for Greening the Financial System (NGFS) [2020], and of GHG emissions from sectoral households' consumption or firms' production/consumption. This model helps us analyze the impact of a carbon price on sectoral production by firms and on households' consumption in each sector. We obtain that at the market equilibrium, the macroeconomic problem is reduced to a non-linear system of output and consumption. Moreover, when the households' utility function is logarithmic in consumption, output and consumption are uniquely defined and precisely described by productivity, the carbon price and the model parameters. Then, for each sector, we can determine labor and intermediary inputs using the relationship of the latter with output and consumption. The sectoral structure also allows us to quantify the interactions between sectors both in terms of productivity and carbon price. The model we build in this first step is close to the one developed in Golosov et al. [2014]. However, there are two main differences. Firstly, they obtain an optimal path for their endogenous carbon taxes while, in our case, the carbon price is exogenous. Secondly, the sectors in their model are allocated between sectors related to energy and a single sector representing the rest of the economy, while our model allows for any type of sectoral organization provided that a proper calibration of the involved parameters can be performed. In addition, our model is also close to the multisectoral model proposed in Devulder and Lisack [2020], with the difference that ours is dynamic and stochastic, and that we appeal to a Cobb Douglas production function instead of a Constant Elasticity of Substitution (CES) one. Moreover, the model developed in this first step also differs from the *REgional Model of Investment and Development* (REMIND) model described in Reis and Augusto [2013] as (1) it is a stochastic multisectoral model and (2) the productivity is exogenous. Finally, our model follows in the wake of the G-cubed version proposed in the NGFS (Network for Greening the Financial System) Occasional Paper Network for Greening the Financial System (NGFS) [2022]. G-Cubed is a hybrid Dynamic Stochastic and Computable General Equilibrium model that integrates emissions and energy data in a multi-sectoral and multi-regional economy, developed by McKibbin and Wilcoxen [2013]. However, compared with our model whose solutions are explicit and in closed form, G-cubed's complexity does not allow to obtain tractable macroeconomic trajectories for subsequent use in credit risk. Furthermore, the adaptations of G-cubed proposed in Network for Greening the Financial System (NGFS) [2022] only account for the direct CO2 emissions that companies produce when they consume fossil fuels, whereas we account here for direct, indirect, and sector-specific GHG emissions from companies as well as direct emissions from households. The setting is also inspired by basic classical monetary models presented in the seminal textbook by Galí [2015], and by Miranda-Pinto and Young [2019]'s sectoral model.

3.1 The model

We first consider two optimization problems: one for the representative firms and one for a representative household. We obtain first-order conditions, namely the optimal behavior of the firm and the consumer as a response to the various variables at hand. Then, relying on market clearing conditions, we derive the equations that the sectoral consumption and output processes must satisfy. Finally, in the last section, we solve these equations by making assumptions on the values taken by the set of involved parameters.

Each sector $i \in \mathcal{I}$ has a representative firm which produces a single good, so that we can associate sector, firm and good. The (log-)productivity process has stationary dynamics as stated in the following standing assumption.

Standing Assumption 3.1.1. We define the $\mathbb{R}^I$-valued processes Θ and $\mathcal{A}$ which evolve according to

$$\begin{cases} \mathcal{A}_t &= \mathcal{A}_{t-1} + \Theta_t, \\ \Theta_t &= \mu + \Gamma\Theta_{t-1} + \varepsilon\mathcal{E}_t, \end{cases} \quad \text{for all } t \in \mathbb{N}^*,$$

with the constants $\mu, \mathcal{A}_0 \in \mathbb{R}^I$ and where the matrix $\Gamma \in \mathbb{R}^{I\times I}$ has eigenvalues all strictly less than 1 in absolute value, $0 < \varepsilon \leq 1$ is an intensity of noise parameter that is fixed: it will be used in Section 4.1 to obtain a tractable proxy of the firm value.

Moreover, $(\mathcal{E}_t)_{t\in\mathbb{Z}}$ are independent and identically distributed (iid) with for $t \in \mathbb{Z}$, $\mathcal{E}_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$ with $\Sigma \in \mathbb{R}^{I\times I}$. We also have $\Theta_0 \sim \mathcal{N}(\overline{\mu}, \varepsilon^2\overline{\Sigma})$ with $\overline{\mu} := (\mathbf{I}_I - \Gamma)^{-1}\mu$, and $\mathrm{vec}(\overline{\Sigma}) := (\mathbf{I}_{I\times I} - \Gamma \bigotimes \Gamma)^{-1}\mathrm{vec}(\Sigma)$, where, for $M \in \mathbb{R}^{d\times d}$, $\mathrm{vec}(M) := [M^{11}, \ldots, M^{d1}, M^{21}, \ldots, M^{d2}, \ldots, M^{1d}, \ldots, M^{dd}]^\top$. The processes $(\mathcal{E}_t)_{t\in\mathbb{N}}$ and the random variable Θ_0 are independent.

We also introduce the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{N}}$ with $\mathcal{G}_0 := \sigma(\Theta_0)$ and for $t \geq 1$, $\mathcal{G} := \sigma\left(\{\Theta_0, \mathcal{E}_s : s \in (0, t] \cap \mathbb{N}^*\}\right)$.

The processes Θ^i and $\mathcal{A}^i$ play a major role in our factor productivity model since, for any $i \in \mathcal{I}$, the total factor productivity of sector i is defined as

$$A^i_t := \exp\left(\mathcal{A}^i_t\right),$$

so that Θ^i is the log-productivity growth and $\mathcal{A}^i$ is the cumulative log-productivity growth. In the rest of the paper, the terminology "productivity" will be used within a context that will allow the reader to understand if the term refers to Θ^i, $\mathcal{A}^i$, or A^i. To summarize, the log-productivity growth is a Vector Autoregressive (VAR) Process. The literature on VAR is rich, with detailed results and proofs in Hamilton [2020], or Kilian and Lütkepohl [2017]. We provide in A.1 additional results that will be useful later on.

Remark 3.1.2.

1. Obviously, for any $t \in \mathbb{N}$, $\mathcal{A}_t = \mathcal{A}_0 + \sum_{u=1}^{t} \Theta_u$. For later use, we define

$$\mathcal{A}_t^\circ := \mathcal{A}_t - \mathcal{A}_0,$$

and observe that $(\mathcal{A}_t^\circ, \Theta_t)_{t\geq 0}$ is a Markov process.

2. Since the eigenvalues of Γ are all strictly less than 1 in absolute value, $(\Theta_t)_{t\in\mathbb{N}}$ is wide-sense stationary i.e. for $t, u \in \mathbb{N}$, the first and the second orders moments ($\mathbb{E}[\Theta_t]$ and $\mathbb{E}[\Theta_t\Theta_{t+u}]$) do not depend on t. Then, given the law of Θ_0, we have for any $t \in \mathbb{N}$, $\Theta_t \sim \mathcal{N}(\overline{\mu}, \varepsilon^2\overline{\Sigma})$.

3. For later use, we also observe the following: let $\mathcal{Z}_0 \sim \mathcal{N}(0, \overline{\Sigma})$ s.t. $\Theta_0 = \overline{\mu} + \varepsilon\mathcal{Z}_0$ and for $t \geq 1$, $\mathcal{Z}_t = \Gamma\mathcal{Z}_{t-1} + \mathcal{E}_t$. Then

$$\Theta_t = \overline{\mu} + \varepsilon\mathcal{Z}_t \text{ and } \mathcal{Z}_t \sim \mathcal{N}(0, \overline{\Sigma}). \tag{3.1.1}$$

For the whole economy, we introduce a deterministic and exogenous carbon price in euro per ton. It allows us to model the impact of the transition pathways on the whole economy. We note δ the carbon price process and we shall then assume the following setting.

Standing Assumption 3.1.3. We introduce the carbon price and the carbon intensities (defined as the quantities of GHG in tons emitted for each euro of production/consumption) processes.

1. Let $0 \leq t_\circ < t_\star$ be given. The sequence δ satisfies
   - for $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in \mathbb{R}_+$, namely the carbon price is constant;
   - for $t \in (t_\circ, t_\star)$, $\delta_t \in \mathbb{R}_+$, the carbon price may evolve;
   - for $t \geq t_\star$, $\delta_t = \delta_{t_\star} \in \mathbb{R}_+$, namely the carbon price is constant.

2. We also introduce carbon intensities as the sequences τ, ζ, and κ being respectively $\mathbb{R}_+^I$, $\mathbb{R}_+^{I\times I}$, and $\mathbb{R}_+^I$-deterministic processes, and representing respectively carbon intensities on firm's output, on firm's intermediary consumption, and on household's consumption (expressed in ton of CO_2-equivalent per euro), and satisfying for all $\mathfrak{y} \in \{\tau^1, \ldots, \tau^I, \zeta^{11}, \zeta^{12}, \ldots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \ldots, \kappa^I\}$ and $t \in \mathbb{N}$,

$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_\mathfrak{y} t)}{\theta_\mathfrak{y}}\right) & \text{if } t \leq t_\star \\ \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0}\frac{1-\exp(-\theta_\mathfrak{y} t_\star)}{\theta_\mathfrak{y}}\right) & \text{else,} \end{cases} \tag{3.1.2}$$

with $\mathfrak{y}_0, -g_{\mathfrak{y},0}, \theta_\mathfrak{y} > 0$. For each $t \in \mathbb{N}$, we call $\mathfrak{y}_t\delta_t$ the *emissions cost rate* at time t.

3. For each $i \in \mathcal{I}$ and for each $t \in \mathbb{N}$,

$$\delta_t \max_{i\in\mathcal{I}} \tau_0^i < 1. \tag{3.1.3}$$

In the first item of the assumption above, we interpret $t_\circ$ as the start of the transition and $t_\star$ as its end. Before the transition, the carbon price is constant (possibly zero) around a stationary level. Then, at the beginning of the transition, which lasts over $(t_\circ, t_\star)$, the carbon price can be dynamic depending on the objectives we want to reach. After $t_\star$, the carbon price becomes constant again. The second item, inspired by the DICE model (see Nordhaus [1993], Traeger [2014]), means that the carbon intensity $\mathfrak{y}$ is exogenous and decreases by a rate $(g_{\mathfrak{y},\cdot})$ which also decreases[1]. Moreover, $\mathfrak{y}_0$ represents emissions per unit of output/consumption in the beginning, $g_{\mathfrak{y},0}$ the initial decarbonization rate, and $\theta_{\mathfrak{y}}$ the growth rate of the decarbonization rate. In the following, we will note for all $t \in \mathbb{N}$,

$$\mathfrak{d}_t := (\tau_t \delta_t, \zeta_t \delta_t, \kappa_t \delta_t). \tag{3.1.4}$$

We now describe the firm and household programs that will allow us to derive the necessary equations that must be satisfied by the output and consumption in each sector. The proposed framework assumes a representative firm in each sector which maximizes its profits by choosing, at each time and for a given productivity, the quantities of labor and intermediary inputs. This corresponds to a sequence of static problems. Then, a representative household solves a dynamic optimization problem to decide how to allocate its consumption expenditures among the different goods and hours worked and among the different sectors.

3.1.1 The firm's point of view

Aiming to work with a simple model, we follow [Galí, 2015, Chapter 2]. It then appears that the firm's problem corresponds to an optimization performed at each period, depending on the state of the world. This problem will depend, in particular, on the productivity and the carbon price process introduced above. Moreover, it will also depend on P^i and W^i, two $\mathbb{G}$-adapted positive stochastic processes representing respectively the price of good i and the wage paid in sector $i \in \mathcal{I}$. We start by considering the associated deterministic problem below, when time and randomness are fixed.

Solution for the deterministic problem We denote $\overline{a} \in (0, +\infty)^I$ the level of technology in each sector, $\overline{p} \in (0, \infty)^I$ the price of the goods produced by each sector, $\overline{w} \in (0, \infty)^I$ the nominal wage in each sector, $\overline{\tau} \in \mathbb{R}_+^I$ and $\overline{\zeta} \in \mathbb{R}_+^{I \times I}$ the carbon intensities of firms' production and consumption of goods, and $\overline{\delta}$ the carbon price. For $i \in \mathcal{I}$, we consider a representative firm of sector i, with technology described by the

[1] In fact, carbon intensities decrease in developed countries like France or the US while increase in developing/emerging countries such as India or Nigeria. For France, we plot on Figures A.4 and A.5), the evolution of carbon intensities of firms and of households per sector.

production function

$$\mathbb{R}_+ \times \mathbb{R}_+^I \ni (n, z) \mapsto F_{\overline{a}}^i(n, z) = \overline{a}^i n^{\psi^i} \prod_{j \in \mathcal{I}} (z^j)^{\boldsymbol{\lambda}^{ji}} \in \mathbb{R}_+,$$

where n represents the number of hours of work in the sector, and z^j the firm's consumption of the intermediary input produced by sector j. The coefficients $\psi \in (\mathbb{R}_+^*)^I$ and $\boldsymbol{\lambda} \in (\mathbb{R}_+^*)^{I \times I}$ are elasticities with respect to the corresponding inputs. Overall, we assume a constant return to scale, namely

$$\psi^i + \sum_{j \in \mathcal{I}} \boldsymbol{\lambda}^{ji} = 1, \qquad \text{for each } i \in \mathcal{I}. \tag{3.1.5}$$

The management of firm i then solves the classical problem of profit maximization

$$\widehat{\Pi}^i_{(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta})} := \sup_{(n,z) \in \mathbb{R}_+ \times \mathbb{R}_+^I} \Pi^i(n, z), \tag{3.1.6}$$

where, omitting the dependency in $(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta})$,

$$\Pi^i(n, z) := F_{\overline{a}}^i(n, z)\overline{p}^i - \overline{\tau}^i F_{\overline{a}}^i(n, z)\overline{p}^i\overline{\delta} - \overline{w}^i n - \sum_{j \in \mathcal{I}} z^j \overline{p}^j + z^j \overline{\zeta}^{ji} \overline{p}^j \overline{\delta}.$$

Remark 3.1.4. This definition of the representative firm's profit allows for the modeling of the costs of externalities that GHG emissions constitute. Indeed, for each sector i,

- $F_{\overline{a}}^i(n, z)$ represents the output in volume, $F_{\overline{a}}^i(n, z)\overline{p}^i$ the output in value (euro), and $\overline{\tau}^i F_{\overline{a}}^i(n, z)\overline{p}^i$ the GHG emissions, in tons, generated to produce the output. Consequently $\overline{\tau}^i F_{\overline{a}}^i(n, z)\overline{p}^i\overline{\delta}$ represents the firm's GHG emissions cost.
- For each sector j, z^j represents the intermediary input from j into i in volume, $z^j\overline{p}^j$ the intermediary input in value (euro), and $z^j\overline{\zeta}^{ji}\overline{p}^j$ the GHG emissions, in tons, generated when sector i consumes goods/services from j. Consequently $z^j\overline{\zeta}^{ji}\overline{p}^j\overline{\delta}$ represents the cost paid by firms i due to their emissions when they consume goods/services from j to produce

Note that $F_{\overline{a}}^i(n, z)\overline{p}^i$ represents the firm's gross revenues and $\overline{\tau}^i F_{\overline{a}}^i(n, z)\overline{p}^i\overline{\delta}$ represents the firm's GHG emissions cost[2], so that $F_{\overline{a}}^i(n, z)(1 - \overline{\tau}^i\overline{\delta})\overline{p}^i$ stands for the firm's revenues after emissions cost. Moreover, observe that $\overline{w}^i n$ characterizes the firm's total compensations and that $\sum_{j \in \mathcal{I}} z^j(1 + \overline{\zeta}^{ji}\overline{\delta})\overline{p}^j$ is the firm's total expenses in intermediary inputs whose emissions are also charged. Condition (3.1.3) in Standing Assumption 3.1.3 implies that $\overline{\tau}^i\overline{\delta} < 1$, therefore assures that firms do not spend all the revenues from their production into GHG emissions costs. Now, we would like to solve the optimization problem for the firms, namely determine the optimal demands $\mathfrak{n}$ and $\mathfrak{z}$ as functions of $(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta})$. Because we will lift these optimal quantities in a dynamical stochastic setting, we impose that they are expressed as measurable functions. We thus introduce:

[2] $F_{\overline{a}}^i(n, z)$ represents the output in volume, $F_{\overline{a}}^i(n, z)\overline{p}^i$ the output in value (euro), and $\overline{\tau}^i F_{\overline{a}}^i(n, z)\overline{p}^i$ the GHG emissions, in tons, generated to produce the output.

Definition 3.1.5. An admissible solution to problem (3.1.6) is a pair of measurable functions

$$(\mathfrak{n}, \mathfrak{z}) : (0,+\infty)^I \times (0,+\infty)^I \times (0,+\infty)^I \times \mathbb{R}_+^I \times \mathbb{R}_+^{I\times I} \times \mathbb{R}_+ \to [0,+\infty)^I \times [0,+\infty)^{I\times I},$$

such that, for each sector i, denoting $\overline{n} := \mathfrak{n}^i(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta})$ and $\overline{z} := \mathfrak{z}^{\cdot i}(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta})$,

$$F_{\overline{a}}^i(\overline{n}, \overline{z})(1 - \overline{\tau}^i\overline{\delta})\overline{p}^i - \overline{w}^i\overline{n} - \sum_{j\in\mathcal{I}} \overline{z}^j(1 + \overline{\zeta}^{ji}\overline{\delta})\overline{p}^j = \widehat{\Pi}^i_{(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta},\overline{\delta})},$$

and $F_{\overline{a}}^i(\overline{n}, \overline{z}) > 0$ (non-zero production), according to (3.1.6).

Remark 3.1.6.

1. The solution obviously depends also on the coefficients ψ and $\boldsymbol{\lambda}$. These are however fixed and we will not study the dependence of the solution with respect to them.
2. For each sector $i \in \mathcal{I}$, we assume a unique representative firm. Therefore, if the latter decide not to produce, then the whole sector will not produce either. In this case, as a fraction of its output is used as inputs for other sectors (goods market clearing conditions in Definition 3.1.10), those sectors will not be able to produce either. Hence the non-zero production hypothesis.

Proposition 3.1.7. *There exist admissible solutions in the sense of Definition 3.1.5. Any admissible solution is given by, for all $i \in \mathcal{I}$, $\mathfrak{n}^i > 0$ and for all $(i,j) \in \mathcal{I}^2$,*

$$\mathfrak{z}^{ji} = \frac{\boldsymbol{\lambda}^{ji}}{\psi^i} \frac{\overline{w}^i}{(1 + \overline{\zeta}^{ji}\overline{\delta})\overline{p}^j} \mathfrak{n}^i > 0. \tag{3.1.7}$$

Moreover, it holds that $\widehat{\Pi}^i_{(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta},\overline{\delta})} = 0$ (according to (3.1.6)) and

$$\mathfrak{n}^i = \psi^i F_{\overline{a}}^i(\mathfrak{n}^i, \mathfrak{z}^{\cdot i}) \frac{(1 - \overline{\tau}^i\overline{\delta})\overline{p}^i}{\overline{w}^i}, \tag{3.1.8a}$$

$$\mathfrak{z}^{ji} = \boldsymbol{\lambda}^{ji} F_{\overline{a}}^i(\mathfrak{n}^i, \mathfrak{z}^{\cdot i}) \frac{(1 - \overline{\tau}^i\overline{\delta})\overline{p}^i}{(1 + \overline{\zeta}^{ji}\overline{\delta})\overline{p}^j}. \tag{3.1.8b}$$

Proof. We study the optimization problem for the representative firm $i \in \mathcal{I}$. Since $\psi^i > 0$ and $\boldsymbol{\lambda}^{ji} > 0$, for all $j \in \mathcal{I}$, as soon as $n = 0$ or $z^j = 0$, for some $j \in \mathcal{I}$, the production is equal to 0. From problem (3.1.6), we obtain that necessarily $n \neq 0$ and $z^j \neq 0$ for all j in this case. So an admissible solution, which has non-zero production, has positive components.
Setting $\overline{n} = \mathfrak{n}^i(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta}) > 0$ and $\overline{z} = \mathfrak{z}^{\cdot i}(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}, \overline{\delta}) > 0$, the optimality of $(\overline{n}, \overline{z})$ yields

$$\partial_n \Pi^i(\overline{n}, \overline{z}) = 0 \text{ and for any } j \in \mathcal{I}, \quad \partial_{z^j} \Pi^i(\overline{n}, \overline{z}) = 0.$$

We then compute

$$\psi^i \frac{F_{\overline{a}}^i(\overline{n}, \overline{z})}{\overline{n}}(1 - \overline{\tau}^i\overline{\delta})\overline{p}^i - \overline{w}^i = 0 \text{ and for any } j \in \mathcal{I}, \quad \boldsymbol{\lambda}^{ji} \frac{F_{\overline{a}}^i(\overline{n}, \overline{z})}{\overline{z}^j}(1 - \overline{\tau}^i\overline{\delta})\overline{p}^i - (1 + \overline{\zeta}^{ji}\overline{\delta})\overline{p}^j = 0,$$

which leads to (3.1.7), (3.1.8a), and (3.1.8b). □

Dynamic setting In Section 3.1.3 below, we characterize the dynamics of the output and consumption processes using market equilibrium arguments. There, the optimal demand by the firm for intermediary inputs and labor is lifted to the stochastic setting where the admissible solutions then write as functions of the productivity, carbon price, goods/services prices, and wage processes, see Definition 3.1.10.

In addition, the dynamic aspect of the firm problem comes from productivity. An obvious extension to make the firm problem "strongly dynamic" would be to introduce capital as an additional factor of production, and then the law of motion of capital over the time(see McKibbin and Wilcoxen [2013], Miranda-Pinto and Young [2019]). A second possibility would be to extend to a New Keynesian model in such a way that between two dates t and $t+1$, firms adjust price levels with a fraction $1-\theta$ of reoptimizing firms and a fraction θ of non-reoptimizing firms with $\theta \in [0;1]$ (see Galí [2015][Chapter 3]. In addition to studying the impact of the carbon price on production, consumption, labor, and intermediate inputs, the inclusion of capital in the model will allow us to explore the effects of the carbon price on capital and investment. Extending to a Keynesian model will also allow for the analysis of the impact of the carbon price on aggregate prices and inflation rate.

3.1.2 The household's point of view

Let $(r_t)_{t\in\mathbb{N}}$ be the (exogenous) deterministic interest rate, valued in $\mathbb{R}_+$. At each time $t \in \mathbb{N}$ and for each sector $i \in \mathcal{I}$, in addition to the price P_t^i of the goods produced in sector i and the wage W_t^i paid in sector i, introduced at the beginning of Section 3.1.1, we denote

- C_t^i the quantity consumed of the single good in the sector i, valued in $\mathbb{R}_+^*$;
- H_t^i the number of hours of work in sector i, valued in $\mathbb{R}_+^*$.

We also introduce a time preference parameter $\beta \in [0,1)$ and a utility function $U : (0,\infty)^2 \to \mathbb{R}$ given, for $\varphi \geq 0$, by $U(x,y) := \frac{x^{1-\sigma}}{1-\sigma} - \frac{y^{1+\varphi}}{1+\varphi}$ if $\sigma \in [0,1)\cup(1,+\infty)$ and by $U(x,y) := \log(x) - \frac{y^{1+\varphi}}{1+\varphi}$, if $\sigma = 1$. We also suppose that

$$\mathfrak{P} := \sup_{t\in\mathbb{N}, i\in\mathcal{I}} \mathbb{E}\left[\left(\frac{P_t^i}{W_t^i}\right)^{1+\varphi}\right] < +\infty. \tag{3.1.9}$$

For any $C, H \in \mathscr{L}_+^1(\mathbb{G}, (0,\infty)^I)$, we introduce the wealth process

$$Q_t := (1+r_{t-1})Q_{t-1} + \sum_{i\in\mathcal{I}} W_t^i H_t^i - \sum_{i\in\mathcal{I}} P_t^i C_t^i - \sum_{i\in\mathcal{I}} \kappa_t^i P_t^i C_t^i \delta_t, \qquad \text{for any } t \geq 0,$$

with the convention $Q_{-1} := 0$ and $r_{-1} := 0$. Note that we do not indicate the dependence of Q upon C and H to alleviate the notations.

Remark 3.1.8. The budget constraint of the household allows for the modeling of the costs of externalities that GHG emissions constitute. Indeed, for each sector i and date t, C_t^i represents the household's consumption in the sector i in volume, $P_t^i C_t^i$ the household's consumption in value (euro), and $\kappa_t^i P_t^i C_t^i$ the GHG emissions, in tons, generated by households when they consume goods/service i. Consequently, $\kappa_t^i P_t^i C_t^i \delta_t$ is the cost paid by households due to their emissions when they consume goods i.

For $t \in \mathbb{N}$ and $i \in \mathcal{I}$, $P_t^i C_t^i$ represents the household's consumption in the sector i and $\kappa_t^i P_t^i C_t^i \delta_t$ is the cost paid by households due to their emissions when they consume goods i, so $\sum_{i\in\mathcal{I}} P_t^i C_t^i (1 + \kappa_t^i \delta_t)$ is the household's total expenses. Moreover, $W_t^i H_t^i$ is the household's labor income in the sector i, $(1 + r_{t-1})Q_{t-1}$ the household's capital income, and $(1 + r_{t-1})Q_{t-1} + \sum_{i\in\mathcal{I}} W_t^i H_t^i$ the household's total revenue.

We define $\mathscr{A}$ as the set of all couples (C, H) with $C, H \in \mathscr{L}_+^1(\mathbb{G}, (0,\infty)^I)$ such that

$$\begin{cases} \mathbb{E}\left[\sum_{i\in\mathcal{I}} \sum_{t=0}^{\infty} \beta^t |U(C_t^i, H_t^i)|\right] < \infty, \\ \lim_{T\uparrow\infty} \mathbb{E}[Q_T | \mathcal{G}_t] \geq 0, \qquad \text{for all } t \geq 0. \end{cases}$$

The second condition is a solvency constraint that prevents it from engaging in Ponzi-type schemes. The representative household consumes the I goods of the economy and provides labor to all the sectors. For any $(C, H) \in \mathscr{A}$, let

$$\mathcal{J}(C, H) := \sum_{i\in\mathcal{I}} \mathcal{J}_i(C^i, H^i), \qquad \text{with} \qquad \mathcal{J}_i(C^i, H^i) := \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(C_t^i, H_t^i)\right], \text{for all } i \in \mathcal{I}.$$

The representative household seeks to maximize its objective function by solving

$$\max_{(C,H)\in\mathscr{A}} \quad \mathcal{J}(C, H). \tag{3.1.10}$$

We choose above a separable utility function as Miranda-Pinto and Young [2019] do, meaning that the representative household optimizes its consumption and hours of work for each sector independently but under a global budget constraint. The following proposition provides first order conditions to (3.1.10).

Proposition 3.1.9. *Assume that* (3.1.10) *has a solution* $(C, H) \in \mathscr{A}$*. Then, for all* $i, j \in \mathcal{I}$*, the household's optimality condition reads, for any* $t \in \mathbb{N}$*,*

$$\frac{P_t^i}{W_t^i} = \frac{1}{1 + \kappa_t^i \delta_t} (H_t^i)^{-\varphi} (C_t^i)^{-\sigma}, \tag{3.1.11a}$$

$$\frac{P_t^i}{P_t^j} = \frac{1 + \kappa_t^j \delta_t}{1 + \kappa_t^i \delta_t} \left(\frac{C_t^i}{C_t^j}\right)^{-\sigma}. \tag{3.1.11b}$$

Note that the discrete-time processes C and H cannot hit zero by definition of $\mathscr{A}$, so that the quantities above are well defined.

Proof. Suppose that $\sigma = 1$. We first check that $\mathscr{A}$ is non empty. Assume that, for all $t \in \mathbb{N}$ and $i \in \mathcal{I}$, $\tilde{C}^i_t = 1$ and $\tilde{H}^i_t = \frac{P^i_t(1+\kappa^i_t\delta_t)}{W^i_t}$, then

$$\mathbb{E}\left[\sum_{i\in\mathcal{I}}\sum_{t=0}^{\infty}\beta^t|U(\tilde{C}^i_t,\tilde{H}^i_t)|\right] \leq \sum_{i\in\mathcal{I}}\sum_{t=0}^{\infty}\beta^t\left(1+\frac{1}{1+\varphi}\mathbb{E}\left[\left(\frac{P^i_t(1+\kappa^i_t\delta_t)}{W^i_t}\right)^{1+\varphi}\right]\right).$$
$$\leq \sum_{i\in\mathcal{I}}\sum_{t=0}^{\infty}\beta^t\left(1+\frac{\mathfrak{P}(1+\kappa^i_t\delta_t)^{1+\varphi}}{1+\varphi}\right) < +\infty,$$

using (3.1.9). We also observe that Q built from $\tilde{H}, \tilde{C}$ satisfies $Q_t = 0$, for $t \in \mathbb{N}$. Thus $(\tilde{H}, \tilde{C}) \in \mathscr{A}$. Let now $(\widehat{C}, \widehat{H}) \in \mathscr{A}$ be such that $\mathcal{J}(\widehat{C}, \widehat{H}) = \max_{(C,H)\in\mathscr{A}} \mathcal{J}(C, H)$. We fix $s \in \mathbb{N}$ and $i \in \mathcal{I}$. Let $\eta = \pm 1$, $0 < h < 1$, $A^s \in \mathcal{G}_s$, $\Delta^{(i,s)} := (\mathbf{1}_{\{i=k,s=t\}})_{k\in\mathcal{I},t\in\mathbb{N}}$ and $\theta^{(i,s)} := \frac{1}{2}(1 \wedge \frac{W^i_s}{P^i_s(1+\kappa^i_s\delta_s)})\hat{C}^i_s \wedge \hat{H}^i_s \wedge 1 > 0$. Set

$$\overline{C} := \widehat{C} + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\Delta^{(i,s)} \text{ and } \overline{H} := \widehat{H} + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\Delta^{(i,s)}\frac{P^i(1+\kappa^i\delta_s)}{W^i}.$$

We observe that for $(j,t) \neq (i,s)$, $\overline{C}^j_t = \widehat{C}^j_t$ and $\overline{H}^j_t = \widehat{H}^j_t$ and we compute

$$\overline{C}^i_s \geq \widehat{C}^i_s - \theta^{(i,s)} \geq \frac{1}{2}\widehat{C}^i_s > 0.$$

Similarly, we obtain $\overline{H}^i_s > 0$. We also observe that $\overline{C} \leq \frac{3}{2}\widehat{C}$ and $\overline{H} \leq \frac{3}{2}\widehat{H}$. Finally, we have

$$\sum_{j\in\mathcal{I}} W^j_t\overline{H}^j_t - \sum_{j\in\mathcal{I}} P^j_t(1+\kappa^j_t\delta_t)\overline{C}^j_t = \sum_{j\in\mathcal{I}} W^j_t\widehat{H}^j_t - \sum_{j\in\mathcal{I}} P^j_t(1+\kappa^j_t\delta_t)\widehat{C}^j_t.$$

This allows us to conclude that $(\overline{C}, \overline{H}) \in \mathscr{A}$.

We have, by optimality of $(\widehat{C}, \widehat{H})$,

$$\mathcal{J}(\widehat{C}, \widehat{H}) - \mathcal{J}(\overline{C}, \overline{H}) = \sum_{j\in\mathcal{I}} \mathcal{J}_j(\widehat{C}^j, \widehat{H}^j) - \sum_{j\in\mathcal{I}} \mathcal{J}_j(\overline{C}^j, \overline{H}^j) \geq 0.$$

However, for all $(t,j) \neq (s,i)$, $\overline{C}^j_t = \widehat{C}^j_t$ and $\overline{H}^j_t = \widehat{H}^j_t$, then

$$\mathbb{E}\left[\beta^s U(\widehat{C}^i_s, \widehat{H}^i_s)\right] - \mathbb{E}\left[\beta^s U\left(\widehat{C}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}, \widehat{H}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\right)\right] \geq 0,$$

i.e.

$$\frac{1}{h}\mathbb{E}\left[U(\widehat{C}^i_s, \widehat{H}^i_s) - U\left(\widehat{C}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}, \widehat{H}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\right)\right] \geq 0.$$

Letting h tend to 0, we obtain

$$\mathbb{E}\left[\eta\theta^{(i,s)}\mathbf{1}_{A^s}\frac{\partial U}{\partial x}(\widehat{C}^i_s, \widehat{H}^i_s) + \eta\theta^{(i,s)}\mathbf{1}_{A^s}\frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\frac{\partial U}{\partial y}(\widehat{C}^i_s, \widehat{H}^i_s)\right] \geq 0.$$

Since the above holds for all $A^s \in \mathcal{G}_s$, $\eta = \pm 1$ and since $\theta^{(i,s)} > 0$, then

$$\frac{\partial U}{\partial x}(\widehat{C}^i_s, \widehat{H}^i_s) + \frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\frac{\partial U}{\partial y}(\widehat{C}^i_s, \widehat{H}^i_s) = 0,$$

leading to (3.1.11a).
For $j \in \mathcal{I} \setminus \{i\}$ and $\theta^{(i,j,s)} := \frac{1}{2}\left(1 \wedge \frac{P^j_s(1+\kappa^j_s\delta_s)}{P^i_s(1+\kappa^i_s\delta_s)}\right)(1 \wedge \widehat{C}^i_s \wedge \widehat{C}^j_s) > 0$, setting now

$$\overline{C} := \widehat{C} + \eta h \mathbf{1}_{A^s}\theta^{(i,j,s)}\left(\Delta^{(i,s)} - \Delta^{(j,s)}\frac{P^i(1+\kappa^i\delta_s)}{P^j(1+\kappa^j\delta_s)}\right) \quad \text{and} \quad \overline{H} := \widehat{H},$$

and using similar arguments as above, we obtain (3.1.11b).
When $\sigma \neq 1$, we carry out an analogous proof. □

3.1.3 Market equilibrium

We now consider that firms and households interact on the labor and goods markets.

Definition 3.1.10. A market equilibrium is a $\mathbb{G}$-adapted positive random process $(\overline{W}, \overline{P})$ such that

1. Condition (3.1.9) holds true for $(\overline{W}, \overline{P})$.

2. The goods' and labor's market clearing conditions are met, namely, for each sector $i \in \mathcal{I}$, and for all $t \in \mathbb{N}$,

$$Y^i_t = C^i_t + \sum_{j\in\mathcal{I}} Z^{ij}_t \qquad \text{and} \qquad H^i_t = N^i_t,$$

where $N_t = \overline{n}(A_t, \overline{W}_t, \overline{P}_t, \kappa_t, \zeta_t, \delta_t)$, $Z_t = \overline{z}(A_t, \overline{W}_t, \overline{P}_t, \kappa_t, \zeta_t, \delta_t)$, $Y = F_A(N, Z)$ with $(\overline{n}, \overline{z})$ an admissible solution (3.1.8a)-(3.1.8b) to (3.1.6), from Proposition 3.1.7 while C and H satisfy (3.1.11a)-(3.1.11b) for $(\overline{W}, \overline{P})$.

In the case of the existence of a market equilibrium, we can derive equations that must be satisfied by the output process Y and the consumption process C.

Proposition 3.1.11. *Assume that there exists a market equilibrium as in Definition 3.1.10. Then, for $t \in \mathbb{N}$, $i \in \mathcal{I}$, it must hold that*

$$\begin{cases} Y^i_t &= C^i_t + \sum_{j\in\mathcal{I}} \Lambda^{ij}(\mathfrak{d}_t)\left(\frac{C^j_t}{C^i_t}\right)^{-\sigma} Y^j_t, \\ Y^i_t &= A^i_t\left[\Psi^i(\mathfrak{d}_t)(C^i_t)^{-\sigma}Y^i_t\right]^{\frac{\psi^i}{1+\varphi}} \prod_{j\in\mathcal{I}}\left[\Lambda^{ji}(\mathfrak{d}_t)\left(\frac{C^i_t}{C^j_t}\right)^{-\sigma} Y^i_t\right]^{\boldsymbol{\lambda}^{ji}}, \end{cases} \tag{3.1.12}$$

where $\mathfrak{d}_t$ *is defined in* (3.1.4) *and where* Ψ *and* Λ *are given, for* $(\overline{\tau}, \overline{\zeta}, \overline{\kappa}, \overline{\delta}) \in \mathbb{R}_+^I \times \mathbb{R}_+^{I\times I} \times \mathbb{R}_+^I \times \mathbb{R}_+$*, by*

$$\Psi(\overline{\mathfrak{d}}) := \left(\psi^i \frac{1-\overline{\tau}^i\overline{\delta}}{1+\overline{\kappa}^i\overline{\delta}}\right)_{i\in\mathcal{I}}, \tag{3.1.13}$$

$$\Lambda(\overline{\mathfrak{d}}) := \left(\boldsymbol{\lambda}^{ji} \frac{1-\overline{\tau}^i\overline{\delta}}{1+\overline{\zeta}^{ji}\overline{\delta}} \frac{1+\overline{\kappa}^j\overline{\delta}}{1+\overline{\kappa}^i\overline{\delta}}\right)_{j,i\in\mathcal{I}}, \tag{3.1.14}$$

with $\overline{\mathfrak{d}} := (\overline{\tau}\overline{\delta}, \overline{\zeta\delta}, \overline{\kappa}\overline{\delta})$.

Proof. Let $i, j \in \mathcal{I}$ and $t \in \mathbb{N}$. Combining Proposition 3.1.7 and Proposition 3.1.9, we obtain

$$Z_t^{ji} = \boldsymbol{\lambda}^{ji} \frac{1-\tau_t^i\delta_t}{1+\zeta_t^{ji}\delta_t} \frac{1+\kappa_t^j\delta_t}{1+\kappa_t^i\delta_t} \left(\frac{C_t^i}{C_t^j}\right)^{-\sigma} Y_t^i. \tag{3.1.15}$$

From Propositions 3.1.7 and 3.1.9 again, we also have

$$N_t^i = \psi^i \frac{1-\tau_t^i\delta_t}{1+\kappa_t^i\delta_t} (H_t^i)^{-\varphi} (C_t^i)^{-\sigma} Y_t^i.$$

The labor market clearing condition in Definition 3.1.10 yields

$$N_t^i = \left[\psi^i \frac{1-\tau_t^i\delta_t}{1+\kappa_t^i\delta_t} (C_t^i)^{-\sigma} Y_t^i\right]^{\frac{1}{1+\varphi}}. \tag{3.1.16}$$

By inserting the expression of N_t^i given in (3.1.16)and Z_t^{ji} given in (3.1.15) into the production function F, we obtain the second equation in (3.1.12).

The first equation in (3.1.12) is obtained by combining the market clearing condition with (3.1.15) (at index (i, j) instead of (j, i)). □

3.1.4 Output and consumption dynamics and associated growth

For each time $t \in \mathbb{N}$ and noise realization, the system (3.1.12) is nonlinear with $2I$ equations and $2I$ variables, and its well-posedness is hence relatively involved. Moreover, it is computationally heavy to solve this system for each carbon price trajectory and productivity scenario. We thus consider a special value for the parameter σ which allows to derive a unique solution in closed form. From now on, and following [Golosov et al., 2014, page 63], we assume that $\sigma = 1$, namely $U(x, y) := \log(x) - \frac{y^{1+\varphi}}{1+\varphi}$ on $(0, \infty)^2$.

Theorem 3.1.12. *Assume that*

1. $\sigma = 1$,
2. $\mathbf{I}_I - \boldsymbol{\lambda}$ *is not singular,*

3. $\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top$ *is not singular for all* $t \in \mathbb{N}$.

Then for all $t \in \mathbb{N}$*, there exists an unique* (C_t, Y_t) *satisfying* (3.1.12)*. Moreover, with* $\mathfrak{e}_t^i := \frac{Y_t^i}{C_t^i}$ *for* $i \in \mathcal{I}$*, we have*

$$\mathfrak{e}_t = \mathfrak{e}(\mathfrak{d}_t) := (\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top)^{-1}\mathbf{1}, \tag{3.1.17}$$

and using $\mathcal{B}_t = (\mathcal{B}_t^i)_{i\in\mathcal{I}} := [\mathcal{A}_t^i + v^i(\mathfrak{d}_t)]_{i\in\mathcal{I}}$ *with*

$$v^i(\mathfrak{d}_t) := \log\left((\mathfrak{e}_t^i)^{-\frac{\varphi\psi^i}{1+\varphi}}\left(\Psi^i(\mathfrak{d}_t)\right)^{\frac{\psi^i}{1+\varphi}}\prod_{j\in\mathcal{I}}\left(\Lambda^{ji}(\mathfrak{d}_t)\right)^{\boldsymbol{\lambda}^{ji}}\right), \tag{3.1.18}$$

and $\mathfrak{d}_t$ *defined in*(3.1.4)*. We obtain*

$$C_t = \exp\left((\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\mathcal{B}_t\right). \tag{3.1.19}$$

Proof. Let $t \in \mathbb{N}$. When $\sigma = 1$, the system (3.1.12) becomes for all $i \in \mathcal{I}$,

$$\begin{cases} Y_t^i &= C_t^i + \sum_{j\in\mathcal{I}}\Lambda^{ij}(\mathfrak{d}_t)\left(\frac{C_t^i}{C_t^j}\right)Y_t^j, \\ Y_t^i &= A_t^i\left[\Psi^i(\mathfrak{d}_t)\mathfrak{e}_t^i\right]^{\frac{\psi^i}{1+\varphi}}\prod_{j\in\mathcal{I}}\left[\Lambda^{ji}(\mathfrak{d}_t)C_t^j\mathfrak{e}_t^i\right]^{\boldsymbol{\lambda}^{ji}}. \end{cases} \tag{3.1.20}$$

For any $i \in \mathcal{I}$, dividing the first equation in (3.1.20) by C_t^i, we get

$$\mathfrak{e}_t^i = 1 + \sum_{j\in\mathcal{I}}\Lambda^{ij}(\mathfrak{d}_t)\mathfrak{e}_t^j,$$

which corresponds to (3.1.17), thanks to (3.1.5). Using $\sum_{j\in\mathcal{I}}\boldsymbol{\lambda}^{ji} = 1-\psi^i$ and $Y_t^i = \mathfrak{e}_t^i C_t^i$ in the second equation in (3.1.20), we compute

$$C_t^i = A_t^i(\mathfrak{e}_t^i)^{-\frac{\varphi\psi^i}{1+\varphi}}\left[\Psi^i(\mathfrak{d}_t)\right]^{\frac{\psi^i}{1+\varphi}}\prod_{j\in\mathcal{I}}\left[\Lambda^{ji}(\mathfrak{d}_t)\right]^{\boldsymbol{\lambda}^{ji}}\prod_{j\in\mathcal{I}}(C_t^j)^{\boldsymbol{\lambda}^{ji}}.$$

Applying log and writing in matrix form, we obtain $(\mathbf{I}_I - \boldsymbol{\lambda})\log(C_t) = \mathcal{B}_t$, implying (3.1.19). □

Corollary 3.1.13 (Output, labor, and intermediary inputs)**.** *With the same assumptions, for all* $t \in \mathbb{N}$*, the output is*

$$Y_t = \mathrm{Diag}(\mathfrak{e}_t)C_t, \tag{3.1.21}$$

the labor is

$$N_t = \left[\mathrm{Diag}(\Psi(\mathfrak{d}_t))\mathfrak{e}_t\right]^{\frac{1}{1+\varphi}}, \tag{3.1.22}$$

and the intermediary inputs

$$Z_t = \Lambda(\mathfrak{d}_t) \odot C_t\mathfrak{e}_t^\top. \tag{3.1.23}$$

Proof. For $t \in \mathbb{N}$ and $i, j \in \mathcal{I}$, Y_t^i comes from the definition of $\mathfrak{e}$ in (3.1.17), while N_t^i (respectively Z_t^{ji}) comes directly from (3.1.16) (respectively (3.1.15)) by setting $\sigma = 1$. □

Remark 3.1.14. The matrix $\boldsymbol{\lambda}$ is generally not diagonal, and therefore, from (3.1.19), the sectors (in output and in consumption) are linked to each other through their respective productivity process. Similarly, charging carbon emissions of one sector affects all the other ones.

Remark 3.1.15. For any $t \in \mathbb{N}$, $i \in \mathcal{I}$, we observe that

$$\mathcal{B}_t^i = \mathcal{A}_t^i + v^i(\mathfrak{d}_t),$$

where $v^i(\cdot)$ is defined using (3.1.18). Namely, $\mathcal{B}_t$ is the sum of the (random) productivity term and a term involving the carbon intensities as well as the carbon price. The economy is therefore subject to fluctuations of two different natures: *the first one comes from the productivity process while the second one comes from the carbon price process.*

We now look at the growths of the macroeconomics variables.

Proposition 3.1.16. *For any $t \in \mathbb{N}^*$, let $\Delta_t^{\varpi} := \log(\varpi_t) - \log(\varpi_{t-1})$, for $\varpi \in \{Y, C, N, Z\}$. Then, with the same assumptions as in Theorem 3.1.12, the consumption growth is*

$$\Delta_t^C = (\mathbf{I}_I - \boldsymbol{\lambda})^{-1} \left[\Theta_t + v(\mathfrak{d}_t) - v(\mathfrak{d}_{t-1})\right], \tag{3.1.24}$$

where v defined in (3.1.18)*. The output growth is*

$$\Delta_t^Y = (\mathbf{I}_I - \boldsymbol{\lambda})^{-1} \left[\Theta_t + \mathfrak{v}(\mathfrak{d}_t) - \mathfrak{v}(\mathfrak{d}_{t-1})\right], \tag{3.1.25}$$

where $\mathfrak{v}(\mathfrak{d}_t) = v(\mathfrak{d}_t) + (\mathbf{I}_I - \boldsymbol{\lambda}) \log(\mathfrak{e}_t)$. The labor growth is

$$\Delta_t^N = \frac{1}{1+\varphi} \left[\log(\Psi(\mathfrak{d}_t)) - \log(\Psi(\mathfrak{d}_{t-1})) + \log(\mathfrak{e}_t) - \log(\mathfrak{e}_{t-1})\right], \tag{3.1.26}$$

and the intermediary inputs growth is

$$\Delta_t^Z = (\mathbf{I}_I - \boldsymbol{\lambda})^{-1} \left[\Theta_t \mathbf{1}^\top + v^Z(\mathfrak{d}_t) - v^Z(\mathfrak{d}_{t-1})\right], \tag{3.1.27}$$

where $v^Z(\mathfrak{d}_t) = v(\mathfrak{d}_t)\mathbf{1}^\top + (\mathbf{I}_I - \boldsymbol{\lambda})\mathbf{1} \log(\mathfrak{e}_t)^\top + (\mathbf{I}_I - \boldsymbol{\lambda}) \log \Lambda(\mathfrak{d}_t)$.

Proof. Let $t \in \mathbb{N}^*$, we have from (3.1.19),

$$\begin{aligned}
\Delta_t^C &= (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\mathcal{B}_t - (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\mathcal{B}_{t-1}, \\
&= (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}(\mathcal{B}_t - \mathcal{B}_{t-1}) \\
&= (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}(\mathcal{A} + v(\mathfrak{d}_t) - \mathcal{A}_{t-1} - v(\mathfrak{d}_{t-1})),
\end{aligned}$$

and (3.1.24) follows. From (3.1.21),

$$\begin{aligned}\Delta_t^Y &= \log\left(\mathrm{Diag}(\mathfrak{e}_t)C_t\right) - \log\left(\mathrm{Diag}(\mathfrak{e}_{t-1})C_{t-1}\right),\\ &= \Delta_t^C + \log\left(\mathfrak{e}_t\right) - \log\left(\mathfrak{e}_{t-1}\right),\\ &= (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\left[\Theta_t + v(\mathfrak{d}_t) - v(\mathfrak{d}_{t-1})\right] + \log\left(\mathfrak{e}_t\right) - \log\left(\mathfrak{e}_{t-1}\right),\end{aligned}$$

and by posing $v^Y(\mathfrak{d}_t) = v(\mathfrak{d}_t) + (\mathbf{I}_I - \boldsymbol{\lambda})\log\left(\mathfrak{e}_t\right)$, (3.1.25) follows. From (3.1.23),

$$\begin{aligned}\Delta_t^Z &= \log\Lambda(\mathfrak{d}_t) \odot C_t\mathfrak{e}_t^\top - \log\Lambda(\mathfrak{d}_{t-1}) \odot C_{t-1}\mathfrak{e}_{t-1}^\top,\\ &= \log\Lambda(\mathfrak{d}_t) - \log\Lambda(\mathfrak{d}_{t-1}) + \Delta_t^C\mathbf{1}^\top + \mathbf{1}(\log\mathfrak{e}_t - \log\mathfrak{e}_{t-1})^\top,\\ &= \log\Lambda(\mathfrak{d}_t) - \log\Lambda(\mathfrak{d}_{t-1}) + \mathbf{1}(\log\mathfrak{e}_t - \log\mathfrak{e}_{t-1})^\top + (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\left[\Theta_t + v(\mathfrak{d}_t) - v(\mathfrak{d}_{t-1})\right]\mathbf{1}^\top,\end{aligned}$$

and by posing $v^Z(\mathfrak{d}_t) = v(\mathfrak{d}_t)\mathbf{1}^\top + (\mathbf{I}_I - \boldsymbol{\lambda})\mathbf{1}\log\left(\mathfrak{e}_t\right)^\top + (\mathbf{I}_I - \boldsymbol{\lambda})\log\Lambda(\mathfrak{d}_t)$, (3.1.27) follows. Finally, (3.1.26) is directly derived from (3.1.22). □

We are now interested in the laws of logarithmic growth of random variables. Since the carbon price is deterministic, these laws directly come from the growth expressions and from the stationary distribution of the log-productivity growth.

Theorem 3.1.17. *Let us recall that $\overline{\mu}$ and $\varepsilon^2\overline{\Sigma}$ are the mean and the variance of the stationary process Θ (Remark 3.1.2), v, $\mathfrak{v}$ and v^Z are defined in Proposition 3.1.16 and $\mathfrak{e}$ in* (3.1.17)*. For any $t \in \mathbb{N}^*$, we have*

1. *With the same assumptions as in Theorem 3.1.12, we have for $\varpi \in \{Y, C\}$.*

$$\Delta_t^\varpi \sim \mathcal{N}\left(m_t^\varpi, \widehat{\Sigma}\right), \qquad \text{for } \varpi \in \{Y, C\},$$

with

$$\begin{aligned}\widehat{\Sigma} &= \varepsilon^2(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\overline{\Sigma}(\mathbf{I}_I - \boldsymbol{\lambda}^\top)^{-1},\\ m_t^C &= (I - \boldsymbol{\lambda})^{-1}\left[\overline{\mu} + v(\mathfrak{d}_t) - v(\mathfrak{d}_{t-1})\right],\\ m_t^Y &= (I - \boldsymbol{\lambda})^{-1}\left[\overline{\mu} + \mathfrak{v}(\mathfrak{d}_t) - \mathfrak{v}(\mathfrak{d}_{t-1})\right].\end{aligned}$$

2. *Δ_t^N is deterministic.*

3. *We also have*

$$\frac{1}{I}\Delta_t^Z\mathbf{1} \sim \mathcal{N}\left(m_t^Z, \widehat{\Sigma}\right),$$

with

$$m_t^Z = (I - \boldsymbol{\lambda})^{-1}\left[\overline{\mu} + \frac{1}{I}\left(v^Z(\mathfrak{d}_t) - v^Z(\mathfrak{d}_{t-1})\right)\mathbf{1}\right].$$

Proof. For the two first items, we use Proposition 3.1.16 introducing the logarithmic growth with Remark 3.1.2 giving the stationary law of Θ. For the last one, we remark that for a vector $V \in \mathbb{R}^I$, we have $V\mathbf{1}^\top\mathbf{1} = IV$. □

From the previous result, we observe that output and consumption growth processes have a stationary variance but a time-dependent mean. Moreover, output growth and consumption growth have the same variance because $\mathfrak{e}$ is deterministic (3.1.17), the latter being a consequence of the goods market clearing conditions. In the context of our standing assumption 3.1.3, we can also make the following observation:

Corollary 3.1.18. *Let $t \in \mathbb{N}^*$. If $t \leq t_\circ$ (before the transition scenario), the carbon price is zero or $t \geq t_\star$ (after the transition), the carbon price is constant, with the same assumptions as in Theorem 3.1.12, we have*

$$\Delta_t^C = \Delta_t^Y = (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\Theta_t, \tag{3.1.28}$$

while

$$\Delta_t^N = 0, \quad \textit{and} \quad \Delta_t^Z = (\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\Theta_t \mathbf{1}^\top.$$

Standing Assumption 3.1.3, Theorem 3.1.17, and Corollary 3.1.18 show that our economy follows three regimes:

- Before the climate transition, if the carbon price is zero, the economy is a stationary state led by productivity.
- During the transition, the economy is in a transitory state led by productivity and carbon price.
- After the transition, we reach constant carbon price and carbon intensities, therefore the economy returns in a stationary state ruled by productivity.

Furthermore, when the carbon price/intensities are constant, the labor growth which here does not depend on the productivity growth is zero: this is a consequence of the clearing of the labor market (the number of hours of work asked by firms are equal to the number of hours of work offered by households) as well as the logarithmic form of the utility function in the consumption (namely $\sigma = 1$).

A possible extension at this step would be to consider a case where the carbon price is stochastic. This would better correspond to the reality given the agents do not know a priori what is the trajectory of the transition. Le Guenedal and Tankov [2022b] consider this point by taking a doubly stochastic Poisson process to model the uncertainty of the climate policy on firm value dynamics. If we do not change any assumption in our setting except the carbon price that we consider stochastic, only Theorem 3.1.17, which gives the laws of the logarithm growths, will change. We must distinguish between two cases:

1. If the carbon price δ and the productivity Θ are independent, then the laws given Theorem 3.1.17 should simply conditioned on the carbon price, namely for $t \in \mathbb{N}^*$, $\Delta_t^\varpi|(\delta_0, \ldots, \delta_t) \sim \mathcal{N}\left(m_t^\varpi, \widehat{\Sigma}\right)$ for $\varpi \in \{Y, C, \frac{1}{I} Z\mathbf{1}\}$ and $\Delta_t^N|(\delta_0, \ldots, \delta_t)$ remains deterministic. Moreover, it is impossible to obtain the complete law due to the form of v.

2. if, on the other hand δ and Θ are not independent (we could think for example that an increasing of the carbon price will decrease the productivity), therefore, it is impossible to obtain either the complete or the conditional law also due to the form of v.

In both cases, it is always possible to run simulations, even if this may prove less useful for our later uses.

3.1.5 Sensitivities of the growths to carbon price

Let $t \in \mathbb{N}$, we are interested here in the sensitivity of the carbon price to the macroeconomic variables. For example, if one decides to charge only the direct GHG emissions of firms of a single sector i, we would like to analyze the effects on output, labor, consumption, and intermediary inputs of sector i, but also on any sector $j \in \mathcal{I} - \{i\}$. Let introduce first the following lemma.

Lemma 3.1.19.

$$\frac{\partial v(\mathfrak{d})}{\partial \delta} = -\frac{\varphi}{1+\varphi}\mathrm{Diag}(\psi)\frac{\partial}{\partial \overline{\delta}}\log\left(\mathfrak{e}(\mathfrak{d})\right) + \frac{1}{1+\varphi}\mathrm{Diag}(\psi)\frac{\partial}{\partial \overline{\delta}}\log\left(\Psi(\mathfrak{d})\right) + \mathrm{Diag}(\lambda^\top \frac{\partial}{\partial \overline{\delta}}\log\Lambda(\mathfrak{d})),$$

where

$$\frac{\partial \log \Psi(\mathfrak{d})}{\partial \delta} = \left(-\frac{\overline{\tau}^i}{1-\overline{\tau}^i\overline{\delta}} - \frac{\overline{\kappa}^i}{1+\overline{\kappa}^i\overline{\delta}}\right)_{i\in\mathcal{I}}$$

$$\frac{\partial \log \Lambda(\mathfrak{d})}{\partial \delta} = \left(-\frac{\overline{\tau}^i}{1-\overline{\tau}^i\overline{\delta}} - \frac{\overline{\zeta}_t^{ji}}{1+\overline{\zeta}^{ji}\overline{\delta}} + \frac{\overline{\kappa}^j}{1+\overline{\kappa}^j\overline{\delta}} - \frac{\overline{\kappa}^i}{1+\overline{\kappa}^i\overline{\delta}}\right)_{j,i\in\mathcal{I}},$$

and

$$\frac{\partial \log \mathfrak{e}(\mathfrak{d})}{\partial \delta} = -\mathrm{Diag}(\mathfrak{e}(\mathfrak{d}))^{-1}(\mathbf{I}_I - \Lambda(\mathfrak{d})^\top)^{-1}\left(\mathbf{I}_I - \frac{\partial}{\partial \overline{\delta}}\Lambda(\mathfrak{d})^\top\right)(\mathbf{I}_I - \Lambda(\mathfrak{d})^\top)^{-1}\mathbf{1}.$$

Proof. Let From (3.1.13),

$$\log\Psi(\overline{\mathfrak{d}}) = \left(\log\psi^i + \log\left(1-\overline{\tau}^i\overline{\delta}\right) - \log\left(1+\overline{\kappa}^i\overline{\delta}\right)\right)_{i\in\mathcal{I}},$$

therefore

$$\frac{\partial \log \Psi(\mathfrak{d})}{\partial \overline{\delta}} = \left(-\frac{\overline{\tau}^i}{1-\overline{\tau}^i\overline{\delta}} - \frac{\overline{\kappa}^i}{1+\overline{\kappa}^i\overline{\delta}}\right)_{i\in\mathcal{I}} = -\mathrm{Diag}\left(\frac{\overline{\tau}}{1-\overline{\tau}\overline{\delta}} + \frac{\overline{\kappa}}{1+\overline{\kappa}\overline{\delta}}\right)$$

From (3.1.14),

$$\log\Lambda(\overline{\mathfrak{d}}) = \left(\log\boldsymbol{\lambda}^{ji} + \log\left(1-\overline{\tau}^i\overline{\delta}\right) - \log\left(1+\overline{\zeta}^{ji}\overline{\delta}\right) + \log\left(1+\overline{\kappa}^j\overline{\delta}\right) - \log\left(1+\overline{\kappa}^i\overline{\delta}\right)\right)_{j,i\in\mathcal{I}},$$

therefore

$$\begin{aligned}\frac{\partial\Lambda(\mathfrak{d})}{\partial\overline{\delta}} &= \left(\boldsymbol{\lambda}^{ji}\frac{-\overline{\tau}^i(1+\overline{\kappa}^j\overline{\delta})}{(1+\overline{\zeta}^{ji}\overline{\delta})(1+\overline{\kappa}^i\overline{\delta})} + \boldsymbol{\lambda}^{ji}\frac{\overline{\kappa}^j(1-\overline{\tau}^i\overline{\delta})}{(1+\overline{\zeta}^{ji}\overline{\delta})(1+\overline{\kappa}^i\overline{\delta})}\right.\\ &\qquad\left.-\boldsymbol{\lambda}^{ji}\overline{\zeta}^{ji}\frac{(1-\overline{\tau}^i\overline{\delta})(1+\overline{\kappa}^j\overline{\delta})}{(1+\overline{\zeta}^{ji}\overline{\delta})^2(1+\overline{\kappa}^i\overline{\delta})} - \boldsymbol{\lambda}^{ji}\overline{\kappa}^i\frac{(1-\overline{\tau}^i\overline{\delta})(1+\overline{\kappa}^j\overline{\delta})}{(1+\overline{\zeta}^{ji}\overline{\delta})(1+\overline{\kappa}^i\overline{\delta})^2}\right)_{j,i\in\mathcal{I}}\\ &= \left(\Lambda(\mathfrak{d})^{ji}\left[-\frac{\overline{\tau}^i}{1-\overline{\tau}^i\overline{\delta}} - \frac{\overline{\zeta}^{ji}}{1+\overline{\zeta}^{ji}\overline{\delta}} + \frac{\overline{\kappa}^j}{1+\overline{\kappa}^j\overline{\delta}} - \frac{\overline{\kappa}^i}{1+\overline{\kappa}^i\overline{\delta}}\right]\right)_{j,i\in\mathcal{I}}\end{aligned}$$

and

$$\frac{\partial\log\Lambda(\mathfrak{d})}{\partial\overline{\delta}} = \left(-\frac{\overline{\tau}^i}{1-\overline{\tau}^i\overline{\delta}} - \frac{\overline{\zeta}_t^{ji}}{1+\overline{\zeta}^{ji}\overline{\delta}} + \frac{\overline{\kappa}^j}{1+\overline{\kappa}^j\overline{\delta}} - \frac{\overline{\kappa}^i}{1+\overline{\kappa}^i\overline{\delta}}\right)_{j,i\in\mathcal{I}}.$$

We then have

$$\frac{\partial\log\mathfrak{e}(\mathfrak{d})}{\partial\delta} = \left(\frac{\partial\log\mathfrak{e}^i(\mathfrak{d})}{\partial\delta}\right)_{i\in\mathcal{I}} = \left(\frac{1}{\mathfrak{e}^i(\mathfrak{d})}\frac{\partial\mathfrak{e}^i(\mathfrak{d})}{\partial\overline{\delta}}\right)_{i\in\mathcal{I}} = \mathrm{Diag}(\mathfrak{e}(\mathfrak{d}))^{-1}\frac{\partial\mathfrak{e}(\mathfrak{d})}{\partial\overline{\delta}}.$$

From (3.1.17),

$$\frac{\partial\mathfrak{e}(\mathfrak{d})}{\partial\overline{\delta}} = \frac{\partial}{\partial\overline{\delta}}(\mathbf{I}_I-\Lambda(\mathfrak{d})^\top)^{-1}\mathbf{1} = -(\mathbf{I}_I-\Lambda(\mathfrak{d})^\top)^{-1}(\mathbf{I}_I-\frac{\partial}{\partial\overline{\delta}}\Lambda(\mathfrak{d})^\top)(\mathbf{I}_I-\Lambda(\mathfrak{d})^\top)^{-1}\mathbf{1}.$$

We write v define in (3.1.18) in vectorial way, we have

$$v(\mathfrak{d}) = -\frac{\varphi}{1+\varphi}\mathrm{Diag}(\psi)\log\left(\mathfrak{e}(\mathfrak{d})\right) + \frac{1}{1+\varphi}\mathrm{Diag}(\psi)\log\left(\Psi(\mathfrak{d})\right) + \mathrm{Diag}(\boldsymbol{\lambda}^\top\log\Lambda(\mathfrak{d})),$$

the conclusion follows. □

We can then compute the sensitivity of the growths to the carbon price. It should be understood at time and randomness fixed. We have the following proposition, whose the proof is straithfoward using and Proposition 3.1.16 and Lemma 3.1.19.

