Index theory via deformation groupoids

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Index theory via deformation groupoids

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Index Theory via deformation groupoids

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Abstract

The goal of this memoir is to present and describe a handful of different index theorems and index related problems associated to different geometric situations. The common point of all the results presented here is the use of groupoids and more particularly deformation groupoids, and the use of algebraic topology tools (like K-theory, Cyclic (co)homology and others) to state and proof some analytic index problems. As I will try to settle down, there are many different situations that can be unified and treated in a relatively common setting by adopting the groupoid perspective. But beyond the interest of getting an unified treatment for several problems, I will show how groupoids and deformation groupoids can be used to get explicit original answers to complex problems in index theory.

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CHAPTER 1

Introduction

1.1. A little story about Index theory

The historical starting point of what it is called today "Index Theory" is the so called Atiyah-Singer index theorem that I will now explain. Given a compact smooth manifold M, an elliptic pseudodifferential operator D has finite dimensional Kernel and Cokernel. Under these circunstances the Fredholm index of D is defined as

$$Ind\, D := dim\, Ker\, D - dim\, Coker\, D \tag{1}$$

which is then an integer. By the 50's decade of the last century it was known by many mathematicians that these indices had very interesting properties (homotopy invariance, invariant under compact operator pertubation, etc.) and that for some very interesting examples, these indices were related with geometric/topologic invariants of the manifold. For example the famous Gauss-Bonnet theorem and the Hirzebruch signature theorem were a couple of known examples by end of the 50's. In fact, at the beggining of that decade, Gelfand asked the question of whether these indices admit a topological computation and hence a topological formula. The complete answer to this problem was given by Atiyah and Singer in the 60's, the Atiyah-Singer index theorem provided a fundamental link between differential geometry, partial differential equations, differential topology, operator algebras, and has links to many other fields. But beyond the index formula and the very numerous applications that Atiyah and Singer found, the techniques and tools that they developed are at the base of several modern mathematical branches. Of particular importance for our exposition is the development of K-theory (topological), indeed, one of the key observations of Atiyah and Singer is that the map $D \mapsto Ind\, D$ is entirely codified by a group morphism, called the analytic index morphism of M,

$$ind_a^M : K^0(T^*M) \to \mathbb{Z} \tag{2}$$

between K-theory groups. To be more precise, if $Ell(M)$ designs the set of elliptic pseudodifferential operators on M there is a commutative diagram

$$\begin{array}{ccc} Ell(M) & \xrightarrow{Ind} & \mathbb{Z} \\ {\scriptstyle\sigma}\downarrow & \nearrow_{ind_a^M} & \\ K^0(T^*M) & & \end{array} \tag{3}$$

where $\sigma : Ell(M) \to K^0(T^*M)$ associates to an elliptic operator its principal symbol class and where σ is a surjective map. In fact, Atiyah and Singer showed that $K^0(T^*M)$ is essentially a quotient of $Ell(M)$. Moreover, once the Fredholm index codified in a K-theory group morphism they used the entire machinerie of this generalized cohomology theory to get in particular a topological expression of the index. Even more, these series of tools/methods allowed Atiyah and Singer to go further into the study of higher index problems, for example, they completely solved the case of families of pseudodifferential elliptic operators, this time the indices were elements of the topological K-theory group of the paremeter space of the family, the topological index they got is a far generalization of the so called topological Riemann-Roch theorem. These seminal works appeared in a series of papers [**6**], [**7**], [**8**], [**9**], [**10**] in the 60's where they started as well different directions of index problems, for example for real operators, for the equivariant case under the action of a compact group, for mention some of them. Index theory was born.

Now, soon after these seminal papers, Index theory grew fast in several different directions. For example for manifolds with boundary by Atiyah and Bott ([**2**]), and by Atiyah, Patodi and Singer ([**3**],[**4**],[**5**]); for regular foliations by Connes, and by Connes and Skandalis ([**42**]), for homogeneous spaces by Atiyah ([**11**]) and by Connes and Moscovici [**41**], for discrete groups by Baum and Connes ([**13**],[**16**]); just to mention some directions and some authors.

In many of the different directions that emerged from the original papers it was clear that the use of classic spaces and classic tools for them were not enough to solve some of the index problems or even for properly state these problems. The developement of C^*-algebras and K-theory for C^*-algebras was crucial to understand new phenomena, related problems and to get some answers in some cases. Also, the development of Connes noncommutative geometry gave a completely new perspective that allowed to deal with geometric situations that were before inaccesible with classic methods, in particular in Connes NCG the use and further developments of groupoids was essential to do index theory for groupoids and with groupoids as a tool.

The goal of this memoir is to present and describe a handful of different index theorems and index related problems associated to different geometric situations. The common point of all the results presented here is the use of groupoids and more particularly deformation groupoids, and the use of algebraic topology tools (like K-theory, Cyclic (co)homology and others) to state and proof some analytic index problems. As I will try to settle down, there are many different situations that can be unified and treated in a relatively common setting by adopting the groupoid perspective. But beyond the interest of getting an unified treatment for several problems, I will show how groupoids and deformation groupoids can be used to get explicit original answers to complex problems in index theory.

I proceed now to resume the contents of each chapter:

1.2. Topological Index theorems using groupoids

In chapter 2, I will start by stating and proving the classic Atiyah-Singer Index formula by means of deformation groupoids. This proof goes back to Connes where he introduced the tangent groupoid, a particular deformation groupoid called now a days the Connes tangent groupoid. As I will try to explain this proof not only gives a conceptually simple proof of the mentioned formula but it extends directly to other cases, I will mention in particular the so called families Index theorem (and families index formula) by Atiyah and Singer, and the longitudinal Index theorem by Connes and Skandalis for regular foliations. I will use these famous theorems to motivate part of my own work that I will start explaining in the next chapter. A small original contribution to the Connes proof is that I explain and give a proof that the Connes-Thom isormorphism that appears in Connes proof is actually a usual Thom isomorphism (up to some Fourier isomorphism). This last part was and is highly accepted among experts but as far as I am aware there is no written proof of it, I hence added this proof which is a part of an ongoing work in collaboration with Jean-Marie Lescure and that we think it will be useful for many different situations.

1.3. Index theory for manifolds with corners

In chapter 3, I discuss my works on index theory for manifolds with corners. I start by introducing the appropriate groupoids. In fact, in this setting, by using good Lie groupoids one can approach some index theoretical problems as in the smooth case of Atiyah-Singer by following the Connes approach. Hence, after introducing these groupoids and some associated index morphisms relevant for the index theory in this context, I resume, in section 3.2, my paper [**29**] in collaboration with Jean-Marie Lescure and Bertrand Monthubert in which, for the case of a manifold with boundary (baby case of a manifold with corners), we were able to apply the same kind of Connes approach explained in the previous section in order to get a cohomological formula for the Fredholm index of a fully elliptic b-pseudodifferential operator. In section 3.3, I explain and resume my papers [**33**] and [**34**] in collaboration with Jean-Marie Lescure and Mario Velasquez, in which we computed the K-theory group of the algebra of b-compact operators associated to a general manifold with corners in terms of very computable homology groups associated to the manifold with corners. As I recall, these computations are relevant to understand the global geometric/topologic obstructions to the Fredholm Perturbartion property of a given b-elliptic operator.

1.4. Wrong way functoriality and Index theory

In chapter 4, I will discuss the link between the wrong way functoriality constructions in K-theory and Index theory. I will explain in particular how to use geometric deformation groupoids to realize such wrong way maps and how to use them to show their main property, the functorality. As I will explain this

functoriality is equivalent when it makes sense to a topological index computation. One of the main interests is that this wrong way functoriality may be applied in different situations in which one does not have necessarilly topological index maps. In this section I will focus in the case of wrong way maps associated to maps between smooth manifolds under some appropriate action of a Lie groupoid, the proof of the wrong way functoriality in this context was done, using deformation groupoids, in my paper [**35**] in collaboration with Bai-Ling Wang. In fact, in that paper we proved the functoriality in the more general context of twisted Lie groupoids, I won't enter on the twisted setting in the present work. As we also showed in ref.cit., the functoriality allows to define a geometric group by assembling the different wrong way maps and to define, geometrically, the so called Baum-Connes assembly map that goes from this geometric group to the K-theory of the C^*-algebra associated to the Lie groupoid, I will also explain this in this chapter and its links with index theory. In the last section I illustrate, by an example, how one can add more "singular cycles" to the definition of the geometric cycles forming the left hand side of the Baum-Connes assembly map. For example, thanks again to the use of deformation groupoids, one can add orbifolds cycles in a very canonical way.

1.5. Topological K-theory for discrete groups and Index theory

In chapter 5, I detail the content of my paper [**30**] in collaboration with Bai-Ling Wang and Hang Wang where we studied further the wrong way functoriality as the last chapter but for the case of discrete groups. Indeed, in this very interesting case, thanks to previous well known results, we can go further in the program of computing index formulas. In resumé, for any countable discrete group Γ (without any further assumptions on it) we first construct an explicit morphism from the Left-Hand side of the Baum-Connes assembly map, $K^*_{top}(\Gamma)$, to the periodic cyclic homology of the group algebra, $HP_*(\mathbb{C}\Gamma)$. This morphism, called the Chern-Baum-Connes assembly map, allows in particular to give a proper and explicit formulation for a Chern-Connes pairing

$$K^*_{top}(\Gamma) \times HP^*(\mathbb{C}\Gamma) \longrightarrow \mathbb{C}.$$

As part of our results we prove that this model for the Baum-Connes left hand side is indeed isomorphic to the analytic model for the left-hand side of the assembly map. Several theorems are needed to formulate the Chern-Baum-Connes assembly map. In particular we establish a delocalised Riemann-Roch theorem, the wrong way functoriality for periodic delocalised cohomology for Γ-proper actions, the construction of a Chern morphism between the Left-Hand side of Baum-Connes and a delocalised cohomology group associated to Γ which is an isomorphism once tensoring with $\mathbb{C}$, and the construction of an explicit cohomological assembly map between the delocalised cohomology group associated to Γ and the homology group $H_*(\Gamma, F\Gamma)$ (where $F\Gamma$ is the complex vector space freely generated by the set of elliptic elements in Γ). This last group $H_*(\Gamma, F\Gamma)$ identifies, by the work of Burghelea, as a direct factor of the cyclic periodic homology

of the group algebra. As part of our results we prove that this model is indeed isomorphic to the analytic model for the left-hand side of the assembly map. We conclude this chapter by recalling how to give an explicit index theoretical formula for the above mentioned pairing (without any further assumptions on Γ) in terms of pairings of invariant forms, associated to geometric cycles and given in terms of delocalized Chern and Todd classes, and currents naturally associated to group cocycles using Burghelea's computation. This gives a complete solution, for discrete countable groups, to the problem of defining and computing a geometric pairing between the left hand side of the Baum-Connes assembly map, given in terms of geometric cycles associated to proper actions of the discrete group on manifolds, and cyclic periodic cohomology of the group algebra.

1.6. The Chern-Baum-Connes assembly map for Lie groupoids

In section 6, I explain some of the results that are possible to generalize from the previous section to more general Lie groupoids, this is mainly the content of my paper [**28**]. In resumé, for a Lie groupoid $\mathscr{G}$, for which there are Thom isomorphisms in periodic cyclic (co)homology for appropriate $\mathscr{G}$-vector bundles, we construct a morphism from the left hand side of the Baum-Connes asssembly map of $\mathscr{G}$ to the periodic cyclic homology group of the Lie groupoid convolution algebra,

$$K^*_{top}(\mathscr{G}) \xrightarrow{Ch_{BC}} HP_*(C^\infty_c(\mathscr{G})).$$

This morphism, that we call here the Chern-Baum-Connes assembly map (CBC-map), is realized by assembling, via geometric pushforward maps in periodic cyclic homology, Chern-Connes character morphisms associated to $\mathscr{G}$-proper $spin^c$ cocompact manifolds together with an explicit cohomological assembly map in periodic cyclic homology. The main result behind is the wrong way functoriality of the pushforward morphisms in periodic cyclic homology. Next, we show that the CBC-map factors as the Baum-Connes assembly map followed by the Chern-Connes character for $\mathscr{G}$ if this later morphism extends to the C^*-algebra K-theory. More generally we show that if for a given periodic cyclic cocycle on $C^\infty_c(\mathscr{G})$ the associated pairing extends to the $K-$theory of the groupoid C^*-algebra then the composition of the Baum-Connes assembly map followed by the associated extended morphism coincides with the Connes pairing of the given periodic cyclic cycle with the Chern-Baum-Connes assembly map.

CHAPTER 2

Topological index theorems using groupoids

2.1. Classic Atiyah-Singer à la Connes

I will now explain how to proof the classic Atiyah-Singer Index theorem using groupoids and deformation groupoids, this will be the main motivation for explaining later how to use deformation groupoid techniques in other related problems and results in geometrical analysis and in algebraic topology, and in particular how it has been the main starting and guiding point in my own research. The statement of Atiyah and Singer's index theorem is as follows.

THEOREM 2.1.1 (Atiyah-Singer index formula). *Let M be a closed smooth manifold and let D be an elliptic pseudodifferential operator on M (acting between sections of smooth vector bundles over M). Then the following equality holds*

$$Ind_a(D) = \int_{T^*M} ch([\sigma_D]) \wedge Td_{\mathbb{C}}(M). \tag{4}$$

I will now recall and explain some of the ingredients in the last statement in order to get to the proof.

(i) **The analytic index morphism as a deformation index using the Connes tangent groupoid.** In the statement above $Ind_a(D) := dim\, Ker\, D - dim\, Coker\, D \in \mathbb{Z}$ is the Fredholm index of the elliptic operator D. In fact, as shown by Atiyah-Singer, this index only depends on the principal symbol class $[\sigma_D] \in K^0(T^*M)$ and it is computed by the image of the so called analytic index morphism associated to M:

$$Ind_a^M : K^0(T^*M) \longrightarrow \mathbb{Z}. \tag{5}$$

I will now recall/explain how to interpret this morphism by using the tangent groupoid of M, a particular case of a deformation groupoid introduced by Connes in the early 80's.

The Connes tangent groupoid of a smooth manifold M is, as we will see below, a particular example of a deformation Lie groupoid.

$$G_M^{tan} := TM \bigsqcup M \times M \times (0,1] \rightrightarrows M \times [0,1] \tag{6}$$

One of the main features of this groupoid C^*-algebra is that it fits in a short exact sequence

$$0 \longrightarrow C^*(M \times M \times (0,1]) \xrightarrow{i_0} C^*(G_M^{tan}) \xrightarrow{e_0} C^*(TM) \longrightarrow 0 \tag{7}$$

where e_0 is the morphism induced by the restriction of G_M^{tan} to the closed subgroupoid $TM \rightrightarrows M$ and where i_0 is the morphism induced by the inclusion in G_M^{tan} of the open dense saturated subgroupoid $M \times M \times (0,1] \rightrightarrows M \times (0,1]$. Now, since the algebra $C^*(M \times M \times (0,1])$ is easily seen to be contractible, because of the presence of the interval $(0,1]$, and since the $K-$theory of C^*-algebras has good cohomological properties (in particular homotopy invariance and long term exact sequence), one obtains that the morphism in K-theory induced by e_0 (and that we denote the same to avoid too much notation),

$$K_*(C^*(G_M^{tan})) \xrightarrow[\cong]{e_0} K_*(C^*(TM)), \tag{8}$$

is an isomorphism. By considering the morphism induced by the restriction of G_M^{tan} to the closed subgroupoid $M \times M \times \{1\} \rightrightarrows M$ we can state the following classic but fundamental result whose proof is a particular case of proposition B.2.1 shown in the general case of Lie groupoids in the appendix B.2.

THEOREM 2.1.2. *For M a compact manifold, we have that the following equality*

$$Ind_a^M = \mathcal{M} \circ e_1 \circ (e_0)^{-1} \circ \mathcal{F} \tag{9}$$

as morphisms from $K^*(T^*M)$ to $\mathbb{Z}$, where $\mathcal{F} : K^*(T^*M) \cong K_*(C^*(TM))$ is the isomorphism induced by the C^*-algebra isomorphism $C_0(T^*M) \cong C^*(TM)$ given by the fiberwise Fourier transform and where $\mathcal{M} : K_*(C^*(M \times M)) \cong \mathbb{Z}$ is the isomorphism induced from the Morita equivalence of C^*-algebras $C^*(M \times M)$ and $\mathbb{C}$. In other words the following diagram is commutative

$$\begin{array}{ccccc} K^*(T^*M) & & \xrightarrow{\quad Ind_a^M \quad} & & \mathbb{Z} \\ \Big\downarrow{\scriptstyle \mathcal{F}}\ {\scriptstyle \cong} & & & & \Big\uparrow{\scriptstyle \cong}\ {\scriptstyle \mathcal{M}} \\ K_*(C^*(TM)) & \xleftarrow[e_0]{\cong} & K_*(C^*(G_M^{tan})) & \xrightarrow[e_1]{} & K_*(C^*(M \times M)) \end{array} \tag{10}$$

(ii) **The Connes-Thom topological index morphism using a groupoid cocycle.**

Let $j : M \hookrightarrow \mathbb{R}^n$ be a smooth embedding with n even. Consider the following groupoid morphism

$$M \times M \xrightarrow{h} \mathbb{R}^n \tag{11}$$

given by $h(x,y) = j(y) - j(x)$, where $\mathbb{R}^n$ is seen here as an additive group. By functoriality of the deformation to the normal cone construction, that gives as a particular example the tangent groupoid, the

groupoid morphism

$$G_M^{tan} \xrightarrow{h^{tan}} \mathbb{R}^n \tag{12}$$

given by $h^{tan}(x,y,t) = th(x,y)$ for $t \neq 0$ and by $h^{tan}(x,V) = d_x j(V)$ for $V \in T_xM$, is a Lie groupoid morphism. It is direct exercise to show that the associated semi-direct product groupoid is free and proper. In particular this semi-direct product groupoid

$$G_M^{tan} \rtimes \mathbb{R}^n \rightrightarrows M \times [0,1] \times \mathbb{R}^n \tag{13}$$

is Morita equivalent, as Lie groupoids, to its orbit space $Orb(G_M^{tan})$ seen as a unit Lie groupoid $Orb(G_M^{tan}) \rightrightarrows Orb(G_M^{tan})$. Now, a direct computation gives a well known expression for this orbit space, indeed there is a canonical diffeomorphism

$$Orb(G_M^{tan}) \cong D(\mathbb{R}^n, M) \tag{14}$$

where

$$D(\mathbb{R}^n, M) = N(\mathbb{R}^n, M) \bigsqcup \mathbb{R}^n \times (0,1] \tag{15}$$

is the deformation to the normal cone of the embedding j.

Now, for any groupoid morphism $h : G \to \mathbb{R}^n$ there is a natural isomorphism

$$\mathcal{T} : K_*(C^*(G)) \to K_*(C^*(G \rtimes \mathbb{R}^n)) \tag{16}$$

called the Connes-Thom isomorphism and constructed for the first time by Connes in [**38**], in a more general setting, and revisited for groupoids in [**29**] using explicit deformation groupoids.

In particular we have the following, trivially commutative, diagram

(17)

$$\begin{array}{ccccc}
K_*(C^*(TM)) & \xleftarrow[e_0]{\cong} & K_*(C^*(G_M^{tan})) & \xrightarrow[e_1]{} & K_*(C^*(M \times M)) \\
\mathcal{T}\downarrow\cong & & \mathcal{T}\downarrow\cong & & \mathcal{T}\downarrow\cong \\
K_*(C^*(TM \rtimes \mathbb{R}^n)) & \xleftarrow[e_0]{\cong} & K_*(C^*(G_M^{tan} \rtimes \mathbb{R}^n)) & \xrightarrow[e_1]{} & K_*(C^*((M \times M) \rtimes \mathbb{R}^n)) \\
\mathcal{M}\downarrow\cong & & \mathcal{M}\downarrow\cong & & \mathcal{M}\downarrow\cong \\
K^*(N(\mathbb{R}^n, M)) & \xleftarrow[e_0]{\cong} & K^*(D(\mathbb{R}^n, M)) & \xrightarrow[e_1]{} & K^*(\mathbb{R}^n),
\end{array}$$

where the isomorphisms $\mathcal{M}$ stand for the morphisms induced by the mentioned groupoid Morita equivalences.

In the diagram above one recognizes by theorem 2.1.2, up to Fourier and Morita induced isomorphisms, the analytic index morphism of M at the top of the diagram. As we shall see, the rest of the diagram, that is the composition of the left vertical morphism followed by the bottom morphism followed by the right vertical morphism corresponds precisely to the topological index morphism (up to Fourier and Morita).

(iii) **Connes-Thom versus Thom and the proof of Atiyah-Singer's theorem.** In the last diagram, it is an easy algebraic topology exercise to show that the morphism

$$K^*(N(\mathbb{R}^n, M)) \xleftarrow[e_0]{\cong} K^*(D(\mathbb{R}^n, M)) \xrightarrow[e_1]{} K^*(\mathbb{R}^n) \tag{18}$$

coincides with the morphism

$$K^*(N(\mathbb{R}^n, M)) \xrightarrow{i!} K^*(\mathbb{R}^n) \tag{19}$$

induced by an open emdedding of the normal bundle $N(\mathbb{R}^n, M)$ in $\mathbb{R}^n$ in topological K-theory.

The rest of the diagram is not completely an exercise, or not an easy one in any case. Indeed, it is highly accepted among experts that the left vertical morphism coincides, up to the induced Fourier isomorphism, with the classic Thom isomorphism in topological K-theory and that the right vertical morphism coincides, up to the induced Morita isomorphism, with the classic Bott isomorphism in topological K-theory. As far as I am aware there is no written proof of this fact. In appendix C, I will write down a complete proof of this fact, proof which I found together with Jean-Marie Lescure in the setting of manifolds with corners. The statement is as follows, at least for the particular case in question.

PROPOSITION 2.1.3 (Carrillo Rouse, Lescure, 2014). *Under the notation above the following two diagrams,*

$$\begin{array}{ccc} K^*(T^*M) & \xrightarrow[\cong]{\mathcal{F}} & K_*(C^*(TM)) \\ {\scriptstyle Th}\downarrow\cong & & {\scriptstyle \mathcal{T}}\downarrow\cong \\ K^*(N(\mathbb{R}^n, M)) & \xleftarrow[\mathcal{M}]{\cong} & K_*(C^*(TM \rtimes \mathbb{R}^n)) \end{array} \tag{20}$$

where Th stands for the classic Thom isomorphism associated to the K-oriented map $T^*M \to M \to N(\mathbb{R}^n, M)$ (composition of the canonical projection followed with the zero section); and

$$\begin{array}{ccc} K_*(C^*(M \times M)) & \xrightarrow[\cong]{\mathcal{M}} & \mathbb{Z} \\ {\scriptstyle \mathcal{T}}\downarrow\cong & & {\scriptstyle B}\downarrow\cong \\ K_*(C^*((M \times M) \rtimes \mathbb{R}^n)) & \xrightarrow[\mathcal{M}]{\cong} & K^*(\mathbb{R}^n) \end{array} \tag{21}$$

where B stands for the Bott periodicity isomorphism, are commutative.

We recall that the topological index morphism defined by Atiyah and Singer, associated to the embedding j, is the morphism

$$Ind_{top}^M : K^0(T^*M) \to \mathbb{Z} \tag{22}$$

given as the composition of the Thom isomorphism Th, followed by the shriek map $i!$ associated to the open embedding of the normal bundle $N(\mathbb{R}^n, M)$ in $\mathbb{R}^n$, followed by the Bott inverse morphism B^{-1}.

As an immediate corollary of the proposition above and of the commutativity of diagram (33) we have that

THEOREM 2.1.4 (Atiyah-Singer index theorem). *Let M be a compact manifold. There is an equality*

$$Ind_a^M = Ind_{top}^M \tag{23}$$

as morphisms from $K^*(T^*M)$ to $\mathbb{Z}$.

As already explained by Atiyah and Singer in the 60's, the above theorem immediately implies the index formula in 2.1.1 by a comparison of the topological index morphism in K-theory above with its analog in de Rham cohomology by using the Chern character morphism.

2.2. Index theorems for families and foliations

In this section I want to explain two other "classic" cases under the mirror of groupoid techniques and particularly with deformation groupoids, these two cases are the case of families, already treated in the 60's by Atiyah and Singer, and the case of regular foliations, treated by the first time by Connes and Skandalis. Of course the foliation case includes the families case. Besides, giving this interesting example will serve of motivation for the next chapters.

To present the general idea let us consider a Lie groupoid $G \rightrightarrows M$. Suppose there is a groupoid morphism

$$h : G \to \mathbb{R}^n \tag{24}$$

where n is even and $\mathbb{R}^n$ is considered as above as an additive group. By functoriality of the deformation to the normal cone construction one gets a Lie groupoid morphism

$$h^{tan} : G^{tan} \to \mathbb{R}^n \tag{25}$$

explicitly given by $h^{tan}(\gamma, t) = th(\gamma)$ for $(\gamma, t) \in G \times (0, 1]$ and by $h^{tan}(x, V) = A(h)(x, V)$ for $(x, V) \in AG = \bigsqcup A_x G$ where $A(h)$ is the Lie algebroid functor applied to h (so it is given by the differential of h in the normal direction to the groupoid's units). In particular we have a trivially commutative diagram

$$\begin{array}{ccccc}
 & & \overset{Ind_a^G}{\frown} & & \\
K_*(C^*(AG)) & \xleftarrow[e_0]{\cong} & K_*(C^*(G^{tan})) & \xrightarrow[e_1]{} & K_*(C^*(G)) \\
\mathcal{T}\downarrow\cong & & \mathcal{T}\downarrow\cong & & \mathcal{T}\downarrow\cong \\
K_*(C^*(AG \rtimes \mathbb{R}^n)) & \xleftarrow[e_0]{\cong} & K_*(C^*(G^{tan} \rtimes \mathbb{R}^n) & \xrightarrow[e_1]{} & K_*(C^*(G \rtimes \mathbb{R}^n))
\end{array} \tag{26}$$

where the isomorphisms $\mathcal{T}$ denote again Connes-Thom isomorphisms. Now, in order to this abstract index theorem to be useful one needs first that an interesting morphism h as above exists, and by interesting I mean that the second row in the diagram above gives a different insight on the computation of the analytic index morphism, for example in the Atiyah-Singer theorem for closed manifolds the second row semi-direct product groupoids are all free and proper and hence Morita equivalent to their orbit space and moreover these orbit spaces are explicitly computable so one can get an index formula. As I will now explain, in the case of regular foliations there are good morphisms h that allow to give a groupoid proof of the longitudinal Connes-Skandalis index theorem. In the next chapter we will see in more detail other very interesting examples where this method works at some extent, next we will see other ways of getting index theorems and/or computing index formulas when one does not have such nice cocycles.

Let (M, F) be a regular foliation, that is $F \subset TM$ is an integrable sub-vector bundle. Under these circumstances there is a well known Lie groupoid (the so called holonomy groupoid, see for example [**103**] or [**76**])

$$\mathcal{H}(M, F) \rightrightarrows M \tag{27}$$

that is s-connected and integrates the foliation, more precisely the Lie algebroid of $\mathcal{H}(M, F)$ is $A\mathcal{H}(M, F) = F$ with anchor map given by the inclusion of F in TM. For getting the cocycle h as above we consider as Atiyah-Singer and as Connes-Skandalis, an embedding $j : M \hookrightarrow \mathbb{R}^n$ for which we will assume n even. With this in hand we let

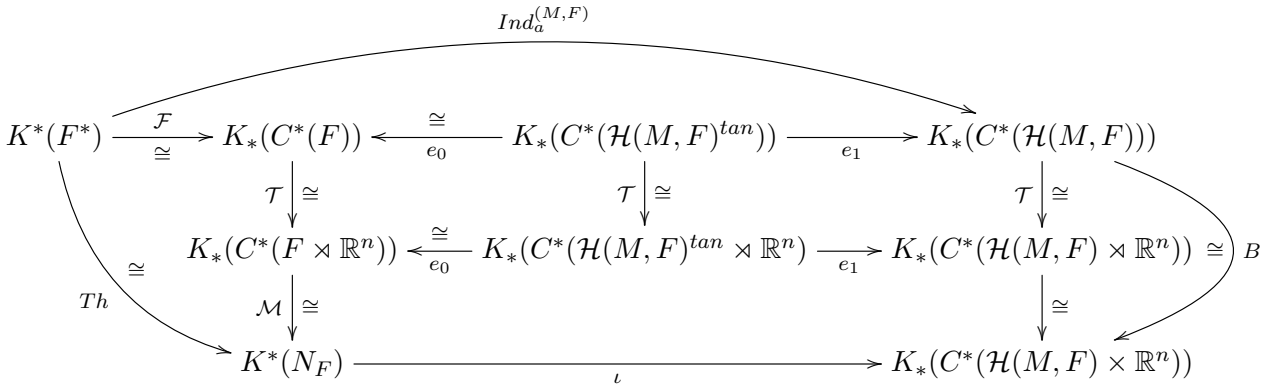
$$h : \mathcal{H}(M, F) \to \mathbb{R}^n \tag{28}$$

as $h(\gamma) = j(s(\gamma)) - j(t(\gamma))$ where s, t are the source and target Lie groupoid structural maps for $\mathcal{H}(M, F)$. Now, a direct computation gives that in this case $A(h) : F \to \mathbb{R}^n$ is just given by the differential of j restricted to F. In particular, a complete analog computation as the smooth manifold case shows that the associated semi-direct crossed product groupoid

$$F \rtimes \mathbb{R}^n \rightrightarrows M \times \mathbb{R}^n \tag{29}$$

is free and proper and even more its orbit space is diffeomorphic to the total space of the normal bundle $N_F \to M$ to F in $\mathbb{R}^n$. On the other hand the semi-direct product groupoid $\mathcal{H}(M, F) \rtimes \mathbb{R}^n$ is not necessarily free and proper but isomorphic to the product groupoid $\mathcal{H}(M, F) \times \mathbb{R}^n$. Applying (26) in this case

together with the previous remarks and applying again the result in the Connes-Thom versus Thom appendix we get the following commutative diagram
(30)



where the isomorphism Th at the left is the Thom isomorphism associated to the K-oriented map $F^* \to M \to N_F$ (canonical projection follwoed by the zero section), the isomorphism B at the right is the Bott isomorphism and where ι is just the morphism fitting the diagram.

The commutatvity of the diagram above, and hence the fact that the longitudinal analytic index in the top can be decomposed as the Thom isomorphism followed by ι followed by the Bott inverse map, is precisely the content of the so called **Connes-Skandalis Longitudinal Index theorem** for regular foliations.

