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Louis-Pierre Chaintron

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# Finite-dimensional and measure-valued constrained dynamics, through the lens of large deviation and control theory

École doctorale 532, Mathématiques et STIC (MSTIC)

Mathématiques

Thèse préparée au sein du DMA (ENS-PSL), du CERMICS (ENPC)  
et de l'équipe Inria MÈDISIM

Thèse soutenue le 18 décembre 2024, par  
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*Ludwig Boltzman, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the work, died similarly in 1933. Now it is our turn to study statistical mechanics. Perhaps it will be wise to approach the subject cautiously. (Opening lines of States of Matter, by D.L. Goodstein).*



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## Remerciements

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*J'ai tendu des cordes de clocher à clocher ; des  
guirlandes de fenêtre à fenêtre ; des chaînes d'or  
d'étoile à étoile, et je danse.*

— Un grand physicien

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# CHAPTER 1

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## Introduction

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*No one really knows what entropy really is, so in a debate you will always have the advantage.*

— John von Neumann

This thesis is devoted to the study of constrained dynamics in finite and infinite-dimensional settings, using tools from large deviation and control theory. The main goal is to modify a given dynamical system to produce trajectories that satisfy some constraints, which can be thought of as measurements. Some covered examples are the stochastic filtering problem, extensions of the Gibbs principle from statistical mechanics, and constrained versions of the Schrödinger bridge problem. This introductory chapter presents how to extend classical tools for constraining Ordinary Differential Equations (ODEs) to measure-valued dynamics.

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The need for constrained dynamics is ubiquitous in real life applications. When modelling a dynamical phenomenon, a discrepancy is often measured between the model prediction and the targeted quantity. Knowing this discrepancy, it is then natural to look for an improved version of the model that takes into account the measurement, by imposing this measurement as a *constraint* on the model. Famous examples originate from statistical mechanics, like the Gibbs principle presented in Section 1.3.3.1 which searches for the most likely configuration of a particle system when only knowing average quantities on it. Another example in Section 1.3.3.1 is the Schrödinger bridge problem, which looks for the most likely evolution of a stochastic process when only knowing its law at initial and terminal times. This last problem enjoys thriving applications with the recent use of *diffusion models* in machine learning Ho et al. [2020]; Song et al. [2020]; Albergo et al. [2023]. Let us also mention the famous filtering problem of finding the conditional distribution of a system in real-time given a noisy measurement on it. More broadly, our framework includes many kinds of deterministic and stochastic estimation problems. A further motivation of this thesis was the study of conditioned systems of interacting diffusions for cardiac modelling, following the works Chaintron et al. [2023a,c], which were done before starting this thesis – see Section 1.4.4 for more precisions.

The present chapter introduces the context of this thesis by progressively presenting motivations and technical elements. A detailed literature review is done along the way. The main objective is to show *how to extend classical tools for constraining ODEs to measure-valued dynamics*. Section 1.1 describes some deterministic approaches for constraining finite-dimensional toy-model dynamics. Section 1.1 serves as a pretext for introducing many tools from optimisation and control theory and paves the way for many analogies with measure-valued dynamics, which are developed in Section 1.3. This presentation is strongly inspired from Gentil et al. [2020]. Section 1.2 describes stochastic approaches that randomise the dynamics before conditioning it. Many tools from large deviation theory are introduced there, together with the stochastic filtering problem. A main contribution of this thesis is the adaptation of these methods to measure-valued dynamics, as presented in Section 1.3. The ODE formalism from Sections 1.1-1.2 allows for many analogies with measure-valued dynamics at the level of both statements and proofs. The large deviation theory induces connections between conditioning problems and entropy minimisation problems on path space. The Girsanov theory described in Section 1.3.3.2 converts these minimisation problems into stochastic control problems. Recent developments presented in Section 1.3.3.3 allow for geometric interpretations of these results using Wasserstein gradient flows and the formalism of Newton equations on the Wasserstein space. We emphasise that Sections 1.1-1.2-1.3 are expository, their purpose being to introduce general ideas using a framework that is not always rigorous, contrary to the subsequent chapters.

A recap of the different contributions of this thesis is done in Section 1.4, summarising the different chapters. The chapters are preprints or articles, which are reproduced without any change. In particular, a detailed presentation of motivations and existing literature can be found at the beginning of each chapter. Thus, some notations may change from a chapter to another, always with a clear specification of it. Chapter 2 presents some results on stochastic filtering for reflected diffusions in the small noise limit. Chapter 3 develops a method for proving large deviations and central limit theorems for mean-field interacting diffusions, improving existing results along the way. Chapters 4-5 contain a detailed study of (extensions of) the Gibbs principle on path space with infinitely many constraints, using tools from the mean-field control theory. In particular, a quantitative stability result is established. Chapter 6 eventually proves new regularity estimates on Hamilton-Jacobi equations using viscosity solution methods. Some perspectives for future

works are listed in Section 1.4.4.

## 1.1 Deterministic correction for ODEs

In this section, we describe several strategies for constraining dynamics using a finite-dimensional toy-model. This toy-model is given by an ODE that we want to modify to produce trajectories that satisfy a given constraint. We can think of this as improving a physical model by including some measurement performed on the target phenomenon. Depending on the chosen strategy, the constrained dynamics will be given by a first-order or a second-order dynamics. This toy-model is well-suited for introducing many tools from optimisation and control theory, paving the way for measure-valued extensions in Section 1.3.

Let us start with an ordinary differential equation (ODE) in  $\mathbb{R}^d$ ,

$$\dot{x}_t = b(x_t), \quad (1.1)$$

that we want to modify to produce a trajectory  $(\bar{x}_t)_{t \geq 0}$  that stays in a sub-domain  $\mathcal{N} \subset \mathbb{R}^d$ . The initial condition  $x_0$  is imposed. To stay consistent with the probabilistic framework adopted throughout this thesis, the time variable  $t$  is written as an index. We can choose the drift  $b$  as being a globally Lipschitz vector field for (1.1) to be well-posed. However, we will not be concerned with regularity aspects throughout this section, keeping computations at a formal level. Similarly, we write (1.1) in  $\mathbb{R}^d$  for the sake of simplicity, but any complete  $d$ -dimensional manifold  $\mathcal{M}$  would suit as well. We can think of  $\mathcal{N}$  as a sub-manifold of  $\mathbb{R}^d$ . Recurrent examples will be  $\mathcal{N} \in \{G, \mathcal{N}_\psi^0, \mathcal{N}_\psi^-\}$ , where  $G \subset \mathbb{R}^d$  is a closed bounded domain and

$$\mathcal{N}_\psi^0 := \{x \in \mathbb{R}^d, \psi(x) = 0\}, \quad \mathcal{N}_\psi^- := \{x \in \mathbb{R}^d, \psi(x) \leq 0\},$$

for a regular function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ . The standard assumption for  $\mathcal{N}_\psi^0$  to be a smooth sub-manifold is  $\psi$  being smooth with never vanishing gradient. Similarly, suitable assumptions ensure that  $\mathcal{N}_\psi^-$  is a sub-manifold of  $\mathbb{R}^d$  with boundary.

Generically, we look for corrected trajectories of the kind

$$\dot{x}_t^\omega = b(x_t^\omega) + \omega_t, \quad (1.2)$$

for  $\omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ , which is e.g. locally square-integrable. Equation (1.2) can be seen as a *controlled* ODE where  $\omega$  is the *control parameter*. We shall now specify some optimisation criterion that will select the corrected curve  $(\bar{x}_t)_{t \geq 0}$ , among all the  $x^\omega$  that stay in  $\mathcal{N}$  at each time. We distinguish between two plausible criteria, which produce different curves:

- *Optimality at each time.* A first natural idea is to require the minimal change for the drift  $b$  at each-time, so that the corrected curve  $(\bar{x}_t)_{t \geq 0}$  satisfies

$$\dot{\bar{x}}_t = \operatorname{argmin}_{v \in T_{\bar{x}_t} \mathcal{N}} |v - b(\bar{x}_t)|^2, \quad (1.3)$$

at each time, when  $\mathcal{N}$  is a manifold with tangent space  $T_{\bar{x}_t} \mathcal{N}$  at  $\bar{x}_t$ . Thus,  $\dot{\bar{x}}_t$  is the minimal correction of  $b(\bar{x}_t)$  (in the least-square sense), so that  $\bar{x}_t$  stays in  $\mathcal{N}$  at each time. This simple natural setting yields a first-order dynamics similar to (1.1), which is described in Section 1.1.1.

- *Optimality on a time-window.* If we want to correct (1.2) on a fixed time-window  $[0, T]$  only, it is natural to look at

$$(\bar{x}_t)_{0 \leq t \leq T} := \operatorname{argmin}_{\substack{(x_t)_{0 \leq t \leq T} \\ \forall t \in [0, T], x_t \in \mathcal{N}}} \int_0^T |\dot{x}_t - b(x_t)|^2 dt, \quad (1.4)$$

corresponding to the control problem (1.2) with quadratic cost under state constraints. This standard problem produces second-order dynamics much richer than (1.1), as described in Section 1.1.2. This framework also allows for constraints on the whole path of the kind  $\Psi((x_t)_{0 \leq t \leq T}) = 0$ , which are more general than imposing  $x_t \in \mathcal{N}$  at each time.

The main drawback of (1.4) compared to (1.3) is the *loss of recursivity*. If  $T < T'$ , the solution produced by solving (1.4) on  $[0, T']$  may not be compatible with the one obtained by optimising on  $[0, T]$  (although the dynamic programming provides a partial replacement, see Section 1.1.2.2). However, (1.4) has the advantage of a better trajectorial accuracy, since the curve produced by (1.3) can be fairly different from a typical trajectory of the dynamical system (1.1). We also emphasise the greater generality of constraints compatible with the path-approach (1.4). For instance, it is possible to impose both the initial and terminal states in (1.2).

The Euclidean distance is used for (1.3)-(1.4) throughout this section, but it could be replaced by a general Riemannian distance up to minor changes.

**Remark 1.1.1** (Non-additive control problem). *From the objective of constraining (1.1), it is natural to look at controlled curves of type (1.2), but this setting can be embedded into non-linear control problems of the kind*

$$\dot{x}_t^\omega = b(x_t^\omega) + \sigma(x_t^\omega)\omega_t \quad \text{or} \quad \dot{x}_t^\omega = b(x_t^\omega, \omega_t).$$

However, we preferred keeping (1.2) for simplifying this expository section.

### 1.1.1 First order approach

We here give examples to illustrate (1.3). First, we notice that this dynamics corresponds to the ODE in  $\mathcal{N}$ ,

$$\dot{\bar{x}}_t = \pi_{T_{\bar{x}_t}\mathcal{N}} [b(\bar{x}_t)], \quad (1.5)$$

where  $\pi_{T_{\bar{x}_t}\mathcal{N}}$  denotes the orthogonal projection on the tangent space at  $\bar{x}_t$ . A simple way to visualise (1.5) is through its explicit Euler discretisation with time step  $h > 0$ , given by

$$\bar{y}_{t+h} = \operatorname{argmin}_{y \in \mathcal{N}} |y - \bar{y}_t - hb(\bar{y}_t)|^2. \quad (1.6)$$

Roughly speaking, the initial dynamics (1.1) is followed at each time-step, before adding a projection step. This definition is non-ambiguous under suitable assumptions on  $\mathcal{N}$  like convexity. To alleviate the presentation, we will not precise these assumptions in the following. An implicit discretisation could be written as well.

#### 1.1.1.1 Gradient flows

A relevant example is the case  $b(x) = -\nabla U(x)$  in (1.1) for some smooth potential  $U$ . In this case, (1.5) corresponds by definition to

$$\dot{\bar{x}}_t = -\nabla_{\mathcal{N}} U(\bar{x}_t),$$

where  $\nabla_{\mathcal{N}}$  is the Riemannian gradient in the sub-manifold  $\mathcal{N}$ . Thus, *the gradient flow structure is preserved*. Gradient flows are known to have the unconditionally stable backward discretisation

$$y_{t+h} = \operatorname{argmin}_{y \in \mathbb{R}^d} \frac{|y - y_t|^2}{2h} + U(y),$$

which reduces the dynamics to a step-wise variational formula [Ambrosio et al., 2005, Chapter 2]. A nice feature of (1.5) is then its compatibility with this discretisation, a useful scheme for (1.5) being

$$\bar{y}_{t+h} = \operatorname{argmin}_{y \in \mathcal{N}} \frac{|y - \bar{y}_t|^2}{2h} + U(y).$$

This interesting alternative to (1.6) is compatible with the gradient flow structure.

### 1.1.1.2 Sub-differential dynamics

Let us think of  $\mathcal{N}$  as a smooth convex domain  $G \subset \mathbb{R}^d$  with outward normal vector  $n(x)$  at  $x \in \partial G$ . We assume that  $G$  is closed with non-empty interior. The tangent space at  $x$  is then the hyperplane orthogonal to  $n(x)$ , so that computing the projection rewrites (1.5) as

$$\dot{\bar{x}}_t = b(\bar{x}_t) - \mathbb{1}_{\bar{x}_t \in \partial G} [b(\bar{x}_t) \cdot n(\bar{x}_t)] n(\bar{x}_t).$$

In particular, nothing changes while  $\bar{x}_t$  is in the interior of  $G$ , a correction being applied only at the boundary. The correction appears as a *sticky reflection*, the dynamics being stuck at the boundary as long as it tries to escape  $G$ . The problem of correcting (1.1) in this way is sometimes referred to as *the deterministic Skorokhod problem*, by analogy with Skorokhod's problem of reflecting stochastic diffusions Skorokhod [1961] – see also Chapter 2 for further illustrations. This behaviour can be re-casted as a *sub-differential dynamics*. The sub-differential at  $x$  of a convex function  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  is defined as

$$\partial F(x) := \{p \in \mathbb{R}^d, \forall y \in \mathbb{R}^d, F(y) \geq p \cdot (y - x) + F(x)\}.$$

It is the set of *sub-gradients* of  $F$  at  $x$ . A sub-differential dynamics is then a continuous dynamics of the kind

$$\dot{x}_t \in \partial F(x_t).$$

This enters the broader setting of dynamics driven by maximal monotone operators Brezis [1973], or the framework of variational inequalities Kinderlehrer and Stampacchia [2000]. Our dynamics constrained to stay in  $G$  enters the above setting by choosing  $F = \chi_G$  as being the convex characteristic function of  $G$ , defined by  $\chi_G(x) = 0$  if  $x \in G$ , and  $\chi_G(x) = +\infty$  otherwise. Such dynamics have many physical applications, modelling e.g. dry friction or elasto-plasticity Duvaut and Lions [1976]; Bastien et al. [2000].

### 1.1.1.3 Equality constraints and Lagrange multiplier

Let us detail the case where  $\mathcal{N} = \mathcal{N}_\psi^0$  is the 0-level set of a smooth function  $\psi$ . The tangent space  $T_x \mathcal{N}_\psi^0$  is the hyperplane orthogonal to  $\nabla \psi(x)$ , so that  $n(x) = |\nabla \psi(x)|^{-1} \nabla \psi(x)$  and

$$\dot{\bar{x}}_t = b(\bar{x}_t) - |\nabla \psi(\bar{x}_t)|^{-2} [b(\bar{x}_t) \cdot \nabla \psi(\bar{x}_t)] \nabla \psi(\bar{x}_t), \quad (1.7)$$

under the non-degeneracy assumption  $\nabla \psi(\bar{x}_t) \neq 0$ .

Alternatively, this expression can be obtained from the constrained minimisation (1.3) using the formalism of Lagrange multipliers:

$$\inf_{v \cdot \nabla \psi(\bar{x}_t) = 0} \frac{1}{2} |v - b(\bar{x}_t)|^2 = \inf_{v \in \mathbb{R}^d} \left[ \frac{1}{2} \sup_{\lambda \in \mathbb{R}} |v - b(\bar{x}_t)|^2 + \lambda v \cdot \nabla \psi(\bar{x}_t) \right],$$

equality holding because the supremum equals  $+\infty$  if  $v \cdot \nabla \psi(\bar{x}_t) \neq 0$ . The duality relation

$$\inf_{v \in \mathbb{R}^d} \sup_{\lambda \in \mathbb{R}} \left[ \frac{1}{2} |v - b(\bar{x}_t)|^2 + \lambda v \cdot \nabla \psi(\bar{x}_t) \right] \geq \sup_{\lambda \in \mathbb{R}} \inf_{v \in \mathbb{R}^d} \left[ \frac{1}{2} |v - b(\bar{x}_t)|^2 + \lambda v \cdot \nabla \psi(\bar{x}_t) \right], \quad (1.8)$$



is then always true and becomes an equality if a saddle point  $(\bar{v}_t, \bar{\lambda}_t)$  exists with  $\bar{\lambda}_t \bar{v}_t \cdot \nabla \psi(x_t) = 0$ . This condition is called the *complementary slackness* condition. Writing the first-order optimality conditions yields

$$\begin{cases} \dot{\bar{x}}_t = b(\bar{x}_t) - \bar{\lambda}_t \nabla \psi(\bar{x}_t), \\ \dot{\bar{x}}_t \cdot \nabla \psi(\bar{x}_t) = 0. \end{cases} \quad (1.9)$$

This is a particular instance of a *differential algebraic system of equations* Kunkel [2006], the ODE being coupled to an implicit equation. It is then standard to compute  $\bar{\lambda}_t$  by writing that

$$0 = \frac{d}{dt} \psi(\bar{x}_t) = b(\bar{x}_t) \cdot \nabla \psi(\bar{x}_t) - \bar{\lambda}_t |\nabla \psi(\bar{x}_t)|^2.$$

If  $\nabla \psi(\bar{x}_t) \neq 0$ , this gives an expression for  $\bar{\lambda}_t$  as a feedback of  $\bar{x}_t$ , recovering (1.7). In the optimisation setting (1.3),  $\nabla \psi(\bar{x}_t) \neq 0$  is called a *constraint qualification* condition.

In the case of inequality constraints  $\mathcal{N} = \mathcal{N}_\psi^-$ , the same reasoning can be adapted. The constraint in (1.8) becomes  $v \cdot \nabla \psi(\bar{x}_t) \leq 0$  when  $\psi(\bar{x}_t) = 0$ , to ensure that  $\bar{x}_t$  stays in  $\mathcal{N}_\psi^-$ . The main difference is that we impose  $\lambda \geq 0$  instead of  $\lambda \in \mathbb{R}$ , so that the supremum in (1.8) is  $+\infty$  only if  $v \cdot \nabla \psi(\bar{x}_t) > 0$ . The complementary slackness condition  $\bar{\lambda}_t v \cdot \nabla \psi(\bar{x}_t) = 0$  now means that the correction activates (i.e.  $\bar{\lambda}_t \neq 0$ ) only when touching the boundary. The corrected dynamics reads

$$\begin{cases} \dot{\bar{x}}_t = b(\bar{x}_t) - |\nabla \psi(\bar{x}_t)|^{-2} [b(\bar{x}_t) \cdot \nabla \psi(\bar{x}_t)] \nabla \psi(\bar{x}_t), & \text{if } \psi(\bar{x}_t) = 0, \\ \dot{\bar{x}}_t = b(\bar{x}_t), & \text{if } \psi(\bar{x}_t) < 0, \end{cases}$$

under the qualification assumption that  $\nabla \psi(\bar{x}_t) \neq 0$  when  $\psi(\bar{x}_t) = 0$ .

### 1.1.2 Second order approach

We now turn to the pathwise procedure (1.4). To fix ideas, we focus on the specific example  $\mathcal{N} = \mathcal{N}_\psi^-$ , so that (1.4) amounts to solving

$$\inf_{\substack{(x_t)_{0 \leq t \leq T} \\ \forall t \in [0, T], \psi(x_t) \leq 0}} \int_0^T \frac{1}{2} |\dot{x}_t - b(x_t)|^2 dt.$$

This enters the classical framework of Lagrangian mechanics, the integral cost being then referred to as an *action functional*. To solve this constrained minimisation problem, we use Lagrange multipliers as previously. The constraint is imposed on the continuous curve  $t \in [0, T] \mapsto \psi(x_t)$ . Since the Lagrange multiplier must be taken in the dual space of the constraint, we end up looking at the variational problem

$$\inf_{(x_t)_{0 \leq t \leq T}} \int_0^T \frac{1}{2} |\dot{x}_t - b(x_t)|^2 dt + \int_0^T \psi(x_t) \bar{\lambda}(dt), \quad (1.10)$$

where the multiplier  $\bar{\lambda}$  is a positive Radon measure on  $[0, T]$  and the initial condition  $x_0$  is imposed. Along the optimal curve  $\bar{x} := (\bar{x}_t)_{0 \leq t \leq T}$ ,  $\bar{\lambda}$  satisfies the complementary slackness condition

$$\psi(\bar{x}_t) = 0, \quad \text{for } \bar{\lambda} \text{ -a.e } t \in [0, T]. \quad (1.11)$$

Existence for such a multiplier can be a tough question, which requires a constraint qualification assumption. Existence often follows from a suitable application of the Hahn-Banach theorem. For sufficient conditions and existence theorems for  $(\bar{x}, \bar{\lambda})$ , we refer to Barbu and

Precupanu [2012]. For some dynamics analogous to (1.1) in infinite dimension, existence of a Lagrange multiplier is studied in Chapter 4, see Section 1.3.3 below for further explanations.

To alleviate the presentation, we now assume that  $\bar{\lambda}$  has a (non-negative) density  $(\lambda_t)_{0 \leq t \leq T}$  w.r.t. the Lebesgue measure. In the following, we derive an equation for  $\bar{x}$  and we describe how bounds can be obtained on  $\bar{\lambda}$ , yielding regularity properties for  $\bar{x}$ .

### 1.1.2.1 Newton equation

To get an equation on the optimal curve  $\bar{x} = (\bar{x}_t)_{0 \leq t \leq T}$ , we must write first-order optimality conditions for (1.10). This is a standard problem from calculus of variations, see e.g. Ekeland and Temam [1999]. The classical method for action functionals uses the optimality of  $\bar{x}$  compared to small perturbations of  $\bar{x}$ , first with fixed endpoints and then perturbing the endpoints. After integration by parts, this provides the second-order equation

$$\ddot{\bar{x}}_t = \frac{1}{2} \nabla |b|^2(\bar{x}_t) + \lambda_t \nabla \psi(\bar{x}_t), \quad (1.12)$$

with initial condition  $x_0$  and terminal relation  $\dot{\bar{x}}_T = b(\bar{x}_T)$ . Indeed, a second-order equation requires two boundary conditions to be well-posed. This equation must be seen as a perturbation of the *acceleration* of the initial process (1.1). Indeed, differentiating (1.1) in time precisely yields the equation  $\dot{x}_t = \frac{1}{2} \nabla |b|^2(x_t)$  on the acceleration. The terminal condition  $\dot{\bar{x}}_T = b(\bar{x}_T)$  is then natural since it is inherited from (1.1).

Compared to the first-order setting, we notice that the non-negative multiplier  $\lambda_t$  enters the dynamics with a + sign contrary to (1.9). This is an artefact of the second-order dynamics due to the forward-backward structure of the Hamiltonian system, see the Hamiltonian description below. From (1.11), we know that  $\lambda_t \psi(\bar{x}_t) = 0$  for Lebesgue-a.e.  $t \in [0, T]$ . On intervals where  $\psi(\bar{x}_t)$  is identically 0, we can compute  $\lambda_t$  by writing that

$$0 = \frac{d^2}{dt^2} \psi(\bar{x}_t) = \dot{\bar{x}}_t \cdot \nabla^2 \psi(\bar{x}_t) \dot{\bar{x}}_t + \frac{1}{2} \nabla |b|^2(\bar{x}_t) \cdot \nabla \psi(\bar{x}_t) + \lambda_t |\nabla \psi(\bar{x}_t)|^2,$$

so that the corrected equation reads

$$\begin{aligned} \ddot{\bar{x}}_t = & \frac{1}{2} \nabla |b|^2(\bar{x}_t) - \mathbb{1}_{\psi(\bar{x}_t)=0} \frac{1}{2} |\nabla \psi(\bar{x}_t)|^{-2} [\nabla |b|^2 \cdot \nabla \psi(\bar{x}_t)] \nabla \psi(\bar{x}_t) \\ & - \mathbb{1}_{\psi(\bar{x}_t)=0} |\nabla \psi(\bar{x}_t)|^{-2} [\dot{\bar{x}}_t \cdot \nabla^2 \psi(\bar{x}_t) \dot{\bar{x}}_t] \nabla \psi(\bar{x}_t), \end{aligned} \quad (1.13)$$

still under the qualification condition  $\nabla \psi(\bar{x}_t) \neq 0$ . We emphasise that this expression can only be obtained if regularity properties on  $\bar{\lambda}$  are known beforehand. Otherwise, this cannot be achieved because  $\bar{\lambda}$  does not always have a density.

**Remark 1.1.2** (Geometric interpretation). *In a Riemannian geometry framework, Equation (1.13) on  $\ddot{\bar{x}}_t$  has a clear interpretation. To make things simpler, let us consider the case of equality constraints  $\mathcal{N} = \mathcal{N}_\psi^0$ . This change removes the characteristic functions in (1.13) since  $\psi(\bar{x}_t) = 0$  at each time.*

*Differentiating (1.1) had given  $\dot{x}_t = \frac{1}{2} \nabla |b(x_t)|^2$ .  $\dot{x}_t$  describes the acceleration of the curve  $t \mapsto x_t$  in the ambient manifold  $\mathcal{M} = \mathbb{R}^d$ . To obtain the corrected curve  $\bar{x}_t$ , we project this second-order equation in the sub-manifold  $\mathcal{N}$  to formally get that*

$$Proj_{\mathcal{N}}[\ddot{\bar{x}}_t] = \pi_{T_{\bar{x}_t} \mathcal{N}}[\frac{1}{2} \nabla |b(\bar{x}_t)|^2] =: \frac{1}{2} \nabla_{\mathcal{N}} |b(\bar{x}_t)|^2. \quad (1.14)$$

*The l.h.s. corresponds to the acceleration of  $t \mapsto \bar{x}_t$  in the sub-manifold  $\mathcal{N}$ . The second-order projection (1.14) is the analogous of the first-order projection (1.5) in Section 1.1.1.*

Let us further formalise. The intrinsic acceleration in  $\mathcal{M}$  is defined as the covariant derivative  $\nabla_{\dot{\bar{x}}_t}^{\mathcal{M}} \dot{\bar{x}}_t$  of the velocity vector  $\dot{\bar{x}}_t$  in the direction  $\dot{\bar{x}}_t$ .  $\nabla^{\mathcal{M}}$  is the Levi-Civita connection on the Riemannian manifold  $\mathcal{M}$  (we simply have  $\nabla_X^{\mathbb{R}^d} Y = DY \cdot X$  in  $\mathbb{R}^d$ ). The intrinsic acceleration in the sub-manifold  $\mathcal{N}$  is then  $\nabla_{\dot{\bar{x}}_t}^{\mathcal{N}} \dot{\bar{x}}_t$ , where  $\nabla^{\mathcal{N}}$  is the induced connection on  $\mathcal{N}$ :  $\nabla^{\mathcal{N}}$  is defined as the orthogonal projection of  $\nabla^{\mathcal{M}}$  on the tangent space of  $\mathcal{N}$ . The projected equation (1.14) now reads

$$\nabla_{\dot{\bar{x}}_t}^{\mathcal{N}} \dot{\bar{x}}_t = \frac{1}{2} \nabla_{\mathcal{N}} |b(\bar{x}_t)|^2.$$

To recover the equation on the acceleration  $\ddot{\bar{x}}_t := \nabla_{\dot{\bar{x}}_t}^{\mathcal{M}} \dot{\bar{x}}_t$  in  $\mathcal{M}$ , we need the difference

$$\mathbb{I}(X, Y) := \nabla_X^{\mathcal{M}} Y - \nabla_X^{\mathcal{N}} Y,$$

also known as the second fundamental form of the sub-manifold  $\mathcal{N}$  [Lee, 2018, Chapter 8]. Comparing with (1.13) shows that

$$\mathbb{I}(\dot{\bar{x}}_t, \dot{\bar{x}}_t) = -|\nabla\psi(\bar{x}_t)|^{-2} [\dot{\bar{x}}_t \cdot \nabla^2\psi(\bar{x}_t) \dot{\bar{x}}_t] \nabla\psi(\bar{x}_t),$$

since the first line in (1.13) only corresponds to the projection of the gradient. This provides the geometric interpretation of (1.13). This further shows that the multiplier  $\lambda_t$  in (1.12) gathers two operations: projection of the acceleration and computation of the second fundamental form.

**Example 1.1.3** (Geodesics and bridges). Let us recall a fundamental advantage of (1.4): it allows for pathwise constraints on the whole trajectory  $(x_t)_{0 \leq t \leq T}$ , possibly more involved than constraining  $x_t$  at each  $t$ . A natural constraint, would be to impose both endpoints of the curve, looking at the interpolation cost

$$d^2(x, y) := \inf_{\substack{x_0=x, x_T=y, \\ \forall t \in [0, T], x_t \in \mathcal{N}}} \int_0^T |\dot{x}_t - b(x_t)|^2 dt. \quad (1.15)$$

When  $\mathcal{N} = \mathbb{R}^d$  and  $b \equiv 0$ , this simply produces the square Euclidean distance between  $x$  and  $y$ . When  $b \equiv 0$  and  $\mathcal{N}$  is a general sub-manifold of  $\mathbb{R}^d$ ,  $d^2(x, y)$  is the square induced distance on  $\mathcal{N}$ , the minimisers being the geodesic curves in  $\mathcal{N}$ . When  $b \equiv 0$  and  $\mathcal{N} = \mathcal{N}_\psi^0$ , (1.13) then corresponds to the geodesic equation in  $\mathcal{N}$ , given by the Christoffel symbols of the Levi-Civita connection  $\nabla^{\mathcal{N}}$  (see Remark 1.1.2 and [Lee, 2018, Chapter 4]).

When  $b \neq 0$ , (1.15) defines an interpolation cost w.r.t. the reference dynamics (1.1). For discussions of such interpolations and connections with the Schrödinger bridge problem, we refer to Section 1.3.2 and [Gentil et al., 2020, Section 2]. Problem (1.15) is similar to Zermelo's navigation problem of finding the quickest path between two points given an external forcing field, which may slower the travel. The prototype example is a ship on the sea subject to the wind. In this setting, the drift  $b$  is considered as a perturbation of the straight-line rather than a dynamics.  $d(x, y)$  is then known as a Randers metric Cheng and Shen [2013].

### 1.1.2.2 Hamilton-Jacobi-Bellman equation

Since the initial dynamics (1.1) was a first-order ODE, it is natural to look for a first-order formulation of (1.12). This gives the forward-backward Hamiltonian system (with imposed  $x_0$ )

$$\begin{cases} \dot{\bar{x}}_t = b(\bar{x}_t) + p_t, \\ \dot{p}_t = -\nabla b(\bar{x}_t) \cdot p_t + \lambda_t \psi(\bar{x}_t), \quad p_T = 0. \end{cases} \quad (1.16)$$

Differentiating the equation satisfied by  $\dot{x}_t$  w.r.t.  $t$  recovers back (1.12). (1.16) fits the setting of (1.2), the optimal control being  $\omega = p$ . A standard way to obtain (1.16) is to rewrite (1.10) as

$$\inf_{\substack{(x_t)_{0 \leq t \leq T}, (\omega_t)_{0 \leq t \leq T} \\ \forall t \in [0, T], \dot{x}_t = b(x_t) + \omega_t}} \int_0^T \frac{1}{2} |\omega_t|^2 + \lambda_t \psi(x_t) dt,$$

seeing the dynamics (1.2) as an equality constraint for the above minimisation.  $p_t$  then corresponds to a Lagrange multiplier for this constraint at time  $t$ . (1.16) is known as the *Pontryagin Maximum Principle* (PMP) in the control literature, see e.g. [Fleming and Soner, 2006, Section I.6]. Alternatively, it is customary to introduce the *value function*,

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad \varphi_t(x) := \inf_{\substack{(\omega_s)_{t \leq s \leq T} \\ x_t^\omega = x}} \int_t^T \frac{1}{2} |\omega_s|^2 + \lambda_s \psi(x_s^\omega) ds.$$

The value  $\varphi_t(x)$  is the *cost* for controlling  $x^\omega$  from  $x$  at time  $t$  till time  $T$ . Following [Fleming and Soner, 2006, Section I.4], we can show that  $\varphi$  satisfies the dynamic programming or Bellman principle

$$\forall \tau \in [0, T - t], \quad \varphi_t(x) = \inf_{(\omega_s)_{t \leq s \leq t+\tau}} \int_t^{t+\tau} \frac{1}{2} |\omega_s|^2 + \lambda_s \psi(x_s^\omega) ds + \varphi_{t+\tau}(x_{t+\tau}^\omega).$$

This simply means that an optimal path from  $t$  till  $T$  remains optimal from  $t + \tau$  till  $T$ . As a consequence, a partial differential equation (PDE) can be derived for  $\varphi$  [Fleming and Soner, 2006, Chapter I.5],

$$\partial_t \varphi_t + b \cdot \nabla \varphi_t - \frac{1}{2} |\nabla \varphi_t|^2 = -\lambda_t \psi, \quad \varphi_T = 0, \quad (1.17)$$

which is known as the *Hamilton-Jacobi-Bellman* equation (HJB). The optimal control then reads  $p_t = -\nabla \varphi_t(\bar{x}_t)$ . Indeed, using (1.17) to differentiate  $p_t$  recovers the previous relation in (1.16). The terminal condition  $\varphi_T = 0$  is compatible with the relation  $\dot{\bar{x}}_T = b(\bar{x}_T)$ , which was computed in Section 1.1.2.1. All this can be made rigorous if some regularity property is known beforehand on the solution of (1.17). Otherwise, some extra care is needed, using the theory of viscosity solutions Crandall and Lions [1983]; Fleming and Soner [2006].

### 1.1.2.3 Estimates on the multiplier

We now briefly sketch how the control formalism of Section 1.1.2.2 enables estimates on the density  $\lambda_t$  of the multiplier. By optimality of  $(\bar{x}_t)_{0 \leq t \leq T}$  in (1.10), using (1.16) and  $p_t = -\nabla \varphi(\bar{x}_t)$ , we have

$$\int_0^T [\psi(\bar{x}_s) - \psi(x_s^\omega)] \lambda_s ds \leq \frac{1}{2} \int_0^T [|\omega_s|^2 - |\nabla \varphi_s(\bar{x}_t)|^2] ds, \quad (1.18)$$

for any square-integrable control  $(\omega_t)_{0 \leq t \leq T}$ . Let us fix  $0 \leq t < t' \leq T$ . If we manage to build  $\omega$  such that

$$\begin{cases} \psi(\bar{x}_s) - \psi(x_s^\omega) > 0, & \text{for } s \in (t, t'), \\ \bar{x}_s = x_s^\omega, & \text{for } s \notin (t, t'), \end{cases}$$

then (1.18) provides a bound on  $\int_t^{t'} \lambda_s ds$ . If  $t$  and  $t'$  can be chosen arbitrarily close, this bound implies in turn a local estimate for  $\lambda_t$ . This reduces the problem to building a control

$\omega_s^h$  arbitrarily close to  $\omega_s^0 := -\nabla\varphi_s(\bar{x}_s)$  such that  $\psi(x_s^h) < \psi(x_s^0)$ , where  $x^h := x^{\omega^h}$ . We now present a non-rigorous scheme of this construction, which will be crucial in Chapter 5.

**Step 1. Gradient flow.** A natural geometric idea is to use the gradient flow  $(T_s)_{s \geq 0}$  of  $\psi$ , defined as the solution of

$$\forall s \geq 0, \forall x \in \mathbb{R}^d, \quad \partial_s T_s(x) = -(\nabla\psi \circ T_s)(x), \quad T_0(x) = x.$$

On the interval  $[t, t']$ , we then push the optimal process  $(x_s^0)_{0 \leq s \leq T}$  along this flow, defining

$$\forall s \in [t, t'], \quad x_s^h := T_{h_s}(x_s^0).$$

for a smooth function  $s \mapsto h_s$  with  $h_t = h_{t'} = 0$ . We have

$$\dot{x}_s^h = -\dot{h}_s \nabla\psi(x_s^h) + \nabla T_{h_s} \cdot [b(x_s^0) - \nabla\varphi_s(x_s^0)],$$

hence the related control reads

$$\omega_s^h := -\dot{h}_s \nabla\psi(x_s^h) + \nabla T_{h_s} \cdot [b(x_s^0) - \nabla\varphi_s(x_s^0)] - b(x_s^h).$$

We now use this competitor in (1.18), and we turn to the computation of the r.h.s.

**Step 2. Expansion.** Since  $T_h = \text{Id} - h\nabla^2\psi + o(h)$  (writing  $h$  for  $\sup_{t \leq s \leq t'} |h_s|$ ), we get

$$\omega_s^h = -\nabla\varphi_s(x_s^0) - \dot{h}_s \nabla\psi(x_s^h) + h_s \nabla b \cdot \nabla\psi(x_s^0) + h_s \nabla^2\psi \cdot [\nabla\varphi_s - b](x_s^0) + o(h),$$

hence

$$\begin{aligned} |\omega_s^h|^2 - |\nabla\varphi_s(x_s^0)|^2 &= |\dot{h}_s \nabla\psi(x_s^h)|^2 + 2\dot{h}_s \nabla\psi(x_s^0) \cdot \nabla\varphi_s(x_s^0) \\ &\quad - 2h_s \nabla\varphi_s \cdot [\nabla b \cdot \nabla\psi + \nabla^2\psi \cdot \nabla\varphi_s - \nabla^2\psi \cdot b](x_s^0) + o(h). \end{aligned}$$

This suggests to choose  $h_s$  so that  $|\dot{h}_s|^2$  is of order  $h$ . The leading order term is now  $\dot{h}_s \nabla\psi(x_s^0) \cdot \nabla\varphi_s(x_s^0)$  and we have to get rid of it.

**Step 3. Integration by parts.** The trick is to use the integral in (1.18), noticing that

$$\nabla\psi(x_s^0) \cdot \nabla\varphi_s(x_s^0) = b(x_s^0) \cdot \nabla\psi(x_s^0) - \frac{d}{ds}\psi(x_s^0).$$

Integrating by parts and using that  $h_t = h_{t'} = 0$ ,

$$\int_t^{t'} \dot{h}_s \nabla\psi(x_s^0) \cdot \nabla\varphi_s(x_s^0) ds = \int_t^{t'} h_s \left[ \frac{d^2}{ds^2}\psi(x_s^0) - \frac{d}{ds} b(x_s^0) \cdot \nabla\psi(x_s^0) \right] ds.$$

From Section 1.1.2.1, the behaviour of  $\lambda_t$  can be grasped by looking at the second-order derivative of  $\psi(x_s^0)$ , hence the above computation was expected. Moreover, this second derivative is 0 on intervals where  $\lambda_s \neq 0$  using (1.11). At this stage, the last task for showing that the r.h.s. of (1.18) is of order  $h$  would be to prove bounds on  $\nabla\varphi_s$ .

**Step 4. Conclusion.** Looking at the l.h.s. of (1.18),

$$\psi(x_s^0) - \psi(x_s^h) = h_s |\nabla\psi(x_s^0)|^2 + o(h).$$

To lower bound  $\psi(x_s^0) - \psi(x_s^h)$ , we thus need the usual qualification condition:

$$\nabla\psi(x_s^0) \neq 0 \quad \text{if} \quad \psi(x_s^0) = 0.$$

Gathering everything in (1.18) eventually yields a bound on  $\int_t^{t'} \lambda_s ds$ . Up to regularisation procedures, this bound can be used to show that  $\bar{\lambda}$  in (1.10) has a bounded density.

The above construction shares analogies with [Erbar et al., 2015, Section 4]. Its adaptation to infinite-dimensional settings will be a key-ingredient in Chapter 5.

### 1.1.2.4 Relaxed formulation

Let us sketch an alternative approach to (1.4). A natural penalisation of the constraint is

$$\inf_{(x_t)_{0 \leq t \leq T}} \int_0^T \frac{1}{2} |\dot{x}_t - b(x_t)|^2 dt + \int_0^T \frac{\kappa}{2} |\psi(x_t)|_+^2 dt, \quad (1.19)$$

for some large  $\kappa > 0$  and  $|\psi(x_t)|_+ := \max(0, \psi(x_t))$ . The constraint is no more forced, but it is “softly” introduced in the problem with a strength  $\kappa$ . As  $\kappa \rightarrow +\infty$ , (1.19) recovers the constrained formulation (1.4) (a  $\Gamma$ -convergence result can be proved).

Let  $(x_t^\kappa)_{0 \leq t \leq T}$  be an optimal curve for (1.19). As in Section 1.1.2.1, writing optimality condition yields

$$\ddot{x}_t^\kappa = \frac{1}{2} \nabla |b(x_t^\kappa)|^2 + \kappa |\psi(x_t^\kappa)|_+ \nabla \psi(x_t^\kappa).$$

The comparison with (1.12) is striking. As  $\kappa \rightarrow +\infty$ ,  $(x_t^\kappa)_{0 \leq t \leq T}$  is expected to converge towards  $(\bar{x}_t)_{0 \leq t \leq T}$ , and we deduce that  $\kappa |\psi(x_t^\kappa)|_+$  is an approximation of the Lagrange multiplier  $\lambda_s$ . We underline that any smooth positive penalisation  $J_\kappa(\psi(x_t))$  could have been used instead of the quadratic one, yielding the same approximation result. This actually holds for any type of minimisation problem under constraints.

In some regular situations, sending  $\kappa \rightarrow +\infty$  is not needed because (1.12) with large enough – but finite –  $\kappa$  already selects the optimal solution. We refer to Cannarsa et al. [2018] for an example in deterministic control under constraints, and to Daudin [2023b] for an infinite-dimensional extension.

At this stage, the constraint-free problem (1.10) with Lagrange multiplier  $\bar{\lambda}$  appears as an analogous of (1.19).  $\bar{\lambda}$  can be seen as the optimal weight giving the importance of the constraint w.r.t. the dynamics. There is a fidelity trade-off between the constraint and the dynamics. We will refer to any minimisation problem similar to (1.19) as a *relaxed problem*, for any type of penalisation. This terminology of relaxed problem directly extends to general pathwise constraints of type  $\Psi((x_t)_{0 \leq t \leq T}) = 0$ .

### 1.1.3 Mortensen observer

A recurrent drawback of the second-order approach (1.4) is its incapacity to produce recursive dynamics: the optimiser for (1.4) on  $[0, T]$  does not help for solving (1.4) on  $[0, T']$  with  $T' > T$ . This is because the solution  $\varphi_t$  to the HJB equation in Section 1.1.2.2 depends on  $T$  through its terminal condition.

To circumvent this, an approach was proposed in Mortensen [1968] in the context of deterministic estimation. The purpose is to estimate the state of a physical system  $x^\omega$  modelled by (1.2),  $\omega$  being an unknown parameter. The known data is a measurement  $\dot{y}_s = h(x_s^\omega)$  on the system. In the terminology of Section 1.1.2.4, we rely on a relaxed formulation

$$\inf_{x_0^\omega, \omega} J(x_0^\omega) + \int_0^T \frac{1}{2} |\omega_s|^2 + \frac{1}{2} |\dot{y}_s - h(x_s^\omega)|^2 ds,$$

the optimisation being a trade-off between model accuracy and observation accuracy. Contrary to (1.4), the initial condition is now an optimisation parameter. We could impose it as well, but this extra degree of freedom comes for free in Mortensen’s approach. The penalisation  $J(x_0^\omega)$  can be used to incorporate a priori knowledge on the initial condition, for instance  $J(x_0^\omega) = \frac{1}{2} (x_0^\omega - \hat{x}_0)^2$  where  $\hat{x}_0$  is a plausible initial condition for the system. We emphasise that this setting is fully compatible with the penalisation approach (1.19) in Section 1.1.2.4.

The key-idea of Mortensen is to introduce the *cost-to-come value function*

$$V_t(x) := \inf_{\substack{x_0^\omega, \omega \\ x_t^\omega = x}} J(x_0^\omega) + \int_0^t \frac{1}{2} |\omega_s|^2 + \frac{1}{2} |\dot{y}_s - h(x_s^\omega)|^2 ds, \quad (1.20)$$

which is the cost for being at  $x$  at time  $t$ , seen as the *likelihood of being at  $x$  at  $t$* . The main difference compared to  $\varphi$  in Section 1.1.2.2 is that  $x$  is the arrival point at  $t$  of the controlled curve and no more the initial point, which is now free. The estimator at time  $t$  is then defined by

$$\hat{x}_t \in \operatorname{argmin}_{x \in \mathbb{R}^d} V_t(x),$$

up to usual verifications to make this definition rigorous.  $\hat{x}_t$  is the most likely position for the system at time  $t$ , because  $\hat{x}_t$  has the lowest cost. This estimator is non-anticipative in time (causality); it is an *observer* in the terminology of Krener [1999]. Up to penalising as in Section 1.1.2.4, this provides an alternative to (1.4) for correcting (1.1).

**Remark 1.1.4** (Minimisers). *If the initial condition  $x_0$  were imposed in (1.20), then the minimising curve would be given by*

$$\ddot{x}_s^\omega = \frac{1}{2} \nabla |b|^2(x_s^\omega) - \nabla h(x_s^\omega) \cdot [\dot{y}_s - h(x_s^\omega)], \quad x_0^\omega = x_0, \quad x_t^\omega = x,$$

as in Section 1.1.2.1. This is reminiscent of the interpolations introduced in Example 1.1.3.

As in Section 1.1.2.2,  $V$  satisfies a HJB equation, which is now forward in time,

$$\partial_t V_t + b \cdot \nabla V_t + \frac{1}{2} |\nabla V_t|^2 - \frac{1}{2} |\dot{y}_t - h|^2 = 0, \quad V_0 = J. \quad (1.21)$$

In particular,  $V_{t'}$  for  $t' > t$  can now be computed by starting back from  $V_t$ . At this stage, it becomes possible to compute a recursive ODE for  $\hat{x}_t$ , by differentiating in time the relation  $\nabla V_t(\hat{x}_t) = 0$ :  $\hat{x}_t$  is fully determined from time-to-time by the 0-level set of  $\nabla V_t$ . Assuming that  $\nabla^2 V_t(\hat{x}_t)$  is invertible, the HJB equation yields

$$\dot{\hat{x}}_t = b(\hat{x}_t) - [\nabla^2 V_t(\hat{x}_t)]^{-1} \nabla h(\hat{x}_t) \cdot [\dot{y}_t - h(\hat{x}_t)]. \quad (1.22)$$

Noticeably, this perturbed dynamics is of type (1.2). When  $b$  and  $h$  are linear, explicit expressions are given in Example 1.1.5 below. In non-linear settings, it is uneasy to make all this rigorous. We refer to Rockafellar and Wolenski [2000a,b] for a setting with strong convexity assumptions, and to Breiten and Schröder [2023] for a quadratic setting with a specific structure. The Hessian term being uneasy to compute, this approach is computationally costly in spite of recent works towards numerical implementation Moireau [2018]; Breiten et al. [2023].

The whole structure of Mortensen's observer is deeply connected to stochastic filtering, see Section 1.2.3. An infinite-dimensional generalisation is proposed in Example 1.3.7.

**Example 1.1.5** (Deterministic Kalman filter). *In the linear-quadratic setting*

$$b(x) = Ax, \quad h(x) = Cx, \quad J(x) = \frac{1}{2} [x - \hat{x}_0] \cdot P_0^{-1} [x - \hat{x}_0],$$

the HJB equation has an explicit quadratic solution

$$V_t(x) = \frac{1}{2} [x - \hat{x}_t] \cdot P_t^{-1} [x - \hat{x}_t] + \frac{1}{2} \int_0^t |\dot{y}_s - h(\hat{x}_s)|^2 ds, \quad (1.23)$$

where

$$\dot{\hat{x}}_t = A\hat{x}_t + P_t C^\top [\dot{y}_t - C\hat{x}_t], \quad (1.24)$$

and the symmetric matrix  $P_t$  solves the Riccati equation

$$\dot{P}_t = \text{Id} + AP_t + P_t A^\top - P_t^\top C C^\top P_t. \quad (1.25)$$

This is the ideal scenario for numerics, since  $P_t$  can be computed beforehand independently of  $\dot{y}$ , while  $\hat{x}_t$  can be computed on the fly by solving an ODE. Following Fleming [1997], this deterministic setting is the analogous of the Kalman-Bucy stochastic filter, see Example 1.2.4 in Section 1.2.3.

## 1.2 Conditioning and filtering

The main idea of Section 1.1 was to embed the ODE (1.1) into the wider class of dynamics (1.2), before selecting the closest dynamics in this class satisfying the constraint. If (1.1) were a stochastic dynamics, a natural alternative would be to impose the constraint by conditioning. In this section, we describe a stochastic-based strategy by adding noise to (1.1), before conditioning and then removing the noise. Links with the different examples of Section 1.1 are emphasised throughout the text. This setting is well-suited for introducing some tools related to large deviations and the setting of stochastic filtering.

We will be mainly interested in randomised versions of (1.1) of the kind

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t, \quad (1.26)$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion, and the amplitude parameter  $\varepsilon > 0$  is small. An advantage of (1.26) is the *regularising effect of the noise*: (1.26) is known to be well-posed under less restrictive assumptions than (1.1), see Flandoli [2011] for a detailed survey. The random variable  $\sqrt{\varepsilon}B_t$  has a Gaussian density w.r.t. the Lebesgue measure

$$p_t^\varepsilon(x) := \frac{1}{(2\pi\varepsilon t)^{d/2}} \exp\left[-\frac{x^2}{2\varepsilon t}\right].$$

As  $\varepsilon \rightarrow 0$ , the law  $\mathcal{L}(\sqrt{\varepsilon}B_t)$  of  $\sqrt{\varepsilon}B_t$  converges to the Dirac mass at 0. More precisely, we can write this concentration on the density at the exponential-in- $\varepsilon$  scale:

$$p_t^\varepsilon(x) \asymp e^{-x^2/2\varepsilon t}, \quad \text{as } \varepsilon \rightarrow 0,$$

the  $\asymp$  symbol meaning that taking  $\varepsilon$  times the logarithm of each side gives the same  $\varepsilon \rightarrow 0$  limit. The Laplace principle then reads, for every measurable  $A \subset \mathbb{R}^d$ ,

$$\mathbb{P}(\sqrt{\varepsilon}B_t \in A) \asymp e^{-\inf_{x \in A} x^2/2\varepsilon t}.$$

This can be formalised by proving that  $(\mathcal{L}(\sqrt{\varepsilon}B_t))_{\varepsilon > 0}$  satisfies the Large Deviation Principle (LDP) with rate function  $x \mapsto \frac{x^2}{2t}$  Dembo and Zeitouni [2009].

The Freidlin-Wentzell theory Freidlin and Wentzell [1998] extends this result to the law of the full random path  $(X_t^\varepsilon)_{0 \leq t \leq T}$ . More precisely, these laws satisfy the LDP with rate function

$$I_T : (x_t)_{0 \leq t \leq T} \in C([0, T], \mathbb{R}^d) \mapsto \begin{cases} \int_0^T \frac{1}{2} |\dot{x}_t - b(x_t)|^2 dt, & \text{if } (x_t)_{0 \leq t \leq T} \in AC([0, T], \mathbb{R}^d), \\ +\infty, & \text{otherwise,} \end{cases}$$

where  $AC([0, T], \mathbb{R}^d)$  denotes the space of absolutely continuous curves. Roughly speaking,

$$\mathbb{P}((X_t^\varepsilon)_{0 \leq t \leq T} \simeq (x_t)_{0 \leq t \leq T}) \asymp e^{-\varepsilon^{-1} I_T((x_t)_{0 \leq t \leq T})}. \quad (1.27)$$



As  $\varepsilon \rightarrow 0$ ,  $\mathcal{L}((X_t^\varepsilon)_{0 \leq t \leq T})$  concentrates on the unique minimiser of  $I_T$ , which corresponds to the dynamics (1.1). In the following sections, we use this LDP to study the  $\varepsilon \rightarrow 0$  limit of conditional laws. A notable example is the case of stochastic filtering developed in Section 1.2.3, which is closely related to Section 1.1.3.

**Remark 1.2.1** (Quadratic structure). *The quadratic structure of the rate function  $I_T$  is a consequence of the Gaussian structure of the noise. A multiplicative noise of the form  $\sqrt{\varepsilon}\sigma(X_t^\varepsilon)dB_t$  would replace the Euclidean norm in  $I_T$  by a Riemannian metric with weight  $[\sigma\sigma^\top]^{-1}$ . Similarly, considering another Markov process instead of  $(B_t)_{0 \leq t \leq T}$  would yield another rate function. The Brownian noise is well-suited to recover the setting of Section 1.1 as  $\varepsilon \rightarrow 0$ . However, Section 1.1 itself could be adapted by considering some other cost functions, which may be related to other rate functions.*

**Remark 1.2.2** (Norton dynamics). *In the following, we impose the constraint by conditioning the law of  $X^\varepsilon$ . An alternative would be to directly project the dynamics of  $X^\varepsilon$  onto the sub-manifold  $\mathcal{N}$ , as we did for the deterministic dynamics (1.5). In the case of equality constraints  $\mathcal{N} = \mathcal{N}_\psi^0$ , this approach is known as the Norton dynamics Evans and Morris [1985]; Blassel and Stoltz [2024]. See Lelièvre et al. [2008] for a related discretisation approach.*

### 1.2.1 Point-wise conditioning

We here present how the discrete-time dynamics (1.6) introduced in Section 1.1.1 can be recovered as the  $\varepsilon \rightarrow 0$  limit of a conditioning procedure. The (explicit) Euler-Maruyama discretisation for (1.26) reads

$$Y_{t+h}^\varepsilon - Y_t^\varepsilon = hb(Y_t^\varepsilon) + \sqrt{h\varepsilon}G,$$

where  $G \sim \mathcal{N}(0, \text{Id})$  is a standard Gaussian variable. Setting  $\varepsilon = 0$  recovers the explicit Euler scheme for (1.1). The law of  $Y_{t+h}^\varepsilon$  knowing that  $Y_t^\varepsilon = y$  has a density

$$K_h^\varepsilon(x|y) := \frac{1}{(2\pi\varepsilon h)^{d/2}} \exp\left[-\frac{(x - y - hb(y))^2}{2\varepsilon h}\right].$$

To constrain the dynamics to stay in  $\mathcal{N}$ , we condition the Gaussian variable  $G$  so that  $Y_{t+h}^\varepsilon$  stays in  $\mathcal{N}$  at each step. This amounts to replacing the kernel  $K_h^\varepsilon$  by

$$\bar{K}_h^\varepsilon(x|y) := Z_\varepsilon^{-1} \mathbb{1}_{x \in \mathcal{N}} K_h^\varepsilon(x|y),$$

where  $Z_\varepsilon > 0$  is a normalising constant. As  $\varepsilon \rightarrow 0$ , the measure  $\bar{K}_h^\varepsilon(x|y)dx$  concentrates on

$$\operatorname{argmin}_{x \in \mathcal{N}} |x - y - hb(y)|^2.$$

This precisely recovers the discrete scheme (1.6). Sending  $h \rightarrow 0$  then recovers the projected dynamics (1.5).

**Remark 1.2.3** (Gradient flow from large deviations). *In the gradient case  $b = -\nabla U$ , let us show how the discrete scheme from Section 1.1.1.1 can be deduced from the Freidlin-Wentzell rate function. We consider the initial time-step, starting from  $X_0^\varepsilon = y$ . As  $\varepsilon \rightarrow 0$ , the LDP tells that  $\mathcal{L}(X_h^\varepsilon)$  concentrates on  $\operatorname{argmin}_{x \in \mathbb{R}^d} J_h(x)$ , where*

$$J_h(x) := \inf_{\substack{(x_s)_{0 \leq s \leq h} \\ x_0 = y, x_h = x}} I_h((x_s)_{0 \leq s \leq h}).$$

However, the gradient structure allows us to rewrite the action functional as

$$I_h((x_s)_{0 \leq s \leq h}) = U(x_h) - U(x_0) + \int_0^h \frac{1}{2} |\dot{x}_s|^2 + \frac{1}{2} |\nabla U(x_s)|^2 ds. \quad (1.28)$$

When  $h$  is small, the constraints  $x_0 = y$  and  $x_h = x$  impose that  $\dot{x}_s \simeq h^{-1}(x - y)$ . At the leading order, we hence get that

$$J_h(x) \simeq U(x) - U(y) + \frac{(x - y)^2}{2h},$$

recovering the minimisation scheme from Section 1.1.1.1. A rigorous way to state this is to prove the  $\Gamma$ -convergence of  $x \mapsto J_h(x) - (x - y)^2/2h$  towards  $U - U(y)$ . This result is extended to an infinite-dimensional setting in Erbar et al. [2015].

## 1.2.2 Pathwise conditioning

We now condition the full path  $X^\varepsilon := (X_t^\varepsilon)_{0 \leq t \leq T}$  rather than the increments at each step. This allows for general constraints of the kind  $\Psi((x_t)_{0 \leq t \leq T}) = 0$ . From (1.27), we can formally compute the law of  $X^\varepsilon$  conditionally on  $\Psi(X^\varepsilon) = 0$ . Indeed, for any measurable  $A \subset C([0, T], \mathbb{R}^d)$ ,

$$\begin{aligned} \mathbb{P}(X^\varepsilon \in A | \Psi(X^\varepsilon) = 0) &= \frac{\mathbb{P}(\{X^\varepsilon \in A\} \cap \{\Psi(X^\varepsilon) = 0\})}{\mathbb{P}(\Psi(X^\varepsilon) = 0)} \\ &\asymp \exp \left[ -\varepsilon^{-1} \inf_{x \in A, \Psi(x)=0} I_T(x) + \varepsilon^{-1} \inf_{x', \Psi(x')=0} I_T(x') \right]. \end{aligned}$$

This suggests that the family of the conditional laws  $\mathcal{L}(X^\varepsilon | \Psi(X^\varepsilon) = 0)$  satisfies the LDP with rate function  $x \mapsto I_T(x) - \inf_{x', \Psi(x')=0} I_T(x')$ . As  $\varepsilon \rightarrow 0$ , this shows that  $\mathcal{L}(X^\varepsilon | \Psi(X^\varepsilon) = 0)$  concentrates on the minimisers for

$$\inf_{\substack{x=(x_t)_{0 \leq t \leq T} \\ \Psi(x)=0}} \int_0^T |\dot{x}_t - b(x_t)|^2 dt.$$

This precisely recovers the deterministic strategy (1.4). The minimisers are now interpreted as the *most likely configurations of  $X^\varepsilon$  knowing  $\Psi(X^\varepsilon) = 0$* , in the  $\varepsilon \rightarrow 0$  limit. The developments of Section 1.1.2 can then be adapted.

Making this rigorous often requires extra care, because the event  $\{\Psi(X^\varepsilon) = 0\}$  may have 0-probability hence the conditioning is ill-defined. To circumvent this, a standard strategy is to thicken the constraint into  $\{|\Psi(X^\varepsilon)| \leq \delta\}$  with  $\delta > 0$ , before taking the  $\varepsilon \rightarrow 0$  limit and then  $\delta \rightarrow 0$ . In the Gibbs principle setting of Section 1.3.3, we refer to Stroock and Zeitouni [1991] for this thickening. It is also possible to modify  $\Psi$  into a well-chosen  $\Psi^\varepsilon$  that converges as  $\varepsilon \rightarrow 0$ , or to make  $\delta$  depend on  $\varepsilon$  Cattiaux and Gozlan [2007]. The case  $\mathcal{N} = \mathcal{N}_\psi^-$  of inequality constraints of type

$$\forall t \in [0, T], \quad \psi(X_t^\varepsilon) \leq 0,$$

avoids this difficulty, as in Chapter 4.

### 1.2.3 Stochastic filtering

We now describe the problem of small noise stochastic filtering and its connection with Mortensen's observer from Section 1.1.3. The filtering problem consists in estimating the state of  $X_t^\varepsilon$  defined by (1.26), given the realisation of the observation process

$$dY_s^\varepsilon = h(X_s^\varepsilon)ds + \sqrt{\varepsilon}dB'_s,$$

from time 0 to time  $t$ ,  $(B'_s)_{0 \leq s \leq t}$  being a Brownian motion independent from the one driving  $X^\varepsilon$  Kushner [1967]; Kallianpur and Striebel [1968]; Jazwinski [1970]. The noise in (1.26) models an uncertainty on the dynamics of the signal, whereas  $(B'_s)_{0 \leq s \leq t}$  is an observation noise. In a physical setting, it seems reasonable to assume that these noises are independent.

#### 1.2.3.1 Zakai equation

The mathematical object of interest is the conditional law  $\pi_t^\varepsilon$  of  $X_t^\varepsilon$  given  $(Y_s^\varepsilon)_{0 \leq s \leq t}$ , which is the random probability measure defined by

$$\forall \varphi \in C_b(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \varphi d\pi_t^\varepsilon = \mathbb{E}[\varphi(X_t^\varepsilon) | \sigma(Y_s^\varepsilon, 0 \leq s \leq t)].$$

Under suitable assumptions,  $\pi_t^\varepsilon$  is the solution of a non-linear stochastic PDE driven by  $Y_t^\varepsilon$ , known as the *Kushner-Stratonovich* equation Kushner [1967]. Alternatively, a standard approach computes the density of the joint path-law of  $(X_s, Y_s)_{0 \leq s \leq t}$  w.r.t. the law of  $(B_s, B'_s)_{0 \leq s \leq t}$  using the Girsanov transform. From this explicit expression, the conditional density can be computed using the Bayes formula. This yields the definition of a non-normalised positive random measure  $\rho_t^\varepsilon$  such that

$$\pi_t^\varepsilon = [\rho_t^\varepsilon(\mathbb{R}^d)]^{-1} \rho_t^\varepsilon,$$

this relation being known as the *Kallianpur-Striebel formula* Kallianpur and Striebel [1968]. The main point is that  $\rho_t^\varepsilon$  can be computed as the solution of the linear *Zakai equation* Zakai [1969]; Pardoux [1980],

$$d\rho_t^\varepsilon = (L^\varepsilon)^* \rho_t^\varepsilon dt + \varepsilon^{-1} \rho_t^\varepsilon h \cdot dY_t^\varepsilon, \quad (1.29)$$

where  $(L^\varepsilon)^*$  is the (formal)  $L^2$ -adjoint of the infinitesimal generator  $L^\varepsilon := b \cdot \nabla + \frac{\varepsilon}{2} \Delta$  of the Markov process  $X^\varepsilon$  defined by (1.26).

**Example 1.2.4** (Kalman-Bucy filter). *Let us consider the linear case of Example 1.1.5,*

$$b(x) = Ax, \quad h(x) = Cx,$$

*initialising  $X_0^\varepsilon$  from the Gaussian law  $\mathcal{N}(\hat{x}_0, \varepsilon P_0)$ . It turns out that the processes  $(X_t^\varepsilon, Y_t^\varepsilon)$  stay Gaussian at each time, so that the conditional law  $\pi_t^\varepsilon$  is Gaussian with parameters  $(\hat{X}_t^\varepsilon, P_t^\varepsilon)$ . Standard computations Kalman and Bucy [1961]; Jazwinski [1970]; Davis and Davis [1977] then show that  $P_t^\varepsilon = \varepsilon P_t$  is deterministic and  $\hat{X}_t^\varepsilon$  solves the linear SDE*

$$d\hat{X}_t^\varepsilon = A\hat{X}_t^\varepsilon + P_t H^\top [dY_t^\varepsilon - H\hat{X}_t^\varepsilon dt], \quad \hat{X}_0^\varepsilon = \hat{x}_0,$$

*where  $P_t$  is the solution to the Riccati equation (1.25). As a consequence,*

$$\pi_t^\varepsilon(dx) = Z_\varepsilon^{-1} \exp \left[ -\frac{[x - \hat{X}_t^\varepsilon] \cdot P_t^{-1} [x - \hat{X}_t^\varepsilon]}{2\varepsilon} \right] dx,$$

*where  $Z_\varepsilon > 0$  is a normalising constant. As  $\varepsilon \rightarrow 0$ , this shows that  $\pi_t^\varepsilon$  concentrates on the Mortensen observer  $\hat{x}(t)$  satisfying (1.24), defined as the minimiser of the quadratic function (1.23). We refer to Carrillo et al. [2022]; Burger et al. [2023] for approximation procedures using particles and gradient flow interpretations.*

### 1.2.3.2 Small noise limit

In non-linear situations, the  $\varepsilon \rightarrow 0$  limit and the related large deviations are due to Hijab [1984]; James and Baras [1988]; Fleming [1997]. Under suitable assumptions Pardoux [1980], the solution  $\rho_t^\varepsilon$  of the Zakai equation has a density  $q_t^\varepsilon(x)$  w.r.t. to the Lebesgue measure. The theory of pathwise filtering Davis [1981] then introduces the random function  $p_t^\varepsilon$  defined by

$$p_t^\varepsilon(x) := [-\varepsilon^{-1}Y_t^\varepsilon h(x)]q_t^\varepsilon(x), \quad (1.30)$$

which solves the robust Zakai equation

$$\partial_t p_t^\varepsilon = -[b - \nabla h Y_t^\varepsilon] \cdot \nabla p_t^\varepsilon + \frac{\varepsilon}{2} \Delta p_t^\varepsilon - \frac{1}{\varepsilon} \left\{ \frac{1}{2} |h|^2 + Y_t^\varepsilon \cdot L^\varepsilon h - \frac{1}{2} |\nabla h \cdot Y_t^\varepsilon|^2 + \varepsilon \nabla \cdot [b - \nabla h Y_t^\varepsilon] \right\}. \quad (1.31)$$

The randomness of  $p_t^\varepsilon$  only comes from the  $Y_t^\varepsilon$ -dependent coefficients. For each continuous realisation  $y = (y_t)_{t \geq 0}$  of  $(Y_t^\varepsilon)_{t \geq 0}$ , (1.31) has a unique (deterministic) solution  $p^\varepsilon : (t, x) \mapsto p^\varepsilon(t, x)$ , which continuously depends on  $y$  [Fleming and Pardoux, 1982, Lemma 3.2]. The random function  $p_t^\varepsilon$  can then be recovered by solving (1.31) for each continuous path  $y$ . The seminal work Clark and Crisan [2005] further shows that  $\pi_t^\varepsilon$  continuously depends on  $(Y_s^\varepsilon)_{0 \leq s \leq t}$ .

This approach is used in James and Baras [1988] to study the  $\varepsilon \rightarrow 0$  limit: a  $C^1$  realisation  $(y_t)_{t \geq 0}$  of the process  $(Y_t^\varepsilon)_{t \geq 0}$  is frozen, before studying the  $\varepsilon \rightarrow 0$  behaviour of the related solution  $p^\varepsilon$  of (1.31). Alternatively,  $q^\varepsilon$  can be deduced back from  $p^\varepsilon$  using (1.30), for each continuous realisation  $y = (y_t)_{t \geq 0}$  of  $(Y_t^\varepsilon)_{t \geq 0}$ , so that  $q^\varepsilon$  is a continuous function of  $y$ . In [Fleming, 1997, Section 5], a  $C^1$  realisation  $y$  is frozen to obtain that

$$\partial_t q^\varepsilon(t, x) = (L_t^\varepsilon)^* q^\varepsilon(x) - \frac{1}{\varepsilon} \left[ \frac{1}{2} |h(t, x)|^2 - \dot{y}_t \cdot h(x) \right] q^\varepsilon(t, x), \quad (1.32)$$

corresponding to the Stratonovich form of (1.29) with  $(Y_t^\varepsilon)_{t \geq 0}$  replaced by  $(y_t)_{t \geq 0}$ . We then define

$$\tilde{q}_t^\varepsilon(x) := \exp \left[ -\frac{1}{2\varepsilon} \int_0^t |\dot{y}_s|^2 ds \right] q^\varepsilon(t, x), \quad V_t^\varepsilon(x) := -\varepsilon \log \tilde{q}_t^\varepsilon(x).$$

The exponential factor is a normalising term, which does not affect the minimisation of  $x \mapsto V_t^\varepsilon(x)$ . The following key-result from [Fleming, 1997, Lemma 5.1] then makes the connection with the cost-to-come function  $V$  from Section 1.1.3.

**Theorem 1.2.5** (Small noise limit). *For every compact set  $K \subset \mathbb{R}^d$ , if*

$$\sup_{x \in K} |-\varepsilon \log q_0^\varepsilon(x) - J(x)| \leq C_K \varepsilon^{1/2}, \quad (1.33)$$

for some  $C_K > 0$  independent of  $\varepsilon$ , then for every  $t > 0$ ,

$$\sup_{(s, x) \in [0, t] \times K} |V_s^\varepsilon(x) - V_s(x)| \leq C'_K \varepsilon^{1/2},$$

for some  $C'_K > 0$  that only depends on  $(t, K)$ .

This result precisely says that  $\tilde{q}^\varepsilon$  concentrates on the Mortensen observer from Section 1.1.3 as  $\varepsilon \rightarrow 0$ . This can be rephrased as a large deviation principle with rate function  $V$ , see e.g. [James and Baras, 1988, Lemma 6.1]. Theorem 1.2.5 relies on the Hamilton-Jacobi equation for  $V_t^\varepsilon$ ,

$$\partial_t V_t^\varepsilon + b \cdot \nabla V_t^\varepsilon + \frac{1}{2} |\nabla V_t^\varepsilon|^2 - \frac{1}{2} |\dot{y}_t - h|^2 = \varepsilon \nabla \cdot b + \frac{\varepsilon}{2} \Delta V_t^\varepsilon, \quad V_0^\varepsilon = -\log q_0^\varepsilon,$$

before taking the vanishing-viscosity limit to recover (1.21). This logarithmic transform of (1.31) is known as the Cole-Hopf transform. Following [Fleming, 1997, Section 5],  $V_t^\varepsilon$  can be interpreted as the value function

$$V_t^\varepsilon(x) := \inf_{\substack{(u_s)_{0 \leq s \leq t} \\ Y_t^{\varepsilon, u} = x}} \mathbb{E} \left[ V_0^\varepsilon(Y_0^{\varepsilon, u}) + \int_0^t \frac{1}{2} |u_s|^2 + \varepsilon \nabla \cdot b(Y_s^{\varepsilon, u}) + \frac{1}{2} |\dot{y}_s - h(Y_s^{\varepsilon, u})|^2 ds \right], \quad (1.34)$$

the controlled curve being defined through the backward equation

$$dY_s^{\varepsilon, u} = b(Y_s^{\varepsilon, u}) ds + u_s ds + \sqrt{\varepsilon} d\beta_s,$$

where  $(\beta_s)_{0 \leq s \leq t}$  is a backward Brownian motion, see [Fleming, 1997, Lemma 5.1] for more precisions. Formally, the  $\varepsilon \rightarrow 0$  limit in (1.34) recovers the deterministic problem of controlling the ODE (1.2) backward in time. From the usual Cauchy-Lispchitz theory, forward and backward ODEs are equivalent, hence this recovers the cost-to-come function (1.20) as desired. Note, however, the *importance of time-reversal* in backward control interpretations for forward filtering equations, see also Section 1.4.1.

### 1.3 Constrained dynamics on measures

We now turn to the infinite-dimensional setting, replacing the toy-model ODE (1.1) by a Partial Differential Equation (PDE). In some particular smooth settings, we could try to directly adapt the methods from Section 1.1-1.2. However, PDE structures are more complex than ODE ones, hence further restrictions are needed. In the following, we will be mainly interested in PDEs describing the flow of marginal laws of a continuous stochastic process. The main examples will be Fokker-Planck equations of type

$$\partial_t \mu_t = \nabla \cdot [-\mu_t b + \frac{1}{2} \nabla \cdot [\mu_t \sigma \sigma^\top]], \quad (1.35)$$

which gives the flow of time-marginals  $\mu_t = \mathcal{L}(X_t)$  of the diffusion process

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t, \quad (1.36)$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion. We will also consider the non-linear version

$$\partial_t \mu_t = \nabla \cdot [-\mu_t b(\cdot, \mu_t) + \frac{1}{2} \nabla \cdot [\mu_t \sigma \sigma^\top]], \quad (1.37)$$

corresponding to the McKean-Vlasov process

$$dX_t = b(X_t, \mathcal{L}(X_t)) dt + \sigma(X_t) dB_t, \quad (1.38)$$

which interacts with its own law. In many cases, the relevant object to understand the constrained flow will be the path-law of  $(X_t)_{0 \leq t \leq T}$  in  $\mathcal{P}(C([0, T], \mathbb{R}^d))$ . Before going further into stochastic analysis, let us introduce the crucial setting of *Wasserstein gradient flows*. This setting will enable us to include many other examples and to draw strong parallels with Section 1.1.

**Wasserstein distance.** For  $p \geq 1$  and any metric space  $(E, d_E)$  endowed with its Borel  $\sigma$ -algebra,  $\mathcal{P}_p(E)$  is defined as the set of probability measures  $\mu \in \mathcal{P}(E)$  with finite  $\int_E d_E^p(x, x_0) d\mu(x)$ , for some (and thus any)  $x_0 \in E$  Villani et al. [2009]. The  $p$ -Wasserstein distance  $W_p$  is then defined on  $\mathcal{P}_p(E)$  as

$$W_p^p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|^p].$$

In the following, we fix  $p = 2$ . Relevant examples will be  $E = \mathbb{R}^d$  or  $E = C([0, T], \mathbb{R}^d)$ . The Benamou-Brenier Benamou and Brenier [2000] formula gives the variational representation

$$W_2^2(\mu, \nu) := \inf_{(\mu_s, v_s)_{0 \leq s \leq 1}} \int_0^1 \int_{\mathbb{R}^d} |v_s|^2 d\mu_s ds,$$

where the infimum runs over curves  $(\mu_s, v_s)_{0 \leq s \leq 1}$  such that  $\mu_s \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $(\mu_0, \mu_1) = (\mu, \nu)$ , and satisfying the transport equation

$$\partial_s \mu_s + \nabla \cdot [\mu_s v_s] = 0, \quad (1.39)$$

in the sense of distributions. The square velocity of the curve  $(\mu_s)_{0 \leq s \leq 1}$  at  $t \in [0, 1]$  can be defined as

$$|\dot{\mu}_t|_{\mu_t}^2 := \inf \left\{ \int_{\mathbb{R}^d} |v_t|^2 d\mu_t, (v_s)_{0 \leq s \leq 1} \text{ s.t. (1.39)} \right\}.$$

Although uniqueness does not hold for  $(v_s)_{0 \leq s \leq 1}$  in (1.39), this definition is non-ambiguous and

$$W_2^2(\mu, \nu) = \inf_{\substack{(\mu_s)_{0 \leq s \leq 1} \\ \mu_0 = \mu, \mu_1 = \nu}} \int_0^1 |\dot{\mu}_s|_{\mu_s}^2 ds. \quad (1.40)$$

This strongly reminds us of the geodesic distance on a Riemannian manifold. By analogy with the Riemannian setting, we can then define Wasserstein geodesics, gradients, Hessians... see Gentil et al. [2020] for a formal presentation.

**Gradient flow.** Alternatively,  $W_2$ -gradient flows can be defined from the general theory of gradient flows in metric spaces, see Ambrosio et al. [2005] for a detailed rigorous exposition. At the end of the day, computations show that a curve  $(\mu_s)_{0 \leq s \leq 1}$  in  $\mathcal{P}_2(\mathbb{R}^d)$  is the  $W_2$ -gradient flow of a functional  $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (-\infty, +\infty]$  if

$$\partial_s \mu_s + \nabla \cdot \left[ \mu_s \nabla \frac{\delta \mathcal{F}}{\delta \mu} \right] = 0,$$

in the sense of distributions, where the *linear functional derivative*  $\frac{\delta \mathcal{F}}{\delta \mu}(\mu) : x \mapsto \frac{\delta \mathcal{F}}{\delta \mu}(\mu, x)$  is defined (when it exists) by

$$\forall \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} [\mathcal{F}((1 - \varepsilon)\mu + \varepsilon\nu) - \mathcal{F}(\mu)] = \int_{\mathbb{R}^d} \frac{\delta \mathcal{F}}{\delta \mu}(\mu) d[\nu - \mu].$$

In particular,  $\frac{\delta \mathcal{F}}{\delta \mu}(\mu, x)$  can be computed by taking  $\nu = \delta_x$ . Since this only gives  $\frac{\delta \mathcal{F}}{\delta \mu}(\mu)$  up to an additive constant, it is customary to impose  $\int_{\mathbb{R}^d} \frac{\delta \mathcal{F}}{\delta \mu}(\mu) d\mu = 0$ . See e.g. Ambrosio et al. [2005]; Cardaliaguet et al. [2019] for a detailed construction of differential calculus on measures.

**Example 1.3.1** (Linear function). *For linear functions  $\mathcal{F}(\mu) := \int_{\mathbb{R}^d} U d\mu$ , where  $U : \mathbb{R}^d \rightarrow \mathbb{R}$  is measurable with sub-quadratic growth, the linear derivative simply reads  $\frac{\delta \mathcal{F}}{\delta \mu}(\mu) = U - \int_{\mathbb{R}^d} U d\mu$ .*

From Jordan et al. [1998], a seminal example of such gradient flows is (1.35) with  $b(x) = -\nabla U(x)$  and  $\sigma \equiv \text{Id}$ , which turns out to be the  $W_2$ -gradient flow of

$$\mathcal{F}(\mu) := \int_{\mathbb{R}^d} U d\mu + \frac{1}{2} \int_{\mathbb{R}^d} \log \frac{d\mu}{d\text{Leb}} d\mu, \quad (1.41)$$

with the convention  $\mathcal{F}(\mu) = +\infty$  if  $\mu$  has no density w.r.t. the Lebesgue measure. The Fokker-Planck equation (1.35) then reads

$$\dot{\mu}_t = -\nabla_{W_2} \mathcal{F}(\mu_t), \quad (1.42)$$

which can be seen as an infinite dimensional ODE. To constrain the dynamics (1.35), Section 1.1 can now be adapted using this formalism. An analogous rewriting can be done for the non-linear equation (1.37), when e.g.  $b(x, \mu) = -\nabla[W * \mu]$ . The function  $D_\mu \mathcal{F} := \nabla \frac{\delta \mathcal{F}}{\delta \mu}$  is often referred to as the *intrinsic derivative* or *Lions-derivative* Cardaliaguet et al. [2019], so that (1.42) reads

$$\partial_t \mu_t + \nabla \cdot [\mu_t D_\mu \mathcal{F}(\mu_t)] = 0,$$

in the sense of distributions. In the setting of Example 1.3.1, this yields  $D_\mu \mathcal{F} = \nabla U$ .

**Remark 1.3.2** (More general gradient flows). *In the following, we develop the above example of  $W_2$ -gradient flows. Further examples could be obtained by considering gradient flows w.r.t. other metrics, see e.g. [Gentil et al., 2020, Section 3.2]. We could also include non-constant  $\sigma$  by considering a Riemannian metric on  $\mathbb{R}^d$  weighted by  $\sigma \sigma^\top$  instead of the Euclidean distance.*

### 1.3.1 First order approach

We now aim at constraining the Fokker-Planck equations (1.35)-(1.37) to stay in the sub-domain

$$\mathcal{N}_\Psi^- := \{\mu \in \mathcal{P}_2(\mathbb{R}^d), \Psi(\mu) \leq 0\},$$

for some (regular)  $\Psi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ . When these dynamics fit in (1.42), the analogy with Section 1.1.1.1 suggests the corrected dynamics

$$\dot{\bar{\mu}}_t = -\nabla_{\mathcal{N}_\Psi^-} \mathcal{F}(\bar{\mu}_t), \quad (1.43)$$

where  $\nabla_{\mathcal{N}_\Psi^-}$  is the gradient corresponding to the geodesic distance induced by (1.40) on  $\mathcal{N}_\Psi^-$ .

**Discrete-time approach.** Following Ambrosio et al. [2005], a discrete-time scheme for (1.42) is

$$\nu_{t+h} = \operatorname{argmin}_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} \frac{W_2^2(\nu, \nu_t)}{2} + \mathcal{F}(\nu),$$

under suitable assumptions on  $\mathcal{F}$ , like convexity, for this to be well-posed. In the current Wasserstein setting, this is known as the Jordan-Kinderlehrer-Otto (JKO) scheme for the Fokker-Planck equation (1.35) Jordan et al. [1998]. The corresponding scheme for the corrected curve  $(\bar{\mu}_t)_{t \geq 0}$  reads

$$\bar{\nu}_{t+h} = \operatorname{argmin}_{\nu \in \mathcal{N}_\Psi^-} \frac{W_2^2(\nu, \bar{\nu}_t)}{2} + \mathcal{F}(\nu),$$

which is the direct analogous of (1.6). This simply corresponds to the JKO scheme in the sub-manifold  $\mathcal{N}_\Psi^-$ . In a one-dimensional setting with  $\Psi(\mu) = \int_{\mathbb{R}} x d\mu(x)$ , an exhaustive study of this construction is performed in Eberle et al. [2017]. For constraining (1.42) to stay on a sphere in  $\mathcal{P}_2(\mathbb{R}^d)$ , we refer to the seminal works Carlen and Gangbo [2003]; Tudorascu [2008].

**Constrained gradient flow.** The  $W_2$ -structure provides a suitable way to compute projections on the sub-manifold  $\mathcal{N}_\Psi^-$ . Following Section 1.1.1.3, (1.43) can be rewritten as

$$\dot{\mu}_t = -\nabla_{W_2} \mathcal{F}(\bar{\mu}_t) - \lambda_t \nabla_{W_2} \Psi(\bar{\mu}_t),$$

using a non-negative Lagrange multiplier  $\lambda_t$  satisfying the slackness condition  $\lambda_t \Psi(\bar{\mu}_t) = 0$ . As for (1.9), the multiplier can be computed writing that

$$0 = \frac{d}{dt} \Psi(\mu_t) = \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t) d[\partial_t \mu_t] = \int_{\mathbb{R}^d} [-D_\mu \mathcal{F}(\bar{\mu}_t) \cdot D_\mu \Psi(\bar{\mu}_t) - \lambda_t |D_\mu \Psi(\bar{\mu}_t)|^2] d\mu_t,$$

on intervals where  $\lambda_t$  is identically 0. The analogous of the constraint qualification condition in Section 1.1.1.3 is now that  $D_\mu \Psi(\bar{\mu}_t) \neq 0$  on a set with positive  $\bar{\mu}_t$ -measure.

**Mean reflected SDE.** The above projection procedure can be used to constrain equations (1.35)-(1.37) without requiring the gradient structure (1.42) anymore. In terms of SDEs, this amounts to replacing (1.36) by

$$d\bar{X}_t = b(\bar{X}_t)dt - \lambda_t D_\mu \Psi(\mathcal{L}(\bar{X}_t), \bar{X}_t)dt + \sigma(\bar{X}_t)dB_t, \quad (1.44)$$

where  $\lambda_t$  is computed as above. We assumed till now that the Lagrange multiplier has a density w.r.t. the Lebesgue measure. As in the setting of Section 1.1.2.3, this is not always true and the right setting requires a non-negative Radon measure  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  satisfying

$$\forall t \in [0, T], \quad \Psi(\mathcal{L}(\bar{X}_t)) \leq 0, \quad \text{and} \quad \int_0^T \Psi(\mathcal{L}(\bar{X}_t)) \bar{\lambda}(dt) = 0. \quad (1.45)$$

The problem of constraining (1.36) in  $\mathcal{N}_\Psi^-$  using a (deterministic)  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  satisfying (1.44)-(1.45) is known as *mean reflection for SDEs*. It can be seen as an infinite-dimensional generalisation of the Skorokhod problem described in Section 1.1.1.2. This problem was initially introduced in Briand et al. [2018] for backward stochastic differential equations (BSDE) and further developed in Briand et al. [2020b]. This last work also proposes an approximating scheme using an interacting  $N$ -particle system with almost sure constraints. This system can be seen as a reflected SDE in  $\mathbb{R}^{dN}$  which enters the framework of Remark 1.2.2. An exhaustive study of these reflected dynamics is done in Briand et al. [2020a], which also includes mean reflection for SDEs with an additional common noise (in this last setting, the multiplier  $\bar{\lambda}$  becomes random).

The specific constraint  $\Psi(\mu) := 1 - \mu(G)$  for some measurable  $G \subset \mathbb{R}^d$  amounts to the support constraint  $\mu(G) = 1$ . The corresponding problem of constraining (1.35) can be seen as an infinite-dimensional extension of the sub-differential dynamics from Section 1.1.1.2. This problem is tackled using the above  $W_2$ -approach in Jabir [2017], introducing *Wasserstein sub-differential dynamics*. See also Ambrosio et al. [2005] for a generic notion of Wasserstein sub-gradient. In all these examples, the initial dynamics (1.36) is a classical SDE, whereas the corrected dynamics is a McKean-Vlasov process of type (1.44), corresponding to a non-linear Fokker-Planck equation of type (1.37). This was expected since the constraint acts on the law of the process, rather than the process itself. Constraining directly non-linear equations of type (1.37) is then a natural extension, see Xiangdong and Shaopeng [2023] for an exhaustive study of this setting.

**Example 1.3.3** (Some applications). *The idea of simulating constrained dynamics using a projected  $N$ -particle approximation had been used in Samaey et al. [2011] for simulating polymeric fluids. As mentioned above, such particle systems are reminiscent of the Norton dynamics, which is a useful simulation tool in molecular dynamics, see Remark 1.2.2. We also refer to Debrabant et al. [2017]; Lelièvre et al. [2020] for a related, but different, approach using moment constraints for accelerating numerical simulations.*



### 1.3.2 Newton equation on measures

Looking back at the dynamics (1.42), the gradient flow  $(\mu_s)_{0 \leq s \leq T}$  is the unique minimiser for

$$\inf_{(\nu_s)_{0 \leq s \leq T}} \mathcal{F}(\nu_T) - \mathcal{F}(\nu_0) + \int_0^T \left[ \frac{1}{2} |\dot{\nu}_s|_{\nu_s}^2 + \int_{\mathbb{R}^d} \frac{1}{2} |D_\mu \mathcal{F}(\nu_s)|^2 d\nu_s \right] ds, \quad (1.46)$$

the initial law  $\mu_0$  being imposed. This property is an instance of the characterisation of gradient flows as curves with maximal slope with respect to  $\mathcal{F}$ , see Ambrosio et al. [2005]. Problem (1.46) can be seen as an infinite-dimensional generalisation of the variational formula studied in Section 1.1.2, in the specific case of gradient dynamics. Looking at Section 1.1.2.1, it is then natural to write optimality conditions for (1.46) as a Newton equation in  $\mathcal{P}_2(\mathbb{R}^d)$  satisfied by  $(\mu_s)_{0 \leq s \leq T}$ . This equation has been first introduced by the prominent work Conforti [2019] in the specific case (1.41), and further developed in Gentil et al. [2020], as well as the related geometric framework. A main challenge is to make sense of the acceleration  $\ddot{\mu}_t$ , defined as the covariant derivative  $D_{\dot{\mu}_t} \dot{\mu}_t$  in  $\mathcal{P}_2(\mathbb{R}^d)$  of  $\dot{\mu}_t$  in the direction  $\dot{\mu}_t$ , see Remark 1.1.2. From [Gentil et al., 2020, Section 3.3], formal computations give the covariant derivative of  $\dot{\nu}_t := \nabla \Phi_t$  at  $\mu_t$  in the direction  $\dot{\mu}_t = \nabla \Theta_t$  as

$$D_{\dot{\mu}_t} \dot{\nu}_t = \text{Proj}_{T_{\mu_t} \mathcal{P}_2(\mathbb{R}^d)} [\nabla \partial_t \Phi_t + \nabla^2 \Phi_t \nabla \Theta_t],$$

using the orthogonal projection in  $L^2(\mu_t)$  on the tangent space

$$T_{\mu_t} \mathcal{P}_2(\mathbb{R}^d) := \overline{\{\nabla \chi, \chi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu_t)}.$$

The Newton equation obtained by taking the (covariant) derivative of (1.42) reads

$$\ddot{\mu}_s = \frac{1}{2} \nabla_{W_2} |D_\mu \mathcal{F}(\mu_s)|^2, \quad (1.47)$$

as computed in [Gentil et al., 2020, Proposition 22]. In the case (1.41), the computations in Conforti [2019] yield

$$4 |D_\mu \mathcal{F}(\mu_s)|^2 = \int_{\mathbb{R}^d} |\nabla [\log \mu_s + 2V]|^2 d\mu_s.$$

This quantity is well-known as the (weighted) *Fisher information* of  $\mu_s$ . This setting corresponds to the famous Schrödinger bridge problem, see Section 1.3.3.1 below.

**Remark 1.3.4** ( $\mathcal{F}$ -interpolation). *Following Gentil et al. [2020], we can extend the setting of gradient flows by defining the interpolation cost*

$$\mathcal{A}_{\mathcal{F}}(\mu, \nu) := \inf_{\substack{(\nu_s)_{0 \leq s \leq T} \\ \nu_0 = \mu, \nu_T = \nu}} \int_0^T \left[ \frac{1}{2} |\dot{\nu}_s|_{\nu_s}^2 + \int_{\mathbb{R}^d} \frac{1}{2} |D_\mu \mathcal{F}(\nu_s)|^2 d\nu_s \right] ds,$$

for any  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ . This notion of interpolation can be seen as an infinite-dimensional generalisation of (1.15) in Example 1.1.3. As in the finite-dimensional setting, the optimal curves for  $\mathcal{A}_{\mathcal{F}}(\mu, \nu)$  satisfy the Newton equation (1.47). Finer properties, like convexity and contraction inequalities, of these interpolations are studied in Gentil et al. [2020]. A notable example is the case (1.41) for which this interpolation corresponds to the Schrödinger bridge introduced in Section 1.3.3.1. When  $\mathcal{F} \equiv 0$ ,  $\mathcal{A}_{\mathcal{F}}(\mu, \nu)$  reduces to the square distance  $\frac{1}{2} W_2^2(\mu, \nu)$ , the minimisers being the Wasserstein geodesics, also known as McCann's displacement interpolations McCann [1995].

**Remark 1.3.5** (Link with Schrödinger’s equation). Equation (1.47) corresponds to the time-derivative of the Fokker-Planck equation  $\partial_t \mu_t = \nabla \cdot [\mu_t \nabla U + \frac{1}{2} \nabla \mu_t]$ . An interesting connection with Schrödinger’s equation had been shown in von Renesse [2012] by changing the sign into  $\ddot{\mu}_s = -\frac{1}{2} \nabla_{W_2} |D_\mu \mathcal{F}(\mu_s)|^2$ . This last equation corresponds to the time-derivative of a Schrödinger-type equation. This amounts to considering “imaginary time” by performing the formal change of variable  $t = i\tau$  – namely Wick’s rotation. This formalism allows for many analogies with the behaviour of finite-dimensional Newton equations (steady states, oscillations...).