Proposition 3.1.20. *For any $t \in \mathbb{N}^*$, with the same assumptions as in Proposition 3.1.16, the sensitivity to the consumption growth is*

$$\frac{\partial}{\partial\delta}\Delta_\cdot^C = (\mathbf{I}_I-\boldsymbol{\lambda})^{-1}\frac{\partial}{\partial\overline{\delta}}v(\mathfrak{d}),$$

the sensitivity to the output is

$$\frac{\partial}{\partial\delta_t}\Delta_\cdot^Y = (\mathbf{I}_I-\boldsymbol{\lambda})^{-1}\frac{\partial}{\partial\overline{\delta}}v(\mathfrak{d}) + \frac{\partial}{\partial\overline{\delta}}\log\left(\mathfrak{e}(\mathfrak{d})\right),$$

the sensitivity to the labor is

$$\frac{\partial}{\partial\delta_t}\Delta_\cdot^N = \frac{1}{1+\varphi}\left[\frac{\partial}{\partial\overline{\delta}}\log\left(\Psi(\mathfrak{d})\right) + \frac{\partial}{\partial\overline{\delta}}\log\left(\mathfrak{e}(\mathfrak{d})\right)\right],$$

and the sensitivity to the intermediary inputs

$$\frac{\partial}{\partial\delta_t}\Delta_\cdot^Z = \frac{\partial}{\partial\overline{\delta}}\log\Lambda(\mathfrak{d}_t) + \mathbf{1}\frac{\partial}{\partial\overline{\delta}}\log\left(\mathfrak{e}(\mathfrak{d})\right)^\top + \left((\mathbf{I}_I-\boldsymbol{\lambda})^{-1}\frac{\partial}{\partial\overline{\delta}}v(\mathfrak{d})\right)\mathbf{1}^\top.$$

Unless a particular form is given to $\boldsymbol{\lambda}$ and $\Lambda(\mathfrak{d})$ (and this would require an economic justification as much on the elasticities as on the intensities and the carbon price), we cannot obtain a precise form for $(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}$ ans $(\mathbf{I}_I - \Lambda(\mathfrak{d}))^{-1}$. It therefore seems quite difficult to analytically describe the impacts of the carbon price. Otherwise, we would have liked to quantify ideally and precisely the impacts on all the macroeconomic variables and in all the sectors if:

- only the direct GHG emissions of one only sector are charged;
- the GHG emissions of household when they consume goods/services from one only sector are charged;
- the (indirect) GHG emissions of firms when they consume goods/services from one only sector are charged.

In the absence of an analytical result, we will settle for numerical simulations later.

3.1.6 Evolution of the sectoral contributions and of the GHG emissions

We conclude our modeling with two measures that allow us to analyze the impact of the carbon price on the relative evolution of sectors on one hand, and on GHG emissions on the other hand.

Evolution of the sectoral contributions

The aim is to calculate for each variable and for each date, the share of each sector relative to the sum of all sectors. We obtain the following corollary which directly derives from Theorem 3.1.12 and Corollary 3.1.13.

Corollary 3.1.21. *With the same assumptions as in Theorem 3.1.12, for $\varpi \in \{Y, C, N\}$, for any $t \in \mathbb{N}$ the vector of the contributions of ϖ relative the whole economy writes*

$$\mathfrak{P}(\varpi_t) = \frac{1}{\mathbf{1}^\top \varpi_t} \varpi_t.$$

This measure allows us to verify how the carbon price affects the distribution of economic variables relatively to the whole economy. For example, does the share of firms production in the most polluting sectors decrease? We illustrate this example in Figure 3.4.

Greenhouse gases

The objective of the carbon price is to reduce GHG emissions or at least the carbon intensity in the economy. Since we are working on the French economy in which carbon intensities are decreasing as a trend, it is appropriate to check GHG emissions behave in our model. We obtain them in the following definition.

Definition 3.1.22. With the same assumptions as in Theorem 3.1.12, for any $t \in \mathbb{N}$, the vector quantity of GHGs generated by companies during their production is

$$Y_t \odot \tau_t.$$

The vector quantity of GHGs generated by companies when they consume intermediate inputs is

$$Z_t \odot \zeta_t.$$

The vector quantity of GHGs generated by households when they consume is

$$C_t \odot \kappa_t.$$

Since price is no longer included in the formula, these quantities are expressed in ton per euros. We illustrate an example in Figure 3.5

3.2 Estimation and calibration

Assume that the time unit is year. We will calibrate the model parameters on a set of data ranging from year $\mathfrak{t}_0$ to $\mathfrak{t}_1$. In practice, $\mathfrak{t}_0 = 1978$ and $\mathfrak{t}_1 = t_\circ = 2021$. For each sector $i \in \mathcal{I}$ and $\mathfrak{t}_0 \leq t < \mathfrak{t}_1 = t_\circ$, we observe the output Y_t^i, the labor N_t^i, the aggregate price P_t^i, the intermediary inputs $(Z_t^{ji})_{j\in\mathcal{I}}$, and the consumption C_t^i (recall that the transition starts at year $t_\circ$). For the sake of clarity, we will omit the dependence of each estimated parameter on $\mathfrak{t}_0$ and $\mathfrak{t}_1$.

3.2.1 Definition of carbon price

We assume here that the carbon price is deterministic. The regulator fixes the transition time horizon $t_\star \in \mathbb{N}^*$, the carbon price at the beginning of the transition $\delta_{t_\circ} > 0$, at the end of the transition $\delta_{t_\star} > \delta_{t_\circ}$, and the annual evolution rate $\eta_\delta > 0$. Then, for all $t \in \mathbb{N}$,

$$\delta_t = \begin{cases} \delta_{t_\circ}, & \text{if } t < t_\circ, \\ \delta_{t_\circ}(1+\eta_\delta)^{t-t_\circ}, & \text{if } t \in \{t_\circ, \dots, t_\star\}, \\ \delta_{t_\star} = \delta_{t_\circ}(1+\eta_\delta)^{t_\star - t_\circ}, & \text{otherwise.} \end{cases}$$

Time $t = t_\circ$ is the first year of the transition. Moreover, we assume that the carbon price increases continuously between $t_\circ$ to $t_\star$. However, there are several scenarios that could be considered, including a price that would increase until a certain year before leveling off or even decreasing. The framework can be adapted to various sectors as well as scenarios.

3.2.2 Calibration of carbon intensities

Note that GHG emissions is in tonnes of CO2-equivalent while output and consumption are in euros. For each time $\mathfrak{t}_0 \leq t \leq \mathfrak{t}_1$ and for all sector $i \in \mathcal{I}$, we compute the carbon intensities.

- If $E_t^{i,F}$ is the GHG emissions (similar to Scope 1 emissions) by all the firms of the sector i at t, then the carbon intensity on firm's production is set such that

$$\tau_t^i = \frac{E_t^{i,F}}{Y_t^i P_t^i}. \tag{3.2.1}$$

- If $E_t^{i,\mathcal{I}}$ is the GHG emitted by households through their consumption in sector i at t, then the carbon intensity on households final consumption is set such that

$$\kappa_t^i = \frac{E_{i,t}^H}{P_t^i C_t^i}. \tag{3.2.2}$$

- If $E_t^{ji,\mathcal{I}}$ is the GHG emitted by firm in sector i through their consumption in sector j at t, then the carbon intensity on firms' intermediary consumption, for each sector i and j, is set such that

$$\zeta_t^{ji} = \frac{E_t^{ji,F}}{P_t^j Z_t^{ji}}. \tag{3.2.3}$$

  To obtain "intermediary emissions", we first estimate the indirect emissions of each sector by using the input-output analysis as Desnos et al. [2023] and by assuming that the indirect emissions are proportional to the contribution of j to i.

For each $\varphi \in \{\tau^1, \ldots, \tau^I, \zeta^{11}, \zeta^{12}, \ldots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \ldots, \kappa^I\}$, we have the realized carbon intensity from (3.2.1), (3.2.2), or (3.2.3). Therefore, the calibration of $\varphi_0, g_{\varphi,0}$, and θ_φ will appeal to (3.1.2) in Standing Assumption 3.1.3. More precisely, by applying the log function to (3.1.2), we get

$$\log \varphi_t = \frac{g_{\varphi,0}}{\theta_\varphi} + \log \varphi_0 - \frac{g_{\varphi,0}}{\theta_\varphi} \exp(-\theta_\varphi t).$$

If we then set $g_t := \log \varphi_t - \log \varphi_{t-1} = \frac{g_{\varphi,0}}{\theta_\varphi}(\exp \theta_\varphi - 1) \exp(-\theta_\varphi t)$ and recall that $g_{\varphi,0}, \theta_\varphi > 0$, we compute, after applying the log function,

$$\log g_t = \log \frac{g_{\varphi,0}}{\theta_\varphi}(\exp \theta_\varphi - 1) - \theta_\varphi t.$$

We can therefore obtain θ_φ and $g_{\varphi,0}$ thanks to the ordinary least squares regression of $\log g_t$ on t, as well as $\widehat{\varphi_0} = \frac{\sum_{t=\mathfrak{t}_0}^{\mathfrak{t}_1} \varphi_t \exp\left[\widehat{g_{\varphi,0}} \frac{1-\exp(-\widehat{\theta_\varphi} t)}{\widehat{\theta_\varphi}}\right]}{\sum_{t=\mathfrak{t}_0}^{\mathfrak{t}_1} \exp\left[2\widehat{g_{\varphi,0}} \frac{1-\exp(-\widehat{\theta_\varphi} t)}{\widehat{\theta_\varphi}}\right]}$ thanks to the least squares optimization on (3.1.2).

3.2.3 Calibration of economic parameters

As in Galí [2015], we assume a unitary Frisch elasticity of labor supply so $\varphi = 1$ and the utility of consumption is logarithmic so $\sigma = 1$. Similarly, for any $i, j \in \mathcal{I}$, the shares of inputs, $\boldsymbol{\lambda}^{ij}$, are estimated as euro payments from sector j to sector i expressed as a fraction of the value of production in sector j. The parameter ψ^i is estimated as euro compensation in sector i expressed as a fraction of the value of production in sector i. This gives us $(\widehat{\boldsymbol{\lambda}}^{ij})_{i,j\in\mathcal{I}}$ and $(\widehat{\psi^i})_{i\in\mathcal{I}}$. We can then compute the functions Ψ in (3.1.13) and Λ in (3.1.14). We can also compute the sectoral output growth $\left(\Delta_t^Y = (\log(Y_t^i) - \log(Y_{t-1}^i))_{j\in\mathcal{I}}\right)_{t\in\mathfrak{t}_0,\dots,\mathfrak{t}_1-1}$ directly from data.

When the carbon price is zero, the carbon emissions rate $\mathfrak{v}_t$ is zero for $\mathfrak{t}_0 \leq t \leq \mathfrak{t}_1$. It then follows from (3.1.28) in Corollary 3.1.18 that, for each $t \in \{\mathfrak{t}_0, \dots, \mathfrak{t}_1 - 1\}$, the computed output growth Δ_t^Y is equal to $\Delta_t^Y = (\mathbf{I}_I - \widehat{\boldsymbol{\lambda}})^{-1}\widehat{\Theta}_t$ when $\mathbf{I}_I - \widehat{\boldsymbol{\lambda}}$ is not singular. Hence, $\widehat{\Theta}_t = (\mathbf{I}_I - \widehat{\boldsymbol{\lambda}})\Delta_t^Y$ and we can easily compute the estimations $\widehat{\mu}$, $\widehat{\Gamma}$, and $\widehat{\Sigma}$, and then $\widehat{\overline{\mu}}$ and $\widehat{\overline{\Sigma}}$ of the VAR(1) parameters μ, Γ, Σ, $\overline{\mu}$, and $\overline{\Sigma}$ (all defined in Standing Assumption 3.1.1). We check by the same token that $\widehat{\Theta}$ follows a VAR stationary process.

3.3 Results and discussion

3.3.1 Data

We work on data related to the French economy:

1. Annual consumption, labor, output (displayed on Figure A.1 and Figure A.2), and intermediary inputs come from INSEE[3] from 1978 to 2021 (see INSEE [2023] for details) and are expressed in billion euros. We consider a time horizon of ten years with $t_\circ = 2021$ as starting point, a time step of one year and $t_\star = 2030$ as ending point. In addition, we will be extending the curves to 2034 to see what happens after the transition, even though the results will be calculated and analyzed during the transition.

2. The 38 INSEE sectors are grouped into four categories: *Very High Emitting*, *Very Low Emitting*, *Low Emitting*, and *High Emitting*, based on their carbon intensities. We indeed compute the average carbon intensity of output for each sector from 2008 to 2021 in kilograms of CO_2-equivalent per euro (as shown in Figure 3.1). Carbon intensities by activity thus appear to form 4 separable groups with intensities ranging over [0, 0.05],]0.05, 0.3],]0.3, 0.5] and]0.5, 1], leading to $I = 4$. This separation seems natural and is confirmed by applying an unsupervised learning model (such as Gaussian Mixture model). We can see,

[3]The French National Institute of Statistics and Economic Studies

from Figure A.7 that this separability persists almost every year from 2008 to 2021. Each group's composition is detailed in A.2.

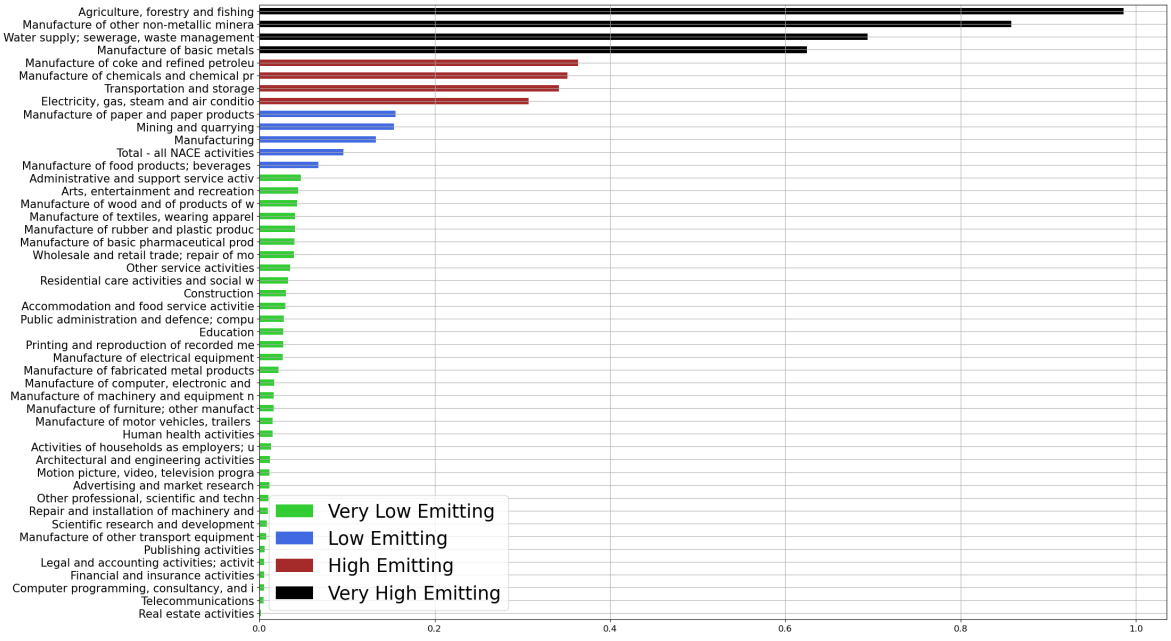


Figure 3.1: Average air emissions intensities (in kilograms of CO2-equivalent per euro) by NACE from 2008 to 2021

3. The carbon intensities are calibrated on the realized emissions Eur [2023] (expressed in tonnes of CO2-equivalent) between 2008 and 2021 (see Figures A.4 and A.5).

   Regarding the households GHG emissions, Eurostat only provides data for transport, heating and cooking, as well as emissions that fall under the category "other". Following our sectors classification, we put transport, heating and cooking in the High Emitting sector. Then, we divide the Eurostat sectors falling under the category "other" between *Very High Emitting*, *Very Low Emitting*, and *Low Emitting*, proportionally to their contribution to the households consumption.

3.3.2 Calibration

Calibration of economics parameters

For the parameters σ and φ, we use the same values as in Galí [2015]: a unitary log-utility $\sigma = 1$ and a unitary Frisch elasticity of labor supply $\varphi = 1$. We have the parameters of the multisectoral model $(\widehat{\psi}_i)_{i\in\mathcal{I}}$ and $(\widehat{\boldsymbol{\lambda}}_{ji})_{i,j\in\mathcal{I}}$ in Table 3.1 and in Table 3.2.

According to Tables 3.1 and 3.2, the assumed identity $\psi^i + \sum_{j\in\mathcal{I}} \boldsymbol{\lambda}^{ji} = 1$ would be expected to hold in the case where other production factors such as capital stock are

Output	**Very High**	**High**	**Low**	**Very Low**
Elasticity of labor supply	0.183	0.215	0.161	0.331

Table 3.1: Elasticity of labor supply $\widehat{\psi}$

Input / Output	**Very High**	**High**	**Low**	**Very Low**
Very High	0.273	0.028	0.266	0.052
High	0.130	0.304	0.061	0.043
Low	0.064	0.129	0.242	0.033
Very Low	0.157	0.159	0.143	0.312

Table 3.2: Elasticity of intermediary inputs $\widehat{\boldsymbol{\lambda}}$

Emissions Level	φ_0	$g_{\varphi,0}$	$\theta_\varphi(\%)$
$\tau_{\textit{Very High}}$	0.473	-0.013	0.001
$\tau_{\textit{High}}$	0.377	-0.049	0. 001
$\tau_{\textit{Low}}$	0.07	-0.039	3.7
$\tau_{\textit{Very Low}}$	0.024	-0.028	0.001

Table 3.3: Carbon intensities parameters

included in our setup. However, our setup avoids the inclusion of capital accumulation (as in the white paper Devulder and Lisack [2020] authored by the Banque de France), as well as imports and exports in order to simplify the already involved analysis. Still, our numerical application shows that our setup allows to capture in average 82% of the sum. We then obtain the productivity parameters in Table 3.4, 3.5, 3.6, while the carbon intensities parameters are in Table 3.3 and A.1. It is worth noting that for each intensity φ, $g_{\varphi,0}$ is negative and θ_φ is positive, which means that carbon intensities are decreasing in France.

Emissions Level	**Very High**	**High**	**Low**	**Very Low**
$\times 10^{-3}$	2.649	3.826	-4.691	4.288

Table 3.4: Parameter μ of the productivity

Emissions Level	**Very High**	**High**	**Low**	**Very Low**
Very High	-0.191	-0.061	0.108	-0.005
High	0.017	0.404	0.282	-0.067
Low	0.302	0.190	-0.552	0.290
Very Low	0.177	0.021	0.623	0.539

Table 3.5: Parameter Γ of the productivity

The eigenvalues of $\widehat{\Gamma}$ are $\{-0.790, -0.145, 0.692, 0.443\}$ which are all strictly less than 1 in absolute value, therefore $\widehat{\Theta}$ is weak-stationary as assumed.

Emissions Level	**Very High**	**High**	**Low**	**Very Low**
Very High	0.329	0.020	0.011	0.082
High	0.020	0.134	0.013	0.030
Low	0.011	0.013	0.071	-0.012
Very Low	0.082	0.030	-0.012	0.066

Table 3.6: Parameter$\Sigma \times 10^3$ of the productivity

In our simulation, we consider four deterministic transition scenarios giving four deterministic carbon price trajectories. The scenarios used come from the NGFS simulations, whose descriptions are given on the NGFS website NGFS [2022] as follows:

- ***Net Zero 2050*** *is an ambitious scenario that limits global warming to* $1.5°C$ *through stringent climate policies and innovation, reaching net zero CO2 emissions around 2050. Some jurisdictions such as the US, EU and Japan reach net zero for all GHG by this point.*
- ***Divergent Net Zero*** *reaches net-zero by 2050 but with higher costs due to divergent policies introduced across sectors and a quicker phase out of fossil fuels.*
- ***Nationally Determined Contributions (NDCs)*** *includes all pledged policies even if not yet implemented.*
- ***Current Policies*** *assumes that only currently implemented policies are preserved, leading to high physical risks.*

For each scenario, we compute the average annual growth of the carbon price as displayed in the fourth column of Table 3.7.

Scenario	**2020 Carbon Price (€/ton)**	**2030 Carbon Price (€/ton)**	**Average Annual Growth Rate (%)**
Current Policies	39.05	39.05	0.
NDCs	39.05	76.46	6.42
Net Zero 2050	39.05	162.67	13.24
Divergent Net Zero	96.43	395.21	10.63

Table 3.7: Carbon price in 2020 and 2030, and average annual growth over ten years

Calibration of carbon intensities

The evolution of carbon prices between 2020 and 2030 are shown on Figure 3.2. Moreover, we compute the evolution of the *carbon intensities on production*, τ, the *carbon intensities on final consumption*, κ, and the *carbon intensities on the firms' intermediary consumption*, ζ, for each sector based on the realized emissions. Recall that the carbon price is expressed in euro per ton and the carbon intensity in tons per

euro so that their product, that we called *emissions cost rate*, is dimensionless. We report the annual average per scenario in Table 3.9, Table 3.10, and Table 3.11.

Given that carbon intensities are slightly decreasing, emissions cost rate will not follow exactly the same trends as carbon prices.

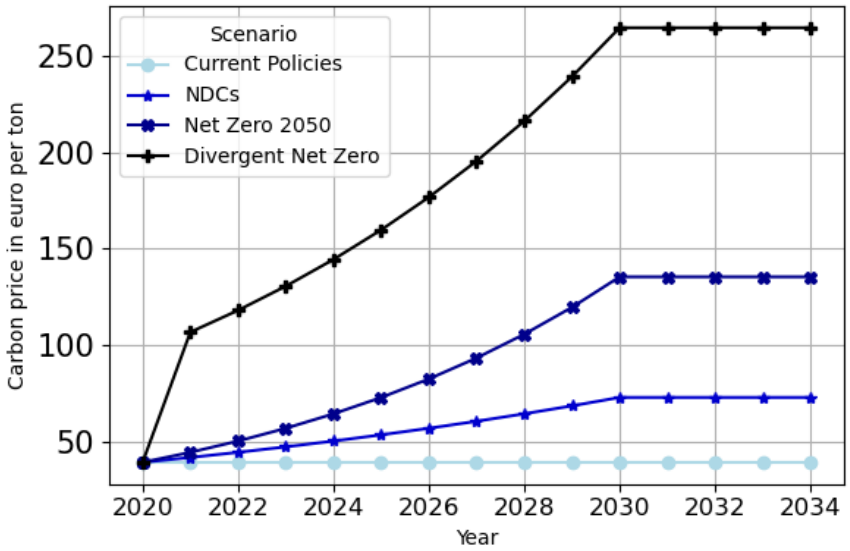


Figure 3.2: Annual carbon price per scenario

In order to ensure that the condition (3.1.3) is satisfied, it is sufficient to compute the product of the maximum of the carbon price $\delta_{t_\star}$ (as $t \mapsto \delta_t$ is non-decreasing) and the maximum of the output carbon intensity τ_0^i (as $t \mapsto \tau_t^i$ is decreasing), for each sector i and for each scenario.

$\tau_0^i \delta_{t_\star}$	**Very High**	**High**	**Low**	**Very Low**
Current Policies	0.017	0.014	0.003	0.000
NDCs	0.031	0.026	0.004	0.001
Net Zero 2050	0.054	0.045	0.009	0.002
Divergent Net Zero	0.108	0.089	0.017	0.005

Table 3.8: Maximum firms' carbon intensities multiplied by carbon price in 2030 per scenario

The highest level of emissions cost rate for households' consumption comes from the *High Emitting* group (involved for transport, cooking and heating).

On firms' production side, the *Very High Emitting* group is the highest charged (because agriculture and farming emit large amounts of GHG like methane), and is naturally followed by the *High Emitting* one which emits significant amounts of CO2.

On the *emissions cost rate* of firms' intermediary consumption, we observe expected patterns. For example, the *emissions cost rate* applied on goods/services produced by the *Very High Emitting* sector and consumed by the *Low Emitting* one is very high.

Emissions level	**Very High**	**High**	**Low**	**Very Low**
Current Policies	0.007	2.233	0.007	0.007
NDCs	0.010	3.031	0.010	0.010
Net Zero 2050	0.014	4.273	0.014	0.014
Divergent Net Zero	0.031	9.235	0.031	0.031

Table 3.9: Average annual *emissions cost rate* $\delta\kappa$ on households' consumption from each sector between 2020 and 2030 (in %)

Emissions level	**Very High**	**High**	**Low**	**Very Low**
Current Policies	1.483	0.644	0.169	0.058
NDCs	2.047	0.870	0.232	0.080
Net Zero 2050	2.933	1.219	0.331	0.113
Divergent Net Zero	6.301	2.641	0.713	0.244

Table 3.10: Average annual *emissions cost rate* $\delta\tau$ on firms' production in each sector between 2020 and 2030 (in %)

This is explained by the fact that many inputs used by sectors belonging to the *Low Emitting* group (such as *Manufacture of food products, beverages and tobacco products*) consume the output from *Agriculture, forestry and fishing* which belongs to the *Very High Emitting* group. Precisely, the inputs from *Very High Emitting* into *Low Emitting* represent in this case, up to 3.68% of all intermediate inputs (see Figure A.6). Similar comments can be done for the other sectors. These results thus show that sectors are not only affected by their own emissions, but also by the emissions from the sectors from which they consume products. Moreover, we observe a relation between the level of *emissions cost rate* applied to intra-sectoral consumption and the corresponding level of elasticity displayed in Table 3.2.

We now calibrate our model on the historical data assuming no carbon price as detailed in Section 3.2.3 and perform simulations.

3.3.3 Simulations and discussion

Output growth

After $M = 5000$ simulations, we compute the mean of the annual output growth and related 95% confidence interval for each sector and each scenario. Results are displayed on Figure 3.3. Additionally, we compute the average annual output growth over the ten-year period, as illustrated in Table 3.12.

It follows from the *Total* column in Table 3.12 that the average annual growth between 2020 and 2030 is decreasing. The *Divergent Net Zero* is the economic worst case (the best one for the climate) where the carbon ton would cost 395.21€ in 2030. The *Current Policies* is the economic best case (the worst one for the climate) where the carbon ton would cost 39.05€ in 2030. The difference of the annual output growth

Emissions level / Output	**Very High**	**High**	**Low**	**Very Low**
Current Policies	0.255	0.088	0.095	0.032
NDCs	0.347	0.119	0.131	0.044
Net Zero 2050	0.491	0.166	0.188	0.061
Divergent Net Zero	1.061	0.360	0.404	0.132

(a) Input: *Very High*

Emissions level / Output	**Very High**	**High**	**Low**	**Very Low**
Current Policies	0.031	0.347	0.122	0.047
NDCs	0.042	0.471	0.162	0.064
Net Zero 2050	0.059	0.666	0.223	0.091
Divergent Net Zero	0.128	1.439	0.487	0.197

(b) Input: *High*

Emissions level / Output	**Very High**	**High**	**Low**	**Very Low**
Current Policies	0.117	0.022	0.151	0.014
NDCs	0.156	0.03	0.203	0.019
Net Zero 2050	0.216	0.041	0.282	0.026
Divergent Net Zero	0.471	0.089	0.613	0.057

(c) Input: *Low*

Emissions level / Output	**Very High**	**High**	**Low**	**Very Low**
Current Policies	0.078	0.061	0.077	0.130
NDCs	0.107	0.084	0.106	0.178
Net Zero 2050	0.152	0.119	0.152	0.251
Divergent Net Zero	0.328	0.257	0.326	0.543

(d) Input: *Very Low*

Table 3.11: Average annual *emissions cost rate* $\delta\zeta$ on firms' intermediary inputs from each sector between 2020 and 2030 (in %)

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Total**
NDCs	-0.248	-0.245	-0.062	-0.018	-0.128
Net Zero 2050	-0.712	-0.692	-0.181	-0.051	-0.362
Divergent Net Zero	-1.187	-0.978	-0.310	-0.099	-0.554

Table 3.12: Average annual output growth evolution with respect to the *Current Policies* scenario between 2020 and 2030 (in %)

between the worst and the best scenarios is of about -0.554%.

The four scenarios are clearly discriminating. In the *Divergent Net Zero* scenario, our model shows, on the last subplot in Figure 3.3, a drop in output growth, with respect to the *Current Policies* scenario, that starts at 0.405% in 2020 and increases every year until a 0.746% drop is reached in 2030. Cumulatively, from 2020 to 2030, a drop of 5.539% is witnessed.

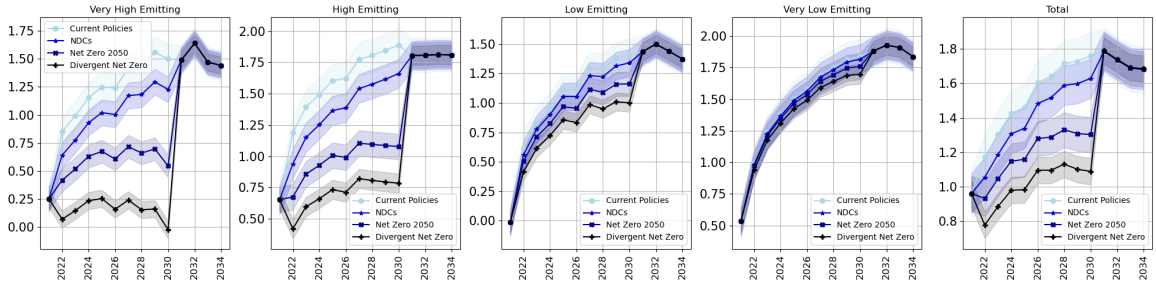


Figure 3.3: Mean and 95% confidence interval of the annual output growth

We can compare this value to 2.270% which is the GDP drop between the *Net Zero 2050* and *Current Policies* scenarios obtained with the REMIND model in Luderer et al. [2015]. The difference observed with REMIND can be explained by the fact that our model does not specify how the revenues generated by charging GHG emissions and collected by the regulator are reinvested or redistributed. We could, for example, head the investment towards low-carbon energies, which would have the effect of reducing the GHG emissions costs on these sectors. Moreover, in our model, the carbon price is assumed to increase uniformly (which implies that emissions would increase indefinitely - which is not desirable) from 2021 to 2030, while in REMIND an adjustment of the carbon price growth rate is being made in 2025. Furthermore, productivity is totally exogenous in our model while there are exogenous labor productivity and endogenous technological change for green energies in REMIND, which is expected to have a downward effect on the evolution of the carbon price. However, we recall that our model has the benefit to be stochastic and multisectoral.

Now, it follows from both Figure 3.3 and Table 3.12 that the introduction of the carbon price is less adverse for the *Very Low Emitting* and *Low Emitting* groups than for the *High Emitting* and *Very High Emitting* ones. The slowdown is highest for the *Very High Emitting* group, which was anticipated given that the emissions cost on firms was the highest. Moreover, the slowdown could be accelerated by the climate transition, not only because this sector emits GHG, but also because its intermediary inputs are from the *High Emitting* and *Very High Emitting* sectors. On the other hand, the *Very Low Emitting* sector continues its strong growth because it emits less and because France is driven by the service industry. In addition, the consumption in the two most polluting sectors suffers from a slowdown higher than the whole consumption slowdown and lower than in the two least polluting ones.

Finally, from figure 3.4 [4], due to deindustrialization and the reduction in agricultural production, the share of production from sectors *Very High Emitting* and *Low Emitting* is tending to decline in the French economy. This decline accelerates with the severity of the transition. However, it is evident that the French economy is shifting toward a service-based economy. This phenomenon becomes more pronounced as the carbon price increases. It could imply that companies are increasingly leaving the most

[4]We ignore years 2020 and 2021 due to COVID-19 jumps.

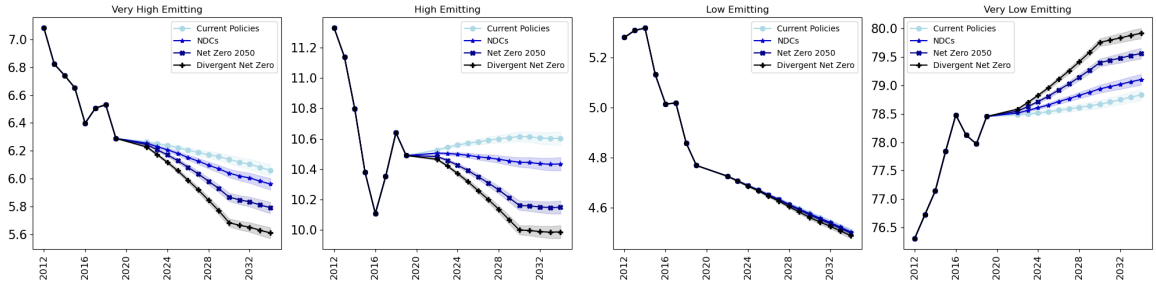


Figure 3.4: Mean and 95% confidence interval of the contributions (in %) of each sector in the total output

polluting (and therefore most taxed) sectors. To mitigate these effects, we could intelligently reinvest the collected carbon taxes.

Greenhouse gases

We observe in Figure 3.5 [5] a slow reduction in GHG emissions concurrent with deindustrialization, the reduction of agricultural production and probably thanks to the efforts that are beginning to be made. With the introduction of a carbon price, this downward trend accelerates. However, for the sector *Very High Emitting*, GHG emissions continue to increase for soft transition scenarios. Finally, it should be noted

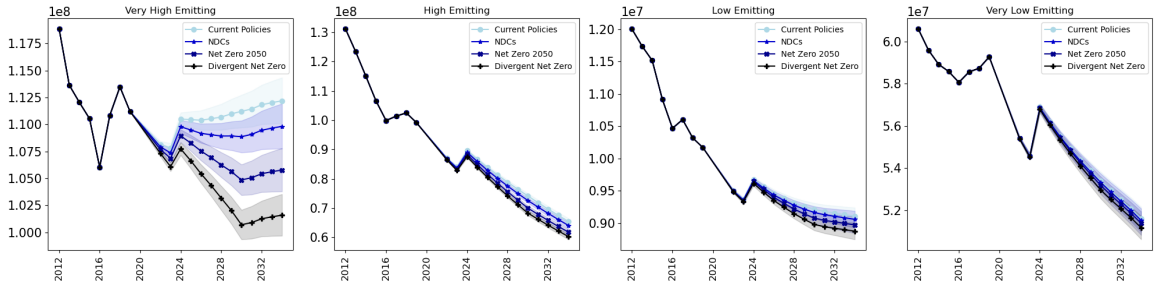


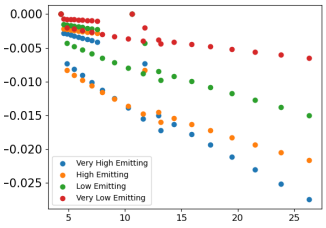
Figure 3.5: Mean and 95% confidence interval of the direct GHG emissions (in ton of CO2-equivalent) of firms in each sector

that these trends in GHG emissions can also be explained by the structure chosen for our carbon intensities.

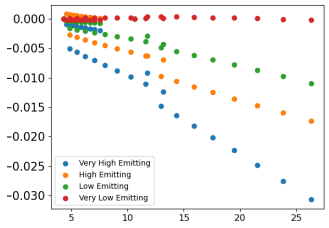
Sensitivities of the macroeconomics variables to the carbon price

In ths section, we compute the different sensitivities, precisely, for each economic variables (consumption, output, labour, and intermediary inputs), we quantity the impact of a variation of the carbon price on the growths. Transition scenarios are not taken into account here. As might be expected, when the price of carbon rises, the fall in growth is significant. The more polluting the sector, the greater the drop.

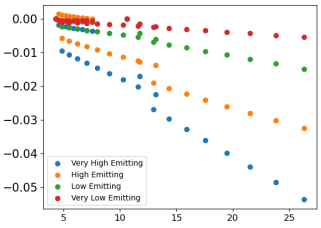
[5] We ignore years 2020 and 2021 due to COVID-19 jumps.



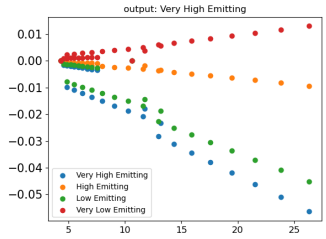
(a) Labour



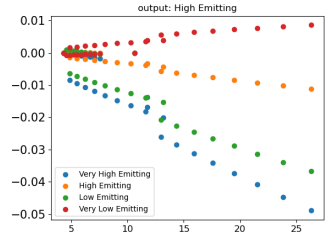
(b) Consumption



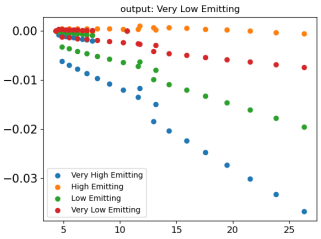
(c) Output



(d) Output: *Very High Emitting*



(e) output: *High Emitting*



(f) output: Very Low Emitting

Figure 3.6: Sensitivities: x: carbon price in €, y: growth in %

These results help us to measure a kind of uncertainties. Given that neither the agents nor the regulator do not exactly know the trajectory of the carbon price. It is valuable to know how the variables evolve if the regulator later deviate from the trajectory defined a priori by the transition scenario.

Conclusion

In this chapter, we study how the introduction of a carbon price would propagate into a close economy. To this aim, we build a dynamic stochastic multisectoral model in which firms (resp. households) are charged for the GHG they emit when they consume intermediary inputs from other sectors and when they produce goods/services (resp. for the GHG they emit when they consume goods/services). This step opens the way to numerous extensions. For example exogenous and deterministic scenarios as well as homogeneous agents are assumed while one could consider agent-based (see Gualdi et al. [2015], Dawid et al. [2012],)or mean-field games models (see Cousin et al. [2011], Gomes et al. [2016]) where a central planner decides on the carbon price and agents (companies or households) optimize production, prices, and consumption according to the carbon price/tax level.

Chapter 4

Loss modelling in discrete time, with carbon emissions cost and without collateral

This chapter is based on the paper Bouveret et al. [2023][Section 3&4] under revisions.

This chapter is divided in two main sections. In a first section, we define the firm value by using the discounted cash flows methodology Kruschwitz and Löffler [2020]. By remarking that cash flows are a proportion of sales and by assuming that the latter are a proportion of the sectoral output, we have that the cash flow growth is a linear function of the (sectoral) output growth. This allows us to describe the firm's cash flows and firm value as functions of productivity and carbon price. Then, by assuming that the noise term in the productivity is small, we obtain a closed-form formula of the firm value.

In a second section, we use the firm value in structural credit risk model. For different climate transition scenarios, we then calculate the evolution of the annual probability of default, the expected loss, and the unexpected loss of a credit portfolio. This is close to the analyses in Garnier [2021] and Bourgey et al. [2021]. However, Garnier [2021] relies on the Vasicek-Merton model with a centered Gaussian systemic factor, while we appeal to a microeconomic definition of the firm value as in Bourgey et al. [2021]. Contrary to the latter, (1) we emphasize how firms are affected by macroeconomic factors (e.g., productivity and carbon price processes) but do not allow them to optimize their transition strategy, and (2) besides discussing the impacts of a carbon price on the probability of default, we also investigate their impacts on losses. We finally introduce an indicator to describe the sensitivity of the (un)expected loss of a portfolio to a carbon price. his allows us to analyze how the risk measures would vary, should we deviate from a carbon price given by our supposedly deterministic scenarios.

Contents

4.1 A firm valuation model

When an economy is in good health, the probabilities of default are relatively low, but when it enters a recession, the number of failed firms increases significantly. The same phenomenon is observed on the loss given default. This relationship between default rate and business cycle has been extensively studied in the literature: Nickell et al. [2000] quantifies the dependency between business cycles and rating transition probabilities while Bangia et al. [2002] shows that the loss distribution of credit portfolios varies highly with the health of the economy, and Castro [2013] uses an econometric model to show the link between macroeconomic conditions and the banking credit risk in European countries.

Following these works, Pesaran et al. [2006] uses an econometric model to empirically characterize the time series behaviour of probabilities of default and of recovery rates. The goal of that work is "to show how global macroeconometric models can be linked to firm-specific return processes which are an integral part of Merton-type credit risk models so that quantitative answers to such questions can be obtained". This simply implies that macroeconomic variables are used as systemic factors introduced in the Merton model. The endogenous variables typically include *real GDP, inflation, interest rate, real equity prices, exchange rate and real money balances.* One way to choose the macroeconomic variables is to run a LASSO regression between the logit function ($p \mapsto \log\left(\frac{p}{1-p}\right)$ on $(0,1)$) of observed default rates of firms and a set of macroeconomic variables. We perform such an analysis on a segment of S&P's data in B.1.

In addition to these statistical works, Baker et al. [2005] show through three different models that, in a steady state economy, economic growth and asset returns are linearly

related. On the one hand, economic growth is equivalent to productivity growth. On the other hand, the physical capital rate of gross profit, the net rate of return on a balanced financial portfolio and the net rate of return on equities are supposed to behave similarly. In particular, in the Solow [1956] model, the *physical capital rate of gross profit* is proportional to the return-to-capital parameter, to the productivity growth, and inversely proportional to the gross saving. In the Diamond [1965] model, the *net rate of return on a balanced financial portfolio* is proportional to the labor productivity growth. In the Ramsey model (see Romer [2012]) with a log utility function, the *net rate of return on equities* is proportional to the reduction in labor productivity growth.

4.1.1 The firm value from the Discounted Cash Flow method

Inspired by the aforementioned works, we introduce the following assumption describing the cash flows dynamics. Consider a portfolio of $N \in \mathbb{N}^*$ firms.

Assumption 4.1.1. For each $n \in \{1, \dots, N\}$, the $\mathbb{R}^N$-valued process on the cash flows growth of firm n denoted by $(\omega_t^n)_{t\in\mathbb{N}^*}$ is linear in the economic factors (the output growth of the sector introduced in (3.1.25)), specifically we set for all $t \in \mathbb{N}$,

$$\omega_t^n = \tilde{\mathfrak{a}}^{n\cdot} \Delta_t^Y + \mathfrak{b}_t^n, \tag{4.1.1}$$

for $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$, where the idiosyncratic noise $(\mathfrak{b}_t)_{t\in\mathbb{N}} := (\mathfrak{b}_t^n)_{t\in\mathbb{N}, 1\leq n\leq N}$ is i.i.d. with law $\mathcal{N}(0, \mathrm{diag}(\sigma_{\mathfrak{b}^n}^2))$ with $\sigma_{\mathfrak{b}^n} > 0$ for $n \in \{1, \dots, N\}$. Moreover, $(\Delta_t^Y)_{t\in\mathbb{N}^*}$ and $(\mathfrak{b}_t)_{t\in\mathbb{N}}$ are independent.

Remark 4.1.2. For each $n \in \{1, \dots, N\}$, the definition (4.1.1) can be rewritten, with $\mathfrak{a}^{n\cdot} := \tilde{\mathfrak{a}}^{n\cdot}(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}$, as

$$\omega_t^n = \mathfrak{a}^{n\cdot} \left(\Theta_t + \mathfrak{v}(\mathfrak{d}_t) - \mathfrak{v}(\mathfrak{d}_{t-1})\right) + \mathfrak{b}_t^n, \tag{4.1.2}$$

using (3.1.25). We call $\mathfrak{a}^{n\cdot}$ and $\tilde{\mathfrak{a}}^{n\cdot}$ *factor loadings*, quantifying the extent to which ω^n is related to Δ^Y. Moreover, the economic motivation behind (4.1.1) comes from the fact that if firm n belongs to sector i, then its production is proportional to the sectoral output and its cash flows are proportional to its production (as in the DKW model Barth et al. [2001], Dechow et al. [1998]). Thus, we obtain a relation between the cash flows of firm n and the total output of sector i. The assumption $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ stems from the fact that a company is not restricted to one activity sector only in general. However, since we are considering the emission sector here, we expect that each firm n only belongs to one sector (i for example). Therefore $\mathfrak{a}^{nj} = 0$ for all $i \neq j$ and hence $|\mathfrak{a}^{ni}| = \max_{j\in\mathcal{I}} |\mathfrak{a}^{nj}|$.

We define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}}$ by $\mathcal{F}_t = \sigma\left(\mathcal{G}_t \cup \sigma\left\{\mathfrak{b}_s : s \in [0,t] \cap \mathbb{N}\right\}\right)$ for $t \in \mathbb{N}$, denote $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ and, for all $1 \leq n \leq N$,

$$\mathcal{W}_t^n := \sum_{u=1}^{t} \mathfrak{b}_u^n. \tag{4.1.3}$$

In addition to the empirical results on the dependency between default indicators and business cycles, firm valuation models provide additional explanatory arguments. On the one hand, the structural credit risk model models says that default metrics (such as probability of default) depend on the firm value; on the other hand, valuation models help express the firm value as a function of economic cycles. Reis and Augusto [2013] organize valuations models in five groups: "models based on the discount of cash flows, models of dividends, models related to the firm value, models based on accounting elements creation, and sustaining models in real options".

Definition 4.1.3. Considering the Discounted Cash Flows method and following Kruschwitz and Löffler [2020], the firm value is the sum of the present value of all future cash flows. For any time $t \geq 0$ and firm $n \in \{1, \dots, N\}$, we note F_t^n the free cash flows [1] of n at t, and $r > 0$ the discount rate[2]. Then, the value V_t^n of the firm n, at time t, is

$$V_t^n := \mathbb{E}_t \left[\sum_{s=0}^{+\infty} e^{-rs} F_{t+s}^n \right]. \tag{4.1.4}$$

From Definition 4.1.3 above and from Assumption 4.1.1, we can write

$$F_{t+1}^n = F_t^n \exp\{\omega_{t+1}^n\}, \qquad \text{for } t \in \mathbb{N},$$

with F_0^n and $\frac{1}{F_0^n}$ both belonging to $\mathcal{L}^\infty(\mathcal{F}_0)$.
The following proposition studies the well-posedness of the firm value.

Proposition 4.1.4. *Assume that* $|\Gamma| < 1$ *and that*

$$\rho := \max_{1 \leq n \leq N} \left\{ \mathfrak{a}^{n\cdot} \overline{\mu} + \frac{1}{2} \sigma_{\mathfrak{b}^n}^2 + \frac{\varepsilon^2}{2} |\mathfrak{a}^{n\cdot}|^2 |\sqrt{\Sigma}|^2 (1 - |\Gamma|)^{-2} \right\} < r. \tag{4.1.5}$$

Then, for any $t \in \mathbb{N}$ *and* $1 \leq n \leq N$, V_t^n *is well defined and for some* $p > 1$, *which does not depend on* t *nor on* n *but on* ρ *and* r, $\|V_t^n\|_p \leq C_p \|F_t^n\|_q < +\infty$, *for some* $q > 1$ *that depends on* p, ρ *and* r.

Proof. Let $t \in \mathbb{N}$, $n \in \{1, \dots, N\}$. For $s > 0$, from Assumption 4.1.1, we observe that

$$F_{t+s}^n = F_t^n \exp\left(\sum_{u=1}^{s} w_{t+u}^n \right). \tag{4.1.6}$$

Let $K \in \mathbb{N}^*$ and define

$$V_t^{n,K} := \mathbb{E}_t \left[\sum_{s=0}^{K} e^{-rs} F_{t+s}^n \right]. \tag{4.1.7}$$

[1] Which is defined as cash flow beyond what is necessary to maintain assets in place and to finance expected new investments.

[2] Here, to simplify, we take r constant, deterministic, and independent of the companies. However, in a more general setting, it should be a stochastic process depending on the firm.

We now show that $\lim_{K\to+\infty} V^{n,K}$ exists, in particular that $\mathbb{E}_t\left[e^{-rs}F^n_{t+s}\right]$ is summable. To this end, we first observe that

$$V^{n,K}_t = F^n_t\left(1+\sum_{s=1}^{K} e^{-rs}\mathbb{E}_t\left[\exp\left(\sum_{u=1}^{s} w^n_{t+u}\right)\right]\right).$$

We now give an upper bound for $\|\exp\left(\sum_{u=1}^{s} w^n_{t+u}\right)\|_p$ for some $p>1$. We observe that, using (4.1.2),

$$\sum_{u=1}^{s} w^n_{t+u} = \mathfrak{a}^{n\cdot}\left(\sum_{u=1}^{s}\Theta_{t+u}+\mathfrak{v}(\mathfrak{d}_{t+s})-\mathfrak{v}(\mathfrak{d}_t)\right)+\sum_{u=1}^{s}\mathfrak{b}^n_{t+u}. \tag{4.1.8}$$

From Assumption 3.1.1 and (3.1.1), it follows

$$\Theta_{t+u} = \overline{\mu}+\varepsilon\left(\Gamma^u\mathcal{Z}_t+\sum_{v=1}^{u}\Gamma^{u-v}\mathcal{E}_{t+v}\right).$$

We define $\Upsilon_k := \sum_{v=0}^{k}\Gamma^v$ and observe that

$$|\Upsilon_k| \le (1-|\Gamma|)^{-1}. \tag{4.1.9}$$

Since

$$\sum_{u=1}^{s}\sum_{v=1}^{u}\Gamma^{u-v}\mathcal{E}_{t+v} = \sum_{v=1}^{s}\Upsilon_{s-v}\mathcal{E}_{t+v},$$

we compute

$$\sum_{u=1}^{s}\Theta_{t+u} = \overline{\mu}s+\varepsilon\Gamma\Upsilon_{s-1}\mathcal{Z}_t+\varepsilon\sum_{v=1}^{s}\Upsilon_{s-v}\mathcal{E}_{t+v}.$$

Then (4.1.8) reads

$$\sum_{u=1}^{s} w^n_{t+u} = \varepsilon\mathfrak{a}^{n\cdot}\Gamma\Upsilon_{s-1}\mathcal{Z}_t+s\mathfrak{a}^{n\cdot}\overline{\mu}+\varepsilon\sum_{v=1}^{s}\mathfrak{a}^{n\cdot}\Upsilon_{s-v}\mathcal{E}_{t+v}+\mathfrak{a}^{n\cdot}\left(\mathfrak{v}(\mathfrak{d}_{t+s})-\mathfrak{v}(\mathfrak{d}_t)\right)+\sum_{u=1}^{s}\mathfrak{b}^n_{t+u}.$$

Observe that under Assumption 3.1.3, there exists a constant $\mathfrak{C}>0$ such that

$$\sup_{n,s,t}\exp\left(\mathfrak{a}^{n\cdot}\left(\mathfrak{v}(\mathfrak{d}_{t+s})-\mathfrak{v}(\mathfrak{d}_t)\right)\right) \le \mathfrak{C}. \tag{4.1.10}$$

Thus, using the independence of $\mathcal{Z}_t, (\mathcal{E}_{t+v})_{v\ge1}, (\mathfrak{b}^n_{t+v})_{v\ge1}$, we obtain

$$\mathbb{E}_t\left[\exp\left(p\sum_{u=1}^{s} w^n_{t+u}\right)\right] \le \mathfrak{C}^p\exp\left(p\varepsilon\mathfrak{a}^{n\cdot}\Gamma\Upsilon_{s-1}\mathcal{Z}_t+ps\mathfrak{a}^{n\cdot}\overline{\mu}\right)\mathbb{E}\left[\exp\left(p\varepsilon\sum_{v=1}^{s}\mathfrak{a}^{n\cdot}\Upsilon_{s-v}\mathcal{E}_{t+v}+p\sum_{u=1}^{s}\mathfrak{b}^n_{t+u}\right)\right]. \tag{4.1.11}$$

Since

$$\mathbb{E}\left[\exp\left(p\sum_{u=1}^{s}\mathfrak{b}^n_{t+u}\right)\right] = \exp\left(\frac{p^2}{2}s\sigma^2_{\mathfrak{b}^n}\right), \tag{4.1.12}$$

we compute

$$\mathbb{E}\left[\exp\left(p\varepsilon\mathfrak{a}^{n\cdot}\Upsilon_{s-v}\mathcal{E}_{t+v}\right)\right]=\exp\left(\frac{\varepsilon^2p^2}{2}|\mathfrak{a}^{n\cdot}\Upsilon_{s-v}\sqrt{\Sigma}|^2\right)\leq\exp\left(\frac{\varepsilon^2p^2}{2}|\mathfrak{a}^{n\cdot}|^2|\sqrt{\Sigma}|^2(1-|\Gamma|)^{-2}\right). \tag{4.1.13}$$

One could also have found above a finer upper bound. Combining (4.1.12)-(4.1.13) with (4.1.11), we obtain

$$\mathbb{E}_t\left[\exp\left(p\sum_{u=1}^{s}w_{t+u}^n\right)\right]\leq\mathfrak{C}^p\exp\left(p\varepsilon\mathfrak{a}^{n\cdot}\Gamma\Upsilon_{s-1}\mathcal{Z}_t+p^2\rho s\right).$$

Using similar computations as above, we also get (because Υ_{s-1} is bounded and $\mathcal{Z}_t$ is stationary and Gaussian)

$$\mathbb{E}\left[\exp\left(p\varepsilon\mathfrak{a}^{n\cdot}\Gamma\Upsilon_{s-1}\mathcal{Z}_t\right)\right]\leq C_p, \tag{4.1.14}$$

and hence

$$\left\|\exp\left(\sum_{u=1}^{s}w_{t+u}^n\right)\right\|_p\leq C_pe^{p\rho s}.$$

Under (4.1.5), we then obtain

$$\sum_{s\geq0}e^{-rs}\left\|\exp\left(\sum_{u=1}^{s}w_{t+u}^n\right)\right\|_p<+\infty,$$

for some $p>1$. Set $1<\tilde{p}:=\frac{p}{1+\epsilon}$, for $\epsilon>0$ small enough. Then, using Hölder's inequality (with $\frac{1}{\tilde{p}}=\frac{1}{p}+\frac{1}{p/\epsilon}$),

$$\mathbb{E}\left[|V_t^{n,K}|^{\tilde{p}}\right]\leq C_p\mathbb{E}\left[|F_t^n|^{\frac{p}{\epsilon}}\right]<+\infty,$$

since $\|F_t^n\|_q<\infty$ for any $q\geq1$. □

Remark 4.1.5. The inequality (4.1.5) guarantees the non-explosion of the expected discounted future cash flows of the firm. Moreover, we could remove the condition $|\Gamma|<1$. Indeed, we know that, by Assumption 3.1.1, Γ has eigenvalues with absolute value strictly lower than one. However, we would need to alter condition (4.1.5) by using a matrix norm $|\cdot|_s$ (subordinated) such that $|\Gamma|_s<1$. The condition would then involve equivalence of norm constants between $|\cdot|$ and $|\cdot|_s$.

We now derive a more explicit expression for V_t^n. Describing it as a function of the underlying processes driving the economy does not lead to an easily tractable formula, but allows us to write it as a fixed-point problem which can be solved by numerical methods such as Picard iteration (see Berinde and Takens [2007]) or by deep learning methods (sec Hammad et al. [2022]). To facilitate the forthcoming credit risk analysis, we approximate $\frac{V_t^n}{F_t^n}$ by the first term of an expansion in terms of the noise intensity ε appearing in Θ (Assumption 3.1.1). An expanded expression of the firm value is

$$V_t^n=F_t^n\left(1+\sum_{s=1}^{+\infty}e^{-rs}\mathbb{E}_t\left[\exp\left(\mathfrak{a}^{n\cdot}\left(\mathfrak{v}(\mathfrak{d}_{t+s})-\mathfrak{v}(\mathfrak{d}_t)+\sum_{u=1}^{s}\Theta_{t+u}\right)+\sum_{u=1}^{s}\mathfrak{b}_{t+u}^n\right)\right]\right).$$

4.1.2 A proxy of the firm value

Let us introduce, for a firm n and $t \in \mathbb{N}$, the quantity

$$\mathcal{V}_t^n := F_t^n \left(1 + \sum_{s=1}^{+\infty} e^{-rs} \mathbb{E}_t \left[\exp\left(\mathfrak{a}^{n\cdot} \left(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t) + s\overline{\mu} \right) + \sum_{u=1}^{s} \mathfrak{b}_{t+u}^n \right) \right] \right). \quad (4.1.15)$$

We recall that Θ depends on ε according to the Standing Assumption 3.1.1, therefore ω and F^n also depend on ε according to Assumption 4.1.1 and Assumption 4.1.1 respectively. This gives the dependence of V^n on ε. From (4.1.15), $\frac{\mathcal{V}_t^n}{F_t^n}$ almost corresponds to the definition of $\frac{V_t^n}{F_t^n}$ but with the noise term coming from the economic factor in the definition of Θ set to zero, for the dates after t, according to (4.1.4), (4.1.6) and (4.1.2). We first make the following observation.