The particular case of families. When in the example above the foliation is given by the fibers of a surjective submersion $\phi : M \to B$ we can further in the above diagram. Indeed, in this case the holonomy groupoid in question writes

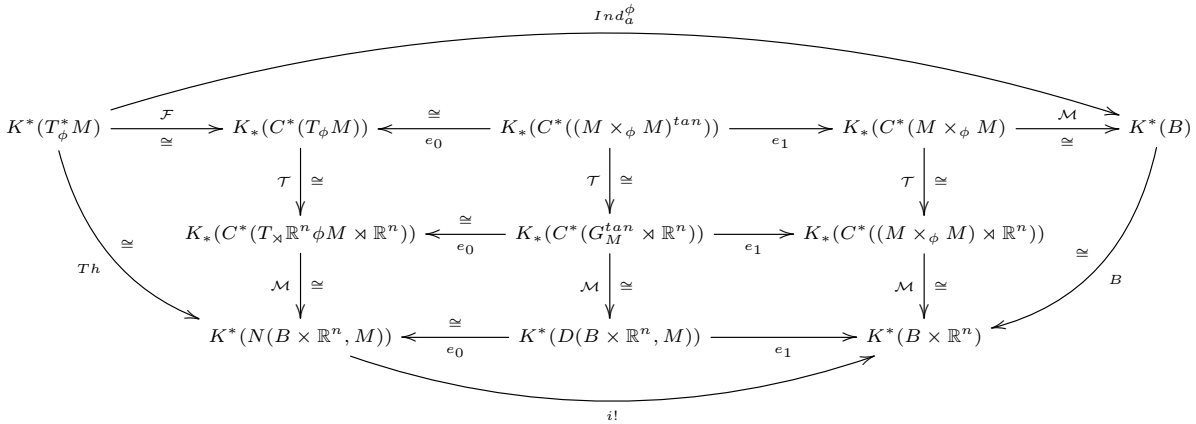
$$M \times_\phi M \rightrightarrows M \tag{31}$$

and it is Morita equivalent to B (as a unit groupoid $B \rightrightarrows B$), but even more the semidirect product groupoid

$$(M \times_\phi M)^{tan} \rtimes \mathbb{R}^n \rightrightarrows M \times [0,1] \times \mathbb{R}^n \tag{32}$$

is free and proper (direct computation) and its orbit space can be identified with the deformation to tne normal cone space $D(B \times \mathbb{R}^n, M)$ associated to the injective immersion $M \overset{(\phi,j)}{\to} B \times \mathbb{R}^n$. Hence, in this case we get the following

commutative diagram
(33)



where the morphism in the bottom is the pushforward morphism in topological K-theory associated to (an) the embedding of $N(B\times\mathbb{R}^n, M)$ in $B\times\mathbb{R}^n$.

The commutatvity of the diagram above, and hence the fact that the families analytic index in the top can be decomposed as the Thom isomorphism followed by $i!$ followed by the Bott inverse map, is precisely the content of the so called **Atiyah-Singer families Index theorem**.

As for the case of a single closed smooth manifold, the above K-theoretical theorem gives the following theorem, by comparing topological K-theory with de Rham cohomology via the Chern character.

THEOREM 2.2.1 (Atiyah-Singer index formula for families). *For $D\in\Psi^*_\phi(M)$ a vertical elliptic operator, the following equality holds*

$$ch_B(Ind_a^\phi([\sigma_D])) = \phi!(ch_M([\sigma_D])\wedge Td_{\mathbb{C}}(T^*_\phi M)) \tag{34}$$

where ch_B denotes the Chern character of B, ch_M the Chern character of $T^*_\phi M$ and where $\phi! : H^*_c(T^*_\phi M)\to H^*_c(B)$ denotes the morphism in de Rham cohomology with compact supports given by integration of forms along the fibers of ϕ.

CHAPTER 3

Index theorems for manifolds with corners

3.1. Basics on Index theory for manifolds with corners

We will consider compact manifolds X with embedded corners [**71**]: we thus may fix once for all a smooth compact manifold $\tilde{X}$ and smooth maps $\rho_1, ..., \rho_n : \tilde{X} \longrightarrow \mathbb{R}$ such that:

(i) $X = \bigcap_{1 \leq j \leq n} \rho_j^{-1}([0, +\infty)) \subset \tilde{X}$

(ii) Setting $H_j = \rho_j^{-1}(\{0\}) \cap X$, $j = 1, \ldots, n$, we require that $\{d\rho_{j_1}, ..., d\rho_{j_k}\}$ has maximal rank at any point of $H_{j_1} \cap \ldots \cap H_{j_k}$ for any $1 \leq j_1 < \cdots < j_k \leq n$.

We assume for simplicity that all the boundary hypersurfaces H_j of X are connected, as well as X itself.

The so-called Puff groupoid [**78**] is then defined by:

$$G(\tilde{X}, (\rho_i)) = \{(x, y, \lambda_1, ..., \lambda_n) \in \tilde{X} \times \tilde{X} \times \mathbb{R}^n : \rho_i(x) = e^{\lambda_i} \rho_i(y)\}. \tag{35}$$

This is a Lie subgroupoid of $\tilde{X} \times \tilde{X} \times \mathbb{R}^k$. The Puff groupoid is not s-connected, denote by $G_c(\tilde{X}, (\rho_i))$ its s-connected component.

Definition 3.1.1 (The b-groupoid). *The $b-$groupoid $\Gamma_b(X)$ of X is by definition the restriction to X of the s-connected Puff groupoid (35) considered above, that is*

$$\Gamma_b(X) := G_c(\tilde{X}, (\rho_i))|_X \rightrightarrows X \tag{36}$$

The $b-$groupoid was introduced by B. Monthubert in [**78**] in order to give a groupoid description for the Melrose's algebra of b-pseudodifferential operators. We summarize below the main properties we will be using about this groupoid:

Theorem 3.1.2 (Monthubert [**78**]). *Let X be a manifold with corners as above, we have that*

(i) *$\Gamma_b(X)$ is a $C^{0,\infty}$-amenable groupoid.*

(ii) *It has Lie algebroid $A(\Gamma_b(X)) =^b TX$, the b-tangent bundle of Melrose.*

(iii) *Its C^*-algebra (reduced or maximal is the same since amenability) coincides with the algebra of b-compact operators. The canonical isomorphism*

$$C^*(\Gamma_b(X)) \cong \mathcal{K}_b(X) \tag{37}$$

is given as usual by the Schwartz Kernel theorem.

(iv) *The pseudodifferential calculus of $\Gamma_b(X)$ coincides with compactly supported b-calculus of Melrose.*

See also [**61**] or [**79**] for further details on the above theorem, for the Lie algebroid part an explicit explanation can be found in [**47**].

REMARK 3.1.3. *To simplify, in the present paper, we only discuss the case of scalar operators. The case of operators acting on sections of vector bundles is treated as classically by considering bundles of homomorphisms.*

We will now introduce the several index morphisms we will be using, mainly the Analytic and the Fredholm index. In all this section, X denotes a compact and connected manifold with embedded corners.

3.1.1. Ellipticity and Analytical Index morphisms. The analytical index morphism (of the manifold with embedded corners X) takes it values in the group $K_0(\mathcal{K}_b(X))$. It can be defined in two ways. First, we may consider the connecting homomorphism I of the exact sequence in K-theory associated with the short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{K}_b(X) \longrightarrow \overline{\Psi_b^0(X)} \xrightarrow{\sigma_b} C({}^bS^*X) \longrightarrow 0. \tag{38}$$

Then, if $[\sigma_b(D)]_1$ denotes the class in $K_1(C({}^bS^*X))$ of the principal symbol $\sigma_b(D)$ of an elliptic b-pseudodifferential D, we define the analytical index $\mathrm{Ind}_{\mathrm{an}}(D)$ of D by

$$\mathrm{Ind}_{\mathrm{an}}(D) = I([\sigma_b(D)]_1) \in K_0(\mathcal{K}_b(X)).$$

Secondly, we can in a first step produce a K_0-class $[\sigma_b(D)]$ out of $\sigma_b(D)$:

$$[\sigma_b(D)] = \delta([\sigma_b(D)]_1) \in K_0(C_0({}^bT^*X)) \tag{39}$$

where δ is the connecting homomorphism of the exact sequence relating the vector and sphere bundles:

$$0 \longrightarrow C_0({}^bT^*X) \longrightarrow C_0({}^bB^*X) \longrightarrow C({}^bS^*X) \longrightarrow 0. \tag{40}$$

Next, we consider the exact sequence coming with the adiabatic deformation of $\Gamma_b(X)$:

$$0 \longrightarrow C^*(\Gamma_b(X) \times (0,1]) \longrightarrow C^*(\Gamma_b^{tan}(X)) \xrightarrow{r_0} C^*({}^bTX) \longrightarrow 0, \tag{41}$$

in which the ideal is K-contractible. Using the shorthand notation $K^0_{top}({}^bT^*X)$ for $K_0(C^*({}^bTX))$, we set:

$$Ind_X^a = r_1 \circ r_0^{-1} : K^0_{top}({}^bT^*X) \longrightarrow K_0(\mathcal{K}_b(X)) \tag{42}$$

where $r_1 : K_0(C^*(\Gamma_b^{tan}(X))) \to K_0(C^*(\Gamma_b(X)))$ is induced by the restriction morphism to $t = 1$. Applying a mapping cone argument to the exact sequence (38) gives a commutative diagram

$$\begin{array}{ccc} K_1(C({}^bS^*X)) & \xrightarrow{\quad I \quad} & K_0(\mathcal{K}_b(X)) \\ & {\scriptstyle\delta}\searrow \quad \nearrow{\scriptstyle Ind_X^a} & \\ & K^0_{top}({}^bT^*X) & \end{array} \tag{43}$$

Therefore we get, as announced:

$$\mathrm{Ind}_{\mathrm{an}}(D) = Ind_X^a([\sigma_b(D)]) \tag{44}$$

The map Ind_X^a will be called the *Analytic Index morphism* of X. A closely related homomorphism is the *Boundary analytic Index morphism*, in which the restriction to $X \times \{1\}$ is replaced by the one to $\partial X \times \{1\}$, that is, we set:

$$Ind_X^\partial = r_\partial \circ r_0^{-1} : K_0(C_0({}^bT^*X)) \longrightarrow K_0(C^*(\Gamma_b(X)|_{\partial X})), \tag{45}$$

where r_∂ is induced by the homomorphism $C^*(\Gamma_b^{tan}(X)) \longrightarrow C^*(\Gamma_b(X))|_{\partial X}$. We have of course

$$Ind_X^\partial = r_{1,\partial} \circ Ind_X^a \tag{46}$$

if $r_{1,\partial}$ denotes the map induced by the homomorphism $C^*(\Gamma_b(X)) \longrightarrow C^*(\Gamma_b(X)|_{\partial X})$. Since $r_{1,\partial}$ induces an isomorphism between K_0 groups (proposition 5.6 in [**33**]), both indices have the same meaning.

3.1.2. The noncommutative tangent space, full ellipticity and the Fredholm Index morphism. The second index morphism will be related to the longitudinal fully ellipticity. First consider the Fully Elliptic deformation groupoid

$$(\Gamma_b(X))^{FE} \rightrightarrows X_\partial \tag{47}$$

defined to be the open saturated subgroupoid of $\Gamma_b(X)^{tan}$ given by its restriction to $X_\partial := (X \times [0,1]) \setminus \partial X \times \{1\}$. Then it induces a K-theory morphism

$$K_0(C^*(T_{nc}X) \xrightarrow{{}^bInd_{Fred}} K_0(C_r^*(\overset{\circ}{X} \times \overset{\circ}{X})) \cong \mathbb{Z} \tag{48}$$

given as the composition

$$K_0(C^*(T_{nc}X)) \xleftarrow[r]{\approx} K_0(C^*((\Gamma_b(X))^{FE}) \xrightarrow{e_1} K_0(C_r^*(\overset{\circ}{X} \times \overset{\circ}{X})) \cong \mathbb{Z} \tag{49}$$

where

$$T_{nc}(X) \rightrightarrows X_\partial$$

is by definition the saturated closed subgroupoid of $(\Gamma_b)^{FE}$ given as the complement of the open dense saturated subgroupoid

$$(\overset{\circ}{X} \times \overset{\circ}{X}) \times (0,1] \rightrightarrows \overset{\circ}{X} \times (0,1].$$

We will relate the above index with the full symbol σ_F, for this we briefly recall how this symbol is defined. Consider the following commutative diagram of

C^*-algebras

(50)

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & C^*(\overset{\circ}{X}\times\overset{\circ}{X}) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & C^*(\Gamma_b(X)) & \longrightarrow & \overline{{}^b\Psi^0(X)} & \xrightarrow{\sigma} & C({}^bS^*X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^*(\Gamma_b(X)|_{\partial(X)}) & \longrightarrow & \overline{\Psi^0(\partial(X)} & \xrightarrow{\sigma} & C({}^bS^*X|_{\partial(X)}) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

By direct diagram chasing there is an induced short exact sequence

(51)

$$0 \longrightarrow C^*(\overset{\circ}{X}\times\overset{\circ}{X}) \longrightarrow \overline{{}^b\Psi^0}(X) \xrightarrow{\sigma_F} C({}^bS^*X) \times_{C({}^bS^*X_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)}) \longrightarrow 0$$

In fact, an operator $D \in \overline{{}^b\Psi^0}(X)$ is said to be fully elliptic if $\sigma_F(D)$ is invertible. In [**64**] (the statement also appears in [**73**]) it is proved that full ellipticity is equivalent to Fredholmness on any b-Sobolev spaces $H_b^s(X)$.

We want next to show that the connecting morphism in K-theory associated to the above short exact sequence, that by definition computes the Fredholm index of a fully elliptic operator, factorizes through the deformation index above defined by the Fully Elliptic groupoid. For this we will define

$$\delta_{nc} : K_1(C({}^bS^*X) \times_{C({}^bS^*X_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)}) \to K_0(C^*(T_{nc}(X))$$

and call it the Noncommutative symbol morphism. Consider the following commutative diagram of C^*-algebras

(52)

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & C^*((\Gamma_b(X))^{FE}) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & C^*((\Gamma_b(X))^{tan}) & \longrightarrow & \overline{\Psi^0((\Gamma_b(X))^{tan})} & \xrightarrow{\sigma} & C({}^bS^*X\times[0,1]) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^*(\Gamma_b(X)|_{\partial(X)}) & \longrightarrow & \overline{\Psi^0(\Gamma_b(X)|_{\partial(X)})} & \xrightarrow{\sigma} & C({}^bS^*F|_{\partial(X)}) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

that induces a short exact sequence
(53)

$$0 \longrightarrow C^*((\Gamma_b(X))^{FE}) \longrightarrow \overline{\Psi^0}((\Gamma_b(X))^{tan})) \xrightarrow{\sigma_{F'}} C({}^bS^*X \times [0,1]) \times_{C({}^bS^*X|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)}) \longrightarrow 0$$

Remember that in $K-$theory the evaluation at $t = 1$ in $[0, 1]$ induces the indentity

$$K_1(C({}^bS^*X \times [0,1]) \times_{C({}^bS^*X|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)})) \longrightarrow K_1(C({}^bS^*X) \times_{C({}^bS^*X|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)})).$$

Hence we can consider the connecting morphism of the above short exact sequence as a map

$$K_1(C({}^bS^*X) \times_{C({}^bS^*X|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)})) \xrightarrow{\delta_F} K_0(C^*((\Gamma_b(X))^{FE})) \tag{54}$$

Definition 3.1.4 (Noncommutative symbol). *The Noncommutative symbol morphism*

$$\delta_{nc} : K_1(C({}^bS^*X) \times_{C({}^bS^*F|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)})) \to K_0(C^*(T_{nc}(X))) \tag{55}$$

is the morphism given by the composition of the connecting morphism δ_F followed by the isomorphism $r : K_0(C^*((\Gamma_b(X))^{FE})) \to K_0(C^*(T_{nc}(X)))$ induced from the restriction as a closed subgroupoid.

By definition of fully ellipticity every Fully elliptic operator has a Noncommutative symbol in $K_0(C^*(T_{nc}(X)))$.

The following proposition justifies the terminology.

Proposition 3.1.5. *The deformation index induced by the Fully Elliptic groupoid fits the following commutative diagram*
(56)

$$\begin{array}{ccc} K_1(C({}^bS^*X) \times_{C({}^bS^*F|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)})) & \xrightarrow{\quad\delta\quad} & K_0(C_r^*(\overset{\circ}{X} \times \overset{\circ}{X})) \\ & {\scriptstyle \delta_{nc}} \searrow \quad \nearrow {\scriptstyle {}^bInd_{Fred}} & \\ & K_0(C^*(T_{nc}(X))) & \end{array}$$

where δ is the connecting morphism associated to the full principal short exact sequence (51) and δ_{nc} is the non commutative symbol constructed above.

Proof. It is enough to apply the six term short exact sequence in K-theory to the following commutative diagram
(57)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*((\Gamma_b(X))^{FE}) & \longrightarrow & \overline{\Psi^0}((\Gamma_b(X))^{tan})) & \xrightarrow{\sigma_{F'}} & C({}^bS^*X \times [0,1]) \times_{C({}^bS^*X|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)}) & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle e_1} & & \downarrow {\scriptstyle e_1} & & \downarrow {\scriptstyle e_1} & & \\ 0 & \longrightarrow & C^*(\overset{\circ}{X} \times \overset{\circ}{X})) & \longrightarrow & \overline{{}^b\Psi^0}(\Gamma_b(X)) & \xrightarrow{\sigma_F} & C({}^bS^*X) \times_{C({}^bS^*X|_{\partial(X)})} \overline{{}^b\Psi^0}(\Gamma_b(X)|_{\partial(X)}) & \longrightarrow & 0 \end{array}$$

□

Definition 3.1.6. *The index morphism ${}^bInd_{Fred}$ is called the Fredholm index morphism associated to the manifold with corners X.*

3.1.3. Obstruction to full ellipticity and Fredholm perturbation property. In order to analyse the obstruction to full ellipticity, we introduce Fredholm Perturbation Properties, following the terminology introduced in [**83**].

DEFINITION 3.1.7. *Let $D \in \Psi_b^m(X)$ be elliptic. We say that D satisfies:*

- *the Fredholm Perturbation Property $(\mathcal{FP})$ if there is $R \in \Psi_b^{-\infty}(X)$ such that $D+R$ is fully elliptic.*
- *the stably Fredholm Perturbation Property $(\mathcal{SFP})$ if $D \oplus 1_H$ satisfies $(\mathcal{FP})$ for some identity operator 1_H.*
- *the stably homotopic Fredholm Perturbation Property $(\mathcal{HFP})$ if there is a fully elliptic operator D' with $[\sigma_b(D')] = [\sigma_b(D)] \in K_0(C^*({}^bTX))$.*

We also say that X satisfies the *(resp. stably) Fredholm Perturbation Property* if any elliptic b-operator on X satisfies $(\mathcal{FP})$ (resp. $(\mathcal{SFP})$).

Property $(\mathcal{FP})$ is stronger than property $(\mathcal{SFP})$ which in turn is equivalent to property $(\mathcal{HFP})$ by [**49**, Proposition 4.3]. In [**82**], Nazaikinskii, Savin and Sternin characterized $(\mathcal{HFP})$ for arbitrary manifolds with corners using an index map associated with their dual manifold construction. In [**33**] the result of [**82**] is rephrased in terms of deformation groupoids with the non trivial extra contribution of changing $(\mathcal{HFP})$ by $(\mathcal{SFP})$ thanks to [**49**, Proposition 4.3]:

THEOREM 3.1.8. *Let D be an elliptic b-pseudodifferential operator on a compact manifold with corners X. Then D satisfies $(\mathcal{SFP})$ if and only if $Ind_X^{\partial}([\sigma_b(D)]) = 0$ in $K_0(C^*(\Gamma_b(X)|_{\partial X}))$.*
In particular, if D satisfies $(\mathcal{FP})$ then its boundary analytic index vanishes.

This motivates the computation of $K_*(C^*(\Gamma_b(X))) \cong K_*(C^*(\Gamma_b(X)|_{\partial X}))$ but also the computation of the index morphisms introduced above. As far as I am aware, before the computations we did with Jean-Marie Lescure, the only general computations known were for smooth manifolds and smooth manifolds with boundary. In fact, for the latter the obstruction group always vanishes so the immediate interesting question is to compute the associated Fredholm index, this is the content of the next section.

3.2. Manifolds with boundary and cohomological formula for APS indices

In a series of papers [**3, 4, 5**], Atiyah, Patodi and Singer investigated the case of non-local elliptic boundary value problems. They showed that under some boundary conditions (the now so-called APS boundary conditions), Dirac operators (among a larger class of first order differential operators) become Fredholm on suitable spaces and they computed the index. To the characteristic form from the closed smooth case they added a correction term, called the eta invariant, which is determined by an appropriate restriction of the operator to the boundary : this is a spectral invariant, measuring the asymmetry of the spectrum. However, a cohomological formula expressing the APS index as an integration of some characteristic form on the boundary is impossible. Roughly speaking,

the non locality of the chosen boundary condition prevents us to express this spectral correction term with a local object, like a density on the boundary. In other words, applying directly the methods of Atiyah-Singer cannot yield the result in the APS index problem. However, we will see that the use of some noncommutative spaces, associated with groupoids, will give us the possibility to understand the computation of the APS index using classic algebraic topology methods (K-theory and cohomology).

In [**29**], we follow Connes' groupoid approach to obtain a cohomological formula for the index of a fully elliptic (pseudo)differential operator on a closed manifold with boundary. For the case of such a manifold, let us describe woth more details the b-groupoid and other associated groupoids.

Let X be a manifold with boundary. We denote, as usual, $\overset{\circ}{X}$ the interior which is a smooth manifold and ∂X its boundary, that we suppose for simplicity to have a single connected component. Let

$$\Gamma_b(X) \rightrightarrows X \tag{58}$$

be the groupoid of the b-calculus, where

$$\Gamma_b(X) = \overset{\circ}{X} \times \overset{\circ}{X} \bigsqcup \partial X \times \partial X \times \mathbb{R},$$

with groupoid structure given as a family of pair groupoids and the (additive) group $\mathbb{R}$. It is a continuous family groupoid with the topology explicity described in [**78**]

Consider

$$\Gamma_b(X)^{tan} = A(\Gamma_b(X)) \bigsqcup \Gamma_b(X) \times (0,1] \rightrightarrows X \times [0,1]$$

its tangent groupoid.

Take now the open subgroupoid of $\Gamma_b(X)^{tan}$ obtained by restriction to $X_{\mathcal{F}} := X \times [0,1] \setminus (\partial X \times \{1\})$

$$\Gamma_b(X)_{\mathcal{F}} = A(\Gamma_b(X)) \times \{0\} \bigsqcup \overset{\circ}{X} \times \overset{\circ}{X} \times (0,1] \bigsqcup \partial X \times \partial X \times \mathbb{R} \times (0,1) \rightrightarrows X_{\mathcal{F}}.$$

By definition $\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]$ is a saturated, open dense subgroupoid of $\Gamma_b(X)_{\mathcal{F}}$. This leads to a complementary closed subgroupoid of $\Gamma_b(X)_{\mathcal{F}}$:

$$\mathcal{T}_{nc}X \rightrightarrows X_{\partial}, \tag{59}$$

where $X_{\partial} := X_{\mathcal{F}} \setminus \overset{\circ}{X} \times (0,1] = X \underset{\partial X \times \{0\}}{\cup} \partial X \times [0,1)$.

The groupoid $\mathcal{T}_{nc}X$, called here *"The noncommutative tangent space of X"*, was introduced in [**45**] in the framework of pseudomanifolds with isolated singularities and used for Poincaré duality purpose. It was later used again in [**46**] to derive an index theorem and reintroduced in [**79**] in the framework of manifolds with boundary. Note also that $\mathcal{T}_{nc}X$ is denoted by $T^{\mathsf{FC}}X$ in [**48**] where it is generalized to the case of stratified spaces, or equivalently, manifolds with (iterated) fibred corners.

Deformation groupoids like $\Gamma_b(X)_{\mathcal{F}}$ induce index morphisms. Indeed, its algebra comes equipped with a restriction morphism to the algebra of $\mathcal{T}_{nc}X$ and an evaluation morphism to the algebra of $\overset{\circ}{X} \times \overset{\circ}{X}$ (for $t = 1$). Indeed, we have a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]) \longrightarrow C^*(\Gamma_b(X)_{\mathcal{F}}) \xrightarrow{e_0} C^*(\mathcal{T}_{nc}X) \longrightarrow 0 \tag{60}$$

where the algebra $C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1])$ is contractible. Hence applying the K-theory functor to this sequence we obtain an index morphism

$${}^bInd_{Fred} = (e_1)_* \circ (e_0)_*^{-1} : K_0(C^*(\mathcal{T}_{nc}X)) \longrightarrow K_0(C^*(\overset{\circ}{X} \times \overset{\circ}{X})) \approx \mathbb{Z}. \tag{61}$$

As we saw in the last section, this index computes indeed the Fredholm index of those elliptic operators on X satisfying the APS boundary condition, and hence we call it *The Fredholm index morphism of* X.

A first task in order to follow the Atiyah-Singer approach would be to compute the morphism $ind_{\mathcal{F}}$ by topological means. For instance, using an appropriate embedding into a space in which the computation could follow in an easier way. This idea has been already followed up in [**46**] in the framework of manifolds with conical singularities, using a KK-equivalent version of the noncommutative tangent space $\mathcal{T}_{nc}X$. There, the authors use embeddings into euclidean spaces to extend the construction of the Atiyah-Singer topological index map, thanks to a "Thom isomorphism" between the noncommutative tangent space of the singular manifold and of its singular normal bundle, and then get an index theorem in the framework of K-theory. Here, we follow a different approach and we are going to extend the Atiyah-Singer topological index map using Connes' ideas on tangent groupoid actions on euclidean spaces; moreover we investigate the cohomological counterpart of the K-theoretic statement of the index theorem. Note also that the index map considered here coincide, through KK-equivalences, with the index maps considered in [**61**] and in [**46**].

Consider an appropriate embedding (*i.e.*, respecting the boundary)

$$i : X \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+ \tag{62}$$

of X into $\mathbb{R}^{N-1} \times \mathbb{R}_+$. Following Connes, we use it to define a continuous family groupoid morphism

$$h : \Gamma_b(X) \to \mathbb{R}^N \tag{63}$$

where we see $\mathbb{R}^N$ as an additive group and we assume N even. This morphism induces an action of $\Gamma_b(X)$ on $X \times \mathbb{R}^N$ and an induced deformation action of $\Gamma_b(X)_{\mathcal{F}}$ on $X_{\mathcal{F}} \times \mathbb{R}^N$ (coming from an induced morphism $h_{\mathcal{F}}$). We have the following result:

Proposition 3.2.1. *The semi-direct groupoid* $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}} := \Gamma_b(X)_{\mathcal{F}} \rtimes_{h_{\mathcal{F}}} \mathbb{R}^N$ *is a free proper groupoid.*

In [**29**] we explain how the Connes-Thom isomorphism links the K-theory of a groupoid with the K-theory of a semi-direct groupoid as above. For instance,

the new semi-direct groupoid $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ defines as well an index morphism and this one is linked with the index (61) by a natural isomorphism, the so called Connes-Thom isomorphism, thus giving the following commutative diagram
(64)

$$\begin{array}{ccccc}
 & & \overset{ind_{\mathcal{F}}}{\frown} & & \\
K_0(C^*(\mathcal{T}_{nc}X)) & \xleftarrow[\approx]{e_0} & K_0(C^*(\Gamma_b(X)_{\mathcal{F}})) & \xrightarrow{e_1} & K_0(C^*(\overset{\circ}{X}\times\overset{\circ}{X}))\approx\mathbb{Z} \\
\downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx & & \downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx & & \downarrow{\scriptstyle \mathscr{C}\mathscr{T}}\approx \\
K_0(C^*((\mathcal{T}_{nc}X)_{h_0})) & \xleftarrow[\approx]{e_0} & K_0(C^*((\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}})) & \xrightarrow{e_1} & K_0(C^*((\overset{\circ}{X}\times\overset{\circ}{X})_{h_1})) \\
 & & \underset{ind_{h_{\mathcal{F}}}}{\smile} & &
\end{array}$$

where h_0 and h_1 denote the respective restricted actions of $\mathcal{T}_{nc}X$ and $\overset{\circ}{X}\times\overset{\circ}{X}$ on $\mathbb{R}^N$.

Now, the proposition above tells us that the orbit space of $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ is a nice space and moreover that this semi-direct groupoid is Morita equivalent to its orbit space. This means that the index morphism $ind_{h_{\mathcal{F}}}$ can be computed, modulo Morita equivalences, as the deformation index morphism of some space. More precisely, denoting by $\mathscr{B}_{h_{\mathcal{F}}}$ the orbit space of $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$, by $\mathscr{B}_{h_0}$ the orbit space of $(\mathcal{T}_{nc}X)_{h_0}$ and by $\mathscr{B}_{h_1}$ the orbit space of $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ we have an index morphism between K-theory of spaces (topological K-groups, no more C^*-algebras if one does not like it!)

$$ind_{\mathscr{B}_{h_{\mathcal{F}}}}: \qquad K^0(\mathscr{B}_{h_0}) \xleftarrow[\approx]{e_0} K^0(\mathscr{B}_{h_{\mathcal{F}}}) \xrightarrow{e_1} K^0(\mathscr{B}_{h_1}) \tag{65}$$

from which we would be able to compute the Fredholm index.

In order to describe them we need to introduce a new space, but let us first motivate this by looking at the situation when $\partial X = \emptyset$ (following [**39**] II.5). In this case, the orbit space of $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ can be identified with the deformation to the normal cone of the embedding $X \hookrightarrow \mathbb{R}^N$, that is a C^∞-cobordism between the normal bundle to X in $\mathbb{R}^N$, $N(X)$, and $\mathbb{R}^N$ itself:

$$\mathscr{B}_{AS} := N(X) \bigsqcup (0,1] \times \mathbb{R}^N. \tag{66}$$

In this picture we also see the orbit space of $(TX)_{h_0}$ which identifies with $N(X)$ and the orbit space of $(X \times X)_{h_1}$ which identifies with $\mathbb{R}^N$.

Still, in this boundaryless case ($\partial X = \emptyset$), this space $\mathscr{B}_{AS}$ gives in K-theory a deformation index morphism

$$ind_{\mathscr{B}_{AS}}: \qquad K^0(N(X)) \xleftarrow[e_0]{\approx} K^0(\mathscr{B}_{AS}) \xrightarrow{e_1} K^0(\mathbb{R}^N)$$

which is easily seen to be the shriek map associated to the identification of $N(X)$ as an open subset of $\mathbb{R}^N$.

In the boundary case, the normal bundle is not the right space, we know for instance that the APS index cannot be computed by an integration over this space due to the non-locality of the APS boundary condition. One has then to compute the orbit spaces, in fact the orbit space of $(\mathcal{T}_{nc}X)_{h_0}$ identifies with the singular normal bundle:

$$\mathscr{N}_{sing}(X) := N(X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1) \tag{67}$$

which is the C^∞-manifold obtained by gluing the normal bundle $N(X)$[1] associated to the embedding (62) and

$$D_\partial := N(\partial X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1)$$

the deformation to the normal cone associated to the embedding $\partial X \hookrightarrow \mathbb{R}^{N-1}$, along their common boundary (the gluing depending on a choice of a defining function of the boundary of X). The orbit space of $(\overset{\circ}{X} \times \overset{\circ}{X})_{h_1}$ is easily identified with $\mathbb{R}^N$.