From (1.46), let us go back to the problem of constraining (1.42) to stay in  $\mathcal{N}_\Psi^-$ . Comparing with Section 1.1.2, the analogous of the second order procedure (1.4) now reads

$$\inf_{\substack{(\nu_s)_{0 \leq s \leq T} \\ \forall t \in [0, T], \Psi(\nu_t) \leq 0}} \int_0^T \left[ \frac{1}{2} |\dot{\nu}_s|_{\nu_s}^2 + \int_{\mathbb{R}^d} \frac{1}{2} |D_\mu \mathcal{F}(\nu_s)|^2 d\nu_s \right] ds,$$

the initial condition  $\nu_0$  being possibly imposed. *The study of this procedure is a main contribution of this thesis, which is summarised in Section 1.4.3.2.* A convenient framework will be the one of entropy minimisation, the connection with the above formulation being detailed in Section 1.3.3.3. In particular, it enables us to handle more general dynamics, which need not be gradient, including (1.35)-(1.37).

### 1.3.3 Conditioning approach and entropy

Let us now present an infinite-dimensional analogous of the probabilistic approach of Section 1.2. We mostly focus on the pathwise approach of Section 1.2.2. The main idea is to randomise (1.35)-(1.37) in a suitable way, before using conditioning to impose the constraint

$$\forall t \in [0, T], \quad \Psi(\mu_t) \leq 0,$$

and eventually letting the noise amplitude vanish. In the specific case where the infinite-dimensional dynamics of interest can be written as an ODE in some Hilbert space, the procedure of Section 1.2 can be directly applied using the infinite-dimensional Freidlin-Wentzell theory Budhiraja et al. [2008]. Otherwise, PDE structures require more care. There has been much recent interest in adding noise to (1.35), often motivated by regularisation-by-noise effects as in Stannat [2002]; Renesse and Sturm [2007]; Konarovskiy [2017]; Dello Schiavo [2022]; Delarue and Hammersley [2022]. In our present setting, we are rather interested in the behaviour of the randomisation with respect to conditioning. Motivated by the famous Gibbs principle in statistical mechanics, we approximate the solution of (1.35) using the empirical measure of independent copies of (1.36),

$$dX_t^i = b(X_t^i)dt + \sigma(X_t^i)dB_t^i, \quad i \geq 1, \quad (1.48)$$

where  $(B^i)_{i \geq 1}$  is a sequence of independent Brownian motions in  $\mathbb{R}^d$ . For  $N \geq 1$  and  $t \in [0, T]$ , we define the *empirical measure of the  $N$ -particle system at time  $t$*  as

$$\pi(\vec{X}_t^N) := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i} \in \mathcal{P}(\mathbb{R}^d),$$

using the notation  $\vec{X}_t^N := (X_t^1, \dots, X_t^N)$ . From the law of large numbers, almost surely,

$$\forall t \in [0, T], \quad \pi(\vec{X}_t^N) \xrightarrow[N \rightarrow +\infty]{\text{weak}} \mu_t,$$

giving the consistency of our approximation of (1.35). From Ito's formula,  $\pi(\vec{X}_t^N)$  a.s. satisfies

$$d \int_{\mathbb{R}^d} \varphi d\pi(\vec{X}_t^N) = \left[ \int_{\mathbb{R}^d} L\varphi d\pi(\vec{X}_t^N) \right] dt + \frac{1}{N} \sum_{i=1}^N \nabla\varphi(X_t^i) \cdot \sigma(X_t^i) dB_t^i,$$

for any smooth test function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ , the generator being defined by  $L\varphi := b \cdot \nabla\varphi + \frac{1}{2} \text{Tr}[\sigma\sigma^\top \nabla^2\varphi]$ . The last term on the r.h.s is a martingale whose quadratic variation is of order  $N^{-1}$ . This motivates the informal re-writing

$$d\pi(\vec{X}_t^N) = L^* \pi(\vec{X}_t^N) dt + N^{-1/2} dM_t, \quad (1.49)$$

where  $(M_t)_{0 \leq t \leq T}$  is intended to be a measure-valued martingale term of order 1, and  $L^*$  is the formal  $L^2$ -adjoint of  $L$ . As expected, (1.49) formally recovers (1.35) as  $N \rightarrow +\infty$ . Equation (1.49) is known as the *Dean-Kawasaki stochastic PDE*, see e.g. Kawasaki [1994]; Dean [1996]; Konarovskyi et al. [2019]; Fehrman and Gess [2022]. This writing yields a striking infinite-dimensional analogy (due to Dawson and Gärtner [1989]) with the Freidlin-Wentzell diffusion (1.26), replacing  $b(X_t^\varepsilon)$  by  $L^* \pi(\vec{X}_t^N)$  and  $\varepsilon$  by  $N^{-1}$ . In particular, this motivates our choice of approximation for (1.35), by comparison with Section 1.2. This as well extends to non-linear dynamics of type (1.37), replacing the independent copies  $X^i$  by the interacting particles

$$dX_t^{i,N} = b(X_t^{i,N}, \pi(\vec{X}_t^N)) dt + \sigma(X_t^{i,N}) dB_t^i, \quad 1 \leq i \leq N. \quad (1.50)$$

In this case, the convergence of  $\pi(\vec{X}_t^N)$  towards the solution of (1.37) is known as a *propagation of chaos* result Sznitman [1991], see the survey articles Chaintron and Diez [2022a,b] on this topic. The related Dean-Kawasaki equation can be written as well, replacing  $L$  in (1.49) by the measure-dependent generator  $L_{\pi(\vec{X}_t^N)}\varphi(x) := b(x, \pi(\vec{X}_t^N)) \cdot \nabla\varphi(x) + \frac{1}{2} \text{Tr}[\sigma\sigma^\top \nabla^2\varphi(x)]$ .

Following the procedure in Section 1.2, we now look for a constrained measure  $\bar{\mu}_t \in \mathcal{N}_\Psi^-$ , obtained as the  $N \rightarrow +\infty$  limit of  $\pi(\vec{X}_t^N)$  conditionally on the event

$$\forall t \in [0, T], \quad \Psi(\pi(\vec{X}_t^N)) \leq 0. \quad (1.51)$$

Since (1.51) involves the full path  $(\pi(\vec{X}_t^N))_{0 \leq t \leq T}$ , the natural probabilistic object is the *pathwise empirical measure*,

$$\pi(\vec{X}_{[0,T]}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{X_{[0,T]}^{i,N}} \in \mathcal{P}(C([0, T], \mathbb{R}^d)),$$

using the explicit notation  $X_{[0,T]}^{i,N} := (X_t^{i,N})_{0 \leq t \leq T}$ . The measure  $\pi(\vec{X}_t^N)$  can be recovered as the time-marginal law of  $\pi(\vec{X}_{[0,T]}^N)$  at time  $t$ . As in Section 1.2.2, large deviations provide a convenient framework to study the  $N \rightarrow +\infty$  behaviour of  $\pi(\vec{X}_{[0,T]}^N)$  conditionally on (1.51).

### 1.3.3.1 Gibbs principle and Schrödinger bridge problems

Let us briefly describe two famous problems related to large deviations of mean-field systems.

**Gibbs principle.** Let us consider a system of  $N$  identically distributed random variables  $(X^{i,N})_{1 \leq i \leq N}$  taking values in some Polish space  $E$ . For instance,  $E = \mathcal{P}(C([0, T], \mathbb{R}^d))$

in the particle settings (1.48)-(1.50). We assume that the  $X^{i,N}$  are either independent or interacting through their empirical measure  $\pi(\vec{X}^N) \in \mathcal{P}(E)$ . Next, let us imagine that some measurement is performed on the system's configuration revealing that  $\pi(\vec{X}^N)$  belongs to a subset  $A \subset \mathcal{P}(E)$ . It is then natural to look for the most likely configuration of the system or the law of  $X^{1,N}$  conditionally on  $\{\pi(\vec{X}^N) \in A\}$ . Both questions can be answered when the sequence of the  $\mathcal{L}(\pi(\vec{X}^N))$  satisfies a LDP with rate function  $I$ . This answer is part of a meta-principle, called the *Gibbs conditioning principle*: an informal statement of this principle is

$$\lim_{N \rightarrow +\infty} \mathcal{L}(X^{1,N} | \pi(\vec{X}^N) \in A) = \operatorname{argmin}_A I.$$

In words, the conditional distribution converges to the minimiser (if well-defined) of the rate function among all distributions compatible with the observation. Some reference textbooks about the Gibbs conditioning principle are Lanford [1973]; Ruelle [1965]; Dembo and Zeitouni [2009]; Ellis [2006]; Dupuis and Ellis [2011].

When the  $(X^{i,N})_{1 \leq i \leq N}$  are independent with common law  $\nu \in \mathcal{P}(E)$ , the LDP for  $\mathcal{L}(\pi(\vec{X}^N))$  is the famous Sanov theorem (see e.g. Dembo and Zeitouni [2009]), the rate function  $I$  being the *relative entropy* functional

$$H(\mu|\nu) := \begin{cases} \int_E \log \frac{d\mu}{d\nu} d\mu, & \text{if } \mu \ll \nu, \\ +\infty, & \text{otherwise.} \end{cases}$$

This function is always non-negative, strictly convex and has compact level sets for the weak topology on probability measures. Moreover,  $\nu$  is the unique zero of  $\mu \mapsto H(\mu|\nu)$ . When  $E = \mathcal{P}(C([0, T], \mathbb{R}^d))$  and  $\nu$  is the path-law of the diffusion process (1.36), this covers the case of the independent particles (1.48). The measurement then corresponds to

$$A := \{\mu_{[0, T]} \in \mathcal{P}(C([0, T], \mathbb{R}^d)), \forall t \in [0, T], \Psi(\mu_t) \leq 0\}.$$

When the set  $A$  is convex (this holds if  $\Psi$  is convex in the above example),  $\operatorname{argmin}_{\mu \in A} H(\mu|\nu)$  is well-defined and known as the *entropic projection* (or  $I$ -projection) of  $\nu$  onto the convex set  $A$ , see the seminal works Csiszár [1975, 1984]. This projection enjoys many geometric properties connected to information theory.

**Example 1.3.6** (Tower property). *A main property of entropic projections is the tower property [Csiszár, 1975, Theorem 2.3]. It states that the entropic projection of  $\operatorname{argmin}_{\mu \in A} H(\mu|\nu)$  onto a convex subset  $A' \subset A$  coincides with the entropic projection of  $\nu$  onto  $A'$ . In our constrained setting, defining*

$$A_{T, T'} := \{\mu_{[0, T']} \in \mathcal{P}(C([0, T'], \mathbb{R}^d)), \forall t \in [0, T], \Psi(\mu_t) \leq 0\}, \quad 0 \leq T \leq T',$$

*the tower property tells us that the projection onto  $A_{T', T'}$  can be computed by projecting the projection onto  $A_{T, T'}$ , onto  $A_{T', T'}$ . This restores a kind of recursivity property at the pathwise level.*

When the  $(X^{i,N})_{1 \leq i \leq N}$  correspond to the interacting system (1.50), the LDP is more involved Tanaka [1984]; Dawson and Gärtner [1987]; Budhiraja et al. [2012]; Fischer [2014]. The rate function  $I$  is now  $\mu_{[0, T]} \mapsto H(\mu_{[0, T]} | \Gamma(\mu_{[0, T]}))$ , where  $\Gamma(\mu_{[0, T]})$  is the path-law of the solution to

$$dX_t = b(X_t, \mu_t)dt + \sigma(X_t)dB_t,$$

the interaction in the drift  $b$  being frozen at  $\mu_{[0, T]}$ . Under mild (Lipschitz) assumptions on the coefficients, we can check that the path-law of the McKean-Vlasov process (1.38) is the

unique zero of  $\mu_{[0,T]} \mapsto H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]}))$ . Because of the non-linearity, proving this LDP is more intricate than in the independent case. *Its study is a major part of Chapter 3, see Section 1.4.2.*

**Schrödinger bridge.** Equivalently, the distribution  $\operatorname{argmin}_A I$  can be seen as the most likely configuration of the system conditionally on the observation. This viewpoint was the one adopted by E. Schrödinger in a celebrated thought experiment Schrödinger [1932] motivating the formulation of the *Schrödinger bridge problem*, which is the prototype of a stochastic mass transport problem. Given a reference path-law  $\nu_{[0,T]}$ , this problem looks for the closest dynamics to this reference, knowing the initial and terminal laws  $\mu_{\text{ini}}$  and  $\mu_{\text{fin}}$ . Motivated by the Gibbs principle, Schrödinger naturally chose the relative entropy as a distance criterion, leading to the formulation

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}(C([0,T], \mathbb{R}^d)) \\ \mu_0 = \mu_{\text{ini}}, \mu_T = \mu_{\text{fin}}}} H(\mu_{[0,T]}|\nu_{[0,T]}).$$

This pathwise interpolation problem reminds us of Remark 1.3.4, see Section 1.3.3.3 below for further connections. We refer to Dawson et al. [1990]; Cattiaux and Léonard [1994]; Föllmer and Gantert [1997] for connections with large deviations, and to Zambrini [1986, 1987]; Krener [1988] for links with quantum Euclidean mechanics. See also the survey article Léonard [2014] and references therein. This formulation paves the way for natural generalisations, replacing the relative entropy by any rate function  $I$ . The case  $\mu_{[0,T]} \mapsto H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]}))$  is known as the *mean-field Schrödinger problem* Backhoff et al. [2020].

**Generic constraints.** At this stage, we obtained a generic procedure analogous to (1.4) to constrain dynamics on measures that correspond to the time-marginals of a path-law  $\nu_{[0,T]}$ . The action functional used in the finite-dimensional setting from Section 1.1.2 is now replaced by the relative entropy w.r.t.  $\nu_{[0,T]}$  (or any convenient rate function), leading to

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]}|\nu_{[0,T]}). \quad (1.52)$$

**Studying this constraint procedure is the main goal of Chapters 4-5, see Section 1.4.3.** Any kind of pathwise constraint could be imposed as well. In the following, we will mostly assume that  $\nu_{[0,T]}$  is the path-law of a diffusion process of type (1.48). A crucial question is then to describe path-measures that have a finite entropy w.r.t.  $\nu_{[0,T]}$ , see Section 1.3.3.2 below. For elements on this question when  $\nu_{[0,T]}$  is the path-law of a general Markov process, we refer to Kraaij [2018] and references therein.

**Example 1.3.7** (Support constraints and Mortensen observer). *A notable example is  $\Psi(\mu) = 1 - \mu(G)$  for an open domain  $G \subset \mathbb{R}^d$ , as mentioned in Section 1.3.1. If  $\nu_{[0,T]}$  is the law of  $(X_t)_{0 \leq t \leq T}$  given by (1.36), then the minimiser  $\bar{\nu}_{[0,T]}$  of (1.52) is the law of  $(X_t)_{0 \leq t \leq T}$  conditionally on*

$$\forall t \in [0, T], \quad X_t \in G,$$

*allowing for explicit computations Pinsky [1985]. In particular, Doob's  $h$ -transform provides that  $\bar{\nu}_{[0,T]}$  is the law of the solution to*

$$dY_t = b(Y_t)dt + \sigma \sigma^\top \nabla \log u_t(Y_t)dt + \sigma(Y_t)dB_t,$$

*where  $u_t(x) := \mathbb{P}(\tau_G^x > T - t)$ ,  $\tau_G^x$  being the exit time from  $G$  when  $X_t$  is started at  $x$ . Moreover,  $u_t$  solves the Dirichlet problem*

$$\begin{cases} \partial_t u_t + Lu_t = 0, & \text{in } G, \\ u_t|_{\partial G} = 0. \end{cases}$$

Looking back at Section 1.1.3, the analogous of Mortensen's observer in this context is

$$\bar{\nu}_T := \operatorname{argmin}_{\substack{\mu \in \mathcal{P}(\mathbb{R}^d) \\ \mu(G)=1}} \inf_{\substack{\mu_{[0,T]}, \mu_T = \mu \\ \forall t \in [0,T], \mu_t(G)=1}} H(\mu_{[0,T]} | \nu_{[0,T]}),$$

corresponding to the marginal law of  $\bar{\nu}_{[0,T]}$  at the terminal time  $T$ . Since  $\bar{\nu}_T$  is the law of  $X_T$  conditionally on  $\{T < \tau_G\}$ , a direct computation yields

$$\begin{cases} \partial_T \bar{\nu}_T = L^* \bar{\nu}_T - [\frac{d}{dT} \log \mathbb{P}(T < \tau_G)] \bar{\nu}_T, & \text{in } G, \\ \bar{\nu}_T |_{\partial G} = 0. \end{cases}$$

This closed-form PDE is the infinite-dimension analogous of the Mortensen dynamics (1.22). Remarkably, this allows us to condition on intervals  $[0, T]$  with increasing  $T$  without recomputing the full path-law  $\bar{\nu}_{[0,T]}$  at each step. The existence of such a closed form equation for more general constraints is a challenging open question.

### 1.3.3.2 Girsanov theory and stochastic control

Looking at (1.52), the next step is to characterise measures which have a finite entropy w.r.t. the path-law  $\nu_{[0,T]}$  of the diffusion process (1.48). This question is answered by the famous Girsanov theory under finite entropy condition Jacod and Shiryaev [2013]; Föllmer [1988]; Léonard [2012], and allows for connections with stochastic control.

**Girsanov theory.** If  $\mu_{[0,T]} \in \mathcal{P}(C([0, T], \mathbb{R}^d))$  satisfies  $H(\mu_{[0,T]} | \nu_{[0,T]}) < +\infty$  and uniqueness holds for the solution of (1.48), [Léonard, 2012, Theorems 2.1-2.3] provide an adapted process  $(c_t)_{0 \leq t \leq T}$  such that the canonical process  $(X_t)_{0 \leq t \leq T}$  on  $\Omega = C([0, T], \mathbb{R}^d)$  satisfies

$$dX_t = b(X_t)dt + \sigma \sigma^\top(X_t) c_t dt + \sigma(X_t) dB_t, \quad \mu_{[0,T]}\text{-a.s.},$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion under  $\mu_{[0,T]}$  and

$$H(\mu_{[0,T]} | \nu_{[0,T]}) = H(\mu_0 | \nu_0) + \mathbb{E} \int_0^T \frac{1}{2} |\sigma^\top(X_t) c_t|^2 dt.$$

This powerful tool reduces entropy minimisation problems to stochastic control problems, connecting with the formalism of Section 1.1.2.

**Constrained stochastic control.** On a filtered probability space  $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ , we consider the controlled dynamics

$$dX_t^\alpha = b(X_t^\alpha)dt + \sigma(X_t^\alpha) \alpha_t dt + \sigma(X_t^\alpha) dB_t,$$

for any adapted square-integrable process  $(\alpha_t)_{0 \leq t \leq T}$ . Problem (1.52) boils down to the control problem with law constraints

$$\inf_{\forall t \in [0,T], \Psi(\mathcal{L}(X_t^\alpha)) \leq 0} H(\mathcal{L}(X_0^\alpha) | \nu_0) + \mathbb{E} \int_0^T \frac{1}{2} |\alpha_t|^2 dt, \quad (1.53)$$

the minimisation being over  $\alpha$  and  $(X_0^\alpha, \alpha)$ . This type of control problem is non-standard because of the constraints that act on the distribution of the controlled process, rather than the process itself. A control problem close to (1.53) has been recently studied in Daudin [2022, 2023b], see also references therein for applications in stochastic finance. These works rely on PDE methods close to the *mean-field game* theory Huang et al. [2006]; Lasry and Lions [2007]; Cardaliaguet et al. [2019], rather than our pathwise entropy approach. A

non-linear version of (1.53) can be obtained by replacing the above controlled dynamics by the McKean-Vlasov dynamics

$$dX_t^\alpha = b(X_t^\alpha, \mathcal{L}(X_t^\alpha))dt + \sigma(X_t^\alpha)\alpha_t dt + \sigma(X_t^\alpha)dB_t.$$

This precisely amounts (see Fischer [2014]) to replacing the relative entropy in (1.52) by the mean-field entropy  $\mu_{[0,T]} \mapsto H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]}))$  defined in Section 1.3.3.1. **An extensive study of this non-linear problem is done in Chapter 5, see Section 1.4.3.2.** Similarly, the Schrödinger bridge problem can be recovered from (1.53) by replacing the inequality constraints by  $\mathcal{L}(X_0^\alpha) = \mu_{\text{ini}}$  and  $\mathcal{L}(X_T^\alpha) = \mu_{\text{fin}}$ . This formulation was introduced by P.-L. Lions in his courses at Collège de France as the *planning problem* for mean-field games, see also Graber et al. [2019]; Bakaryan et al. [2021] and the recent variation Bertucci [2023].

**PDE formulation.** Looking for a PDE formulation of (1.53) in the spirit of Section 1.3.2, we only consider feedback controls  $\alpha_t = h_t(X_t^\alpha)$  for deterministic vector fields  $h_t$ , yielding

$$\inf_{\substack{(\mu_0^h, h) \\ \forall t \in [0, T], \Psi(\mu_t^h) \leq 0}} H(\mu_0^h | \nu_0) + \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} |h_t|^2 d\mu_t dt, \quad (1.54)$$

the controlled PDE being

$$\partial_t \mu_t^h = \nabla \cdot [-\mu_t^h b - \mu_t^h h_t + \frac{1}{2} \nabla \cdot [\mu_t^h \sigma \sigma^\top]],$$

in the sense of distributions. Following the seminal work Dawson and Gärtner [1987], we can get rid of the control field  $h$  by introducing the norm on distributions  $\mathbb{T}$  defined by

$$\|\mathbb{T}\|_\mu^2 := \sup_{\varphi \in C_c^\infty(\mathbb{R}^d)} \langle \mathbb{T}, \varphi \rangle - \frac{1}{2} \langle \mu, |\nabla \varphi|^2 \rangle,$$

for any  $\mu \in \mathcal{P}(\mathbb{R}^d)$ , the value  $+\infty$  being allowed. Formally, this norm is realised by some  $\varphi$  solving the Poisson problem  $-\nabla \cdot [\mu \nabla \varphi] = \mathbb{T}$ . Alternatively, we can check that  $\|\mathbb{T}\|_\mu^2$  coincides with the square-norm in the negative Sobolev space  $H^{-1}(\mu)$ . Using the contraction principle from the large deviation theory, Dawson and Gärtner [1987] proves that for any absolutely continuous  $\mathcal{P}(\mathbb{R}^d)$ -valued curve  $(\mu_t)_{0 \leq t \leq T}$ ,

$$\inf_{\substack{\mu'_{[0,T]} \in \mathcal{P}(C([0,T], \mathbb{R}^d)) \\ (\mu'_t)_{0 \leq t \leq T} = (\mu_t)_{0 \leq t \leq T}}} H(\mu'_{[0,T]} | \nu_{[0,T]}) = H(\mu_0 | \nu_0) + \int_0^T \|\partial_t \mu_t - L^* \mu_t\|_{\mu_t}^2 dt,$$

the dual generator  $L^*$  being the one in (1.49). Similarly, for the non-linear case (1.37),

$$\inf_{\substack{\mu'_{[0,T]} \in \mathcal{P}(C([0,T], \mathbb{R}^d)) \\ (\mu'_t)_{0 \leq t \leq T} = (\mu_t)_{0 \leq t \leq T}}} H(\mu'_{[0,T]} | \Gamma(\mu'_{[0,T]})) = H(\mu_0 | \nu_0) + \int_0^T \|\partial_t \mu_t - L_{\mu_t}^* \mu_t\|_{\mu_t}^2 dt. \quad (1.55)$$

This formula is the infinite-dimensional analogous of the Lagrangian formulation used in Section 1.1.2. In particular, the above quantity is the large deviation functional for the LDP satisfied by the  $\mathcal{L}((\pi(\vec{X}_t^N))_{0 \leq t \leq T})$  in  $\mathcal{P}(C([0, T], \mathcal{P}(\mathbb{R}^d)))$  Dawson and Gärtner [1987]. The PDE formulation corresponding to (1.52) then reads

$$\inf_{\substack{(\mu_t)_{0 \leq t \leq T} \\ \forall t \in [0, T], \Psi(\mu_t) \leq 0}} H(\mu_0 | \nu_0) + \int_0^T \|\partial_t \mu_t - L_{\mu_t}^* \mu_t\|_{\mu_t}^2 dt,$$

obtained as an infinite-dimensional analogous of the second-order procedure (1.4). As previously, we refer to Section 1.4.3.2 and Chapter 5 for a detailed analysis of this problem. To further emphasise the analogy with the relaxed formulation (1.10) in Section 1.1.2, we notice that the relaxed formulation for the above problem reads

$$\inf_{(\mu_t)_{0 \leq t \leq T}} H(\mu_0 | \nu_0) + \int_0^T \|\partial_t \mu_t - L_{\mu_t}^* \mu_t\|_{\mu_t}^2 dt + \int_0^T \Psi(\mu_t) \bar{\lambda}(dt),$$

for a suitable Lagrange multiplier  $\bar{\lambda} \in \mathcal{M}_+([0, T])$ . Adapting the computations in Section 1.1.2.3 to this setting is a crucial ingredient in Chapter 5. Similarly to Section 1.1.2.4, a penalisation approach can be used to approximate  $\bar{\lambda}$ , this strategy being applied in Daudin [2023b].

**Remark 1.3.8** (Link with gradient flows). *As in Remark 1.2.3, when (1.37) corresponds to the gradient flow dynamics (1.41)-(1.42), the JKO scheme described in Section 1.3.1 can be recovered from the large deviation functional (1.55). Let us first sketch an informal argument from [Adams et al., 2013, Section 6.1]. If we look at independent Brownian motions  $\vec{B}^N = (B^1, \dots, B^N)$ ,*

$$\mathbb{P}(\vec{B}_h^N = \vec{y}^N | \vec{B}_0^N = \vec{x}^N) \asymp \exp \left[ -\frac{1}{2h} \sum_{i=1}^N |x^i - y^i|^2 \right],$$

as  $h \rightarrow 0$ . More rigorously, a LDP holds with rate function  $\vec{y}^N \mapsto \frac{1}{2} |\vec{y}^N - \vec{x}^N|^2$ . Let us denote by  $[\vec{x}^N]$  the equivalence class of  $\vec{x}^N$  in the quotient space  $(\mathbb{R}^d)^N / \mathfrak{S}_N$  of indistinguishable  $N$ -tuples. The mapping  $\vec{x}^N \mapsto [\vec{x}^N]$  being continuous, the contraction principle for large deviations yields

$$\mathbb{P}([\vec{B}_h^N] = [\vec{y}^N] | [\vec{B}_0^N] = [\vec{x}^N]) \asymp \exp \left[ -\inf_{\tau \in \mathfrak{S}_N} \frac{1}{2h} \sum_{i=1}^N |x^i - y^{\tau(i)}|^2 \right].$$

Remarkably, the embedding

$$\begin{cases} (\mathbb{R}^d)^N / \mathfrak{S}_N & \rightarrow \mathcal{P}_2(\mathbb{R}^d), \\ [\vec{x}^N] & \mapsto \pi(\vec{x}^N), \end{cases}$$

is an isometry for the  $W_2$ -distance, suggesting that

$$\mathbb{P}(\pi(\vec{B}_h^N) = \mu | \pi(\vec{B}_0^N) = \nu) \asymp \exp \left[ -\frac{N}{h} W_2^2(\mu, \nu) \right],$$

provided that  $\mu, \nu$  are  $N$ -empirical measures. This motivates the use of the  $W_2$ -distance for handling empirical measures of Brownian particles. The entropy functional then gets involved to go beyond  $N$ -empirical measures  $\mu, \nu$ . More rigorously, a  $\Gamma$ -convergence result like the one mentioned in Remark 1.2.3 is proved in Adams et al. [2011]; Erbar et al. [2015], stating that

$$\mu \mapsto \left[ \inf_{\substack{(\mu_t)_{0 \leq t \leq h} \\ \mu_0 = \nu, \mu_h = \mu}} \int_0^h \|\partial_t \mu_t - L_{\mu_t}^* \mu_t\|_{\mu_t}^2 dt \right] - \frac{1}{2h} W_2^2(\mu, \nu) \xrightarrow[h \rightarrow 0]{\Gamma} \mu \mapsto \mathcal{F}(\mu) - \mathcal{F}(\nu),$$

where  $\mathcal{F}$  is defined by (1.41).



### 1.3.3.3 Time-reversal and optimal transport

Let us now sketch the connection between entropy minimisation problems and the transport setting from Section 1.3.2. To do so, we briefly describe some fundamental results about time-reversal of diffusion processes Cattiaux et al. [2023].

*Time-reversal.* Let  $\nu_{[0,T]}$  be the path-law of the solution to

$$dX_t^\nu = b_t^\nu(X_t^\nu)dt + \sigma_t(X_t^\nu)dB_t,$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion and  $b_t^\nu$  (resp.  $\sigma_t$ ) is a deterministic vector (resp. matrix) field. Let  $\mu_{[0,T]}$  be any path measure with  $H(\mu_{[0,T]}|\nu_{[0,T]}) < +\infty$ . From Section 1.3.3.2,  $\mu_{[0,T]}$  is the law of the solution to a similar SDE whose drift term is a perturbation of  $b_t^\nu$ . We further assume that this drift is a deterministic vector field  $b_t^\mu$ , so that  $\mu_{[0,T]}$  is the path-law of the solution to

$$dX_t^\mu = b_t^\mu(X_t^\mu)dt + \sigma_t(X_t^\mu)dB_t,$$

with  $b_t^\mu = b_t^\nu + \sigma_t \sigma_t^\top c_t$ . For the sake of simplicity, we assume that  $\sigma_t$  is invertible. This yields

$$H(\mu_{[0,T]}|\nu_{[0,T]}) = H(\mu_0|\nu_0) + \mathbb{E} \int_0^T \frac{1}{2} |\sigma_t^{-1} [b_t^\mu - b_t^\nu](X_t^\mu)|^2 dt. \quad (1.56)$$

Let us now introduce the law  $\overleftarrow{\nu}_{[0,T]}$  of the time-reversal  $(X_{T-t}^\nu)_{0 \leq t \leq T}$ . We similarly define  $\overleftarrow{\mu}_{[0,T]}$ . Under regularity assumptions on  $(b^\nu, \sigma)$ , a fundamental result tracing back to Haussmann and Pardoux [1986] is that  $\overleftarrow{\nu}_{[0,T]}$  is the path-law of the solution to

$$d\overleftarrow{X}_t^\nu = \overleftarrow{b}_t^\nu(\overleftarrow{X}_t^\nu)dt + \overleftarrow{\sigma}_t(\overleftarrow{X}_t^\nu)dB_t,$$

with  $\overleftarrow{X}_t^\nu = X_{T-t}^\nu$ ,  $\overleftarrow{\sigma}_t(x) := \sigma_{T-t}(x)$  and

$$\overleftarrow{b}_t^\nu(x) := -b_{T-t}^\nu(x) + \nu_{T-t}^{-1}(x) \nabla \cdot [\nu_{T-t} \sigma_{T-t}^\top \sigma_{T-t}^\top(x)],$$

where  $\mu_t(x)$  denotes the density of  $\mu_t$  w.r.t. the Lebesgue measure and the divergence is taken in the sense of distributions. The time-reversed drift  $\overleftarrow{b}_t^\mu$  can be similarly computed. A formal computation then gives the time-reversal formula

$$[b_t^\mu + \overleftarrow{b}_{T-t}^\mu] - [b_t^\nu + \overleftarrow{b}_{T-t}^\nu] = \sigma_t \sigma_t^\top \nabla \log \rho_t,$$

where  $\rho_t := \mu_t/\nu_t$  is the density of  $\mu_t$  w.r.t.  $\nu_t$ . Under suitable assumptions, this formula is proved in [Cattiaux et al., 2023, Theorem 1.16]. As a consequence, the Girsanov theory implies that

$$\begin{aligned} H(\overleftarrow{\mu}_{[0,T]}|\overleftarrow{\nu}_{[0,T]}) &= H(\mu_T|\nu_T) + \mathbb{E} \int_0^T \frac{1}{2} |\sigma_{T-t}^{-1} [\overleftarrow{b}_t^\mu - \overleftarrow{b}_t^\nu](\overleftarrow{X}_t^\mu)|^2 dt \\ &= H(\mu_T|\nu_T) + \mathbb{E} \int_0^T \frac{1}{2} |\sigma_t^{-1} [b_t^\nu - b_t^\mu](X_t^\mu) + \sigma_t^\top \nabla \log \rho_t(X_t^\mu)|^2 dt. \end{aligned}$$

Taking the half-sum with (1.56) now yields

$$\begin{aligned} [H(\mu_{[0,T]}|\nu_{[0,T]}) + H(\overleftarrow{\mu}_{[0,T]}|\overleftarrow{\nu}_{[0,T]})]/2 &= [H(\mu_0|\nu_0) + H(\mu_T|\nu_T)]/2 \\ &+ \mathbb{E} \int_0^T \frac{1}{2} |\sigma_t^{-1} [b_t^\nu - b_t^\mu](X_t^\mu) + \frac{1}{2} \sigma_t^\top \nabla \log \rho_t(X_t^\mu)|^2 dt + \mathbb{E} \int_0^T \frac{1}{8} |\sigma_t^\top \nabla \log \rho_t(X_t^\mu)|^2 dt. \end{aligned}$$

Since time-reversal is a one-to-one mapping, we have  $H(\mu_{[0,T]}|\nu_{[0,T]}) = H(\overleftarrow{\mu}_{[0,T]}|\overleftarrow{\nu}_{[0,T]})$ .

Let us further introduce the *current velocity*  $v_t^\nu := -b_t^\nu + \frac{1}{2}\nu_t^{-1}\nabla \cdot [\nu_t\sigma_t\sigma_t^\top]$ , so that

$$\partial_t\nu_t + \nabla \cdot [\nu_tv_t^\nu] = 0,$$

in the sense of distributions. If  $v_t^\nu$  is a gradient field, this recovers the notion of Wasserstein velocity introduced in Section 1.3.2. We similarly define  $v_t^\mu$ . This yields

$$\begin{aligned} H(\mu_{[0,T]}|\nu_{[0,T]}) &= \frac{1}{2}[H(\mu_0|\nu_0) + H(\mu_T|\nu_T)] + \mathbb{E} \int_0^T \frac{1}{2}|\sigma_t^{-1}[v_t^\mu - v_t^\nu](X_t^\mu)|^2 dt \\ &\quad + \mathbb{E} \int_0^T \frac{1}{8}|\sigma_t^\top \nabla \log \frac{d\mu_t}{d\nu_t}(X_t^\mu)|^2 dt. \end{aligned} \quad (1.57)$$

The first term on the r.h.s. only depends on marginal laws at times 0 and  $T$ . The second term is the weighted  $L^2$ -norm of the *relative current velocity* of  $\mu$  w.r.t.  $\nu$ . In Nelson's formalism of stochastic mechanics Nelson [2020], this term is interpreted as a *kinetic action* by analogy with classical mechanics, whereas the last term is the *osmotic action*. This last term is the integrated (weighted) *Fisher information* of  $\mu_t$  w.r.t.  $\nu_t$ : it only depends on marginal laws. It can also be written as the  $L^2$ -norm of the *relative osmotic velocity* using Nelson's formalism of *stochastic velocities*, see e.g. [Cattiaux et al., 2023, Section 4]. To get rid of the  $\sigma_t$ -factors, let us point out that a common framework – which traces back to Dawson and Gärtner [1987] – replaces the Euclidean norm by a Riemannian metric weighted by  $\sigma_t\sigma_t^\top$ . We refrained to do so for preserving the consistency of our notations.

**Back to gradient flows.** Let us now recover the gradient flow formalism from Section 1.3.2. We first assume that  $\sigma_t \equiv \text{Id}$ , before noticing that

$$\begin{aligned} |v_t^\mu - v_t^\nu|^2 + \frac{1}{4}|\nabla \log \frac{d\mu_t}{d\nu_t}|^2 &= |v_t^\mu|^2 + \frac{1}{4}|\nabla[\log \mu_t + 2U]|^2 \\ &\quad - 2v_t^\mu \cdot v_t^\nu + |v_t^\nu|^2 - \frac{1}{4}|\nabla[\log \nu_t + 2U]|^2 - \nabla[2\log \mu_t - \log \nu_t + 2U] \cdot \nabla[2\log \mu_t - \log \nu_t + 2U], \end{aligned} \quad (1.58)$$

for any smooth potential  $U : \mathbb{R}^d \rightarrow \mathbb{R}$ . We now assume that  $b_t^\nu = -\nabla U$ , so that  $(\nu_t)_{0 \leq t \leq T}$  is the gradient flow (1.42) of the free energy  $\mathcal{F}$  defined by (1.41). Under suitable integrability conditions on  $U$ , the stationary measure of this flow is  $\nu_\infty(dx) := Z_\infty^{-1}e^{-2U(x)}dx$ ,  $Z_\infty$  being a normalising constant. We then notice that

$$\mathcal{F}(\mu) = \frac{1}{2}[H(\mu|\nu_\infty) - \log Z_\infty],$$

so that (1.41)-(1.42) appear as the gradient flow of the relative entropy w.r.t. to  $\nu_\infty$ . We eventually impose that  $\nu_0 = \nu_\infty$ . Since  $\nu_{[0,T]}$  is  $\nu_\infty$ -reversible, this implies that  $\nu_{[0,T]} = \overleftarrow{\nu}_{[0,T]}$ . In particular,  $\nu_t = \nu_\infty$  at any time, so that  $v_t^\nu = 0$  using the expression of  $(b_t^\nu, \nu_\infty)$ . Since all the terms in the second line of (1.58) vanish, the decomposition (1.57) becomes

$$H(\mu_{[0,T]}|\nu_{[0,T]}) = \frac{1}{2}[H(\mu_0|\nu_\infty) + H(\mu_T|\nu_\infty)] + \mathbb{E} \int_0^T \frac{1}{2}|v_t^\mu(X_t^\mu)|^2 + \frac{1}{8}|\nabla[\log \mu_t + 2U](X_t^\mu)|^2 dt.$$

Noticing that  $\int_{\mathbb{R}^d} |v_t^\mu|^2 d\mu_t$  corresponds to  $|\dot{\mu}_s|_{\mu_s}^2$ , the above quantity precisely recovers the variational formula (1.46) from Section 1.3.2, up to an additive term that only depends on  $(\mu_0, \mu_T)$ . This recovers the setting of Conforti [2019] described in Section 1.3.2, so that the Newton equation in  $\mathcal{P}_2(\mathbb{R}^d)$  follows. Similarly, the above formula for  $H(\mu_{[0,T]}|\nu_{[0,T]})$  coincides (up to a constant) with the interpolation cost  $\mathcal{A}_{\mathcal{F}}(\mu_0, \mu_T)$  defined in Remark 1.3.4 with the choice (1.41) for  $\mathcal{F}$ .

**Conclusion.** At this stage, we completed the links between the finite-dimensional settings in Sections 1.1-1.2 and the measure setting in Section 1.3. The analogous of the first-order strategy (1.3) is the constrained gradient formalism in Section 1.3.1, together with its non-gradient extension. The Lagrangian in the second-order approach (1.4) in Section 1.1.2 has been replaced by the relative entropy (1.52). The Mortensen observer can be defined in this context, see Example 1.3.7. The links with conditioning described in Section 1.2 when adding a small Gaussian noise corresponds to the Gibbs principle from Section 1.3.3. This as well includes non-linear dynamics of type (1.37). In particular, the Dawson-Gärtner formula (1.55) extends the classical Lagrangian mechanics to spaces of probability measures. At the end of the day, optimality conditions can be written in terms on Newton equations on the Wasserstein space.

**Remark 1.3.9** (Geometric interpretation). *We can look for a geometric interpretation of the second-order procedure (1.52) on measures. As in Remark 1.1.2, the Newton equation obtained in writing optimality conditions for (1.46) tells us that the time-marginals  $(\bar{\mu}_t)_{0 \leq t \leq T}$  of a minimiser  $\bar{\mu}_{[0,T]}$  for (1.52) formally satisfy the Newton equation*

$$\ddot{\bar{\mu}}_t = \frac{1}{2} \nabla_{\mathcal{N}_{\Psi}^-} |D_{\mu} \mathcal{F}(\bar{\mu}_t)|^2,$$

*the gradient being taken w.r.t. to the Riemannian distance induced on the sub-manifold  $\mathcal{N}_{\Psi}^-$  of  $\mathcal{P}_2(\mathbb{R}^d)$ . In particular, the l.h.s. is the covariant derivative in  $\mathcal{N}_{\Psi}^-$  of  $\dot{\bar{\mu}}_t$  in the direction  $\dot{\bar{\mu}}_t$ . As in Remark 1.1.2, the computation of the Lagrange multiplier  $\bar{\lambda}$  performed in Chapter 5 amounts to identifying the second fundamental form of the Wasserstein sub-manifold  $\mathcal{N}_{\Psi}^-$ .*

## 1.4 Recap of contributions

Let us now summarise the contributions of this thesis, in the light of the constraint procedures introduced in Sections 1.1-1.2-1.3. Section 1.4.1 studies the small noise filtering for reflected diffusions in the spirit of Section 1.2.3. Section 1.4.2 establishes new results in large deviation theory needed by Section 1.3.3. Section 1.4.3 studies the second-order procedure on measures introduced in Sections 1.3.2-1.3.3.2. Some perspectives for future works are listed in Section 1.4.4.

### 1.4.1 Small noise filtering for reflected diffusions

This part is concerned with the filtering problem

$$\begin{cases} dX_t^\varepsilon + \partial \chi_G(X_t^\varepsilon)(dt) \ni b_t(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t, \\ dY_t^\varepsilon = h_t(X_t^\varepsilon) + \sqrt{\varepsilon}dB'_t, \end{cases} \quad (1.59)$$

where  $(B_t)_{t \geq 0}$  and  $(B'_t)_{t \geq 0}$  are independent Brownian motions and  $b_t$ ,  $h_t$  are Lipschitz vector fields in  $\mathbb{R}^d$ . The signal process  $X_t^\varepsilon$  is the solution of a SDE that is reflected on the boundary of a smooth domain  $G \subset \mathbb{R}^d$  in the Skorokhod sense, see Section 1.1.1.2 and Chapter 2 for finer descriptions. We are interested in the  $\varepsilon \rightarrow 0$  limit of the law  $\pi_t^\varepsilon$  of  $X_t^\varepsilon$  conditionally on the observation  $(Y_s^\varepsilon)_{0 \leq s \leq t}$  up to time  $t$ . More precisely, we aim at proving a LDP with rate function

$$V(t, x) := \inf_{\substack{(x_0^\omega, \omega) \\ x_t^\omega = x}} J(x_0^\omega) + \int_0^t \frac{1}{2} |\omega_s|^2 + \frac{1}{2} |\dot{y}_t - h_s(x_s^\omega)|^2 ds, \quad (1.60)$$

where  $(x_s^\omega)_{0 \leq s \leq t}$  satisfies the deterministic dynamics

$$\dot{x}_s^\omega + \partial \chi_G(x_s^\omega) \ni b_s(x_s^\omega) + \omega_s, \quad (1.61)$$

for a  $L^2$ -control parameter  $(\omega_s)_{0 \leq s \leq t}$ . The curve  $(y_s)_{0 \leq s \leq t}$  stands for a fixed  $C^1$  realisation of  $(Y_s^\varepsilon)_{0 \leq s \leq t}$ , see Section 1.2.3.2 for a justification of this procedure. The function  $V$  is introduced as the *cost-to-come* function in Section 1.1.3, its minimiser (if well-defined) being the *Mortensen observer* Mortensen [1968]. In the setting of Section 1.2.3.2, the study of the conditional density is brought back to the one of the solution  $\tilde{q}^\varepsilon$  of

$$\begin{cases} \partial_t \tilde{q}^\varepsilon(t, x) = \nabla \cdot [-\tilde{q}^\varepsilon(t, x) b_t(x) + \frac{\varepsilon}{2} \nabla \tilde{q}^\varepsilon(t, x)] - \frac{1}{2\varepsilon} |\dot{y}(t) - h_t(x)|^2 \tilde{q}^\varepsilon(t, x), & x \in G \\ -\tilde{q}^\varepsilon(t, x) b_t(x) \cdot n(x) + \frac{\varepsilon}{2} \frac{\partial \tilde{q}^\varepsilon}{\partial n}(t, x) = 0, & x \in \partial G, \end{cases} \quad (1.62)$$

where  $n(x)$  is the outward normal vector at  $x \in \partial G$ . When the differential inclusion (1.59) is replaced by an equality, there is no more boundary condition and  $-\varepsilon \log \tilde{q}^\varepsilon$  is known to converge towards  $V$  uniformly on compact sets James and Baras [1988]; Fleming [1997], as stated in Theorem 1.2.5.

The purpose of Chapter 2 is to address the same question for the reflected dynamics (1.59) and to establish large deviation results. Contrary to the equality setting, the  $\varepsilon \rightarrow 0$  limit of (1.59) has deep implications. For instance,  $X_t^\varepsilon$  always spends a Lebesgue-negligible time on  $\partial G$  when  $\varepsilon > 0$ , whereas the  $\varepsilon = 0$  limit can stay *stuck at the boundary* and evolve on it. *Reversibility effects* are also involved in the definition of  $V$ , since two curves following (1.61) can be equal at time  $t$  without having the same initial condition. In particular, (1.61) is *not well-posed when imposing a terminal condition*: the initial data must be specified instead.

This study was initiated during the summer school CEMRACS 2022, yielding the article Chaintron et al. [2023b] in collaboration with Alvaro Gonzales, Laurent Mertz and Philippe Moireau. This article studied the small noise limit for the filtering problem (1.59) in the 1D simplified setting  $b \equiv 0$  and  $G = \mathbb{R}_{\geq 0}$ . Chapter 2 tackles the general setting (1.59),  $G$  being a convex open domain which is smooth and bounded. This work is a collaboration with Laurent Mertz, Philippe Moireau and Hasnaa Zidani. Our first result reveals the delicate nature of the boundary conditions for  $V$ .

**Theorem 1.4.1** (Viscosity solution). *Let  $V$  be defined by (1.60).*

(i)  *$V$  is a viscosity sub-solution of the HJB equation*

$$\begin{cases} \partial_t V(t, x) + b_t(x) \cdot \nabla V(t, x) + \frac{1}{2} |\nabla V(t, x)|^2 - \frac{1}{2} |\dot{y}(t) - h_t(x)|^2 = 0, & x \in G, \\ b(t, x) \cdot n(x) + \frac{\partial V}{\partial n}(t, x) = 0, & x \in \partial G. \end{cases}$$

(ii)  *$V$  is a viscosity super-solution of the HJB equation*

$$\begin{cases} \partial_t V(t, x) + b_t(x) \cdot \nabla V(t, x) + \frac{1}{2} |\nabla V(t, x)|^2 - \frac{1}{2} |\dot{y}(t) - h_t(x)|^2 = 0, & x \in G, \\ b(t, x) \cdot n(x) + \frac{1}{2} \frac{\partial V}{\partial n}(t, x) = 0, & x \in \partial G. \end{cases}$$

In general, the two boundary conditions do not coincide. However, if  $b \cdot n \leq 0$  on  $\partial G$ , we can check that any viscosity sub-solution of the HJB equation in (i) is also a viscosity sub-solution of the HJB equation in (ii). This HJB equation then has a unique viscosity solution from [Barles and Lions, 1991, Theorem 3]. The case  $b \cdot n \leq 0$  corresponds to an initial model  $\dot{x}_t = b(x_t)$  that leaves the domain  $G$  stable without needing the reflection at the boundary. If  $b \cdot n \leq 0$  does not hold, Theorem 1.4.1 has far-reaching consequences since *it prevents the uniform convergence of  $-\varepsilon \log \tilde{q}^\varepsilon$  towards  $V$ , contrary to the standard case of differential equality*. Indeed, the vanishing viscosity limit in the log-transform of (1.62) would imply that the equations (i) and (ii) coincide in Theorem 1.4.1: this is not true in general. However, we still managed to prove the targeted result of large deviations for the conditional law.

**Theorem 1.4.2** (Laplace principle). *If*

$$\sup_{x \in \overline{G}} |-\varepsilon \log \tilde{q}^\varepsilon(0, x) - J(x)| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

then for every continuous  $\Phi : \overline{G} \rightarrow \mathbb{R}$ ,

$$\forall t \geq 0, \quad -\varepsilon \log \int_G e^{-\Phi(x)/\varepsilon} \tilde{q}^\varepsilon(t, x) dx \xrightarrow{\varepsilon \rightarrow 0} \inf_{x \in \overline{G}} \Phi(x) + V(t, x).$$

As  $\varepsilon \rightarrow 0$ , this tells us that the conditional density concentrates on the minimisers of  $x \mapsto V(t, x)$ , recovering the Mortensen observer from Section 1.1.3. Chapter 2 further contains an approximation result of  $V$  by the cost-to-come function obtained by replacing (1.61) by the penalised dynamics

$$\dot{x}_t^{\kappa, \omega} + \kappa [x_t^{\kappa, \omega} - \pi_{\overline{G}}(x_t^{\kappa, \omega})] = b_t(x_t^{\kappa, \omega}) + \omega_t,$$

where  $\pi_{\overline{G}}$  denotes the orthogonal projection onto  $\overline{G}$ . The uniform rate of convergence  $\kappa^{-1/4}$  is obtained for the cost-to-come as  $\kappa \rightarrow +\infty$ . A discretisation strategy and numerical illustrations are further given at the end of Chapter 2.

#### 1.4.2 Quasi-continuity method for mean-field diffusions: large deviations and central limit theorem

The content of Chapter 3 is a preprint proving large deviation results, which were used in Section 1.3.3. Let us consider a mean-field system of interacting diffusions in  $\mathbb{R}^d$ ,

$$dX_t^{i, N} = b_t(X^{i, N}, \pi(\vec{X}^N))dt + \sigma_t(X^{i, N}, \pi(\vec{X}^N))dB_t^i, \quad 1 \leq i \leq N, \quad (1.63)$$

starting from i.i.d. initial conditions  $X_0^i$ , the  $B^i := (B_t^i)_{0 \leq t \leq T}$  being i.i.d. Brownian motions in  $\mathbb{R}^d$ . These particles interact through their empirical measure

$$\pi(\vec{X}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{X^{i, N}} \in \mathcal{P}(C([0, T], \mathbb{R}^d)),$$

using the notation  $\vec{X}^N := (X^{i, N})_{1 \leq i \leq N}$ . The coefficients  $b$  and  $\sigma$  are globally Lipschitz functions which are path-dependent in a non-anticipative way and  $\sigma$  is globally bounded but possibly degenerate. As  $N \rightarrow +\infty$ ,  $\pi(\vec{X}^N)$  converges to the path-law  $\mathcal{L}(\overline{X})$  of the McKean-Vlasov equation

$$d\overline{X}_t = b_t(\overline{X}, \mathcal{L}(\overline{X}))dt + \sigma_t(\overline{X}, \mathcal{L}(\overline{X}))dB_t^1, \quad X_0 \sim X_0^1,$$

such a mean-field limit being known as a *propagation of chaos* result Sznitman [1991]; Chaintron and Diez [2022a, b].

When the coefficients are no more path-dependent, the Dean-Kasawaki stochastic PDE (1.49) formally reads

$$d\pi(\vec{X}_t^N) = L_{\pi(\vec{X}_t^N)}^* \pi(\vec{X}_t^N)dt + N^{-1/2}dM_t,$$

for a suitable measure-dependent generator  $L_{\pi(\vec{X}_t^N)} \pi(\vec{X}_t^N)$ , as detailed in Section 1.3.3. This provides a striking analogy with the well-known Freidlin-Wentzell dynamics (1.26),  $X_t^\varepsilon$  being replaced by the empirical measure  $\pi(\vec{X}_t^N)$ . It is tempting to go beyond analogies by using this structure within proofs. Indeed, using the contraction principle when  $\sigma \equiv \text{Id}$ ,

the Freidlin-Wentzell theory classically computes the large deviations of  $(X_t^\varepsilon)_{0 \leq t \leq T}$  by taking the continuous image of the large deviations of  $(\sqrt{\varepsilon}B_t)_{0 \leq t \leq T}$ .

When  $\sigma \equiv \text{Id}$ , a pathwise construction due to Tanaka [1984] builds  $\bar{X}$  and the  $X^{i,N}$  in a unified framework. In particular, the empirical measure  $\pi(\bar{X}^N)$  of the particles is a continuous function of the one of the driving noises  $\pi(\bar{B}^N)$ . This powerful approach rephrases the mean-field limit as a mere continuity result and allows for computing large deviations by contraction as in the Freidlin-Wentzell approach.

When  $\sigma \neq \text{Id}$ , the contraction approach cannot be directly used for  $(X_t^\varepsilon)_{0 \leq t \leq T}$  because the stochastic integral is not a continuous function of the Brownian motion – we refer to the introduction of Chapter 3 for alternatives using rough path theory. However, a solution was proposed by Azencott et al. [1980] known as the *quasi-continuity method*, the main idea being to approximate the noise by a smoother path and to show that the resulting error is negligible at the large deviation scale. The objectives of Chapter 3 are three-fold:

- We extend Tanaka’s construction to non-constant  $\sigma$  using an ***Euler scheme for (1.63) for which we prove strong consistency estimates***: this provides a quasi-continuity method in the setting (1.63) of mean-field diffusions.
- We compute the first-order fluctuations of  $\pi(\bar{X}^N)$  by proving a ***pathwise central limit theorem*** (CLT), i.e. the convergence of  $N^{-1/2}[\pi(\bar{X}^N) - \mathcal{L}(\bar{X})]$  towards a Gaussian field. The proof relies on a first-order expansion of the path-wise construction together with uniform fluctuation estimates on the Euler scheme.
- We prove a ***large deviation principle*** (LDP), estimating  $\mathbb{P}(\pi(\bar{X}^N) \in A)$  at the exponential scale for any measurable  $A \subset \mathcal{P}(C([0, T], \mathbb{R}^d))$ . The proof relies on the contraction principle and uniform estimates at the exponential scale on the Euler scheme.

As a by-product of our path-wise construction, we prove mean-field limits, central limit theorems and large deviations for discretised systems with *general driving noises*, which may not be Brownian. *Exchangeability is not required* either. These results are already useful as such, the practitioner being interested in the behaviour of the numerically implemented model, rather than the theoretical one.

In the Brownian case, we extend these results to the continuum by proving uniform fluctuation estimates. Following the weak convergence approach Dupuis and Ellis [2011], the exponential approximation relies on a stochastic control interpretation of exponential moments. Thus, our estimates also have an interest from a mean-field control perspective. They provide a *uniform-in- $N$  discrete-time approximation for mean-field control with quadratic cost*. The CLT and LDP results for diffusions are not the first ones of their kind, but they are new at the considered level of generality, see Chapter 3 for a detailed review of existing literature.

### 1.4.3 Constrained dynamics on measures

This section is devoted to the pathwise minimisation problem (1.52):

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]} | \nu_{[0,T]}), \quad (1.64)$$

given some reference measure  $\nu_{[0,T]} \in \mathcal{P}(C([0, T], \mathbb{R}^d))$ . This problem was motivated in Section 1.3.3.1 as finding the most likely configuration of a large system of  $N$  i.i.d. particles given the observation that  $\Psi(\pi(\bar{X}_t^N)) \leq 0$  for  $t \in [0, T]$ . We are mainly concerned with two questions:

- What are the regularity properties of the corrected dynamics?
- Is the corrected dynamics stable w.r.t. perturbations of  $\Psi$ ?

The regularity question is natural for practical purposes and numerical applications. It is reasonable to expect a regular corrected dynamics when starting from a regular  $\nu_{[0,T]}$ . In all physical experiments, measurements are affected by uncertainties of different nature. Thus, the stability question is of paramount importance too.

The contents of Chapters 4-5 are two preprints in collaboration with Giovanni Conforti and Julien Reygner. Chapter 4 studies abstract minimisation problems generalising (1.64), including Schrödinger bridges with additional inequality constraints. Sufficient conditions are given for minimisers to be Gibbs measures involving a Lagrange multiplier and a general stability result is proved. Chapter 5 focuses on the case where  $\nu_{[0,T]}$  is the path-law of a diffusion process. Fine regularity properties are proved for the dynamics and the Lagrange multiplier, together with quantitative stability results w.r.t.  $\Psi$ . The results also apply when replacing  $H(\mu_{[0,T]}|\nu_{[0,T]})$  by the mean-field entropy  $H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]}))$ , as motivated in Section 1.3.3.1.

The content of Chapter 6 is the article Chaintron [2023], which was recently accepted in *Annales de la faculté des sciences de Toulouse*. This work proves gradient estimates for unbounded solutions of viscous Hamilton-Jacobi PDEs using viscosity solution methods. These estimates are a main ingredient of Chapters 4-5. They have an independent interest in stochastic control, allowing for lighter regularity assumptions on coefficients which are natural in a probabilistic setting.

### 1.4.3.1 Gibbs principle with infinitely many constraints: optimality conditions and stability

Let  $E$  be a Polish space and  $\phi : E \rightarrow \mathbb{R}_{\geq 0}$  be continuous. We define the set

$$\mathcal{P}_\phi(E) := \{\mu \in \mathcal{P}(E), \int_E \phi d\mu < +\infty\},$$

which we endow with the topology of weak convergence plus the convergence of moments against  $\phi$ . Let us fix a reference measure  $\nu \in \mathcal{P}_\phi(E)$ . Let  $\mathcal{F} : \mathcal{P}_\phi(E) \rightarrow (-\infty, +\infty]$  be lower semi-continuous and

$$\mathcal{D}_\mathcal{I} := \{\mu \in \mathcal{P}_\phi(E), H(\mu|\nu) + \mathcal{F}(\mu) < +\infty\}.$$

Let  $(\zeta_s)_{s \in \mathcal{S}}$  be any family of continuous functions  $E \rightarrow \mathbb{R}$ , such that

$$\forall s \in \mathcal{S}, \forall x \in E, \quad |\zeta_s(x)| \leq C_s[1 + \phi(x)],$$

for some constant  $C_s > 0$ . Let  $(\Psi_t)_{t \in \mathcal{T}}$  be a family of lower semi-continuous functions  $\mathcal{P}_\phi(E) \rightarrow \mathbb{R}$ ,  $\mathcal{T}$  being a compact topological space. We define the (closed) constrained sets

$$\mathcal{A}^\zeta := \{\mu \in \mathcal{P}_\phi(E), \forall s \in \mathcal{S}, \int_E \zeta_s d\mu = 0\}, \quad \mathcal{A}_\Psi := \{\mu \in \mathcal{P}_\phi(E), \forall t \in \mathcal{T}, \Psi_t(\mu) \leq 0\}.$$

A main result of Chapter 4 is the following one, in the spirit of Csiszár [1975, 1984]; Léonard [2000].

**Theorem 1.4.3** (Gibbs density for minimisers). *Under regularity assumptions detailed in Chapter 4, let  $\bar{\mu}$  be a minimiser for*

$$\inf_{\mu \in \mathcal{A}^\zeta \cap \mathcal{A}_\Psi} H(\mu|\nu) + \mathcal{F}(\mu),$$

*which satisfies a constraint qualification assumption. The following results hold:*

1. Any  $\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  is absolutely continuous w.r.t.  $\bar{\mu}$ .
2. There exist  $(\bar{\zeta}, \bar{\lambda}) \in L^1(E, d\bar{\mu}) \times \mathcal{M}_+(\mathcal{T})$  and a measurable set  $\bar{S} \subset E$  such that  $\bar{\mu}(\bar{S}) = 1$ ,

$$\frac{d\bar{\mu}}{d\nu}(x) = \frac{1}{\bar{Z}} \mathbb{1}_{\bar{S}}(x) \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right] \quad (1.65)$$

with respect to  $\nu$ ,  $\bar{Z}$  being a finite normalising constant.

3. The above  $\bar{\zeta}$  belongs to the closure of  $\text{Span}(\zeta_s, s \in \mathcal{S})$  in  $L^1(E, d\bar{\mu})$ , hence  $\int_E \bar{\zeta} d\bar{\mu} = 0$ , and the complementary slackness condition is satisfied:

$$\Psi_t(\bar{\mu}) = 0 \quad \text{for } \bar{\lambda}\text{-a.e. } t \in \mathcal{T}. \quad (1.66)$$

The above statement uses the formalism of linear derivatives introduced in Section 1.3. The proof of Theorem 1.4.3 relies on a suitable linearisation procedure and the Hahn-Banach theorem. A conditional LDP is proved as well in Chapter 4. When  $\mathcal{F}$  and  $\Psi$  are convex, uniqueness holds for  $\bar{\mu}$  and (1.65)-(1.66) turn out to be sufficient conditions to characterise  $\bar{\mu}$ . Theorem 1.4.3 includes (1.64) as the particular case  $E = C([0, T], \mathbb{R}^d)$ ,  $\mathcal{S} = \emptyset$  and  $\Psi_t(\mu_{[0, T]}) = \Psi(\mu_t)$ . It is noteworthy that this correction procedure induces non-trivial behaviours, even when  $\nu_{[0, T]}$  is a simple Gaussian process, see examples in Chapter 4. If  $E$  is a product space like  $G \times F$  or  $C([0, T], \mathbb{R}^d)$ , the equality constraints allow us to impose as many marginal laws as we wish. In particular, *this includes the Schrödinger bridge problem* introduced in Section 1.3.3.1 as well as constrained versions of it.

A *stability result* for  $\bar{\mu}$  is proved in Chapter 4 when perturbing  $(\nu, \mathcal{F}, \Psi)$ . Other technical results are also proved, paving the way for Chapter 5.

#### 1.4.3.2 Regularity and stability for the Gibbs conditioning principle on path space via McKean-Vlasov control

Let us now describe our results on (1.64). Let  $\Gamma(\mu_{[0, T]})$  denote the path-law of the solution to

$$dX_t = b(X_t, \mu_t)dt + \sigma(X_t)dB_t, \quad X_0 \sim \nu_0,$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion and the coefficients are continuous functions  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ . The generalisation of (1.64) we are interested in is

$$\inf_{\substack{\mu_{[0, T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)) \\ \forall t \in [0, T], \Psi(\mu_t) \leq 0}} H(\mu_{[0, T]} | \Gamma(\mu_{[0, T]})),$$

where  $\Psi : \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$  is lower semi-continuous. Let us assume that for every  $\alpha > 0$ ,  $x \mapsto e^{\alpha|x|}$  is  $\nu_0$ -integrable. For simplifying the presentation, we restrict ourselves to

$$b(x, \mu) = b(x), \quad \Psi(\mu) = \int_{\mathbb{R}^d} \psi d\mu,$$

so that  $\Gamma(\mu_{[0, T]}) =: \nu_{[0, T]}$  does not depend on  $\mu_{[0, T]}$ , recovering (1.64). However, we emphasise that *the above restriction is not made in Chapter 5*. We further assume that  $b, \sigma, \psi, \nabla \psi$  are globally Lipschitz, and that  $\sigma$  is bounded and invertible with bounded  $\sigma^{-1}$ .



**Theorem 1.4.4** (Structure of optimisers). *Let us assume that  $\bar{\mu}_{[0,T]}$  is an optimal measure satisfying the constraint qualification  $\nabla\psi \neq 0$ ,  $\bar{\mu}_t$ -a.s. Then, there exists a Radon measure  $\bar{\lambda} \in \mathcal{M}_+([0, T])$ ,  $\bar{\varphi} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\bar{Z} > 0$  such that  $\bar{\mu}_0(dx) = \bar{Z}^{-1} e^{-\bar{\varphi}_0(x)} \nu_0(dx)$  and*

$$\begin{cases} \partial_t \bar{\mu}_t - \nabla \cdot [-\bar{\mu}_t b + \bar{\mu}_t \sigma \sigma^\top \nabla \bar{\varphi}_t + \frac{1}{2} \nabla \cdot [\bar{\mu}_t \sigma \sigma^\top]] = 0, \\ -\bar{\varphi}_t + \int_t^T b \cdot \nabla \bar{\varphi}_s - \frac{1}{2} |\sigma^\top \nabla \bar{\varphi}_s|^2 + \frac{1}{2} \text{Tr}[\sigma \sigma^\top \nabla^2 \bar{\varphi}_s] ds + \int_{[t, T]} \psi \bar{\lambda}(ds) = 0. \end{cases} \quad (1.67)$$

Furthermore, the complementary slackness condition holds,

$$\int_{\mathbb{R}^d} \psi d\bar{\mu}_t = 0, \quad \text{for } \bar{\lambda}\text{-a.e. } t \in [0, T],$$

and we have the pathwise representation

$$\frac{d\bar{\mu}_{[0, T]}}{d\nu_{[0, T]}}(x_{[0, T]}) = \bar{Z}^{-1} \exp \left[ - \int_0^T \psi(x_t) \bar{\lambda}(dt) \right].$$

**Theorem 1.4.5** (Density for the multiplier). *In the setting of Theorem 1.4.4, there exists  $(\bar{\lambda}, \bar{\varphi})$  satisfying (1.67) such that  $x \mapsto \nabla \bar{\varphi}_t(x)$  is Lipschitz-continuous and*

$$\bar{\lambda}(dt) = \bar{\lambda}_0 \delta_0(dt) + \bar{\lambda}_t dt + \bar{\lambda}_T \delta_T(dt),$$

for  $t \mapsto \bar{\lambda}_t$  in  $L^\infty(0, T)$ , if we assume that  $\nu_0$  has a differentiable density (still denoted  $\nu_0$ ) w.r.t. the Lebesgue measure, and that  $\varepsilon_0 > 0$  exists such that **either (i) or (ii)** holds, where:

$$(i) \quad \nabla^2 \psi \text{ is Lipschitz and } \int_{\mathbb{R}^d} [(\log \nu_0)^{1+\varepsilon_0} + |\nabla \log \nu_0|^{2+\varepsilon_0}] d\nu_0 < +\infty.$$

$$(ii) \quad \nabla b, \nabla \sigma, \nabla^2 \sigma \text{ are Lipschitz and } \int_{\mathbb{R}^d} [(\log \nu_0)^{1+\varepsilon_0} + |\nabla \log \nu_0|^{4+\varepsilon_0} + |\nabla^2 \log \nu_0|^{2+\varepsilon_0}] d\nu_0 < +\infty.$$

If (ii) holds,  $x \mapsto \nabla^2 \bar{\varphi}_t(x)$  is Lipschitz-continuous and the densities of time-marginals satisfy

$$\left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^d} [\log \bar{\mu}_t + |\nabla \log \bar{\mu}_t|^4 + |\nabla^2 \log \bar{\mu}_t|^2] d\bar{\mu}_t \right] + \int_0^T \int_{\mathbb{R}^d} |\nabla^3 \log \bar{\mu}_t|^2 d\bar{\mu}_t dt < +\infty. \quad (1.68)$$

The results on the density of  $\bar{\lambda}$  can be seen as a **regularity trade-off between the constraint and the coefficients**. They rely on the new technical estimate (1.68) for diffusions, together with the analogous of estimates in Section 1.1.2.3. The proof of Theorem 1.4.5 further shows that  $t \in (0, T] \mapsto \nabla \varphi_t(x)$  is continuous and  $t \in [0, T] \mapsto \int_{\mathbb{R}^d} \psi \bar{\mu}_t$  is  $C^1$  under (i), whereas  $t \in (0, T] \mapsto \nabla^2 \varphi_t(x)$  is continuous under (ii). Depending on where the regularity is assumed, on the constraints or on the coefficients, we thus get more smoothness on either the constrained curve or the optimal control. Requiring more derivatives on coefficients would improve the regularity in  $(t, x)$ .

**Theorem 1.4.6** (Quantitative stability). *In the setting of Theorem 1.4.4, we assume that (ii) holds, that  $\psi$  is  $C^4$  with bounded derivatives and that  $\int_{\mathbb{R}^d} e^{\varepsilon_0 |x|^2} \nu_0(dx) < +\infty$ . For  $\varepsilon \geq 0$ , let  $\mu_{[0, T]}^\varepsilon$  denote the unique minimiser for*

$$\inf_{\substack{\mu_{[0, T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)) \\ \forall t \in [0, T], \Psi(\mu_t) \leq \varepsilon}} H(\mu_{[0, T]} | \nu_{[0, T]}).$$

Let  $(\varphi_t^\varepsilon, \lambda_t^\varepsilon, \lambda_0^\varepsilon, \lambda_T^\varepsilon)$  be related to  $\mu_{[0, T]}^\varepsilon$  by Theorems 1.4.4-1.4.5. We assume that the number of non-trivial intervals where  $\int_{\mathbb{R}^d} \psi d\mu_t^0$  is identically zero is finite. Then, there exists  $C > 0$  independent of  $\varepsilon$  such that uniformly in  $\varepsilon \in [0, 1]$ ,

(i) Entropic stability:  $H(\mu_{[0,T]}^\varepsilon | \mu_{[0,T]}^0) \leq C\varepsilon$ .

(ii) Multiplier stability:  $|\lambda_0^\varepsilon - \lambda_0^0| + \|\lambda^0 - \lambda^\varepsilon\|_{L^1(0,T)} + |\lambda_T^\varepsilon - \lambda_T^0| \leq C\varepsilon^{1/4}$ .

(iii) Control stability:  $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |\nabla \varphi_t^\varepsilon(x) - \nabla \varphi_t^0(x)| \leq C\varepsilon^{1/4}$ .

The Gaussian bound is only used to simplify the convergence rate (otherwise, additional  $\log \varepsilon$  factors would have been needed). Similarly, the assumption on the finite number of intervals can be removed if we deteriorate the rate, see Chapter 5.

### 1.4.3.3 Existence and global Lipschitz estimates for unbounded classical solutions of a Hamilton-Jacobi equation

The main objective of Chapter 6 is the following result.

**Theorem 1.4.7.** *If  $b$ ,  $\sigma$  and  $f$  are Lipschitz-continuous functions, with  $\sigma$  bounded and  $\sigma\sigma^\top$  uniformly elliptic, then the HJB equation*

$$\partial_t u(t, x) + b \cdot \nabla u(t, x) + \frac{1}{2} \text{Tr}[\sigma\sigma^\top \nabla^2 u(t, x)] - \frac{1}{2} |\sigma^\top \nabla u(t, x)|^2 + f(t, x) = 0, \quad (1.69)$$

with globally Lipschitz terminal data  $u(T, x) = g(x)$ , has a unique linear growth  $C^{1,2}$  solution  $u$ , which has a uniformly bounded gradient.

The need for classical solutions of (1.69) is a recurrent issue in stochastic control, as detailed in the introduction of Chapter 6. In particular, Theorem (1.4.7) is used in Chapters 4-5 for studying (1.67). If we assume that

$$\forall (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \quad |f(t, x) - f(t, y)| \leq \lambda(t)|x - y|, \quad \lambda \in L^1(0, T),$$

it is proved in Chapter 6 that the bound on  $\nabla u$  only depends on  $f$  through  $\|\lambda\|_{L^1(0,T)}$ . This fact is crucial in (1.67) since we only have a priori bounds on the total mass  $\bar{\lambda}([0, T])$  of the multiplier.

The Lipschitz assumptions in Theorem 1.4.7 are natural in a probabilistic setting, the global bound on  $\sigma$  being required for the solution to be Lipschitz. Surprisingly, it seems that this useful result had not been stated before in the literature. The main difficulty compared to standard settings is that  $\sigma \neq \text{Id}$ , implying that the  $x$ -differential of the Hamiltonian does not have a linear growth w.r.t.  $\nabla u$ . A second major task is the long literature review in the introduction of Chapter 6. The proof combines several tricks from the viscosity solution theory to obtain this result at the edges of what the setting allows for.

## 1.4.4 Perspectives

In the light of Sections 1.3-1.4, let us now describe some possible perspectives and research axes for future works.

**Quantitative convergence.** Section 1.4.3.2 has proved many regularity properties on the minimiser  $\bar{\mu}_{[0,T]}$  when  $\nu_{[0,T]}$  is the law of the diffusion process (1.50). From the large deviation results proved in Chapter 4,  $\bar{\mu}_{[0,T]}$  is the limit of the empirical measure  $\pi(\bar{X}_{[0,T]}^N)$  of i.i.d.  $\nu_{[0,T]}$ -distributed particles conditionally on

$$\forall t \in [0, T], \quad \Psi(\pi(\bar{X}_t^N)) \leq 0. \quad (1.70)$$

It is then natural to look for a quantitative rate of convergence for the conditioned particles. For some quantitative versions of the general Gibbs principle, we refer to Dembo and Zeitouni [1996]; Dembo and Kuelbs [1998]; Cattiaux and Gozlan [2007].

Let us introduce the controlled  $N$ -particle system

$$dY_t^{\alpha, i, N} = b(Y_t^{\alpha, i, N})dt + \sigma(Y_t^{\alpha, i, N})\alpha_t^{i, N}dt + \sigma(Y_t^{\alpha, i, N})dB_t^i, \quad 1 \leq i \leq N,$$

the control process being  $\bar{\alpha}_t^N := (\alpha_t^{i, N})_{1 \leq i \leq N}$ . From Example 1.3.7 and the control interpretation in Section 1.3.3.2, the path-law of  $\bar{X}_{[0, T]}^N$  conditionally on (1.70) is the law of the optimally controlled process for  $V^N(0, \bar{x}^N)$ , where

$$V^N(t, \bar{x}^N) := \inf_{\substack{(\bar{\alpha}_s^N)_{t \leq s \leq T}, \bar{Y}_t^{\alpha, N} = \bar{x}^N \\ \forall s \in [t, T], \Psi(\pi(\bar{Y}_s^{\alpha, N})) \leq 0 \text{ a.s.}}} \mathbb{E} \int_t^T \frac{1}{2N} |\bar{\alpha}_s^N|^2 ds.$$

Moreover,  $V^N(t, \bar{x}^N) = -\log \mathbb{P}(\tau_{\bar{\Psi}}^{\bar{x}^N} < T - t)$  following Example 1.3.7, where the stopping time  $\tau_{\bar{\Psi}}^{\bar{x}^N}$  is the first time  $s$  at which  $\Psi(\pi(\bar{X}_s^N)) \geq 0$  when  $\bar{X}^N$  starts at  $\bar{x}^N$ . As  $N \rightarrow +\infty$  and  $\pi(\bar{x}^N) \rightarrow \mu$ ,  $V^N(t, \bar{x}^N)$  converges at each  $t \in [0, T]$  towards the value function

$$V(t, \mu) := \inf_{\substack{(\alpha_s)_{t \leq s \leq T}, X_t^\alpha \sim \mu \\ \forall s \in [t, T], \Psi(\mathcal{L}(X_s^\alpha)) \leq 0}} \mathbb{E} \int_t^T \frac{1}{2} |\alpha_s|^2 ds,$$

using the notations from Section 1.3.3.2. This convergence result is proved in Daudin [2023a]. The function  $V$  is expected to be a viscosity solution of

$$\partial_t V(t, \mu) - \frac{1}{2} \int_{\mathbb{R}^d} |D_\mu V(t, \mu, x)|^2 d\mu(x) + \frac{1}{2} \int_{\mathbb{R}^d} [\nabla \cdot D_\mu V(t, \mu, x)] d\mu(x) = 0,$$

in the spirit of Daudin and Seeger [2024]; Conforti et al. [2024], with some additional boundary condition to be determined. This equation can be formally obtained as the limit of the HJB equation satisfied by  $V^N$  which can be deduced from Example 1.3.7. Quantifying the convergence of  $V^N$  towards  $V$  can be linked to the regularity properties of this equation and the related *master equation*, see Cardaliaguet et al. [2019, 2023]; Daudin et al. [2023].

**Long-time behaviour.** Another natural question is the study of (1.64) as  $T \rightarrow +\infty$ . In the control formulation (1.53), we conjecture that the value function normalised by  $T$  converges towards the value function

$$\eta := \inf_{\alpha, \forall t \geq 0, \Psi(\mathcal{L}(X_t^\alpha)) \leq 0} \limsup_{T \rightarrow +\infty} \frac{1}{2T} \mathbb{E} \int_0^T |\alpha_t|^2 dt,$$

corresponding to an ergodic control problem, by analogy with the finite-dimensional setting Lasry and Lions [1989]. When  $\Psi(\mu) = \int_{\mathbb{R}^d} \psi d\mu$ , we further conjecture the existence of a stationary solution with law  $\mu_\infty$  corresponding to the feedback control process  $\alpha_t = -\sigma^\top \nabla u(X_t^\alpha)$ ,  $u$  being the solution of the stationary HJB equation

$$-\eta + b \cdot \nabla u + \frac{1}{2} \text{Tr}[\sigma \sigma^\top \nabla^2 u] - \frac{1}{2} |\sigma^\top \nabla u|^2 = -\lambda \psi,$$

for a Lagrange multiplier  $\lambda \geq 0$ . The probability density  $\varphi := Z_\varphi^{-1} e^{-u}$  then satisfies

$$b \cdot \nabla \varphi + \frac{1}{2} \text{Tr}[\sigma \sigma^\top \nabla^2 \varphi] = -\eta \varphi + \lambda \psi,$$

which corresponds to a “twisted eigenvalue problem”. Introducing the dual (normalised) eigenfunction  $\rho$  such that

$$\nabla \cdot [-\rho b \cdot \nabla \varphi + \frac{1}{2} \nabla \cdot [\rho \sigma \sigma^\top]] = -\eta \rho + \lambda \rho \psi,$$

we eventually conjecture that  $\rho \geq 0$  and  $\mu_\infty$  has a density proportional to  $\varphi \rho$  w.r.t. the Lebesgue measure. For support constraints as in Example 1.3.7, boundary conditions are needed and the above conjectures correspond to known results on *Q-processes* and *quasi-stationary distributions* Pinsky [1985]; Budhiraja et al. [2022].

**Particle approximation.** Section 1.4.3 has shed lights on many properties of the corrected measure  $\bar{\mu}_{[0,T]}$  obtained from (1.64), but it does not tell us how to numerically simulate  $\bar{\mu}_{[0,T]}$ . In our mean-field setting, it is natural to look for a particle system whose empirical measure approximates  $\bar{\mu}_{[0,T]}$ . For the sake of simplicity, we assume that  $\Psi(\mu) = \int_{\mathbb{R}^d} \psi d\mu$  is linear. Let us introduce the solution  $\bar{\mu}_{[0,T]}^\kappa$  to the relaxed problem

$$\inf_{\mu_{[0,T]}} H(\mu_{[0,T]} | \nu_{[0,T]}) + \int_0^T \frac{\kappa |\Psi(\mu_t)|_+^2}{2} dt.$$

As  $\kappa \rightarrow +\infty$ ,  $\bar{\mu}_{[0,T]}^\kappa$  is expected to converge towards  $\bar{\mu}_{[0,T]}$ . Linearising at  $\bar{\mu}_{[0,T]}^\kappa$  yields

$$\inf_{\mu_{[0,T]}} H(\mu_{[0,T]} | \nu_{[0,T]}) + \int_0^T \kappa |\Psi(\bar{\mu}_t^\kappa)|_+ \left[ \int_{\mathbb{R}^d} \psi d\mu_t \right] dt,$$

the minimiser being  $\bar{\mu}_{[0,T]}^\kappa$  which can be written as a Gibbs measure.

Let us now introduce a system of  $N$  particles  $(X^1, \dots, X^N)$ , each particle being equipped with a Poisson death clock and a Poisson birth clock. The evolution is the following one:

- Between two clock rings, each particle evolves independently of the others like the Markov process with law  $\nu_{[0,T]}$ .
- For every  $1 \leq i \leq N$ , the death clock of  $X_t^i$  rings at rate  $\kappa |\Psi(\pi(\vec{X}_t^N))|_+ |\psi(X_t^i)|_+$ . If the death clock of  $X_t^i$  rings at time  $t$ , the particle  $X^i$  is killed and another one  $X^j$  is chosen at random. The *whole history*  $(X_s^j)_{0 \leq s \leq t}$  of  $X^j$  is then copied to produce a new particle  $X^i$  which has the same history as  $X^j$ . From now on,  $X^i$  and  $X^j$  will evolve independently of each other, except for the coupling through birth/death rates.
- For every  $1 \leq i \leq N$ , the birth clock of  $X_t^i$  rings at rate  $\kappa |\Psi(\pi(\vec{X}_t^N))|_+ |\psi(X_t^i)|_-$ . If the birth clock of  $X_t^i$  rings at time  $t$ , another particle  $X^j$  is chosen at random and killed. The *whole history*  $(X_s^i)_{0 \leq s \leq t}$  of  $X^i$  is then copied to produce a new particle  $X^j$  which has the same history as  $X^i$ . From now on,  $X^i$  and  $X^j$  will evolve independently of each other, except for the coupling through birth/death rates.

This defines a mean-field system of branching particles, which interact through their branching rates. This kind of particle system is reminiscent of Fleming-Viot algorithms used to simulate processes conditioned to stay in a bounded domain as in Example 1.3.7, see e.g. Villemonais [2014] and references therein. We conjecture that the pathwise empirical measure  $\pi(\vec{X}_{[0,T]}^N)$  of this system at time  $T$  converges towards  $\bar{\mu}_{[0,T]}^\kappa$  as  $N \rightarrow +\infty$ .

An alternative approach would be to discretise the time and to project iteratively each marginal constraint as in the Sinkhorn algorithm, in the spirit of [Benamou et al., 2019, Section 5].

**Application to cardiac simulation.** This part is motivated by the previous works Chaintron et al. [2023a,c] on stochastic modelling for the cardiac muscle, which were done before starting this thesis. The first article was published in collaboration with Matthieu Caruel and François Kimmig. The second one was published in collaboration with François Kimmig, Matthieu Caruel and Philippe Moireau. The elementary contractile unit responsible for muscle contraction is called *sarcomere*. For detailed biological context and illustrations, we refer to the aforementioned articles and references therein. These works were focusing on stochastic modelling of the *cross-bridges*, which are myosin filaments that attach and detach on actin sites, generating contraction along the way. Assembling the  $N$  cross-bridges generates coupled equations which involve the elongation  $Y_t$  of the sarcomere as well as the ones  $X_t^{i,N}$  of the cross-bridges,  $1 \leq i \leq N$ . We refer to [Caruel, 2011, Chapter 2.5] for mechanical modelling of the sarcomere and to [Caruel, 2011, Chapter 6] for the derivation of these equations. When considering  $N$  cross-bridges connected to an external load  $F$ , the energy landscape reads

$$W^N(\vec{x}^N, y, f) := \frac{k}{2} \sum_{i=1}^N (x^i - y)^2 - Fy,$$

where  $k$  is a stiffness parameter. To simplify the system without loss of its fundamental properties, it is customary to eliminate the variable  $y$  by assuming the equilibrium condition  $\frac{\partial W^N}{\partial y} = 0 \Leftrightarrow y = (Nk)^{-1}F + N^{-1} \sum_{i=1}^N x^i$ . The resulting equations for the cross-bridges in an environment at temperature  $T$  read, for  $1 \leq i \leq N$ ,

$$dX_t^{i,N} = -\partial_i W^N(\vec{X}_t^N, y, (Nk)^{-1}F + N^{-1} \sum_{j=1}^N X_t^{j,N}) dt + \sqrt{2\eta^{-1}k_B T} dB_t^i,$$

where  $k_B$  is the Boltzmann constant,  $\eta$  is a damping coefficient and the  $(B_t^i)_{t \geq 0}$  are independent Brownian motions. This gives a mean-field system of interacting diffusions of the Curie-Weiss type, as studied in Pra and Hollander [1996]. For numerical simulations, reduced models are needed in the different elongation regimes studied in Caruel and Truskinovsky [2016], when imposing that the reaction coordinate  $N^{-1} \sum_{j=1}^N X_t^{j,N}$  stays either big or small. A possible approach is given by the particle approximation in the previous paragraph.

**Mortensen observer on measures.** Extending the settings of Section 1.1.3 and Example 1.3.7, it is natural to define the Mortensen observer on measures

$$\bar{\nu}_T := \operatorname{argmin}_{\substack{\mu \in \mathcal{P}(\mathbb{R}^d) \\ \Psi(\mu) \leq 0}} \inf_{\substack{\mu_{[0,T]}, \mu_T = \mu \\ \forall t \in [0,T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]} | \nu_{[0,T]}),$$

corresponding to the endpoint of the optimal path-measure  $\bar{\mu}_{[0,T]}$ . This definition is non-ambiguous if e.g.  $\Psi$  is convex. The study of  $\bar{\nu}_T$  is a challenging question. In particular, it would be useful to obtain a closed-form equation for the evolution of  $\bar{\nu}_T$  as in Example 1.3.7.

**Wasserstein sub-manifolds.** Remark 1.3.9 in Section 1.3.3.3 suggested that the flow  $(\bar{\mu}_t)_{0 \leq t \leq T}$  of time-marginals of the minimiser in (1.64) formally satisfies the Newton equation

$$\ddot{\bar{\mu}}_t = \frac{1}{2} \nabla_{\mathcal{N}_{\bar{\Psi}}^-} |D_{\mu} \mathcal{F}(\bar{\mu}_t)|^2,$$

the gradient being taken w.r.t. to the Riemannian distance induced on the sub-manifold  $\mathcal{N}_{\bar{\Psi}}^-$  of  $\mathcal{P}_2(\mathbb{R}^d)$ . The l.h.s. is the covariant derivative in  $\mathcal{N}_{\bar{\Psi}}^-$  of  $\dot{\bar{\mu}}_t$  in the direction  $\dot{\bar{\mu}}_t$ . An interesting challenge would be to give a rigorous meaning to this equation in the spirit of

Conforti [2019]; Gentil et al. [2020], building further tools of Riemannian geometry in this setting of Wasserstein sub-manifold. In particular, as in Remark 1.1.2, the computation of the Lagrange multiplier  $\bar{\lambda}$  in Chapter 5 shall correspond to a notion of second fundamental form of the sub-manifold  $\mathcal{N}_{\Psi}^-$ .

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# Introduction (Français)

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*Cocorico.*  
— Un coq en Armorique

Cette thèse porte sur l'étude de dynamiques contraintes dans différents contextes, en dimension finie et à valeurs dans des espaces de mesures. Les méthodes utilisées combinent des outils issus de la théorie des grandes déviations, du calcul des variations, du contrôle, des systèmes de particules en interaction et du filtrage stochastique. L'objectif principal est de modifier un système dynamique donné pour produire de nouvelles trajectoires satisfaisant certaines contraintes. Parmi les exemples abordés se trouvent le problème du filtrage stochastique, des extensions du principe de Gibbs en mécanique statistique et des versions contraintes du problème du pont de Schrödinger. Au travers de ces exemples, on montre comment les outils classiques pour contraindre des équations différentielles ordinaires (EDOs) s'étendent naturellement aux dynamiques à valeurs mesures.

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Les dynamiques contraintes interviennent dans de nombreuses applications du quotidien. Lorsqu'on modélise un phénomène physique, il est fréquent de mesurer un écart entre la prédiction du modèle et la quantité ciblée. Connaissant cet écart, il devient naturel de chercher une version améliorée du modèle qui prenne en compte la mesure, en imposant cette mesure comme une *contrainte* sur le modèle. Des exemples célèbres proviennent de la mécanique statistique, comme le principe de Gibbs introduit en Section 1.3.3.1, qui recherche la configuration la plus probable d'un système de particules lorsqu'on ne connaît que des quantités moyennes à son sujet. Un autre exemple est donné par le problème du pont de Schrödinger en Section 1.3.3.1, qui cherche l'évolution la plus probable d'un processus stochastique dont la loi n'est connue qu'aux instants initial et terminal. Ce dernier problème connaît des applications florissantes avec l'utilisation récente, notamment en apprentissage automatique, des *modèles de diffusion* Ho et al. [2020]; Song et al. [2020]; Albergo et al. [2023]. Mentionnons également le célèbre problème du filtrage qui recherche la loi conditionnelle d'un système en temps réel à partir d'une mesure incomplète et bruitée. Le cadre de cette thèse inclut de nombreux problèmes d'estimation déterministes et stochastiques. Une motivation supplémentaire provenait des travaux Chaintron et al. [2023a,c] portant sur la modélisation du muscle cardiaque, où les contraintes proviennent directement des conditions de mesure expérimentale (cf. Section 1.4.4 pour plus de détails).

Le Chapitre 1 présente des éléments de motivation et de contexte, en introduisant progressivement certains outils techniques. Une analyse de littérature s'y trouve également. L'objectif principal est de montrer *comment étendre les outils classiques pour contraindre des EDOs à des dynamiques à valeurs mesures*. La Section 1.1 décrit certaines approches déterministes pour modifier des dynamiques en dimension finie et les illustre sur des exemples-jouet. Cette section introduit de nombreux outils de la théorie de l'optimisation et du contrôle, tout en ouvrant la voie pour de nombreuses analogies avec les dynamiques à valeurs mesures, développées en Section 1.3. Cette présentation est fortement inspirée de Gentil et al. [2020]. La Section 1.2 décrit des approches stochastiques qui brisent la dynamique pour pouvoir la conditionner. De nombreux outils issus de la théorie des grandes déviations sont introduits ici, ainsi que le problème du filtrage stochastique. Une des contributions majeures de cette thèse est l'adaptation de ces méthodes aux dynamiques à valeurs mesures, comme présenté en Section 1.3. Le formalisme EDO des Sections 1.1-1.2 autorise de nombreuses analogies avec les dynamiques à valeurs mesures, tant au niveau des énoncés que des preuves. La théorie des grandes déviations induit des connexions entre les problèmes de conditionnement et de minimisation de l'entropie sur l'espace des trajectoires. La théorie de Girsanov décrite en Section 1.3.3.2 transforme ces problèmes de minimisation en problèmes de contrôle stochastique. Les développements récents présentés en Section 1.3.3.3 permettent des interprétations géométriques de ces résultats en utilisant les flots de gradient de Wasserstein et le formalisme des équations de Newton dans l'espace de Wasserstein. Insistons sur le fait que le contenu des Sections 1.1-1.2-1.3 peut être assez formel, le but étant d'introduire les idées-clef dans un cadre parfois non-rigoureux, contrairement aux chapitres suivants.

Un résumé des différentes contributions de cette thèse est donné en Section 1.4, synthétisant les différents chapitres. Ces chapitres sont des prépublications ou des articles, reproduits sans modification. En particulier, une présentation détaillée des motivations et de la littérature existante est disponible en tête de chaque chapitre. Certaines notations peuvent changer d'un chapitre à l'autre, toujours avec une spécification claire de ces changements. Le Chapitre 2 présente des résultats sur le filtrage stochastique de diffusions réfléchies dans la limite petit bruit. Le Chapitre 3 développe une nouvelle méthode pour établir des principes de grandes déviations et théorèmes de la limite centrale pour des diffusions en interaction, permettant une amélioration des résultats connus. Les Chapitres 4-5

contiennent une étude détaillée du principe de Gibbs (et de ses extensions) sur l'espace des chemins avec un nombre infini de contraintes, par des méthodes de contrôle champ-moyen. Un résultat de stabilité quantitative est notamment prouvé. Le Chapitre 6 établit finalement de nouvelles estimées de régularité pour les équations de Hamilton-Jacobi par des méthodes de solutions de viscosité.