Lemma 4.1.6. *For any $n \in \{1, \dots, N\}$, assume that $\varrho_n := \frac{1}{2}\sigma_{\mathfrak{b}_n}^2 + \mathfrak{a}^{n\cdot}\overline{\mu} - r < 0$. Then $\mathcal{V}_t^n$ is well defined for all $t \in \mathbb{N}$ and*

$$\mathcal{V}_t^n = F_0^n \mathfrak{R}_t^n(\mathfrak{d}) \exp\left(\mathfrak{a}^{n\cdot} (\mathcal{A}_t^\circ - \mathfrak{v}(\mathfrak{d}_0)) \right) \exp\left(\mathcal{W}_t^n \right), \quad (4.1.16)$$

where $\mathcal{W}$ is defined in (4.1.3) *and*

$$\mathfrak{R}_t^n(\mathfrak{d}) := \sum_{s=0}^{\infty} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} \mathfrak{v}(\mathfrak{d}_{t+s}) \right). \quad (4.1.17)$$

Moreover, with $t_\circ$ and $t_\star$ defined in Standing Assumption 3.1.3, we obtain the explicit form

$$\mathfrak{R}_t^n(\mathfrak{d}) = \begin{cases} \dfrac{e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star})}}{1 - e^{\varrho_n}}, & \text{if } t \geq t_\star, \\ \displaystyle\sum_{s=0}^{t_\star - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} \mathfrak{v}(\mathfrak{d}_{t+s}) \right) + \frac{e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star}) + \varrho_n(t_\star - t + 1)}}{1 - e^{\varrho_n}}, & \text{if } t_\circ \leq t < t_\star, \\ e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\circ})} \dfrac{1 - e^{\varrho_n(t_\circ - t + 1)}}{1 - e^{\varrho_n}} + \displaystyle\sum_{s=t_\circ - t + 1}^{t_\star - t} e^{\varrho_n s} e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})} + \frac{e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star}) + \varrho_n(t_\star - t + 1)}}{1 - e^{\varrho_n}}, & \text{otherwise.} \end{cases}$$

Proof. Let $t \in \mathbb{N}$ and introduce, for $K > t_\star$,

$$\mathcal{V}_t^{n,K} := F_t^n \left(1 + \sum_{s=1}^{K} e^{-rs} \mathbb{E}_t \left[\exp\left(s\mathfrak{a}^{n\cdot}\overline{\mu} + \mathfrak{a}^{n\cdot} \left(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t) \right) + \sum_{u=1}^{s} \mathfrak{b}_{t+u}^n \right) \right] \right). \quad (4.1.18)$$

Similar computations as (in fact easier than) the ones performed in the proof of Proposition 4.1.4 show that $\mathcal{V}_t^n = \lim_{K \to +\infty} \mathcal{V}_t^{n,K}$ is well defined in $\mathcal{L}^q(\mathcal{H}, \mathbb{E})$ for any $q \geq 1$. Furthermore,

$$\begin{aligned} \mathcal{V}_t^{n,K} &= F_t^n \left(1 + \sum_{s=1}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} \left(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t) \right) \right) \right) \\ &= F_t^n \left(1 + e^{-\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_t)} \sum_{s=1}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} \mathfrak{v}(\mathfrak{d}_{t+s}) \right) \right), \end{aligned}$$

where ϱ_n is defined in the lemma, and from Assumptions 4.1.1 and 4.1.1,

$$F_t^n = F_0^n \exp\left(\sum_{u=1}^{t} w_u^n\right) = F_0^n e^{\mathfrak{a}^{n\cdot}(\mathfrak{v}(\mathfrak{d}_t)-\mathfrak{v}(\delta_0))} \exp\left(\mathfrak{a}^{n\cdot}\mathcal{A}_t^\circ + \mathcal{W}_t^n\right).$$

We then have

$$F_t^n\left(1+e^{-\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_t)}\sum_{s=1}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right)\right) = F_0^n e^{-\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_0)}\exp\left(\mathfrak{a}^{n\cdot}\mathcal{A}_t^\circ + \mathcal{W}_t^n\right)\sum_{s=0}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right).$$

(1) If $t < t_\circ$, then

$$\begin{aligned}
\mathfrak{R}_t^{n,K}(\mathfrak{d}) &:= \sum_{s=0}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right)\\
&= \sum_{s=0}^{t_\circ - t} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) + \sum_{s=t_\circ-t+1}^{t_\star-t} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) + \sum_{s=t_\star-t+1}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right)\\
&= e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\circ})}\frac{1-e^{\varrho_n(t_\circ-t+1)}}{1-e^{\varrho_n}} + \sum_{s=t_\circ-t+1}^{t_\star-t} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) + e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star})+\varrho_n(t_\star-t+1)}\frac{1-e^{\varrho_n(K-t_\star+t)}}{1-e^{\varrho_n}}.
\end{aligned}$$

(2) If $t_\circ \le t < t_\star$, then

$$\begin{aligned}
\sum_{s=0}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) &= \sum_{s=0}^{t_\star-t} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) + \sum_{s=t_\star-t+1}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right)\\
&= \sum_{s=0}^{t_\star-t} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) + e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star})+\varrho_n(t_\star-t+1)}\frac{1-e^{\varrho_n(K-t_\star+t)}}{1-e^{\varrho_n}}.
\end{aligned}$$

(3) If $t \ge t_\star$, then

$$\sum_{s=0}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_{t+s})\right) = \sum_{s=0}^{K} e^{\varrho_n s}\exp\left(\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star})\right) = e^{\mathfrak{a}^{n\cdot}\mathfrak{v}(\delta_{t_\star})}\frac{1-e^{\varrho_n(K+1)}}{1-e^{\varrho_n}}.$$

Finally, $e^{\varrho_n(K+1)}$ and $e^{\varrho_n(K-t_\star+t)}$ converge to 0 for $\varrho_n < 0$ as K tends to infinity and the result follows. □

The following proposition shows that $\frac{\mathcal{V}_t^n}{F_t^n}$ and $\frac{V_t^n}{F_t^n}$ become closer as ε gets to 0.

Proposition 4.1.7. *Assume that* $|\Gamma| < 1$ *and that* (4.1.5) *is satisfied, then*

$$\mathbb{E}\left[\left|\frac{V_t^n}{F_t^n} - \frac{\mathcal{V}_t^n}{F_t^n}\right|\right] \le C\varepsilon,$$

for some positive constant C *(depending on* t, ρ*).*

Proof. For $K \in \mathbb{N}^*$, recall the expressions of $V_t^{n,K}$ in (4.1.7) and $\mathcal{V}_t^{n,K}$ in (4.1.18) and note that

$$\mathbb{E}\left[\left|\frac{V_t^n}{F_t^n} - \frac{\mathcal{V}_t^n}{F_t^n}\right|\right] \le \mathbb{E}\left[\left|\frac{V_t^n}{F_t^n} - \frac{V_t^{n,K}}{F_t^n}\right|\right] + \mathbb{E}\left[\left|\frac{V_t^{n,K}}{F_t^n} - \frac{\mathcal{V}_t^{n,K}}{F_t^n}\right|\right] + \mathbb{E}\left[\left|\frac{\mathcal{V}_t^{n,K}}{F_t^n} - \frac{\mathcal{V}_t^n}{F_t^n}\right|\right] \quad (4.1.19)$$

Using Hölder's inequality and Proposition 4.1.4, one gets that the first term in the right hand side of the above inequality goes to zero as K goes to $+\infty$. Similarly, using Hölder's inequality and (the beginning of the proof of) Lemma 4.1.6, one shows that the last term in the right hand side of the above inequality goes to zero as K goes to infinity. It remains thus to study the middle term to obtain the desired result. Observe that

$$\frac{V_t^{n,K} - \mathcal{V}_t^{n,K}}{F_t^n} = \left(\sum_{s=1}^{K} e^{-rs} \mathbb{E}_t \left[\exp\left(s\mathfrak{a}^{n\cdot}\overline{\mu} + \mathfrak{a}^{n\cdot}(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t)) + \sum_{u=1}^{s} \mathfrak{b}_{t+u}^n \right) \Delta_s \right] \right),$$

with

$$\Delta_s := \exp\left\{ \varepsilon \sum_{u=1}^{s} \mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u} \right\} - 1, \tag{4.1.20}$$

using (4.1.8) and (3.1.1). We first compute, by independence,

$$\begin{aligned}
&\left| \mathbb{E}_t \left[\exp\left(s\mathfrak{a}^{n\cdot}\overline{\mu} + \mathfrak{a}^{n\cdot}(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t)) + \sum_{u=1}^{s} \mathfrak{b}_{t+u}^n \right) \Delta_s \right] \right| \\
&= \mathbb{E}\left[\exp\left(s\mathfrak{a}^{n\cdot}\overline{\mu} + \mathfrak{a}^{n\cdot}(\mathfrak{v}(\mathfrak{d}_{t+s}) - \mathfrak{v}(\mathfrak{d}_t)) + \sum_{u=1}^{s} \mathfrak{b}_{t+u}^n \right) \right] \left| \mathbb{E}_t\left[\Delta_s\right] \right| \\
&\leq \mathfrak{C} \exp\left(s\mathfrak{a}^{n\cdot}\overline{\mu} + \frac{1}{2} s\sigma^2_{\mathfrak{b}^n} \right) \mathbb{E}_t\left[|\Delta_s| \right],
\end{aligned}$$

using (4.1.10). We then obtain

$$\|(V_t^{n,K} - \mathcal{V}_t^{n,K})/F_t^n\|_1 \leq \left(\sum_{s=1}^{K} \mathfrak{C} e^{\varrho_n s} \mathbb{E}\left[|\Delta_s| \right] \right),$$

where ϱ_n is defined in Lemma 4.1.6. We can rewrite (4.1.20) as

$$\Delta_s = \varepsilon \int_0^1 \exp\left(\varepsilon\lambda \sum_{u=1}^{s} \mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u} \right) \sum_{u=1}^{s} \mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u} d\lambda.$$

For $p > 1$, using Hölder's inequality, we deduce from the previous expression

$$\mathbb{E}\left[|\Delta_s| \right] \leq \varepsilon \mathbb{E}\left[\left| \int_0^1 \exp\left(\varepsilon\lambda \sum_{u=1}^{s} \mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u} \right) \mathrm{d}\lambda \right|^p \right]^{\frac{1}{p}} \mathbb{E}\left[\left| \sum_{u=1}^{s} \mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u} \right|^q \right]^{\frac{1}{q}}, \tag{4.1.21}$$

with q the conjugate exponent to p.

We first compute by convexity

$$\mathbb{E}\left[\left| \sum_{u=1}^{s} \mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u} \right|^q \right] \leq s^{q-1} \sum_{u=1}^{s} \mathbb{E}\left[|\mathfrak{a}^{n\cdot} \mathcal{Z}_{t+u}|^q \right] \leq C_q s^q,$$

where the last inequality follows since $\mathcal{Z}_{t+u} \sim \mathcal{N}(0, \overline{\Sigma})$.

We now turn to the first term in the right hand side of (4.1.21),
Using Jensen's inequality, we have

$$\mathbb{E}\left[\left|\int_0^1 \exp\left(\varepsilon\lambda\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\mathcal{Z}_{t+u}\right)\mathrm{d}\lambda\right|^p\right] \leq \int_0^1 \mathbb{E}\left[\exp\left(\varepsilon\lambda p\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\mathcal{Z}_{t+u}\right)\right]\mathrm{d}\lambda.$$

Since $\mathcal{Z}_{t+u} = \Gamma^u\mathcal{Z}_t + \sum_{v=1}^{u}\Gamma^{u-v}\mathcal{E}_{t+v}$, we write

$$\mathbb{E}_t\left[\exp\left(p\varepsilon\lambda\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\mathcal{Z}_{t+u}\right)\right] = \exp\left(p\varepsilon\lambda\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\Gamma^u\mathcal{Z}_t\right)\times\mathbb{E}_t\left[\exp\left(p\varepsilon\lambda\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\sum_{v=1}^{u}\Gamma^{u-v}\mathcal{E}_{t+v}\right)\right].$$

By (4.1.9), $|\Upsilon_k| \leq (1-|\Gamma|)^{-1}$ where $\Upsilon_k := \sum_{v=0}^{k}\Gamma^v$. We compute

$$\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\sum_{v=1}^{u}\Gamma^{u-v}\mathcal{E}_{t+v} = \sum_{v=1}^{s}\mathfrak{a}^{n\cdot}\Upsilon_{s-v}\mathcal{E}_{t+v}.$$

Using (4.1.13) and recalling that $\lambda \in [0,1]$, we get

$$\mathbb{E}_t\left[\exp\left(p\varepsilon\lambda\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\sum_{v=1}^{u}\Gamma^{u-v}\mathcal{E}_{t+v}\right)\right] \leq \exp\left(s\frac{\varepsilon^2p^2}{2}|\mathfrak{a}^{n\cdot}|^2|\sqrt{\Sigma}|^2(1-|\Gamma|)^{-2}\right).$$

Thus, appealing to (4.1.14), we get

$$\mathbb{E}\left[\exp(\varepsilon p\lambda\sum_{u=1}^{s}\mathfrak{a}^{n\cdot}\mathcal{Z}_{t+u})\right] \leq C_{p,\epsilon}\exp\left(s\frac{\varepsilon^2p^2}{2}|\mathfrak{a}^{n\cdot}|^2|\sqrt{\Sigma}|^2(1-|\Gamma|)^{-2}\right).$$

Finally, combining the above inequalities, we obtain

$$\mathbb{E}\left[|\Delta_s|\right] \leq C_{p,\epsilon}\varepsilon s\exp\left(s\frac{\varepsilon^2p}{2}|\mathfrak{a}^{n\cdot}|^2|\sqrt{\Sigma}|^2(1-|\Gamma|)^{-2}\right),$$

and then

$$\sum_{s=1}^{K}\mathfrak{C}e^{\varrho_n s}\mathbb{E}\left[|\Delta_s|\right] \leq \varepsilon\sum_{s=1}^{K}C_{p,\epsilon}se^{p(\rho-r)s}.$$

For $p-1>0$ small enough, we thus get

$$\|(V_t^{n,K}-\mathcal{V}_t^{n,K})/F_t^n\|_1 \leq \sum_{s=1}^{K}\mathfrak{C}e^{\varrho_n s}\mathbb{E}\left[|\Delta_s|\right] \leq C\varepsilon.$$

The proof is thus concluded letting K goes to infinity in (4.1.19). □

It follows from Lemma 4.1.6 that at time $t \in \mathbb{N}$, the (proxy of the) firm value $\mathcal{V}_t^n$ is a function of the productivity processes $\mathcal{A}_t$, the carbon price process δ, the carbon intensities processes τ, ζ, κ, the parameters F_0^n, $\mathfrak{a}^{n\cdot}$, $\sigma_{\mathfrak{b}^n}^2, \varepsilon$ and the different parameters introduced in Section 3. In addition, by applying the log function to (4.1.16), we get

$$\log\mathcal{V}_t^n = \log F_0^n\mathfrak{R}_t^n(\mathfrak{d}) - \mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_0) + \mathfrak{a}^{n\cdot}\mathcal{A}_t^\circ + \mathcal{W}_t^n.$$

It appears that our model is close to the Vasicek one Vasicek [2002] (which assumes that the asset value depends on a common risk factor and an idiosyncratic one, both being two independent standard Gaussian random variables). However, the main differences are that (1) our systemic factor $\mathcal{A}$, standing for the cumulative log-productivity growth, is not a standard Gaussian random variable but a non-stationary and non-centered Gaussian one, (2) our idiosyncratic factor $\mathcal{W}$, representing the noise of cumulative cash flows growth, is a Gaussian random variable but a non-stationary and centered one, and (3) we introduce an additional term depending on climate transition risk through the carbon price process.

Remark 4.1.8. We could also be interested in the different sensitivities of cash flows and the value of the firm to the carbon price. But as in the case of macroeconomic variables, we would obtain expressions that are not very tractable.

4.1.3 Laws of the firm value

Moreover, we can identify the law of $\mathcal{V}_t^n|\mathcal{G}_t$.

Corollary 4.1.9. *For all $t \in \mathbb{N}$,*

$$(\log \mathcal{V}_t^n)_{1\leq n\leq N}|\mathcal{G}_t \sim \mathcal{N}\left(\log(F_0) + \mathfrak{m}(\delta, t, \mathcal{A}_t^\circ), \mathrm{diag}[t\sigma^2_{\mathfrak{b}_n}]\right),$$

with for $n \in \{1, \dots, N\}$,

$$\mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ) := \mathfrak{a}^{n\cdot}\left(\mathcal{A}_t^\circ - \mathfrak{v}(\mathfrak{d}_0)\right) + \log(\mathfrak{R}_t^n(\mathfrak{d})).$$

Proof. Let $t \geq 1$ and $n \in \{1, \dots, N\}$, we have from (4.1.16)

$$\mathcal{V}_t^n = F_0^n \mathfrak{R}_t^n(\mathfrak{d}) \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}_t^\circ - \mathfrak{v}(\mathfrak{d}_0))\right) \exp\left(\sum_{u=1}^t \mathfrak{b}_u^n\right),$$

then

$$\log(\mathcal{V}_t^n) = \log(F_0^n) + \log(\mathfrak{R}_t^n(\mathfrak{d})) + \mathfrak{a}^{n\cdot}(\mathcal{A}_t^\circ - \mathfrak{v}(\mathfrak{d}_0)) + \sum_{u=1}^t \mathfrak{b}_u^n.$$

Therefore $\log(\mathcal{V}_t^n)|\mathcal{G}_t \sim \mathcal{N}\left(\log(F_0^n\mathfrak{R}_t^n(\mathfrak{d})) + \mathfrak{a}^{n\cdot}(\mathcal{A}_t^\circ - \mathfrak{v}(\mathfrak{d}_0)), t\sigma^2_{\mathfrak{b}^n}\right)$ and the conclusion follows. □

The following remark gives the law of the firm value at time $t+T$ conditionally on $\mathcal{G}_t$, with $t, T \in \mathbb{N}$.

Remark 4.1.10. Let $(\Upsilon_u)_{u\in\mathbb{N}}$ be as in A.1. For $t, T \in \mathbb{N}$ and $1 \leq n \leq N$, denote

$$\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t) := \mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ) + \log\left(\frac{\mathfrak{R}_{t+T}^n(\mathfrak{d})}{\mathfrak{R}_t^n(\mathfrak{d})}\right) + \mathfrak{a}^{n\cdot}\Gamma\Upsilon_{T-1}\Theta_t + \mathfrak{a}^{n\cdot}\left(\sum_{u=1}^T \Upsilon_{u-1}\right)\mu, \tag{4.1.22}$$

and

$$\mathcal{L}^n(t,T) := \sigma^2_{\mathfrak{b}^n}(t+T) + \varepsilon^2 \sum_{u=1}^{T} (\mathfrak{a}^{n\cdot}\Upsilon_{T-u})\Sigma(\mathfrak{a}^{n\cdot}\Upsilon_{T-u})^\top. \tag{4.1.23}$$

We have

$$\log(\mathcal{V}^n_{t+T})|\mathcal{G}_t \sim \mathcal{N}\left(\log(F^n_0) + \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \Theta_t), \mathcal{L}^n(t,T)\right).$$

Proof of Remark 4.1.10. Let $t, T \geq 1$, we have from (4.1.16),

$$\begin{aligned}\mathcal{V}^n_{t+T} &= F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}^\circ_{t+T} - \mathfrak{v}(\mathfrak{d}_0))\right) \exp\left(\sum_{u=1}^{t+T} \mathfrak{b}^n_u\right) \\ &= F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) \exp\left(-\mathfrak{a}^{n\cdot}\mathfrak{v}(\mathfrak{d}_0)\right) \exp\left(\mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T}\right) \exp\left(\mathcal{W}^n_{t+T}\right)\end{aligned}$$

But $\mathcal{A}^\circ_{t+T} = \mathcal{A}^\circ_t + \sum_{u=t+1}^{t+T} \Theta_u$, then

$$\begin{aligned}\mathcal{V}^n_{t+T} &= F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}^\circ_t - \mathfrak{v}(\mathfrak{d}_0))\right) \exp\left(\mathfrak{a}^{n\cdot} \sum_{u=t+1}^{t+T} \Theta_u\right) \exp\left(\mathcal{W}^n_{t+T}\right) \\ &= F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}^\circ_t - \mathfrak{v}(\mathfrak{d}_0))\right) \exp\left(\mathfrak{a}^{n\cdot} \sum_{u=1}^{T} \Theta_{t+u} + \mathcal{W}^n_{t+T}\right).\end{aligned}$$

But recall from Remark 3.1.2 and A.1, for all $u \in \{1, \ldots, T\}$,

$$\Theta_{t+u} = \Gamma^u \Theta_t + \Upsilon_{u-1}\mu + \varepsilon \sum_{v=1}^{u} \Gamma^{u-v}\mathcal{E}_{t+v},$$

then

$$\begin{aligned}\sum_{u=1}^{T} \Theta_{t+u} &= \sum_{u=1}^{T} \Gamma^u \Theta_t + \sum_{u=1}^{T} \Upsilon_{u-1}\mu + \varepsilon \sum_{u=1}^{T}\sum_{v=1}^{u} \Gamma^{u-v}\mathcal{E}_{t+v} \\ &= \Gamma\Upsilon_{T-1}\Theta_t + \left(\sum_{u=1}^{T} \Upsilon_{u-1}\right)\mu + \varepsilon \sum_{v=1}^{T} \Upsilon_{T-v}\mathcal{E}_{t+v}.\end{aligned}$$

From Assumptions 3.1.1 and 4.1.1, we have

$$\left(\sum_{u=1}^{T} \mathfrak{a}^{n\cdot}\Theta_{t+u} + \mathcal{W}^n_{t+T} \middle| \mathcal{G}_t\right) \sim$$
$$\mathcal{N}\left(\mathfrak{a}^{n\cdot}\Gamma\Upsilon_{T-1}\Theta_t + \mathfrak{a}^{n\cdot}\left(\sum_{u=1}^{T} \Upsilon_{u-1}\right)\mu, \varepsilon^2 \sum_{u=1}^{T} (\mathfrak{a}^{n\cdot}\Upsilon_{T-u})\Sigma(\Upsilon_{T-u}\mathfrak{a}^{n\cdot})^\top + \sigma^2_{\mathfrak{b}^n}(t+T)\right),$$

and the conclusion follows. □

In the following, we will work directly with $\mathcal{V}^n_t$ instead of V^n_t, as it appears to be a tractable proxy (its law can be easily identified). Indeed, this is justified when the noise term in the productivity process is small as shown in the following result Baker et al. [2005].

4.2 A credit risk model

4.2.1 General information on credit risk

In their credit risk assessment, Basel Committee on Banking Supervision [2017] introduces four parameters: the probability of *default* (PD) measures the default risk associated with each borrower, the exposure at default (EAD) measures the outstanding debt at the time of default, the loss given default (LGD) denotes the expected percentage of EAD that is lost if the debtor defaults, and the effective maturity T represents the duration of the credit. With these four parameters, we can compute the portfolio loss L, with a few assumptions:

Assumption 4.2.1. Consider a portfolio of $N \in \mathbb{N}^*$ credits. For $1 \leq n \leq N$,

(1) Firm n has issued two classes of securities: equity and debt;

(2) $(\mathrm{EAD}_t^n)_{t\in\mathbb{N}^*}$ is a $\mathbb{R}_*^+$-valued deterministic process;

(3) $(\mathrm{LGD}_t^n)_{t\in\mathbb{N}^*}$ is a $(0,1]$-valued deterministic process;

(4) the default barrier $\mathcal{D}^n \in \mathbb{R}^+$ is a deterministic scalar that we will use to define the conditions under which a borrower is considered to be in default. We will also denote $B^n := \dfrac{\mathcal{D}^n}{F_0^n}$ as debt on cash flow ratio,

(5) the value of the firm n at time t is assumed to be a tradable asset given by $\mathcal{V}_t^n$ defined in (4.1.16).

Even if the LGD and the EAD are assumed here to be deterministic, we could take them to be stochastic. In particular, they could (or should) depend on the climate transition scenario: (1) the LGD could be impacted by the premature write down of assets - that is stranded assets - due to the climate transition, while (2) the EAD could depend on the bank's balance sheet, which can be modified according to the bank's policy or to the credit conversion factor of the obligor (if related to climate transition). This will be the object of future research.

According to Kruschwitz and Löffler [2020], there are two ways to handle the default of a company: for a given financing policy, a levered firm is

1. *in danger of illiquidity* if the cash flows do not suffice to fulfill the creditors' payment claims (interest and net redemption) as contracted,

2. *over-indebted* if the market value of debt exceeds the firm's market value.

We recall that for all $n \in \{1, \ldots, N\}$, we consider $\mathcal{V}_t^n$, defined in (4.1.15), to be the proxy value of firm n at time t and its conditional law given in Corollary 4.1.9. We consider the second definition proposed above: a firm defaults when it is *over-indebted*, that is in fact the same approach used in the structural credit risk models. Therefore, the default of entity n occurs when $\mathcal{V}_t^n$ falls below a given barrier $\mathcal{D}^n$, related to the net debt, given in Assumption 4.2.1(3).

Definition 4.2.2. For $t \geq 1$, the potential loss of the portfolio at time t is defined as

$$L_t^N := \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}}. \tag{4.2.1}$$

We take the point of view of the bank managing its credit portfolio and which has to compute various risk measures impacting its daily/monthly/quarterly/yearly routine, some of which may be required by regulators. We are also interested in understanding and visualizing how these risk measures evolve in time and particularly how they change due to carbon price paths, i.e. due to transition scenarios. This explains why all these measures are defined below with respect to the information available at t, namely the $\mathbb{F}$-filtration.

We now study statistics of the process $(L_t^N)_{t\geq 0}$, typically its mean, variance, and quantiles, under various transition scenarios. This could be achieved through (intensive) numerical simulations, however we shall assume that the portfolio is fine grained so that the idiosyncratic risks can be averaged out. The above quantities can then be approximated by only taking into account the common risk factors. We thus make the following assumption:

Assumption 4.2.3. For all $t \in \mathbb{N}^*$, the family $(\mathrm{EAD}_t^n)_{n=1,\dots,N}$ is a sequence of positive constants such that

1. $\sum_{n\geq 1} \mathrm{EAD}_t^n = +\infty$;
2. there exists $\upsilon > 0$ such that $\frac{\mathrm{EAD}_t^n}{\sum_{n=1}^N \mathrm{EAD}_t^n} = \mathcal{O}(N^{-(\frac{1}{2}+\upsilon)})$, as N tends to infinity.

The following theorem, similar to the one introduced in [Gordy, 2003a, Propositions 1&2] and used when a portfolio is perfectly fine grained, shows that we can approximate the portfolio loss by the conditional expectation of losses given the systemic factor. For all $t \in \mathbb{N}$, define

$$\mathrm{L}_t^{\mathbb{G},N} := \mathbb{E}\left[L_t^N \middle| \mathcal{G}_t\right] = \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathrm{PD}_t^n,$$

and

$$\mathrm{PD}_t^n := \Phi\left(\frac{\log(B^n) - \mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t}}\right), \tag{4.2.2}$$

where $\mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ)$ is defined in Corollary 4.1.9.

Theorem 4.2.4. *Under Assumptions 4.2.1 and 4.2.3, $L_t^N - \mathrm{L}_t^{\mathbb{G},N}$ converges to zero almost surely as N tends to infinity, for each $t \in \mathbb{N}$.*

This implies that, at each time $t \in \mathbb{N}$, in the limit, we only require the knowledge of $\mathrm{L}_t^{\mathbb{G},N}$ to approximate the distribution of L_t^N. In the following, we will use $\mathrm{L}_t^{\mathbb{G},N}$ as a proxy for L_t^N.

Proof. Let $t \in \mathbb{N}$. We have

$$\begin{aligned}
\mathrm{L}_t^{\mathbb{G},N} &= \mathbb{E}\left[L_t^N \middle| \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}} \middle| \mathcal{G}_t\right] \quad \text{from (4.2.1)} \\
&= \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbb{E}\left[\mathbf{1}_{\{\mathcal{V}_t^n \leq \mathcal{D}^n\}} \middle| \mathcal{G}_t\right] \quad \text{from (1) and (3) in Assumption 4.2.1} \\
&= \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathbb{P}\left[\mathcal{V}_t^n \leq \mathcal{D}^n\}|\mathcal{G}_t\right] \\
&= \sum_{n=1}^{N} \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \Phi\left(\frac{\log(\mathcal{D}^n) - \log(F_0^n) - \mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t}}\right) \text{ from Corollary 4.1.9.}
\end{aligned}$$

The rest of the proof requires a version of the strong law of large numbers (Appendix of [Gordy, 2003a, Propositions 1, 2]), where the systematic risk factor is $\mathcal{A}_t^\circ$. □

For stress testing, it is fundamental to estimate through some statistics of loss, bank's capital evolution. In particular, some key measures for the bank to understand the (dynamics of the) risk in its portfolios of loans are the loss and the probability of *default* conditionally to the information generated by the risk factors. We would like to understand how these key measures are distorted when GHG emissions of firms and of households are charged. To this aim, we rely on the results derived in Section 3 and Section 4.1. Precisely, given a portfolio of $N \in \mathbb{N}^*$ counterparts, each of which belonging to any sector, for a date $t \in \mathbb{N}$ and a time horizon $T \in \mathbb{N}$, we would like to know these risk measures at t of the portfolio at time horizon T.

Definition 4.2.5. Let $t \geq 0$ be the time at which the risk measures are computed over a period $T \geq 1$. As classically done (as shown in Figure 4.1 and detailed in Yeh et al. [2005]), the potential loss is divided into three components:

- The conditional Expected Loss (EL) is the amount that an institution expects to lose on a credit exposure seen at t and over a given time horizon T. It has to be quantified/included into the products and charged to the clients, and reads

$$\mathrm{EL}_t^{N,T} := \mathbb{E}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} \middle| \mathcal{G}_t\right]. \tag{4.2.3}$$

  In the normal course of business, a financial institution should set aside an amount equal to the EL as a provision or reserves, even if it should be covered from the portfolio's earnings.

- The conditional Unexpected Loss (UL) is the amount by which potential credit losses might exceed the EL. The UL should be covered by capital requirements. For $\alpha \in (0,1)$,

$$\mathrm{UL}_{t,\alpha}^{N,T} := \mathrm{VaR}_t^{\alpha,N,T} - \mathrm{EL}_t^{N,T}, \quad \text{where} \quad 1-\alpha = \mathbb{P}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]. \tag{4.2.4}$$

- The Stressed Loss (or Expected Shortfall or ES) is the amount by which potential credit losses might exceed the capital requirement $\mathrm{VaR}_t^{\alpha}(L_s^N)$:

$$\mathrm{ES}_{t,\alpha}^{N,T} := \mathbb{E}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} \middle| \mathrm{L}_{t+T}^{\mathbb{G},N} \geq \mathrm{VaR}_t^{\alpha,N,T}, \mathcal{G}_t\right], \qquad \text{for } \alpha \in (0,1).$$

  This loss is mitigated through economic capital.

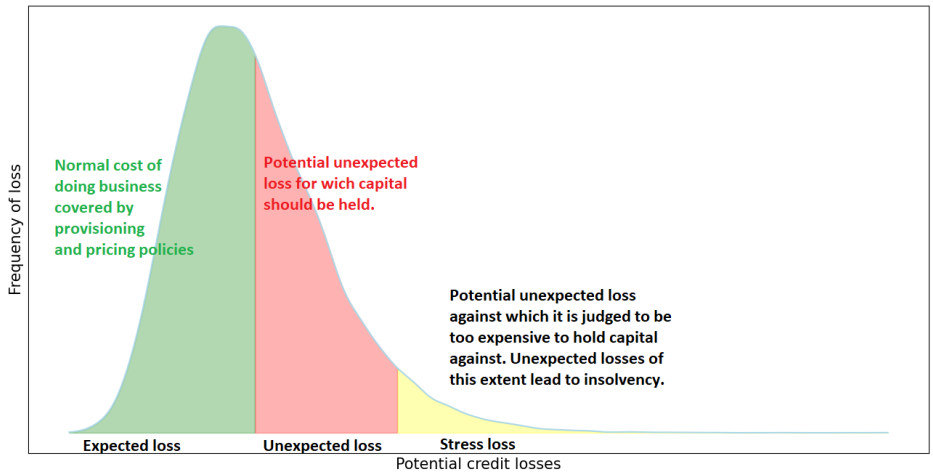


Figure 4.1: An example of loss distribution. Source: Page 8 in Yeh et al. [2005].

In the following sections, we write the expression of the portfolio EL and UL as functions of the parameters and of the processes introduced above, and introduce the entity's probability of *default*.

4.2.2 Expected loss

The following proposition computes the probability of default of each firm and the portfolio expected loss.

Proposition 4.2.6. *Let* $(\Upsilon_u)_{u\in\mathbb{N}}$ *and* $(\mathfrak{R}_u^n(\mathfrak{d}))_{u\in\mathbb{N}}$ *(with* $n \in \{1,\dots,N\}$*) be as in A.1 and* (4.1.17) *respectively. For* $(a,\theta) \in \mathbb{R}^I \times \mathbb{R}^I$, $t \in \mathbb{N}$, $T \in \mathbb{N}^*$, *and* $n \in \{1,\dots,N\}$, *define*

$$\mathfrak{L}^n(\mathfrak{d},t,T,a,\theta) := \Phi\left(\frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d},t,T,a,\theta)}{\sqrt{\mathcal{L}^n(t,T)}}\right),$$

where $\mathcal{K}^n(\mathfrak{d},t,T,a,\theta)$ *and* $\mathcal{L}^n(t,T)$ *are defined in Remark 4.1.10. Then, the (conditional) probability of default of the entity* n *at time* t *over the time horizon* T *is*

$$\mathrm{PD}_{t,T,\mathfrak{d}}^n := \mathbb{P}\left(\mathcal{V}_{t+T}^n \leq \mathcal{D}^n | \mathcal{G}_t\right) = \mathfrak{L}^n(\mathfrak{d},t,T,\mathcal{A}_t^\circ,\Theta_t), \tag{4.2.5}$$

and the (conditional) EL of the portfolio at time t over the time horizon T reads

$$\mathrm{EL}_{t,\mathfrak{d}}^{N,T} := \mathrm{EL}_t^{N,T} = \sum_{n=1}^{N} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n \cdot \mathfrak{L}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t). \tag{4.2.6}$$

Proof. Let $t \in \mathbb{N}$ and $T \in \mathbb{N}^*$. For $1 \leq n \leq N$, (4.1.10) gives the law of $\log(\mathcal{V}_{t+T}^n)|\mathcal{G}_t$, and (4.2.5) follows. Moreover,

$$\begin{aligned}
\mathrm{EL}_{t,\mathfrak{d}}^{N,T} &= \mathbb{E}\left[\mathrm{L}_{t+T}^{\mathbb{G},N}|\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{N} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n \cdot \Phi\left(\frac{\log(\mathcal{D}^n) - \log(F_0^n) - \mathfrak{m}^n(\mathfrak{d}, t+T, \mathcal{A}_{t+T}^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\right)\middle|\mathcal{G}_t\right] \\
&= \sum_{n=1}^{N} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n \cdot \mathbb{E}\left[\Phi\left(\frac{\log(B^n) - \mathfrak{m}^n(\mathfrak{d}, t+T, \mathcal{A}_{t+T}^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\right)\middle|\mathcal{G}_t\right],
\end{aligned}$$

where the last equality comes from Assumption 4.2.1(1)-(3). However,

$$\begin{aligned}
\mathfrak{m}^n(\mathfrak{d}, t+T, \mathcal{A}_{t+T}^\circ) &= \mathfrak{a}^{n\cdot}\left(\mathcal{A}_{t+T}^\circ - \mathfrak{v}(\mathfrak{d}_0)\right) + \log(\mathfrak{R}_{t+T}^n(\mathfrak{d})) \\
&= \mathfrak{a}^{n\cdot}\left(\mathcal{A}_t^\circ + \sum_{u=t+1}^{t+T} \Theta_u - \mathfrak{v}(\mathfrak{d}_0)\right) + \log(\mathfrak{R}_{t+T}^n(\mathfrak{d})) \\
&= \mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ) + \log(\mathfrak{R}_{t+T}^n(\mathfrak{d})) - \log(\mathfrak{R}_t^n(\mathfrak{d})) + \mathfrak{a}^{n\cdot}\sum_{u=1}^{T} \Theta_{t+u}.
\end{aligned}$$

For all $\theta \in \mathbb{R}^d$, according to (A.1.1),

$$\left(\sum_{u=1}^{T} \Theta_{t+u}\middle|\Theta_t = \theta\right) \sim \mathcal{N}\left(\Gamma\Upsilon_{T-1}\theta + \left(\sum_{u=1}^{T}\Upsilon_{u-1}\right)\mu, \varepsilon^2\sum_{u=1}^{T}\Upsilon_{T-u}\Sigma(\Upsilon_{T-u})^\top\right),$$

Let $n \in \{1, \ldots, N\}$, therefore,

$$\left(\mathfrak{a}^{n\cdot}\sum_{u=1}^{T} \Theta_{t+u}\middle|\mathcal{G}_t\right) \sim \mathcal{N}\left(\mathfrak{a}^{n\cdot}\Gamma\Upsilon_{T-1}\Theta_t + \mathfrak{a}^{n\cdot}\left(\sum_{u=1}^{T}\Upsilon_{u-1}\right)\mu, \varepsilon^2\sum_{u=1}^{T}(\mathfrak{a}^{n\cdot}\Upsilon_{T-u})\Sigma(\mathfrak{a}^{n\cdot}\Upsilon_{T-u})^\top\right).$$

Then

$$\left(\frac{\log(B^n) - \mathfrak{m}^n(\mathfrak{d}, t+T, \mathcal{A}_{t+T}^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\middle|\mathcal{G}_t\right) \sim \frac{\mathcal{S}^n(T)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\mathcal{X}^n + \frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}, \tag{4.2.7}$$

where $(\mathcal{X}^n)_{1\leq n\leq N} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_N)$, and where $\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)$ is defined in (4.1.22) and where

$$\mathcal{S}^n(T) := \varepsilon\sqrt{\sum_{u=1}^{T}(\mathfrak{a}^{n\cdot}\Upsilon_{T-u})\Sigma(\mathfrak{a}^{n\cdot}\Upsilon_{T-u})^\top}.$$

We then have

$$\begin{aligned}
&\mathbb{E}\left[\Phi\left(\frac{\mathcal{S}^n(T)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\mathcal{X}^n + \frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\right)\middle|\mathcal{G}_t\right] \\
&\quad = \mathbb{E}_{\mathcal{X}^n}\left[\Phi\left(\frac{\mathcal{S}^n(T)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\mathcal{X}^n + \frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\right)\right] \\
&\quad = \int_{-\infty}^{+\infty}\Phi\left(\frac{\mathcal{S}^n(T)}{\sigma_{\mathfrak{b}^n}\sqrt{(t+T)}}x + \frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^n}\sqrt{(t+T)}}\right)\phi(x)\mathrm{d}x \\
&\quad = \Phi\left(\frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)}{\sqrt{\mathcal{L}^n(t,T)}}\right),
\end{aligned}$$

where $\mathcal{L}^n(t,T)$ is defined in (4.1.23), and the conclusion follows.

The last equality comes from the following result found in [Roncalli, 2020, Page 1063]: if Φ and ϕ are the Gaussian cumulative distribution and density functions, then for $a, b \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty}\Phi(a+bx)\phi(x)\mathrm{d}x = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right).$$

□

4.2.3 Unexpected loss

At time $t \in \mathbb{N}$ and over a given time horizon $T \in \mathbb{N}^*$, it follows from the definition of UL in (4.2.4) that we need to compute the quantile of the (proxy of the) loss distribution $\mathrm{L}_{t+T}^{\mathbb{G},N}$. For $\alpha \in (0,1)$, we obtain from Theorem 4.2.4,

$$\begin{aligned}
1-\alpha &= \mathbb{P}\left[\mathrm{L}_{t+T}^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T}\middle|\mathcal{G}_t\right] \\
&= \mathbb{P}\left[\sum_{n=1}^{N}\mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n \cdot \Phi\left(\frac{\log(B^n) - \mathfrak{m}^n(\mathfrak{d}, t+T, \mathcal{A}_{t+T}^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\right) \leq \mathrm{VaR}_t^{\alpha,N,T}\middle|\mathcal{G}_t\right].
\end{aligned}$$

However, it follows from (4.2.7),

$$1-\alpha = \mathbb{P}_{\mathcal{X}^1,\ldots,\mathcal{X}^N}\left[\sum_{n=1}^{N}\mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n \cdot \Phi\left(\frac{\mathcal{S}^n(T)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\mathcal{X}^n + \frac{\log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^n}\sqrt{t+T}}\right) \leq \mathrm{VaR}_t^{\alpha,N,T}\right] \quad (4.2.8)$$

Since the quantile function is not linear, one cannot find an analytical solution. Therefore, a numerical solution is needed. Recall that we must simulate $(\mathcal{X}^1, \ldots, \mathcal{X}^N)$ to find $\mathrm{VaR}_t^{\alpha,N,T}$, which will also be a function of the random variables $(\mathcal{A}_t^\circ, \Theta_t)$, of dimension $2I$. This can be solved for example by Monte Carlo Gordy and Juneja [2010] or by deep learning techniques Barrera et al. [2022].

4.2.4 Projection of one-year risk measures of the sub-portfolios

At this stage, we use (4.2.5) to compute, for each $n \in \{1, \dots, N\}$, the probability of *default* of a given firm n at maturity T, stressed by the (deterministic) carbon price δ. We can also calculate EL using (4.2.6) and UL using (4.2.8). We first need the parameters, especially $\mathbf{a}^n$, $\sigma^2_{\mathfrak{b}^n}$, F_0^n, and $\mathcal{D}^n$. We can distinguish two ways to determine them:

1. **Firm's view:** $\mathbf{a}^n$, $\sigma^2_{\mathfrak{b}^n}$ and F_0^n are calibrated on the firm's historical free cash flows, while $\mathcal{D}^n$ relates to the principal of its loans.

2. **Portfolio's view:** if we assume that there is just one risk class in the portfolio so that all the firms have the same $\mathbf{a}^n$, $\sigma^2_{\mathfrak{b}^n}$, and B^n (and not $\mathcal{D}^n$), then knowing the historical default of the portfolio, we can use a log-likelihood maximization as in Gordy and Heitfield [2002] to determine them.

Let us introduce the following assumption related to the portfolio view.

Assumption 4.2.7. For each $1 \leq i \leq I$, since there is only one risk class in the sub-portfolio g_i, we have for any $n \in g_i$, $\mathbf{a}^n = \mathbf{a}^{n_i}$, $\sigma^2_{\mathfrak{b}^n} = \sigma^2_{\mathfrak{b}^{n_i}}$, and $B^n = B^{n_i}$.

In our setting, since each firm of the sub-portfolio i belongs to the sector i, the risk factor of the sub-portfolio i is $(\Delta^Y)^i$ after calling (4.1.1). In practice, banks need to compute the one-year probability of *default*. For clarity, we thus simplify the expressions for the risk measures by setting $T = 1$ from now on.

Corollary 4.2.8. *Under Assumption 4.2.7, for $t \in \mathbb{N}$ and $1 \leq m \leq I$ and for each $n \in g_m$, the one-year (conditional) probability of default of firm n at time t is*

$$\mathrm{PD}^n_{t,1,\mathfrak{d}} = \mathrm{PD}^{n_i}_{t,1,\mathfrak{d}} = \Phi\left(\frac{\log(B^{n_i}) - \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}^\circ_t, \Theta_t)}{\sqrt{\mathcal{L}^{n_i}(t,1)}}\right). \tag{4.2.9}$$

Expected loss

The following corollary, whose proof follows from Corollary 4.2.8, gives a simplified formula for EL.

Proposition 4.2.9. *Under Assumption 4.2.7, the one-year (conditional) EL of the sub-portfolio g_m with $m \in \{1, \dots, I\}$ at time t is (with $\mathrm{PD}^{n_i}_{t,1,\mathfrak{d}}$ defined in Corollary 4.2.8)*

$$\mathrm{EL}^{g_m,1}_{t,\mathfrak{d}} = \left(\sum_{n \in g_m} \mathrm{EAD}^n_{t+1} \cdot \mathrm{LGD}^n_{t+1}\right) \cdot \mathrm{PD}^{n_i}_{t,1,\mathfrak{d}}. \tag{4.2.10}$$

Unexpected loss

We saw in (4.2.8) that determining the UL is not possible analytically and is numerically intensive (since quantiles depend on rare events and because of the dimension of the macroeconomic factors). However, Assumption 4.2.3 allows to further simplify the formula for the UL.

Corollary 4.2.10. *Under Assumption 4.2.7, the one-year (conditional) UL of the sub-portfolio g_m with $m \in \{1, \dots, I\}$ at time t is*

$$\mathrm{UL}_{t,\mathfrak{d},\alpha}^{g_m,1} = \left(\sum_{n \in g_m} \mathrm{EAD}_{t+1}^n \cdot \mathrm{LGD}_{t+1}^n \right) \left[\Phi \left(\frac{\mathcal{S}^{n_i}(1)\Phi^{-1}(1-\alpha) + \log(B^{n_i}) - \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^{n_i}} \sqrt{t+1}} \right) - \mathrm{PD}_{t,1,\mathfrak{d}}^{n_i} \right]. \tag{4.2.11}$$

Proof. From (4.2.8), we have

$$1 - \alpha = \mathbb{P}_{\mathcal{X}^1, \dots, \mathcal{X}^N} \left[\sum_{n=1}^N \mathrm{EAD}_{t+1}^n \cdot \mathrm{LGD}_{t+1}^n \cdot \Phi \left(\frac{\mathcal{S}^n(1)\mathcal{X}^n + \log(B^n) - \mathcal{K}^n(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^n} \sqrt{t+1}} \right) \leq \mathrm{VaR}_t^{\alpha, g_m, 1} \right],$$

but with Assumption 4.2.7,

$$\begin{aligned}
1 - \alpha &= \mathbb{P}_{\mathcal{X}^{n_i}} \left[\Phi \left(\frac{\mathcal{S}^{n_i}(1)\mathcal{X}^{n_i} + \log(B^{n_i}) - \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^{n_i}} \sqrt{t+1}} \right) \leq \frac{\mathrm{VaR}_t^{\alpha, g_m, 1}}{\sum_{n \in g_m} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n} \right] \\
&= \mathbb{P}_{\mathcal{X}^{n_i}} \left[\frac{\mathcal{S}^{n_i}(1)\mathcal{X}^{n_i} + \log(B^{n_i}) - \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^{n_i}} \sqrt{t+1}} \leq \Phi^{-1} \left(\frac{\mathrm{VaR}_t^{\alpha, g_m, T}}{\sum_{n=1}^N \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T}^n} \right) \right] \\
&= \mathbb{P}_{\mathcal{X}^{n_i}} \left[\mathcal{X}^{n_i} \leq \frac{1}{\mathcal{S}^{n_i}(1)} \left(\sigma_{\mathfrak{b}^{n_i}} \sqrt{t+1} \Phi^{-1} \left(\frac{\mathrm{VaR}_t^{\alpha, g_m, 1}}{\sum_{n \in g_m} \mathrm{EAD}_{t+1}^n \cdot \mathrm{LGD}_{t+1}^n} \right) - \log(B^{n_i}) + \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t) \right) \right].
\end{aligned}$$

By recalling that $\mathcal{X}^{n_i} \sim \mathbb{N}(0,1)$, we have

$$1 - \alpha = \Phi \left(\frac{1}{\mathcal{S}^{n_i}(1)} \left(\sigma_{\mathfrak{b}^{n_i}} \sqrt{t+1} \Phi^{-1} \left(\frac{\mathrm{VaR}_t^{\alpha, g_m, 1}}{\sum_{n \in g_m} \mathrm{EAD}_{t+1}^n \cdot \mathrm{LGD}_{t+1}^n} \right) - \log(B^{n_i}) + \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t) \right) \right).$$

Therefore,

$$\mathrm{VaR}_t^{\alpha, g_m, 1} = \left(\sum_{n \in g_m} \mathrm{EAD}_{t+1}^n \cdot \mathrm{LGD}_{t+T}^n \right) \cdot \Phi \left(\frac{\mathcal{S}^{n_i}(1)\Phi^{-1}(1-\alpha) + \log(B^{n_i}) - \mathcal{K}^{n_i}(\mathfrak{d}, t, 1, \mathcal{A}_t^\circ, \Theta_t)}{\sigma_{\mathfrak{b}^{n_i}} \sqrt{t+1}} \right),$$

the conclusion follows. □

4.2.5 Sensitivity of losses to a carbon price

We would like to quantify the variation of losses for a given variation in the carbon price.

Definition 4.2.11. For our portfolio of N firms and for $\alpha \in (0,1)$, we introduce the sensitivity of expected and unexpected losses to a carbon price, at time $t \in \mathbb{N}$ over the time horizon $T \in \mathbb{N}^*$, and for a given sequence of carbon prices δ, respectively denoted $\Gamma_{t,\delta}^{N,T,\mathrm{EL}}(\mathfrak{U})$ and $\Gamma_{t,\delta,\alpha}^{N,T,\mathrm{UL}}(\mathfrak{U})$, as being,

$$\Gamma_{t,\delta}^{N,T,\mathrm{EL}}(\mathfrak{U}) := \lim_{\vartheta \to 0} \frac{\mathrm{EL}_{t,\mathfrak{d}'}^{N,T} - \mathrm{EL}_{t,\mathfrak{d}}^{N,T}}{\vartheta} \qquad \text{and} \qquad \Gamma_{t,\delta,\alpha}^{N,T,\mathrm{UL}}(\mathfrak{U}) := \lim_{\vartheta \to 0} \frac{\mathrm{UL}_{t,\mathfrak{d}+\vartheta\mathfrak{U},\alpha}^{N,T} - \mathrm{UL}_{t,\mathfrak{d},\alpha}^{N,T}}{\vartheta},$$

where for all $t \in \mathbb{N}$ $\mathfrak{d}'_t := (\tau_t(\delta_t + \vartheta\mathfrak{U}_t), \zeta_t(\delta_t + \vartheta\mathfrak{U}_t), \kappa_t(\delta_t + \vartheta\mathfrak{U}_t))$, and where $\mathfrak{U} \in (\mathbb{R}_+)^{\mathbb{N}}$ is chosen so that there exists a neighborhood v of the origin so that for all $\vartheta \in v$ and $t \in \mathbb{N}$, $\tau_t(\delta_t + \vartheta\mathfrak{U}_t) < 1$.

These sensitivities can be computed and understood in two different ways depending of the direction $\mathfrak{U}$: either in relation to the entire carbon price trajectory or at a given date. In the same way, we could introduce sensitivities to other variables (productivity or carbon intensities) or parameters (elasticities, discount rate, standard deviation of cash flows, etc.). We could also (and will so in a future note) give the results for a stochastic carbon price in the transition period. In this case, if the productivity Θ and the carbon price δ are independent, it is enough to add in the previous results, the expectation conditionally to δ.

4.2.6 Joint modelling of PD and LGD

In the previous section, we assume, as introduced in Assumption 4.2.1, that LGD is known and deterministic. That is a strong assumption. There are both empirical and theoretical works showning the LGD could depends both on economic cycles Carey and Gordy [2004], Pesaran et al. [2009] and on the debtors' debt structures Schuermann and Wyman [2004]. In the same time, as Altman et al. [2005] showed in their empirical work, there are close relationship between probabilities of default and recovery rates. Recalling that LGD $= 1 -$ Recoveryrates, several authors have proposed joint modeling of PD and LGD. Here we will use the toy model proposed by Fermanian [2020].

Definition 4.2.12. For $t \geq 1$ and for each $n \in \{1, \ldots, N\}$, the potential loss of the loan n at time t is defined as the difference between the debt amount and the assets, we have

$$L_{n,t} := \frac{\mathrm{EAD}_t^n}{\mathcal{D}^n} \cdot (\mathcal{D}^n - \mathcal{V}_t^n)_+. \tag{4.2.12}$$

In the definition, $\frac{\mathrm{EAD}_t^n}{\mathcal{D}^n}$ represents the fraction of the total firm debt presents in our portfolio while $(\mathcal{D}^n - \mathcal{V}_t^n)_+$ is the total debtor loss.

We can therefore write whose assumptions and proof are similar with those of Theorem 4.2.4.

Theorem 4.2.13. *Under Assumptions 4.2.1 (except item 3) and 4.2.3, we pose*

$$\overline{L}_t^N := \sum_{n=1}^N L_{n,t} = \sum_{n=1}^N \frac{\mathrm{EAD}_t^n}{\mathcal{D}^n} \cdot (\mathcal{D}^n - \mathcal{V}_t^n)_+.$$

and

$$\overline{\mathrm{L}}_t^{\mathbb{G},N} := \mathbb{E}\left[\overline{L}_t^N \middle| \mathcal{G}_t\right] = \sum_{n=1}^N \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathrm{PD}_t^n,$$

where PD_t^n *is defined in* (4.2.2) *and*

$$\begin{aligned}\mathrm{LGD}_t^n &:= \mathbb{E}\left[\frac{(\mathcal{D}^n - \mathcal{V}_t^n)}{\mathcal{D}^n}\middle| \mathcal{D}^n \geq \mathcal{V}_t^n, \mathcal{G}_t\right] \\ &= 1 - \frac{1}{\mathrm{PD}_t^n}\exp\left(-\sigma_{\mathfrak{b}^n}\sqrt{t}\Phi^{-1}(\mathrm{PD}_t^n) + \frac{1}{2}t\sigma_{\mathfrak{b}^n}^2\right)\Phi\left(\Phi^{-1}(\mathrm{PD}_t^n) - \sigma_{\mathfrak{b}^n}\sqrt{t}\right),\end{aligned} \tag{4.2.13}$$

therefore $\overline{L}_t^N - \overline{\mathrm{L}}_t^{\mathbb{G},N}$ *converges to zero almost surely as* N *tends to infinity, for each* $t \in \mathbb{N}$

Proof. Let $t \in \mathbb{N}$, we have from (4.2.12),

$$\begin{aligned}\overline{\mathrm{L}}_t^{\mathbb{G},N} &= \mathbb{E}\left[\sum_{n=1}^N \frac{\mathrm{EAD}_t^n}{\mathcal{D}^n}\cdot(\mathcal{D}^n - \mathcal{V}_t^n)_+\middle|\mathcal{G}_t\right] \\ &= \sum_{n=1}^N \mathrm{EAD}_t^n \cdot \mathbb{E}\left[\frac{(\mathcal{D}^n - \mathcal{V}_t^n)_+}{\mathcal{D}^n}\middle|\mathcal{G}_t\right] \\ &= \sum_{n=1}^N \mathrm{EAD}_t^n \cdot \mathbb{E}\left[\frac{(\mathcal{D}^n - \mathcal{V}_t^n)\mathbf{1}_{\mathcal{D}^n \geq \mathcal{V}_t^n}}{\mathcal{D}^n}\middle|\mathcal{G}_t\right] \\ &= \sum_{n=1}^N \mathrm{EAD}_t^n \cdot \mathbb{E}\left[\frac{(\mathcal{D}^n - \mathcal{V}_t^n)}{\mathcal{D}^n}\middle|\mathcal{D}^n \geq \mathcal{V}_t^n, \mathcal{G}_t\right]\mathbb{P}[\mathcal{D}^n \geq \mathcal{V}_t^n|\mathcal{G}_t] = \sum_{n=1}^N \mathrm{EAD}_t^n \cdot \mathrm{LGD}_t^n \cdot \mathrm{PD}_t^n.\end{aligned}$$

In addition

$$\begin{aligned}\mathrm{LGD}_t^n &= \mathbb{E}\left[\frac{(\mathcal{D}^n - \mathcal{V}_t^n)}{\mathcal{D}^n}\middle|\mathcal{D}^n \geq \mathcal{V}_t^n, \mathcal{G}_t\right] = \frac{1}{\mathrm{PD}_t^n}\mathbb{E}\left[\left(1 - \frac{\mathcal{V}_t^n}{\mathcal{D}^n}\right)\mathbf{1}_{\frac{\mathcal{V}_t^n}{\mathcal{D}^n}\leq 1}\middle|\mathcal{G}_t\right] \\ &= 1 - \frac{1}{\mathrm{PD}_t^n}\mathbb{E}\left[\frac{\mathcal{V}_t^n}{\mathcal{D}^n}\mathbf{1}_{\frac{\mathcal{V}_t^n}{\mathcal{D}^n}\leq 1}\middle|\mathcal{G}_t\right] = 1 - \frac{1}{\mathrm{PD}_t^n}\mathbb{E}\left[e^{\log\left(\frac{\mathcal{V}_t^n}{\mathcal{D}^n}\right)}\mathbf{1}_{\log\left(\frac{\mathcal{V}_t^n}{\mathcal{D}^n}\right)\leq 0}\middle|\mathcal{G}_t\right] \\ &= 1 - \frac{1}{\mathrm{PD}_t^n}\exp\left(\mathfrak{m}^n(\mathfrak{d},t,\mathcal{A}_t^\circ) - \log(B^n) + \frac{1}{2}t\sigma_{\mathfrak{b}^n}^2\right)\Phi\left(\frac{\log(B^n) - \mathfrak{m}^n(\mathfrak{d},t,\mathcal{A}_t^\circ)}{\sigma_{\mathfrak{b}^n}\sqrt{t}} - \sigma_{\mathfrak{b}^n}\sqrt{t}\right).\end{aligned}$$

and by remarking that $\sigma_{\mathfrak{b}^n}\sqrt{t}\Phi^{-1}(\mathrm{PD}_t^n) = \log(B^n) - \mathfrak{m}^n(\mathfrak{d},t,\mathcal{A}_t^\circ)$. Then, as in Theorem 4.2.4, the rest of the proof requires a version of the strong law of large numbers (Appendix of [Gordy, 2003a, Propositions 1, 2]), where the systematic risk factor is $\mathcal{A}_t^\circ$. □

In the same way, we obtain the LGD as a function of the PD. The transmission of the economy's productivity and the climate transition, which affect the PD, are consequently reflected in the LGD. Afterwards, for $t, T \in \mathbb{N}$ and $n \in \{1, \ldots, N\}$, the (conditional) probability of *over-indebtedness* of the entity n at time t over the time

horizon T introduced in (4.2.3) and that we denote now $\overline{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}}$ becomes

$$\begin{aligned}\overline{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}} &= \mathbb{E}\left[\overline{\mathrm{L}}^{\mathbb{G},N}_{t+T}|\mathcal{G}_t\right] \\ &= \sum_{n=1}^{N}\mathrm{EAD}^n_t\cdot\mathbb{E}\left[\mathbb{E}\left[\left(1-\frac{\mathcal{V}^n_{t+T}}{\mathcal{D}^n}\right)\mathbf{1}_{\frac{\mathcal{V}^n_{t+T}}{\mathcal{D}^n}\leq 1}\middle|\mathcal{G}_{t+T}\right]\middle|\mathcal{G}_t\right] \\ &= \sum_{n=1}^{N}\mathrm{EAD}^n_t\cdot\mathbb{E}\left[\left(1-\frac{\mathcal{V}^n_{t+T}}{\mathcal{D}^n}\right)\mathbf{1}_{\frac{\mathcal{V}^n_{t+T}}{\mathcal{D}^n}\leq 1}\middle|\mathcal{G}_t\right].\end{aligned}$$

Finally, given Remark 4.1.10 and that $\sqrt{\mathcal{L}^n(t,T)}\Phi^{-1}(\mathrm{PD}^n_{t,T,\mathfrak{d}}) = \log(B^n) - \mathcal{K}^n(\mathfrak{d},t,T,\mathcal{A}^\circ_t,\Theta_t)$, we obtain

$$\overline{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}} = \sum_{n=1}^{N}\mathrm{EAD}^n_t\cdot\mathrm{LGD}^n_{t,T,\mathfrak{d}}\cdot\mathrm{PD}^n_{t,T,\mathfrak{d}},$$

where

$$\mathrm{LGD}^n_{t,T,\mathfrak{d}} := 1-\frac{1}{\mathrm{PD}^n_{t,T,\mathfrak{d}}}\exp\left(-\sqrt{\mathcal{L}^n(t,T)}\Phi^{-1}(\mathrm{PD}^n_t)+\frac{1}{2}\mathcal{L}^n(t,T)\right)\Phi\left(\Phi^{-1}(\mathrm{PD}^n_{t,T,\mathfrak{d}})-\sqrt{\mathcal{L}^n(t,T)}\right). \tag{4.2.14}$$

Equation (4.2.14) above gives the (conditional) LGD of the entity n at time t over the time horizon T.

4.3 Estimation and calibration

As in chapter 3, we assume that the time unit is year and we calibrate the model parameters on a set of data ranging from year $\mathfrak{t}_0$ to $\mathfrak{t}_1$. Similarly, $\mathfrak{t}_0 = 1978$ and $\mathfrak{t}_1 = t_\circ = 2021$. For each sector $i \in \mathcal{I}$ and $\mathfrak{t}_0 \leq t < \mathfrak{t}_1 = t_\circ$, we observe the output Y^i_t, the labor N^i_t, the aggregate price P^i_t, the intermediary inputs $(Z^{ji}_t)_{j\in\mathcal{I}}$, and the consumption C^i_t (recall that the transition starts at year $t_\circ$). For the sake of clarity, we will omit the dependence of each estimated parameter on $\mathfrak{t}_0$ and $\mathfrak{t}_1$. We start by estimating the various parameters of the macroeconomic model and defining the transition scenario as described in the previous section 3.2.