Finally the orbit space of $(\Gamma_b(X)_{\mathcal{F}})_{h_{\mathcal{F}}}$ is homeomorphic to a space looking as

$$\mathscr{B}_{\mathscr{F}} := \mathscr{N}_{sing}(X) \bigsqcup (0,1] \times \mathbb{R}^N, \tag{68}$$

where more precisely we prove the following

Proposition 3.2.2. *The locally compact space $\mathscr{B}_{\mathscr{F}}$ admits an oriented C^∞-manifold with boundary structure of dimension $N+1$.*

The last proposition is essential to explicitly compute the index (65) above once the explicit identifications are performed, indeed $\mathscr{B}_{\mathscr{F}}$ is an oriented cobordism from $\mathscr{N}_{sing}(X)$ to $\mathbb{R}^N$, we can hence apply a Stoke's theorem argument to obtain the following:

Proposition 3.2.3. *The index morphism of the deformation space $\mathscr{B}_{\mathcal{F}}$ can be computed by means of the following commutative diagram:*

$$\begin{array}{ccccc} & & ind_{\mathscr{B}_{\mathscr{F}}} & & \\ & \overset{\frown}{\quad} & & \overset{\frown}{\quad} & \\ K^0(\mathscr{N}_{sing}(X)) & \xleftarrow[\approx]{(e_0)_*} & K^0(\mathscr{B}_{\mathscr{F}}) & \xrightarrow{(e_1)_*} & K^0(\mathbb{R}^N) \\ & {\scriptstyle \int_{\mathscr{N}_{sing}(X)} ch(\cdot)} \searrow & & \swarrow {\scriptstyle \int_{\mathbb{R}^N} ch(\cdot)} & \\ & & \mathbb{R} & & \end{array}$$

The index theorem is the following:

Theorem 3.2.4. *[K-theoretic APS] Let X be a manifold with boundary, consider an embedding of X in $\mathbb{R}^N$ as in 62. The Fredholm index morphism $ind_{Fred} : K_0(C^*(\mathcal{T}_{nc}X)) \to \mathbb{Z}$ decomposes as the composition of the following three morphisms*

[1] which is an honest vector bundle over X.

(i) *A Connes-Thom isomorphism $\mathscr{C}\mathscr{T}$:*

$$K_0(C^*(\mathcal{T}_{nc}X)) \xrightarrow{\mathscr{C}\mathscr{T}} K^0(\mathscr{N}_{sing}(X)),$$

(ii) *The index morphism of the deformation space $\mathscr{B}_{\mathcal{F}}$:*

$$K^0(\mathscr{N}_{sing}(X)) \xrightarrow{ind_{\mathscr{B}_{\mathcal{F}}}} K^0(\mathbb{R}^N)$$

(iii) *the usual Bott periodicity isomorphism:*

$$K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}.$$

In other terms, the following diagram is commutative

$$\begin{array}{ccc} K_0(C^*(\mathcal{T}_{nc}X)) & \xrightarrow{\quad ind_{Fred} \quad} & \mathbb{Z} \\ {\scriptstyle \mathscr{C}\mathscr{T}}\downarrow \approx & & \approx \uparrow {\scriptstyle Bott} \\ K^0(\mathscr{N}_{sing}(X)) & \xrightarrow[\quad ind_{\mathscr{B}_{\mathcal{F}}} \quad]{} & K^0(\mathbb{R}^N) \end{array}$$

As discussed above, the three morphisms of the last theorem are computable, and then, exactly as in the classic Atiyah-Singer theorem the last theorem allows to conclude that, given an embedding $i : X \hookrightarrow \mathbb{R}^N$ as above, any fully elliptic operator D on X with "non commutative symbol" $[\sigma_D] \in K_0(C^*(\mathcal{T}_{nc}X))$ gives rise to the following formula:

Cohomological formula for the Fredholm index

$$Index_{Fred}(D) = \int_{\mathscr{N}_{sing}(X)} Ch(\mathscr{C}\mathscr{T}([\sigma_D])) \tag{69}$$

where $\int_{\mathscr{N}_{sing}(X)}$ is the integration with respect to the fundamental class of $\mathscr{N}_{sing}(X)$. In the last section of [**29**] we perform an explicit description for $\mathscr{C}\mathscr{T}([\sigma_D]) \in K^0(\mathscr{N}_{sing}(X))$.

The manifold $\mathscr{N}_{sing}(X)$ (see (67) above) already reflects an interior contribution and a boundary contribution. In particular, picking up a differential form ω_D on $\mathscr{N}_{sing}(X)$ representing $Ch(\mathscr{C}\mathscr{T}([\sigma_D])$, we obtain:

$$Index_{Fred}(D) = \int_{\mathscr{N}(X)} \omega_D + \int_{D_\partial} \omega_D. \tag{70}$$

The first integral above involves the restriction of ω_D to $\mathscr{N}(X)$, which is related to the ordinary principal symbol of D. More precisely, the principal symbol $\sigma_{pr}(D)$ of D provides a K-theory class of $C^*(A^*(\Gamma_b(X)))$, that is a compactly supported K-theory class of the dual of the Lie algebroid of $\Gamma_b(X)$ or in other words of the b-cotangent bundle ${}^bT^*X$, and by functoriality of both the Chern character and Thom-Connes maps, we have

$$[(\omega_D)|_{\mathscr{N}(X)}] = Ch(\mathscr{C}\mathscr{T}([\sigma_{pr}(D)]).$$

The second integral can thus be viewed as a correction term, which contains the eta invariant appearing in APS formula and which also depends on the choice of the representative $\omega_D \in Ch(\mathscr{C}\mathscr{T}([\sigma_D]))$.

Our approach and results are not only a groupoid interpretation of the Atiyah-Patodi-Singer formula. A serious comparison between both formulas has to be done. For instance, as we mentioned above, there is a relation between the second integral on (70) and the so called eta invariant. For deeply understanding this, we need to explicitly describe the Chern character of the $\mathscr{C}\mathscr{T}([\sigma_D] \in K^0(\mathscr{N}_{sing}(X))$, for which one might need to use the Chern character computations done mainly by Bismut in [**22**]. Also, the second integral comes from the part of the b-groupoid corresponding to the boundary,

$$\partial X \times \partial X \times \mathbb{R} \times (0,1),$$

and this groupoid's algebra is related with the suspended algebra of Melrose ([**72**]), a relation between this integral and the Melrose and Melrose-Nistor ([**70**]), becomes then very interesting to study.

3.3. Index theory for Manifolds with corners of general codimension

In general, as I tried to motivate at the begining of this section, one of the first problems one can encounter is to understand the obstruction group $K_0(C^*(\Gamma_b(X)))$ and the associated indices. For manifolds with corners of general codimension this is part of the program I started to work out with Jean-Marie Lescure and with Mario Velasquez. As we showed, the computation of the above mentioned K-theory groups was done in terms of a very naturally and computable homology theory. I explain now these computations.

3.3.1. Conormal homology and Geometric obstructions for Fredholm perturbation problems. Conormal homology is introduced (under a different name) and studied in [**24**]. In [**33**], a slighty different presentation of this homology is given, after the observations that it coincides with the E^2 page of the spectral sequence computing $K_*(C^*(\Gamma_b(X)))$ and that it should provide easily computable obstructions to various Fredholm perturbations properties. We just briefly recall the definition of the chain complex and of the differential of conormal homology (see [**24, 33**] for more details).

With the same notation as above, the chain complex $C_*(X)$ is the $\mathbb{Z}$-module where $C_p(X)$ is generated by

$$\{f \otimes \varepsilon \ ; \ f \in F_p \text{ and } \varepsilon \text{ is an orientation of } N_f\}. \tag{71}$$

Here $N_f = (T_f X/Tf)^*$ is the conormal bundle of $f \subset X$. Note that this bundle is always trivializable with $e_i = d\rho_i, \ i \in I(f)$ as a preferred global basis, and oriented by $\epsilon_{I(f)} = \wedge_{i\in I(f)} e_i$ or its opposite. We define the differential $\delta_* : C_*(X) \to C_{*-1}(X)$ by

$$\delta_p(f \otimes \epsilon) = \sum_{\substack{g\in F_{p-1},\\ f\subset \overline{g}}} g \otimes e_{i_{(g,f)}} \lrcorner \epsilon \tag{72}$$

where the index $i(f,g)$ corresponds to the defining function $\rho_{i(f,g)}$ defining f in g and where $\lrcorner$ denotes the contraction of exterior forms. The *conormal homology* of X, denoted by $H_*^{\mathrm{cn}}(X)$ is defined to be the homology of $(C_*(X), \delta_*)$. Even and odd groups are called *periodic conormal homology*:

$$H_0^{\mathrm{pcn}}(X) = \oplus_{p\geq 0} H_{2p}^{\mathrm{cn}}(X) \text{ and } H_1^{\mathrm{pcn}}(X) = \oplus_{p\geq 0} H_{2p+1}^{\mathrm{cn}}(X). \tag{73}$$

The case up to codim 3:

Let us recall the group isomorphisms

$$K^*(\Gamma_b(X)) \xrightarrow[\cong]{T} H_*^{pcn}(X) \tag{74}$$

we gave in [**33**] for mwc X of codimension up to 3 for $* = 0, 1$ (even, odd). The presentation we will give below slightly differs from the original one presented in [**33**], in fact in ref.cit. the isomorphisms were given from the periodic conormal groups to the K-theory groups. We will give some detailed review since we will use it in the sequel.

1A. $Codim(X) = 0$**:** The only face of codimension 0 is X (we are always assuming X to be connected). The isomorphism

$$K_0(A_0) \xrightarrow[\cong]{T_0} H_0^{cn}(X) \tag{75}$$

is simply given by sending the rank one projector p_X chosen above to X.

1B. $Codim(X) = 1$**:** Consider the canonical short exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_1/A_0 \longrightarrow 0$$

That gives the following exact sequence in K-theory

$$0 \longrightarrow K_1(A_1) \longrightarrow K_1(A_1/A_0) \xrightarrow{d_1} K_0(A_0) \longrightarrow 0$$

from which $K_1(A_1) \cong \ker d_1$. By theorem 5.27 and corollary 5.35 in [**33**], we have the following commutative diagram

$$\begin{array}{ccc} K_1(A_1/A_0) & \xrightarrow{d_1} & K_0(A_0) \\ {\scriptstyle T_{1,0}}\downarrow\cong & & \cong\downarrow{\scriptstyle T_0} \\ C_1^{cn}(X) & \xrightarrow[\delta_1]{} & C_0^{cn}(X). \end{array}$$

Then there is a unique isomorphism

$$K_1(\Gamma_b(X)) = K_1(A_1) \xrightarrow[\cong]{T_1} Ker\,\delta_1 = H_1^{pcn}(X) \tag{76}$$

fitting the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(A_1) & \longrightarrow & K_1(A_1/A_0) & \xrightarrow{d_1} & K_0(A_0) & \longrightarrow & 0 \\ & & {\scriptstyle T_1}\downarrow\cong & & {\scriptstyle T_{1,0}}\downarrow\cong & & {\scriptstyle T_0}\downarrow\cong & & \\ 0 & \longrightarrow & Ker\,\delta_1 & \longrightarrow & C_1^{cn}(X) & \xrightarrow[\delta_1]{} & C_0^{cn}(X) & \longrightarrow & 0. \end{array}$$

1C. *Codim*$(X) = 2$**:** Let us start by defining the isomorphism for the even group $K_0(A_2)$. In fact, as shown in proposition 5.40 in [**33**] the morphism

$$K_0(A_2) \xrightarrow{r} K_0(A_2/A_0) \tag{77}$$

induced by restriction to the boundary is an isomorphism. Now, let us consider the canonical short exact sequence

$$0 \longrightarrow A_1/A_0 \longrightarrow A_2/A_0 \longrightarrow A_1/A_0 \longrightarrow 0$$

that gives the following exact sequence in K-theory

$$0 \longrightarrow K_0(A_2/A_0) \longrightarrow K_0(A_2/A_1) \xrightarrow{d_2} K_1(A_1/A_0) \longrightarrow 0$$

from which $K_0(A_2/A_0) \cong \ker d_2$. Again, by 5.27 a,d 5.35 in [**33**], we have the following commutative diagram

$$\begin{array}{ccc} K_0(A_2/A_1) & \xrightarrow{d_2} & K_1(A_1/A_0) \\ {\scriptstyle T_{2,1}}\downarrow\cong & & \cong\downarrow{\scriptstyle T_{1,0}} \\ C_2^{cn}(X) & \xrightarrow[\delta_2]{} & C_1^{cn}(X). \end{array}$$

Then there is a unique isomorphism

$$K_0(\Gamma_b(X)|X_1) = K_0(A_2/A_0) \xrightarrow[\cong]{T_2} Ker\,\delta_2 = H_2^{cn}(X) \tag{78}$$

fitting the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A_2/A_1) & \longrightarrow & K_0(A_2/A_1) & \xrightarrow{d_2} & K_1(A_1/A_0) & \longrightarrow & 0 \\ & & {\scriptstyle T_2}\downarrow\cong & & {\scriptstyle T_{2,1}}\downarrow\cong & & {\scriptstyle T_{1,0}}\downarrow\cong & & \\ 0 & \longrightarrow & Ker\,\delta_2 & \longrightarrow & C_2^{cn}(X) & \xrightarrow[\delta_2]{} & C_1^{cn}(X) & \longrightarrow & 0. \end{array}$$

Let us now recall the isomorphism $T_2 : K_1(A_2) \to H_1^{pcn}(X)$ for X of codimension 2 in odd K-theory. In this case

$$H_1^{pcn}(X) = H_1^{cn}(X) = \frac{Ker\,\delta_1}{Im\,\delta_2}.$$

Consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A_2/A_1 & \longrightarrow & A_2/A_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_2 & \longrightarrow & A_2/A_0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_1/A_0 & \longrightarrow & 0. \end{array} \tag{79}$$

By applying K-theory to it we get (at least the opart that concerns us here)

(80)
$$\begin{array}{ccccccc} & & K_0(A_2/A_1) & \xrightarrow[=]{Id} & K_0(A_2/A_1) & & \\ & & \downarrow & & \downarrow{\scriptstyle d_2} & & \\ 0 & \longrightarrow & K_1(A_1) & \longrightarrow & K_1(A_1/A_0) & \xrightarrow{d_1} & K_0(A_0) \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =\ Id} \\ 0 & \longrightarrow & K_1(A_2) & \longrightarrow & K_1(A_2/A_0) & \longrightarrow & K_0(A_0) \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

As shown in [**33**] p.554-555 the isomorphism

(81)
$$K_1(A_2) \xrightarrow{T_2} \frac{Ker\,\delta_1}{Im\,\delta_2}$$

is then obtained, by a classic chase diagram argument using the previous diagram, as the unique isomorphism fitting the following commutative diagram

(82)
$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ K_1(A_2) & \xrightarrow[\cong]{T_2} & \frac{Ker\,\delta_1}{Im\,\delta_2} \\ \uparrow & & \uparrow{\scriptstyle q} \\ K_1(A_1) & \xrightarrow[\cong]{T_1} & Ker\,\delta_1 \\ \uparrow & & \uparrow{\scriptstyle \delta_2} \\ K_0(A_2/A_1) & \xrightarrow[T_{2,1}]{\cong} & C_2^{cn}(X) \end{array}$$

where $q : Ker\delta_1 \to \frac{Ker\delta_1}{Im\delta_2}$ denotes the canonical quotient projection.

1D. $Codim(X) = 3$**:** Consider the following commutative diagram of short exact sequences

(83)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A_3/A_2 & \longrightarrow & A_3/A_2 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A_1/A_0 & \longrightarrow & A_3/A_0 & \longrightarrow & A_3/A_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A_1/A_0 & \longrightarrow & A_2/A_0 & \longrightarrow & A_2/A_1 & \longrightarrow & 0. \end{array}$$

By applying K-theory to it we get (at least the opart that concerns us here)

(84)
$$\begin{array}{ccccccc} & & K_1(A_3/A_2) & \xrightarrow[=]{Id} & K_1(A_3/A_2) & & \\ & & \downarrow & & \downarrow{\scriptstyle d_3} & & \\ 0 & \longrightarrow & K_0(A_2/A_0) & \longrightarrow & K_0(A_2/A_1) & \xrightarrow{d_2} & K_1(A_1/A_0) \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Id} \\ 0 & \longrightarrow & K_0(A_3/A_0) & \longrightarrow & K_0(A_3/A_1) & \longrightarrow & K_1(A_1/A_0) \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

and then we get an isomorphism

(85)
$$K_0(A_3/A_0) \cong \frac{Ker\, d_2}{Im\, d_3}$$

and by indcution as before the isomorphism

(86)
$$K_0(A_3/A_0) \cong H_2^{cn}(X) := \frac{Ker\, \delta_2}{Im\, \delta_3}.$$

For conclude the even case we just recall that in general the morphism $K_0(A_d) \to K_0(A_d/A_0)$ induced by restriction to the boundary, is an isomorphism for every $d \geq 1$. We get then an isomorphism

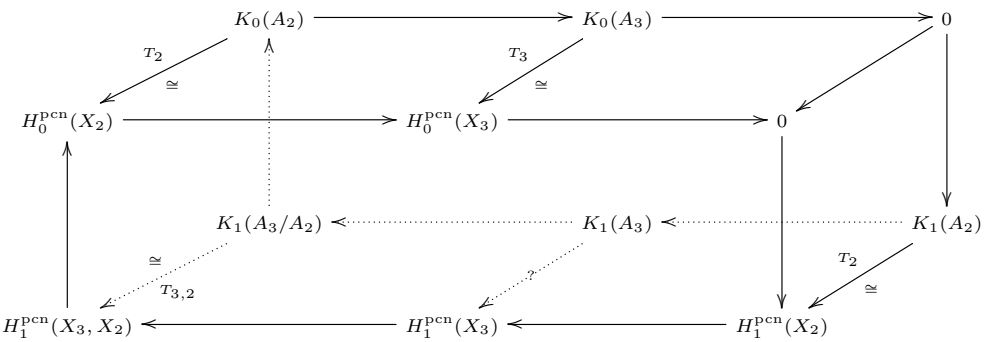
(87)
$$T_3 : K_0(A_3) \xrightarrow{\cong} H_2^{cn}(X).$$

To get the isomorphism from $K_1(A_3)$, consider the short exact sequence:

$$0 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_3/A_2 \longrightarrow 0.$$

We compare its associated six term short exact sequence in K-theory with the one in conormal homology to get

(88)



where we need now to define an isomorphism ?. In fact, if we can define a morphim such that the diagrams are commutative then by a simple Five lemma

argument it would be an isomorphism. Let us first check that the diagram

$$\begin{array}{ccc} K_1(A_3/A_2) & \xrightarrow{\partial} & K_0(A_2) \\ {\scriptstyle T_{3,2}}\downarrow\cong & & \cong\downarrow{\scriptstyle T_2} \\ H_1^{\mathrm{pcn}}(X_3,X_2) & \xrightarrow[\partial]{} & H_0^{\mathrm{pcn}}(X_2) \end{array} \tag{89}$$

is commutative. For this consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_1 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_3/A_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2/A_1 & \longrightarrow & A_3/A_1 & \longrightarrow & A_3/A_2 & \longrightarrow & 0. \end{array} \tag{90}$$

that implies that the connecting morphism $K_1(A_3/A_2) \xrightarrow{\partial} K_0(A_2)$ followed by the morphism $K_0(A_2) \to K_0(A_2/A_1)$ coincides with the connecting morphism $K_1(A_3/A_2) \xrightarrow{\partial} K_0(A_2/A_1)$. Now, the two latter morphisms are compatible with the analogs in conormal homology via the isomorphisms described above and the morphism $K_0(A_2) \to K_0(A_2/A_1)$ is injective (since $K_0(A_1) = 0$), hence the commutativity of diagram 89 above follows.

For defining ?, it is now enough to choose a splitting for the morphism

$$K_1(A_3) \to \operatorname{im} j \to 0,$$

where j is the canonical morphism $j : K_1(A_3) \to K_1(A_3/A_2)$ (remember all the groups $K_*(A_p/A_{p-1})$ are torsion free).

The general codimension case:

For X a mwc of any codimension we cannot proceed as above, in fact, an integral (with integer coefficients) computation as above seems very difficult or perhaps impossible in general from codimension 4 as shown by the recent examples computed by Velasquez and Schick. Still, with Jean-Marie Lescure and Mario Velasquez we constructed an explicit morphism

$$T_{ev/odd} : K_{ev/odd}(\mathcal{K}_b(X)) \longrightarrow H^{cn}_{ev/odd}(X) \otimes \mathbb{Q} \tag{91}$$

that becomes an isomorphism when tensoring with $\mathbb{Q}$ and that factors through an integer computation in the low codimensional cases explained above. I will sketch the main parts to get this isomorphism, the main motivation was the proof of Atiyah-Singer explained in the last section, and as I will expalin further below it gives not only the computation of the K-theory groups above but it gives as well some index theorems for manifolds with corners.

The first ingredient is an embedding

$$\iota : \widetilde{X} \hookrightarrow \mathbb{R}^{N-n}$$

with N even and n still denoting the number of boundary hypersurfaces of X. Consider the groupoid morphism

$$h : \widetilde{X} \times \widetilde{X} \times \mathbb{R}^n \to \mathbb{R}^{N-n} \times \mathbb{R}^n = \mathbb{R}^N \tag{92}$$

given by

$$h(x, y, (\lambda_i)_i) = (\iota(x) - \iota(y), (\lambda_i)_i).$$

The monomorphism[2] of groupoids h induces a semi-direct product groupoid

$$(\widetilde{X} \times \widetilde{X} \times \mathbb{R}^n) \rtimes \mathbb{R}^N \rightrightarrows \widetilde{X} \times \mathbb{R}^N. \tag{93}$$

A very simple and direct computation gives that this groupoid is free (that is, has trivial isotropy subgroups) and proper, the freeness comes from the fact h is a monomorphism of Lie groupoids (indeed $h(x, y, (\lambda_i)_i) = 0$ implies $x = y$ and $\lambda_i = 0$ for all i), and the properness of the map

$$(\widetilde{X} \times \widetilde{X} \times \mathbb{R}^n) \rtimes \mathbb{R}^N \xrightarrow{(t,s)} (\widetilde{X} \times \mathbb{R}^N)^2$$

can be verified by a direct computation.

Now, as shown by Tu in [**95**] proposition 2.10, a topological groupoid $G \rightrightarrows Z$ is proper iff the associated map (t, s) is closed and the stabilizers are quasi-compact. In particular, since $\Gamma_b(X)$ is a closed subgroupoid of $\widetilde{X} \times \widetilde{X} \times \mathbb{R}^n$ and since the induced groupoid morphism

$$\Gamma_b(X) \xrightarrow{h} \mathbb{R}^N \tag{94}$$

is a groupoid monomorphism, since it is the restriction to a subgroupoid of a monomorphism, we obtain that:

Proposition 3.3.1. *The semi-direct product groupoid*

$$\Gamma_b(X) \rtimes \mathbb{R}^N \rightrightarrows X \times \mathbb{R}^N \tag{95}$$

is free and proper.

By [**95**] (section 2), the space of orbits $X \times \mathbb{R}^N / \Gamma_b(X) \rtimes \mathbb{R}^N$ is then Hausdorff and locally compact. We let:

Definition 3.3.2 (The orbit space O_X). *We denote by*

$$O_X := Orb(\Gamma_b(X) \rtimes \mathbb{R}^N) \tag{96}$$

the Orbit space associated with the groupoid $\Gamma_b(X) \rtimes \mathbb{R}^N \rightrightarrows X \times \mathbb{R}^N$.

By classic groupoid's results recalled for instance in [**29**] section 2, we have the following

Proposition 3.3.3. *There is an isomorphism*

$$CT_h : K_*(C^*(\Gamma_b(X))) \longrightarrow K^*_{top}(O_X) \tag{97}$$

given by the composition of the Connes-Thom isomorphism

$$CT : K_*(C^*(\Gamma_b(X))) \longrightarrow K_*(C^*(\Gamma_b(X) \rtimes \mathbb{R}^N))$$

[2] A groupoid morphism $h : G \to \mathbb{R}^N$ is a monomorphism if $h(\gamma) = 0$ implies γ is a unit of G.

and the isomorphism

$$\mu : K_*(C^*(\Gamma_b(X) \rtimes \mathbb{R}^N)) \longrightarrow K^*_{top}(O_X)$$

induced from the groupoid Morita equivalence between $\Gamma_b(X) \rtimes \mathbb{R}^N$ *and* O_X *(seen as a trivial unit groupoid).*

Later on the paper we will need to apply the Chern character morphism to the topological space O_X (to its topological K-theory) and for this we will justify in this section that this space has indeed the homotopy type of a CW-space, in fact this space inherits from $X \times \mathbb{R}^N$ a manifold with corners structure as we will now explain.

Consider the s-connected Puff groupoid, recalled in (35),

$$G_c(\tilde{X}, (\rho_i)) \rightrightarrows \tilde{X}. \tag{98}$$

It is a Lie groupoid and the semi-direct product groupoid

$$G_c(\tilde{X}, (\rho_i)) \rtimes_h \mathbb{R}^N \rightrightarrows \tilde{X} \times \mathbb{R}^N \tag{99}$$

induced by the morphism h defined in (92) is a free proper Lie groupoid by exactly the same arguments applied to $\Gamma_b(X)$ above. By classic results on Lie groupoid theory, the orbit space

$$O_{\tilde{X}} := Orb(G_c(\tilde{X}, (\rho_i)) \rtimes_h \mathbb{R}^N) \tag{100}$$

inherits from $\tilde{X} \times \mathbb{R}^N$ a structure of a C^∞-manifold. A good reference for this is the nice extended survey of Crainic and Mestre, [**44**], that clarifies and explains very interesting results on Lie groupoid theory that were confusing in the literature, in particular they explain the role of the linearization theorem for proper Lie groupoids (theorem 2 in ref.cit.) on the local structure of such groupoids and on their orbit spaces.

We will now give the defining functions on $O_{\tilde{X}}$ whose positive parts will define O_X. For this, denote, as in sections above, a vector $v = (v', v'') \in \mathbb{R}^{N-n} \times \mathbb{R}^n$. A simple and direct computation shows that, for $i = 1, ...n$, the C^∞-map

$$(x, v) \mapsto \rho_i(x) e^{v''_i} \tag{101}$$

induces a well defined C^∞-map

$$\tilde{\rho}_i : O_{\tilde{X}} \to \mathbb{R}. \tag{102}$$

Using the map (112) and the induced homeomorphisms on the faces (110) one can get that

$$O_X = \bigcap_{i=1,...,n} \{\tilde{\rho}_i \geq 0\}. \tag{103}$$

Finally, a simple computation yields

$$d_{(x,A)} R_i(W, V) = e^{A''_i} d_x \rho_i(W) + e^{A''_i} V''_i \rho_i(x) \tag{104}$$

where $R_i = \tilde{\rho}_i \circ p$ (with $p : \tilde{X} \times \mathbb{R}^N \to O_{\tilde{X}}$ the quotient map), $(W, V) \in T_x\tilde{X} \times T_A\mathbb{R}^N$, and since p is a submersion we obtain that $\{d\tilde{\rho}_{j_1}, ..., d\tilde{\rho}_{j_k}\}$ has

maximal rank at any point of $\tilde{H}_{j_1} \cap \ldots \cap \tilde{H}_{j_k}$ for any $1 \leq j_1 < \cdots < j_k \leq n$, where $H_j = \tilde{\rho}_j^{-1}(\{0\}) \cap O_X$.

In conclusion we obtain that O_X is a manifold with embedded corners defined by the defining functions $\tilde{\rho}_1, ..., \tilde{\rho}_n$, the set of its faces of a given codimension is in bijection with the set of faces of X the corresponding codimension. As proved above, each face is homeomorphic to an euclidean space.

Now, it is a classic fact, for example see corollary 1 in [**74**], that any topological separable manifold as O_X has the homotopy type of a countable CW-complex. This is all we will need in the following sections.

The space O_X is a quotient space $X \times \mathbb{R}^N / \sim$ where the relation is given as follows $(x, A) \sim (y, B)$ iff there is $\gamma = ((x, y), (\lambda_i)_i) \in \Gamma_b(X)$ with $B = h(\gamma) + A$. We denote by $\pi : X \times \mathbb{R}^N \longrightarrow O_X$ the quotient map. This map is open [**95**, Prop. 2.11].

The space X is naturally filtrated. Indeed, denote by F_p the set of connected faces of codimension p (and d the codimension of X). For a given face $f \in F_p$, we define the index set $I(f)$ of f to be the unique tuple $(i_1, \ldots, i_p)$ such that $1 \leq i_1 < \ldots < i_p \leq n$ and

$$f \subset H_{i_1} \cap \ldots \cap H_{i_p} \tag{105}$$

where we recall that $H_j = \rho_j^{-1}(\{0\}) \subset X$. The filtration of X is then given by:

$$X_j = \bigcup_{\substack{f \in F \\ d-j \leq \mathrm{codim}(f) \leq d}} f \tag{106}$$

Then:

$$F_d = X_0 \subset X_1 \subset \cdots \subset X_d = X. \tag{107}$$

and setting $Y_p = \pi(X_p \times \mathbb{R}^N)$, we get a filtration of O_X:

$$Y_0 \subset Y_1 \subset \cdots \subset Y_{d-1} \subset Y_d = O_X \tag{108}$$

For any index set I we let:

(109)

$$\mathbb{R}^N_I := \{(y, x) \in \mathbb{R}^{N-n} \times \mathbb{R}^n_+ \ ; \ x_i = 0 \text{ if } i \in I \text{ and } x_i > 0 \text{ otherwise } \} \subset \mathbb{R}^N$$

and we write $\mathbb{R}^N_f$ instead of $\mathbb{R}^N_{I(f)}$. We are going to define a map

$$Q : X \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n_+,$$

smooth and compatible with the equivalence relation on $X \times \mathbb{R}^N$, whose quotient map $q : O_X \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n_+$ induces homeomorphisms:

$$\pi(f \times \mathbb{R}^N) \simeq \mathbb{R}^N_f \tag{110}$$

for any face f and

$$\pi((f \cup g) \times \mathbb{R}^N) \simeq \mathbb{R}^N_{(f,g)} := \mathbb{R}^N_f \ \cup \ \mathbb{R}^N_g \tag{111}$$

for any pair $(f, g) \in F_p \times F_{p-1}$ such that $f \subset \overline{g}$.

For that purpose, we write for convenience e^A for $(e^{A_1}, \dots, e^{A_k})$ and $\rho.v$ for $(\rho_1 v_1, \dots, \rho_k v_k)$ for all k and $A, \rho, v \in \mathbb{R}^k$. Also, we use the notation $v = (v', v'') \in R^{N-n} \times \mathbb{R}^n$ for any $v \in \mathbb{R}^N$. We then define

$$x \in X,\ v \in \mathbb{R}^N, \quad Q(x,v) = (\iota(x) + v', \rho(x).e^{v''}). \tag{112}$$

It is easy to check that $Q : X \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n_+$ is a surjective submersion, compatible with the equivalence relation. We denote by $q : O_X \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n_+$ the quotient map. For any $f \in F_*$, one can check that

(113)

$$f \times \mathbb{R}^N = Q^{-1}(\mathbb{R}^N_f) \text{ and } \forall x, y \in f, v, w \in \mathbb{R}^N,\ Q(x,v) = Q(y,w) \iff (x,v) \sim (y,w).$$

It follows that $q_{|_f}$ and $q_{|_{f \cup g}}$ provide the homeomorphisms (110) and (111). We have proved:

Proposition 3.3.4. *For any pair $(f,g) \in F_{d-q} \times F_{d-q-1}$ such that $f \subset \overline{g}$, we have a commutative diagram:*

$$\begin{array}{ccc} (Y_q \setminus Y_{q-1})_f & \longrightarrow & (Y_{q+1} \setminus Y_{q-1})_{f \cup g} \\ {\scriptstyle q_{|_f}}\downarrow \approx & & \approx \downarrow {\scriptstyle q_{|_{f \cup g}}} \\ \mathbb{R}^N_f & \longrightarrow & \mathbb{R}^N_{(f,g)} \end{array} \tag{114}$$

where the vertical maps are homeomorphisms, the horizontal maps are the inclusions and:

$$(Y_q \setminus Y_{q-1})_f := \pi(f \times \mathbb{R}^N) \quad ; \quad (Y_{q+1} \setminus Y_{q-1})_{f \cup g} := q((f \cup g) \times \mathbb{R}^N) \tag{115}$$

This will be used to compute the singular cohomology groups of O_X, which requires the understanding of the inclusions $Y_q \setminus Y_{q-1} \hookrightarrow Y_{q+1} \setminus Y_{q-1}$ and more specifically how they look like around a given face $f \in F_{d-q}$ with respect to a given $g \in F_{d-q-1}$ with $f \subset \overline{g}$.