## 1.5 Estimation et filtrage à petit bruit pour des dynamiques réfléchies

Le problème du filtrage cherche à estimer l'état d'un système modélisé par une EDS

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t^1, \quad (1.71)$$

à partir d'une **observation incomplète et bruitée**

$$dY_t^\varepsilon = h(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t^2,$$

les mouvements Browniens  $B^1$  et  $B^2$  étant indépendants. Le but du filtrage stochastique est de calculer la loi  $\pi_t^\varepsilon$  de  $X_t^\varepsilon$  conditionnellement aux observations passées  $(Y_s^\varepsilon)_{0 \leq s \leq t}$  : le **filtre**  $\pi_t^\varepsilon \in \mathcal{P}(\mathbb{R}^d)$  est donc une **mesure aléatoire**. Le filtre  $\pi_t^\varepsilon$  peut être calculé en normalisant la mesure positive  $\rho_t^\varepsilon$  solution de l'**équation de Zakai** Zakai [1969]; Pardoux [1980].

Pour calculer la limite  $\varepsilon \rightarrow 0$ , James and Baras [1988]; Fleming [1997] fixent une réalisation  $(y_s)_{0 \leq s \leq t}$  de l'observation, supposée  $C^1$  par simplicité (même si ce point demande justification Davis [1981]; Clark and Crisan [2005]). Après renormalisation de la solution de l'EDP, cette procédure produit une famille de densités de probabilité  $(q_t^\varepsilon)_{\varepsilon > 0}$ , qui est un analogue déterministe de  $(\pi_t^\varepsilon)_{\varepsilon > 0}$  à observation fixée. Sous certaines conditions de régularité, Hijab [1984]; James and Baras [1988] montrent alors que  $(q_t^\varepsilon(x)dx)_{\varepsilon > 0}$  satisfait un **principe de grandes déviations** pour la fonction de taux

$$x \mapsto V(t, x) = \inf_{\substack{(x_s)_{0 \leq s \leq t} \in AC([0, t], \mathbb{R}^d) \\ x_t = x}} J(x_0) + \int_0^t \frac{1}{2} |\dot{x}_s - b(x_s)|^2 + \frac{1}{2} |\dot{y}_s - h(x_s)|^2 ds,$$

où on minimise sur les courbes absolument continues à valeurs dans  $\mathbb{R}^d$ . La pénalisation initiale  $J$  provient du choix des lois initiales  $(q_0^\varepsilon)_{\varepsilon > 0}$ . Ceci définit la **fonction valeur**  $V$  d'un problème de **contrôle progressif en temps**. Si  $V$  est strictement convexe,  $C^2$  et coercive, on montre que  $V$  admet un unique minimiseur  $\hat{x}_t$  qui vérifie

$$\dot{\hat{x}}_t = b(\hat{x}_t) - [\nabla^2 V(t, \hat{x}_t)]^{-1} \nabla h(\hat{x}_t) [\dot{y}_t - h(\hat{x}_t)].$$

Ceci définit un estimateur, ou **observateur**, analogue déterministe de  $\pi_t^\varepsilon$  dû à Mortensen Mortensen [1968]. Cet estimateur présente l'intérêt d'être **calculable récursivement** si  $V$  est connue, et coïncide avec le filtre de Kalman Kalman and Bucy [1961] dans le cas linéaire gaussien. L'approche de James and Baras [1988]; Fleming [1997] consiste à prouver que  $(t, x) \mapsto -\varepsilon \log q_t^\varepsilon(x)$  converge uniformément sur tout compact vers  $(t, x) \mapsto V(t, x)$ , ce qui correspond à une limite de viscosité évanescence dans l'équation de Hamilton-Jacobi satisfaite par  $-\varepsilon \log q_t^\varepsilon$ .

Le Chapitre 2 s'intéresse à l'analogie de cette procédure lorsqu'on remplace (1.71) par la dynamique

$$dX_t^\varepsilon + \partial \chi_G(X_t^\varepsilon)(dt) \ni b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t^1, \quad (1.72)$$



qui est **réfléchie au sens de Skorokhod** sur le bord d'un ouvert convexe, borné et régulier  $G$ . Dans la dynamique ci-dessus,  $\partial\chi_G(X_t^\varepsilon)$  est le **sous-différentiel** de l'indicatrice convexe du domaine  $G$  (cf. Pardoux and Răşcanu [2014] pour l'étude de ce type de dynamiques). La fonction  $V$  dans (1.5) est alors remplacée par

$$W(t, x) := \inf_{\substack{x_0^\omega, \omega \\ x_t^\omega = x}} J(x_0) + \int_0^t \frac{1}{2} |\omega_s|^2 + \frac{1}{2} |\dot{y}_s - h(x_s^\omega)|^2 ds,$$

où  $(x_s^\omega)_{0 \leq s \leq t}$  est la dynamique déterministe contrôlée

$$\dot{x}_s^\omega + \partial\chi_G(x_s^\omega) \ni b(x_s^\omega) + \omega_s,$$

pour  $\omega \in L^2((0, t), \mathbb{R}^d)$ . Avec un léger abus de notation, on continue d'appeler  $q_t^\varepsilon$  la solution renormalisée de l'équation de Zakai pour une réalisation  $C^1$  fixée  $(y_s)_{0 \leq s \leq t}$  de l'observation. Suivant la méthode de James and Baras [1988]; Fleming [1997], il est naturel de calculer la limite  $\varepsilon \rightarrow 0$  en montrant que  $W$  est une **solution de viscosité** de l'équation de Hamilton-Jacobi

$$\partial_t W(t, x) + b(t, x) \cdot \nabla W(t, x) + \frac{1}{2} |\nabla W(t, x)|^2 - \frac{1}{2} |\dot{y}_t - h(x)|^2 = 0, \quad x \in G, \quad (1.73)$$

complétée par une **condition au bord** adéquate, obtenue en prenant la limite de l'équation satisfaite par  $-\varepsilon \log q^\varepsilon$ . L'article Chaintron et al. [2023b] en collaboration avec Alvaro Gonzales, Laurent Mertz et Philippe Moireau, montre la convergence de  $-\varepsilon \log q^\varepsilon$  vers  $W$  dans le cas 1D  $G = \mathbb{R}_+$  et  $b \equiv 0$ , à la suite d'un travail initié lors de l'école d'été CEMRACS 2021. L'équation (1.73) est alors complétée par la condition de Neumann  $\partial_x W(t, 0) = 0$ . Les théorèmes suivants issus du Chapitre 2 traitent le cas général et font partie d'un travail en collaboration avec Laurent Mertz, Philippe Moireau et Hasnaa Zidani.

**Theorem 1.5.1.** *Au sens de Barles and Lions [1991],*

(i)  *$W$  est une sous-solution de viscosité de (1.73) avec la condition au bord*

$$b(t, x) \cdot n(x) + \frac{\partial W}{\partial n}(t, x) = 0, \quad x \in \partial G.$$

(ii)  *$W$  est une sur-solution de viscosité de (1.73) avec la condition différente*

$$b(t, x) \cdot n(x) + \frac{1}{2} \frac{\partial W}{\partial n}(t, x) = 0, \quad x \in \partial G,$$

où  $n(x)$  désigne le vecteur normal sortant en  $x \in \partial G$ .

On peut montrer que ces deux conditions au bord au sens de viscosité coïncident lorsque  $b(x) \cdot n(x) \leq 0$  pour tout  $x \in \partial G$ , et suffisent alors à garantir le principe de comparaison et l'unicité de la solution – on retombe sur une notion classique de **solution de viscosité contrainte** Soner [1986]; Capuzzo-Dolcetta and Lions [1990]. En dehors de ce cas, la **question de l'unicité reste ouverte**, et montrer le principe de comparaison fera l'objet de travaux futurs. En particulier, on ne peut plus espérer une convergence uniforme sur tout compact de  $-\varepsilon \log q^\varepsilon$  vers  $W$ . Le résultat de grandes déviations (1.5) reste cependant vrai.

**Theorem 1.5.2.** *Sous des hypothèse adéquates sur  $(q_0^\varepsilon)_{\varepsilon > 0}$ , pour tout  $t \geq 0$ ,  $(q_t^\varepsilon(x) dx)_{\varepsilon > 0}$  satisfait un principe de grandes déviations pour la fonction de taux  $x \mapsto W(t, x)$ .*

Nous avons également montré un résultat d'**approximation quantitative** lorsqu'on approche (1.72) par une **dynamique pénalisée** du type (1.71). Comprendre finement le défaut de convergence de  $-\varepsilon \log q^\varepsilon$  dans le cas réfléchi fera l'objet de travaux futurs.

## 1.6 Fluctuations et grandes déviations de diffusions en interaction

La notion de propagation du chaos remonte aux travaux fondateurs de Boltzmann en mécanique statistique. Dans Boltzmann [1872], Boltzmann introduisit l'idée qu'il est impossible et inutile de garder trace de toutes les particules (ou plutôt molécules)  $\vec{X}^N := (X^{1,N}, \dots, X^{N,N})$  qui composent un gaz,  $N$  étant considérablement grand, mais qu'une **description statistique** est suffisante. Il est alors naturel d'introduire la densité empirique des particules

$$\pi(\vec{X}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}},$$

et de se demander si elle converge vers un objet limite lorsque  $N \rightarrow +\infty$ . Dans un **système champ-moyen**, les particules sont **échangeables** et n'interagissent entre elles qu'au travers de  $\pi(\vec{X}^N)$ . En particulier, lorsque  $N \rightarrow +\infty$ , les corrélations individuelles entre les particules disparaissent : ce phénomène correspond au **chaos moléculaire** introduit par Boltzmann, hypothèse nécessaire à la dérivation de l'équation qui porte son nom.

Dans le cadre mathématique introduit par Kac [1956]; McKean [1969]; Sznitman [1991], les particules sont des trajectoires  $X^{i,N} = (X_t^{i,N})_{0 \leq t \leq T}$  dans  $C([0, T], \mathbb{R}^d)$ , dont l'évolution est stochastique. Une dynamique récurrente dans ce document est le système

$$dX_t^{i,N} = b_t(X^{i,N}, \pi(\vec{X}^N))dt + \sigma_t(X^{i,N}, \pi(\vec{X}^N))dB_t^{i,N}, \quad 1 \leq i \leq N, \quad (1.74)$$

les  $(B_t^{i,N})_{1 \leq i \leq N}$  étant des mouvements Browniens indépendants. Les coefficients  $b$  et  $\sigma$  sont typiquement Lipschitz, et dépendent des trajectoires de manière **non-anticipative** : par exemple,  $b_t(X^{i,N}, \pi(\vec{X}^N))$  ne dépend de  $X^{i,N}$  qu'au travers de  $(X_s^{i,N})_{0 \leq s \leq t}$ . Dans un cadre similaire, le travail fondateur Sznitman [1991] a montré la convergence faible de  $\pi(\vec{X}^N)$  vers la loi trajectorielle du **processus de McKean-Vlasov** solution de

$$d\bar{X}_t = b_t(\bar{X}, \text{Loi}(\bar{X}))dt + \sigma_t(\bar{X}, \text{Loi}(\bar{X}))dB_t, \quad (1.75)$$

pourvu que la moyenne empirique  $\pi(\vec{X}_0^N)$  des conditions initiales converge dans  $\mathcal{P}(\mathbb{R}^d)$  vers  $\text{Loi}(\bar{X}_0)$ . La convergence de la loi trajectorielle  $\pi(\vec{X}^N)$  vers  $\text{Loi}(\bar{X})$  implique celle des mesures empiriques  $\pi(\vec{X}_t^N)$  vers la loi marginale  $\text{Loi}(\bar{X}_t)$  à tout  $t \in [0, T]$ , d'où le terme de **propagation du chaos** Chaintron and Diez [2022a,b]. Des questions naturelles sont alors les **fluctuations** dans la convergence de (1.74) vers (1.75), ainsi que les **grandes déviations** associées.

Si les coefficient  $b$  et  $\sigma$  dans (1.74) ne dépendent que du présent, un calcul formel montre que la mesure empirique  $\pi(\vec{X}_t^N)$  de (1.74) à l'instant  $t$  satisfait

$$d\pi(\vec{X}_t^N) = Q(\pi(\vec{X}_t^N))dt + N^{-1/2}dM_t^N,$$

où le terme  $M_t^N$  correspond à une martingale bornée, et l'opérateur  $Q$  est donné par l'EDP non-linéaire satisfaite à la limite. Cette équation, vue comme une EDP stochastique, est connue sous le nom d'**équation de Dean-Kawasaki** Kawasaki [1994]; Dean [1996]; Fehrman and Gess [2022]. Elle peut être considérée comme un analogue en dimension infinie de la dynamique de **Freidlin-Wentzell**

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t, \quad (1.76)$$

exemple classique en théorie des grandes des grandes déviations. Le but du Chapitre 3 est de pousser cette analogie au niveau même des preuves, afin d'établir des résultats de convergence, de **fluctuations et de grandes déviations** pour (1.74).

Dans l'article fondateur Tanaka [1984], Tanaka construit, pour chaque  $P$  dans  $\mathcal{P}(C([0, T], \mathbb{R}^d))$ , une application  $X^P : C([0, T], \mathbb{R}^d) \rightarrow C([0, T], \mathbb{R}^d)$  telle que

$$\forall \omega \in C([0, T], \mathbb{R}^d), \forall t \in [0, T], \quad X_t^P(\omega) = \int_0^t b_s(X^P(\omega), X_{\#}^P P) ds + \omega_t,$$

et montre la continuité de  $\Psi : P \mapsto X_{\#}^P P$ . Cette application permet d'obtenir le système de particules (1.74) de manière trajectorielle en posant

$$X^{i,N} = X^{\pi(\vec{X}^N)}(B^{i,N}), \quad \pi(\vec{X}^N) = \Psi(\pi(\vec{B}^N)).$$

L'application  $\Psi$  étant continue, la limite champ moyen découle alors de la convergence de la mesure empirique des Browniens indépendants. De plus, Tanaka [1984] obtient un **théorème central limite** pour  $\pi(\vec{X}^N)$  en calculant le développement à l'ordre 1 de  $\Psi$ . De même, les grandes déviations de  $\pi(\vec{X}^N)$  découlent de celles de  $\pi(\vec{B}^N)$  par **contraction**, comme c'est le cas pour la dynamique classique (1.76).

Lorsqu'on ajoute une matrice de diffusion  $\sigma$  non constante à (1.76), les choses se compliquent car  $X^\varepsilon$  n'est plus une fonction continue de  $(B_t)_{0 \leq t \leq T}$ . Une solution a été proposée par Azencott Azencott et al. [1980], connue sous le nom de **quasi-continuité** : Azencott et al. [1980] montre que l'erreur commise en régularisant la trajectoire du Brownien est négligeable à l'échelle exponentielle. Le Chapitre 3 étend cette méthode à (1.74) en utilisant le système de particules discrétisé en temps

$$dX_t^{h,i,N} = b_{t_h}(X^{h,i,N}, \pi(\vec{X}^{h,N}))dt + \sigma_{t_h}(X^{h,i,N}, \pi(\vec{X}^{h,N}))dB_t^{i,N}, \quad 1 \leq i \leq N, \quad (1.77)$$

où  $h > 0$  est un pas de discrétisation et  $t_h := h\lfloor t/h \rfloor$ . La convergence et les fluctuations de (1.77) sont **uniformes en  $h$** , ce qui permet de faire commuter les limites  $N \rightarrow +\infty$  et  $h \rightarrow 0$ . Ces résultats sont précisés dans les théorèmes suivants, où (1.74) est identifié à  $\vec{X}^{0,N}$  (i.e. (1.77) pour  $h = 0$ ) avec un léger abus. On suppose que  $\mathbb{E}[e^{\alpha|X_0^{i,N}|}]$  est finie pour tout  $\alpha > 0$ .

**Theorem 1.6.1** (Grandes déviations uniformes). *Sous des hypothèses Lipschitz sur les coefficients précisées dans [Chaintron, 2024, Theorems 2.7-2.9], pour tout  $h \in [0, 1]$ , la suite des Loi( $\pi(\vec{X}^{h,N})$ ) satisfait un principe de grandes déviations dans  $\mathcal{P}_p(C([0, T], \mathbb{R}^d))$ ,  $p \in [1, 2)$ , pour la bonne fonction de taux*

$$I_h : P \in \mathcal{P}_p(C([0, T], \mathbb{R}^d)) \mapsto H(P|\Gamma_h(P)),$$

où  $H$  est l'entropie relative usuelle et  $\Gamma_h(P)$  est la loi trajectorielle de la solution de l'EDS

$$dY_t^h = b_{t_h}(Y^h, P)dt + \sigma_{t_h}(Y^h, P)dB_t.$$

De plus, pour toute  $F : \mathcal{P}_p(C([0, T], \mathbb{R}^d)) \rightarrow \mathbb{R}$  bornée Lipschitz,

$$\sup_{N \geq 1} |N^{-1} \log \mathbb{E}[\exp[NF(\pi(\vec{X}^{h,N}))]] - N^{-1} \log \mathbb{E}[\exp[NF(\pi(\vec{X}^{0,N}))]]| \xrightarrow{h \rightarrow 0} 0,$$

et la fonction de taux  $I_h$   $\Gamma$ -converge vers  $I_0$  lorsque  $h \rightarrow 0$ .

La preuve de l'estimée uniforme repose sur des arguments de **contrôle stochastique champ-moyen**, en utilisant la formule de représentation de Boué-Dupuis Boué and Dupuis [1998]. Dans l'énoncé suivant,  $\bar{X}^{h,i}$  désigne la limite de  $X^{h,i,N}$  lorsque  $N \rightarrow +\infty$ , qui est solution d'une équation de McKean-Vlasov analogue à (1.75) dirigée par  $B^{i,N}$ .

**Theorem 1.6.2** (Fluctuations uniformes). *Sous des hypothèses de structure et de différentiabilité des coefficients précisées dans [Chaintron, 2024, Theorem 2.11], il existe  $C > 0$  telle que pour tout  $h \in (0, 1]$ ,*

$$\sup_{N \geq 1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{h,i,N} - \bar{X}_t^{h,i} - [X_t^{0,i,N} - \bar{X}_t^{0,i}]| \right] \leq C |h \log h|^{1/2},$$

De plus, pour tout  $\varphi \in C_b^{1,1}(C([0, T], \mathbb{R}^d), \mathbb{R})$  et  $h \in [0, 1]$ ,

$$\sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N \varphi(X^{h,i,N}) - \mathbb{E}[\varphi(\bar{X}^{h,1})] \right]$$

converge en loi vers une gaussienne centrée de variance  $\sigma_{h,\varphi}^2$  explicite.

Ces théorèmes s'inscrivent dans la suite des résultats de fluctuations et grandes déviations Jourdain and Méléard [1998]; Budhiraja et al. [2012]; Coghi et al. [2020]. . . pour (1.74). Le caractère uniforme en la discrétisation est nouveau. Pour le cas continu  $h = 0$ , il n'existait pas de principe de **grandes déviations trajectorielle en topologie de Wasserstein** au niveau de généralité de (1.74), de même pour les fluctuations. La méthode de Tanaka autorise de nombreuses généralisations, notamment pour des systèmes **non-échangeable en environnement aléatoire** Coghi et al. [2020].

## 1.7 Stabilité du principe de Gibbs et contrôle sous contraintes

Dans l'esprit de la section précédente, considérons un système de  $N$  particules aléatoires  $(X^{i,N})_{1 \leq i \leq N}$  dans un ensemble  $E$ , indépendantes ou qui interagissent via leur moyenne empirique  $\pi(\vec{X}^N) \in \mathcal{P}(E)$ . Supposons qu'une **mesure sur ce système** révèle que  $\pi(\vec{X}^N)$  reste confinée dans un sous-ensemble  $A \subset \mathcal{P}(E)$ . Il est alors naturel de chercher la configuration la plus probable du système connaissant cette mesure. Lorsque la suite des Loi( $\pi(\vec{X}^N)$ ) satisfait un principe de grandes déviations pour la fonction de taux  $I$ , une réponse est donnée par le **principe conditionnel de Gibbs**, qui montre que

$$\lim_{N \rightarrow +\infty} \text{Loi}(X^{1,N} | \pi(\vec{X}^N) \in A) \xrightarrow{N \rightarrow +\infty} \operatorname{argmin}_{\mu \in A} I(\mu).$$

Si  $E = C([0, T], \mathbb{R}^d)$  est un espace de trajectoires et les particules sont données par (1.74), un tel principe de grandes déviations est montrée par le Théorème 1.6.1 en Partie 1.6. Un des principaux objectifs de ma thèse était l'étude de ce principe conditionnel lorsque

$$A = \{\mu_{[0,T]} \in \mathcal{P}(C([0, T], \mathbb{R}^d)), \forall t \in [0, T], \Psi(\mu_t) \leq 0\}, \quad (1.78)$$

où  $\Psi : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$  est une fonction **non-nécessairement convexe**, vue comme une contrainte sur les lois marginales en temps. En d'autres termes, il s'agit d'une version du **principe de Gibbs avec une infinité de contraintes**. Un exemple typique est le cas linéaire  $\Psi(\mu) = \int_{\mathbb{R}^d} \psi d\mu$ , qui revient à imposer qu'une statistique du système (typiquement son énergie moyenne) ne dépasse pas un certain seuil au cours du temps. Des questions naturelles sont alors les suivantes :

- À quel **processus contraint** correspond la loi optimale ?
- Quelle est la **régularité** de ce processus par rapport à celles du système et de la contrainte ?

- Le processus est-il **stable** lorsqu'on perturbe la contrainte ?

La motivation physique de cette dernière question provient du fait qu'une mesure expérimentale est souvent teintée d'incertitude, donc on souhaite donc que le modèle soit stable par rapport aux erreurs de mesure.

La Section 1.7.1 ci-dessous résume le Chapitre 4, travail en collaboration avec Giovanni Conforti et Julien Reygner. Ce chapitre étudie une large classe de problèmes de minimisation dans un espace abstrait  $\mathcal{P}(E)$  sous une infinité de contraintes, et montre notamment des résultats de stabilité faible. Le cas  $E = C([0, T], \mathbb{R}^d)$  de **trajectoires en interaction champ-moyen** est traité dans le Chapitre 5 en collaboration avec Giovanni Conforti, incluant notamment des résultats de **stabilité quantitative** présentés en Section 1.7.2. Ces résultats nécessitent des estimées de régularité sur une équation de Hamilton-Jacobi montrées dans le Chapitre 6 par des méthodes de **solutions de viscosité**. Ce dernier chapitre est résumé en Section 1.7.3.

### 1.7.1 Principe de Gibbs avec une infinité de contraintes

Le Chapitre 4 considère un espace Polonais  $E$  général, ainsi que  $\phi \in C_b(E, \mathbb{R}_+)$  et l'ensemble

$$\mathcal{P}_\phi(E) := \{\mu \in \mathcal{P}(E), \int_E \phi d\mu < +\infty\},$$

muni de la topologie de la convergence faible, plus la convergence contre  $\phi$  (le cas particulier  $\phi(x) = d^p(x, x_0)$  correspond à la topologie de Wasserstein- $p$ ). Étant données une **mesure de référence**  $\nu \in \mathcal{P}_\phi(E)$  et  $\mathcal{F} : \mathcal{P}_\phi(E) \rightarrow (-\infty, +\infty]$  semi-continue inférieurement, le domaine de définition de la fonction de taux  $I(\mu) := H(\mu|\nu) + \mathcal{F}(\mu)$  est

$$\mathcal{D}_{\mathcal{I}} := \{\mu \in \mathcal{P}_\phi(E), H(\mu|\nu) + \mathcal{F}(\mu) < +\infty\}.$$

On introduit ensuite une famille de fonctions continues  $(\zeta_s)_{s \in \mathcal{S}}$  telle que

$$\forall s \in \mathcal{S}, \exists C_s > 0 : \forall x \in E, |\zeta_s(x)| \leq C_s[1 + \phi(x)],$$

ainsi qu'une famille  $(\Psi_t)_{t \in \mathcal{T}}$  de fonctions semi-continues inférieurement  $\mathcal{P}_\phi(E) \rightarrow \mathbb{R}$ ,  $\mathcal{T}$  étant un espace topologique compact. On définit enfin les **ensembles contraints** (qui sont des fermés)

$$\mathcal{A}^\zeta := \{\mu \in \mathcal{P}_\phi(E), \forall s \in \mathcal{S}, \int_E \zeta_s d\mu = 0\}, \quad \mathcal{A}^\Psi := \{\mu \in \mathcal{P}_\phi(E), \forall t \in \mathcal{T}, \Psi_t(\mu) \leq 0\}.$$

Le théorème suivant [Chaintron et al., 2024, Theorem 2.13] établit les conditions d'optimalité dans un cadre abstrait autorisant une infinité de contraintes d'égalité linéaires et une infinité de contraintes d'inégalité possiblement non-convexes.

**Theorem 1.7.1** (Conditions d'optimalité). *Soit  $\bar{\mu}$  un minimiseur pour le problème*

$$\inf_{\mu \in \mathcal{A}^\zeta \cap \mathcal{A}^\Psi} H(\mu|\nu) + \mathcal{F}(\mu). \quad (1.79)$$

*Sous des hypothèses de régularité et une condition de qualification détaillées dans [Chaintron et al., 2024, Section 2.3],*

1. *Tout  $\mu$  dans  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^\zeta$  est absolument continue par rapport à  $\bar{\mu}$ .*
2. *Il existe  $(\bar{\zeta}, \bar{\lambda}) \in L^1(E, d\bar{\mu}) \times \mathcal{M}_+(\mathcal{T})$  et un ensemble mesurable  $\bar{S} \subset E$  tel que  $\bar{\mu}(\bar{S}) = 1$  et*

$$\frac{d\bar{\mu}}{d\nu}(x) = \frac{1}{\bar{Z}} \mathbb{1}_{\bar{S}}(x) \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right], \quad (1.80)$$

*où  $\bar{Z} \in (0, +\infty)$  est une constante de normalisation.*

3. La fonction  $\bar{\zeta}$  ci-dessus appartient à la fermeture de  $\text{Vect}(\zeta_s, s \in \mathcal{S})$  dans  $L^1(E, d\bar{\mu})$ , donc  $\int_E \bar{\zeta} d\bar{\mu} = 0$ , et la condition de relâchement suivante est satisfaite :

$$\Psi_t(\bar{\mu}) = 0 \quad \text{pour } \bar{\lambda}\text{-presque tout } t \in \mathcal{T}. \quad (1.81)$$

Les fonctions  $\frac{\delta \mathcal{F}}{\delta \mu}$  et  $\frac{\delta \Psi}{\delta \mu}$  sont les dérivées linéaires (Gâteaux) de  $\mathcal{F}$  et  $\Psi$  par rapport à  $\mu$ . La preuve repose sur une utilisation adéquate du théorème de Hahn-Banach, dans l'esprit de Csiszár [1975, 1984]; Léonard [2000]. Un principe de **grandes déviations pour la loi conditionnelle** de  $\pi(\bar{X}^N)$  est également montré dans [Chaintron et al., 2024, Theorem 2.7]. Ce théorème inclut (1.78) qui correspond au cas  $E = C([0, T], \mathbb{R}^d)$ ,  $\mathcal{S} = \emptyset$  et  $\Psi_t(\mu_{[0, T]}) = \Psi(\mu_t)$ . Plusieurs **exemples gaussiens** sont explicitement détaillés dans [Chaintron et al., 2024, Section 3.4], donnant lieu à des comportements originaux. Si  $E$  est un ensemble produit du type  $E = F \times G$  ou  $E = C([0, T], \mathbb{R}^d)$ , les contraintes d'égalité permettent d'imposer autant de lois marginales que l'on souhaite. En particulier, ce cadre couvre le problème du **pont de Schrödinger** Léonard [2014] ainsi que des extensions comportant davantage de contraintes.

Lorsque les fonctions  $\mathcal{F}$  et  $\Psi_t$  sont convexes, il y a unicité du minimiseur  $\bar{\mu}$  et les conditions (1.80)-(1.81) suffisent à le caractériser [Chaintron et al., 2024, Theorem 2.17]. Dans ce cadre, on obtient le résultat de stabilité suivant [Chaintron et al., 2024, Theorem 2.22].

**Theorem 1.7.2** (Stabilité faible). *Soit  $((\nu_k, F_k, \Psi))_{k \geq 1}$  une suite qui tend vers  $(\nu, \mathcal{F}, \Psi)$  en satisfaisant les hypothèses de régularité détaillées dans [Chaintron et al., 2024, Section 2.4]. La suite des minimiseurs  $(\mu_k)_{k \geq 1}$  converge alors vers le minimiseur  $\bar{\mu}$  de (1.79). De plus, la suite des multiplicateurs de Lagrange  $(\lambda_k)_{k \geq 1}$  associée à  $(\mu_k)_{k \geq 1}$  par (1.80) est pré-compacte dans  $\mathcal{M}_+(\mathcal{T})$ , et toute valeur adhérence est un multiplicateur  $\bar{\lambda}$  satisfaisant (1.80) pour  $\bar{\mu}$ .*

Plusieurs autres résultats techniques sont prouvés dans les Chapitres 4 et 6, en préparation des résultats de **stabilité quantitative** du Chapitre 5.

### 1.7.2 Régularité et stabilité quantitative par le contrôle champ-moyen

Cette partie présente les résultats du Chapitre 5, qui étudie la stabilité quantitative du principe de Gibbs pour (1.78) et des **diffusions en interaction**. Pour  $\mu_{[0, T]} \in \mathcal{P}(C([0, T], \mathbb{R}^d))$ , soit  $\Gamma(\mu_{[0, T]})$  la loi trajectorielle de la solution de l'EDS

$$dX_t = b(X_t, \mu_t)dt + \sigma(X_t)dB_t, \quad X_0 \sim \nu_0,$$

où  $(B_t)_{0 \leq t \leq T}$  est un mouvement Brownien et les coefficients sont des fonctions continues  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  et  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ . Dans l'esprit du Théorème 1.6.1, on s'intéresse au problème

$$\inf_{\substack{\mu_{[0, T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)) \\ \forall t \in [0, T], \Psi(\mu_t) \leq 0}} H(\mu_{[0, T]} | \Gamma(\mu_{[0, T]})),$$

pour  $\Psi : \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$  semi-continue inférieurement. Ce problème correspond Budhiraja et al. [2012]; Fischer [2014] au problème de contrôle sous contraintes en lois

$$\inf_{\substack{\alpha, X_0^\alpha \\ \forall t \in [0, T], \Psi(\text{Loi}(X_t^\alpha)) \leq 0}} H(\text{Loi}(X_0^\alpha) | \nu_0) + \mathbb{E} \int_0^T \frac{1}{2} |\alpha_t|^2 dt,$$

de la dynamique de McKean-Vlasov

$$dX_t^\alpha = b(X_t^\alpha, \text{Loi}(X_t^\alpha))dt + \sigma(X_t^\alpha)\alpha_t dt + \sigma(X_t^\alpha)dB_t.$$

On suppose que pour tout  $\alpha > 0$ ,  $x \mapsto e^{\alpha|x|}$  est  $\nu_0$ -intégrable. Pour simplifier la présentation des résultats tout en restant dans un cadre rigoureux, on se restreint ici à

$$b(x, \mu) = b(x), \quad \Psi(\mu) = \int_{\mathbb{R}^d} \psi d\mu,$$

même si le Chapitre 5 traite le cas non-linéaire général. En particulier,  $\Gamma(\mu_{[0,T]}) =: \nu_{[0,T]}$  ne dépend maintenant plus de  $\mu_{[0,T]}$ . On suppose également que  $b$ ,  $\sigma$ ,  $\psi$ ,  $\nabla\psi$  sont globalement Lipschitz, et  $\sigma$  est bornée inversible et d'inverse bornée.

**Theorem 1.7.3** (Structure des minimiseurs). *Soit  $\bar{\mu}_{[0,T]}$  un minimiseur satisfaisant la contrainte de qualification  $\nabla\psi \neq 0$ ,  $\bar{\mu}_t$ -a.s. Il existe alors une mesure de Radon positive  $\bar{\lambda} \in \mathcal{M}_+([0, T])$ ,  $\bar{\varphi} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  et  $\bar{Z} > 0$  tels que  $\bar{\mu}_0(dx) = \bar{Z}^{-1} e^{-\bar{\varphi}_0(x)} \nu_0(dx)$  et*

$$\begin{cases} \partial_t \bar{\mu}_t - \nabla \cdot [-\bar{\mu}_t b + \bar{\mu}_t \sigma \sigma^\top \nabla \bar{\varphi}_t + \frac{1}{2} \nabla \cdot [\bar{\mu}_t \sigma \sigma^\top]] = 0, \\ -\bar{\varphi}_t + \int_t^T b \cdot \nabla \bar{\varphi}_s - \frac{1}{2} |\sigma^\top \nabla \bar{\varphi}_s|^2 + \frac{1}{2} \text{Tr}[\sigma \sigma^\top \nabla^2 \bar{\varphi}_s] ds + \int_{[t, T]} \psi \bar{\lambda}(ds) = 0. \end{cases} \quad (1.82)$$

De plus la contrainte de relâchement suivante est satisfaite

$$\int_{\mathbb{R}^d} \psi d\bar{\mu}_t = 0, \quad \text{pour } \bar{\lambda}\text{-presque tout } t \in [0, T],$$

et la densité trajectorielle admet l'expression

$$\frac{d\bar{\mu}_{[0,T]}}{d\nu_{[0,T]}}(x_{[0,T]}) = \bar{Z}^{-1} \exp \left[ - \int_0^T \psi(x_t) \bar{\lambda}(dt) \right].$$

Ce théorème et le suivant généralisent des résultats de Daudin [2022, 2023] pour un problème de contrôle stochastique (cas  $b \equiv 0$  et  $\sigma \equiv \text{Id}$ ) avec contraintes en lois régulières et convexes.

**Theorem 1.7.4** (Densité du multiplicateur). *Dans le cadre du Théorème 1.7.3, il existe  $(\bar{\lambda}, \bar{\varphi})$  satisfaisant (1.82) tel que  $x \mapsto \nabla \bar{\varphi}_t(x)$  soit Lipschitz et*

$$\bar{\lambda}(dt) = \bar{\lambda}_0 \delta_0(dt) + \bar{\lambda}_t dt + \bar{\lambda}_T \delta_T(dt),$$

pour  $t \mapsto \bar{\lambda}_t$  in  $L^\infty(0, T)$ , si on suppose que  $\nu_0$  admet une densité par rapport à Lebesgue presque partout différentiable (toujours notée  $\nu_0$ ), et qu'il existe  $\varepsilon_0 > 0$  tel que **ou bien (i) ou bien (ii)** est satisfaite :

- (i)  $\nabla^2 \psi$  est Lipschitz et  $\int_{\mathbb{R}^d} [(\log \nu_0)^{1+\varepsilon_0} + |\nabla \log \nu_0|^{2+\varepsilon_0}] d\nu_0 < +\infty$ .
- (ii)  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla^2 \sigma$  sont Lipschitz et  $\int_{\mathbb{R}^d} [(\log \nu_0)^{1+\varepsilon_0} + |\nabla \log \nu_0|^{4+\varepsilon_0} + |\nabla^2 \log \nu_0|^{2+\varepsilon_0}] d\nu_0 < +\infty$ .

Si (ii) est satisfaite,  $x \mapsto \nabla^2 \bar{\varphi}_t(x)$  est de plus Lipschitz et la densité des marginales en temps  $\bar{\mu}_t$  satisfait

$$\left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^d} [\log \bar{\mu}_t + |\nabla \log \bar{\mu}_t|^4 + |\nabla^2 \log \bar{\mu}_t|^2] d\bar{\mu}_t \right] + \int_0^T \int_{\mathbb{R}^d} |\nabla^3 \log \bar{\mu}_t|^2 d\bar{\mu}_t dt < +\infty. \quad (1.83)$$

Ces résultats sur la densité de  $\bar{\lambda}$  sont un **compromis entre la régularité des coefficients et celle de la contrainte**. Les ingrédient-clefs sont l'estimée (1.83) pour des processus de diffusion, ainsi qu'une procédure géométrique [Chaintron and Conforti, 2024, Section 3.1] qui construit des mesures admissibles en poussant la mesure optimale le long du flot-gradient de la contrainte. La preuve de ce théorème montre également que  $t \in (0, T] \mapsto \nabla \varphi_t(x)$  est continue et  $t \in [0, T] \mapsto \int_{\mathbb{R}^d} \psi \bar{\mu}_t$  est  $C^1$  sous (i), et que  $t \in (0, T] \mapsto \nabla^2 \varphi_t(x)$  est continue sous (ii). En fonction de l'**origine de la régularité** (contrainte ou coefficients), on obtient plus de régularité sur la courbe contrainte ou le contrôle optimal. On pourrait en obtenir encore davantage en supposant que les coefficients ont davantage de dérivées bornées.

**Theorem 1.7.5** (Stabilité quantitative). *Dans le cadre du Theorem 1.7.3, on suppose que (ii) est satisfaite, que  $\psi$  est  $C^4$  à dérivées bornées et que  $\int_{\mathbb{R}^d} e^{\varepsilon_0 |x|^2} \nu_0(dx) < +\infty$ . Pour  $\varepsilon \geq 0$ , soit  $\mu_{[0, T]}^\varepsilon$  le minimiseur de*

$$\inf_{\substack{\mu_{[0, T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)) \\ \forall t \in [0, T], \Psi(\mu_t) \leq \varepsilon}} H(\mu_{[0, T]} | \nu_{[0, T]}).$$

Soit  $(\varphi_t^\varepsilon, \lambda_t^\varepsilon, \lambda_0^\varepsilon, \lambda_T^\varepsilon)$  donné par l'application à  $\mu_{[0, T]}^\varepsilon$  du Théorème 1.7.3-1.7.4. On suppose que le nombre d'intervalles non-triviaux où  $\int_{\mathbb{R}^d} \psi d\mu_t^0$  est identiquement nulle est fini. Il existe alors  $C > 0$  telle que pour tout  $\varepsilon \in [0, 1]$ ,

$$(i) \text{ Stabilité entropique : } H(\mu_{[0, T]}^\varepsilon | \mu_{[0, T]}^0) \leq C\varepsilon.$$

$$(ii) \text{ Stabilité du multiplicateur : } |\lambda_0^\varepsilon - \lambda_0^0| + \|\lambda^0 - \lambda^\varepsilon\|_{L^1(0, T)} + |\lambda_T^\varepsilon - \lambda_T^0| \leq C\varepsilon^{1/4}.$$

$$(iii) \text{ Stabilité du contrôle : } \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |\nabla \varphi_t^\varepsilon(x) - \nabla \varphi_t^0(x)| \leq C\varepsilon^{1/4}.$$

L'hypothèse de moment gaussien pour  $\nu_0$  ne sert qu'à simplifier les taux de convergence (sans elle, on aurait des facteurs logarithmiques supplémentaires). De même, l'hypothèse sur le nombre fini d'intervalles peut être omise si on accepte un taux moins bon [Chaintron and Conforti, 2024, Remark 4.10].

### 1.7.3 Existence et estimées Lipschitz pour la solution d'une équation de HJB

L'objectif principal du Chapitre 6 est le résultat suivant.

**Theorem 1.7.6.** *Si  $b$ ,  $\sigma$  et  $f$  sont des fonctions Lipschitz, avec  $\sigma$  bornée et  $\sigma\sigma^\top$  uniformément elliptique, alors l'équation de HJB*

$$\partial_t u + b \cdot \nabla u + \frac{1}{2} \text{Tr}[\sigma\sigma^\top \nabla^2 u] - \frac{1}{2} |\sigma^\top \nabla u|^2 + f = 0, \quad (1.84)$$

avec la donnée initiale Lipschitz  $u(T, x) = g(x)$ , admet une unique solution  $C^{1,2}$  à croissance linéaire, qui est par ailleurs globalement Lipschitz.

De telles solutions classiques sont très utiles pour la résolution de problèmes de **contrôle stochastique**. Ce résultat est crucial dans les articles Chaintron et al. [2024]; Chaintron and Conforti [2024] ainsi que le raffinement suivant : si de plus

$$\forall (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \quad |f(t, x) - f(t, y)| \leq \lambda(t)|x - y|, \quad \lambda \in L^1(0, T),$$

alors la borne sur  $\nabla u$  only ne dépend de  $f$  qu'au travers  $\|\lambda\|_{L^1(0, T)}$ . Cette propriété permet d'obtenir des estimées sur la solution de (1.82) **ne dépendant que de la masse totale  $\bar{\lambda}([0, T])$**  du multiplicateur.



Les hypothèses Lipschitz ci-dessus sont naturelles dans un cadre probabiliste, et la borne sur  $\sigma$  est nécessaire pour que  $\nabla u$  soit bornée. Étonnamment, ce résultat utile ne semble pas avoir été prouvée dans la littérature antérieure, comme longuement détaillée dans l'introduction de Chaintron [2023]. La difficulté principale est que  $\sigma \neq \text{Id}$ , ce qui implique une croissance quadratique en  $\nabla u$  pour la dérivée en  $x$  du hamiltonien. La preuve de ce résultat combine plusieurs techniques et astuces autour des **solutions de viscosité**, donnant une estimée qui semble à la limites des hypothèses.

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## CHAPTER 2

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# Constrained non-linear estimation and links with stochastic filtering

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*Ma mission première est de te maintenir en vie.*

— Philippe Moireau, CIRM 2022

This work is a collaboration with Laurent Mertz, Philippe Moireau, and Hasnaa Zidani. This article studies the problem of estimating the state of a non-smooth sub-differential dynamics in a domain given some real-time observation on it. We show that the small noise limit of the corresponding stochastic filtering problem recovers the deterministic smooth framework of Mortensen, which was recently extended to sub-differential dynamics. Contrary to the smooth setting already studied by Baras and James, intricate phenomena appear at the boundary of the domain, because of the non-reversibility of our non-smooth dynamics. In particular, the value function is a viscosity solution of a Hamilton-Jacobi equation with Neumann-type boundary conditions that are different for the sub-solution and the super-solution. Using a dual formulation, we establish a large deviation result in the small noise limit. We further prove a quantitative approximation result, when replacing the non-smooth dynamics by a smooth penalised one. We eventually provide a numerical illustration in a 1D configuration with a discretisation strategy that benefits from an adapted Bellman principle introduced at the discrete level.

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## 2.1 Introduction

Sequential estimation aims to combine a dynamical system with some measurements as they become available, to reduce potential uncertainties in the dynamics and thus produce a model prediction that is more consistent with available data. Such a goal can be pursued for a wide range of dynamical systems: finite dimensional (ODE) or infinite dimensional (PDE), linear or non-linear dynamics, unconstrained (smooth dynamics) or constrained (non-smooth dynamics combined with variational inequality), deterministic (observer theory) or stochastic (filtering theory). In the present paper, we focus on constrained dynamics in finite dimension, and we link the framework of stochastic filtering with the deterministic framework aiming at defining observer dynamics. Stochastic filtering for unconstrained ODEs has been known since the 1960s with the seminal work of Kalman and Bucy [1961] for linear dynamics, and then generalized for nonlinear dynamics, see Kushner [1967]; Duncan [1967]; Zakai [1969]; Jazwinski [1970]... These results were then extended to some constrained dynamics, in particular for the Skorokhod problem with the series of works by Pardoux [1978a,b]; Hucke [1990]. As for the deterministic view, the observer theory based on Minimum Energy Estimation has been known since the pioneering work of Mortensen [1968], see also the presentation proposed by Fleming [1997]. While the unconstrained case has been well understood since James and Baras [1988b], with an asymptotic connection to stochastic filtering introduced in Hijab [1984] and further justified in Fleming [1997], the case of constrained dynamics was not studied until a recent attempt Chaintron et al. [2023] for a simple one-dimensional dynamics. The constraint is there introduced using the formalism of non-smooth sub-differential dynamics Moreau [1971]; Tyrrell Rockafellar [1970]. A main difficulty of this setting is the loss of time reversibility: in contrast with smooth dynamics, the non-smooth dynamics is well-posed in forward time only, making the connection harder between stochastic filtering and deterministic estimation. Chaintron et al. [2023] was able to make a few strides to reconcile both points of view.

In this paper, we generalize the works of Willems [2004]; James and Baras [1988b]; Chaintron et al. [2023] to non-smooth dynamics associated with trajectories that must remain in a bounded domain. In particular, we fully connect the deterministic representation to stochastic filtering by extending the results of James and Baras [1988b] in a suitable way: we show that the deterministic view corresponds to the small noise limit of the stochastic framework by proving a large deviation result. Following James and Baras [1988b]; Fleming [1997], we rely on a viscosity solution setting to deal with the underlying Hamilton-Jacobi-Bellman (HJB) equations. However, due to the non-reversibility of non-smooth dynamics, our approach differs from them. In particular, the sub-solution and the super-solution satisfy different Neumann-type boundary conditions in the viscosity sense, and the comparison principle is unknown to our knowledge. To circumvent this difficulty, the small noise limit is established using a dual formulation. We also show how the penalized estimator developed in Chaintron et al. [2023] to deal with the constraint converges to the fully constrained estimator in a quantitative way, and we provide numerical illustrations.

### 2.1.1 Problem statement

Let  $G \subset \mathbb{R}^n$  be a bounded open domain that is convex with  $C^2$  boundary. We consider a class of non-smooth dynamical systems of the form

$$\dot{x}_\omega(s) + \partial\chi_G(x_\omega(s)) \ni b(s, x_\omega(s)) + \sigma(s, x_\omega(s))\omega(s), \quad s > 0, \quad (2.1)$$

with an initial condition  $x_\omega(0) \in \overline{G}$ .

Here, the state variable at time  $s$  is denoted  $x_\omega(s)$ , while the functions  $b : \mathbb{R}_+ \times \overline{G} \rightarrow \mathbb{R}^n$  and  $\sigma : \mathbb{R}_+ \times \overline{G} \rightarrow \mathbb{R}^{n \times r}$  are Lipschitz-continuous functions. The term  $\omega(s) \in \mathbb{R}^r$  represents the state disturbance,  $\chi_G : \mathbb{R}^n \rightarrow \{0, +\infty\}$  is the characteristic function of the domain  $G$ , and  $\partial\chi_G(x_\omega(s))$  is the subdifferential of the convex function  $\chi_G$  at  $x_\omega(s)$ . Recall that this subdifferential corresponds to the normal cone  $\mathcal{N}_{\overline{G}}(x_\omega(s))$  to  $\overline{G}$  at  $x_\omega(s)$ . This means that when the state  $x_\omega(s)$  reaches the boundary of  $G$ , the dynamics are reflected, ensuring that  $x_\omega(s)$  remains within  $\overline{G}$ . In particular, the dynamics are driven by the vector field  $b(s, x_\omega(s)) + \sigma(s, x_\omega(s))\omega(s)$ , but whenever  $x_\omega(s)$  approaches the boundary of  $G$ , the normal cone prevents it from leaving the domain, reflecting the trajectory back into the domain in the sense of Skorokhod.

In the sequel, we assume that the disturbance belongs to the Lebesgue space  $L^2(0, +\infty; \mathbb{R}^m)$  that consists of all measurable functions  $w : (0, +\infty) \rightarrow \mathbb{R}^r$  that are square integrable, i.e. the norm

$$\|w\|_{L^2(0, +\infty; \mathbb{R}^m)} := \left( \int_0^{+\infty} \|w(s)\|^2 ds \right)^{\frac{1}{2}}$$

is finite, and where functions which agree almost everywhere are identified.

For square-integrable disturbances  $\omega$ , given any initial data  $x_\omega(0) \in \overline{G}$ , the existence and uniqueness of an absolutely continuous solution to the system (2.1) are established in [Clarke et al., 2008, Section 4.1] and [Pardoux and Răşcanu, 2014, Section 4.2.3], ensuring that the differential inclusion (2.1) holds for almost every  $s \geq 0$ .

We consider a measurement procedure  $h : \mathbb{R}_+ \times G \rightarrow \mathbb{R}^m$ , so that observations associated with a trajectory of the dynamics (2.1) are given by

$$\forall t \geq 0, \quad \dot{y}(t) = h(t, x_\omega(t)) + \eta(t), \quad (2.2)$$

where  $\eta(t) \in \mathbb{R}^m$  is the observation *disturbance*, which is assumed to be square-integrable. Our purpose is to build an estimator at time  $t$  for the state of a dynamics described by (2.1) given the measurement  $(\dot{y}(s))_{0 \leq s \leq t}$  produced by (2.2). We want this estimator to be *causal* in the sense of Krener [1998], meaning that the computed state only depends on the past measurements.

**Remark 2.1.1.** *In most deterministic observation problems, the observation is usually denoted  $y$ . Here, we denote by  $y$  a primitive of the observation for consistency with the stochastic filtering setting. As in practice, only  $\dot{y}$  will appear in the estimator equations, defining the observation with a time-derivative is a mere notation convention.*

An iconic case of non-smooth dynamics in an unbounded domain is  $G = \mathbb{R}_+$ . It was studied in Chaintron et al. [2023]. Solutions in this case are given by Skorokhod [1961]:

$$x(t) + \Delta(t) = x(0) + \int_0^t \omega(s) ds,$$

$$\text{where } \Delta(t) = \min_{0 \leq s \leq t} \min \left( 0; x(0) + \int_0^s \omega(\tau) d\tau \right),$$

and illustrated in Figure 2.1.

Before describing our results, we briefly review some existing methods in linear and non-linear estimation.

### 2.1.2 Unconstrained linear dynamics: Kalman filter

In the linear case, the system is characterized by the following dynamics:

$$b(t, x) = Ax, \quad \sigma(t, x) = \Sigma, \quad h(t, x) = Hx, \quad \psi(x) = \frac{1}{2}[x - \hat{x}_0]^\top P_0^{-1}[x - \hat{x}_0],$$

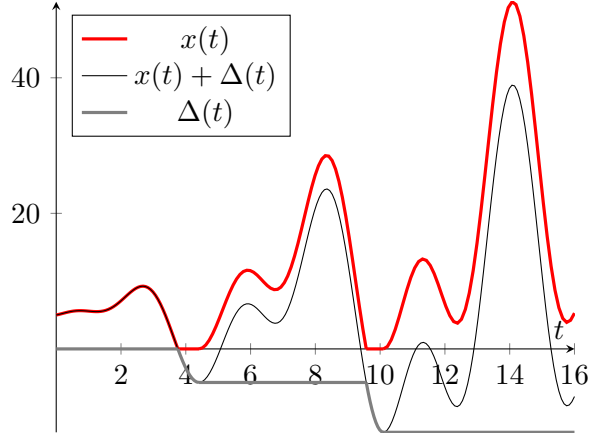


Figure 2.1: Example of trajectory

where  $A \in \mathbb{R}^{n \times n}$ ,  $\Sigma \in \mathbb{R}^{n \times r}$ ,  $H \in \mathbb{R}^{m \times n}$ ,  $P_0 \in S_n^{++}(\mathbb{R})$  is a positive-definite matrix, and  $\hat{x}_0 \in \mathbb{R}^n$ . The unconstrained linear dynamics, with a linear observation operator, serves as a classical example for defining a sequential estimator in both deterministic and stochastic systems. This was first introduced by Kalman and Bucy [1961], and a full treatment of stochastic filtering in this case can be found in Jazwinski [1970]; Davis and Davis [1977]. For a deterministic perspective, we refer to Willems [2004]. Essentially, both approaches lead to the definition of the estimator given by

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + P(t)H^\top [\dot{y}(t) - H\hat{x}(t)], & t > 0, \\ \hat{x}(0) = \hat{x}_0, \end{cases} \quad (2.3)$$

where the symmetric positive matrix  $P \in S_n^+(\mathbb{R})$  is a solution of the following Riccati equation

$$\begin{cases} \dot{P}(t) = A(t)P + P(t)A^\top + \Sigma\Sigma^\top - P(t)^\top HH^\top P(t), & t > 0, \\ P(0) = P_0. \end{cases} \quad (2.4)$$

Equation (2.3) provides a recursive estimator that can be computed in real time, the matrix  $P(t)$  being pre-computed beforehand.

### 2.1.3 Non-linear dynamics: Mortensen observer

If we now consider a general non-linear unconstrained dynamics of the form

$$\dot{x}_\omega(s) = b(s, x_\omega(s)) + \sigma(s, x_\omega(s))\omega(s), \quad s > 0, \quad (2.5)$$

the Mortensen estimator Mortensen [1968] generalises the Kalman estimator in the deterministic setting. Given an observation  $\dot{y} \in L^2((0, t); \mathbb{R}^m)$ , we introduce a value function called *cost-to-come* as the function defined by

$$\mathcal{U}(t, x) \triangleq \inf_{\substack{\omega \in L^2((0, t); \mathbb{R}^m) \\ x_\omega(t) = x}} \psi(x_\omega(0)) + \int_0^t \ell(s, x_\omega(s), \omega(s)) ds, \quad (2.6)$$

where  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is Lipschitz and such that  $\psi(x) \rightarrow +\infty$  as  $|x| \rightarrow +\infty$ ,

$$\ell(s, x, \omega) \triangleq \frac{1}{2}|\omega|^2 + \frac{1}{2}|\dot{y}(s) - h(s, x)|^2.$$



The Mortensen observer  $\hat{x}$  is defined as a minimiser of  $x \mapsto \mathcal{U}(t, x)$ . If uniqueness holds for this minimiser, we simply define

$$\forall t \geq 0, \quad \hat{x}(t) \triangleq \operatorname{argmin}_{x \in \mathbb{R}^n} \mathcal{U}(t, x). \quad (2.7)$$

For a well-posedness result in a non-linear setting, we refer to Breiten and Schröder [2023]. If moreover  $\mathcal{U}(t, x)$  is  $C^2$  at  $(t, \hat{x}(t))$  with invertible Hessian, the optimality conditions for  $\hat{x}(t)$  yield

$$\dot{\hat{x}}(t) = b(t, \hat{x}(t)) + [\nabla^2 \mathcal{U}(t, \hat{x}(t))]^{-1} \nabla h(t, \hat{x}(t)) [\dot{y}(t) - h(t, \hat{x}(t))], \quad t > 0. \quad (2.8)$$

This recursive feed-back structure extends the one of the Kalman estimator. Indeed, (2.8) precisely corresponds to (2.3) in the linear setting of Section 2.1.2. In this linear setting, the cost-to-come reads

$$\mathcal{U}(t, x) = \frac{1}{2} [x - \hat{x}(t)] \cdot P^{-1}(t) [x - \hat{x}(t)] + \int_0^t \frac{1}{2} |\dot{y}(s) - Hx(s)|^2 ds,$$

enabling us to recover the Kalman estimator.

### 2.1.4 Outline of the article

We adapt Mortensen's approach to the setting of non-smooth dynamics. The cost-to-come function for the constrained setting is defined in Section 2.2.1 and characterised as a viscosity solution with intricate boundary conditions. An approximation procedure is further introduced to bring the problem back to the setting of Section 2.1.3, the sub-differential being replaced by a penalisation term that pulls the dynamics back in  $\bar{G}$  when it escapes. A quantitative rate of convergence is obtained, the proof being done in Section 2.3.1. We then describe the link with non-smooth stochastic filtering. Existing links between stochastic filtering and deterministic estimation are recalled in Sections 2.2.3.1-2.2.3.2. Our results for the non-smooth setting are stated in Section 2.2.3.3 and proved in Section 2.4. A numerical illustration is given in Section 2.5.

## 2.2 Description of the results

Following James and Baras [1988a]; Rockafellar and Wolenski [2000], we introduce the following value function.

**Definition 2.2.1** (Cost-to-come). *The cost-to-come to the point  $x \in \bar{G}$  at time  $t \in [0, t]$ , given an observation  $\dot{y} \in L^2((0, t); \mathbb{R}^m)$ , is the function defined by*

$$\mathcal{V}(t, x) \triangleq \inf_{(x_\omega(0), \omega) \in \mathcal{A}_{t,x}^G} \psi(x_\omega(0)) + \int_0^t \ell(s, x_\omega(s), \omega(s)) ds, \quad (2.9)$$

where

$$\ell(x, \omega, s) \triangleq \frac{1}{2} |\omega|^2 + \frac{1}{2} |\dot{y}(s) - h(s, x)|^2,$$

and the admissible set is defined by

$$\mathcal{A}_{t,x}^G \triangleq \{(x_\omega(0), \omega) \in \bar{G} \times L^2((0, t); \mathbb{R}^m) : x_\omega \text{ follows (2.1) with } x_\omega(t) = x\}.$$

The admissible set is always non-empty. By standard arguments for value functions in control theory (see e.g. [Clarke et al., 2008, Chapter 4, Section 7] for non-smooth dynamics),  $\mathcal{V}$  is a continuous function. Before studying  $\mathcal{V}$  itself, we describe an approximating procedure that brings the problem back to smooth unconstrained dynamics.

**Remark 2.2.2** (Inward pointing drift). *Let us assume that  $b(x) \cdot n(x) \leq 0$  for every  $x \in \partial G$ ,  $n(x)$  being the outward normal vector at  $x$ . For simplicity, we further assume that  $n = r$  and  $\sigma \equiv \text{Id}$ . In this setting, the control problem (2.9) of the differential inclusion (2.1) reduces to a standard control problem for a differential equation under state constraints. Indeed, introducing the solution  $z_\omega$  of*

$$\dot{z}_\omega(s) = b(s, z_\omega(s)) + \omega(s), \quad (2.10)$$

we notice that

$$\mathcal{V}(t, x) = \inf_{\substack{\omega, z_\omega(t)=x \\ \forall s \in [0, t], z_\omega(s) \in \bar{G}}} \psi(z_\omega(0)) + \int_0^t \ell(s, z_\omega(s), \omega(s)) \, ds.$$

The  $\leq$  inequality stems from the fact that any trajectory of (2.10) that stays in  $\bar{G}$  is a trajectory of (2.1). The  $\geq$  inequality results from the fact that any trajectory of (2.1) can be realised by a trajectory of (2.10) for a  $\omega$  that has a lower  $L^2$ -norm, using that  $b \cdot n \leq 0$ .

### 2.2.1 Penalisation approach

In this section, we introduce an estimation problem for an approximation of (2.1) defined in the whole space  $\mathbb{R}^n$ . With a slight abuse of notations, we assume that our coefficients  $b$ ,  $\sigma$ ,  $\psi$  and  $h$  are defined and Lipschitz-continuous on  $\mathbb{R}^n$ . Let  $\pi_{\bar{G}}$  denote the orthogonal projection on the closed convex set  $\bar{G}$ . For  $\kappa > 0$ , we introduce the penalisation  $f_\kappa : G \rightarrow \mathbb{R}^n$  of  $\partial\chi_G$  defined by  $f_\kappa(x) \triangleq \kappa[x - \pi_{\bar{G}}(x)]$ . Since  $G$  is convex, we can follow the Moreau-Yosida regularisation Moreau [1971], which was extended to more general domains in Thibault [2008]; Jourani and Vilches [2017]. We also refer to de Pinho et al. [2019] for an alternative exponential penalisation that is smooth. We thus introduce the penalised dynamics

$$\dot{x}_\omega^\kappa(t) + f_\kappa(x_\omega^\kappa(t)) = b(t, x_\omega^\kappa(t)) + \sigma(t, x_\omega^\kappa(t))\omega(t), \quad t > 0. \quad (2.11)$$

The penalised cost-to-come is defined as

$$\mathcal{V}^\kappa(t, x) \triangleq \inf_{\substack{\omega \in L^2(0, t), \\ x_\omega^\kappa(t) = x}} \psi(x_\omega^\kappa(0)) + \int_0^t \ell(s, x_\omega^\kappa(s), \omega(s)) \, ds. \quad (2.12)$$

By standard arguments in control theory [Fleming and Soner, 2006, Section II.10],  $\mathcal{V}^\kappa$  is a continuous function. Our first main results are the following.

**Theorem 2.2.3** (Quantitative convergence of  $\mathcal{V}^\kappa$ ). *For every  $t > 0$ ,*

$$\sup_{(s, x) \in [0, t] \times \bar{G}} |\mathcal{V}^\kappa(s, x) - \mathcal{V}(s, x)| \leq C\kappa^{-1/4},$$

for some constant  $C > 0$  that only depends on  $t$ .

**Corollary 2.2.3.1** (Convergence of observers). *For every  $t > 0$ , any limit point of a family  $(\hat{x}^\kappa(t))_{\kappa > 0}$  of minimisers for  $x \mapsto \mathcal{V}^\kappa(t, x)$  is a minimiser of  $x \mapsto \mathcal{V}(t, x)$ .*

Theorem 2.2.3 and its corollary are proved in Section 2.3.1. These consistency results show that the estimation problem for (2.11) is indeed a good approximation of the estimation problem for (2.1), and provide the convergence of Mortensen observers.

## 2.2.2 Hamilton-Jacobi-Bellman equation

Since our estimation procedure is based on the functions  $\mathcal{V}^\kappa$  and  $\mathcal{V}$ , we need a way to characterise them. Using the standard dynamic programming approach, the cost-to-come is the unique viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation. The notion of viscosity solution is standard to deal with the lack of regularity of solutions to HJB equations. The needed definitions are reminded in Appendix 2.A. For the sake of simplicity, we assume that  $n = r$  and  $\sigma \equiv \text{Id}$ . The following result corresponds to [James and Baras, 1988b, Theorem 3.1]. It is proved in [Fleming, 1997, Section 2], under the additional assumption that  $h$  is bounded. However, the result still holds under our running assumptions, an appropriate comparison principle for unbounded viscosity solutions being provided by Da Lio and Ley [2006].

**Theorem 2.2.4** (Viscosity solution).  *$\mathcal{V}^\kappa$  is the unique viscosity solution in  $[0, T] \times \mathbb{R}^n$  of the following HJB equation in the sense of Definition 2.A.1,*

$$\partial_t \mathcal{V}^\kappa(t, x) + (b(t, x) - f_\kappa(x)) \cdot \nabla \mathcal{V}^\kappa(t, x) + \frac{1}{2} |\nabla \mathcal{V}^\kappa(t, x)|^2 - \frac{1}{2} |\dot{y}(t) - h(t, x)|^2 = 0,$$

with the initial condition  $\mathcal{V}^\kappa(0, x) = \psi(x)$ .

Formally taking the  $\kappa \rightarrow +\infty$  limit, we prove that this equation still holds for  $\mathcal{V}$  within the open domain  $G$ ,

$$\partial_t \mathcal{V}(t, x) + b(t, x) \cdot \nabla \mathcal{V}(t, x) + \frac{1}{2} |\nabla \mathcal{V}(t, x)|^2 - \frac{1}{2} |\dot{y}(t) - h(t, x)|^2 = 0. \quad (2.13)$$

However, intricate boundary conditions now appear.

**Theorem 2.2.5** (Viscosity solution). *In the sense of Definition 2.A.2,*

(i)  *$\mathcal{V}$  is a viscosity sub-solution of the HJB equation (2.13) with the boundary condition*

$$b(t, x) \cdot n(x) + \frac{\partial \mathcal{V}}{\partial n}(t, x) = 0, \quad x \in \partial G. \quad (2.14)$$

(ii)  *$\mathcal{V}$  is a viscosity super-solution of the HJB equation (2.13) with the different boundary condition*

$$b(t, x) \cdot n(x) + \frac{1}{2} \frac{\partial \mathcal{V}}{\partial n}(t, x) = 0, \quad x \in \partial G. \quad (2.15)$$

In the boundary conditions (2.14)-(2.15),  $n(x)$  denotes the outward normal vector at the point  $x \in \partial G$ . Theorem 2.2.5 is proved in Section 2.3.2. The notion of viscosity solution used in Theorem 2.2.5 is detailed in Appendix 2.A; it is the standard notion of viscosity solution with Neumann boundary condition Lions [1985]; Barles and Lions [1991], in which the boundary condition is relaxed by allowing the equation to hold at the boundary. In general, the boundary conditions (2.14) and (2.15) do not coincide. Up to our knowledge, proving a comparison principle for this setting is an open question, which will be the subject of future works.

However, if  $b \cdot n \leq 0$  on  $\partial G$ , Definition 2.A.2 shows that any viscosity sub-solution of (2.13) with the boundary condition (2.14) is also a viscosity sub-solution of (2.13) with the boundary condition (2.15). The comparison principle is then proved in [Barles and Lions, 1991, Theorem 3] and uniqueness holds for the viscosity solution of the HJB equation (2.13)-(2.15). Furthermore, Remark 2.3.4 shows in this case that  $\mathcal{V}$  is a viscosity super-solution of (2.13) on the closed domain  $[0, T] \times \overline{G}$ , with no further boundary condition. This recovers the notions of *constrained viscosity solution* introduced in Soner [1986]; Ishii

and Koike [1996]. This was expected, since the assumption  $b \cdot n \leq 0$  reduces the control problem (2.9) to the standard control problem of (2.10) under state constraints using Remark 2.2.2, and (2.10) satisfies the *inward pointing condition* from Soner [1986]. The related comparison principle is proved in [Capuzzo-Dolcetta and Lions, 1990, Theorem III.2].

### 2.2.3 Links with stochastic filtering

In general, viscosity solutions of HJB equations lack regularity; this may hinder an efficient computation of their minimisers. To circumvent this, a standard approach is to smooth them by adding a (small) laplacian term. Interestingly, Hijab [1984]; James and Baras [1988b]; Fleming [1997] proved that this smoothing procedure is connected to the renowned stochastic filtering problem Kallianpur and Striebel [1968]; Kushner [1967]; Jazwinski [1970]... In our constrained setting, this amounts to replacing the model dynamics (2.1) by the stochastic reflected dynamics

$$dX_t^\varepsilon + \partial\chi_G(X_t^\varepsilon)(dt) \ni b(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(t, X_t^\varepsilon)dB_t, \quad (2.16)$$

on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ , the law of  $X_0^\varepsilon$  being  $q_0^\varepsilon(x)dx$ . The integral of the deterministic model noise  $\omega$  in (2.1) has been replaced by an adapted Brownian motion  $(B_t)_{0 \leq t \leq T}$  in  $\mathbb{R}^r$ . Under our current Lipschitz assumptions, well-posedness for (2.16) is proved in [Pardoux and Răşcanu, 2014, Section 4.2.2]. We complement (2.16) with the observation process defined by

$$dY_t^\varepsilon = h(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t', \quad Y_0^\varepsilon = 0, \quad (2.17)$$

where  $(B_t')_{0 \leq t \leq T}$  is an adapted Brownian motion in  $\mathbb{R}^m$  independent of  $(B_t)_{0 \leq t \leq T}$ . Equation (2.17) is the stochastic analogous of (2.2). The purpose of stochastic filtering is to compute the law  $\pi_t^\varepsilon$  of  $X_t^\varepsilon$  knowing the observation up to time  $t$ , i.e. the law of  $X_t^\varepsilon$  conditionally to the filtration  $\sigma(Y_s^\varepsilon, 0 \leq s \leq t)$ :

$$\forall \varphi \in C_b(\overline{G}, \mathbb{R}), \quad \int_{\overline{G}} \varphi d\pi_t^\varepsilon = \mathbb{E}[\varphi(X_t^\varepsilon) | \sigma(Y_s^\varepsilon, 0 \leq s \leq t)].$$

Hence,  $\int_{\overline{G}} \varphi d\pi_t^\varepsilon$  is the optimal estimator in the least-square sense of  $\varphi(X_t^\varepsilon)$  given  $(Y_s^\varepsilon)_{0 \leq s \leq t}$ . From [Pardoux, 1978a, Section 3],  $\pi_t^\varepsilon$  is expected to have a density w.r.t. the Lebesgue measure, and thus  $\pi_t^\varepsilon$  does not charge the boundary  $\partial G$ . We are now interested in the  $\varepsilon \rightarrow 0$  behavior of  $\pi_t^\varepsilon$ . In particular, we show that  $\pi_t^\varepsilon$  concentrates on minimisers of  $x \mapsto \mathcal{V}(t, x)$  as  $\varepsilon \rightarrow 0$ , corresponding to the Mortensen observer  $\hat{x}(t)$  when it is uniquely defined. We first review the existing results for non-constrained systems in Sections 2.2.3.1-2.2.3.2, before stating our new results for the non-smooth case in Section 2.2.3.3.

#### 2.2.3.1 Kalman-Bucy filter

We go back to the linear setting of Section 2.1.2,

$$b(t, x) = Ax, \quad \sigma(t, x) = \Sigma, \quad h(t, x) = Hx,$$

and we initialise  $X_0^\varepsilon$  from the Gaussian law  $\mathcal{N}(\hat{x}_0, \varepsilon P_0)$ . The processes defined by (2.16) and (2.17) are then Gaussian at each time, so that the same goes for the conditioned process: it is thus sufficient to compute the conditional mean and covariance matrix  $(\hat{X}_t^\varepsilon, P_t^\varepsilon)$ . Classically Kalman and Bucy [1961]; Jazwinski [1970]; Davis and Davis [1977], the covariance  $P_t^\varepsilon = \varepsilon P(t)$  is deterministic, and

$$d\hat{X}_t^\varepsilon = A\hat{X}_t^\varepsilon dt + P(t)H^\top [dY_t^\varepsilon - H\hat{X}_t^\varepsilon dt], \quad \hat{X}_0^\varepsilon = \hat{x}_0, \quad (2.18)$$

where  $P(t)$  is the solution to the Riccati equation (2.4). As a consequence,

$$\pi_t^\varepsilon(dx) = Z_\varepsilon^{-1} \exp \left[ - \frac{[x - \hat{X}_t^\varepsilon] \cdot P^{-1}(t)[x - \hat{X}_t^\varepsilon]}{2\varepsilon} \right] dx,$$

where  $Z_\varepsilon$  is a normalisation constant. As  $\varepsilon \rightarrow 0$ , this density concentrates on the solution  $\hat{x}(t)$  of the deterministic equation (2.3). If we freeze a  $C^1$  realisation  $(y(t))_{t \geq 0}$  of the process  $(Y_t^\varepsilon)_{t \geq 0}$ , Equation (2.18) exactly correspond to (2.3), and the related quadratic cost-to-come corresponds to the logarithm of the density of  $\pi_t^\varepsilon$ . The Mortensen observer in this linear setting thus appears as a maximum-likelihood estimator.

### 2.2.3.2 Small noise filtering for the penalised dynamics

Let us apply the results of James and Baras [1988b] to (a variation of) the penalised dynamics (2.11) from Section 2.2.1, in the case  $n = r$  and  $\sigma \equiv \text{Id}$ . In this setting, the analogous of (2.16)-(2.17) reads

$$\begin{cases} dX_t^{\kappa,\varepsilon} = b(t, X_t^{\kappa,\varepsilon})dt - f_\kappa(X_t^{\kappa,\varepsilon})dt + \sqrt{\varepsilon}dB_t, \\ dY_t^{\kappa,\varepsilon} = h(t, X_t^{\kappa,\varepsilon})dt + \sqrt{\varepsilon}dB'_t, \end{cases}$$

where  $(X_t^{\kappa,\varepsilon})_{t \geq 0}$  is a diffusion process in  $\mathbb{R}^n$  with initial law  $q_0^{\kappa,\varepsilon}$ . The results of James and Baras [1988b] assume that the coefficients are smooth bounded functions with bounded derivatives, and we assume the same for  $b$ ,  $f_\kappa$  and  $h$  in this sub-section, up to replacing the penalisation  $f_\kappa$  from Section 2.2.1 by a suitable regularisation. Since the present sub-section is mainly here for illustration and comparison purposes, we did not try to improve these assumptions.

The filtering density  $\pi_t^{\kappa,\varepsilon}$  can be computed as the solution of a non-linear stochastic PDE known as the Kushner-Stratonovich equation Kushner [1967]. Alternatively,  $\pi_t^{\kappa,\varepsilon}(dx)$  can be computed by normalising the positive (random) measure  $q_t^{\kappa,\varepsilon}(x)dx$  that solves the Zakai equation Zakai [1969]; Pardoux [1980]:

$$dq_t^{\kappa,\varepsilon}(x) = (L_t^{\kappa,\varepsilon})^* q_t^{\kappa,\varepsilon}(x)dt + \frac{1}{\varepsilon} q_t^{\kappa,\varepsilon} h(t, x) \cdot dY_t^{\kappa,\varepsilon}, \quad (2.19)$$

where  $(L_t^{\kappa,\varepsilon})^*$  is the formal  $L^2$ -adjoint of the infinitesimal generator  $L_t^{\kappa,\varepsilon}$  of the Markov process  $(X_t^{\kappa,\varepsilon})_{t \geq 0}$ , which is given on  $C^2$  test functions  $\varphi : \mathbb{R}^\omega \rightarrow \mathbb{R}$  by

$$L_t^{\kappa,\varepsilon}\varphi(x) = [b(t, x) - f_\kappa(x)] \cdot \nabla\varphi(x) + \frac{\varepsilon}{2}\Delta\varphi(x).$$

Following the theory of pathwise filtering Davis [1981], we introduce the random function  $p_t^{\kappa,\varepsilon}$  defined by

$$p_t^{\kappa,\varepsilon}(x) \triangleq \left[ - \frac{1}{\varepsilon} Y_t^{\kappa,\varepsilon} h(t, x) \right] q_t^{\kappa,\varepsilon}(x), \quad (2.20)$$

which solves the robust Zakai equation

$$\begin{aligned} \partial_t p_t^{\kappa,\varepsilon}(x) = & -[b(t, x) - f_\kappa(x) - \nabla h(t, x)Y_t^{\kappa,\varepsilon}] \cdot \nabla p_t^{\kappa,\varepsilon} + \frac{\varepsilon}{2}\Delta p_t^{\kappa,\varepsilon} - \frac{1}{\varepsilon} \left\{ \frac{1}{2}|h(t, x)|^2 \right. \\ & \left. + Y_t^{\kappa,\varepsilon} \cdot [\partial_t + L_t^{\kappa,\varepsilon}]h(t, x) - \frac{1}{2}|\nabla h(t, x)Y_t^{\kappa,\varepsilon}|^2 + \varepsilon \nabla \cdot [b(t, x) - f_\kappa(x) - \nabla h(t, x)Y_t^{\kappa,\varepsilon}] \right\}. \end{aligned} \quad (2.21)$$

For each continuous realisation  $y = (y(t))_{t \geq 0}$  of the process  $(Y_t^{\kappa,\varepsilon})_{t \geq 0}$ , Equation (2.21) has a unique (deterministic) solution  $p^{\kappa,\varepsilon}$  which continuously depends on  $y$  [Fleming and

Pardoux, 1982, Lemma 3.2]. This allows us to recover the random function  $p_t^{\kappa,\varepsilon}$  by solving (2.21) for each continuous path  $y$ . This result motivates the approach of James and Baras [1988b]: they freeze a  $C^1$  realisation  $(y(t))_{t \geq 0}$  of the process  $(Y_t^{\kappa,\varepsilon})_{t \geq 0}$ , before to study the related solution  $p^{\kappa,\varepsilon}$  of (2.21) to make a connection with  $\mathcal{V}^\kappa$  as  $\varepsilon \rightarrow 0$ . Alternatively, Equation (2.20) gives a meaning to  $q^{\kappa,\varepsilon}$  from  $p^{\kappa,\varepsilon}$  for each continuous realisation  $y = (y(t))_{t \geq 0}$  of  $(Y_t^{\kappa,\varepsilon})_{t \geq 0}$ :  $q^{\kappa,\varepsilon}$  is now a continuous function of  $y$ . [Fleming, 1997, Section 5] then freezes a  $C^1$  realisation  $y$  to obtain that

$$\partial_t q^{\kappa,\varepsilon}(t, x) = (L_t^{\kappa,\varepsilon})^* q^{\kappa,\varepsilon}(t, x) dt - \frac{1}{\varepsilon} \left[ \frac{1}{2} |h(t, x)|^2 - \dot{y}(t) \cdot h(t, x) \right] q^{\kappa,\varepsilon}(t, x), \quad (2.22)$$

which corresponds to the Stratonovich form of (2.19) where  $(Y_t^{\kappa,\varepsilon})_{t \geq 0}$  was replaced by  $(y(t))_{t \geq 0}$ . Let now

$$\tilde{q}^{\kappa,\varepsilon}(t, x) \triangleq \exp \left[ -\frac{1}{2\varepsilon} \int_0^t |\dot{y}(s)|^2 ds \right] q^{\kappa,\varepsilon}(t, x), \quad \mathcal{V}^{\kappa,\varepsilon}(t, x) \triangleq -\varepsilon \log \tilde{q}^{\kappa,\varepsilon}(t, x).$$

The exponential factor is a normalisation term that does not affect the minimisation of  $x \mapsto \mathcal{V}^{\kappa,\varepsilon}(t, x)$ . The following result [Fleming, 1997, Lemma 5.1] makes the connection with the penalised cost-to-come  $\mathcal{V}^\kappa$  defined in (2.6).

**Theorem 2.2.6** (Small noise limit). *For every compact set  $K \subset \mathbb{R}^n$ , if*

$$\sup_{x \in K} |-\varepsilon \log q_0^{\kappa,\varepsilon}(x) - \psi(x)| \leq C_K \varepsilon^{1/2}, \quad (2.23)$$

for some  $C_K > 0$  independent of  $\varepsilon$ , then for every  $t > 0$ ,

$$\sup_{(s,x) \in [0,t] \times K} |\mathcal{V}^{\kappa,\varepsilon}(s, x) - \mathcal{V}^\kappa(s, x)| \leq C'_K \varepsilon^{1/2},$$

for some constant  $C'_K > 0$  that only depends on  $(t, \kappa, K)$ .

As a consequence (see e.g. [James and Baras, 1988b, Lemma 6.1]), we get the following large deviation result.

**Corollary 2.2.6.1** (Laplace principle). *If (2.23) holds for every compact set, then for every bounded continuous  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ ,*

$$\forall t \geq 0, \quad -\varepsilon \log \int_{\mathbb{R}^n} e^{-\Phi(x)/\varepsilon} \tilde{q}^{\kappa,\varepsilon}(t, x) dx \xrightarrow{\varepsilon \rightarrow 0} \inf_{x \in \mathbb{R}^n} \Phi(x) + \mathcal{V}^\kappa(t, x).$$

This statement is equivalent to a large deviation principle for  $\tilde{q}^{\kappa,\varepsilon}(t, x) dx$ , see e.g. Dembo [2009]. As  $\varepsilon \rightarrow 0$ , this tells that the non-normalised density  $q_t^{\varepsilon,\kappa}$  concentrates on the minimisers of  $x \mapsto \mathcal{V}^\kappa(t, x)$ . When uniqueness holds for this minimiser, this precisely corresponds to the Mortensen observer (2.7) as defined in Section 2.1.3. This extends the observation that we made in the Gaussian setting of Section 2.2.3.1. A stronger large deviation result for the conditional density  $q_t^{\kappa,\varepsilon}$  given  $y$  is proved in Hijab [1984].

### 2.2.3.3 Small noise filtering for reflected dynamics

We now extend the previous results to the setting of reflected dynamics:

$$\begin{cases} dX_t^\varepsilon + \partial \chi_G(X_t^\varepsilon)(dt) \ni b(t, X_t^\varepsilon) dt + \sqrt{\varepsilon} dB_t, \\ dY_t^\varepsilon = h(t, X_t^{\kappa,\varepsilon}) dt + \sqrt{\varepsilon} dB_t', \end{cases}$$

the initial law of  $x$  being  $q_0^\varepsilon(x)dx$ . As previously, the filtering density  $\pi_t^\varepsilon(dx)$  can be computed by normalising the solution  $q_t^\varepsilon(x)dx$  of the Zakai equation. In this setting, the Zakai equation was studied in Pardoux [1978a,b]; Hücke [1990]. Equation (2.19) is now completed with a “no-flux” boundary condition:

$$\begin{cases} dq_t^\varepsilon(x) = \nabla \cdot [-q_t^\varepsilon(x)b(t,x) + \frac{\varepsilon}{2}\nabla q_t^\varepsilon(x)] + \frac{1}{\varepsilon}q_t^\varepsilon(x)h(x) \cdot dY_t^\varepsilon, & x \in G, \\ -q_t^\varepsilon(x)b(t,x) \cdot n(x) + \frac{\varepsilon}{2}\frac{\partial q_t^\varepsilon}{\partial n}(x) = 0, & x \in \partial G, \end{cases}$$

As previously, we freeze a  $C^1$  realisation  $y = (y(t))_{t \geq 0}$  to bring the study back to a deterministic function  $q^\varepsilon$  that continuously depends on  $y$ . After normalising

$$\tilde{q}^\varepsilon(t,x) \triangleq \exp \left[ -\frac{1}{2\varepsilon} \int_0^t |\dot{y}(s)|^2 ds \right] q^\varepsilon(t,x),$$

we end up with the following equation:

$$\begin{cases} \partial_t \tilde{q}^\varepsilon(t,x) = \nabla \cdot [-\tilde{q}^\varepsilon(t,x)b(t,x) + \frac{\varepsilon}{2}\nabla \tilde{q}^\varepsilon(t,x)] - \frac{1}{2\varepsilon}|\dot{y}(t) - h(t,x)|^2 \tilde{q}^\varepsilon(t,x), \\ -\tilde{q}^\varepsilon(t,x)b(t,x) \cdot n(x) + \frac{\varepsilon}{2}\frac{\partial \tilde{q}^\varepsilon}{\partial n}(t,x) = 0, & x \in \partial G. \end{cases} \quad (2.24)$$

We assume that  $q_0^\varepsilon$  is defined and  $C^1$  in a neighbourhood of  $\partial G$ , so that existence and uniqueness for a solution of (2.24) in  $C([0,T] \times \overline{G}) \cap C^{1,2}((0,T] \times G)$  stems from [Friedman, 2008, Chapter 5, Corollary 2]. If we define  $\mathcal{V}^\varepsilon \triangleq -\varepsilon \log \tilde{q}^\varepsilon$  as previously, we obtain the boundary condition

$$b(t,x) \cdot n(x) + \frac{1}{2} \frac{\partial \mathcal{V}^\varepsilon}{\partial n}(t,x)(x) = 0, \quad x \in \partial G,$$

which differs from the one of the sub-solution in Theorem 2.2.5. This suggests that the analogous of Theorem 2.2.6 is no more true in the present setting (except if  $b \cdot n \leq 0$  on  $\partial G$ ). However, we still managed to prove the large deviation result corresponding to Corollary 2.2.6.1.

**Theorem 2.2.7** (Laplace principle). *If*

$$\sup_{x \in \overline{G}} |-\varepsilon \log q_0^\varepsilon(x) - \psi(x)| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

then for every continuous  $\Phi : \overline{G} \rightarrow \mathbb{R}$ ,

$$\forall t \geq 0, \quad -\varepsilon \log \int_G e^{-\Phi(x)/\varepsilon} \tilde{q}^\varepsilon(t,x) dx \xrightarrow{\varepsilon \rightarrow 0} \inf_{x \in \overline{G}} \Phi(x) + \mathcal{V}(t,x).$$

As  $\varepsilon \rightarrow 0$ , this tells that the non-normalised density  $q_t^\varepsilon$  concentrates on the minimisers of  $x \mapsto \mathcal{V}(t,x)$ , recovering the desired Mortensen observer. Theorem 2.2.7 is proved in Section 2.4.

**Remark 2.2.8** (Loss of the boundary condition). *Interestingly, the boundary condition for  $\mathcal{V}^\varepsilon$  differs from the one of the sub-solution in Theorem 2.2.5. This reminds us of a fundamental difference between  $X^\varepsilon$  and  $x_\omega$  around the boundary  $\partial G$ . Indeed, the time spent by  $X^\varepsilon$  on  $\partial G$  has 0 Lebesgue-measure, whereas  $x_\omega$  can have a continuous dynamics on  $\partial G$ . This is also related to the non-reversibility of non-smooth dynamics.*

## 2.3 Study of the value function

This section is devoted to the proofs of Theorems 2.2.3 and 2.2.5, i.e. the uniform convergence of  $\mathcal{V}^\varepsilon$  and the viscosity solution description of  $\mathcal{V}$ .

### 2.3.1 Convergence of the penalised problem

**Lemma 2.3.1** (Control of the penalisation). *For every  $(x_\omega(0), \omega)$  in  $\overline{G} \times L^2((0, t), \mathbb{R}^r)$ :*

- (i)  $\sup_{\kappa > 0} \sup_{0 \leq s \leq t} |x_\omega^\kappa(s)| \leq C,$
- (ii)  $\sup_{\kappa > 0} \sup_{0 \leq s \leq t} \int_0^t |f_\kappa(x_\omega^\kappa(s))| ds \leq C,$
- (iii)  $\sup_{0 \leq s \leq t} \text{dist}(x_\omega^\kappa(s), \overline{G}) \leq C\kappa^{-1/2},$

for a constant  $C > 0$  that only depends on  $(t, \|\omega\|_{L^2})$ .

*Proof.* First, we write that

$$|x_\omega^\kappa(t)| \leq |x_\omega(0)| + \int_0^t |f_\kappa(x_\omega^\kappa(s))| + |b(s, x_\omega^\kappa(s)) + \sigma(s, x_\omega^\kappa(s))\omega(s)| ds. \quad (2.25)$$

We define  $p(x) \triangleq \frac{1}{2} \text{dist}(x, \overline{G})^2 = \frac{1}{2} (x - \pi_{\overline{G}}(x))^2$ . Since  $\overline{G}$  is closed and convex,  $p$  is  $C^1$  with  $\nabla p = \kappa^{-1} f_\kappa$ , so that

$$\begin{aligned} p(x_\omega^\kappa(t)) + \kappa \int_0^t |\nabla p(x_\omega^\kappa(s))|^2 ds \\ = \int_0^t \nabla p(x_\omega^\kappa(s)) \cdot [b(s, x_\omega^\kappa(s)) + \sigma(s, x_\omega^\kappa(s))\omega(s)] ds, \end{aligned}$$

and then

$$\left( \int_0^t |\nabla p(x_\omega^\kappa(s))|^2 ds \right)^{1/2} \leq \kappa^{-1} \left( \int_0^t |b(s, x_\omega^\kappa(s)) + \sigma(s, x_\omega^\kappa(s))\omega(s)|^2 ds \right)^{1/2}, \quad (2.26a)$$

$$p(x_\omega^\kappa(t)) \leq \kappa^{-1} \left( \int_0^t |b(s, x_\omega^\kappa(s)) + \sigma(s, x_\omega^\kappa(s))\omega(s)|^2 ds \right)^{1/2}. \quad (2.26b)$$

Plugging the square of (2.26a) into the square of (2.25), the Gronwall lemma yields (i) using that coefficients are Lipschitz. Plugging (i) into (2.26a)-(2.26b) then yields (ii)-(iii).  $\square$

**Lemma 2.3.2** (Convergence of penalised curves). *For  $(x_\omega(0), \omega)$  in  $\overline{G} \times L^2((0, t), \mathbb{R}^r)$ ,*

$$\sup_{0 \leq s \leq t} |x_\omega^\kappa(s) - x_\omega(s)| \leq C\kappa^{-1/4},$$

for a constant  $C > 0$  that only depends on  $(t, \|\omega\|_{L^2})$ .

*Proof.* We define the curves  $y$ ,  $\overline{y}^\kappa$  and  $y^\kappa$  by

$$\begin{aligned} y(t) &\triangleq x_\omega(0) + \int_0^t b(s, x_\omega(s)) + \sigma(s, x_\omega(s))\omega(s) ds, \\ \overline{y}^\kappa(t) &\triangleq x_\omega(0) + \int_0^t b(s, x_\omega^\kappa(s)) + \sigma(s, x_\omega^\kappa(s))\omega(s) ds, \\ y^\kappa(t) &\triangleq \pi_{\overline{G}}(x_\omega^\kappa(t)) - x_\omega^\kappa(t) + \overline{y}^\kappa(t), \end{aligned}$$

together with the correction terms

$$\varphi(t) \triangleq x_\omega(t) - y(t), \quad \varphi^\kappa(t) \triangleq - \int_0^t f_\kappa(x_\omega^\kappa(s)) ds.$$



We then apply [Tanaka, 1979, Lemma 2.2] to  $\pi_{\bar{G}}(x_\omega^\kappa) = y^\kappa + \varphi^\kappa$  and  $x_\omega = y + \varphi$ :

$$\begin{aligned} & |\pi_{\bar{G}}(x_\omega^\kappa(t)) - x_\omega(t)|^2 \\ & \leq |y^\kappa(t) - y(t)|^2 + 2 \int_0^t [y^\kappa(t) - y(t) - y^\kappa(s) + y(s)] d(\varphi^\kappa - \varphi)(s) \\ & \leq |y^\kappa(t) - y(t)|^2 + 2 \int_0^t [y^\kappa(t) - \bar{y}^\kappa(t) - y^\kappa(s) + \bar{y}^\kappa(s)] d(\varphi^\kappa - \varphi)(s) \\ & \quad + 2 \int_0^t [\bar{y}^\kappa(t) - y(t) - \bar{y}^\kappa(s) + y(s)] d(\varphi^\kappa - \varphi)(s). \end{aligned}$$

The first integral is bounded by

$$2 \sup_{0 \leq s \leq t} |\pi_{\bar{G}}(x_\omega^\kappa(s)) - x_\omega^\kappa(s)| \int_0^t d|\varphi^\kappa|(s) + d|\varphi|(s) \leq C\kappa^{-1/2},$$

where we used Lemma 2.3.1-(iii), and Lemma 2.3.1-(ii) to bound  $\int_0^t d|\varphi^\kappa|(s) = \int_0^t |f_\kappa(x_\omega^\kappa(s))| ds$  uniformly in  $\kappa$ . Integrating by parts, the second integral becomes

$$\begin{aligned} \int_0^t [\varphi^\kappa(s) - \varphi(s)] d(\bar{y}^\kappa - y)(s) &= \int_0^t [x_\omega^\kappa(s) - x_\omega(s) - \bar{y}^\kappa(s) + y(s)] d(\bar{y}^\kappa - y)(s) \\ &= \int_0^t [x_\omega^\kappa(s) - x_\omega(s)] d(\bar{y}^\kappa - y)(s) - \frac{1}{2} |\bar{y}^\kappa(t) - y(t)|^2. \end{aligned}$$

We now notice that  $\bar{y}^\kappa - y$  is absolutely continuous w.r.t. the Lebesgue measure, and we bound the remaining integral by

$$\begin{aligned} & 2 \int_0^t |x_\omega^\kappa(s) - x_\omega(s)|^2 + |b(s, x_\omega^\kappa(s)) \\ & \quad + \sigma(s, x_\omega^\kappa(s))\omega(s) - b(s, x_\omega(s)) - \sigma(s, x_\omega(s))\omega(s)|^2 ds, \end{aligned}$$

where we also used that  $ab \leq 2(a^2 + b^2)$ . Lemma 2.3.1-(iii) gives  $C' > 0$  such that

$$|x_\omega^\kappa(s) - x_\omega(s)| \leq C'\kappa^{-1/2} + |\pi_{\bar{G}}(x_\omega^\kappa(s)) - x_\omega(s)|.$$

Gathering all the terms and using that coefficients are Lipschitz, the Gronwall lemma gives the desired result.  $\square$

**Corollary 2.3.2.1** (Time-regularity). *For every  $(x_\omega(0), \omega)$  in  $\bar{G} \times L^2((0, t), \mathbb{R}^r)$ ,*

$$(i) \quad \forall s \in [0, t], \quad \int_s^t |\dot{x}_\omega(r)|^2 dr \leq C(t-s) + \int_s^t |\omega(r)|^2 dr,$$

$$(ii) \quad \forall r, s \in [0, t], \quad |x_\omega(r) - x_\omega(s)| \leq C|r-s|^{1/2},$$

for a constant  $C > 0$  that only depends on  $(t, \|\omega\|_{L^2})$ .