4.3.1 Estimation of firm and of the credit model parameters

Recall that we have a portfolio with $N \in \mathbb{N}^*$ firms (or credit) at time $t_\circ$. For each firm $n \in \{1,\ldots,N\}$, we have its historical cash flows $(F^n_t)_{t\in\mathfrak{t}_0,\ldots,\mathfrak{t}_1-1}$, hence its log-cash flow growths. For any $t \in \{\mathfrak{t}_0,\ldots,\mathfrak{t}_1-1\}$ and $1 \leq m \leq I$, we denote by r^m_t (resp. d^m_t) the number of firms in g_m rated at the beginning of the year t (resp. defaulted during the year t). In particular, $r_{\mathfrak{t}_0} = \#g_m$. Within each group g_m, all the firms behave in the same way as there is only one risk class. Since each sub-portfolio constitutes a single risk class, recall Assumption 4.2.7, we have for each $n \in g_m$, $\mathbf{a}^n = \mathbf{a}^{n_i}$, $\sigma_{\mathfrak{b}^n} = \sigma_{\mathfrak{b}^{n_i}}$, and $B^n = B^{n_i}$. We then proceed as follows:

1. Knowing the output growth $\left(\Delta_t^Y\right)_{t\in\{\mathfrak{t}_0,\dots,\mathfrak{t}_1-1\}}$, we calibrate the factor loading $\widehat{\mathbf{a}}_{n_i}$ and the standard deviation $\widehat{\sigma}_{n_i}$, according to Assumptions 4.1.1 and (4.1.1), appealing to the regression

$$\sum_{n\in g_m}\omega_t^n = (\#g_m)\mathbf{a}^{n_i}\Delta_t^Y + \sqrt{\#g_m}\sigma_{\mathfrak{b}^{n_i}}\mathfrak{u}_t \quad \text{where} \quad \mathfrak{u}_t\sim\mathcal{N}(0,1), \text{ for all } t\in\{\mathfrak{t}_0,\dots,\mathfrak{t}_1-1\}.$$

2. We then estimate the barrier B^{n_i} by MLE as detailed in Gordy and Heitfield in [Gordy and Heitfield, 2002, Section 3]:

   we compute

$$\widehat{B}^{n_i} := \underset{B^{n_i}\in\mathbb{R}^+}{\arg\max}\,\mathcal{L}(B^{n_i}),$$

   where $\mathcal{L}(B^{n_i})$ is the log-likelihood function defined by

$$\mathcal{L}(B^{n_i}) := \sum_{t=\mathfrak{t}_0}^{\mathfrak{t}_1-1}\log\left(\int_{\mathbb{R}^{2I}}\mathbb{P}\left[D^{n_i}=d_t^m|(a,\theta)\right]\mathrm{d}\mathbb{P}[(\mathcal{A}_t^\circ,\Theta_t)\le(a,\theta)]\right),$$

   and where

$$\mathbb{P}[D^{n_i}=d_t^m|(\mathcal{A}_t^\circ,\Theta_t)] = \binom{r_t^m}{d_t^m}\left(\mathrm{PD}_{t,1,0}^{n_i}\right)^{d_t^m}\left(1-\mathrm{PD}_{t,1,0}^{n_i}\right)^{r_t^m-d_t^m},$$

   with D^{n_i} the Binomial random variable standing for the conditional number of defaults, and $\mathrm{PD}_{t,1,0}^{n_i}$ in Corollary 4.2.8, depending on $\sigma_{\mathfrak{b}^{n_i}}=\widehat{\sigma}_{\mathfrak{b}^{n_i}}$, $\mathbf{a}^{n_i}=\widehat{\mathbf{a}}^{n_i}$, for $t\in\{\mathfrak{t}_0,\dots,\mathfrak{t}_1-1\}$, $\delta_t=0$ and on B^{n_i}.

4.3.2 Expected and unexpected losses

Suppose that we have chosen or estimated all the economic parameters $(\varphi,\sigma,\psi,\boldsymbol{\lambda},\mu,\Gamma,\Sigma)$ and firm specific parameters $((B^n,\mathbf{a}^n,F_0^n,\sigma_{\mathfrak{b}^n})_{1\le n\le N})$, thanks to the previous equations. Starting from a trajectory of the carbon price δ, then, for all $t\in\{t_\circ,\dots,t_\star\}$, PD, EL and UL are computed by Monte Carlo simulations following the formulas below. We simulate $M\in\mathbb{N}^*$ paths of $(\Theta_{t_\circ}^m,\dots,\Theta_{t_\star}^m)$ indexed by $m\in\{1,\dots,M\}$, as a VAR(1) process, and we derive $((\mathcal{A}_{t_\circ}^\circ)^m,\dots,(\mathcal{A}_{t_\star}^\circ)^m)$. For any $t\in\{t_\circ,\dots,t_\star\}$:

- for each $1\le i\le I$ and for each $n\in g_i$, from (4.2.9), the estimated one-year probability of *default* of firm n is

$$\widehat{\mathrm{PD}}_{t,1,\mathfrak{d}}^{n,M} = \widehat{\mathrm{PD}}_{t,1,\mathfrak{d}}^{n_i,M} := \frac{1}{M}\sum_{m=1}^{M}\Phi\left(\frac{\log(B^{n_i})-\mathcal{K}^{n_i}(\mathfrak{d},t,1,(\mathcal{A}_t^\circ)^m,\Theta_t^m)}{\sqrt{\mathcal{L}^{n_i}(t,1)}}\right), \quad (4.3.1)$$

- the one-year expected loss is, from (4.2.10),

$$\widehat{\mathrm{EL}}_{t,\mathfrak{d}}^{N,T} := \sum_{n=1}^{N}\mathrm{EAD}_{t+1}^n\cdot\mathrm{LGD}_{t+1}^n\cdot\widehat{\mathrm{PD}}_{t,1,\mathfrak{d}}^{n,M}, \quad (4.3.2)$$

- the one-year unexpected loss is, from (4.2.11),

$$\widehat{\mathrm{UL}}^{N,T}_{t,\delta,\alpha} := q_{\alpha,M}\left(\left\{\sum_{n=1}^{N} \mathrm{EAD}^n_{t+1}\cdot \mathrm{LGD}^n_{t+1}\cdot \Phi\left(\frac{\log(B^n)-\mathcal{K}^n(\mathfrak{d},t,1,(\mathcal{A}^\circ_t)^m,\Theta^m_t)}{\sqrt{\mathcal{L}^n(t,1)}}\right)\right\}_{1\le m\le M}\right) - \widehat{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}}, \quad (4.3.3)$$

  where $q_{\alpha,M}(\{Y^1,\dots,Y^M\})$ denotes the empirical α-quantile of the distribution of Y.

If we want to compute the EL and UL of each sub-portfolio g_i with $1 \le i \le I$, we must sum on $g_m = i$ instead of $\{1,\dots,N\}$.

4.3.3 Summary of the process

More concretely, the goal is to project, for a given portfolio, the $T = 1$ year probability of *default*, as well as the expected and unexpected losses between year $t_\circ$ and year $t_\star$. To achieve that, we use (1) the number of firms rated r_t and defaulted d_t between $\mathfrak{t}_0$ and $\mathfrak{t}_1 - 1$, (2) all the firms' cash flows $(F^n_t)_{1\le n\le N}$ between $\mathfrak{t}_0$ and $\mathfrak{t}_1 - 1$, (3) the macroeconomic variables as well as the carbon intensities by sector observed between $\mathfrak{t}_0$ and $\mathfrak{t}_1 - 1$, and (4) the carbon price dynamics $(\delta_t)_{t\in\{t_\circ,\dots,t_\star\}}$ given by the regulator. We proceed as follows:

1. From the macroeconomic historical data, we estimate the productivity parameters $\widehat{\Gamma}$, $\widehat{\mu}$ and $\widehat{\Sigma}$, as well as the elasticities $\widehat{\psi}$ and $\widehat{\boldsymbol{\lambda}}$ as described in Subsection 3.2.3.

2. For each $m \in \{1,\dots,I\}$, we estimate the parameters B^{n_i}, $\sigma_{\mathfrak{b}^{n_i}}$, $\mathbf{a}^{n_i}$ using Subsections 4.3.1, yielding $\widehat{B}^{n_i}$, $\widehat{\sigma}_{\mathfrak{b}^{n_i}}$, $\widehat{\mathbf{a}}^{n_i}$.

3. We compute the carbon price dynamics $(\delta_t)_{t_\circ\le t\le t_\star}$ and the carbon intensities $(\tau_t)_{t_\circ\le t\le t_\star}$, $(\zeta_t)_{t_\circ\le t\le t_\star}$, and $(\kappa_t)_{t_\circ\le t\le t_\star}$ as defined in Subsection 3.2.1 as well as the emissions cost rate $(\mathfrak{d}_t)_{t_\circ\le t\le t_\star}$ defined in (3.1.4) and the output carbon cost function $\mathfrak{v}$ defined in (3.1.18).

4. We fix a large enough integer M, and simulate M paths of the productivity process $(\Theta^p_t)_{t_\circ\le t\le t_\star,1\le p\le M}$, then we derive $((\mathcal{A}^\circ_t)^p)_{t_\circ\le t\le t_\star,1\le p\le M}$ as defined in Assumption 3.1.1. For each $n \in \{1,\dots,N\}$, we compute the one-year probability of *default* $\widehat{\mathrm{PD}}^{n,M}_{t,1,\mathfrak{d}}$, for each $t_\circ \le t \le t_\star$, using (4.3.1).

5. We compute the expected (resp. unexpected) losses $\widehat{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}}$ (resp. $\widehat{\mathrm{UL}}^{N,T}_{t,\delta,\alpha}$), for each $t_\circ \le t \le t_\star$, using (4.3.2) (resp. (4.3.3)).

6. We fix the direction $\mathfrak{U}$ and a small step ϑ, and repeat 3.-4.-5. replacing $\mathfrak{d}$ by $\mathfrak{d} + \vartheta\mathfrak{U}$. Finally, we approach the sensitivity of the losses with respect to the carbon price δ by finite differences, i.e. for each $t_\circ \le t \le t_\star$,

$$\widehat{\Gamma}^{N,T,\mathrm{EL}}_{t,\delta}(\mathfrak{U}) := \frac{1}{\vartheta}\left(\widehat{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}'} - \widehat{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}}\right) \quad \text{and} \quad \widehat{\Gamma}^{N,T,\mathrm{UL}}_{t,\delta,\alpha}(\mathfrak{U}) := \frac{1}{\vartheta}\left(\widehat{\mathrm{UL}}^{N,T}_{t,\delta',\alpha} - \widehat{\mathrm{UL}}^{N,T}_{t,\mathfrak{d},\alpha}\right), \quad (4.3.4)$$

with $\mathfrak{d}'$ defined in Definition 4.2.11. In the sequel, we choose the direction $\mathfrak{U} \in (\mathbb{R}_+)^{t_\star+1}$ which is equal to 1 at t and 0 everywhere else, for each time t, and a step $\vartheta = 1\%$.

4.4 Results

4.4.1 Simulations and discussion

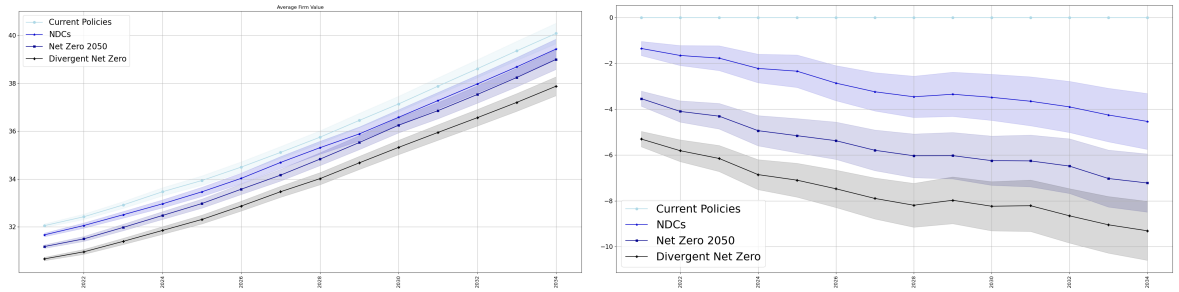
As in Section 3.3, we consider

- 4 sectors/sub-portfolios: *Very High Emitting*, *Very Low Emitting*, *Low Emitting*, and *High Emitting*.
- 4 transition scenarios: **Net Zero 2050**, **Divergent Net Zero**, **Nationally Determined Contributions (NDCs)** and **Current Policies**.

For confidentiality reasons, we cannot publish in this work the data from a BPCE credit portfolio on which these models have been tested and calibrated. We will therefore use fictitious data that we will describe.

Firm valuation

Here, we consider a representative firm characterized by its cashflow $F_{t_\circ-1}$ at $t_\circ - 1$, with standard deviation $\sigma_{\mathfrak{b}}$ and by the contribution $\mathfrak{a}$ of sectoral consumption growth to its cash flows growth. We would like to know how the value of this company evolves during the transition period and with the carbon price introduced in the economy. Consider $F_{t_\circ-1} = €1{,}000{,}000$, $\sigma_{\mathfrak{b}} = 5.0\%$, $\mathfrak{a} = [0.25, 0.25, 0.25, 0.25]$ (each sector has the same contribution to the growth of the cash flows of the firm), the interest rate $r = 5\%$. For $M = 5000$ simulations of the productivity processes $(\Theta_t, \mathcal{A}_t)_{t_\circ \le t \le t_\star}$, we compute the firm value using (4.1.16). We can analyze both the average evolution of the firm value per year and per scenario (Figure 4.2) and the empirical distribution of the firm value per scenario (Figure 4.3).



(a) Annual firm value per scenario in million euros per year

(b) Annual firm value growth per scenario in % per year

Figure 4.2: Firm value

We see that even if the value of the firm grows each year, this growth is affected by the severity of the transition scenario (Figure 4.2a). The presence of a carbon price in the economy clearly reduces the firm value yearly by -2.440% for *NDCs*, -5.009% for *Net Zero 2050*, and -7.412% for *Divergent Net Zero*. The faster the transition, the steeper the drop over 10 years. We start thus from a decrease of 1.351% in 2021 to 3.483% in 2030 for *NDCs*, from 3.541% to 6.248% for *Net Zero 2050*, and from 5.307% to 8.238% for *Divergent Net Zero* (Figure 4.2b).

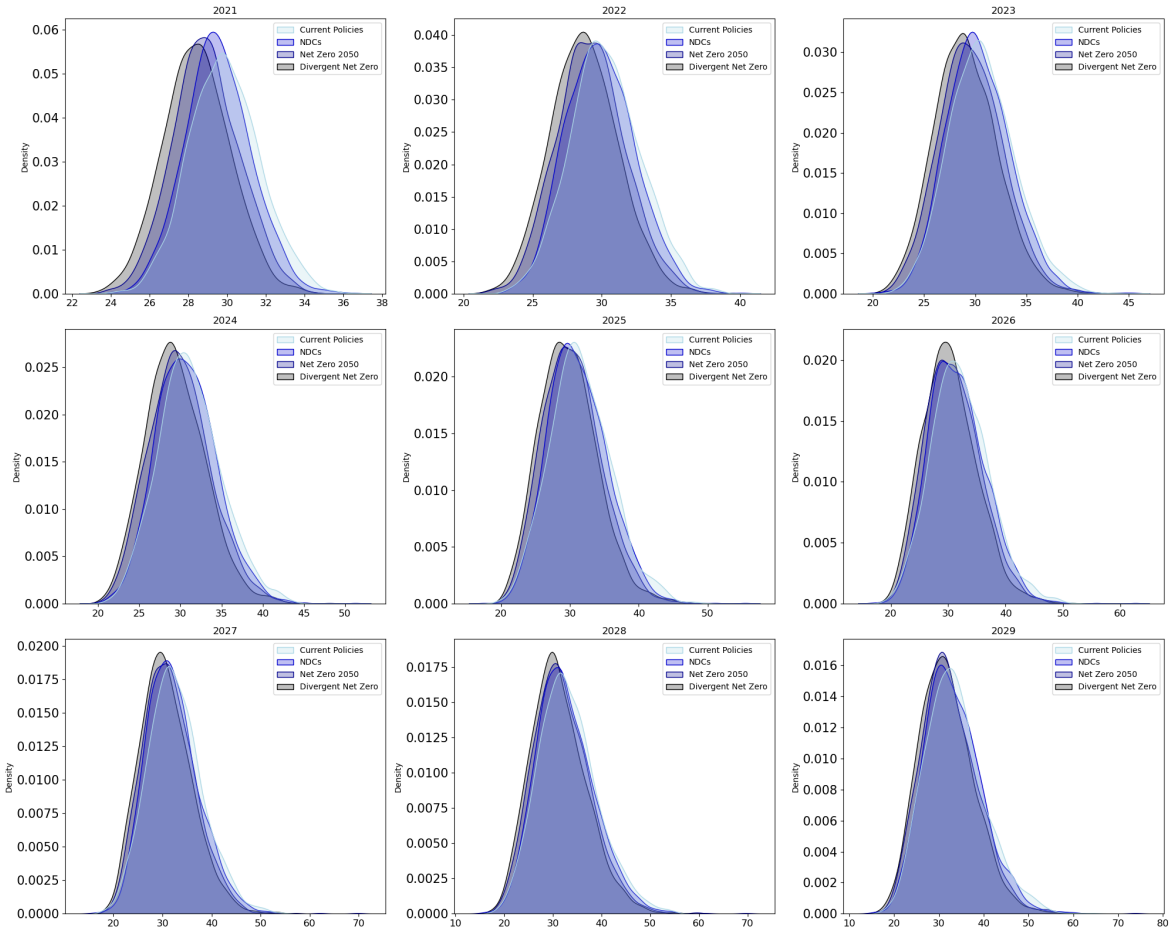


Figure 4.3: Firm value distribution per scenario and per year

The introduction of the transition scenario distorts the density function of the firm value, and in particular, moves it to the left.

Credit risk Consider a fictitious portfolio of $N = 16$ firms described in Table 4.1 below. This choice is made to ease the reproducibility of the result since the default data are proprietary data of BPCE. Note that the firms 1 to 4, 5 to 8, 9 to 12, and 13 to 16 respectively belong to the *Very High Emitting*, *High Emitting*, *Low Emitting*, and *Very Low Emitting* groups.

n°	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\sigma_b{}^n$	0.05	0.05	0.06	0.06	0.06	0.07	0.07	0.07	0.08	0.08	0.08	0.09	0.09	0.09	0.10	0.10
F_0^n	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
B^n	2.95	2.94	2.93	2.92	3.06	3.02	2.98	2.94	2.94	2.93	2.92	2.90	2.99	2.96	2.94	2.92
$\mathfrak{a}^n$ **(Very High)**	1.0	0.75	0.50	0.25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\mathfrak{a}^n$ **(High)**	0.0	0.0	0.0	0.0	1.0	0.75	0.50	0.25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\mathfrak{a}^n$ **(Low)**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.75	0.50	0.25	0.0	0.0	0.0	0.0
$\mathfrak{a}^n$ **(Very Low)**	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.75	0.50	0.25

Table 4.1: Characteristics of the portfolio

Probabilities of default

We use the parameters of the portfolio and firms as detailed in Table 4.1 to compute the annual PDs over ten years using the closed-form formulae (4.3.1). We then report, in Figure 4.4, the average annual PD and its annual evolution.

The remarks raised for the output growth remain valid, only the monotony changes: we can clearly distinguish the fourth various climate transition scenario. The probability of *default* grows each year, which is consistent as uncertainty increases with time. Even in the *Current Policies* scenario, the PD goes from 4.306% in 2021 to 5.193% in 2030.

Moreover, the increase is emphasized when the transition scenario gets tougher from

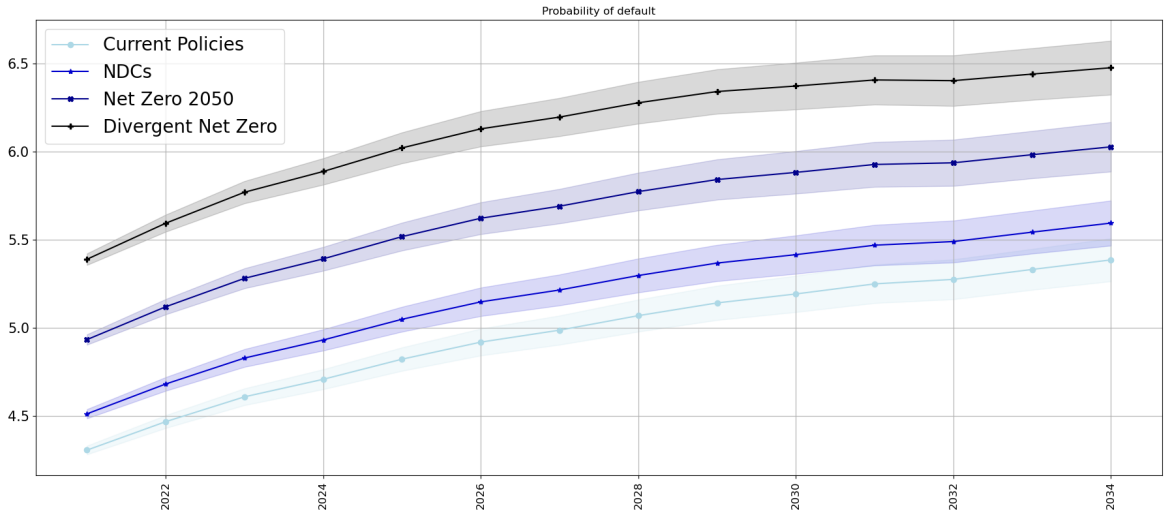


Figure 4.4: Average annual probability of *default* and 95% confident interval of the portfolio, per scenario and year in %

an economic point of view. Between the worst-case (*Divergent Net Zero*) scenario and the best-case (*Current Policies*) one, the difference in the average PD reaches 1.152% in 2030 and gradually decreases after the transition. Over the transition period of 10 years, the annual average PD is 4.906% for the *Current Policies* scenario, 5.124% for the *NDCs* scenario, 5.577% for the *Net Zero 2050* scenario, and 6.058% for the *Divergent Net Zero* scenario. It is no surprise that the introduction of a carbon price increases the portfolio's average probability of *default*.

In Figure 4.5 above and Table 4.2, we can also observe that, for each sub-portfolio,

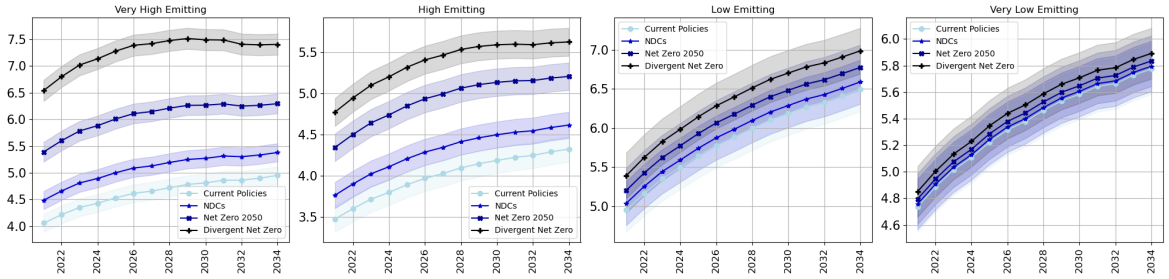


Figure 4.5: Average annual probability of *default* and 95% confident interval, per scenario and per sub-portfolio

the evolution of the PD depends on the sector that is at the origin of the growth of its cash flows. As expected, the PD grows throughout the years, and the growth is even more abrupt when the sub-portfolio is polluting. Diversification also has a positive effect on the portfolio: the average PD of the overall portfolio is higher than the average PD of the least polluting sub-portfolios, and lower than the average PD of the most polluting portfolios. Bouchet and Le Guenedal [2020] and Bourgey et al.

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Portfolio**
NDCs	0.451	0.305	0.096	0.021	0.218
Net Zero 2050	1.419	0.925	0.280	0.062	0.671
Divergent Net Zero	2.625	1.375	0.490	0.120	1.152

Table 4.2: Average annual PD

[2021] also worked on the impact of a carbon price on credit portfolios. They computed the PDs from 2020 up to 2050/2060. More precisely, Bouchet and Le Guenedal [2020] presents the percentage of companies by sector and by scenario whose probabilities of default are above 99%, and Bourgey et al. [2021] focuses on default intensities and probabilities of default. However, they fixed the time at which the PDs are computed and varied the time horizon (maturity) while in our case, we are doing the other way around. Moreover, they do not comment much about the uncertainties on the dynamics of the balance sheet, productivity, carbon intensities, and the carbon price, while such uncertainties are expected to significantly increase with the transition time horizon, and therefore to substantially impact any credit risk metrics. As the average length of small and medium-sized enterprises loans is of about seven years, we prefer to focus on short-term risk measures.

Expected and unexpected losses

We compute the EL and UL using (4.3.2) and (4.3.3), assuming that LGD and EAD are constant over the years and $\text{LGD}^n = 45\%$ and $\text{EAD}^n = €10$ million for each firm n described in Table 4.1. The annual exposure of the notional portfolio of the $N = 16$ firms thus remains fixed and is equal to €160 millions, while each sub-portfolio exposure

is of €40 millions. We then express losses as a percentage of the firm's or portfolio's exposure. Table 4.3 and Table 4.4 show the average annual EL and UL.

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Portfolio**
Current Policies	0.329	0.034	0.097	0.160	0.119
NDCs	0.504	0.050	0.107	0.186	0.161
Net Zero 2050	1.066	0.100	0.128	0.246	0.292
Divergent Net Zero	2.057	0.138	0.155	0.327	0.512

Table 4.3: Average annual EL as a percentage of exposure

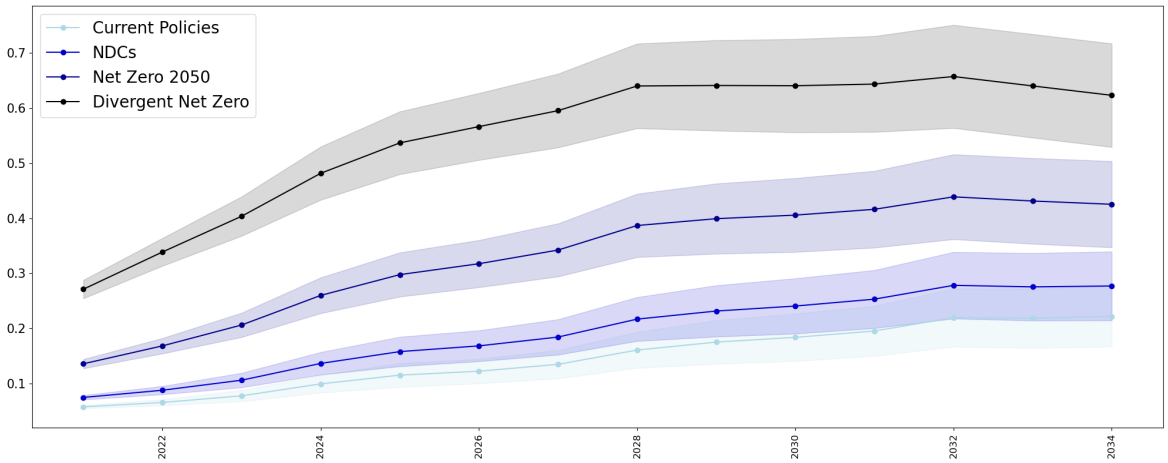


Figure 4.6: EL of the portfolio in % of the exposure per scenario and per year

We observe in Table 4.3 and Figure 4.6 that, as expected (notably because the LGD is assumed to be deterministic and constant), the different scenarios remain clearly differentiated for the EL. The latter as a percentage of the portfolio's exposure increases with the year and the carbon price. For the portfolio as a whole, we see that the average annual EL increases from 330% between the two extreme scenarios. Moreover, still focusing on the two extreme scenarios, the average annual EL increases by 525% for the *Very High Emitting* portfolio while it increases by 143% for the *Very Low Emitting* portfolio. The EL being covered by the provisions coming from the fees charged to the client, an increase in the EL implies an increase in credit cost. Therefore companies from the most polluting sectors should be charged more than those from the least polluting sectors.

Similarly for the UL, we observe the difference between the scenarios from Table 4.4 and Figure 4.7. For the portfolio as a whole, we see that the average annual UL increases by 377% between the two extreme scenarios. Moreover, still focusing on the two extreme scenarios, the average annual UL increases by 284% for the *Very High Emitting* portfolio while it increases by 83% for the *Very Low Emitting* portfolio.

The UL being covered by the economic capital coming from the capital gathered by the shareholders, an increase in the UL implies a decrease in the bank's profitability.

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Portfolio**
Current Policies	1.191	0.066	0.147	0.277	0.109
NDCs	1.691	0.098	0.163	0.316	0.161
Net Zero 2050	2.964	0.193	0.197	0.400	0.307
Divergent Net Zero	4.585	0.264	0.239	0.507	0.520

Table 4.4: Average annual UL as a percentage of exposure

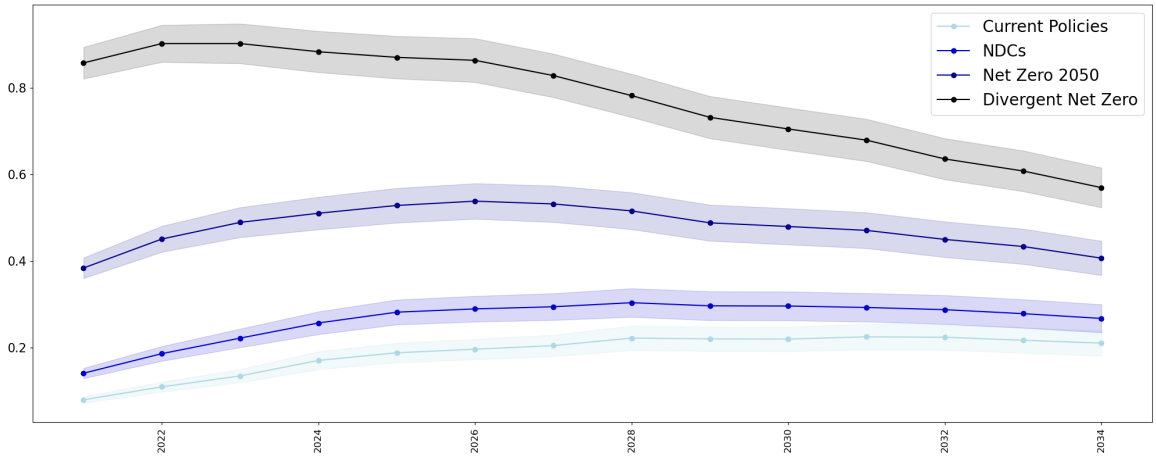


Figure 4.7: UL of the portfolio in % of the exposure per scenario and per year

Therefore, in some way, granting loans to companies from the most polluting sectors will affect banks more negatively than doing so to companies from the least polluting sectors. We therefore observe that the introduction of a carbon price will not only increase the banking fees charged to the client (materialized by the provisions via the expected loss) but will also reduce the bank's profitability (via the economic capital that is calculated from the unexpected loss). Finally, for more in-depth analysis, Figure B.3 (respectively Figure B.4) shows the distortions of the distribution of the EL (respectively the UL) per scenario and per year.

Losses' sensitivities to carbon price

Finally, we compute the sensitivity of our portfolio losses to carbon price using (4.3.4). Since the scenarios are deterministic, this quantity allows us to measure some form of model uncertainty. Indeed, for a given scenario, it allows to capture the level by which the computed loss would vary should that assumed deterministic scenario deviate by a certain percentage. For each time t, we choose the direction $\mathfrak{U} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^\top \in \mathbb{R}_+^{t_\star+1}$, and a step $\vartheta = 1\%$. A carbon price change of 1% will cause a change in the EL of $\widehat{\Gamma}_{t,\delta}^{N,T}(\text{EL})$ and a change in the UL of $\widehat{\Gamma}_{t,\delta,\alpha}^{N,T}(\text{UL})$. We report the results in Table 4.5 and Table 4.6.

For example, over the next ten years, if the price of carbon varies by 1% around the scenario *NDCs*, the portfolio's EL will vary by 1.402% while the portfolio's UL will

change by 1.148% around this scenario.

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Portfolio**
Current Policies	1.561	1.581	1.191	0.827	1.280
NDCs	1.777	1.687	1.261	0.904	1.402
Net Zero 2050	2.142	1.864	1.386	1.035	1.631
Divergent Net Zero	2.668	2.096	1.562	1.215	1.973

Table 4.5: Average annual EL sensitivity to carbon price in %

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Portfolio**
Current Policies	1.299	1.290	1.042	0.547	1.135
NDCs	1.463	1.365	1.102	0.583	1.148
Net Zero 2050	1.726	1.485	1.206	0.634	1.197
Divergent Net Zero	2.070	1.632	1.352	0.681	1.472

Table 4.6: Average annual UL sensitivity to carbon price in %

The greater the sensitivity, the more polluting the sector is. This is to be expected as carbon prices are higher in these sectors. In addition, the sensitivity of the portfolio is smaller than that in the most polluting sectors, and greater than that in the least polluting ones. Finally, we notice that the variation of the EL is slightly more sensitive than the variation of the UL. This means that the bank's provisions will increase a bit more than the bank's capital, or that the growth of the carbon price will impact customers more than shareholders.

LGD as a function of PD

In this section, we analyse LGDs of our portfolio and sub-portfolios using the joint modelling proposed in Section 4.2.6 and expecially in (4.2.14). Given that LGD here is just a function of PD, the LGD modeling at this point is not microscopically linked to the carbon price. Nevertheless, it is already an interesting starting point to estimate the impact of a carbon price on recovery.

We analyze LGD on the same portfolio described in Table 4.1 we use for PD. We compute the average LGD for the portfolio as well as for each sub-portfolio and we summarize the result in Table 4.7 and Figure 4.8 below. The PD was slightly increasing over time. It is observed that the LGD also increases over time, regardless of whether it is the portfolio or one of the sub-portfolios. Moreover, it was observed both in the table and in the figure that the introduction of a carbon price has the effect of increasing the LGD (and therefore, of decreasing the bank's recovery rate). This increase in the LGD is even more significant for the most polluting sectors and for the most severe scenarios. Such conclusions could also have been drawn using the PD and finely analyzing (4.2.13). Even though this provides a first approach, it is not based on a micro-founded modeling of the LGD.

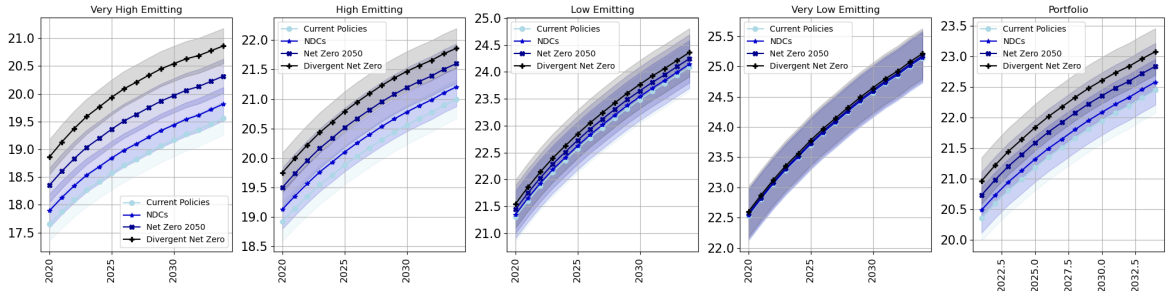


Figure 4.8: Average annual LGDand 95% confident interval, per scenario and per sub-portfolio

Emissions level	**Very High**	**High**	**Low**	**Very Low**	**Portfolio**
NDCs	0.264	0.218	0.055	0.014	0.138
Net Zero 2050	0.775	0.622	0.158	0.039	0.399
Divergent Net Zero	1.334	0.891	0.273	0.076	0.644

Table 4.7: Average annual LGD

Conclusion

In this chapter, we study how the introduction of a carbon price would propagate in a credit portfolio. To this aim, we use the Discounted Cash Flows methodology to compute the firm value and introduce the latter in a structural credit risk model to project PD, EL and UL. We finally introduce losses' sensitivities to carbon price to measure the uncertainty of the losses to the transition scenarios. We obtain that the carbon price distorts the distribution of the firm's value, increases banking fees (materialized by the level of provisions computed from the expected loss) charged to clients or supported by bank operating incomes, and reduces banks' solvency (translated by the economic capital calculated from the unexpected loss). This chapter also opens the way to numerous extensions. The LGD is assumed to be deterministic, constant, and independent of the carbon price. In a brief extension, we have also proposed a joint modelling of PD and LGD, but ignoring the potential impact of guarantees. In the next chapter, we will analyze how the LGD is affected by the stranding of assets. We furthermore assume that EAD and thus bank balance sheets remain static over the years while the transition will require huge investments. In our forthcoming research, one could introduce capital in the model. Finally, we have adopted a sectoral view, while one could alternatively assess the credit risk at the counterpart level and thus penalize or reward companies according to their individual and not sectoral emissions.

Part II

Loss with collateral in continuous time

Chapter 5

Portfolio loss in continuous time, with carbon emissions cost and with stochastic collateral

This chapter is based on two preprints Sopgoui [2024b] and Sopgoui [2024a] that will be submitted soon.

In this chapter, we propose models to quantify the distortion a credit portfolio (expected and unexpected) losses, when the obligor companies as well as their guarantees belong to an economy subject to the climate transition. The economy is driven by its productivity which is a multidimensional Ornstein-Uhlenbeck process while the climate transition is declined thanks to the carbon price, a continuous deterministic process. We define each loan's loss at default as the difference between Exposure at Default (EAD) and the liquidated collateral, which will help us to define the Loss Given Default (LGD) – the expected percentage of exposure that is lost if a debtor defaults. We consider two types of collateral: *financial asset* such as invoices, cash, or investments or *physical asset* such as real estate, business equipment, or inventory. First, if it is a *financial asset*, we model the later by the continuous time version of the discounted cash flows methodology, where the cash flows SDE is driven by the instantaneous output growth, the instantaneous growth of a carbon price function, and an arithmetic Brownian motion. Secondly, for *physical asset*, we focus on the example of a *property in housing market*. Therefore, we obtain its value as the difference between the price of an equivalent efficient building following an exponential Ornstein-Uhlenbeck as well as the actualized renovation costs and the actualized sum of the future additional energy costs due to the inefficiency of the building, before an optimal renovation date which depends on the carbon price process. Finally, we obtain how the loss' risk measures of a credit portfolio are skewed in the context of climate transition through carbon price and/or energy performance of buildings when both the obligors and their guarantees are affected.

Contents

Introduction

When an obligor (firm, government, or individual) defaults, the creditor (bank) stands to lose its money. One way to ensure the stability of the banks' business and more generally the soundness of the whole financial system, is ideally, to anticipate when the default will happen and how much a bank could lose. In order to achieve that, we use the four parameters: PD, EAD, LGD, and the effective maturity T that we introduced in section 4.2. By using, among others, these parameters, banks can compute various risk measures (such as Expected, Unexpected and Stressed Losses) which help later on to determine provisions, as well as economic and regulatory capital. An essential part of a bank risk division is to estimate how these risk measures change with various factors such as time and economic conditions.

Let us focus for example on LGD. When a debtor defaults, banks can lose part or whole of its exposure. The fraction of the loss relative to EAD is LGD while the recovery rate is the fraction of EAD recovered so that $LGD = 1 - Recovery$. So modelling LGD or modelling recovery are equivalent. According to Chalupka and Kopecsni [2008], there are three ways to handle LGD: "*Market LGD* is observed from market prices of defaulted bonds or marketable loans soon after the actual default event. *Workout LGD* is derived from a set of estimated cash flows resulting from a workout and collection process, properly discounted to a date of default. Thirdly, implied *market LGD* is derived from risky but not defaulted bond prices using a theoretical

asset pricing model". In the IRB approach, LGD refers to *Workout LGD* and there are several techniques to model it. In economic modeling, as detailed by Bastos [2010], Roncalli [2020], LGD is a (linear or non linear) function of many factors which can be factors external to the issuer, specific to the issuer or specific to the debt issuance. That function can be obtained/calibrated through logistic regression, regression trees, or neural networks. In stochastic modeling, it is assumed that LGD follows a given distribution (parametric or non-parametric). In this case, LGD is commonly modeled by a Beta distribution as Roncalli [2020][Page 193] and Chalupka and Kopecsni [2008]. Its parameter are then estimated on historical data. Fermanian [2020], for his part, proposes a joint modelling of PD and LGD by writing the potential loss at default as the difference between the debt amount (EAD) and the assets at the default date.

There are secured and unsecured loans. In these approaches, even for secured loans, there is one parameter essential that is sometimes overlooked: the collateral. However, not all borrowers put up collateral when taking out loans. It is even worse, there is even some evidence that loans with collateral attached may be riskier for lenders (see Berger and Udell [1990]). If the loan is secured, when the counterpart defaults, the bank liquidates the collateral (guarantee), and if the EAD is not reached, it can recover the remainder by liquidating other assets (called residual recovery). These guarantees can be tangible assets (buildings, business equipments, inventories, etc.) or intangible assets (cash deposits, public bonds, securities, etc.) as noted by Berger and Udell [1990], Blazy and Weill [2013]. In the presence of collateral, the recovery (i.e. $1 - \text{LGD}$) is therefore made up of the value of the collateral at the date of default and the value of the residual recovery Frontczak and Rostek [2015], and Pelizza and Schenk-Hoppé [2020]. We will model here two examples of guarantees: either a security or a (commercial or residential) building, which both, will be affected by the climate transition at its liquidation.

A security can represent ownership in a corporation in the form of stock, a creditor relationship with a governmental body or a corporation represented by owning that entity's bond; or rights to ownership as represented by an option. A security generate a stream of cash flows. The proxy of the security value is the infinite sum of the present value of the future cash flows. As we already know from the literature and from what we propose in chapter 3 and chapter 4, the security (notably if it is a firm) value will be affected by transition risk. We will therefore revisit the results of the two previous chapters where carbon price dynamics affect the firm value and credit risk measures such as probability of default, expected and unexpected losses. In particular, we redesign the multisectoral model with carbon price, the firm valuation model, and the credit risk model proposed in continuous time.

In the same way, a commercial or residential building price will be affected by transition for example through the Energy Performance (or Energy Efficiency) as mentioned in Aydin et al. [2020], Franke and Nadler [2019]. In the absence of the transition, a real estate market model was proposed by Fabozzi et al. [2012] to price real estate derivatives, and then used for the calculation of the LGD by Frontczak

and Rostek [2015]. These works model the price of a property as an exponential Ornstein-Uhlenbeck process. At the same time, the value of a property can be seen as the discounted sum of an income (rent, for example) over its residual lifetime. We will use these two approaches here. To introduce the climate transition, we draw inspiration from Ter Steege and Vogel [2021] who write the price difference per square meter between two properties with different energy efficiency as the sum of the discounted value of (expected) energy cost differences. We will enhance their work by additionally considering the renovation costs to improve energy efficiency, and the optimal renovation date which will depend on the trajectory of energy prices (writing as a function of the carbon price).

The rest of the current chapter is organized as follows. We revisit in Section 5.2 the results of chapters 3 and 4 in a continuous time setting, namely a multisectoral economic model with carbon price, a firm valuation model, and a credit risk model. In section 5.3, we define the loss at default as the difference between EAD and the liquidated collateral, which will help us to define LGD. If the collateral is a *financial asset*, we model it in Subsection 5.3.2 by the continuous time version of the discounted cash flows, where the cash flows SDE is driven by the instantaneous consumption growth, the instantaneous growth of a carbon price's function and a Brownian motion. If the collateral is a *building*, we model it in Subsection 5.3.3 both by an exponential Ornstein-Uhlenbeck (O.U.) and by a discounted sum of an income over its residual lifetime. The last section is dedicated to estimations, simulations, and discussion.

5.1 The problem

We consider a bank credit portfolio composed of $N \in \mathbb{N}^*$ firms in a closed economy (in other words no import and no export). In credit risk assessment, one of the first steps is to create homogeneous sub-portfolios of firms. As we are dealing here with climate transition risk, we would like to classify firms by carbon intensity: firms with similar carbon intensities belong to a same homogeneous sub-portfolio. It should be noted that in the absence of a climate transition, firms are traditionally clustered in terms of industry, geography, size, and credit rating, for example.

We thus assume $I \in \mathbb{N}^*$ ($I \leq N$) homogeneous carbon emission sectors in the economy. Nevertheless, as we rarely have the firm individual carbon emissions/intensities, we assume that each company has the carbon intensity of its industry sector. This amounts to grouping "industry sectors" into I "carbon emission sectors". From now on, sectors are to be interpreted as carbon emission sectors.

Definition 5.1.1. We divide our portfolio into I disjunct sub-portfolios $g_1, \ldots, g_I$ so that each sub-portfolio represents a single risk class and the firms in each sub-portfolio belong to a single carbon emission sectors. From now on, we denote $\mathcal{I}$ the set of sectors with cardinal $I \in \mathbb{N}^*$. We also fix $n_i := \min\{n \in \{1, \ldots, N\} \text{ such that } n \in g_i\}$ for each $i \in \mathcal{I}$. Therefore, firm n_i is a representative of the group i.

We would like to know how the whole portfolio loss and sub-portfolios losses would be affected should the regulator introduce a carbon price in the economy, in order to mitigate the effects of climate change. This precisely amounts to quantifying the distortion over time of credit risk measures created by the introduction of a carbon price. For example, if the government decides to charge firms and households GHG emissions between 2025 and 2035, a bank would like to estimate today how the probability of a company to default in 2030 is impacted.

The bank's potential loss caused by a firm depends essentially on the default date and on the liquidation of the guarantees if they exist. The firm as well as the guarantee belong to the same economy subject to the climate transition. Thus, we build in the first stage a dynamic, stochastic, and multisectoral economic model in which direct and indirect GHG emissions from companies as well as direct GHG emissions from households are charged. We choose a representative firm in each sector and a representative household for the whole economy. By observing that each firm belongs to a sector and its cash flows are a proportion of its sales. The latter are themselves a proportion of the sectoral output, we obtain the cash flows dynamics that we use to model the value of firms in an environment where GHG emissions are charged. Then, starting from a default model in which a company defaults if its value falls below its debt, we calculate the probability of default of each firm. Finally, we compute the distortion of the (associated statistics of) loss – defined as the difference between the exposure and the liquidated collateral if exists – by the introduction of a carbon price.

5.2 Main assumptions and results of chapters 3 and 4 in continuous time

In this section, we revisit the framework developed in chapters 3 and 4 in continuous time. Precisely, we decline, in continuous time, the two standing assumptions as well as the three main results respectively on the dynamic stochastic multisectoral model with carbon emissions costs, on the firm valuation model, and on the structural credit risk model. Most of the proofs can be derived from the discrete time so that we will either skip them or detail them in appendix.

5.2.1 A Multisectoral Model with Carbon price

Each sector $i \in \mathcal{I}$ has a representative firm which produces a single good, so that we can associate sector, firm and good. We introduce the following standing assumption which describes the productivity, which is considered to have stationary Ornstein-Uhlenbeck dynamics.

Standing Assumption 5.2.1. We define the $\mathbb{R}^I$-valued process $\mathcal{A}$ which evolves

according to

$$\begin{cases} \mathrm{d}\mathcal{Z}_t &= -\Gamma\mathcal{Z}_t\mathrm{d}t + \Sigma\mathrm{d}B_t^{\mathcal{Z}} \\ \mathrm{d}\mathcal{A}_t &= (\mu + \varsigma\mathcal{Z}_t)\,\mathrm{d}t \end{cases} \quad \text{for all } t \in \mathbb{R}_+, \tag{5.2.1}$$

where $(B_t^{\mathcal{Z}})_{t\in\mathbb{R}^*}$ is a I-dimensional Brownian Motion, and where the constants $\mu, \mathcal{A}_0 \in \mathbb{R}^I$, the matrices $\Gamma, \Sigma \in \mathbb{R}^{I\times I}$, $\mathcal{Z}_0 \sim \mathcal{N}\left(0, \Sigma\Sigma^\top\right)$, and $0 < \varsigma \leq 1$ is an intensity of noise parameter that is fixed: it will be used later to obtain a tractable proxy of the firm value. Moreover, Σ is a positive definite matrix and $-\Gamma$ is a Hurwitz matrix i.e. its eigenvalues have strictly negative real parts.

The processes $\mathcal{Z}^i$ and $\mathcal{A}^i$ play a major role in our factor productivity model since, for any $i \in \mathcal{I}$, the total factor productivity of sector i is defined as

$$A^i := \exp\left(\mathcal{A}^i\right),$$

so that $\mathcal{Z}^i$ is the log-productivity growth and $\mathcal{A}^i$ is the cumulative log-productivity growth. In the rest of the paper, the terminology "productivity" will be used within a context that will allow the reader to understand if the term refers to $\mathcal{Z}^i$, $\mathcal{A}^i$, or A^i.

We also introduce the following filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{R}^*}$ with $\mathcal{G}_0 := \sigma(\mathcal{Z}_0)$ and for $t > 0$, $\mathcal{G}_t := \sigma\left(\left\{\mathcal{Z}_0, B_s^{\mathcal{Z}} : s \leq t\right\}\right)$.

Remark 5.2.2 (O.U. process)**.** We have the following results on O.U. that we will use later:

1. According to Gobet and She [2016][Proposition 1], if one assumes that $\mathcal{Z}_0$ and $B^{\mathcal{Z}}$ are independent and $\mathcal{Z}_0$ is square integrable, then, there exists a unique square integrable solution to the I-dimentional Ornstein-Uhlenbeck process $\mathcal{Z}$ satisfying $\mathrm{d}\mathcal{Z}_t = -\Gamma\mathcal{Z}_t\mathrm{d}t + \Sigma\mathrm{d}B_t^{\mathcal{Z}}$, represented as

$$\mathcal{Z}_t = e^{-\Gamma t}\left(\mathcal{Z}_0 + \int_0^t e^{\Gamma u}\Sigma\mathrm{d}B_u^{\mathcal{Z}}\right), \quad \text{for all } t \in \mathbb{R}_+.$$

   Additionally, for any $t, h \geq 0$, the distribution of $\mathcal{Z}_{t+h}$ conditional on $\mathcal{G}_t$ is Gaussian $\mathcal{N}\left(M_t^{\mathcal{Z},h}, \Sigma_t^{\mathcal{Z},h}\right)$, with the mean vector

$$M_t^{\mathcal{Z},h} := \mathbb{E}[\mathcal{Z}_{t+h}|\mathcal{G}_t] = e^{-\Gamma h}\mathcal{Z}_t,$$

   and the covariance matrix

$$\Sigma_t^{\mathcal{Z},h} := \mathbb{V}[\mathcal{Z}_{t+h}|\mathcal{G}_t] = \int_0^h e^{-\Gamma u}\Sigma\Sigma^\top e^{-\Gamma^\top u}\mathrm{d}u.$$

2. Since $-\Gamma$ is a Hurwitz matrix, then if we note $\lambda_\Gamma := \max_{\lambda\in\lambda(\Gamma)} Re(\lambda)$, there exists $c_\Gamma > 0$ so that $\|e^{-\Gamma t}\| < c_\Gamma e^{-\lambda_\Gamma t}$ for all $t \geq 0$. Therefore, according to Gobet and She [2016][Proposition 2], $\mathcal{Z}$ has a unique stationary distribution which is Gaussian with mean 0 and covariance $\int_0^{+\infty} e^{-\Gamma u}\Sigma\Sigma^\top e^{-\Gamma^\top u}\mathrm{d}u$.

3. We can show in C.1.3 that for any $t, h \geq 0$, we have

$$\mathcal{A}_{t+h} = \mathcal{A}_t + \int_t^{t+h} (\mu + \varsigma \mathcal{Z}_s) \mathrm{d}s = \mu h + \varsigma \int_t^{t+h} \mathcal{Z}_s \mathrm{d}s,$$

and conditionally on $\mathcal{G}_t$, $\mathcal{A}_{t+h}$ has an I-dimensional normal distribution with the mean vector

$$M_t^{\mathcal{A},h} := \mu h + \varsigma \Upsilon_h \mathcal{Z}_t + \mathcal{A}_t,$$

with

$$\Upsilon_h := \int_0^h e^{-\Gamma s} \mathrm{d}s = \Gamma^{-1}(\mathbf{I}_I - e^{-\Gamma h}), \tag{5.2.2}$$

and the covariance matrix

$$\Sigma_t^{\mathcal{A},h} := \varsigma^2 \Gamma^{-1} \left(\int_0^h \left(e^{-\Gamma u} - \mathbf{I}_I \right) \Sigma \Sigma^\top \left(e^{-\Gamma u} - \mathbf{I}_I \right) \mathrm{d}u \right) (\Gamma^{-1})^\top = \varsigma^2 \int_0^h \Upsilon_u \Sigma \Sigma^\top \Upsilon_u^\top \mathrm{d}u.$$

4. For later use, we define

$$\mathcal{A}_t^\circ := \mathcal{A}_t - \mathcal{A}_0,$$

and observe that $(\mathcal{A}_t^\circ, \mathcal{Z}_t)_{t \geq 0}$ is a Markov process.

Firms emit GHG when they consume intermediary input from other sectors and when they produce output. Likewise, households emit GHG when they consume. All these emissions are charged through a carbon price dynamics. For the whole economy, we introduce a deterministic and exogenous carbon price in euro/dollar per ton. It allows us to model the impact of the transition pathways on the whole economy. We will note δ the complete carbon price process. We shall then assume the following setting.

Standing Assumption 5.2.3. We introduce the carbon price and the carbon intensities (the quantity of GHG in tons emits for each unit of production/consumption) processes:

1. Let $0 \leq t_\circ < t_\star$ be given. The sequence δ satisfies

   - for $t \in [0; t_\circ]$, $\delta_t = \delta_0 \in (\mathbb{R}_+)^I$, namely the carbon price is constant;
   - for $t \in (t_\circ, t_\star)$, $\delta_t \in (\mathbb{R}_+)^I$, the carbon price may evolve;
   - for $t \geq t_\star$, $\delta_t = \delta_{t_\star} \in (\mathbb{R}_+)^I$, namely the carbon price is constant.

   We assume moreover that $t \mapsto \delta_t$ is $\mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$.

2. We also introduce carbon intensities as the sequences τ, ζ, and κ being respectively $\mathbb{R}_+^I$, $\mathbb{R}_+^{I \times I}$, and $\mathbb{R}_+^I$-processes, and representing respectively carbon intensities on firm's output, on firm's intermediary consumption, and on

household's consumption, and satisfying for all $t \in \mathbb{R}_+$ and $\mathfrak{y} \in \{\tau^1, \dots, \tau^I, \zeta^{11}, \zeta^{12}, \dots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \dots, \kappa^I\}$,

$$\mathfrak{y}_t = \begin{cases} \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0} \frac{1-\exp(-\theta_{\mathfrak{y}} t)}{\theta_{\mathfrak{y}}}\right) & \text{if } 0 \le t \le t_\star \\ \mathfrak{y}_0 \exp\left(g_{\mathfrak{y},0} \frac{1-\exp(-\theta_{\mathfrak{y}} t_\star)}{\theta_{\mathfrak{y}}}\right) & \text{else,} \end{cases} \tag{5.2.3}$$

with $\mathfrak{y}_0, -g_{\mathfrak{y},0}, \theta_{\mathfrak{y}} > 0$. For each $t \ge 0$, we call $\mathfrak{y}_t \delta_t$ the *emissions cost rate* at time t.

3. For each $i \in \mathcal{I}$ and for each $t \in \mathbb{R}_+$,

$$\delta_t \max_{i \in \mathcal{I}} \tau_0^i < 1.$$

In the following, we will note for all $t \ge 0$,

$$\mathfrak{d}_t := (\tau_t \delta_t, \zeta_t \delta_t, \kappa_t \delta_t). \tag{5.2.4}$$

An example of carbon price process We assume the regulator fixes $t_\circ \ge 0$ when the transition starts and the transition horizon time $t_\star > t_\circ$, the carbon price at the beginning of the transition $P_{carbon} > 0$, at the end of the transition $\delta_{t_\star} > P_{carbon}$, and the annual growth rate $\eta_\delta > 0$. Then, for all $t \ge 0$,

$$\delta_t = \begin{cases} P_{carbon}, & \text{if } t \le t_\circ, \\ P_{carbon} e^{\eta_\delta (t - t_\circ)}, & \text{if } t \in (t_\circ, t_\star], \\ \delta_{t_\star} = P_{carbon} e^{\eta_\delta (t_\star - t_\circ)}, & \text{otherwise.} \end{cases} \tag{5.2.5}$$

In the example above that will be used in the rest of this work, we assume that the carbon price increases. However, there are several scenarios that could be considered, including a carbon price that would increase until a certain year before leveling off or even decreasing. We also assume an unique carbon price for the entire economy whereas we could proceed differently. For example, the carbon price could increase for production when stabilize or disappear on households in order to avoid social movements and so on. The framework can be adapted to various sectors as well as scenarios.

In our framework, a representative firm in each sector which maximizes its profits by choosing, at each time and for a given productivity, the quantities of labor and intermediary inputs, while, a representative household solves a dynamic optimization problem to decide how to allocate its consumption expenditures among the different goods and hours worked and among the different sectors. We assume that the utility function $U(x, y) := \log x - \frac{y^{1+\varphi}}{1+\varphi}$ with $\varphi \ge 0$. Moreover, $\boldsymbol{\lambda}$ (respectively ψ) are matrix in $(\mathbb{R}_+^*)^{I \times I}$ (respectively vector in $(\mathbb{R}_+^*)^I$) of the elasticities of intermediary inputs (respectively labor). We also assume a constant return to scale, namely

$$\psi^i + \sum_{j \in \mathcal{I}} \boldsymbol{\lambda}^{ji} = 1, \qquad \text{for each } i \in \mathcal{I}. \tag{5.2.6}$$

Since the productivity and the carbon price processes are continuous, the firms and households problems are well posed and their solutions exist. More details are given in C.2. In the following proposition, we give the expression of the output.

Proposition 5.2.4. *For* $(\overline{\tau}, \overline{\zeta}, \overline{\kappa}, \overline{\delta}) \in \mathbb{R}_+^I \times \mathbb{R}_+^{I\times I} \times \mathbb{R}_+^I \times \mathbb{R}_+$*, let us note*

$$\Psi(\overline{\mathfrak{d}}) := \left(\psi^i \frac{1-\overline{\tau}^i\overline{\delta}}{1+\overline{\kappa}^i\overline{\delta}}\right)_{i\in\mathcal{I}} \quad \text{and} \quad \Lambda(\overline{\mathfrak{d}}) := \left(\boldsymbol{\lambda}^{ji}\frac{1-\overline{\tau}^i\overline{\delta}}{1+\overline{\zeta}_t^{ji}\overline{\delta}}\frac{1+\overline{\kappa}^j\overline{\delta}}{1+\overline{\kappa}^i\overline{\delta}}\right)_{j,i\in\mathcal{I}}, \tag{5.2.7}$$

with $\overline{\mathfrak{d}} := (\overline{\tau}\overline{\delta}, \overline{\zeta\delta}, \overline{\kappa}\overline{\delta})$*. Assume that*

1. $\mathbf{I}_I - \boldsymbol{\lambda}$ *is not singular,*
2. $\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top$ *is not singular for all* $t \in \mathbb{R}_+$*.*

Then, for all $t \in \mathbb{R}_+$*, there exists an unique couple of consumption and output* (C_t, Y_t) *solving the (dynamic stochastic) multisectoral model. Moreover, for all* $t \in \mathbb{R}_+$*.*

1. *if* $\mathfrak{e}_t^i := \frac{Y_t^i}{C_t^i}$ *for* $i \in \mathcal{I}$*, we have*

$$\mathfrak{e}_t = \mathfrak{e}(\mathfrak{d}_t) := (\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top)^{-1}\mathbf{1},$$

2. *Using* $\mathcal{B}_t = (\mathcal{B}_t^i)_{i\in\mathcal{I}} := [\mathcal{A}_t^i + v^i(\mathfrak{d}_t)]_{i\in\mathcal{I}}$ *with for* $(\overline{\tau}, \overline{\zeta}, \overline{\kappa}, \overline{\delta}) \in \mathbb{R}_+^I \times \mathbb{R}_+^{I\times I} \times \mathbb{R}_+^I \times \mathbb{R}_+$*,*

$$v^i(\overline{\mathfrak{d}}) := \log\left((\mathfrak{e}(\overline{\mathfrak{d}})^i)^{-\frac{\varphi\psi^i}{1+\varphi}}\left(\Psi^i(\overline{\mathfrak{d}})\right)^{\frac{\psi^i}{1+\varphi}}\prod_{j\in\mathcal{I}}\left(\Lambda^{ji}(\overline{\mathfrak{d}})\right)^{\boldsymbol{\lambda}^{ji}}\right)_{i\in\mathcal{I}} + ((\mathbf{I}_I - \boldsymbol{\lambda})\log(\mathfrak{e}(\overline{\mathfrak{d}}))), \tag{5.2.8}$$

We obtain

$$Y_t = \exp\left((\mathbf{I}_I - \boldsymbol{\lambda})^{-1}\mathcal{B}_t\right). \tag{5.2.9}$$

3. *Furthermore, since* $\mathfrak{d} \in \mathcal{C}^1(\mathbb{R}_+, [0,1)^I \times (\mathbb{R}_+)^{I\times I} \times (\mathbb{R}_+)^I)$*, we directly have* $\Psi(\overline{\mathfrak{d}}.), \Lambda(\overline{\mathfrak{d}}.) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$*. Moreover,* $\overline{\mathfrak{d}} \mapsto (\mathbf{I}_I - \Lambda(\overline{\mathfrak{d}})^\top)^{-1}$ *on* $\mathbb{R}_+^I \times \mathbb{R}_+^{I\times I} \times \mathbb{R}_+^I$*) is differentiable, then* $(\mathbf{I}_I - \Lambda(\mathfrak{d}.)^\top)^{-1}) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$*.*

The output Y is also positive, we can then introduce, from the third item, for all $t \geq 0$, $\mathrm{d}\log(Y_t)$ representing the instantaneous consumption growth. This proposition is an equivalent of Theorem 3.1.12 in continuous time.