We can expect the same difficulties in computing $K^*(O_X)$ as the ones encountered in computing $K_*(C^*(\Gamma_b(X))$ (using the spectral sequence associated with the corresponding filtrations). Instead, the spectral sequence argument becomes simpler for the (singular) cohomology of O_X with compact support. In this section we follow the notations used in [**69**, Sec. 2.2]

The spectral sequence associated to the filtration considered in the last subsection will allow to give a very explicit cohomological computation because of proposition 3.3.4.

Explicitly we associate a cohomological spectral sequence to the filtration 108. It can be done by considering the exact couple

$$\mathcal{A} = \bigoplus_{p,n} H^n(Y_p), \mathcal{E} = \bigoplus_{p,n} H^n(Y_p, Y_{p-1}), \tag{116}$$

with the usual maps $i_{p,n} : H^n(Y_{p+1}) \to H^n(Y_p)$, $j_{p,n} : H^n(Y_p) \to H^{n+1}(Y_{p+1}, Y_p)$ and $k_{p,n} : H^{n+1}(Y_{p+1}, Y_p) \to H^{n+1}(Y_{p+1})$.

Denote by $(E_r^{p,q}(X), d_r^{p,q})$ the associated spectral sequence to the exact couple (116), this spectral sequence converges to

$$E_{p,q}^{\infty}(X) \cong F_{p-1}(H^{p+q}(O_X))/F_p(H^{p+q}(O_X)), \tag{117}$$

where $F_p(H^{p+q}(O_X)) = \ker(H^{p+q}(O_X) \xrightarrow{i^*} H^{p+q}(Y_p))$ and $i : Y_p \to O_X$ is the inclusion, see for instance theorem 2.6 in [**69**]. We have the following result.

PROPOSITION 3.3.5. *The spectral sequence $(E_r^{p,q}(O_X), d_r^{p,q})$ collapses at the page two and moreover, for $r = 0, ..., d$*

$$H^{N-r}(O_X) \cong \bigoplus_{p+q=N-r} E_2^{p,q}(O_X). \tag{118}$$

PROOF. From the exact couple we know that

$$E_1^{p,q}(X) = H^{p+q}(Y_p, Y_{p-1})$$

and a simple application of the long exact sequence axiom and of proposition 3.3.4 gives

$$E_1^{p,q}(X) = \begin{cases} \mathbb{Z}^{|F_{d-p}|} & \text{if } q = N \\ 0 & \text{if } q \neq N \end{cases}$$

On the other hand the first differential is defined as

$$d_1^{p,q} = j_{p,n} \circ k_{p,p+q-1} : H^{p+q}(Y_p, Y_{p-1}) \to H^{p+q+1}(Y_{p+1}, Y_p). \tag{119}$$

The page E^2 has only one non trivial row, which is given by the cohomology of the complex:

$$0 \to E_1^{0,N}\mathbb{Z}^{|F_d|} \to \cdots \to E_1^{d-1,N}\mathbb{Z}^{|F_1|} \to E_1^{d,N}\mathbb{Z} \to 0. \tag{120}$$

It implies that

$$E_{\infty}^{p,q}(X) = E_2^{p,q}(X).$$

Now, for the next part of the statement we need to identify the associated graded group with the singular cohomology of O_X with compact support. The proof of that fact is very similar to the identification of cellular cohomology with singular cohomology.

Indeed, let us consider the long exact sequence of the pair (Y_p, Y_{p-1}) in singular cohomology with compact support

$$\cdots \to H^k(Y_p, Y_{p-1}) \to H^k(Y_p) \to H^k(Y_{p-1}) \to H^{k+1}(Y_p, Y_{p-1}) \to \cdots . \tag{121}$$

Because of the fact that Y_p/Y_{p-1} is homeomorphic to a finite disjoint union of $\mathbb{R}^{N-d+p}$ for $p = 0, ..., d$ we have that the morphisms (induced from the canonical inclusions)

$$H^k(Y_p) \to H^k(Y_{p-1}) \tag{122}$$

are isomorphisms for $k > N - d + p$ and for $k \leq N - d + p - 2$, injective for $k = N - d + p - 1$ and surjective for $k = N - d + p$. A direct computation gives that, for $r = 0, .., d$,

$$F_p H^{N-r}(O_X) = H^{N-r}(O_X) \text{ for } p = -1, ..., d - r - 1, \tag{123}$$

and

$$F_pH^{N-r}(O_X) = 0 \text{ for } p = d-r, ..., d. \tag{124}$$

The statement (118) above follows now by (117).

□

We will construct an explicit isomorphism

$$H^{ev/odd}(O_X) \xrightarrow{B} H^{pcn}_{ev/odd}(X) \tag{125}$$

where in the left hand side $H^{ev/odd}$ stands for singular cohomology (with compact supports) with integer coefficients. For this it will be enough, after the last proposition, to explicitly compute the first differentials $d_1^{p,N-d}$, (119) above.

We start by fixing an $\alpha \in H_1(\mathbb{R})$ with $\alpha \mapsto 1$ under the connecting map (which is an isomorphism) associated to the inclusion of $\{0\}$ in $\mathbb{R}_+$. Let now $\beta \in H^1(\mathbb{R})$ be such that $(\alpha, \beta) \mapsto 1$.

Let $f \in F_p$, there is a canonical homeomorphism $\phi_f : \mathbb{R}^N_f \to \mathbb{R}^{N-p}$ where $\mathbb{R}^{N-p}$ is the usual euclidean space. We let $\beta_f \in H^{N-p}(\mathbb{R}^N_f)$ be the generator given by the image of $(\beta, ..., \beta) \in (H^1(\mathbb{R}))^{N-p}$ by the product isomorphism

$$H^1(\mathbb{R}) \otimes \cdots \otimes H^1(\mathbb{R}) \xrightarrow{\cong} H^{N-p}(\mathbb{R}^{N-p}), \tag{126}$$

where in the left hand side there are exactly $N-p$ copies of $H^1(\mathbb{R})$, followed by the isomorphism in cohomology

$$H^{N-p}(\mathbb{R}^{N-p}) \xrightarrow[\cong]{(\phi_f)^*} H^{N-p}(\mathbb{R}^N_f) \tag{127}$$

induced by ϕ_f.

By construction, for every $p = 0, ..., d$ we have a basis $(\beta_f)_{f \in F_p}$ of $H^{N-p}(Y_{d-p}, Y_{d-p-1})$ via the isomorphism induced from proposition 3.3.4. We can now prove the following

Proposition 3.3.6. *With the notations above we have that for $f \in F_{d-p}$ and $g \in F_{d-p-1}$ with $f \subset \overline{g}$ the following holds:*

$$d_1^{p,N-d}(\beta_f) = \sigma(f,g) \cdot \beta_g, \tag{128}$$

where $\sigma(f,g) = (-1)^{j-1}$ with j the place of the coordinate in the multi-index $I(f)$ whose associated index $i(f,g)$ and correspondant defining function $\rho_{i(f,g)}$ defines f in g.

Proof. By construction the differential

$$d_1^{p,N-d} : H^{N-d+p}(Y_p, Y_{p-1}) \to H^{N-d+p+1}(Y_{p+1}, Y_p)$$

is given by the connecting morphism in cohomology associated to the inclusion

$$Y_p \setminus Y_{p-1} \hookrightarrow Y_{p+1} \setminus Y_{p-1}.$$

Hence, by proposition 3.3.4, we are led to compute the connecting morphism

$$H^{N-d+p}(\mathbb{R}^N_f) \xrightarrow{d_{(f,g)}} H^{N-d+p+1}(\mathbb{R}^N_g) \tag{129}$$

associated to the canonical inclusion

$$\mathbb{R}^N_f \hookrightarrow \mathbb{R}^N_{(f,g)}.$$

Now, it is just a simple algebraic topology exercise to show that

$$d_{(f,g)}(\beta_f) = \sigma(f,g)\cdot\beta_g$$

from which we conclude the proof. □

From the last two propositions we obtain the following corollary.

COROLLARY 3.3.7. *For every $p = 0, ..., d$ the isomorphism $H^{N-p}(Y_{d-p}, Y_{d-p-1}) \to C^{cn}_p(X)$ given in a basis by $\beta_f \mapsto f \otimes \epsilon_{I(f)}$ induces an isomorphism*

$$B_p : H^{N-p}(O_X) \longrightarrow H^{cn}_p(X). \tag{130}$$

In particular, since N is even there are induced isomorphisms between the periodic versions

$$B_{ev/odd} : H^{ev/odd}(O_X) \longrightarrow H^{cn}_{ev/odd}(X). \tag{131}$$

I can state the main theorem of [**34**]

THEOREM 3.3.8. *For every connected manifold with corners X there are morphisms*

$$T_{ev/odd} : K_{ev/odd}(\mathcal{K}_b(X)) \longrightarrow H^{cn}_{ev/odd}(X)\otimes\mathbb{Q} \tag{132}$$

inducing rational isomorphisms. Explicitly, T_* is given by the composition of

(i) *The Connes-Thom isomorphism*

$$CT_h : K_*(\mathcal{K}_b(X)) \to K^*_{top}(O_X), \tag{133}$$

(ii) *the Chern character morphism*

$$K^*_{top}(O_X) \xrightarrow{ch} H^{ev/odd}(O_X)\otimes\mathbb{Q}, \tag{134}$$

which is a rational isomorphism and

(iii) *the natural isomorphism*

$$B_*\otimes Id : H^*(O_X)\otimes\mathbb{Q} \longrightarrow H^{pcn}_*(X)\otimes\mathbb{Q}, \tag{135}$$

described above.

CHAPTER 4

Wrong way functoriality, index theory and the Baum-Connes assembly map

The K-theory functor for topological spaces is a contravariant functor, that is for $f : X \to Y$ a continuous map there is a naturally associated morphism

$$f^* : K^*(Y) \to K^*(X)$$

induced from the pullback of vector bundles. As we will se below, for appropriate f one can define a so called shriek map

$$f! : K^*(X) \to K^*(Y)$$

whose main property is going to be the functoriality, that is $(f \circ g)! = f! \circ g!$ whenever it makes sense. This is what it is called "Wrong way functoriality" in the litterature. Now, the construction of these shriek maps is based in some examples already disussed in this work. For example if M is a smooth manifold, its analytic index morphism

$$Ind_a : K^0(T^*M) \to \mathbb{Z} \cong K^0(\{pt\})$$

is going to be the shriek map

$$p! : K^0(T^*M) \to \mathbb{Z} \cong K^0(\{pt\})$$

associated to the smoooth map

$$p : T^*M \to \{pt\}.$$

In the case of smooth manifolds (without actions of groups or other structures) it is easy to see that in order to get the topological index formula from the wrong way functoriality it is enough to have two more examples, the shriek map associated to the zero section map $s_0 : M \to E$ of a spin vector bundle

$$s_0! : K^*(M) \to K^*(E)$$

given as the Thom isomorphism in K-theory, and the shriek map associated to an inclusion $j; U \to M'$ of an open subset U of M',

$$j! : K^*(U) \to K^*(M')$$

given as the canonical extension by zero. Indeed, the topological index theorem of Atiyah-Singer is a particular case of the wrong way functoriality by decomposing the projection p as $p = q \circ j \circ s_0$ where $q!$ is a computable analytic index morphism (it is the Bott inverse morphism for $\mathbb{R}^N$).

As suggested by the previous discussion, for a general smooth map $f : M \to M'$ (K-oriented) one may use essentially the topological index for families of

Atiyah and Singer to get the shriek map $f!$. This is indeed possible and it is still possible in slighter more general situations, for example one may use the Connes-Skandalis longitudinal to construct $f! : K^*(M) \to K^*(M'/F)$ associated to a (generalized) K-oriented smooth map $f : M \to M'/F$ from a manifold M to the space of leaves of a regular foliation (M', F), so here $K^*(M'/F) := K_*(C^*(M', F))$. However, it is not in general possible to count with topological indices to help define these shriek maps. In fact, in the next section I will describe how to use deformation groupoids to get wrong way funtoriality in a very large context that will as well allow, as we will see in the next chapters, to get very interesting applications. The use of deformation groupoids for get shriek maps as described below it is contained in my paper [**35**] and it is largely inspired from a fundamental paper on the subject by Hilsum and Skandalis, [**57**], where they use for deformation groupoids to define some shriek maps associated to some maps between spaces of leaves of regular foliations.

4.1. Wrong way functoriality via deformation groupoids

Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid. A $\mathscr{G}$-manifold P is a smooth manifold P with a momentum map $\pi_P : P \to M$, which is assumed to be an oriented submersion, and a right action of $\mathscr{G}$ on P: $P \rtimes \mathscr{G} \to P$ given by $(p, \gamma) = p \circ \gamma$ such that

$$(p \circ \gamma_1) \circ \gamma_2 = p \circ (\gamma_1 \cdot \gamma_2)$$

for any $(\gamma_1, \gamma_2) \in \mathscr{G}^{(2)}$. Here $P \rtimes \mathscr{G} = \{(p, \gamma) \in P \times \mathscr{G} | \pi_P(p) = r(\gamma)\}$. We will denote by $T^v P$ the vertical tangent bundle associated to π_P. A $\mathscr{G}$-manifold P is called $\mathscr{G}$-proper if the map

$$P \rtimes \mathscr{G} \longrightarrow P \times P$$

defined by $(p, \gamma) \mapsto (p, p \circ \gamma)$ is proper. Then the induced action groupoid

$$P \rtimes \mathscr{G} \rightrightarrows P$$

with $s(p, \gamma) = p, r(p, \gamma) = p \circ \gamma$ is a proper Lie groupoid.

Notation: In what follows, given a Lie groupoid $\mathscr{H}$ we will denote by $K^*(\mathscr{H})$ the K-theory group of its reduced C^*-algebra $K_*(C^*_r(\mathscr{H}))$.

Let P, N be two $\mathscr{G}$-manifolds and $f : P \longrightarrow N$ be a smooth $\mathscr{G}$-equivariant map such that the bundle $T_f := T^v P \oplus f^* T^v N$ has a $\mathscr{G} - spin^c$ structure, that is such that f is K-oriented. Using only geometric deformation groupoids, we will construct a morphism, called the shriek map $f_!$

$$K^*(P \rtimes \mathscr{G}) \xrightarrow{f_!} K^*(N \rtimes \mathscr{G}) \tag{136}$$

The main result of [**35**] is the functoriality of this shriek map. .

The construction of the shriek map (136) follows the lines of Connes construction, II.6 in [**39**]. It is divided in two steps.

Step 1. The first step is the $\mathscr{G}$-equivariant Thom isomorphism associated to the vector bundle $T_f \to P$.

$$K^*(P \rtimes \mathscr{G}) \xrightarrow{\mathscr{T}} K^*(T_f \rtimes \mathscr{G}). \tag{137}$$

Step 2. The second step is to consider the groupoid immersion

$$P \xrightarrow{f\times\triangle} N \times_M (P \times_M P). \tag{138}$$

The associated deformation groupoid is $D_f \rightrightarrows D_f^{(0)}$ where

$$D_f := T_f \times \{0\} \bigsqcup N \times_M (P \times_M P) \times (0,1] \text{ and} \tag{139}$$

$$D_f^{(0)} = f^*T^vN \times \{0\} \bigsqcup N \times_M P \times (0,1] \tag{140}$$

Notice that $N \times_M (P \times_M P)$ and N are Morita equivalent groupoids with the Morita equivalence given by the canonical projection.

The functoriality of the deformation to the normal cone construction yields an action of $\mathscr{G}$ on D_f.

We can hence consider the deformation index morphism associated to $D_f \rtimes \mathscr{G}$:

$$\begin{array}{ccc} K^*(T_f \rtimes \mathscr{G}) & \xrightarrow{D_f} & K^*(N \times_M (P \times_M P) \rtimes \mathscr{G}) \\ & & \cong \big\downarrow m \\ & & K^*(N \rtimes \mathscr{G}) \end{array} \tag{141}$$

where we denoted D_f instead of $D_{f\times\triangle}$ for keeping the notation short, and m is the isomorphism defined by the Morita equivalence between $N \times_M (P \times_M P) \rtimes \mathscr{G}$ and $N \rtimes \mathscr{G}$.

Definition 4.1.1 (Pushforward morphism). *Let P, N be two manifolds and $f : P \longrightarrow N$ be a smooth $\mathscr{G}$ K-oriented $\mathscr{G}$-map. We let*

$$K^*(P \rtimes \mathscr{G}) \xrightarrow{f_!} K^*(N \rtimes \mathscr{G}) \tag{142}$$

be the morphism given by the composition of the morphisms given in the two last steps.

The wrong way functoriality theorem of [**35**] can be stated.

Theorem 4.1.2. *The above push-forward morphism is functorial, that means, if we have a composition of smooth $\mathscr{G}$-maps between $\mathscr{G}-$manifolds as above:*

$$P \xrightarrow{f} N \xrightarrow{g} L, \tag{143}$$

then the following diagram commutes

$$\begin{array}{ccc} K^*(P \rtimes \mathscr{G}) & \xrightarrow{(g\circ f)_!} & K^*(L \rtimes \mathscr{G}) \\ & {\scriptstyle f_!}\searrow \quad \nearrow{\scriptstyle g_!} & \\ & K^*(N \rtimes \mathscr{G}) & \end{array}$$

In fact, the last theorem above is a particular case of theorem 4.2 in [**35**] because in there we stated and prove it for twisted Lie groupoids. The proof in ref.cit is perhaps very long and involved to read but the main idea behind was to have the proof for $\mathscr{G} = \{pt\}$ and use the deformation to the normal cone functoriality to be able to pass to more general cases.

4.2. The Baum-Connes assembly map or assembling pushforwards

As I tried to explain above the wrong way functoriality implies some topological index theorems in some very interesting cases. Now, the functoriality also has a deep consequence, it allows to assembly different pushforwards maps and to assembly different higher indices. This is the content of the so called Baum-Connes assembly map. Here I will describe a geometric version of it that uses the above wrong way functoriality, in the next chapter I will discus further the very interesting case of discret groups for which we have more results at the present time. The starting point is to define the left hand side of the geometric Baum-Connes map.

Definition 4.2.1 (Topological K-theory for Lie groupoids). *Let $\mathscr{G} \rightrightarrows M$. The* topological K-theory group *associated to $\mathscr{G}$ is the abelian group denoted by $K^*_{top}(\mathscr{G})$ with generators given by the cycles (P, x) where*

1. *P is a smooth co-compact $\mathscr{G}$-proper spin^c manifold,*
2. *$\pi_P : P \to M$ is the smooth momentum map which supposed to be an oriented submersion, and*
3. *$x \in K^*(P \rtimes \mathscr{G})$,*

and the equivalence relations

$$(P, x) \sim (P', g_!(x)) \tag{144}$$

where $g : P \to P'$ is a smooth $\mathscr{G}$-equivariant map (in particular $\pi_{P'} \circ g = \pi_P$).

Now, to get the Baum-Connes assembly map it is enough to remark that for P a smooth co-compact $\mathscr{G}$-proper *spin^c* manifold as above the same constructio of the shriek map $\pi_P! : K^*(P \rtimes \mathscr{G}) \to K^*(\mathscr{G})$ makes sense, *i.e.* when $N = \{pt\}$. We can state the theorem resuming this discussion.

Theorem 4.2.2. *[Geometric Baum-Connes assembly map for Lie groupoids] Let (P, x) be a geometric cycle over $\mathscr{G}$. Let $\mu_{\mathscr{G}}(P, x) = (\pi_P)_!(x)$. Then $\mu_{\mathscr{G}}(P, x)$ only depends upon the equivalence class of the cycle (P, x). Hence we have a well defined assembly map*

$$\mu_{\mathscr{G}} : K^*_{top}(\mathscr{G}) \longrightarrow K^*(\mathscr{G}). \tag{145}$$

Let us give an example of computation of the above assembly map.

Proposition 4.2.3. *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid, and let $\mathcal{C}_{\mathscr{G}}$ be the category of proper $\mathscr{G}$-manifolds as above and homotopy classes of smooth $\mathscr{G}$-equivariant maps. Then if Q is a final object for $\mathcal{C}_{\mathscr{G}}$ one has an isomorphism*

$$K^*_{top}(\mathscr{G}) \xrightarrow[\cong]{\mu_Q} K^*(Q \rtimes \mathscr{G}). \tag{146}$$

Moreover the following commutative diagram

$$K^*_{top}(\mathscr{G}) \xrightarrow[\cong]{\mu_Q} K^*(Q \rtimes \mathscr{G}) \tag{147}$$

$$\mu_{\mathscr{G}} \searrow \qquad \swarrow (\pi_Q)_!$$

$$K^*(\mathscr{G})$$

PROOF. Let (P, x) be a geometric cycle over $\mathscr{G}$. By hypothesis there is a $\mathscr{G}$-equivariant map $q_P : P \to Q$ since Q is a final object in $\mathcal{C}_{\mathscr{G}}$. We define μ_Q by

$$\mu_Q([P, x]) = (q_Q)_!(x) \tag{148}$$

which is well defined by theorem 4.1.2 above.

We will explicitly define the inverse. Let $y \in K^*(Q \rtimes \mathscr{G})$, we define $\beta_Q(y) \in K^*_{top}(\mathscr{G})$ to be the class of the cycle (Q, y).

In one direction, $\mu_Q(\beta_Q(y)) = y$ is obvious, and in the other direction, $\beta_Q(\mu_Q([P, x])) = [Q, (q_P)_!(x)] = [P, x]$.

The commutativity of diagram (147) follows immediately from wrong way functoriality. □

EXAMPLES 4.2.4.

(1) *The most basic example in which the last proposition applies is when the groupoid $\mathscr{G} \rightrightarrows M$ is proper with $M/\mathscr{G}$ compact and M spin^c. This covers the case of (spin^c) orbifold groupoids. Then we have an explicit isomorphism*

$$K^*_{top}(\mathscr{G}) \xrightarrow[\cong]{\mu_M} K^*(\mathscr{G}). \tag{149}$$

(2) *A very interesting example where one can apply the computation above is the following (Connes book* [**39**] 10.β*): Let G be a connected Lie group and let L be a maximal compact subgroup of G, by a result of Abels and Borel (*[**1**]*), the homogeneous space $Q = L \setminus G$ is a final object of $\mathcal{C}_G$. Then, assuming $L \setminus G$ has a G-spin^c structure* [1]*, there is an explicit isomorphism*

$$K^*_{top}(G) \xrightarrow[\cong]{\mu_{L\setminus G}} K^*((L \setminus G) \rtimes G,). \tag{150}$$

Note that the action of G on the homogeneous space $L \setminus G$ is transitive, hence the groupoid $L \setminus G \rtimes G$ is transitive as well. Now, we know (proposition 5.14 in [**76**]*) transitive groupoids are Morita equivalent to Lie groups, and more explicitly one Lie group model could be given by an isotropy group. In our case, it is easy to check that the isotopry group of the class of the identity $(L \setminus G \rtimes G)_{[e]}$ identifies canonically with L. Hence, the canonical inclusion $L \hookrightarrow L \setminus G \rtimes G$ given by*

$$l \mapsto ([e], l)$$

is a Morita equivalence of groupoids (proposition 5.14 (iv) in [**76**]*). Using (150) and the Morita equivalence just described, we can obtain an isomorphism*

$$K^*_{top}(G) \xrightarrow[\cong]{\mu_L} K^*(L) \tag{151}$$

[1] If this is not the case some twistings will appear, see [**35**].

such that

$$K^*_{top}(G) \xrightarrow[\cong]{\mu_L} K^*(L) \qquad K^*(G) \tag{152}$$

is a commutative diagram where $i : L \hookrightarrow G$ is the inclusion.

This particular example is very interesting, in particular for its relation with representation theory. Recently, it has been a great amount of interest in this examples for several kind of groups and there has been several authors that have been using deformation groupoids to attack or solve some questions. See for example the recent papers [**37**] *and* [**94**]*, together with the references there in for a general view point.*

4.3. An orbifold refinement of the Baum-Connes assembly map

Another application of the use of deformation groupoids in the definition of the Baum-Connes assembly map is that there is not too much extra cost to add in the definition of the geometric cycles forming the BC left hand side more "singular cylces", for example we can easily add orbifolds as I will now explain.

Consider two proper étale groupoids $\mathfrak{X}_1 \rightrightarrows M_1$ and $\mathfrak{X}_2 \rightrightarrows M_2$. Suppose we have a smooth map

$$\begin{array}{ccc} \mathfrak{X}_1 & \xrightarrow{f} & \mathfrak{X}_2 \\ \downarrow\downarrow & & \downarrow\downarrow \\ M_1 & \xrightarrow[f_0]{} & M_2 \end{array} \tag{153}$$

If the above map is an immersion of groupoids, Hilsum and Skandalis construction in [**57**] gives an element $f! \in KK_*(\mathfrak{X}_1, \mathfrak{X}_2)$ under the assumption that the normal bundle $N(\mathfrak{X}_2, \mathfrak{X}_1)$ admits a *spin*^{c} structure. Their construcion uses deformation groupoids and it inspired our work for the general case. In fact, in the general case one can consider, as in the case of manifolds, the immersion of Lie groupoids

$$f \times \Delta : \mathfrak{X}_1 \to \mathfrak{X}_2 \times (M_1 \times M_1)$$

and get $f! : K^*(\mathfrak{X}_1) \to K^*(\mathfrak{X}_2)$ by using the associated deformation groupoid to $f \times \Delta$. In fact, our construction works, verbatim, in the presence of an external Lie groupoid action of a given Lie groupoid $\mathscr{G} \rightrightarrows M$. Thus, if f is an equivariant $K-$oriented $\mathscr{G}$-map we get a shriek map

$$f! : K^*(\mathfrak{X}_1 \rtimes \mathscr{G}) \to K^*(\mathfrak{X}_2 \rtimes \mathscr{G}) \tag{154}$$

using the exact same deformation groupoid together with the functoriality of the deformation to the normal cone construction to get the associated semi-direct product groupoid given by the action of $\mathscr{G}$.

The wrong way functoriality gives first the assembled group

Definition 4.3.1 (Topological K-theory for Lie groupoids, an orbifold refinement). *Let $\mathscr{G} \rightrightarrows M$. The* Orbifold topological K-theory group *associated to $\mathscr{G}$ is the abelian group denoted by $K^*_{top,orb}(\mathscr{G})$ with generators given by the cycles (P, x) where*

(1) *$\mathfrak{X}$ is a smooth co-compact $\mathscr{G}$-proper spin^c orbifold,*
(2) *$\pi_{\mathfrak{X}} : \mathfrak{X} \to M$ is the smooth momentum map which supposed to be a submersion, and*
(3) *$x \in K^*(\mathfrak{X} \rtimes \mathscr{G})$,*

and the equivalence relations

$$(\mathfrak{X}_1, x) \sim (\mathfrak{X}_2, g_!(x)) \tag{155}$$

where $g : \mathfrak{X}_1 \to \mathfrak{X}_2$ is a smooth $\mathscr{G}$-equivariant map.

One gets then the corresponding Baum-Connes assembly map

Theorem 4.3.2. *[Geometric Baum-Connes assembly map for Lie groupoids, the Orbifold version] Let $(\mathfrak{X}, x)$ be an orbifold cycle over $\mathscr{G}$. Let $\mu^{orb}_{\mathscr{G}}(\mathfrak{X}, x) = (\pi_{\mathfrak{X}})_!(x)$. Then $\mu^{orb}_{\mathscr{G}}(\mathfrak{X}, x)$ only depends upon the equivalence class of the cycle $(\mathfrak{X}, x)$ in $K^*_{top,orb}(\mathscr{G})$. Hence we have a well defined assembly map*

$$\mu^{orb}_{\mathscr{G}} : K^*_{top,orb}(\mathscr{G}) \longrightarrow K^*(\mathscr{G}). \tag{156}$$

We have the following result relating the two assembly maps, its proof is immediate from the defintions and wrong way functoriality.

Proposition 4.3.3. *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid. There is a canonical group morphism*

$$\lambda : K^*_{top}(\mathscr{G}) \to K^*_{top,orb}(\mathscr{G}) \tag{157}$$

sending the class of a manifold cycle (M, x) to the class of the orbiolfd cycle (M, x). It fits the following commutative diagram

$$\begin{array}{ccc} K^*_{top,orb}(\mathscr{G}) & \xrightarrow{\mu^{orb}_{\mathscr{G}}} & K^*(\mathscr{G}) \\ \lambda \uparrow & \nearrow_{\mu_{\mathscr{G}}} & \\ K^*_{top}(\mathscr{G}) & & \end{array} \quad . \tag{158}$$

The details of this section will appear in a forthcoming paper where I will go further and prove (for the moment a conjecture) that the morphism λ above is an isomorphism, I will besides study the orbifold version in the very intersting case of discrete groups.

CHAPTER 5

Topological index theory for discrete groups and the Chern-Baum-Connes pairing formula

Let Γ be a countable discrete group. In [**13, 14, 15**], Baum and Connes initiated a program on index theory for Γ-manifolds that lead to the so-called Baum-Connes assembly map associated to the group Γ and to the ensuing Baum-Connes conjecture that states that the assembly map is an isomorphism. The conjecture is at the present time still open, even though it has been proved (or in some cases just the injectivity or the surjectivity) for a large class of groups. One of the main reasons this conjecture became an important source of research is that it implies some other important conjectures in geometry, topology and analysis, for example, to mention the most famous ones, the Novikov conjecture, the stable Gromov-Lawson conjecture, the modified trace conjecture and the Kadison-Kaplansky conjecture; See for instance [**59, 58, 39**]. The assembly map in question is the group morphism

$$\mu : K^*_{top}(\Gamma) \longrightarrow K_*(C^*_r(\Gamma)) \tag{159}$$

constructed in the last chapter applied to the discrete group Γ. This assembly morphism has been studied extensively in recent years in the context of higher secondary invariants, Gromov-Lawson indices, problems related to (existence and classification of) positive scalar curvature metrics and surgery on manifolds, for example see [**20**], [**21**], [**36**], [**50**], [**56**], [**89**], [**102**] or [**106**], just to mention some recent developments related in some way to our work. For a general overview of the Baum-Connes assembly map the reader may consult [**55**], [**75**], [**98**].