*Proof.* Starting from the penalised dynamics:

$$\dot{x}_\omega^\kappa(r) = -f_\kappa(x_\omega^\kappa(r)) + b(r, x_\omega^\kappa(r)) + \sigma(r, x_\omega^\kappa(r))\omega(r).$$

We now take the square and we integrate. Using Lemma 2.3.1-(i) and (2.26a) to bound the penalisation, we obtain a bound on  $x_\omega^\kappa$  in  $H^1((s, t), \mathbb{R}^n)$  that does not depend on  $\kappa$ . Since  $x_\omega^\kappa$  uniformly converges towards  $x_\omega$ , this bound implies weak convergence towards  $x_\omega$  in  $H^1((s, t), \mathbb{R}^n)$ . The  $H^1$ -norm being lower semi-continuous w.r.t. weak convergence, this proves (i). Using the Cauchy-Schwarz inequality, (ii) is a consequence of (i).  $\square$

*Proof of Theorem 2.2.3.* From Lemma 2.3.2-(i),  $\mathcal{V}^\kappa$  can be bounded by a constant  $M > 0$  uniformly in  $\kappa$ . As a consequence, we can restrict the minimisation (2.12) to controls  $\omega$  with square  $L^2$ -norm lower than  $2M$ . We then plug the result of Lemma 2.3.2 into the minimisation (2.12): since  $\psi$  and  $h$  are Lipschitz-continuous, this completes the proof.  $\square$

*Proof of Corollary 2.2.3.1.* Let  $(\hat{x}^\kappa(t))_{\kappa>0}$  be any family of minimisers for  $x \mapsto \mathcal{V}^\kappa(t, x)$ . As in the above proof, we restrict the minimisation (2.12) to controls  $\omega$  with square  $L^2$ -norm lower than  $2M$ . The coefficients being Lipschitz, it is standard to show that  $\mathcal{V}^\kappa(t, \hat{x}^\kappa(t))$  is realised by some  $\omega^\kappa \in L^2(0, t)$  with  $\|\omega^\kappa\|_{L^2(0, t)}^2 \leq 2M$ . Lemma 2.3.1-(i) then shows that  $(\hat{x}^\kappa(t))_{\kappa>0}$  is bounded, hence pre-compact. Since uniform convergence implies  $\Gamma$ -convergence, the result follows.  $\square$

### 2.3.2 Viscosity solution

The key-ingredient is the following dynamic programming principle.

**Lemma 2.3.3** (Bellman principle). *For any  $x$  in  $\overline{G}$  and  $0 \leq t - \tau \leq t \leq T$ , the dynamic programming holds:*

$$\mathcal{V}(t, x) = \inf_{(x_\omega(0), \omega) \in \mathcal{A}_{t,x}^G} \mathcal{V}(t - \tau, x_\omega(t - \tau)) + \int_{t-\tau}^t \ell(s, x_\omega(s), \omega(s)) ds.$$

The proof of this result is very standard in control theory, and we omit it. We now turn to the proof of Theorem 2.2.5.

*Proof for the sub-solution part.* Given  $t > 0$  and  $x \in \overline{G}$ , let  $\varphi : \mathbb{R}_+ \times \overline{G} \rightarrow \mathbb{R}$  be a  $C^1$  test function such that  $\mathcal{V} - \varphi$  has a local maximum at  $(t, x)$ . We now use that the control additively enters the dynamics ( $\sigma \equiv \text{Id}$ ). If  $x \in G$ , for every  $\tilde{\omega} \in \mathbb{R}^n$  we can find  $(x_\omega(0), \omega) \in \mathcal{A}_{t,x}^G$  such that  $\omega$  is continuous,  $\omega(t) = \tilde{\omega}$  and  $x_\omega(s) \in G$  for  $s < t$ . If  $x \in \partial G$ , this is still possible provided that  $[b(t, x) + \tilde{\omega}] \cdot n(x) \geq 0$ . From the local maximum condition,

$$\begin{aligned} \varphi(t, x) - \varphi(t - \tau, x_\omega(t - \tau)) &\leq \mathcal{V}(t, x) - \mathcal{V}(t - \tau, x_\omega(t - \tau)) \\ &\leq \int_{t-\tau}^t \ell(s, x_\omega(s), \omega(s)) ds, \end{aligned}$$

where the second inequality stems from Lemma 2.3.3. By construction of  $\omega$ ,  $s \mapsto x_\omega(s)$  is differentiable at  $t$ . Dividing by  $\tau$  and taking the  $\tau \rightarrow 0^+$  limit gives that

$$\partial_t \varphi(t, x) + [b(t, x) + \tilde{\omega}] \cdot \nabla \varphi(t, x) - \frac{1}{2} |\tilde{\omega}|^2 - \frac{1}{2} |\dot{y}(t) - h(t, x)|^2 \leq 0.$$

If  $x \in G$ , maximising over  $\tilde{\omega} \in \mathbb{R}^n$  gives the sub-solution property; moreover, the maximum is realised by  $\tilde{\omega} = \nabla \varphi(t, x)$ . If  $x \in \partial G$ , we can assume that  $[b(t, x) + \nabla \varphi(t, x)] \cdot n(x) \geq 0$  from Definition 2.A.2. This allows us to take  $\tilde{\omega} = \nabla \varphi(t, x)$  to realise the maximum. In both cases, we obtained the sub-solution property.  $\square$

*Proof for the super-solution part.* Given  $(t, x)$  in  $\mathbb{R}_+ \times \overline{G}$ , we consider a  $C^1$  test function  $\varphi : \mathbb{R}_+ \times \overline{G} \rightarrow \mathbb{R}$  such that  $\mathcal{V} - \varphi$  has a local minimum at  $(t, x)$ . Positive numbers  $\delta, h > 0$  exist such that

$$|t - t'| \leq \delta \text{ and } |x - x'| \leq h \Rightarrow \mathcal{V}(t', x') - \varphi(t', x') \geq \mathcal{V}(t, x) - \varphi(t, x). \quad (2.28)$$

Fix now  $\varepsilon > 0$ . Since  $\mathcal{V}$  is bounded on  $\overline{G}$  by some  $M > 0$ , we can restrict the minimisation (2.9) to controls  $\omega$  with square  $L^2$ -norm lower than  $2M$ . From Corollary 2.3.2.1-(ii),

$s \mapsto x_\omega(s)$  is continuous on  $[0, t]$  uniformly in  $(x_\omega(0), \omega) \in \mathcal{A}_{t,x}^G$  such that  $\|\omega\|_{L^2(0,t)}^2 \leq 2M$ . This provides  $\eta > 0$  such that for every  $(x_\omega(0), \omega) \in \mathcal{A}_{t,x}^G$  with  $\|\omega\|_{L^2(0,t)}^2 \leq 2M$ ,

$$0 \leq \tau \leq \eta \Rightarrow |x_\omega(t - \tau) - x| \leq h.$$

Let  $(\tau_k)_{k \geq 1}$  be a sequence that converges to 0 with  $0 < \tau_k \leq \min(\delta, \eta)$ . In Lemma 2.3.3, it is sufficient to minimise over  $(x_\omega(0), \omega) \in \mathcal{A}_{t,x}^G$  with  $\|\omega\|_{L^2(0,t)}^2 \leq 2M$ . By definition of the infimum, there exists  $(x_{\omega_k}(0), \omega_k) \in \mathcal{A}_{t,x}^G$  such that  $\|\omega_k\|_{L^2(0,t)}^2 \leq 2M$  and

$$\mathcal{V}(t, x) + \varepsilon\tau_k \geq \mathcal{V}(t - \tau_k, x_{\omega_k}(t - \tau_k)) + \int_{t-\tau_k}^t \ell(s, x_{\omega_k}(s), \omega_k(s)) ds. \quad (2.29)$$

Using (2.28), we get that

$$\begin{aligned} \varphi(t, x) - \varphi(t - \tau_k, x_{\omega_k}(t - \tau_k)) &\geq \mathcal{V}(t, x) - \mathcal{V}(t - \tau_k, x_{\omega_k}(t - \tau_k)) \\ &\geq -\varepsilon\tau_k + \int_{t-\tau_k}^t \ell(s, x_{\omega_k}(s), \omega_k(s)) ds. \end{aligned}$$

Since  $\varphi$  is  $C^1$  and  $s \mapsto x_{\omega_k}(s)$  is absolutely continuous, this yields

$$\int_{t-\tau_k}^t \partial_s \varphi(s, x_{\omega_k}(s)) + \dot{x}_{\omega_k}(s) \cdot \nabla \varphi(s, x_{\omega_k}(s)) - \ell(s, x_{\omega_k}(s), \omega_k(s)) ds \geq -\varepsilon\tau_k. \quad (2.30)$$

We now carefully handle the boundary.

**Case  $x \in G$ :** the uniform in  $k$  continuity of  $x_{\omega_k}$  provides that for large enough  $k$ ,

$$\forall s \in [t - \tau_k, t], \quad x_{\omega_k}(s) \in G, \quad \text{hence} \quad \dot{x}_{\omega_k}(s) = b(s, x_{\omega_k}(s)) + \omega_k(s).$$

Moreover,

$$\begin{aligned} b(s, x_{\omega_k}(s)) \cdot \nabla \varphi(s, x_{\omega_k}(s)) + \frac{1}{2} |\nabla \varphi(s, x_{\omega_k}(s))|^2 \\ \geq [b(s, x_{\omega_k}(s)) + \omega_k(s)] \cdot \nabla \varphi(s, x_{\omega_k}(s)) - \frac{1}{2} |\omega_k(s)|^2, \end{aligned}$$

hence from (2.30),

$$\begin{aligned} \int_{t-\tau_k}^t \left[ \partial_s \varphi(s, x_{\omega_k}(s)) + b(s, x_{\omega_k}(s)) \cdot \nabla \varphi(s, x_{\omega_k}(s)) + \frac{1}{2} |\nabla \varphi(s, x_{\omega_k}(s))|^2 \right. \\ \left. - \frac{1}{2} |\dot{y}(s) - h(s, x_{\omega_k}(s))|^2 \right] ds \geq -\varepsilon\tau_k. \quad (2.31) \end{aligned}$$

Using again that  $s \mapsto x_{\omega_k}(s)$  is continuous at  $s = t$  uniformly in  $k$ , we divide by  $\tau_k$  and we take the  $k \rightarrow +\infty$  limit to obtain that

$$\partial_t \varphi(t, x) + b(t, x) \cdot \nabla \varphi(t, x) + \frac{1}{2} |\nabla \varphi(t, x)|^2 - \frac{1}{2} |\dot{y}(t) - h(t, x)|^2 \geq -\varepsilon.$$

Since this holds for every  $\varepsilon > 0$ , this gives the super-solution property.

**Case  $x \in \partial G$  and  $\nabla \varphi(t, x) \cdot n(x) > 0$ :** since  $G$  has a  $C^2$  boundary, there exists a neighbourhood  $U$  of  $x$  in  $\mathbb{R}^n$  such that

$$G \cap U = \{y \in U, \gamma(y) < 0\}, \quad \partial G \cap U = \{y \in U, \gamma(y) = 0\},$$

for a  $C^2$  function  $\gamma : U \rightarrow \mathbb{R}$  with  $n(x) = \nabla\gamma(x)$ . We now decompose

$$\begin{aligned} \dot{x}_{\omega_k}(s) \cdot \nabla\varphi(s, x_{\omega_k}(s)) &= [\dot{x}_{\omega_k}(s) \cdot \nabla\gamma(x_{\omega_k}(s))] [\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma(x_{\omega_k}(s))] \\ &\quad + \pi_{\nabla\gamma(x_{\omega_k}(s))}^\perp(\dot{x}_{\omega_k}(s)) \cdot \pi_{\nabla\gamma(x_{\omega_k}(s))}^\perp(\nabla\varphi(s, x_{\omega_k}(s))), \end{aligned}$$

where  $\pi_{\nabla\gamma}^\perp$  denotes the orthogonal projection on the hyperplane with normal vector  $\nabla\gamma$ . To alleviate notations, we write  $\nabla\gamma$  instead of  $\nabla\gamma(x_{\omega_k}(s))$ . We have  $\pi_{\nabla\gamma}^\perp(\dot{x}_{\omega_k}(s)) = \pi_{\nabla\gamma}^\perp(b(s, x_{\omega_k}(s))) + \pi_{\nabla\gamma}^\perp(\omega_k(s))$  hence

$$\begin{aligned} &\pi_{\nabla\gamma}^\perp(b(s, x_{\omega_k}(s))) \cdot \pi_{\nabla\gamma}^\perp(\nabla\varphi(s, x_{\omega_k}(s))) + \frac{1}{2} |\pi_{\nabla\gamma}^\perp(\nabla\varphi(s, x_{\omega_k}(s)))|^2 \\ &\geq [\pi_{\nabla\gamma}^\perp(b(s, x_{\omega_k}(s))) + \pi_{\nabla\gamma}^\perp(\omega_k(s))] \cdot \pi_{\nabla\gamma}^\perp(\nabla\varphi(s, x_{\omega_k}(s))) - \frac{1}{2} |\pi_{\nabla\gamma}^\perp(\omega_k(s))|^2. \end{aligned} \quad (2.32)$$

On the other hand, the uniform in  $k$  continuity guarantees that for large enough  $k$ ,

$$\forall s \in [t - \tau_k, t], \quad \nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma(x_{\omega_k}(s)) \geq 0,$$

so that, for almost every  $s \in [t - \tau_k, t]$ ,

$$[\dot{x}_{\omega_k}(s) \cdot \nabla\gamma] [\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma] \leq [b(s, x_{\omega_k}(s)) \cdot \nabla\gamma + \omega_k(s) \cdot \nabla\gamma] [\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma],$$

and we can reason as in (2.32). Gathering all the terms in (2.30) gives (2.31), and we conclude as before.

**Case  $x \in \partial G$  and  $\nabla\varphi(t, x) \cdot n(x) = 0$ :** the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\left| \int_{t-\tau_k}^t [\dot{x}_{\omega_k}(s) \cdot \nabla\gamma] [\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma] ds \right| \\ &\leq \left( \int_{t-\tau_k}^t [\dot{x}_{\omega_k}(s) \cdot \nabla\gamma]^2 ds \right)^{1/2} \left( \int_{t-\tau_k}^t [\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma]^2 ds \right)^{1/2}, \end{aligned} \quad (2.33)$$

where we write  $\nabla\gamma$  instead of  $\nabla\gamma(x_{\omega_k}(s))$  as in the previous case. From Corollary 2.3.2.1-(i), the first integral on the r.h.s can be bounded in terms of  $\int_{t-\tau_k}^t |\omega_k(s)|^2 ds$ . From (2.29), using the continuity of  $\mathcal{V}$  together with the continuity of  $x_{\omega_k}$  uniformly in  $k$ , this integral goes to 0 as  $k \rightarrow +\infty$ .

In the current viscosity setting, we can always assume that  $\varphi$  is  $C^2$ . Moreover,  $\gamma$  is  $C^2$  too. Using  $\nabla\varphi(t, x) \cdot \nabla\gamma(x) = 0$  and Corollary 2.3.2.1-(ii), this yields  $|\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma(x_{\omega_k}(s))|^2 \leq C\tau_k$ , for  $C > 0$  independent of  $(s, k)$ . From (2.33), this implies

$$\tau_k^{-1} \int_{t-\tau_k}^t [\dot{x}_{\omega_k}(s) \cdot \nabla\gamma(x_{\omega_k}(s))] [\nabla\varphi(s, x_{\omega_k}(s)) \cdot \nabla\gamma(x_{\omega_k}(s)) (x_{\omega_k}(s))] ds \xrightarrow[k \rightarrow +\infty]{} 0.$$

From (2.30)-(2.32) and  $\nabla\varphi(t, x) \cdot n(x) = 0$ , reasoning as in the previous case concludes.

**Case  $x \in \partial G$  and  $\nabla\varphi(t, x) \cdot n(x) < 0$ :** we notice that

$$\int_{t-\tau_k}^t \dot{x}_{\omega_k}(s) \nabla\gamma(x_{\omega_k}(s)) ds = \gamma(x) - \gamma(x_{\omega_k}(t - \tau_k)) \geq 0,$$

hence

$$\begin{aligned} &\int_{t-\tau_k}^t [\dot{x}_{\omega_k}(s) \cdot \nabla\gamma(x_{\omega_k}(s))] [\nabla\varphi(x_{\omega_k}(s)) \cdot \nabla\gamma(x_{\omega_k}(s))] ds \\ &\leq \int_{t-\tau_k}^t [\dot{x}_{\omega_k}(s) \cdot \nabla\gamma(x_{\omega_k}(s))] [\nabla\varphi(x_{\omega_k}(s)) \cdot \nabla\gamma(x_{\omega_k}(s)) - \nabla\varphi(t, x) \cdot \nabla\gamma(x)] ds. \end{aligned}$$

The integral on the r.h.s. can be handled as we did for (2.33) to get that

$$\limsup_{k \rightarrow +\infty} \tau_k^{-1} \int_{t-\tau_k}^t [\dot{x}_{\omega_k}(s) \cdot \nabla \gamma(x_{\omega_k}(s))] [\nabla \varphi(x_{\omega_k}(s)) \cdot \nabla \gamma(x_{\omega_k}(s))] ds \leq 0.$$

Going back to (2.30)-(2.32), we reason as before, we send  $k \rightarrow +\infty$  and then  $\varepsilon \rightarrow 0$ . This gives the incomplete property:

$$\partial_t \varphi(t, x) + \pi_{n(x)}^\perp(b(t, x)) \cdot \pi_{n(x)}^\perp(\nabla \varphi(t, x)) + \frac{1}{2} |\pi_{n(x)}^\perp(\nabla \varphi(t, x))|^2 - \frac{1}{2} |\dot{y}(t) - h(t, x)|^2 \geq 0.$$

The boundary condition being imposed in the viscosity sense, Definition 2.A.2 allows us to assume that  $b(t, x) \cdot n(x) + \frac{1}{2} \nabla \varphi(t, x) \cdot n(x) \leq 0$ , hence

$$[b(t, x) \cdot n(x)] [\nabla \varphi(t, x) \cdot n(x)] + \frac{1}{2} |\nabla \varphi(t, x) \cdot n(x)|^2 \geq 0,$$

because  $\nabla \varphi(t, x) \cdot n(x) \leq 0$ . Adding this to the incomplete property concludes.  $\square$

**Remark 2.3.4** (Super-solution property at the boundary). *When  $b \cdot n \leq 0$  on  $\partial G$ , we deduce from the last case of the above proof that  $\mathcal{V}$  is actually a viscosity super-solution of*

$$\partial_t \mathcal{V}(t, x) + b(t, x) \cdot \nabla \mathcal{V}(t, x) + \frac{1}{2} |\nabla \mathcal{V}(t, x)|^2 - \frac{1}{2} |\dot{y}(t) - h(t, x)|^2 = 0,$$

on the full  $[0, T] \times \overline{G}$  for every  $T > 0$  (no more boundary condition). The related comparison principle then enters the scope of [Capuzzo-Dolcetta and Lions, 1990, Theorem III.2].

## 2.4 Small noise filtering for reflected dynamics

This section is devoted to the proof of Theorem 2.2.7. Let us fix a continuous  $\Phi : \overline{G} \rightarrow 0$  and  $t > 0$ . By regularisation and density, we can assume that  $\Phi$  is defined and  $C^1$  on a neighbourhood of  $\overline{G}$ . This allows us to apply [Friedman, 2008, Chapter 5, Corollary 2] to get that

$$\begin{cases} \partial_s \Phi^\varepsilon(s, x) + b(s, x) \cdot \nabla \Phi^\varepsilon(s, x) + \frac{\varepsilon}{2} \Delta \Phi^\varepsilon(s, x) - \frac{1}{2\varepsilon} |\dot{y}(s) - h(s, x)|^2 \Phi^\varepsilon(s, x) = 0, \\ \Phi^\varepsilon(t, x) = e^{-\Phi(x)/\varepsilon}, \quad x \in \overline{G}, \\ \frac{\partial \Phi^\varepsilon}{\partial n}(s, x) = 0, \quad (s, x) \in [0, t] \times \partial G. \end{cases}$$

has a unique solution  $\Phi^\varepsilon$  in  $C([0, t] \times \overline{G}) \cap C^{1,2}([0, t] \times G)$ . From [Hucke, 1990, Lemma 3.2], we get the duality relation

$$\int_G \Phi^\varepsilon(0, x) \tilde{q}^\varepsilon(t, x) dx = \int_G \Phi^\varepsilon(t, x) \tilde{q}^\varepsilon(0, x) dx. \quad (2.34)$$

The log-transform  $V_\Phi^\varepsilon \triangleq -\varepsilon \log \Phi^\varepsilon$  then satisfies

$$\begin{cases} \partial_s V_\Phi^\varepsilon(s, x) + b(s, x) \cdot \nabla V_\Phi^\varepsilon(s, x) + \frac{\varepsilon}{2} \Delta V_\Phi^\varepsilon(s, x) - \frac{1}{2} |\nabla V_\Phi^\varepsilon(s, x)|^2 \\ \quad + \frac{1}{2} |\dot{y}(s) - h(s, x)|^2 = 0, \quad (s, x) \in [0, t] \times G, \\ V_\Phi^\varepsilon(t, x) = \Phi(x), \quad x \in \overline{G}, \\ \frac{\partial V_\Phi^\varepsilon}{\partial n}(s, x) = 0, \quad (s, x) \in [0, t] \times \partial G. \end{cases}$$

From this, we proceed as in [Fleming, 1997, Section 5] to give a control representation for  $V_\Phi^\varepsilon$ . On a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s)_{0 \leq s \leq t})$ , let  $(B_s)_{0 \leq s \leq t}$  be an adapted

Brownian motion. For  $0 \leq s \leq t$ ,  $x \in \bar{G}$  and any square-integrable adapted process  $(\alpha_r)_{s \leq r \leq t}$ , we define the controlled reflected dynamics

$$dY_r^{\varepsilon, \alpha} + \chi_G(Y_r^{\varepsilon, \alpha})(ds) \ni b(s, Y_r^{\varepsilon, \alpha})dr + \alpha_r dr + \sqrt{\varepsilon}dB_r, \quad s \leq r \leq t,$$

with initial condition  $Y_s^{\varepsilon, \alpha} = x$ . Since  $V_\Phi^\varepsilon$  is  $C^{1,2}$ , a standard verification argument now gives that

$$V_\Phi^\varepsilon(s, x) = \inf_{Y_s^{\varepsilon, \alpha} = x} \mathbb{E} \left[ \int_s^t \frac{1}{2} |\alpha_r|^2 + \frac{1}{2} |\dot{y}(r) - h(r, Y_r^{\varepsilon, \alpha})|^2 dr + \Phi(Y_t^{\varepsilon, \alpha}) \right]. \quad (2.35)$$

The next result is the analogous of [Fleming, 1997, Lemma 5.1] or Theorem 2.2.6.

**Lemma 2.4.1.** *There exists  $C > 0$  independent of  $\varepsilon$  such that*

$$\sup_{(s, x) \in [0, t] \times \bar{G}} |V_\Phi^\varepsilon(s, x) - V_\Phi^0(s, x)| \leq C\varepsilon^{1/4}.$$

*Proof.* Since  $G$  is bounded, choosing e.g.  $\alpha = 0$  in (2.35), we get that  $V_\Phi^\varepsilon$  is uniformly bounded by some  $M > 0$  independent of  $\varepsilon$ . Thus, we can restrict the minimisation to control processes  $(\alpha_r)_{s \leq r \leq t}$  with square  $L^2$ -norm lower than  $2M$ . From [Pardoux and Răşcanu, 2014, Proposition 4.16-I] we get that

$$\mathbb{E} \left[ \sup_{s \leq r \leq t} |Y_r^{\varepsilon, \alpha}|^2 \right] \leq C_M,$$

and from [Pardoux and Răşcanu, 2014, Proposition 4.16-II] that

$$\mathbb{E} \left[ \sup_{s \leq r \leq t} |Y_r^{\varepsilon, \alpha} - Y_r^{0, \alpha}|^2 \right] \leq C_M \varepsilon^{1/2},$$

for  $C_M > 0$  independent of  $\varepsilon$ . We then plug this into the minimisation (2.35): since  $\Phi$  and  $h$  are Lipschitz-continuous, we conclude by using the Cauchy-Schwarz inequality for the  $\frac{1}{2} |\dot{y}(r) - h(r, Y_r^{\varepsilon, \alpha})|^2$  term.  $\square$

*Proof of Theorem 2.2.7.* Going back to (2.34), the assumption on  $q_0^\varepsilon(x) = \tilde{q}^\varepsilon(0, x)$  and Lemma 2.4.1 give that

$$-\varepsilon \log \int_G e^{-\Phi(x)/\varepsilon} q^\varepsilon(t, x) dx = -\varepsilon \log \int_G e^{-[V_\Phi^0(0, x) + \psi(x) + r^\varepsilon(x)]/\varepsilon} dx,$$

for some continuous  $r^\varepsilon$  that uniformly converges to 0 as  $\varepsilon \rightarrow 0$ . The standard Laplace principle (see e.g. [James and Baras, 1988b, Lemma 6.1]) now shows that the r.h.s. converges towards  $\inf_{x \in \bar{G}} V_\Phi^0(0, x) + \psi(x)$ . However,

$$\begin{aligned} & \inf_{x \in \bar{G}} V_\Phi^0(0, x) + \psi(x) \\ &= \inf_{x_\omega(0), \omega} \Phi(x_\omega(t)) + \psi(x_\omega(0)) + \int_0^t \frac{1}{2} |\omega(s)|^2 + \frac{1}{2} |\dot{y}(s) - h(s, x_\omega(s))|^2 ds \\ &= \inf_{x \in \bar{G}} \Phi(x) + \mathcal{V}(t, x), \end{aligned}$$

concluding the proof.  $\square$

## 2.5 Numerical illustrations

### 2.5.1 Discrete-time setting

Finally, we propose a numerical illustration of the estimation problem in a simplified configuration corresponding to the Skorokhod problem presented in Section 2.1.1, albeit with constrained on 0 and 1. Therefore,  $m = n = 1$ ,  $\bar{G} = [0, 1]$ ,  $b \equiv 0$  and  $\sigma = \text{Id}$ . Here, we pursue a discretization strategy based on the principle discretize-then-optimize as reviewed in Moireau [2023]. A time-discretization of the dynamics (2.1) is given by

$$\begin{cases} x_{\zeta, \omega}^{k+1} + \Delta t \partial \chi_G(x_{\zeta, \omega}^{k+1}) \ni x_{\zeta, \omega}^k + \omega^{k+1}, \\ x_{\zeta, \omega}^0 = \zeta, \end{cases} \quad (2.36)$$

and we set  $y^k = y(t_k)$ .

### 2.5.2 Convergent discretization

A discrete-time cost-to-come is defined by

$$\mathcal{V}_{\Delta t}^k(x) = \inf_{(\zeta, \omega) \in \mathcal{A}_{k,x}^G} \psi(\zeta) + \frac{1}{2} \sum_{j=0}^{k-1} \Delta t |y^j - h(x_{\zeta, \omega}^j)|^2 + \frac{1}{2} \sum_{j=1}^k \frac{1}{\Delta t} |\omega^j|^2, \quad (2.37)$$

where here

$$\mathcal{A}_{k,x}^G \triangleq \{(\zeta, (\omega^j)_j) \in G \times \mathbb{R}^k : x_{\zeta, \omega}^k \text{ follows (2.36) with } x_{\zeta, \omega}^0 = x\}.$$

Under convexity assumptions on  $x \mapsto |y^k - h(x)|^2$  for all  $k$ , the discrete-time cost-to-come is strictly convex, and the minimizer is denoted by

$$\hat{x}^k = \operatorname{argmin}_{x \in G} \mathcal{V}_{\Delta t}^k(x). \quad (2.38)$$

This definition provides a discretization for the continuous-time estimator as soon as we can compute  $\mathcal{V}_{\Delta t}^k$ . From Bellman's principle, it is straightforward to verify that the discrete-time cost-to-come can be computed from a splitting procedure that first computes

$$\tilde{\mathcal{V}}_{\Delta t}^k(x) = \mathcal{V}_{\Delta t}^k(x) + \frac{1}{2} \Delta t |y^k - h(x)|^2, \quad (2.39)$$

followed by

$$\mathcal{V}_{\Delta t}^{k+1}(x) = \inf_{z \in G} \left\{ \tilde{\mathcal{V}}_{\Delta t}^k(z) + \frac{1}{2\Delta t} |x - z|^2 \right\}. \quad (2.40)$$

This inf-convolution can be computed using the Hopf-Lax formula, by solving

$$\begin{cases} \partial_s v(s, x) + \frac{1}{2} |\partial_x v(s, x)|^2 = 0, & (s, x) \in (0, \Delta t) \times (0, 1) \\ \partial_s v(s, 0) = \partial_s v(s, 1) = 0, \\ v(0, x) = \tilde{\mathcal{V}}_{\Delta t}^k(x), \end{cases} \quad (2.41)$$

and defining

$$\mathcal{V}_{\Delta t}^{k+1}(x) = v(\Delta t, x). \quad (2.42)$$

Regarding the spatial discretization, the procedure is rather straightforward. The domain  $G = [0, 1]$  is discretized with a regular grid  $(x_i)_{0 \leq i \leq N}$  such that

$$0 = x_0 < x_i = \frac{i}{\Delta x} < x_N = 1.$$

Then, we introduce the following space-time discretization. Let  $(\mathcal{V}_{\Delta t, \Delta x}^{k,i})_{0 \leq i \leq N}$  be defined on the grid by

- Correction:

$$\tilde{\mathcal{V}}_{\Delta t, \Delta x}^{k,i} = \mathcal{V}_{\Delta t, \Delta x}^{k,i} + \frac{1}{2} \Delta t |y^k - h(x_i)|^2.$$

- Prediction:

$$\mathcal{V}_{\Delta t, \Delta x}^{k+1,i} = S_{\Delta t, \Delta x} \mathcal{V}_{\Delta t, \Delta x}^{k,i},$$

where  $S_{\Delta t, \Delta x}$  is the discrete flow associated with a discretization of (2.41).

Here, we propose to rely on the Hopf-Cole formula to define the scheme associated that gives  $S_{\Delta t, \Delta x}$ . Indeed, we know that  $v_\varepsilon$  is the solution of

$$\begin{cases} \partial_s v_\varepsilon(s, x) + \frac{1}{2} |\partial_x v_\varepsilon(s, x)|^2 = \frac{\varepsilon}{2} \Delta v_\varepsilon(s, x), & (s, x) \in (0, \Delta t) \times (0, 1), \\ \partial_s v_\varepsilon(s, 0) = \partial_s v_\varepsilon(s, 1) = 0, & s \in (0, \Delta t), \\ v_\varepsilon(0, x) = \tilde{\mathcal{V}}_{\Delta t}^k(x), & x \in (0, 1), \end{cases} \quad (2.43)$$

is a vanishing viscosity approximation of  $v$ . Moreover,  $p_\varepsilon \triangleq \exp(-v_\varepsilon/\varepsilon)$  is the solution of

$$\begin{cases} \partial_s p_\varepsilon(s, x) = \frac{\varepsilon}{2} \Delta p_\varepsilon(s, x), & (s, x) \in (0, \Delta t) \times (0, 1), \\ \partial_s p_\varepsilon(s, 0) = \partial_s p_\varepsilon(s, 1) = 0, & s \in (0, \Delta t), \\ p_\varepsilon(0, x) = \exp\left(-\frac{\tilde{\mathcal{V}}_{\Delta t}^k(x)}{\varepsilon}\right), & x \in (0, 1). \end{cases} \quad (2.44)$$

We then choose an explicit time-discretization (2.44) for a time-discretization of the interval

$$0 = s_0 < s_n = n\Delta s < s_{N_{\Delta s}} = \Delta t,$$

of the form  $S_{\Delta t, \Delta x} \mathcal{V}_{\Delta t, \Delta x}^{k,i} \triangleq \varepsilon \log(p_\varepsilon^{N_{\Delta s}, i})$ , for an adequate choice of  $\varepsilon$  as a function  $\Delta s$  and an adequate choice of  $\Delta s$  a function of  $\Delta x$ , where

$$\begin{cases} \frac{p_\varepsilon^{n+1,i} - p_\varepsilon^{n,i}}{\Delta s} = \frac{\varepsilon p_\varepsilon^{n,i+1} - 2p_\varepsilon^{n,i} + p_\varepsilon^{n,i-1}}{2\Delta x^2}, & 0 < i < N_{\Delta x}, 0 \leq n \leq N_{\Delta s}, \\ p_\varepsilon^{n,0} = p_\varepsilon^{n,1}, & 0 \leq n < N_{\Delta s}, \\ p_\varepsilon^{n, N_{\Delta x}} = p_\varepsilon^{n, N_{\Delta x}-1}, & 0 \leq n \leq N_{\Delta s}, \\ p_\varepsilon^{0,i} = \exp\left(-\frac{\Pi(\tilde{\mathcal{V}}_{\Delta t}^k)(x_i)}{\varepsilon}\right), & 0 < i < N_{\Delta x}, \end{cases} \quad (2.45)$$

where  $\Pi$  is the linear interpolator on the grid  $(x_i)_{0 \leq i \leq N}$ . Choosing  $\varepsilon = O(\sqrt{\Delta s})$  and  $\varepsilon \Delta t \leq \Delta x^2$  then provides a stable, monotone – the consistency being easy to verify –, hence convergent, approach to compute the continuous-time  $\mathcal{V}$ .

Note finally that as an alternative to (2.45), which may suffer from floating-point error mitigation, we could rely on a direct monotone approximation of (2.43) with a Lax-Friedrichs like scheme of the form

$$\begin{cases} \frac{v_\varepsilon^{n+1,i} - v_\varepsilon^{n,i}}{\Delta s} + \frac{1}{2} \left| \frac{v_\varepsilon^{n,i+1} - v_\varepsilon^{n,i-1}}{2\Delta x} \right|^2 \\ = \frac{\varepsilon v_\varepsilon^{n,i+1} - 2v_\varepsilon^{n,i} + v_\varepsilon^{n,i-1}}{2\Delta x^2}, & 0 < i < N_{\Delta x}, 0 \leq n < N_{\Delta s} - 1, \\ v_\varepsilon^{n,0} = v_\varepsilon^{n,1}, \\ v_\varepsilon^{n, N_{\Delta x}} = v_\varepsilon^{n, N_{\Delta x}-1}, \\ v_\varepsilon^{0,i} = \Pi(\tilde{\mathcal{V}}_{\Delta t}^k)(x_i). \end{cases}$$



### 2.5.3 Numerical result

As an illustration, we consider a configuration where  $T = 1$ ,  $\Delta t = 10^{-2}$  and  $\Delta x = 10^{-2}$ . Moreover, we choose  $\Delta s = 10^{-3}\Delta t$  and  $\varepsilon = 510^{-2}$  for a discretization of the inf-convolution using the Hopf-Lax scheme (2.45). The target trajectory starts at  $\frac{1}{2}$  with a model disturbance

$$\omega = \cos(2\pi t) + \cos(4\pi t) = \cos(8\pi t).$$

The observations of the trajectory are produced using  $h(x) = x$ , and an additional observation noise defined at the discrete level as a random variable with law  $\mathcal{N}(0, 0.01)$ .

The numerical results are presented in Figure 2.2, where we plot the target trajectory  $k \mapsto x_\omega^k$ , and the estimated trajectory built from  $k \mapsto \hat{x}^k$  with an additional confidence interval built from the region  $\{(x, k) \mid \mathcal{V}_{\Delta t}^k(x) - \min_{x'} \mathcal{V}_{\Delta t}^k(x') \leq 2\}$ , which generalizes the 95% confidence interval for classical Gaussian models. Moreover, in Figure 2.3, we plot the associated value function  $\mathcal{V}_{\Delta t}^k$  re-centered around 0 by plotting the time-interpolation function built from  $x \mapsto \mathcal{V}_{\Delta t}^k(x) - [\min_{x'} \mathcal{V}_{\Delta t}^k(x')]$ , where  $k \in \{0, \dots, N_{\Delta t}\}$ .

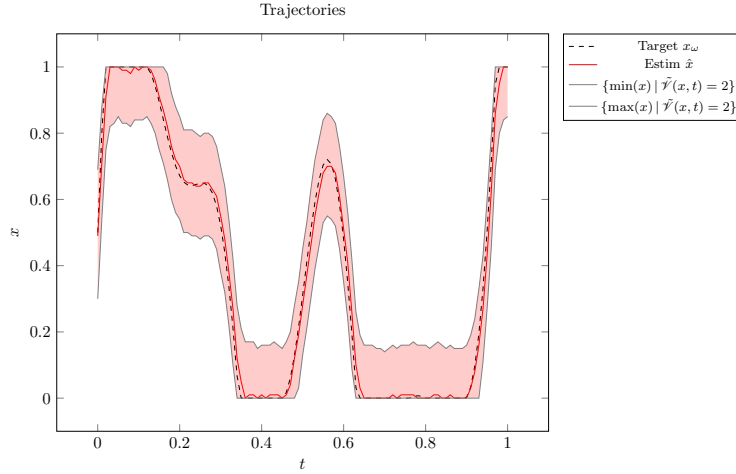


Figure 2.2: Target and estimated trajectory with confidence interval

## Appendix

### 2.A Viscosity solutions

We recall here the classical notions of viscosity solution that are used throughout the article. We consider Hamilton-Jacobi equations of the kind

$$\partial_t \mathcal{V}(t, x) + H(t, x, \nabla \mathcal{V}(t, x)) = 0, \quad (2.46)$$

with an initial condition  $\mathcal{V}(0, x) = \psi(x)$ .

**Definition 2.A.1** (Viscosity solution). *Let  $\mathcal{V} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuous function.*

- $\mathcal{V}$  is a viscosity sub-solution of (2.46) if for every  $(t, x, \varphi) \in [0, T] \times \mathbb{R}^d \times C^1([0, T] \times \mathbb{R}^d)$  such that  $\mathcal{V} - \varphi$  has a local maximum at  $(t, x)$ ,

$$\partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x)) \leq 0.$$

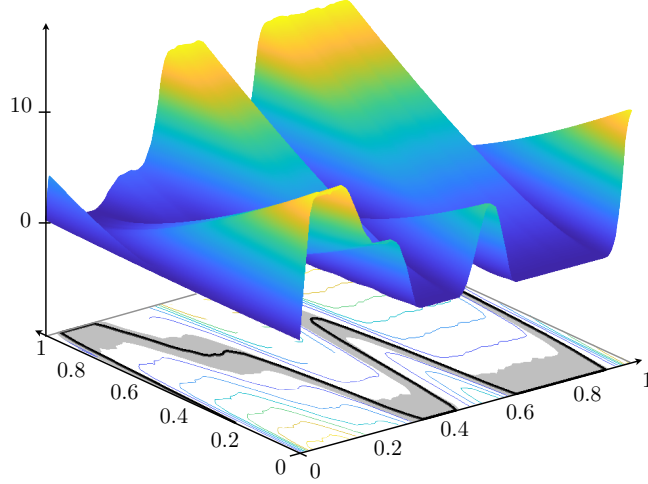


Figure 2.3: Cost-to-come value function of the estimation problem as a function of space and time

- $\mathcal{V}$  is a viscosity super-solution of (2.46) if for every  $(t, x, \varphi) \in [0, T] \times \mathbb{R}^d \times C^1([0, T] \times \mathbb{R}^d)$  such that  $\mathcal{V} - \varphi$  has a local minimum at  $(t, x)$ ,

$$\partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x)) \geq 0.$$

- $\mathcal{V}$  is a viscosity solution of (2.46) if it is both a viscosity sub-solution and a viscosity super-solution.

Under mild assumptions, existence and uniqueness for such a viscosity solution is proved in Crandall and Lions [1983]. We now consider (2.46) in a Lipschitz domain  $G$ , and we add the boundary condition

$$b(t, x) \cdot n(x) + \frac{\partial \mathcal{V}}{\partial n}(t, x) = 0, \quad (2.47)$$

for some  $b : [0, T] \times \overline{G} \rightarrow \mathbb{R}$ ,  $n(x)$  being the outward normal vector at  $x \in \partial G$ .

**Definition 2.A.2** (Viscosity solution with boundary condition). *Let  $\mathcal{V} : [0, T] \times \overline{G} \rightarrow \mathbb{R}$  be a continuous function.*

- $\mathcal{V}$  is a viscosity sub-solution of (2.46)-(2.47) if for every  $(t, x, \varphi) \in [0, T] \times \mathbb{R}^d \times C^1([0, T] \times \overline{G})$  such that  $\mathcal{V} - \varphi$  has a local maximum at  $(t, x)$ ,

$$\begin{cases} \partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x)) \leq 0, & \text{if } x \in G, \\ \min [(b(t, x) + \nabla \varphi(t, x)) \cdot n(x), \partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x))] \leq 0, & \text{if } x \in \partial G. \end{cases}$$

- $\mathcal{V}$  is a viscosity super-solution of (2.46) if for every  $(t, x, \varphi) \in [0, T] \times \mathbb{R}^d \times C^1([0, T] \times \mathbb{R}^d, \mathbb{R})$  such that  $\mathcal{V} - \varphi$  has a local minimum at  $(t, x)$ ,

$$\begin{cases} \partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x)) \geq 0, & \text{if } x \in G, \\ \max [(b(t, x) + \nabla \varphi(t, x)) \cdot n(x), \partial_t \varphi(t, x) + H(t, x, \nabla \varphi(t, x))] \geq 0, & \text{if } x \in \partial G. \end{cases}$$

- $\mathcal{V}$  is a viscosity solution of (2.46)-(2.47) if it is both a viscosity sub-solution and a viscosity super-solution.

Under mild assumptions, existence and uniqueness for such a viscosity solution is proved in Barles and Lions [1991].

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## CHAPTER 3

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# Quasi-continuity method for mean-field diffusions: large deviations and central limit theorem

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*Entropy is a figure of speech then... a metaphor.*  
— T. Pynchon, *The Crying of Lot 49*

A pathwise large deviation principle in the Wasserstein topology and a pathwise central limit theorem are proved for the empirical measure of a mean-field system of interacting diffusions. The coefficients are path-dependent. The framework allows for degenerate diffusion matrices, which may depend on the empirical measure, including mean-field kinetic processes. The main tool is an extension of a pathwise construction due to Tanaka to non-constant diffusion matrices. This can be seen as a mean-field analogous of Azencott's quasi-continuity method for the Freidlin-Wentzell theory. As a by-product, uniform-in-time-step fluctuation and large deviation estimates are proved for a discrete-time version of the mean-field system. Uniform-in-time-step convergence is also proved for the value function of some mean-field control problems with quadratic cost.

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### 3.1 Introduction

We consider a mean-field system of interacting diffusions in  $\mathbb{R}^d$ ,

$$dX_t^{i,N} = b_t(X^{i,N}, \pi(\vec{X}^N))dt + \sigma_t(X^{i,N}, \pi(\vec{X}^N))dB_t^i, \quad 1 \leq i \leq N, \quad (3.1)$$

starting from i.i.d. initial conditions  $X_0^i$ , the  $B^i := (B_t^i)_{0 \leq t \leq T}$  being i.i.d. Brownian motions in  $\mathbb{R}^d$ . These particles interact through their empirical measure

$$\pi(\vec{X}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}} \in \mathcal{P}(\mathcal{C}([0, T], \mathbb{R}^d)),$$

using the notation  $\vec{X}^N := (X^{i,N})_{1 \leq i \leq N}$ . The coefficients  $b$  and  $\sigma$  are globally Lipschitz functions which are path-dependent in a non-anticipative way, and  $\sigma$  is globally bounded (precise conditions are listed in Assumption (A.1) below). As  $N \rightarrow +\infty$ ,  $\pi(\vec{X}^N)$  converges to the path-law  $\mathcal{L}(\bar{X})$  of the McKean-Vlasov equation

$$d\bar{X}_t = b_t(\bar{X}, \mathcal{L}(\bar{X}))dt + \sigma_t(\bar{X}, \mathcal{L}(\bar{X}))dB_t^1, \quad X_0 \sim X_0^1.$$

Such a mean-field limit is often referred to as a *propagation of chaos* result Sznitman [1991]. Under suitable assumptions on  $(b, \sigma)$ , the mean-field limit holds with rate  $N^{-1/2}$  for the mean-square convergence, see e.g. [Sznitman, 1991, Theorem 1.4]. When  $\sigma \equiv \text{Id}$ , a pathwise construction due to Tanaka [1984] builds  $\bar{X}$  and the  $X^{i,N}$  in a unified framework. In particular, the empirical measure  $\pi(\vec{X}^N)$  of the particles is a continuous function of the one of the driving noises  $\pi(\vec{B}^N)$ . This powerful approach rephrases the mean-field limit as a mere continuity result. Furthermore, Tanaka's construction allows for computing the fluctuations of  $\pi(\vec{X}^N)$  directly from the ones of  $\pi(\vec{B}^N)$ . From there, the objectives of this article are three-fold:

- We extend Tanaka's construction to non-constant  $\sigma$  using an Euler scheme for (3.1) for which we prove strong consistency estimates.
- We compute the first-order fluctuations of  $\pi(\vec{X}^N)$  by proving a *pathwise central limit theorem* (CLT), i.e. the convergence of  $N^{-1/2}[\pi(\vec{X}^N) - \mathcal{L}(\bar{X})]$  towards a Gaussian field.
- We prove a *large deviation principle* (LDP), estimating  $\mathbb{P}(\pi(\vec{X}^N) \in A)$  at the exponential scale for any measurable  $A \subset \mathcal{P}(\mathcal{C}([0, T], \mathbb{R}^d))$ , which may not contain  $\mathcal{L}(\bar{X})$ .

These three points correspond to Theorems 3.2.4-3.2.8-3.2.10 below. The CLT quantifies the normal fluctuations of  $\pi(\vec{X}^N)$  in the  $N \rightarrow +\infty$  limit, whereas the LDP quantifies large deviations. The CLT and the LDP are first proved for the Euler approximation of (3.1), before being transferred to the continuous-time system. The CLT is proved for binary interactions of type  $b(x, P) = \int \tilde{b}(x, y)dP(y)$ . The LDP is proved in the Wasserstein- $p$  topology, for every  $p \in [1, 2)$ . These CLT and LDP results are not the first ones of their kind, but they are new at the considered level of generality, see Section 3.1.1 below for a detailed review of existing literature. The main challenge of our work is to include non-constant  $\sigma$  that depend on the measure argument and may be degenerate. In particular, our results cover kinetic systems like mean-field interacting Langevin dynamics. We refer to our extended construction as a *quasi-continuity* result, by analogy with the seminal Freidlin-Wentzell theory Freidlin and Wentzell [1998] and the Azencott quasi-continuity method Azencott et al. [1980].



To motivate this terminology, let us recall the analogy made in the introduction of Dawson and Gärtner [1989]. To simplify the presentation, we assume that  $b_t(x, P) = b(x_t, P_t)$  and  $\sigma \equiv \text{Id}$ . From Ito's formula, the empirical measure  $\pi(\vec{X}_t^N)$  of the particles at time  $t$  a.s. satisfies

$$d \int_{\mathbb{R}^d} \varphi d\pi(\vec{X}_t^N) = \left[ \int_{\mathbb{R}^d} L_{\pi(\vec{X}_t^N)} \varphi d\pi(\vec{X}_t^N) \right] dt + \frac{1}{N} \sum_{i=1}^N \nabla \varphi(X_t^{i,N}) \cdot dB_t^i,$$

for any smooth test function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ , the measure-dependent generator being defined by  $L_\pi \varphi(x) := b(x, \pi) \cdot \nabla \varphi(x) + \frac{1}{2} \Delta \varphi(x)$ . The last term on the r.h.s is a martingale whose quadratic variation is of order  $N^{-1}$ . This motivates the informal re-writing

$$d\pi(\vec{X}_t^N) = L_{\pi(\vec{X}_t^N)}^* \pi(\vec{X}_t^N) dt + N^{-1/2} dM_t, \quad (3.2)$$

where  $(M_t)_{0 \leq t \leq T}$  is intended to be a measure-valued martingale term of order 1, and  $L_\pi^*$  is the formal  $L^2$ -adjoint of  $L_\pi$ . Equation (3.2) is known as the Dean-Kawasaki stochastic partial differential equation, see e.g. Kawasaki [1994]; Dean [1996]; Konarovskyi et al. [2019]; Fehrman and Gess [2022]. This writing draws an (infinite-dimensional) analogy with the famous Freidlin-Wentzell diffusion

$$dX_t^\varepsilon = b(X_t^\varepsilon) dt + \sqrt{\varepsilon} dB_t, \quad (3.3)$$

where  $b$  is now a vector field in  $\mathbb{R}^d$  and  $B := (B_t)_{0 \leq t \leq T}$  is a Brownian motion. Let us describe how our approach enables us to push this analogy beyond qualitative statements, at the very level of proofs. We first recall some basic notions about the Freidlin-Wentzell theory.

**Pathwise Freidlin-Wentzell theory.** If  $b$  is globally Lipschitz in (3.3), setting  $\mathcal{C}^d := \mathcal{C}([0, T], \mathbb{R}^d)$ , the Cauchy-Lipschitz theory provides a map  $X : \mathcal{C}^d \rightarrow \mathcal{C}^d$ , such that for every  $\omega = (\omega_t)_{0 \leq t \leq T}$ ,  $X(\omega)$  is the solution to the ordinary differential equation (ODE),

$$\forall t \in [0, T], \quad X_t(\omega) = \int_0^t b(X_s(\omega)) ds + \omega_t.$$

This allows us to build the diffusion (3.3) by simply setting  $X^\varepsilon := X(\sqrt{\varepsilon}B)$ . As  $\varepsilon \rightarrow 0$ , the a.s. pathwise convergence of  $X^\varepsilon$  towards  $X^0$  is then a simple consequence of the continuity of  $X$ . Looking back at (3.2), this convergence corresponds to the  $N \rightarrow +\infty$  limit. Classically, the Freidlin-Wentzell LDP is obtained from the contraction principle, using the continuous map  $X$  to push the large deviations of  $(\sqrt{\varepsilon}B)_{\varepsilon > 0}$  given by Schilder's theorem. The first-order expansion follows by formally writing that

$$X^\varepsilon - X^0 = \sqrt{\varepsilon} Y + o(\sqrt{\varepsilon}),$$

where the Gaussian process  $Y = (Y_t)_{0 \leq t \leq T}$  solves the SDE  $dY_t = Db(X_t^0) \cdot Y_t dt + dB_t$ , differentiating  $X$  using the standard theory of flow maps, see e.g. [Friedman, 1975, Chapter 5.5] or [Freidlin and Wentzell, 1998, Chapter 2].

**Azencott's quasi-continuity method.** The above construction does not allow for non-constant diffusion matrices  $\sigma$  in (3.3) because the stochastic integral is not a continuous function of the driving path. More precisely, a measurable map  $X : \mathcal{C}^d \rightarrow \mathcal{C}^d$  could be defined as above on a set of full Wiener-measure but  $X$  would not be continuous, preventing us from computing large deviations by contraction. In the seminal work Azencott et al. [1980], a solution is developed which is by now referred to as the quasi-continuity method.

The main idea is to approximate the Brownian path by a curve in the Sobolev space  $H^1([0, T], \mathbb{R}^d)$ , before defining a continuous contraction map  $H^1 \rightarrow H^1$ . The key-result is then an approximation estimate stating that the made error is negligible at the exponential scale in terms of  $\varepsilon$ . Thus, the large deviations can still be computed by contraction. A similar argument can be found in [Deuschel and Stroock, 2001, Lemma 1.4.21], using an Euler-approximation of the stochastic integral, and showing that the made error is negligible.

An alternative approach relies on the rough-path theory which was first introduced by Lyons [1998]. Extending the notion of stochastic process, this theory makes the stochastic integral be a continuous function of the driving noise. The flow map is then built on a space of rough paths, yielding the large deviations by continuous contraction Ledoux et al. [2002]. We emphasise that such a result is very strong, because it provides large deviations at the level of rough paths in addition to the process's level. As a consequence, there is a technical price to pay to use this sophisticated machinery.

**Tanaka's pathwise construction.** Going back to the setting of (3.2), the key-element of Tanaka [1984] is the construction of  $X^P : \mathcal{C}^d \rightarrow \mathcal{C}^d$ , for every  $P \in \mathcal{P}(\mathcal{C}^d)$ , such that

$$\forall \omega \in \mathcal{C}^d, \forall t \in [0, T], \quad X_t^P(\omega) = \int_0^t b(X_s(\omega), (X_s^P)_\# P) ds + \omega_t.$$

We emphasise that the above equation is non-local, because the drift term depends on the whole map  $X^P$  through the push-forward. Tanaka [1984] then proves that the map  $\Psi : P \mapsto X^P_\# P$  is continuous in a suitable topology, before building the particle system in a pathwise way as

$$X^{i,N} := X^{\pi(\vec{B}^N)}(B^i), \quad \pi(\vec{X}^N) = \Psi(\pi(\vec{B}^N)),$$

the dependence on the empirical measure  $\pi(\vec{B}^N)$  of the driving noises being continuous. Looking at (3.2)-(3.3), there is a striking analogy with the pathwise Freidlin-Wentzell theory. As a by-product, Tanaka [1984] directly obtains the large deviations by contraction from the Sanov theorem, and the CLT is obtained by performing a first-order expansion of  $\Psi$  – this is reminiscent of the delta method in statistics. This framework is very convenient for mean-field limits of any kind: extensions are developed in Coghi et al. [2020] for non-exchangeable systems, general driving noises, systems with common noise, càdlàg processes, reflected processes... Similarly to the Freidlin-Wentzell theory, a crucial limitation is the difficulty to handle non-constant diffusion matrices  $\sigma$ , because of the need for stochastic integration.

Up to our knowledge, there is no existing rough-path approach for extending Tanaka's construction. However, we mention Bailleul et al. [2021] which proves the mean-field limit for interacting rough paths, and Deuschel et al. [2018] for a related LDP. This last LDP was proved for  $\sigma \equiv \text{Id}$  using the Girsanov transform and a standard large deviation result for Gibbs measures: this prevents it from going beyond the case of non-degenerate  $\sigma$  that do not depend on the measure argument.

**Quasi-continuity method for mean-field systems.** Motivated by Tanaka's and Azencott's approaches, it is natural to extend Tanaka's construction to handle non-constant  $\sigma$ . We first design a pathwise construction for a Euler-Maruyama approximation of (3.1) with general driving noises that may not be Brownian (Theorem 3.2.4). Exchangeability is not required either. This allows us to prove mean-field limits, central limit theorems and large deviations for discretised systems at a great level of generality. These novel results are already useful as such, the practitioner being interested in the behaviour of the

numerically implemented model, rather than the theoretical one. When the driving noises are i.i.d. Brownian motions, we extend these results to the continuous setting, by showing:

- A uniform-in-time-step fluctuation estimate to recover the CLT (Theorem 3.2.10).
- A uniform approximation at the exponential scale to recover the LDP (Theorem 3.2.8).

These uniform estimates are quantitative and new up to our knowledge. They are of independent interest from a numerical analysis perspective. Following the weak convergence approach Dupuis and Ellis [2011], the exponential approximation relies on a stochastic control interpretation of exponential moments. Thus, our estimates also have an interest from a mean-field control perspective. They provide a uniform-in- $N$  discrete-time approximation for mean-field control with quadratic cost. Detailed statements of our results are given in Section 3.2.

### 3.1.1 Review of related literature

For general results about mean-field limits and propagation of chaos, we refer to the survey articles Chaintron and Diez [2022a,b] and references therein. For the specific case of the path-dependent setting, we refer to e.g. Lacker [2018]; Bernou and Liu [2022]; Lacker [2023], which cover slightly less general settings. There is a lot of references for discretised versions of systems like (3.1), among which Bossy and Talay [1997]; Antonelli and Kohatsu-Higa [2002]; Bao and Huang [2019]; Bernou and Liu [2022] and references therein. To the best of our knowledge, our LDP and CLT results in the discrete-time setting are new.

**Tanaka’s pathwise construction.** Up to our knowledge, there are surprisingly few instances of Tanaka’s pathwise construction in the mean-field literature. We mention [Backhoff et al., 2020, Section 3.1] where the LDP is proved in the Wasserstein- $p$  topology ( $p \in [1, 2)$ ) in a path-dependent setting with  $\sigma \equiv \text{Id}$  and binary interactions of type  $b(x, P) = \int_{\mathcal{C}^d} \tilde{b}(x, y) dP(y)$ . An exhaustive treatment of the  $\sigma \equiv \text{Id}$  setting is done in Coghi et al. [2020] with extensions to many related situations (general noises, jump processes...). The path-construction is also used in Delarue et al. [2020] with  $\sigma \equiv \text{Id}$  to prove the LDP in the Wasserstein-1 topology, before extending it to prove the LDP in a common noise setting. This last framework is particularly intricate because the rate function no more has compact level sets.

**Central limit theorem.** Until recently, there has been much interest in proving the CLT for mean-field systems close to (3.1). The first related work seems to be McKean [1975], computing fluctuations for interacting particles in the two-state space. A seminal work is then Braun and Hepp [1977] for a deterministic Vlasov system, which inspired Tanaka’s CLT Tanaka [1984] on path space (with  $\sigma \equiv \text{Id}$ ). The fundamental works Sznitman [1984, 1985] prove the CLT on path space for binary interactions, starting from a Girsanov formula to compute symmetric statistics and multiple Wiener integrals. Many following works re-used this approach. An alternative method for large deviations and CLT for Gibbs measures is Bolthausen [1986], which relies on embeddings in suitable Banach spaces. This method was further developed and applied to diffusion systems in Arous and Brunaud [1990]; Pra and Hollander [1996]. A by now classical approach Fernandez and Méléard [1997]; Méléard [1998]; Jourdain and Méléard [1998] computes the fluctuations of the random curve  $t \mapsto \pi(\vec{X}_t^N)$  in weighted Sobolev spaces. Some tightness estimates are first proved, before identifying the limit as an infinite-dimensional Ornstein-Uhlenbeck process. Such results are weaker than the pathwise CLT, but this approach can take benefit of PDE structures to get sharper estimates. A fairly general CLT for  $t \mapsto \pi(\vec{X}_t^N)$  is

thus proved in Jourdain and Tse [2021], by careful analysis of a measure-valued flow. The very recent work Bernou and Duerinckx [2024] uses analogous methods to prove (among many other results) a uniform-in-time and quantitative version of this CLT. In the setting of mean-field games with common noise, we also mention the CLT result Delarue et al. [2018]. To the best of our knowledge, our pathwise CLT for a system as general as (3.1) is new.

**Large deviations.** There is an abundant literature for large deviations of the empirical measure of systems like (3.1). The seminal work Tanaka [1984] covers the case  $\sigma \equiv \text{Id}$  with binary interactions  $b(x, \pi) = \int_{\mathcal{C}^d} \tilde{b}(x, y) d\pi(y)$ . When  $\tilde{b}(x, y) = -\nabla V(x - y)$ , the Girsanov transform computes the density of the law of  $\vec{X}^N$  w.r.t. to the law of the corresponding system of i.i.d. drift-less particles. The gradient structure further removes the stochastic integral term from the density given by Girsanov's theorem. The resulting density being a Gibbs measure, large deviations are then a standard result. This method is applied in e.g Arous and Brunaud [1990]; Del Moral and Guionnet [1998], and in Pra and Hollander [1996] in a more general setting including environment noises. For an extension of this method to non-gradient binary interactions, we refer to Del Moral and Zajic [2003].

For more general drifts  $b$ , the LDP is a harder task: it was initially proved in Dawson and Gärtner [1987] under Lipschitz conditions and non-degeneracy of  $\sigma$ , using an extended version of the weak topology on measures. The result is strengthened in Budhiraja et al. [2012] under much weaker assumptions (martingale problem framework), notably allowing for path-dependent settings. In both works, the initial data are assumed to be deterministic, and there is no direct way to handle random initial data from there. We also refer to Orrieri [2020]; Budhiraj and Conroy [2022], which prove the LDP for (3.2) in a setting combining both mean-field and small-noise limits.

Computing the large deviation rate function as a relative entropy can be intricate, depending on the chosen approach for proving the LDP. The link with stochastic control traces back to Föllmer [1988]. We refer to Cattiaux and Léonard [1995]; Pra and Hollander [1996]; Cattiaux and Léonard [1996] for seminal results in this direction. An exhaustive account of these links is given by Léonard [2012]; Fischer [2014].

The question of establishing the LDP in the Wasserstein- $p$  topology is more recent. A necessary and sufficient condition for Sanov's theorem to hold in the Wasserstein topology is proved in Wang et al. [2010]. In the Brownian setting, this forces  $p \in [1, 2)$ . We already referred to Backhoff et al. [2020]; Delarue et al. [2020]. For large deviations in the strong topology, we refer to Dawson and Del Moral [2005]. For Gibbs measures in the Wasserstein topology, we also mention Léonard [1987]; Reygner [2018]; Dupuis et al. [2020]; Liu and Wu [2020]. To the best of our knowledge, our LDP in the Wasserstein topology for a system as general as (3.1) is a new result.

### 3.1.2 Notations

- $\mathcal{P}(E)$  denotes the space of probability measures over a Polish space  $E$ , the default topology on it being the one of weak convergence.
- $\mathcal{L}(X)$  denotes the law in  $\mathcal{P}(E)$  of a  $E$ -valued random variable  $X$ .
- $T_{\#}P$  denotes the push-forward (or image) measure of  $P \in \mathcal{P}(E)$  by a measurable  $T : E \rightarrow F$ .
- When  $E$  and  $F$  are Banach spaces,  $\mathcal{C}^1(E, F)$  denotes the space of Fréchet-differentiable functions  $E \rightarrow F$ .  $\mathcal{C}^{1,1}(E, F)$  is the sub-space of functions in  $\mathcal{C}^1(E, F)$  with globally

Lipschitz derivative  $DF$ .  $\mathcal{C}_b^{1,1}(E, F)$  is the space of bounded functions in  $\mathcal{C}^{1,1}(E, F)$  with bounded derivative.

- $\delta_x$  denotes the Dirac measure at some point  $x \in E$ .
- Given  $N$  variables  $x^1, \dots, x^N$  in some product space  $E^N$ , we will use the notations  $\vec{x}^N := (x^1, \dots, x^N)$  and

$$\pi(\vec{x}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{x^i} \in \mathcal{P}(E).$$

- $\pi(\vec{x}^N, \vec{y}^N)$  will similarly denote the empirical measure

$$\pi(\vec{x}^N, \vec{y}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{(x^i, y^i)} \in \mathcal{P}(E \times E),$$

with a slight abuse of notation.

- $H(P|Q)$  denotes the relative entropy of  $P$  w.r.t.  $Q$ , for  $P, Q \in \mathcal{P}(E)$ , defined as  $H(P|Q) = \mathbb{E}_P[\log \frac{dP}{dQ}]$  if  $P \ll Q$ , and  $H(P|Q) = +\infty$  otherwise.
- $\delta_P F(P) : x \mapsto \delta_P F(P, x)$  denotes the linear functional derivative at  $P$  of a function  $F : \mathcal{P}(E) \rightarrow \mathbb{R}$ , see Definition 3.1.1 below. The convention is adopted that  $\int_E \delta_P F(P) dP = 0$ .
- $\mathcal{P}_p(E)$  denotes the set of measures  $P \in \mathcal{P}(E)$  such that  $\int_E d_E^p(x, x_0) dP(x) < +\infty$  for the distance  $d_E$  on  $E$ ,  $x_0 \in E$  and  $p \geq 1$ .
- $L_{\mathbb{P}}^p(\Omega, E)$  denotes the space of  $E$ -valued random variables  $X : \Omega \rightarrow E$ , on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $\mathcal{L}(X)$  belongs to  $\mathcal{P}_p(E)$ .
- $W_p$  denotes the Wasserstein distance on  $\mathcal{P}_p(E)$ , defined by

$$W_p^p(P, Q) := \inf_{X \sim P, Y \sim Q} \mathbb{E}[d_E^p(X, Y)].$$

- $d, d' \geq 1$  are fixed integers, and  $T > 0$  is a given finite time horizon.
- $\lfloor t \rfloor$  is the image of  $t \in \mathbb{R}$  by the floor function. We similarly write  $\lceil t \rceil$  for the ceiling function. We recall that  $\lfloor t \rfloor$  and  $\lceil t \rceil$  are integers.
- $\mathcal{C}^d$  denotes the space  $\mathcal{C}([0, T], \mathbb{R}^d)$  of continuous paths, endowed with the uniform topology. A typical path in  $\mathcal{C}^d$  will be denoted by  $x$ , and a typical one in  $\mathcal{C}^{d'}$  by  $\gamma$ .
- $\mathcal{C}_M^d$  denotes the space of paths  $x \in \mathcal{C}^d$  with  $\sup_{0 \leq t \leq T} |x_t| \leq M$ , for some  $M > 0$ .
- $x_{\wedge t}$  denotes the path  $(x_{s \wedge t})_{0 \leq s \leq T}$ , for any  $t \in [0, T]$  and  $x \in \mathcal{C}^d$ . We recall that  $s \wedge t$  is the minimum of  $s$  and  $t$ .
- $P_{\wedge t}$  denotes the push-forward measure of  $P \in \mathcal{P}(\mathcal{C}^d)$  by the map  $x \mapsto x_{\wedge t}$ .
- $P_t \in \mathcal{P}(\mathbb{R}^d)$  denotes the marginal-law at time  $t$  of a path-measure  $P \in \mathcal{P}(\mathcal{C}^d)$ .
- $P \in \mathcal{C}([0, T], \mathcal{P}(\mathbb{R}^d))$  denotes the related curve  $(P_t)_{0 \leq t \leq T}$  of time-marginals.

- $\Sigma = (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$  denotes a reference probability system in the terminology of [Fleming and Soner, 2006, Chapter 4]:  $(\Omega, \mathcal{F}_T, \mathbb{P})$  is a probability space,  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is a filtration satisfying the usual conditions, and  $(B_t)_{0 \leq t \leq T}$  is a  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -Brownian motion.
- $|a|$  denotes the Frobenius norm  $\sqrt{\text{Tr}[aa^\top]}$  of a matrix  $a$ .
- $\nabla \cdot a$  is the vector field whose  $i$ -entry is the divergence of the vector field  $(a^{i,j})_{1 \leq j \leq d}$ , for a matrix field  $a = (a^{i,j})_{1 \leq i, j \leq d}$ . We similarly define differential operators on matrices.

**Definition 3.1.1** (Linear functional derivative). *A map  $F : \mathcal{P}_p(E) \rightarrow \mathbb{R}$  is differentiable at  $P$  if there exists a measurable map*

$$\delta_P F(P) : \begin{cases} E \rightarrow \mathbb{R}, \\ x \mapsto \delta_P F(P, x), \end{cases}$$

such that for every  $Q$  in  $\mathcal{P}_p(E)$ ,  $\delta_P F(P)$  is  $Q$ -integrable and satisfies

$$\varepsilon^{-1} [F((1 - \varepsilon)P + \varepsilon Q) - F(P)] \xrightarrow{\varepsilon \rightarrow 0^+} \int_E \delta_P F(P) d[Q - P].$$

The map  $\delta_P F(P)$  being defined up to an additive constant, we adopt the usual convention that  $\int_E \delta_P F(P) dP = 0$ . This map is called the linear functional derivative of  $F$  at  $P$ . We notice that this definition does not depend on the behaviour of  $F$  outside of an arbitrary small neighbourhood of  $P$ .

By direct integration, the definition implies

$$\forall P, Q \in \mathcal{P}_p(E), \quad F(P) - F(Q) = \int_0^1 \int_E \delta_P F((1 - r)Q + rP) d[P - Q] dr,$$

provided that the integral on the r.h.s. is well-defined.

**Example 3.1.2** (Linear case). *In the particular case  $F(P) = \int_E f dP$  for some measurable  $f : E \rightarrow \mathbb{R}$  with polynomial growth of order  $p$ , we have  $\delta_P F(P, x) = f(x) - \int_E f dP$ .*

## 3.2 Statement of the main results

Let us fix  $p \geq 1$ , together with positive integers  $d$  and  $d'$ . We will restrict ourselves to  $p \in [1, 2)$  for large deviation results in Sections 3.2.2-3.2.3-3.2.4. The coefficients are continuous functions

$$b : [0, T] \times \mathcal{C}^d \times \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}^d, \quad \sigma : [0, T] \times \mathcal{C}^d \times \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}^{d \times d'}.$$

For  $P \in \mathcal{P}_p(\mathcal{C}^d)$ , we recall that  $P_{\wedge t} \in \mathcal{P}_p(\mathcal{C}^d)$  denotes the push-forward measure of  $P$  under the map  $x \in \mathcal{C}^d \mapsto x_{\wedge t} := (x_{s \wedge t})_{0 \leq s \leq T} \in \mathcal{C}^d$ . We make the following assumption on the coefficients.

**Assumption (A.1)** (Coefficients). *There exist  $L_b, L_\sigma > 0$  and*

$$\bar{b} : [0, T] \times \mathcal{C}^d \times \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}^d, \quad \bar{\sigma} : [0, T] \times \mathcal{C}^d \times \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}^{d \times d'},$$

which are respectively globally  $L_b$ - and  $L_\sigma$ -Lipschitz and such that, for every  $(t, x, P)$ ,

$$b_t(x, P) = \bar{b}_t(x_{\wedge t}, P_{\wedge t}), \quad \sigma_t(x, P) = \bar{\sigma}_t(x_{\wedge t}, P_{\wedge t}).$$

Moreover,  $\sigma$  is globally bounded by some  $M_\sigma > 0$ .

In particular, this includes standard Lipschitz coefficients of the kind  $b_t(x, P) = b_t(x_t, P_t)$ . Our framework further includes pathwise settings which are non-Markov but still non-anticipative like models with delay. The Lipschitz-continuity w.r.t.  $t$  will only be needed in the time-continuous settings of Sections 3.2.3-3.2.4, but we keep (A.1) as such for the sake of simplicity.

**Lemma 3.2.1.** *Under (A.1), for any Brownian motion  $(B_t)_{0 \leq t \leq T}$  and initial data  $\bar{X}_0$  with finite  $\mathbb{E}[|\bar{X}_0|^p]$ , strong existence and pathwise-uniqueness hold for solutions of the the path-dependent McKean-Vlasov equation*

$$d\bar{X}_t = b_t(\bar{X}, \mathcal{L}(\bar{X}))dt + \sigma_t(\bar{X}, \mathcal{L}(\bar{X}))dB_t$$

that satisfy  $\mathbb{E}[\sup_{0 \leq t \leq T} |\bar{X}_t|^p] < +\infty$ .

This result is contained in [Djete et al., 2022, Appendix A]. See Lacker et al. [2022]; Bernou and Liu [2022] for similar results.

### 3.2.1 Discretised system with general driving noise

Let  $h > 0$  be a fixed discretisation parameter. Given  $t \geq 0$ , we define  $t_h := h\lfloor t/h \rfloor$ . On a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , we study the discretised McKean-Vlasov SDE

$$\begin{cases} dX_{t_h}^h = b_{t_h}(X^h, \mathcal{L}(X^h))dt + \sigma_{t_h}(X^h, \mathcal{L}(X^h))dW_t, \\ X_0^h = \zeta, \end{cases} \quad (3.4)$$

the input being the random variable

$$(\zeta, W) : \Omega \rightarrow \mathbb{R}^d \times \mathcal{C}^d,$$

which belongs to  $L_{\mathbb{P}}^p(\Omega, \mathbb{R}^d \times \mathcal{C}^d)$ . The driving noise  $W$  can be any continuous random path, which does not have to be a Brownian motion. In particular, (3.4) does not involve any stochastic integral, the differential being a mere notation. To be non-ambiguous, we specify the notion of solution for (3.4). For every  $P \in \mathcal{P}_p(\mathcal{C}^d)$ ,  $(\zeta, W) \in \mathbb{R}^d \times \mathcal{C}^d$ ,  $X \in \mathcal{C}^d$ , we define the path  $f^{h,P}(\zeta, W, X)$  in  $\mathcal{C}^d$  by

$$\begin{aligned} f_t^{h,P}(\zeta, W, X) &= \zeta + b_{t_h}(X, P)[t - t_h] + \sigma_{t_h}(X, P)[W_t - W_{t_h}] \\ &\quad + \sum_{i=0}^{\lfloor t/h \rfloor - 1} hb_{ih}(X, P) + \sigma_{ih}(X, P)[W_{(i+1)h} - W_{ih}], \end{aligned}$$

for every  $t$  in  $[0, T]$ . We use the usual convention that an empty sum equals 0.

**Definition 3.2.2.** *A pathwise solution of (3.4) with input  $(\zeta, W) \in L_{\mathbb{P}}^p(\Omega, \mathbb{R}^d \times \mathcal{C}^d)$ , is a random variable  $X^h : \Omega \rightarrow \mathcal{C}^d$  that belongs to  $L_{\mathbb{P}}^p(\Omega, \mathcal{C}^d)$  such that for  $\mathbb{P}$ -a.e.  $\omega$  in  $\Omega$ ,*

$$\forall t \in [0, T], \quad X_t^h(\omega) = f_t^{h, \mathcal{L}(X^h)}(\zeta(\omega), W(\omega), X^h(\omega)).$$

Closely related to (3.4) is the discretised system of interacting particles  $\vec{X}^{h,N} = (X^{h,i,N})_{1 \leq i \leq N}$  defined by

$$\begin{cases} dX_t^{h,i,N} = b_{t_h}(X^{h,i,N}, \pi(\vec{X}^{h,N}))dt + \sigma_{t_h}(X^{h,i,N}, \pi(\vec{X}^{h,N}))dW_t^{i,N}, \\ X_0^{h,i,N} = \zeta^{i,N}, \end{cases} \quad (3.5)$$

for  $1 \leq i \leq N$ , with given inputs

$$(\vec{\zeta}^N, \vec{W}^N) : \omega \mapsto (\zeta^{i,N}(\omega), W^{i,N}(\omega))_{1 \leq i \leq N},$$

in  $L_{\mathbb{P}}^p(\Omega, (\mathbb{R}^d \times \mathcal{C}^d)^N)$ . We emphasise that *no independence nor exchangeability are required*.

**Remark 3.2.3** (Heterogeneous mean-field). *This formalism includes heterogeneous particle systems of the type*

$$dX_t^{h,i,N} = b_{t_h}(X^{h,i,N}, R^{i,N}, \pi(\vec{X}^{h,N}, R^{i,N}))dt + \sigma_{t_h}(X^{h,i,N}, R^{i,N}, \pi(X^{h,i,N}, R^{i,N}))dW_t^{i,N},$$

for any family  $(R^{i,N})_{1 \leq i \leq N, N \geq 1}$  of random paths in  $L_{\mathbb{P}}^p(\Omega, \mathcal{C}^k)$  under natural Lipschitz assumptions on  $(b, \sigma)$ . Indeed, this amounts to replacing the random path  $X^{h,i,N} \in \mathcal{C}^d$  by  $Z^{h,i,N} := (X^{h,i,N}, R^{i,N}) \in \mathcal{C}^{d+k}$ . The following results can then be written in this setting, see [Coghi et al., 2020, Section 3.4]. This extends some results of Pra and Hollander [1996] for McKean-Vlasov processes that interact with random media. See also [Coghi et al., 2020, Section 3.3] for adding a common noise to (3.5).

We consider time-discretised systems because the  $h \rightarrow 0$  limit would require some notion of stochastic integral that is not available at this level of generality. However, when  $\sigma \equiv \text{Id}$ , all the following results hold for  $h = 0$ , because the flow maps are still continuous functions of the driving noise. Adapting our proofs to this setting can be done readily, and we refer to Coghi et al. [2020] for an exhaustive discussion of this setting.

**Theorem 3.2.4** (Pathwise construction). *Under (A.1),*

- (i) *There exists a pathwise unique strong solution  $\vec{X}^{h,N}$  of (3.5).*
- (ii) *There exists a unique pathwise solution  $X^h$  of (3.4) in the sense of Definition 3.2.2, and*

$$\Psi_h : \begin{cases} \mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^{d'}) \rightarrow \mathcal{P}_p(\mathcal{C}^d), \\ \mathcal{L}(\zeta, W) \mapsto \mathcal{L}(X^h), \end{cases}$$

*is continuous.*

- (iii)  *$\mathbb{P}$ -almost surely,  $\pi(\vec{X}^{h,N}) = \Psi_h(\pi(\vec{\zeta}^N, \vec{W}^N))$ .*

The above setting reduces the mean-field limit to a simple continuity result. No exchangeability nor other specific structures are required. Since the map  $\Psi_h$  is continuous, the following result is a direct consequence of the contraction principle [Dupuis and Ellis, 2011, Theorem 1.3.2]. For general definitions about large deviations, see Deuschel and Stroock [2001]; Dupuis and Ellis [2011].

**Corollary 3.2.4.1** (Mean-field limit and large deviations).

- (i) *If  $W_p(\pi(\vec{\zeta}^N, \vec{W}^N), \mathcal{L}(\zeta, W)) \rightarrow 0$ ,  $\mathbb{P}$ -a.s. as  $N \rightarrow +\infty$ , then*

$$W_p(\pi(\vec{X}^{h,N}), \mathcal{L}(X^h)) \rightarrow 0, \quad \mathbb{P}\text{-a.s. as } N \rightarrow +\infty.$$

- (ii) *If the sequence of the  $\mathcal{L}(\pi(\vec{\zeta}^N, \vec{W}^N))$  satisfies the LDP with good rate function  $I : \mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^{d'}) \rightarrow [0, +\infty]$ , then the sequence of the  $\mathcal{L}(\pi(\vec{X}^{h,N}))$  satisfies the LDP with good rate function*

$$P \in \mathcal{P}_p(\mathcal{C}^d) \mapsto \inf_{\substack{R \in \mathcal{P}(\mathbb{R}^d \times \mathcal{C}^{d'}) \\ \Psi_h(R) = P}} I(R).$$

Theorem 3.2.4 is proved in Section 3.3.2. For proving the CLT, we need further assumptions on the coefficients.

**Assumption (A.2)** (Strengthening of (A.1)). *(A.1) holds and for every  $(t, x) \in [0, T] \times \mathcal{C}^d$ ,*



- (i)  $x \mapsto \bar{b}_t(x, P), \bar{\sigma}_t(x, P)$  are Fréchet-differentiable and  $(x, P) \mapsto D_x \bar{b}_t(x, P), D_x \bar{\sigma}_t(x, P)$  are globally Lipschitz uniformly in  $t$ .
- (ii)  $P \mapsto \bar{b}_t(x, P), \bar{\sigma}_t(x, P)$  have linear derivatives  $y \mapsto \delta_P \bar{b}_t(x, P, y), \delta_P \bar{\sigma}_t(x, P, y)$ , in the sense of Definition 3.1.1, which are Fréchet-differentiable w.r.t.  $y$ .
- (iii)  $(x, P, y) \mapsto D_y \delta_P \bar{b}_t(x, P, y), D_y \delta_P \bar{\sigma}_t(x, P, y)$  are globally Lipschitz uniformly in  $t$ .

Under (A.1)-(A.2), we notice that  $b_t(x, P)$  and  $\sigma_t(x, P)$  are differentiable w.r.t.  $(x, P)$  too, with derivatives only depending on  $x_{\wedge t}$  and  $P_{\wedge t}$ . For the sake of simplicity, we further assume that all the particles start at 0: we remove the dependence of  $\Psi_h$  on the initial condition. To state a result at this level of generality, we restrict ourselves to bounded noises taking values in the space  $\mathcal{C}_{M'}^d$  of paths with norm bounded by  $M' > 0$ . This restriction is not needed if  $\sigma \equiv \text{Id}$ , see [Coghi et al., 2020, Section 5].

**Theorem 3.2.5** (Central Limit Theorem). *Under (A.2), let  $(W^i)_{i \geq 1}$  be an i.i.d. sequence of  $\mathcal{C}_{M'}^d$ -valued variables with common law  $R$ . Let  $\vec{X}^{h,N}$  be the related system of particles starting at 0 given by (3.5), and let  $X^h$  be the solution of (3.4) driven by  $W^1$ . Then, for every  $\varphi \in \mathcal{C}_b^{1,1}(\mathcal{C}^d, \mathbb{R})$ , the random variable*

$$\sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N \varphi(X^{h,i,N}) - \mathbb{E}[\varphi(X^h)] \right]$$

converges in law towards a centred Gaussian variable whose variance  $\sigma_\varphi^2$  is explicit in terms of  $\Psi_h(R)$ , see Corollary 3.3.7.2 below.

Theorem 3.2.5 is proved in Section 3.3.3.

### 3.2.2 Discretised setting with Brownian noise

First, we apply Corollary 3.2.4.1 to Brownian noises. To clearly separate this Brownian setting from the generic one of the previous section, we will write  $(X_0^i)_{1 \leq i \leq N}$  for the initial conditions instead of  $(\zeta^{i,N})_{1 \leq i \leq N}$ . Since we aim for a LDP in the Wasserstein topology, Wang et al. [2010] tells us that we must restrict ourselves to  $p \in [1, 2)$  and that we need the following assumption.

**Assumption (A.3)** (Initial exponential moments).  $p \in [1, 2)$  and the particles are initialised from a i.i.d. sequence  $(X_0^i)_{i \geq 1}$  such that

$$\forall \alpha > 0, \quad \mathbb{E}[e^{\alpha |X_0^1|^p}] < +\infty.$$

For any  $P$  in  $\mathcal{P}_p(\mathcal{C}^d)$ , we define  $\Gamma_h(P)$  as being the path-law of the solution to

$$dY_t^h = b_{t_h}(Y^h, P)dt + \sigma_{t_h}(Y^h, P)dB_t, \quad Y_0^h = X_0^1, \quad (3.6)$$

where  $(B_t)_{0 \leq t \leq T}$  is a  $\mathbb{R}^d$ -valued Brownian motion. Since  $\sigma$  is globally bounded and  $b$  has linear growth, (A.1)-(A.3) imply that

$$\forall \alpha > 0, \quad \mathbb{E}[\exp[\alpha \sup_{0 \leq t \leq T} |Y_t^h|^p]] < +\infty,$$

as required by [Wang et al., 2010, Theorem 1.1].

**Proposition 3.2.6** (Discretised LDP). *Under (A.1)-(A.3), let  $(B^i)_{i \geq 1}$  be an i.i.d. sequence in  $\mathcal{C}^{d'}$  of Brownian motions. Let  $\vec{X}^{h,N}$  be the related system of particles starting from  $(X_0^i)_{i \geq 1}$ . Then, the sequence of the  $\mathcal{L}(\pi(\vec{X}^{h,N}))$  satisfies the LDP with good rate function*

$$I_h : P \in \mathcal{P}_p(\mathcal{C}^d) \mapsto H(P|\Gamma_h(P)).$$

We recall that the relative entropy  $H$  is defined in Section 3.1.2. Proposition 3.2.6 is proved in Section 3.3.2. We then extend Theorem 3.2.5 to the Brownian setting. We rely on an approximation procedure by truncating the noises. To do so, we need uniform estimates for fluctuation processes given by Proposition 3.3.9. In particular, we need uniform propagation of chaos estimates with rate  $N^{-1/2}$ . For such estimates to hold, it is convenient to restrict ourselves to binary interactions.

**Assumption (A.4)** (Binary interactions). *There exist functions  $\tilde{b} : [0, T] \times \mathcal{C}^d \times \mathcal{C}^d \rightarrow \mathbb{R}^d$  and  $\tilde{\sigma} : [0, T] \times \mathcal{C}^d \times \mathcal{C}^d \rightarrow \mathbb{R}^{d \times d'}$  such that*

$$b_t(x, P) = \int_{\mathcal{C}^d} \tilde{b}_t(x \wedge t, y) dP_{\wedge t}(y), \quad \sigma_t(x, P) = \int_{\mathcal{C}^d} \tilde{\sigma}_t(x \wedge t, y) dP_{\wedge t}(y).$$

Moreover,  $\tilde{b}$ ,  $\tilde{\sigma}$  are  $\mathcal{C}^{1,1}$  and  $\tilde{\sigma}$  is bounded, so that  $b$ ,  $\sigma$  satisfy (A.1)-(A.2).

Let  $(B_t)_{0 \leq t \leq T}$ ,  $(\tilde{B}_t)_{0 \leq t \leq T}$  be independent Brownian motions in  $\mathbb{R}^d$ . Let  $X^h$  denote the strong solution of the discretised McKean-Vlasov equation

$$dX_{t_h}^h = b_{t_h}(X^h, \mathcal{L}(X^h))dt + \sigma_{t_h}(X^h, \mathcal{L}(X^h))dB_t, \quad X_0^h = 0,$$

which corresponds to (3.4) driven by  $B$ . We recall that strong existence and pathwise-uniqueness are provided by Theorem 3.2.4. Let  $\tilde{X}^h$  denote the solution of the same equation driven by  $\tilde{B}$ . We introduce the solution  $\delta X^h$  of the McKean-Vlasov SDE

$$\begin{aligned} d\delta X_{t_h}^h &= [D_x \sigma_{t_h}(X^h, \mathcal{L}(X^h)) \cdot \delta X^h + \delta_P \sigma_{t_h}(X^h, \mathcal{L}(X^h), \tilde{X}^h)] dB_t + \{D_x b_{t_h}(X^h, \mathcal{L}(X^h)) \cdot \delta X^h \\ &\quad + \delta_P b_{t_h}(X^h, \mathcal{L}(X^h), \tilde{X}^h) + \mathbb{E}[D_y \delta_P b_{t_h}(X^h, \mathcal{L}(X^h), X^h) \cdot \delta X^h | (\tilde{B}_s)_{0 \leq s \leq t_h}]\} dt \\ &\quad + \mathbb{E}[D_y \delta_P \sigma_{t_h}(X^h, \mathcal{L}(X^h), X^h) \cdot \delta X^h | (\tilde{B}_s)_{0 \leq s \leq t_h}] dB_t, \quad \delta X_0^h = 0. \end{aligned} \quad (3.7)$$

Well-posedness for (3.7) is a direct induction (similar to Lemma 3.3.2 below). This can be seen as a discretised version of a McKean-Vlasov SDE with common noise like the one studied in [Djete et al., 2022, Appendix A].

**Proposition 3.2.7.** *Under (A.4), let  $(B^i)_{i \geq 1}$  be an i.i.d. sequence in  $\mathcal{C}^{d'}$  of Brownian motions. Let  $\vec{X}^{h,N}$  be the related system of particles starting from 0 given by (3.5). Then, for every  $\varphi \in \mathcal{C}_b^{1,1}(\mathcal{C}^d, \mathbb{R})$ , the random variable*

$$\sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N \varphi(X^{h,i,N}) - \mathbb{E}[\varphi(X^h)] \right]$$

converges in law towards a centred Gaussian variable with variance  $\sigma_{h,\varphi}^2$  given by

$$\sigma_{h,\varphi}^2 := \mathbb{E}\{[\varphi(\tilde{X}^h) + \mathbb{E}[D\varphi(X^h) \cdot \delta X^h - \varphi(X^h) | \tilde{B}]]^2\}.$$

The underlying Gaussian field can be described in terms of an infinite-dimensional SDE, see [Tanaka, 1984, Section 4]. Proposition 3.2.7 is proved in Section 3.3.4, using results from Appendix 3.A.

### 3.2.3 Continuous setting with Brownian noise

We now extend Proposition 3.2.6 to the continuous setting. For  $P \in \mathcal{P}_p(\mathcal{C}^d)$ , we define  $\Gamma_0(P)$  as being the path-law of the solution to

$$dY_t = b_t(Y, P)dt + \sigma_t(Y, P)dB_t, \quad Y_0 = X_0^1,$$

where  $(B_t)_{0 \leq t \leq T}$  is a  $\mathbb{R}^d$ -valued Brownian motion. We notice that  $\Gamma_0(P)$  can be seen as the  $h \rightarrow 0$  limit of  $\Gamma_h(P)$  given by (3.6). In the following, we will use the same convention for extending definitions to  $h = 0$ .

**Theorem 3.2.8** (Exponential approximation and LDP). *Let us assume that  $\sigma_t(x, P) = \sigma_t(P)$  does not depend on  $x$ . Under (A.1)-(A.3), let  $(B^i)_{i \geq 1}$  be an i.i.d. sequence in  $\mathcal{C}^d$  of Brownian motions. For  $h \in [0, 1]$ , let  $\vec{X}^{h,N}$  be the related system of particles starting from  $(X_0^i)_{i \geq 1}$ . Then, for every bounded Lipschitz  $F : \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}$ ,*

$$\sup_{N \geq 1} \left| N^{-1} \log \mathbb{E} \left[ \exp \left[ NF(\pi(\vec{X}^{h,N})) \right] \right] - N^{-1} \log \mathbb{E} \left[ \exp \left[ NF(\pi(\vec{X}^{0,N})) \right] \right] \right| \xrightarrow{h \rightarrow 0} 0.$$

Moreover, the sequence of the  $\mathcal{L}(\pi(\vec{X}^{0,N}))$  satisfies the LDP with good rate function

$$I_0 : P \in \mathcal{P}_p(\mathcal{C}^d) \mapsto H(P|\Gamma_0(P)).$$

Moreover,  $I_h$   $\Gamma$ -converges towards  $I_0$  as  $h \rightarrow 0$ .

Theorem 3.2.11 below provides the LDP when  $\sigma$  does depend on  $x$  but not on  $P$ . The above result can be rephrased as the quantitative convergence of the value function of a mean-field control problem with quadratic cost, see Proposition 3.4.3 below. Theorem 3.2.8 is proved in Section 3.4.1. Some technical results about rate functions are deferred to Appendix 3.B (in particular, links with stochastic control).

**Remark 3.2.9** (Mean-field limit and scheme convergence). *From Lemma 3.4.2-(iv) and Lemma 3.4.5, we have the consistency results, for every  $h \in [0, 1]$ ,  $N \geq 1$  and  $1 \leq i \leq N$ ,*

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{h,i,N} - X_t^{0,i,N}|^2 \right] \leq C|h \log h|, \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{h,i,N} - \bar{X}_t^{h,i}|^2 \right] \leq CN^{-1},$$

for  $C > 0$  independent of  $(h, N)$ . In particular, the  $h \rightarrow 0$  and  $N \rightarrow +\infty$  limits commute with quantitative rates of convergence, see Lemma 3.4.5. Theorem 3.2.8 proves the same commutation at the level of large deviations, whereas Theorem 3.2.10 proves it for normal fluctuations.