5.2.2 A Firm Valuation Model

Consider a fix $n \in \{1, \ldots, N\}$. For any time $t \in \mathbb{R}_+$ and firm n, we note F_t^n the free cash flows of n at t, and $r > 0$ the discount rate, we introduce the following assumption:

Assumption 5.2.5. The $\mathbb{R}$-valued process on the instantaneous growth of the cash flows of firm n denoted by $\mathrm{d}\log F^n_{t,\mathfrak{d}}$ is linear in the economic factors (output growth by sector), specifically we set for all $t \in \mathbb{R}_+$,

$$\mathrm{d}\log F^n_{t,\mathfrak{d}} = \tilde{\mathfrak{a}}^{n\cdot}\mathrm{d}\log Y_t + \sigma_n \mathrm{d}\mathcal{W}^n_t = \mathfrak{a}^{n\cdot}(\mathrm{d}\mathcal{A}_t + \mathrm{d}v(\mathfrak{d}_t)) + \sigma_n \mathrm{d}\mathcal{W}^n_t, \tag{5.2.10}$$

for $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ and $\mathfrak{a}^{n\cdot} = \tilde{\mathfrak{a}}^{n\cdot}(\mathbf{I}_I - \boldsymbol{\lambda})^{-1}$, where $(\mathcal{W}^n_t)_{t\in\mathbb{R}_+}$ is a $\mathbb{R}^N$-Brownian motion with $\sigma_n > 0$. Moreover, $B^{\mathcal{Z}}$ and $\mathcal{W}^n$ are independent.

We define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by $\mathcal{F}_t = \sigma\left(\mathcal{G}_t \cup \sigma\left\{\mathfrak{b}_s : s \in [0,t]\right\}\right)$ for $t \geq 0$, and we denote $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$.

Recall that the economic motivation behind (5.2.10) comes from the fact that if firm n belongs to sector i, then its production is proportional to the sectoral output and its cash flows are proportional to its production (as in the Dechow-Kothari-Watts model in Barth et al. [2001]). Thus, we obtain a relation between the cash flows of firm n and the total output of sector i. The assumption $\tilde{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ stems from the fact that a company is not restricted to one sector only in general. However, since we are considering the emission sector here, we expect that each firm n only belongs to one sector (i for example). Therefore $\mathfrak{a}^{nj} = 0$ for all $i \neq j$ and hence $|\mathfrak{a}^{ni}| = \max_{j\in\mathcal{I}} |\mathfrak{a}^{nj}|$.

Let $r \geq 0$ representing the interest rate, by the continuous form of the discounted cash flows valuation, the value $V^n_{t,\mathfrak{d}}$ of the firm n, at time t, is

$$V^n_{t,\mathfrak{d}} := \mathbb{E}_t\left[\int_t^{+\infty} e^{-r(s-t)} F^n_{s,\mathfrak{d}} \mathrm{d}s\right]. \tag{5.2.11}$$

For $n \in \{1,\dots,N\}$ and $t \geq 0$, describing $V^n_{t,\mathfrak{d}}$ as a function of the underlying processes driving the economy does not lead to an easily tractable formula. To facilitate the forthcoming credit risk analysis, when ς (introduced in Standing Assumption 5.2.1) is closed to 0, we approach $V^n_{t,\mathfrak{d}}$ by the quantity

$$\mathcal{V}^n_{t,\mathfrak{d}} := F^n_{t,\mathfrak{d}} \int_t^{+\infty} e^{-r(s-t)} \mathbb{E}_t\left[\exp\left((s-t)\mathfrak{a}^{n\cdot}\mu + \mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) + \sigma_n(\mathcal{W}^n_s - \mathcal{W}^n_t)\right)\mathrm{d}s\right], \tag{5.2.12}$$

that we describe as a proxy the firm n value at time t. We will work directly with $\mathcal{V}^n_{t,\mathfrak{d}}$ instead of $V^n_{t,\mathfrak{d}}$. We have the following proposition, whose proof is given in C.2.

Proposition 5.2.6. *For any $n \in \{1,\dots,N\}$ and for all $t \in \mathbb{R}_+$.*

1. *Assume that $\varrho_n := \frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n\cdot}\mu - r < 0$, then $\mathcal{V}^n_{t,\mathfrak{d}}$ is well defined and*

$$\mathcal{V}^n_{t,\mathfrak{d}} = F^n_0 \mathfrak{R}^n_t(\mathfrak{d}) \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}_t - v(\mathfrak{d}_0))\right) \exp\left(\sigma_n \mathcal{W}^n_t\right), \tag{5.2.13}$$

where

$$\mathfrak{R}^n_t(\mathfrak{d}) := \int_0^{\infty} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right)\mathrm{d}s. \tag{5.2.14}$$

2. *Moreover, with $t_\circ$ and $t_\star$ defined in Standing Assumption 5.2.3, we obtain the following explicit form,*

$$\mathfrak{R}_t^n(\mathfrak{d}) = \begin{cases} -\dfrac{e^{\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\star})}}{\varrho_n}, & \text{if } t \geq t_\star, \\ \displaystyle\int_0^{t_\star - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s - \frac{e^{\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\star}) + \varrho_n (t_\star - t)}}{\varrho_n}, & \text{if } 0 \leq t < t_\star, . \end{cases}$$

3. *Assume that*

$$\rho_n := \frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n\cdot}\mu + \frac{1}{2}\varsigma^2 \frac{c_\Gamma^2}{\lambda_\Gamma^2} \|\mathfrak{a}^{n\cdot}\|^2 \|\Sigma\|^2 < r. \tag{5.2.15}$$

therefore $V_{t,\mathfrak{d}}^n$ is well defined and there exists a constant C such that $\mathbb{E}\left[\left|\frac{V_{t,\mathfrak{d}}^n}{F_{t,\mathfrak{d}}^n} - \frac{\mathcal{V}_{t,\mathfrak{d}}^n}{F_{t,\mathfrak{d}}^n}\right|\right] \leq C\varsigma$, for all $\varsigma > 0$.

The previous proposition is an equivalent of Lemma 4.1.6 and Proposition 4.1.7 in continuous time. The following corollary gives (conditional) laws of the (proxy) of the firm value $\mathcal{V}_{\cdot,\mathfrak{d}}^n$.

Corollary 5.2.7. *For all $t, T \geq 0$.*

1. *We note $\mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ) := \log\left(F_0^n \mathfrak{R}_t^n(\mathfrak{d})\right) + \left(\mathfrak{a}^{n\cdot}(\mathcal{A}_t^\circ - v(\mathfrak{d}_0))\right)$ and we have*

$$\log \mathcal{V}_{t,\mathfrak{d}}^n | \mathcal{G}_t \sim \mathcal{N}\left(\mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t^\circ), t\sigma_n^2\right).$$

2. *We note $\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \mathcal{Z}_t) := \log\left(F_0^n \mathfrak{R}_{t+T}^n(\mathfrak{d})\right) + \mathfrak{a}^{n\cdot}(\mu T + \varsigma \Upsilon_T \mathcal{Z}_t + \mathcal{A}_t^\circ - v(\mathfrak{d}_0))$ and $\mathcal{L}^n(t, T) := \mathfrak{a}^{n\cdot} \Sigma_t^{\mathcal{A},T} \mathfrak{a}^{n\cdot} + (t+T)\sigma_n^2$, we have*

$$\log \mathcal{V}_{t+T,\mathfrak{d}}^n | \mathcal{G}_t \sim \mathcal{N}\left(\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}_t^\circ, \mathcal{Z}_t), \mathcal{L}^n(t, T)\right).$$

5.2.3 A Credit Risk Model without collateral

To conclude this section, we present the probability of default. As Basel Committee on Banking Supervision [2017], we introduce four credit risk parameters: We can compute the portfolio loss L, with some assumptions:

Assumption 5.2.8. Consider a portfolio of $N \in \mathbb{N}^*$ credits. For $1 \leq n \leq N$,

(1) Firm n has issued two classes of securities: equity and debt.

(2) $(\mathrm{EAD}_t^n)_{t \geq 0}$ is a $\mathbb{R}_+^*$-valued continuous and deterministic process, and for all $t \geq 0$, the family $(\mathrm{EAD}_t^n)_{n=1,\dots,N}$ is a sequence of positive constants such that

   (a) $\displaystyle\sum_{n \geq 1} \mathrm{EAD}_t^n = +\infty$;

   (b) there exists $\upsilon > 0$ such that $\frac{\mathrm{EAD}_t^n}{\sum_{n=1}^N \mathrm{EAD}_t^n} = \mathcal{O}(N^{-(\frac{1}{2}+\upsilon)})$, as N tends to infinity.

(3) $(\mathrm{LGD}_t^n)_{t\geq 0}$ is a $(0,1]$-valued continuous and deterministic process;

(4) $(\mathcal{D}_t^n)_{t\geq 0}$ is a $\mathbb{R}_+$-valued continuous and deterministic process, representing the debt of firm n at time t. We will also denote $D_t^n := \dfrac{\mathcal{D}_t^n}{\mathbb{E}[F_{t,0}^n]}$ representing the debt to cash flows ratio.

(5) The value of the firm n at time t is assumed to be a tradable asset given by $V_{t,\mathfrak{d}}^n$ defined in (5.2.13).

According to Kruschwitz and Löffler [2020], there are two ways to handle the default of a company: for a given financing policy, a levered firm is

- *in danger of illiquidity* if the cash flows do not suffice to fulfill the creditors' payment claims (interest and net redemption) as contracted,
- *over-indebted* if the market value of debt exceeds the firm's market value.

We follow in the present work the second definition of default proposed: a firm default when it is *over-indebted*, that is in fact the same approach used in the structural credit risk models. We retain this definition in this continuous setting in the same way as Bourgey et al. [2022]. Actually, the term *default* may even be considered an abuse of language. So, to avoid any confusion in the following, we will use the terms *default* and *over-indebted* without distinction. Therefore, the *over-indebtedness* of entity n occurs at time t when the firm value $\mathcal{V}_{t,\mathfrak{d}}^n$ falls below a given barrier $\mathcal{D}_t^n$, related to the net debt, namely on the event $\left\{\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n\right\}$.

However, it should be noted that in a continuous time setting, it can be interesting to work in the Black and Cox [1976] model. Here, the default event depends on the trajectory of the firm value process $\mathcal{V}$. Therefore, at a given time t, the firm defaults if it has been *over-indebted* at least one time during the period $[0,t]$, that is $\left\{\exists s \in [0,t] \text{ such that } \mathcal{V}_{s,\mathfrak{d}}^n < \mathcal{D}_s^n\right\}$. Thus, the default time is given by

$$\tau^n := \inf\left\{t \geq 0, \quad \mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n\right\}.$$

Then, if we are interested in the probability of the firm n defaulting before t conditionally to $\mathcal{G}_t$ that is noted $PD_{t,\mathfrak{d}}^n$, we have

$$PD_{t,T,\mathfrak{d}}^n = \mathbb{P}\left(\tau^n \leq t | \mathcal{G}_t\right) = \mathbb{P}\left(\inf_{0\leq s\leq t} \mathcal{V}_{s,\mathfrak{d}}^n < \mathcal{D}_s^n \middle| \mathcal{G}_t\right) = \mathbb{P}\left(\inf_{0\leq s\leq t} \log \mathcal{V}_{s,\mathfrak{d}}^n < \log \mathcal{D}_s^n \middle| \mathcal{G}_t\right).$$

But for $0 \leq s \leq t$ and from (5.2.13),

$$\log \mathcal{V}_{s,\mathfrak{d}}^n = \log\left(F_{0,\mathfrak{d}}^n \mathfrak{R}_s^n(\mathfrak{d})\right) + \mathfrak{a}^{n\cdot}(\mathcal{A}_s^\circ - v(\mathfrak{d}_0)) + \sigma_n \mathcal{W}_s^n,$$

therefore $\log \mathcal{V}_{\cdot,\mathfrak{d}}^n$ is a Gaussian process. However, as Azaıs and Wschebor [2000] summarizes, the computation of the distribution function of the random variable $\inf_{0\leq s\leq t} \log \mathcal{V}_{s,\mathfrak{d}}^n$ is by means of a closed formula is known only for a very restricted

number of stochastic processes as the Brownian Motion, the Brownian Bridge, the Brownian Motion with a linear drift, and the stationary Gaussian processes with relatively simple with covariance. This is not the case here.

At each time $t \geq 0$, we are interested in the probability that firm n is *over-indebted* at a certain date $t+T$, we note $\mathrm{PD}^n_{t,T,\mathfrak{d}}$ and we have

$$\mathrm{PD}^n_{t,T,\mathfrak{d}} := \mathbb{P}\left(\mathcal{V}^n_{t+T,\mathfrak{d}} \leq \mathcal{D}^n_{t+T} \middle| \mathcal{G}_t\right). \tag{5.2.16}$$

We have the following proposition whose proof is a direct application of Corollary 5.2.7.

Proposition 5.2.9 (Probability of default)**.** *For $t \geq 0$, $T \geq 0$, and $n \in \{1, \ldots, N\}$, the (conditional) probability of default of the entity n at time t over the horizon T is*

$$\mathrm{PD}^n_{t,T,\mathfrak{d}} = \Phi\left(\frac{\log(\mathcal{D}^n_{t+T}) - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t)}{\sqrt{\mathcal{L}^n(t,T)}}\right), \tag{5.2.17}$$

where $\mathcal{K}^n(\mathfrak{d}, t, T, a, \theta)$ and $\mathcal{L}^n(t,T)$ are defined in Corollary 5.2.7.

The previous results tell us that the probability of the *over-indebtedness* depends on:

1. parameters specific to the climate transition
    - the carbon price δ,
    - the carbon intensities τ, ζ, κ,
2. parameters specific to the company (the contract),
    - the factors loading $\mathfrak{a}^{n,\cdot}$ and the standard deviation of the cash flows σ_n,
    - the time t when it is computed,
    - the potential date of the *over-indebtedness* $t+T$,
    - the *over-indebtedness*'s barrier $\mathcal{D}$,
3. parameters specific to the economy to which the company belongs to:
    - the productivity $\mathcal{Z}$ and $\mathcal{A}$ (and their parameters) of the economy,
    - the interest rate r.

Therefore, by assuming that EAD and LGD are deterministic and independent of the carbon price, we could obtain the expressions of the EL and UL. In the next section, we will express LGD as a function of some guarantees which are affected by the climate transition.

5.3 LGD with stochastic collaterals in continuous time

We are in the same framework as in the previous section, but Assumption 5.2.8(2) is not satisfied anymore, therefore the bank could require from each counterpart $1 \leq n \leq N$ a (single) collateral C^n to secure its debt. Collateral can take the form of a *physical asset* such as real estate, business equipment, or inventory, or it can be a *financial asset* such as invoices, cash, or investments. If a firm is *over-indebted* at time t, we assume that the liquidation ends at $t+a$ with $a \in \mathbb{R}_+$ where a is the *liquidation delay*. Moreover, $k \in [0,1)$ represents the fraction of the collateral used to cover liquidations auctions, as well as other legal and administrative procedures.

A firm is *over-indebted* at time $t \geq 0$ if the market value of its debt $\mathcal{D}_t^n$ exceed its market value $\mathcal{V}_{t,\mathfrak{d}}^n$, namely $\{\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n\}$. At time t, if the company n in the portfolio defaults i.e. $\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n$, the bank recovers $(1-k)e^{-ra}C_{t+a}^n$ after the collateral liquidation. In general, the liquidations do not cover all the debt, i.e. $\mathrm{EAD}_t^n \geq (1-k)e^{-ra}C_{t+a}^n$, the bank deploys further actions to recover an additional fraction. We note that fraction $\gamma \in [0,1)$. Therefore, the bank recovers $\gamma(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n)_+$ by other tools.

The potential loss that would be recorded due to the firm default event is the difference between the debt amount EAD_t^n and the amount gets after the recovery processes at the time horizon. Consequently, if there is not default ($\mathcal{V}_{t,\mathfrak{d}}^n \geq \mathcal{D}_t^n$) or there is default and the collateral liquidated exceed the exposure ($(1-k)e^{-ra}C_{t+a}^n \geq \mathrm{EAD}_t^n$), the loss noted $L_{n,t}$ is zero, and if there is default and if the exposure exceed the collateral liquidated, the loss is

$$L_{n,t} = \mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n - \gamma(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n) = (1-\gamma)\left(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n\right),$$

where the constant r is the discount rate. The loss of the portfolio at time t, is in fact, defined as

$$L_t^N := \sum_{n=1}^{N} L_{n,t} = \sum_{n=1}^{N} (1-\gamma)(\mathrm{EAD}_t^n - (1-k)e^{-ra}C_{t+a}^n)_+ \cdot \mathbf{1}_{\{\mathcal{V}_{t,\mathfrak{d}}^n < \mathcal{D}_t^n\}}.$$

The following result is similar with the one introduced in Proposition 4.2.4. It gives a proxy of the loss of the portfolio.

Theorem 5.3.1 (Definition of PD and LGD)**.** *For all $t \in \mathbb{R}_+$, define*

$$\mathrm{L}_t^{\mathbb{G},N} := \sum_{n=1}^{N} L_{n,t}^{\mathbb{G}} \quad \textit{with} \quad L_{n,t}^{\mathbb{G}} = \mathbb{E}\left[L_{n,t} | \mathcal{G}_t\right] = \mathrm{EAD}_t^n \cdot \mathrm{LGD}_{t,\mathfrak{d}}^n \cdot \mathrm{PD}_{t,\mathfrak{d}}^n, \tag{5.3.1}$$

where

$$\mathrm{PD}^n_{t,\mathfrak{d}} := \mathbb{P}\left(\mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t | \mathcal{G}_t\right) = \Phi\left(\frac{\log(\mathcal{D}^n_t) - \mathfrak{m}^n(\mathfrak{d}, t, \mathcal{A}_t)}{\sigma^n\sqrt{t}}\right),$$
$$\mathrm{LGD}^n_{t,\mathfrak{d}} := (1-\gamma)\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{C^n_{t+a}}{\mathrm{EAD}^n_t}\right)_+ \middle| \mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t, \mathcal{G}_t\right]. \quad (5.3.2a)$$

Under Assumptions 5.2.8, we have $L^N_t - \mathrm{L}^{\mathbb{G},N}_t$ *converges to zero almost surely as* N *tends to infinity, for all* $t \in \mathbb{R}_+$.

Proof. Let $t \in \mathbb{R}_+$,

$$\mathrm{L}^{\mathbb{G},N}_t = \mathbb{E}\left[L^N_t \middle| \mathcal{G}_t\right] = \mathbb{E}\left[\sum_{n=1}^N (1-\gamma)(\mathrm{EAD}^n_t - (1-k)e^{-ra}C^n_{t+a})_+ \cdot \mathbf{1}_{\{\mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t\}} \middle| \mathcal{G}_t\right].$$

$(\mathrm{EAD}^n_t)_{n\in\{1,\dots,N\}}$ is deterministic, we have:

$$\begin{aligned}\mathrm{L}^{\mathbb{G},N}_t &= \sum_{n=1}^N \mathrm{EAD}^n_t \cdot \mathbb{E}\left[(1-\gamma)\left(1-(1-k)e^{-ra}\frac{C^n_{t+a}}{\mathrm{EAD}^n_t}\right)_+ \cdot \mathbf{1}_{\{\mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t\}} \middle| \mathcal{G}_t\right] \\ &= \sum_{n=1}^N \mathrm{EAD}^n_t \cdot \mathbb{E}\left[(1-\gamma)\left(1-(1-k)e^{-ra}\frac{C^n_{t+a}}{\mathrm{EAD}^n_t}\right)_+ \middle| \mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t, \mathcal{G}_t\right] \cdot \mathbb{P}\left[\mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t | \mathcal{G}_t\right].\end{aligned}$$

The rest of the proof requires a version of the strong law of large numbers (Appendix of [Gordy, 2003b, Propositions 1, 2]), where the systematic risk factor is $\mathcal{G}_t$. □

Explicitly, in the above theorem, we assume that our portfolio is perfectly fine grained, so that we can approximate L^N_t – the portfolio loss – by $L^{\mathbb{G},N}_t$ – the conditional expectation of loss given the systemic factor–. By construction, the loss given default noted LGD is the percentage of the total exposure that is lost when a *over-indebtedness* occurs. The literature on LGD modeling is fairly extensive. We can distinguish namely, economic modeling Bastos [2010], Roncalli [2020] and stochastic modeling Roncalli [2020][Page 193], Chalupka and Kopecsni [2008]. As the definition of PD does not change compare to what we did in Section 5.2 and as EAD is given, we will focus on LGD modeling. We can first remark that $0 \leq \mathrm{LGD}^n_{t,\mathfrak{d}} \leq 1-\gamma$, then the presence of a collateral necessarily reduces LGD.

Key quantities for the bank to understand the (dynamics of the) risk in the portfolio are the (expected and unexpected) losses and probability of default conditionally to the (information generated by the) risk factors. Precisely, for a date t and a horizon T, a bank would like to know some risk measures at t of its portfolio maturing at horizon T.

Definition 5.3.2 (Projected losses)**.** Let $t \geq 0$ be the time when the risk measure is computed for a period $T \geq 0$. As classically done, the potential loss is separated into three components:

- The (conditional) Expected Loss (EL) reads

$$\mathrm{EL}^{N,T}_t := \mathbb{E}\left[\mathrm{L}^{\mathbb{G},N}_{t+T} \middle| \mathcal{G}_t\right]. \quad (5.3.3)$$

- The Unexpected Loss (UL) reads for $\alpha \in (0,1)$,

$$\mathrm{UL}_t^{N,T}(\alpha) := \mathrm{VaR}_t^{\alpha,N,T} - \mathrm{EL}_t^{N,T}, \quad \text{where} \quad 1-\alpha = \mathbb{P}\left[L_{t+T}^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]. \tag{5.3.4}$$

- The Stressed Loss (or Expected Shortfall or ES) reads $\mathrm{VaR}_t^{\alpha}(L_s^N)$:

$$\mathrm{ES}_t^{N,T}(\alpha) := \mathbb{E}\left[L_{t+T}^N \middle| L_{t+T}^N \geq \mathrm{VaR}_t^{\alpha,N,T}, \mathcal{G}_t\right], \qquad \text{for } \alpha \in (0,1).$$

From now, if the collateral exists, we focus on two types: a financial asset and a property in housing market. Let $n \in \{1,\dots,N\}$.

5.3.1 When there is not collateral

When firm n does not have a collateral, therefore $C^n = 0$ and from (5.3.2a), LGD is

$$\mathrm{LGD}_{t,\mathfrak{d}}^n = 1-\gamma.$$

5.3.2 When collateral is a financial asset

Here we assume that the collateral of the firm n is an investment in a financial asset. Precisely, we assume that that investment is a proportion $\alpha^n \in (0,1]$ of a given firm located in the economy described in Section 5.2. Consequently, it is subjected to the same constraints in terms of productivity and of carbon transition scenarios as firm n. As any investment, it should generate a stream of cash flows so that at each time, we can compute its value by using the discounted cash flows model introduced in (5.2.11).

Let note the collateral cash flows $(\overline{F}_t^n)_{t\in\mathbb{R}_+}$, its dynamics is similar to the firm cash flows introduced in Assumption 5.2.5. We have for all $t \in \mathbb{R}_+$,

$$\mathrm{d}\overline{F}_{t,\mathfrak{d}}^n = \overline{\mathfrak{a}}^{n\cdot}((\mu + \varsigma\mathcal{Z}_t)\mathrm{d}t + \mathrm{d}v(\mathfrak{d}_t)) + \overline{\sigma}_n \mathrm{d}\overline{\mathcal{W}}_t^n, \tag{5.3.5}$$

where $\overline{\mathfrak{a}}^{n\cdot} \in \mathbb{R}^I$ and where $(\overline{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is a $\mathbb{R}^N$-Brownian motion with $\overline{\sigma}_n > 0$. Moreover, $B^{\mathcal{Z}}$ (noise of productivity), $\overline{\mathcal{W}}^n$ (noise of collateral), and $(\mathcal{W}^n)_{n\in\{1,\dots,N\}}$ (noise of debtors) are independent. We also note $\overline{\tilde{\mathfrak{a}}}^{n\cdot} = \overline{\mathfrak{a}}^{n\cdot}(\mathbf{I}_I - \boldsymbol{\lambda})$.

Remark 5.3.3. We have assumed that $\overline{\mathcal{W}}^n$ and $\mathcal{W}^n$ are not correlated, but this is not always the case. For example, if the depreciation of the firm value heading to its *over-indebtedness* implies the depreciation of the collateral value, then we should have a positive correlation.

Inspired by (5.2.11), the collateral value at time t is

$$C_{t,\mathfrak{d}}^n := \alpha^n \mathbb{E}_t\left[\int_{s=t}^{+\infty} e^{-rs}\overline{F}_{s,\mathfrak{d}}^n \mathrm{d}s\right],$$

and by (5.2.12), the approached collateral value as

$$\mathcal{C}^n_{t,\mathfrak{d}} := \alpha^n \overline{F}^n_{t,\mathfrak{d}} \int_t^{+\infty} e^{-r(s-t)} \mathbb{E}_t \left[\exp\left((s-t)\overline{\mathfrak{a}}^{n\cdot}\mu + \overline{\mathfrak{a}}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) + \sigma_n(\overline{\mathcal{W}}^n_s - \overline{\mathcal{W}}^n_t)\right) \mathrm{d}s \right]. \tag{5.3.6}$$

Therefore, the following proposition and its proof are inspired by Lemma 5.2.6 and corollary 5.2.7, gives a proxy of the collateral value.

Proposition 5.3.4. *For any $n \in \{1, \dots, N\}$ and for all $t \in \mathbb{R}_+$*

1. *Assume that $\overline{\varrho}_n := \frac{1}{2}\overline{\sigma}^2_{\mathfrak{b}_n} + \overline{\mathfrak{a}}^{n\cdot}\mu - r < 0$. Given the carbon emissions costs sequence $\mathfrak{d}$, the proxy of collateral value defined in (5.3.6), is well defined and*

$$\mathcal{C}^n_{t,\mathfrak{d}} = \alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_t(\mathfrak{d}) \exp\left(\overline{\mathfrak{a}}^{n\cdot}(\mathcal{A}^\circ_t - v(\mathfrak{d}_0))\right) \exp\left(\overline{\sigma}_n \overline{\mathcal{W}}^n_t\right), \tag{5.3.7}$$

where

$$\overline{\mathfrak{R}}^n_t(\mathfrak{d}) := \int_0^\infty e^{\overline{\varrho}_n s} \exp\left(\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s.$$

2. *Moreover, we note $\overline{\mathfrak{m}}^n(\mathfrak{d}, t, \mathcal{A}^\circ_t) := \log\left(\alpha^n \overline{F}^n_0\right) + \log \overline{\mathfrak{R}}^n_t(\mathfrak{d}) + \left(\overline{\mathfrak{a}}^{n\cdot}(\mathcal{A}^\circ_t - v(\mathfrak{d}_0))\right)$ and we have*

$$\log \mathcal{C}^n_{t,\mathfrak{d}} | \mathcal{G}_t \sim \mathcal{N}\left(\overline{\mathfrak{m}}^n(\mathfrak{d}, t, \mathcal{A}^\circ_t), t\overline{\sigma}^2_n\right),$$

and we note $\overline{\mathcal{K}}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t) := \log\left(\alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T}(\mathfrak{d})\right) + \overline{\mathfrak{a}}^{n\cdot}(\mu T + \varsigma \Upsilon_T \mathcal{Z}_t + \mathcal{A}^\circ_t - v(\mathfrak{d}_0))$ and $\overline{\mathcal{L}}^n(t, T) := \overline{\mathfrak{a}}^{n\cdot}\Sigma^{\mathcal{A}}_{t,t+T}\overline{\mathfrak{a}}^{n\cdot} + (t+T)\overline{\sigma}^2_n$, and we have

$$\log \mathcal{C}^n_{t+T} | \mathcal{G}_t \sim \mathcal{N}\left(\overline{\mathcal{K}}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t), \overline{\mathcal{L}}^n(t, T)\right).$$

3. *Assume that*

$$\overline{\rho}_n := \frac{1}{2}\overline{\sigma}^2_n + \overline{\mathfrak{a}}^{n\cdot}\mu + \frac{1}{2}\varsigma^2 \frac{c^2_\Gamma}{\lambda^2_\Gamma} \|\overline{\mathfrak{a}}^{n\cdot}\|^2 \|\Sigma\|^2 < r,$$

therefore $C^n_{t,\mathfrak{d}}$ is well defined and there exists a constant $\overline{C}$ such that $\mathbb{E}\left[\left|\frac{C^n_{t,\mathfrak{d}}}{\overline{F}^n_{t,\mathfrak{d}}} - \frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\overline{F}^n_{t,\mathfrak{d}}}\right|\right] \leq \overline{C}\varsigma$, for all $\varsigma > 0$.

Proof. Let $n \in \{1, \dots, N\}$ and $t \in \mathbb{R}_+$, (5.3.7) directly comes from Proposition 5.2.6. The proofs of the three points are equivalent to C.2. Let us develop the conditional laws. From (5.3.7), we have

$$\log \mathcal{C}^n_t = \log \alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_t(\mathfrak{d}) + \overline{\mathfrak{a}}^{n\cdot}(\mathcal{A}^\circ_t - v(\mathfrak{d}_0)) + \overline{\sigma}_n \overline{\mathcal{W}}^n_t.$$

Because $\overline{\mathcal{W}}^n$ is a Brownian motion, $\overline{\mathcal{W}}^n_t \sim \mathcal{N}(0, t)$ and, $\overline{\mathcal{W}}^n$ and $B^{\mathcal{Z}}$ are independent, we obtain $\log \mathcal{C}^n_t | \mathcal{G}_t \sim \mathcal{N}\left(\overline{\mathfrak{m}}^n(\mathfrak{d}, t, \mathcal{A}^\circ_t), t\overline{\sigma}^2_n\right)$. Let also $T \in \mathbb{R}_+$, we have

$$\log \mathcal{C}^n_{t+T} = \log \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T}(\mathfrak{d}) + \overline{\mathfrak{a}}^{n\cdot}(\mathcal{A}^\circ_{t+T} - v(\mathfrak{d}_0)) + \overline{\mathcal{W}}^n_{t+T}.$$

From Remark 5.2.2, $\mathcal{A}_{t+T} | \mathcal{G}_t \sim \mathcal{N}\left(M^{\mathcal{A},T}_t, \Sigma^{\mathcal{A},T}_t\right)$ and because $\overline{\mathcal{W}}^n$ is a Brownian motion, $\overline{\mathcal{W}}^n_{t+T} \sim \mathcal{N}(0, t+T)$. Moreover, $\overline{\mathcal{W}}^n$ and $B^{\mathcal{Z}}$ are independent. We have

$$\log \mathcal{C}^n_{t+T} | \mathcal{G}_t \sim \mathcal{N}\left(\log \alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T}(\mathfrak{d}) + \overline{\mathfrak{a}}^{n\cdot}(M^{\mathcal{A},T}_t - v(\mathfrak{d}_0)), \overline{\mathfrak{a}}^{n\cdot}\Sigma^{\mathcal{A},T}_t \overline{\mathfrak{a}}^{n\cdot} + (t+T)\overline{\sigma}^2_n\right).$$

The conclusion follows. □

From the (proxy of the) collateral value $\mathcal{C}$, we can then derive a precised expression of LGD based on Theorem 5.3.1. We have:

Theorem 5.3.5. *When $a = 0$ (no liquidation delay), the Loss Given Default of the obligor n over-indebted at time $t \in \mathbb{R}_+$, conditional on $\mathcal{G}_t$ is*

$$\mathrm{LGD}^n_{t,\mathfrak{d}} = (1-\gamma)\left[\Phi\left(\frac{w^n_t}{\overline{\sigma}_{\mathfrak{b}_n}\sqrt{t}}\right) - \exp\left(-w^n_t + \frac{1}{2}t\overline{\sigma}^2_n\right)\Phi\left(\frac{w^n_t}{\overline{\sigma}_n\sqrt{t}} - \overline{\sigma}_n\sqrt{t}\right)\right], \quad (5.3.8)$$

where

$$w^n_t := \log\left(\frac{\mathrm{EAD}^n_t}{1-k}\right) - \overline{\mathfrak{m}}^n(\mathfrak{d}, t, \mathcal{A}_t). \quad (5.3.9)$$

Proof. From (5.3.2a) and when $a = 0$, we have

$$\begin{aligned}
\mathrm{LGD}^n_{t,\mathfrak{d}} &= (1-\gamma)\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+a,\mathfrak{d}}}{\mathrm{EAD}^n_t}\right)_+ \middle| \mathcal{V}^n_{t,\mathfrak{d}} < \mathcal{D}^n_t, \mathcal{G}_t\right] \\
&= (1-\gamma)\mathbb{E}\left[\left(1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t}\right)_+ \middle| \mathcal{G}_t\right] \quad \text{because} \quad \mathcal{C}^n_{t,\mathfrak{d}}|\mathcal{G}_t \text{ and } F^n_{t,\mathfrak{d}}|\mathcal{G}_t \text{ are independent} \\
&= (1-\gamma)\mathbb{E}\left[\left(1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t}\right)\mathbf{1}_{\{1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t}\geq 0\}} \middle| \mathcal{G}_t\right] \\
&= (1-\gamma)\left[\mathbb{P}\left(1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t} \geq 0 \middle| \mathcal{G}_t\right) - \frac{(1-k)}{\mathrm{EAD}^n_t}\mathbb{E}\left[\mathcal{C}^n_{t,\mathfrak{d}}\mathbf{1}_{\{1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t}\geq 0\}} \middle| \mathcal{G}_t\right]\right].
\end{aligned}$$

But, from Corollary 5.2.7, we have $\log \mathcal{C}^n_{t,\mathfrak{d}}|\mathcal{G}_t \sim \mathcal{N}\left(\overline{\mathfrak{m}}^n(\mathfrak{d}, t, \mathcal{A}_t), t\overline{\sigma}^2_n\right)$. We also consider w^n_t defined in (5.3.9), therefore

$$\mathbb{P}\left(1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t} \geq 0 \middle| \mathcal{G}_t\right) = \Phi\left(\frac{w^n_t}{\overline{\sigma}_{\mathfrak{b}_n}\sqrt{t}}\right).$$

We also have

$$\mathbb{E}\left[\mathcal{C}^n_{t,\mathfrak{d}}\mathbf{1}_{\{1-(1-k)\frac{\mathcal{C}^n_{t,\mathfrak{d}}}{\mathrm{EAD}^n_t}\geq 0\}} \middle| \mathcal{G}_t\right] = \exp\left(-w^n_t + \frac{1}{2}t\overline{\sigma}^2_n\right)\Phi\left(\frac{w^n_t}{\overline{\sigma}_n\sqrt{t}} - \overline{\sigma}_n\sqrt{t}\right).$$

The conclusion follows. □

We can also remark that the situation where there is no collateral corresponds to $\overline{F}^n_0 = 0$. We then have

$$\overline{F}^n_0 \to 0 \implies \log\left(\overline{F}^n_0\right) \to -\infty \implies \overline{\mathfrak{m}}^n(\mathfrak{d}, t, \mathcal{A}^\circ_t) \to -\infty \implies w^n_t \to +\infty \implies \mathrm{LGD}^n_{t,\mathfrak{d}} \to 1-\gamma.$$

For each $t, T \in \mathbb{R}_+$, we introduce now the (conditional) LGD of the entity n at time t on the horizon T, namely

$$\mathrm{LGD}^n_{t,T,\mathfrak{d}} := (1-\gamma)\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\mathfrak{d}}}{\mathrm{EAD}^n_{t+T}}\right)_+ \middle| \mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}, \mathcal{G}_t\right].$$

It is precisely about calculating at date t the proportion of the exposure that the bank would lose if the counterpart n is *over-indebted* at date $t+T$.

For $x, y \in \mathbb{R}$, we note $\Phi_2(x, y; \rho)$ is the cumulative distribution function of the bi-variate Gaussian vector (X, Y) with correlation ρ on the space $[-\infty, x] \times [-\infty, y]$.

Proposition 5.3.6 (Projected PD and LGD)**.** *For each $t, T \in \mathbb{R}_+$, the (conditional)* LGD *of the entity n at time t on the horizon T, reads*

$$\begin{aligned}\mathrm{LGD}^n_{t,T,\mathfrak{d}} = \frac{1-\gamma}{\mathrm{PD}^n_{t,T,\mathfrak{d}}} \Big[&\Phi_2\left(\overline{\omega}^n_{t,T,a}, \Phi^{-1}(\mathrm{PD}^n_{t,T,\mathfrak{d}}); \rho^n_{t,T,a}\right) - \\ &\exp\left(\frac{1}{2}\overline{\mathcal{L}}^n(t,T+a) - \sqrt{\overline{\mathcal{L}}^n(t,T+a)}\overline{\omega}^n_{t,T,a}\right)\Phi_2\left(\overline{\omega}^n_{t,T,a} - \sqrt{\overline{\mathcal{L}}^n(t,T+a)}, \Phi^{-1}(\mathrm{PD}^n_{t,T,\mathfrak{d}}) - \rho^n_{t,T,a}\sqrt{\overline{\mathcal{L}}^n(t,T+a)}; \rho^n_{t,T,a}\right)\Big],\end{aligned} \tag{5.3.10}$$

where

$$\rho^n_{t,T,a} := \frac{\varsigma^2 \mathfrak{a}^{n\cdot}\Gamma^{-1}\left(\int_0^T \left(e^{-\Gamma u} - \mathbf{I}_I\right)\Sigma\Sigma^\top\left(e^{-\Gamma(u+a)} - \mathbf{I}_I\right)\mathrm{d}u\right)(\overline{\mathfrak{a}}^{n\cdot}\Gamma^{-1})^\top}{\sqrt{\mathcal{L}^n(t,T)\overline{\mathcal{L}}^n(t,T+a)}},$$

and

$$\overline{\omega}^n_{t,T,a} := \frac{\log\frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}} - \overline{\mathcal{K}}^n(\mathfrak{d}, t, T+a, \mathcal{A}^\circ_t, \mathcal{Z}_t)}{\sqrt{\overline{\mathcal{L}}^n(t,T+a)}},$$

and where $\mathrm{PD}^n_{t,T,\mathfrak{d}}$ defined in Proposition 5.2.9.

Proof. Let $t, T \in \mathbb{R}_+$, from (5.3.1) and (5.3.3),

$$\mathrm{EL}^{N,T}_t := \mathbb{E}\left[\mathrm{L}^{\mathbb{G},N}_{t+T}\middle|\mathcal{G}_t\right] = \mathbb{E}\left[\sum_{n=1}^N L^{\mathbb{G}}_{n,t+T}\middle|\mathcal{G}_t\right] = \sum_{n=1}^N \mathbb{E}\left[L^{\mathbb{G}}_{n,t+T}\middle|\mathcal{G}_t\right].$$

But for $1 \le n \le N$, we have

$$\begin{aligned}
\mathbb{E}\left[L^{\mathbb{G}}_{n,t+T}\middle|\mathcal{G}_t\right] &= \mathbb{E}\left[\mathrm{EAD}^n_{t+T}\mathrm{LGD}^n_{t+T,\mathfrak{d}}\mathrm{PD}^n_{t+T,\mathfrak{d}}\middle|\mathcal{G}_t\right] \\
&= \mathrm{EAD}^n_{t+T}\mathbb{E}\left[\mathrm{LGD}^n_{t+T,\mathfrak{d}}\mathrm{PD}^n_{t+T,\mathfrak{d}}\middle|\mathcal{G}_t\right] \quad \text{as } \mathrm{EAD}^n_{t+T} \text{ is deterministic} \\
&= \mathrm{EAD}^n_{t+T}\mathbb{E}\left[\mathbb{E}\left[(1-\gamma)\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\mathfrak{d}}}{\mathrm{EAD}^n_{t+T}}\right)_+ \cdot \mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}\}}\middle|\mathcal{G}_{t+T}\right]\middle|\mathcal{G}_t\right] \\
&= (1-\gamma)\mathrm{EAD}^n_{t+T}\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\mathfrak{d}}}{\mathrm{EAD}^n_{t+T}}\right)_+ \cdot \mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}\}}\middle|\mathcal{G}_t\right] \\
&= (1-\gamma)\mathrm{EAD}^n_{t+T}\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\mathfrak{d}}}{\mathrm{EAD}^n_{t+T}}\right)\cdot \mathbf{1}_{1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\mathfrak{d}}}{\mathrm{EAD}^n_{t+T}} \ge 0}\mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}\}}\middle|\mathcal{G}_t\right] \\
&= (1-\gamma)\mathrm{EAD}^n_{t+T}\mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\mathfrak{d}}}{\mathrm{EAD}^n_{t+T}}\right)\cdot \mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T},\quad \mathcal{C}^n_{t+T+a,\mathfrak{d}} \le \frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}}\}}\middle|\mathcal{G}_t\right] \\
&= (1-\gamma)\mathrm{EAD}^n_{t+T}\left(\mathbb{P}\left[\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}, \mathcal{C}^n_{t+T+a,\mathfrak{d}} \le \frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}}\}\middle|\mathcal{G}_t\right]\right. \\
&\qquad \left. -\frac{(1-k)e^{-ra}}{\mathrm{EAD}^n_{t+T}}\mathbb{E}\left[\mathcal{C}^n_{t+T+a,\mathfrak{d}}\cdot \mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}, \mathcal{C}^n_{t+T+a,\mathfrak{d}} \le \frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}}\}}\middle|\mathcal{G}_t\right]\right).
\end{aligned}$$

However, from (5.2.13),

$$\log \mathcal{V}^n_{t+T,\mathfrak{d}} = \log\left(F^n_0\mathfrak{R}^n_{t+T}(\mathfrak{d})\right) + \mathfrak{a}^{n\cdot}(\mathcal{A}^\circ_{t+T} - v(\mathfrak{d}_0)) + \sigma_n\mathcal{W}^n_{t+T},$$

and from (5.3.7)

$$\log \mathcal{C}^n_{t+T+a,\mathfrak{d}} = \log \alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T+a}(\mathfrak{d}) + \overline{\mathfrak{a}}^{n\cdot}(\mathcal{A}^\circ_{t+T+a} - v(\mathfrak{d}_0)) + \overline{\sigma}_n \overline{\mathcal{W}}^n_{t+T+a}.$$

Therefore,

$$F^n_{t+T}|\mathcal{G}_t \sim \mathcal{LN}\left(\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t), \mathcal{L}^n(t,T)\right),$$

$$\mathcal{C}^n_{t+T+a}|\mathcal{G}_t \sim \mathcal{LN}\left(\overline{\mathcal{K}}^n(\mathfrak{d}, t, T+a, \mathcal{A}^\circ_t, \mathcal{Z}_t), \overline{\mathcal{L}}^n(t,T)\right),$$

and

$$cov(\log F^n_{t+T}, \log \mathcal{C}^n_{t+T+a}|\mathcal{G}_t) = \mathbb{E}[\log F^n_{t+T} \log \mathcal{C}^n_{t+T+a}|\mathcal{G}_t] - \mathbb{E}[\log F^n_{t+T}|\mathcal{G}_t]\mathbb{E}[\log \mathcal{C}^n_{t+T+a}|\mathcal{G}_t].$$

However,

$$\begin{aligned}
&\mathbb{E}[\log F^n_{t+T} \log \mathcal{C}^n_{t+T+a}|\mathcal{G}_t] \\
&\quad= \mathbb{E}\left[\left(\log\left(F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_0)}\right) + \mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T} + \sigma_n \mathcal{W}^n_{t+T}\right)\right. \\
&\qquad\left.\left(\log\left(\alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T+a}(\mathfrak{d}) e^{-\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_0)}\right) + \overline{\mathfrak{a}}^{n\cdot}\mathcal{A}^\circ_{t+T+a} + \overline{\sigma}_n \overline{\mathcal{W}}^n_{t+T+a}\right)\Big|\mathcal{G}_t\right] \\
&\quad= \mathbb{E}\left[\log\left(F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_0)}\right)\log\left(\alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T}(\mathfrak{d}) e^{-\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_0)}\right) + \mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T}\overline{\sigma}_n\overline{\mathcal{W}}^n_{t+T+a}\right. \\
&\qquad+ \log\left(F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_0)}\right)(\overline{\mathfrak{a}}^{n\cdot}\mathcal{A}^\circ_{t+T+a} + \overline{\sigma}_n\overline{\mathcal{W}}^n_{t+T+a}) + \overline{\sigma}_n\overline{\mathcal{W}}^n_{t+T+a}\sigma_n\mathcal{W}^n_{t+T} \\
&\qquad+ \log\left(\alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T+a}(\mathfrak{d}) e^{-\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_0)}\right)\mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T} + \mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T}\overline{\mathfrak{a}}^{n\cdot}\mathcal{A}^\circ_{t+T+a} \\
&\qquad\left.+ \log\left(\alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T+a}(\mathfrak{d}) e^{-\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_0)}\right)\sigma_n\mathcal{W}^n_{t+T} + \overline{\mathfrak{a}}^{n\cdot}\mathcal{A}^\circ_{t+T+a}\sigma_n\mathcal{W}^n_{t+T}\Big|\mathcal{G}_t\right] \\
&\quad= \log\left(F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_0)}\right)\log \alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T+a}(\mathfrak{d}) e^{-\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_0)} \\
&\qquad+ \log\left(F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d}) e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_0)}\right)\overline{\mathfrak{a}}^{n\cdot}\mathbb{E}[\mathcal{A}^\circ_{t+T+a}|\mathcal{G}_t] \\
&\qquad+ \log\left(\alpha^n \overline{F}^n_0 \overline{\mathfrak{R}}^n_{t+T+a}(\mathfrak{d}) e^{-\overline{\mathfrak{a}}^{n\cdot} v(\mathfrak{d}_0)}\right)\mathfrak{a}^{n\cdot}\mathbb{E}[\mathcal{A}^\circ_{t+T}|\mathcal{G}_t] + \mathbb{E}[\mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T}\overline{\mathfrak{a}}^{n\cdot}\mathcal{A}^\circ_{t+T+a}|\mathcal{G}_t].
\end{aligned}$$

By also developing $\mathbb{E}[\log F^n_{t+T}|\mathcal{G}_t]\mathbb{E}[\log \mathcal{C}^n_{t+T}|\mathcal{G}_t]$, we obtain

$$\begin{aligned}
cov(\log F^n_{t+T}, \log \mathcal{C}^n_{t+T}|\mathcal{G}_t) &= cov(\mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T}, \overline{\mathfrak{a}}^{n\cdot}\mathcal{A}^\circ_{t+T}|\mathcal{G}_t) \\
&= \varsigma^2 \mathfrak{a}^{n\cdot}\Gamma^{-1}\left(\int_0^T \left(e^{-\Gamma u} - \mathbf{I}_I\right)\Sigma\Sigma^\top\left(e^{-\Gamma(u+a)} - \mathbf{I}_I\right)\mathrm{d}u\right)(\overline{\mathfrak{a}}^{n\cdot}\Gamma^{-1})^\top := cv_{t,T,a}.
\end{aligned}$$

We obtain

$$\begin{bmatrix}\log \mathcal{V}^n_{t+T} \\ \log \mathcal{C}^n_{t+T+a}\end{bmatrix}|\mathcal{G}_t \sim \mathcal{N}\left(\begin{bmatrix}\mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t) \\ \overline{\mathcal{K}}^n(\mathfrak{d}, t, T+a, \mathcal{A}^\circ_t, \mathcal{Z}_t)\end{bmatrix}, \begin{bmatrix}\mathcal{L}^n(t,T) & cv_{t,T,a} \\ cv_{t,T,a} & \overline{\mathcal{L}}^n(t,T+a)\end{bmatrix}\right).$$

We have

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}, \mathcal{C}^n_{t+T+a,\mathfrak{d}} \le \frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}}\}\Big|\mathcal{G}_t\right] \\
&\quad= \Phi_2\left(\frac{\log \frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}} - \overline{\mathcal{K}}^n(\mathfrak{d}, t, T+a, \mathcal{A}^\circ_t, \mathcal{Z}_t)}{\sqrt{\overline{\mathcal{L}}^n(t,T+a)}}, \frac{\log \mathcal{D}^n_{t+T} - \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t)}{\sqrt{\mathcal{L}^n(t,T)}}; \frac{cv_{t,T,a}}{\sqrt{\mathcal{L}^n(t,T)\overline{\mathcal{L}}^n(t,T+a)}}\right),
\end{aligned}$$

and

$$\frac{(1-k)e^{-ra}}{\mathrm{EAD}^n_{t+T}}\mathbb{E}\left[\mathcal{C}^n_{t+T+a,\mathfrak{d}}\cdot\mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}}<\mathcal{D}^n_{t+T},\mathcal{C}^n_{t+T+a,\mathfrak{d}}\leq\frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}}\}}\middle|\mathcal{G}_t\right]$$
$$=\mathbb{E}\left[e^{\log\mathcal{C}^n_{t+T+a,\mathfrak{d}}}\cdot\mathbf{1}_{\left\{\frac{\log\mathcal{C}^n_{t+T+a,\mathfrak{d}}-\overline{\mathcal{K}}^n(\mathfrak{d},t,T+a,\mathcal{A}^\circ_t,\mathcal{Z}_t)}{\sqrt{\overline{\mathcal{L}}^n(t,T+a)}}\leq\overline{\omega}^n_{t,T,a},\frac{\log\mathcal{V}^n_{t+T,\mathfrak{d}}-\mathcal{K}^n(\mathfrak{d},t,T,\mathcal{A}^\circ_t,\mathcal{Z}_t)}{\sqrt{\mathcal{L}^n(t,T)}}<\Phi^{-1}(\mathrm{PD}^n_{t,T,\mathfrak{d}})\right\}}\middle|\mathcal{G}_t\right]$$

However, according to (C.1.2), $\mathbb{E}[e^{\sigma X}\mathbf{1}_{X\leq x,Y\leq y}]=e^{\frac{1}{2}\sigma^2}\Phi_2\left(x-\sigma,y-\rho\sigma;\rho\right)$, therefore,

$$\frac{(1-k)e^{-ra}}{\mathrm{EAD}^n_{t+T}}\mathbb{E}\left[e^{\log\mathcal{C}^n_{t+T+a,\mathfrak{d}}}\cdot\mathbf{1}_{\{\mathcal{V}^n_{t+T,\mathfrak{d}}<\mathcal{D}^n_{t+T},\mathcal{C}^n_{t+T+a,\mathfrak{d}}\leq\frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}}\}}\middle|\mathcal{G}_t\right]$$
$$=\exp\left(\tfrac{1}{2}\overline{\mathcal{L}}^n(t,T+a)-\sqrt{\overline{\mathcal{L}}^n(t,T+a)}\overline{\omega}^n_{t,T,a}\right)\Phi_2\left(\overline{\omega}^n_{t,T,a}-\sqrt{\overline{\mathcal{L}}^n(t,T+a)},\Phi^{-1}(\mathrm{PD}^n_{t,T,\mathfrak{d}})-\frac{cv_{t,T,a}}{\sqrt{\mathcal{L}^n(t,T)}};\frac{cv_{t,T,a}}{\sqrt{\mathcal{L}^n(t,T)\overline{\mathcal{L}}^n(t,T+a)}}\right);$$

Moreover, from Proposition 5.2.9,

$$\mathbb{P}\left[\mathcal{V}^n_{t+T,\mathfrak{d}}<\mathcal{D}^n_{t+T}\middle|\mathcal{G}_t\right]=\Phi\left(\frac{\log(\mathcal{D}^n_{t+T})-\mathcal{K}^n(\mathfrak{d},t,T,\mathcal{A}^\circ_t,\mathcal{Z}_t)}{\sqrt{\mathcal{L}^n(t,T)}}\right),$$

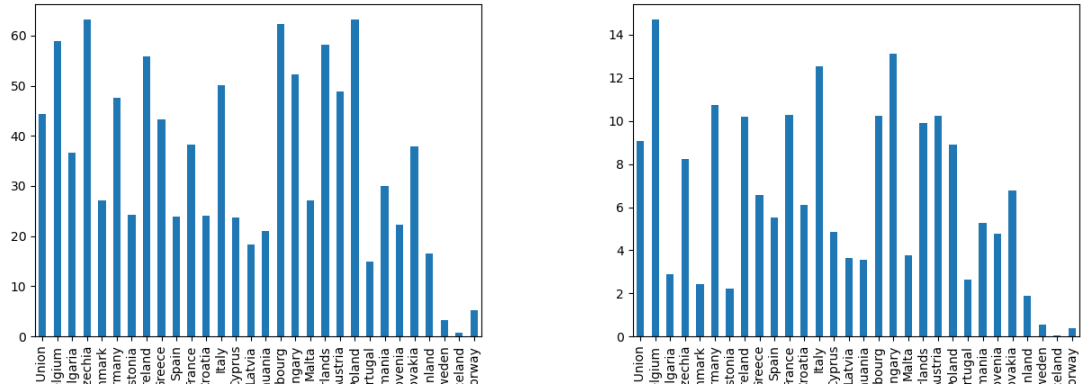
and from Corollary 5.3.4, we conclude the proof. □

We remark that the carbon price introduced in our economy affect both PD through the obligor cash flows and LGD through the collateral cash flows. See more remarks in Section 5.3.5.

5.3.3 When collateral is commercial or residential property

In this section, we assume that loans are backed by either residential or commercial building. The problem here is then to model the real estate market in the presence of the climate transition risk. Pelizza and Schenk-Hoppé [2020] and Frontczak and Rostek [2015] use exponential Ornstein–Uhlenbeck processes to model the stochastic dynamics of real estate prices, while Moody's Investors Service (2022) use a Geometric Brownian Motion. However, Fabozzi et al. [2012] found an exponential Ornstein–Uhlenbeck is more suitable to model the real estate markets and precisely the housing price index. At the same time, as Ghysels et al. [2007], we can see the value of a property as the discounted sum of an income over its residual lifetime. For the efficient buildings, we use the first approach while for the inefficient ones, we adopt the second approach.

Residential or commercial buildings are one of the biggest GHG emitters. Using Eurostat GHG emissions data Eurostat [2022], we see in Figure 5.1a that, between 2008 and 2021, around 45% of the total households emissions came from residential heating and cooling. And as shown in Figure 5.1b, they represent around 9% of the total emissions, in the European Union (EU) zone. This is why renovation of buildings constitutes a central challenge in the climate policies. The Energy Performance of Buildings Directive (EPBD) from European Parliament and Council [2002] and European Union [2022], introduced in 2002 by the European Commission and revised



(a) total households emissions

(b) total economy emissions

Figure 5.1: Part in % of households GHG emissions on heating and cooking

in 2010 and later, is a key instrument to increase the energy performance of buildings across the EU. Similarly to the carbon price, it is a way to implement climate transition. It consists in ranking buildings in terms of their energy efficiency (EE)[1] by using letters from A to G (where A is the most efficient while G is the less). Aydin et al. [2020] found that EE is capitalized quite precisely into home prices in the Dutch housing market. de Ayala et al. [2016] found in a study on 1507 homes in Spain that dweelings labelled A, B or C are valued at between 5.4% and 9.8% higher price compared to D, E, F or G rated home. Franke and Nadler [2019] also highlight that, in the rental decision-making, EE achieves a high importance score similar to that of rent, price and location respectively. One can easily imagine a scenario in which a company defaults, using a real estate asset as collateral. However, at the beginning of the contract, the EE of that building was not considered in the regulations, then in the collateral price. But that is no longer the case. The bank may now face a rather significant variation in its losses due to the depreciation of the collateral caused by poor EE. How could we value the collateral by taking into account this new reality?

According to Ter Steege and Vogel [2021], in discrete time, the price difference per square meter between two properties, one of which with the highest EE label as the reference point (A+), should be solely explained by the sum of the discounted value of (expected) energy cost (noted EC) differences:

$$P_t^j - P_t^{A+} = -\sum_{h=1}^{T} \frac{EC_{t+h}^j - EC_{t+h}^{A+}}{(1+r)^h}.$$

This equation takes the perspective of a potential buyer who weighs the options between buying the efficient property with score $A+$ at a higher price and enjoying the lower energy costs, and buying the inefficient one with score j at a discount that reflects the expected increased energy costs at time T. Moreover, energy costs EC_t^j

[1] in ton of CO_2 emissions per square meter per year or kilowatt hour per square meter per year

are the simple product of the expected energy price and a constant factor measuring the EE.

We extend the Ter Steege and Vogel [2021] work here by assuming that:

1. in the absence of climate transition, the housing price follows an Exponential Ornstein-Uhlenbeck process;

2. each dwelling consumes a given quantity of energy per square meter, which is used to determine its energy efficiency, noted α^n and expressed in kilowatts per square meter (KWh/m^2);

3. as a consequence, the dwelling price is depreciated (or appreciated) by the actualized sum of the future energy costs;

4. once a certain level of energy efficiency $\alpha^\star$ is reached, the market is insensitive to this factor

5. the energy price is a deterministic function $\mathfrak{f}$ of two variables, the first variable is the carbon price and the second is the source of energy,

6. during the life of the property, the owner may undergo renovations which moves the energy efficiency from α^n to $\alpha^\star$, and whose cost per square meter is a function $\mathfrak{c}$ of its energy efficiency,

7. the date of renovation $\mathfrak{t}_n$ of a dwelling is unknown, but is to be optimized.

8. after renovations, the price of the building becomes insensitive to energy costs.

Assumption 5.3.7 (Housing price without climate transition)**.** We consider here two ways to model a housing price:

1. The market value of the building indexed by n at $t \geq 0$, is given by

$$C_t^n := R_n C_0^n e^{K_t}, \tag{5.3.11}$$

where

$$\mathrm{d}K_t = (\dot{\chi}_t + \nu(\chi_t - K_t))\,\mathrm{d}t + \overline{\sigma}\mathrm{d}\overline{B}_t, \tag{5.3.12a}$$
$$\mathrm{d}\overline{B}_t = \rho^\top \mathrm{d}B_t^{\mathcal{Z}} + \sqrt{1 - \|\rho\|^2}\mathrm{d}\overline{\mathcal{W}}_t, \tag{5.3.12b}$$

where $(\overline{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion independent to $B^{\mathcal{Z}}$ introduced in Standing Assumption 5.2.1 and driving the productivity of the economy. Moreover, C_0^n, r, $R_n, \overline{\sigma} > 0$, $\rho \in \mathbb{R}_+^I$, and $\chi \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$. We introduce the following filtration $\mathbb{U} := (\mathcal{U}_t)_{t\in\mathbb{R}^*}$ with for $t \geq 0$, $\mathcal{U}_t := \sigma\left(\left\{\overline{W}_s, B_s^{\mathcal{Z}} : s \leq t\right\}\right)$.

2. Owning a building allows to generate an income/dividends cash-flow process $(D_t^n)_{t\geq 0}$ which is continuous and $\mathbb{U}$-adapted, therefore for all $t \geq 0$, an another way to write the price C_t^n of the building is

$$C_t^n = R_n \mathbb{E}\left[\int_0^{+\infty} e^{-\bar{r}s} D_{t+s}^n \mathrm{d}s \middle| \mathcal{U}_t\right], \tag{5.3.13}$$

with $\bar{r} > 0$.

We can do the following observations:

- C_0^n in (5.3.11), is the value per square meter of the building at time 0 (in euros/m^2 for example) while K is the log of housing price index whose dynamics is (5.3.12a) inspired by Frontczak and Rostek [2015], Fabozzi et al. [2012]. It characterizes the returns of the housing market which are correlated and fluctuates over time around a long-term average χ. In Figure 5.2, we plot historical housing price

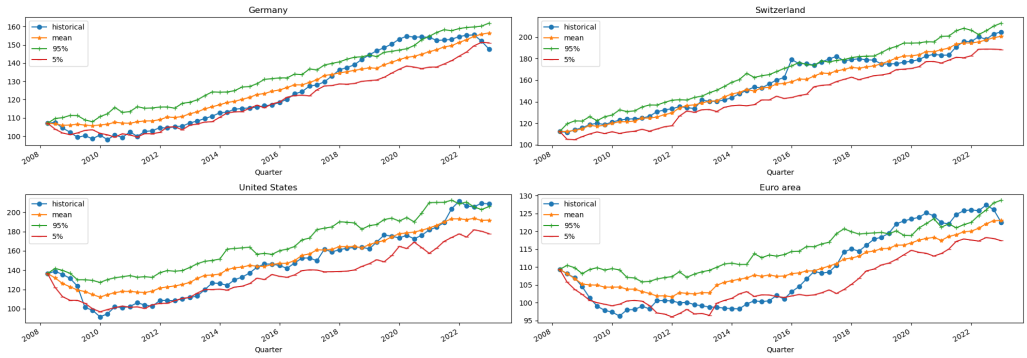


Figure 5.2: Mean and 95% confidence interval of the housing price index of four countries

index of four countries calibrated on (5.3.12a) with the average long-term level χ linear in time. This means that $\chi_t \varrho t + \vartheta$, for all $t \in \mathbb{R}_+$, is a good choice.

- Equation (5.3.13) refers to the fact that owning a building leads to additional income such as depreciation, maintenance costs, opportunity costs of capital (rent received or saved), or flow of various taxes (see Ghysels et al. [2007]).