Now, in the original papers of Baum and Connes, ref.cit., the cycles for the left hand side consist of classes of couples (M, x), where M is a Γ-proper, $spin^c$, cocompact manifold and x is an element of the equivariant K-theory group[1] $K^*_\Gamma(M)$ and the equivalence relation is generated by being equal up to a Γ-family index, to be more precise, if $f : M \to N$ is an appropriate Γ-equivariant map between manifolds as above, then we shall have $(M, x) \sim (N, f_!(x))$ where the shriek map $f_!$ is a Γ-family K-theoretical index morphism. In the original papers of Baum and Connes the assembly was then defined to be

$$[M, x] \mapsto \pi_M!(x) \in K_*(C^*_r(\Gamma))$$

[1] A model for this can be the K-theory group of the C^*-algebra $C^*_r(M \rtimes \Gamma)$ associated to the action groupoid $M \rtimes \Gamma$.

where $\pi_M!(x)$ is the Γ-higher index associated to the projection $\pi_M : M \to \{*\}$. Now, even if Baum and Connes gave the fundamental ideas and directions to follow, not all proofs were completed, but just sketched. For example, in those papers the well-definedness of the left hand side groups was sketched and not proved. The fact that it is well-defined is a consequence of the wrong way functoriality theorem proved in a more general context in [**35**]. This wrong way functoriality also implies that the originally defined Baum-Connes assembly map is well-defined.

After the original formulation mentioned in the work of Baum and Connes, several different directions occurred; In fact, the assembly map was formalised some years later by Baum, Connes and Higson in [**16**] using the powerful tool of Kasparov's KK-theory to properly define a "Left hand side" and the assembly map, with the time other models for this left hand side also entered into the game (Baum-Douglas model, Lück model). The emergence of KK-theory led to deep and powerful results on the injectivity and surjectivity of the assembly map and allowed to establish several properties. One of the main motivations of the original ideas of Baum-Connes was to be able to give geometric formulas for the higher index, to be able to use classical topological tools for computing higher analytical indices, however actual index formulae computations of the assembly map are rare, for example in [**13**] Baum and Connes sketched a construction of a Chern character from the left hand side that would eventually lead to an actual pairing with higher currents or higher traces. In [**30**] we overcome this problem and fill this gap by constructing a Chern character type morphism

$$K^*_{top}(\Gamma) \xrightarrow{ch^{top}_{\mu}} HP_*(\mathbb{C}\Gamma) \tag{160}$$

that will be called the Chern-Baum-Connes assembly map because it is constructed by assembling as Baum-Connes, and as will be detailed below, different Chern characters. We hence complete the program on the Chern character for discrete groups as stated in [**13**] and [**14**] by Baum and Connes. Besides, this geometric approach allows us to continue this program by first comparing the models we use in our work with the more common analytic models used in the literature and second by giving a complete solution, for countable discrete groups, to the problem of defining and computing a geometric pairing between the left hand side of the Baum-Connes assembly map, given in terms of geometric cycles associated to proper actions of the discrete group, and the cyclic periodic cohomology of the group algebra. We now explain in more details the contents of our work.

For a Γ-proper manifold M, there are periodic delocalised cohomology groups $H^*_{\Gamma,deloc}(M)$, even and odd, that might be defined by periodizing the de Rham cohomology of the so-called inertia groupoid (for more details see appendix A or section 3 in [**30**]). In fact, one justification of why this is the correct group to consider is because there is a Γ-Chern character morphism

$$ch_M^\Gamma : K_\Gamma^*(M) \longrightarrow H^*_{\Gamma,deloc}(M) \tag{161}$$

given explicitly as the composition of the Connes-Chern character[2]

$$Ch : K_\Gamma^*(M) \longrightarrow HP_*(C_c^\infty(M \rtimes \Gamma)). \tag{162}$$

followed by an isomorphism

$$HP_*(C_c^\infty(M \rtimes \Gamma)) \xrightarrow[\cong]{TX} H^*_{\Gamma,deloc}(M). \tag{163}$$

explicitly described in [**97**] by Tu and Xu, and recalled in one of the appendices in [**30**]. In fact, Tu and Xu proved even more in [**97**] that the Chern character (161) is an isomorphism up to tensoring with $\mathbb{C}$.

From our perspective there are two major technical problems in index theory for Γ-proper actions on smooth manifolds:

(i) Relate different Γ-proper actions and their pairings as above. Explicitly this means being able to prove (and give sense to) Riemann-Roch type theorems and pushforward functoriality for such actions and then being able to assemble these results to give an explicit formulation of a pairing

$$K^*_{top}(\Gamma) \times HP^*(\mathbb{C}\Gamma) \to \mathbb{C} \tag{164}$$

that only depends on the group Γ and that contains/exhausts all possible index pairings for Γ-proper smooth cocompact manifolds.

(ii) Compute explicit index formulas for the pairing above. In particular, as we will discuss below, the cyclic periodic cohomology groups $HP^*(\mathbb{C}\Gamma)$ can be computed explicitly, by the work of Burghelea [**25**], in terms of group cohomology groups associated to the centralizers of elliptic and non elliptic elements. The problem of computing an explicit formula for the pairing above can be stated as giving a geometric/topological expression for the pairings

$$\langle [M, x], \tau \rangle \in \mathbb{C} \tag{165}$$

for a geometric cycle $[M, x] \in K^*_{top}(\Gamma)$ and a group cocycle τ in a group cohomology (that only depends on its conjugacy class) of some elliptic element (as we will see it is zero for non-elliptic elements). There are several particular cases computed in the literature, for free and proper actions and localised at the unit cocycle by Connes and Moscovici [**40**], for particular Connes-Moscovici cocyles by Ponge and Wang in [**92, 91**], for some delocalised traces but under some extra hypothesis on the extensibility of the cocycles by the two last named authors in [**101**], for mention some of them, see also [**88, 86**]. We are certainly aware of several other computations but as far as we know there is

[2]Defined a priori in $K_*(C_c^\infty(M \rtimes \Gamma))$ but $C_c^\infty(M \rtimes \Gamma) \subset C^*(M \rtimes \Gamma)$ is stable under holomorphic calculus since the action is proper.

not a general and systematic procedure or formula that computes the above pairing.

In this chapter we will recall how to completely solve the two problems above. The complete proofs can be found in my paper [**30**]. In order to be more precise, we need to introduce some terminology and explain what are the results leading to the solution of these problems.

5.1. Pushforward maps in periodic delocalised cohomology and delocalised Riemman-Roch theorem

Let $f : M \longrightarrow N$ be a Γ-equivariant C^∞-map between two Γ-proper cocompact manifolds M and N. We defined in [**30**] definition 3.2 (see definition 6.2.1 for a definition in a slighter general case) a pushforward (shriek) map in periodic delocalised cohomology

$$f_! : H^{*-r_f}_{\Gamma,deloc}(M) \longrightarrow H^*_{\Gamma,deloc}(N), \tag{166}$$

where r_f is the rank of the vector bundle $T_f = TM \oplus f^*TN$, and under the assumption that T^*_f admits a $spin^c$-structure, in other words, f is Γ-equivariant and K-oriented. The construction of these shriek maps follows the same lines of the shriek maps in K-theory constructed in [**35**] (following the ideas proposed originally in [**15**] and [**39**]). Indeed we use f to construct an appropriate deformation groupoid that allows to define $f_!$ as a composition of a Thom isomorphism followed by a deformation morphism, see Section 3 in [**30**] for more details.

A very important and fundamental point to remark is that to be able to apply constructions similar to those used in K-theory one needs, besides suitable groupoids, appropriate smooth subalgebras of the C^*-algebras of the corresponding groupoids, in particular appropriate enough to have the right exact sequences that lead to classical arguments in algebraic topology (the homotopy invariance, periodicity, six term exact sequences, etc.). In our case we explained in Appendices A and B in [**30**] which are the main groupoids used in this paper and which algebras we are using in each case when applying cohomology (mainly periodic cyclic (co)homology) to them, see also appendix D. In Appendix C of [**30**] we construct and study the deformation index type morphisms applied to those algebras/groupoids in periodic cyclic (co)homology.

The first main result of this chapter is that these cohomological pushforward maps are compatible with the K-theoretical pushforward maps (defined in [**35**]) via a twisted Chern character. The result states as follows (Theorem 3.4 in [**30**]):

THEOREM 5.1.1 (Delocalised Riemman-Roch). *Let $f : M \longrightarrow N$ be a Γ-equivariant, K-oriented C^∞-map between two Γ-proper cocompact manifolds M*

and N. The following diagram is commutative

$$\begin{array}{ccc} K_\Gamma^{*-r_f}(M) & \xrightarrow{f_!} & K_\Gamma^*(N) \\ {\scriptstyle ch_{Td_M^\Gamma}}\downarrow & & \downarrow{\scriptstyle ch_{Td_N^\Gamma}} \\ H_{\Gamma,deloc}^{*-r_f}(M) & \xrightarrow[f_!]{} & H_{\Gamma,deloc}^*(N). \end{array} \tag{167}$$

In the above statement the twisted Chern character $ch_{Td_M^\Gamma}$ is the morphism

$$ch_{Td_M^\Gamma} : K_\Gamma^*(M) \longrightarrow H_{\Gamma,deloc}^*(M) \tag{168}$$

given by $x \mapsto ch_M^\Gamma(x) \wedge Td_M^\Gamma$, where Td_M^Γ is a delocalised Todd class, recalled/defined in Section 3.2 in [**30**].

5.2. Delocalised cohomology wrong way functoriality, Chern character for discrete groups

In order to assembly the above delocalised Riemann-Roch diagram, we need to establish first the functoriality of the pushforward maps in delocalised cohomology. This again is done by using appropriate deformation groupoids, appropriate smooth subalgebras associated to them and classic cohomological properties that allows to adapt to this setting the K-theoretical analog proof. The result (Theorem 4.1 [**30**]) states as follows:

THEOREM 5.2.1 (Pushforward functoriality in delocalised cohomology). *Let $f : M \longrightarrow N$ and $g : N \longrightarrow L$ be Γ-equivariant, K-oriented C^∞-maps between Γ-proper cocompact manifolds as above. Then*

$$(g \circ f)_! = g_! \circ f_! \tag{169}$$

as morphisms from $H_{\Gamma,deloc}^*(M)$ to $H_{\Gamma,deloc}^{*+r_{g\circ f}}(L)$ ($r_{g\circ f} = r_f + r_g \ mod\, 2$).

One of the first things one can do with such a result is to assemble the groups $H_{\Gamma,deloc}^*(M)$ by using the pushforward maps and in an essential way the pushforward functoriality. For $* = 0, 1 \ mod\, 2$, we can consider the abelian groups

$$H_{top}^*(\Gamma) = \varinjlim_{f_!} H_{\Gamma,deloc}^*(M). \tag{170}$$

where the limit is taken over the Γ-proper cocompact *spin*^{c} manifolds of dimension equal to $*$ modulo 2.

Equivalently it can be defined as the abelian free group with generators (M, ω) with M a Γ-proper cocompact *spin*^{c} manifold and $\omega \in H_{\Gamma,deloc}^*(M)$ with dimension of M equal to $*$ modulo 2, and the equivalence relation generated by $\omega \sim f_!(\omega)$ for $f : M \to N$ as above.

As a direct consequence of the two theorems above together with the pushforward functoriality in K-theory (Theorem 4.2 in [**35**]) we can assemble the twisted Chern characters and get the following Chern character for any discrete group Γ (Theorem 4.5 in [**30**]):

THEOREM 5.2.2 (Chern character for discrete groups). *For any discrete group* Γ*, there is a well defined Chern character morphism*

$$ch^{\Gamma} : K^*_{top}(\Gamma) \longrightarrow H^*_{top}(\Gamma) \tag{171}$$

given by

$$ch^{\Gamma}([M,x]) = [M, ch^{\Gamma}_M(x) \wedge Td^{\Gamma}_M]. \tag{172}$$

Furthermore, it is an isomorphism once tensoring with $\mathbb{C}$*.*

5.3. The Chern-Baum-Connes assembly map and the Chern-Connes pairing formula for discrete groups

In [**100**] Voigt constructs a Chern character from bivariant equivariant Kasparov theory to the equivariant bivariant cohomology theory defined previously by Baum and Schneider in [**12**]. He obtained in particular a Chern character from the analytic left hand side model of Baum and Connes and some delocalised Baum and Schneider cohomology group. In principle his Chern character and ours should coincide for formal reasons (at least it is expected to be like that), it would be very interesting to explore this relation. Another related Chern character was constructed by Matthey in [**68**] following the related works of Lück in [**66**] and [**65**], in these papers the analytic model is used as well. Even if we do not compare our Chern morphisms with that by Matthey[3], we follow in Section 6 the ideas and results of Matthey on the delocalization of the Baum-Connes assembly map. Now, there is however a big difference on the approaches and techniques developed for these in principle different Chern characters but also in the actual computation in terms of cycles, the Chern character in the Theorem above is computed explicitly in terms of cycles and of the delocalised Todd classes thanks to the delocalised Riemann-Roch theorem and the wrong way functorialities in K-theory and in equivariant cohomology. This will play a major role in the actual computation of index pairing formulas developed below.

To describe what follows we need to give more details on the periodic delocalized cohomology groups $H^*_{\Gamma,deloc}(M)$ for a Γ-proper cocompact $spin^c$ manifold M. In fact, since the action of Γ on M is proper, and for $* = 0, 1 \ mod 2$, the associated periodic delocalised cohomology groups can be given, as explained in appendix D in [**30**], by

$$H^*_{\Gamma,deloc}(M) := \bigoplus_{\langle g\rangle \in \langle\Gamma\rangle^{fin}} \prod_{k=* \, mod\, 2} H^k_c(M_g \rtimes \Gamma_g), \tag{173}$$

where $\langle\Gamma\rangle^{fin}$ stands for the set of conjugacy classes of finite order elements (called elliptic elements as well) of Γ with a fixed finite order element g of its conjugacy class, $\Gamma_g = \{h \in \Gamma : hg = gh\}$ is the centralizer of g; and where $M_g \rtimes \Gamma_g$ stands for the action groupoid associated to the canonical action by conjugation of the centralizer $\Gamma_g = \{h \in \Gamma : hg = gh\}$ on the fixed point submanifold $M_g = \{x \in M : x \cdot g = x\}$, and $H^k_c(M_g \rtimes \Gamma_g)$ stands for the compactly supported

[3]It would be very interesting but it is out of the particular subject of the present article.

de Rham groupoid cohomology as reviewed in the next section. For these groups, we prove in Section 6 Lemma 6.1 in [**30**] that the canonical integrations along the fibers of the canonical groupoid projections

$$M_g \rtimes \Gamma_g \longrightarrow \Gamma_g, \tag{174}$$

$$\pi_{M_g}! : \Omega^n_c(M_g \times \Gamma^p_g) \longrightarrow \mathbb{C}\Gamma^p_g, \tag{175}$$

with $p \in \mathbb{N}$ and $n = dim\, M_g$, given by

$$\pi_{M_g}!(\omega)(\gamma) := \int_{M_g} \omega(m, \gamma) \tag{176}$$

for $\gamma \in \Gamma^p_g$ and for $\omega \in \Omega^n_c(M_g \times \Gamma^p_g)$, induce a well-defined morphism

$$H^*_{\Gamma, deloc}(M) \xrightarrow{\pi_M!} H_*(\Gamma, F\Gamma), \tag{177}$$

for every M of dimension $n = *, mod\, 2$, and where

$$H_*(\Gamma, F\Gamma) := \left(\bigoplus_{\langle g \rangle \in \langle \Gamma \rangle^{fin}} \bigoplus_{k = * \, mod\, 2} H_k(\Gamma_g; \mathbb{C}) \right) \tag{178}$$

is the group homology of Γ with coefficients in the complex vector space $F\Gamma$ generated by the elliptic elements (See Appendix C of Matthey [**68**] for a discussion of this isomorphism that uses the Shapiro's Lemma).

Section 6 of [**30**]is mainly dedicated to prove, through several lemmas and propositions of independently interest, the following theorem.

THEOREM 5.3.1. *The maps*

$$H^*_{\Gamma, deloc}(M) \xrightarrow{\pi_M!} H_*(\Gamma, F\Gamma), \tag{179}$$

induce a well-defined cohomological assembly map

$$\mu_{F\Gamma} : H^*_{top}(\Gamma) \longrightarrow H_*(\Gamma, F\Gamma) \tag{180}$$

given by

$$\mu_{F\Gamma}([M, \omega]) = \pi_M!(\omega). \tag{181}$$

There are also other models in the literature for a cohomological assembly map as in Theorem 5.3.1. See for example [**43**] and [**53**] for more formal constructions. The novelty of our construction (see Section 5) is that it is given explicitly in terms of cohomological pushforward maps and based on the wrong way functoriality. It would be of course interesting to compare the above assembly to the previous ones, since for example in the above cited papers the authors get very interesting properties related to index theory. In our case, the assembly map from Theorem above will allow, as explained below, to give index formulas for the pairing between periodic cyclic cohomology and topological K-theory for discrete groups.

Now, the periodic cyclic homology groups of $\mathbb{C}\Gamma$ have been computed. Indeed Burghelea showed in [**25**] that there are isomorphisms

$$HP^*(\mathbb{C}\Gamma) \cong \left(\prod_{\langle g \rangle \in \langle \Gamma \rangle^{fin}} \bigoplus_{k=* \, mod \, 2} H^k(\Gamma_g; \mathbb{C}) \right) \bigoplus \prod_{\langle g \rangle \in \langle \Gamma \rangle^{\infty}} T^*(g, \mathbb{C}) \tag{182}$$

where $\langle \Gamma \rangle^{\infty}$ stands for the set of conjugacy classes of infinite order elements) of Γ, $g \in \Gamma$ is a fixed element (of infinite order for the second factor) of its conjugacy class; and where $T^*(g, \mathbb{C})$ are some limit groups depending on the cohomology groups of N_g, the quotient group of the centralizer $\Gamma_g = \{h \in \Gamma : hg = gh\}$ by the subgroup generated by g, but that will not be explicitly described in this paper or its sequel since this factor will not enter in our index formula computations. In fact, by Burghelea's work [**25**] there is a morphism

$$B : H_*(\Gamma, F\Gamma) \longrightarrow HP_*(\mathbb{C}\Gamma) \tag{183}$$

which is an isomorphism onto its image as a direct factor. In particular, we can consider the assembly map

$$\mu_\Gamma : H^*_{top}(\Gamma) \longrightarrow HP_*(\mathbb{C}\Gamma) \tag{184}$$

given by the composition of the cohomological assembly map $\mu_{F\Gamma}$ of the theorem above followed by Burghelea's morphism B.

We are ready to state the final theorem that combines all the above mentioned results and gives an explicit formula for the index pairing. For this, note that there are canonical morphisms (for every g of finite order)

$$H^k(\Gamma_g; \mathbb{C}) \xrightarrow{\pi_g^*} H^k(M_g \rtimes \Gamma_g) \tag{185}$$

induced from the canonical groupoid projection $\pi_g : M_g \rtimes \Gamma_g \to \Gamma_g$. As we justify in Section 6 in [**30**] these pullback morphisms are in duality with the morphisms induced by integration along the fibers described in the last theorem. This finally leads to the following theorem (see Theorems 6.7 and 6.8 in [**30**]) that gives an index theoretical geometric formula for the pairing.

THEOREM 5.3.2 (The Chern assembly map). *For any discrete group Γ, there is well-defined morphism*

$$K^*_{top}(\Gamma) \xrightarrow{ch^\Gamma_\mu} HP_*(\mathbb{C}\Gamma) \tag{186}$$

given by the composition of the Chern character

$$ch^\Gamma : K^*_{top}(\Gamma) \longrightarrow H^*_{top}(\Gamma) \tag{187}$$

followed by the cohomological assembly

$$\mu_\Gamma : H^*_{top}(\Gamma) \longrightarrow HP_*(\mathbb{C}\Gamma). \tag{188}$$

The morphism induces a well-defined pairing

$$K^*_{top}(\Gamma) \times HP^*(\mathbb{C}\Gamma) \to \mathbb{C} \tag{189}$$

computed in every g-component (with respect to Burghelea's decomposition above) by

$$\langle [M,x],\tau_g\rangle = \langle ch_M^g(x)\wedge Td_g^M, \pi_g^*(\tau_g)\rangle \tag{190}$$

where the right hand side corresponds the pairing between Γ_g-invariant forms and currents on M_g. The morphism ch_μ^Γ will be called the Chern-Baum-Connes assembly map of the group Γ.

We remark that the Chern-Baum-Connes assembly map in the Theorem above is explicitly computed in terms of cycles and delocalised Todd classes. It would be interesting to compare our Chern assembly with the ones obtained by other constructions. For example, as observed by Engel and Marcinkowski in [**53**], one can use Yu's algebraic assembly map [**105**] to get as well a Chern-Baum-Connes assembly map. One can also use the cohomological assembly map studied by Cortiñas and Tartaglia in [**43**]. As far as we know there are no comparison results about these, in principle different, models for the Chern assembly map.

Also, relating the K-theoretical assembly map with the cohomological assembly map is also part of the program of studying secondary invariants. For example, in [**89**], Piazza, Schick and Zenobi used noncommutative de Rham theory and delocalised versions of it to produce some long surgery exact sequences in homology and they obtained very deep results on the study of higher rho invariants by mapping the K-theoretical surgery sequence to the homological one. It would be also very interesting to relate our constructions to theirs and study secondary invariants. For example, in [**90**], Piazza and Zenobi used groupoid techniques in a crucial way to study some related problems for singular spaces. Another related work in this direction can be found in [**36**].

There are several reasons for which we decided to state the final result above in terms of a pairing with cyclic periodic cohomology instead of just finishing the paper with the delocalized group homology version:

(i) An important part of the techniques developed in this paper apply to more general groups or groupoids. Computations such as the one by Burghelea for discrete groups are not so common, but several examples of geometric cyclic cocycles are known in many different situations. One could expect to extend some index theoretic formulas to some other settings by using the techniques of the present paper.

(ii) For geometric/topological applications, it would be interesting to explore the injectivity of the cohomological assembly maps $\mu_{F\Gamma}$ and μ_Γ. In principle, there are no analytic obstructions for this kind of problems. In fact, there are at least two models of similar cohomological assembly maps of this kind, both using periodic cyclic theory, in which the injectivity was proved under general assumptions on the group; See [**43**] and [**105**]. A comparison of their work with our cohomological assembly maps would be very interesting.

(iii) Given a periodic cyclic class $\tau \in HP^*(\mathbb{C}\Gamma)$, by pairing with K-theory, it gives rise to a morphism $\varphi_\tau : K_*(\mathbb{C}\Gamma) \to \mathbb{C}$; Sometimes it makes sense to replace the domain by the K-theory groups of larger algebras (for example, the algebra $\mathcal{R}\Gamma$ considered by Connes and Moscovici in [**40**]). If the morphism φ_τ extends to $K_*(C_r^*(\Gamma))$ (or at least to appropriate smooth subalgebras) then one can consider the pairing

$$\langle \mu_{BC}(x), \tau \rangle \tag{191}$$

with $x \in K^*_{top}(\Gamma)$ and μ_{BC} the K-theoretical Baum-Connes assembly map. Note that extending the morphisms φ_τ is a very hard problem. For example, for classes localized at identity, this would imply the Novikov conjecture on the homotopy invariance of higher signatures. On the other hand, for $x \in K^*_{top}(\Gamma)$ and any τ as above, our pairing, $\langle ch^\Gamma_\mu(x), \tau \rangle$, is defined (and computed explicitly by Theorem 6.8 in [**30**]) and in all known cases we have

$$\langle ch^\Gamma_\mu(x), \tau \rangle = \langle \mu_{BC}(x), \tau \rangle, \tag{192}$$

whenever the right hand side makes sense and whenever it has been computed. This rises a very interesting question, in general, if φ_τ extends, do we have the equality (192)?

(iv) In recent years, very interesting articles have appeared on the study of higher secondary invariants, surgery sequences and cyclic cocycle pairings and the very interesting and fruitful relations between these three topics were developed; For example, just to mention some of them: [**20**], [**21**], [**36**], [**89**] and [**104**]. All these works are certainly closely related to our present paper and it would be highly interesting to explore some relations with these results.

CHAPTER 6

Chern-Baum-Connes assembly map for Lie groupoids

In this chapter we resume the contents of my paper [**28**]. For a given Lie groupoid $\mathscr{G} \rightrightarrows P$, we study of the hypothetical existence of a commutative diagram of assembled morphisms

$$\begin{array}{ccc} K^*_{top}(\mathscr{G}) & \xrightarrow{\mu_{BC}} & K_*(C^*_r(\mathscr{G})) \\ {\scriptstyle Ch_{top}}\downarrow & & \downarrow{\scriptstyle Ch_{CO}} \\ H^*_{top}(\mathscr{G}) & \xrightarrow[\mu_{HP}]{} & HP_*(C^\infty_c(\mathscr{G})) \end{array} \tag{193}$$

where Ch_{CO} would be the hypothetical extension of the Chern-Connes character

$$Ch_{CO} : K_*(C^\infty_c(\mathscr{G})) \to HP_*(C^\infty_c(\mathscr{G})).$$

In this work we will use geometric deformation groupoids and associated pushforward morphism in periodic cyclic (co)homology to construct two morphisms above Ch_{top} and μ_{HP} for a large class of Lie groupoids[1]. Additionally, the composition will yield a morphism

$$Ch_{BC} : K^*_{top}(\mathscr{G}) \to HP_*(C^\infty_c(\mathscr{G}))$$

that we will call the *Chern-Baum-Connes assembly map* and that will satisfy that the above diagram (193) commutes if the Chern-Connes character for $\mathscr{G}$ extends to the C^*-algebra $K-$theory groups as above.

Now, the existence of the extension morphism Ch_{CO} in the right of the diagram (193) is a very hard analytic problem, unknown in general for discrete groups and for many interesting Lie groupoids, and possibly not true for some other examples. For having a more flexible version of the extension problem above we will see in our main theorem below, theorem 6.0.3, that if for a given periodic cyclic cocycle on $C^\infty_c(\mathscr{G})$ the associated pairing extends to the $K-$theory of the reduced C^*-algebra then the composition of the Baum-Connes assembly map followed by the associated extended morphism coincides with the Connes pairing of the given periodic cyclic cycle with the Chern-Baum-Connes assembly map.

To develop this program, an important and fundamental point to remark is that to be able to apply constructions in periodic cyclic theory similar to those used in K-theory one needs, besides suitable groupoids, appropriate smooth

[1]That we conjecture to be all the Lie groupoids, see next remark, but that contains at least all foliation groupoids (Morita equivalent to an étale groupoid), discrete and compact groups for example.

subalgebras of the C^*-algebras of the corresponding groupoids. In particular appropriate enough to have the right exact sequences that lead to classic arguments in algebraic topology (use of homotopy invariance, periodicity, six term exact sequences, etc.). In appendix D we explain which are the main groupoids used in this paper and which algebras we are using in each case when applying periodic cyclic (co)homology to them. In section 6.1 we construct and study the deformation index type morphisms applied to those algebras/groupoids in periodic cyclic (co)homology.

Next, to give a detailed description of the contents of the paper we start by introducing the class of Lie groupoids that we will consider, which we believe to be all Lie groupoids, see remark below. Let $\mathscr{G} \rightrightarrows P$ be a Lie groupoid, we will say that $\mathscr{G}$ satisfies the Thom isomorphism property in periodic cyclic (co)homology or simply that it is a HP-Thom Lie groupoid if the following two properties are satisfied

(i) For every $E \to M$ $\mathscr{G}$-*spin*^{c} proper vector bundle there is an unique isomorphism

$$\mathcal{T}_E^{\mathscr{G}} : HP_*(C_c^\infty(M \rtimes \mathscr{G})) \longrightarrow HP_{*+r_E}(\mathscr{S}_c(E \rtimes \mathscr{G})) \tag{194}$$

where $* + r_E = * + rank(E)$ mod 2 (see section **??** for the definition of $\mathscr{S}_c(E \rtimes \mathscr{G})$), such that the following diagram commutes

$$\begin{array}{ccc} K_*(C_c^\infty(M \rtimes \mathscr{G})) & \xrightarrow[\cong]{\mathcal{T}_E^{\mathscr{G}}} & K_{*+r_E}(\mathscr{S}_c(E \rtimes \mathscr{G})) \\ {\scriptstyle Ch_{CO}}\downarrow & & \downarrow{\scriptstyle Ch_{CO}} \\ HP_*(C_c^\infty(M \rtimes \mathscr{G})) & \xrightarrow[\mathcal{T}_E^{\mathscr{G}}]{\cong} & HP_{*+r_E}(\mathscr{S}_c(E \rtimes \mathscr{G})) \end{array} \tag{195}$$

where the morphism $\mathcal{T}_E^{\mathscr{G}}$ on the top is the Thom isomorphism[2] in K-theory of the correspondent algebras and where the vertical morphisms stand for the correspondent Chern-Connes characters.

(ii) For every $E \to M$ $\mathscr{G}$-*spin*^{c} proper vector bundle there is an isomorphism

$$\mathcal{T}_{\mathscr{G}}^E : HP_{*+r_E}(\mathscr{S}_c(E \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty(M \rtimes \mathscr{G})) \tag{196}$$

where $* + r_E = * + rank(E)$ mod 2, such that the following property is satisfied

- For every $\omega \in HP_*(M \rtimes \mathscr{G})$ and every $\delta \in HP_{*+r_E}(\mathscr{S}_c(E \rtimes \mathscr{G}))$ the following equality of Connes pairings holds

$$\langle \omega, \mathcal{T}_{\mathscr{G}}^E(\delta) \rangle = \langle \mathcal{T}_E^{\mathscr{G}}(\omega), \delta \rangle. \tag{197}$$

Remark 6.0.1. *Some important points to remark about this Thom isomorphism hypothesis in periodic cyclic (co)homology:*

[2]In principle the Thom isomorphism in K-theory is defined for the correspondent C^*-algebras, but in this particular case the smooth algebras are stable under holomorphic calculus since we are assuming proper action.

- *If it exists, the above Thom isomorphism in periodic cyclic homology is more like a twisted Thom isomorphism. Indeed, in the cases where geometrical computations are avalaible (e.g. classic spaces case or discrete groups case for instance), the cohomological Thom isomorphism does not commute with the K-theoretical Thom isomorphism via the Chern character. However, it does commute under the presence of appropriate Todd classes. In this paper we will use for simplicity the terminology Thom isomorphism for the above morphism.*
- *The Thom isomorphism in K-theory above holds in all generality, it can be obtained by some descent procedure from the classic topological K-theory Thom isomorphism for spaces by using for example the work of Le Gall on groupoid equivariant KK-theory, see* [**62**]. *For more details the reader can see appendix A.2 in* [**35**].
- *As far as the author knows, there is no proof of the Periodic Cyclic (co)homology Thom isomorphism for general Lie groupoids. It is certainly highly expected to be true in all generality.*
- *Some interesting cases in which the above property of being HP-Thom is known to hold are*
  - (i) *Foliation groupoids, that is, Lie groupoids which are Morita equivalent to étale groupoids. This can be seen using for example the work of Tu and Xu for proper etale groupoids,* [**97**]*, together with the Lie groupoid Morita invariance of the Periodic Cyclic (co)homology, see for example* [**87**]. *This case covers the cases of discrete groups and holonomy groupoids of regular foliations.*
  - (ii) *Compact Lie groups, for example by following the work of Block and Getzler in* [**23**].