We now extend Proposition 3.2.7 to the continuous setting. As previously, let  $(B_t)_{0 \leq t \leq T}$ ,  $(\tilde{B}_t)_{0 \leq t \leq T}$  be independent Brownian motions in  $\mathbb{R}^d$ . Let  $X$  denote the strong solution of the McKean-Vlasov equation given by Lemma 3.2.1,

$$dX_t = b_t(X, \mathcal{L}(X))dt + \sigma_t(X, \mathcal{L}(X))dB_t, \quad X_0 = 0,$$

which corresponds to the  $h = 0$  extension of (3.4) driven by  $B$ . Let  $\tilde{X}$  denote the solution of the same equation driven by  $\tilde{B}$ . We then introduce the solution  $\delta X$  of

$$\begin{aligned} d\delta X_t &= [D_x \sigma_t(X, \mathcal{L}(X)) \cdot \delta X + \delta_P \sigma_t(X, \mathcal{L}(X), \tilde{X})]dB_t + \{D_x b_t(X, \mathcal{L}(X)) \cdot \delta X \\ &\quad + \delta_P b_t(X, \mathcal{L}(X), \tilde{X}) + \mathbb{E}[D_y \delta_P b_t(X, \mathcal{L}(X), X) \cdot \delta X | (\tilde{B}_s)_{0 \leq s \leq t}]\}dt \\ &\quad + \mathbb{E}[D_y \delta_P \sigma_t(X, \mathcal{L}(X), X) \cdot \delta X | (\tilde{B}_s)_{0 \leq s \leq t}]dB_t, \quad \delta X_0 = 0. \end{aligned} \quad (3.8)$$

Well-posedness for such a path-dependent McKean-Vlasov SDE with common noise can be found in [Djete et al., 2022, Appendix A].

**Theorem 3.2.10.** *Under (A.4), let  $(B^i)_{i \geq 1}$  be an i.i.d. sequence in  $\mathcal{C}^d$  of Brownian motions. Let  $\vec{X}^{h,N}$  be the related system of particles starting from 0 given by (3.5). Let further  $\bar{X}^{h,i}$  denote the solution of (3.4) with driving noise  $B^i$ . There exists  $C > 0$  such that*

$$\sup_{N \geq 1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{h,i,N} - \bar{X}_t^{h,i} - [X_t^{0,i,N} - \bar{X}_t^{0,i}]| \right] \leq C |h \log h|^{1/2},$$

for every  $h \in [0, 1]$ . Moreover, for every  $\varphi \in \mathcal{C}_b^{1,1}(\mathcal{C}^d, \mathbb{R})$ ,

$$\sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N \varphi(X^{0,i,N}) - \mathbb{E}[\varphi(\bar{X}^{0,1})] \right]$$

converges in law towards a centred Gaussian variable with variance  $\sigma_\varphi^2$  given by

$$\sigma_\varphi^2 := \mathbb{E} \{ [\varphi(\tilde{X}) + \mathbb{E}[D\varphi(X) \cdot \delta X - \varphi(X) | \tilde{B}]]^2 \}.$$

The underlying Gaussian field can be described in terms of an infinite-dimensional SDE, see [Tanaka, 1984, Section 4]. Theorem 3.2.10 is proved in Section 3.4.3.

### 3.2.4 Contraction

Since the map

$$\begin{cases} \mathcal{P}_p(\mathcal{C}^d) & \rightarrow \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)), \\ P & \mapsto P := (P_t)_{0 \leq t \leq T}, \end{cases}$$

is continuous, Proposition 3.2.6 and Theorem 3.2.8 induce by contraction the LDP for the sequence of the  $\mathcal{L}(\pi(\vec{X}^{h,N}))$  with good rate function

$$\bar{I}_h : P \in \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)) \mapsto \inf_{\substack{Q \in \mathcal{P}_p(\mathcal{C}^d) \\ Q = P}} H(Q | \Gamma_h(Q)),$$

where  $h \in [0, 1]$ . The random curve  $\pi(\vec{X}^{0,N})$  precisely corresponds to (3.2) in the Introduction, this framework being natural by comparison with (3.3). In this setting, the following result allows for a dependence of  $\sigma$  on  $x$ .

**Assumption (A.5).** *The coefficients are globally Lipschitz functions independent of the history  $b_t(x, P) = b_t(x_t, P_t)$  and  $\sigma_t(x, P) = \sigma_t(x_t)$ . Moreover,  $\sigma$  is globally bounded.*

**Theorem 3.2.11.** *Under (A.3)-(A.5), let  $(B^i)_{i \geq 1}$  be an i.i.d. sequence in  $\mathcal{C}^d$  of Brownian motions. For  $h \in [0, 1]$ , let  $\vec{X}^{h,N}$  be the related system of particles starting from  $(X_0^i)_{i \geq 1}$ . Then, for every bounded Lipschitz  $F : \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)) \rightarrow \mathbb{R}$ ,*

$$\sup_{N \geq 1} \left| N^{-1} \log \mathbb{E} \left[ \exp \left[ NF(\pi(\vec{X}^{h,N})) \right] \right] - N^{-1} \log \mathbb{E} \left[ \exp \left[ NF(\pi(\vec{X}^{0,N})) \right] \right] \right| \xrightarrow{h \rightarrow 0} 0.$$

Moreover, the sequence of the  $\mathcal{L}(\pi(\vec{X}^{h,N}))$  satisfies the LDP with good rate function  $\bar{I}_h$ .

This estimate is reminiscent (but different) of log-efficiency estimates developed in the work Bezemek and Heldman [2024] for importance sampling. From Lemma 3.B.1, we can intuit the representation formula

$$\bar{I}_0(P) = H(P_0 | \mathcal{L}(X_0^1)) + \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} |v_t|^2 dP_t dt,$$

if there exists  $v \in L^2([0, T] \times \mathbb{R}^d, \mathbb{R}^{d'}, dt \otimes dP_t)$  such that

$$\partial_t P_t = \nabla \cdot \left[ -P_t b_t(\cdot, P_t) - P_t \sigma_t(\cdot, P_t) v_t + \frac{1}{2} \nabla \cdot [P_t \sigma_t \sigma_t^\top(\cdot, P_t)] \right],$$

in the sense of distributions, and  $\bar{I}_0(P) = +\infty$  otherwise. When  $d = d'$ , this formula is proved in [Delarue et al., 2020, Lemma 6.17] when  $\sigma \equiv \text{Id}$ , and in [Dawson and Gärtner, 1987, Lemma 4.8] under a non-degeneracy assumption on  $\sigma$ . Theorem 3.2.11 is proved in Section 3.4.2.

### 3.3 Constructions on the path space

This section is devoted to the discretised setting. The construction of the map  $\Psi_h$  is performed in Section 3.3.1. The results about mean-field limit and large deviations are proved in Section 3.3.2. The central limit theorem is eventually proved in Section 3.3.3.

#### 3.3.1 Construction of the discretised system

We follow the presentation of [Coghi et al., 2020, Section 2], adapting their notations to our setting. Before studying the non-linear SDE (3.4), we show well-posedness for the related ODE obtained by freezing the random inputs  $(\zeta(\omega), W(\omega))$  and the measure argument  $\mathcal{L}(X)$ .

**Lemma 3.3.1** (Measure-frozen equation).

- (i) For every  $(x_0, \gamma, P) \in \mathbb{R}^d \times \mathcal{C}^{d'} \times \mathcal{P}_p(\mathcal{C}^d)$ , the map  $x \mapsto f^{h,P}(x_0, \gamma, x)$  has a unique fixed-point  $S^{h,P}(x_0, \gamma)$  in  $\mathcal{C}^d$ .
- (ii) For any  $\gamma \in \mathcal{C}^{d'}$ , there exists  $L(\gamma)$  such that for every  $(x_0, P) \in \mathbb{R}^d \times \mathcal{P}_p(\mathcal{C}^d)$  and  $(x'_0, \gamma', P') \in \mathbb{R}^d \times \mathcal{C}^{d'} \times \mathcal{P}_p(\mathcal{C}^d)$ ,  $|S^{h,P}(x_0, \gamma) - S^{h,P'}(x'_0, \gamma')| \leq L(\gamma)[|x_0 - x'_0| + |\gamma - \gamma'| + W_p(P, P')]$ , and  $\gamma \mapsto L(\gamma)$  is bounded on bounded sets.
- (iii) For  $(\zeta, W) \in L^p_{\mathbb{P}}(\Omega, \mathbb{R}^d \times \mathcal{C}^{d'})$  and  $P \in \mathcal{P}_p(\mathcal{C}^d)$ , the path-dependent SDE

$$dY_t = b_{t_h}(Y, P)dt + \sigma_{t_h}(Y, P)dW_t, \quad Y_0 = \zeta, \quad (3.9)$$

has a pathwise-unique strong solution in  $L^p_{\mathbb{P}}(\Omega, \mathcal{C}^d)$ , given by  $S^{h,P}(\zeta, W)$ .

The proof of Lemma 3.3.1 is simple, because (3.9) is an explicit Euler scheme for a SDE.

*Proof.* (i) If  $(S_s^{h,P}(x_0, \gamma))_{0 \leq s \leq ih}$  is uniquely defined for some  $i \geq 0$ , then

$$S_t^{h,P}(x_0, \gamma) = S_{ih}^{h,P}(x_0, \gamma) + (t - ih)b_{ih}(S_{\wedge ih}^{h,P}(x_0, \gamma), P) + \sigma_{ih}(S_{\wedge ih}^{h,P}(x_0, \gamma), P)[\gamma_t - \gamma_{ih}],$$

for  $ih \leq t \leq (i+1)h$ , uniquely extending the definition till time  $(i+1)h$ . All this makes sense because  $b_{ih}(x, P)$  and  $\sigma_{ih}(x, P)$  only depend on  $(x_s)_{0 \leq s \leq ih}$  using (A.1). By induction, this proves existence and uniqueness for the fixed-point  $S^{h,P}(x_0, \gamma)$ .

(ii) We consider  $(x_0, \gamma, P), (x'_0, \gamma', P') \in \mathbb{R}^d \times \mathcal{C}^{d'} \times \mathcal{P}_p(\mathcal{C}^d)$ . By induction, if

$$\sup_{0 \leq s \leq ih} |S_s^{h,P}(x_0, \gamma) - S_s^{h,P'}(x'_0, \gamma')| \leq L_i[|x_0 - x'_0| + |\gamma - \gamma'| + W_p(P_{\wedge ih}, P'_{\wedge ih})], \quad (3.10)$$

for some  $L_i > 0$ , then (A.1) yields, for  $ih \leq t \leq (i+1)h$ ,

$$\begin{aligned} |S_t^{h,P}(x_0, \gamma) - S_t^{h,P'}(x'_0, \gamma')| &\leq |S_{ih}^{h,P}(x_0, \gamma) - S_{ih}^{h,P'}(x'_0, \gamma')| + 2M_\sigma \sup_{ih \leq s \leq t} |\gamma_s - \gamma'_s| \\ &+ [hL_b + 2L_\sigma \sup_{ih \leq s \leq (i+1)h} |\gamma_s|] \left[ \sup_{0 \leq s \leq ih} |S_s^{h,P}(x_0, \gamma) - S_s^{h,P'}(x'_0, \gamma')| + W_p(P_{\wedge ih}, P'_{\wedge ih}) \right]. \end{aligned}$$

As a consequence, using that  $W_p(P_{\wedge ih}, P'_{\wedge ih}) \leq W_p(P_{\wedge (i+1)h}, P'_{\wedge (i+1)h})$ , (3.10) holds till time  $(i+1)h$  for  $L_{i+1} := L_i[1+hL_b+2L_\sigma \sup_{ih \leq s \leq (i+1)h} |\gamma_s|] + \max\{2M_\sigma, hL_b+2L_\sigma \sup_{ih \leq s \leq (i+1)h} |\gamma_s|\}$ .

(iii) As a consequence of (ii),  $Y : \omega \mapsto S^{h,P}(\zeta(\omega), W(\omega))$  is measurable. Using (A.1) and taking expectations, there exists  $C > 0$  such that

$$\mathbb{E} \sup_{ih \leq t \leq (i+1)h} |Y_t|^p \leq C \left[ 1 + \mathbb{E} \sup_{0 \leq s \leq ih} |Y_s|^p + M_\sigma^p \mathbb{E} \sup_{ih \leq t \leq (i+1)h} |W_t - W_{ih}|^p + \int_{\mathcal{C}^d} \sup_{0 \leq s \leq ih} |x_s|^p dP_{\wedge ih}(x) \right].$$

By induction,  $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^p]$  is thus finite if  $\mathbb{E}[|\zeta|^p]$  and  $\mathbb{E}[\sup_{0 \leq t \leq T} |W_t|^p]$  are finite. The fact that  $Y$  is the pathwise-unique strong solution of (3.9) then stems from (i).  $\square$

We now turn to the discretised McKean-Vlasov equation (3.4).

**Lemma 3.3.2** (Pathwise solution map). *For any  $R \in \mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^{d'})$ , there exists a unique map*

$$\bar{S}^{h,R} : \begin{cases} \mathbb{R}^d \times \mathcal{C}^{d'} & \rightarrow \mathcal{C}^d, \\ (x_0, \gamma) & \mapsto \bar{S}^{h,R}(x_0, \gamma), \end{cases}$$

that is everywhere well-defined, that belongs to  $L^p_R(\mathbb{R}^d \times \mathcal{C}^{d'}, \mathcal{C}^d)$ , and that satisfies

$$\forall (x_0, \gamma, t) \in \mathbb{R}^d \times \mathcal{C}^{d'} \times [0, T], \quad \bar{S}_t^{h,R}(x_0, \gamma) = f_t^{h, \bar{S}_\#^{h,R} R}(x_0, \gamma, \bar{S}^{h,R}(x_0, \gamma)). \quad (3.11)$$

Moreover, for  $(\zeta, W) \in L^p_{\mathbb{P}}(\Omega, \mathbb{R}^d \times \mathcal{C}^{d'})$ , (3.4) has a unique solution in the sense of Definition 3.2.2, given by  $\bar{S}^{h, \mathcal{L}(\zeta, W)}(\zeta, W)$ .

In particular,  $\bar{S}_\#^{h,R} R$  belongs to  $\mathcal{P}_p(\mathcal{C}^d)$ . In the case  $R = \mathcal{L}(\zeta, W)$ ,  $\bar{S}_\#^{h,R} R$  is the law of the solution of (3.4). Following [Coghi et al., 2020, Section 2],  $\bar{S}_\#^{h,R} R$  can be characterised as the unique fixed-point of the map

$$P \mapsto S_\#^{h,P} R.$$

In [Tanaka, 1984, Section 2.1], [Coghi et al., 2020, Lemma 13] and [Backhoff et al., 2020, Lemma 3.1], the fixed-point is built by showing convergence for an iterative scheme. The result is much easier for our explicit Euler scheme.

*Proof.* We first show that (3.4) has a pathwise unique strong solution, before using this solution to build  $\bar{S}^{h,R}(x_0, \gamma)$ .

By induction, if  $(X_s^h)_{0 \leq s \leq ih}$  is  $\mathbb{P}$ -a.s. uniquely defined for some  $i \geq 0$ , then

$$X_t^h = X_{ih}^h + (t - ih)b_{ih}(X_{\wedge ih}^h, \mathcal{L}(X_{\wedge ih}^h)) + \sigma_{ih}(X_{\wedge ih}^h, \mathcal{L}(X_{\wedge ih}^h))[W_t - W_{ih}],$$

for  $ih \leq t \leq (i+1)h$ , uniquely extending the definition till time  $(i+1)h$ . All this makes sense because  $b_{ih}(x, P)$  and  $\sigma_{ih}(x, P)$  only depend on  $x_{\wedge ih}$  and  $P_{\wedge ih}$  using (A.1). By induction, the above Euler scheme builds  $(X_{\wedge ih}^h, \mathcal{L}(X_{\wedge ih}^h))$  for every  $i \geq 0$ , proving existence and pathwise uniqueness for (3.4). In particular, this proves existence and uniqueness for  $\mathcal{L}(X^h)$ .

To obtain an everywhere-defined solution, we set  $\bar{S}^{h,R}(x_0, \gamma) := S^{h, \mathcal{L}(X^h)}(x_0, \gamma)$ , where  $\mathcal{L}(X^h)$  is the path-law of the solution of (3.4) for any  $(\zeta, W)$  with law  $R$ .  $\bar{S}^{h,R}(\zeta, W)$  a.s. satisfies (3.9) with  $P = \mathcal{L}(X^h)$ , so that  $\bar{S}^{h,R}(\zeta, W)$  is the pathwise solution of (3.4). Consequently,  $\bar{S}_\#^{h,R} R = \mathcal{L}(X^h)$ , and  $\bar{S}^{h,R}(x_0, \gamma)$  satisfies (3.11) for every  $(x_0, \gamma)$ . Uniqueness stems from using Lemma 3.3.1-(i) with  $P = \bar{S}_\#^{h,R} R$ .  $\square$

We now state the main result of this section.

**Proposition 3.3.3** (Continuous fixed-point map). *The map*

$$\Psi_h : \begin{cases} \mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^d) & \rightarrow \mathcal{P}_p(\mathcal{C}^d), \\ R & \mapsto \bar{S}_\#^{h,R} R, \end{cases}$$

is continuous. Moreover, for  $(\zeta, W) \in L^p_{\mathbb{F}}(\Omega, \mathbb{R}^d \times \mathcal{C}^d)$ ,  $\Psi_h(\mathcal{L}(\zeta, W))$  is the law of the unique solution of (3.4).

When the matrix  $\sigma$  is constant, we can show that  $\Psi_h$  is globally Lipschitz-continuous (see Lemma 3.4.1 below), as in [Coghi et al., 2020, Theorem 7] and [Backhoff et al., 2020, Lemma 3.4]. In a setting that includes common noise, [Delarue et al., 2020, Lemma 6.16] could also be adapted to show that  $\Psi_h$  is uniformly continuous, but still requiring that  $\sigma$  is constant. To handle general  $\sigma$ , we rely on a compactness approach.

*Proof.* In view of Lemma 3.3.2, we only have to prove the continuity of  $\Psi_h$ . We use the sequential characterisation in the metric space  $(\mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^d), W_p)$ . Let  $(R_k)_{k \geq 1}$  be a sequence in  $\mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^d)$  that converges towards some  $R$ . Since the  $W_p$ -convergence implies weak convergence [Villani et al., 2009, Definition 6.8], the Skorokhod representation theorem provides a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  that supports a sequence  $(\zeta^k, W^k)_{k \geq 1}$ , where  $(\zeta^k, W^k)$  is  $R_k$ -distributed and a.s. converges towards some  $R$ -distributed  $(\zeta, W)$ . Let  $X^{h,k}$  denotes the pathwise solution of (3.4) driven by  $(\zeta^k, W^k)$ , and let  $X^h$  denote the one driven by  $(\zeta, W)$ .

**Step 1.** *Uniform in  $k$  bounds.* As in the proof of Lemma 3.3.1-(iii), (A.1) yields

$$\sup_{ih \leq s \leq (i+1)h} |X_s^{h,k}|^p \leq C \left[ 1 + \sup_{0 \leq s \leq ih} |X_s^{h,k}|^p + \sup_{ih \leq s \leq (i+1)h} |W_s^k|^p + \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{0 \leq s \leq ih} |X_s^{h,k}|^p \right],$$

for every  $i \geq 0$  and a constant  $C > 0$  that does not depend on  $k$ . From [Villani et al., 2009, Definition 6.8-(i)],  $(\mathbb{E}_{\tilde{\mathbb{P}}} [|\zeta^k|^p])_{k \geq 1}$  and  $(\mathbb{E}_{\tilde{\mathbb{P}}} [\sup_{0 \leq t \leq T} |W_t^k|^p])_{k \geq 1}$  are converging sequences, hence bounded ones. We now take expectations to get by induction that

$$\sup_{k \geq 1} \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{0 \leq t \leq T} |X_t^{h,k}|^p < +\infty, \quad (3.12)$$

and then  $\tilde{\mathbb{P}}$ -a.s.,

$$\sup_{0 \leq t \leq T} |X_t^{h,k}|^p \leq C \left[ 1 + |\zeta^k|^p + \sup_{0 \leq t \leq T} |W_t^k|^p \right], \quad (3.13)$$

for some  $C > 0$  independent of  $k$ . For  $s, t \in [0, T]$  with  $t_h \leq s \leq t \leq t_h + h$ , using (A.1),

$$|X_t^{h,k} - X_s^{h,k}| \leq (t - s) |b_{t_h}(X^{h,k}, \mathcal{L}(X^{h,k}))| + M_\sigma |W_t^k - W_s^k|.$$

Using (A.1) and the bound (3.12),  $\mathbb{E}_{\tilde{\mathbb{P}}} [\sup_{0 \leq t \leq T} |b_t(X^{h,k}, \mathcal{L}(X^{h,k}))|]$  is bounded uniformly in  $k$ . Taking expectations, we deduce that for every  $\delta \in (0, h]$ ,

$$\mathbb{E}_{\tilde{\mathbb{P}}} \sup_{|t-s| \leq \delta} |X_t^{h,k} - X_s^{h,k}| \leq C\delta + C\mathbb{E}_{\tilde{\mathbb{P}}} \sup_{|t-s| \leq \delta} |W_t^k - W_s^k|, \quad (3.14)$$

for a constant  $C > 0$  that does not depend on  $k$  and  $\delta$ .

**Step 2.** *Weak relative compactness.* For  $\varepsilon, M > 0$ , we decompose

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{|t-s| \leq \delta} |W_t^k - W_s^k|^p &\leq \varepsilon^p + (2M)^p \tilde{\mathbb{P}} \left[ \sup_{|t-s| \leq \delta} |W_t^k - W_s^k| > \varepsilon \right] \\ &\quad + 2^p \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sup_{0 \leq t \leq T} |W_t^k|^p \mathbb{1}_{\sup_{0 \leq t \leq T} |W_t^k| \geq M} \right]. \end{aligned}$$

Since  $(R_k)_{k \geq 1}$  converges in  $(\mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^{d'}), W_p)$ , we deduce from [Villani et al., 2009, Definition 6.8-(iii)] that  $M_\varepsilon > 0$  exists such that

$$\sup_{k \geq 1} \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{0 \leq t \leq T} |W_t^k|^p \mathbb{1}_{\sup_{0 \leq t \leq T} |W_t^k| \geq M_\varepsilon} \leq \varepsilon.$$

Since  $(R_k)_{k \geq 1}$  converges in  $W_p$ , it is tight, and [Billingsley, 2013, Theorem 7.3] provides that

$$\sup_{k \geq 1} \tilde{\mathbb{P}} \left[ \sup_{|t-s| \leq \delta} |W_t^k - W_s^k| > \varepsilon \right] \leq \varepsilon / M_\varepsilon^p,$$

for every small enough  $\delta > 0$ . Gathering everything, we get that

$$\sup_{k \geq 1} \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{|t-s| \leq \delta} |W_t^k - W_s^k|^p \xrightarrow{\delta \rightarrow 0} 0.$$

From (3.14), using  $p \geq 1$  and Jensen's inequality, we obtain that

$$\sup_{k \geq 1} \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{|t-s| \leq \delta} |X_t^{h,k} - X_t^h| \xrightarrow{\delta \rightarrow 0} 0.$$

Using this together with (3.12), [Billingsley, 2013, Theorem 7.3] now gives that the  $\bar{S}_{\#}^{h,R_k} R_k = \mathcal{L}(X^{h,k})$  form a relatively compact sequence in  $\mathcal{P}(\mathcal{C}^d)$  for the weak topology.

**Step 3.**  *$W_p$ -convergence.* Since  $R_k$  converges in the  $W_p$ -topology,  $(|\zeta^k|^p)_{k \geq 1}$  and  $(|W^k|^p)_{k \geq 1}$  are uniformly integrable [Villani et al., 2009, Definition 6.8-(iii)]. From (3.13), we then deduce that  $(\sup_{0 \leq t \leq T} |X_t^{h,k}|^p)_{k \geq 1}$  is uniformly integrable. Using **Step 2.** and the characterisation [Villani et al., 2009, Definition 6.8-(iii)], this implies that  $(\mathcal{L}(X^{h,k}))_{k \geq 1}$  is relatively compact in  $(\mathcal{P}_p(\mathcal{C}^d), W_p)$ . Let  $P$  be any limit point in  $\mathcal{P}_p(\mathcal{C}^d)$ .

Using the  $\tilde{\mathbb{P}}$ -a.s. convergence of  $(\zeta^k, W^k)$ , and the continuity of  $(x_0, \gamma, P) \mapsto S^{h,P}(x_0, \gamma)$ , which is implied by Lemma 3.3.1-(ii),  $\bar{S}^{h,R_k}(\zeta^k, W^k) = S^{h,\mathcal{L}(X^{h,k})}(\zeta^k, W^k)$  has a sub-sequence that  $\tilde{\mathbb{P}}$ -a.s. converges towards  $S^{h,P}(\zeta, W)$ . Along such a sub-sequence, we deduce that  $\mathcal{L}(X^{h,k}) = \mathcal{L}(\bar{S}^{h,R_k}(\zeta^k, W^k))$  weakly converges towards  $\mathcal{L}(S^{h,P}(\zeta, W))$ , so that  $P = S_{\#}^{h,P} R$ . This implies that  $S^{h,P}(\zeta, W)$  is a solution of (3.4). The pathwise uniqueness in Lemma 3.3.2 then gives that  $S^{h,P}(\zeta, W) = \bar{S}^{h,R}(\zeta, W)$ ,  $\tilde{\mathbb{P}}$ -a.s., so that  $P = \bar{S}_{\#}^{h,R} R = \Psi_h(R)$ . The unique limit point in  $W_p$  of the relatively compact sequence  $(\bar{S}_{\#}^{h,R_k} R_k)_{k \geq 1}$  is thus  $\Psi_h(R)$ ; hence  $\Psi_h(R_k) = \bar{S}_{\#}^{h,R_k} R_k$  converges towards  $\Psi_h(R)$ .  $\square$

To prove the central limit theorem, we will need a stronger continuity result.

**Lemma 3.3.4** (Lipschitz-continuous restriction). *For any  $M' > 0$ , the restriction of  $\Psi_h$  to  $\mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}_{M'}^{d'})$  is Lipschitz-continuous.*



*Proof.* Let us consider  $R$  and  $R'$  in  $\mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}_{M'}^d)$ . On any probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ , let  $(\zeta, W)$  and  $(\zeta', W')$  be respectively  $R$  and  $R'$ -distributed variables.  $W$  and  $W'$  are  $\tilde{\mathbb{P}}$ -a.s. bounded by  $M'$ , hence using (A.1),

$$\begin{aligned} \sup_{ih \leq t \leq (i+1)h} |\bar{S}_t^{h,R}(\zeta, W) - \bar{S}_t^{h,R'}(\zeta', W')|^p &\leq C_{i,M'} \left[ \sup_{0 \leq t \leq ih} |\bar{S}_t^{h,R}(\zeta, W) - \bar{S}_t^{h,R'}(\zeta', W')|^p \right. \\ &\quad \left. + \sup_{ih \leq t \leq (i+1)h} |W_t - W'_t|^p + \mathbb{E}_{\tilde{\mathbb{P}}} \sup_{0 \leq t \leq ih} |\bar{S}_t^{h,R}(\zeta, W) - \bar{S}_t^{h,R'}(\zeta', W')|^p \right], \end{aligned}$$

for  $C_{i,M'} > 0$  that only depends on  $(i, M')$ . Taking expectations, we get by induction that

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sup_{0 \leq t \leq T} |\bar{S}_t^{h,R}(\zeta, W) - \bar{S}_t^{h,R'}(\zeta', W')|^p \right] \leq L_{M'} \mathbb{E}_{\tilde{\mathbb{P}}} [|\zeta - \zeta'|^p] + L_{M'} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |W_t - W'_t|^p \right],$$

for some  $L_{M'} > 0$ . We then use  $(\bar{S}^{h,R}(\zeta, W), \bar{S}^{h,R'}(\zeta', W'))$  as a coupling to get the bound

$$W_p^p(\Psi_h(R), \Psi_h(R')) = W_p^p(\bar{S}_{\#}^{h,R} R, \bar{S}_{\#}^{h,R'} R') \leq L_{M'} \mathbb{E}_{\tilde{\mathbb{P}}} [|\zeta - \zeta'|^p] + L_{M'} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sup_{0 \leq t \leq T} |W_t - W'_t|^p \right].$$

Minimising over all possible couplings eventually gives that

$$W_p^p(\Psi_h(R), \Psi_h(R')) \leq L_{M'} W_p^p(R, R'),$$

concluding the proof.  $\square$

**Remark 3.3.5** (Inverse map). *If we assume that  $d = d'$  and  $\sigma_t(x, P)$  is always invertible, we can define the solution map  $U^{h,P}(\gamma)$  for the ODE*

$$\begin{aligned} \forall t \in [0, T], \quad x_t = x_{[t/h]h-h} + \sigma_{t_h}^{-1}(\gamma, P) &\left[ \gamma_t - \gamma_0 - b_{t_h}(\gamma, P)[t - t_h] \right. \\ &\quad \left. - \sum_{i=0}^{[t/h]-2} h b_{ih}(\gamma, P) + \sigma_{ih}(\gamma_{ih}, P)[x_{(i+1)h} - x_{ih}] \right], \end{aligned}$$

with the convention  $x_{-h} = 0$ . If  $\sigma^{-1}$  is uniformly bounded, we can show as in Lemma 3.3.1 that  $U_{\#}^{h,P} P$  belongs to  $\mathcal{P}_p(\mathcal{C}^d)$  if  $P$  belongs to  $\mathcal{P}_p(\mathcal{C}^d)$ . Moreover, by construction,

$$\forall (x_0, \gamma) \in \mathbb{R}^d \times \mathcal{C}^d, \quad (x_0, S^{h,P}(x_0, U^{h,P}(\gamma))) = (x_0, U^{h,P}(S^{h,P}(x_0, \gamma))) = (x_0, \gamma).$$

From this, similarly to [Coghi et al., 2020, Lemma 31] or [Backhoff et al., 2020, Proof of Theorem 3.1], we can show that  $\Psi_h$  is one-to-one with

$$\forall P \in \mathcal{P}_p(\mathcal{C}^d), \quad \Psi_h^{-1}(P) = (X_0, U^{h,P})_{\#} P,$$

for  $X_0 : \mathcal{C}^d \rightarrow \mathbb{R}^d, (x_t)_{0 \leq t \leq T} \mapsto x_0$ .

### 3.3.2 Mean-field limit and large deviations

We now analyse the discretised particle system (3.5) using the the results of Section 3.3.1, and we prove the mean-field limit and the large deviations from (3.4).

*Proof of Theorem 3.2.4.* Following [Coghi et al., 2020, Section 3.1], we can see (3.5) as a special instance of the generalised McKean-Vlasov equation (3.4). Indeed, let us consider the space  $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ , where  $\Omega_N := \{1, \dots, N\}$ ,  $\mathcal{F}_N := 2^{\Omega_N}$  is the power set of  $\Omega_N$ , and  $\mathbb{P}_N := \frac{1}{N} \sum_{i=1}^N \delta_i$ . On this space, a  $N$ -tuple  $\vec{x}^N$  in some product space  $E^N$  can be identified to the  $E$ -valued random variable  $i \in \Omega_N \mapsto x^i \in E$ . In this setting, the empirical measure  $\pi(\vec{x}^N)$  is precisely the law of  $\vec{x}^N$  under  $\mathbb{P}_N$ . Going back to (3.5), for each  $\omega \in \Omega$ , we apply this to the  $N$ -tuples

$$(\vec{\zeta}^N, \vec{W}^N)(\omega) = (\zeta^{i,N}(\omega), W^{i,N}(\omega))_{1 \leq i \leq N} \quad \text{and} \quad \vec{X}^N(\omega) = (X^{i,N}(\omega))_{1 \leq i \leq N}.$$

For fixed  $\omega \in \Omega$ , the empirical measure  $\pi((\vec{\zeta}^N, \vec{W}^N)(\omega))$  is the law of  $(\vec{\zeta}^N, \vec{W}^N)(\omega)$  under  $\mathbb{P}_N$ . Similarly,  $\pi(\vec{X}^N(\omega))$  is the law of  $\vec{X}^N(\omega)$  under  $\mathbb{P}_N$ . The empirical measure  $\pi(\vec{X}^N(\omega))$  is precisely the one that appears in (3.5), so that the particle system (3.5) appears as an instance of the generalised McKean-Vlasov equation (3.4) on the probability space  $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ . Using the notations of Proposition 3.3.2, we eventually get that

$$\mathcal{L}(X^h) = \Psi_h(\mathcal{L}(\zeta, W)) \quad \text{and} \quad \pi(\vec{X}^{h,N}) = \Psi_h(\pi(\vec{\zeta}^N, \vec{W}^N)) \quad \mathbb{P}\text{-a.s.},$$

together with,  $\mathbb{P}$ -a.s.,

$$X^h = S^{h, \mathcal{L}(X^h)}(\zeta, W) \quad \text{and} \quad X^{h,i,N} = S^{h, \pi(\vec{X}^{h,N})}(\zeta^i, W^i),$$

for  $1 \leq i \leq N$ . In particular, the continuity of  $\Psi_h$  and Lemma 3.3.1-(ii) guarantee that  $\omega \mapsto \vec{X}^{h,N}(\omega)$  is measurable. This unified formalism gives the desired results (i)-(ii)-(iii), using Proposition 3.3.2 and the continuity proved in Proposition 3.3.3.  $\square$

*Proof of Proposition 3.2.6.* From the Sanov theorem in the  $W_p$ -topology [Wang et al., 2010, Theorem 1.1], the sequence of the  $\mathcal{L}(\pi(\vec{X}_0^N, \vec{B}^N))$  satisfies the LDP with good rate function  $R' \in \mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^{d'}) \mapsto H(R'|R)$ , where  $R := \mathcal{L}(X_0^1, B^1)$ . By contraction [Dupuis and Ellis, 2011, Theorem 1.3.2], Corollary 3.2.4.1 proves that the  $\mathcal{L}(\pi(\vec{X}^{h,N}))$  satisfies the LDP with good rate function

$$I_h : P \mapsto \inf_{\substack{R' \in \mathcal{P}_p(\mathbb{R}^d \times \mathcal{C}^{d'}) \\ \Psi_h(R') = P}} H(R'|R).$$

From Lemma 3.3.2,  $\Psi_h(R') = P$  implies  $\Psi_h(R') = S_{\#}^{h,P} R'$ . [Fischer, 2014, Lemma A.1] now gives that

$$I_h(P) = H(P|S_{\#}^{h,P} R).$$

By definition of  $\Gamma_h$ , we notice that  $S_{\#}^{h,P} R = \Gamma_h(P)$ , concluding the proof.  $\square$

### 3.3.3 CLT for the discretised system

This section adapts the results of [Tanaka, 1984, Section 2] and [Coghi et al., 2020, Section 5] to our discretised setting. The CLT will result from the following key-result which is a mere rewriting of [Tanaka, 1984, Theorem 1.1] or [Coghi et al., 2020, Theorem 35], using the linear derivative from Definition 3.1.1.

**Theorem 3.3.6** (Tanaka). *Given a Polish space  $E$ , let  $f : E \times \mathcal{P}_p(E) \rightarrow \mathbb{R}$  be a bounded function such that  $P \mapsto f(x, P)$  has a linear functional derivative  $y \mapsto \delta_P f(x, P, y)$  that is bounded and Lipschitz in  $P$  uniformly in  $(x, y)$ . Then, given any i.i.d. sequence  $(X^i)_{i \geq 1}$  with common law  $P \in \mathcal{P}(E)$ , the random variable*

$$\sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N f(X^i, \pi(\vec{X}^N)) - \int_E f(x, P) dP(x) \right]$$

converges in law towards a centred Gaussian variable with variance  $\sigma_f^2$  given by

$$\sigma_f^2 := \int_E \left[ f(y, P) + \int_E [\delta_P f(x, P, y) - f(x, P)] dP(x) \right]^2 dP(y).$$

Throughout this section, we assume that (A.2) holds. All the particles now start at 0, so that we can remove the dependence on the initial condition. As required by Theorem 3.2.5, we restrict ourselves to bounded driving paths  $\gamma \in \mathcal{C}_{M'}^d$ . As a consequence of (A.1), the paths  $S^{h,P}(\gamma)$  stay in  $\mathcal{C}_M^d$  for some  $M > 0$  that only depend on  $(h, M')$  (see Lemma 3.3.1-(ii)). To alleviate notations, we remove all the  $h$ -exponents in the remainder of this section, writing

$$\Psi : \mathcal{P}_p(\mathcal{C}_{M'}^d) \rightarrow \mathcal{P}_p(\mathcal{C}_M^d),$$

for the fixed-point map from Proposition 3.3.3. Let us fix  $R \in \mathcal{P}_p(\mathcal{C}_{M'}^d)$ . We recall that for  $\gamma \in \mathcal{C}_{M'}^d$ ,

$$\forall t \in [0, T], \quad S_t^{\Psi(R)}(\gamma) = \int_0^t b_{s_h}(S^{\Psi(R)}(\gamma), S_{\#}^{\Psi(R)}R) ds + \sigma_{s_h}(S^{\Psi(R)}(\gamma), S_{\#}^{\Psi(R)}R) d\gamma_s. \quad (3.15)$$

The linear derivative of  $\bar{S}^R = S^{\Psi(R)}$  w.r.t.  $R$  can be computed (if well-defined) writing that

$$\delta_P \bar{S}^R(\gamma, \tilde{\gamma}) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} [\bar{S}^{(1-\varepsilon)R + \varepsilon \delta \tilde{\gamma}}(\gamma) - \bar{S}^R(\gamma)], \quad (3.16)$$

where we recall the convention  $\int_{\mathcal{C}_{M'}^d} \delta_P \bar{S}^R(\gamma, \gamma') dR(\gamma') = 0$  from Definition 3.1.1. After careful differentiation of each term in (3.15), this suggests that for every  $t \in [0, T]$ ,

$$\begin{aligned} \delta_P \bar{S}_t^R(\gamma, \tilde{\gamma}) &= \int_0^t [D_x b_{s_h}(\bar{S}^R(\gamma), \bar{S}_{\#}^R R) \cdot \delta_P \bar{S}^R(\gamma, \tilde{\gamma}) + \delta_P b_{s_h}(\bar{S}^R(\gamma), \bar{S}_{\#}^R R, \bar{S}^R(\tilde{\gamma}))] ds \\ &\quad + \int_0^t \int_{\mathcal{C}^d} D_y \delta_P b_{s_h}(\bar{S}^R(\gamma), \bar{S}_{\#}^R R, \bar{S}^R(\gamma')) \cdot \delta_P \bar{S}^R(\gamma', \tilde{\gamma}) dR(\gamma') ds \\ &\quad + \int_0^t [D_x \sigma_{s_h}(\bar{S}^R(\gamma), \bar{S}_{\#}^R R) \cdot \delta_P \bar{S}_t^R(\gamma, \tilde{\gamma}) + \delta_P \sigma_{s_h}(\bar{S}^R(\gamma), \bar{S}_{\#}^R R, \bar{S}^R(\tilde{\gamma}))] d\gamma_s \\ &\quad + \int_0^t \int_{\mathcal{C}^d} D_y \delta_P \sigma_{s_h}(\bar{S}^R(\gamma), \bar{S}_{\#}^R R, \bar{S}^R(\gamma')) \cdot \delta_P \bar{S}_t^R(\gamma', \tilde{\gamma}) dR(\gamma') d\omega_s. \end{aligned}$$

To rigorously build  $\delta_P \bar{S}^R$  we follow the approach of [Tanaka, 1984, Section 2.1]. For  $(P, x, \tilde{x}, X) \in \mathcal{P}_p(\mathcal{C}_M^d) \times \mathcal{C}_M^d \times \mathcal{C}_M^d \times \mathcal{C}(\mathcal{C}_M^d, \mathcal{C}^d)$ , let us define the path  $g^P(x, \tilde{x}, X)$  in  $\mathcal{C}^d$  by

$$\begin{aligned} g_t^P(x, \tilde{x}, X) &:= \int_0^t \left[ D_x b_{s_h}(x, P) \cdot X(x) + \delta_P b_{s_h}(x, P, \tilde{x}) + \int_{\mathcal{C}^d} D_y \delta_P b_{s_h}(x, P, x') \cdot X(x') dP(x') \right] ds \\ &\quad + \int_0^t \left[ D_x \sigma_{s_h}(x, P) \cdot X(x) + \delta_P \sigma_{s_h}(x, P, \tilde{x}) + \int_{\mathcal{C}^d} D_y \delta_P \sigma_{s_h}(x, P, x') \cdot X(x') dP(x') \right] d\gamma_s. \end{aligned}$$

We then introduce an intermediary ODE.

**Lemma 3.3.7** (Differential map). *We assume (A.2).*

(i) *For every  $(P, \tilde{x}) \in \mathcal{P}_p(\mathcal{C}_M^d) \times \mathcal{C}_M^d$ , the equation*

$$\forall x \in \mathcal{C}_M^d, \quad U_t^P(x, \tilde{x}) = g_t^P(x, \tilde{x}, U^P(\cdot, \tilde{x})),$$

*has unique solution  $x \mapsto U_t^P(x, \tilde{x})$  in  $\mathcal{C}(\mathcal{C}_M^d, \mathcal{C}^d)$ .*

(ii) The map  $(P, x, \tilde{x}) \mapsto U_t^P(x, \tilde{x})$  is globally bounded and Lipschitz-continuous.

The global bound on  $U^P(x, \tilde{x})$  is only possible because we restricted ourselves to  $\mathcal{C}_M^d$ .

*Proof.* These results are the analogues of [Tanaka, 1984, Lemma 2.2] in a simpler setting (we consider discretised dynamics instead of ODEs). Alternatively, (i) can be obtained by readily adapting the induction in the proof of Lemma 3.3.2, while (ii) is a direct adaptation of the induction in the proof of Lemma 3.3.4.  $\square$

We can now define  $\delta_P \bar{S}^R$  by setting

$$\delta_P \bar{S}^R(\gamma, \tilde{\gamma}) := U^{\bar{S}^R}(\bar{S}^R(\gamma), \bar{S}^R(\tilde{\gamma})). \quad (3.17)$$

We notice that  $\delta_P \bar{S}^R$  defined by (3.17) indeed satisfies the equation computed above. From this and (3.15), it is a standard ODE-like computation to verify that (3.16) indeed holds (see e.g. [Coghi et al., 2020, Lemma 42]).

**Corollary 3.3.7.1** (Lipschitz-continuity). *Under (A.2), the map  $(R, \gamma, \tilde{\gamma}) \mapsto \delta_P \bar{S}^R(\gamma, \tilde{\gamma})$  is globally bounded and Lipschitz-continuous on  $\mathcal{P}_p(\mathcal{C}_{M'}^{d'}) \times \mathcal{C}_{M'}^{d'} \times \mathcal{C}_{M'}^{d'}$ .*

*Proof.* From (3.17), this result is a mere concatenation of Lemma 3.3.1-(ii), Lemma 3.3.4 and Lemma 3.3.7-(ii).  $\square$

The following result is a detailed version of Theorem 3.2.5, obtained as a consequence of Theorem 3.3.6.

**Corollary 3.3.7.2.** *Under (A.2), let  $(W^i)_{i \geq 1}$  be an i.i.d. sequence of  $\mathcal{C}_{M'}^{d'}$ -valued variables with common law  $R$ . Let  $\bar{X}^{h,N}$  be the related system of particles starting from 0 given by (3.5), and let  $X^h$  be a related solution of (3.4). Then, for every  $\varphi \in \mathcal{C}_b^{1,1}(\mathcal{C}^d, \mathbb{R})$ , the random variable*

$$\sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N \varphi(X^{h,i,N}) - \mathbb{E}[\varphi(X^h)] \right]$$

converges in law towards a centred Gaussian variable with variance  $\sigma_\varphi^2$  given by

$$\sigma_\varphi^2 := \int_{\mathcal{C}_{M'}^{d'}} \left[ \varphi(\bar{S}^R(\tilde{\gamma})) + \int_{\mathcal{C}_{M'}^{d'}} [D\varphi(\bar{S}^R(\gamma)) \cdot \delta_P \bar{S}^R(\gamma, \tilde{\gamma}) - \varphi(\bar{S}^R(\gamma))] dR(\gamma) \right]^2 dR(\tilde{\gamma}).$$

*Proof.* The function  $f : \mathcal{C}_{M'}^{d'} \times \mathcal{P}_p(\mathcal{C}_{M'}^{d'}) \rightarrow \mathbb{R}$  defined by  $f(\gamma, R) := \varphi(\bar{S}^R(\gamma))$  has a linear derivative given by  $\delta_P f(\gamma, R, \tilde{\gamma}) = D\varphi(\bar{S}^R(\gamma)) \cdot \delta_P \bar{S}^R(\gamma, \tilde{\gamma})$ . Using Lemma 3.3.4 and Corollary 3.3.7.1,  $f$  satisfies the assumptions of Theorem 3.3.6, which now gives the result.  $\square$

### 3.3.4 Extension to the Brownian setting

Throughout this section, we assume that (A.4) holds. Our purpose is to extend the results of the previous section for proving Proposition 3.2.7. Let  $(B^i)_{i \geq 1}$  be a countable sequence in  $\mathcal{C}^{d'}$  of i.i.d. Brownian motions. We rely on an approximation procedure by stopping the  $B^i$  when they reach a given threshold, and by showing that the fluctuations of the related particle system can be uniformly controlled. For  $M > 0$ , we introduce the stopping time

$$\tau_M^i := \inf\{t \in [0, T], |B_t^i| \geq M\},$$

with the convention  $\tau_M^i = T$  if the set on the r.h.s. is empty. The related particle system given by (3.5) is

$$dX_t^{h,M,i,N} = b_{t_h}(X^{h,M,i,N}, \pi(\vec{X}^{h,M,N}))dt + \sigma_{t_h}(X^{h,M,i,N}, \pi(\vec{X}^{h,M,N}))dB_{t \wedge \tau_M^i}^i,$$

with  $X_0^{h,M,i,N} = 0$ ,  $1 \leq i \leq N$ . Since we discretised with the time step  $h > 0$ , we recall that stochastic integration is not needed to make sense of the above SDE. The limit particle system (3.5) when  $\zeta^i = 0$  and  $W^i = B^i$  corresponds to  $\vec{X}^{h,\infty,N}$ . For each  $1 \leq i \leq N$ , let us introduce the McKean-Vlasov process  $\bar{X}^{h,M,i}$  solution of

$$d\bar{X}_t^{h,M,i} = b_{t_h}(\bar{X}^{h,M,i}, \mathcal{L}(\bar{X}^{h,M,i}))dt + \sigma_{t_h}(\bar{X}^{h,M,i}, \mathcal{L}(\bar{X}^{h,M,i}))dB_{t \wedge \tau_M^i}^i, \quad \bar{X}_0^{h,M,i} = 0,$$

which corresponds to (3.4) for the stopped Brownian motions. Let us introduce the difference process  $\delta^{h,M,i,N} := X^{h,M,i,N} - \bar{X}^{h,M,i}$ , together with  $\delta^{h,\infty,i,N} := X^{h,\infty,i,N} - \bar{X}^{h,\infty,i}$  when  $M = +\infty$ . We first need that the mean-field limit holds uniformly in  $M$ , and that the obtained approximation is uniform in  $N$ .

**Lemma 3.3.8** (Uniform limit). *There exists  $C_h > 0$  such that for every  $N \geq 1$ ,  $M > 0$ ,  $1 \leq i \leq N$ ,*

- (i)  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,M,i,N}|^2] \leq C_h$ .
- (ii)  $\mathbb{E}[\sup_{0 \leq t \leq T} |\delta_t^{h,M,i,N}|^2] \leq C_h N^{-1}$ .
- (iii)  $\mathbb{E}[\sup_{0 \leq t \leq T} |\delta_t^{h,\infty,i,N}|^2] \leq C_h N^{-1}$ .
- (iv)  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,M,i,N} - X_t^{h,\infty,i,N}|^2] \leq C_h \mathbb{E}[\mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq t \leq T} |B_t^1|^2]$ .
- (v)  $\mathbb{E}[\sup_{0 \leq t \leq T} |\bar{X}_t^{h,M,i} - \bar{X}_t^{h,\infty,i}|^2] \leq C_h \mathbb{E}[\mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq t \leq T} |B_t^1|^2]$ .

We then strengthen these results for controlling the fluctuations.

**Proposition 3.3.9.** *There exists  $C_h > 0$  such that for every  $M > 0$ ,*

$$\sup_{N \geq 1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\delta_t^{h,M,i,N} - \delta_t^{h,\infty,i,N}| \right] \leq C_h \mathbb{E}^{1/2} [\mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq t \leq T} |B_t^1|^2].$$

With these results at hand, Proposition 3.2.7 will easily follow. The proofs of the approximation results rely on the classical coupling method from [Sznitman, 1991, Theorem 1.4]. To alleviate notations, we drop the exponents  $(h, N)$  in the remainder of this section (although the dependence on  $N$  is crucial). We will repeatedly use the technical estimates proved in Appendix 3.A. In the following,  $C_h$  is a generic constant that may change from line to line, but staying independent of  $(M, N)$ .

*Proof of Lemma 3.3.8.* (i) By symmetry, all the  $X^{M,i}$  have the same law. The bound is then a straight-forward induction very close to **Step 1.** in the proof of Proposition 3.3.3, using that  $\mathbb{E}[\sup_{0 \leq t \leq T} |B_{t \wedge \tau_M^i}^i|^2]$  is bounded uniformly in  $M$ .

(ii) By definition of  $X^{M,i}$  and  $\bar{X}^{M,i}$ , for  $0 \leq t \leq T$ ,

$$\begin{aligned} \delta_t^{M,i} &= \int_0^t [b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))] ds \\ &\quad + \int_0^t [\sigma_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))] dB_{s \wedge \tau_M^i}^i. \end{aligned}$$

We then take the square of this expression and we use Jensen's inequality, before taking the supremum in time and expectations to get

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\delta_t^{M,i}|^2 \right] &\leq 2t \int_0^t \mathbb{E} |b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))|^2 ds \\ &\quad + 2\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s [\sigma_{r_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{r_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))] dB_{r \wedge \tau_M^i}^i \right|^2. \end{aligned} \quad (3.18)$$

For the drift term, we split

$$\begin{aligned} b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i})) &= b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \pi(\vec{X}^M)) \\ &\quad + [b_{s_h}(\bar{X}^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))]. \end{aligned}$$

Using (i), (A.4), and Lemma 3.A.3 with the i.i.d. variables  $(\bar{X}^i, \bar{Y}^i) = (\bar{X}^{M,i}, 0)$ , we get that

$$\mathbb{E} |b_{s_h}(\bar{X}^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))|^2 \leq C_h N^{-1}.$$

Further using the Lipschitz assumption (A.4) on  $b$ , Jensen's inequality, and the fact that  $((X^{M,j}, \bar{X}^{M,j}))_{1 \leq j \leq M}$  is identically distributed, we deduce that

$$\mathbb{E} |b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))|^2 \leq C_h \mathbb{E} \left[ \sup_{0 \leq r \leq s_h} |X_r^{M,i} - \bar{X}_r^{M,i}|^2 \right] + C_h N^{-1}.$$

To handle the second line in (3.18), we rely on stochastic integration. From the optional stopping theorem,  $(B_{t \wedge \tau_M^i}^i)_{0 \leq t \leq T}$  is a square-integrable martingale; its quadratic variation is the process  $(t \wedge \tau_M^i)_{0 \leq t \leq T}$ . The Burkholder-Davis-Gundy (BDG) inequality then yields

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s [\sigma_{r_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{r_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))] dB_{r \wedge \tau_M^i}^i \right|^2 \\ \leq C_h \mathbb{E} \int_0^{t \wedge \tau_M^i} |\sigma_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))|^2 ds \\ \leq C_h \int_0^t \mathbb{E} |\sigma_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))|^2 ds. \end{aligned}$$

We can now handle the r.h.s. as we did for the drift term. Gathering the terms in (3.18),

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\delta_t^{M,i}|^2 \right] \leq C_h N^{-1} + C_h \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} |\delta_r^{M,i}|^2 \right] ds.$$

The conclusion then follows from the Gronwall Lemma.

(iii) If  $\tau_M^i = T$  in the above proof of (ii), we notice that all the computations remain valid when replacing  $(B_{t \wedge \tau_M^i}^i)_{0 \leq t \leq T}$  by  $(B_t^i)_{0 \leq t \leq T}$ . This yields the result.

(iv) As for proving (ii), for  $0 \leq t \leq T$ ,

$$\begin{aligned} X_t^{M,i} - X_t^{\infty,i} &= \int_0^t [\sigma_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{s_h}(X^{\infty,i}, \pi(\vec{X}^\infty))] dB_{s \wedge \tau_M^i}^i \\ &\quad - \int_0^t \sigma_{s_h}(X^{\infty,i}, \pi(\vec{X}^\infty))] dB_s^i - B_{s \wedge \tau_M^i}^i + \int_0^t [b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(X^{\infty,i}, \pi(\vec{X}^\infty))] ds. \end{aligned}$$

Setting  $B_s^{i,M} := B_s^i - B_{\tau_M^i}^i$ , the second integral on the r.h.s. is bounded by

$$M_\sigma \left| \mathbb{1}_{\tau_M^i < t} B_t^{i,M} - \mathbb{1}_{\tau_M^i < t_h} B_{t_h}^{i,M} \right| + \sum_{j=0}^{\lfloor t/h \rfloor - 1} M_\sigma \left| \mathbb{1}_{\tau_M^i < (j+1)h} B_{(j+1)h}^{i,M} - \mathbb{1}_{\tau_M^i < jh} B_{jh}^{i,M} \right|, \quad (3.19)$$

and further by  $C_h \mathbb{1}_{\tau_M^i < T} \sup_{0 \leq s \leq t} |B_s^i|$ . The result then follows by reproducing the estimates and the Gronwall argument in the proof of (ii).

(v) By symmetry, we recall that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,M,i,N} - X_t^{h,\infty,i,N}|^2]$  does not depend on  $i$ . The result follows by taking the  $N \rightarrow +\infty$  limit in the uniform estimate (iv), using (ii)-(iii).  $\square$

*Proof of Proposition 3.3.9.* For every  $0 \leq t \leq t' \leq T$ , we have the decomposition

$$\delta_{t'}^{M,i} - \delta_{t'}^{\infty,i} = \delta_t^{M,i} - \delta_t^{\infty,i} + \int_t^{t'} b_s^{M,i} ds + \int_t^{t'} \sigma_s^{M,i} dB_s^i + \int_t^{t'} \bar{\sigma}_s^{M,i} d[B_{s \wedge \tau_M^i}^i - B_s^i],$$

where

$$\begin{aligned} b_s^{M,i} &:= b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i})) - [b_{s_h}(X^{\infty,i}, \pi(\vec{X}^\infty)) - b_{s_h}(\bar{X}^{\infty,i}, \mathcal{L}(\bar{X}^{\infty,i}))], \\ \sigma_s^{M,i} &:= \sigma_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i})) - [\sigma_{s_h}(X^{\infty,i}, \pi(\vec{X}^\infty)) - \sigma_{s_h}(\bar{X}^{\infty,i}, \mathcal{L}(\bar{X}^{\infty,i}))], \\ \bar{\sigma}_s^{M,i} &:= \sigma_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - \sigma_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i})). \end{aligned}$$

Taking absolute values, supremum in time and expectations gives

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq s \leq t'} |\delta_s^{M,i} - \delta_s^{\infty,i}| \right] &\leq \mathbb{E} [|\delta_t^{M,i} - \delta_t^{\infty,i}|] + \int_t^{t'} \mathbb{E} [|b_s^{M,i}|] ds + \mathbb{E} \sup_{t \leq s \leq t'} \left| \int_t^s \sigma_r^{M,i} dB_r^i \right| \\ &\quad + \mathbb{E} \sup_{t \leq s \leq t'} \left| \int_t^s \bar{\sigma}_r^{M,i} d[B_{r \wedge \tau_M^i}^i - B_r^i] \right|. \quad (3.20) \end{aligned}$$

In the following,  $C_h$  is a generic constant, which may change from line to line but staying independent of  $(M, N)$  and  $(t, t')$ . To handle the  $b_s^{M,i}$ -term, we use the splitting

$$\begin{aligned} b_s^{M,i} &= b_{s_h}(X^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \pi(\vec{X}^M)) - [b_{s_h}(X^{\infty,i}, \pi(\vec{X}^\infty)) - b_{s_h}(\bar{X}^{\infty,i}, \pi(\vec{X}^\infty))] \\ &\quad + b_{s_h}(\bar{X}^{M,i}, \pi(\vec{X}^M)) - b_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i})) - [b_{s_h}(\bar{X}^{\infty,i}, \pi(\vec{X}^\infty)) - b_{s_h}(\bar{X}^{\infty,i}, \mathcal{L}(\bar{X}^{\infty,i}))]. \end{aligned}$$

Let  $\mathbf{b}_s^1$  (resp.  $\mathbf{b}_s^2$ ) denote the first (resp. the second) line. To control  $\mathbf{b}_s^1$ , we use (A.4) and Lemma 3.A.2 applied to  $(X^i, Y^i) = (X^{M,i}, X^{\infty,i})$  and  $(\bar{X}^i, \bar{Y}^i) = (\bar{X}_s^{M,i}, \bar{X}_s^{\infty,i})$  to get

$$\begin{aligned} |\mathbf{b}_s^1| &\leq \frac{1}{N} \sum_{j=1}^N \|\tilde{b}_{s_h}\|_{\text{Lip}} \left[ \sup_{0 \leq r \leq s} |\delta_r^{M,i} - \delta_r^{\infty,i}| + \sup_{0 \leq r \leq s} |\delta_r^{M,j} - \delta_r^{\infty,j}| \right] \\ &\quad + \|D\tilde{b}_{s_h}\|_{\text{Lip}} R_s^{i,j} \left[ \sup_{0 \leq r \leq s} |\delta_r^{\infty,i}| + \sup_{0 \leq r \leq s} |\delta_r^{\infty,j}| \right], \end{aligned}$$

where

$$R_s^{i,j} := \sup_{0 \leq r \leq s} |X_r^{M,i} - X_r^{\infty,i}| + \sup_{0 \leq r \leq s} |\bar{X}_r^{M,i} - \bar{X}_r^{\infty,i}| + \sup_{0 \leq r \leq s} |X_r^{M,j} - X_r^{\infty,j}| + \sup_{0 \leq r \leq s} |\bar{X}_r^{M,j} - \bar{X}_r^{\infty,j}|$$

We then take supremum in time and expectations, and we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.3.8 to get

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq r \leq s} |\mathbf{b}_r^1| \right] &\leq C_h \mathbb{E} \left[ \sup_{0 \leq r \leq s} |\delta_r^{M,i} - \delta_r^{\infty,i}| \right] + \frac{C_h}{N} \sum_{j=1}^N \mathbb{E} \left[ \sup_{0 \leq r \leq s} |\delta_r^{M,j} - \delta_r^{\infty,j}| \right] \\ &\quad + \frac{C_h}{\sqrt{N}} \mathbb{E}^{1/2} \left[ \mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq r \leq T} |B_r^1|^2 \right]. \end{aligned}$$

We recall that  $(\delta^{M,j})_{1 \leq j \leq M}$  is identically distributed. The supremum in time was not needed to estimate  $\mathbb{E}[|\mathbf{b}_s^1|]$ , but it will be useful for performing the same computation with the  $\sigma_s^1$ -term below. We control  $\mathbf{b}_s^2$  using Lemma 3.A.3 applied to the i.i.d. processes  $(\bar{X}^j, \bar{Y}^j) = (\bar{X}^{M,j}, \bar{X}^{\infty,j})$ ,  $1 \leq j \leq N$ . From the bounds in Lemma 3.3.8,

$$\sup_{0 \leq r \leq s} \mathbb{E}[|\mathbf{b}_r^2|^2] \leq C_h N^{-1} \mathbb{E} \left[ \mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq r \leq T} |B_r^1|^2 \right].$$

For the  $\bar{\sigma}_s^{M,i}$ -term in (3.20), we bound the integral using (3.19) with  $\sup_{t \leq s \leq t'} |\bar{\sigma}_s^{M,i}|$  instead of  $M_\sigma$ . We then split

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq s \leq t'} |\bar{\sigma}_s^{M,i}|^2 \right] &\leq 2 \mathbb{E} \left[ \sup_{0 \leq s \leq T} |\sigma_{s_h}(X^{M,i}, \pi(\bar{X}^{\vec{M}})) - \sigma_{s_h}(\bar{X}^{M,i}, \pi(\bar{X}^{\vec{M}}))|^2 \right] + \\ &\quad 2 \mathbb{E} \left[ \sup_{0 \leq s \leq T} |\sigma_{s_h}(\bar{X}^{M,i}, \pi(\bar{X}^{\vec{M}})) - \sigma_{s_h}(\bar{X}^{M,i}, \mathcal{L}(\bar{X}^{M,i}))|^2 \right]. \end{aligned}$$

Using (A.1) and Lemma 3.3.8-(ii), the first term is bounded by  $C_h N^{-1}$ . From (A.4) and Lemma 3.A.3, the same bound holds for the second term. From the Cauchy-Schwarz inequality,

$$\mathbb{E} \sup_{t \leq s \leq t'} \left| \int_t^s \bar{\sigma}_r^{M,i} d[B_r^i - B_r^i] \right| \leq \frac{C_h}{\sqrt{N}} \mathbb{E}^{1/2} \left[ \mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq r \leq T} |B_r^1|^2 \right].$$

For the  $\sigma_s^{M,i}$ -term in (3.20), we split  $\sigma_s^{M,i} = \sigma_s^1 + \sigma_s^2$  as we did for  $b_s^{M,i}$ . Using the BDG inequality,

$$\mathbb{E} \sup_{t \leq s \leq t'} \left| \int_t^s \sigma_r^1 dB_r^i \right| \leq C_h \mathbb{E} \left[ \left( \int_t^{t'} |\sigma_s^1|^2 ds \right)^{1/2} \right] \leq C_h (t' - t)^{1/2} \mathbb{E} \left[ \sup_{t \leq s \leq t'} |\sigma_s^1| \right].$$

We then handle  $\mathbb{E}[\sup_{t \leq s \leq t'} |\sigma_s^1|]$  as we did for  $\mathbb{E}[\sup_{t \leq s \leq t'} |\mathbf{b}_s^1|]$ . Using the BDG inequality and Jensen's inequality,

$$\mathbb{E} \sup_{t \leq s \leq t'} \left| \int_0^s \sigma_r^2 dB_r^i \right| \leq C_h \left( \int_t^{t'} \mathbb{E}[|\sigma_s^2|^2] ds \right)^{1/2},$$

and we handle  $\mathbb{E}[|\sigma_s^2|^2]$  as we did for  $\mathbb{E}[|\mathbf{b}_s^2|^2]$ . Let us gather all the terms in (3.20), using that

$$\forall s \in [t, t'], \quad \sup_{0 \leq r \leq s} |\delta_r^{M,i} - \delta_r^{\infty,i}| \leq \sup_{0 \leq r \leq t} |\delta_r^{M,i} - \delta_r^{\infty,i}| + \sup_{t \leq r \leq s} |\delta_r^{M,i} - \delta_r^{\infty,i}|. \quad (3.21)$$

Setting  $f^M(t, t') := \sqrt{N} \mathbb{E}[\sup_{t \leq s \leq t'} |\delta_s^{M,i} - \delta_s^{\infty,i}|]$ , for any  $0 \leq t \leq t' \leq T$ ,

$$f^M(t, t') \leq C_h f^M(0, t) + C_h \mathbb{E}^{1/2} \left[ \mathbb{1}_{\tau_M^1 < T} \sup_{0 \leq s \leq T} |B_s^1|^2 \right] + C_h (t' - t)^{1/2} f^M(t, t') + C_h \int_t^{t'} f^M(t, s) ds,$$

for a constant  $C_h$  that does not depend on  $(t, t')$ . Using (3.21), Lemma 3.A.1 now gives the desired bound on  $f^M(0, T)$ .  $\square$



*Proof of Proposition 3.2.7.* Let us fix  $\varphi$  in  $\mathcal{C}^{1,1}(\mathcal{C}^d, \mathbb{R})$ , before defining

$$\delta_\varphi^M := N^{-1} \sum_{i=1}^N [\varphi(X^{M,i}) - \mathbb{E}[\varphi(\bar{X}^{M,i})]].$$

We then decompose

$$\begin{aligned} \delta_\varphi^M - \delta_\varphi^\infty &= N^{-1} \sum_{i=1}^N [\varphi(X^{M,i}) - \varphi(\bar{X}^{M,i})] - N^{-1} \sum_{i=1}^N [\varphi(X^{\infty,i}) - \varphi(\bar{X}^{\infty,i})] \\ &\quad + N^{-1} \sum_{i=1}^N [\varphi(\bar{X}^{M,i}) - \mathbb{E}[\varphi(\bar{X}^{M,i})]] - N^{-1} \sum_{i=1}^N [\varphi(\bar{X}^{\infty,i}) - \mathbb{E}[\varphi(\bar{X}^{\infty,i})]]. \end{aligned}$$

As previously, we control the first line using Lemma 3.A.2, Lemma 3.3.8 and Proposition 3.3.9. Similarly, we handle the second line using Lemma 3.A.3 and Lemma 3.3.8. At the end of the day, we obtain that

$$\sup_{N \geq 1} \sqrt{N} \mathbb{E}[|\delta_\varphi^M - \delta_\varphi^\infty|] \xrightarrow{M \rightarrow +\infty} 0.$$

As a consequence, the convergence given by Corollary 3.3.7.2 holds uniformly in  $M$ , with variance  $\sigma_{M,\varphi}^2$ . Let  $(B_t)_{0 \leq t \leq T}$ ,  $(\tilde{B}_t)_{0 \leq t \leq T}$  be independent Brownian motions in  $\mathbb{R}^d$ . Let  $X^M$  denote the strong solution of the discretised McKean-Vlasov (3.4) with driving noise  $(B_{t \wedge \tau_M})_{0 \leq t \leq T}$ . Similarly, let  $\tilde{X}^M$  denote the strong solution of (3.4) with driving noise  $(\tilde{B}_{t \wedge \tilde{\tau}_M})_{0 \leq t \leq T}$ , the definition of  $\tilde{\tau}_M$  being straightforward. Let us then introduce the solution  $\delta X^M$  of the McKean-Vlasov SDE with common noise

$$\begin{aligned} d\delta X_{t_h}^M &= [D_x \sigma_{t_h}(X^M, \mathcal{L}(X^M)) \cdot \delta X^M + \delta_P \sigma_{t_h}(X^M, \mathcal{L}(X^M), \tilde{X}^h)] dB_{t \wedge \tau_M} \\ &\quad + [D_x b_{t_h}(X^M, \mathcal{L}(X^M)) \cdot \delta X^M + \delta_P b_{t_h}(X^M, \mathcal{L}(X^M), \tilde{X}^M)] dt \\ &\quad + \mathbb{E}[D_y \delta_P b_{t_h}(X^M, \mathcal{L}(X^M), X^M) \cdot \delta X^M | (\tilde{B}_s)_{0 \leq s \leq t_h}] dt \\ &\quad + \mathbb{E}[D_y \delta_P \sigma_{t_h}(X^M, \mathcal{L}(X^M), X^M) \cdot \delta X^M | (\tilde{B}_s)_{0 \leq s \leq t_h}] dB_{t \wedge \tau_M}, \quad \delta X_0^M = 0. \end{aligned}$$

Well-posedness for the above equation is given by Lemma 3.3.7. This can be seen as a discretised version of a McKean-Vlasov SDE with common noise. From Corollary 3.3.7.2 and the equation (3.17) satisfied by  $\bar{S}^{h,M,R}$ , we deduce that

$$\sigma_{M,\varphi}^2 := \mathbb{E}\{[\varphi(\tilde{X}^M) + \mathbb{E}[D\varphi(X^M) \cdot \delta X^M - \varphi(X^M) | \tilde{B}]]^2\}.$$

Using synchronous coupling and a Gronwall argument (as in the above proofs), we can show that

$$\mathbb{E}[\sup_{0 \leq t \leq T} |\delta X_t^M - \delta X_t^h|] \xrightarrow{M \rightarrow +\infty} 0,$$

where  $\delta X^h$  is the solution of (3.7) with the same  $(B, \tilde{B})$ . From this, we can deduce that  $\sigma_{M,\varphi}^2$  converges to  $\sigma_{h,\varphi}^2$  as required by Proposition 3.2.7. This completes the proof.  $\square$

## 3.4 Extension to the continuous setting

### 3.4.1 Large deviations and stochastic control

Throughout this section, we assume (A.1)-(A.3) and  $p \in [1, 2)$ . We recall that the notion of reference system  $\Sigma = (\Omega, (\mathcal{F}_t)_{t \leq 0 \leq T}, \mathbb{P}, (B_t^N)_{0 \leq t \leq T})$  is defined in Section 3.1.2. For such a  $\Sigma$ ,  $h \in [0, 1]$  and  $N \geq 1$ , we define the controlled system  $\vec{X}^{h,N,\vec{u}^N}$  by

$$\begin{aligned} dX_t^{h,i,N,\vec{u}^N} &= b_{t_h}(X^{h,i,N,\vec{u}^N}, \pi(\vec{X}^{h,N,\vec{u}^N})) dt + \sigma_{t_h}(X^{h,i,N,\vec{u}^N}, \pi(\vec{X}^{h,N,\vec{u}^N})) u_t^{i,N} dt \\ &\quad + \sigma_{t_h}(X^{h,i,N,\vec{u}^N}, \pi(\vec{X}^{h,N,\vec{u}^N})) dB_t^i, \quad \vec{X}^{h,i,N,\vec{u}^N} = X_0^{h,i,N}, \quad 1 \leq i \leq N, \end{aligned} \quad (3.22)$$

where  $\vec{X}_0^{h,N} \in L^p_{\mathbb{P}}(\Omega, \mathbb{R}^d)$  and  $\vec{u}^N = (\vec{u}_t^N)_{0 \leq t \leq T}$  is any  $(\mathbb{R}^d)^N$ -valued square-integrable progressively measurable process. Such a process will be called an *admissible control*. Following [Budhiraja et al., 2012, Section 3, page 9], strong existence and pathwise uniqueness for (3.22) is given by the Girsanov transform if  $\int_0^T |\vec{u}_t^N|^2 dt \leq M$  a.s. for some  $M > 0$ , and still holds otherwise using a localisation argument.

Let  $F : \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}$  be a bounded and Lipschitz-continuous function. Using the notations

$$\mathcal{E}(\vec{u}^N) := \frac{1}{N} \sum_{i=1}^N \mathcal{E}(u^{i,N}), \quad \mathcal{E}(u^{i,N}) := \int_0^T \frac{1}{2} |u_t^{i,N}|^2 dt,$$

we introduce the stochastic control problem  $\mathcal{F}^{h,N} := \inf_{\Sigma, \vec{X}_0^{h,N}, \vec{u}^N} J_{\Sigma}^h(\vec{X}_0^{h,N}, \vec{u}^N)$ , where

$$J_{\Sigma}^h(\vec{X}_0^{h,N}, \vec{u}^N) := N^{-1} H(\mathcal{L}(\vec{X}_0^{h,N}) | \mathcal{L}(X_0^1)^{\otimes N}) + \mathbb{E}[\mathcal{E}(\vec{u}^N) + F(\pi(\vec{X}^{h,N}, \vec{u}^N))],$$

and  $\mathcal{L}(X_0^1)$  is given by (A.3). Let us fix a reference system  $\Sigma$ . We first restrict the class of controls  $(\vec{X}_0^{h,N}, \vec{u}^N)$ . We recall that  $(X_0^{h,i,N}, u^{i,N})_{1 \leq i \leq N}$  is *exchangeable* if the law of  $(X_0^{h,\tau(i),N}, u^{\tau(i),N})_{1 \leq i \leq N}$  is the same for any permutation  $\tau$  of  $\{1, \dots, N\}$ .

**Lemma 3.4.1** (Exchangeability). *We can restrict the minimisation defining  $\mathcal{F}^{h,N}$  to controls  $(\vec{X}_0^{h,N}, \vec{u}^N)$  such that for every  $1 \leq i \leq N$ ,*

$$(i) \ (X_0^{h,i,N}, u^{i,N})_{1 \leq i \leq N} \text{ is exchangeable and } \mathbb{E}[\mathcal{E}(u^{1,N})] \leq 2\|F\|_{\infty}.$$

$$(ii) \ \mathbb{E}[|X_0^{h,1,N}|^p] \leq C, \text{ where } C \text{ only depends on } (p, \|F\|_{\infty}, \mathcal{L}(X_0^1)).$$

*Proof.* By choosing  $\mathcal{L}(\vec{X}_0^{h,N}) = \mathcal{L}(X_0^1)^{\otimes N}$  and  $\vec{u}^N \equiv 0$ , we get that  $\mathcal{F}^{h,N} \leq \|F\|_{\infty}$ . This allows us to restrict ourselves to admissible controls  $\vec{u}^N$  with  $\mathbb{E}[\mathcal{E}(\vec{u}^N)] \leq 2\|F\|_{\infty}$ . Let us fix such a  $(\vec{X}_0^{h,N}, \vec{u}^N)$ . Let  $\tau$  be a uniformly distributed random permutation of  $\{1, \dots, N\}$  that is independent of  $\vec{B}^N$ ,  $\vec{X}_0^{h,N}$ , and  $\vec{u}^N$ . Defining  $X_0^{h,\tau,i,N} := X_0^{h,\tau(i),N}$  and  $u^{\tau,i,N} := u^{\tau(i),N}$ , we notice that  $(X_0^{h,\tau,i,N}, u^{\tau,i,N})_{1 \leq i \leq N}$  is exchangeable. The symmetry of (3.22) then implies that  $\pi(X^{h,\tau,N}, \vec{u}^{\tau,N})$  has the same law as  $\pi(X^{h,N}, \vec{u}^N)$ . For any permutation  $\tau'$ , we define

$$f_{\tau'}^N : (x_1, \dots, x_N) \mapsto (x_{\tau'(1)}, \dots, x_{\tau'(N)}).$$

The contraction property of entropy [Fischer, 2014, Lemma A.1] gives that

$$H(\mathcal{L}(\vec{X}_0^{h,\tau',N}) | \mathcal{L}(X_0^1)^{\otimes N}) = H((f_{\tau'}^N)_{\#} \mathcal{L}(\vec{X}_0^{h,N}) | (f_{\tau'}^N)_{\#} \mathcal{L}(X_0^1)^{\otimes N}) \leq H(\mathcal{L}(\vec{X}_0^{h,N}) | \mathcal{L}(X_0^1)^{\otimes N}).$$

Since  $\mathcal{L}(\vec{X}_0^{h,\tau,N}) = \mathbb{E}[(f_{\tau}^N)_{\#} \mathcal{L}(\vec{X}_0^{h,N})]$  and  $P \mapsto H(P|Q)$  is convex, Jensen's inequality then implies that

$$H(\mathcal{L}(\vec{X}_0^{h,\tau,N}) | \mathcal{L}(X_0^1)^{\otimes N}) \leq \mathbb{E}[H((f_{\tau}^N)_{\#} \mathcal{L}(\vec{X}_0^{h,N}) | \mathcal{L}(X_0^1)^{\otimes N})] \leq H(\mathcal{L}(\vec{X}_0^{h,N}) | \mathcal{L}(X_0^1)^{\otimes N}).$$

From all this, we can restrict ourselves to assuming (i). Similarly, we can assume that

$$N^{-1} H(\mathcal{L}(\vec{X}_0^{h,N}) | \mathcal{L}(X_0^1)^{\otimes N}) \leq 2\|F\|_{\infty}.$$

Using the symmetry assumption (i), the tensorisation property [Csiszár, 1984, Equation (2.10)] yields

$$H(\mathcal{L}(X_0^{h,1,N}) | \mathcal{L}(X_0^1)) \leq N^{-1} H(\mathcal{L}(\vec{X}_0^{h,N}) | \mathcal{L}(X_0^1)^{\otimes N}).$$

Let us use the dual representation [Léonard, 2012, Proposition 3.1-(iii)] for the relative entropy

$$H(\mathcal{L}(X_0^{h,1,N})|\mathcal{L}(X_0^1)) = \sup_{\substack{\phi \text{ measurable} \\ \mathbb{E}[\phi(X_0^1)] < +\infty}} \mathbb{E}[\phi(X_0^{h,1,N})] - \log \mathbb{E}[e^{\phi(X_0^1)}], \quad (3.23)$$

with  $\phi(x) := |x|^p$ ,  $\mathbb{E}[e^{|\phi(X_0^1)|^p}]$  being finite from (A.3). Gathering inequalities now allows us to assume the bound (ii) on  $\mathbb{E}[|X_0^{h,1,N}|^p]$ .  $\square$

**Lemma 3.4.2** (Uniform approximation). *There exists  $C > 0$  such that for every  $N \geq 1$ ,  $h \in [0, 1]$ , every  $(\vec{X}_0^{h,N}, \vec{u}^N)$  satisfying (i)-(ii) in Lemma 3.4.1, and  $1 \leq i \leq N$ ,*

- (i)  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,N,\vec{u}^N}|^p] \leq C$ .
- (ii)  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,N,\vec{u}^N} - X_{t_h}^{h,i,N,\vec{u}^N}|^p] \leq C|h \log h|^{p/2}$ .
- (iii)  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{0,i,N,\vec{u}^N} - X_{t_h}^{0,i,N,\vec{u}^N}|^p] \leq C|h \log h|^{p/2}$ .
- (iv)  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,N,0} - X_{t_h}^{0,i,N,0}|^p] \leq C|h \log h|^{p/2}$ .

In the above estimates, we assume that  $\vec{X}_0^{h,N,\vec{u}^N} = \vec{X}_0^{0,N,\vec{u}^N} = \vec{X}_0^{h,N}$ .

*Proof.* In the following proof,  $C > 0$  is a constant that may change from line to line, but always satisfying the requirements of Lemma 3.4.2. With a slight abuse, we use the convention  $t_0 = t$ . We write  $X^{h,i} = X^{h,i,N,\vec{u}^N}$  for conciseness.

(i) Using (A.1), for  $1 \leq i \leq N$ ,  $0 \leq t \leq T$ ,

$$|b_{t_h}(X^{h,i}, \pi(\vec{X}^h))|^p \leq C[1 + \sup_{0 \leq s \leq t_h} |X_s^{h,i}|^p + N^{-1} \sum_{j=1}^N \sup_{0 \leq s \leq t_h} |X_s^{h,j}|^p]. \quad (3.24)$$

From Lemma 3.4.1-(i), all the  $X^{h,i}$  have the same law. Moreover,  $\vec{u}^N$  is square-integrable and  $\sigma$  is bounded. When  $h > 0$  a direct induction shows that  $\mathbb{E}[\sup_{0 \leq s \leq t} |X_s^{h,i}|^p]$  is finite under (A.1). When  $h = 0$ , this is still true as a classical consequence of the Gronwall lemma. Integrating in (3.22), taking the  $p$ -power and using Jensen's inequality, before taking supremum in time and expectations, we get that

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s^{h,i}|^p \right] &\leq C + CT^{p-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} |X_r^{h,i}|^p \right] ds + CM_\sigma^p \int_0^t \mathbb{E}[|u_s^{i,N}|^p] ds, \\ &\quad + C \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \sigma_{r_h}(X^{i,h}, \pi(\vec{X}^h)) dB_r^i \right|^p \right]. \end{aligned}$$

Since  $p \in [1, 2]$ , the penultimate term is bounded by  $C[1 + \mathbb{E}[\mathcal{E}(u^{i,N})]]$ . From the BDG inequality and (A.1),

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \sigma_{r_h}(X^{i,h}, \pi(\vec{X}^h)) dB_r^i \right|^p \right] \leq C \mathbb{E} \left[ \left( \int_0^t |\sigma_{s_h}(X^{i,h}, \pi(\vec{X}^h))|^2 ds \right)^{p/2} \right] \leq C.$$

Gathering the terms, the Gronwall lemma now gives the desired uniform-in- $h$  bound.

(ii)-(iii) Let  $h'$  be in  $\{0, h\}$ . For  $1 \leq i \leq N$  and  $0 \leq t \leq T$ ,

$$\begin{aligned} X_t^{h',i} - X_{t_h}^{h',i} &= \int_{t_h}^t b_{s_{h'}}(X^{h',i}, \pi(\vec{X}^{h',N})) ds + \int_{t_h}^t \sigma_{s_{h'}}(X^{h',i}, \pi(\vec{X}^{h',N})) u_s^{i,N} ds \\ &\quad + \int_{t_h}^t \sigma_{s_{h'}}(X^{h',i}, \pi(\vec{X}^{h',N})) dB_s^i. \end{aligned}$$

We then take the  $p$ -power and supremum in time. Taking supremum in time and expectations in (3.24), (A.1) and (i) yield

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{t_h}^t b_{s_{h'}}(X^{h',i}, \pi(\vec{X}^{h',N})) ds \right|^p \leq Ch^p.$$

Similarly, using  $p \in [1, 2]$ , Jensen's inequality and the bound on  $\sigma$ ,

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{t_h}^t \sigma_{s_{h'}}(X^{h',i}, \pi(\vec{X}^{h',N})) u_s^{i,N} ds \right|^p \leq Ch^{p/2} \mathbb{E} \left[ \left( \int_0^T |u_t^{i,N}|^2 dt \right)^{p/2} \right] \leq Ch^{p/2} \mathbb{E}^{p/2} [2\mathcal{E}(u^{i,N})],$$

and  $\mathbb{E}[\mathcal{E}(u^{i,N})]$  is bounded using Lemma 3.4.1-(i). Since  $\sigma$  is bounded, the estimate [Fischer and Nappo, 2009, Theorem 1] for the continuity modulus of Ito processes yields

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{t_h}^t \sigma_{s_{h'}}(X^{h',i}, \pi(\vec{X}^{h',N})) dB_s^i \right|^p \leq C|h \log h|^{p/2}.$$

Gathering all the terms gives the desired bound.

(iv) We now write  $X^{h,i} = X^{h,i,N,0}$ , for  $h \in [0, 1]$ . For every  $0 \leq t \leq t' \leq T$  and  $1 \leq i \leq N$ ,

$$X_{t'}^{h,i} - X_{t'}^{0,i} = X_t^{h,i} - X_t^{0,i} + \int_t^{t'} b_s^{h,i} ds + \int_t^{t'} \sigma_s^{h,i} dB_s^i,$$

where

$$b_s^{h,i} := b_{s_h}(X^{h,i}, \pi(\vec{X}^{h,N})) - b_s(X^{0,i}, \pi(\vec{X}^{0,N})),$$

and  $\sigma_s^{h,i}$  is similarly defined. We then take the  $p$ -power of each side and supremum in time. We decompose as

$$b_s^{h,i} = b_{s_h}(X_{\wedge s_h}^{h,i}, \pi(\vec{X}_{\wedge s_h}^{h,N})) - b_{s_h}(X_{\wedge s_h}^{0,i}, \pi(\vec{X}_{\wedge s_h}^{0,N})) + b_{s_h}(X_{\wedge s_h}^{0,i}, \pi(\vec{X}_{\wedge s_h}^{0,N})) - b_s(X_{\wedge s}^{0,i}, \pi(\vec{X}_{\wedge s}^{0,N})).$$

Hence, using (A.1),

$$\begin{aligned} |b_s^{h,i}|^p &\leq C \sup_{0 \leq r \leq s_h} |X_r^{h,i} - X_r^{0,i}|^p + CN^{-1} \sum_{j=1}^N \sup_{0 \leq r \leq s_h} |X_r^{h,j} - X_r^{0,j}|^p \\ &\quad + Ch^p + C \sup_{0 \leq r \leq s} |X_r^{0,i} - X_{r_h}^{0,i}|^p + CN^{-1} \sum_{j=1}^N \sup_{0 \leq r \leq s_h} |X_r^{0,j} - X_{r_h}^{0,j}|^p. \end{aligned} \quad (3.25)$$

We recall that  $((X^{h,j}, X^{0,j}))_{1 \leq j \leq N}$  is exchangeable, and we now take expectations. Using the BDG inequality,

$$\mathbb{E} \left[ \sup_{t \leq s \leq t'} \left| \int_0^s \sigma_r^{h,i} dB_r^i \right|^p \right] \leq C \mathbb{E} \left[ \left( \int_t^{t'} |\sigma_s^{h,i}|^2 ds \right)^{p/2} \right] \leq C(t' - t)^{p/2} \mathbb{E} \left[ \sup_{t \leq s \leq t'} |\sigma_s^{h,i}|^p \right],$$

and we handle  $|\sigma_s^{h,i}|^p$  as we did for  $|b_s^{h,i}|^p$ . We further use (iii) to bound the expectation of the two last terms in (3.25). Noticing that

$$\forall s \in [t, t'], \quad \sup_{0 \leq r \leq s} |X_r^{h,i} - X_r^{0,i}| \leq \sup_{0 \leq r \leq t} |X_r^{h,i} - X_r^{0,i}| + \sup_{t \leq r \leq s} |X_r^{h,i} - X_r^{0,i}|, \quad (3.26)$$

we set  $f(t, t') := \mathbb{E} \left[ \sup_{t \leq s \leq t'} |X_s^{h,i} - X_s^{0,i}|^p \right]$  and we gather all the terms to obtain

$$\forall 0 \leq t \leq t' \leq T, \quad f(t, t') \leq Cf(0, t) + C|h \log h|^{p/2} + C(t' - t)^{p/2} f(t, t') + C \int_t^{t'} f(t, s) ds,$$

for a constant  $C > 0$  that does not depend on  $(t, t')$ . The desired bound on  $f(0, T)$  now follows from (3.26) and Lemma 3.A.1.  $\square$

**Proposition 3.4.3** (Quantitative convergence for mean-field control). *Under (A.1)-(A.3), if  $\sigma_t(x, P) = \sigma_t(P)$  does not depend on  $x$ , then*

$$\sup_{N \geq 1} |\mathcal{F}^{h,N} - \mathcal{F}^{0,N}| \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* In the following,  $C > 0$  is a constant that may change from line to line, but always satisfying the requirements of Proposition 3.4.3.

**Step 1.** *Coupling of controls.* For  $\varepsilon > 0$ , let  $(\Sigma, \vec{X}_0^{h,N}, \vec{u}^{h,N})$ , satisfying (i)-(ii) in Lemma 3.4.1, be  $\varepsilon$ -optimal for  $\mathcal{F}^{h,N}$ , meaning that  $J_\Sigma^h(\vec{X}_0^{h,N}, \vec{u}^{h,N}) \leq \mathcal{F}^{h,N} + \varepsilon$ . For any  $(\vec{X}_0^{0,N}, \vec{u}^{0,N})$ ,

$$\mathcal{F}^{0,N} - \mathcal{F}^{h,N} \leq \varepsilon + J_\Sigma^0(\vec{X}_0^{0,N}, \vec{u}^{0,N}) - J_\Sigma^h(\vec{X}_0^{h,N}, \vec{u}^{h,N}). \quad (3.27)$$

We choose  $\vec{X}_0^{0,N} = \vec{X}_0^{h,N}$  and  $\vec{u}^{0,N} = \vec{u}^{h,N}$ . For conciseness, we write  $X^{h,i}$  instead of  $X^{h,i,N,\vec{u}^{h,N}}$ , and  $u^i$  for  $u^{h,i,N}$ . For  $M > 0$ , let us control  $\mathbb{E}[\mathbb{1}_{\mathcal{E}(\vec{u}^{h,N}) \leq M} \sup_{0 \leq t \leq T} |X_t^{h,i} - X_t^{0,i}|^p]$  as in Lemma 3.4.2-(iv). Setting  $\vec{v}^{h,N} := \mathbb{1}_{\mathcal{E}(\vec{u}^{h,N}) \leq M} \vec{u}^{h,N}$ , we notice that

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\mathcal{E}(\vec{u}^{h,N}) \leq M} \sup_{0 \leq t \leq T} |X_t^{h,i,\vec{u}^{h,N}} - X_t^{0,i,\vec{u}^{h,N}}|^p] &= \mathbb{E}[\mathbb{1}_{\mathcal{E}(\vec{u}^{h,N}) \leq M} \sup_{0 \leq t \leq T} |X_t^{h,i,\vec{v}^{h,N}} - X_t^{0,i,\vec{v}^{h,N}}|^p] \\ &\leq \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,\vec{v}^{h,N}} - X_t^{0,i,\vec{v}^{h,N}}|^p]. \end{aligned}$$

Thus, it is sufficient to perform the estimate while assuming that  $\mathcal{E}(\vec{u}^{h,N}) \leq M$  a.s.; this corresponds to replacing  $\vec{u}^{h,N}$  by  $\vec{v}^{h,N}$ , removing  $\mathbb{1}_{\mathcal{E}(\vec{u}^{h,N}) \leq M}$  from the expectation. As in the proof of Lemma 3.4.2-(iv), for  $0 \leq t \leq t'$ , we decompose

$$X_{t'}^{h,i} - X_{t'}^{0,i} = X_t^{h,i} - X_t^{0,i} + \int_t^{t'} b_s^{h,i} ds + \int_t^{t'} [\sigma_{s_h}(\pi(\vec{X}^h)) - \sigma_s(\pi(\vec{X}^0))] u_s^i ds + \int_t^{t'} \sigma_s^{h,i} dB_s^i.$$

We then take  $p$ -power, supremum in time and expectations, and we get as previously that

$$\mathbb{E}[|b_s^{h,i}|^p] \leq Ch^p + C\mathbb{E}[\sup_{0 \leq r \leq s} |X_r^{h,i} - X_r^{0,i}|^p + \sup_{0 \leq r \leq s} |X_r^{0,i} - X_{r_h}^{0,i}|^p],$$

$$\mathbb{E}\left[\sup_{t \leq s \leq t'} \left| \int_t^s \sigma_r^{h,i} dB_r^i \right|^p\right] \leq Ch^p + C(t' - t)^{p/2} \mathbb{E}\left[\sup_{0 \leq r \leq s} |X_r^{h,i} - X_r^{0,i}|^p + \sup_{0 \leq r \leq s} |X_r^{0,i} - X_{r_h}^{0,i}|^p\right].$$

Using  $p \in [1, 2)$  with Hölder's and Jensen's inequalities, we further get

$$\begin{aligned} \left| \int_t^{t'} [\sigma_{s_h}(\pi(\vec{X}^h)) - \sigma_s(\pi(\vec{X}^0))] u_s^i ds \right|^p &\leq \left[ \int_t^{t'} |\sigma_{s_h}(\pi(\vec{X}^h)) - \sigma_s(\pi(\vec{X}^0))|^{p'} ds \right]^{p/p'} \int_t^{t'} |u_s^i|^p ds, \\ &\leq (t' - t)^{p/2} \sup_{t \leq s \leq t'} |\sigma_{s_h}(\pi(\vec{X}^h)) - \sigma_s(\pi(\vec{X}^0))|^p \left[ \int_0^{t'} t'^i |u_s^i|^2 ds \right]^{p/2}, \end{aligned}$$

denoting by  $p'$  the conjugate exponent of  $p$ . From Jensen's inequality,

$$\frac{1}{N} \sum_{i=1}^N \left[ \int_t^{t'} |u^i|^2 ds \right]^{p/2} \leq [2\mathcal{E}(\vec{u}^{h,N})]^{p/2} \leq (2M)^{p/2}.$$

Using the same splitting as (3.25), we finally get

$$\mathbb{E}\left[\sup_{t \leq s \leq t'} |\sigma_{s_h}(\pi(\vec{X}^h)) - \sigma_s(\pi(\vec{X}^0))|^p\right] \leq Ch^p + C\mathbb{E}\left[\sup_{0 \leq r \leq s} |X_r^{h,i} - X_r^{0,i}|^p + \sup_{0 \leq r \leq s} |X_r^{0,i} - X_{r_h}^{0,i}|^p\right].$$

Using (3.26) and setting  $f(t, t') := \mathbb{E}[\sup_{t \leq s \leq t'} |X_s^{h,i} - X_s^{0,i}|^p]$ , as in the proof of Lemma 3.4.2-(iv), we get that for every  $0 \leq t \leq t' \leq T$ ,

$$f(t, t') \leq Cf(0, t) + C[1 + M]|h \log h|^{p/2} + C[1 + M](t' - t)^{p/2}f(t, t') + C \int_t^{t'} f(t, s) ds,$$

for  $C > 0$  independent of  $(t, t')$ , and we conclude as previously using Lemma 3.A.1.