- Moreover, recall that $B^{\mathcal{Z}}$ is the noise of the productivity process of the economy, the definition of $\overline{B}^n$ in (5.3.12b) allows then to link the real estate market and the productivity (to verify this, one could look at the supply and use tables of INSEE [2023][2] to see the links between real estate activities and others economic sectors).

[2]The French National Institute of Statistics and Economic Studies

We are interested now by the law of the solution of (5.3.12a). We have for $0 \leq t \leq T$,

$$\begin{aligned} K_T &= \chi_T - (\chi_0 - K_0)\, e^{-\nu T} + \overline{\sigma} \int_0^T e^{-\nu(T-s)} \mathrm{d}\overline{B}_s^n \\ &= \chi_T - (\chi_t - K_t)\, e^{-\nu(T-t)} + \overline{\sigma}\rho^\top \int_t^T e^{-\nu(T-s)} \mathrm{d}B_s^{\mathcal{Z}} + \overline{\sigma}\sqrt{1 - \|\rho\|^2} \int_t^T e^{-\nu(T-s)} \mathrm{d}\overline{\mathcal{W}}_s. \end{aligned}$$

We therefore get the distribution of K_T conditional on $\mathcal{G}_t$ (and not conditional on K_t) of is Gaussian

$$\begin{aligned} \mathcal{N}\Bigg(\chi_T - (\chi_0 - K_0)\, e^{-\nu T} + \overline{\sigma}\rho \int_0^t e^{-\nu(T-s)} \mathrm{d}B_s^{\mathcal{Z}} \\ \frac{(\overline{\sigma}\|\rho\|)^2)}{2\nu} \left(1 - e^{-2\nu(T-t)}\right) + \frac{(\overline{\sigma})^2(1 - \|\rho\|^2)}{2\nu} \left(1 - e^{-2\nu T}\right) \Bigg). \end{aligned} \tag{5.3.14}$$

We can rewrite (5.3.11) is for $0 \leq t$

$$C_t^n = R_n C_0^n \exp\left(\chi_t - (\chi_0 - K_0)\, e^{-\nu t} + \overline{\sigma} \int_0^t e^{-\nu(t-s)} \mathrm{d}\overline{B}_s^n \right).$$

The following corollary gives the conditional distribution of the collateral. Its proof is straightforward and directly comes from (5.3.14).

Corollary 5.3.8. *For $0 \leq t \leq T$, the law of $C_{t+T}^n = R_n C_0^n \exp(K_{t+T})$ conditional on $\mathcal{G}_t$ is log-Normal $\mathcal{LN}(m_{t,T}^n, v_{t,T}^n)$ with*

$$m_{t,T}^n := \log(R_n C_0^n) + \chi_{t+T} - (\chi_0 - K_0)\, e^{-\nu(t+T)} + \overline{\sigma}\rho^\top \int_0^t e^{-\nu(t+T-s)} \mathrm{d}B_s^{\mathcal{Z}}, \tag{5.3.15}$$

and

$$v_{t,T}^n := \frac{(\overline{\sigma}\|\rho\|)^2)}{2\nu} \left(1 - e^{-2\nu T}\right) + \frac{(\overline{\sigma})^2(1 - \|\rho\|^2)}{2\nu} \left(1 - e^{-2\nu(t+T)}\right). \tag{5.3.16}$$

We now turn our attention to the pricing of a building, taking into account its energy efficiency. We would like to compute the value of a dwelling at time time t. We use the actualized sum of the cash flows before the renovation date (taking into account the additional energy costs due to inefficiency of the building), at the renovation date, and after the renovation date (when the building becomes efficient). Moreover, the agent chooses rationally the date of renovation which maximizes the value of his property. We have

Definition 5.3.9 (Housing price with climate transition)**.** The market value of the building serving as the collateral to firm n at $t \geq 0$, given the carbon price sequence δ, is represented by

$$\mathcal{C}_{t,\delta}^n := R_n \operatorname*{ess\,sup}_{\theta \geq t} \mathbb{E}\left[\begin{array}{c} \int_t^\theta \left[D_s^n - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)} \mathrm{d}u - \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\theta - t)} \\ + \int_\theta^{+\infty} e^{-\bar{r}(s-t)} D_{t+s}^n \mathrm{d}s \end{array} \middle| \mathcal{U}_t \right], \tag{5.3.17}$$

where

- $\mathfrak{c}$ is a continuous function from $(\mathbb{R}_+)^2$ to $\mathbb{R}_+^*$,
- for each source of energy $\mathfrak{p}$, $\mathfrak{f}(.,\mathfrak{p})$ is a derivable function from $\mathbb{R}_+$ to $\mathbb{R}_+^*$,
- $r, \alpha^n, \alpha^\star > 0$ with $\alpha^n > \alpha^\star$.

Moreover, if the optimal renovation date noted $\mathfrak{t}_n$ exists in $[t, +\infty]$, we have

$$\mathcal{C}^n_{t,\delta} := R_n \mathbb{E}\left[\begin{array}{c} \int_t^{\mathfrak{t}_n} \left[D^n_u - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)} \mathrm{d}u - \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\mathfrak{t}_n - t)} \\ + \int_{\mathfrak{t}_n}^{+\infty} e^{-\bar{r}(u-t)} D^n_u \mathrm{d}u \end{array} \middle| \mathcal{U}_t \right].$$

At time t, D^n_t represents the income at t when the building is "perfectly" efficient (i.e. whose the energy efficiency is $\alpha^\star$) while $D^n_t - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_t, \mathfrak{p})$ is the income of a non renovated building whose consequently undergoes the climate transition so that its income is the difference of the income of an equivalent efficient building minus the energy costs.

Therefore, the term $\left[D^n_u - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})\right] e^{-\bar{r}(u-t)}$ is the actualized income at t of the property before the optimal renovation date $\mathfrak{t}_n$ while the term $D^n_u e^{-\bar{r}(u-t)}$ is the actualized income at t of the property after $\mathfrak{t}_n$. Furthermore, the term $\mathfrak{c}(\alpha^n, \alpha^\star)e^{-r(\mathfrak{t}_n - t)}$ is the actualized renovation costs which are performed at time $\mathfrak{t}_n$. We note finally that for all $t \geq 0$, $C^n_t - \mathcal{C}^n_{t,\delta} \leq R_n \mathfrak{c}(\alpha^n, \alpha^\star)$ i.e. the agent can always renovate the home right away, but that might not be optimal.

An example of the energy price function We can assume that the price of each type of energy $\mathfrak{p}$ is a linear function (introduced in Assumption 5.3.9) of the carbon price, therefore

$$\mathfrak{f} : (\delta_t, \mathfrak{p}) \mapsto \mathfrak{f}^{\mathfrak{p}}_1 \delta_t + \mathfrak{f}^{\mathfrak{p}}_0 \qquad t \geq 0, \tag{5.3.18}$$

with $\mathfrak{f}^{\mathfrak{p}}_1, \mathfrak{f}^{\mathfrak{p}}_0 > 0$ and δ is the carbon price defined in the Standing Assumption 5.2.3 or an example given in (5.2.5).

An example of the renovation costs function We can consider that the costs of renovation of a dwelling $\mathfrak{c}$, to move its energy efficiency from x to y, is

$$\mathfrak{c} : (x, y) \mapsto c_0 |x - y|^{1+c_1}, \tag{5.3.19}$$

with $c_0 > 0$ and $c_1 \geq -1$. This choice of $\mathfrak{c}$ allows us to model that when a building has a bad energy efficiency, its renovation is costly.

The expression (5.3.17) can be simplified in the following proposition.

Theorem 5.3.10. *Assume that the following conditions are satisfied:*

1. *the carbon price function $\delta : t \mapsto \delta_t$ is non decreasing on $\mathbb{R}_+$ and deterministic;*
2. *the energy price $\mathfrak{f}(\cdot, \mathfrak{p})$ is non decreasing on $\mathbb{R}_+$ for all $\mathfrak{p}$.*

Then, the market value of the building serving as the collateral to firm n at $t \geq 0$, given the carbon price sequence δ, is given by

$$\mathcal{C}^n_{t,\delta} = C^n_t - R_n X^n_{t,\delta}, \tag{5.3.20}$$

where

$$X^n_{t,\delta} := \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\mathfrak{t}_n - t)} + (\alpha^n - \alpha^\star) \int_t^{\mathfrak{t}_n} \mathfrak{f}(\delta_u, \mathfrak{p}) e^{-\bar{r}(u-t)} \mathrm{d}u, \tag{5.3.21}$$

and where the optimal date of renovations $\mathfrak{t}_n \in [t, +\infty]$ is given by

$$\mathfrak{t}_n = \begin{cases} t & \text{if } \mathfrak{f}(\delta_\theta, \mathfrak{p}) - \bar{r}\frac{\mathfrak{c}(\alpha^n, \alpha^\star)}{\alpha^n - \alpha^\star} > 0 \text{ for all } \theta \in [t, \infty) \\ +\infty & \text{if } \mathfrak{f}(\delta_\theta, \mathfrak{p}) - \bar{r}\frac{\mathfrak{c}(\alpha^n, \alpha^\star)}{\alpha^n - \alpha^\star} < 0 \text{ for all } \theta \in [t, \infty) \\ \theta^\star & \text{the unique solution of } \mathfrak{f}(\delta_\theta, \mathfrak{p}) = \bar{r}\frac{\mathfrak{c}(\alpha^n, \alpha^\star)}{\alpha^n - \alpha^\star} \text{ on } \theta \in [t, \infty) \end{cases}$$

.

Proof. Let $n \in \{1, \ldots, N\}$ and $t \geq 0$, the difference between a building with energy efficiency α^n and an equivalent one with efficiency α^* is

$$\begin{aligned} C^n_t - \mathcal{C}^n_{t,\delta} = R_n \mathbb{E}\left[\int_t^{+\infty} e^{-\bar{r}(s-t)} D^n_s \mathrm{d}s \middle| \mathcal{U}_t \right] \\ - R_n \operatorname*{ess\,sup}_{\theta \geq t} \mathbb{E}\left[\begin{array}{c} \int_t^\theta \left[D^n_s - (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p}) \right] e^{-\bar{r}(u-t)} \mathrm{d}u - \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\theta - t)} \\ + \int_\theta^{+\infty} e^{-\bar{r}(s-t)} D^n_{t+s} \mathrm{d}s \end{array} \middle| \mathcal{U}_t \right], \end{aligned}$$

after a few calculations, we have

$$C^n_t - \mathcal{C}^n_{t,\delta} = R_n \operatorname*{ess\,inf}_{\theta \geq t} \mathbb{E}\left[\mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\theta - t)} + \int_t^\theta (\alpha^n - \alpha^\star) \mathfrak{f}(\delta_u, \mathfrak{p}) e^{-\bar{r}(u-t)} \mathrm{d}u \middle| \mathcal{U}_t \right] \tag{5.3.25}$$

According to Standing Assumption 5.2.3, δ is deterministic. We can then write

$$C^n_t - \mathcal{C}^n_{t,\delta} \geq R_n \mathbb{E}\left[\inf_{\theta \geq t} \left\{ \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\theta - t)} + \int_t^\theta (\alpha^n - \alpha^\star) \mathfrak{f}(\delta_u, \mathfrak{p}) e^{-\bar{r}(u-t)} \mathrm{d}u \right\} \middle| \mathcal{U}_t \right] \tag{5.3.26}$$

The function under the "inf" that we note

$$H : \theta \mapsto \mathfrak{c}(\alpha^n, \alpha^\star) e^{-\bar{r}(\theta - t)} + \int_t^\theta (\alpha^n - \alpha^\star) \mathfrak{f}(\delta_u, \mathfrak{p}) e^{-\bar{r}(u-t)} \mathrm{d}u,$$

is twice differentiable on $[t, +\infty]$, its first order derivative is

$$H' : \theta \mapsto \left(-\bar{r}\mathfrak{c}(\alpha^n, \alpha^\star) + (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_\theta, \mathfrak{p}) \right) e^{-\bar{r}(\theta - t)},$$

and its second order derivative is

$$H'' : \theta \mapsto -\bar{r} \left[-\bar{r}\mathfrak{c}(\alpha^n, \alpha^\star) + (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_\theta, \mathfrak{p}) \right] e^{-\bar{r}(\theta - t)} + (\alpha^n - \alpha^\star)\delta'_\theta \mathfrak{f}'(\delta_\theta, \mathfrak{p}) e^{-\bar{r}(\theta - t)}.$$

1. If a solution noted $\theta^\star$ of (5.3.27) following

$$\mathfrak{f}(\delta_{\theta^\star}, \mathfrak{p}) = \bar{r}\frac{\mathfrak{c}(\alpha^n, \alpha^\star)}{\alpha^n - \alpha^\star}, \tag{5.3.27}$$

exists in $[t+\infty)$ then, by remarking that $H''(\theta^\star) = (\alpha^n - \alpha^\star)\delta'_\theta \mathfrak{f}'(\delta_{\theta^\star}, \mathfrak{p})e^{-\bar{r}(\theta^\star - t)} \geq 0$, we obtain that $\theta^\star$ is a minimum. Moreover, according to (5.3.25), $C_t^n - \mathcal{C}_{t,\delta}^n \leq R_n H(\theta^\star)$. Combining with (5.3.26), we conclude that

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\theta \geq t} \mathbb{E}\left[\mathfrak{c}(\alpha^n, \alpha^\star)e^{-\bar{r}(\theta - t)} + \int_t^\theta (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})e^{-\bar{r}(u-t)}\mathrm{d}u \middle| \mathcal{U}_t\right] \\
&\quad = \mathbb{E}\left[\inf_{\theta \geq t}\left\{\mathfrak{c}(\alpha^n, \alpha^\star)e^{-\bar{r}(\theta - t)} + \int_t^\theta (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})e^{-\bar{r}(u-t)}\mathrm{d}u\right\} \middle| \mathcal{U}_t\right] \\
&\quad = \mathfrak{c}(\alpha^n, \alpha^\star)e^{-\bar{r}(\mathfrak{t}_n - t)} + \int_t^{\mathfrak{t}_n} (\alpha^n - \alpha^\star)\mathfrak{f}(\delta_u, \mathfrak{p})e^{-\bar{r}(u-t)}\mathrm{d}u,
\end{aligned}$$

2. If (5.3.27) does not have a solution on $[t, \infty)$, then

   (a) if for all $\theta \in [t, \infty)$, $\mathfrak{f}(\delta_\theta, \mathfrak{p}) - \bar{r}\frac{\mathfrak{c}(\alpha^n, \alpha^\star)}{\alpha^n - \alpha^\star} > 0$, we have $H' > 0$ on $[t, \infty)$. We can then write for all $\theta \in [t, \infty)$, $H(t) \leq H(\theta)$ i.e. the "inf" of H exists and is reached in t. Finally, combining to (5.3.25) implying that $C_t^n - \mathcal{C}_{t,\delta}^n \leq R_n H(t)$, we conclude that $C_t^n - \mathcal{C}_{t,\delta}^n = R_n H(t)$ and (5.3.22) follows.

   (b) if for all $\theta \in [t, \infty)$, $\mathfrak{f}(\delta_\theta, \mathfrak{p}) - \bar{r}\frac{\mathfrak{c}(\alpha^n, \alpha^\star)}{\alpha^n - \alpha^\star} < 0$, we have $H' < 0$ on $[t, \infty)$. We can then write for all $\theta \in [t, \infty)$, $\lim_{x \mapsto +\infty} H(x) \leq H(\theta)$ i.e. the "inf" of H is reached in $+\infty$.
   Assume that $(\theta_m)_{m \in \mathbb{N}}$ is a non decreasing and non negative sequence so that $\lim_{m \to +\infty} \theta_m = +\infty$. According to (5.3.25), $C_t^n - \mathcal{C}_{t,\delta}^n \leq R_n H(\theta_m)$ for all $m \in \mathbb{N}$. And because H is continuous, we have $C_t^n - \mathcal{C}_{t,\delta}^n \leq R_n \lim_{x \mapsto +\infty} H(x)$. We conclude that that $C_t^n - \mathcal{C}_{t,\delta}^n = R_n \lim_{\theta \mapsto +\infty} H(\theta)$ and (5.3.23) follows.

□

The date of renovations $\mathfrak{t}_n$ (obtained by (5.3.22), (5.3.24), and(5.3.23)) shows that the optimal date chosen to renovate the building mainly depends on the carbon price policy δ. We also remark that the shock price due to the climate transition $\mathcal{C}_{\cdot,\delta}^n - C_\cdot^n$ is deterministic. This is because the carbon price as well as renovation costs are deterministic. Note that $t \mapsto X_{t,\delta}^n$ is continuous. If, at date t, we realize that the optimal renovation date is $\mathfrak{t}_n$, the best thing we can do is to spend $\mathfrak{c}(\alpha^n, \alpha^\star)$ at date t in order to renovate. We also have the following remarks.

Remark 5.3.11. In our model, the usual price of housing is (partly) offset by the costs $X_{\cdot,\delta}^n$ associated with the climate transition. The dwelling price could also be negative. However, we can imagine many others ways to decline the effects of transition on real estate, for example,

$$\begin{cases} C_t^n = R_n C_0^n \exp K_t^n \\ K_t^n = (\dot{\chi}_t + \nu(\chi_t - K_t^n))\,\mathrm{d}t + \overline{\sigma}^n \mathrm{d}\overline{B}_t^n - \mathrm{d}X_{t,\delta}^n, \end{cases}$$

where X^n is a jump diffusion process.

1. We could for example assume that X^n follows a homogenous Markov process: each year t, the energy efficiency jumps from state s^n_{t-1} to state s^n_t, where $s^n_{t-1}, s^n_t \in \{A, B, C, D, E, F\}$, so that the price increases or decreases. A heat sieve that is renovated, for example, would therefore see its rating improved and then, its price jump. We would calibrate "easily" the transition from historical data.

2. We could introduce the climate transition policy by the jump term X^n, inspired by Le Guenedal and Tankov [2022a]. That climate policy is characterized, for all $t \geq 0$, by a process $X^n_{t,\delta} = \sum_{i=1}^{N_t} R^n_i$ where the Poisson process N_t has a constant arrival rate $\lambda > 0$ and $(R^n_i)_{i\geq 1}$ is a sequence of i.i.d. random variables, independent from $B^{\mathcal{Z}}$. The choice of $(R_i)_{i\geq 1}$ expresses the fact that the climate transition could affect real estate price positively (if for example energy renovation work is carried out in a building), or negatively (if for example regulations on housing emissions are tightened.)

An example of the optimal renovation time With the example of the carbon price in (5.2.5), the example of the energy price in (5.3.18), and the example of the renovation costs in (5.3.19), the optimal renovation time, solution of (5.3.24) is given by

$$\mathfrak{t}_n = t_\circ + \frac{1}{\eta_\delta} \log\left(\frac{c_0 r |\alpha^n - \alpha^\star|^{c_1} - \mathfrak{f}^{\mathfrak{p}}_0}{\mathfrak{f}^{\mathfrak{p}}_1 P_{carbon}}\right). \tag{5.3.28}$$

We can clearly remark that the optimal renovation date depends on the climate transition policy (P_{carbon} and η_δ), on the energy prices ($\mathfrak{f}^{\mathfrak{p}}_0$ and $\mathfrak{f}^{\mathfrak{p}}_1$), on the renovation costs (c_0 and c_1), and on the energy efficiencies (α^n and $\alpha^\star$).

By using the housing price under the climate transition as given in Proposition 5.3.10, we can then derive a precised expression of LGD when the collateral exists and is a building. We have:

Theorem 5.3.12. *Let $1 \leq n \leq N$. When $a = 0$ (no liquidation delay), the Loss Given Default of the obligor n is over-indebted at time $t \in \mathbb{R}_+$, conditional on $\mathcal{G}_t$, is*

$$\mathrm{LGD}^n_{t,\delta} = (1-\gamma)\left[\left(1 + (1-k)\frac{R_n X^n_{t,\delta}}{\mathrm{EAD}^n_t}\right)\Phi\left(\frac{w^n_t}{\sqrt{v^n_{t,t}}}\right) - \exp\left(-w^n_t + \frac{1}{2}v^n_{t,t}\right)\Phi\left(\frac{w^n_t}{\sqrt{v^n_{t,t}}} - \sqrt{v^n_{t,t}}\right)\right], \tag{5.3.29}$$

where

$$w^n_t := \log\left(\frac{\mathrm{EAD}^n_t}{(1-k)R_n} + X^n_{t,\delta}\right) - m^n_{t,0}, \tag{5.3.30}$$

and with $m^n_{t,0}$ and $v^n_{t,t}$ defined in Corollary 5.3.8, and $X^n_{t,\delta}$ defined in (5.3.21).

We can also verify that when there is not collateral corresponding to $C_0^n = 0$. We then have

$$C_0^n \to 0 \implies \log\left(R_n C_0^n\right) \to -\infty \implies m_{t,0}^n \to -\infty \implies w_t^n \to +\infty \implies \mathrm{LGD}_{t,\delta}^n \to 1-\gamma.$$

It is even worse when the costs associated with the transition explode, LGD also explodes as

$$X_{t,\delta}^n \to +\infty \implies w_t^n \to +\infty \implies \mathrm{LGD}_{t,\delta}^n \to +\infty.$$

Proof. Let $t \geq 0$ and $1 \leq n \leq N$, similarly to (5.3.2a) and to the beginning of the proof of Theorem 5.3.5, we have

$$\begin{aligned}
\mathrm{LGD}_{t,\delta}^n &= (1-\gamma)\mathbb{P}\left(1-(1-k)\frac{\mathcal{C}_{t,\delta}^n}{\mathrm{EAD}_t^n} \geq 0 \middle| \mathcal{G}_t\right) - (1-\gamma)\frac{1-k}{\mathrm{EAD}_t^n}\mathbb{E}\left[\mathcal{C}_{t,\delta}^n \mathbf{1}_{\left\{1-(1-k)\frac{\mathcal{C}_{t,\delta}^n}{\mathrm{EAD}_t^n}\geq 0\right\}} \middle| \mathcal{G}_t\right] \\
&= (1-\gamma)\left(1+(1-k)\frac{R_n X_{t,\delta}^n}{\mathrm{EAD}_t^n}\right)\mathbb{P}\left[C_0^n e^{K_t} \leq X_{t,\delta}^n + \frac{\mathrm{EAD}_t^n}{(1-k)R_n} \middle| \mathcal{G}_t\right] \\
&\quad - (1-\gamma)\frac{1-k}{\mathrm{EAD}_t^n}\mathbb{E}\left[R_n C_0^n e^{K_t}\mathbf{1}_{\left\{C_0^n e^{K_t} \leq X_{t,\delta}^n + \frac{\mathrm{EAD}_t^n}{(1-k)R_n}\right\}} \middle| \mathcal{G}_t\right].
\end{aligned}$$

However, from Corollary 5.3.8, we have the law of $\mathcal{C}_{t,\mathfrak{d}}^n$ conditional on $(\mathcal{G}_t)$ is log-Normal $\mathcal{LN}(m_{t,0}^n, v_{t,t}^n)$ with

$$m_{t,0}^n = \log R_n C_0^n + \chi_t - (\chi_0 - K_0)\, e^{-\nu t} + (\overline{\sigma}\|\rho\|)^2 \int_0^t e^{-\nu(t-s)} \mathrm{d}B_s^{\mathcal{Z}},$$

and

$$v_{t,t}^n = \frac{(\overline{\sigma})^2(1-\|\rho\|^2)}{2\nu}\left(1-e^{-2\nu t}\right).$$

By proceeding similarly to what we did in the proof of Theorem 5.3.5, we obtain,

$$\mathrm{LGD}_\delta^n = (1-\gamma)\left[\left(1+(1-k)\frac{R_n X_{t,\delta}^n}{\mathrm{EAD}_t^n}\right)\Phi\left(\frac{w_t^n}{\sqrt{v_{t,t}^n}}\right) - \exp\left(-w_t^n + \frac{1}{2}v_{t,t}^n\right)\Phi\left(\frac{w_t^n}{\sqrt{v_{t,t}^n}} - \sqrt{v_{t,t}^n}\right)\right],$$

where

$$w_t^n := \log\left(\frac{\mathrm{EAD}_t^n}{(1-k)R_n} + X_{t,\delta}^n\right) - m_{t,0}^n$$

where $X_{t,\delta}^n$ is defined in (5.3.21). The conclusion follows. □

Once again, we want to compute the (conditional) Loss Given Default of the entity n at time t on the horizon T. We can formalize that in the following proposition:

Proposition 5.3.13 (Projected LGD)**.** *For each $t, T \geq 0$ and $1 \leq n \leq N$, the (conditional) Loss Given Default of the entity n at time t on the horizon T, reads*

$$\begin{aligned}
\mathrm{LGD}_{t,T,\delta}^n = \frac{1-\gamma}{\mathrm{PD}_{t,T,\mathfrak{d}}^n}\Bigg[&\left(1+(1-k)e^{-ra}\frac{R_n X_{t+T+a,\delta}^n}{\mathrm{EAD}_{t+T}^n}\right)\Phi_2\left(\overline{\omega}_{t,T,a}^n, \Phi^{-1}(\mathrm{PD}_{t,T,\mathfrak{d}}^n); \rho_{t,T,a}^n\right) \\
&- \exp\left(\frac{1}{2}v_{t,T+a}^n - \sqrt{v_{t,T+a}^n}\overline{\omega}_{t,T,a}^n\right)\Phi_2\left(\overline{\omega}_{t,T,a}^n - \sqrt{v_{t,T+a}^n}, \Phi^{-1}(\mathrm{PD}_{t,T,\mathfrak{d}}^n) - \rho_{t,T,a}^n\sqrt{v_{t,T+a}^n}; \rho_{t,T,a}^n\right)\Bigg],
\end{aligned} \tag{5.3.31}$$

where

$$\rho^n_{t,T,a} := \overline{\sigma}\varsigma \frac{\mathfrak{a}^{n\cdot}\Gamma^{-1}\left(\int_0^T e^{-\nu(u+a)}\left(\mathbf{I}_I - e^{-\Gamma u}\right)\Sigma \mathrm{d}u\right)\rho}{\sqrt{\mathcal{L}^n(t,T)\times v^n_{t,T+a}}},$$

$$\overline{w}^n_{t,T,a} := \frac{\log\left(\frac{\mathrm{EAD}^n_{t+T}}{(1-k)R_n e^{-ra}} + X^n_{t+T+a,\delta}\right) - m^n_{t,T+a}}{\sqrt{v^n_{t,T+a}}},$$

and with $m^n_{t,t+T+a}$ *and* $v^n_{t,t+T+a}$ *defined in Corollary 5.3.8 and* $\mathrm{PD}^n_{t,T,\mathfrak{d}}$ *defined in Proposition 5.2.9.*

Proof. Let $t, T \geq 0$ and $1 \leq n \leq N$, from (5.3.2a), we have

$$\begin{aligned}
\frac{\mathrm{LGD}^n_{t,T,\delta}}{1-\gamma} &= \mathbb{E}\left[\left(1-(1-k)e^{-r(t+T+a-\tau^n)}\frac{\mathcal{C}^n_{\tau^n,\delta}}{\mathrm{EAD}^n_{\tau^n}}\right)_+ \middle| \tau^n = t+T, \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\left(1-(1-k)e^{-ra}\frac{\mathcal{C}^n_{t+T+a,\delta}}{\mathrm{EAD}^n_{t+T}}\right)_+ \middle| \mathcal{G}_t\right] \\
&= \left(1+(1-k)e^{-ra}\frac{R_n X^n_{t+T+a,\delta}}{\mathrm{EAD}^n_{t+T}}\right)\mathbb{P}\left[C^n_0 e^{K_{t+T+a}} \leq X^n_{t+T+a,\delta} + \frac{\mathrm{EAD}^n_{t+T}}{(1-k)R_n e^{-ra}}, \mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T} \middle| \mathcal{G}_t\right] \\
&\quad - \frac{(1-k)e^{-ra}}{\mathrm{EAD}^n_{t+T}}\mathbb{E}\left[R_n C^n_0 e^{K_t}\mathbf{1}_{\left\{C^n_0 e^{K_{t+T+a}} \leq X^n_{t+T+a,\delta} + \frac{\mathrm{EAD}^n_{t+T}}{(1-k)R_n e^{-ra}}, \mathcal{V}^n_{t+T,\mathfrak{d}} < \mathcal{D}^n_{t+T}\right\}} \middle| \mathcal{G}_t\right]
\end{aligned}$$

Remark that

$$\begin{aligned}
\log C^n_0 e^{K_{t+T+a}} &= \log\left(C^n_0\right) + \chi_{t+T+a} - (\chi_0 - K_0)e^{-\nu(t+T+a)} \\
&\quad + \overline{\sigma}\rho^\top \int_0^{t+T+a} e^{-\nu(t+T+a-s)}\mathrm{d}B^{\mathcal{Z}}_s + \overline{\sigma}\sqrt{1-\|\rho\|^2}\int_0^{t+T+a} e^{-\nu(t+T+a-s)}\mathrm{d}\overline{\mathcal{W}}_s,
\end{aligned}$$

and recall that

$$\log \mathcal{V}^n_{t+T,\mathfrak{d}} = \log\left(F^n_0 \mathfrak{R}^n_{t+T}(\mathfrak{d})\right) + \mathfrak{a}^{n\cdot}\left(\mathcal{A}^\circ_{t+T} - v(\mathfrak{d}_0)\right) + \sigma_n \mathcal{W}^n_{t+T}.$$

Therefore,

$$\begin{aligned}
cov\left(\log C^n_0 e^{K_{t+T+a}}, \log F^n_{t+T} \middle| \mathcal{G}_t\right) &= cov\left(\mathfrak{a}^{n\cdot}\mathcal{A}^\circ_{t+T}, \overline{\sigma}\rho^\top \int_0^{t+T+a} e^{-\nu(t+T+a-s)}\mathrm{d}B^{\mathcal{Z}}_s \middle| \mathcal{G}_t\right) \\
&= \overline{\sigma}\mathfrak{a}^{n\cdot} cov\left(\mathcal{A}^\circ_{t+T}, \int_0^{t+T+a} e^{-\nu(t+T+a-s)}\mathrm{d}B^{\mathcal{Z}}_s \middle| \mathcal{G}_t\right)\rho \\
&= \overline{\sigma}\varsigma\mathfrak{a}^{n\cdot}\Gamma^{-1}\left(\int_0^T e^{-\nu(u+a)}\left(\mathbf{I}_I - e^{-\Gamma u}\right)\Sigma \mathrm{d}u\right)\rho := cv_{t,T,a}.
\end{aligned}$$

Consequently, we can write

$$\begin{bmatrix} \log F^n_{t+T} \\ \log C^n_0 e^{K_{t+T+a}} \end{bmatrix} | \mathcal{G}_t \sim \mathcal{N}\left(\begin{bmatrix} \mathcal{K}^n(\mathfrak{d}, t, T, \mathcal{A}^\circ_t, \mathcal{Z}_t) \\ m^n_{t,T+a} \end{bmatrix}, \begin{bmatrix} \mathcal{L}^n(t,T) & cv_{t,T,a} \\ cv_{t,T,a} & v^n_{t,T+a} \end{bmatrix}\right).$$

Finally, by proceeding as in the proof of Proposition 5.3.6, the conclusion follows by remarking that

$$\overline{w}^n_{t,T,a} = \frac{\log\left(\frac{\mathrm{EAD}^n_{t+T}}{(1-k)R_n e^{-ra}} + X^n_{t+T+a,\delta}\right) - m^n_{t,t+T+a}}{\sqrt{v^n_{t,T+a}}}.$$

□

We can remark that $\mathrm{LGD}^n_{t,\delta}$ as well as $\mathrm{LGD}^n_{t,T,\delta}$ are also functions of the optimal renovation time $\mathfrak{t}_n$. Furthermore, if both the financial asset and the housing price are affected by the climate transition through their dependence on the carbon price sequence δ, the financial asset depends also on the carbon price intensities (of firms production/consumption and of households consumption) (τ, ζ, κ) which are not specific to a given company but to the economy as a whole. The housing price is clearly affected by specific climate factors, namely the energy efficiency α^n and the renovation date $\mathfrak{t}_n$.

5.3.4 Expected and Unexpected losses

Let us recall that we have a portfolio with N loans. We assume that loans from 1 to N_1 are unsecured, loans from $N_1 + 1$ to N_2 are secured by a financial asset as collateral, and loans from $N_2 + 1$ to N are secured by a commercial or residential property as collateral.

We write the expression of the portfolio EL and UL as functions of the parameters and of the processes introduced above, and introduce the entity's probability of default.

We can therefore give expressions of EL and UL. Let $t, T \geq 0$, the (conditional) Expected Loss of the portfolio at time t on the horizon T defined in (5.3.3), reads

$$\begin{aligned}\mathrm{EL}^{N,T}_t &= \mathbb{E}\left[\mathrm{L}^{\mathbb{G},N}_{t+T}\middle|\mathcal{G}_t\right] = \sum_{n=1}^{N} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\mathfrak{d}} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}} \\ &= \sum_{n=1}^{N_1} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\mathfrak{d}} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}} + \sum_{n=N_1+1}^{N_2} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\mathfrak{d}} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}} \\ &\quad + \sum_{n=N_2+1}^{N} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\delta} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}}.\end{aligned} \tag{5.3.32}$$

We can then compute each term conditionally to $\mathcal{G}_t$.

1. Given that $\mathrm{LGD}^n_{t,T,\mathfrak{d}} = 1-\gamma$ for $1 \leq n \leq N_1$, to compute $\sum_{n=1}^{N} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\mathfrak{d}} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}}$, all you have to do is calculate $\mathrm{PD}^n_{t,T,\mathfrak{d}}$.

2. Given that $N_1 + 1 \leq n \leq N_2$, the collaterals are financial assets, therefore, to compute $\sum_{n=N_1+1}^{N_2} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\mathfrak{d}} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}}$, we compute first $\mathrm{PD}^n_{t,T,\mathfrak{d}}$. Then we compute $\mathrm{LGD}^n_{t,T,\mathfrak{d}}$ directly through (5.3.10).

3. Given that $N_1 + 1 \leq n \leq N_2$, the collaterals are properties, therefore, to compute $\sum_{n=N_2+1}^{N} \mathrm{EAD}^n_{t+T} \cdot \mathrm{LGD}^n_{t,T,\delta} \cdot \mathrm{PD}^n_{t,T,\mathfrak{d}}$, we compute first $\mathrm{PD}^n_{t,T,\mathfrak{d}}$. Then we compute $\mathrm{LGD}^n_{t,T,\delta}$ directly through (5.3.31).

For $\alpha \in (0, 1)$, the (conditional) Unexpected Loss of the portfolio at time t on the horizon T, cannot be obtained in closed-form as EL. Precisely, there is not a closed-form expression neither of $\mathrm{UL}^{\alpha,N,T}_t$ nor of $\mathrm{VaR}^{\alpha,N,T}_t$. But we can describe how to compute

$\mathrm{VaR}_t^{\alpha,N,T}$ given that $\mathbb{P}\left[L_{t+T}^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]$ as introduced in (5.3.4). First, let us note that from Theorem 5.3.1, we have

$$\begin{aligned} L_{t+T}^{\mathbb{G},N} &= \sum_{n=1}^{N} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\mathfrak{d}}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n \\ &= \sum_{n=1}^{N_1} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\mathfrak{d}}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n + \sum_{n=N_1+1}^{N_2} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\mathfrak{d}}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n \\ &\qquad + \sum_{n=N_2+1}^{N} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\delta}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n. \end{aligned}$$

We can then describe each term's law conditionally to $\mathcal{G}_t$.

1. $\mathrm{LGD}_{t+T,\mathfrak{d}}^n = 1-\gamma$ and from (5.2.16), we have $\mathrm{PD}_{t+T,\mathfrak{d}}^n$ which depends on $\mathcal{A}_{t+T}$. Then to simulate law of $\sum_{n=1}^{N_1} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\mathfrak{d}}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n$ conditional on $\mathcal{G}_t$, just simulate $\mathcal{A}_{t+T}|\mathcal{G}_t$.

2. From (5.3.8), we have $\mathrm{LGD}_{t+T,\mathfrak{d}}^n$ which depends on $\mathcal{A}_{t+T}$ through w_{t+T}^n defined in (5.3.9). We said in the previous item that $\mathrm{PD}_{t+T,\mathfrak{d}}^n$ depends on $\mathcal{A}_{t+T}$. Therefore, to simulate law of $\sum_{n=N_1+1}^{N_2} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\mathfrak{d}}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n$ conditional on $\mathcal{G}_t$, just simulate $\mathcal{A}_{t+T}|\mathcal{G}_t$.

3. From (5.3.29), we have $\mathrm{LGD}_{t+T,\mathfrak{d}}^n$ which depends on $\int_0^{t+T} e^{-\nu(t+T-s)}\mathrm{d}B_s^{\mathcal{Z}}$ through w_{t+T}^n defined in (5.3.30). We said in the previous item that $\mathrm{PD}_{t+T,\mathfrak{d}}^n$ depends on $\mathcal{A}_{t+T}$. Therefore, to simulate law of $\sum_{n=N_2+1}^{N} \mathrm{EAD}_{t+T}^n \cdot \mathrm{LGD}_{t+T,\delta}^n \cdot \mathrm{PD}_{t+T,\mathfrak{d}}^n$ conditional on $\mathcal{G}_t$, just simulate $\mathcal{A}_{t+T}|\mathcal{G}_t$ and $\int_0^{t+T} e^{-\nu(t+T-s)}\mathrm{d}B_s^{\mathcal{Z}}|\mathcal{G}_t$ (which are in fact the same because both $\mathcal{A}_{t+T}$ and $\int_0^{t+T} e^{-\nu(t+T-s)}\mathrm{d}B_s^{\mathcal{Z}}$ are $\mathcal{G}_{t+T}$-measurable).

5.3.5 Remarks on the determinants of LGD

The results (5.3.8) and (5.3.31) tell us that, in the case the collateral is an investment or a building, Loss Given Default depends on:

1. the carbon price δ for both (5.3.8) and (5.3.31),

2. parameters specific to the company (the contract),

   - the time t when it is computed,
   - the date of default $t+T$,
   - the Exposure at Default EAD,

3. parameters specific to the collateral,

   - its liquidation time $t+T+a$,
   - the liquidation costs k,

- the correlation of its cash flows with the environment $\overline{\mathfrak{a}}$,
- the standard deviation of its cash flows $\overline{\sigma}_{\mathfrak{b}}$,
- the fraction of recovery from other means γ,

4. the nature of the collateral:
   - if it is a financial asset, then parameters related to the carbon intensities τ, ζ, κ,
   - if it is a building, then parameters related to the energy efficiency α, type of energy $\mathfrak{p}$, and renovation costs $\mathfrak{c}$,
5. parameters specific to the economy to which the collateral belongs to:
   - the (cumulative) productivity $\mathcal{A}$ (and its parameters) of the economy,
   - the interest rates r and $\bar{r}$.

Some of these typical risk drivers are reported by Chalupka and Kopecsni [2008].

We could also look at the sensitivities of the LGD to each of these variables and parameters. However, the expressions of LGD we obtained are not very tractable so that it would be difficult to get detailed expressions of theses sensitivities. If necessary, they can be calculated using numerical methods.

5.4 Numerical experiments, estimation and calibration

In this section, we describe how the parameters of multisectoral model, of the firm valuation model, and of the credit risk model are estimated given the historical macroeconomic variables (consumption, labour, output, GHG emissions, housing prices, etc.) as well as the historical credit portfolio data (firms rated and defaulted, collateral, etc.) In a second step, we give the expression of the risk measures (PD, LGD, EL, and UL) introduced in the previous sections, that we compute using Monte Carlo simulations.

5.4.1 Calibration and estimation

We will calibrate the model parameters on a set of data ranging from time $\mathfrak{t}_0$ to $\mathfrak{t}_1$. In practice, $\mathfrak{t}_0 = 1978$ and $\mathfrak{t}_1 = t_\circ = 2021$. From now on, we will discretize the observation interval into $M \in \mathbb{N}^*$ steps $t_m = \mathfrak{t}_0 + \frac{\mathfrak{t}_1 - \mathfrak{t}_0}{M} m$ for $0 \leq m \leq M$. We note $\mathcal{T}^M := \{t_0, t_1, \ldots, t_M\}$. We will not be interested in convergence results here.

Estimation of carbon intensities

For each sector $i \in \mathcal{I}$ and for $0 \leq m \leq M$, we observe the output $Y^i_{t_m}$, the labor $N^i_{t_m}$, the intermediary input $(Z^{ji}_{t_m})_{j\in\mathcal{I}}$, and the consumption $C^i_{t_m}$ (recall that the transition starts at year $t_\circ$). For the sake of clarity, we will omit the dependence of each estimated parameter on M.

To calibrate each carbon intensity $\mathfrak{y} \in \{\tau^1, \dots, \tau^I, \zeta^{11}, \zeta^{12}, \dots, \zeta^{II-1}, \zeta^{II}, \kappa^1, \dots, \kappa^I\}$, we follow exactly the same process already presented in chapter 3. The main difference is that after calibration, we can compute $\mathfrak{y}$ for each $t \in \mathbb{R}_+$. Afterwards, if we consider the example of the carbon price introduces in (5.2.5), we can compute the *emissions cost rate* $\widehat{\mathfrak{d}}_t$.

Estimation of economic parameters

As in Galí [2015], we assume a unitary Frisch elasticity of labor supply so $\varphi = 1$ and the utility of consumption is logarithmic so $\sigma = 1$, while we calibrate $(\boldsymbol{\lambda}^{ij})_{i,j\in\mathcal{I}}$ and $(\chi^i)_{i\in\mathcal{I}}$in the same way as in chapter 3. We can then compute the functions χ and Λ defined in Proposition 5.2.4, followed by the function $\widehat{v}^i$ as defined in (5.2.8). We can also compute the output growth $\left(\Delta^Y_{t_m} = (\log(Y^i_{t_m}) - \log(Y^i_{t_{m-1}}))_{j\in\mathcal{I}}\right)_{1\leq m\leq M}$ directly from data.

Without carbon tax in any sector, it follows from (5.2.9) in Corollary 5.2.4 that, for each $1 \leq m \leq M$, the computed consumption growth $\Delta^Y_{t_m}$ is equal to $\Delta^Y_{t_m} = \frac{\mathfrak{t}_1-\mathfrak{t}_0}{M}(\mathbf{I}_I - \widehat{\boldsymbol{\lambda}})^{-1}\widehat{\Theta}_{t_m}$ when $\mathbf{I}_I - \widehat{\boldsymbol{\lambda}}$ is not singular; hence $\widehat{\Theta}_{t_m} = \frac{M}{\mathfrak{t}_1-\mathfrak{t}_0}(\mathbf{I}_I - \widehat{\boldsymbol{\lambda}})\Delta^Y_{t_m}$. We can then compute the estimations $\widehat{\mu}$, $\widehat{\Gamma}$, $\widehat{\Sigma}$ and $\widehat{\varsigma}$, parameters μ, Γ, Σ, and ς (all defined in Standing Assumption 5.2.1).

As $\mathcal{Z}$ is a centered O.-U., μ correspond to the mean. We have

$$\widehat{\mu} = \frac{1}{M}\sum_{m=1}^{M}\widehat{\Theta}_{t_m}.$$

Then, we can take ς so that $V[Z_t] = 1$ for all $t \in \mathbb{R}$, then $\varsigma^2 = V[Z_t]$, then

$$\widehat{\varsigma} = \sqrt{\frac{1}{M-1}\sum_{m=1}^{M}(\widehat{\Theta}_{t_m} - \widehat{\mu})^\top(\widehat{\Theta}_{t_m} - \widehat{\mu})}.$$

For all $1 \leq m \leq M$, we then have $\widehat{\mathcal{Z}}_{t_m} := \frac{\widehat{\Theta}_{t_m}-\widehat{\mu}}{\widehat{\varsigma}}$. If we discretize the first equation of (5.2.1), we also have ,

$$\widehat{\mathcal{Z}}_{t_m} = \left(\mathbf{I} - \frac{\mathfrak{t}_1-\mathfrak{t}_0}{M}\Gamma\right)\widehat{\mathcal{Z}}_{t_{m-1}} + \mathcal{E}_{t_m}, \quad \text{with} \quad \mathcal{E}_{t_m} = \Sigma(B^{\mathcal{Z}}_{t_m} - B^{\mathcal{Z}}_{t_{m-1}}) \sim \mathcal{N}\left(0, \frac{\mathfrak{t}_1-\mathfrak{t}_0}{M}\Sigma\right). \tag{5.4.1}$$

The discrete process $(\widehat{\mathcal{Z}}_{t_m})_{1\leq m\leq M}$ is then a VAR process. The estimations $\widehat{\Sigma}$ and $\widehat{\Gamma}$ of Σ and Γ respectively are directly got.

Estimation of firm and of the credit model parameters

Recall that we have a portfolio with $N \in \mathbb{N}^*$ firms (or credit) at time $t_\circ$. For each firm $n \in \{1, \dots, N\}$, we have its historical cash flows $(F^n_{t_m})_{1 \le m \le M}$, hence its log-cash flow growths. For any $t \in \mathcal{T}^M$ and $1 \le i \le I$, we denote by r^i_t (resp. d^i_t) the number of firms in g_i rated at the beginning of the year t (resp. defaulted during the year t). In particular, $r_{\mathfrak{t}_0} = \#g_i$. Within each group g_i, all the firms behave in the same way as there is only one risk class. Since each sub-portfolio constitutes a single risk class, we have for each $n \in g_m$, $\mathfrak{a}^n = \mathfrak{a}^{n_i}$, $\sigma_{\mathfrak{b}^n} = \sigma_{\mathfrak{b}^{n_i}}$, and $B^n = B^{n_i}$. We then proceed as follows:

1. Knowing the output growth $\left(\Delta^Y_t\right)_{t \in \mathcal{T}^M}$, we calibrate the factor loading $\mathfrak{a}_{n_i}$ and the standard deviation σ_{n_i}, according to Assumption 5.2.5, appealing to the regression

$$\sum_{n \in g_i} \log F^n_{t_m} - \log F^n_{t_{m-1}} = (\#g_i)\mathfrak{a}^{n_i}\Delta^Y_{t_k} + \sqrt{\frac{\mathfrak{t}_1 - \mathfrak{t}_0}{M}\#g_i}\sigma_{\mathfrak{b}^{n_i}}\mathfrak{u}_{t_m},$$

   where $\mathfrak{u}_{t_m} \sim \mathcal{N}(0,1)$, for all $1 \le m \le M$.

2. We then estimate the barrier B^{n_i} by MLE as detailed in [Gordy and Heitfield, 2002, Section 3]:

   we compute

$$\widehat{B}^{n_i} := \underset{B^{n_i} \in \mathbb{R}^+}{\arg\max}\, \mathcal{L}(B^{n_i}),$$

   where $\mathcal{L}(B^{n_i})$ is the log-likelihood function defined by

$$\mathcal{L}(B^{n_i}) := \sum_{m=1}^{M} \log\left(\int_{\mathbb{R}^{2I}} \mathbb{P}[D^{n_i} = d^i_{t_m} | (a,\theta)] d\mathbb{P}[(\mathcal{A}^\circ_{t_m}, \mathcal{Z}_{t_m}) \le (a,z)]\right),$$

   and where

$$\mathbb{P}[D^{n_i} = d^i_{t_m} | (\mathcal{A}^\circ_{t_m}, \mathcal{Z}_{t_m})] = \binom{r^i_{t_m}}{d^i_{t_m}} (\mathrm{PD}^{n_i}_{t_m,1,0})^{d^i_{t_m}} \left(1 - \mathrm{PD}^{n_i}_{t_m,1,0}\right)^{r^i_{t_m} - d^i_{t_m}},$$

   with D^{n_i} the Binomial random variable standing for the conditional number of defaults, and $\mathrm{PD}^{n_i}_{t_m,1,0}$ in Proposition 5.2.17, depending on $\sigma_{\mathfrak{b}^{n_i}} = \widehat{\sigma}_{\mathfrak{b}^{n_i}}$, $\mathfrak{a}^{n_i} = \widehat{\mathfrak{a}}^{n_i}$, for $1 \le m \le M$, $\delta_{t_m} = 0$ and on B^{n_i}.

Calibration of collateral

Recall that we have a portfolio with $N \in \mathbb{N}^*$ firms (or credit) at time $t_\circ$. For each firm $n \in \{1, \dots, N\}$, if the collateral is

A financial asset We have its historical cash flows $(\overline{F}^n_{t_m})_{0\leq m\leq M}$, hence its log-cash flow growths. Recall that, even if two firms belong to the same sub-portfolio, there is no reason that their collaterals behave in the same way. We also know the output growth $\left(\Delta^Y_{t_m}\right)_{1\leq m\leq M}$. We then have,

1. the proportion α^n of the investment representing the collateral is known.
2. we calibrate the factor loading $\widehat{\overline{\mathfrak{a}}}_n$ and the standard deviation $\widehat{\overline{\sigma}}_n$, according to (5.3.5), appealing to the regression: for all $1 \leq m \leq M$,

$$\log \overline{F}^n_{t_m} - \log \overline{F}^n_{t_{m-1}} = \overline{\mathfrak{a}}^n \Delta^Y_{t_m} + \sqrt{\frac{\mathfrak{t}_1 - \mathfrak{t}_0}{M}} \overline{\sigma}_n \mathfrak{u}_{t_m} \quad \text{where} \quad \mathfrak{u}_{t_m} \sim \mathcal{N}(0,1).$$

A commercial or residential property We assume that in the past, carbon price did not have impact on the dwelling price so that for all $t \in \mathcal{T}^M$, $X^n_{t,\delta}$ defined in (5.3.21) is zero. Moreover, C^n_0 defined in (5.3.20), the value of the collateral at 0, is known. All that remains is to calibrate the parameters of the process K defined in (5.3.12a) and (5.3.12b). Let us consider a real estate index $(REI_{t_m})_{0\leq m\leq M}$, then for each $1 \leq m \leq M$, $K_{t_m} := \log(REI_{t_m})$. For calibration, we proceed exactly as Fabozzi et al. [2012]. Let assume that the long-term average of the real estate index χ, introduced in Assumption 5.3.7, is linear as Frontczak and Rostek [2015] do, therefore for all $t \in \mathbb{R}_+$, $\chi_t = \varrho t + \vartheta$. The estimation of the parameters (ϱ, ϑ) is realised prior to the others.

- ϱ and ϑ) can be estimated with a minimisation procedure, possibly nonlinear, by

$$(\widehat{\varrho}, \widehat{\vartheta}) = argmin_{(\varrho,\vartheta)} \left\{ \sum_{k=0}^{M} (K_{t_m} - \varrho t_m - \vartheta)^2 \right\}.$$

- the estimation of the mean-reversion parameter ν (introduced in (5.3.12a)),

$$\widehat{\nu} := \log \frac{\sum_{m=1}^{M} K^2_{t_{m-1}}}{\sum_{m=1}^{M} K_{t_m} K_{t_{m-1}}},$$

- the estimation of the volatility parameter $\overline{\sigma}$ (introduced in (5.3.12a)),

$$\widehat{\overline{\sigma}}^2 := \frac{1}{M} \sum_{m=1}^{M} (K_{t_m} - K_{t_{m-1}})^2,$$

- From (5.3.12a), we have $1 < m \leq M$, the increments of $\overline{B}$ corresponds to

$$u^{\overline{B}}_{t_m} := \frac{1}{\widehat{\overline{\sigma}}} \left((Kt_m - K_{t_{m-1}}) - \left(\widehat{\varrho} + \widehat{\nu}(\widehat{\varrho} t_{m-1} + \widehat{\vartheta} - K_{t_{m-1}}) \right) \frac{\mathfrak{t}_1 - \mathfrak{t}_0}{M} \right) \sim \mathcal{N}\left(0, \frac{\mathfrak{t}_1 - \mathfrak{t}_0}{M}\right),$$

  and from $\mathcal{E}_{t_m}$ defined in (5.4.1), the increments of $B^{\mathcal{Z}}$ corresponds to

$$u^{B^{\mathcal{Z}}}_{t_m} := \widehat{\Sigma}^{-1} \mathcal{E}_{t_m} \in \sim \mathcal{N}\left(0, \frac{\mathfrak{t}_1 - \mathfrak{t}_0}{M} \mathbf{I}_I\right).$$

We see from (5.3.12b), that

$$\widehat{\rho}^{\top} := \left[\frac{1}{M}\sum_{m=1}^{M}\left(u_{t_m}^{B^{\mathcal{Z}}} - \frac{1}{M}\sum_{i=1}^{M}u_{t_i}^{B^{\mathcal{Z}}}\right)\left(u_{t_m}^{\overline{B}} - \frac{1}{M}\sum_{i=1}^{M}u_{t_i}^{\overline{B}}\right)\right] \times \left[\frac{1}{M}\sum_{m=1}^{M}\left(u_{t_m}^{B^{\mathcal{Z}}} - \frac{1}{M}\sum_{i=1}^{M}u_{t_i}^{B^{\mathcal{Z}}}\right)\left(u_{t_m}^{B^{\mathcal{Z}}} - \frac{1}{M}\sum_{i=1}^{M}u_{t_i}^{B^{\mathcal{Z}}}\right)^{\top}\right]^{-1}.$$

- the other parameters $\bar{r}, R_n, \alpha^n, \alpha^\star$ useful to compute X^n are given. Recall that examples of $\mathfrak{f}$ and $\mathfrak{c}$ are defined in (5.3.18) and in (5.3.19).

5.4.2 Approximations

In this section as well, the idea here is not to (re)demonstrate or improve convergence results.

Of the productivities $\mathcal{Z}$ and $\mathcal{A}$

Let $K \in \mathbb{N}$, for $0 \leq k \leq K$, we note $u_k = t_\circ + \frac{t_\star - t_\circ}{K}k$ for $0 \leq k \leq K$. We would like to simulate $\mathcal{Z}_{u_k}$ and $\mathcal{A}_{u_k}$. For $\mathcal{Z}$, we adopt the Euler-Maruyana Maruyama [1955], Kanagawa [1988] scheme: we have $\mathcal{Z}_{t_\circ}$ and

$$\begin{cases} \widehat{\mathcal{Z}}_{u_k} &= \widehat{\mathcal{Z}}_{u_{k-1}} - \widehat{\Gamma}\widehat{\mathcal{Z}}_{u_{k-1}}\frac{t_\star - t_\circ}{K} + \widehat{\Sigma}\eta_{u_k}, \qquad \eta_{u_k}\,\mathcal{N}\left(0, \frac{t_\star - t_\circ}{K}\mathbf{I}_I\right) \quad \text{and} \quad k = 1, \ldots, K \\ \widehat{\mathcal{Z}}_t &= \widehat{\mathcal{Z}}_{u_{k-1}}, \qquad u_{k-1} \leq t \leq u_k \quad \text{and} \quad k = 1, \ldots, K \end{cases}. \tag{5.4.2}$$

Then, given that $\mathcal{A}_{t_\circ}$ is known and $\mathcal{A}_t = \int_{t_\circ}^t (\mu + \varsigma\mathcal{Z}_u)\,\mathrm{d}u = \mathcal{A}_{t_\circ} + \mu(t - t_\circ) + \varsigma\int_{t_\circ}^t \mathcal{Z}_u \mathrm{d}u$, we have

$$\int_{t_\circ}^t \widehat{\mathcal{Z}}_u \mathrm{d}u = \int_{u_{k-1}}^t \widehat{\mathcal{Z}}_u \mathrm{d}u + \sum_{i=1}^{k}\int_{u_{i-1}}^{u_i}\widehat{\mathcal{Z}}_u \mathrm{d}u = (t - u_{k-1})\widehat{\mathcal{Z}}_{u_{k-1}} + \frac{t_\star - t_\circ}{K}\sum_{i=1}^{k}\widehat{\mathcal{Z}}_{u_{i-1}}$$

and then for each $k = 1, \ldots, K$ and $u_{k-1} \leq t \leq u_k$,

$$\widehat{\mathcal{A}}_t = \mathcal{A}_{t_\circ} + \widehat{\mu}(t - t_\circ) + \widehat{\varsigma}\left((t - u_{k-1})\widehat{\mathcal{Z}}_{u_{k-1}} + \frac{t_\star - t_\circ}{K}\sum_{i=1}^{k}\widehat{\mathcal{Z}}_{u_{i-1}}\right). \tag{5.4.3}$$

Remark 5.4.1. We could also adapt the exact simulation of the multidimensional Ornstein–Uhlenbeck $\mathcal{Z}$ based on Li and Wu [2019] or de la Cruz and Jimenez [2020].

Of the probability of *default* PD and of LGD

For $n \in \{1, \ldots, N\}$ and $t_\circ \leq t \leq t_\star$, We would like to compute $\mathrm{PD}^n_{t,T,\mathfrak{d}}$ as defined in (5.2.17) as well as $\mathrm{LGD}^n_{t,T,\mathfrak{d}}$ defined in (5.3.10) and $\mathrm{LGD}^n_{t,T,\delta}$ in (5.3.31). After simulating $\mathcal{Z}_t$ and $\mathcal{A}_t$ as described in 5.4.2, we get $\widehat{\mathcal{Z}}_t$ and $\widehat{\mathcal{A}}_t$. Then, for each $1 \leq i \leq I$ and for each $n \in g_i$, we have

1. from (5.2.17), the estimated probability of default of firm n is

$$\widehat{\mathrm{PD}}^{n}_{t,T,\mathfrak{d}} = \Phi\left(\frac{\log(\mathcal{D}^n_{t+T}) - \widehat{\mathcal{K}}^n(\widehat{\mathfrak{d}}, t, T, \widehat{\mathcal{A}}_t, \widehat{\mathcal{Z}}_t)}{\widehat{\mathcal{L}}^n(t,T)}\right), \tag{5.4.4}$$

with

$$\widehat{\mathcal{K}}^n(\widehat{\mathfrak{d}}, t, T, \widehat{\mathcal{A}}_t, \widehat{\mathcal{Z}}_t) = \log\left(F^n_0 \widehat{\mathfrak{R}}^n_{t+T}(\widehat{\mathfrak{d}})\right) + \widehat{\mathfrak{a}}^{n\cdot}(\widehat{\mu}T + \widehat{\varsigma}\Upsilon_T\widehat{\mathcal{Z}}_t + \widehat{\mathcal{A}}_t - \widehat{v}(\widehat{\mathfrak{d}}_0)), \tag{5.4.5}$$

and

$$\widehat{\mathcal{L}}^n(t,T) := \widehat{\varsigma}^2\widehat{\mathfrak{a}}^{n\cdot}\widehat{\Gamma}^{-1}\left(\frac{T}{L}\sum_{l=0}^{L}\left(e^{-\widehat{\Gamma}u_l} - \mathbf{I}_I\right)\widehat{\Sigma}\widehat{\Sigma}^\top\left(e^{-\widehat{\Gamma}u_l} - \mathbf{I}_I\right)\right)(\widehat{\mathfrak{a}}^{n\cdot}\widehat{\Gamma}^{-1})^\top + (t+T)\widehat{\sigma}^2_n, \tag{5.4.6}$$

where F^n_0 and $\mathcal{D}^n_{t+T}$ are know, $\widehat{\mathfrak{d}}$ defined in Section 5.4.1, $\widehat{\mathfrak{R}}^n_{t+T}(\widehat{\mathfrak{d}}))$ in (5.2.14) in Theorem 5.2.6, $\widehat{\mathcal{A}}_t$ is given in (5.4.3), $\widehat{\mathcal{Z}}_t$ is given in (5.4.2), $\widehat{\Gamma}, \widehat{\varsigma}, \widehat{v}$ in Section 5.4.1, $\widehat{\mathfrak{a}}^{n\cdot}, \widehat{\sigma}^n$ in Section 5.4.1, $\widehat{\Upsilon}_T := \widehat{\Gamma}^{-1}(\mathbf{I}_I - e^{-\widehat{\Gamma}T})$ and with $u_l := \frac{Tl}{L}, l = 0, \ldots, L$.