As for the K-theoretical case and the discrete group case developed in [**35**] and [**30**], the above Thom property, together with the use of appropriate deformation groupoids algebras plays a fundamental role to have:

- A pushforward morphism

$$HP_*(C_c^\infty(M \rtimes \mathscr{G})) \xrightarrow{f!} HP_*(C_c^\infty(N \rtimes \mathscr{G})) \tag{198}$$

  for appropriate maps $f : M \to N$ between $\mathscr{G}-$manifolds, definition 6.2.1;
- the pushforward functoriality for the morphisms above, theorem 6.2.2;
- an assembled group

$$H^*_{top}(\mathscr{G}) := \varinjlim_{f!} HP_*(C_c^\infty(M \rtimes \mathscr{G})), \tag{199}$$

  with manifolds M being $\mathscr{G}$-proper $spin^c$ cocompact, and an assembled Chern character (corollary 6.2.4)

$$Ch_{top} : K^*_{top}(\mathscr{G}) \longrightarrow H^*_{top}(\mathscr{G}) \tag{200}$$

given by

$$Ch_{top}([M,x]) = [M, Ch^M_{CO}(x)]. \tag{201}$$

where $Ch^M_{CO} : K_*(C^\infty_c(M \rtimes \mathscr{G})) \to HP_*(C^\infty_c(M \rtimes \mathscr{G}))$ is the Chern-Connes character;

- a cohomological assembly map (theorem 6.2.5)

$$\mu_{HP} : H^*_{top}(\mathscr{G}) \longrightarrow HP_*(C^\infty_c(\mathscr{G})), \tag{202}$$

that is explicitly given by

$$\mu_{HP}([M,\tau]) = \pi_M!(\tau). \tag{203}$$

where $\pi_M : M \to P$ is the $\mathscr{G}-$momentum map given by the $\mathscr{G}$-action on M; and

- the commutativity of diagram (193) if the Chern-Connes of $\mathscr{G}$ extends to the K-theory of the C^*-algebra.

We summarize the last results in the following theorem.

THEOREM 6.0.2. *For any HP-Thom Lie groupoid $\mathscr{G}$ there is a well defined morphism*

$$K^*_{top}(\mathscr{G}) \xrightarrow{Ch_{BC}} HP_*(C^\infty_c(\mathscr{G})) \tag{204}$$

given by the composition of the Chern character

$$Ch_{top} : K^*_{top}(\mathscr{G}) \longrightarrow H^*_{top}(\mathscr{G}) \tag{205}$$

followed by the cohomological assembly

$$\mu_{HP} : H^*_{top}(\mathscr{G}) \longrightarrow HP_*(C^\infty_c(\mathscr{G})), \tag{206}$$

inducing a well-defined pairing

$$K^*_{top}(\mathscr{G}) \times HP^*(C^\infty_c(\mathscr{G})) \to \mathbb{C} \tag{207}$$

given by

$$\langle [M,x], \tau \rangle := \langle \pi_M!(Ch^M_{CO}(x)), \tau \rangle. \tag{208}$$

Moreover, the commutativity of diagram (193) holds if the Chern-Connes character morphism for $\mathscr{G}$ extends to $K_*(C^*_r(\mathscr{G}))$.

We call the morphism Ch_{BC} above the Chern-Baum-Connes assembly map of the Lie groupoid $\mathscr{G}$.

The construction of the assembly map above includes of course the case of discrete groups. Now, for discrete groups, the assembly map we constructed in [**?**] is in principle different, in fact in that paper we use the fact that an explicit computation for the periodic cyclic (co)homology groups is known in terms of group cohomology groups to give a more geometric description in that case and a formula for the index pairing. We will study elsewhere the relation between these two assembly maps, this would be very interesting since for the assembly map studied in this paper we have an explicit relation with the Baum-Connes

map, theorem above and below, while in [**30**] we obtained explicit index formulas for the Connes pairing.

We conclude this introduction with one of the main result of this chapter, theorem 6.2.7.

THEOREM 6.0.3. *Let $\mathscr{G} \rightrightarrows P$ be a Lie groupoid satisfying the Thom isomorphism property in periodic cyclic (co)homology and let $\tau \in HP_*(C_c^\infty(\mathscr{G}))$. Suppose that the morphism*

$$\tilde{\varphi}_\tau : K_*(C_c^\infty(\mathscr{G})) \longrightarrow \mathbb{C} \tag{209}$$

induced from the pairing with $HP_*(C_c^\infty(\mathscr{G}))$ (via the Chern-Connes character) extends to a morphism

$$\varphi_\tau : K_*(C_r^*(\mathscr{G})) \longrightarrow \mathbb{C}. \tag{210}$$

Then, for every $[M, x] \in K^*_{top}(\mathscr{G})$ we have

$$\varphi_\tau(\mu_{BC}([M, x])) = \langle Ch_{BC}([M, x]), \tau \rangle. \tag{211}$$

One of the main interest of the theorem above is that for every (HP-Thom) Lie groupoid, and for every periodic cyclic cocycle, the right hand side of the formula above always makes sense.

6.1. Deformation morphisms in periodic cyclic (co)homology

Let $f : M \longrightarrow N$ be a $\mathscr{G}$-equivariant C^∞-map between two $\mathscr{G}-$manifolds M and N. We consider the $\mathscr{G}-$vector bundle $T_f := T_\pi M \oplus f^* T_\pi N$ over M as the previous section. We are going to use deformation groupoids to construct a morphism

$$I_f : HP_*(\mathscr{S}(T_f \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty((M \times_P M) \times_P N) \rtimes \mathscr{G})). \tag{212}$$

Consider the deformation groupoid

$$D_f := T_f \bigsqcup ((M \times_P M) \times_P N) \times (0, 1] \tag{213}$$

over

$$f^* T_\pi N \bigsqcup (M \times_P N) \times (0, 1] \tag{214}$$

given as above as the normal groupoid associated to the $\mathscr{G}$-map

$$M \stackrel{\Delta \times f}{\longrightarrow} (M \times_P M) \times_P N.$$

As we already discussed above, the action of $\mathscr{G}$ on $(M \times_P M) \times_P N$ (diagonal on $M \times_P M$) leaves invariant the image of M by $\Delta \times f$ and then we have, by functoriality of the deformation to the normal cone construction, an induced semi-direct product groupoid

$$D_f \rtimes \mathscr{G} \rightrightarrows f^* T_\pi N \bigsqcup (M \times_P N) \times (0, 1].$$

The following result follows directly from proposition 4.6 and lemma 5.5 in [**32**].

PROPOSITION 6.1.1. *For a $\mathscr{G}$-map $f : M \to N$ as above the restriction to zero, or equivalently to the closed subgroupoid $T_f \rtimes \mathscr{G}$ of $D_f \rtimes \mathscr{G}$, induces a short exact sequence*

$$0 \longrightarrow J \longrightarrow \mathscr{S}_c(D_f \rtimes \mathscr{G}) \xrightarrow{e_0} \mathscr{S}_c(T_f \rtimes \mathscr{G}) \longrightarrow 0 \tag{215}$$

where $J = \lambda \cdot \mathscr{S}_c(D_f \rtimes \mathscr{G})$ for $\lambda : D_f \rtimes \mathscr{G} \to [0,1]$ the canonical projection.

As an immediate consequence of the homotopy invariance and of the six term short exact sequence for periodic cyclic homology we have the following corollary.

COROLLARY 6.1.2. *With the notations of the last proposition we have that*

$$HP_*(J) = 0 \tag{216}$$

and

$$(e_0)_* : HP_*(\mathscr{S}_c(D_f \rtimes \mathscr{G})) \to HP_*(\mathscr{S}_c(T_f \rtimes \mathscr{G})) \tag{217}$$

is an isomorphism.

DEFINITION 6.1.3. *For a $\mathscr{G}$-map as above we consider the morphism*

$$I_f : HP_*(\mathscr{S}_c(T_f \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty((M \times_P M) \times_P N) \rtimes \mathscr{G})). \tag{218}$$

given as the composition of $(e_0)_*^{-1}$ followed by $(e_1)_*$. We call it the deformation morphism associated to the the $\mathscr{G}$-map f.

6.2. Wrong way functoriality in Periodic cyclic homology and the Chern-Baum-Connes assembly map

6.2.1. Wrong way functoriality in Periodic Cyclic theory. In this section we will only work with Lie groupoids satisfying the Thom isomorphism property as defined in the introduction.

DEFINITION 6.2.1 (Pushward maps in Periodic cyclic homology). *Let $\mathscr{G} \rightrightarrows P$ be a HP-Thom Lie groupoid. Let $f : M \longrightarrow N$ be a $\mathscr{G}$-equivariant, K-oriented, C^∞-map between two $\mathscr{G}-$proper cocompact manifolds M and N. Let $r_f := rank(T_f)$. We define the morphism*

$$f_! : HP_{*-r_f}(C_c^\infty(M \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty(N \rtimes \mathscr{G})). \tag{219}$$

as the composition of the following morphisms

- *The Thom isomorphism in Periodic cyclic homology*

$$Th : HP_{*-r_f}(C_c^\infty(M \rtimes \mathscr{G})) \longrightarrow HP_*(\mathscr{S}_c(T_f^* \rtimes \mathscr{G})), \tag{220}$$

- *the isomorphism*

$$F : HP_*(\mathscr{S}_c(T_f^* \rtimes \mathscr{G})) \longrightarrow HP_*(\mathscr{S}_c(T_f \rtimes \mathscr{G})) \tag{221}$$

  induced from the Fourier algebra isomorphism,

- *the deformation morphism*

$$I_f : HP_*(\mathscr{S}_c(T_f \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty(((M \times_P M) \times_P N) \rtimes \mathscr{G})), \tag{222}$$

and

- *the isomorphism induced from the Morita equivalence (isomorphism by Proposition 3.6 in* [**87**]*) of groupoids* $((M \times_P M) \times_P N) \rtimes \mathscr{G} \sim N \rtimes \mathscr{G}$

$$\mathcal{M} : HP_*(C_c^\infty(((M \times_P M) \times_P N) \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty(N \rtimes \mathscr{G})). \tag{223}$$

As in the case of K-theory groups, we can prove that the above-defined pushforward morphisms are functorial. This is one of our main results, stated as follows:

THEOREM 6.2.2 (Pushforward functoriality in Periodic cyclic homology). *Let* $\mathscr{G} \rightrightarrows P$ *be a HP-Thom Lie groupoid. Let* $f : M \longrightarrow N$ *and* $g : N \longrightarrow L$ *be* $\mathscr{G}$ *K-oriented* C^∞*-maps between* $\mathscr{G}-$*proper cocompact manifolds. Then*

$$(g \circ f)_! = g_! \circ f_! \tag{224}$$

as morphisms from $HP_*(C_c^\infty(M \rtimes \mathscr{G}))$ *to* $HP_{*+r_{g\circ f}}(C_c^\infty(L \rtimes \mathscr{G}))$ *(*$r_{g\circ f} = r_f + r_g \ mod\, 2$*).*

The proof of the theorem above follows exactly the same steps of the proof of theorem 4.1 in [**30**] or Theorem 4.2 in [**35**] and is based on (co)homological properties of the periodic cyclic homology groups as homotopy invariance, Thom isomorphism (here by hypothesis), Lie groupoid Morita invariance, and naturality.

The above wrong way functoriality theorem allows us to consider the following assembled groups.

DEFINITION 6.2.3. *For a HP-Thom Lie groupoid* $\mathscr{G} \rightrightarrows P$ *and for* $* = 0, 1 \ mod\, 2$ *we can consider the abelian group*

$$H^*_{top}(\mathscr{G}) = \varinjlim_{f_!} HP_*(M \rtimes \mathscr{G}). \tag{225}$$

where the limit is taken over the $\mathscr{G}$ *proper cocompact spin*^{c} *manifolds with submersive moment map and of fiber dimension (the rank its vertical tangent bundle) equal to* $*$ *modulo* 2*.*

Equivalently it can be defined as the free abelian group with generators (M, τ) *with* M *a* $\mathscr{G}$ *proper cocompact spin*^{c} *manifold and* $\tau \in HP_*(C_c^\infty(M \rtimes \mathscr{G}))$*, and the equivalence relation generated by* $\tau \sim f_!(\tau)$ *for* $f : M \to N$ *as above.*

The following is an immediate corollary of the push-forward functoriality theorems in K-theory (Theorem 4.2 in [**35**]) and in periodic cyclic homology, Theorem 6.2.2 above.

COROLLARY 6.2.4. *For any HP-Thom Lie groupoid* $\mathscr{G}$*, there is a well-defined Chern character morphism*

$$Ch_{top} : K^*_{top}(\mathscr{G}) \longrightarrow H^*_{top}(\mathscr{G}) \tag{226}$$

given by

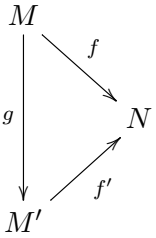
$$Ch_{top}([M,x]) = [M, Ch^M_{CO}(x)]. \tag{227}$$

Now, the push-forward construction above and its functoriality might be slightly more general. In particular, N does not have to be $\mathscr{G}$-proper to define a similar morphism in cyclic periodic homology. Indeed, if $f : M \to N$ is $\mathscr{G}$-map between two $\mathscr{G}$-manifolds with M proper and T_f carrying a $\mathscr{G}$-proper, $\mathscr{G} - spin^c$ vector bundle structure, then we can consider the pushforward map

$$f_! : HP_*(C^\infty_c(M \rtimes \mathscr{G})) \to HP_*(C^\infty_c(N \rtimes \mathscr{G})) \tag{228}$$

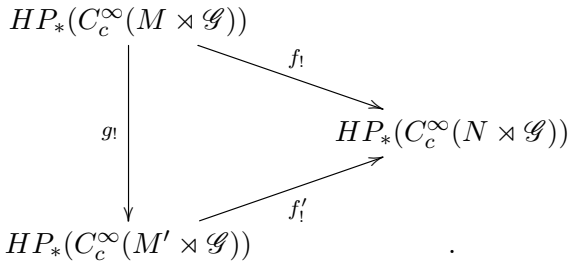
defined as in Definition 6.2.1. The pushforward functoriality theorem can be applied in the same way to prove that if

(229)



is a commutative diagram of $\mathscr{G}$-equivariant maps between two $\mathscr{G}$-manifolds with M, M' proper and T_f, $T_{f'}$ having $\mathscr{G} - spin^c$ vector bundle structures, then the following diagram is commutative

(230)



.

Besides the case when N is also $\mathscr{G}$-proper there is also the very interesting case when N is a point. In particular we can define $\pi_M!$ for the moment map π_M of M a $\mathscr{G}$-proper $spin^c$ cocompact manifold as above.

We have the following theorem whose proof follows the same steps as Theorem 6.2.2 together with the observation leading to (230).

THEOREM 6.2.5. *The assembly of the above morphisms μ^{top}_M gives rise to a well defined morphism*

$$\mu_{top} : H^*_{top}(\mathscr{G}) \longrightarrow HP_*(C^\infty_c(\mathscr{G})), \tag{231}$$

that is explicitly given by

$$\mu_{top}([M,\tau]) = \pi_M!(\tau). \tag{232}$$

where $\pi_M : M \to P$ is the $\mathscr{G}-$momentum map given by the $\mathscr{G}$-action on M.

6.2.2. The Chern-Baum-Connes assembly map for Lie groupoids. We summarise the last results in the following theorem.

THEOREM 6.2.6. *For any HP-Thom Lie groupoid $\mathscr{G}$ there is a well defined morphism*

$$K^*_{top}(\mathscr{G}) \xrightarrow{Ch_{BC}} HP_*(C^\infty_c(\mathscr{G})) \tag{233}$$

given by the composition of the Chern character

$$ch^{top} : K^*_{top}(\mathscr{G}) \longrightarrow H^*_{top}(\mathscr{G}) \tag{234}$$

followed by the cohomological assembly

$$\mu_{top} : H^*_{top}(\mathscr{G}) \longrightarrow HP_*(C^\infty_c(\mathscr{G})), \tag{235}$$

inducing a well-defined pairing

$$K^*_{top}(\mathscr{G}) \times HP^*(C^\infty_c(\mathscr{G})) \to \mathbb{C} \tag{236}$$

given by

$$\langle [M,x], \tau \rangle := \langle \pi_M!(Ch^M_{CO}(x)), \tau \rangle. \tag{237}$$

Moreover, the commutativity of diagram (193) holds if the Chern-Connes character morphism for $\mathscr{G}$ extends to $K_*(C^*_r(\mathscr{G}))$.

We call the morphism Ch_{BC} above the Chern-Baum-Connes assembly map of the Lie groupoid $\mathscr{G}$.

The construction of the assembly map above includes of course the case of discrete groups. Now, for discrete groups, the assembly map we develop in [**?**] is in principle different. Indeed, in that paper we use the fact that an explicit computation for the periodic cyclic (co)homology groups is known in terms of group cohomology groups to give a more geometric description in that case and a formula for the index pairing. We will study elsewhere the relation between these two assemblies. This is interesting because for the Chern-Baum-Connes assembly studied in this paper we can answer the question raised in the introduction of comparing this assembly map with the Baum-Connes assembly map whenever this makes sense. Indeed, we have the following theorem.

THEOREM 6.2.7. *Let $\mathscr{G} \rightrightarrows P$ be a Chern-Thom Lie groupoid and let $\tau \in HP_*(C^\infty_c(\mathscr{G}))$. Suppose that the morphism*

$$\tilde{\varphi}_\tau : K_*(C^\infty_c(\mathscr{G})) \longrightarrow \mathbb{C} \tag{238}$$

induced from the pairing with $HP_*(C^\infty_c(\mathscr{G}))$ (via the Chern-Connes character) extends to a morphism

$$\varphi_\tau : K_*(C^*_r(\mathscr{G})) \longrightarrow \mathbb{C}. \tag{239}$$

Then, for every $[M,x] \in K^*_{top}(\mathscr{G})$ we have

$$\varphi_\tau(\mu_{BC}([M,x])) = \langle Ch_{BC}([M,x]), \tau \rangle. \tag{240}$$

PROOF. Let $M \xrightarrow{\pi_M} P$ be a $\mathscr{G}$-proper $spin^c$ manifold, $x \in K^*(M \rtimes \mathscr{G})$ and $\tau \in HP_*(C_c^\infty(\mathscr{G}))$ satisfying the extension property in the statement above. We have to show that

$$\varphi_\tau(\pi_M!(x)) = \langle \pi_M!(Ch_{CO}^M(x), \tau) \rangle \tag{241}$$

where on the left hand side $\pi_M!$ stands for the K-theoretical pushforward (the pairing makes sense by the extension hypothesis), and where on the right hand side $\pi_M!$ stands for the cyclic periodic homology pushforward and $Ch_{CO}^M(x)$ for the Chern-Connes character of x.

Let us consider the morphism

$$\pi_M^* : HP^*(C_c^\infty(\mathscr{G})) \longrightarrow HP^*(C_c^\infty(M \rtimes \mathscr{G})) \tag{242}$$

given as the composition of the following morphisms

- The isomorphism

$$HP^*(C_c^\infty(\mathscr{G})) \xrightarrow[\approx]{\mathcal{M}} HP^*(C_c^\infty((M \times_P M) \rtimes \mathscr{G})) \tag{243}$$

  induced from the canonical groupoid Morita equivalence,
- the morphism

$$HP^*(C_c^\infty((M \times_P M) \rtimes \mathscr{G})) \xrightarrow{(e_1)^*} HP^*(\mathsf{S}_c(G_M^{tan} \rtimes \mathscr{G})) \tag{244}$$

  induced from the evaluation at $t = 1$ morphism,
- the isomorphism

$$HP^*(\mathsf{S}_c(G_M^{tan} \rtimes \mathscr{G})) \xrightarrow{(e_0^*)^{-1}} HP^*(\mathsf{S}_c(T^\pi M \rtimes \mathscr{G})) \tag{245}$$

  induced from the evaluation at $t = 0$ morphism,
- the isomorphism

$$HP^*(\mathsf{S}_c(T^\pi M \rtimes \mathscr{G})) \xrightarrow[\approx]{\mathcal{F}} HP^*(\mathsf{S}_c(T_\pi^* M \rtimes \mathscr{G})) \tag{246}$$

  induced from the Fourier algebra isomorphism $\mathsf{S}_c(T_\pi M \rtimes \mathscr{G}) \approx \mathsf{S}(T_\pi^* M \rtimes \mathscr{G})$,
- the equivariant Thom isomorphism

$$HP^*(\mathsf{S}_c(T_\pi^* M \rtimes \mathscr{G})) \xrightarrow[\approx]{Th} HP^*(C_c^\infty(M \rtimes \mathscr{G})) \tag{247}$$

By a direct computation, for getting the equality (241), it is enough to prove the following two points

A. For every $\omega \in HP_*(M \rtimes \mathscr{G})$ and for every $\delta \in HP^*(C_c^\infty(\mathscr{G}))$ the following equality holds

$$\langle \omega, \pi_M^*(\delta) \rangle = \langle \pi_M!(\omega), \delta \rangle. \tag{248}$$

B. For every $x \in K^*(M \times \mathscr{G})$ the following equality holds

$$\varphi_\tau(\pi_M!(x)) = \langle x, \pi_M^*(\tau) \rangle \tag{249}$$

where $\langle x, \pi_M^*(\tau) \rangle := \langle Ch_{CO}^M(x), \pi_M^*(\tau) \rangle$.

Proof of A. Since both morphisms, $\pi_M!$ and π_M^* are defined by compositions of Thom isomorphisms, Morita isomorphisms and morphisms induced from algebra morphisms, the equality (248) follows immediately from the following general properties of Connes' pairing:

Naturality: Given a morphism of algebras $\phi : A \to B$ we have that

$$\langle \phi_*(a), b\rangle = \langle a, \phi^*(b)\rangle \tag{250}$$

for every $a \in HP_*(A)$ and every $b \in HP^*(B)$. This property is quite standard and holds in all generality.

Thom invariance: Given a $\mathscr{G}$-*spin*^{c} vector bundle $E \to M$ we have that for every $\omega \in HP_*(M \rtimes \mathscr{G})$ and every $\delta \in HP_{*+r_E}(\mathscr{S}_c(E \rtimes \mathscr{G}))$ the following equality of Connes pairings holds

$$\langle \omega, \mathcal{T}^E_{\mathscr{G}}(\delta)\rangle = \langle \mathcal{T}^{\mathscr{G}}_E(\omega), \delta\rangle. \tag{251}$$

This property holds by hypothesis since we are assuming $\mathscr{G}$ to be a HP-Thom Lie groupoid.

Morita invariance: For a $\mathscr{G}$-manifold M as above if we denote by

$$M_{HP_*} : HP_*(C_c^\infty((M \times_P M) \rtimes \mathscr{G})) \longrightarrow HP_*(C_c^\infty(\mathscr{G})) \tag{252}$$

and by

$$M_{HP^*} : HP^*(C_c^\infty(\mathscr{G})) \longrightarrow HP^*(C_c^\infty((M \times_P M) \rtimes \mathscr{G})) \tag{253}$$

the canonical isomorphisms induced from the Lie groupoid Morita equivalence

$$(M \times_P M) \rtimes \mathscr{G} \longrightarrow \mathscr{G},$$

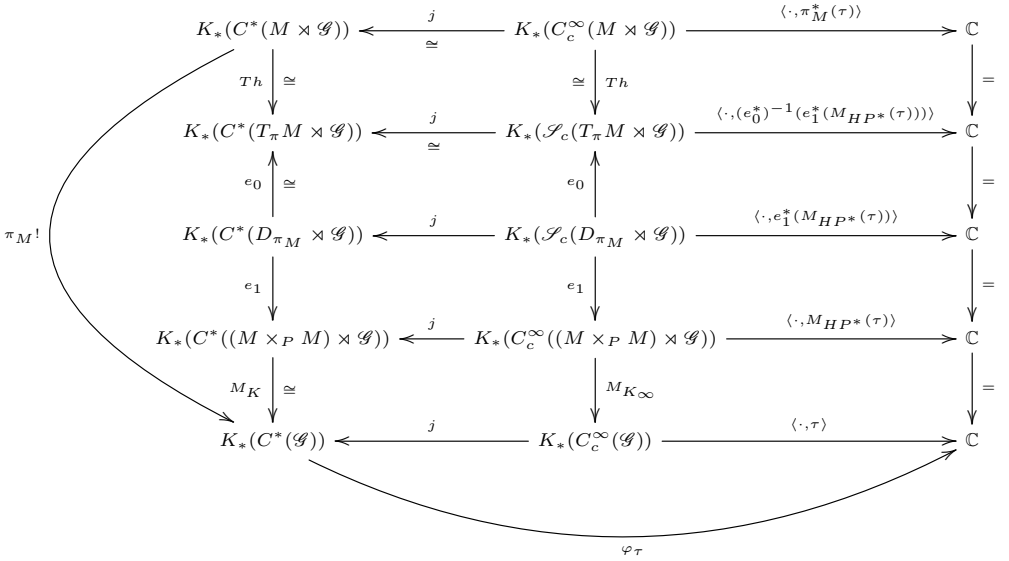
we have that the following equality holds

$$\langle M_{HP_*}(\mu), \rho\rangle = \langle \mu, M_{HP^*}(\rho)\rangle \tag{254}$$

for every $\mu \in HP_*(C_c^\infty((M \times_P M) \rtimes \mathscr{G}))$ and every $\rho \in HP^*(C_c^\infty(\mathscr{G}))$. This property holds, by a direct computation, at least in this case since the morphism induced at the level of cycles consists in integrating the $(M \times_P M)$ part along the diagonal. The fact that these are isomorphism follows from proposition 3.6 in [**87**].

Proof of B. By the properties above, naturality, Morita invariance, and Thom invariance of the Connes pairing, we have the following commutative diagram

(255)



where the morphisms denoted by j are the morphisms induced by the canonical inclusion of algebras, and where we are using the hypothesis that φ_τ extends the pairing morphism given by τ, *i.e.* the commutativity of the last bottom diagram. Also we recall that the two first morphisms j on the top are isomorphisms because M is a $\mathscr{G}$-proper *spin*^{c} manifold. Now, the proof of **B** follow by direct diagram chasing if the morphisms

$$(e_0)_* : K_*(\mathscr{S}_c(D_{\pi_M} \rtimes \mathscr{G})) \longrightarrow K_*(\mathscr{S}_c(T_\pi M \rtimes \mathscr{G})) \tag{256}$$

are surjective. This last statement follows from the fact that we are assuming that $T_\pi M$ is a $\mathscr{G}$-*spin*^{c} proper vector bundle, in particular with an invariant $\mathscr{G}-$metric. Indeed, one can use the local structure of the deformation to the normal cone D_{π_M} around $T_\pi M$ as in appendix C of [**30**]. This completes the proof of B and of the theorem. □

One of the main interests of the theorem above is that the right hand side of the formula above always makes sense, for every (HP-Thom) Lie groupoid and for every periodic cyclic cocycle.

APPENDIX A

Lie groupoids and differentiable stacks

A.1. Basics on Lie groupoids and differentiable stacks

In this section, we review the notion of Lie groupoids.

Let us recall what a groupoid is:

DEFINITION A.1.1. *A groupoid consists of the following data: two sets $\mathscr{G}$ and $\mathscr{G}^{(0)}$, and maps*

(1) *$s, r : \mathscr{G} \to \mathscr{G}^{(0)}$ called the source map and target map respectively,*
(2) *$m : \mathscr{G}^{(2)} \to \mathscr{G}$ called the product map (where $\mathscr{G}^{(2)} = \{(\gamma, \eta) \in \mathscr{G} \times \mathscr{G} : s(\gamma) = r(\eta)\}$),*

together with two additional maps, $u : \mathscr{G}^{(0)} \to \mathscr{G}$ (the unit map) and $i : \mathscr{G} \to \mathscr{G}$ (the inverse map), such that, if we denote $m(\gamma, \eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, we have

(i) *$r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.*
(ii) *$\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$, $\forall \gamma, \eta, \delta \in \mathscr{G}$ whenever this makes sense.*
(iii) *$\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta$, $\forall \gamma, \eta \in \mathscr{G}$ with $s(\gamma) = x$ and $r(\eta) = x$.*
(iv) *$\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma))$, $\forall \gamma \in \mathscr{G}$.*

For simplicity, we denote a groupoid by $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$. A strict morphism f from a groupoid $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ to a groupoid $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ is given by maps

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{f} & \mathscr{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathscr{H}^{(0)} & \xrightarrow[f_0]{} & \mathscr{G}^{(0)} \end{array}$$

which preserve the groupoid structure, i.e., f commutes with the source, target, unit, inverse maps, and respects the groupoid product in the sense that $f(h_1 \cdot h_2) = f(h_1) \cdot f(h_2)$ for any $(h_1, h_2) \in \mathscr{H}^{(2)}$.

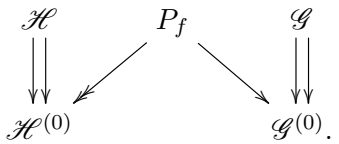
In this work we will only deal with Lie groupoids, that is, a groupoid in which $\mathscr{G}$ and $\mathscr{G}^{(0)}$ are smooth manifolds, and s, r, m, u are smooth maps (with s and r submersions, see [**67, 85**]).

A.1.1. The Hilsum-Skandalis category. Lie groupoids form a category with strict morphisms of groupoids. It is now a well-established fact in Lie groupoid's theory that the right category to consider is the one in which Morita equivalences correspond precisely to isomorphisms. We review some basic definitions and properties of generalized morphisms between Lie groupoids, see [**96**] section 2.1, or [**57, 81, 76**] for more detailed discussions.

DEFINITION A.1.2 (Generalized homomorphisms). *Let $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ and $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ be two Lie groupoids. A generalized groupoid morphism, also called a Hilsum-Skandalis morphism, from $\mathscr{H}$ to $\mathscr{G}$ is given by principal $\mathscr{G}$-bundle over $\mathscr{H}$, that is, a right principal $\mathscr{G}$-bundle over $\mathscr{H}^{(0)}$ which is also a left $\mathscr{H}$-bundle over $\mathscr{G}^{(0)}$ such that the the right $\mathscr{G}$-action and the left $\mathscr{H}$-action commute, formally denoted by*

$$f : \mathscr{H} \dashrightarrow \mathscr{G}$$

or by



if we want to emphsize the bi-bundle P_f involved.

Notice that a generalized morphism (or Hilsum-Skandalis morphism), $f : \mathscr{H} \dashrightarrow \mathscr{G}$, is given by one of the three equivalent data:

(i) A locally trivial right principal $\mathscr{G}$-bundle P_f over $\mathscr{H}$ as Definition A.1.2.

(ii) A 1-cocycle $f = \{(\Omega_i, f_{ij})\}_{i\in I}$ on $\mathscr{H}$ with values in $\mathscr{G}$. Here a $\mathscr{G}$-valued 1-cocycle on $\mathscr{H}$ with respect to an indexed open covering $\{\Omega_i\}_{i\in I}$ of $\mathscr{H}^{(0)}$ is a collection of smooth maps

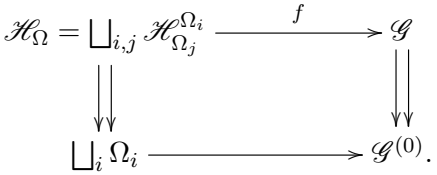
$$f_{ij} : \mathscr{H}_{\Omega_j}^{\Omega_i} \to \mathscr{G},$$

satisfying the following cocycle condition: $\forall\gamma \in \mathscr{H}_{ij}$ and $\forall\gamma' \in \mathscr{H}_{jk}$ with $s(\gamma) = r(\gamma')$, we have

$$f_{ij}(\gamma)^{-1} = f_{ji}(\gamma^{-1}) \text{ and } f_{ij}(\gamma) \cdot f_{jk}(\gamma') = f_{ik}(\gamma \cdot \gamma').$$

We will denote this data by $f = \{(\Omega_i, f_{ij})\}_{i\in I}$.