At the end of the day, we deduce that for every  $M > 0$ ,

$$\forall h \in [0, 1], \quad \mathbb{E}[\mathbb{1}_{\mathcal{E}(\bar{u}^{h,N}) \leq M} \sup_{0 \leq t \leq T} |X_t^{h,i} - X_t^{0,i}|^p] \leq C_M |h \log h|^{p/2},$$

for a constant  $C_M > 0$  independent of  $(h, N)$ .

**Step 2. Difference of the costs.** For  $M > 0$ ,  $F$  being  $W_p$ -Lipschitz,

$$\mathbb{1}_{\mathcal{E}(\bar{u}^{h,N}) \leq M} |F(\pi(\bar{X}^0)) - F(\pi(\bar{X}^h))|^p \leq \frac{\|F\|_{\text{Lip}}^p}{N} \sum_{i=1}^N \mathbb{1}_{\mathcal{E}(\bar{u}^{h,N}) \leq M} \sup_{0 \leq t \leq T} |X_t^{0,i} - X_t^{h,i}|^p.$$

Using **Step 1.**, this implies

$$J_{\Sigma}^0(\bar{X}_0^{h,N}, \bar{u}^{h,N}) - J_{\Sigma}^h(\bar{X}_0^{h,N}, \bar{u}^{h,N}) \leq C_M |h \log h|^{1/2} + 2\|F\|_{\infty} \mathbb{P}(\mathcal{E}(\bar{u}^{h,N}) > M).$$

We bound the last term by Markov's inequality and Lemma 3.4.1-(i). Plugging this into (3.27),

$$\mathcal{F}^{0,N} - \mathcal{F}^{h,N} \leq \varepsilon + C_M |h \log h|^{1/2} + CM^{-1},$$

and we send  $\varepsilon$  to 0. Since  $C$  does not depend on  $(h, M, N)$  and  $C_M$  does not depend on  $(h, N)$ , this implies that  $\limsup_{h \rightarrow 0} \sup_{N \geq 1} \mathcal{F}^{0,N} - \mathcal{F}^{h,N} \leq 0$ .

**Step 3. Conclusion.** Inverting the roles of  $\mathcal{F}^{0,N}$  and  $\mathcal{F}^{h,N}$ , we readily adapt **Steps 1-2.** to obtain that  $\limsup_{h \rightarrow 0} \sup_{N \geq 1} \mathcal{F}^{h,N} - \mathcal{F}^{0,N} \leq 0$ , completing the proof.  $\square$

*End of the proof of Theorem 3.2.8.* For  $h \in [0, 1]$ , as a particular case of Corollary 3.B.3.2,

$$\mathcal{F}^{h,N} = \inf_{R_N \in \mathcal{P}_p((\mathbb{R}^d \times \mathcal{C}^{d'})^N)} N^{-1} H(R_N | \mathcal{L}(X_0^1, B^1)^{\otimes N}) + N^{-1} \int_{\mathbb{R}^d \times \mathcal{C}^d} [F \circ \Psi_h] dR_N.$$

From [Dupuis and Ellis, 2011, Proposition 1.4.2], this implies that

$$\mathcal{F}^{h,N} = -N^{-1} \log \mathbb{E}[\exp[-NF(\Psi_h(\pi(\bar{X}_0^N, \bar{B}^N)))]].$$

Proposition 3.4.3 then yields the uniform-in- $N$  convergence stated in Theorem 3.2.8, as  $h \rightarrow 0$ .

For  $h \in (0, 1]$ , Proposition 3.2.6 gives that the sequence of the  $\mathcal{L}(\Psi_h(\pi(\bar{X}_0^N, \bar{B}^N)))$  satisfies the LDP with good rate function  $P \in \mathcal{P}_p(\mathcal{C}^d) \mapsto H(P | \Gamma_h(P))$ . Varadhan's lemma [Dupuis and Ellis, 2011, Theorem 1.2.1] then gives that

$$-N^{-1} \log \mathbb{E}[\exp[-NF(\Psi_h(\pi(\bar{X}_0^N, \bar{B}^N)))] \xrightarrow{N \rightarrow +\infty} \inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P | \Gamma_h(P)) + F(P).$$

From Proposition 3.B.4, the r.h.s converges to  $\inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P | \Gamma_0(P)) + F(P)$  as  $h \rightarrow 0$ . The uniform-in- $N$  convergence now allows us to invert the  $N \rightarrow +\infty$  and  $h \rightarrow 0$  limits, so that

$$-N^{-1} \log \mathbb{E}[\exp[-NF(\pi(\bar{X}^{0,N}))]] \xrightarrow{N \rightarrow +\infty} \inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P | \Gamma_0(P)) + F(P),$$

for every bounded Lipschitz  $F : \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}$ . From [Dupuis and Ellis, 2011, Corollary 1.2.5], this proves that the sequence of the  $\mathcal{L}(\pi(\bar{X}^{0,N}))$  satisfies the LDP with good rate function  $P \mapsto H(P | \Gamma_0(P))$ , this rate function being indeed good from Corollary 3.B.3.1. The  $\Gamma$ -convergence of  $I_h$  towards  $I_0$  precisely corresponds to the proof of Proposition 3.B.4 with  $F \equiv 0$ .  $\square$

### 3.4.2 Markov setting

In the setting of Theorem 3.2.11, still for  $p \in [1, 2)$ , let now  $F : \mathcal{C}([0, T], \mathcal{P}_p(\mathbb{R}^d)) \rightarrow \mathbb{R}$  denote a bounded Lipschitz-continuous function. We keep all the notations from the previous section, replacing  $F(\pi(\vec{X}^{h,N, \bar{u}^N}))$  by  $F(\pi(\vec{X}^{h,N, \bar{u}^N}))$  in the definition of  $J^h$ . As previously, we fix a reference system  $\Sigma$ , and we start all the processes from the same initial condition  $\vec{X}_0^{h,N}$ .

**Proposition 3.4.4** (Quantitative convergence). *Under (A.3)-(A.5), for every  $M > 0$ , there exists  $C_M > 0$  such that for every  $N \geq 1$ ,*

$$\forall h \in [0, 1], \quad \mathbb{E} \left[ \mathbb{1}_{\mathcal{E}(\bar{u}^N) \leq M} \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^N |X_t^{h,i,N, \bar{u}^N} - X_t^{0,i,N, \bar{u}^N}|^p \right] \leq C_M |h \log h|^{\frac{p(1-p/2)}{2}},$$

for every admissible  $(\vec{X}_0^{h,N}, \bar{u}^N)$  satisfying (i)-(ii) in Lemma 3.4.1.

*Proof.* As in **Step 1.** in the proof of Proposition 3.4.3, we can assume, up to modifying  $\bar{u}^N$ , that  $\mathcal{E}(\bar{u}^N) \leq M$  almost surely, thus removing  $\mathbb{1}_{\mathcal{E}(\bar{u}^N) \leq M}$  from the expectation. Let us fix  $\varepsilon \in (0, 1]$ . We are going to control errors in a suitably weighted norm that allows us to get rid of the controls.

**Step 1. Notations.** In the following, we write  $\vec{X}^{h,N} = \vec{X}^{h,N, \bar{u}^N}$  and we introduce

$$\Lambda_t^{h,i} := X_t^{h,i} - X_{t_h}^{h,i}, \quad \Delta_t^{h,i} := X_t^{h,i} - X_t^{0,i}, \quad \eta_t^N := \exp \left[ - \frac{2^p L_\sigma^p}{N} \sum_{i=1}^N \int_0^t [1 + |u_s^{i,N}|^2] ds \right],$$

together with

$$\bar{\Lambda}_t^h := \frac{1}{N} \sum_{i=1}^N |\Lambda_t^{h,i}|^p, \quad \bar{\Delta}_t^h := \frac{1}{N} \sum_{i=1}^N |\Delta_t^{h,i}|^p.$$

For  $1 \leq i \leq N$ , we have the decomposition

$$d\Delta_t^{h,i} = b_t^{h,i} dt + \sigma_t^{h,i} u_t^{i,N} dt + \sigma_t^{h,i} dB_t^i,$$

where

$$b_t^{h,i} := b_{t_h}(X_{t_h}^{h,i}, \pi(\vec{X}_{t_h}^{h,N})) - b_t(X_t^{0,i}, \pi(\vec{X}_t^{0,N})),$$

and  $\sigma_t^{h,i}$  is similarly defined. Let  $|\Delta_t^{h,i}|_\varepsilon := (|\Delta_t^{h,i}|^2 + \varepsilon^2)^{1/2}$ .

**Step 2. Differentiation.** Using Ito's formula,

$$\begin{aligned} d[\eta_t^N |\Delta_t^{h,i}|_\varepsilon^p] &= -\eta_t^N 2^p L_\sigma^p |\Delta_t^{h,i}|_\varepsilon^{p-2} \sum_{i=1}^N [1 + |u_t^{i,N}|^2] dt + p \eta_t^N |\Delta_t^{h,i}|_\varepsilon^{p-2} \Delta_t^{h,i} \cdot b_t^{h,i} dt \\ &\quad + p \eta_t^N |\Delta_t^{h,i}|_\varepsilon^{p-2} \Delta_t^{h,i} \cdot \sigma_t^{h,i} u_t^{i,N} dt + p \eta_t^N |\Delta_t^{h,i}|_\varepsilon^{p-2} \Delta_t^{h,i} \cdot \sigma_t^{h,i} dB_t^i \\ &\quad + \frac{p}{2} \eta_t^N |\Delta_t^{h,i}|_\varepsilon^{p-2} \text{Tr}[\sigma_t^{h,i} (\sigma_t^{h,i})^\top [\text{Id} + (p-2) |\Delta_t^{h,i}|_\varepsilon^{-2} (\Delta_t^{h,i} \otimes \Delta_t^{h,i})]] dt. \end{aligned}$$

We now integrate in time. First,  $b$  being Lipschitz-continuous in  $(t, x, P_t)$  from (A.5),

$$|b_t^{h,i}|^p \leq C[h^p + |\Delta_t^{h,i}|^p + \bar{\Delta}_t^h + |\Lambda_t^{h,i}|^p + \bar{\Lambda}_t^h],$$

using a decomposition similar to (3.25). Since  $p \in [1, 2)$  and  $|\Delta_t^{h,i}| \leq |\Delta_t^{h,i}|_\varepsilon$ , this yields

$$\begin{aligned} p |\Delta_t^{h,i}|_\varepsilon^{p-2} \Delta_t^{h,i} \cdot b_t^{h,i} &\leq p |\Delta_t^{h,i}|^{p-1} |b_t^{h,i}| \leq (p-1) |\Delta_t^{h,i}|^p + |b_t^{h,i}|^p \\ &\leq C[h^p + |\Delta_t^{h,i}|^p + \bar{\Delta}_t^h + |\Lambda_t^{h,i}|^p + \bar{\Lambda}_t^h], \end{aligned} \quad (3.28)$$

where the second inequality uses Young's inequality. Similarly, using that  $\sigma_t(x, P) = \sigma_t(x)$  and  $|\sigma_t^{h,i}| \leq 2M_\sigma$ ,

$$|\sigma_t^{h,i}(\sigma_t^{h,i})^\top| \leq (2M_\sigma)^2 \wedge [Ch^2 + C|\Delta_t^{h,i}|^2 + C|\Lambda_t^{h,i}|^2] \leq C[h^2 + |\Delta_t^{h,i}|^2] + (4M_\sigma^2) \wedge (C|\Lambda_t^{h,i}|^2).$$

We then use  $(4M_\sigma^2) \wedge (C|\Lambda_t^{h,i}|^2) \leq (4M_\sigma^2)^{1-p/2} C^{p/2} |\Lambda_t^{h,i}|^p$ ,  $|\Delta_t^{h,i}| \leq |\Delta_t^{h,i}|_\varepsilon$  and  $|\Delta_t^{h,i}|_\varepsilon \geq \varepsilon$ ,

$$p|\Delta_t^{h,i}|_\varepsilon^{p-2} |\sigma_t^{h,i}(\sigma_t^{h,i})^\top| \leq C[\varepsilon^{p-2} h^2 + \varepsilon^{p-2} |\Lambda_t^{h,i}|^p + |\Delta_t^{h,i}|^p]. \quad (3.29)$$

**Step 3. Gathering and cancellations.** At this stage, using  $|\Delta_t^{h,i}| \leq |\Delta_t^{h,i}|_\varepsilon$  and  $h^2 \leq h^p$ , we get that for every  $0 \leq t \leq t' \leq T$ ,

$$\begin{aligned} \eta_{t'}^N |\Delta_{t'}^{h,i}|^p &\leq \eta_t^N |\Delta_t^{h,i}|_\varepsilon^p + \int_t^{t'} C \eta_s^N [(1 + \varepsilon^{p-2})h^p + |\Delta_s^{h,i}|^p + \bar{\Delta}_s^h + (1 + \varepsilon^{p-2})|\Lambda_s^{h,i}|^p + \bar{\Lambda}_s^h] ds \\ &\quad + \int_t^{t'} \left\{ -\eta_s^N 2^p L_\sigma^p |\Delta_s^{h,i}|^p N^{-1} \sum_{i=1}^N [1 + |u_s^{i,N}|^2] + p \eta_s^N |\Delta_s^{h,i}|_\varepsilon^{p-2} \Delta_s^{h,i} \cdot \sigma_t^{h,i} u_s^{i,N} \right\} ds \\ &\quad + \int_t^{t'} p \eta_s^N |\Delta_s^{h,i}|_\varepsilon^{p-2} \Delta_s^{h,i} \cdot \sigma_s^{h,i} dB_s^i, \end{aligned}$$

integrating all the pieces from **Step 2.** Reasoning as in (3.28)-(3.29),

$$p|\Delta_s^{h,i}|_\varepsilon^{p-2} \Delta_s^{h,i} \cdot \sigma_s^{h,i} u_s^{i,N} \leq (p-1)|\Delta_s^{h,i}|^p + 2^p L_\sigma^p |\Delta_s^{h,i}|^p |u_s^{i,N}|^p + |u_s^{i,N}|^p [Ch^p + (2M_\sigma)^p \wedge (C|\Lambda_s^{h,i}|^p)].$$

We now sum over  $i$  and we divide by  $N$ . Since  $|u_s^{i,N}|^p \leq 1 + |u_s^{i,N}|^2$ , this cancels the  $2^p L_\sigma^p |\Delta_s^{h,i}|^p |u_s^{i,N}|^p$  terms. We then use that  $\eta_t^N \leq 1$ ,  $|\Delta_t^{h,i}|_\varepsilon \leq \varepsilon + |\Delta_t^{h,i}|$ , and we take supremum in time to obtain

$$\begin{aligned} \sup_{t \leq s \leq t'} \eta_s^N \bar{\Delta}_s^h &\leq C[\varepsilon^p + \bar{\Delta}_t^h] + \int_t^{t'} C \left\{ (1 + \varepsilon^{p-2})[h^p + \sup_{0 \leq r \leq s} \bar{\Lambda}_r^h] + \sup_{0 \leq r \leq s} \bar{\Delta}_r^h + h^p N^{-1} \sum_{i=1}^N |u_s^{i,N}|^p \right\} ds \\ &\quad + \int_t^{t'} N^{-1} \sum_{i=1}^N |u_s^{i,N}|^p [(2M_\sigma)^p \wedge (C|\Lambda_s^{h,i}|^p)] ds + \sup_{t \leq s \leq t'} \int_t^s p \eta_r^N N^{-1} \sum_{i=1}^N |\Delta_r^{h,i}|_\varepsilon^{p-2} \Delta_r^{h,i} \cdot \sigma_r^{h,i} dB_r^i. \end{aligned}$$

Let  $\mathbf{B}_t$  denote the last term on the r.h.s. Since  $\mathcal{E}(\bar{u}^N) \leq M$ , we have

$$\int_t^{t'} h^p N^{-1} \sum_{i=1}^N |u_s^{i,N}|^p ds \leq 2T[1 + M]h^p.$$

Using  $p < 2$ , Hölder's inequality further yields

$$\begin{aligned} &\int_t^{t'} N^{-1} \sum_{i=1}^N |u_s^{i,N}|^p [(2M_\sigma)^p \wedge (C|\Lambda_s^{h,i}|^p)] ds \\ &\leq [2\mathcal{E}(\bar{u}^N)]^{p/2} \left[ \int_t^{t'} N^{-1} \sum_{i=1}^N [(2M_\sigma)^p \wedge (C|\Lambda_s^{h,i}|^p)]^{2/(2-p)} \right]^{1-p/2} \leq CM^{p/2} \left[ \int_t^{t'} \bar{\Lambda}_s^h ds \right]^{1-p/2}. \end{aligned}$$

Jensen's inequality then bounds the expectation of the r.h.s. by  $CM^{p/2} \mathbb{E}^{1-p/2}[T \sup_{0 \leq s \leq t} \bar{\Lambda}_s^h]$ .

**Step 4. Gronwall argument.** Using the BDG inequality, and  $\eta_r^N \leq 1$ ,

$$\begin{aligned} \mathbb{E}[\mathbf{B}_t] &\leq C \mathbb{E} \left[ \left( \int_t^{t'} N^{-1} \sum_{i=1}^N p |\Delta_r^{h,i}|_\varepsilon^{p-2} \Delta_r^{h,i} \cdot \sigma_r^{h,i}|^2 ds \right)^{1/2} \right] \\ &\leq C(t' - t)^{1/2} \mathbb{E} \left[ h^p + \sup_{0 \leq s \leq t} \bar{\Delta}_s^h + \sup_{0 \leq s \leq t} \bar{\Lambda}_s^h \right], \end{aligned}$$



where we reasoned as in (3.28) for the second inequality. Noticing that

$$\forall 0 \leq t \leq s \leq t' \leq T, \quad \sup_{0 \leq r \leq s} \bar{\Delta}_r^h \leq \sup_{0 \leq r \leq t} \bar{\Delta}_r^h + \sup_{t \leq r \leq s} \bar{\Delta}_r^h, \quad (3.30)$$

we set  $f^h(t, t') := \mathbb{E}[\sup_{t \leq s \leq t'} \bar{\Delta}_s^h]$ , and we gather all the pieces from **Step 3.** to get

$$\begin{aligned} e^{-2^p L_\sigma^p(2M+T)} f^h(t, t') &\leq C\varepsilon^p + C f^h(0, t) + C[1 + \varepsilon^{p-2}][(1 + M)h^p + \mathbb{E}[\sup_{0 \leq s \leq T} \bar{\Lambda}_s^h]] \\ &\quad + CM^{p/2} \mathbb{E}^{1-p/2}[\sup_{0 \leq s \leq T} \bar{\Lambda}_s^h] + (t' - t)^{1/2} f^h(t, t') + \int_t^{t'} f^h(t, s) ds, \end{aligned}$$

where we used that  $\eta_t^N \geq e^{-2^p L_\sigma^p(2M+T)}$  a.s., and  $C$  is independent of  $(t, t')$ . Using (3.30) and Lemma 3.A.1, we get

$$f^h(0, T) \leq C_M \varepsilon^p + C_M [1 + \varepsilon^{p-2}][h^p + \mathbb{E}[\sup_{0 \leq s \leq T} \bar{\Lambda}_s^h]] + C_M \mathbb{E}^{1-p/2}[\sup_{0 \leq s \leq T} \bar{\Lambda}_s^h],$$

for  $C_M > 0$  independent of  $(h, N)$ . Since  $\mathbb{E}[\sup_{0 \leq s \leq T} \bar{\Lambda}_s^h]$  is bounded by  $C|h \log h|^{p/2}$  from Lemma 3.4.2-(ii), taking  $\varepsilon = |h \log h|^{1/2}$  concludes.  $\square$

*End of the proof of Theorem 3.2.11.* Reasoning as in **Step 2.** in the proof of Proposition 3.4.3, Proposition 3.4.4 gives that for every  $M > 0$ ,  $C_M > 0$  exists such that

$$\forall h \in (0, 1], \forall N \geq 1, \quad |\mathcal{F}^{h,N} - \mathcal{F}^{0,N}| \leq 2\|F\|_\infty M^{-1} + C_M |h \log h|^{\frac{p(1-p/2)}{2}}.$$

As a consequence,

$$\sup_{N \geq 1} |\mathcal{F}^{h,N} - \mathcal{F}^{0,N}| \xrightarrow{h \rightarrow 0} 0.$$

Inverting the limits, the result is then a direct adaptation of the proof of Theorem 3.2.8.  $\square$

### 3.4.3 Uniform fluctuation estimates

We here extend Proposition 3.2.7 to the continuous setting  $h = 0$ . Throughout this section, we assume that (A.4) holds. The structure of the following proofs is very close to Section 3.3.4, studying the  $h \rightarrow 0$  limit instead of  $M \rightarrow +\infty$ . We essentially emphasise the main differences and the needed adaptations.

Let  $(B^i)_{i \geq 1}$  be a countable sequence in  $\mathcal{C}^d$  of i.i.d. Brownian motions. Let  $\vec{X}^{h,N}$  denote the related particle system (3.5), starting at 0. We will write  $\vec{X}^{0,N}$  for the continuous particle system, where (3.5) now involves stochastic integrals. For each  $1 \leq i \leq N$ , let us introduce the McKean-Vlasov process  $\bar{X}^{h,i}$  solution of (3.4) with the driving noise  $B^i$ , starting at 0. We write  $\bar{X}^{0,i}$  for the related McKean-Vlasov process. Let us introduce the difference process  $\delta^{h,i} := X^{h,i,N} - \bar{X}^{h,i}$ . We first need a uniform in  $h$  mean-field limit.

**Lemma 3.4.5** (Uniform limit). *There exists  $C > 0$  such that for every  $N \geq 1$ ,  $h \in (0, 1]$ , and  $1 \leq i \leq N$ ,*

$$(i) \quad \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,N}|^2] \leq C.$$

$$(ii) \quad \mathbb{E}[\sup_{0 \leq t \leq T} |\delta_t^{h,i,N}|^2] \leq CN^{-1}.$$

$$(iii) \quad \mathbb{E}[\sup_{0 \leq t \leq T} |\delta_t^{0,i}|^2] \leq CN^{-1}.$$

$$(iv) \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,N} - X_t^{0,i,N}|^2] \leq C|h \log h|.$$

$$(v) \mathbb{E}[\sup_{0 \leq t \leq T} |\bar{X}_t^{h,i} - \bar{X}_t^{0,i}|^2] \leq C|h \log h|.$$

We then strengthen these results for controlling the fluctuations.

**Proposition 3.4.6.** *There exists  $C > 0$  such that for every  $M > 0$ ,*

$$\sup_{N \geq 1} \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\delta_t^{h,M,i,N} - \delta_t^{h,\infty,i}| \right] \leq C|h \log h|^{1/2}.$$

With these results at hand, Theorem 3.2.10 will easily follow. As previously, we rely on the classical coupling method from [Sznitman, 1991, Theorem 1.4]. To alleviate notations, we drop the exponents  $N$  in the remainder of this section (although the dependence on  $N$  is crucial). We will repeatedly use the technical estimates proved in Appendix 3.A. In the following,  $C$  is a generic constant that may change from line to line, but staying independent of  $(h, N)$ .

*Proof of Lemma 3.4.5.* (i) is obtained by noticing that the proof of Lemma 3.4.2-(i) still works with  $p = 2$ .

(ii) We adapt the classical coupling argument, as in e.g. Bernou and Liu [2022]. By definition of  $X^{h,i}$  and  $\bar{X}^{h,i}$ , for  $0 \leq t \leq T$ ,

$$\begin{aligned} \delta_t^{h,i} &= \int_0^t [b_{s_h}(X^{h,i}, \pi(\bar{X}^h)) - b_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))] ds \\ &\quad + \int_0^t [\sigma_{s_h}(X^{h,i}, \pi(\bar{X}^h)) - \sigma_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))] dB_s^i. \end{aligned}$$

We then take the square, supremum in time, and expectations. Using (A.2), reasoning as for proving Lemma 3.3.8-(ii) yields

$$\mathbb{E} |b_{s_h}(X^{h,i}, \pi(\bar{X}^h)) - b_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))|^2 \leq C \mathbb{E} \left[ \sup_{0 \leq r \leq s_h} |X_r^{h,i} - \bar{X}_r^{h,i}|^2 \right] + CN^{-1}.$$

Using the BDG inequality,

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s [\sigma_{r_h}(X^{h,i}, \pi(\bar{X}^h)) - \sigma_{r_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))] dB_r^i \right|^2 \\ \leq C \int_0^t |\sigma_{s_h}(X^{h,i}, \pi(\bar{X}^h)) - \sigma_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))|^2 ds, \end{aligned}$$

and we similarly handle the r.h.s. We then gather everything to get that

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\delta_s^{h,i}|^2 \right] \leq CN^{-1} + C \int_0^t \mathbb{E} \left[ \sup_{0 \leq r \leq s} |\delta_r^{h,i}|^2 \right] ds.$$

The conclusion follows from the Gronwall Lemma.

(iii) If  $h = 0$  in the above proof of (ii), we notice that all the computations remain valid when replacing  $s_h$  by  $s$ . This yields the result.

(iv) is obtained by noticing that the proof of Lemma 3.4.2-(iv) still works with  $p = 2$ .

(v) By symmetry, we recall that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,i,N} - X_t^{0,i,N}|^2]$  does not depend on  $i$ . The result follows by taking the  $N \rightarrow +\infty$  limit in the uniform estimate (iv), using (ii)-(iii).  $\square$

*Proof of Proposition 3.4.6.* For every  $0 \leq t \leq t' \leq T$ , we have the decomposition

$$\delta_{t'}^{h,i} - \delta_t^{0,i} = \delta_t^{h,i} - \delta_t^{0,i} + \int_t^{t'} b_s^{h,i} ds + \int_t^{t'} \sigma_s^{h,i} dB_s^i,$$

where

$$\begin{aligned} b_s^{h,i} &:= b_{s_h}(X^{h,i}, \pi(\vec{X}^h)) - b_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i})) - [b_s(X^{0,i}, \pi(\vec{X}^\infty)) - b_s(\bar{X}^{\infty,i}, \mathcal{L}(\bar{X}^{0,i}))], \\ \sigma_s^{h,i} &:= \sigma_{s_h}(X^{h,i}, \pi(\vec{X}^h)) - \sigma_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i})) - [\sigma_s(X^{0,i}, \pi(\vec{X}^\infty)) - \sigma_s(\bar{X}^{\infty,i}, \mathcal{L}(\bar{X}^{0,i}))]. \end{aligned}$$

We then take absolute values, supremum in times and expectations. For the drift, we split

$$\begin{aligned} b_s^{h,i} &= b_{s_h}(X^{h,i}, \pi(\vec{X}^h)) - b_{s_h}(\bar{X}^{h,i}, \pi(\vec{X}^h)) - [b_s(X^{h,i}, \pi(\vec{X}^h)) - b_s(\bar{X}^{h,i}, \pi(\vec{X}^h))] \\ &\quad + b_{s_h}(\bar{X}^{h,i}, \pi(\vec{X}^h)) - b_{s_h}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i})) - [b_s(\bar{X}^{h,i}, \pi(\vec{X}^h)) - b_s(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))] \\ &\quad + b_s(X^{h,i}, \pi(\vec{X}^h)) - b_s(\bar{X}^{h,i}, \pi(\vec{X}^h)) - [b_s(X^{i,0}, \pi(\vec{X}_s^0)) - b_s(\bar{X}^{i,0}, \pi(\vec{X}_s^0))] \\ &\quad + b_s(\bar{X}^{h,i}, \pi(\vec{X}^h)) - b_s(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i})) - [b_s(\bar{X}_s^{i,0}, \pi(\vec{X}_s^0)) - b_s(\bar{X}^{i,0}, \mathcal{L}(\bar{X}^{i,0}))]. \end{aligned}$$

For each line of  $b_s^{h,i}$ , we now reason as we did for proving Proposition 3.3.9. For the first line, we use the differentiability assumption (A.4) to write

$$b_{s_h}(X^{h,i}, \pi(\vec{X}^h)) - b_s(X^{h,i}, \pi(\vec{X}^h)) = \int_0^1 (s_h - s) \partial_s b_{(1-r)s_h + rs}(X^{h,i}, \pi(\vec{X}^h)) dr,$$

and we similarly rewrite  $b_{s_h}(\bar{X}^{h,i}, \pi(\vec{X}^h)) - b_s(\bar{X}^{h,i}, \pi(\vec{X}^h))$ . Using the  $\mathcal{C}^{1,1}$  assumption (A.4) on  $b$  and Lemma 3.4.5, this yields

$$\sup_{0 \leq s \leq T} \mathbb{E}[|b_{s_h}(X^{h,i}, \pi(\vec{X}^h)) - b_{s_h}(\bar{X}^{h,i}, \pi(\vec{X}^h)) - [b_s(X^{h,i}, \pi(\vec{X}^h)) - b_s(\bar{X}^{h,i}, \pi(\vec{X}^h))]|] \leq ChN^{-1/2}.$$

We similarly handle the second line of  $b_s^{h,i}$ , using Lemma 3.A.3 to get that

$$\max_{0 \leq r \leq 1} \mathbb{E}[|\partial_s b_{(1-r)s_h + rs}(\bar{X}^{h,i}, \pi(\vec{X}^h)) - \partial_s b_{(1-r)s_h + rs}(\bar{X}^{h,i}, \mathcal{L}(\bar{X}^{h,i}))|] \leq CN^{-1/2}.$$

To handle the third line in  $b_s^{h,i}$ , we use Lemma 3.A.2 with  $(X^i, Y^i) = (X^{h,i}, X^{0,i})$  and  $(\bar{X}^i, \bar{Y}^i) = (\bar{X}_s^{h,i}, \bar{X}_s^{0,i})$ , together with Lemma 3.4.5. To handle the last line, we use Lemma 3.A.3 with the i.i.d. processes  $(\bar{X}^i, \bar{Y}^i) = (\bar{X}^{h,i}, \bar{X}^{0,i})$ ,  $1 \leq i \leq N$ . Gathering everything yields

$$\sup_{t \leq s \leq t'} \mathbb{E}[|b_s^{h,i}|] \leq CN^{-1/2} |h \log h|^{1/2} + C \mathbb{E}[\sup_{0 \leq s \leq t'} |\delta_s^{h,i}|].$$

We use a similar splitting for  $\sigma_s^{h,i}$  and we reason as in the proof of Proposition 3.3.9 to obtain

$$\mathbb{E} \sup_{t \leq s \leq t'} \left| \int_t^s \sigma_r^{h,i} dB_r^i \right| \leq CN^{-1/2} |h \log h|^{1/2} + C(t' - t)^{1/2} \mathbb{E}[\sup_{0 \leq s \leq t'} |\delta_s^{h,i}|].$$

Setting  $f^h(t, t') := \sqrt{N} \mathbb{E}[\sup_{t \leq s \leq t'} |\delta_s^{h,i}|]$ , we gather terms as previously to get that

$$\forall 0 \leq t \leq t' \leq T, \quad f^h(t, t') \leq Cf(0, t) + C|h \log h|^{1/2} + C(t' - t)^{1/2} f^h(t, t') + C \int_t^{t'} f^h(t, s) ds,$$

and we conclude using Lemma 3.A.1, as in the proof of Proposition 3.3.9.  $\square$

End of the proof of Theorem 3.2.10. Let us fix  $\varphi$  in  $\mathcal{C}^{1,1}(\mathcal{C}^d, \mathbb{R})$ , before defining

$$\delta_\varphi^h := N^{-1} \sum_{i=1}^N [\varphi(X^{h,i}) - \mathbb{E}[\varphi(\bar{X}^{h,i})]].$$

We then decompose

$$\begin{aligned} \delta_\varphi^h - \delta_\varphi^0 &= N^{-1} \sum_{i=1}^N [\varphi(X^{h,i}) - \varphi(\bar{X}^{h,i})] - N^{-1} \sum_{i=1}^N [\varphi(X^{0,i}) - \varphi(\bar{X}^{0,i})] \\ &\quad + N^{-1} \sum_{i=1}^N [\varphi(\bar{X}^{h,i}) - \mathbb{E}[\varphi(\bar{X}^{h,i})]] - N^{-1} \sum_{i=1}^N [\varphi(\bar{X}^{0,i}) - \mathbb{E}[\varphi(\bar{X}^{0,i})]]. \end{aligned}$$

As previously, we control the first line using Lemma 3.A.2, Lemma 3.4.6 and Proposition 3.4.6. Similarly, we handle the second line using Lemma 3.A.3 and Lemma 3.4.6. At the end of the day, we obtain that

$$\sup_{N \geq 1} \sqrt{N} \mathbb{E}[|\delta_\varphi^h - \delta_\varphi^0|] \xrightarrow{h \rightarrow 0} 0.$$

As a consequence, the convergence given by Proposition 3.2.7 holds uniformly in  $h$ . Let  $(B_t)_{0 \leq t \leq T}$ ,  $(\tilde{B}_t)_{0 \leq t \leq T}$  be independent Brownian motions in  $\mathbb{R}^d$ . Let  $X^h$  denote the strong solution of the discretised McKean-Vlasov (3.4) with driving noise  $B$ . Similarly, let  $\tilde{X}^h$  denote the strong solution of (3.4) with driving noise  $\tilde{B}$ . Let us then introduce the solution  $\delta X^h$  of the McKean-Vlasov SDE with common noise given by (3.7),

$$\begin{aligned} d\delta X_{t_h}^h &= [D_x \sigma_{t_h}(X^h, \mathcal{L}(X^h)) \cdot \delta X^h + \delta_P \sigma_{t_h}(X^h, \mathcal{L}(X^h), \tilde{X}^h)] dB_t \\ &\quad + [D_x b_{t_h}(X^h, \mathcal{L}(X^h)) \cdot \delta X^h + \delta_P b_{t_h}(X^h, \mathcal{L}(X^h), \tilde{X}^h)] dt \\ &\quad + \mathbb{E}[D_y \delta_P b_{t_h}(X^h, \mathcal{L}(X^h), X^h) \cdot \delta X^h | (\tilde{B}_s)_{0 \leq s \leq t_h}] dt \\ &\quad + \mathbb{E}[D_y \delta_P \sigma_{t_h}(X^h, \mathcal{L}(X^h), X^h) \cdot \delta X^h | (\tilde{B}_s)_{0 \leq s \leq t_h}] dB_t, \quad \delta X_0^h = 0. \end{aligned}$$

As previously, synchronous coupling and a Gronwall argument give that

$$\mathbb{E}[\sup_{0 \leq t \leq T} |\delta X_t^h - \delta X_t|] \xrightarrow{h \rightarrow 0} 0,$$

where  $\delta X$  is the solution of (3.8) with the same  $(B, \tilde{B})$ . From this, we conclude that  $\sigma_{h,\varphi}^2$  given by Proposition 3.2.7 converges to  $\sigma_\varphi^2$  as required by Theorem 3.2.10.  $\square$

## Appendix

### 3.A Some useful estimates

First, let us state a variation on the Gronwall lemma, which is used throughout the article.

**Lemma 3.A.1.** *Let  $f : [0, T] \times [0, T] \rightarrow \mathbb{R}$ ,  $g : [0, T] \rightarrow \mathbb{R}$  be continuous functions, and  $\alpha, C > 0$  be such that*

$$\forall 0 \leq t \leq t' \leq T, \quad f(t, t') \leq g(t) + Cf(0, t) + C(t' - t)^\alpha f(t, t') + C \int_t^{t'} f(t, s) ds,$$

together with

$$\forall 0 \leq t \leq s \leq t', \quad f(0, s) \leq f(0, t) + f(t, s).$$

Then, there exists  $C' > 0$  that only depends on  $(\alpha, C, T)$ , such that  $\sup_{0 \leq t \leq T} f(0, t) \leq C'[f(0, 0) + \sup_{0 \leq t \leq T} g(t)]$ .

*Proof.* Let us fix  $t_0 \in (0, T]$  such that  $1 - Ct_0^\alpha \geq 1/2$ . For any  $k \geq 0$  with  $kt_0 \leq T$ , we get that for every  $t \in [kt_0, (k+1)t_0 \wedge T]$ ,

$$f(kt_0, t) \leq 2g(t) + 2Cf(0, kt_0) + 2C \int_{kt_0}^t f(kt_0, s) ds.$$

The usual Gronwall lemma then provides the bound

$$\sup_{kt_0 \leq t \leq (k+1)t_0 \wedge T} f(kt_0, t) \leq 2e^{2Ct_0} [Cf(0, kt_0) + \sup_{0 \leq t \leq T} g(t)].$$

Since

$$f(0, (k+1)t_0) \leq f(0, kt_0) + f(kt_0, (k+1)t_0),$$

the desired bound on  $f(0, T)$  follows by (finite) induction.  $\square$

Let now  $(E, |\cdot|)$  be a Banach space. For  $k \geq 1$ , let  $F : E \times E \rightarrow \mathbb{R}^k$  be a  $C^{1,1}$  function. For  $P \in \mathcal{P}_1(E)$ , we define

$$\forall x \in E, \quad F(x, P) := \int_E F(x, y) dP(y),$$

with a slight abuse of notation.

**Lemma 3.A.2.** *Let  $(X^i)_{1 \leq i \leq N}$ ,  $(\bar{X}^i)_{1 \leq i \leq N}$ ,  $(Y^i)_{1 \leq i \leq N}$ ,  $(\bar{Y}^i)_{1 \leq i \leq N}$  be triangular arrays of  $E$ -valued random variables. Then, for every  $1 \leq i \leq N$ ,*

$$\begin{aligned} & |F(X^i, \pi(\vec{X}^N)) - F(\bar{X}^i, \pi(\vec{\bar{X}}^N)) - [F(Y^i, \pi(\vec{Y}^N)) - F(\bar{Y}^i, \pi(\vec{\bar{Y}}^N))]| \\ & \leq \frac{1}{N} \sum_{j=1}^N \|F\|_{\text{Lip}} [|X^i - \bar{X}^i - [Y^i - \bar{Y}^i]| + |X^j - \bar{X}^j - [Y^j - \bar{Y}^j]|] \\ & \quad + \|DF\|_{\text{Lip}} [|Y^i - \bar{Y}^i| + |Y^j - \bar{Y}^j|] [|X^i - Y^i| + |X^j - Y^j| + |\bar{X}^i - \bar{Y}^i| + |\bar{X}^j - \bar{Y}^j|] \end{aligned}$$

*Proof.* Since  $F$  is differentiable,

$$\begin{aligned} F(X^i, \pi(\vec{X}^N)) - F(\bar{X}^i, \pi(\vec{\bar{X}}^N)) &= \frac{1}{N} \sum_{j=1}^N \int_0^1 (X^i - \bar{X}^i) \cdot \partial_1 F((1-r)(\bar{X}^i, \bar{X}^j) + r(X^i, X^j)) dr \\ & \quad + \frac{1}{N} \sum_{j=1}^N \int_0^1 (X^j - \bar{X}^j) \cdot \partial_2 F((1-r)(\bar{X}^i, \bar{X}^j) + r(X^i, X^j)) dr, \end{aligned}$$

and the same decomposition holds writing  $Y$  instead of  $X$ . We then subtract the decomposition for  $Y$  to the above one. The difference of the first terms reads

$$\begin{aligned} & \frac{1}{N} \sum_{j=1}^N \int_0^1 [X^i - \bar{X}^i - (Y^i - \bar{Y}^i)] \cdot \partial_1 F((1-r)(\bar{X}^i, \bar{X}^j) + r(X^i, X^j)) \\ & + (Y^i - \bar{Y}^i) \cdot [\partial_1 F((1-r)(\bar{X}^i, \bar{X}^j) + r(X^i, X^j)) - \partial_1 F((1-r)(\bar{Y}^i, \bar{Y}^j) + r(Y^i, Y^j))] dr. \end{aligned}$$

We similarly handle the difference of the second terms. The result follows using the assumed bounds on  $DF$ .  $\square$

**Lemma 3.A.3.** *Let  $(\bar{X}^i, \bar{Y}^i)_{1 \leq i \leq N}$  be an i.i.d. sequence of  $E \times E$ -valued variables with marginal laws  $(\bar{\pi}_X, \bar{\pi}_Y)$ . There exists  $C > 0$  that only depends on  $\|F\|_{\text{Lip}}$  such that*

$$\max_{1 \leq i \leq N} \mathbb{E} |F(\bar{X}^i, \pi(\bar{X}^{\vec{N}})) - F(\bar{X}^i, \bar{\pi}_X) - [F(\bar{Y}^i, \pi(\bar{Y}^{\vec{N}})) - F(\bar{Y}^i, \bar{\pi}_Y)]|^2 \leq CN^{-1} \mathbb{E}[|\bar{X}^1 - \bar{Y}^1|^2].$$

*Proof.* By definition,

$$\begin{aligned} & F(\bar{X}^i, \pi(\bar{X}^{\vec{N}})) - F(\bar{X}^i, \bar{\pi}_X) - [F(\bar{Y}^i, \pi(\bar{Y}^{\vec{N}})) - F(\bar{Y}^i, \bar{\pi}_Y)] \\ &= \frac{1}{N} \sum_{j=1}^N F(\bar{X}^i, \bar{X}^j) - F(\bar{Y}^i, \bar{Y}^j) - [F(\bar{X}^i, \bar{\pi}_X) - F(\bar{Y}^i, \bar{\pi}_Y)] =: \frac{1}{N} \sum_{j=1}^N F_{i,j}^N. \end{aligned}$$

Following the classical argument in [Sznitman, 1991, Theorem 1.4], we take the expectation of the square of this sum. By independence of the  $(X^i, Y^i)$ , we have for  $k \neq l$ ,

$$\mathbb{E}[F_{i,k}^N F_{i,l}^N] = \mathbb{E}[\mathbb{E}[F_{i,k}^N F_{i,l}^N | (X^i, Y^i)]] = \mathbb{E}[\mathbb{E}[F_{i,k}^N | (X^i, Y^i)] \mathbb{E}[F_{i,l}^N | (X^i, Y^i)]].$$

By definition of  $(\bar{\pi}_X, \bar{\pi}_Y)$ ,  $\mathbb{E}[F_{i,k}^N | (X^i, Y^i)] = 0$  as soon as  $k \neq i$ . Since  $(k, l) \neq (i, i)$  because  $k \neq l$ , this implies  $\mathbb{E}[F_{i,k}^N F_{i,l}^N] = 0$ . Thus, we only keep the diagonal terms:

$$\mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N F_{i,j}^N \right|^2 = \frac{1}{N^2} \sum_{j=1}^N \mathbb{E}[|F_{i,j}^N|^2] \leq \frac{C'}{N} \{ \mathbb{E}[|\bar{X}^1 - \bar{Y}^1|^2] + [\mathbb{E}[|\bar{X}^1 - \bar{Y}^1|]]^2 \},$$

for some  $C' > 0$ . Using Jensen's inequality, the result follows with  $C := 2C'$ .  $\square$

## 3.B Some results on rate functions

### 3.B.1 Representation formulae

The following result is a slight variation of [Fischer, 2014, Theorem 5.2]. We recall that the notion of reference system  $\Sigma = (\Omega, (\mathcal{F}_t)_{t \leq 0 \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$  is defined in Section 3.1.2. For such a  $\Sigma$ ,  $h \in [0, 1]$  and any square-integrable progressively measurable process  $u = (u_t)_{0 \leq t \leq T}$  on  $\Sigma$ , we say that  $X^{h,u} := (X_t^{h,u})_{0 \leq t \leq T}$  is a solution of the McKean-Vlasov SDE

$$dX_s^{h,u} = b_{s_h}(X^{h,u}, \mathcal{L}(X^{h,u})) ds + \sigma_{s_h}(X^{h,u}, \mathcal{L}(X^{h,u})) u_s ds + \sigma_{s_h}(X^{h,u}, \mathcal{L}(X^{h,u})) dB_s, \quad (3.31)$$

if  $X^{h,u}$  is  $(\mathcal{F}_t)_{t \leq 0 \leq T}$ -adapted and the integrated version of (3.31) holds  $\mathbb{P}$ -a.s. Let  $W$  denote the Wiener measure on the canonical space  $\Omega = \mathcal{C}^d$ , with initial law  $\mathcal{L}(X_0^1)$ , and let  $P \in \mathcal{P}_p(\mathcal{C}^d)$ . From (3.6),  $\Gamma_h(P)$  is the law of the strong solution under  $W$  to

$$dY_t^{h,P} = b_{t_h}(Y^{h,P}, P) dt + \sigma_{t_h}(Y^{h,P}, P) d\omega_t, \quad Y_0^{h,P} = \omega_0,$$

where  $(\omega_t)_{0 \leq t \leq T}$  is the canonical process. Indeed, strong existence and pathwise-uniqueness for this SDE are standard under (A.1)-(A.3).

**Lemma 3.B.1.** *Under (A.1)-(A.3), for  $h \in [0, 1]$  and  $P \in \mathcal{P}_p(\mathcal{C}^d)$ ,*

$$H(P|\Gamma_h(P)) = \inf_{\Sigma} \inf_{\mathcal{L}(X^{h,u})=P} H(\mathcal{L}(X_0^{h,u})|\mathcal{L}(X_0^1)) + \mathbb{E} \int_0^T \frac{1}{2} |u_t|^2 dt.$$

where we minimise over  $(X^{h,u}, u)$  satisfying (3.31) in the reference system  $\Sigma$ , with the convention that an infimum over an empty set equals  $+\infty$ .

*Proof.* We notice that  $\Gamma_h(P) = Y_{\#}^{h,P}W$  is the push-forward of  $W$  by  $Y^{h,P}$ . Following the proof of [Fischer, 2014, Theorem 5.2], we use the contraction property of entropy [Fischer, 2014, Lemma A.1] to write that

$$H(P|\Gamma_h(P)) = \inf_{\substack{R \in \mathcal{P}(\mathcal{C}^d) \\ P = Y_{\#}^{h,P}R}} H(R|W).$$

For  $R \in \mathcal{P}(\mathcal{C}^d)$  with finite  $H(R|W)$ , we now claim that

$$H(R|W) = H(R_0|\mathcal{L}(X_0^1)) + \inf_{\Sigma} \inf_{\mathcal{L}(Z^u)=R} \mathbb{E} \int_0^T \frac{1}{2} |u_t|^2 dt, \quad (3.32)$$

where we minimise over processes  $Z^u = (Z_t^u)_{0 \leq t \leq T}$  satisfying  $\mathcal{L}(Z^u) = R$ , and  $\mathbb{P}$ -a.s.

$$\forall t \in [0, T], \quad Z_t^u = Z_0^u + \int_0^t u_s ds + B_t.$$

Equation (3.32) can be obtained as a consequence of the Girsanov transform [Léonard, 2012, Theorem 2.3], or by adding an initial condition in [Fischer, 2014, Lemma B.1] ( $Z^u$  is no more a non-linear process, contrary to  $X^{h,u}$ ). As a consequence,

$$H(P|\Gamma_h(P)) = \inf_{\Sigma} \inf_{P = Y_{\#}^{h,P}R} \inf_{\mathcal{L}(Z^u)=R} H(R_0|\mathcal{L}(X_0^1)) + \inf_{\mathcal{L}(Z^u)=R} \mathbb{E} \int_0^T \frac{1}{2} |u_t|^2 dt.$$

This yields the desired result, by noticing that  $Y^{h,P}(Z^u)$  is a strong solution of (3.31) if  $P = Y_{\#}^{h,P}R$ , and that uniqueness in law holds for the solution of

$$dX_s^{h,u} = b_{s_h}(X^{h,u}, P)ds + \sigma_{s_h}(X^{h,u}, P)u_s ds + \sigma_{s_h}(X^{h,u}, P)dB_s,$$

using the Girsanov transform as for (3.22). This unique law is precisely  $Y_{\#}^{h,P}\mathcal{L}(Z^u)$ .  $\square$

We now prove a useful tightness result. The next result shows pre-compactness for sequences of processes of type (3.31), for which strong existence may not always hold.

**Lemma 3.B.2.** *For  $p \in [1, 2)$  and every  $k \geq 1$ , on a reference system  $\Sigma_k$ , let us assume that*

$$dX_t^k = b_{t_{h_k}}(X^k, \mathcal{L}(X^k))dt + \sigma_{t_{h_k}}(X^k, \mathcal{L}(X^k))u_t^k dt + \sigma_{t_{h_k}}(X_t^k, \mathcal{L}(X^k))dB_t^k,$$

has a strong solution with path-law  $P_k$ ,  $(u_t^k)_{0 \leq t \leq T}$  being a control. We assume (A.1)-(A.3) and

$$\sup_{k \geq 1} H(\mathcal{L}(X_0^k)|\mathcal{L}(X_0^1)) + \mathbb{E} \int_0^T \frac{1}{2} |u_t^k|^2 dt < +\infty.$$

Then, for any sequence  $(h_k)_{k \geq 1}$  in  $[0, 1]$ ,  $(P_k)_{k \geq 1}$  is pre-compact in  $\mathcal{P}_p(\mathcal{C}^d)$ .

*Proof.* As in Lemma 3.4.2-(i), a Gronwall argument gives that

$$\sup_{0 \leq s \leq T} |X_s^k|^p \leq C \left[ 1 + |X_0^k|^p + \int_0^T |u_s^k|^p ds + \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_{s_{h_k}}(X^k, \mathcal{L}(X^k)) dB_s^k \right|^p \right], \quad (3.33)$$

for  $C > 0$  independent of  $k$ . From the BDG inequality,

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_{s_{h_k}}(X^k, \mathcal{L}(X^k)) dB_s^k \right|^p \leq \mathbb{E} \left[ \left( \int_0^T |\sigma_{s_{h_k}}(X^k, \mathcal{L}(X^k))|^2 ds \right)^{p/2} \right] \leq T^{p/2} M_{\sigma}. \quad (3.34)$$

Taking expectations in (3.33), we get the moment bound

$$\sup_{k \geq 1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^k|^p \right] < +\infty.$$

For  $0 \leq s \leq t \leq T$ ,

$$X_t^k - X_s^k = \int_s^t b_{r_{h_k}}(X^k, \mathcal{L}(X^k)) dr + \int_s^t \sigma_{r_{h_k}}(X^k, \mathcal{L}(X^k)) u_r^k dr + \int_s^t \sigma_{r_{h_k}}(X^k, \mathcal{L}(X^k)) dB_r^k.$$

We use the sub-linear growth of  $b$  and the moment bound to control the first term. The last term can be controlled using (3.34) again. For the middle term, we use the Cauchy-Schwarz inequality to write that

$$\mathbb{E} \left| \int_s^t \sigma_{r_{h_k}}(X^k, \mathcal{L}(X^k)) u_r^k dr \right| \leq (t-s)^{1/2} M_\sigma \mathbb{E} \left[ \int_0^T |u_r^k|^2 dr \right].$$

At the end of the day, we deduce that

$$\sup_{k \geq 1} \mathbb{E} \left[ \sup_{|s-t| \leq \delta} |X_t^k - X_s^k| \right] \xrightarrow{\delta \rightarrow 0} 0.$$

Together with the moment bound, this shows that  $(P_k)_{k \geq 1}$  is tight [Billingsley, 2013, Chapter 2, Theorem 7.3], and thus relatively compact for the weak convergence of measures. To obtain pre-compactness in  $\mathcal{P}_p(\mathcal{C}^d)$ , it is now sufficient to show uniform integrability for  $(\sup_{0 \leq t \leq T} |X_t^k|^p)_{k \geq 1}$  [Villani et al., 2009, Definition 6.8-(iii)]. To do so, we show that each term on the r.h.s. of (3.33) is uniformly integrable. The dual representation (3.23) for relative entropy reads

$$H(\mathcal{L}(X_0^k) | \mathcal{L}(X_0^1)) = \sup_{\substack{\phi \text{ measurable} \\ \mathbb{E}[\phi(X_0^1)] < +\infty}} \mathbb{E}[\phi(X_0^k)] - \log \mathbb{E}[e^{\phi(X_0^1)}],$$

Applying this to  $\phi(x) = \alpha |x|^p \mathbb{1}_{|x| \geq M}$  for every  $\alpha, M > 0$ , uniform integrability for  $(|X_0^k|^p)_{k \geq 1}$  follows from (A.3) and the bound on  $(H(\mathcal{L}(X_0^k) | \mathcal{L}(X_0^1)))_{k \geq 1}$ . For the uniform integrability of

$$\int_0^T |u_s^k|^p ds \quad \text{and} \quad \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_{s_{h_k}}(X^k, \mathcal{L}(X^k)) dB_s^k \right|^p,$$

we use that  $p < 2$  and that

$$\int_0^T |u_s^k|^2 ds \quad \text{and} \quad \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_{s_{h_k}}(X^k, \mathcal{L}(X^k)) dB_s^k \right|^2$$

are bounded uniformly in  $k$ , using Ito's isometry and the bound on  $\sigma$  for the second term. This concludes the proof.  $\square$

**Remark 3.B.3.** *We notice that the same proof actually works if the coefficients  $b = b^k$  and  $\sigma = \sigma^k$  depend on  $k$ , provided the linear growth of  $b^k$  and the bound on  $\sigma^k$  are uniform in  $k$ .*

Combined with Lemma 3.B.1, Lemma 3.B.2 then allows us to extend the compactness result from [Fischer, 2014, Remark 5.2] to the topology on  $\mathcal{P}_p(\mathcal{C}^d)$ .

**Corollary 3.B.3.1.** *Under (A.1)-(A.3), for every  $p \in [1, 2)$  and  $h \in [0, 1]$ ,  $P \mapsto H(P | \Gamma_h(P))$  has compact level sets in  $\mathcal{P}_p(\mathcal{C}^d)$ ; it is a good rate function.*



*Proof.* Let  $(P_k)_{k \geq 1}$  be a sequence in  $\mathcal{P}_p(\mathcal{C}^d)$  such that  $(H(P|\Gamma_h(P)))_{k \geq 1}$  is bounded. From Lemma 3.B.1, for every  $k \geq 1$ , there exist a reference system  $\Sigma_k$ , a progressively measurable process  $u^k$  and a strong solution  $X^{h,k}$  on  $\Sigma_k$  with path-law  $P_k$  for

$$dX_t^{h,k} = b_{t_h}(X^{h,k}, \mathcal{L}(X^{h,k}))dt + \sigma_{t_h}(X^{h,k}, \mathcal{L}(X^{h,k}))u_t^k dt + \sigma_{t_h}(X_t^{h,k}, \mathcal{L}(X^{h,k}))dB_t^k,$$

satisfying

$$\sup_{k \geq 1} H(\mathcal{L}(X_0^k) | \mathcal{L}(X_0^1)) + \mathbb{E} \int_0^T \frac{1}{2} |u_t^k|^2 dt < +\infty.$$

Lemma 3.B.2 then shows that  $(P_k)_{k \geq 1}$  is pre-compact in  $\mathcal{P}_p(\mathcal{C}^d)$ .  $\square$

### 3.B.2 Gamma-convergence

As a consequence of Lemma 3.B.1, we get the following result, which can be seen as an extension of [Budhiraja and Dupuis, 2000, Theorem 3.6] to our setting.

**Corollary 3.B.3.2.** *For every  $h \in [0, 1]$  and every bounded measurable  $F : \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}$ ,*

$$\inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P|\Gamma_h(P)) + F(P) = \inf_{\Sigma, u, X_0^u} F(\mathcal{L}(X^{h,u})) + H(\mathcal{L}(X_0^u) | \mathcal{L}(X_0^1)) + \mathbb{E} \int_0^T \frac{1}{2} |u_t|^2 dt.$$

**Proposition 3.B.4.** *For  $p \in [1, 2)$  and every bounded continuous  $F : \mathcal{P}_p(\mathcal{C}^d) \rightarrow \mathbb{R}$ ,*

$$\inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P|\Gamma_h(P)) + F(P) \xrightarrow{h \rightarrow 0} \inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P|\Gamma_0(P)) + F(P).$$

*Proof.* The lower limit is quite direct, while the upper limit requires extra care to build approximate minimisers.

**Step 1. Lower limit.** For every  $h \in (0, 1]$ , Corollary 3.B.3.1 provides a minimiser  $P_h$  for

$$\bar{\mathcal{F}}_h := \inf_{P \in \mathcal{P}_p(\mathcal{C}^d)} H(P|\Gamma_h(P)) + F(P).$$

Since  $\bar{\mathcal{F}}_h$  is bounded by  $\|F\|_\infty$  independently of  $h$ , Lemma 3.B.2 provides pre-compactness in  $\mathcal{P}_p(\mathcal{C}^d)$  for  $(P_h)_{h \geq 0}$ . From any sub-sequence of  $(P_h)_{h \geq 0}$ , we can thus extract a sub-sequence which converges towards some  $\bar{P}$  in  $\mathcal{P}_p(\mathcal{C}^d)$ . Along this sub-sequence, it is a standard stability result in the spirit of Lemma 3.4.2-(iv) that  $\Gamma_h(P_h)$  weakly converges towards  $\Gamma_0(\bar{P})$ . Since  $F$  is continuous and  $(P, Q) \mapsto H(P|Q)$  is lower semi-continuous, this shows that  $\liminf_{h \rightarrow 0} \bar{\mathcal{F}}^h \geq \bar{\mathcal{F}}^0$ .

**Step 2. Discretised minimiser.** Reciprocally, let us fix  $P$  with finite  $H(P|\Gamma_0(P))$ . From the Girsanov transform [Léonard, 2012, Theorem 2.1], there exists a square-integrable process  $(u_t)_{0 \leq t \leq T}$  on the canonical space  $\mathcal{C}^d$  such that the canonical process  $(X_t^0)_{0 \leq t \leq T}$  satisfies

$$H(P|\Gamma_0(P)) = \mathbb{E}_P \int_0^T \frac{1}{2} |u_t|^2 dt, \quad dX_t^0 = b_t(X^0, P)dt + \sigma_t(X^0, P)u_t dt + \sigma_t(X^0, P)dB_t^P,$$

where  $(B_t^P)_{0 \leq t \leq T}$  is a Brownian motion under  $P$ . For  $M > 0$ , let  $u_t^M := u_t \mathbb{1}_{\mathcal{E}_t(u) \leq M}$ , where  $\mathcal{E}_t(u) := \int_0^t |u_s|^2 ds$ . From Lemma 3.3.2, for every  $h \in (0, 1]$ , the McKean-Vlasov equation

$$dX_t^{h,M} = b_{t_h}(X^{h,M}, \mathcal{L}(X^{h,M}))dt + \sigma_{t_h}(X^{h,M}, \mathcal{L}(X^{h,M}))u_t^M dt + \sigma_{t_h}(X^{h,M}, \mathcal{L}(X^{h,M}))dB_t^P,$$

with  $X_0^{h,M} = X^0$ , has a pathwise-unique strong solution under  $P$ . Lemma 3.B.2 shows that  $(\mathcal{L}_P(X^{h,M}))_{h \geq 0}$  is pre-compact in  $\mathcal{P}_p(\mathcal{C}^d)$ . From any sub-sequence of  $(\mathcal{L}_P(X^{h,M}))_{h \geq 0}$ , we can thus extract a sub-sequence that converges towards some  $P^M$  in  $\mathcal{P}_p(\mathcal{C}^d)$ . Up to reindexing, let us assume that  $\mathcal{L}_P(X^{h,M}) \rightarrow P^M$  as  $h \rightarrow 0$ . Using the Girsanov transform, the SDE

$$dX_t^M = b_t(X^M, P^M)dt + \sigma_t(X^M, P^M)u_t^M dt + \sigma_t(X^M, P^M)dB_t^P, \quad X_0^M = X^0$$

has a pathwise-unique strong solution under  $P$ . We now make the change of measure

$$\frac{dQ}{dP}(X^0) = \exp \left[ - \int_0^T u_t^M dB_t^P - \frac{1}{2} \int_0^T |u_t^M|^2 dt \right]. \quad (3.35)$$

The Girsanov theorem provides a Brownian motion  $(B_t^Q)_{0 \leq t \leq T}$  under  $Q$  such that

$$\begin{aligned} dX_t^{h,M} &= b_{t_h}(X^{h,M}, \mathcal{L}_P(X^{h,M}))dt + \sigma_{t_h}(X^{h,M}, \mathcal{L}_P(X^{h,M}))dB_t^Q, \quad X_0^{h,M} = X^0, \\ dX_t^M &= b_t(X^M, P^M)dt + \sigma_t(X^M, P^M)dB_t^Q, \quad X_0^M = X^0, \end{aligned}$$

these equations holding  $Q$ -a.s. As previously, a coupling argument yields the stability result  $\mathbb{E}_Q[\sup_{0 \leq t \leq T} |X_t^{h,M} - X_t^M|^p] \rightarrow 0$  as  $h \rightarrow 0$ . By (3.35) and dominated convergence, we deduce that  $\mathcal{L}_P(X^{h,M})$  weakly converges towards  $\mathcal{L}_P(X^M)$ , proving that  $P^M = \mathcal{L}_P(X^M)$ . Thus,  $X^M$  is the strong solution of a McKean-Vlasov equation under  $P$ .

**Step 3. Coupling in  $L^2$ .** Subtracting and integrating, (A.1) and the moment bound from Lemma 3.4.2-(i) yield

$$\sup_{0 \leq s \leq t} |X_s^0 - X_s^M| \leq C + C \int_0^t \sup_{0 \leq r \leq s} |X_r^0 - X_r^M| + |u_s^M| ds + C \sup_{0 \leq s \leq t} \left| \int_0^s \sigma_r^M dB_r^P \right|,$$

for some  $\sigma^M$  with  $|\sigma_r^M| \leq 2M_\sigma$  and some  $C > 0$  independent of  $h$  that may change from line to line. Since this holds for every  $t \in [0, T]$ , Gronwall's lemma yields

$$\sup_{0 \leq t \leq T} |X_t^0 - X_t^M| \leq C + C \int_0^T |u_t^M| dt + C \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_s^M dB_s^P \right|,$$

Taking the square and using the BDG inequality, this shows that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^{h,M} - X_t^M|^2]$  is finite. Since  $p \in [1, 2)$ , the Jensen inequality yields

$$(W_p(\mathcal{L}(X^0), \mathcal{L}(X^M)))^2 \leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{h,M} - X_t^M|^2 \right],$$

and we can now use coupling arguments in  $L^2$ . At this stage, we use the definition of  $u^M$  and the proof of [Fischer, 2014, Proposition C.1] to obtain that  $\mathbb{E}_P[\sup_{0 \leq t \leq T} |X_{t \wedge \tau_M}^0 - X_{t \wedge \tau_M}^M|^2] = 0$ , where  $\tau_M := \inf\{t \in [0, T], \mathcal{E}_t(u) \geq M\}$ . From Lemma 3.B.2,  $(\mathcal{L}_P(X^M))_{M > 0}$  is pre-compact in  $\mathcal{P}_p(\mathcal{C}^d)$ . Sending  $M \rightarrow +\infty$ , we eventually deduce that  $P^M = \mathcal{L}_P(X^M)$  converges towards  $P = \mathcal{L}_P(X^0)$ .

**Step 4. Upper limit.** Using Corollary 3.B.3.2,

$$\bar{\mathcal{F}}_h \leq F(\mathcal{L}(X^{h,M})) + \mathbb{E}_P \int_0^T |u_t^M|^2 dt.$$

Taking the  $h \rightarrow 0$  limit,

$$\limsup_{h \rightarrow 0} \bar{\mathcal{F}}_h \leq F(P^M) + \mathbb{E}_P \int_0^T |u_t^M|^2 dt.$$

Sending  $M \rightarrow +\infty$

$$\limsup_{h \rightarrow 0} \overline{\mathcal{F}}_h \leq F(P) + \mathbb{E}_P \int_0^T |u_t|^2 dt = F(P) + H(P|\Gamma_0(P)).$$

Since this holds for every  $P$ ,  $\limsup_{h \rightarrow 0} \overline{\mathcal{F}}^h \leq \overline{\mathcal{F}}^0$ , completing the proof.  $\square$

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## CHAPTER 4

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# Gibbs principle with infinitely many constraints: optimality conditions and stability

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*He would challenge them to explain the second law  
of thermodynamics. The response was invariably a  
cold negative silence.*

— E. E. Daub, *Maxwell's demon*

This work is a collaboration with Giovanni Conforti and Julien Reygner. We extend the Gibbs conditioning principle to an abstract setting combining infinitely many linear equality constraints and non-linear inequality constraints, which need not be convex. A conditional large deviation principle (LDP) is proved in a Wasserstein-type topology, and optimality conditions are written in this abstract setting. This setting further includes versions of the Schrödinger bridge problem with marginal non-linear inequality constraints at every time. In the particular case of convex constraints, stability results are proved when perturbing both the constraints and the reference measure. We then specify our results when the reference measure is the path-law of a continuous diffusion process, whose law is constrained at each time. We obtain a complete description of the constrained process, involving a modified drift that satisfies a HJB equation reminiscent of stochastic control.

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## 4.1 Introduction

### 4.1.1 Motivation

A fundamental question in statistical mechanics is to estimate the most likely configuration of a large system of exchangeable particles, given some macroscopic observation on it. More precisely, let us consider a  $N$ -tuple of exchangeable random variables  $\vec{X}^N := (X^{1,N}, \dots, X^{N,N})$  in some abstract Polish space  $E$ . We assume that the  $X^{i,N}$  are either independent or interact through their empirical measure

$$\pi(\vec{X}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}} \in \mathcal{P}(E).$$

The purpose is then to estimate  $\pi(\vec{X}^N)$  as  $N \rightarrow +\infty$ , given the knowledge that  $\{\pi(\vec{X}^N) \in A\}$  for some measurable  $A \subset E$ . This amounts to determining the behaviour of a *typical particle* in the system given the *observation* that  $\{\pi(\vec{X}^N) \in A\}$ . In a physical context, measurements are often submitted to uncertainties. Therefore, a closely related question is the stability of the computed behaviour when perturbing the observation.

These questions can be specified when  $\mathcal{L}(\pi(\vec{X}^N))$  satisfy the large deviation principle (LDP) with rate function  $\mathcal{I}$ . Informally speaking, the LDP states that

$$\mathbb{P}(\pi(\vec{X}^N) \in B) \asymp \exp \left[ -N \inf_{\mu \in B} \mathcal{I}(\mu) \right], \quad (4.1)$$

at the exponential-in- $N$  scale, for any sufficiently regular  $B \subset E$ . In this setting, the *Gibbs conditioning principle* suggests the heuristic

$$\mathcal{L}(X^{1,N} | \pi(\vec{X}^N) \in A) \xrightarrow{N \rightarrow +\infty} \operatorname{argmin}_A \mathcal{I}, \quad (4.2)$$

provided that  $\operatorname{argmin}_A \mathcal{I}$  is well-defined, and  $\operatorname{argmin}_A \mathcal{I}$  is also the weak limit of  $\pi(\vec{X}^N)$  conditionally on  $\{\pi(\vec{X}^N) \in A\}$ . More precisely, a LDP can be established for the conditional law. Some reference textbooks on this approach are Lanford [1973]; Ruelle [1965]; Dembo and Zeitouni [2009]; Dupuis and Ellis [2011]; Ellis [2006]. To explicitly compute  $\operatorname{argmin}_A \mathcal{I}$ , some further knowledge of  $\mathcal{I}$  is needed.

The scope of this article is to establish (4.2), to compute minimisers of  $\mathcal{I}$ , and to prove stability properties when  $\mathcal{I}$  is the sum of relative entropy and an interaction term:

$$\mathcal{I}(\mu) := H(\mu|\nu) + \mathcal{F}(\mu),$$

the definition of the relative entropy  $H(\mu|\nu)$  being recalled in Section 4.1.5, and  $A = \mathcal{A}_{\Psi}^{\zeta}$ , where

$$\mathcal{A}_{\Psi}^{\zeta} := \left\{ \mu \in \mathcal{P}(E), \forall s \in \mathcal{S}, \int_E \zeta_s d\mu = 0, \forall t \in \mathcal{T}, \Psi_t(\mu) \leq 0 \right\}$$

is given by (possibly) infinitely many linear equality constraints and non-linear inequality constraints. Precise assumptions on  $\mathcal{F}$ ,  $(\zeta_s)_{s \in \mathcal{S}}$  and  $(\Psi_t)_{t \in \mathcal{T}}$  are detailed in Section 4.2, under which we prove that minimisers are Gibbs measures, whose density involves suitable Lagrange multipliers. Our approach relies on the Hahn-Banach theorem, combining tools from functional analysis in the spirit of Csiszár [1984]; Léonard [2000]; Pennanen and Perkkiö [2019] with differential calculus on  $\mathcal{P}(E)$ . In particular,  $\mathcal{F}$  and the  $\Psi_t$  need not be convex. When  $\mathcal{F} \equiv 0$ , the LDP (4.1) is the well-known Sanov theorem Dembo and Zeitouni [2009] for independent particles. The case  $\mathcal{F} \neq 0$  allows for mean-field interaction, modelling particles that are distributed according to mean-field Gibbs measures as in

Léonard [1987]; Arous and Brunaud [1990]; Wang et al. [2010]; Dupuis et al. [2015]. When  $\mathcal{S} = \emptyset$ , our results extend the standard Gibbs principle to settings with infinitely many constraints, see Section 4.1.2 below. When  $E$  is a product space, we can further impose marginal laws for  $\operatorname{argmin}_{\mathcal{A}_{\Psi}^{\zeta}} \mathcal{I}$  through the linear equality constraints. This last setting includes the famous Schrödinger bridge problem, with additional inequality constraints, see Section 4.1.3 below. A notable example, which is studied in Section 4.3, is given by the space of continuous paths  $E = C([0, T], \mathbb{R}^d)$ .

When  $\mathcal{F}$  and the  $\Psi_t$  are convex, uniqueness holds for the minimiser  $\operatorname{argmin}_{\mathcal{A}_{\Psi}^{\zeta}} \mathcal{I}$ , corresponding to the notion of *entropic projection* introduced by Csiszár [1975, 1984]. Stability results for entropic projections have enjoyed many recent developments Ghosal et al. [2022]; Eckstein and Nutz [2022]; Nutz and Wiesel [2023]; Chiarini et al. [2023]; Divol et al. [2024], mainly motivated by the surge of interest around the Schrödinger problem and its applications in machine learning. In our abstract setting, we prove two kinds of apparent results:

- A quantitative stability result when changing  $\Psi_t$  into  $\Psi_t - \varepsilon$  for small  $\varepsilon > 0$ .
- A weak stability result when perturbing  $\nu, \mathcal{F}, (\Psi_t)_{t \in \mathcal{T}}$  at the same time, when  $\mathcal{S} = \emptyset$ .

These new stability results hold in  $\mathcal{P}(E)$  at a great level of generality, under minimal assumptions on  $\nu, \mathcal{F}, (\Psi_t)_{t \in \mathcal{T}}$  that are stated in Section 4.2.4.

Let us now illustrate our results with the examples of the Gibbs conditioning principle and the Schrödinger bridge problem. For the clarity of exposition, the next two subsections provide an overview of our results only in the particular case  $\mathcal{F} \equiv 0$  with linear constraints  $\Psi_t(\mu) = \int_E \psi_t d\mu$ .

### 4.1.2 The Gibbs conditioning principle

When  $\mathcal{I}(\mu) = H(\mu|\nu)$  and the constraints are given by a finite number of moments against  $\psi_1, \dots, \psi_T : E \rightarrow \mathbb{R}$ , namely

$$A = \left\{ \mu \in \mathcal{P}(E), \quad \forall t \in \{1, \dots, T\}, \int_E \psi_t d\mu \leq 0 \right\},$$

then it is well-known that the density of  $\bar{\mu} := \operatorname{argmin}_{\mu \in A} H(\mu|\nu)$  is given by

$$\frac{d\bar{\mu}}{d\nu}(x) = \bar{Z}^{-1} \exp \left[ - \sum_{t=1}^T \bar{\lambda}_t \psi_t(x) \right], \quad (4.3)$$

where  $\bar{Z}$  is a normalising constant, and the  $\bar{\lambda}_1, \dots, \bar{\lambda}_T$  are non-negative Lagrange multipliers. A similar result holds when imposing  $\int_E \psi_t d\mu = 0$  instead of inequalities, corresponding to the *canonical ensemble* in statistical physics. In addition to the aforementioned textbooks on the Gibbs principle, we mention the prominent contributions Borel [1906]; Diaconis and Freedman [1987]; Stroock and Zeitouni [1991] for precise statements, and Dembo and Zeitouni [1996]; Dembo and Kuelbs [1998]; Cattiaux and Gozlan [2007] for quantitative versions of (4.1) in this setting.

Our abstract results now allow for an infinite number of constraints  $(\psi_t)_{t \in \mathcal{T}}$ , provided a continuous dependence on  $t$  in the compact space  $\mathcal{T}$ . In particular, Theorem 4.2.13 provides existence for a positive Radon measure  $\bar{\lambda} \in \mathcal{M}_+(\mathcal{T})$ , which generalises the previous multipliers, such that

$$\frac{d\bar{\mu}}{d\nu}(x) = \bar{Z}^{-1} \exp \left[ - \int_{\mathcal{T}} \psi_t(x) \bar{\lambda}(dt) \right].$$

In the particular setting  $E = C([0, T], \mathbb{R}^d)$  and  $\mathcal{T} = [0, T]$ ,  $x = (x_t)_{t \in [0, T]}$  being a continuous path, a natural choice is  $\psi_t(x) = \psi(x_t)$ , for some continuous  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ . In this case, if  $\nu$  is the path-law of the solution to the stochastic differential equation (SDE),  $(B_t)_{t \geq 0}$  being a Brownian motion,

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t,$$

under standard Lipschitz assumptions, Theorem 4.3.6 further identifies  $\bar{\mu}$  as being the path-law of the solution to the SDE

$$d\bar{X}_t = b_t(\bar{X}_t)dt - \sigma_t \sigma_t^\top \nabla \varphi_t(\bar{X}_t)dt + \sigma_t(\bar{X}_t)dB_t, \quad \bar{X}_0 \sim \bar{Z}^{-1} e^{-\varphi_0(x)} dx,$$

where  $\varphi$  is the weak solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$-\varphi_t + \int_t^T b_s \cdot \nabla \varphi_s - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s|^2 + \frac{1}{2} \text{Tr}[\sigma_s \sigma_s^\top \nabla^2 \varphi_s] ds + \int_{[t, T]} \psi_s \bar{\lambda}(ds) = 0,$$

for which we prove well-posedness in a suitable sense detailed in Section 4.3.3. This HJB approach is reminiscent of mean-field control problems under constraints that were recently studied in Daudin [2022, 2023b,a].

### 4.1.3 Schrödinger bridge with additional constraints

When  $E = C([0, T], \mathbb{R}^d)$  and  $\mathcal{T} = [0, T]$ , another famous example is the Schrödinger bridge problem Schrödinger [1932], which is the prototype of a stochastic mass transport problem. For this problem,

$$A = \{ \mu \in \mathcal{P}(C([0, T], \mathbb{R}^d)), \mu_0 = \mu_{\text{ini}}, \mu_T = \mu_{\text{fin}} \},$$

where the marginal laws  $\mu_{\text{ini}}, \mu_{\text{fin}} \in \mathcal{P}(\mathbb{R}^d)$  are imposed. The literature on Schrödinger bridges has recently enjoyed thriving developments. Some seminal references are Csizsár [1975]; Cattiaux and Léonard [1995]; Cattiaux and Léonard [1996]. For more recent results, we refer to the survey article Léonard [2013], the lecture notes Nutz [2021], and references therein. We also mention Backhoff et al. [2020] for an extension to a more general rate function  $\mathcal{I}$ . Denoting  $\bar{\mu} := \text{argmin}_{\mu \in A} H(\mu | \nu)$ , a standard result Nutz [2021] is the existence of measurable functions  $\xi_0, \xi_T : \mathbb{R}^d \rightarrow \mathbb{R}$ , called *Schrödinger potentials*, such that

$$\frac{d\bar{\mu}}{d\nu}(x) = \exp [ - \xi_0(x_0) - \xi_T(x_T) ],$$

under suitable assumptions on  $(\nu, \mu_{\text{ini}}, \mu_{\text{fin}})$ . Our results extend this decomposition to the case

$$A = \{ \mu \in \mathcal{P}(C([0, T], \mathbb{R}^d)), \mu_0 = \mu_{\text{ini}}, \mu_T = \mu_{\text{fin}}, \forall t \in [0, T], \int_{\mathbb{R}^d} \psi d\mu_t \leq 0 \}.$$

In this setting, as a consequence of Theorem 4.2.13, Theorem 4.3.3 provides  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  and measurable  $\xi_0, \xi_T : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\frac{d\bar{\mu}}{d\nu}(x) = \exp \left[ - \xi_0(x_0) - \xi_T(x_T) - \int_{[0, T]} \psi(x_t) \bar{\lambda}(dt) \right].$$

In fact, Theorem 4.3.3 states a stronger result, since it allows for non-linear and non-convex constraints  $\Psi(\mu_t) \leq 0$  instead of  $\int_{\mathbb{R}^d} \psi d\mu_t \leq 0$ .

#### 4.1.4 Outline

This article is organised as follows. Our main abstract results are stated in Section 4.2. The optimality conditions are presented in Section 4.2.3, whereas the stability results are given in Section 4.2.4. Section 4.3 develops the case of continuous paths  $E = C([0, T], \mathbb{R}^d)$ . Our results on the Schrödinger bridge problem with additional constraints are stated in Section 4.3.2. The specific case of Gibbs principle for diffusion processes is further studied in Section 4.3.3, and some examples for Gaussian processes can be found in Section 4.3.4. The proofs of the results are written in Sections 4.4-4.5. Appendix 4.B finally presents an alternative proof of Theorem 4.2.13 (from Section 4.2.3) in the linear setting of Section 4.1.2, with improved assumptions.

#### 4.1.5 Notations

- $\vec{x}^N$  denotes a generic element of a product set  $E^N$ , for an integer  $N \geq 1$ . We also write  $\vec{x}^N = (x^i)_{1 \leq i \leq N}$ .
- $\mathcal{P}(E)$  denotes the set of probability measures over a Polish space  $E$  endowed with its Borel  $\sigma$ -algebra.
- $\delta_x$  denotes the Dirac measure at some point  $x$  in  $E$ .
- $\pi$  denotes the function that maps a vector  $\vec{x}^N$  in  $E^N$  to the empirical measure

$$\pi(\vec{x}^N) := \frac{1}{N} \sum_{i=1}^N \delta_{x^i} \in \mathcal{P}(E).$$

- $\langle \mu, f \rangle$  denotes the integral (when it exists) of a measurable function  $f$  against the measure  $\mu$ . The convention will often be adopted that  $\langle \mu, f \rangle = +\infty$  if the integral is not well-defined.
- $\mu(\cdot|A)$  stands for the conditional measure  $\mu(A)^{-1} \mathbb{1}_A \mu$ , for  $\mu$  in  $\mathcal{P}(E)$  and  $A$  measurable with  $\mu(A) > 0$ .
- $\mathcal{M}_+(\mathcal{T})$  denotes the convex cone of positive Radon measures. In this setting, a Radon measure is defined as a signed finite measure that is both inner and outer regular as defined in [Rudin, 1970, Definition 2.15].
- $\mathcal{P}_\phi(E)$  denotes the set of measures  $\mu \in \mathcal{P}(E)$  with  $\langle \mu, \phi \rangle < +\infty$  for some measurable  $\phi : E \rightarrow \mathbb{R}$ . When  $\phi(x) = d(x, x_0)^p$  for some distance  $d$  on  $E$ , some  $x_0$  in  $E$  and some  $p \geq 1$ , we will often write  $\mathcal{P}_p(E)$  instead.
- $W_p$  denotes the Wasserstein distance on  $\mathcal{P}_p(E)$ , defined by

$$W_p(\mu, \mu') := \left[ \inf_{X \sim \mu, Y \sim \mu'} \mathbb{E}[|X - Y|^p] \right]^{1/p}.$$

- $T > 0$  is a given real number, and  $d \geq 1$  is an integer.
- $x_{[0, T]} \in C([0, T], \mathbb{R}^d)$  denotes a continuous function  $x_{[0, T]} : [0, T] \rightarrow \mathbb{R}^d$ .
- $\mu_{[0, T]}$  denotes a path measure in  $\mathcal{P}(C([0, T], \mathbb{R}^d))$ . Its marginal measure at time  $t$  will be denoted by  $\mu_t$ .

- $\mu$ . denotes a continuous curve  $t \mapsto \mu_t$  in  $C([0, T], \mathcal{P}(\mathbb{R}^d))$ .
- $X_t$  denotes the coordinate map  $x_{[0, T]} \mapsto x_t$ . It can be seen as a random variable on the canonical space  $\Omega = C([0, T], \mathbb{R}^d)$ .
- $\frac{\delta\Psi}{\delta\mu}(\mu) : x \mapsto \frac{\delta\Psi}{\delta\mu}(\mu, x)$  denotes the linear functional derivative at  $\mu$  (when it exists) of a function  $\Psi : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}$ . See Definition 4.A.1 below for more details. The convention is adopted throughout the paper that  $\langle \mu, \frac{\delta\Psi}{\delta\mu}(\mu) \rangle = 0$ .
- $\cdot^\top$  and  $\text{Tr}[\cdot]$  respectively denote the transpose and the trace of matrices.

## 4.2 Abstract setting and main results

This section contains our main two theorems for the generalisation of the Gibbs principle. We first present our abstract setting in Section 4.2.1, allowing for interaction terms and infinitely many linear equality constraints and nonlinear inequality constraints. We then establish a conditional LDP (Theorem 4.2.7), taking only inequality constraints into account, in Section 4.2.2. Next, we establish a Gibbs-like formula for the minimiser of the rate function of the conditional LDP (Theorem 4.2.13) in Section 4.2.3, and study more properties, in particular stability, of this minimiser in the case where the rate function and the inequality constraints are convex, in Section 4.2.4.

### 4.2.1 Global setting

#### 4.2.1.1 The rate function

Let  $E$  be a complete separable metric space, endowed with its Borel  $\sigma$ -algebra. The space  $\mathcal{P}(E)$  of probability measures over  $E$  is endowed with the *weak topology* [Billingsley, 2013, Chapter 2]. We recall that a sequence  $(\mu_k)_{k \geq 1}$  weakly converges towards  $\mu$  in  $\mathcal{P}(E)$  if and only if

$$\langle \mu_k, \varphi \rangle \xrightarrow{k \rightarrow +\infty} \langle \mu, \varphi \rangle,$$

for every bounded continuous  $\varphi : E \rightarrow \mathbb{R}$ . The space  $\mathcal{P}(E)$  is endowed with its Borel  $\sigma$ -algebra. For any continuous  $\phi : E \rightarrow \mathbb{R}_+$ , we define the set

$$\mathcal{P}_\phi(E) := \{\mu \in \mathcal{P}(E), \langle \mu, \phi \rangle < +\infty\}.$$

The topology on  $\mathcal{P}_\phi(E)$  is now defined through its converging sequences.

**Definition 4.2.1** (Weak convergence in  $\mathcal{P}_\phi(E)$ ). *A sequence  $(\mu_k)_{k \geq 1}$  weakly converges towards  $\mu$  in  $\mathcal{P}_\phi(E)$  if and only if*

$$\langle \mu_k, \phi \rangle \xrightarrow{k \rightarrow +\infty} \langle \mu, \phi \rangle \quad \text{and} \quad \langle \mu_k, \varphi \rangle \xrightarrow{k \rightarrow +\infty} \langle \mu, \varphi \rangle,$$

for every bounded continuous  $\varphi : E \rightarrow \mathbb{R}$ .

This topology on  $\mathcal{P}_\phi(E)$  is stronger than the one induced by the weak topology on  $\mathcal{P}(E)$ . It was used for instance in Léonard [1987]. When  $\phi(x) = 1$  for every  $x$  in  $E$ , we notice that  $\mathcal{P}(E)$  and  $\mathcal{P}_\phi(E)$  coincide, together with their topology. When  $\phi(x) = d(x, x_0)^p$  for some  $x_0$  in  $E$  and  $p \geq 1$ , this topology corresponds to the one defined in [Villani et al., 2009, Definition 6.8]. This latter topology can be metrised using the  $p$ -Wasserstein distance  $W_p$  [Villani et al., 2009, Theorem 6.9].

The dual representation formula

$$H(\mu|\nu) = \sup_{\substack{\Phi \text{ measurable} \\ \langle \nu, e^\Phi \rangle < \infty}} \langle \mu, \Phi \rangle - \log \langle \nu, e^\Phi \rangle \quad (4.4)$$

of the relative entropy [Léonard, 2012, Proposition 3.1-(iii)] yields the following statement.

**Lemma 4.2.2** (Integrability condition). *Let  $\phi : E \rightarrow \mathbb{R}_+$  be a continuous function and  $\nu \in \mathcal{P}(E)$ . If there exists  $\alpha > 0$  such that  $\langle \nu, e^{\alpha\phi} \rangle < \infty$ , then any probability measure  $\mu \in \mathcal{P}(E)$  such that  $H(\mu|\nu) < +\infty$  is in  $\mathcal{P}_\phi(E)$ .*

We now fix a continuous function  $\phi : E \rightarrow \mathbb{R}_+$  and a reference probability measure  $\nu \in \mathcal{P}(E)$ .

**Lemma 4.2.3** (Relative entropy as a good rate function on  $\mathcal{P}_\phi(E)$ ). *If the condition*

$$\forall \alpha > 0, \quad \langle \nu, e^{\alpha\phi} \rangle < \infty \quad (4.5)$$

*holds, then  $\mu \mapsto H(\mu|\nu)$  is lower semi-continuous on  $\mathcal{P}_\phi(E)$ , with compact level sets.*

In the terminology of [Dembo and Zeitouni, 2009, Section 1.2], under the condition (4.5), the relative entropy with respect to  $\nu$  is a *good rate function* on  $\mathcal{P}_\phi(E)$ .

*Proof.* The compactness of the level sets of  $H$  for the usual weak topology is part of the Sanov theorem. From (4.4) applied to  $\Phi = \alpha\phi\mathbb{1}_{|\phi| \geq M}$  for every  $\alpha, M > 0$ ,  $\phi$  is uniformly integrable over any level set of  $H$ . The compactness for the weak topology on  $\mathcal{P}_\phi(E)$  then follows from [Billingsley, 2013, Theorem 3.5].  $\square$

As is stated in the proof of Lemma 4.2.3, by Sanov's theorem, the relative entropy describes the large deviations of systems of independent particles. To take interaction between particles into account, we now consider a measurable map  $\mathcal{F} : \mathcal{P}_\phi(E) \rightarrow (-\infty, +\infty]$ , and define the function  $\mathcal{I} : \mathcal{P}_\phi(E) \rightarrow (-\infty, +\infty]$  by

$$\mathcal{I}(\mu) := H(\mu|\nu) + \mathcal{F}(\mu), \quad (4.6)$$

the domain of  $\mathcal{I}$  being the set

$$\mathcal{D}_{\mathcal{I}} := \{\mu \in \mathcal{P}_\phi(E), H(\mu|\nu) < +\infty, \mathcal{F}(\mu) < +\infty\}. \quad (4.7)$$

**Assumption (A.1)** (Rate function). *Condition (4.5) holds and  $\mathcal{I}$  is lower semi-continuous on  $\mathcal{P}_\phi(E)$ .*

This assumption tells that  $\mathcal{I}$  is a *rate function* on  $\mathcal{P}_\phi(E)$ . Of course, by Lemma 4.2.3,  $\mathcal{I}$  is lower semi-continuous on  $\mathcal{P}_\phi(E)$  as soon as  $\mathcal{F}$  is lower semi-continuous on  $\mathcal{P}_\phi(E)$ . Then a sufficient condition for  $\mathcal{I}$  to be a *good rate function* on  $\mathcal{P}_\phi(E)$  is given as follows. The proof is omitted.

**Lemma 4.2.4** (Sufficient condition for  $\mathcal{I}$  to be a good rate function). *Under Assumption (A.1), if  $\mathcal{F}$  is bounded from below on  $\mathcal{P}_\phi(E)$ , then  $\mathcal{I}$  has compact level sets in  $\mathcal{P}_\phi(E)$ .*

### 4.2.1.2 Constraints

The first goal of this article is to study the minimisation of the rate function  $\mathcal{I}$  on the space  $\mathcal{P}_\phi(E)$ , subject to two types of constraints: linear equality constraints, and nonlinear inequality constraints.

Linear equality constraints are encoded by a family  $(\zeta_s)_{s \in \mathcal{S}}$  of functions  $E \rightarrow \mathbb{R}$ , where  $\mathcal{S}$  is an arbitrary set. They will be assumed to satisfy the following properties.

**Assumption (A.2)** (On the linear equality constraints). *For any  $s \in \mathcal{S}$ ,  $\zeta_s$  is continuous on  $E$ , and there exists  $C_s^\zeta \in [0, \infty)$  such that*

$$\forall x \in E, \quad |\zeta_s(x)| \leq C_s^\zeta [1 + \phi(x)].$$

We define the associated constrained subset of  $\mathcal{P}_\phi(E)$  by

$$\mathcal{A}^\zeta := \{\mu \in \mathcal{P}_\phi(E), \forall s \in \mathcal{S}, \langle \mu, \zeta_s \rangle = 0\}.$$

Nonlinear inequality constraints are encoded by a family  $(\Psi_t)_{t \in \mathcal{T}}$  of functions  $\mathcal{P}_\phi(E) \rightarrow \mathbb{R}$ , where  $\mathcal{T}$  is a compact metric space. The basic assumption on these functions is the following.

**Assumption (A.3)** (On the nonlinear inequality constraints). *For any  $t \in \mathcal{T}$ , the function  $\Psi_t$  is lower semi-continuous on  $\mathcal{P}_\phi(E)$ .*

We define the associated constrained subset of  $\mathcal{P}_\phi(E)$  by

$$\mathcal{A}_\Psi := \{\mu \in \mathcal{P}_\phi(E), \forall t \in \mathcal{T}, \Psi_t(\mu) \leq 0\}.$$

**Remark 4.2.5** (Linear inequality constraints). *A particular case of inequality constraints is given by linear functions  $\Psi_t(\mu) = \langle \mu, \psi_t \rangle$  for some function  $\psi_t : E \rightarrow \mathbb{R}$ . Then Assumption (A.3) holds as soon as, for any  $t \in \mathcal{T}$ ,  $\psi_t$  is lower semi-continuous and there exists  $C_t^\psi \in [0, \infty)$  such that*

$$\forall x \in E, \quad [\psi_t(x)]_- \leq C_t^\psi [1 + \phi(x)].$$

Our assumptions on the linear equality and nonlinear inequality constraints yield the following statement.