2. If the collateral of loan n is a financial asset, from (5.3.10),

$$\widehat{\mathrm{LGD}}^n_{t,T,\mathfrak{d}} = \frac{1-\gamma}{\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}}\Big[\Phi_2\left(\widehat{\varpi}^n_{t,T,a}, \Phi^{-1}\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}); \widehat{\rho}^n_{t,T,a}\right) - \exp\left(\tfrac{1}{2}\widehat{\overline{\mathcal{L}}}^n(t,T+a)\right)\Phi_2\left(\widehat{\varpi}^n_{t,T,a} - \sqrt{\widehat{\overline{\mathcal{L}}}^n(t,T+a)}, \Phi^{-1}(\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}) - \widehat{\rho}^n_{t,T,a}\sqrt{\widehat{\overline{\mathcal{L}}}^n(t,T+a)}; \widehat{\rho}^n_{t,T,a}\right)\Big], \tag{5.4.7}$$

where $\widehat{\overline{\mathcal{K}}}$ and $\widehat{\overline{\mathcal{L}}}$ are computed in the same way $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{L}}$ were in (5.4.5) and (5.4.6). Moreover,

$$\widehat{\rho}^n_{t,T,a} := \frac{\widehat{\varsigma}^2\widehat{\mathfrak{a}}^{n\cdot}\widehat{\Gamma}^{-1}\left(\frac{T}{L}\sum_{l=0}^{L}\left(e^{-\widehat{\Gamma}u_l} - \mathbf{I}_I\right)\widehat{\Sigma}\widehat{\Sigma}^\top\left(e^{-\widehat{\Gamma}(u_l+a)} - \mathbf{I}_I\right)\right)(\widehat{\mathfrak{a}}^{n\cdot}\widehat{\Gamma}^{-1})^\top}{\sqrt{\widehat{\mathcal{L}}^n(t,T)\widehat{\overline{\mathcal{L}}}^n(t,T+a)}},$$

and

$$\widehat{\overline{\omega}}^n_{t,T,a} := \frac{\log\frac{\mathrm{EAD}^n_{t+T}}{(1-k)e^{-ra}} - \widehat{\overline{\mathcal{K}}}^n(\widehat{\mathfrak{d}}, t, T+a, \widehat{\mathcal{A}}^\circ_t, \widehat{\mathcal{Z}}_t)}{\sqrt{\widehat{\overline{\mathcal{L}}}^n(t,T+a)}},$$

with a, k, γ, EAD^n_t, α^n, and $\overline{F}^n_0$ are known, $\widehat{\mathfrak{d}}$ defined in Section 5.4.1, $\widehat{\mathcal{A}}_t$ is given in (5.4.3), $\widehat{\mu}, \widehat{\varsigma}, \widehat{v}$ in Section 5.4.1, and $\widehat{\overline{\mathfrak{a}}}^{n\cdot}, \widehat{\overline{\sigma}}_n$ in Section 5.4.1 and $\overline{\mathfrak{R}}^n_{t+T}(\widehat{\mathfrak{d}})$ in (5.2.14). Finally, $u_l := \frac{Tl}{L}, l = 0, \ldots, L$.

3. If the collateral of loan n is a commercial or residential property, we compute in order (5.3.15), (5.3.16), (5.3.30), and (5.3.31). Since $\chi_t = \varrho t + \vartheta$ and C^n_0 are known, we have

$$\widehat{m}^n_{t,T+a} := \log(R_n C^n_0) + (\widehat{\varrho}t + \widehat{\vartheta}) - (\widehat{\vartheta} - K_0)e^{-\widehat{\nu}(t+T+a)} + \widehat{\overline{\sigma}}\widehat{\rho}^\top\sum_{k=0}^{L} e^{-\widehat{\nu}((t+T+a)-\frac{kt}{L})}\eta_{u_{\frac{kT}{L}}}$$

where $\eta_{\frac{kt}{L}} \sim \mathcal{N}\left(0, \frac{t}{L}\mathbf{I}_I\right)$ and $k = 0, \ldots, L$, with $L \in \mathbb{N}^*$. We also have

$$\widehat{v}^n_{t,T+a} := \frac{(\widehat{\overline{\sigma}}\|\widehat{\rho}\|)^2}{\widehat{\nu}}\left(1 - e^{-2\widehat{\nu}(T+a)}\right) + \frac{(\widehat{\overline{\sigma}})^2(1 - \|\widehat{\rho}\|^2)}{\widehat{\nu}}\left(1 - e^{-2\widehat{\nu}(t+T+a)}\right).$$

Therefore, we have

$$\widehat{\mathrm{LGD}}^n_{t,T,\delta} = \frac{1-\gamma}{\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}}\left[\left(1 + (1-k)e^{-ra}\frac{R_n\widehat{X}^n_{t+T+a,\delta}}{\mathrm{EAD}^n_{t+T}}\right)\Phi_2\left(\widehat{\overline{w}}^n_{t,T,a}, \Phi^{-1}(\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}); \widehat{\rho}^n_{t,T,a}\right)\right.$$
$$\left. - \exp\left(\frac{1}{2}\widehat{v}^n_{t,T+a} - \sqrt{\widehat{v}^n_{t,T+a}}\widehat{\overline{w}}^n_{t,T,a}\right)\Phi_2\left(\widehat{\overline{w}}^n_{t,T,a} - \sqrt{\widehat{v}^n_{t,T+a}}, \Phi^{-1}(\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}) - \widehat{\rho}^n_{t,T,a}\sqrt{\widehat{v}^n_{t,T+a}}; \widehat{\rho}^n_{t,T,a}\right)\right], \tag{5.4.8}$$

where

$$\widehat{\rho}^n_{t,T,a} := \widehat{\overline{\sigma}}\widehat{\varsigma}\frac{\widehat{\mathfrak{a}}^{n\cdot}\widehat{\Gamma}^{-1}\left(\frac{T}{L}\sum_{l=0}^{L} e^{-\widehat{\nu}(u_l+a)}\left(\mathbf{I}_I - e^{-\widehat{\Gamma}u_l}\right)\widehat{\Sigma}\right)\widehat{\rho}}{\sqrt{\widehat{\mathcal{L}}^n(t,T) \times \widehat{v}^n_{t,t+T+a}}},$$

$$\widehat{\overline{w}}^n_{t,T,a} := \log\left(\frac{\mathrm{EAD}^n_{t+T}}{(1-k)R_n e^{-ra}} + \widehat{X}^n_{t+T+a,\delta}\right) - \widehat{m}^n_{t,T+a},$$

and $\widehat{X}$ is obtained by considering that from (5.3.21),

$$\widehat{X}^n_{t,\delta} = \mathfrak{c}(\alpha^n, \alpha^\star)e^{-r(\mathfrak{t}_n - t)} + (\alpha^n - \alpha^\star)\frac{(\mathfrak{t}_n - t)}{P}\sum_{p=1}^{P}\mathfrak{f}(\delta_{v_p}, \mathfrak{p})e^{-r(v_p - t)},$$

and where γ, k, r, R_n, and EAD^n_t are known, $\mathfrak{t}_n$ given by (5.3.28), $u_l := \frac{(t_\star - t)l}{L}, l = 0, \ldots, L$, and $v_p := \frac{(\mathfrak{t}_n - t)p}{P}, l = 0, \ldots, P$.

Of the (un)expected losses EL and UL

For EL, the result is direct by using $\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}$ in (5.4.4), $\widehat{\mathrm{LGD}}^n_{t,T,\mathfrak{d}}$ in (5.4.7), and $\widehat{\mathrm{LGD}}^n_{t,T,\delta}$ in (5.4.8), we have from (5.3.32),

$$\begin{aligned}\widehat{\mathrm{EL}}^{N,T}_t := &\sum_{n=1}^{N_1}(1-\gamma)\mathrm{EAD}^n_{t+T}\cdot\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}} + \sum_{n=N_1+1}^{N_2}\mathrm{EAD}^n_{t+T}\cdot\widehat{\mathrm{LGD}}^n_{t,T,\mathfrak{d}}\cdot\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}\\ &+ \sum_{n=N_2+1}^{N}\mathrm{EAD}^n_{t+T}\cdot\widehat{\mathrm{LGD}}^n_{t,T,\delta}\cdot\widehat{\mathrm{PD}}^n_{t,T,\mathfrak{d}}.\end{aligned} \tag{5.4.9}$$

For UL, we use

$$\begin{aligned}\widehat{L}^{\mathbb{G},N}_{t+T} = &\sum_{n=1}^{N_1}(1-\gamma)\mathrm{EAD}^n_{t+T}\cdot\widehat{\mathrm{PD}}^n_{t+T,\mathfrak{d}} + \sum_{n=N_1+1}^{N_2}\mathrm{EAD}^n_{t+T}\cdot\widehat{\mathrm{LGD}}^n_{t+T,\mathfrak{d}}\cdot\widehat{\mathrm{PD}}^n_{t+T,\mathfrak{d}}\\ &+ \sum_{n=N_2+1}^{N}\mathrm{EAD}^n_{t+T}\cdot\widehat{\mathrm{LGD}}^n_{t+T,\delta}\cdot\widehat{\mathrm{PD}}^n_{t+T,\mathfrak{d}},\end{aligned} \tag{5.4.10}$$

by noting that $\mathrm{PD}^n_{t+T,\mathfrak{d}} = \mathrm{PD}^n_{t+T,0,\mathfrak{d}}$, $\mathrm{LGD}^n_{t+T,\mathfrak{d}} = \mathrm{LGD}^n_{t+T,0,\mathfrak{d}}$, and $\mathrm{LGD}^n_{t+T,\delta} = \mathrm{LGD}^n_{t+T,0,\delta}$. Therefore, as $\widehat{L}^{\mathbb{G},N}_{t+T}$ depends on $(\widehat{\mathrm{PD}}^n_{t+T,\mathfrak{d}}, \widehat{\mathrm{LGD}}^n_{t+T,\mathfrak{d}}, \widehat{\mathrm{LGD}}^n_{t+T,\delta})$

which depends on $(\widehat{\mathcal{A}}_{t+T}, \widehat{\mathcal{Z}}_{t+T})$. However, we want to compute $\mathrm{VaR}_t^{\alpha,N,T}$ so that $\mathbb{P}\left[L_{t+T}^{\mathbb{G},N} \leq \mathrm{VaR}_t^{\alpha,N,T} \middle| \mathcal{G}_t\right]$. Then, we simulate $D \in \mathbb{N}^*$ couples noted $(\widehat{\mathcal{A}}^p_{t+T|t}, \widehat{\mathcal{Z}}^p_{t+T|t})_{1\leq p\leq D}$ so that $\widehat{\mathcal{Z}}^p_{t+T|t} =^d \mathcal{Z}_{t+T}|\mathcal{G}_t$ and $\widehat{\mathcal{A}}^p_{t+T|t} =^d \mathcal{A}_{t+T}|\mathcal{G}_t$. That is straightforward and

$$\widehat{\mathcal{Z}}^p_{t+T|t}|\mathcal{G}_t \sim \mathcal{N}\left(e^{-\widehat{\Gamma}T}\widehat{\mathcal{Z}}_t, \frac{T}{L}\sum_{l=0}^{L} e^{-\widehat{\Gamma}u_l}\widehat{\Sigma}\widehat{\Sigma}^\top e^{-\widehat{\Gamma}^\top u_l}\right),$$

and

$$\widehat{\mathcal{A}}^p_{t+T|t}|\mathcal{G}_t \sim \mathcal{N}\left(\widehat{\mu}T + \widehat{\varsigma}\widehat{\Upsilon}_T\widehat{\mathcal{Z}}_t + \widehat{\mathcal{A}}_t, \widehat{\varsigma}^2\widehat{\Gamma}^{-1}\left[\frac{T}{L}\sum_{l=0}^{L}\left(e^{-\widehat{\Gamma}u_l} - \mathbf{I}_I\right)\widehat{\Sigma}\widehat{\Sigma}^\top\left(e^{-\widehat{\Gamma}u_l} - \mathbf{I}_I\right)\right](\widehat{\Gamma}^{-1})^\top\right),$$

with $u_l := \frac{Tl}{L}, l = 0, \ldots, L$. We also need to simulate $\mathfrak{h}_{t+T}|\mathcal{G}_t := \int_0^{t+T} e^{-\nu(t+T-s)}\mathrm{d}B_s^{\mathcal{Z}}|\mathcal{G}_t$ (which comes from $m^n_{t+T,0}$ in (5.3.15)). As $\mathfrak{h}_{t+T}|\mathcal{G}_t \sim \mathcal{N}\left(\int_0^t e^{-\nu(t+T-s)}\mathrm{d}B_s^{\mathcal{Z}}, \frac{1-e^{-2\nu T}}{2\nu}\mathbf{I}_I\right)$, we have

$$\widehat{\mathfrak{h}}^p_{t+T}|\mathcal{G}_t \sim \mathcal{N}\left(\sum_{k=0}^{L} e^{-\widehat{\nu}((t+T)-\frac{kt}{L})}\eta_{u_{\frac{kT}{L}}}, \frac{1-e^{-2\widehat{\nu}T}}{2\widehat{\nu}}\mathbf{I}_I\right), \quad \eta_{\frac{kt}{L}} \sim \mathcal{N}\left(0, \frac{t}{L}\mathbf{I}_I\right).$$

Then, the unexpected loss is

$$\widehat{\mathrm{UL}}^{N,T}_{t,\delta,\alpha} := q_{\alpha,D}\left(\left\{(\widehat{L}^{\mathbb{G},N}_{t+T})^1, (\widehat{L}^{\mathbb{G},N}_{t+T})^2, \ldots, (\widehat{L}^{\mathbb{G},N}_{t+T})^D\right\}\right) - \widehat{\mathrm{EL}}^{N,T}_{t,\mathfrak{d}}, \tag{5.4.11}$$

where $(\widehat{L}^{\mathbb{G},N}_{t+T})^p$ is obtained by replacing $(\widehat{\mathcal{A}}_{t+T}, \widehat{\mathcal{Z}}_{t+T})$ in (5.4.10) by $(\widehat{\mathcal{A}}^p_{t+T|t}, \widehat{\mathcal{Z}}^p_{t+T|t})$, and where $q_{\alpha,M}(\{Y^1, \ldots, Y^D\})$ denotes the empirical α-quantile of the distribution of Y.

5.5 Simulations and discussion

In this section, we describe the data used to calibrate the different parameters, we perform some simulations, and we comment the results.

5.5.1 Data

As in chapters 3 and 4, we work on data related to the French economy.

1. Due to data availability (precisely, we do not find public monthly/quaterly data for the intermediary inputs), we consider an annual frequency.

2. Annual consumption, labor, output, and intermediary inputs come from INSEE[3] from 1978 to 2021 (see INSEE [2023] for details) and are expressed in billion euros, therefore $\mathfrak{t}_0 = 1978$, $\mathfrak{t}_1 = 2021$, and $M = 44$.

[3]The French National Institute of Statistics and Economic Studies

3. For the climate transition, we consider a time horizon of ten years with $t_\circ = 2021$ as starting point, a time step of one year and $t_\star = 2030$ as ending point. In addition, we will be extending the curves to 2034 to see what happens after the transition, even though the results will be calculated and analyzed during the transition.

4. The 38 INSEE sectors are grouped into four categories: *Very High Emitting*, *Very Low Emitting*, *Low Emitting*, and *High Emitting*, based on their carbon intensities.

5. The carbon intensities are calibrated on the realized emissions from Eur [2023] (expressed in tonnes of CO2-equivalent) between 2008 and 2021.

6. Metropolitan France housing price index comes from *OECD data* and are from 1980 to 2021 (see OECD Stat [2024] for details) in *Base 2015*. We renormalize in *Base 2021*.

5.5.2 Definition of the climate transition

We consider four deterministic transition scenarios giving four deterministic carbon price trajectories. The scenarios used come from the NGFS simulations, whose descriptions are given by NGFS [2022] as follows:

- ***Net Zero 2050*** *is an ambitious scenario that limits global warming to* $1.5°C$ *through stringent climate policies and innovation, reaching net zero* CO_2 *emissions around 2050. Some jurisdictions such as the US, EU and Japan reach net zero for all GHG by this point.*

- ***Divergent Net Zero*** *reaches net-zero by 2050 but with higher costs due to divergent policies introduced across sectors and a quicker phase out of fossil fuels.*

- ***Nationally Determined Contributions (NDCs)*** *includes all pledged policies even if not yet implemented.*

- ***Current Policies*** *assumes that only currently implemented policies are preserved, leading to high physical risks.*

For each scenario, we compute the carbon price $P_{carbon,0}$ in t_0 and the evolution rate η_δ as defined in (5.2.3). We can then compute the carbon price, whose evolution

	Current Policies	**NDCs**	**Divergent Net Zero**	**Net Zero 2050**
$P_{carbon,0}$ **(in euro/ton)**	30.957	33.321	32.963	34.315
η_δ **(in %)**	1.693	7.994	12.893	17.935

Table 5.1: Carbon price parameters

is plotted in Figure C.1a, at each date using (5.2.5).

For the energy price, we consider electricity as the unique source of energy. Then, we assume a linear relation between the electricity and the carbon price inspired by Abrell et al. [2023], where a variation of the carbon price is linked withe the variation of the electricity by a the pass-through rate noted k. This means that $\mathfrak{f}_1^{elec}$ and $\mathfrak{f}_2^{elec}$ define in (5.3.18) are respectively k and $P_{elec,0} - k \times P_{carbon,0}$. For France, we take the electricity price $P_{elec,0} = 0.2161$ euro per Kilowatt-hour and $k = 0.55$ (see Abrell et al. [2023]) ton per Kilowatt-hour. Its evolution is plotted in Figure C.1b.

For the renovation costs to improve a building for the energy efficiency α to $\alpha^\star$ as defined in (5.3.19), we take $c_0 = 0.01$ euro per kilowatt-hour and per square meter (€/KWh.m^2) and $c_1 = 0.1$.

5.5.3 Estimations

The carbon intensities

We use the realized GHG emissions as well as the macroeconomic variables and their frequency being the same as in chapter 3, we use the same estimations. But after that, we can compute the carbon intensities at each date in $\mathbb{R}_+$ using (5.2.3).

Economic parameters

We keep the values of ϕ, σ, $(\chi^i)_{i\in\mathcal{I}}$, and $(\boldsymbol{\lambda}^{ji})_{i,j\in\mathcal{I}}$ already estimated. For the productivity process, we switched from a vector autoregressive model to an Ornstein-Uhlenbeck. We therefore calibrate μ, ς, Σ, and Γ as detailed in Section 5.4.1.

Emissions Level	**Very High**	**High**	**Low**	**Very Low**
$\times 10^{-3}$	5.602	8.475	3.834	12.099

Table 5.2: Parameter μ of the productivity

Emissions Level	**Very High**	**High**	**Low**	**Very Low**
Very High	-0.201	-0.056	0.113	-0.036
High	0.091	0.420	0.214	0.015
Low	-0.103	-0.003	-0.122	0.160
Very Low	0.493	0.168	0.290	0.652

Table 5.3: Parameter Γ of the productivity

The eigenvalues of Γ are $\{1.544, 1.057, 0.636, 0.014\}$ which implies that $-\Gamma$ is a Hurwitz matrix, therefore $\mathcal{Z}$ is weak-stationary as assumed. Moreover, $\varsigma = 0.026$.

Emissions Level	Very High	High	Low	Very Low
Very High	0.473	0.013	0.072	0.092
High	0.013	0.208	0.039	0.037
Low	0.072	0.039	0.059	0.020
Very Low	0.092	0.037	0.020	0.068

Table 5.4: Parameter Σ of the productivity

The housing pricing index (HPI)

We write the housing price index K in *Base 2021* and we apply the logarithm function. This means that $K_{t_0} = 0$ as shown in Figure 5.3. We can therefore calibrate $\varrho, \vartheta, \nu, \overline{\sigma}$, and ρ. The values are presented in Table 5.4 below.

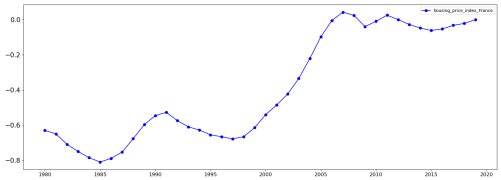


Figure 5.3: Log of the HPI in Base 2021 from 1980 to 2021

Parameter	Value
ϱ	0.024
ϑ	-0.884
ν	0.026
$\overline{\sigma}$	0.050
ρ	[-0.019, -0.042, -0.017, 0.015]

Figure 5.4: Housing price index parameters

5.5.4 Simulations and discussions

In the previous work in

discrete time, we simulate for different climate transition scenario between $t_\circ = 2021$ and $t_\star = 2030$, the annual evolution of (1) the output growth per sector (2) the output share per sector in the total output, (3) the firms direct GHG emissions per sector, (4) a given firm value and distribution, (5) the probabilities of default of fictive sub-portfolio of 4 firms each and of the resulting portfolio, (6) the expected and the unexpected losses of the previous (sub-)portfolios when the LGD are constant and deterministic, (7) the sensitivities of the losses to the carbon price.

In the current simulations, since we are keeping the same data at the same frequency (annual), the main change is then to replace the VAR process by the O.-U. process. Therefore, the comments already made for (1) to (5) concerning the trends, the impact of the carbon price, the difference of scenarios, the relation between sectors, etc. do not change. We will focus here on the LGD and on the losses, with different type of collateral.

Impact of the carbon price on a firm value

We consider 4 firms so that firm 1, 2, 3, and 4 respectively belong to the *Very High Emitting*, *High Emitting*, *Low Emitting*, and *Very Low Emitting* groups. Each firm

is characterized by its cash flows $F_{t_\circ-1}$ at $t_\circ - 1$, the standard deviation of its cash flows $\sigma_{\mathfrak{b}}$, and the contribution $\mathfrak{a}$ of sectoral output growth to its cash flows growth as detailed in table 5.5. The chosen interest rate $r = 5\%$. For $M = 500$ simulations of the

Firm	1	2	3	4
$\sigma_{\mathfrak{b}^n}$	0.05	0.05	0.05	0.05
F_0^n	1.0	1.0	1.0	1.0
$\mathfrak{a}^n$**(Very High)**	1.0	0.0	0.0	0.0
$\mathfrak{a}^n$**(High)**	0.0	1.0	0.0	0.0
$\mathfrak{a}^n$**(Low)**	0.0	0.0	1.0	0.0
$\mathfrak{a}^n$**(Very Low)**	0.0	0.0	0.0	1.0

Table 5.5: Characteristics of the firms

productivity processes $(\mathcal{Z}_t, \mathcal{A}_t)_{t_\circ \leq t \leq t_\star}$, we compute the firm value using (5.2.13). We can analyze both the average evolution of the firm value per year and per scenario.

Emissions level	**Firm 1**	**Firm 2**	**Firm 3**	**Firm 4**
NDCs	-2.264	-2.266	-0.574	-0.162
Divergent Net Zero	-5.098	-5.051	-1.314	-0.374
Net Zero 2050	-9.712	-9.464	-2.573	-0.742

Table 5.6: Average annual growth slowdown with respect to the *Current Policies* scenario between 2021 and 2030 (in %)

For ease of reading, we normalize the firm's value to 1 at the beginning of the transition $t_\circ$. By analyzing both Table 5.6 and Figure 5.5 (and Figure C.2), all the firms continues to grow but, as one might expect, the decline in the growth of the value of each company is determined by whether it belongs to a more or less polluting sector.

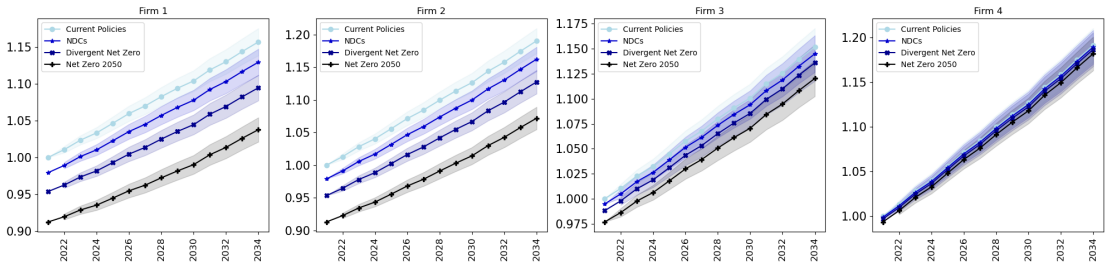


Figure 5.5: Firm value per scenario and per year

Impact of the carbon price on a building value

In order to illustrate the impact of the carbon price on the housing market, we consider here 5 buildings located in the French economy whose characteristics: the price at $t_\circ$,

C_0^n, the energy efficiency α^n, the surface R_n, are given in Table 5.7

Building	1	2	3	4	5
C_n^0	4000	4000	4000	4000	4000
α^n	320.	253.	187.	120.	70.
R_n	25.0	25.0	25.0	25.0	25.

Table 5.7: Characteristics of the building

Precisely, we consider 5 apartments of 25 square meters whose price of the square meter fixed to 4000 euros in $t_\circ = 2021$ is the same for all, but whose the energy efficiency are different. Moreover, we assume that the optimal energy efficiency equals to $\alpha^\star = 70$ kilowatt hour per square meter per year (see Total Energies [2024]) is reached. We can compute and summarize in table 5.8 the optimal renovation date whose the expression is given in (5.3.28).

Emissions level	**Building 1**	**Building 2**	**Building 3**	**Building 4**	**Building 5**
Current Policies	89.32	115.16	144.325	185.32	304.14
NDCs	14.21	18.32	22.96	29.48	48.38
Divergent Net Zero	8.81	11.36	14.24	18.28	29.99
Net Zero 2050	6.33	8.17	10.23	13.14	21.57

Table 5.8: Optimal renovation date (in years) per scenario and per building

We observe in Table 5.8 that the optimal renovation date increases:

- when the building is efficient (α^n decreases) and when $c_1 < 0$: there is no point in renovating an efficient building.
- when the climate transition speeds up: energy costs become unbearable if we do not renovate quickly,
- when the renovation costs increases: if renovation costs become too high, it is better to bear the energy costs.

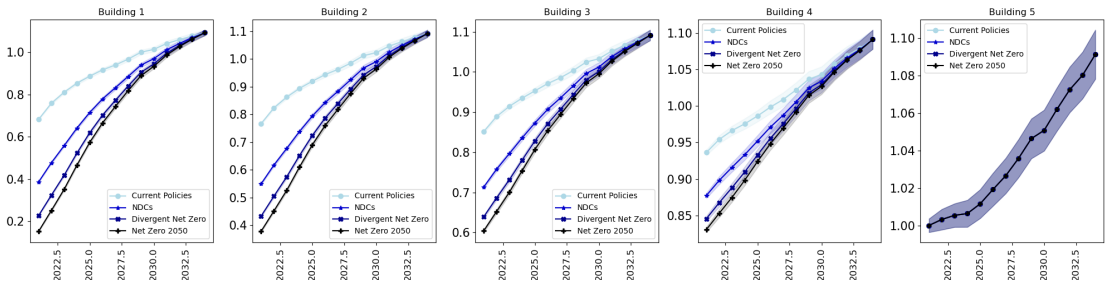


Figure 5.6: Apartment value per scenario and per year

As in the case of the firm values, we normalize the building values by the price of the most efficient building (Building 5) at the beginning of the transition $t_\circ$ for ease of reading.

We can therefore observe that, since in the scenario *Current Policies*, the price of electricity does not really increase (see Figure C.1b), the optimal renovation dates are very large (much larger than the potential lifespan of the building). Therefore, if there is no climate transition, it is not necessary to renovate (for this reason at least). A direct consequence of low-cost energy and a very distant renovation date is that the price of housing continues to grow as it has historically.

Emissions level	**Building 1**	**Building 2**	**Building 3**	**Building 4**	**Building 5**
Current Policies	-1.506	-0.870	-0.506	-0.209	0.000
NDCs	-22.646	-15.526	-9.351	-3.848	-0.080
Divergent Net Zero	-28.007	-20.721	-13.480	-6.151	-0.179
Net Zero 2050	-29.776	-23.055	-15.744	-7.614	-0.263

Table 5.9: Average annual slowdown of the housing price with respect to the *Current Policies* scenario between 2021 and 2030 (in %)

For each scenario and each building, each point of the curve represents the value of the building at date t if the optimal renovation date is (if $t \leq \mathfrak{t}_n$) or was (if $t > \mathfrak{t}_n$) $\mathfrak{t}_n$. Almost all the building prices continue to grow with time as shown on Figure 5.6, but these growths are more or less affected by their energy efficiency. Moreover, given that the impact of the transition on price through energy and renovation costs, the later are stronger in the beginning. In fact, the more time passes, the closer we get to the end the climate transition ($t_\star = 2030$ in our scenarios) as well as the potential date of renovations. If we look at Figure C.3, we remark that when the valuation date is later than the optimal renovation date, the best thing to do is to renovate directly. This stabilizes or even reverses the price decline. Moreover, by adding the energy costs before renovations, we could reach 20 to 30% of depreciation when the carbon price (so the energy price) is pretty high. This seems consistent with the idea that a thermal sieve loses all its value and could become impossible to sell, because of the enormous costs involved in owning it.

Loss Given Default

When there is no guarantee, we assume as in the previous work that LGD is equal to 45% so that $\gamma = 0.55$. To illustrate the case where there is guarantee, we consider, both if the collateral is a *financial asset* and a *building*, EAD starts at 200 and growths annually as the economic total output growth in the *Current Policies* scenario (see Table 5.10 below).

If the collateral is a financial asset We compute here for $M = 500$ simulations of the productivity processes $(\mathcal{Z}, \mathcal{A})$, the loss given default of 4 loans with the same

Year	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034
EAD	200.	202.8	206.6	209.9	213.1	216.2	219.5	222.6	226.2	229.6	233.1	236.3	239.9	243.9

Table 5.10: EAD per year

exposure but with 4 different financial assets collateral described in Table 5.5.

Emissions level	**No collateral**	**Firm 1**	**Firm 2**	**Firm 3**	**Firm 4**
Current Policies	45.	32.934	31.960	34.561	29.281
NDCs	45.	33.177	32.184	34.609	29.357
Divergent Net Zero	45.	33.485	32.471	34.673	29.459
Net Zero 2050	45.	33.995	32.940	34.784	29.640

Table 5.11: Average annual LGD per scenario between 2021 and 2030 (in %)

Both in Table 5.11 and in Figure 5.7, we can first see that the presence of guarantees reduce LGD. Without collateral, we assume 45%, and with collateral, for all scenarios and for different characteristics of firms, LGD is less than 45%.

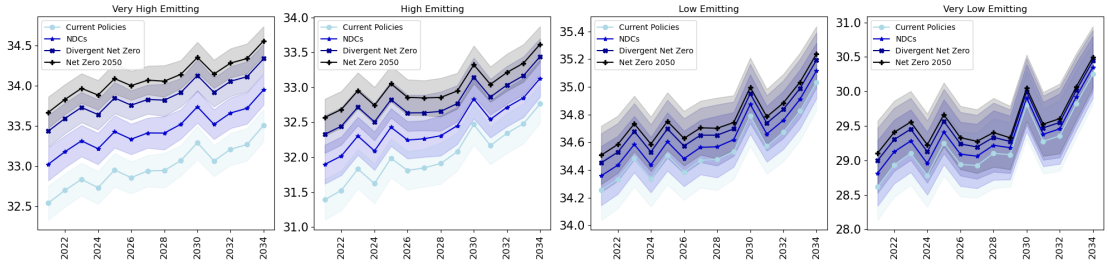


Figure 5.7: LGD with a financial asset as collateral

However, the decreasing of LGD depends on the scenarios. When the scenario becomes tougher, the impact of the presence of the collateral on LGD is lessened. This is logical and due to the fact that the value of the liquidated asset loses value when the price of carbon rises. The decreasing of the LGD also depends on the distinctive characteristics of the guarantees. Precisely, each firm in Table 5.5, serving as collateral, belongs to a unique and distinct sector (through $\mathfrak{a}$), which go from the more to the less polluting. Therefore the more the collateral is in a polluting sector, the less it reduces LGD.

If the collateral is a building We use the $M = 500$ trajectories of the productivity processes $(\mathcal{Z}, \mathcal{A})$ simulated above. We compute the loss given default of 4 loans with the same exposure but with the 4 buildings described in Table 5.5 as collateral.

All the comments made for a financial asset as collateral are valid here: the presence of a collateral reduces LGD, that increases when the climate transition scenario becomes tougher.

There are two main differences. First, the more the building is energetically inefficient, the more LGD increases (it is the same above when the financial asset

Emissions level	**No collateral**	**Building 1**	**Building 2**	**Building 3**	**Building 4**
Current Policies	45.	36.020	36.152	35.928	36.095
NDCs	45.	38.383	37.752	36.922	36.499
Divergent Net Zero	45.	38.939	38.303	37.377	36.751
Net Zero 2050	45.	39.102	38.524	37.615	36.908

Table 5.12: Average annual LGD per scenario between 2021 and 2030 (in %)

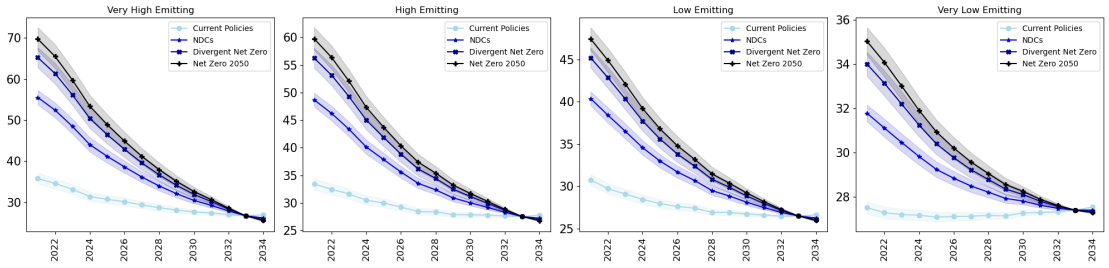


Figure 5.8: LGD with a building as collateral

belongs to a very polluting sector). Secondly, LGD decreases when time increases. This is a consequence of the dynamics of the impact of the carbon price of the housing market as described in Section 5.5.4: as we approach the optimal renovation date, the prices of energy-inefficient buildings rise and converge progressively towards the prices of energy-efficient buildings as seen on Figure 5.6. LGD follows the same behaviour logically but with an inverse monotony.

Expected and unexpected loss

To this aim, to keep things simple, we will consider a credit portfolio of $N = 12$ loans contracted by the firms described in Table 5.13 below.

We can remark that, for each $k = 0, \ldots, 2$, firms $4k+1$, $4k+2$, $4k+3$, and $4k+4$ respectively belong to the *Very High Emitting*, *High Emitting*, *Low Emitting*, and *Very Low Emitting* groups. Moreover, we assume that

- the loans of the firms 1, 2, 3, and 4 are not collateralized;
- the loans of the firms 5, 6, 7, and 8 are collateralized by financial assets described in Table 5.5;
- the loans of the firms 9, 10, 11, and 12 are collateralized by a building described in Table 5.7.

We want to calculate the expected (respectively unexpected) loss noted EL (respectively UL) for each loan $n = 1, \ldots, 12$, by using (5.4.9) (respectively (5.4.11)).

Table 5.14 (respectively Table 5.15) shows average annual EL (respectively UL) normalized to the EL without collateral observed in the scenario *Current Policies*.

Loans	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
EAD_n	200.	200.	200.	200.	200.	200.	200.	200.	200.	200.	200.	200.
F_0^n	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
B^n	3.76	3.98	3.75	4.41	3.76	3.98	3.75	4.41	3.76	3.98	3.75	4.41
$\sigma_{\mathfrak{b}^n}$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\mathfrak{a}^n$ ***(Very High)***	1.	0.	0.	0.	1.	0.	0.	0.	1.	0.	0.	0.
$\mathfrak{a}^n$ ***(High)***	0.	1.	0.	0.	0.	1.	0.	0.	0.	1.	0.	0.
$\mathfrak{a}^n$ ***(Low)***	0.	0.	1.	0.	0.	0.	1.	0.	0.	0.	0.	0.
$\mathfrak{a}^n$ ***(Very Low)***	0.	0.	0.	1.	0.	0.	0.	1.	0.	0.	0.	0.
Collateral type	No	No	No	No	Fa	Fa	Fa	Fa	Ho	Ho	Ho	Ho
$\overline{F}_0^n$					1.	1.	1.	1.				
$\overline{\sigma}_{\mathfrak{b}^n}$					0.05	0.05	0.05	0.05				
$\overline{\mathfrak{a}}^n$ ***(Very High)***					1.	0.	0.	0.				
$\overline{\mathfrak{a}}^n$ ***(High)***					0.	1.	0.	0.				
$\overline{\mathfrak{a}}^n$ ***(Low)***					0.	0.	1.	0.				
$\overline{\mathfrak{a}}^n$ ***(Very Low)***					0.	0.	0.	1.				
C_n^0									4000.	4000.	4000.	4000.
R_n									25.	25.	25.	25.
α^n									320.	253.	187.	120.

Table 5.13: Characteristics of the portfolio (No = no collateral, Fa = Financial asset collateral, Ho = housing collateral)

Emissions level	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Current Policies	1.00	1.00	1.00	1.00	0.79	0.77	0.81	0.71	0.58	0.59	0.58	0.58
NDCs	1.28	1.17	1.04	1.02	1.01	0.90	0.85	0.72	0.91	0.79	0.66	0.62
Divergent Net Zero	1.74	1.41	1.10	1.05	1.38	1.10	0.90	0.75	1.29	0.99	0.72	0.65
Net Zero 2050	2.85	1.91	1.21	1.11	2.27	1.47	0.98	0.79	2.13	1.37	0.81	0.70

Table 5.14: Average annual EL per scenario between 2021 and 2030 (in %)

We can make two key observations that were to be expected from the PD and LGD calculations:

1. Whether collateral is involved or not, we can see that EL and UL increase as the transition hardens. This is to be expected, since PD and LGD behave in the same way.

2. When a loan is collateralized, it significantly reduces the bank's expected and unexpected losses. And for collateralized loans, these losses increase if the collateral has a high carbon footprint: in particular, if the collateral is a financial asset whose value growth is driven by a polluting sector or if it is a building that is not energy efficient.

Emissions level	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Current Policies	1.00	1.00	1.00	1.00	0.81	0.81	0.84	0.82	0.59	0.57	0.56	0.62
NDCs	1.18	1.01	1.00	1.00	0.95	0.82	0.84	0.82	0.85	0.67	0.62	0.64
Divergent Net Zero	1.42	1.02	1.00	1.00	1.16	0.83	0.84	0.82	1.05	0.71	0.64	0.65
Net Zero 2050	1.80	1.02	1.0	0.99	1.49	0.85	0.85	0.82	1.34	0.73	0.65	0.66

Table 5.15: Average annual UL per scenario between 2021 and 2030 (in %)

Conclusion

Following chapters 3 and 4, we developed here a framework to quantify the impacts of the carbon price on a credit portfolio (expected and unexpected) losses, when the obligor companies as well as their guarantees belong to an economy subject to the climate transition declined by carbon price. We start by describing a closed economy, driven by a productivity following a multidimensional Ornstein-Uhlenbeck and subject to a climate transition modeled through a dynamic and deterministic carbon price, by a dynamic stochastic multisectoral. Then, by using the discounted cash flow methodology with the cash flows, following a stochastic differential equation, depending on the productivity as well as the carbon price, we evaluate the obligor value that helps us later on to compute its probability of *default*. We then turn to the bank's loss in the event of a borrower's *over-indebtedness* and if its loan is collateralized. When that is the case, the potential loss of the bank is written as the difference between the debt amount (EAD) and the collateral liquidated. We finally distinguish two types of collateral: either a financial asset or a building, both belonging to the economy so affected by the productivity and the carbon price. This work opens the door to many extensions as a finer modeling of the real estate market, taking into account other types of guarantees, modeling the unsecured loans that we assumed constant.

Conclusion

We studied how the introduction of carbon taxes would propagate in a credit portfolio. This amounted to quantifying the distortions in the portfolio risk measures when the borrowing firms belong to an economy, driven by a dynamic and stochastic productivity and subject to climate transition modelled thanks to a deterministic carbon price process. We were inspired by the approach proposed by Allen et al. [2020].

In a first part, we assumed that the loans are unsecured and the time is discrete. To this aim, we first build a dynamic stochastic multisectoral model in which we introduced greenhouse gases emissions costs, and whose the resolution allowed us to describe the dynamics of some macroeconomic variables. We later use the Discounted Cash Flows (DCF) method to compute the value a firm operating in the economy. We finally introduced the firm value into a structural credit risk models to project some risk measures (PD, EL, and UL). We also introduce losses' sensitivities to carbon taxes to measure the uncertainty of the losses to the transition scenarios.

In a second part, we considered that time is continuous and some loans are secured by collaterals living to the same economy as borrowers and experiencing the same shocks (productivity and climate transition). We then assume two types of collateral. When it was a financial asset, we obtained its value by the continuous DCF method. When it was a building, we obtained its value taking into account the carbon price process, the energy costs, the renovation costs, as well as the price of energy-efficient building. We then used the value of these guarantees to project the portfolio's risk measures (PD, EL, LGD, and UL).

At each stage of our work, we proposed simulations based on real or fictitious data. This work provides a preliminary methodology to calculate the evolution of credit risk measures of a multisectoral credit portfolio, starting from a given climate transition scenario described by the carbon price.

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Appendix A

A Multisectoral Model with Carbon emissions costs

A.1 Vector Autoregressive Model (VAR):

Detailed proofs in Hamilton [2020], and Kilian and Lütkepohl [2017]. Assume that $(\Theta_t)_{t\in\mathbb{N}}$ follows a VAR, i.e. for all $t \in \mathbb{N}^*$,

$$\Theta_t = \mu + \Gamma\Theta_{t-1} + \mathcal{E}_t, \quad \text{where for} \quad t \in \mathbb{Z}, \quad \mathcal{E}_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

with $\mu \in \mathbb{R}^I$ and where the matrix $\Gamma \in \mathbb{R}^{I\times I}$ has eigenvalues all strictly less than 1 in absolute value. We have the following result that can be easily show in VAR's literature.

- $(\Theta_t)_{t\in\mathbb{N}}$ is weak-stationary.
- If $\Theta_0 \sim \mathcal{N}(\overline{\mu}, \overline{\Sigma})$ with $\overline{\mu} := (\mathbf{I}_I - \Gamma)^{-1}\mu$, and $\text{vec}(\overline{\Sigma}) = (\mathbf{I}_{I\times I} - \Gamma \bigotimes \Gamma)^{-1}\text{vec}(\Sigma)$, then for $t \in \mathbb{Z}$, $\mathcal{E}_t \sim \mathcal{N}(\mathbf{0}, \Sigma)$ with $\Sigma \in \mathbb{R}^{I\times I}$.
- For $t, T \in \mathbb{N}$, we note $\Upsilon_t := \sum_{v=0}^{t} \Gamma^v$, then

$$\sum_{u=1}^{T}\sum_{v=1}^{u} \Gamma^{u-v}\mathcal{E}_{t+v} = \sum_{u=1}^{T} \Upsilon_{T-u}\mathcal{E}_{t+u},$$

- For $t, u \in \mathbb{N}$,

$$\Theta_t = \overline{\mu} + \sum_{v=1}^{\infty} \Gamma^v\mathcal{E}_{t-v} \qquad \text{and} \qquad \Theta_{t+T} = \Gamma^T\Theta_t + \Upsilon_{T-1}\mu + \sum_{v=1}^{T} \Gamma^{T-v}\mathcal{E}_{t+v}.$$

- For $t, T \in \mathbb{N}$,

$$\left(\sum_{u=1}^{T} \Theta_{t+u} \middle| \Theta_t\right) \sim \mathcal{N}\left(\Gamma\Upsilon_{T-1}\Theta_t + \left(\sum_{u=1}^{T} \Upsilon_{u-1}\right)\mu, \sum_{u=1}^{T} \Upsilon_{T-u}\Sigma(\Upsilon_{T-u})^\top\right), \tag{A.1.1}$$

  and in particular $(\Theta_{t+1}|\Theta_t) \sim \mathcal{N}(\mu + \Gamma\Theta_t, \Sigma)$.

A.2 Sectoral groups

We use the output and GHG emissions by sector to compute the carbon intensity (which is the tons of GHG emitted per euro of output) per sector. Then we compute their annual average and we group the sectors together if their annual average carbon intensities are close.

1. Very High Emitting
    - Manufacture of basic metals and fabricated metal products, except machinery and equipment
    - Water supply; sewerage, waste management and remediation activities
    - Manufacture of rubber and plastics products, and other non-metallic mineral products
    - Agriculture, forestry and fishing
2. High Emitting
    - Electricity, gas, steam and air conditioning supply
    - Transportation and storage
    - Manufacture of chemicals and chemical products
    - Manufacture of coke and refined petroleum products
3. Low Emitting
    - Manufacture of food products, beverages and tobacco products
    - Manufacture of wood and paper products, and printing
    - Mining and quarrying
4. Very Low Emitting
    - Other Service Activities
    - Arts, Entertainment and Recreation
    - Social Work Activities
    - Human Health Activities
    - Education
    - Public
    - Administration and Defence; Compulsory Social Security
    - Administrative and Support Service Activities
    - Advertising and Market Research; Other Professional, Scientific and Technical Activities; Veterinary Activities
    - Scientific Research and Development Legal and Accounting Activities; Activities of Head Offices; Management Consultancy Activities; Architecture and Engineering Activities; Technical Testing and Analysis
    - Real Estate Activities
    - Financial and Insurance Activities
    - Computer Programming, Consultancy and Related Activities; Information Service Activities
    - Telecommunications

- Publishing, Audiovisual and Broadcasting Activities
- Accommodation and Food Service Activities
- Wholesale and Retail Trade, Repair of Motor Vehicles and Motorcycles
- Construction
- Manufacture of Furniture; Other Manufacturing; Repair and Installation of Machinery and Equipment
- Manufacture of Transport Equipment
- Manufacture of Machinery and Equipment N.E.C.
- Manufacture of Electrical Equipment
- Manufacture of Computer, Electronic and Optical Products
- Manufacture of Basic Pharmaceutical Products and Pharmaceutical Preparations
- Manufacture of Textiles, Wearing Apparel and Leather Products

A.3 Estimation of carbon intensities of intermediary inputs

Emissions Level	φ_0	$g_{\varphi,0}$	$\theta_\varphi(\%)$
$\kappa_{\textbf{\textit{Very High}}}$	0.003	-0.026	0.001
$\kappa_{\textbf{\textit{High}}}$	1.123	-0.040	0.001
$\kappa_{\textbf{\textit{Low}}}$	0.003	-0.026	0.001
$\kappa_{\textbf{\textit{Very Low}}}$	0.003	-0.026	0.001
$\zeta_{\textbf{\textit{Very High, Very High}}}$	0.124	-0.043	1.5
$\zeta_{\textbf{\textit{Very High, High}}}$	0.017	-0.045	0.001
$\zeta_{\textbf{\textit{Very High, Low}}}$	0.088	-0.065	0.001
$\zeta_{\textbf{\textit{Very High, Very Low}}}$	0.034	-0.042	3.6
$\zeta_{\textbf{\textit{High, Very High}}}$	0.051	-0.049	0.001
$\zeta_{\textbf{\textit{High, High}}}$	0.177	-0.046	1.1
$\zeta_{\textbf{\textit{High, Low}}}$	0.022	-0.081	0.001
$\zeta_{\textbf{\textit{High, Very Low}}}$	0.026	-0.030	0.001
$\zeta_{\textbf{\textit{Low, Very High}}}$	0.037	-0.055	11.1
$\zeta_{\textbf{\textit{Low, High}}}$	0.117	-0.079	0.001
$\zeta_{\textbf{\textit{Low, Low}}}$	0.111	-0.065	0.3
$\zeta_{\textbf{\textit{Low, Very Low}}}$	0.026	-0.018	0.001
$\zeta_{\textbf{\textit{Very Low, Very High}}}$	0.019	-0.052	0.1
$\zeta_{\textbf{\textit{Very Low, High}}}$	0.025	-0.05	2.1
$\zeta_{\textbf{\textit{Very Low, Low}}}$	0.016	-0.088	0.001
$\zeta_{\textbf{\textit{Very Low, Very Low}}}$	0.059	-0.034	0.001

Table A.1: Carbon intensities parameters

A.4 Figures

Plots of historical data

We plot the data described in Section 3.3.1.

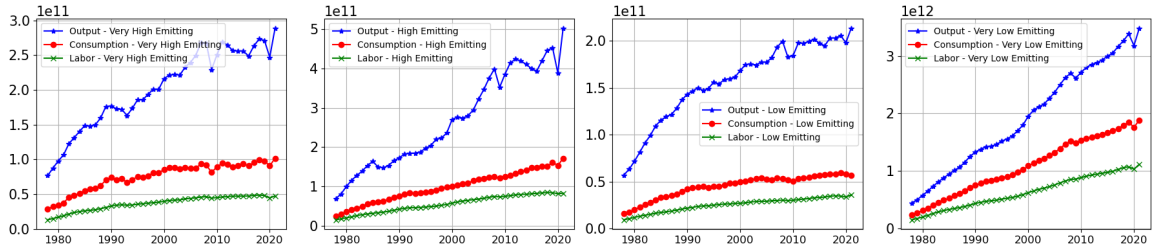


Figure A.1: Nominal consumption, labor, and output (described in item 2)

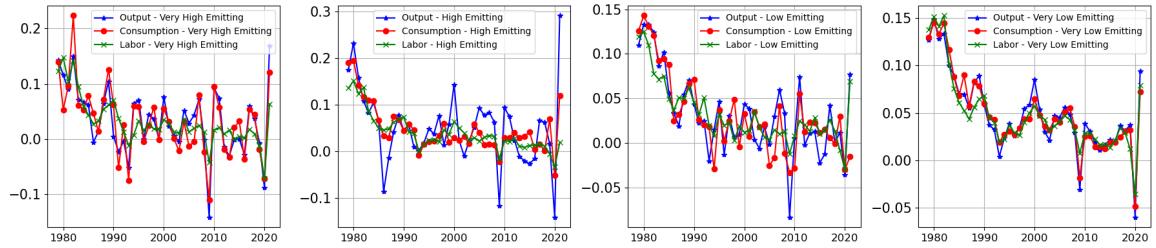


Figure A.2: Consumption, labor, and output growth (described in item 1)

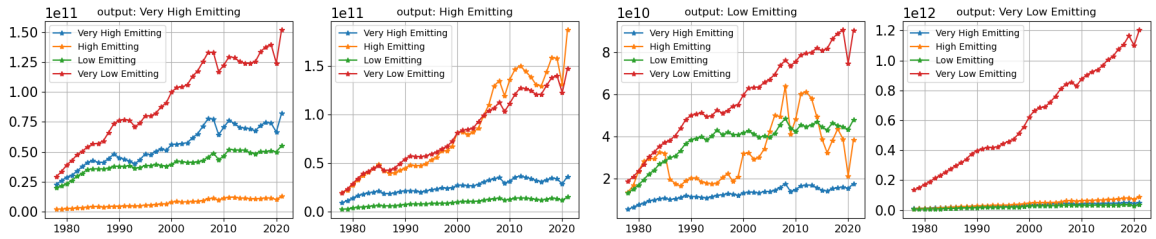


Figure A.3: Nominal intermediary inputs (described in item 1)

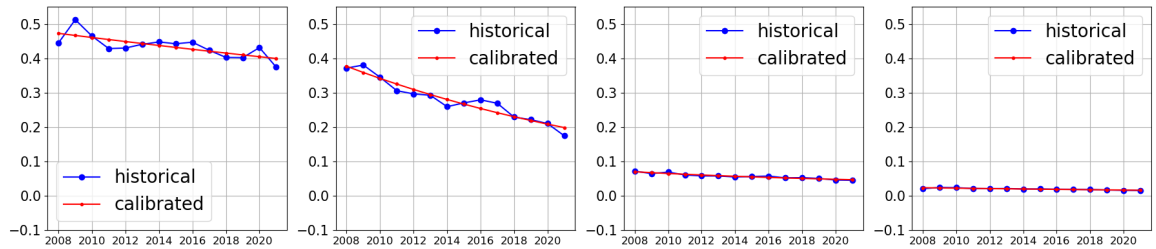


Figure A.4: Carbon intensities of firms production (described in item 3)

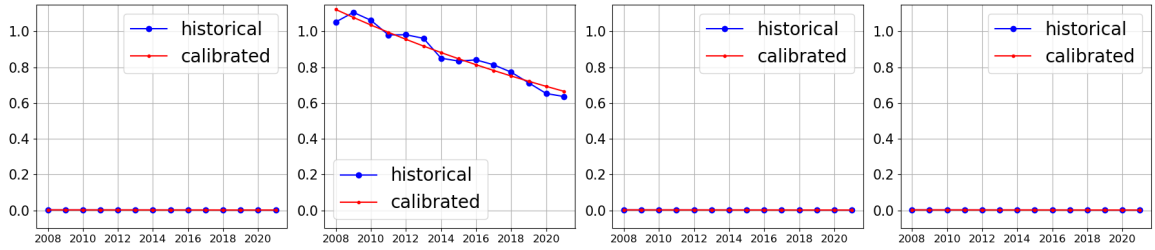


Figure A.5: Carbon intensities of households consumption (described in item 3)

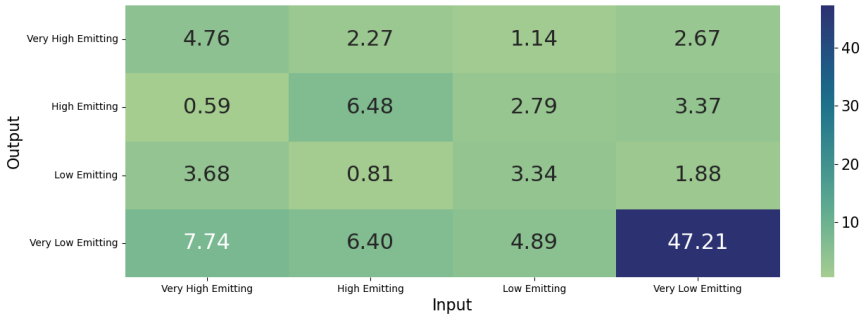


Figure A.6: Proportion of each intermediate input relative to the total intermediate inputs in the economy

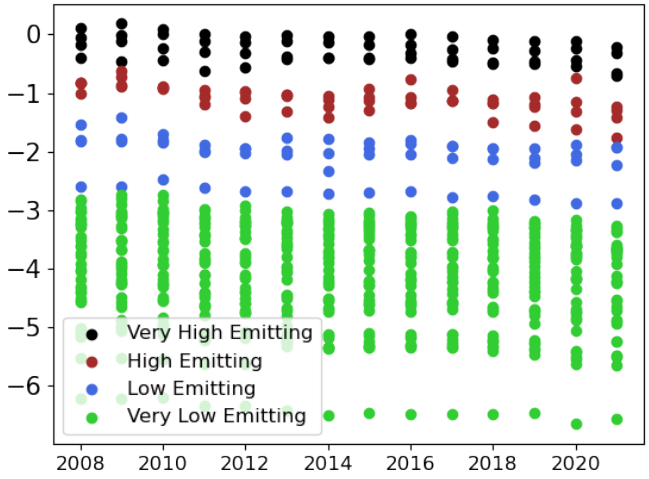


Figure A.7: Log of "air emissions intensities (in kilograms of CO_2-equivalent per euro)" by NACE from 2008 to 2021

Plots of macroeconomics variables per year and per carbon price scenario

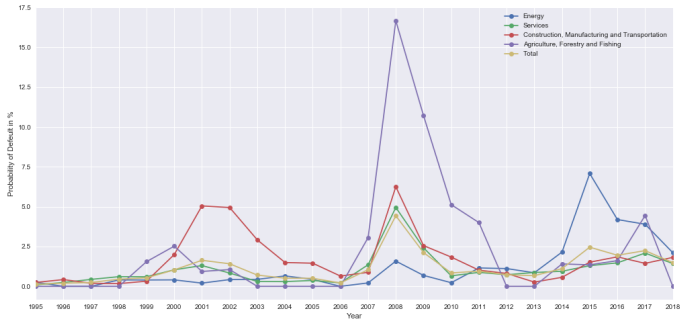


Figure A.8: Historical data of a chosen portfolio - France - from 1995 to 2018 (described in item B.1)

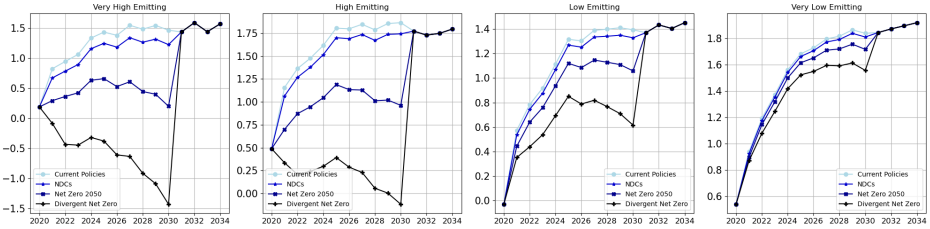


Figure A.9: Average annual output growth

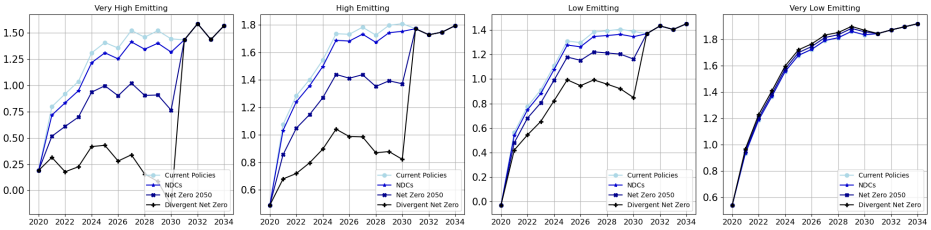


Figure A.10: Average annual consumption growth

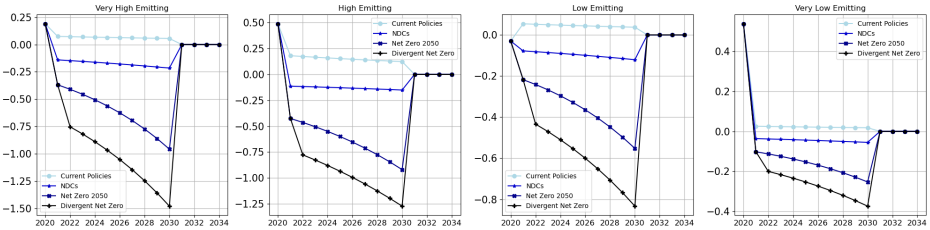


Figure A.11: Average annual labour growth

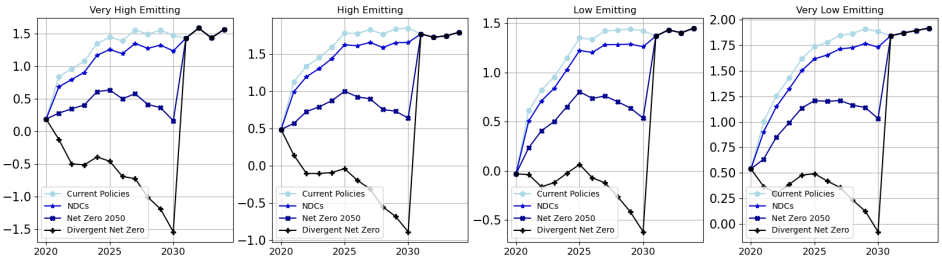


Figure A.12: Average annual growth of intermediary input from *Very High Emitting*

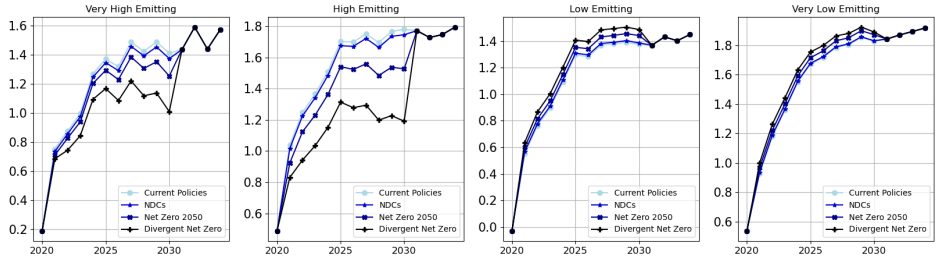


Figure A.13: Average annual growth of intermediary input from *High Emitting*

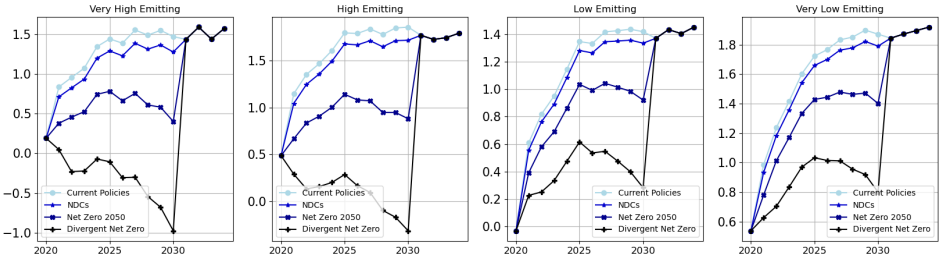


Figure A.14: Average annual growth of intermediary input from *Low Emitting*

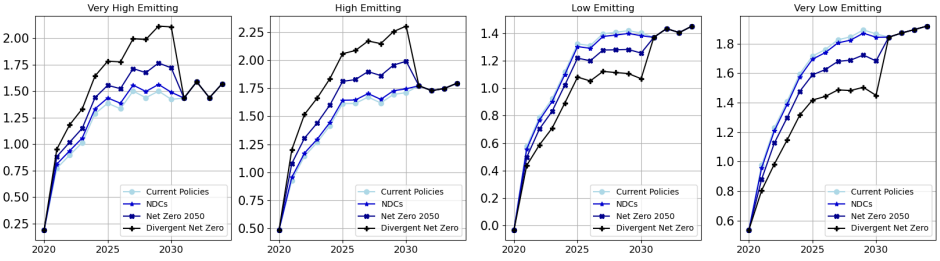


Figure A.15: Average annual growth of intermediary input from *Very Low Emitting*

Appendix B

Loss modelling in discrete time, with carbon emissions cost and without collateral

B.1 Factor selection by LASSO regression

We perform LASSO regression questioning the relationship between credit risk (described by the logit of the default rate) and economics conditions (described by the macroeconomic variables as we assumed in Section 4.2), we use S&P ratings for data on the ratings and default, on a yearly basis from 1995 to 2019, of 7046 large US companies belonging to 13 sectors. We can analyze and use them to compute the historical probability of *default* (displayed Figure A.8) and the migration matrix by sector. The USA macroeconomic time series can be found in the World Bank database and in the FRED Saint-Louis database Fed [2023].