(iii) A strict morphism of groupoids



for an open cover $\Omega = \{\Omega_i\}$ of $\mathscr{H}^{(0)}$.

Associated to a $\mathscr{G}$-valued 1-cocycle on $\mathscr{H}$, there is a canonical defined principal $\mathscr{G}$-bundle over $\mathscr{H}$. In fact, any principal $\mathscr{G}$-bundle over $\mathscr{H}$ is locally trivial (Cf. [**76**]).

EXAMPLE A.1.3. (i) *(Strict morphisms) Consider a (strict) morphism of groupoids*

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{f} & \mathscr{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathscr{H}^{(0)} & \xrightarrow[f_0]{} & \mathscr{G}^{(0)} \end{array}$$

Using the equivalent definitions 2. or 3. above, it is obviously a generalized morphism by taking $\Omega = \{\mathscr{H}^{(0)}\}$. *In terms of the language of principal bundles, the bi-bundle is simply given by*

$$P_f := \mathscr{H}^{(0)} \times_{f_0,t} \mathscr{G},$$

with projections $t_f : P_f \to \mathscr{H}^{(0)}$, *projection in the first factor, and* $s_f : P_f \to \mathscr{G}^{(0)}$, *projection using the source map of* $\mathscr{G}$. *The actions are the obvious ones, that is, on the left,* $h \cdot (a, g) := (t(h), f(h) \circ g)$ *whenever* $s(h) = a$ *and, on the right,* $(a, g) \cdot g' := (a, g \circ g')$ *whenever* $s(g) = t(g')$.

(ii) *(Classic principal bundles) Let* X *be a manifold and* G *be a Lie group. By definition a generalized morphism between the unit groupoid* $X \rightrightarrows X$ *(that is a manifold seen as a Lie groupoid all structural maps are the identity) and the Lie group* $G \rightrightarrows \{e\}$ *seen as a Lie groupoid is given by a* G*-principal bundle over* X.

As the name suggests, generalized morphism generalizes the notion of strict morphisms and can be composed. Indeed, if P and P' are generalized morphisms from $\mathscr{H}$ to $\mathscr{G}$ and from $\mathscr{G}$ to $\mathscr{L}$ respectively, then

$$P \times_{\mathscr{G}} P' := P \times_{\mathscr{G}^{(0)}} P'/(p, p') \sim (p \cdot \gamma, \gamma^{-1} \cdot p')$$

is a generalized morphism from $\mathscr{H}$ to $\mathscr{L}$. Consider the category $Grpd_{HS}$ with objects Lie groupoids and morphisms given by isomorphism classes of generalized morphisms. There is a functor

$$Grpd \longrightarrow Grpd_{HS} \tag{257}$$

where $Grpd$ is the strict category of groupoids.

DEFINITION A.1.4 (Morita equivalent groupoids). *Two groupoids are called Morita equivalent if they are isomorphic in* $Grpd_{HS}$.

We list here a few examples of Morita equivalence groupoids.

EXAMPLE A.1.5 (Pullback groupoid). *Let* $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ *be a Lie groupoid and let* $\phi : M \to \mathscr{G}^{(0)}$ *be a map such that* $t \circ pr_2 : M \times_{\mathscr{G}^{(0)}} \mathscr{G} \to \mathscr{G}^{(0)}$ *is a submersion (for instance if* ϕ *is a submersion), then the pullback groupoid* $\phi^*\mathscr{G} := M \times_{\mathscr{G}^{(0)}} \mathscr{G} \times_{\mathscr{G}^{(0)}} M \rightrightarrows M$ *is Morita equivalent to* $\mathscr{G}$, *the strict morphism* $\phi^*\mathscr{G} \to \mathscr{G}$ *being a generalized isomorphism. For more details on this example the reader can see* [**76**] *examples 5.10(4).*

EXAMPLE A.1.6 (The basic example: the Čech groupoid). *Given a Lie groupoid $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ and an open covering $\{\Omega_i\}_i$ of $\mathscr{H}^{(0)}$, the canonical strict morphism of groupoids $\mathscr{H}_\Omega \longrightarrow \mathscr{H}$ is a Morita equivalence. It corresponds to the pullback groupoid by the canonical submersion $\sqcup_i \Omega_i \to \mathscr{H}^{(0)}$.*

EXAMPLE A.1.7 (Foliations $\sim$ étale groupoids). *One main example to have in mind will be the holonomy groupoid associated to a regular foliation. Let M be a manifold of dimension n. Let F be a subvector bundle of the tangent bundle TM. We say that F is integrable if $C_c^\infty(F) := \{X \in C_c^\infty(M, TM) : \forall x \in M, X_x \in F_x\}$ is a Lie subalgebra of $C_c^\infty(M, TM)$. This induces a partition of M in embedded submanifolds (the leaves of the foliation), given by the solution of integrating F.*

The holonomy groupoid of (M, F) is a Lie groupoid

$$\mathscr{G}_M \rightrightarrows M$$

with Lie algebroid $A\mathscr{G} = F$ and minimal in the following sense: any Lie groupoid integrating the foliation, that is having F as Lie algebroid, contains an open subgroupoid which maps onto the holonomy groupoid by a smooth morphism of Lie groupoids. The holonomy groupoid was constructed by Ehresmann [**51**] *and Winkelnkemper* [**103**] *(see also* [**26**], [**54**], [**85**]*).*

A.2. Lie groupoid cohomology

In this section we will describe some (co)homology groups associated to Lie groupoids and some properties we use on this work. The first three subsections are reviews of classic material, we are going to follow the nice paper/survey by K. Behrend "Cohomology of Stacks", [**17**], for more details and proofs the reader can look at the more complete source [**18**].

A.2.1. The de Rham cohomology. Let $G \rightrightarrows M$ be a Lie groupoid, we denote as usual by $G^{(p)}$ the manifold of p-composables arrows. In particular, $M = G^{(0)}$ and $G^{(1)} = G$. For every $p \in \mathbb{N}$ there are $p+1$ structural maps

$$G^{(p)} \xrightarrow{\partial_i} G^{(p-1)}$$

for $i = 0, .., p$, given by:

$$\partial_0(\gamma_1, ..., \gamma_p) = (\gamma_2, ..., \gamma_p),$$

$$\partial_p(\gamma_1, ..., \gamma_p) = (\gamma_1, ..., \gamma_{p-1}),$$

and, for $i = 1, ..., p-1$,

$$\partial_i(\gamma_1, ..., \gamma_p) = (\gamma_1, ..., \gamma_i \circ \gamma_{i+1}, ..., \gamma_p).$$

A direct computation shows that $\partial^2 = 0$ where

$$\partial : \Omega^q(G^{(p-1)}) \longrightarrow \Omega^q(G^{(p)})$$

is given by $\partial := \sum_{i=0,...,p} (-1)^i \partial_i^*$. For a fixed q, $(\Omega^q(G^{(\cdot)}), \partial)$ is then a complex. The exterior de Rham derivative $d : \Omega^q \to \Omega^{(q+1)}$ connects the various complexes

with each other. These lead to a double complex

(258)
$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^2(G^{(0)}) & \xrightarrow{\partial} & \Omega^2(G^{(1)}) & \xrightarrow{\partial} & \Omega^2(G^{(2)}) & \xrightarrow{\partial} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^1(G^{(0)}) & \xrightarrow{\partial} & \Omega^1(G^{(1)}) & \xrightarrow{\partial} & \Omega^1(G^{(2)}) & \xrightarrow{\partial} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\Omega^0(G^{(0)}) & \xrightarrow{\partial} & \Omega^0(G^{(1)}) & \xrightarrow{\partial} & \Omega^0(G^{(2)}) & \xrightarrow{\partial} & \cdots
\end{array}$$

with associated total complex

(259)
$$C^n_{dR}(G) := \bigoplus_{p+q=n} \Omega^q(G^{(p)})$$

and total differential $\delta : C^n_{dR}(G) \to C^{n+1}_{dR}(G)$ given by

(260)
$$\delta(\omega) = \partial(\omega) + (-1)^p d(\omega),$$

for $\omega \in \Omega^q(G^{(p)})$.

Definition A.2.1 (The de Rham cohomology of a Lie groupoid). *The complex $(C^\bullet_{dR}(G), \delta)$ is called the de Rham complex of the Lie groupoid $G \rightrightarrows M$. Its cohomology groups*

(261)
$$H^n_{dR}(G) := H^n(C^\bullet_{dR}(G), \delta)$$

are called the de Rham cohomology groups of $G \rightrightarrows M$.

Remark A.2.2. *As proved in* [**17**]*, the above groups are invariant under Morita equivalence of Lie groupoids and hence they define groups associated to the corresponding stacks.*

A.2.2. Cohomology with compact supports. Let $G \rightrightarrows M$ be a Lie groupoid. Consider the following two numbers

$$r = dim\, G^{(1)} - dim\, G^{(0)} \text{ and } n = 2dim\, G^{(0)} - dim\, G^{(1)} = dim\, G^{(0)} - r.$$

Let $\Omega_c^q(G^{(p)})$ be the space of differential forms on $G^{(p)}$ with compact support. Consider the double complex

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial_!} & \Omega_c^{n+3r}(G^{(2)}) & \xrightarrow{\partial_!} & \Omega_c^{n+2r}(G^{(1)}) & \xrightarrow{\partial_!} & \Omega_c^{n+r}(G^{(0)}) \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle -d} & & \uparrow{\scriptstyle d} \\
\cdots & \xrightarrow{\partial_!} & \Omega_c^{n+3r-1}(G^{(2)}) & \xrightarrow{\partial_!} & \Omega_c^{n+2r-1}(G^{(1)}) & \xrightarrow{\partial_!} & \Omega_c^{n+r-1}(G^{(0)}) \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle -d} & & \uparrow{\scriptstyle d} \\
\cdots & \xrightarrow{\partial_!} & \Omega_c^{n+3r-2}(G^{(2)}) & \xrightarrow{\partial_!} & \Omega_c^{n+2r-2}(G^{(1)}) & \xrightarrow{\partial_!} & \Omega_c^{n+r-2}(G^{(0)}) \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle -d} & & \uparrow{\scriptstyle d} \\
 & & \vdots & & \vdots & & \vdots
\end{array} \tag{262}$$

where the vertical differential is the usual exterior derivative d and where the horizontal differential is defined in terms of

$$\partial_! := \sum_i (-1)^i (\partial_i)_! \tag{263}$$

with $(\partial_i)_! : \Omega_c^{q+r}(G^{(p)}) \to \Omega_c^q(G^{(p-1)})$ is obtained by integration along the fibers of the structural map ∂_i.

In fact, for $\gamma \in \Omega_c^q(G^{(p)})$, the horizontal differential is given by

$$\gamma \mapsto (-1)^p \partial_! \gamma.$$

The associated total complex is given by

$$C_c^\nu(G) := \bigoplus_{j=0,\ldots,\nu} \Omega_c^{n+(j+1)r-\nu+j}(G^{(j)}) \tag{264}$$

with total differential

$$\delta(\gamma) = \partial_!(\gamma) + (-1)^p d(\gamma), \quad for \ \ \gamma \in \Omega_c^q(G^{(p)}). \tag{265}$$

Definition A.2.3 (The de Rham cohomology with compact supports of a Lie groupoid). *The complex $(C_c^\bullet(G), \delta)$ is called the compactly supported de Rham complex of the Lie groupoid $G \rightrightarrows M$. Its homology groups*

$$H_c^\nu(G) := H_\nu(C_c^\bullet(G), \delta) \tag{266}$$

are called the compactly supported de Rham cohomology groups of $G \rightrightarrows M$.

A.2.2.1. *Currents and Poincaré duality.* Given $\gamma \in \Omega^q(G^{(p)})$ and $\omega \in \Omega_c^{pr-q+dim\, G^{(0)}}(G^{(p)})$, the form

$$\gamma \cap \omega \in \Omega^{dim\, G^{(0)}}(G^{(0)}) \tag{267}$$

defined in [**17**] page 12, allows to define for every such an $\gamma \in \Omega^q(G^{(p)})$ a current

$$C_\gamma : \Omega_c^{pr-q+dim\, G^{(0)}}(G^{(p)}) \to \mathbb{R} \tag{268}$$

given by

$$C_\gamma(\omega) := \int_{G^{(0)}} \gamma \cap \omega \tag{269}$$

EXAMPLE A.2.4. *Let $G = M \rtimes \Gamma \rightrightarrows M$ be the transformation Lie groupoid associated to the action (by diffeomorphisms) of a discrete countable group Γ on a smooth manifold M with $\dim M = n$. Let $p \in \mathbb{N}$ and let $f \in \mathbb{C}\Gamma^p$. We set $\gamma := \pi^* f \in \Omega_c^0(M \times \Gamma^p)$ where $\pi : M \times \Gamma^p \to \Gamma^p$ is the projection on the second factor. Let $\omega \in \Omega_c^n(M \times \Gamma^p)$, in this example the form $\gamma \cap \omega \in \Omega_c^n(M)$ is explicitly given by*

$$\pi_1!(\omega \wedge f) \tag{270}$$

where $\pi_1 : M \times \Gamma^p \to M$ is the projection in the first coordinate. In other words, for $m \in M$ we have

$$(\omega \cap \pi^* f)(m) = \sum_{g \in \Gamma^p} f(g) \cdot \omega(m, g). \tag{271}$$

We can consider in particular the current defined by $f \in \mathbb{C}\Gamma^p$,

$$C_f(\omega) := \int_M \omega \cap \pi^* f. \tag{272}$$

Following from Behrend's paper [**17**], one can check that the above pairing gives a well defined pairing on cohomology

$$H^k_{dR}(G) \times H_c^{n-k}(G) \to \mathbb{R}, \tag{273}$$

which is perfect (Proposition 19 in [**17**]), hence giving Poincaré Duality, under the assumption that both $G^{(1)}$ and $G^{(0)}$ have compatible (via s and t) finite good covers. This assumption holds whenever the groupoid is associated to a proper cocompact action.

A.2.3. The de Rham cohomology with fiberwise rapidly decreasing support. For some particular groupoids one might consider a slight generalization of the above compactly supported cohomology and pairing. We will detail here a particular example we use in this work. Let $E \to M$ be a Γ-equivariant, proper vector bundle. We consider the action groupoid

$$G := E \rtimes \Gamma \rightrightarrows E.$$

Let $\Omega^q_{sch}(G^{(p)})$ be the space of differential forms on $G^{(p)} \cong E \times \Gamma^p$ with compact horizontal support (*i.e.* in the direction of the base M) and with a Schwartz condition along the fibers of E. The differentials $\partial_!$ of Equation (263) above are still well defined in $\Omega^*_{sch}(G^{(p)})$. We can hence consider the analog of the de Rham cohomology with compact supports, we call it the de Rham cohomology with Schwartz support

$$H^\nu_{sch}(E \rtimes \Gamma) := H^\nu(C^\bullet_{sch}(E \rtimes \Gamma), \delta).$$

A.3. Differentiable stacks

A differentiable stack, denoted as $[M/G]$, is the Morita equivalence class of some Lie groupoid $G \rightrightarrows M$, see [**19**] or [**18**] for further details.

Several of the objects we define and use in this work are differentiable stacks invariants in the sense that they can be defined for a Lie groupoid and that

Morita equivalent groupoids give rise to the same (or isomorphic) invariant. For example the K-theory groups of the groupoid C^*-algebras, the periodic cyclic groups of the groupoid convolution algebras or the Lie groupoid cohomology are examples of such invariants. But also the topological groupoid K-theory groups defined in chapter 4 together with the geometric Baum-Connes assembly, see [**35**]. Another example of this is the Chern-Baum -Connes assembly map, the proof of this last statement follows the same lines of the proof in [**35**] and will be developed somewhere else.

Now, even if not explicitly stated in this work or in the most part of my articles, the differentiable stack point of view has been a guiding point in my own research. This is why I wanted to end this appendix with this small note/section.

APPENDIX B

Deformation groupoids

B.1. Deformation Lie groupoids

In this paper we deal with Lie groupoids for which we are going to use the notation $G \rightrightarrows M$, see [**67, 85**] for more details on Lie groupoids.

In this section we want to recall some particular groupoids that will be used in this paper, their construction/definition is based on the deformation to the normal cone construction to be recalled in the following.

Let M be a C^∞ manifold and $X \subset M$ be a C^∞ submanifold. We denote by $\mathscr{N}(M,X)$ the normal bundle to X in M. We define the following set

$$\mathscr{D}(M,X) := (\mathscr{N}(M,X) \times 0) \bigsqcup (M \times \mathbb{R}^*)\,. \tag{274}$$

The purpose of this section is to recall how to define a C^∞-structure on $\mathscr{D}(M,X)$. This is more or less classical, for example it was extensively used in [**57**].

Let us first consider the case where $M = \mathbb{R}^p \times \mathbb{R}^{n-p}$ and $X = \mathbb{R}^p \times \{0\}$ (here we identify X canonically with $\mathbb{R}^p$). We denote by $q = n - p$ and by $\mathscr{D}^n_p$ for $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$ as above. In this case we have that $\mathscr{D}^n_p = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$ (as a set). Consider the bijection $\psi : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \to \mathscr{D}^n_p$ given by

$$\psi(x,\xi,t) = \begin{cases} (x,\xi,0) & \text{if } t = 0 \\ (x,t\xi,t) & \text{if } t \neq 0 \end{cases} \tag{275}$$

whose inverse is given explicitly by

$$\psi^{-1}(x,\xi,t) = \begin{cases} (x,\xi,0) & \text{if } t = 0 \\ (x,\frac{1}{t}\xi,t) & \text{if } t \neq 0 \end{cases}$$

We can consider the C^∞-structure on $\mathscr{D}^n_p$ induced by this bijection.

We pass now to the general case. A local chart $(\mathscr{U},\phi)$ of M at x is said to be an X-slice if

1) $\mathscr{U}$ is an open neighborhood of x in M and $\phi : \mathscr{U} \to U \subset \mathbb{R}^p \times \mathbb{R}^q$ is a diffeomorphsim such that $\phi(x) = (0,0)$.
2) Setting $V = U \cap (\mathbb{R}^p \times \{0\})$, then $\phi^{-1}(V) = \mathscr{U} \cap X$, denoted by $\mathscr{V}$.

With these notations understood, we have $\mathscr{D}(U,V) \subset \mathscr{D}^n_p$ as an open subset. For $x \in \mathscr{V}$ we have $\phi(x) \in \mathbb{R}^p \times \{0\}$. If we write $\phi(x) = (\phi_1(x), 0)$, then

$$\phi_1 : \mathscr{V} \to V \subset \mathbb{R}^p$$

is a diffeomorphism. Define a function

$$\tilde{\phi} : \mathscr{D}(\mathscr{U},\mathscr{V}) \to \mathscr{D}(U,V) \tag{276}$$

by setting $\tilde{\phi}(v,\xi,0) = (\phi_1(v), d_N\phi_v(\xi), 0)$ and $\tilde{\phi}(u,t) = (\phi(u),t)$ for $t \neq 0$. Here $d_N\phi_v : N_v \to \mathbb{R}^q$ is the normal component of the derivative $d_v\phi$ for $v \in \mathscr{V}$. It is clear that $\tilde{\phi}$ is also a bijection. In particular, it induces a C^∞ structure on $\mathscr{D}^{\mathscr{U}}_{\mathscr{V}}$. Now, let us consider an atlas $\{(\mathscr{U}_\alpha, \phi_\alpha)\}_{\alpha\in\Delta}$ of M consisting of $X-$slices. Then the collection $\{(\mathscr{D}(\mathscr{U}_\alpha, \mathscr{V}_\alpha), \tilde{\phi}_\alpha)\}_{\alpha\in\Delta}$ is a C^∞-atlas of $\mathscr{D}(M,X)$ (Proposition 3.1 in [**32**]).

Definition B.1.1 (Deformation to the normal cone). *Let $X \subset M$ be as above. The set $\mathscr{D}(M,X)$ equipped with the C^∞ structure induced by the atlas of X-slices is called the deformation to the normal cone associated to the embedding $X \subset M$.*

One important feature about the deformation to the normal cone is the functoriality. More explicitly, let $f : (M,X) \to (M',X')$ be a C^∞ map $f : M \to M'$ with $f(X) \subset X'$. Define $\mathscr{D}(f) : \mathscr{D}(M,X) \to \mathscr{D}(M',X')$ by the following formulas:

1) $\mathscr{D}(f)(m,t) = (f(m),t)$ for $t \neq 0$,
2) $\mathscr{D}(f)(x,\xi,0) = (f(x), d_N f_x(\xi), 0)$, where $d_N f_x$ is by definition the map

$$(\mathscr{N}(M,X))_x \xrightarrow{d_N f_x} (\mathscr{N}(M',X'))_{f(x)}$$

induced by $T_xM \xrightarrow{df_x} T_{f(x)}M'$ and $\mathscr{N}(M',X')$ is the normal bundle of X' in M'.

Then $\mathscr{D}(f) : \mathscr{D}(M,X) \to \mathscr{D}(M',X')$ is a C^∞-map (Proposition 3.4 in [**32**]). In the language of categories, the deformation to the normal cone construction defines a functor

$$\mathscr{D} : \mathscr{C}^\infty_2 \longrightarrow \mathscr{C}^\infty, \tag{277}$$

where $\mathscr{C}^\infty$ is the category of C^∞-manifolds and $\mathscr{C}^\infty_2$ is the category of pairs of C^∞-manifolds.

We briefly discuss here the deformation groupoid of an immersion of groupoids which is called the normal groupoid in [**57**].

Given an immersion of Lie groupoids $\mathscr{G}_1 \xrightarrow{\varphi} \mathscr{G}_2$, let $\mathscr{G}_1^N = \mathscr{N}(\mathscr{G}_2, \mathscr{G}_1)$ be the total space of the normal bundle to φ, and $(\mathscr{G}_1^{(0)})^N$ be the total space of the normal bundle to $\varphi_0 : \mathscr{G}_1^{(0)} \to \mathscr{G}_2^{(0)}$. Consider $\mathscr{G}_1^N \rightrightarrows (\mathscr{G}_1^{(0)})^N$ with the following structure maps: The source map is the derivation in the normal direction $d_N s : \mathscr{G}_1^N \to (\mathscr{G}_1^{(0)})^N$ of the source map (seen as a pair of maps) $s : (\mathscr{G}_2, \mathscr{G}_1) \to (\mathscr{G}_2^{(0)}, \mathscr{G}_1^{(0)})$ and similarly for the target map.

The deformation to the normal cone construction allows us to consider a C^∞ structure on

$$\mathscr{G}_\varphi := \left(\mathscr{G}_1^N \times \{0\}\right) \bigsqcup (\mathscr{G}_2 \times \mathbb{R}^*),$$

such that $\mathscr{G}_1^N \times \{0\}$ is a closed saturated submanifold and so $\mathscr{G}_2 \times \mathbb{R}^*$ is an open submanifold. The following result is an immediate consequence of the functoriality of the deformation to the normal cone construction.

PROPOSITION B.1.2 (Hilsum-Skandalis, 3.1, 3.19 [**57**]). *Consider an immersion $\mathscr{G}_1 \xrightarrow{\varphi} \mathscr{G}_2$ as above. Let $\mathscr{G}_{\varphi_0} := \left((\mathscr{G}_1^{(0)})^N \times \{0\}\right) \bigsqcup \left(\mathscr{G}_2^{(0)} \times \mathbb{R}^*\right)$ be the deformation to the normal cone of the pair $(\mathscr{G}_2^{(0)}, \mathscr{G}_1^{(0)})$. The groupoid*

$$\mathscr{G}_\varphi \rightrightarrows \mathscr{G}_{\varphi_0} \tag{278}$$

with structure maps compatible with the ones of the groupoids $\mathscr{G}_2 \rightrightarrows \mathscr{G}_2^{(0)}$ and $\mathscr{G}_1^N \rightrightarrows (\mathscr{G}_1^{(0)})^N$, is a Lie groupoid with C^∞-structures coming from the deformation to the normal cone.

The Lie groupoid above will be called the normal groupoid associated to ϕ, besides the article of Hilsum and Skandalis in which it appeared for the first time, these kind of groupoids were extensively used in [**35**], see also the recent article [**77**] for more details on these groupoids.

One of the motivations of these kind of groupoids is to be able to define deformation indices. Indeed, restricting the deformation to the normal cone construction to the closed interval $[0,1]$ and since the groupoid $\mathscr{G}_2 \times (0,1]$ is an open saturated subgroupoid of $\mathscr{G}_\varphi$ (see 2.4 in [**57**] or [**93**] for more details), we have a short exact sequence of C^*-algebras

$$0 \to C^*(\mathscr{G}_2 \times (0,1]) \longrightarrow C^*(\mathscr{G}_\varphi) \xrightarrow{ev_0} C^*(\mathscr{G}_1^N) \to 0, \tag{279}$$

with $C^*(\mathscr{G}_2 \times (0,1])$ contractible. Then the 6-term exact sequence in K-theory provides the isomorphism

$$(ev_0)_* : K_*(C^*(\mathscr{G}_\varphi)) \cong K_*(C^*(\mathscr{G}_1^N)).$$

Hence we can define the index morphism

$$D_\varphi : K_*(C^*(\mathscr{G}_1^N)) \longrightarrow K_*(C^*(\mathscr{G}_2))$$

between the K-theories of the maximal C^*-algebras as the induced deformation index morphism

$$D_\varphi := (ev_1)_* \circ (ev_0)_*^{-1} : K_*(C^*(\mathscr{G}_1^N)) \cong K_*(C^*(\mathscr{G}_\varphi)) \longrightarrow K_*(C^*(\mathscr{G}_2)).$$

As we will see in these appendices, with the use of appropriate algebras, one can construct similar deformation morphisms in cyclic periodic (co)homology.

Another important example that will be used in this paper is constructed from the example of normal groupoid above and the functoriality of the deformation to the normal cone construction. Indeed, one can for instance consider the classic and most famous example of normal groupoid, the so-called tangent or Connes groupoid of a manifold M, it is indeed the normal groupoid associated to the inclusion

$$M \to M \times M$$

of the diagonal, seen as a unit groupoid, in the pair groupoid of the manifold. Up to a (non canonical) identification of the normal bundle of this inclusion with the tangent bundle TM the Connes tangent groupoid takes the following form

$$G_M^{tan} = TM \bigsqcup (M \times M) \times (0,1] \rightrightarrows M \times [0,1].$$

Now, in the presence of an action of a group Γ on M one can consider the associated diagonal action on $M \times M$. Since this action preserves the diagonal it is an immediate consequence of the functoriality of the deformation to the normal cone construction that there is an induced action of Γ on the tangent groupoid G_M^{tan} inducing a semi-direct deformation groupoid

$$G_M^{tan} \rtimes \Gamma \rightrightarrows M \times [0,1]$$

explicitly described in these appendices.

B.2. Pdo calculus for groupoids and deformation groupoids

Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid with compact base M. Besides different C^*-algebras associated to the smooth convolution algebra $C_c^\infty(\mathscr{G})$ there several other convolution algebras and their completions. Of particular importance for index theory, for every Lie groupoid there is an algebra of pseudodifferential operators (in this work, classic homogeneous operators, properly supported)

$$\Psi^\infty(\mathscr{G}) \tag{280}$$

whose regularising part coincides with the smooth convolution algebra,

$$\Psi^{-\infty}(\mathscr{G}) = C_c^\infty(\mathscr{G}), \tag{281}$$

and for which one has en extension of the classic pseudodifferential calculus. One of the main properties is that it contains the quasi-inverses (inverses modulo regularising/smoothing operators) of the elliptic differential operators generated naturally by the vector fields defining the Lie algebroid.

In particular, there is a pdo short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(\mathscr{G}) \longrightarrow \overline{\Psi^0}(\mathscr{G}) \xrightarrow{\sigma} C(S^*\mathscr{G}) \longrightarrow 0. \tag{282}$$

where $\overline{\Psi^0}(\mathscr{G})$ denotes as usual the completion of $\Psi^0(\mathscr{G})$ as bounded operators on some appropriate L^2-spaces. For more details, proofs and further developments on pseudodiffrential calculus and convolution opertaros on Lie groupoids see for example [**84**], [**80**], [**63**], [**99**].

By definition, an operator $D \in \overline{\Psi^0}(\mathscr{G})$ is elliptic if $\sigma(D)$ is invertible. In this case D defines canonically an element in $K_1(C(S^*\mathscr{G}))$, let us denote it by $\delta(D)$. Consider now the K-theory connecting morphism

$$I_{\mathscr{G}} : K_1(C(S^*\mathscr{G})) \longrightarrow K_0(C^*(\mathscr{G})) \tag{283}$$

associated to the short exact sequence above. Associated to an elliptic operator $D \in \overline{\Psi^0}(\mathscr{G})$ as above, the element

$$I_{\mathscr{G}}(\delta(D)) \in K_0(C^*(\mathscr{G}))$$

is called the analytic index of D. It is quite well known that it is given by the classical Connes-Skandalis type idempotent in the presence of a parametrix. However, it is not so classic, even if highly admittted that this index morphism

can be computed in terms of the tangent groupoid of $\mathscr{G}$, we state and prove the following result.

PROPOSITION B.2.1. *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid with compact base M. The following diagram*

$$\begin{array}{ccc} K_1(C(S^*\mathscr{G})) & \xrightarrow{I_{\mathscr{G}}} & K_0(C^*(\mathscr{G})) \\ {\scriptstyle I_{A\mathscr{G}}}\downarrow & \nearrow_{(e_0)_*^{-1}\circ(e_1)_*} & \\ K_0(C^*(A\mathscr{G})) & & \end{array} \tag{284}$$

is commutative, where $I_{A\mathscr{G}}$ is the connecting map associated to the Lie algebroid (considered as a Lie groupod using its vector bundle structure).