**Lemma 4.2.6.** *Under (A.2)-(A.3), the constrained sets  $\mathcal{A}^\zeta$ ,  $\mathcal{A}_\Psi$  and  $\mathcal{A}_\Psi^\zeta := \mathcal{A}^\zeta \cap \mathcal{A}_\Psi$  are closed in  $\mathcal{P}_\phi(E)$ .*

The elements of  $\mathcal{D}_\mathcal{I} \cap \mathcal{A}_\Psi^\zeta$  are the *admissible measures*. Under (A.1)-(A.2)-(A.3), the sequel of this section is dedicated to the study of the minimisation problem

$$\bar{\mathcal{I}}_\Psi^\zeta := \inf_{\mu \in \mathcal{D}_\mathcal{I} \cap \mathcal{A}_\Psi^\zeta} \mathcal{I}(\mu). \tag{4.8}$$

By Lemma 4.2.6, any limit  $\bar{\mu}$  of a minimising sequence for this problem belongs to  $\mathcal{D}_\mathcal{I} \cap \mathcal{A}_\Psi^\zeta$ , and by the semi-continuity assumption (A.1) it satisfies

$$\mathcal{I}(\bar{\mu}) = \bar{\mathcal{I}}_\Psi^\zeta < \infty.$$

We call  $\bar{\mu}$  a *minimiser* for (4.8). If  $\mathcal{D}_\mathcal{I} \cap \mathcal{A}_\Psi^\zeta$  is non-empty and  $\mathcal{I}$  is a good rate function, as in Lemma 4.2.4, then there exists at least one minimiser.



## 4.2.2 Conditional LDP for nonlinear inequality constraints

As is explained in Section 4.1, the first step of the proof of the Gibbs principle is the derivation of a LDP for the conditional distribution of the empirical measure of the particle system under inequality constraints. Thus, in this section, we consider a sequence  $(\Pi^N)_{N \geq 1}$  of probability measures on  $\mathcal{P}_\phi(E)$ , which is assumed to satisfy a LDP with rate function  $\mathcal{I}$  defined in (4.6). Under the assumption that

$$\forall N \geq 1, \quad \Pi^N(\mathcal{A}_\Psi) > 0, \quad (4.9)$$

we may define the conditional distribution  $\Pi^N(\cdot | \mathcal{A}_\Psi)$ . The goal of this section is to state a LDP for the sequence  $\Pi^N(\cdot | \mathcal{A}_\Psi)$ .

### 4.2.2.1 Statement of the result

In order to state the LDP for this sequence, we need to introduce the following assumptions.

**Assumption (A.4)** (Regularity of  $\Psi_t$ ). *The function  $\mu \mapsto \sup_{t \in \mathcal{T}} \Psi_t(\mu)$  is well-defined and continuous on  $\mathcal{P}_\phi(E)$ .*

**Assumption (A.5)** (Constraint qualification). *For any  $\mu \in \mathcal{D}_\mathcal{I} \cap \mathcal{A}_\Psi$ , there exists  $\tilde{\mu} \in \mathcal{D}_\mathcal{I}$  such that:*

- (i) *for every  $\varepsilon > 0$  small enough, we have  $\Psi_t(\mu + \varepsilon(\tilde{\mu} - \mu)) < 0$  for all  $t \in \mathcal{T}$ ;*
- (ii)  *$\limsup_{\varepsilon \rightarrow 0} \mathcal{F}(\mu + \varepsilon(\tilde{\mu} - \mu)) \leq \mathcal{F}(\mu)$ .*

**Theorem 4.2.7** (Conditional LDP). *Let (A.1)-(A.3)-(A.4)-(A.5) hold, and assume that*

$$\bar{\mathcal{I}}_\Psi := \inf_{\mu \in \mathcal{D}_\mathcal{I} \cap \mathcal{A}_\Psi} \mathcal{I}(\mu) < +\infty. \quad (4.10)$$

*Let  $(\Pi^N)_{N \geq 1}$  be a sequence of probability measures on  $\mathcal{P}_\phi(E)$  which satisfies a LDP with rate function  $\mathcal{I}$ , such that (4.9) holds. Then  $(\Pi^N(\cdot | \mathcal{A}_\Psi))_{N \geq 1}$  satisfies a LDP with rate function  $\mathcal{I}_\Psi : \mathcal{P}_\phi(E) \rightarrow [0, +\infty]$  defined by*

$$\mathcal{I}_\Psi(\mu) := \begin{cases} \mathcal{I}(\mu) - \bar{\mathcal{I}}_\Psi, & \text{if } \mu \in \mathcal{A}_\Psi, \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 4.2.7 is proved in Section 4.4.1. This result can be seen as a variation of [La Cour and Schieve, 2015, Theorem 1] or [Chafaï et al., 2021, Theorem 3.1], without requiring the rate function to be good. Under the assumptions of Theorem 4.2.7, if  $\mathcal{I}$  is a good rate function (as in Lemma 4.2.4), then by Lemma 4.2.6, so is  $\mathcal{I}_\Psi$  and therefore  $(\Pi^N(\cdot | \mathcal{A}_\Psi))_{N \geq 1}$  satisfies the LDP with a good rate function. In particular, there is at least one minimiser for (4.10). If this minimiser  $\bar{\mu}$  is moreover unique, then  $\Pi^N(\cdot | \mathcal{A}_\Psi)$  weakly converges towards  $\delta_{\bar{\mu}}$ . Therefore, in this case, the Gibbs principle holds in the general setting of Theorem 4.2.7, namely with infinitely many constraints  $\Psi_t$ , which need not be linear functionals of  $\mu$ , and interaction between particles, which are encoded by the functional  $\mathcal{F}$ . Moreover, it is stated in the abstract metric space  $E$ , for the Wasserstein-like topology on  $\mathcal{P}_\phi(E)$  which is stronger than the usual weak topology.

The detailed study of minimisers for (4.10) is carried out in Sections 4.2.3 and 4.2.4, in the more general setting of (4.8) which takes equality constraints into account. In the sequel of the present section, we discuss how to check the continuity assumption (A.4) when the functions  $\Psi_t$  have a certain regularity property, and then we provide an example

of application of Theorem 4.2.7, when  $\mathcal{F} = 0$ . We therefore obtain a conditional version of Sanov's Theorem, which is stated in Wasserstein topology. Some examples with  $\mathcal{F} \neq 0$  are LDPs for Gibbs measures like Arous and Brunaud [1990] which typically enter the framework of Theorem 4.2.7, and suitable examples in the Wasserstein topology are given in Léonard [1987]; Liu and Wu [2020]; Dupuis et al. [2015]; Chaintron [2024].

#### 4.2.2.2 Checking (A.4) for differentiable constraints

In the sequel of this article, we shall be particularly interested in the case where the functions  $\Psi_t$  are differentiable with respect to  $\mu$ , using the notion of linear functional derivative, whose definition and basic properties are gathered in Appendix 4.A. In this context, explicit, sufficient conditions for the continuity assumption (A.4) may be formulated.

**Proposition 4.2.8** (Assumption (A.4) for differentiable constraints). *Assume that for any  $\mu \in \mathcal{P}_\phi(E)$ , for any  $t \in \mathcal{T}$ , the function  $\Psi_t$  is differentiable at  $\mu$  w.r.t. the set of directions  $\mathcal{P}_\phi(E)$ , in the sense of Definition 4.A.1. If the conditions:*

(i) *for any compact set  $K \subset \mathcal{P}_\phi(E)$ , there exists  $D_K^\Psi \in [0, \infty)$  such that*

$$\forall x \in E, \quad \sup_{(t, \mu) \in \mathcal{T} \times K} \left| \frac{\delta \Psi_t}{\delta \mu}(\mu, x) \right| \leq D_K^\Psi [1 + \phi(x)];$$

(ii) *for any compact set  $K \subset \mathcal{P}_\phi(E)$ , the family of functions  $(x \mapsto \frac{\delta \Psi_t}{\delta \mu}(\mu, x))_{t \in \mathcal{T}, \mu \in K}$  is equi-continuous;*

(iii) *for any  $x \in E$ , the family of functions  $(\mu \mapsto \frac{\delta \Psi_t}{\delta \mu}(\mu, x))_{t \in \mathcal{T}}$  is (sequentially) equi-continuous on  $\mathcal{P}_\phi(E)$ ;*

*are satisfied, then Assumption (A.4) holds.*

Proposition 4.2.8 is proved in Section 4.4.1.

**Remark 4.2.9** (Linear constraints). *In the setting of Remark 4.2.5, the conditions (i,ii,iii) hold as soon as the family  $(\psi_t)_{t \in \mathcal{T}}$  is equi-continuous on  $E$  and there exists  $D^\psi \in [0, \infty)$  such that*

$$\forall x \in E, \quad \sup_{t \in \mathcal{T}} |\psi_t(x)| \leq D^\psi [1 + \phi(x)].$$

#### 4.2.2.3 Application: the conditional Sanov Theorem with linear constraints

A particular setting where Theorem 4.2.7 may be applied is the case where  $\Pi^N$  is the law of the empirical measure of independent random variables distributed according to  $\nu$ . In this case, the LDP is known to hold with  $\mathcal{I}(\mu) = H(\mu|\nu)$  on  $\mathcal{P}_\phi(E)$ , either with  $\phi \equiv 1$  (this is the usual Sanov Theorem), or with  $\phi(x) = d(x, x_0)^p$ ,  $p \geq 1$  under the exponential integrability assumption (4.5) in (A.1) (this is the Sanov Theorem in Wasserstein distance [Wang et al., 2010, Theorem 1.1]).

In this case, and if the inequality constraints are actually linear, as in Remark 4.2.5, then defining

$$E_\eta := \{x \in E : \forall t \in \mathcal{T}, \psi_t(x) \leq -\eta\},$$

we have that a sufficient condition for (A.5)-(i) to hold is

$$\exists \eta > 0 : \quad \nu(E_\eta) > 0, \tag{4.11}$$

because then we may take  $\tilde{\mu} = \nu(\cdot|E_\eta)$ , which satisfies  $\langle \tilde{\mu}, \psi_t \rangle \leq -\eta$  for any  $t$ . This condition also guarantees that the conditions (4.9) and (4.10) are satisfied. Thus, the combination of this observation with Remark 4.2.9 yields the following statement.

**Proposition 4.2.10** (Conditional Sanov Theorem with linear constraints). *Let  $\phi(x) = d(x, x_0)^p$  for  $p = 0$  or  $p \geq 1$ , and let  $\nu \in \mathcal{P}(E)$  be such that the integrability condition (4.5) holds. Let  $\Pi^N$  be the law of the empirical measure of  $N$  independent random variables distributed according to  $\nu$ . Let  $(\psi_t)_{t \in \mathcal{T}}$  be a family of functions  $E \rightarrow \mathbb{R}$  satisfying the conditions of Remark 4.2.9, and let the set  $\mathcal{A}_\Psi$  be defined accordingly. If this family satisfies (4.11), then the sequence  $(\Pi^N(\cdot | \mathcal{A}_\Psi))_{N \geq 1}$  is well-defined and satisfies a LDP on  $\mathcal{P}_\phi(E)$  with good rate function*

$$\begin{cases} H(\mu | \nu) - \inf_{\mu' \in \mathcal{A}_\Psi} H(\mu' | \nu), & \text{if } \mu \in \mathcal{A}_\Psi, \\ +\infty, & \text{otherwise.} \end{cases}$$

In the setting of Proposition 4.2.10, the set  $\mathcal{A}_\Psi$  is convex and the (good) rate function is strictly convex, so it admits a unique minimiser  $\bar{\mu}$  and  $\Pi^N(\cdot | \mathcal{A}_\Psi)$  converges to  $\delta_{\bar{\mu}}$ . A more general discussion on the convex case is postponed to Section 4.2.4.

**Remark 4.2.11** (Convergence for conditioned particles). *Denoting by  $\vec{X}^N = (X^1, \dots, X^N)$  the underlying particle system, whose empirical measure  $\pi(\vec{X}^N)$  has distribution  $\Pi^N$ , this statement implies that the conditional distribution  $\mathcal{L}(X^1 | \pi(\vec{X}^N) \in \mathcal{A}_\Psi)$  of the first particle converges weakly to  $\bar{\mu}$ . This convergence can be quantified: we deduce from [Csiszár, 1984, Theorem 1] that*

$$\forall N \geq 1, \quad H(\mathcal{L}(X^1 | \pi(\vec{X}^N) \in \mathcal{A}_\Psi) | \bar{\mu}) \leq -\frac{1}{N} \log \Pi^N(\mathcal{A}_\Psi) - H(\bar{\mu} | \nu).$$

*More explicit rates are given in Dembo and Zeitouni [1996]; Dembo and Kuelbs [1998]; Cattiaux and Gozlan [2007].*

### 4.2.3 Gibbs density for minimisers

In this section we focus on the minimisation problem (4.8), which takes both equality and inequality constraints into account. We show that minimisers, if they exist, have a density with respect to  $\nu$ , whose form generalises (4.3) to the infinite-dimensional setting. To take into account both the interaction part  $\mathcal{F}$  in the rate function, and the nonlinearity of the inequality constraints  $\Psi_t$ , we need to assume differentiability properties of these functionals, at a given  $\bar{\mu} \in \mathcal{P}_\phi(E)$ .

**Assumption (A.6)** (Differentiability of  $\mathcal{F}$  at  $\bar{\mu}$ ). *The set  $\mathcal{D}_\mathcal{I}$  is convex, and  $\mathcal{F}$  is differentiable at  $\bar{\mu}$  w.r.t. the set of directions  $\mathcal{D}_\mathcal{I}$ .*

**Assumption (A.7)** (Regularity of  $\Psi_t$  and constraint qualification at  $\bar{\mu}$ ).

- (i) *The function  $t \mapsto \Psi_t(\bar{\mu})$  is continuous.*
- (ii) *For any  $t \in \mathcal{T}$ ,  $\Psi_t$  is differentiable at  $\bar{\mu}$  w.r.t. the set of directions  $\mathcal{P}_\phi(E)$ , uniformly in  $t$ , in the sense that for any  $\mu \in \mathcal{P}_\phi(E)$ ,*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \in \mathcal{T}} \left| \frac{\Psi_t((1-\varepsilon)\bar{\mu} + \varepsilon\mu) - \Psi_t(\bar{\mu})}{\varepsilon} - \left\langle \mu - \bar{\mu}, \frac{\delta\Psi_t}{\delta\mu}(\bar{\mu}) \right\rangle \right| = 0.$$

- (iii) *For any  $x \in E$ , the function  $t \mapsto \frac{\delta\Psi_t}{\delta\mu}(\bar{\mu}, x)$  is continuous.*

- (iv) *There exists  $D^\Psi \in [0, \infty)$  such that*

$$\forall x \in E, \quad \sup_{t \in \mathcal{T}} \left| \frac{\delta\Psi_t}{\delta\mu}(\bar{\mu}, x) \right| \leq D^\Psi [1 + \phi(x)].$$

(v) There exist  $\tilde{\mu}$  in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  and  $\tilde{\varepsilon} > 0$  such that

$$\forall t \in \mathcal{T}, \quad \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle < 0.$$

**Remark 4.2.12.** By (A.7)-(i,iii,iv), for any  $\mu \in \mathcal{P}_{\phi}(E)$  and any  $\varepsilon > 0$ , the function  $t \mapsto \Psi_t(\bar{\mu}) + \varepsilon \langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \rangle$  is continuous. In particular, (A.7)-(v) equivalently rewrites

$$\sup_{t \in \mathcal{T}} \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle < 0.$$

The condition (A.7)-(ii) can be verified under equi-continuity assumptions on  $\frac{\delta \Psi_t}{\delta \mu}$  in the spirit of Proposition 4.2.8.

The proof of Theorem 4.2.13 also requires the following technical condition.

**Assumption (A.8).** Every probability measure on  $E$  that has a bounded density with respect to  $\bar{\mu}$  or to the measure  $\tilde{\mu}$  given by (A.7)-(v) belongs to  $\mathcal{D}_{\mathcal{I}}$ .

For instance, Assumption (A.8) holds when  $\mathcal{F} \equiv 0$ , as in Sanov's theorem.

**Theorem 4.2.13** (Gibbs density for minimisers of (4.8)). *Let (A.1)-(A.2)-(A.3) hold, and let  $\bar{\mu}$  be a minimiser for (4.8), which satisfies (A.6)-(A.7)-(A.8). Then, the following holds*

1. Any  $\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  is absolutely continuous w.r.t.  $\bar{\mu}$ .
2. There exist  $(\bar{\zeta}, \bar{\lambda}) \in L^1(E, d\bar{\mu}) \times \mathcal{M}_+(\mathcal{T})$  and a measurable  $\bar{S} \subset E$  such that  $\bar{\mu}(\bar{S}) = 1$ ,

$$\bar{Z} := \int_E \mathbb{1}_{\bar{S}}(x) \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right] \nu(dx) \in (0, \infty), \quad (4.12)$$

and  $\bar{\mu}$  has density

$$\frac{d\bar{\mu}}{d\nu}(x) = \frac{1}{\bar{Z}} \mathbb{1}_{\bar{S}}(x) \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right] \quad (4.13)$$

with respect to  $\nu$ .

3. The function  $\bar{\zeta}$  belongs to the closure of  $\text{Span}(\zeta_s, s \in \mathcal{S})$  in  $L^1(E, d\bar{\mu})$  hence

$$\int_E \bar{\zeta} d\bar{\mu} = 0, \quad (4.14)$$

and the complementary slackness condition is satisfied:

$$\Psi_t(\bar{\mu}) = 0 \quad \text{for } \bar{\lambda}\text{-a.e. } t \in \mathcal{T}. \quad (4.15)$$

The proof of Theorem 4.2.13 is detailed in Section 4.4.2. Remark 4.4.4 there provides a bound on the total mass of  $\bar{\lambda}$  in terms of  $\bar{\mu}$ ,  $\tilde{\mu}$ ,  $\tilde{\varepsilon}$  and  $\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu})$ . In the particular case  $\mathcal{F} \equiv 0$  and  $\mathcal{T} = \emptyset$  (no inequality constraint), we recover the result of [Csiszár, 1975, Theorem 3].

**Remark 4.2.14** (Equivalence between  $\bar{\mu}$  and  $\nu$ ). *In general,  $\bar{\mu}$  is not equivalent to  $\nu$ ; the set  $\bar{S}$  where  $\frac{d\bar{\mu}}{d\nu}$  is positive is thus needed, contrary to the Gibbs density (4.3) in Section 4.1.2. This loss of equivalence only comes from the equality constraints. Indeed, if  $\mathcal{S} = \emptyset$ , we notice that  $\nu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$ , so that  $\nu \ll \bar{\mu}$  and  $\nu(\bar{S}) = 1$  from Theorem 4.2.13-1. If  $\mathcal{S} = \emptyset$ , we of course have  $\bar{\zeta} \equiv 0$ . The need for  $\bar{S}$  for handling equality constraints is well-known in the setting of the Schrödinger bridge, see Section 4.3.2. From Theorem 4.2.13-1, we eventually deduce that a necessary and sufficient condition for  $\bar{\mu} \sim \nu$  is the existence of  $\tilde{\nu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  such that  $\tilde{\nu} \sim \nu$ .*

**Remark 4.2.15** (Assumption (A.7) for linear inequality constraints). *In the case of linear inequality constraints of Remark 4.2.5, Assumption (A.7)-(ii) always holds true. Assumptions (A.7)-(i,iii,iv) are satisfied as soon as for any  $x \in E$ ,  $t \mapsto \psi_t(x)$  is continuous and there exists  $D^\psi \in [0, \infty)$  such that*

$$\forall x \in E, \quad \sup_{t \in \mathcal{T}} |\psi_t(x)| \leq D^\psi [1 + \phi(x)].$$

*Last, Assumption (A.7)-(v) amounts to assuming that there exists  $\tilde{\mu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^\zeta$  such that  $\langle \tilde{\mu}, \psi_t \rangle < 0$  for all  $t \in \mathcal{T}$ . For the present case of linear constraints, the assumptions of Theorem 4.2.13 may be relaxed. In particular, no upper bound on the  $\psi_t$  needs to be assumed. This alternative result is detailed in Appendix 4.B.*

**Remark 4.2.16** (Sufficient condition for (A.7)-(v) without equality constraints). *If  $\mathcal{S} = \emptyset$  (no equality constraints), and  $\bar{\mu}$  gives a positive measure to the set*

$$U := \left\{ x \in E, \forall t \in \mathcal{T}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) < 0 \right\},$$

*we notice that  $\tilde{\mu} := \bar{\mu}(\cdot|U)$  satisfies (A.7)-(v), for any  $\tilde{\varepsilon} > 0$ , if we know that  $\mathcal{F}(\tilde{\mu}) < +\infty$ . The latter condition may follow from (A.8) since  $\tilde{\mu}$  has a bounded density w.r.t.  $\bar{\mu}$ .*

#### 4.2.4 Sufficient conditions and stability in the convex case

In this section, we work under the following assumption.

**Assumption (A.9)** (Convexity). *The functions  $\mathcal{F}$  and  $\Psi_t$ ,  $t \in \mathcal{T}$ , are convex on  $\mathcal{P}_\phi(E)$ .*

Then, the function  $\mu \mapsto H(\mu|\nu)$  being strictly convex, the sets  $\mathcal{D}_{\mathcal{I}}$  and  $\mathcal{A}_{\Psi}^\zeta$  are convex, and there is at most one minimiser for (4.8).

In this setting, we first show a converse statement to Theorem 4.2.13, namely that an admissible measure which satisfies the conditions (4.13)-(4.15) is the minimiser for (4.8).

**Theorem 4.2.17** (Sufficient conditions in the convex case). *Let (A.1)-(A.2)-(A.3)-(A.9) hold. Let  $\bar{\mu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^\zeta$  satisfy the following conditions.*

- (i) *The functions  $\mathcal{F}$  and  $\Psi_t$ ,  $t \in \mathcal{T}$ , are differentiable at  $\bar{\mu}$  w.r.t. the set of directions  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^\zeta$ .*
- (ii) *The function  $(t, x) \mapsto \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x)$  is measurable, and there exists  $D^\Psi \in [0, \infty)$  such that*

$$\forall x \in E, \quad \sup_{t \in \mathcal{T}} \left| \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \right| \leq D^\Psi [1 + \phi(x)].$$

- (iii) *There exist  $\bar{\zeta}$  in the  $L^1(E, d\bar{\mu})$ -closure of  $\text{Span}(\zeta_s, s \in \mathcal{S})$  (which necessarily satisfies (4.14)),  $\bar{\lambda} \in \mathcal{M}_+(\mathcal{T})$  and a measurable subset  $\bar{S} \subset E$  such that  $\bar{\mu}(\bar{S}) = 1$  and (4.12)-(4.13)-(4.15) hold.*

- (iv)  *$\nu(\bar{S}) = 1$  and for any  $\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^\zeta$ ,  $\bar{\zeta} \in L^1(E, d\mu)$  and  $\langle \mu, \bar{\zeta} \rangle = 0$ .*

*Then, the measure  $\bar{\mu}$  is the unique minimiser for (4.8).*

When only equality constraints are considered, corresponding to  $\mathcal{T} = \emptyset$ , Condition (iv) can be verified using [Nutz, 2021, Lemma 2.18].

We now prove stability for (4.8) when moving the inequality constraints  $\Psi_t$  or the reference measure. To this aim, we will use the following sufficient condition for the qualification condition (A.7)-(v) to hold.

**Lemma 4.2.18** (Sufficient condition for qualification). *Under (A.9), if there exists  $\tilde{\mu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  such that*

$$\forall t \in \mathcal{T}, \quad \Psi_t(\tilde{\mu}) < 0,$$

*then the condition (A.7)-(v) holds, with  $\tilde{\varepsilon} = 1$ , at any  $\bar{\mu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^{\zeta}$  at which all the  $\Psi_t$  are differentiable (w.r.t. the set of directions  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$ ).*

Lemma 4.2.18 is an immediate consequence of the convexity of the constraints (see e.g. (4.51) below) and we do not detail it.

We first state a quantitative result when only the inequality constraints are relaxed. To this aim, for any  $\varepsilon \geq 0$ , we introduce the notation

$$\mathcal{A}_{\Psi, \varepsilon} := \{\mu \in \mathcal{P}_{\phi}(E), \forall t \in \mathcal{T}, \Psi_t(\mu) \leq \varepsilon\}, \quad \mathcal{A}_{\Psi, \varepsilon}^{\zeta} := \mathcal{A}^{\zeta} \cap \mathcal{A}_{\Psi, \varepsilon}.$$

**Proposition 4.2.19** (Quantitative stability w.r.t. inequality constraints). *Let (A.1)-(A.2)-(A.3)-(A.9) hold, and assume that  $\mathcal{F}$  is bounded from below on  $\mathcal{P}_{\phi}(E)$ , and that for any  $\mu \in \mathcal{A}^{\zeta}$ , the function  $t \mapsto \Psi_t(\mu)$  is measurable. We consider the minimisation problem*

$$\bar{\mathcal{I}}_{\Psi, \varepsilon}^{\zeta} := \inf_{\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi, \varepsilon}^{\zeta}} \mathcal{I}(\mu). \quad (4.16)$$

*We assume that there is  $\varepsilon_* > 0$  such that (A.6), (A.7)-(i,ii,iii,iv) hold at every  $\bar{\mu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi, \varepsilon_*}^{\zeta}$ , that the condition of Lemma 4.2.18 is satisfied, and that (A.8) holds for every  $\bar{\mu} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi, \varepsilon_*}^{\zeta}$  and for  $\tilde{\mu}$  given by Lemma 4.2.18.*

1. *For every  $\varepsilon \in [0, \varepsilon_*]$ , (4.16) has a unique minimiser  $\bar{\mu}_{\varepsilon}$ , and this minimiser satisfies the conclusions of Theorem 4.2.13 for some  $(\bar{\zeta}_{\varepsilon}, \bar{\lambda}_{\varepsilon}) \in L^1(E, d\bar{\mu}_{\varepsilon}) \times \mathcal{M}_+(\mathcal{T})$  and a measurable set  $\bar{S}_{\varepsilon} \subset E$ .*
2. *Assume in addition that for any  $\varepsilon \in [0, \varepsilon_*]$ ,  $\nu(\bar{S}_{\varepsilon}) = 1$ , and that for any  $\mu \in \mathcal{A}^{\zeta}$ ,  $\bar{\zeta}_{\varepsilon} \in L^1(E, d\mu)$  with  $\langle \mu, \bar{\zeta}_{\varepsilon} \rangle = 0$ . Then the function  $\varepsilon \mapsto \bar{\mathcal{I}}_{\Psi, \varepsilon}^{\zeta}$  is absolutely continuous,*

$$\frac{d}{d\varepsilon} \bar{\mathcal{I}}_{\Psi, \varepsilon}^{\zeta} = -\bar{\lambda}_{\varepsilon}(\mathcal{T}), \quad \text{Lebesgue-a.e.},$$

*and there exists  $C_{\text{stab}} \in [0, \infty)$  such that, for any  $\varepsilon \in [0, \varepsilon_*]$ ,*

$$0 \leq \bar{\mathcal{I}}_{\Psi, 0}^{\zeta} - \bar{\mathcal{I}}_{\Psi, \varepsilon}^{\zeta} \leq C_{\text{stab}} \varepsilon. \quad (4.17)$$

In the case  $\mathcal{F} \equiv 0$ , the estimate (4.17) reads  $0 \leq H(\bar{\mu}_0 | \nu) - H(\bar{\mu}_{\varepsilon} | \nu) \leq C_{\text{stab}} \varepsilon$ . The Pythagorean identity for entropic projections [Csiszár, 1975, Theorem 2.2] then provides the following quantitative stability estimate on the minimiser  $\bar{\mu}_{\varepsilon}$  of (4.16).

**Corollary 4.2.19.1.** *If  $\mathcal{F} \equiv 0$  in the setting of Proposition 4.2.19, then*

$$H(\bar{\mu}_0 | \bar{\mu}_{\varepsilon}) \leq C_{\text{stab}} \varepsilon.$$

We finally address weak stability when  $\nu$ ,  $\mathcal{F}$  and the  $\Psi_t$  are perturbed at the same time. Since interesting results already exist in the setting of equality constraints Ghosal et al. [2022]; Eckstein and Nutz [2022]; Nutz and Wiesel [2023]; Chiarini et al. [2023]; Divol et al. [2024], we only focus on inequality constraints; therefore, we assume that  $\mathcal{S} = \emptyset$ . Let  $(\nu_k)_{k \geq 1}$  and  $\nu$  be in  $\mathcal{P}_{\phi}(E)$ , and  $(\mathcal{F}_k)_{k \geq 1}$  and  $\mathcal{F}$  be functions  $\mathcal{P}_{\phi}(E) \rightarrow (-\infty, +\infty]$ . Let  $\mathcal{I}_k : \mu \mapsto H(\mu | \nu_k) + \mathcal{F}_k(\mu)$  be the related rate function with domain  $\mathcal{D}_{\mathcal{I}_k}$ . For each  $t \in \mathcal{T}$ , let  $(\Psi_{t,k})_{k \geq 1}$  be functions  $\mathcal{P}_{\phi}(E) \rightarrow \mathbb{R}$ .

**Assumption (A.10)** (Setting for stability).

- (i) The functions  $\mathcal{I}$ ,  $(\mathcal{I}_k)_{k \geq 1}$ ,  $(\Psi_t)_{t \in \mathcal{T}}$  and  $(\Psi_{t,k})_{t \in \mathcal{T}, k \geq 1}$  are lower semi-continuous. Moreover, for every  $k \geq 1$ ,  $\mathcal{I}_k$  has compact level sets in  $\mathcal{P}_\phi(E)$ .
- (ii) The functions  $\mathcal{F}$ ,  $(\mathcal{F}_k)_{k \geq 1}$ ,  $(\Psi_t)_{t \in \mathcal{T}}$  and  $(\Psi_{t,k})_{t \in \mathcal{T}, k \geq 1}$  are convex.
- (iii) For every  $k \geq 1$ , any measure that has a bounded density w.r.t.  $\nu_k$  belongs to  $\mathcal{D}_{\mathcal{I}_k}$ .
- (iv)  $\nu_k \rightarrow \nu$  in  $\mathcal{P}_\phi(E)$ , and  $\sup_{k \geq 1} \langle \nu, e^{\alpha \phi} \rangle + \langle \nu_k, e^{\alpha \phi} \rangle$  is finite, for every  $\alpha > 0$ .
- (v) For any  $\mu \in \mathcal{P}_\phi(E)$ ,  $\lim_{k \rightarrow +\infty} \sup_{t \in \mathcal{T}} \Psi_{t,k}(\mu) \leq \sup_{t \in \mathcal{T}} \Psi_t(\mu)$ .
- (vi) If  $(\mu_l)_{l \geq 1}$  converges to  $\mu$  in  $\mathcal{P}_\phi(E)$ , then  $\sup_{k \geq 1} |\sup_{t \in \mathcal{T}} \Psi_{t,k}(\mu_l) - \sup_{t \in \mathcal{T}} \Psi_{t,k}(\mu)| \rightarrow 0$ , and  $|\sup_{t \in \mathcal{T}} \Psi_t(\mu_l) - \sup_{t \in \mathcal{T}} \Psi_t(\mu)| \rightarrow 0$ .
- (vii) There exists  $\tilde{\mu} \in \mathcal{D}_{\mathcal{I}}$  such that  $\sup_{t \in \mathcal{T}} \Psi_t(\tilde{\mu}) < 0$ .

We notice that (A.10)-(i,iv) make  $\mathcal{I}_k$ ,  $\nu_k$  and  $(\Psi_{t,k})_{t \in \mathcal{T}}$  enter the framework of (A.1)-(A.3). We define the sets  $(\mathcal{A}_{\Psi,k})_{k \geq 1}$  accordingly. As previously, (A.10)-(vi) can be verified using Proposition 4.2.8. Similarly, a sufficient condition for the compactness of level sets in (A.10)-(i) is given by Lemma 4.2.4. From (A.10)-(i), the  $(\mathcal{I}_k)_{k \geq 1}$  are good rate functions. Therefore, using the the convexity assumption (A.10)-(ii) (which corresponds to (A.9)), the minimisation problem

$$\bar{\mathcal{I}}_{\Psi,k} := \inf_{\mu \in \mathcal{D}_{\mathcal{I}_k} \cap \mathcal{A}_{\Psi,k}} \mathcal{I}_k(\mu),$$

has a unique minimiser denoted by  $\bar{\mu}_k$ , for every  $k \geq 1$ . Let us now verify the qualification condition in Lemma 4.2.18.

**Lemma 4.2.20** (Qualification). *Under (A.10), there exist  $(\tilde{\mu}_k)_{k \geq 1}$  and  $k_0 \geq 1$  such that  $\tilde{\mu}_k \in \mathcal{D}_{\mathcal{I}_k}$ ,  $(H(\tilde{\mu}_k | \nu_k))_{k \geq 1}$  is bounded, and*

$$\sup_{k \geq k_0} \sup_{t \in \mathcal{T}} \Psi_{t,k}(\tilde{\mu}_k) < 0.$$

To obtain Gibbs measures and show their stability, we need differentiability assumptions.

**Assumption (A.11)** (Local regularity).

- (i) For every large enough  $k$ , the regularity assumptions (A.6) and (A.7)-(i,ii,iii,iv) hold at  $\bar{\mu}_k$  for  $(\mathcal{I}_k, (\Psi_{t,k})_{t \in \mathcal{T}})$ .
- (ii) For every  $k \geq k_0$ , (A.8) holds at  $\bar{\mu}_k$  and  $\tilde{\mu}_k$ , given by Lemma 4.2.20, for  $(\mathcal{I}_k, (\Psi_{t,k})_{t \in \mathcal{T}})$ .
- (iii) There exists  $D^{\mathcal{F},\Psi} \in [0, \infty)$  such that

$$\forall x \in E, \quad \sup_{k \geq 1} \left\{ \left| \frac{\delta \mathcal{F}_k}{\delta \mu}(\bar{\mu}_k, x) \right| + \sup_{t \in \mathcal{T}} \left| \frac{\delta \Psi_{t,k}}{\delta \mu}(\bar{\mu}_k, x) \right| \right\} \leq D^{\mathcal{F},\Psi} [1 + \phi(x)].$$

- (iv) For any sub-sequence  $(\bar{\mu}_{l_k})_{k \geq 1}$  that converges towards some  $\bar{\mu}_\infty$  in  $\mathcal{P}_\phi(E)$ ,  $\bar{\mu}_\infty$  belongs to  $\mathcal{D}_{\mathcal{I}}$ , (A.6) and (A.7)-(i,ii,iii,iv) hold at  $\bar{\mu}_\infty$  for  $(\mathcal{I}, (\Psi_t)_{t \in \mathcal{T}})$ , and for every  $x \in E$ ,

$$\left| \frac{\delta \mathcal{F}_{l_k}}{\delta \mu}(\bar{\mu}_{l_k}, x) - \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_\infty, x) \right| \xrightarrow{k \rightarrow +\infty} 0,$$

$$\sup_{t \in \mathcal{T}} \left| \Psi_{t,l_k}(\bar{\mu}_{l_k}) - \Psi_t(\bar{\mu}_\infty) \right| + \left| \frac{\delta \Psi_{t,l_k}}{\delta \mu}(\bar{\mu}_{l_k}, x) - \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\infty, x) \right| \xrightarrow{k \rightarrow +\infty} 0.$$

Moreover,  $(x \mapsto \frac{\delta \mathcal{F}_{l_k}}{\delta \mu}(\bar{\mu}_{l_k}, x))_{k \geq 1}$ ,  $(x \mapsto \frac{\delta \Psi_{t,l_k}}{\delta \mu}(\bar{\mu}_{l_k}, x))_{t \in \mathcal{T}, k \geq 1}$  are equi-continuous on  $E$ .

The regularity assumptions in (A.11)-(iv) are similar to the ones in Proposition 4.2.8.

**Theorem 4.2.21** (Weak stability).

1. Under (A.10) and (A.11)-(i,ii,iii), for every large enough  $k$ ,  $\bar{\mu}_k$  writes as Gibbs measure (4.13)-(4.15) for some  $\bar{\lambda}_k \in \mathcal{M}_+(\mathcal{T})$ . Moreover, the sequence  $(\bar{\lambda}_k)_k$  is weakly precompact in  $\mathcal{M}_+(\mathcal{T})$ .
2. Under (A.10)-(A.11),  $\mathcal{I}$  has a unique minimiser  $\bar{\mu}$  in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}$ , which is the limit of  $(\bar{\mu}_k)_{k \geq 1}$  in  $\mathcal{P}_{\phi}(E)$ . Moreover, (4.13)-(4.15) hold at  $\bar{\mu}$  for any limit point  $\bar{\lambda}_{\infty}$  of  $(\bar{\lambda}_k)_k$ .

Since we are working with  $\mathcal{S} = \emptyset$ , following Remark 4.2.14, in the statement of Theorem 4.2.21-1, there is no need to for introducing neither the set  $\bar{S}_k$  nor the function  $\bar{\zeta}_k$ .

Theorem 4.2.17, Proposition 4.2.19, Lemma 4.2.20 and Theorem 4.2.21 are proved in Section 4.4.3.

### 4.3 Application to stochastic processes

In this section we fix  $T > 0$  and  $d \geq 1$ . We consider the case where  $E$  is the set of continuous trajectories  $C([0, T], \mathbb{R}^d)$ , endowed with the sup norm. In this setting, we denote by  $x_{[0, T]} = (x_t)_{t \in [0, T]}$  typical elements of  $C([0, T], \mathbb{R}^d)$ , and by  $\mu_{[0, T]}$  typical elements of  $\mathcal{P}(C([0, T], \mathbb{R}^d))$ . If  $\mathbf{X}_{[0, T]}$  is a random variable in  $C([0, T], \mathbb{R}^d)$  with law  $\mu_{[0, T]}$ , then the marginal distribution of  $\mathbf{X}_t$  is denoted by  $\mu_t$ . We focus on inequality constraints of the form

$$\forall t \in \mathcal{T} = [0, T], \quad \Psi_t(\mu_{[0, T]}) = \Psi(\mu_t), \quad (4.18)$$

for some function  $\Psi : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$ . We moreover focus on the case  $\mathcal{F} \equiv 0$ , so that minimisers of (4.8) describe the asymptotic behaviour of large systems of independent and identically distributed continuous stochastic processes, conditionally on the constraint (4.18) on their empirical distribution at all times  $t \in [0, T]$ .

In Section 4.3.1, we provide conditions on the function  $\Psi$  and on the reference measure  $\nu_{[0, T]}$  ensuring that the assumptions of the main theorems of Section 4.2 are satisfied. We next present two examples of applications of these theorems: in Section 4.3.2, we construct and study a constrained version of the Schrödinger bridge, and in Section 4.3.3, we state a Gibbs principle for i.i.d. diffusion processes under constraints of the form (4.18). In particular, we provide a detailed description of the law of the minimiser  $\bar{\mu}_{[0, T]}$  as the law of a diffusion process with tilted initial condition and modified drift coefficient with respect to the reference measure  $\nu_{[0, T]}$ . To illustrate the latter result, we present simple examples in Section 4.3.4.

#### 4.3.1 Constraints on time marginal laws

##### 4.3.1.1 Global setting

Throughout Section 4.3, we work with  $E = C([0, T], \mathbb{R}^d)$  and set  $\phi(x_{[0, T]}) := \sup_{0 \leq t \leq T} |x_t|^p$  for some  $p \in [1, +\infty)$ , so that the topology on  $\mathcal{P}_{\phi}(E) = \mathcal{P}_p(C([0, T], \mathbb{R}^d))$  corresponds to the one induced by the  $p$ -Wasserstein distance  $W_p$ . In this context, we have the following standard properties, which follow from elementary coupling arguments.

**Lemma 4.3.1** (Metric properties of time marginal distributions).



(i) For any  $t \in [0, T]$ , the map

$$\begin{aligned} (\mathcal{P}_p(C([0, T], \mathbb{R}^d)), W_p) &\rightarrow (\mathcal{P}_p(\mathbb{R}^d), W_p) \\ \mu_{[0, T]} &\mapsto \mu_t \end{aligned}$$

is 1-Lipschitz continuous.

(ii) For any  $\mu_{[0, T]} \in \mathcal{P}_p(C([0, T], \mathbb{R}^d))$ , the map  $t \mapsto \mu_t$  is continuous.

We shall always take  $\mathcal{F} \equiv 0$ , so that given a reference probability measure  $\nu_{[0, T]}$  on  $C([0, T], \mathbb{R}^d)$ , Assumption (A.1) holds as soon as

$$\forall \alpha > 0, \quad \mathbb{E}_{\nu_{[0, T]}} \left[ e^{\alpha \sup_{0 \leq t \leq T} |\mathbf{X}_t|^p} \right] < \infty, \quad (4.19)$$

where the notation  $\mathbb{E}_{\nu_{[0, T]}}[\cdot \cdot \cdot]$  means that the process  $\mathbf{X}_{[0, T]}$  has distribution  $\nu_{[0, T]}$ . Moreover, the set  $\mathcal{D}_{\mathcal{I}}$  rewrites

$$\mathcal{D}_{\mathcal{I}} = \left\{ \mu_{[0, T]} \in \mathcal{P}_p(C([0, T], \mathbb{R}^d)), H(\mu_{[0, T]} | \nu_{[0, T]}) < \infty \right\}.$$

Equality constraints are represented by a family  $(\zeta_s)_{s \in \mathcal{S}}$  of functions  $C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$  which satisfy (A.2). Last, we fix a function  $\Psi : \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}$  and define the functions  $\Psi_t$ ,  $t \in [0, T]$ , by (4.18). Then (A.3) holds as soon as  $\Psi$  is lower semi-continuous on  $\mathcal{P}_p(\mathbb{R}^d)$ . Overall, we deduce that in this setting, (A.1)-(A.2)-(A.3) hold under the following condition.

**Assumption (B.1)** (Condition for (A.1)-(A.2)-(A.3)).  $\mathcal{F} \equiv 0$ , (4.19) holds,  $(\zeta_s)_{s \in \mathcal{S}}$  satisfies (A.2) and  $\Psi : \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}$  is lower semi-continuous.

Notice that under (B.1),  $\mathcal{I}(\mu_{[0, T]}) = H(\mu_{[0, T]} | \nu_{[0, T]})$  is always a good rate function thanks to Lemma 4.2.4, so minimisers to (4.8) exist as soon as  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^{\zeta}$  is nonempty.

#### 4.3.1.2 Differentiability of the constraints

In the assumptions of Theorem 4.2.7 (via Proposition 4.2.8) and Theorem 4.2.13, the linear functional derivative of the function  $\Psi_t$  plays a central role. The next lemma, whose proof is straightforward, links the linear functional derivatives of  $\Psi_t$  (defined on  $\mathcal{P}_p(C([0, T], \mathbb{R}^d))$ ) to that of  $\Psi$  (defined on  $\mathcal{P}_p(\mathbb{R}^d)$ ).

**Lemma 4.3.2** (Linear functional derivative of  $\Psi_t$ ). *Let  $\mu \in \mathcal{P}_p(\mathbb{R}^d)$  be such that  $\Psi$  is differentiable at  $\mu$  w.r.t. the set of directions  $\mathcal{P}_p(\mathbb{R}^d)$ , and let  $t \in [0, T]$ . For any  $\mu_{[0, T]} \in \mathcal{P}_p(C([0, T], \mathbb{R}^d))$  such that  $\mu_t = \mu$ ,  $\Psi_t$  is differentiable at  $\mu_{[0, T]}$  w.r.t. the set of directions  $\mathcal{P}_p(C([0, T], \mathbb{R}^d))$ , and for any  $x_{[0, T]} \in C([0, T], \mathbb{R}^d)$ ,*

$$\frac{\delta \Psi_t}{\delta \mu_{[0, T]}}(\mu_{[0, T]}, x_{[0, T]}) = \frac{\delta \Psi}{\delta \mu}(\mu_t, x_t).$$

As a consequence of Lemma 4.3.2, we get that the assumptions of Proposition 4.2.8 are satisfied under the following condition on  $\Psi$ .

**Assumption (B.2)** (Global differentiability of  $\Psi$ ). *For any  $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ ,  $\Psi$  is differentiable at  $\mu$  w.r.t. the set of directions  $\mathcal{P}_p(\mathbb{R}^d)$ . Moreover,*

(i) *for any compact set  $K \subset \mathcal{P}_p(\mathbb{R}^d)$ , there exists  $D_K^{\Psi} \in [0, \infty)$  such that*

$$\forall x \in \mathbb{R}^d, \quad \sup_{\mu \in K} \left| \frac{\delta \Psi}{\delta \mu}(\mu, x) \right| \leq D_K^{\Psi} [1 + |x|^p];$$

- (ii) for any compact set  $K \subset \mathcal{P}_p(\mathbb{R}^d)$ , the family of functions  $(x \mapsto \frac{\delta\Psi}{\delta\mu}(\mu, x))_{\mu \in K}$  is equi-continuous;
- (iii) for any compact set  $\Xi \subset \mathbb{R}^d$ , the family of functions  $(\mu \mapsto \frac{\delta\Psi}{\delta\mu}(\mu, x))_{x \in \Xi}$  is uniformly equi-continuous on  $\mathcal{P}_p(\mathbb{R}^d)$ .

The fact that the assumptions of Proposition 4.2.8 are satisfied under (B.2) essentially follows from Lemma 4.3.2, together with the observation that if  $K_{[0,T]}$  is a compact subset of  $\mathcal{P}_p(C([0,T], \mathbb{R}^d))$ , then  $K := \{\mu_t : t \in [0,T], \mu_{[0,T]} \in K_{[0,T]}\}$  is a compact subset of  $\mathcal{P}_p(\mathbb{R}^d)$ . The latter statement is an easy consequence of Lemma 4.3.1 (i) and (ii).

Hence, since we are working with  $\mathcal{F} \equiv 0$ , Theorem 4.2.7 applies as soon as (B.1)-(B.2) hold (with  $\mathcal{S} = \emptyset$ ), together with the constraint qualification condition that for any  $\mu_{[0,T]} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}$ , there exists  $\tilde{\mu}_{[0,T]} \in \mathcal{D}_{\mathcal{I}}$  such that, for any  $\varepsilon > 0$  small enough,  $\Psi(\mu_t + \varepsilon(\tilde{\mu}_t - \mu_t)) < 0$  for all  $t \in [0, T]$ .

We now turn our attention to the assumptions of Theorem 4.2.13. Since  $\mathcal{F} \equiv 0$ , (A.6) and (A.8) necessarily hold true, at any  $\bar{\mu}_{[0,T]} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^{\zeta}$ . On the other hand, Lemmata 4.3.1 and 4.3.2 again show that given  $\bar{\mu}_{[0,T]}$ , (A.7) is satisfied under the following conditions on  $\Psi$ .

**Assumption (B.3)** (Regularity of  $\Psi$  and constraint qualification at  $\bar{\mu}_{[0,T]}$ ).

- (i) The function  $\Psi$  is continuous on  $\mathcal{P}_p(\mathbb{R}^d)$ .
- (ii) For any  $t \in \mathcal{T}$ ,  $\Psi$  is differentiable at  $\bar{\mu}_t$  w.r.t. the set of directions  $\mathcal{P}_p(\mathbb{R}^d)$ , uniformly in  $t$ , in the sense that for any  $\mu_{[0,T]} \in \mathcal{P}_p(C([0,T], \mathbb{R}^d))$ ,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0,T]} \left| \frac{\Psi((1-\varepsilon)\bar{\mu}_t + \varepsilon\mu_t) - \Psi(\bar{\mu}_t)}{\varepsilon} - \left\langle \mu_t - \bar{\mu}_t, \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t) \right\rangle \right| = 0.$$

- (iii) The function  $(t, x) \mapsto \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t, x)$  is continuous on  $[0, T] \times \mathbb{R}^d$ .

- (iv) There exists  $D^{\Psi} \in [0, \infty)$  such that

$$\forall x \in \mathbb{R}^d, \quad \sup_{t \in [0,T]} \left| \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t, x) \right| \leq D^{\Psi} [1 + |x|^p].$$

- (v) There exist  $\tilde{\mu}_{[0,T]}$  in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  and  $\tilde{\varepsilon} > 0$  such that

$$\forall t \in [0, T], \quad \Psi(\bar{\mu}_t) + \tilde{\varepsilon} \left\langle \tilde{\mu}_t - \bar{\mu}_t, \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t) \right\rangle < 0.$$

This condition allows to apply Theorem 4.2.13 to characterise minimisers of (4.8). Let us finally mention that, as soon as  $\Psi$  is convex, then (A.9) holds and the minimiser is therefore unique. Moreover, in this case, Lemma 4.2.18 implies that (B.3)-(v) holds (with  $\tilde{\varepsilon} = 1$ ) as soon as there exists  $\tilde{\mu}_{[0,T]}$  in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  such that  $\Psi(\tilde{\mu}_t) < 0$  for any  $t \in [0, T]$ .

#### 4.3.1.3 Example of a nonlinear inequality constraint

A typical example of a nonlinear function  $\Psi$  is given as follows. Let  $W : \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function, bounded from below, which satisfies  $W(x) \leq C(1 + |x|^p)$ . For any  $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ , set

$$\Psi(\mu) = \langle \mu, W \star \mu \rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) \mu(dx) \mu(dy). \quad (4.20)$$

Then  $\Psi$  is well-defined, and lower semi-continuous, on  $\mathcal{P}_p(\mathbb{R}^d)$ . Moreover, it is differentiable on  $\mathcal{P}_p(\mathbb{R}^d)$  w.r.t. the set of directions  $\mathcal{P}_p(\mathbb{R}^d)$ , and its linear functional derivative writes

$$\frac{\delta\Psi}{\delta\mu}(\mu, x) = (W + W_-) \star \mu(x) - 2\Psi(\mu), \quad W_-(x) := W(-x).$$

However, in general, it is not convex. For example, if  $p \geq 2$ , then one may take  $W(x) = \frac{1}{2}|x|^2 - M$ , for some  $M \geq 0$ , so the condition that  $\Psi(\mu) \leq 0$  means that the variance of  $\mu$  (understood, for  $d \geq 2$ , as the trace of the covariance matrix) is bounded from above by  $M$ . Then  $\Psi$  is concave.

### 4.3.2 Constrained Schrödinger bridge

Given a reference probability measure  $\nu_{[0,T]} \in \mathcal{P}_p(C([0,T], \mathbb{R}^d))$ , for example the law of the  $d$ -dimensional Brownian motion on  $[0, T]$ , and two probability measures  $\mu^{\text{ini}}, \mu^{\text{fin}} \in \mathcal{P}_p(\mathbb{R}^d)$ , such that  $\mu^{\text{ini}} \ll \nu_0$  and  $\mu^{\text{fin}} \ll \nu_T$ , the standard Schrödinger bridge is the continuous process on  $[0, T]$  whose law  $\bar{\mu}_{[0,T]}$  is the minimiser of the problem

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_p(C([0,T], \mathbb{R}^d)), \\ \mu_0 = \mu^{\text{ini}}, \mu_T = \mu^{\text{fin}}}} H(\mu_{[0,T]} | \nu_{[0,T]}).$$

In this section, we address the *dynamically constrained* version of this problem, namely, for a lower semi-continuous function  $\Psi : \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}$ , we consider the minimisation problem

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_p(C([0,T], \mathbb{R}^d)), \\ \mu_0 = \mu^{\text{ini}}, \mu_T = \mu^{\text{fin}}, \\ \forall t \in [0, T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]} | \nu_{[0,T]}). \quad (4.21)$$

We call minimisers of this problem *constrained Schrödinger bridges*. Our main result for this constrained problem reads as follows.

**Theorem 4.3.3** (Constrained Schrödinger bridge). *Let  $\nu_{[0,T]} \in \mathcal{P}_p(C([0,T], \mathbb{R}^d))$  satisfy (4.19), and  $\Psi : \mathcal{P}_p(\mathbb{R}^d) \rightarrow \mathbb{R}$  be a lower semi-continuous function. Let  $\mu^{\text{ini}}, \mu^{\text{fin}} \in \mathcal{P}_p(\mathbb{R}^d)$  be such that  $\nu_{0,T} \sim \mu^{\text{ini}} \otimes \mu^{\text{fin}}$ , where  $\nu_{0,T}$  denotes the marginal distribution of the pair  $(X_0, X_T)$  under  $\nu_{[0,T]}$ .*

1. *If the infimum in (4.21) is finite, then it is reached at some  $\bar{\mu}_{[0,T]}$  which satisfies  $\bar{\mu}_0 = \mu^{\text{ini}}$ ,  $\bar{\mu}_T = \mu^{\text{fin}}$ , and  $\Psi(\bar{\mu}_t) \leq 0$  for any  $t \in [0, T]$ . Moreover, if  $\Psi$  is convex, then this minimiser is unique.*
2. *If  $\bar{\mu}_{[0,T]}$  is a minimiser at which (B.3) holds, then there exist  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  and measurable  $\bar{\zeta}_0, \bar{\zeta}_T : \mathbb{R}^d \rightarrow \mathbb{R}$  such that*

$$\frac{d\bar{\mu}_{[0,T]}}{d\nu_{[0,T]}}(x_{[0,T]}) = \bar{Z}^{-1} \exp \left[ -\bar{\zeta}_0(x_0) - \bar{\zeta}_T(x_T) - \int_{[0,T]} \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t, x_t) \bar{\lambda}(dt) \right], \quad (4.22)$$

where  $\bar{Z} \in (0, \infty)$  is a normalising constant, and the slackness condition holds

$$\Psi(\bar{\mu}_t) = 0 \quad \text{for } \bar{\lambda}\text{-a.e. } t \in [0, T].$$

The proof of Theorem 4.3.3 follows from the application of Theorem 4.2.13 to the case where the equality constraints are defined as follows: let  $(\phi^k)_{k \geq 1}$  be a countable family of

bounded continuous functions  $\mathbb{R}^d \rightarrow \mathbb{R}$  which separates measures. For  $k \geq 1$ , we define  $\phi_0^k, \phi_T^k : C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$  by

$$\phi_0^k(x_{[0,T]}) := \phi^k(x_0) - \int_{\mathbb{R}^d} \phi^k d\mu^{\text{ini}}, \quad \phi_T^k(x_{[0,T]}) := \phi^k(x_T) - \int_{\mathbb{R}^d} \phi^k d\mu^{\text{fin}}.$$

We finally set  $(\zeta_s)_{s \in \mathcal{S}} := (\phi_0^k)_{k \geq 1} \cup (\phi_T^k)_{k \geq 1}$ . The details of the proof are postponed to Section 4.5.1.

Compared to Theorem 4.2.13, a major improvement of Theorem 4.3.3 is that  $\bar{\mu}_{[0,T]} \sim \nu_{[0,T]}$ , see Remark 4.2.14.

**Remark 4.3.4** (Multi-marginal constraints and support constraints). *We could easily adapt the above result to impose marginal laws at arbitrarily many instants. We could also consider time-dependent constraints  $\Psi(t, \mu_t)$ .*

### 4.3.3 Gibbs principle for diffusion processes

In this section, we focus on the Gibbs principle for i.i.d. diffusion processes under constraints of the form (4.18) on the empirical distribution.

**Assumption (C.1)** (On the reference SDE). *The functions  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  are measurable and locally bounded,  $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$ , and in addition:*

- (i) *uniformly in  $t \in [0, T]$ ,  $x \mapsto b_t(x)$  and  $x \mapsto \sigma_t(x)$  are Lipschitz continuous;*
- (ii) *there exists  $M_\sigma \geq 0$  such that  $|\sigma_t(x)| \leq M_\sigma$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , and  $t \mapsto \sigma_t(x)$  is locally Hölder-continuous;*
- (iii) *for all  $\alpha > 0$ ,  $\int_{\mathbb{R}^d} e^{\alpha|x|^p} \nu_0(dx) < \infty$ .*

Under (C.1), for a given  $d$ -dimensional Brownian motion  $(B_t)_{t \in [0, T]}$ , it is standard that the SDE

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 \sim \nu_0, \quad (4.23)$$

has a pathwise unique strong solution  $X_{[0, T]}$ , whose law in  $C([0, T], \mathbb{R}^d)$  is denoted by  $\nu_{[0, T]}$ . Moreover, if  $p < 2$  then this measure satisfies (4.19).

As a consequence, the law  $\Pi^N$  of the empirical distribution of  $N$  iid copies of the SDE (4.23) satisfies a LDP on  $\mathcal{P}_p(C([0, T], \mathbb{R}^d))$ , with good rate function given by  $H(\mu_{[0, T]} | \nu_{[0, T]})$  Wang et al. [2010]. Thus, if  $\Psi$  is such that the functions  $(\Psi_t)_{t \in [0, T]}$  satisfy the assumptions of Theorem 4.2.7, the conditional law  $\Pi^N(\cdot | \mathcal{A}_\Psi)$  satisfies a LDP with good rate function given by

$$H(\mu_{[0, T]} | \nu_{[0, T]}) - \inf_{\mu'_{[0, T]} \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_\Psi} H(\mu'_{[0, T]} | \nu_{[0, T]})$$

on  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_\Psi$ , and as soon as this good rate function has a unique minimiser  $\bar{\mu}_{[0, T]}$ ,  $\Pi^N(\cdot | \mathcal{A}_\Psi)$  converges weakly to  $\delta_{\bar{\mu}_{[0, T]}}$ . For such a minimiser, if  $\Psi$  satisfies (B.3), Theorem 4.2.13 yields the existence of  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  such that

$$\frac{d\bar{\mu}_{[0, T]}}{d\nu_{[0, T]}}(x_{[0, T]}) = \frac{1}{\bar{Z}} \exp \left[ - \int_{[0, T]} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t, x_t) \bar{\lambda}(dt) \right], \quad \bar{Z} \in (0, +\infty).$$

Starting from this expression, the goal of the present section is to express  $\bar{\mu}_{[0, T]}$  as the law of the solution  $\bar{X}_{[0, T]}$  to some SDE of the form

$$d\bar{X}_t = \bar{b}_t(\bar{X}_t)dt + \sigma_t(\bar{X}_t)dB_t, \quad \bar{X}_0 \sim \bar{\mu}_0, \quad (4.24)$$

and to express the drift coefficient  $\bar{b}_t$  and the probability measure  $\bar{\mu}_0$  in terms of the original SDE (4.23). We shall actually work in the more general setting in which there exist measurable functions  $c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ , and a positive Radon measure  $\lambda \in \mathcal{M}_+([0, T])$ , such that the measure  $\bar{\mu}_{[0, T]}$  satisfies

$$\frac{d\bar{\mu}_{[0, T]}}{d\nu_{[0, T]}}(x_{[0, T]}) = \frac{1}{Z} \exp \left[ - \int_0^T c_t(x_t) dt - \int_{[0, T]} \psi_t(x_t) \lambda(dt) \right], \quad Z \in (0, +\infty), \quad (4.25)$$

which would thus allow us to take a nonzero interaction functional  $\mathcal{F}$  into account. We postpone the detail of our assumptions on  $c$  and  $\psi$  to (C.2) below, and first provide an informal sketch of our argument.

Let us assume that  $\lambda$  has a density  $(\lambda_t)_{0 \leq t \leq T}$  w.r.t. to the Lebesgue measure. Under very weak regularity assumptions, [Léonard, 2022, Theorem 5.24] shows that  $\bar{\mu}_{[0, T]}$  is the law of the solution to (4.24), with

$$\bar{b}_t(x) = b_t(x) + a_t(x)\alpha_t(x),$$

for  $a_t := \sigma_t \sigma_t^\top$  and a measurable  $\alpha : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Moreover,

$$\alpha_t = -\nabla \varphi_t, \quad \frac{d\bar{\mu}_0}{d\nu_0}(x) = Z^{-1} e^{-\varphi_0(x)},$$

where  $\varphi$  is the solution to the Hamilton-Jacobi equation (HJB)

$$\partial_t \varphi_t + b_t \cdot \nabla \varphi_t - \frac{1}{2} |\sigma_t^\top \nabla \varphi_t|^2 + \frac{1}{2} \text{Tr}[a_t \nabla^2 \varphi_t] = -c_t - \psi_t \lambda_t, \quad \varphi_T = 0, \quad (4.26)$$

in a generalised sense. Formally, the Feynman-Kac formula tells that the solution of (4.26) can be represented as

$$\varphi_t(x) := -\log \mathbb{E} \exp \left[ \int_t^T c_s(Z_s^{t,x}) ds + \int_{[t, T]} \psi_s(Z_s^{t,x}) \lambda(ds) \right],$$

where  $(Z_s^{t,x})_{t \leq s \leq T}$  is the solution of  $dZ_s^{t,x} = b_s(Z_s^{t,x}) ds + \sigma_s(Z_s^{t,x}) dB_s$ ,  $t \leq s \leq T$ , with  $Z_t^{t,x} = x$ . We now prove an analogous of this result in a more classical sense, and without assuming existence of a density for  $\lambda$ .

**Assumption (C.2)** (Regularity of the coefficients of (4.26)). *Assumption (C.1) holds with  $p = 1$ ,  $\psi, c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  are measurable and locally bounded, and in addition:*

- (i) *the function  $(t, x) \mapsto \psi_t(x)$  is continuous;*
- (ii) *uniformly in  $t \in [0, T]$ ,  $c_t$  and  $\psi_t$  are Lipschitz-continuous, and  $\sigma_t^{-1}$  is bounded;*
- (iii) *the families  $(x \mapsto \nabla b_t(x))_{0 \leq t \leq T}$ ,  $(x \mapsto \nabla \sigma_t(x))_{0 \leq t \leq T}$ ,  $(x \mapsto \nabla c_t(x))_{0 \leq t \leq T}$ , and  $(x \mapsto \nabla \psi_t(x))_{0 \leq t \leq T}$  are equi-continuous.*

We now fix a positive Radon measure  $\lambda \in \mathcal{M}_+([0, T])$ . To make sense of the analogous of (4.26) when  $\lambda$  may not have a density, we rely on the integrated version

$$-\varphi_t + \int_t^T \left( b_s \cdot \nabla \varphi_s - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s|^2 + \frac{1}{2} \text{Tr}[a_s \nabla^2 \varphi_s] + c_s \right) ds + \int_{[t, T]} \psi_s \lambda(ds) = 0. \quad (4.27)$$

If  $\lambda(\{t\}) \neq 0$ , an arbitrary choice has been made when considering integrals over  $[t, T]$  rather than  $(t, T]$ . However, the set of atoms of  $\lambda$  is at most countable; hence the choice of

the interval does not matter if we only require equality Lebesgue-a.e. For approximation and stability purposes, we introduce a specific notion of solution, which relies on the following construction.

Under the assumptions made on  $\sigma$  in (C.1)-(C.2), a consequence of [Rubio, 2011, Theorem 2.1] is that for any  $0 < s \leq T$  and any continuous  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  with linear growth, the parabolic equation

$$\begin{cases} \partial_t \varphi_t + \frac{1}{2} \text{Tr}[a_t \nabla^2 \varphi_t] = 0, & 0 \leq t \leq s, \\ \varphi_s = \varphi, \end{cases}$$

has a unique solution  $\varphi \in C([0, s] \times \mathbb{R}^d) \cap C^{1,2}((0, s) \times \mathbb{R}^d)$  with linear growth. From this, we define the evolution system  $(S_{t,s})_{0 \leq t \leq s \leq T}$  by

$$S_{t,s}[\varphi](x) = \varphi_t(x),$$

for any continuous  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  with linear growth,  $S_{t,s}[\varphi]$  being a  $C^2$  function with linear growth as soon as  $t < s$ .

**Definition 4.3.5.** *We say that a measurable  $\varphi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a mild solution of (4.27) if for Lebesgue-a.e.  $t \in [0, T]$ ,  $x \mapsto \varphi_t(x)$  is  $C^1$ ,  $(t, x) \mapsto \nabla \varphi_t(x)$  is bounded measurable, and for a.e.  $t \in [0, T]$ ,*

$$\varphi_t = \int_t^T S_{t,s} [b_s \cdot \nabla \varphi_s - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s|^2 + c_s] ds + \int_{[t,T]} S_{t,s} [\psi_s] \lambda(ds). \quad (4.28)$$

*This implies that  $x \mapsto \varphi_t(x)$  is  $C^2$  for Lebesgue-a.e.  $t$ .*

The above definition is a natural extension of the Duhamel formula for perturbations of a linear PDE, which was already used in Daudin [2023b]. Lebesgue-almost sure uniqueness always holds for (4.27) in the sense of Definition 4.3.5 because the difference of two solutions solves a classical linear parabolic equation without source term.

**Theorem 4.3.6.** *Under (C.2), let  $\bar{\mu}_{[0,T]}$  be given by (4.25). Then,  $\bar{\mu}_{[0,T]}$  is the path-law of a solution to the SDE*

$$d\bar{X}_t = b_t(\bar{X}_t)dt - a_t(\bar{X}_t)\nabla \varphi_t(\bar{X}_t)dt + \sigma_t(\bar{X}_t)dB_t, \quad \bar{X}_0 \sim Z^{-1}e^{-\varphi_0(x)}\nu_0(dx),$$

where  $\varphi_t$  is given by

$$\varphi_t(x) := -\log \mathbb{E} \exp \left[ \int_t^T c_s(Z_s^{t,x})ds + \int_{[t,T]} \psi_s(Z_s^{t,x})\lambda(ds) \right], \quad (4.29)$$

where  $(Z_s^{t,x})_{t \leq s \leq T}$  is the path-wise unique solution to

$$dZ_s^{t,x} = b_s(Z_s^{t,x})ds + \sigma_s(Z_s^{t,x})dB_s, \quad t \leq s \leq T, \quad (4.30)$$

with  $Z_t^{t,x} = x$ . Moreover,  $\varphi$  is the mild solution of (4.27) in the sense of Definition 4.3.5.

Theorem 4.3.6 is proved in Section 4.5.2.

**Remark 4.3.7.** *If one wants to apply Theorem 4.3.6 with  $\psi_t(x) = \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t, x)$  in order to describe the measure  $\bar{\mu}_{[0,T]}$  obtained by Theorem 4.2.13, with a function  $\Psi$  that satisfies (B.3), then the global Lipschitz assumption made on  $\psi_t$  in (C.2) implies that  $\frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t, x)$  must grow at most linearly in  $x$ . In the case of a linear constraint, where  $\Psi(\mu) = \langle \mu, \psi \rangle$  for some function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ , this implies that  $\psi$  must grow at most linearly in  $x$ . In this case, and if  $\psi$  is continuous, then (B.3) is satisfied already for  $p = 1$ . Therefore, there is no need to require (C.1) to hold with some  $p > 1$  in the statement of (C.2).*

### 4.3.4 Gaussian examples with inequality constraint on the expectation

In this section, we provide examples of one-dimensional diffusion processes  $X_{[0,T]}$ , for which the results of Theorems 4.2.7 and 4.2.13 hold, with the linear inequality constraint  $\Psi(\mu_{[0,T]}) = \langle \mu_t, \psi \rangle$ , where we fix  $\psi(x) = x$ . In other words, we consider cases where the conditional distribution of  $X_{[0,T]}^{1,N}$  given the event

$$\forall t \in [0, T], \quad \frac{1}{N} \sum_{i=1}^N X_t^{i,N} \leq 0$$

converges as  $N \rightarrow +\infty$  towards the measure  $\bar{\mu}_{[0,T]}$  with density w.r.t. to  $\nu_{[0,T]} := \mathcal{L}(X_{[0,T]})$  given by

$$\frac{d\bar{\mu}_{[0,T]}}{d\nu_{[0,T]}}(x_{[0,T]}) = \frac{1}{Z} \exp \left[ - \int_{[0,T]} x_t \bar{\lambda}(dt) \right],$$

for some Lagrange multiplier  $\bar{\lambda} \in \mathcal{M}_+([0, T])$ . Then by Theorem 4.3.6,  $\bar{\mu}_{[0,T]}$  is the law of a *corrected* diffusion process  $\bar{X}_{[0,T]}$  with modified drift and tilted initial condition. On our examples, we are able to give *explicit* formulas for these corrections, as well as for  $\bar{\lambda}$ .

#### 4.3.4.1 Time-inhomogeneous drifted Brownian motion

Let  $m : [0, T] \rightarrow \mathbb{R}$  be a  $C^2$  function such that  $m(0) = 0$ ,  $\dot{m} \geq 0$  and  $\ddot{m} \leq 0$ , where  $\dot{m} := \frac{d}{dt}m$ . Let  $\nu_{[0,T]}$  be the path-law of the Gaussian process given by

$$dX_t = \dot{m}(t)dt + dB_t, \quad X_0 \sim \mathcal{N}(x_0, \sigma^2).$$

To avoid situations in which  $\nu_{[0,T]}$  already satisfies the constraints, we assume that  $x_0 + m(T) > 0$  and  $\sigma > 0$ . The corrected process given by Theorems 4.2.13-4.3.6 shall be the solution to

$$d\bar{X}_t = \dot{m}(t)dt - \nabla \varphi_t(\bar{X}_t)dt + dB_t, \quad \bar{X}_0 \sim Z^{-1} e^{-\varphi_0(x)} \nu_0(dx), \quad (4.31)$$

where  $\varphi$  solves the HJB equation (4.27). In the current simple case, we can look for solutions of (4.27) with  $\nabla \varphi_t$  independent of  $x$ . Several options appear, depending on the function

$$\bar{m} : t \mapsto x_0 + m(t) - [\sigma^2 + t]\dot{m}(t).$$

Our assumptions on  $m$  make  $\bar{m}$  non-decreasing.

1. If  $\bar{m}(T) < 0$ : we can verify that

$$\bar{\lambda}(dt) = \frac{x_0 + m(T)}{\sigma^2 + T} \delta_T(dt), \quad \nabla \varphi_t(x) = \frac{x_0 + m(T)}{\sigma^2 + T},$$

satisfy the HJB equation (4.27), and that the process  $\bar{X}_{[0,T]}$  given by (4.31) satisfies  $\bar{X}_0 \sim \mathcal{N}(\frac{T}{T+\sigma^2}(x_0 - \sigma^2 \dot{m}(T)), \sigma^2)$ ,  $\mathbb{E}[\bar{X}_t] < 0$  for  $t \in [0, T)$ , and  $\mathbb{E}[\bar{X}_T] = 0$ . From Theorem 4.3.6 and the sufficient condition given by Theorem 4.2.17, this shows that  $\mathcal{L}(\bar{X}_{[0,T]})$  is indeed the corrected law  $\bar{\mu}_{[0,T]}$ . Both the velocity and the initial condition are tilted by the effect of the multiplier at the final time.

2. If  $\bar{m}(0) \leq 0 \leq \bar{m}(T)$ : let  $\bar{\tau}$  be the first time at which  $\bar{m}$  vanishes. We similarly obtain a solution of (4.27) by considering

$$\bar{\lambda}(dt) = -\mathbb{1}_{[\bar{\tau}, T]} \dot{m}(t) dt + \dot{m}(T) \delta_T(dt), \quad \nabla \varphi_t(x) = \begin{cases} \dot{m}(\bar{\tau}), & \text{if } t \leq \bar{\tau}, \\ \dot{m}(t), & \text{if } t > \bar{\tau}. \end{cases}$$

The related process  $\bar{X}_{[0, T]}$  satisfies  $\bar{X}_0 \sim \mathcal{N}(x_0 - \sigma^2 \dot{m}(\bar{\tau}), \sigma^2)$ ,  $\mathbb{E}[\bar{X}_t] < 0$  for  $t \in [0, \bar{\tau})$ , and  $\mathbb{E}[\bar{X}_t] = 0$  for  $t \in [\bar{\tau}, T]$ . As previously, this shows that  $\mathcal{L}(\bar{X}_{[0, T]}) = \bar{\mu}_{[0, T]}$ . Both the velocity and the initial condition are tilted. The continuous part of the multiplier activates at the first time where  $\mathbb{E}[\bar{X}_t]$  vanishes. Its effect is to decrease the correction  $\nabla \varphi_t(x)$  of the velocity, which is generated by the atom at the terminal time. The process is corrected at the second order in time (the acceleration is modified), as it is customary in control theory.

3. If  $\bar{m}(0) > 0$ : similarly, we verify that the corrected process is given by (4.31) with

$$\bar{\lambda}(dt) = \frac{x_0 - \sigma^2 \dot{m}(0)}{\sigma^2} \delta_0(dt) - \dot{m}(t) dt + \dot{m}(T) \delta_T(dt), \quad \nabla \varphi_t(x) = \begin{cases} \frac{x_0}{\sigma^2}, & \text{at } t = 0, \\ \dot{m}(t), & \text{if } t > 0, \end{cases}$$

together with  $\bar{X}_0 \sim \mathcal{N}(0, \sigma^2)$ . The corrected process now satisfies  $\mathbb{E}[\bar{X}_t] = 0$  at each time. The atom at time 0 generates a discontinuity that pushes the initial law on the constraint before starting the dynamics.

The transition between the different cases is continuous. Interestingly,  $\bar{\lambda}$  having a density on  $(0, T)$  is directly related to the (second order) regularity of the input. If we decrease the  $C^2$  regularity of  $m$ , other atoms may appear within  $\bar{\lambda}$ .

#### 4.3.4.2 Ornstein-Uhlenbeck process

We now assume that  $\nu_{[0, T]}$  is the path-law of the Gaussian process given by

$$dX_t = (1 - X_t) dt + dB_t, \quad X_0 \sim \mathcal{N}(x_0, \sigma^2).$$

To avoid situations in which  $\nu_{[0, T]}$  already satisfies the constraints, we assume that  $1 + (x_0 - 1)e^{-T} > 0$  and  $\sigma > 0$ . The corrected process shall still be the solution of (4.31) with  $\varphi$  solving the HJB equation (4.27). In this simple case, we can still look for affine solutions of (4.27) with  $\nabla \varphi_t$  independent of  $x$ . As in Section 4.3.4.1, several options appear, now depending on the functions

$$m_{\tau, \lambda} : t \in \mathbb{R}_{\geq 0} \mapsto 1 + (x_0 - 1 - \sigma^2 \lambda e^{-\tau}) e^{-t} - \lambda e^{-\tau} \sinh(t),$$

for  $\tau \geq 0$  and  $\lambda \in [0, 1]$ . Since  $e^{2t} m_{t, 1}(t)$  is a second-order polynomial in  $e^t$ , we easily get that  $x_0 \leq \sigma^2$  implies the existence of a unique  $\bar{\tau} > 0$  such that  $m_{\bar{\tau}, 1}(\bar{\tau}) = 0$ . Using that  $\cosh(\bar{\tau}) \geq 0$ , this implies that  $x_0 - 1 - \sigma^2 e^{-\bar{\tau}} \leq 0$ . We can then check that  $t \mapsto m_{\bar{\tau}, 1}(t)$  is increasing on  $[0, \bar{\tau}]$ .

1. If  $x_0 \leq \sigma^2$  and  $T < \bar{\tau}$ : from  $m_{\bar{\tau}, 1}(\bar{\tau}) = 0$ , we get  $m_{\bar{\tau}, 1}(T) < 0$ . Since  $T < \bar{\tau}$ , this implies that  $m_{T, 1}(T) < 0$ . Since we assumed that  $m_{T, 0}(T) > 0$ , there exists a unique  $\lambda_T \in (0, 1)$  such that  $m_{T, \lambda_T}(T) = 0$ . Since  $e^t m_{T, \lambda_T}(t)$  is a second-order polynomial in  $e^t$ , we easily deduce that  $m_{T, \lambda_T}(t) < 0$  for every  $t \in [0, T)$ . At this stage, we can verify that

$$\bar{\lambda}(dt) = \lambda_T \delta_T(dt), \quad \nabla \varphi_t(x) = \lambda_T e^{t-T},$$



satisfy the HJB equation (4.27). The process  $\bar{X}_{[0,T]}$  given by (4.31) then satisfies  $\bar{X}_0 \sim \mathcal{N}(x_0 - \lambda_T \sigma^2 e^{-T}, \sigma^2)$ , and  $\mathbb{E}[\bar{X}_t] = m_{T, \lambda_T}(t)$  for every  $t \in [0, T]$ . From Theorem 4.3.6 and the sufficient condition given by Theorem 4.2.17, this shows that  $\mathcal{L}(\bar{X}_{[0,T]})$  is indeed the corrected law  $\bar{\mu}_{[0,T]}$ . This situation is similar to the first case in Section 4.3.4.1.

2.  $x_0 \leq \sigma^2$  and  $\bar{\tau} \leq T$ : we obtain a solution of (4.27) by considering

$$\bar{\lambda}(dt) = \mathbb{1}_{t \in [\bar{\tau}, T]} dt + \delta_T(dt), \quad \nabla \varphi_t(x) = \begin{cases} e^{t-\bar{\tau}}, & \text{if } t \leq \bar{\tau}, \\ 1, & \text{otherwise.} \end{cases}$$

The related process  $\bar{X}_{[0,T]}$  satisfies  $\bar{X}_0 \sim \mathcal{N}(x_0 - \sigma^2 e^{-\bar{\tau}}, \sigma^2)$ ,  $\mathbb{E}[\bar{X}_t] = m_{\bar{\tau}, 1}(t)$  for  $t \in [0, \bar{\tau}]$ , and  $\mathbb{E}[\bar{X}_t] = 0$  for  $t \in [\bar{\tau}, T]$ . As previously, this shows that  $\mathcal{L}(\bar{X}_{[0,T]}) = \bar{\mu}_{[0,T]}$ . The continuous part of the multiplier activates at the first time where  $\mathbb{E}[\bar{X}_t]$  vanishes. Its effect is to decrease the correction  $\nabla \varphi_t(x)$  of the velocity, which is generated by the atom at the terminal time.

3. If  $x_0 > \sigma^2$ : similarly, we verify that the corrected process is given by (4.31) with

$$\bar{\lambda}(dt) = \frac{x_0 - \sigma^2}{\sigma^2} \delta_0(dt) - dt + \delta_T(dt), \quad \nabla \varphi_t \equiv 1 \text{ for } t > 0, \quad \bar{X}_0 \sim \mathcal{N}(x_0, \sigma^2).$$

As previously, the atom at time 0 generates a discontinuity that pushes the initial law on the constraint before starting the dynamics. Then, the correction compensates the +1 component of the velocity, so that  $\mathbb{E}[\bar{X}_t] = 0$  for every  $t \in [0, T]$ .

The transition between the different cases is continuous. Past the first vanishing of  $\mathbb{E}[\bar{X}_t]$ , the corrected process corresponds to a shifted Ornstein-Uhlenbeck process, whose new stationary distribution is  $\mathcal{N}(0, \sigma^2)$ . Interestingly, Sections 4.3.4.1-4.3.4.2 show that quite different behaviours can happen with specific transitions.

## 4.4 Proofs of the results of Section 4.2

### 4.4.1 Proofs of the results of Section 4.2.2

The proof of Theorem 4.2.7 relies on the following preliminary lemma.

**Lemma 4.4.1** (Continuity sets). *Under the assumptions of Theorem 4.2.7, for any open set  $U$  in  $\mathcal{P}_\phi(E)$ ,*

$$\inf_{\mu \in U \cap \mathcal{A}_\Psi} \mathcal{I}(\mu) = \inf_{\mu \in U \cap \mathcal{A}_\Psi} \mathcal{I}(\mu).$$

*Proof.* Since  $U \cap \mathcal{A}_\Psi \subset U \cap \mathcal{A}_\Psi$ , it is sufficient to show that the l.h.s. is lower than the r.h.s. Let us fix  $\mu$  in  $U \cap \mathcal{A}_\Psi$ . There is no loss of generality in assuming that  $\mu \in \mathcal{D}_T$ . Let  $\tilde{\mu}$  be given by the constraint qualification (A.5), and let  $\mu_\varepsilon := (1 - \varepsilon)\mu + \varepsilon\tilde{\mu}$ . From (A.5)-(i),

$$\forall t \in \mathcal{T}, \quad \Psi_t(\mu_\varepsilon) < 0,$$

for every small enough  $\varepsilon > 0$ . By (A.4), the mapping  $\mu \in \mathcal{P}_\phi(E) \mapsto \sup_{t \in \mathcal{T}} \Psi_t(\mu)$  is continuous and  $\mu_\varepsilon$  belongs to the pre-image of the open set  $(-\infty, 0)$ , which is contained in  $\mathcal{A}_\Psi$ . Therefore, for every  $\varepsilon$  small enough,  $\mu_\varepsilon \in \mathcal{A}_\Psi$ . Since  $\mu_\varepsilon$  converges towards  $\mu$  as

$\varepsilon \rightarrow 0$  and  $U$  is open, we eventually get that  $\mu_\varepsilon$  belongs to  $U \cap \mathring{\mathcal{A}}_\Psi$  for every small enough  $\varepsilon$ . Moreover, by convexity of  $H$ ,

$$\mathcal{I}(\mu_\varepsilon) = H(\mu_\varepsilon|\nu) + \mathcal{F}(\mu_\varepsilon) \leq (1 - \varepsilon)H(\mu|\nu) + \varepsilon H(\bar{\mu}|\nu) + \mathcal{F}(\mu_\varepsilon),$$

and (A.5)-(ii) implies that

$$\limsup_{\varepsilon \rightarrow 0} H(\mu_\varepsilon|\nu) + \mathcal{F}(\mu_\varepsilon) \leq H(\mu|\nu) + \mathcal{F}(\mu) = \mathcal{I}(\mu),$$

completing the proof.  $\square$

We may now present the proof of Theorem 4.2.7.

*Proof of Theorem 4.2.7.* For any measurable subset  $A$  of  $\mathcal{P}_\phi(E)$ ,

$$\log \Pi^N(A|\mathcal{A}_\Psi) = \log \Pi^N(A \cap \mathcal{A}_\Psi) - \log \Pi^N(\mathcal{A}_\Psi).$$

If  $A$  is closed, we write

$$\log \Pi^N(A|\mathcal{A}_\Psi) \leq \log \Pi^N(A \cap \mathcal{A}_\Psi) - \log \Pi^N(\mathring{\mathcal{A}}_\Psi).$$

Since, by Lemma 4.2.6,  $A \cap \mathcal{A}_\Psi$  is closed and  $\mathring{\mathcal{A}}_\Psi$  is open, we get that

$$\limsup_{N \rightarrow +\infty} N^{-1} \log \Pi^N(A|\mathcal{A}_\Psi) \leq - \inf_{\mu \in A \cap \mathcal{A}_\Psi} \mathcal{I}(\mu) + \inf_{\mu \in \mathring{\mathcal{A}}_\Psi} \mathcal{I}(\mu),$$

using the LDP satisfied by  $(\Pi^N)_{N \geq 1}$ . Similarly, if  $A$  is open,

$$\log \Pi^N(A|\mathcal{A}_\Psi) \geq \log \Pi^N(A \cap \mathring{\mathcal{A}}_\Psi) - \log \Pi^N(\mathcal{A}_\Psi),$$

and the LDP satisfied by  $(\Pi^N)_{N \geq 1}$  yields

$$\liminf_{N \rightarrow +\infty} N^{-1} \log \Pi^N(A|\mathring{\mathcal{A}}_\Psi) \geq - \inf_{\mu \in A \cap \mathring{\mathcal{A}}_\Psi} \mathcal{I}(\mu) + \bar{\mathcal{I}}_\Psi.$$

We get the desired large deviation upper bound by using Lemma 4.4.1 with  $U = \mathcal{P}_\phi(E)$ , and the lower bound using Lemma 4.4.1 with  $U = A$ .  $\square$

We now turn to the proof of Proposition 4.2.8. We start with an elementary measure theoretical result.

**Lemma 4.4.2.** *Let  $(\mu_k)_{k \geq 1}$  be a sequence that weakly converges towards  $\mu$  in  $\mathcal{P}(E)$ . Let  $(f_t^k)_{t \in \mathcal{T}, k \geq 1}$  be an equi-continuous family in  $C(E, \mathbb{R})$ , such that*

$$\forall x \in E, \quad \sup_{t \in \mathcal{T}} |f_t^k(x) - f_t(x)| \xrightarrow{k \rightarrow +\infty} 0,$$

for some (necessarily equi-continuous) family  $(f_t)_{t \in \mathcal{T}}$  in  $C(E, \mathbb{R})$ . If the following uniform integrability criterion holds

$$\sup_{t \in \mathcal{T}, k \geq 1} \int_E |f_t| \mathbb{1}_{|f_t| > M} d\mu + \int_E |f_t^k| \mathbb{1}_{|f_t^k| > M} d\mu_k \xrightarrow{M \rightarrow +\infty} 0,$$

then

$$\sup_{t \in \mathcal{T}} \left| \int_E f_t^k d\mu_k - \int_E f_t d\mu \right| \xrightarrow{k \rightarrow +\infty} 0.$$

*Proof.* Since  $E$  is separable, the Skorokhod representation theorem provides a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  together with a sequence  $(X_k)_{k \geq 1}$  of random variables on it with  $X_k \sim \mu_k$  that  $\mathbb{P}$ -a.s. converges towards some  $X \sim \mu$ . As a consequence:

$$\sup_{t \in \mathcal{T}} \left| \int_E f_t^k d\mu_k - \int_E f_t d\mu \right| \leq \mathbb{E} \left[ \sup_{t \in \mathcal{T}} |f_t^k(X^k) - f_t(X)| \right]. \quad (4.32)$$

Writing

$$\sup_{t \in \mathcal{T}} |f_t^k(X^k) - f_t(X)| \leq \sup_{t \in \mathcal{T}} |f_t^k(X^k) - f_t^k(X)| + \sup_{t \in \mathcal{T}} |f_t^k(X) - f_t(X)|,$$

we have by equi-continuity that, a.s.,

$$\sup_{t \in \mathcal{T}} |f_t^k(X^k) - f_t^k(X)| \leq \sup_{t \in \mathcal{T}, k' \geq 1} |f_t^{k'}(X^{k'}) - f_t^{k'}(X)| \xrightarrow{k \rightarrow +\infty} 0,$$

while by assumption, a.s.,

$$\sup_{t \in \mathcal{T}} |f_t^k(X) - f_t(X)| \xrightarrow{k \rightarrow +\infty} 0.$$

As a consequence, defining

$$M_k := \sup_{t \in \mathcal{T}} |f_t^k(X^k)| \vee |f_t(X)|,$$

we get, by dominated convergence, for any  $M > 0$ ,

$$\mathbb{E} \left[ \sup_{t \in \mathcal{T}} |f_t^k(X^k) - f_t(X)| \mathbb{1}_{M_k \leq M} \right] \xrightarrow{k \rightarrow +\infty} 0.$$

On the other hand, the uniform integrability assumption implies that

$$\sup_{k \geq 1} \mathbb{E} \left[ \sup_{t \in \mathcal{T}} |f_t^k(X^k) - f_t(X)| \mathbb{1}_{M_k > M} \right] \xrightarrow{M \rightarrow +\infty} 0,$$

concluding the proof.  $\square$

We may now prove Proposition 4.2.8.

*Proof of Proposition 4.2.8.* Let  $(\mu_k)_{k \geq 1}$  be a sequence that weakly converges towards  $\mu$  in  $\mathcal{P}_\phi(E)$ . The condition (i) in Proposition 4.2.8 allows us to apply Lemma 4.A.4 to get, for any  $t \in \mathcal{T}$ :

$$\Psi_t(\mu_k) - \Psi_t(\mu) = \int_0^1 \left\langle \mu_k - \mu, \frac{\delta \Psi_t}{\delta \mu}((1-r)\mu + r\mu_k) \right\rangle dr =: \int_E f_t^k d\mu_k - \int_E f_t d\mu,$$

with

$$\begin{aligned} f_t^k(x) &:= \int_0^1 \left[ \frac{\delta \Psi_t}{\delta \mu}((1-r)\mu + r\mu_k, x) - \left\langle \mu, \frac{\delta \Psi_t}{\delta \mu}((1-r)\mu + r\mu_k) \right\rangle \right] dr, \\ f_t(x) &:= \frac{\delta \Psi_t}{\delta \mu}(\mu, x) - \left\langle \mu, \frac{\delta \Psi_t}{\delta \mu}(\mu) \right\rangle. \end{aligned}$$

We shall check that the functions  $f_t^k$  and  $f_t$  satisfy the assumptions of Lemma 4.4.2, which thus yields

$$\sup_{t \in \mathcal{T}} \left| \Psi_t(\mu_k) - \Psi_t(\mu) \right| \xrightarrow{k \rightarrow +\infty} 0$$

and completes the proof. We first show that the family  $(f_t^k)_{t \in \mathcal{T}, k \geq 1}$  is equi-continuous. Let  $(x_l)_{l \geq 1}$  be a sequence of elements of  $E$  converging to some  $x \in E$ . For any  $t \in \mathcal{T}$  and  $k \geq 1$ ,

$$|f_t^k(x_l) - f_t^k(x)| = \left| \int_0^1 \left[ \frac{\delta \Psi_t}{\delta \mu}((1-r)\mu + r\mu_k, x_l) - \frac{\delta \Psi_t}{\delta \mu}((1-r)\mu + r\mu_k, x) \right] dr \right|,$$

so that, setting  $K = \cup_{k \geq 1} \{(1-r)\mu + r\mu_k, r \in [0, 1]\}$  which is easily seen to be compact,

$$\sup_{t \in \mathcal{T}, k \geq 1} |f_t^k(x_l) - f_t^k(x)| \leq \sup_{t \in \mathcal{T}, \rho \in K} \left| \frac{\delta \Psi_t}{\delta \mu}(\rho, x_l) - \frac{\delta \Psi_t}{\delta \mu}(\rho, x) \right| \xrightarrow{l \rightarrow +\infty} 0,$$

thanks to the condition (ii) in Proposition 4.2.8. Let us now show that  $f_t^k(x) \rightarrow f_t(x)$ , uniformly in  $t \in \mathcal{T}$ . For any  $x \in E$ , we have

$$\sup_{t \in \mathcal{T}} |f_t^k(x) - f_t(x)| \leq \int_0^1 [\varrho_{k,r}(x) + \langle \mu, \varrho_{k,r} \rangle] dr,$$

with

$$\varrho_{k,r}(x) := \sup_{t \in \mathcal{T}} \left| \frac{\delta \Psi_t}{\delta \mu}((1-r)\mu + r\mu_k, x) - \frac{\delta \Psi_t}{\delta \mu}(\mu, x) \right|.$$

The condition (iii) in Proposition 4.2.8 implies that, for any  $r \in [0, 1]$  and  $x \in E$ ,

$$\varrho_{k,r}(x) \xrightarrow{k \rightarrow +\infty} 0.$$

Since, in addition, the condition (i) in Proposition 4.2.8 yields  $\varrho_{k,r}(x) \leq 2D_K^\Psi [1 + \phi(x)]$  with the same compact set  $K$  as above, we deduce from the dominated convergence theorem that

$$\int_0^1 [\varrho_{k,r}(x) + \langle \mu, \varrho_{k,r} \rangle] dr \xrightarrow{k \rightarrow +\infty} 0.$$

It remains to check the uniform integrability condition of Lemma 4.4.2. The latter easily follows from the condition (i) in Proposition 4.2.8 and the fact that, from [Bogachev and Ruas, 2007, Theorem 4.5.6], the weak convergence of  $\mu_k$  in  $\mathcal{P}_\phi(E)$  implies that

$$\sup_{k \geq 1} \int_E \phi \mathbb{1}_{\phi > M} d\mu_k \xrightarrow{M \rightarrow +\infty} 0,$$

so the proof is completed.  $\square$

#### 4.4.2 Proofs of the results of Section 4.2.3

As a preliminary step for the proof of Theorem 4.2.13, we first show that any minimiser for (4.8) is also a minimiser for a linearised version of this problem.