	Coef_	Importance	Percentage
Industry value added growth	-0.433	0.433	73.979
Real GDP per capita growth	-0.073	0.073	12.485
Unemployment rate	0.046	0.046	7.934
Stocks returns	-0.033	0.033	5.602
Export of goods and services	0	0	0
Real GDP growth	0	0	0
Inflation rate	0	0	0
10-year interest rate	0	0	0

Table B.1: Factor selection by LASSO

B.2 Figures

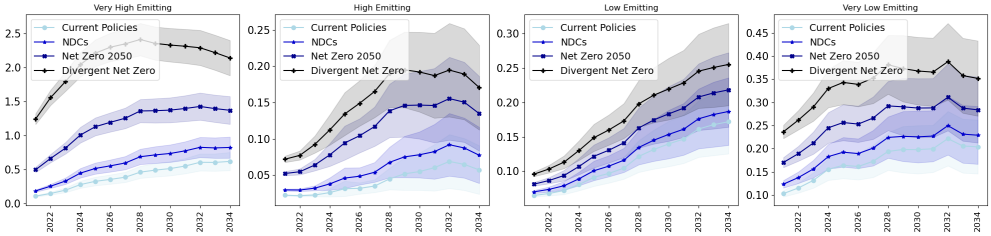


Figure B.1: Average annual EL per scenario for some firms

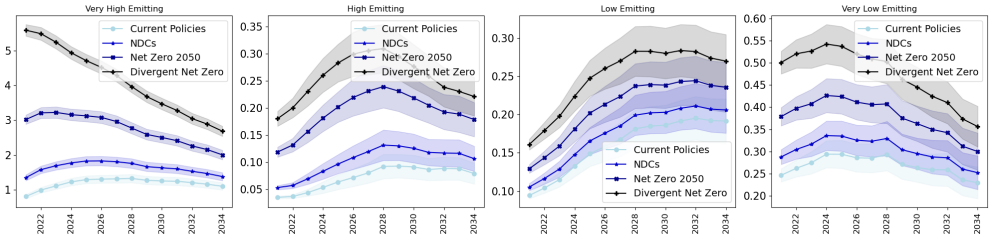


Figure B.2: Average annual UL per scenario for some firms

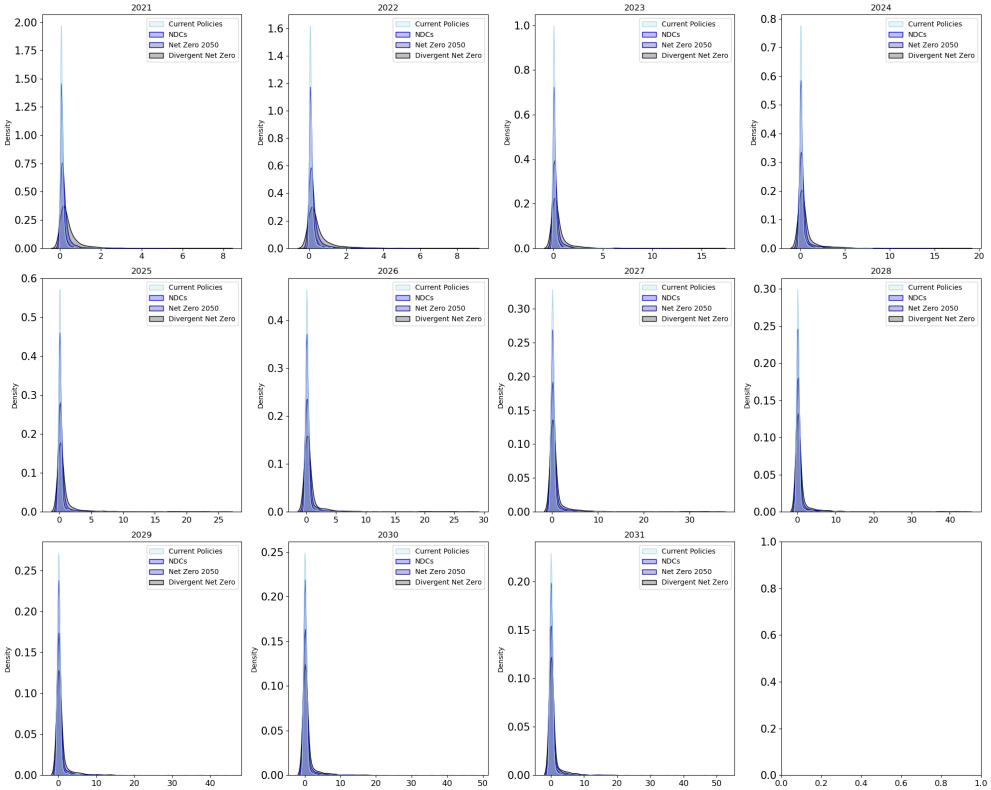


Figure B.3: Annual EL distribution per scenario

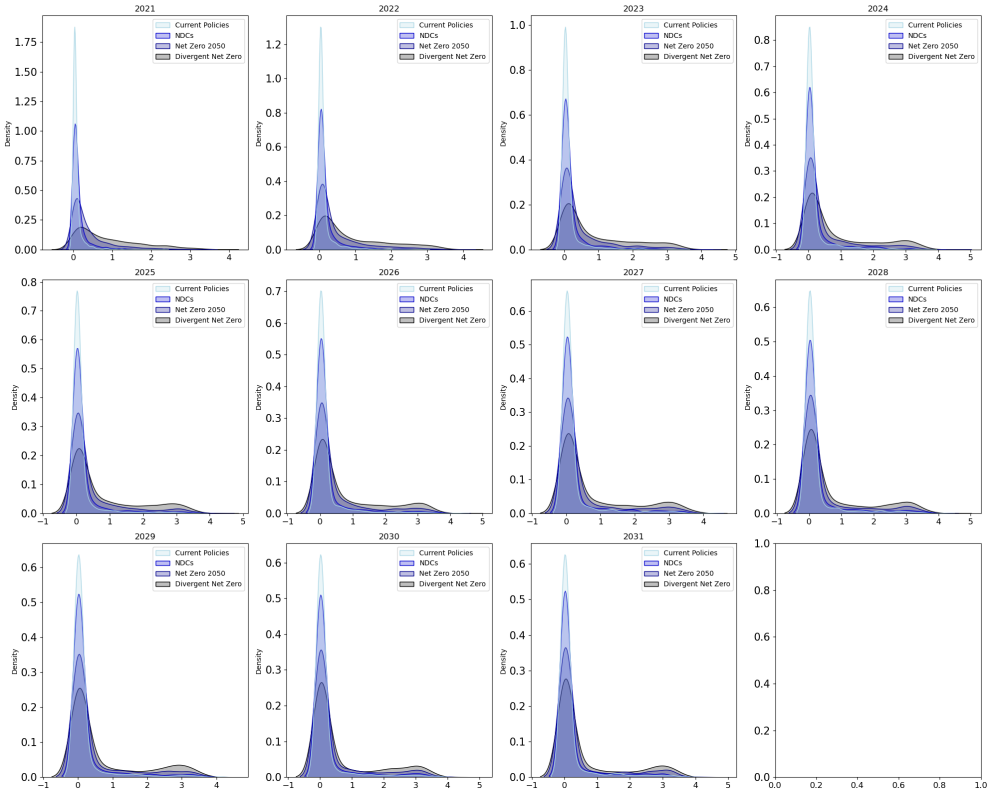


Figure B.4: Annual UL distribution per scenario

Appendix C

Portfolio loss in continuous time, with carbon emissions cost and with stochastic collateral

C.1 Proofs

C.1.1 Hurwitz matrix

Assume that $-\Gamma$ is a Hurwitz matrix, then

1. if we note $\lambda_\Gamma := \max_{\lambda \in \lambda(\Gamma)} Re(\lambda) \geq 0$, there exists $c_\Gamma > 0$ so that $\|e^{-\Gamma t}\| < c_\Gamma e^{-\lambda_\Gamma t}$ for all $t \geq 0$.

2. Moreover, for $t \geq 0$n Υ_t defined in (5.2.2) is such that

$$\|\Upsilon_t\| = \left\| \int_0^t e^{-\Gamma s} \mathrm{d}s \right\| \leq \int_0^t \left\| e^{-\Gamma s} \right\| \mathrm{d}s \leq c_\Gamma \int_0^t e^{-\lambda_\Gamma s} \mathrm{d}s \leq c_\Gamma \min \left\{ \frac{1}{\lambda_\Gamma}, t \right\} \quad (C.1.1)$$

C.1.2 Bivariate Gaussian

Assume that X and Y are two standard Gaussian with correlation coefficient ρ. We then have for $(x, y) \in \mathbb{R}^2$, the cdf,

$$\Phi_2(x, y) := \mathbb{P}[X \leq x, Y \leq y] = \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{1}{2(1-\rho^2)} \left(u^2 + v^2 - 2\rho uv \right) \right) \mathrm{d}u \mathrm{d}v.$$

Let $\sigma > 0$, we want to compute $\mathbb{E}[e^{\sigma X}\mathbf{1}_{X\leq x,Y\leq y}]$. We have

$$\begin{aligned}
\mathbb{E}[e^{\sigma X}\mathbf{1}_{X\leq x,Y\leq y}] &= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x}\int_{-\infty}^{y} e^{\sigma u}\exp\left(-\frac{1}{2(1-\rho^2)}\left(u^2+v^2-2\rho uv\right)\right)\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x} e^{\sigma u-\frac{1}{2(1-\rho^2)}u^2}\int_{-\infty}^{y}\exp\left(-\frac{1}{2(1-\rho^2)}\left(v^2-2\rho uv\right)\right)\mathrm{d}v \quad \mathrm{d}u \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x} e^{\sigma u-\frac{1}{2(1-\rho^2)}u^2}\int_{-\infty}^{y}\exp\left(-\frac{1}{2(1-\rho^2)}\left((v-\rho u)^2-\rho^2u^2\right)\right)\mathrm{d}v \quad \mathrm{d}u \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x} e^{\sigma u-\frac{1}{2}u^2}\int_{-\infty}^{y}\exp\left(-\frac{1}{2(1-\rho^2)}\left((v-\rho u)^2\right)\right)\mathrm{d}v \quad \mathrm{d}u
\end{aligned}$$

But

$$\int_{-\infty}^{y}\exp\left(-\frac{1}{2(1-\rho^2)}\left((v-\rho u)^2\right)\right)\mathrm{d}v = \sqrt{2\pi(1-\rho^2)}\Phi\left(\frac{y-\rho u}{\sqrt{1-\rho^2}}\right),$$

therefore,

$$\begin{aligned}
\mathbb{E}[e^{\sigma X}\mathbf{1}_{X\leq x,Y\leq y}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\sigma u-\frac{1}{2}u^2}\Phi\left(\frac{y-\rho u}{\sqrt{1-\rho^2}}\right)\mathrm{d}u \\
&= \frac{e^{\frac{1}{2}\sigma^2}}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}(u-\sigma)^2}\Phi\left(\frac{y-\rho u}{\sqrt{1-\rho^2}}\right)\mathrm{d}u \\
&= e^{\frac{1}{2}\sigma^2}\int_{-\infty}^{x-\sigma}\phi(u)\Phi\left(\frac{y-\rho\sigma}{\sqrt{1-\rho^2}}+\frac{-\rho}{\sqrt{1-\rho^2}}u\right)\mathrm{d}u.
\end{aligned}$$

However,

$$\int_{-\infty}^{c}\Phi(a+bx)\phi(x)\mathrm{d}x = \Phi_2\left(c,\frac{a}{\sqrt{1+b^2}};\frac{-b}{\sqrt{1+b^2}}\right),$$

we can then conclude that

$$\mathbb{E}[e^{\sigma X}\mathbf{1}_{X\leq x,Y\leq y}] = e^{\frac{1}{2}\sigma^2}\Phi_2\left(x-\sigma,y-\rho\sigma;\rho\right). \tag{C.1.2}$$

C.1.3 Ornstein-Uhlenbeck process

Let $t,h\geq 0$, from the second equation of (5.2.1),

$$\mathcal{A}_{t+h} = \mathcal{A}_t + \mu h + \varsigma\int_t^{t+h}\mathcal{Z}_s\mathrm{d}s.$$

However, for all $t\in[t,t+h]$, from the first equation of (5.2.1),

$$\mathcal{Z}_s = e^{-\Gamma(s-t)}\mathcal{Z}_t + \int_t^s e^{-\Gamma(s-u)}\Sigma\mathrm{d}B_u^{\mathcal{Z}},$$

therefore,

$$\mathcal{A}_{t+h} = \mathcal{A}_t + \mu h + \varsigma\left(\int_t^{t+h}e^{-\Gamma(s-t)}\mathrm{d}s\right)\mathcal{Z}_t + \varsigma\int_t^{t+h}\left(\int_t^s e^{-\Gamma(s-u)}\Sigma\mathrm{d}B_u^{\mathcal{Z}}\right)\mathrm{d}s.$$

We then have

$$\mathcal{A}_{t+h} = \mathcal{A}_t + \mu h + \varsigma \Upsilon_h \mathcal{Z}_t + \varsigma \int_t^{t+h} e^{-\Gamma s} \left(\int_t^s e^{\Gamma u} \Sigma \mathrm{d}B_u^{\mathcal{Z}} \right) \mathrm{d}s,$$

where $(\Upsilon_h)_{h\geq 0}$ is defined in (5.2.2).

Let pose $X_s := e^{-\Gamma s}$ and $Y_s := \int_t^s e^{\Gamma u} \Sigma \mathrm{d}B_u^{\mathcal{Z}}$, for $s \in [t, t+h]$. Then

$$\int_t^{t+h} e^{-\Gamma s} \left(\int_t^s e^{\Gamma u} \Sigma \mathrm{d}B_u^{\mathcal{Z}} \right) \mathrm{d}s = -\Gamma^{-1} \int_t^{t+h} \mathrm{d}X_s \cdot Y_s.$$

By integration by parts, as $[X, Y]_u = 0$ for all $u \geq 0$, we have

$$\begin{aligned} \int_t^{t+h} \mathrm{d}X_s \cdot Y_s &= X_{t+h} \cdot Y_{t+h} - X_t \cdot Y_t - \int_t^{t+h} X_s \cdot \mathrm{d}Y_s \\ &= e^{-\Gamma(t+h)} \int_t^{t+h} e^{\Gamma u} \Sigma \mathrm{d}B_u^{\mathcal{Z}} - \int_t^{t+h} \Sigma \mathrm{d}B_s^{\mathcal{Z}} \\ &= \int_t^{t+h} \left(e^{-\Gamma(t+h-s)} - \mathbf{I}_I \right) \Sigma \mathrm{d}B_s^{\mathcal{Z}}. \end{aligned}$$

Finally

$$\mathcal{A}_{t+h} = \mathcal{A}_t + \mu h + \varsigma \Upsilon_h \mathcal{Z}_t - \varsigma \Gamma^{-1} \int_t^{t+h} \left(e^{-\Gamma(t+h-s)} - \mathbf{I}_I \right) \Sigma \mathrm{d}B_s^{\mathcal{Z}}.$$

The conclusion follows.

C.2 The multisectoral model in continuous time

For all $i \in \mathcal{I}$, let us consider the following $\mathbb{G}$-measurable and positive processes: Y^i the production of sector i, N^i the labor demand in sector i, and for all $j \in \mathcal{I}$, Z^{ji} the consumption by sector i of intermediate inputs produced by sector j.

The firm's point of view

Aiming to work with a simple model, we follow [Galí, 2015, Chapter 2]. It then appears that the firm's problem corresponds to an optimization performed at each period, depending on the state of the world. This problem will depend, in particular, on the productivity and the price processes introduced above. Moreover, it will also depend on P^i and W^i, two $\mathbb{G}$-adapted positive stochastic processes representing respectively the price of good i and the wage paid in sector $i \in \mathcal{I}$. We start by considering the associated deterministic problem below, when time and randomness are fixed.

Solution for the deterministic problem We denote $\overline{a} \in (0,+\infty)^I$ the level of technology in each sector, $\overline{p} \in (0,\infty)^I$ the price of the goods produced by each sector, $\overline{w} \in (0,\infty)^I$ the nominal wage in each sector, $\overline{\tau} \in [0,1)^I$ and $\overline{\zeta} \in [0,1)^{I\times I}$ the price on production and consumption of goods. For $i \in \mathcal{I}$, we consider a representative firm of sector i, with technology described by the production function

$$\mathbb{R}_+ \times \mathbb{R}_+^I \ni (n,z) \mapsto F_{\overline{a}}^i(n,z) = \overline{a}^i n^{\psi^i} \prod_{j\in\mathcal{I}} (z^j)^{\boldsymbol{\lambda}^{ji}} \in \mathbb{R}_+,$$

where n represents the number of hours of work in the sector, and z^j the firm's consumption of intermediary input produced by sector j. The coefficients $\psi \in (\mathbb{R}_+^*)^I$ and $\boldsymbol{\lambda} \in (\mathbb{R}_+^*)^{I\times I}$ are elasticities satisfying (5.2.6). The management of firm i then solves the classical problem of profit maximization

$$\widehat{\Pi}^i_{(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta},\overline{\delta})} := \sup_{(n,z)\in\mathbb{R}_+\times\mathbb{R}_+^I} \Pi^i(n,z), \tag{C.2.1}$$

where, omitting the dependency in $(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta})$,

$$\Pi^i(n,z) := F_{\overline{a}}^i(n,z)\overline{p}^i - \overline{\tau}^i F_{\overline{a}}^i(n,z)\overline{p}^i\overline{\delta} - \overline{w}^i n - \sum_{j\in\mathcal{I}} z^j\overline{p}^j + z^j\overline{\zeta}^{ji}\overline{p}^j\overline{\delta}.$$

Note that $F_{\overline{a}}^i(n,z)(1-\overline{\tau}^i)\overline{p}^i$ represents the firm's revenues after carbon price, that $\overline{w}^i n$ stands for the firm's total compensations, and that $\sum_{j\in\mathcal{I}} z^j(1+\overline{\zeta}^{ji})\overline{p}^j$ is the firm's total intermediary inputs. Now, we would like to solve the optimization problem for the firms, namely determine the optimal demands $\mathfrak{n}$ and $\mathfrak{z}$ as functions of $(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta})$. Because we will lift these optimal quantities in a dynamical stochastic setting, we impose that they are expressed as measurable functions. We thus introduce:

Definition C.2.1. An admissible solution to problem (C.2.1) is a pair of measurable functions

$$(\mathfrak{n},\mathfrak{z}) : (0,+\infty)^I \times (0,+\infty)^I \times (0,+\infty)^I \times [0,1)^I \times [0,1)^{I\times I} \to [0,+\infty)^I \times [0,+\infty)^{I\times I},$$

such that, for each sector i, denoting $\overline{n} := \mathfrak{n}^i(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta})$ and $\overline{z} := \mathfrak{z}^{\cdot i}(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta})$,

$$F_{\overline{a}}^i(\overline{n},\overline{z})(1-\overline{\tau}^i\overline{\delta})\overline{p}^i - \overline{w}^i\overline{n} - \sum_{j\in\mathcal{I}} \overline{z}^j(1+\overline{\zeta}^{ji}\overline{\delta})\overline{p}^j = \widehat{\Pi}^i_{(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta},\overline{\delta})},$$

and $F_{\overline{a}}^i(\overline{n},\overline{z}) > 0$ (non-zero production), according to (C.2.1).

Remark C.2.2. The solution obviously depends also on the coefficients ψ and $\boldsymbol{\lambda}$. But these are fixed once and we will not study the dependence of the solution with respect to them.

Proposition C.2.3. *There exists admissible solutions in the sense of Definition C.2.1. Any admissible solution is given by for all $i \in \mathcal{I}$, $\mathfrak{n}^i > 0$ and for all $(i,j) \in \mathcal{I}^2$,*

$$\mathfrak{z}^{ji} = \frac{\boldsymbol{\lambda}^{ji}}{\psi^i} \frac{\overline{w}^i}{(1+\overline{\zeta}^{ji}\overline{\delta})\overline{p}^j} \mathfrak{n}^i > 0. \tag{C.2.2}$$

Moreover, it holds that $\widehat{\Pi}^i_{(\overline{a},\overline{w},\overline{p},\overline{\tau},\overline{\zeta},\overline{\delta})} = 0$ *(according to* (C.2.1)*) and*

$$\mathfrak{n}^i = \psi^i F^i_{\overline{a}}(\mathfrak{n}^i, \mathfrak{z}^{\cdot i}) \frac{(1-\overline{\tau}^i\overline{\delta})\overline{p}^i}{\overline{w}^i}, \tag{C.2.3a}$$

$$\mathfrak{z}^{ji} = \boldsymbol{\lambda}^{ji} F^i_{\overline{a}}(\mathfrak{n}^i, \mathfrak{z}^{\cdot i}) \frac{(1-\overline{\tau}^i\overline{\delta})\overline{p}^i}{(1+\overline{\zeta}^{ji}\overline{\delta})\overline{p}^j}. \tag{C.2.3b}$$

Proof. We study the optimization problem for the representative firm $i \in \mathcal{I}$. Since $\psi^i > 0$ and $\boldsymbol{\lambda}^{ji} > 0$, for all $j \in \mathcal{I}$, as soon as $n = 0$ or $z^j = 0$, for some $j \in \mathcal{I}$, the production is equal to 0. From problem (C.2.1), we obtain that necessarily $n \neq 0$ and $z^j \neq 0$ for all j in this case. So an admissible solution, which has non-zero production, has positive components.
Setting $\overline{n} = \mathfrak{n}^i(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}) > 0$ and $\overline{z} = \mathfrak{z}^{\cdot i}(\overline{a}, \overline{w}, \overline{p}, \overline{\tau}, \overline{\zeta}) > 0$, the optimality of $(\overline{n}, \overline{z})$ yields

$$\partial_n \Pi^i(\overline{n}, \overline{z}) = 0 \text{ and for any } j \in \mathcal{I}, \quad \partial_{z^j} \Pi^i(\overline{n}, \overline{z}) = 0.$$

We then compute

$$\psi^i \frac{F^i_{\overline{a}}(\overline{n}, \overline{z})}{\overline{n}}(1-\overline{\tau}^i\overline{\delta})\overline{p}^i - \overline{w}^i = 0 \text{ and for any } j \in \mathcal{I}, \boldsymbol{\lambda}^{ji}\frac{F^i_{\overline{a}}(\overline{n}, \overline{z})}{\overline{z}^j}(1-\overline{\tau}^i\overline{\delta})\overline{p}^i - (1+\overline{\zeta}^{ji}\overline{\delta})\overline{p}^j = 0,$$

which leads to (C.2.2), (C.2.3a), and (C.2.3b). □

Dynamic setting In C.2 below, we characterize the dynamics of the output and consumption processes using market equilibrium arguments. There, the optimal demand by the firm for intermediary inputs and labor is lifted to the stochastic setting where the admissible solutions then write as functions of the productivity, carbon price, price of goods/services; and wage processes, see Definition C.2.5. For all $i \in \mathcal{I}$, Y^i representing the production of sector i, N^i representing the labor demand in sector i, and for all $j \in \mathcal{I}$, Z^{ji} representing the consumption by sector i of intermediate inputs produced by sector j are therefore positive and $\mathbb{G}$-adapted processes.

The household's point of view

Let $(r_t)_{t\geq 0}$ be the (exogenous) deterministic interest rate, valued in $\mathbb{R}_+$. At each time $t \geq 0$ and for each sector $i \in \mathcal{I}$, we denote

- C^i_t the quantity consumed of the single good in the sector i, valued in $\mathbb{R}^*_+$;
- H^i_t the number of hours of work in sector i, valued in $\mathbb{R}^*_+$.

We also introduce a time preference parameter $\beta \in [0, 1)$ and a utility function $U : (0, \infty)^2 \to \mathbb{R}$ given, for $\varphi \geq 0$, by $U(x, y) := \frac{x^{1-\sigma}}{1-\sigma} - \frac{y^{1+\varphi}}{1+\varphi}$ if $\sigma \in [0, 1) \cup (1, +\infty)$ and by $U(x, y) := \log(x) - \frac{y^{1+\varphi}}{1+\varphi}$, if $\sigma = 1$. We also suppose that

$$\mathfrak{P} := \sup_{t\geq 0, i\in\mathcal{I}} \mathbb{E}\left[\left(\frac{P^i_t}{W^i_t}\right)^{1+\varphi}\right] < +\infty. \tag{C.2.4}$$

For any $C, H \in \mathscr{L}^1_+(\mathbb{G}, (0,\infty)^I)$, we introduce the wealth process

$$\mathrm{d}Q_t = r_t Q_t \mathrm{d}t + \sum_{i\in\mathcal{I}} W_t^i H_t^i - \sum_{i\in\mathcal{I}} P_t^i C_t^i - \sum_{i\in\mathcal{I}} \kappa_t^i P_t^i C_t^i \delta_t, \qquad \text{for any } t \geq 0,$$

with the convention $Q_0 := 0$ and $r_0 := 0$. Note that we do not indicate the dependence of Q upon C and H to alleviate the notations.

For $t \geq 0$ and $i \in \mathcal{I}$, $P_t^i C_t^i$ represents the household's consumption in the sector i and $\kappa_t^i P_t^i C_t^i \delta_t$ is the cost paid by households due to their emissions when they consume goods i, so $\sum_{i\in\mathcal{I}} P_t^i C_t^i (1 + \kappa_t^i \delta_t)$ is the household's total expenses. Moreover, $W_t^i H_t^i$ is the household's labor income in the sector i, $(1 + r_{t-1})Q_{t-1}$ the household's capital income, and $(1 + r_{t-1})Q_{t-1} + \sum_{i\in\mathcal{I}} W_t^i H_t^i$ the household's total revenue.

We define $\mathscr{A}$ as the set of all couples (C, H) with $C, H \in \mathscr{L}^1_+(\mathbb{G}, (0,\infty)^I)$ such that

$$\begin{cases} \mathbb{E}\left[\sum_{i\in\mathcal{I}} \int_{t=0}^{\infty} \beta^t |U(C_t^i, H_t^i)| \mathrm{d}t\right] < \infty, \\ \lim_{T\uparrow\infty} \mathbb{E}[Q_T | \mathcal{G}_t] \geq 0, \qquad \text{for all } t \geq 0. \end{cases}$$

The representative household consumes the I goods of the economy and provides labor to all the sectors. For any $(C, H) \in \mathscr{A}$, let

$$\mathcal{J}(C, H) := \sum_{i\in\mathcal{I}} \mathcal{J}_i(C^i, H^i), \qquad \text{with} \qquad \mathcal{J}_i(C^i, H^i) := \mathbb{E}\left[\int_{t=0}^{\infty} \beta^t U(C_t^i, H_t^i) \mathrm{d}t\right], \quad \forall i \in \mathcal{I}.$$

The representative household seeks to maximize its objective function by solving

$$\max_{(C,H)\in\mathscr{A}} \quad \mathcal{J}(C, H). \tag{C.2.5}$$

We choose above a separable utility function as Miranda-Pinto and Young [2019] does, meaning that the representative household optimizes its consumption and hours of work for each sector independently but under a global budget constraint. The following proposition provides an explicit solution to (C.2.5).

Proposition C.2.4. *Assume that* (C.2.5) *has a solution* $(C, H) \in \mathscr{A}$*. Then, for all* $i, j \in \mathcal{I}$*, the household's optimality condition reads, for any* $t \geq 0$*,*

$$\frac{P_t^i}{W_t^i} = \frac{1}{1 + \kappa_t^i \delta_t} (H_t^i)^{-\varphi} (C_t^i)^{-\sigma}, \tag{C.2.6a}$$

$$\frac{P_t^i}{P_t^j} = \frac{1 + \kappa_t^j \delta_t}{1 + \kappa_t^i \delta_t} \left(\frac{C_t^i}{C_t^j}\right)^{-\sigma}. \tag{C.2.6b}$$

Note that the discrete-time processes C and H cannot hit zero by definition of $\mathscr{A}$, so that the quantities above are well defined.

Proof. Suppose that $\sigma \neq 1$. We first check that $\mathscr{A}$ is non empty. Assume that, for all $t \geq 0$ and $i \in \mathcal{I}$, $\tilde{C}^i_t = 1$ and $\tilde{H}^i_t = \frac{P^i_t(1+\kappa^i_t)}{W^i_t}$, then

$$\mathbb{E}\left[\sum_{i\in\mathcal{I}}\int_{t=0}^{\infty}\beta^t|U(\tilde{C}^i_t,\tilde{H}^i_t)|\mathrm{d}t\right] \leq \sum_{i\in\mathcal{I}}\int_{t=0}^{\infty}\beta^t\left(\frac{1}{1-\sigma}+\frac{1}{1+\varphi}\mathbb{E}\left[\left(\frac{P^i_t(1+\kappa^i_t\delta_t)}{W^i_t}\right)^{1+\varphi}\right]\right)\mathrm{d}t.$$
$$\leq \sum_{i\in\mathcal{I}}\int_{t=0}^{\infty}\beta^t\left(\frac{1}{1-\sigma}+\frac{\mathfrak{P}(1+\kappa^i_t\delta_t)^{1+\varphi}}{1+\varphi}\right)\mathrm{d}t < +\infty,$$

using (C.2.4). We also observe that Q built from $\tilde{H}, \tilde{C}$ satisfies $Q_t = 0$, for $t \geq 0$. Thus $(\tilde{H}, \tilde{C}) \in \mathscr{A}$.

Let now $(\widehat{C}, \widehat{H}) \in \mathscr{A}$ be such that $\mathcal{J}(\widehat{C}, \widehat{H}) = \max_{(C,H)\in\mathscr{A}} \mathcal{J}(C, H)$.
We fix $s \geq 0$ and $i \in \mathcal{I}$. Let $\eta = \pm 1$, $0 < h < 1$, $A^s \in \mathcal{G}_s$, $\Delta^{(i,s)} := (\mathbf{1}_{\{i=k,s=t\}})_{k\in\mathcal{I},t\geq 0}$ and $\theta^{(i,s)} := \frac{1}{2}(1 \wedge \frac{W^i_s}{P^i_s(1+\kappa^i_s)})\hat{C}^i_s \wedge \hat{H}^i_s \wedge 1 > 0$. Set

$$\overline{C} := \widehat{C} + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\Delta^{(i,s)} \text{ and } \overline{H} := \widehat{H} + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\Delta^{(i,s)}\frac{P^i(1+\kappa^i\delta_s)}{W^i}.$$

We observe that for $(j,t) \neq (i,s)$, $\overline{C}^j_t = \widehat{C}^j_t$ and $\overline{H}^j_t = \widehat{H}^j_t$ and we compute

$$\overline{C}^i_s \geq \widehat{C}^i_s - \theta^{(i,s)} \geq \frac{1}{2}\widehat{C}^i_s > 0.$$

Similarly, we obtain $\overline{H}^i_s > 0$. We also observe that $\overline{C} \leq \frac{3}{2}\widehat{C}$ and $\overline{H} \leq \frac{3}{2}\widehat{H}$. Finally, we have that

$$\sum_{j\in\mathcal{I}} W^j_t\overline{H}^j_t - \sum_{j\in\mathcal{I}} P^j_t(1+\kappa^j_t\delta_t)\overline{C}^j_t = \sum_{j\in\mathcal{I}} W^j_t\widehat{H}^j_t - \sum_{j\in\mathcal{I}} P^j_t(1+\kappa^j_t\delta_t)\widehat{C}^j_t.$$

This allows us to conclude that $(\overline{C}, \overline{H}) \in \mathscr{A}$.

We have, by optimality of $(\widehat{C}, \widehat{H})$,

$$\mathcal{J}(\widehat{C}, \widehat{H}) - \mathcal{J}(\overline{C}, \overline{H}) = \sum_{j\in\mathcal{I}} \mathcal{J}_j(\widehat{C}^j, \widehat{H}^j) - \sum_{j\in\mathcal{I}} \mathcal{J}_j(\overline{C}^j, \overline{H}^j) \geq 0.$$

However, for all $(t,j) \neq (s,i)$, $\overline{C}^j_t = \widehat{C}^j_t$ and $\overline{H}^j_t = \widehat{H}^j_t$, then

$$\mathbb{E}\left[\beta^s U(\widehat{C}^i_s, \widehat{H}^i_s)\right] - \mathbb{E}\left[\beta^s U\left(\widehat{C}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}, \widehat{H}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\right)\right] \geq 0,$$

i.e.

$$\frac{1}{h}\mathbb{E}\left[U(\widehat{C}^i_s, \widehat{H}^i_s) - U\left(\widehat{C}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}, \widehat{H}^i_s + \eta h\theta^{(i,s)}\mathbf{1}_{A^s}\frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\right)\right] \geq 0.$$

Letting h tend to 0, we obtain

$$\mathbb{E}\left[\eta\theta^{(i,s)}\mathbf{1}_{A^s}\frac{\partial U}{\partial x}(\widehat{C}^i_s, \widehat{H}^i_s) + \eta\theta^{(i,s)}\mathbf{1}_{A^s}\frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\frac{\partial U}{\partial y}(\widehat{C}^i_s, \widehat{H}^i_s)\right] \geq 0.$$

Since the above holds for all $A^s \in \mathcal{G}_s$, $\eta = \pm 1$ and since $\theta^{(i,s)} > 0$, then

$$\frac{\partial U}{\partial x}(\widehat{C}^i_s, \widehat{H}^i_s) + \frac{P^i_s(1+\kappa^i_s\delta_s)}{W^i_s}\frac{\partial U}{\partial y}(\widehat{C}^i_s, \widehat{H}^i_s) = 0,$$

leading to (C.2.6a).
For $j \in \mathcal{I} \setminus \{i\}$ and $\theta^{(i,j,s)} := \frac{1}{2}\left(1 \wedge \frac{P^j_s(1+\kappa^j_s\delta_s)}{P^i_s(1+\kappa^i_s\delta_s)}\right)(1 \wedge \widehat{C}^i_s \wedge \widehat{C}^j_s) > 0$, setting now

$$\overline{C} := \widehat{C} + \eta h \mathbf{1}_{A^s}\theta^{(i,j,s)}\left(\Delta^{(i,s)} - \Delta^{(j,s)}\frac{P^i(1+\kappa^i\delta_s)}{P^j(1+\kappa^j\delta_s)}\right) \quad \text{and} \quad \overline{H} := \widehat{H},$$

and using similar arguments as above, we obtain (C.2.6b).
When $\sigma = 1$, we carry out an analogous proof. □

Markets equilibrium

We now consider that firms and households interact on the labor and goods markets.

Definition C.2.5. A market equilibrium is a $\mathbb{G}$-adapted positive random process $(\overline{W}, \overline{P})$ such that

1. Condition (C.2.4) holds true for $(\overline{W}, \overline{P})$.
2. The goods' and labor's market clearing conditions are met, namely, for each sector $i \in \mathcal{I}$, and for all $t \geq 0$,

$$Y^i_t = C^i_t + \sum_{j\in\mathcal{I}} Z^{ij}_t \qquad \text{and} \qquad H^i_t = N^i_t,$$

where $N_t = \overline{n}(A_t, \overline{W}_t, \overline{P}_t, \kappa_t, \zeta_t)$, $Z_t = \overline{z}(A_t, \overline{W}_t, \overline{P}_t, \kappa_t, \zeta_t)$, $Y = F_A(N, Z)$ with $(\overline{n}, \overline{z})$ an admissible solution (C.2.3a)-(C.2.3b) to (C.2.1), from Proposition C.2.3 while C and H satisfy (C.2.6a)-(C.2.6b) for $(\overline{W}, \overline{P})$.

In the case of the existence of a market equilibrium, we can derive equations that must be satisfied by the output production process Y and the consumption process C.

Proposition C.2.6. *Assume that there exists a market equilibrium as in Definition C.2.5. Then, for $t \geq 0$, $i \in \mathcal{I}$, it must hold that*

$$\begin{cases} Y^i_t &= C^i_t + \sum_{j\in\mathcal{I}} \Lambda^{ij}(\mathfrak{d}_t)\left(\frac{C^j_t}{C^i_t}\right)^{-\sigma} Y^j_t, \\ Y^i_t &= A^i_t\left[\Psi^i(\mathfrak{d}_t)(C^i_t)^{-\sigma}Y^i_t\right]^{\frac{\psi^i}{1+\varphi}} \prod_{j\in\mathcal{I}}\left[\Lambda^{ji}(\mathfrak{d}_t)\left(\frac{C^i_t}{C^j_t}\right)^{-\sigma} Y^i_t\right]^{\boldsymbol{\lambda}^{ji}}, \end{cases} \tag{C.2.7}$$

where Ψ and Λ are defined in (5.2.7)*, and $\mathfrak{d}_t$ is defined in* (5.2.4).

Proof. Let $i, j \in \mathcal{I}$ and $t \geq 0$. Combining Proposition C.2.3 and Proposition C.2.4, we obtain

$$Z_t^{ji} = \boldsymbol{\lambda}^{ji} \frac{1 - \tau_t^i \delta_t}{1 + \zeta_t^{ji} \delta_t} \frac{1 + \kappa_t^j \delta_t}{1 + \kappa_t^i \delta_t} \left(\frac{C_t^i}{C_t^j} \right)^{-\sigma} Y_t^i. \tag{C.2.8}$$

From Propositions C.2.3 and C.2.4 again, we also have

$$N_t^i = \psi^i \frac{1 - \tau_t^i \delta_t}{1 + \kappa_t^i \delta_t} (H_t^i)^{-\varphi} (C_t^i)^{-\sigma} Y_t^i.$$

The labor market clearing condition in Definition C.2.5 yields

$$N_t^i = \left[\psi^i \frac{1 - \tau_t^i \delta_t}{1 + \kappa_t^i \delta_t} (C_t^i)^{-\sigma} Y_t^i \right]^{\frac{1}{1+\varphi}}. \tag{C.2.9}$$

Then, by inserting the expression of N_t^i given in (C.2.9)and Z_t^{ji} given in (C.2.8) into the production function F, we obtain the second equation in (C.2.7). The first equation in (C.2.7) is obtained by combining the market clearing condition with (C.2.8) (at index (i, j) instead of (j, i)). □

Output and consumption dynamics and associated growth

For each time $t \geq 0$ and noise realization, the system (C.2.7) is nonlinear with $2I$ equations and $2I$ variables, and its well-posedness is hence relatively involved. Moreover, it is computationally heavy to solve this system for each price trajectory and productivity scenario. We thus consider a special value for the parameter σ which allows to derive a unique solution in closed form. From now on, and following [Golosov et al., 2014, page 63], we assume that $\sigma = 1$, namely $U(x, y) := \log(x) - \frac{y^{1+\varphi}}{1+\varphi}$ on $(0, \infty)^2$.

Theorem C.2.7. *Assume that*

1. $\sigma = 1$,
2. $\mathbf{I}_I - \boldsymbol{\lambda}$ *is not singular,*
3. $\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top$ *is not singular for all* $t \in \mathbb{R}_+$.

Then for all $t \geq 0$, *there exists a unique* (C_t, Y_t) *satisfying* (C.2.7). *Moreover, with* $\mathfrak{e}_t^i := \frac{Y_t^i}{C_t^i}$ *for* $i \in \mathcal{I}$, *we have*

$$\mathfrak{e}_t = \mathfrak{e}(\mathfrak{d}_t) := (\mathbf{I}_I - \Lambda(\mathfrak{d}_t)^\top)^{-1} \mathbf{1}, \tag{C.2.10}$$

and using $\mathcal{B}_t = (\mathcal{B}_t^i)_{i \in \mathcal{I}} := [\mathcal{A}_t^i + \mathfrak{v}^i(\mathfrak{d}_t)]_{i \in \mathcal{I}}$ *with*

$$\mathfrak{v}^i(\mathfrak{d}_t) := \log \left((\mathfrak{e}_t^i)^{-\frac{\varphi \psi^i}{1+\varphi}} \left(\Psi^i(\mathfrak{d}_t) \right)^{\frac{\psi^i}{1+\varphi}} \prod_{j \in \mathcal{I}} \left(\Lambda^{ji}(\mathfrak{d}_t) \right)^{\boldsymbol{\lambda}^{ji}} \right), \tag{C.2.11}$$

we obtain

$$C_t = \exp\left((\mathbf{I}_I - \boldsymbol{\lambda})^{-1} \mathcal{B}_t \right). \tag{C.2.12}$$

Proof. Let $t \geq 0$. When $\sigma = 1$, the system (C.2.7) becomes for all $i \in \mathcal{I}$,

$$\begin{cases} Y_t^i &= C_t^i + \sum_{j\in\mathcal{I}} \Lambda^{ij}(\mathfrak{d}_t) \left(\frac{C_t^i}{C_t^j}\right) Y_t^j, \\ Y_t^i &= A_t^i \left[\Psi^i(\mathfrak{d}_t)\mathfrak{e}_t^i\right]^{\frac{\psi^i}{1+\varphi}} \prod_{j\in\mathcal{I}} \left[\Lambda^{ji}(\mathfrak{d}_t) C_t^j \mathfrak{e}_t^i\right]^{\boldsymbol{\lambda}^{ji}}. \end{cases} \tag{C.2.13}$$

For any $i \in \mathcal{I}$, dividing the first equation in (C.2.13) by C_t^i, we get

$$\mathfrak{e}_t^i = 1 + \sum_{j\in\mathcal{I}} \Lambda^{ij}(\mathfrak{d}_t)\mathfrak{e}_t^j,$$

which corresponds to (C.2.10), thanks to (5.2.6). Using $\sum_{j\in\mathcal{I}} \boldsymbol{\lambda}^{ji} = 1 - \psi^i$ and $Y_t^i = \mathfrak{e}_t^i C_t^i$ in the second equation in (C.2.13), we compute

$$C_t^i = A_t^i (\mathfrak{e}_t^i)^{-\frac{\varphi\psi^i}{1+\varphi}} \left[\Psi^i(\mathfrak{d}_t)\right]^{\frac{\psi^i}{1+\varphi}} \prod_{j\in\mathcal{I}} \left[\Lambda^{ji}(\mathfrak{d}_t)\right]^{\boldsymbol{\lambda}^{ji}} \prod_{j\in\mathcal{I}} (C_t^j)^{\boldsymbol{\lambda}^{ji}}.$$

Applying log and writing in matrix form, we obtain $(\mathbf{I}_I - \boldsymbol{\lambda}) \log(C_t) = \mathcal{B}_t$, implying (C.2.12). □

Remark C.2.8. The matrix $\boldsymbol{\lambda}$ is generally not diagonal, and therefore, from (C.2.12), the sectors (in output and in consumption) are linked to each other through their respective productivity process. Similarly, an introduction of price in one sector affects the other ones.

Remark C.2.9. For any $t \geq 0$, $i \in \mathcal{I}$, we observe that

$$\mathcal{B}_t^i = \mathcal{A}_t^i + v^i(\mathfrak{d}_t), \tag{C.2.14}$$

where $v^i(\cdot)$ is defined using (C.2.11). Namely, $\mathcal{B}_t$ is the sum of the (random) productivity term and a term involving the price. The economy is therefore subject to fluctuations of two different natures: *the first one comes from the productivity process while the second one comes from the price processes.*

We now look at the dynamics of production and consumption growth.

Theorem C.2.10. *For any $t \geq 0$ and for $\varpi \in \{Y, C\}$. With the same assumptions as in Theorem C.2.7,*

$$\mathrm{d}\log \varpi_t \sim \mathcal{N}\left(m_t^\varpi, \widehat{\Sigma}_t\right), \qquad \textit{for } \varpi \in \{Y, C\},$$

with

$$\begin{aligned} \widehat{\Sigma}_t &= \varsigma^2 (\mathbf{I}_I - \boldsymbol{\lambda})^{-1} \overline{\Sigma} (\mathbf{I}_I - \boldsymbol{\lambda}^\top)^{-1} (\mathrm{d}t)^2, \\ m_t^C &= (I - \boldsymbol{\lambda})^{-1} \left[\mu \mathrm{d}t + \mathrm{d}\mathfrak{v}(\mathfrak{d}_t)\right], \\ m_t^Y &= (I - \boldsymbol{\lambda})^{-1} \left[\mu \mathrm{d}t + \mathrm{d}\mathfrak{v}(\mathfrak{d}_t)\right], \end{aligned}$$

and

$$v(\mathfrak{d}_t) := \mathfrak{v}(\mathfrak{d}_t) + (\mathbf{I}_I - \boldsymbol{\lambda}) \log(\mathfrak{e}(\mathfrak{d}_t)),$$

where $\overline{\mu}$ *and* $\varsigma^2\overline{\Sigma}$ *are the mean and the variance of the stationary process* $\mathcal{Z}$ *(Remark 5.2.2),* v *is defined in* (C.2.11) *and* $\mathfrak{e}$ *in* (C.2.10).

Proof. Let $t \geq 0^*$, from (C.2.14), we have, for $i \in \mathcal{I}$,

$$\mathrm{d}\mathcal{B}_t^i = (\mu^i + \varsigma \mathcal{Z}_t^i)\mathrm{d}t + \mathrm{d}v^i(\mathfrak{d}_t).$$

Combining the previous equality with (C.2.12), we get

$$\mathrm{d}\log C_t = (\mathbf{I}_I - \boldsymbol{\lambda})^{-1} \left[(\mu + \varsigma \mathcal{Z}_t)\mathrm{d}t + \mathrm{d}v(\mathfrak{d}_t)\right].$$

Applying Remark 5.2.2 leads to $\mathrm{d}\log C_t \sim \mathcal{N}\left(m_t^C, \widehat{\Sigma}_t\right)$. Using (C.2.10), we observe that, for $i \in \mathcal{I}$,

$$(\mathrm{d}\log Y_t)^i = (\mathrm{d}\log C_t)^i + \mathrm{d}\log(\mathfrak{e}^i(\mathfrak{d}_t)),$$

which, using the previous characterization of the law of $\mathrm{d}\log C_t$, allows to conclude. □

From the previous result, we observe that output and consumption growth processes have a stationary variance but a time-dependent mean.

Proof. of Proposition 5.2.6.

Let $t \geq 0$, $n \in \{1, \ldots, N\}$, and $T > t_\star$.

1. we also introduce,

$$\mathcal{V}_{t,\mathfrak{d}}^{n,K} := F_{t,\mathfrak{d}}^n \int_t^{+\infty} e^{-r(s-t)} \mathbb{E}_t \left[\exp\left((s-t)\mathfrak{a}^{n\cdot}\mu + \mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) + \sigma_n(\mathcal{W}_s^n - \mathcal{W}_t^n)\right)\mathrm{d}s\right].$$

Similar computations as (in fact easier than) the ones performed in the proof of Proposition 4.1.4 show that $\mathcal{V}_{t,\mathfrak{d}}^n = \lim_{K\to+\infty} \mathcal{V}_t^{n,K}$ is well defined in $\mathcal{L}^q(\mathcal{H}, \mathbb{E})$ for any $q \geq 1$. Furthermore,

$$\begin{aligned}\mathcal{V}_{t,\mathfrak{d}}^{n,K} &= F_{t,\mathfrak{d}}^n \int_{s=0}^K e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_{t+s}) - v(\mathfrak{d}_t)\right)\right)\mathrm{d}s \\ &= F_{t,\mathfrak{d}}^n e^{-\mathfrak{a}^{n\cdot}v(\mathfrak{d}_t)} \int_{s=0}^K e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot}v(\mathfrak{d}_{t+s})\right)\mathrm{d}s,\end{aligned}$$

where ϱ_n is defined in the lemma, and from Assumption 5.2.5 and Corollary 5.2.4,

$$\begin{aligned}F_t^n &= F_0^n \exp\left(\int_{u=0}^t \mathfrak{a}^{n\cdot}(\Theta_u \mathrm{d}u + \mathrm{d}v(\mathfrak{d}_u)) + \sigma_n \mathrm{d}\mathcal{W}_t^n \mathrm{d}u\right) \\ &= F_0^n e^{\mathfrak{a}^{n\cdot}(v(\mathfrak{d}_t) - v(\mathfrak{d}_0))} \exp\left(\mathfrak{a}^{n\cdot}\mathcal{A}_t^\circ + \sigma_n \mathcal{W}_t^n\right).\end{aligned}$$

We then have

$$F^n_{t,\mathfrak{d}} e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_t)} \int_{s=0}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s$$
$$= F^n_0 e^{-\mathfrak{a}^{n\cdot} v(\mathfrak{d}_0)} \exp\left(\mathfrak{a}^{n\cdot} \mathcal{A}^\circ_t + \sigma_n \mathcal{W}^n_t\right) \int_{s=0}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s.$$

2. Moreover,

- If $t < t_\circ$, then

$$\begin{aligned}
\mathfrak{R}^{n,K}_t(\mathfrak{d}) &:= \int_{s=0}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s \\
&= \int_{s=0}^{t_\circ - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s + \int_{s=t_\circ - t}^{t_\star - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s \\
&\quad + \int_{s=t_\star - t}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s \\
&= e^{\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\circ})} \frac{1 - e^{\varrho_n (t_\circ - t)}}{-\varrho_n} + \int_{s=t_\circ - t}^{t_\star - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s \\
&\quad + e^{\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\star}) + \varrho_n (t_\star - t)} \frac{1 - e^{\varrho_n (K - t_\star + t)}}{-\varrho_n}.
\end{aligned}$$

- If $t_\circ \leq t < t_\star$, then

$$\begin{aligned}
&\int_{s=0}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s \\
&\quad = \int_{s=0}^{t_\star - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s + \int_{s=t_\star - t+1}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s \\
&\quad = \int_{s=0}^{t_\star - t} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s + e^{\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\star}) + \varrho_n (t_\star - t+1)} \frac{1 - e^{\varrho_n (K - t_\star + t)}}{-\varrho_n}.
\end{aligned}$$

- If $t \geq t_\star$, then

$$\int_{s=0}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t+s})\right) \mathrm{d}s = \int_{s=0}^{K} e^{\varrho_n s} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\star})\right) \mathrm{d}s = e^{\mathfrak{a}^{n\cdot} v(\mathfrak{d}_{t_\star})} \frac{1 - e^{\varrho_n (K+1)}}{-\varrho_n}.$$

Finally, $e^{\varrho_n (K+1)}$ and $e^{\varrho_n (K - t_\star + t)}$ converge to 0 for $\varrho_n < 0$ as K tends to infinity, and the result follows.

3. We denote

$$V^{n,T}_{t,\mathfrak{d}} := \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} F^n_{s,\mathfrak{d}} \mathrm{d}s \right].$$

As $F^n_{s,\mathfrak{d}} = F^n_{t,\mathfrak{d}} \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}_s - \mathcal{A}_t) + \mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) + \sigma_n(\mathcal{W}^n_s - \mathcal{W}^n_t)\right)$, we have from (5.2.10), then from C.1.3,

$$\begin{aligned}
V^{n,T}_{t,\mathfrak{d}} &= \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} F^n_t \exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}_s - \mathcal{A}_t) + \mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) + \sigma_n(\mathcal{W}^n_s - \mathcal{W}^n_t)\right) \mathrm{d}s \right] \\
&= F^n_{t,\mathfrak{d}} \int_t^T e^{\left(\frac{1}{2}\sigma_n^2 - r\right)(s-t)} \exp\left(\mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right)\right) \mathbb{E}_t\left[\exp\left(\mathfrak{a}^{n\cdot}(\mathcal{A}_s - \mathcal{A}_t)\right)\right] \mathrm{d}s \\
&= F^n_{t,\mathfrak{d}} \int_t^T e^{\left(\frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n\cdot}\mu - r\right)(s-t)} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) \exp\left(\varsigma \mathfrak{a}^{n\cdot} \Upsilon_{s-t} \mathcal{Z}_t + \frac{1}{2} \mathfrak{a}^{n\cdot} \Sigma^{\mathcal{A},h}_t (\mathfrak{a}^{n\cdot})^\top\right) \mathrm{d}s.
\end{aligned}$$

Then using Hölder's inequality (with $1 = \frac{1}{p} + \frac{1}{q}$), we have

$$\begin{aligned}
&\|V^{n,T}_{t,\mathfrak{d}}\|_1 \\
&\quad \leq \|F^n_{t,\mathfrak{d}}\|_q \left\| \int_t^T e^{\left(\frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n\cdot}\mu - r\right)(s-t)} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) \exp\left(\varsigma \mathfrak{a}^{n\cdot} \Upsilon_{s-t} \mathcal{Z}_t + \frac{1}{2}\mathfrak{a}^{n\cdot} \Sigma_t^{\mathcal{A},s-t} (\mathfrak{a}^{n\cdot})^\top\right) \mathrm{d}s \right\|_p \\
&\quad \leq \|F^n_{t,\mathfrak{d}}\|_q \int_t^T e^{\left(\frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n\cdot}\mu - r\right)(s-t)} \exp\left(\mathfrak{a}^{n\cdot} v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right) \exp\left(\frac{1}{2}\mathfrak{a}^{n\cdot} \Sigma_t^{\mathcal{A},s-t} (\mathfrak{a}^{n\cdot})^\top\right) \left\|\exp\left(\varsigma \mathfrak{a}^{n\cdot} \Upsilon_{s-t} \mathcal{Z}_t\right)\right\|_p \mathrm{d}s.
\end{aligned}$$

Observe that under Assumption 5.2.3, there exists a constant $\mathfrak{C}_{\mathfrak{d}} > 0$ such that

$$\sup_{n,s,t} \exp\left(\mathfrak{a}^{n\cdot}\left(v(\mathfrak{d}_s) - v(\mathfrak{d}_t)\right)\right) \leq \mathfrak{C}_{\mathfrak{d}}\,.$$

Given that $\mathcal{Z}$ is stationary and Υ_{s-t} is bounded ((C.1.1)), there exists $\mathfrak{C}_{n,p} > 0$ so that $\leq \mathfrak{C}_{n,p}$

$$\left\|\exp\left(\varsigma \mathfrak{a}^{n\cdot} \Upsilon_{s-t} \mathcal{Z}_t\right)\right\|_p = \mathbb{E}\left[\exp\left(\varsigma p \mathfrak{a}^{n\cdot} \Upsilon_{s-t} \mathcal{Z}_t\right)\right]^{\frac{1}{p}} \leq \mathfrak{C}_{n,p}.$$

Moreover,

$$\begin{aligned}
\exp\left(\frac{1}{2}\mathfrak{a}^{n\cdot} \Sigma_t^{\mathcal{A},h} (\mathfrak{a}^{n\cdot})^\top\right) &= \exp\left(\frac{1}{2}\varsigma^2 \int_0^{s-t} \mathfrak{a}^{n\cdot} \Upsilon_u \Sigma\Sigma^\top \Upsilon_u^\top (\mathfrak{a}^{n\cdot})^\top \mathrm{d}u\right) \\
&\leq \exp\left(\frac{1}{2}\varsigma^2 \int_0^{s-t} \|\mathfrak{a}^{n\cdot}\|^2 \|\Sigma\|^2 \|\Upsilon_u\|^2 \mathrm{d}u\right) \\
&\leq \exp\left(\frac{1}{2}\varsigma^2 \frac{c_\Gamma^2}{\lambda_\Gamma^2} \|\mathfrak{a}^{n\cdot}\|^2 \|\Sigma\|^2 (s-t)\right).
\end{aligned}$$

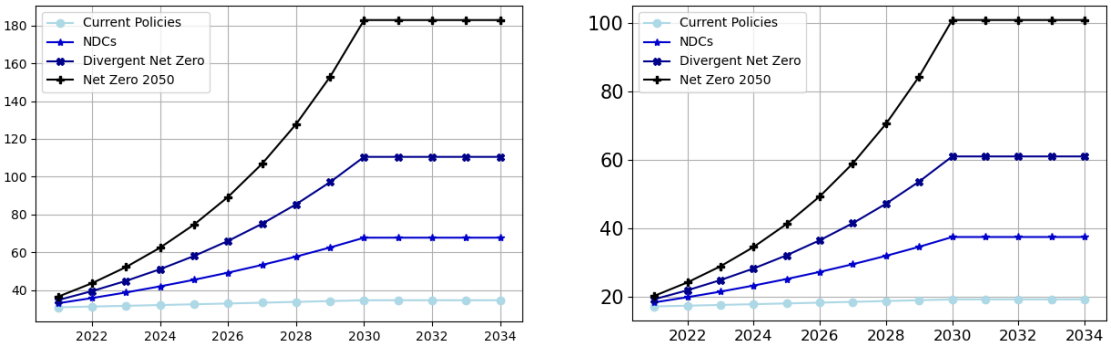
Next, we can write

$$\|V^{n,T}_{t,\mathfrak{d}}\|_1 \leq \mathfrak{C}_{\mathfrak{d}} \mathfrak{C}_{n,p} \|F^n_{t,\mathfrak{d}}\|_q \int_t^T \exp\left(\frac{1}{2}\sigma_n^2 + \mathfrak{a}^{n\cdot}\mu + \frac{1}{2}\varsigma^2 \frac{c_\Gamma^2}{\lambda_\Gamma^2} \|\mathfrak{a}^{n\cdot}\|^2 \|\Sigma\|^2 - r\right)(s-t)\mathrm{d}s,$$

and if (5.2.15) is satisfied and $T \to +\infty$, then $V^{n,K}_{t,\mathfrak{d}}$ converges to $V^n_{t,\mathfrak{d}}$. Finally, similar methods must be used to show $\mathbb{E}\left[\left|\frac{V^n_{t,\mathfrak{d}}}{F^n_{t,\mathfrak{d}}} - \frac{\mathcal{V}^n_{t,\mathfrak{d}}}{F^n_{t,\mathfrak{d}}}\right|\right] \leq C\varsigma$.

□

C.3 Figures



(a) Carbon price (b) Energy price

Figure C.1: per scenario and per year

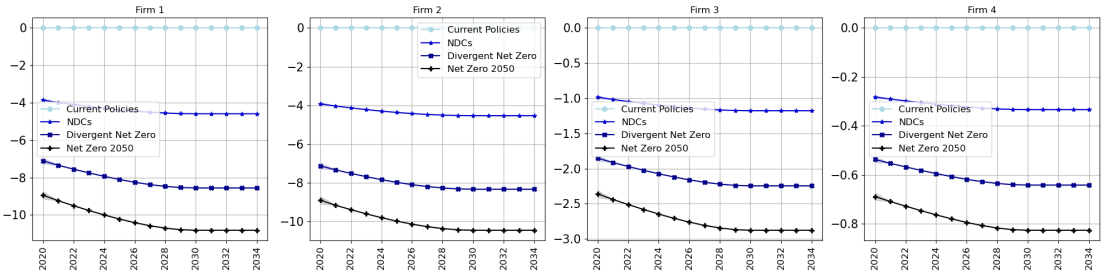


Figure C.2: Firm value slowdown per scenario and per year

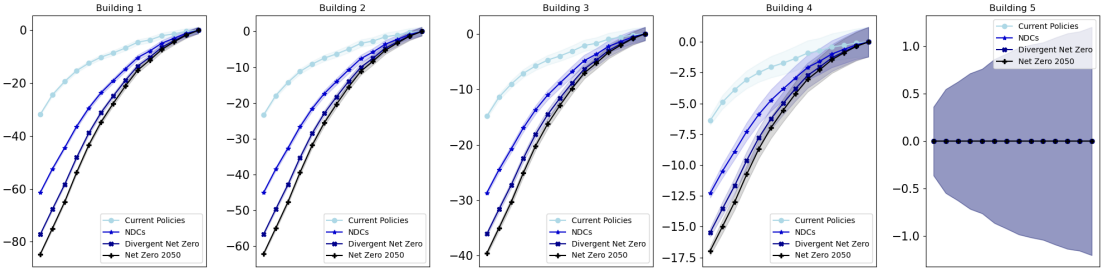


Figure C.3: Apartment value slowdown per scenario and per year