PROOF. Given the Lie groupoid $\mathscr{G} \rightrightarrows M$ we may consider the following two Lie groupoids, the Lie algebroid (considered as a Lie groupod using its vector bundle structure)

$$A\mathscr{G} \rightrightarrows M \tag{285}$$

and the tangent groupoid

$$\mathscr{G}^{tan} \rightrightarrows M \times [0,1]. \tag{286}$$

Now, consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(A\mathscr{G}) & \longrightarrow & \overline{\Psi^0}(A\mathscr{G}) & \longrightarrow & C(S^*\mathscr{G}) & \longrightarrow & 0 \\ & & {\scriptstyle e_0}\uparrow & & {\scriptstyle e_0}\uparrow & & {\scriptstyle e_0}\uparrow & & \\ 0 & \longrightarrow & C^*(\mathscr{G}^{tan}) & \longrightarrow & \overline{\Psi^0}(\mathscr{G}^{tan}) & \longrightarrow & C(S^*\mathscr{G}\times[0,1]) & \longrightarrow & 0 \\ & & {\scriptstyle e_1}\downarrow & & {\scriptstyle e_1}\downarrow & & {\scriptstyle e_1}\downarrow & & \\ 0 & \longrightarrow & C^*(\mathscr{G}) & \longrightarrow & \overline{\Psi^0}(\mathscr{G}) & \longrightarrow & C(S^*\mathscr{G}) & \longrightarrow & 0 \end{array} \tag{287}$$

that links the pdo short exact sequences of these three groupoids. The right vertical line is of this form since the Lie algebroid of $A\mathscr{G} \rightrightarrows M$ identifies with $A\mathscr{G}$ (the anchor is not the same but we do not need this) and the Lie algebroid of $\mathscr{G}^{tan} \rightrightarrows M \times [0,1]$ identifies with $A\mathscr{G} \times [0,1]$ (again the anchor is interesting but we do not need it here). To conclude it is enough to apply the $K-$theory functor of C^*-algebras to the entire diagram above together with the fact that the K-theory morphisms associated to the right vertical morphisms are the identity in $K-$theory by homotopy invariance. □

To put the above proposition in a more familiar way, we consider the following diagram

(288)
$$\begin{array}{ccc} Ell_{\mathscr{G}} & \xrightarrow{\delta} & K_1(C(S^*\mathscr{G})) \\ \downarrow{\scriptstyle p_r} & & \downarrow{\scriptstyle I_{A\mathscr{G}}} \\ K^0_{top}(A^*\mathscr{G}) & \xrightarrow[F]{\cong} & K_0(C^*(A\mathscr{G})) \end{array}$$

where $Ell_{\mathscr{G}}$ denotes the set of elliptic $\mathscr{G}$-operators (one admits operators cting between sections of vector bundles), p_r denotes the map that associates, to such an elliptic operator, the principal symbol class in the topological K-theory group $K^0_{top}(A^*\mathscr{G}) = K_0(C_0(A^*\mathscr{G}))$, and where F stands for the isomorphism induced from the algebra isomorphism $C_0(A^*\mathscr{G}) \cong C^*(A^*\mathscr{G})$ given by the fiberwise Fourier isomorphism. By classic properties of pseudodifferential calculus the diagram above is commutative, even more the vertical maps are surjective. In any case, thanks to proposition above we get that the morphism

(289)
$$Ind_a^{\mathscr{G}} : K^0_{top}(A^*\mathscr{G}) \longrightarrow K_0(C^*(\mathscr{G}))$$

given by the composition of the isomorphism F above followed by the deformation index $((e_0)_*^{-1} \circ (e_1)_*)$ computes indeed the analytic index of elliptic $\mathscr{G}$-operators. We call the morphism $Ind_a^{\mathscr{G}}$ the analytic index morphism of the groupoid $\mathscr{G}$.

APPENDIX C

Connes-Thom vs Thom

Let $\mathscr{G} \rightrightarrows M$ be a locally compact groupoid.

We consider $\mathbb{R}^N$ as an additive group, hence a one unit groupoid. Suppose we have an homomorphism of groupoids

$$\mathscr{G} \xrightarrow{h} \mathbb{R}^N. \tag{290}$$

This gives rise to an action (say, a right one) of $\mathscr{G}$ on the space $M \times \mathbb{R}^N$ and thus to a new semi-direct groupoid:

$$\mathscr{G}_h := (M \times \mathbb{R}^N) \rtimes \mathscr{G} : \mathscr{G} \times \mathbb{R}^N \rightrightarrows M \times \mathbb{R}^N \tag{291}$$

which has the following structural maps:

- The source and target maps are given by
$$s(\gamma, X) = (s(\gamma), X + h(\gamma)) \text{ and } r(\gamma, X) = (r(\gamma), X)$$
- The multiplication is defined on composable arrows by the formula
$$(\gamma, X) \cdot (\eta, X + h(\gamma)) := (\gamma \cdot \eta, X).$$

Then it is obviously a groupoid with unit map $u(m, X) = (m, X)$ ($h(m) = 0$ since h is an homomorphism), and inverse given by $(\gamma, X)^{-1} = (\gamma^{-1}, X + h(\gamma))$ (again since we have a homomorphism, $h(\gamma) + h(\gamma^{-1}) = 0$).

REMARK C.0.1. *For the trivial homomorphism $h_0 = 0$, the associated groupoid is just the product groupoid*

$$\mathscr{G} \times \mathbb{R}^N \rightrightarrows M \times \mathbb{R}^N.$$

If $\mathscr{G}$ is provided with a Haar system, then the semi-direct groupoid $\mathscr{G}_h$ inherits a natural Haar system such that the C^*-algebra $C^*(\mathscr{G}_h)$ is isomorphic to the crossed product algebra $C^*(\mathscr{G}) \rtimes_h \mathbb{R}^N$ where $\mathbb{R}^N$ acts on $C^*(\mathscr{G})$ by automorphisms by the formula: $\alpha_X(f)(\gamma) = e^{i\cdot(X\cdot h(\gamma))} f(\gamma)$, $\forall f \in C_c(\mathscr{G})$, (see [**39**], propostion II.5.7 for details). In particular, in the case N is even, we have a Connes-Thom isomorphism in K-theory ([**39**], II.C)

$$K_0(C^*(\mathscr{G})) \xrightarrow[\approx]{\mathscr{CT}} K_0(C^*(\mathscr{G}_h)) \tag{292}$$

which generalizes the classical Thom isomorphism, and which is natural with respect to morphisms of algebras.

Since we will need to compute explicitly the morphism induced by the homomorphism we propose an alternative construction of Connes-Thom which can be computed in our context. More precisely we want to work directly with

the groupoid algebras $C^*(\mathscr{G}_h)$ without passing through the isomorphism with $C^*(\mathscr{G}) \rtimes_h \mathbb{R}^N$.

Given the morphism $\mathscr{G} \xrightarrow{h} \mathbb{R}^N$ we consider the product groupoid $\mathscr{G} \times [0,1] \rightrightarrows M \times [0,1]$ of the groupoid $\mathscr{G}$ with the *space* $[0,1]$ and we define

$$H : \mathscr{G} \times [0,1] \longrightarrow \mathbb{R}^N,$$

the homomorphism given by

$$H(\gamma, t) := t \cdot h(\gamma).$$

This homomorphism gives a deformation between the trivial homomorphism and h, more precisely in [**29**] lemma 2.8 we proved:

LEMMA C.0.2. *Denote by $\mathscr{G}_H := (\mathscr{G} \times [0,1])_H$ the semi-direct groupoid associated to the homomorphism H. For each $t \in [0,1]$ we denote by $(\mathscr{G}_H)_t$ the restriction subgroupoid $\mathscr{G}_H|_{M \times \{t\} \times \mathbb{R}^N}$. We have the following properties:*

(i) $(\mathscr{G}_H)_0 = \mathscr{G} \times \mathbb{R}^N$
(ii) $(\mathscr{G}_H)_1 = \mathscr{G}_h$
(iii) $(\mathscr{G}_H)|_{(0,1]} \approx \mathscr{G}_h \times (0,1]$ *and in particular* $C^*((\mathscr{G}_H)|_{(0,1]}) \approx C^*(\mathscr{G}_h \times (0,1])$ *is contractible.*

The last lemma gives rise to a short exact sequence of C^*-algebras ([**57, 93**]):

$$0 \to C^*(\mathscr{G}_h \times (0,1]) \to C^*(\mathscr{G}_H) \xrightarrow{e_0} C^*(\mathscr{G} \times \mathbb{R}^N) \to 0, \tag{293}$$

where e_0 is induced by the evaluation at zero. This defines a deformation index morphism

$$\mathscr{D}_h : K_*(C^*(\mathscr{G} \times \mathbb{R}^N)) \to K_*(C^*(\mathscr{G}_h)). \tag{294}$$

The natural map $\mathscr{G}_H \to [0,1]$ gives to $\mathscr{G}_H$ the structure of a continuous field of groupoids over $[0,1]$ and if $\mathscr{G}$ is assumed to be amenable, we get by [**60**] that $C^*(\mathscr{G}_H)$ is the space of continuous sections of a continuous field of C^*-algebras. Then, the deformation index morphism above coincides with the morphism of theorem 3.1 in [**52**].

DEFINITION C.0.3. *Let $\mathscr{G}$ be a groupoid together with a homomorphism h from $\mathscr{G}$ to $\mathbb{R}^N$ (with N even). Consider the morphism in K-theory*

$$K_*(C^*(\mathscr{G})) \xrightarrow{\mathscr{CT}_h} K_*(C^*(\mathscr{G}_h)), \tag{295}$$

given by the composition of the Bott morphism

$$K_*(C^*(\mathscr{G})) \xrightarrow{B} K_*(C^*(\mathscr{G} \times \mathbb{R}^N)),$$

and the deformation index morphism

$$K_*(C^*(\mathscr{G} \times \mathbb{R}^N)) \xrightarrow{\mathscr{D}_h} K_*(C^*(\mathscr{G}_h)).$$

We will refer to this morphism as the Connes-Thom map associated to h.

In fact, Elliot, Natsume and Nest proved that this morphism coincides with the usual Connes-Thom isomorphism, theorem 5.1 in [**52**]. We can restate their result in our framework as follows:

PROPOSITION C.0.4 (Elliot-Natsume-Nest). *Let $(\mathscr{G}, h)$ be an amenable continuous family groupoid (or amenable locally compact groupoid with a continuous Haar system) together with a homomorphism on $\mathbb{R}^N$ (N even). Then the morphism $\mathscr{C}\mathscr{T}_h : K_*(C^*(\mathscr{G})) \to K_*(C^*(\mathscr{G}_h))$ coincides with the Connes-Thom isomorphism. In particular, it satisfies the following properties:*

(i) *Naturality.*
(ii) *If $\mathscr{G}$ is a space (the groupoid equals its set of units), then $\mathscr{C}\mathscr{T}_h$ is the Bott morphism.*

At some point we will need to recover from our Connes-Thom isomorphism the classic Atiyah-Singer integrand, therefore we need to relate, at least in a very particular case, this Connes-Thom isomorphism with the classic Thom isomorphism. We enounce the result in the following proposition:

PROPOSITION C.0.5. *[Thom as Connes-Thom] Let X be a C^∞ manifold (probably with corners) together with an embedding $j : X \hookrightarrow \mathbb{R}^N$ with N an even integer. Let N_X the normal bundle to X associated to the embedding j. Consider the groupoid morphism*

$$h : TX \longrightarrow \mathbb{R}^N$$

given by $h(x, V) = d_x j(V)$, where $TX \rightrightarrows X$ is seen as a groupoid using its vector bundle structure. Then we have

(*i*) *The crossed product groupoid $TX \times_h \mathbb{R}^N$ is morita equivalent to the space N_X. In particular there is a Connes-Thom isomorphism*

$$K^*(TX) \xrightarrow[\approx]{\mathscr{C}\mathscr{T}} K^*(N_X). \tag{296}$$

(*ii*) *Let $N_X \xrightarrow{p} X$ the vector bundle projection. Consider the pullback vector bundle $p^*(N_X \oplus TX)$ over N_X. The total space of this vector bundle identifies canonically with the total space of the vector bundle $\pi^*(N_X \times N_X)$ over TX. We have then a Thom isomorphism*

$$K^*(TX) \xrightarrow[\approx]{\mathscr{T}} K^*(p^*(N_X \oplus TX)). \tag{297}$$

(iii) *We have a vector bundle isomorphism $p^*(N_X \oplus TX) \approx N_X \times \mathbb{R}^N$ and the following commutative diagram*

$$\begin{array}{ccc} K^*(TX) & \xrightarrow{\mathscr{C}\mathscr{T}} & K^*(N_X) \\ {\scriptstyle \mathscr{T}}\downarrow & \nearrow_{B^{-1}} & \\ K^*(N_X \times \mathbb{R}^N) & & \end{array} \tag{298}$$

where B stands for the usual Bott isomorphism. In other words $\mathscr{C}\mathscr{T} = B^{-1} \circ \mathscr{T}$.

PROOF. We will use several morphisms (with the correspondent crossed products) on different groupoids, we start by enounce them. We remark that in all four cases the morphism will be denoted by h to keep the notation short,

more important the morphisms h are all injective and define proper free crossed product groupoids (again by using proposition 2.14 in [**95**]).

(i) The groupoid morphism

$$h : TX \longrightarrow \mathbb{R}^N$$

given by $h(x, V) = V$, where $TX \rightrightarrows X$ is seen as a groupoid using its vector bundle structure.

The correspondant free proper groupoid $TX \times_h \mathbb{R}^N$ is Morita equivalent to the space N_X.

(ii) Consider the pullback groupoid diagram

(299)
$$\begin{array}{ccc} {}^*p^*TX & \xrightarrow{q} & TX \\ \downarrow\downarrow & & \downarrow\downarrow \\ N_X & \longrightarrow & X. \end{array}$$

In particular there is an induced groupoid morphism

$$h :^* p^*TX \to \mathbb{R}^N$$

given by $h(n_1, (x, V), n_2) = V$.

Remember ${}^*p^*TX \xrightarrow{q} TX$ is a Morita equivalence between these two groupoids. Hence, the correspondant free proper groupoid ${}^*p^*TX \times_h \mathbb{R}^N$ is also Morita equivalent to the space N_X.

(iii) The algebroid of the groupoid ${}^*p^*TX$ above is easily seen to be the pullback vector bundle $p^*(TX \oplus N_X) \to N_X$ over N_X. As a groupoid, it has an induced morphism

$$h : p^*(TX \oplus N_X) \to \mathbb{R}^N$$

given by $h(n, V, N) = V + N$.

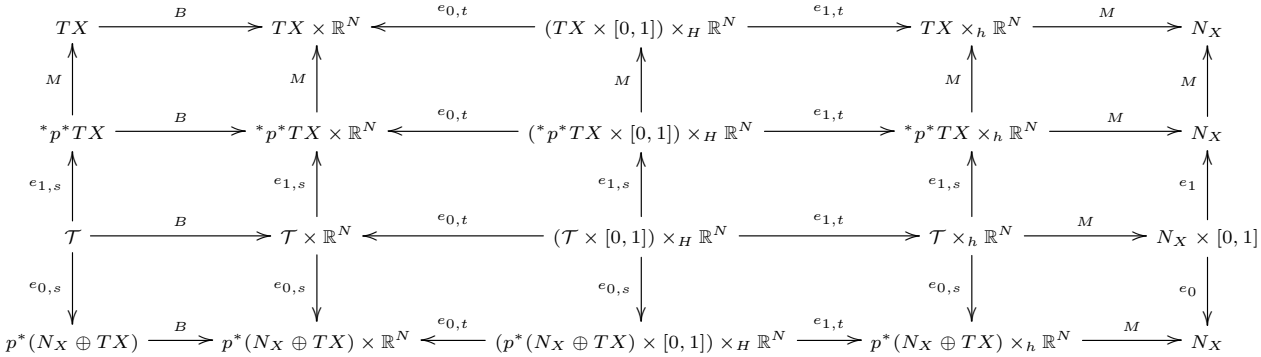
The correspondant free proper groupoid $p^*(TX \oplus N_X) \times_h \mathbb{R}^N$ is also Morita equivalent to the space N_X, an easy computation of the orbit space confirms it.

(iv) From the precedent two points, if we consider the tangent groupoid, $\mathcal{T} \rightrightarrows N_X \times [0, 1]$ of the groupoid ${}^*p^*TX$ above, there is an induced morphism

$$h : \mathcal{T} \to \mathbb{R}^N.$$

The correspondant free proper groupoid $\mathcal{T} \times_h \mathbb{R}^N$ is Morita equivalent to the space $N_X \times [0, 1]$, an easy computation of the orbit space confirms it.

For every morphism h above we have the associated morphism H and the respective Morita equivalences. In fact, it is now immediate that the following diagram between $K-$theory groups is commutative:



where

- at each time (because lack of space) we have only putted the groupoid instead of its K-theory group (ex: $\mathcal{T}$ in the diagram stands for $K_0(C^*(\mathcal{T}))$),
- the morphisms B stand for the respective Bott periodicity isomorphisms,
- the morphisms M stand for the respective Morita isomorphisms (the induced morphism in K-theory by the Morita equivalences of groupoids), and
- the morphisms denoted by $e_{i,s}$ or $e_{i,t}$, for $i = 0, 1$, are the ones induced by the evaluation morphisms at i with respect to variable s (the parameter used in the morphism H) or t (the deformation parameter of the tangent groupoid).

Now we need to recognize in the diagram the morphisms we are looking for:

- The morphism on top line gives by definition the Connes-Thom isomorphism (together with the Morita equivalence)
$$\mathscr{C}\mathscr{T} : K^0(TX) \to K^0(N_X).$$
- By identifying the space $p^*(N_X \oplus TX)$ with $\pi^*(TX \oplus TX)$ (where $\pi : TX \to X$ is the canonical projection) we have by theorem 6.4 in [**46**][1] that the vertical left hand side morphism is the Thom isomorphism
$$K^0(TX) \xrightarrow{\mathscr{T}} K^0(p^*(N_X \oplus TX)).$$
- The vertical morphism on the right hand side gives the identity
$$Id : K^0(N_X) \to K^0(N_X).$$
- Finally, in the bottom line morphism we have to recognize the morphism

(300)
$$K^0(p^*(N_X \oplus TX) \times \mathbb{R}^N) \xleftarrow{e_{0,t}} K^0((p^*(N_X \oplus TX) \times [0,1]) \times_H \mathbb{R}^N) \xrightarrow{e_{1,t}} K^0(p^*(N_X \oplus TX) \times_h \mathbb{R}^N) \xrightarrow{M} K^0(N_X)$$

As a vector bundle over N_X, $p^*(N_X \oplus TX)$ is the trivial bundle $N_X \times \mathbb{R}^N$ (N_X is by definition the normal bundle of X in $\mathbb{R}^N$). Under this

[1] in this paper, the authors explicitely compute the morphism $K^0(p^*(N_X \oplus TX)) \to K^0(TX)$ showing that it is precisely the inverse of the Thom isomorphism

identification, the morphism $h : N_X \times \mathbb{R}^N \to \mathbb{R}^N$ is given by $h(n, X) = X$ and hence the induced crossed product groupoid is nothing more that the translation action groupoid of $\mathbb{R}^N$ on $\mathbb{R}^N$ over N_X. Then, the morphism (300) is the Bott inverse isomorphism

$$K^0(p^*(N_X \oplus TX) \times \mathbb{R}^N) = K^0((N_X \times \mathbb{R}^N) \times \mathbb{R}^N) \xrightarrow{B^{-1}} K^0(N_X)$$

We conclude by the commutativity of the big diagram above that

$$\mathscr{C}\mathscr{T} = B^{-1} \circ \mathscr{T}.$$

□

APPENDIX D

Semidirect product groupoids and algebras used in this article

In this appendix we will describe explicitly the main groupoids we use in the present article as well as the associated algebras. For more details on Lie groupoids, generalized morphisms and particularly deformation Lie groupoids and their use in index theory the reader can see [**35**], [**57**] or [**30**].

D.1. Semi-direct product groupoid algebras

Let $\mathscr{G} \rightrightarrows P$ be a Lie groupoid and M be a $\mathscr{G}$-manifold with moment map $\pi_M : M \to P$ that we will always assume to be a submersion, we can consider the following groupoids and algebras:

- **The semi-direct groupoid $M \rtimes \mathscr{G}$:** It is the groupoid

$$M \rtimes \mathscr{G} \rightrightarrows M \tag{301}$$

with $M \rtimes \mathscr{G} = \{(m, \gamma) \in M \times \mathscr{G} : \pi_M(m) = t(\gamma)\}$ and with structure maps

(i) Source and target maps defined by

$$s(m, g) = m \cdot g, \;\; t(m, g) = m,$$

(ii) groupoid product

$$(m, g) \cdot (mg, h) := (m, gh),$$

(iii) unit map

$$u(m) = (m, \pi_M(m)),$$

where $\pi_M(m) \in \mathscr{G}$ is the unit element associated to the groupoid structure on $\mathscr{G}$, and

(iv) inverse map

$$(m, g)^{-1} := (mg, g^{-1}).$$

For this groupoid we will usually consider the usual groupoid convolution algebra of complex valued compactly supported C^∞-functions $C_c^\infty(M \rtimes \mathscr{G})$.

- **The semi-direct groupoid $(M \times_P M) \rtimes \mathscr{G}$:** It is the groupoid associated to the diagonal action on $M \times_P M$, given by

$$(M \times_P M) \rtimes \mathscr{G} \rightrightarrows M \tag{302}$$

with $(M \times_P M) \rtimes \mathscr{G} = \{(m', m, \gamma) \in M \times M \times \mathscr{G} : \pi_M(m) = \pi_M(m') = t(\gamma)\}$ and with structure maps

(i) Source and target maps defined by

$$s(m, n, \gamma) = n \cdot \gamma, \;\; t(m, , n, \gamma) = m,$$

(ii) groupoid product

$$(m, n, \gamma) \cdot (n\gamma, l, \eta) := (m, l\mathscr{G}^{-1}, \gamma\eta)$$

(iii) unit map

$$u(m) = (m, m, \pi_M(m)),$$

where $\pi_M(m) \in \mathscr{G}$ is the unit element associated to the groupoid structure on $\mathscr{G}$, and

(iv) inverse map

$$(m, n, \gamma)^{-1} := (n\gamma, m\gamma, \gamma^{-1}).$$

For this groupoid we will usually consider the usual groupoid convolution algebra of complex valued compactly supported C^∞-functions $C_c^\infty((M \times_P M) \rtimes \mathscr{G})$.

- **The semi-direct groupoid $T_\pi M \rtimes \mathscr{G}$:** It is the groupoid associated to the infinitesimal action of $\mathscr{G}$ on the vertical tangent space $T_\pi M := Ker d\pi_M$, it is given by

(303) $$T_\pi M \rtimes \mathscr{G} \rightrightarrows M$$

with $T_\pi M \rtimes \mathscr{G} = \{((m, V), \gamma) \in T_\pi M \times \mathscr{G} : \pi_M(m) = t(\gamma)\}$ and with structure maps

(i) Source and target maps defined by

$$s(m, V, \gamma) = m \cdot \gamma, \;\; t(m, V, \gamma) = m,$$

(ii) groupoid product

$$(m, V, \gamma) \cdot (m\gamma, W, \eta) := (m, d_m(R_\gamma)(V) + W, \gamma\eta),$$

where $R_\gamma : \pi_M^{-1}(t(\gamma)) \to \pi_M^{-1}(s(\gamma))$ stands for the right action by γ,

(iii) unit map

$$u(m) = (m, 0_m, \pi_M(m)),$$

where $\pi_M(m) \in \mathscr{G}$ is the unit element associated to the groupoid structure on $\mathscr{G}$, $0_m \in (T_\pi M)_m$ is the origin vector, and

(iv) inverse map

$$(m, V, \gamma)^{-1} := (m\gamma, -d_m R_\gamma(V), \gamma^{-1}).$$

For this groupoid we will usually consider the Schwartz type groupoid convolution algebra of complex valued C^∞- fiber rapidly decreasing functions with compact support in direction of the base and in the $\mathscr{G}$-direction, $\mathscr{S}_c(T_\pi M \rtimes \mathscr{G})$. To be more precise an element in this algebra is a C^∞, complex valued function f on $T_\pi M \times \mathscr{G}$ such that $f(V,\gamma) = 0$ for γ outside a compact subset of $\mathscr{G}$ and such that the functions $f_\gamma : TM_{t(\gamma)} \to \mathbb{C}$, where $TM_{t(\gamma)} = \pi^{-1}(t(\gamma))$, defined by $f_\gamma(V) := f(V,\gamma)$ are in the Schwartz type space $\mathscr{S}_c(TM_{t(\gamma)})$ defined in [**27**] Definition 4.6.

The algebra product is given as the convolution product of $C_c^\infty(T_\pi M \rtimes \mathscr{G})$ where the groupoid under consideration is the groupoid $T_\pi M \rtimes \mathscr{G} \rightrightarrows M$ as above, the product is well-defined since we are taking fiberwise Schwartz functions, see [**27**] for more details.

REMARK D.1.1. *For a $\mathscr{G}-$vector bundle $E \to M$ we can consider the analog algebra $\mathscr{S}_c(E \rtimes \mathscr{G})$.*

- **The semi-direct deformation groupoid $D_f \rtimes \mathscr{G}$:** Let $f : M \longrightarrow N$ be a $\mathscr{G}$-equivariant C^∞-map between two $\mathscr{G}-$ manifolds M and N, we consider the $\mathscr{G}-$vector bundle $T_f := T_\pi M \oplus f^* T_\pi N$ over M and the deformation groupoid

$$D_f : T_f \bigsqcup (M \times_P M) \times_P N \times (0,1] \rightrightarrows f^* T_\pi N \bigsqcup (M \times_P N) \times (0,1] \tag{304}$$

given as the normal groupoid associated to the $\mathscr{G}$-map

$$M \stackrel{\Delta \times f}{\longrightarrow} (M \times_P M) \times_P N.$$

Here the $\mathscr{G}$-action on $(M \times_P M) \times_P N$ preserves the image of M under the previous map $\Delta \times f$. Hence, by functoriality of the deformation to the normal cone construction, we have an action of $\mathscr{G}$ on D_f with moment map $\pi_f : D_f \to P$ given by the composition of the deformation map of the pair couple $((M \times_P M) \times_P N, M) \to (P,P)$. This is canonically induced from the moment maps of M and N, followed by the first projection $D(P,P) = P \times [0,1] \to P$ (where $D(\cdot,\cdot)$ denotes the deformation to the normal cone). This induces a semi-direct product Lie groupoid

$$D_f \rtimes \mathscr{G} \rightrightarrows f^* T_\pi N \bigsqcup (M \times_P N) \times (0,1] \tag{305}$$

with $D_f \rtimes \mathscr{G} = \{(A,\gamma) \in D_f \times \mathscr{G} : \pi_f(A) = t(\gamma)\}$. The groupoid structure is given fiberwisely over $[0,1]$ as the groupoid structure of

$$((M \times_P M) \times_P N) \rtimes \mathscr{G} \rightrightarrows M \times_P N$$

for $t \neq 0$, and at $t = 0$ the groupoid structure is

$$(T_\pi M \oplus f_* T_\pi N) \rtimes \mathscr{G} \rightrightarrows f_* T_\pi N.$$

This groupoid is a LIE groupoid and we could consider its convolution algebra but we will need a slightly larger algebra, indeed for this

groupoid we consider the Schwartz type algebra $\mathscr{S}_c(D_f \rtimes \mathscr{G})$ as an immediate generalisation of the Schwartz type algebra for the tangent groupoid defined in [**27**]. More precisely an element in $\mathscr{S}_c(D_f \rtimes \mathscr{G})$ is a C^∞-function $f : D_f \rtimes \mathscr{G} \to \mathbb{C}$ such that $f_\gamma = f(\cdot, \gamma) : (D_f)_\gamma \to \mathbb{C}$ is zero for all γ outside a compact set of $\mathscr{G}$, and all of them belong to the vector space $\mathscr{S}_c((D_f)_\gamma)$ defined in [**27**] Section 4.2, where $(D_f)_\gamma$ is the deformation to the normal cone associated to the immersion induced by f (since it commutes with the moment maps) of $\pi_M^{-1}(t(\gamma))$ in $\pi^{-1}(t(\gamma))$ where π_M is the $\mathscr{G}$-moment map of M and π the $\mathscr{G}$-moment map of $(M \times_P M) \times_P N$. The convolution product in this algebra is induced by the groupoid product in $D_f \rtimes \mathscr{G}$.

The fundamental property of this algebra is to be an intermediate algebra

$$C_c^\infty(D_f \rtimes \mathscr{G}) \subset \mathscr{S}_c(D_f \rtimes \mathscr{G}) \subset C_r^*(D_f \rtimes \mathscr{G}) \tag{306}$$

such that the canonical evaluation morphisms yield

$$\mathscr{S}_c(D_f \rtimes \mathscr{G})|_{t=0} = \mathscr{S}_c(T_f \rtimes \mathscr{G}) \tag{307}$$

and

$$\mathscr{S}_c(D_f \rtimes \mathscr{G})|_{t=1} = C_c^\infty(((M \times_P M) \times_P N) \rtimes \mathscr{G}). \tag{308}$$

Moreover, as seen below, there is a short exact sequence of the form.

$$0 \longrightarrow J \longrightarrow \mathscr{S}_c(D_f \rtimes \mathscr{G}) \xrightarrow{e_0} \mathscr{S}_c(T_f \rtimes \mathscr{G}) \longrightarrow 0. \tag{309}$$

As we will see in the next section, the above kernel algebra J has trivial Cyclic periodic (co)homology groups and hence the evaluation at $t = 0$ will induce isomorphisms in these (co)homology groups.

D.2. Schwartz type algebras for $\mathscr{G}$-vector bundles and the fiberwise Fourier isomorphism

We already introduced in the last section some algebras that use Schwartz functions in a certain way. In this section we want to explain the reason of using Schwartz algebras in the present article and some important facts related to these algebras.

Given a $\mathscr{G}$-proper equivariant vector bundle $E \to M$ there are two kinds of Schwartz algebras considered in this article. As sets they are the same, the vector space of complex valued C^∞-functions rapidly decreasing in the direction of the vector bundle fibers, with compact support in direction of the base and in the $\mathscr{G}$-direction, denoted by $\mathscr{S}_c(E \rtimes \mathscr{G})$.

Now, there are two different algebra products in these spaces depending on which groupoid one is considering. Indeed, if one considers the semi-direct product groupoid

$$E \rtimes \mathscr{G} \rightrightarrows M$$

associated to the $\mathscr{G}$-vector bundle structure, then there is a convolution product $(\mathscr{S}_c(E \rtimes \mathscr{G}), *)$ using this groupoid structure. On the other hand, one might consider the action groupoid

$$E \rtimes \mathscr{G} \rightrightarrows E$$

associated to the $\mathscr{G}$-space E. In this case there is a product algebra $(\mathscr{S}_c(E \rtimes \mathscr{G}), \cdot)$ that uses the groupoid structure of this action groupoid. These two algebras are not the same at all. In the context of this paper, we use both examples, for the semi-direct product groupoid

$$T_\pi M \rtimes \mathscr{G} \rightrightarrows M,$$

and for the action groupoid

$$T^*_\pi M \rtimes \mathscr{G} \rightrightarrows T^*_\pi M,$$

or for the bundles T_f and T^*_f respectively. As in the classical case when $\mathscr{G}$ is the trivial group we have a Fourier type algebra isomorphism

$$(\mathscr{S}_c(T_\pi M \rtimes \mathscr{G}), *) \cong (\mathscr{S}_c(T^*_\pi M \rtimes \mathscr{G}), \cdot),$$

which indeed is a particular case of the following proposition.

PROPOSITION D.2.1. *Given a $\mathscr{G}$-proper equivariant vector bundle $E \to M$, we have the following two properties:*

(i) *The fiberwise Fourier transform gives an algebra isomorphism*

$$(\mathscr{S}_c(E \rtimes \mathscr{G}), *) \cong (\mathscr{S}_c(E^* \rtimes \mathscr{G}), \cdot). \tag{310}$$

(ii) *The convolution algebra $(\mathscr{S}_c(E \rtimes \mathscr{G}), *)$ is stable under holomorphic calculus as a subalgebra of $(C^*_r(E \rtimes \mathscr{G}), *)$.*

The proof of the above two statements is quite classical, for instance one can follow the same lines as the proof of Proposition 4.5 in [**32**].

Now, one important consequence for us is that, in the above situation, we have the following isomorphisms

$$HP_*(\mathscr{S}_c(E^* \rtimes \mathscr{G}), \cdot) \cong HP_*(\mathscr{S}_c(E \rtimes \mathscr{G}), *) \tag{311}$$

and

$$HP^*(\mathscr{S}_c(E^* \rtimes \mathscr{G}), \cdot) \cong HP^*(\mathscr{S}_c(E \rtimes \mathscr{G}), *), \tag{312}$$

induced by the Fourier algebra isomorphism.

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