**Lemma 4.4.3** (Linearisation of  $\mathcal{F}$  and  $\Psi_t$ ). *Let  $\bar{\mu}$  be a minimiser for (4.8) at which the regularity condition (A.6) and the constraint qualification (A.7) hold. Let  $\tilde{\varepsilon} > 0$  be given by (A.7)-(v).*

1. For any  $\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^\zeta$  that satisfies

$$\forall t \in \mathcal{T}, \quad \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \leq 0, \quad (4.33)$$

we have

$$\left\langle \mu, \left| \log \frac{d\bar{\mu}}{d\nu} \right| \right\rangle < +\infty.$$

2. The measure  $\bar{\mu}$  is a minimiser for

$$\inf_{\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}} \text{ s.t. (4.33) holds } \left\langle \mu, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle,$$

and the minimum value is  $H(\bar{\mu}|\nu)$ .

*Proof.* Let  $\tilde{\varepsilon}$  be given by (A.7)-(v). Let  $\mu$  be any measure in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  that satisfies (4.33) with a strict inequality, namely

$$\forall t \in \mathcal{T}, \quad \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle < 0. \quad (4.34)$$

For  $\varepsilon$  in  $(0, \tilde{\varepsilon}]$ , we define the probability measure  $\bar{\mu}_{\varepsilon} := (1 - \varepsilon)\bar{\mu} + \varepsilon\mu$ , which belongs to  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  by (A.6). By Remark 4.2.12 and the choice of  $\mu$ , we have

$$-\eta := \sup_{t \in \mathcal{T}} \left\{ \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \right\} < 0.$$

Moreover, by (A.7)-(ii), we have, for every small enough  $\varepsilon$  and every  $t \in \mathcal{T}$ ,

$$\begin{aligned} \Psi_t(\bar{\mu}_{\varepsilon}) &\leq \Psi_t(\bar{\mu}) + \varepsilon \left( \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle + \frac{\eta}{2\tilde{\varepsilon}} \right) \\ &\leq \frac{\varepsilon}{\tilde{\varepsilon}} \left( \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle + \frac{\eta}{2} \right) \leq -\frac{\varepsilon\eta}{2\tilde{\varepsilon}}, \end{aligned}$$

where we have used the fact that  $\varepsilon \leq \tilde{\varepsilon}$  and  $\Psi_t(\bar{\mu}) \leq 0$ , for any  $t \in \mathcal{T}$ , at the second line. This shows that  $\bar{\mu}_{\varepsilon} \in \mathcal{A}_{\Psi}$  for every small enough  $\varepsilon$ .

**Step 1. Linearisation.** For small enough  $\varepsilon$ ,  $\bar{\mu}_{\varepsilon}$  is admissible for (4.8), hence

$$H(\bar{\mu}|\nu) + \mathcal{F}(\bar{\mu}) \leq H(\bar{\mu}_{\varepsilon}|\nu) + \mathcal{F}(\bar{\mu}_{\varepsilon}) < +\infty,$$

by optimality of  $\bar{\mu}$ . We then divide by  $\varepsilon$  and we send it to 0. Using (A.6),

$$\varepsilon^{-1}[\mathcal{F}(\bar{\mu}_{\varepsilon}) - \mathcal{F}(\bar{\mu})] \xrightarrow{\varepsilon \rightarrow 0} \left\langle \mu - \bar{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle,$$

and we deduce from Lemma 4.A.5 that

$$0 \leq \int_E \log \frac{d\bar{\mu}}{d\nu} d\mu - H(\bar{\mu}|\nu) + \left\langle \mu - \bar{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle,$$

the integral belonging to  $\mathbb{R} \cup \{-\infty\}$ . However  $H(\bar{\mu}|\nu) < +\infty$ , hence the sign condition gives the finiteness of the integral: we have proved that the first part of Lemma 4.4.3 holds under the condition (4.34). Moreover, recalling that  $\langle \bar{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \rangle = 0$ , we conclude that

$$H(\bar{\mu}|\nu) \leq \inf_{\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta} \text{ s.t. (4.34) holds}} \left\langle \mu, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle.$$

**Step 2. Closure.** Let now  $\mu$  be any measure in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  which satisfies (4.33). Let  $\tilde{\mu}$  be given by (A.7)-(v). For  $\varepsilon$  in  $(0, 1]$ ,  $\mu_{\varepsilon} := (1 - \varepsilon)\mu + \varepsilon\tilde{\mu}$  belongs to  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^{\zeta}$  from (A.6), and since, for any  $t \in \mathcal{T}$ ,

$$\begin{aligned} &\Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \mu_{\varepsilon} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \\ &= (1 - \varepsilon) \left( \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \right) + \varepsilon \left( \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \right), \end{aligned}$$

then  $\mu_\varepsilon$  satisfies (4.34). Since  $\tilde{\mu}$  also satisfies (4.34), we deduce from **Step 1.** that  $\langle \mu, |\log \frac{d\bar{\mu}}{d\nu}| \rangle < +\infty$ , which is the first part of Lemma 4.4.3. Moreover, using the conclusion of **Step 1.** applied to  $\mu_\varepsilon$ , we get

$$\begin{aligned} H(\bar{\mu}|\nu) &\leq \left\langle \mu_\varepsilon, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle \\ &= (1 - \varepsilon) \left\langle \mu, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle + \varepsilon \left\langle \tilde{\mu}, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle. \end{aligned}$$

Sending  $\varepsilon$  to 0 eventually yields

$$\left\langle \bar{\mu}, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle = H(\bar{\mu}|\nu) \leq \left\langle \mu, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle,$$

which completes the proof of the second part of Lemma 4.4.3.  $\square$

We may now present the proof of Theorem 4.2.13.

*Proof of Theorem 4.2.13.* Since, by definition,  $\bar{\mu} \in \mathcal{D}_{\mathcal{I}}$ ,  $\bar{\mu}$  is absolutely continuous w.r.t.  $\nu$ . We fix a (measurable) density  $\frac{d\bar{\mu}}{d\nu}$  and define

$$\bar{S} := \left\{ x \in E, \frac{d\bar{\mu}}{d\nu}(x) > 0 \right\}$$

which by construction is measurable and satisfies  $\bar{\mu}(\bar{S}) = 1$ .

**Step 1. Absolute continuity.** Let  $\mu$  be any measure in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^\zeta$ . From Remark 4.2.12,  $t \mapsto \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \langle \mu - \bar{\mu}, \frac{\delta\Psi_t}{\delta\mu}(\bar{\mu}) \rangle$  is continuous on the compact set  $\mathcal{T}$ , hence bounded. As a consequence,  $\tilde{\mu}_\varepsilon := (1 - \varepsilon)\tilde{\mu} + \varepsilon\mu$  satisfies (4.34) for  $\varepsilon > 0$  small enough, where  $\tilde{\mu}$  is given by the constraint qualification (A.7)-(v). Lemma 4.4.3-1 applied to  $\tilde{\mu}_\varepsilon$  then implies that  $\int_E \log \frac{d\bar{\mu}}{d\nu} d\tilde{\mu}_\varepsilon > -\infty$ , hence  $\frac{d\bar{\mu}}{d\nu}$  is  $\tilde{\mu}_\varepsilon$ -a.s. positive and then  $\mu$ -a.s. positive. Any  $\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}^\zeta$  is thus absolutely continuous w.r.t.  $\bar{\mu}$ , or equivalently  $\mu(\bar{S}) = 1$ , proving Theorem 4.2.13-1. This holds in particular for  $\tilde{\mu}$ .

**Step 2. Computation of the density.** Let us consider the map

$$\Phi : \begin{cases} E \rightarrow \mathbb{R}, \\ x \mapsto \log \frac{d\bar{\mu}}{d\nu}(x) + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}, x) - H(\bar{\mu}|\nu) \end{cases}$$

which belongs to  $L^1(E, d\bar{\mu})$ . We notice that  $\Phi$  is finite on  $\bar{S}$ . From Lemma 4.4.3-1,  $\Phi$  also belongs to  $L^1(E, d\tilde{\mu})$ . We define  $\mathcal{C}$  as the convex cone generated by the set

$$\{\pm\zeta_s, s \in \mathcal{S}\} \cup \left\{ -\Psi_t(\bar{\mu}) - \tilde{\varepsilon} \frac{\delta\Psi_t}{\delta\mu}(\bar{\mu}), t \in \mathcal{T} \right\},$$

i.e. the elements of  $\mathcal{C}$  are the (finite) linear combinations with non-negative coefficients of elements from the above set. We then set  $\mathcal{C}^+ := \mathcal{C} + C_b(E, \mathbb{R}_+)$ . Let us show that  $\Phi$  belongs to the closure of  $\mathcal{C}^+$  in  $L^1(E, d\bar{\mu}) \cap L^1(E, d\tilde{\mu})$ . If it were not the case, using the geometric Hahn-Banach theorem [Barbu and Precupanu, 2012, Corollary 4.5],  $\Phi$  could be strictly separated from the closure of  $\mathcal{C}^+$ , which is a closed convex set, by an affine hyperplane. Since  $L^1(E, d\bar{\mu}) \cap L^1(E, d\tilde{\mu})$  is dense in  $L^1(E, d\bar{\mu})$  and  $L^1(E, d\tilde{\mu})$ , the topological dual of  $L^1(E, d\bar{\mu}) \cap L^1(E, d\tilde{\mu})$  is  $L^\infty(E, d\bar{\mu}) + L^\infty(E, d\tilde{\mu})$  from [Bergh and Löfström, 2012, Theorem 2.7.1]. The strict separation thus provides  $(h_1, h_2) \in L^\infty(E, d\bar{\mu}) \times L^\infty(E, d\tilde{\mu})$  such that

$$\int_E \Phi h_1 d\bar{\mu} + \int_E \Phi h_2 d\tilde{\mu} < \inf_{\varphi \in \mathcal{C}^+} \int_E \varphi h_1 d\bar{\mu} + \int_E \varphi h_2 d\tilde{\mu}. \quad (4.35)$$

The infimum on the r.h.s. is non-positive because the function  $\varphi \equiv 0$  belongs to  $\mathcal{C}^+$ . If this infimum were negative, it would be  $-\infty$  because, for every  $\varphi \in \mathcal{C}^+$ ,  $\alpha\varphi \in \mathcal{C}^+$  for every  $\alpha > 0$ . Since the l.h.s. of (4.35) is finite, the infimum on the r.h.s. thus equals 0.

From **Step 1.**,  $\tilde{\mu}$  is absolutely continuous w.r.t.  $\bar{\mu}$ , so that we can define

$$h := h_1 + h_2 \frac{d\tilde{\mu}}{d\bar{\mu}} \in L^1(E, d\bar{\mu}).$$

Since  $\mathcal{C}^+$  contains  $C_b(E, \mathbb{R}_+)$  and the r.h.s. of (4.35) is non-negative (it equals 0),  $h$  is  $\bar{\mu}$ -a.s. non-negative. Up to modifying the bounded functions  $h_1$  and  $h_2$ , we can then assume that  $h_1$  and  $h_2$  are  $\bar{\mu}$ -a.s. non-negative. Similarly, (4.35) can be divided by any positive constant; hence we can assume that  $\int_E h d\bar{\mu} = 1$ .

Let  $\mu_h$  denote the measure with density  $h$  w.r.t.  $\bar{\mu}$ . Let  $\mu_h^1$  (resp.  $\mu_h^2$ ) denote the probability measure with density w.r.t.  $\bar{\mu}$  (resp.  $\tilde{\mu}$ ) proportional to  $h_1$  (resp.  $h_2$ ). From the definition of  $h$ ,  $\mu_h$  is a convex combination of  $\mu_h^1$  and  $\mu_h^2$ . Since  $\mu_h^1$  (resp.  $\mu_h^2$ ) has a bounded density w.r.t. to  $\bar{\mu}$  (resp.  $\tilde{\mu}$ ) and  $H(\bar{\mu}|\nu)$  (resp.  $H(\tilde{\mu}|\nu)$ ) is finite, this implies that  $H(\mu_h|\nu)$  is finite. Using the same argument and (A.6)-(A.8), we further get that  $\mu_h$  belongs to  $\mathcal{D}_{\mathcal{I}}$ . For every  $s \in \mathcal{S}$ , we now use that the r.h.s. of (4.35) is non-negative when considering the test functions  $\varphi = \zeta_s$  and  $\varphi = -\zeta_s$ , which belong to  $\mathcal{C}^+$ . This shows that  $\langle \mu_h, \zeta_s \rangle = 0$  for every  $s \in \mathcal{S}$ , so that  $\mu_h \in \mathcal{A}^{\zeta}$ . We can thus apply Lemma 4.4.3-2 to  $\mu_h$ , yielding

$$0 \leq \int_E \log \frac{d\bar{\mu}}{d\nu} d\mu_h + \left\langle \mu_h, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle - H(\bar{\mu}|\nu) = \int_E h \Phi d\bar{\mu}.$$

Since the r.h.s. of (4.35) is 0, this gives the desired contradiction.

**Step 3. Lagrange multiplier.** From **Step 2.**, there exists a sequence  $((f_k, g_k))_{k \geq 1}$  in  $\mathcal{C} \times C_b(E, \mathbb{R}_+)$  such that

$$f_k + g_k \xrightarrow[k \rightarrow +\infty]{L^1(\bar{\mu}) \cap L^1(\tilde{\mu})} \Phi.$$

By definition of  $\mathcal{C}$ ,  $\int_E f_k d\bar{\mu} \geq 0$  because  $\bar{\mu} \in \mathcal{A}_{\Psi}^{\zeta}$ . Since  $\int_E \Phi d\bar{\mu} = 0$ , this yields

$$\|g_k\|_{L^1(\bar{\mu})} = \int_E g_k d\bar{\mu} \xrightarrow[k \rightarrow +\infty]{} 0,$$

so that  $\Phi$  belongs to the closure of  $\mathcal{C}$  in  $L^1(E, d\bar{\mu})$ . By definition of  $\mathcal{C}$ , each  $f_k$  can be decomposed as

$$f_k = -\xi_k - \sum_{i=1}^{n_k} \lambda_{i,k} \left[ \Psi_{t_{i,k}}(\bar{\mu}) + \tilde{\varepsilon} \frac{\delta \Psi_{t_{i,k}}}{\delta \mu}(\bar{\mu}) \right], \quad (4.36)$$

where  $\xi_k \in \text{Span}(\zeta_s, s \in \mathcal{S})$ ,  $\{t_{i,k}, 1 \leq i \leq n_k\}$  is a finite subset of  $\mathcal{T}$  and the  $\lambda_{i,k}$  are non-negative real numbers. We then define

$$\lambda_k := \sum_{i=1}^{n_k} \lambda_{i,k} \delta_{t_{i,k}} \in \mathcal{M}_+(\mathcal{T}).$$

By Remark 4.2.12, we have

$$-\eta := \sup_{t \in \mathcal{T}} \left\{ \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \right\} < 0.$$

Hence, integrating (4.36) against  $\tilde{\mu}$  and using that  $g_k \geq 0$ ,

$$\int_E [f_k + g_k] d\tilde{\mu} = \int_E g_k d\tilde{\mu} - \int_{\mathcal{T}} \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \left\langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \lambda_k(dt) \geq \eta \lambda_k(\mathcal{T}).$$

Since  $\int_E [f_k + g_k] d\tilde{\mu}$  converges, this shows that  $(\lambda_k(\mathcal{T}))_{k \geq 1}$  is bounded. Up to extracting a sub-sequence, we can assume that  $(\lambda_k(\mathcal{T}))_{k \geq 1}$  converges towards some  $\tilde{\lambda}(\mathcal{T}) \geq 0$ . If  $\tilde{\lambda}(\mathcal{T}) = 0$ , we set  $\tilde{\lambda} := 0$ . Otherwise,  $(\frac{\lambda_k}{\lambda_k(\mathcal{T})})_{k \in \mathbb{N}}$  is a sequence of probability measures over the compact set  $\mathcal{T}$ ; hence, it is relatively compact by the Prokhorov theorem [Billingsley, 2013, Theorem 5.1]. Consequently, we can always assume that  $(\lambda_k)_{k \in \mathbb{N}}$  weakly converges towards some  $\tilde{\lambda} \in \mathcal{M}_+(\mathcal{T})$  with mass  $\tilde{\lambda}(\mathcal{T})$ . Thus, for any  $x \in E$ , by the continuity assumptions (A.7)-(i,iii),

$$\int_{\mathcal{T}} \left[ \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \right] \lambda_k(dt) \xrightarrow{k \rightarrow +\infty} \int_{\mathcal{T}} \left[ \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \right] \tilde{\lambda}(dt),$$

and by the domination (A.7)-(iv), this convergence holds in  $L^1(E, d\bar{\mu})$ . Since  $f_k$  converges in  $L^1(E, d\bar{\mu})$ , (4.36) now shows that  $\xi_k$  converges in  $L^1(E, d\bar{\mu})$  towards some  $\bar{\zeta}$  in the closure of  $\text{Span}(\zeta_s, s \in \mathcal{S})$ . As a consequence, we may choose  $\bar{\zeta}$  such that, for all  $x \in \bar{\mathcal{S}}$ ,

$$\Phi(x) = -\bar{\zeta}(x) - \int_{\mathcal{T}} \left[ \Psi_t(\bar{\mu}) + \tilde{\varepsilon} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \right] \tilde{\lambda}(dt). \quad (4.37)$$

Since  $\xi_k \in \text{Span}(\zeta_s, s \in \mathcal{S})$  and  $\bar{\mu} \in \mathcal{A}^\zeta$ , we have  $\int_E \xi_k d\bar{\mu} = 0$ ; hence we get  $\int_E \bar{\zeta} d\bar{\mu} = 0$ , by  $L^1(E, d\bar{\mu})$ -convergence of  $\xi_k$  to  $\bar{\zeta}$ . Moreover,  $\int_E \Phi d\bar{\mu} = \int_E \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) d\bar{\mu} = 0$  and  $\int_E \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) d\bar{\mu} = 0$ , so that integrating (4.37) against  $\bar{\mu}$  yields

$$\int_{\mathcal{T}} \Psi_t(\bar{\mu}) \tilde{\lambda}(dt) = 0.$$

Since  $\Psi_t(\bar{\mu}) \leq 0$  for every  $t \in \mathcal{T}$ , we get  $\Psi_t(\bar{\mu}) = 0$  for  $\tilde{\lambda}$ -a.e.  $t \in \mathcal{T}$ . Setting  $\bar{\lambda} := \tilde{\varepsilon} \tilde{\lambda}$ , this proves Theorem 4.2.13-3. We now take the exponential of (4.37) to obtain that

$$\forall x \in \bar{\mathcal{S}}, \quad \frac{d\bar{\mu}}{d\nu}(x) = e^{H(\bar{\mu}|\nu)} \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right],$$

while  $\frac{d\bar{\mu}}{d\nu}(x) = 0$  for  $x \notin \bar{\mathcal{S}}$ . Since  $\int_E \frac{d\bar{\mu}}{d\nu} d\nu = 1$ , we get that

$$\bar{Z} := \int_E \mathbb{1}_{\bar{\mathcal{S}}}(x) \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right] \nu(dx) = e^{-H(\bar{\mu}|\nu)} \in (0, \infty).$$

This implies that  $\bar{\mu}$  is the Gibbs measure (4.13), completing the proof of Theorem 4.2.13-2.  $\square$

**Remark 4.4.4** (Bound on the Lagrange multiplier). *The computations in Step 3. show that*

$$\bar{\lambda}(\mathcal{T}) \leq \tilde{\varepsilon} \eta^{-1} \left\langle \tilde{\mu}, \log \frac{d\bar{\mu}}{d\nu} + \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle - \tilde{\varepsilon} \eta^{-1} H(\bar{\mu}|\nu),$$

where  $\eta := -\sup_{t \in \mathcal{T}} [\Psi_t(\bar{\mu}) + \tilde{\varepsilon} \langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \rangle]$ . This gives a bound on the mass of  $\bar{\lambda}$ .

#### 4.4.3 Proofs of the results of Section 4.2.4

*Proof of Theorem 4.2.17.* Let  $\mu \in \mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_{\Psi}^{\zeta}$ . Since  $H(\mu|\nu) < \infty$  and  $\nu(\bar{\mathcal{S}}) = 1$ , [Nutz, 2021, Lemma 1.4-(b)] ensures that the identity

$$H(\mu|\nu) - H(\mu|\bar{\mu}) = \int_E \left[ -\log \bar{Z} - \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) - \bar{\zeta}(x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}, x) \bar{\lambda}(dt) \right] d\mu(x)$$



holds in  $[-\infty, +\infty)$ . The differentiability assumption (i) on  $\mathcal{F}$  and  $\Psi_t$ , together with domination (ii) and the assumption (iv) on  $\bar{\zeta}$  imply that

$$\left\langle \mu, \left| \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right| \right\rangle < \infty, \quad \langle \mu, |\bar{\zeta}| \rangle < \infty, \quad \int_{\mathcal{T}} \left\langle \mu, \left| \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right| \right\rangle \bar{\lambda}(dt) < \infty,$$

which shows that  $H(\mu|\bar{\mu}) < \infty$  and that

$$H(\mu|\nu) - H(\mu|\bar{\mu}) = -\log \bar{Z} - \left\langle \mu, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle - \int_{\mathcal{T}} \left\langle \mu, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \bar{\lambda}(dt),$$

since  $\langle \mu, \bar{\zeta} \rangle = 0$  from (iv). On the other hand, using that  $\bar{\mu}$  is a Gibbs measure from (iii),

$$H(\bar{\mu}|\nu) = -\log \bar{Z} - \left\langle \bar{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle - \int_{\mathcal{T}} \left\langle \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \bar{\lambda}(dt).$$

We deduce that

$$H(\mu|\nu) = H(\mu|\bar{\mu}) + H(\bar{\mu}|\nu) + \left\langle \bar{\mu} - \mu, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle + \int_{\mathcal{T}} \left\langle \bar{\mu} - \mu, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \bar{\lambda}(dt).$$

By the convexity assumption (A.9), the differentiability (i) of  $\mathcal{F}$  and  $\Psi_t$  at  $\bar{\mu}$ , and (4.51),

$$\mathcal{F}(\mu) \geq \mathcal{F}(\bar{\mu}) + \left\langle \mu - \bar{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle, \quad \Psi_t(\mu) \geq \Psi_t(\bar{\mu}) + \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle.$$

Since  $\mu$  is admissible and the slackness condition (4.15) holds, we have

$$\left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \leq \Psi_t(\mu) - \Psi_t(\bar{\mu}) \leq 0, \quad \text{for } \bar{\lambda}(dt)\text{-a.e. } t \in \mathcal{T},$$

so that

$$\int_{\mathcal{T}} \left\langle \mu - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle \bar{\lambda}(dt) \leq 0$$

and therefore

$$H(\mu|\nu) + \mathcal{F}(\mu) \geq H(\mu|\bar{\mu}) + H(\bar{\mu}|\nu) + \mathcal{F}(\bar{\mu}),$$

which concludes because  $H(\mu|\bar{\mu}) \geq 0$ .  $\square$

*Proof of Proposition 4.2.19.* For any  $\varepsilon > 0$ , the minimisation problem (4.16) is the same as (4.8), with  $\Psi_t$  replaced by  $\Psi_t - \varepsilon$ . Therefore, since  $\mathcal{F}$  is bounded from below, Lemmata 4.2.4 and 4.2.6 for (4.16) show that this problem admits a minimiser  $\bar{\mu}_\varepsilon$ , which is unique by convexity of  $\mathcal{F}$  and  $\Psi_t - \varepsilon$ . Since  $\mathcal{A}_{\Psi_t, \varepsilon}^\zeta \subset \mathcal{A}_{\Psi_t, \varepsilon_*}^\zeta$  for  $\varepsilon \in [0, \varepsilon_*]$ , the regularity assumptions (A.6), (A.7)-(i,ii,iii,iv) and (A.8) are satisfied at  $\bar{\mu}_\varepsilon$  for  $\Psi_t - \varepsilon$ . Last, Lemma 4.2.18 trivially implies that the qualification condition (A.7)-(v) holds for  $\Psi_t - \varepsilon$ , therefore Theorem 4.2.13 applies to  $\bar{\mu}_\varepsilon$ . This proves Theorem 4.2.17-1.

For  $\varepsilon \in [0, \varepsilon_*]$  and  $(\mu, \lambda) \in \mathcal{A}^\zeta \times \mathcal{M}_+(\mathcal{T})$ , we define

$$\mathcal{L}^\varepsilon(\mu, \lambda) := \mathcal{I}(\mu) + \int_{\mathcal{T}} [\Psi_t(\mu) - \varepsilon] \lambda(dt).$$

By the convexity assumption (A.9) and (4.51), for any  $(\mu, \lambda) \in \mathcal{A}^\zeta \times \mathcal{M}_+(\mathcal{T})$ ,

$$\Psi_t(\mu) \geq \Psi_t(\bar{\mu}_\varepsilon) + \left\langle \mu - \bar{\mu}_\varepsilon, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\varepsilon) \right\rangle,$$

and by the domination assumption (A.7)-(iv) for  $\bar{\mu}_\varepsilon$ , there is  $D_\varepsilon^\Psi \in [0, \infty)$  such that

$$\int_{\mathcal{T}} \left| \left\langle \mu - \bar{\mu}_\varepsilon, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\varepsilon) \right\rangle \right| \lambda(dt) \leq \lambda(\mathcal{T}) D_\varepsilon^\Psi \langle \mu, 1 + \phi \rangle.$$

We deduce that

$$\mathcal{L}^\varepsilon(\mu, \lambda) \geq H(\mu|\nu) + \inf \mathcal{F} + \int_{\mathcal{T}} [\Psi_t(\bar{\mu}_\varepsilon) - \varepsilon] \lambda(dt) - \lambda(\mathcal{T}) D_\varepsilon^\Psi \langle \mu, 1 + \phi \rangle.$$

By the continuity assumption (A.7)-(i) for  $\bar{\mu}_\varepsilon$ , the integral on the r.h.s. is finite, and by the dual representation for entropy (4.4),  $H(\mu|\nu) - \lambda(\mathcal{T}) D_\varepsilon^\Psi \langle \mu, 1 + \phi \rangle$  is bounded from below, uniformly in  $\mu \in \mathcal{P}_\phi(E)$ . We conclude that, for any  $\lambda \in \mathcal{M}_+(\mathcal{T})$ ,

$$\inf_{\mu \in \mathcal{A}^\zeta} \mathcal{L}^\varepsilon(\mu, \lambda) > -\infty.$$

Moreover, it follows from the semi-continuity assumption (A.3), the convexity (A.9), (4.51) and Fatou's lemma that  $\mu \mapsto \int_{\mathcal{T}} [\Psi_t(\mu) - \varepsilon] \lambda(dt)$  is lower semi-continuous on  $\mathcal{P}_\phi(E)$ . Therefore, since  $\mathcal{F}$  is bounded from below, by Lemma 4.2.4, the level sets of  $\mu \mapsto \mathcal{L}^\varepsilon(\mu, \lambda)$  are compact in  $\mathcal{P}_\phi(E)$ . Since  $\mathcal{A}^\zeta$  is closed by Lemma 4.2.6, we deduce that there is a minimiser for  $\inf_{\mu \in \mathcal{A}^\zeta} \mathcal{L}^\varepsilon(\mu, \lambda)$ , and this minimiser does not depend on  $\varepsilon$  because of the shape  $\mathcal{L}^\varepsilon$ . Moreover,  $\varepsilon \mapsto \inf_{\mu \in \mathcal{A}^\zeta} \mathcal{L}^\varepsilon(\mu, \lambda)$  is continuously differentiable with constant derivative  $-\lambda(\mathcal{T})$ .

The assumptions on  $\bar{\zeta}_\varepsilon$  and  $\bar{S}_\varepsilon$  now allow us to use the same arguments as in the proof of Theorem 4.2.17 to get that

$$\inf_{\mu \in \mathcal{A}^\zeta} \mathcal{L}^\varepsilon(\mu, \bar{\lambda}_\varepsilon) = \mathcal{L}^\varepsilon(\bar{\mu}_\varepsilon, \bar{\lambda}_\varepsilon),$$

which then yields that  $\sup_\lambda \inf_\mu \mathcal{L}^\varepsilon(\mu, \lambda)$  is reached at  $\bar{\lambda}_\varepsilon$  and takes the value  $\mathcal{L}^\varepsilon(\bar{\mu}_\varepsilon, \bar{\lambda}_\varepsilon) = \bar{I}_{\Psi, \varepsilon}^\zeta$ . The envelope theorem [Milgrom and Segal, 2002, Theorem 2] now states that  $\varepsilon \mapsto \bar{I}_{\Psi, \varepsilon}^\zeta$  is absolutely continuous with derivative a.e. equal to  $-\bar{\lambda}^\varepsilon(\mathcal{T})$ .

To show that  $\bar{\lambda}^\varepsilon(\mathcal{T})$  is bounded from above uniformly in  $\varepsilon$ , we recall that, using Lemma 4.2.18 and Remark 4.4.4,

$$\bar{\lambda}_\varepsilon(\mathcal{T}) \leq \eta^{-1} \left\langle \tilde{\mu}, \log \frac{d\bar{\mu}_\varepsilon}{d\nu} + \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_\varepsilon) \right\rangle,$$

where

$$\eta := -\sup_{t \in \mathcal{T}} \Psi_t(\bar{\mu}) + \left\langle \tilde{\mu} - \bar{\mu}, \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}) \right\rangle.$$

By convexity of  $\Psi$ , the bound still holds with  $\eta := -\sup_{t \in \mathcal{T}} \Psi_t(\tilde{\mu})$ , which is positive from (A.7)-(i) and does not depend on  $\varepsilon$ . Since  $H(\tilde{\mu}|\nu)$  is finite, [Nutz, 2021, Lemma 1.4-(b)] yields

$$\left\langle \tilde{\mu}, \log \frac{d\bar{\mu}_\varepsilon}{d\nu} \right\rangle \leq H(\tilde{\mu}|\nu) < +\infty,$$

while using the convexity (A.9) and (4.51),

$$\left\langle \tilde{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_\varepsilon) \right\rangle \leq \mathcal{F}(\tilde{\mu}) - \mathcal{F}(\bar{\mu}_\varepsilon) \leq \mathcal{F}(\tilde{\mu}) - \inf \mathcal{F} < +\infty.$$

We therefore deduce that there exists  $C_{\text{stab}} \in [0, \infty)$  such that for any  $\varepsilon \in [0, \varepsilon_*]$ ,  $\bar{\lambda}_\varepsilon(\mathcal{T}) \leq C_{\text{stab}}$ . This yields the upper bound in (4.17), while the lower bound simply follows from the fact that  $\mathcal{A}_\Psi \subset \mathcal{A}_{\Psi, \varepsilon}$ .  $\square$

**Remark 4.4.5** (Towards improved bounds on  $\lambda^\varepsilon$ ). *The mass of  $\bar{\lambda}^\varepsilon$  should be very small in regions where  $\Psi_t(\bar{\mu}_\varepsilon) = \varepsilon$  and  $\Psi_t(\bar{\mu}_0) < 0$ , because the  $\varepsilon$ -relaxed problem should not correct much when the non-relaxed problem needs no correction. This can be seen by writing the Lagrangian duality: we have shown in the proof of Proposition 4.2.19 that  $\bar{\mu}_\varepsilon$  is a minimiser for*

$$\inf_{\mu \in \mathcal{A}^\varepsilon} \mathcal{I}(\mu) + \int_{\mathcal{T}} [\Psi_t(\mu) - \varepsilon] \bar{\lambda}^\varepsilon(dt).$$

Let us use  $\bar{\mu}_0$  as a competitor for this problem: using the complementary slackness (4.15) for  $\bar{\mu}_\varepsilon$ , we get

$$- \int_{\mathcal{T}} \Psi_t(\bar{\mu}_0) \bar{\lambda}^\varepsilon(dt) \leq \mathcal{I}(\bar{\mu}_0) - \mathcal{I}(\bar{\mu}_\varepsilon) - \varepsilon \bar{\lambda}^\varepsilon(\mathcal{T}),$$

and the r.h.s. is negligible in front of  $\varepsilon$  because  $\frac{d}{d\varepsilon} \mathcal{I}(\bar{\mu}_\varepsilon) = -\bar{\lambda}^\varepsilon(\mathcal{T})$ . However, deducing local bounds on  $\bar{\lambda}^\varepsilon$  from this seems non-obvious.

*Proof of Lemma 4.2.20.* From (A.10)-(vii), there exists  $\tilde{\mu} \in \mathcal{D}_{\mathcal{I}}$  such that  $\sup_{t \in \mathcal{T}} \Psi_t(\tilde{\mu}) < 0$ . For  $M > 1$ , let us define the probability measure  $\tilde{\mu}^M$  by

$$\frac{d\tilde{\mu}^M}{d\nu}(x) := \frac{1}{Z_M} \left[ \frac{d\tilde{\mu}}{d\nu}(x) \wedge M \right], \quad \text{where} \quad Z_M := \int_E \left[ \frac{d\tilde{\mu}}{d\nu}(x) \wedge M \right] d\nu(x) \in (0, \infty).$$

By dominated convergence,  $\tilde{\mu}^M$  converges towards  $\tilde{\mu}$  in  $\mathcal{P}_\phi(E)$  as  $M \rightarrow +\infty$ . Using the continuity assumption (A.10)-(vi), this implies that  $\sup_{t \in \mathcal{T}} \Psi_t(\tilde{\mu}^M) < 0$ , for every large enough  $M$ . For every small enough  $\varepsilon > 0$ , the Lusin theorem for Borel measures (see e.g. [Bogachev and Ruas, 2007, Theorem 7.1.13]) provides a closed set  $C_{M,\varepsilon}$  such that  $\frac{d\tilde{\mu}^M}{d\nu}|_{C_{M,\varepsilon}}$  is continuous and  $\nu(E \setminus C_{M,\varepsilon}) < \varepsilon$ . The Tietze-Urysohn extension theorem [Munkres, 2018, Theorem 35.1] then provides a continuous  $\rho_{M,\varepsilon} : E \rightarrow [0, M/Z_M]$  that coincides with  $\frac{d\tilde{\mu}^M}{d\nu}$  on  $C_{M,\varepsilon}$ . In particular,  $\rho_{M,\varepsilon}$  converges to  $\frac{d\tilde{\mu}^M}{d\nu}$ ,  $\nu$ -a.e., as  $\varepsilon \rightarrow 0$ . We then define the probability measure  $\tilde{\mu}^{M,\varepsilon}$  by

$$\frac{d\tilde{\mu}^{M,\varepsilon}}{d\nu}(x) := \frac{1}{Z_{M,\varepsilon}} [\varepsilon + \rho_{M,\varepsilon}(x)], \quad \text{where} \quad Z_{M,\varepsilon} := \int_E [\varepsilon + \rho_{M,\varepsilon}(x)] d\nu(x) \in (0, +\infty).$$

By dominated convergence,  $\tilde{\mu}^{M,\varepsilon}$  converges to  $\tilde{\mu}^M$  in  $\mathcal{P}_\phi(E)$ , as  $\varepsilon \rightarrow 0$ . Using (A.10)-(vi) again, we deduce that  $\sup_{t \in \mathcal{T}} \Psi_t(\tilde{\mu}^{M,\varepsilon}) < 0$ , for every small enough  $\varepsilon$ . From now on, we fix  $(M, \varepsilon)$  such that this property holds. Let  $\tilde{\mu}^*$  denote the related measure  $\tilde{\mu}^{M,\varepsilon}$ , and let  $\varrho^*$  denote its density w.r.t.  $\nu$ , which is positive, continuous and bounded.

For any  $k \geq 1$ , we define the probability measure  $\tilde{\mu}_k$  by

$$\frac{d\tilde{\mu}_k}{d\nu_k} := \frac{1}{Z_k} \varrho^*(x), \quad \text{where} \quad Z_k := \int_E \varrho^*(x) d\nu_k(x) \in (0, +\infty).$$

Since  $\nu_k$  converges to  $\nu$  in  $\mathcal{P}_\phi(E)$  from (A.10)-(iv), and  $\varrho^*$  is continuous and bounded,  $\tilde{\mu}_k$  converges to  $\tilde{\mu}^*$  in  $\mathcal{P}_\phi(E)$ . Similarly,  $H(\tilde{\mu}_k|\nu_k)$  converges towards  $H(\tilde{\mu}^*|\nu)$ ; hence  $(H(\tilde{\mu}_k|\nu_k))_{k \geq 1}$  is bounded. Since  $\varrho^*$  is bounded, (A.10)-(iii) gives that  $\tilde{\mu}_k \in \mathcal{D}_{\mathcal{I}_k}$ . We now write

$$\sup_{t \in \mathcal{T}} \Psi_{t,k}(\tilde{\mu}_k) \leq \left| \sup_{t \in \mathcal{T}} \Psi_{t,k}(\tilde{\mu}_k) - \sup_{t \in \mathcal{T}} \Psi_{t,k}(\tilde{\mu}^*) \right| + \sup_{t \in \mathcal{T}} \Psi_{t,k}(\tilde{\mu}^*).$$

From (A.10)-(vi), the first term on the r.h.s. vanishes as  $k \rightarrow +\infty$ . From (A.10)-(v), the limsup of the second term is lower than  $\sup_{t \in \mathcal{T}} \Psi_t(\tilde{\mu}^*)$ . Since this last quantity is negative by construction of  $\tilde{\mu}^*$ , this completes the proof.  $\square$

*Proof of Theorem 4.2.21-1.* From (A.10), Lemma 4.2.20 and the regularity assumptions (A.11)-(i,ii),  $\bar{\mu}_k$  satisfies the assumptions of Theorem 4.2.13, for every large enough  $k$ . As a consequence,  $\bar{\mu}_k$  has the Gibbs form (4.13)-(4.15) for some  $\bar{\lambda}_k \in \mathcal{M}_+(\mathcal{T})$ . It remains to show that  $(\bar{\lambda}_k)_{k \geq 1}$  is weakly pre-compact. Using the Prokhorov theorem as in the proof of Theorem 4.2.13, it is sufficient to show that  $(\bar{\lambda}_k(\mathcal{T}))_{k \geq 1}$  is bounded. Let  $(\tilde{\mu}_k)_{k \geq k_0}$  be given by Lemma 4.2.20. Using Remark 4.4.4,

$$\bar{\lambda}_k(\mathcal{T}) \leq \eta_k^{-1} \left\langle \tilde{\mu}_k, \log \frac{d\bar{\mu}_k}{d\nu_k} + \frac{\delta \mathcal{F}_k}{\delta \mu}(\bar{\mu}_k) \right\rangle,$$

where  $\eta_k := -\sup_{t \in \mathcal{T}} \Psi_{t,k}(\bar{\mu}_k) + \langle \tilde{\mu}_k - \bar{\mu}_k, \frac{\delta \Psi_{t,k}}{\delta \mu}(\bar{\mu}_k) \rangle$ . By convexity of  $\Psi_{t,k}$  and (4.51), the bound still holds with  $\eta := -\sup_{k \geq k_0} \sup_{t \in \mathcal{T}} \Psi_{t,k}(\tilde{\mu}_k)$  instead of  $\eta_k$ , which is positive by Lemma 4.2.20 and independent of  $k$ . By optimality of  $\bar{\mu}_k$ , [Nutz, 2021, Lemma 1.4-(b)] gives the estimate

$$\left\langle \tilde{\mu}_k, \log \frac{d\bar{\mu}_k}{d\nu_k} \right\rangle \leq H(\tilde{\mu}_k | \nu_k),$$

and  $(H(\tilde{\mu}_k | \nu_k))_{k \geq k_0}$  is bounded from Lemma 4.2.20. The bound on  $(H(\tilde{\mu}_k | \nu_k))_{k \geq k_0}$  together with the dual representation for entropy (4.4), the domination (A.11)-(iii) on  $\frac{\delta \mathcal{F}_k}{\delta \mu}(\bar{\mu}_k)$ , and the exponential integrability (A.10)-(iv) then give that  $\langle \tilde{\mu}_k, \frac{\delta \mathcal{F}_k}{\delta \mu}(\bar{\mu}_k) \rangle$  is bounded uniformly in  $k$ . This shows that  $(\bar{\lambda}_k(\mathcal{T}))_{k \geq 1}$  is indeed bounded.  $\square$

*Proof of Theorem 4.2.21-2.* For  $k \geq 1$ , we set

$$P_k(x) := \exp \left[ -\frac{\delta \mathcal{F}_k}{\delta \mu}(\bar{\mu}_k, x) - \int_{\mathcal{T}} \frac{\delta \Psi_{t,k}}{\delta \mu}(\bar{\mu}_k, x) \bar{\lambda}_k(dt) \right], \quad \bar{Z}_k := \int_E P_k d\nu_k,$$

so that  $\frac{d\bar{\mu}_k}{d\nu_k} = \bar{Z}_k^{-1} P_k$ . Using the domination (A.11)-(iii) and the bound on  $(\bar{\lambda}_k(\mathcal{T}))_{k \geq 1}$  given by Theorem 4.2.21-1, there exist  $D^P, \beta > 0$  such that

$$\forall x \in E, \forall k \geq 1, \quad (D^P)^{-1} \exp[-\beta \phi(x)] \leq P_k(x) \leq D^P \exp[\beta \phi(x)]. \quad (4.38)$$

Since  $(\nu_k)_{k \geq 1}$  weakly converges to  $\nu$ , the lower bound in (4.38) shows that  $(\bar{Z}_k^{-1})_{k \geq 1}$  is bounded. Since  $(\nu_k)_{k \geq 1}$  converges in  $\mathcal{P}_\phi(E)$ , it is a tight sequence. Using the upper bound in (4.38) and the exponential integrability (A.10)-(iv), we deduce that  $(\bar{\mu}_k)_{k \geq 1}$  is also tight, and thus pre-compact in  $\mathcal{P}(E)$ . The same argument yields

$$\limsup_{M \rightarrow \infty} \sup_{k \geq 1} \int_{\phi \geq M} \phi d\bar{\mu}_k = 0,$$

which eventually gives that  $(\bar{\mu}_k)_{k \geq 1}$  is pre-compact in  $\mathcal{P}_\phi(E)$ . Up to re-indexing, we can assume that  $(\bar{\mu}_k)_{k \geq 1}$  converges towards some  $\bar{\mu}_\infty$  in  $\mathcal{P}_\phi(E)$ , and that  $(\bar{\lambda}_k)_{k \geq 1}$  weakly converges towards some  $\bar{\lambda}_\infty \in \mathcal{M}_+(\mathcal{T})$ . From (A.11)-(iv),  $\bar{\mu}_\infty \in \mathcal{D}_{\mathcal{I}}$ .

We now show that, for all  $x \in E$ ,  $P_k(x)$  converges towards

$$P_\infty(x) := \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_\infty, x) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\infty, x) \bar{\lambda}_\infty(dt) \right].$$

The convergence of  $\frac{\delta \mathcal{F}_k}{\delta \mu}(\bar{\mu}_k, x)$  to  $\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_\infty, x)$  is given by (A.11)-(iv). Besides, for every  $x \in E$ ,

$$\begin{aligned} & \left| \int_{\mathcal{T}} \frac{\delta \Psi_{t,k}}{\delta \mu}(\bar{\mu}_k, x) \bar{\lambda}_k(dt) - \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\infty, x) \bar{\lambda}_\infty(dt) \right| \\ & \leq \bar{\lambda}_k(\mathcal{T}) \sup_{t \in \mathcal{T}} \left| \frac{\delta \Psi_{t,k}}{\delta \mu}(\bar{\mu}_k, x) - \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\infty, x) \right| + \left| \int_{\mathcal{T}} \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\infty, x) [\bar{\lambda}_k(dt) - \bar{\lambda}_\infty(dt)] \right|, \end{aligned}$$

and the second term goes to 0 using the weak convergence of  $(\lambda_k)_{k \geq 1}$ , because  $t \mapsto \frac{\delta \Psi_t}{\delta \mu}(\bar{\mu}_\infty, x)$  is continuous using (A.11)-(iv). Since  $(\bar{\lambda}_k(\mathcal{T}))_{k \geq 1}$  is bounded, the point-wise convergence to 0 of the first term (along the considered sub-sequence) then follows from (A.11)-(iv).

Let us show that for any bounded continuous  $\varphi : E \rightarrow \mathbb{R}$ ,

$$\int_E \varphi P_k d\nu_k \xrightarrow{k \rightarrow +\infty} \int_E \varphi P_\infty d\nu. \quad (4.39)$$

To do so, we set  $f^k := \varphi P_k$ . The upper bound in (4.38) and the equi-continuity assumption (A.11)-(iv) w.r.t.  $x$  give that  $(f^k)_{k \geq 1}$  is equi-continuous. Moreover, the upper bound in (4.38) and the integrability assumption (A.10)-(iv) ensure that

$$\sup_{k \geq 1} \int_E \mathbb{1}_{|f^k| \geq M} |f^k| d\nu_k \xrightarrow{M \rightarrow +\infty} 0.$$

Lemma 4.4.2 (with no dependence on  $t$ ) then provides (4.39). Taking  $\varphi = 1$  further shows that  $\bar{Z}_k$  converges to  $\bar{Z}_\infty := \int_E P_\infty d\nu$ .

We now write

$$\sup_{t \in \mathcal{T}} \Psi_t(\bar{\mu}_\infty) = \left[ \sup_{t \in \mathcal{T}} \Psi_t(\bar{\mu}_\infty) - \sup_{t \in \mathcal{T}} \Psi_{t,k}(\bar{\mu}_k) \right] + \sup_{t \in \mathcal{T}} \Psi_{t,k}(\bar{\mu}_k).$$

By (A.11)-(iv), the first term on the r.h.s. vanishes as  $k \rightarrow +\infty$ . By definition of  $\bar{\mu}_k$ , the second term is non-positive. Therefore,  $\sup_{t \in \mathcal{T}} \Psi_t(\bar{\mu}_\infty) \leq 0$ . Using (A.11)-(iv) and the bound on  $(\lambda_k(\mathcal{T}))_{k \geq 1}$ ,

$$0 = \int_{\mathcal{T}} \Psi_{t,k}(\bar{\mu}_k) \bar{\lambda}_k(dt) \xrightarrow{k \rightarrow +\infty} \int_{\mathcal{T}} \Psi_t(\bar{\mu}_\infty) \bar{\lambda}_\infty(dt),$$

where we also used that  $t \mapsto \Psi_t(\bar{\mu}_\infty)$  is continuous from (A.11)-(iv). This shows that  $(\bar{\mu}_\infty, \bar{\lambda}_\infty)$  satisfies (4.13)-(4.15). Since  $\bar{\mu}_\infty \in \mathcal{D}_{\mathcal{I}}$  from (A.11)-(iv), Theorem 4.2.17 now proves that  $\bar{\mu}_\infty$  is the unique minimiser  $\bar{\mu}$  of  $\mathcal{I}$  in  $\mathcal{D}_{\mathcal{I}} \cap \mathcal{A}_\Psi$ . In particular,  $\bar{\mu}$  satisfies (4.13)-(4.15) for any limit point  $\bar{\lambda}_\infty$  of  $(\bar{\lambda}_k)_{k \geq 1}$ . Since  $\bar{\mu}_\infty$  was any limit point of  $(\bar{\mu}_k)_{k \geq 1}$ , this finally proves the convergence of the full sequence  $(\bar{\mu}_k)_{k \geq 1}$  towards  $\bar{\mu}$ .  $\square$

## 4.5 Proofs of the results of Section 4.3

### 4.5.1 Proof of Theorem 4.3.3

With the family of functions  $(\zeta_s)_{s \in \mathcal{S}}$  introduced after the statement of Theorem 4.3.3, the minimisation problem (4.21) is exactly (4.8) in the setting of the condition (B.1). As a consequence, the first part of Theorem 4.3.3 is straightforward.

Let us now fix a minimiser  $\bar{\mu}_{[0,T]}$  for (4.21), at which (B.3) holds. Let  $\bar{\zeta}$ ,  $\bar{\lambda}$  and  $\bar{S}$  be given by Theorem 4.2.13. To complete the proof of the second part of Theorem 4.3.3, we need to prove the following two statements.

**Lemma 4.5.1** (Completion of the proof of Theorem 4.3.3).

(i)  $\nu_{[0,T]}(\bar{S}) = 1$ ;

(ii) there exist  $(\bar{\zeta}_0, \bar{\zeta}_T) \in L^1(\mathbb{R}^d, d\mu^{\text{ini}}) \times L^1(\mathbb{R}^d, d\mu^{\text{fin}})$  such that  $\bar{\zeta}(x_{[0,T]}) = \bar{\zeta}_0(x_0) + \bar{\zeta}_T(x_T)$ , for  $\nu_{[0,T]}$ -a.e.  $x_{[0,T]}$ .

*Proof.* To prove the first claim, we consider the static Schrödinger problem

$$\inf_{\substack{\mu \in \mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d) \\ \mu_1 = \mu^{\text{ini}}, \mu_2 = \mu^{\text{fin}}}} H(\mu | \nu_{0,T}).$$

The above infimum is finite, because the law of  $(X_0, X_T)$  when  $X_{[0,T]} \sim \bar{\mu}_{[0,T]}$  has finite entropy w.r.t.  $\nu_{0,T}$ , using the additivity property of entropy [Dembo and Zeitouni, 2009, Theorem D.13]. Since  $\nu_{0,T} \sim \mu^{\text{ini}} \otimes \mu^{\text{fin}}$ , [Nutz, 2021, Theorem 2.1] provides an optimal measure  $\pi^* \sim \nu_{0,T}$  for this static problem. From e.g. [Dembo and Zeitouni, 2009, Theorem D.3], the disintegration  $\nu_{[0,T]}^{x,y}$  of  $\nu_{[0,T]}$  given  $(X_0, X_T) = (x, y)$  is well-defined. The path-measure  $\nu_{[0,T]}^{x_0, x_T}(\mathrm{d}x_{[0,T]})\pi^*(\mathrm{d}x_0, \mathrm{d}x_T)$  is then an admissible measure for (4.21) that is equivalent to  $\nu_{[0,T]}$ . From Theorem 4.2.7-1, this shows that  $\bar{\mu}_{[0,T]}$  is equivalent to  $\nu_{[0,T]}$ , and thus that  $\nu_{[0,T]}(\bar{S}) = 1$ .

We now prove the existence of  $(\bar{\zeta}_0, \bar{\zeta}_T)$ . Let  $(\xi^k)_{k \geq 1}$  be a sequence in  $\text{Span}((\phi_0^k)_{k \geq 1} \cup (\phi_T^k)_{k \geq 1})$  that converges in  $L^1(C([0, T], \mathbb{R}^d), \mathrm{d}\bar{\mu}_{[0,T]})$  towards  $\bar{\zeta}$ . Up to extracting a subsequence, we can assume that this convergence holds  $\bar{\mu}_{[0,T]}$ -a.s. Since  $\bar{\mu}_{[0,T]} \sim \nu_{[0,T]}$ , this convergence also holds  $\nu_{[0,T]}$ -a.s. By definition,  $\xi^k$  only depends on  $(X_0, X_T)$ ; hence, the convergence holds  $\nu_{0,T}$ -a.s. Since  $\nu_{0,T} \sim \mu^{\text{ini}} \otimes \mu^{\text{fin}}$ , the convergence eventually holds  $\mu^{\text{ini}} \otimes \mu^{\text{fin}}$ -a.s. As a consequence, [Nutz, 2021, Corollary 2.12] gives that  $\bar{\zeta} = \bar{\zeta}_0 + \bar{\zeta}_T$  as desired, completing the proof.  $\square$

#### 4.5.2 Proof of Theorem 4.3.6

The main obstacle in the proof of Theorem 4.3.6 being the time-regularity of the coefficients, we are going to smooth them to produce solutions  $(\varphi^k)_{k \geq 1}$  of approximated equations. For every  $t \in [0, T]$ , we will then show local compactness estimates for  $(\nabla \varphi_t^k)_{k \geq 1}$  in  $C^1(\mathbb{R}^d, \mathbb{R}^d)$ . To identify the limit as a function of  $t$ , the representation formula (4.29) will be needed. We first prove a general stability result.

**Proposition 4.5.2** (Limit HJB equation). *Under (C.2)-(i), let  $(b^k)_{k \geq 1}$ ,  $(\sigma^k)_{k \geq 1}$ ,  $(c^k)_{k \geq 1}$ ,  $(\psi^k)_{k \geq 1}$  be measurable functions  $[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ ,  $[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ , and  $(\lambda^k)_{k \geq 1}$  be a sequence in  $\mathcal{M}_+([0, T])$ , satisfying the following conditions.*

- (i)  $t \mapsto b_t^k(0)$ ,  $t \mapsto \sigma_t^k(0)$ ,  $t \mapsto c_t^k(0)$ ,  $t \mapsto \psi_t^k(0)$  are bounded uniformly in  $k \geq 1$ .
- (ii)  $b^k$ ,  $\sigma^k$ ,  $c^k$ ,  $\psi^k$  satisfy (C.1)-(i,ii) and (C.2)-(ii) uniformly in  $k \geq 1$ .
- (iii) For Lebesgue-a.e.  $t \in [0, T]$ , for every  $x \in \mathbb{R}^d$ ,  $(b_t^k(x), \sigma_t^k(x), c_t^k(x))$  converges towards  $(b_t(x), \sigma_t(x), c_t(x))$  as  $k \rightarrow +\infty$ .
- (iv)  $(\psi^k)_{k \geq 1}$  converges towards  $\psi$  uniformly on every compact set of  $[0, T] \times \mathbb{R}^d$ .
- (v)  $(\lambda^k)_{k \geq 1}$  weakly converges towards  $\lambda$  in  $\mathcal{M}_+([0, T])$ .

Let  $\varphi^k$  be given by

$$\varphi_t^k(x) := -\log \mathbb{E} \exp \left[ \int_t^T c_s^k(Z_s^{t,x,k}) \mathrm{d}s + \int_{[t,T]} \psi_s^k(Z_s^{t,x,k}) \lambda^k(\mathrm{d}s) \right], \quad (4.40)$$

where  $(Z_s^{t,x,k})_{t \leq s \leq T}$  is the path-wise unique solution to

$$\mathrm{d}Z_s^{t,x,k} = b_s^k(Z_s^{t,x,k}) \mathrm{d}s + \sigma_s^k(Z_s^{t,x,k}) \mathrm{d}B_s, \quad t \leq s \leq T, \quad Z_t^{t,x,k} = x.$$

For every  $k \geq 1$ , we assume that  $\varphi^k$  is the mild solution of

$$-\varphi_t^k + \int_t^T \left( b_s^k \cdot \nabla \varphi_s^k - \frac{1}{2} |(\sigma_s^k)^\top \nabla \varphi_s^k|^2 + \frac{1}{2} \text{Tr}[\sigma_s^k (\sigma_s^k)^\top \nabla^2 \varphi_s^k] + c_s^k \right) ds + \int_{[t,T]} \psi_s^k \lambda^k(ds) = 0, \quad (4.41)$$

in the sense of Definition 4.3.5, and that  $(\nabla \varphi^k)_{k \geq 1}$  is uniformly bounded. For every  $t \in [0, T]$  and every compact set  $K \subset \mathbb{R}^d$ , we assume that  $(\nabla \varphi_t^k)_{k \geq 1}$  is pre-compact in  $C(K, \mathbb{R}^d)$ . Then, for  $t = 0$  and every  $t \in (0, T]$  with  $\lambda(\{t\}) = 0$ ,  $(\varphi_t^k)_{k \geq 1}$  point-wise converges towards  $\varphi_t$  given by (4.29), and  $\varphi$  is the solution of (4.27) in the sense of Definition 4.3.5.

*Proof.* Using (i,ii), the coefficients have linear growth uniformly in  $k$ , and  $(\sigma^k)_{k \geq 1}$  is uniformly bounded. For every  $(t, x) \in [0, T] \times \mathbb{R}^d$ , this implies that

$$\forall \alpha > 0, \quad \sup_{k \geq 1} \mathbb{E} \exp \left[ \alpha \sup_{t \leq s \leq T} |Z_s^{t,x,k}| \right] < +\infty.$$

From (ii,iii), a standard coupling argument (a variation of the proof of [Friedman, 1975, Chapter 5, Theorem 5.2]) implies that

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |Z_s^{t,x,k} - Z_s^{t,x}| \right] \xrightarrow{k \rightarrow +\infty} 0,$$

where  $(Z_s^{t,x})_{t \leq s \leq T}$  is given by (4.30). Let us fix  $t = 0$  or any  $t \in (0, T]$  that is not an atom of  $\lambda$ . Since the number of atom is always countable, this includes Lebesgue a.e.  $t \in [0, T]$ . For such a  $t$ , (v) and [Billingsley, 2013, Theorem 2.7] imply that

$$\forall \phi \in C([0, T], \mathbb{R}), \quad \int_{[t,T]} \phi(s) \lambda^k(ds) \xrightarrow{k \rightarrow +\infty} \int_{[t,T]} \phi(s) \lambda(ds).$$

As a consequence, using (ii,iii,iv) and the continuity (C.2)-(i) of  $\psi$ , (4.40) shows that  $(\varphi_t^k)_{k \geq 1}$  point-wise converges towards  $\varphi_t$  given by (4.29). The pre-compactness of  $(x \mapsto \nabla \varphi_t^k(x))_{k \geq 1}$  on every compact set further shows that  $\varphi_t$  is  $C^1$ . We then deduce that  $(\nabla \varphi_t^k)_{k \geq 1}$  converges towards  $\nabla \varphi_t$ , uniformly on every compact set. The uniform bound on  $\nabla \varphi_t$  is inherited from the one on  $(\nabla \varphi_t^k)_{k \geq 1}$ .

From Definition 4.3.5, for Lebesgue-a.e.  $t \in [0, T]$ ,

$$-\varphi_t^k + \int_t^T S_{t,s}^k [b_s^k \cdot \nabla \varphi_s^k - \frac{1}{2} |(\sigma_s^k)^\top \nabla \varphi_s^k|^2 + c_s^k] ds + \int_{[t,T]} S_{t,s}^k [\psi_s^k] \lambda^k(ds) = 0. \quad (4.42)$$

The Feynman-Kac representation formula (see e.g. [Friedman, 1975, Chapter 6.5]) yields

$$\forall x \in \mathbb{R}^d, \quad S_{t,s}^k [\phi](x) = \mathbb{E}[\phi(\xi_s^{t,x,k})],$$

for every continuous  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  with linear growth, where  $\xi_s^{t,x,k}$  is the solution of  $d\xi_s^{t,x,k} = \sigma_s^k(\xi_s^{t,x,k}) dB_s$ ,  $\xi_t^{t,x,k} = x$ . As for  $Z_s^{t,x,k}$ , we get that

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |\xi_s^{t,x,k} - \xi_s^{t,x}| \right] \xrightarrow{k \rightarrow +\infty} 0,$$

where the definition of  $\xi_s^{t,x}$  involves  $\sigma$  instead of  $\sigma^k$ . Setting  $\phi_s^k := b_s^k \cdot \nabla \varphi_s^k - \frac{1}{2} |(\sigma_s^k)^\top \nabla \varphi_s^k|^2 + c_s^k$ , the pre-compactness assumption, the bound on  $(\nabla \varphi^k)_{k \geq 1}$  and (ii) show that  $(x \mapsto \phi_s^k(x))_{k \geq 1}$  is equi-continuous. By dominated convergence, this provides the point-wise

convergence of  $S_{t,s}^k[\phi_s^k]$  as  $k \rightarrow +\infty$ , and then of the middle term in (4.42). Similarly, we split

$$\begin{aligned} \int_{[t,T]} S_{t,s}^k[\psi_s^k](x)\lambda^k(ds) - \int_{[t,T]} S_{t,s}[\psi_s](x)\lambda(ds) = \\ \int_{[t,T]} \mathbb{E}[\psi_s^k(\xi_s^{t,x,k}) - \psi_s(\xi_s^{t,x})]\lambda^k(ds) + \int_{[t,T]} \mathbb{E}[\psi_s(\xi_s^{t,x})][\lambda^k - \lambda](ds). \end{aligned}$$

Since  $(\lambda^k([0, T]))_{k \geq 1}$  is bounded, the first term on the r.h.s. goes to 0 using the path-wise convergence of  $\xi_s^{t,x,k}$  and the uniform convergence (iv) of  $\psi^k$  on compact sets. From the continuity (C.2)-(i) of  $\psi$  and the weak convergence of  $(\lambda^k)_{k \geq 1}$ , the second term goes to 0 too as  $k \rightarrow +\infty$ . Taking the limit in (4.42) now shows that  $\varphi$  is the solution of (4.27) in the sense of Definition 4.3.5.  $\square$

**Lemma 4.5.3.** *Under (C.2)-(i) and (i,ii,iii,iv,v) in Proposition 4.5.2, let us assume that  $t \mapsto b_t^k(x)$ ,  $\sigma_t^k(x)$ ,  $c_t^k(x)$ ,  $\psi_t^k(x)$  are  $C^1$ , and that  $\lambda^k(dt) = \lambda_t^k dt + \bar{\lambda}_T \delta_T(dt)$  for some  $t \mapsto \lambda_t^k$  in  $C^1([0, T], \mathbb{R}_+)$ . Then, the function  $\varphi^k$  given by (4.40) belongs to  $C([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T) \times \mathbb{R}^d)$  and is the solution of (4.41) in the sense of Definition 4.3.5. If  $(\nabla \varphi_t^k)_{k \geq 1}$  is furthermore pre-compact in  $C(K, \mathbb{R}^d)$ , for every compact  $K \subset \mathbb{R}^d$  and  $t \in [0, T]$ , then the conclusion of Theorem 4.3.6 holds.*

*Proof.* From [Chaintron, 2023, Theorem 2.2], the Hamilton-Jacobi-Bellman (HJB) equation

$$\partial_t u_t^k + b_t^k \cdot \nabla u_t^k + \frac{1}{2} \text{Tr}[a_t^k \nabla^2 u_t^k] - \frac{1}{2} |(\sigma_t^k)^\top \nabla u_t^k|^2 = -c_t^k - \lambda_t^k \psi_t^k, \quad u_T^k = \bar{\lambda}_T \psi_T^k, \quad (4.43)$$

where  $a_t^k := \sigma_t^k (\sigma_t^k)^\top$ , has a unique solution  $u^k$  in  $C([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T) \times \mathbb{R}^d)$ , and

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |\nabla u_t^k(x)| \leq C,$$

for a constant  $C$  that does not depend on  $k$ , because  $(\lambda^k([0, T]))_{k \geq 1}$  is bounded. Using the Cole-Hopf transform  $v^k := e^{-u^k}$ ,  $v^k$  is a  $C^{1,2}$  solution of the linear parabolic equation

$$\partial_t v_t^k + b_t^k \cdot \nabla v_t^k + \frac{1}{2} \text{Tr}[a_t^k \nabla^2 v_t^k] = c_t^k v_t^k + \lambda_t^k \psi_t^k v_t^k.$$

From the Feynman-Kac representation formula [Friedman, 1975, Chapter 6.5], for  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,

$$v_t^k(x) = \mathbb{E} \exp \left[ \bar{\lambda}_T \psi_T^k(Z_T^{t,x,k}) + \int_t^T [c_s^k(Z_s^{t,x,k}) + \psi_s^k(Z_s^{t,x,k}) \lambda_s^k] ds \right]. \quad (4.44)$$

From (4.40), this yields  $u^k = \varphi^k$ . Classically, the solution  $u^k$  of (4.43) is the mild solution of (4.41) in the sense of Definition 4.3.5. Under the pre-compactness assumption,  $(\varphi^k)_{k \geq 1}$  thus satisfies the assumptions of Proposition 4.5.2, yielding that  $\varphi$  given by (4.29) is the solution of (4.27) in the sense of Definition 4.3.5. Let us now finish the proof of Theorem 4.3.6 by identifying the process with law  $\bar{\mu}_{[0,T]}$ .

On the canonical space  $\Omega = C([0, T], \mathbb{R}^d)$ , the canonical process satisfies

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)d\mathbf{B}_t, \quad \nu_{[0,T]\text{-a.s.}}$$



where  $(\mathbf{B}_t)_{0 \leq t \leq T}$  is a Brownian motion under  $\nu_{[0,T]}$ . Let  $\tilde{\mu}_{[0,T]}$  be the measure over  $C([0,T], \mathbb{R}^d)$  defined by

$$\frac{d\tilde{\mu}_{[0,T]}}{d\nu_{[0,T]}}(\mathbf{X}_{[0,T]}) := \exp \left[ - \int_0^T \nabla \varphi_t(\mathbf{X}_t) \cdot \sigma_t(\mathbf{X}_t) d\mathbf{B}_t - \frac{1}{2} \int_0^T |\sigma_t^\top(\mathbf{X}_t) \nabla \varphi_t(\mathbf{X}_t)|^2 dt \right]. \quad (4.45)$$

The bounds on  $\sigma$  and  $\nabla \varphi$ , inherited from the one  $(\nabla \varphi^k)_{k \geq 1}$ , guarantee that the Novikov condition holds

$$\mathbb{E}_{\nu_{[0,T]}} \exp \left[ \frac{1}{2} \int_0^T |\sigma_t^\top(\mathbf{X}_t) \nabla \varphi_t(\mathbf{X}_t)|^2 dt \right] < +\infty,$$

so that the Girsanov transform (see e.g. [Léonard, 2012, Theorem 2.3]) tells us that  $\tilde{\mu}_{[0,T]}$  is a probability measure such that

$$d\mathbf{X}_t = b_t(\mathbf{X}_t)dt - a_t(\mathbf{X}_t)\nabla \varphi_t(\mathbf{X}_t)dt + \sigma_t(\mathbf{X}_t)d\tilde{\mathbf{B}}_t, \quad \tilde{\mu}_{[0,T]}\text{-a.s.},$$

the process  $(\tilde{\mathbf{B}}_t)_{0 \leq t \leq T}$  being a Brownian motion under  $\tilde{\mu}_{[0,T]}$ . Moreover, uniqueness in law holds for the above SDE.

Under  $\nu_{[0,T]}$ , let now  $X^k$  denote the strong solution of the SDE

$$dX_t^k = b_t^k(X_t^k)dt + \sigma_t^k(X_t^k)d\mathbf{B}_t, \quad X_0^k = \mathbf{X}_0.$$

Using (4.43), Ito's formula applied to  $\varphi^k = u^k$  yields

$$\begin{aligned} \varphi_0^k(X_0^k) - \int_0^T [c_t^k(X_t^k) + \psi_t^k(X_t^k)\lambda_t^k]dt - \bar{\lambda}_T \psi_T^k(X_T^k) = \\ - \int_0^T \nabla \varphi_t^k(X_t^k) \cdot \sigma_t^k(X_t^k) d\mathbf{B}_t - \frac{1}{2} \int_0^T |(\sigma_t^k)^\top(X_t^k) \nabla \varphi_t^k(X_t^k)|^2 dt. \end{aligned}$$

As previously, we can show that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^k - \mathbf{X}_t|] \rightarrow 0$ . We then take the  $k \rightarrow +\infty$  limit (in probability) in the above equality, as we did in the proof of Proposition 4.5.2, to get

$$\frac{d\tilde{\mu}_{[0,T]}}{d\nu_{[0,T]}}(\mathbf{X}_{[0,T]}) = \exp \left[ \varphi_0(\mathbf{X}_0) - \int_0^T c_t(\mathbf{X}_t)dt - \int_{[0,T]} \psi_t(\mathbf{X}_t)\lambda(dt) \right].$$

Noticing that  $\bar{\mu}(dx_{[0,T]}) = Z^{-1} e^{-\varphi_0(x_0)} \tilde{\mu}(dx_{[0,T]})$ , this concludes.  $\square$

To complete the proof of Theorem 4.3.6, it remains to build approximations of the coefficients that are  $C^1$  in time and satisfy the assumptions of Lemma 4.5.3.

**Lemma 4.5.4.** *By convolution with a non-negative approximation of unity with compact support, let  $(\lambda^k)_{k \geq 1}$  be a sequence of functions in  $C^1([0,T], \mathbb{R}_+)$  that weakly converges towards  $\lambda$ . Similarly, let  $t \mapsto c_t(x)$ ,  $t \mapsto b_t(x)$ ,  $t \mapsto \psi_t(x)$  be regularisations by convolution of  $t \mapsto c_t^k(x)$ ,  $t \mapsto b_t^k(x)$ ,  $t \mapsto \psi_t^k(x)$ , which are now  $C^1$  in time. Then (i,ii,iii,iv,v) in Proposition 4.5.2 are satisfied.*

*Proof.* The convolution in time does not change the  $x$ -dependence:  $x \mapsto c_t^k(x)$ ,  $x \mapsto b_t^k(x)$  and  $x \mapsto \psi_t^k(x)$  are globally Lipschitz uniformly in  $(k,t)$ . From (C.1)-(C.2), the coefficients were locally bounded; hence, for every  $x \in \mathbb{R}^d$ , for Lebesgue-a.e.  $t \in [0,T]$ ,  $(b_t^k(x), \sigma_t^k(x), c_t^k(x))$  converges to  $(b_t(x), \sigma_t(x), c_t(x))$ . Using the uniform Lipschitz regularity, this implies that for Lebesgue-a.e.  $t \in [0,T]$  and every  $x \in \mathbb{R}^d$ ,  $(b_t^k(x), \sigma_t^k(x), c_t^k(x))$  converges to  $(b_t(x), \sigma_t(x), c_t(x))$ , giving (iii) in Proposition 4.5.2. From (C.2)-(i),  $(t,x) \mapsto \psi_t(x)$  is uniformly continuous and bounded on every compact set; hence,  $\psi^k$  converges towards  $\psi$  uniformly on every compact set. The uniform bounds on  $\sigma_t^k$  are similarly obtained. We thus verified items (ii,iii,iv,v) in Proposition 4.5.2. Item (i) further holds because the coefficients are locally bounded.  $\square$

To conclude the proof of Theorem 4.3.6 using Lemma 4.5.3, it remains to show pre-compactness.

**Lemma 4.5.5** (Equi-continuity for the gradients). *Under (C.2), for every  $t$  in  $[0, T]$  and any compact  $K \subset \mathbb{R}^d$ ,  $(x \mapsto \nabla \varphi_t^k(x))_{k \geq 1}$  is pre-compact in  $C(K, \mathbb{R})$ , where  $\varphi^k$  is defined by (4.40).*

*Proof.* From the proof of Lemma 4.5.3,  $(\nabla \varphi_t^k)_{k \geq 1}$  is uniformly bounded. From the Arzelà-Ascoli theorem, pre-compactness will follow if we show that  $(x \mapsto \nabla \varphi_t^k(x))_{k \geq 1}$  is equi-continuous.

Let us fix  $x \in \mathbb{R}^d$ . From (C.2)-(iii), the coefficients  $b^k$  and  $\sigma^k$  have derivatives w.r.t.  $x$ , which are equi-continuous w.r.t.  $k$ . From [Friedman, 1975, Chapter 5, Theorem 5.3],  $x \mapsto Z_s^{t,x,k}$  is a.s. differentiable and the gradient process  $\nabla Z^{t,x,k}$  is the solution of the linear SDE

$$\begin{cases} d(\nabla Z_s^{t,x,k}) = \nabla b_s^k(Z_s^{t,x,k}) \nabla Z_s^{t,x,k} ds + \nabla \sigma_s^k(Z_s^{t,x,k}) \nabla Z_s^{t,x,k} dB_s, \\ Z_t^{t,x,k} = \text{Id}. \end{cases} \quad (4.46)$$

We now fix  $(t, x)$  in  $[0, T] \times \mathbb{R}^d$ . From the Lipschitz assumptions, the derivatives of the coefficients are globally bounded. We then define

$$\chi_t^k(x) := \int_t^T [c_s^k(Z_s^{t,x,k}) + \lambda_s^k \psi_s^k(Z_s^{t,x,k})] ds,$$

whose gradient w.r.t.  $x$  writes

$$\nabla \chi_t^k(x) := \int_t^T [\nabla Z_s^{t,x,k} \nabla c_s^k(Z_s^{t,x,k}) + \lambda_s^k \nabla Z_s^{t,x,k} \nabla \psi_s^k(Z_s^{t,x,k})] ds.$$

Setting  $v^k := \mathbb{E}[e^{\chi_t^k(x)}]$ , we can write that

$$\nabla \varphi_t^k(x) = -\frac{\nabla v_t^k(x)}{v_t^k(x)} = -\frac{\mathbb{E}[e^{\chi_t^k(x)} \nabla \chi_t^k(x)]}{v_t^k(x)}. \quad (4.47)$$

To prove the equi-continuity of  $\nabla \varphi_t^k$  at  $x$ , let us consider a sequence  $(x_l)_{l \geq 1}$  that converges in  $\mathbb{R}^d$  towards  $x$  as  $l \rightarrow +\infty$ . We want to show that  $\nabla \varphi_t^k(x_l)$  converges towards  $\nabla \varphi_t^k(x)$  uniformly in  $k$ . From (4.44), we obtain as previously that  $(v_t^k(0))_{k \geq 1}$  converges towards a positive number, so that  $(\varphi_t^k(0))_{k \geq 1}$  is bounded; the uniform bound on  $\nabla \varphi^k$  then gives that  $\varphi_t^k$  has linear growth uniformly in  $k$ . As a consequence,  $v_t^k(x_l)$  is bounded from above, and bounded from below by a positive constant, uniformly in  $(k, l)$ . Similarly,  $\nabla v_t^k(x_l)$  is uniformly bounded, and  $v_t^k(x_l)$  converges towards  $v_t^k(x)$  uniformly in  $k$ . The convergence of  $\nabla v_t^k(x_l)$  remains to be studied. First,

$$\begin{aligned} |\nabla v_t^k(x_l) - \nabla v_t^k(x)| &= |\mathbb{E}[(e^{\chi_t^k(x_l)} - e^{\chi_t^k(x)}) \nabla \chi_t^k(x) + e^{\chi_t^k(x_l)} (\nabla \chi_t^k(x_l) - \nabla \chi_t^k(x))]|, \\ &\leq \mathbb{E}[\max(e^{\chi_t^k(x_l)}, e^{\chi_t^k(x)}) |\chi_t^k(x_l) - \chi_t^k(x)| |\nabla \chi_t^k(x)|] \\ &\quad + \mathbb{E}[|\nabla \chi_t^k(x_l) - \nabla \chi_t^k(x)| e^{\chi_t^k(x_l)}]. \end{aligned}$$

We now show how to control the second term on the r.h.s., the first one being similar (and even easier because it does not involve  $\nabla \chi_t^k(x_l)$ ). Since  $(x_l)_{l \geq 1}$ ,  $\nabla b^k$ ,  $\nabla \sigma^k$  and  $\sigma^k$  are bounded uniformly in  $k$ , it is standard from (4.46) that for any  $\alpha > 0$  and  $q \geq 1$ ,

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |\nabla Z_s^{t,x_l,k}|^q \right] \quad \text{and} \quad \mathbb{E} \exp \left[ \alpha \sup_{t \leq s \leq T} |Z_s^{t,x_l,k}| \right]$$

are bounded uniformly in  $(k, l)$ . From the linear growth of  $c^k$  and  $\psi^k$  (uniformly in  $k$ ) together with the uniform bound on  $\|\lambda^k\|_{L^1(0,T)}$ , we deduce analogous uniform bounds on  $\mathbb{E}[|\nabla\chi_t^k(x_l)|^q]$  and  $\mathbb{E}[e^{\alpha|\chi_t^k(x_l)}|]$ . For  $M > 0$ , we then write

$$\begin{aligned} \mathbb{E}[|\nabla\chi_t^k(x_l) - \nabla\chi_t^k(x)|e^{\lambda_t^k(x_l)}] &= \mathbb{E}[|\nabla\chi_t^k(x_l) - \nabla\chi_t^k(x)|e^{\lambda_t^k(x_l)}\mathbb{1}_{\sup_{t \leq s \leq T} |Z_s^{t,x_l,k}| + |Z_s^{t,x,k}| \leq M}] \\ &\quad + \mathbb{E}[|\nabla\chi_t^k(x_l) - \nabla\chi_t^k(x)|e^{\lambda_t^k(x_l)}\mathbb{1}_{\sup_{t \leq s \leq T} |Z_s^{t,x_l,k}| + |Z_s^{t,x,k}| > M}]. \end{aligned}$$

The moment bounds ensure that the second term can be made arbitrarily small by choosing a large enough  $M$ , uniformly in  $(k, l)$ . Moreover,

$$\begin{aligned} |\nabla\chi_t^k(x_l) - \nabla\chi_t^k(x)| &\leq T \sup_{t \leq s \leq T} |\nabla Z_s^{t,x_l,k} \nabla c_s^k(Z_s^{t,x_l,k}) - \nabla Z_s^{t,x,k} \nabla c_s^k(Z_s^{t,x,k})| \\ &\quad + \|\lambda^k\|_{L^1(0,T)} \sup_{t \leq s \leq T} |\nabla Z_s^{t,x_l,k} \nabla \psi_s^k(Z_s^{t,x_l,k}) - \nabla Z_s^{t,x,k} \nabla \psi_s^k(Z_s^{t,x,k})|. \end{aligned}$$

When  $M$  is fixed, the conditioning ensures that  $Z_s^{t,x_l,k}$  remain in a compact set  $K$  that does not depend on  $(k, l)$ . Since convolution in time does not affect the space regularity, the continuous functions  $x \mapsto \nabla c_s^k(x)$  and  $x \mapsto \nabla \psi_s^k(x)$  are uniformly continuous on  $K$ , uniformly in  $(s, k, l)$ . From [Friedman, 1975, Chapter 5, Lemma 3.3],

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |Z_s^{t,x_l,k} - Z_s^{t,x,k}|^2 \right] \xrightarrow{l \rightarrow +\infty} 0, \quad (4.48)$$

and this holds uniformly in  $k$ , because the Lipschitz constant of  $b^k$  and  $\sigma^k$  are independent of  $k$ . Using the Cauchy-Schwarz inequality, we can now conclude if we prove that

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |\nabla Z_s^{t,x_l,k} - \nabla Z_s^{t,x,k}|^2 \right] \xrightarrow{l \rightarrow +\infty} 0,$$

uniformly in  $k$ . Following [Friedman, 1975, Chapter 5, Theorem 5.2] for this proof, we use (4.46) to write that for every  $t \leq s \leq T$ ,

$$\begin{aligned} \nabla Z_s^{t,x_l,k} - \nabla Z_s^{t,x,k} &= \gamma_s^{k,x_l} + \delta_s^{k,x_l} + \int_t^s \nabla b_r^k(Z_r^{t,x_l,k}) [\nabla Z_r^{t,x_l,k} - \nabla Z_r^{t,x,k}] dr \\ &\quad + \int_t^s \nabla \sigma_r^k(Z_r^{t,x_l,k}) [\nabla Z_r^{t,x_l,k} - \nabla Z_r^{t,x,k}] dB_r, \end{aligned} \quad (4.49)$$

where

$$\begin{aligned} \gamma_s^{k,x_l} &:= \int_t^s [\nabla b_r^k(Z_r^{t,x_l,k}) - \nabla b_r^k(Z_r^{t,x,k})] \nabla Z_r^{t,x,k} dr, \\ \delta_s^{k,x_l} &:= \int_t^s [\nabla \sigma_r^k(Z_r^{t,x_l,k}) - \nabla \sigma_r^k(Z_r^{t,x,k})] \nabla Z_r^{t,x,k} dB_r. \end{aligned}$$

We recall that  $\nabla b^k$  and  $\nabla \sigma^k$  are bounded uniformly in  $k$ . We now take the square of (4.49), and we use the Jensen and the Burkholder-Davis-Gundy (BDG) inequalities to get that for every  $t \leq s \leq T$ ,

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq r \leq s} |\nabla Z_r^{t,x_l,k} - \nabla Z_r^{t,x,k}|^2 \right] &\leq 4\mathbb{E} \left[ \sup_{t \leq r \leq T} |\gamma_r^{k,x_l}|^2 + |\delta_r^{k,x_l}|^2 \right] \\ &\quad + C \int_t^s \mathbb{E} \left[ \sup_{t \leq r \leq u} |\nabla Z_r^{t,x_l,k} - \nabla Z_r^{t,x,k}|^2 \right] du, \end{aligned} \quad (4.50)$$

for a constant  $C > 0$  that does not depend on  $(k, l)$ . Using Gronwall's lemma, we can conclude if we show that the first term on the r.h.s. of (4.50) goes to 0 as  $l \rightarrow +\infty$ , uniformly in  $k$ . As previously, we write that

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \leq r \leq T} |\gamma_r^{k, x_l}|^2 \right] &= \mathbb{E} \left[ \sup_{t \leq r \leq T} |\gamma_r^{k, x_l}|^2 \mathbb{1}_{\sup_{t \leq s \leq T} |Z_s^{t, x_l, k}| + |Z_s^{t, x, k}| + |\nabla Z_r^{t, x, k}| \leq M} \right] \\ &\quad + \mathbb{E} \left[ \sup_{t \leq r \leq T} |\gamma_r^{k, x_l}|^2 \mathbb{1}_{\sup_{t \leq s \leq T} |Z_s^{t, x_l, k}| + |Z_s^{t, x, k}| + |\nabla Z_r^{t, x, k}| > M} \right], \end{aligned}$$

and the second term can be made arbitrarily small by choosing a large enough  $M$ , uniformly in  $(k, l)$ . For the first term, when  $M$  is fixed,  $Z_s^{t, x_l, k}$ ,  $Z_s^{t, x, k}$  and  $\nabla Z_s^{t, x_l, k}$  remain in a compact set  $K$  that does not depend on  $(k, l)$ , and  $x \mapsto \nabla b_s^k(x)$  is uniformly continuous on  $K$ , uniformly in  $(s, k, l)$ . The uniform vanishing of  $\mathbb{E}[\sup_{t \leq r \leq T} |\gamma_r^{k, x_l}|^2]$  then results from (4.48). Using the BDG inequality, the same reasoning gives the uniform vanishing of  $\mathbb{E}[\sup_{t \leq r \leq T} |\delta_r^{k, x_l}|^2]$ , completing the proof.  $\square$

Iterating the method of the above proof for differentiating  $x \mapsto \nabla Z^{t, x, k}$  within (4.44), we get the following corollary.

**Corollary 4.5.5.1.** *In the setting of Lemma 4.5.3, if the coefficients  $b^k$ ,  $\sigma^k$ ,  $c^k$  and  $\psi^k$  are  $C^\infty$  in  $x$  with bounded derivatives, then  $\varphi^k$  is  $C^\infty$  in  $x$ .*

## Appendix

### 4.A Linear functional derivative

This appendix gathers some useful tools for differentiating functions on measures. Let  $E$  be a complete metric space, endowed with its Borel  $\sigma$ -algebra.

**Definition 4.A.1** (Linear functional derivative). *Let  $\mathcal{C}$  be a convex subset of  $\mathcal{P}(E)$  and let  $\mu \in \mathcal{C}$ . A map  $F : \mathcal{C} \rightarrow \mathbb{R}$  is differentiable at  $\mu$  w.r.t. to the set of directions  $\mathcal{C}$  if there exists a measurable map*

$$\frac{\delta F}{\delta \mu}(\mu) : \begin{cases} E \rightarrow \mathbb{R}, \\ x \mapsto \frac{\delta F}{\delta \mu}(\mu, x), \end{cases}$$

such that for every  $\mu'$  in  $\mathcal{C}$ ,  $\frac{\delta F}{\delta \mu}(\mu)$  is  $\mu'$ -integrable and satisfies

$$\varepsilon^{-1} [F((1 - \varepsilon)\mu + \varepsilon\mu') - F(\mu)] \xrightarrow{\varepsilon \rightarrow 0^+} \left\langle \mu' - \mu, \frac{\delta F}{\delta \mu}(\mu) \right\rangle.$$

To alleviate notations, the dependence on  $\mathcal{C}$  is not emphasised in the notation  $\frac{\delta F}{\delta \mu}(\mu)$ . The map  $\frac{\delta F}{\delta \mu}(\mu)$  being defined up to an additive constant, we adopt the usual convention that  $\langle \mu, \frac{\delta F}{\delta \mu}(\mu) \rangle = 0$ . This map is called the linear functional derivative of  $F$  at  $\mu$  (w.r.t. the set of directions  $\mathcal{C}$ ). We notice that this definition does not depend on the behaviour of  $F$  outside of an arbitrary small neighbourhood of  $\mu$ .

**Example 4.A.2** (Linear case). *In the particular case  $F(\mu) = \langle \mu, f \rangle$  for some measurable  $f : E \rightarrow \mathbb{R}$  that is  $\mu$ -integrable for every  $\mu$  in  $\mathcal{C}$ , we have  $\frac{\delta F}{\delta \mu}(\mu, x) = f(x) - \langle \mu, f \rangle$ .*

**Remark 4.A.3** (Convex case). *If  $F$  is convex on  $\mathcal{C}$  and differentiable at  $\mu \in \mathcal{C}$  w.r.t.  $\mathcal{C}$ , Definition 4.A.1 implies that*

$$\forall \mu' \in \mathcal{C}, \quad F(\mu') \geq F(\mu) + \left\langle \mu' - \mu, \frac{\delta F}{\delta \mu}(\mu) \right\rangle. \quad (4.51)$$

**Lemma 4.A.4** (Integration). *Let  $\mu, \mu' \in \mathcal{C}$ . Assume that for any  $r \in [0, 1]$ ,  $F$  is differentiable at  $(1-r)\mu' + r\mu$  w.r.t. the set of directions  $\mathcal{C}$ , and that*

$$\int_0^1 \left| \left\langle \mu - \mu', \frac{\delta F}{\delta \mu}((1-r)\mu' + r\mu) \right\rangle \right| dr < \infty.$$

Then

$$F(\mu) - F(\mu') = \int_0^1 \left\langle \mu - \mu', \frac{\delta F}{\delta \mu}((1-r)\mu' + r\mu) \right\rangle dr.$$

Notably, the relative entropy does not fit in the setting of Definition 4.A.1, because the directional derivative may be  $-\infty$ . However, the following classical result still holds, whose proof traces back to [Csiszár, 1975, Lemma 2.1].

**Lemma 4.A.5** (Derivative of  $H$ ). *Let  $\mu$  and  $\mu'$  be measures in  $\mathcal{P}(E)$  with  $H(\mu|\nu), H(\mu'|\nu) < +\infty$ . For  $\varepsilon \in [0, 1]$ , we define  $\mu_\varepsilon := (1-\varepsilon)\mu + \varepsilon\mu'$ . Then,*

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} H(\mu_\varepsilon|\nu) = \int_E \log \frac{d\mu}{d\nu} d\mu' - H(\mu|\nu),$$

the integral on the r.h.s. being in  $\mathbb{R} \cup \{-\infty\}$ .

## 4.B A variant of Theorem 4.2.13 for linear constraints

In this section we only work with linear inequality constraints as in Remarks 4.2.5-4.2.15. We present a set of relaxed assumptions on the family  $(\psi_t)_{t \in \mathcal{T}}$  which ensures that the conclusion of Theorem 4.2.13 remains in force, in the absence of equality constraints ( $\mathcal{S} = \emptyset$ ).

**Proposition 4.B.1** (Lagrange multiplier for linear constraints). *Let us consider  $\nu$  in  $\mathcal{P}(E)$ , a continuous  $\phi : E \rightarrow \mathbb{R}_+$ , and a measurable  $\mathcal{F} : \mathcal{P}_\phi(E) \rightarrow \mathbb{R}$ . Let  $(\psi_t)_{t \in \mathcal{T}}$  be a family of measurable functions  $E \rightarrow \mathbb{R}$ . We assume that there exists  $C_\psi \geq 0$  such that,*

$$\forall x \in E, \quad \inf_{t \in \mathcal{T}} \psi_t(x) \geq -C_\psi [1 + \phi(x)]. \quad (4.52)$$

Then the  $\Psi_t : \mathcal{P}_\phi(E) \rightarrow (-\infty, +\infty], \mu \mapsto \Psi_t(\mu)$  are measurable functions. Let the set  $\mathcal{A}_\Psi$  be defined accordingly. We further assume that for  $\nu$ -a.e.  $x$  in  $E$ , the map  $t \in \mathcal{T} \mapsto \psi_t(x)$  is continuous, and that there exist  $\tilde{\mu} \in \mathcal{P}_\phi(E)$  and  $\eta > 0$  such that

$$\forall t \in \mathcal{T}, \quad \langle \tilde{\mu}, \psi_t \rangle \leq -\eta. \quad (4.53)$$

In this setting, let  $\bar{\mu} \in \mathcal{A}_\Psi$  be a minimiser for (4.8) (with  $\mathcal{S} = \emptyset$ ). We assume that  $\mathcal{F}$  is differentiable at  $\bar{\mu}$  w.r.t. the set of directions  $\mathcal{P}_\phi(E)$  in the sense of Definition 4.A.1, and that  $C_{\mathcal{F}} \in \mathbb{R}$  exists such that for  $\nu$ -a.e.  $x$  in  $E$ ,

$$\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}, x) \geq -C_{\mathcal{F}} [1 + \phi(x)].$$

We eventually assume that

$$\langle \nu, e^{\bar{\alpha}\phi} \rangle < +\infty, \quad (4.54)$$

for some  $\bar{\alpha} \in \mathbb{R}$  with  $C_{\mathcal{F}} + C_\psi C(\nu, \mathcal{F}, \eta, \tilde{\mu}, \bar{\mu}) < \bar{\alpha}$ , where

$$C(\nu, \mathcal{F}, \eta, \tilde{\mu}, \bar{\mu}) := \eta^{-1} \left[ \log \left\langle \nu, \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right] \right\rangle + H(\tilde{\mu}|\nu) + \left\langle \tilde{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle \right].$$

Then, there exists a positive Radon measure  $\bar{\lambda}$  on  $\mathcal{T}$  such that

$$\frac{d\bar{\mu}}{d\nu}(x) = \bar{Z}^{-1} \exp \left[ -\frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}, x) - \int_{\mathcal{T}} \psi_t(x) \bar{\lambda}(dt) \right], \quad (4.55)$$

where  $\bar{Z} \in (0, +\infty)$  is a normalising constant. Moreover,  $\bar{\lambda}(\mathcal{T}) \leq C(\nu, \mathcal{F}, \eta, \tilde{\mu}, \bar{\mu})$  and the complementary slackness condition is satisfied:

$$\langle \bar{\mu}, \psi_t \rangle = 0 \quad \text{for } \bar{\lambda}\text{-a.e. } t \in \mathcal{T}. \quad (4.56)$$

Notice that, contrarily to the remainder of this work, here the functions  $\Psi_t$  are not required to be l.s.c. on  $\mathcal{P}_\phi(E)$ , but under (4.52) they are so if the  $\psi_t$  are l.s.c. on  $E$ ; furthermore, they are allowed to take the value  $+\infty$  on  $\mathcal{P}_\phi(E)$ . The main strength of Proposition 4.B.1 is to remove the upper bound condition in Remark 4.2.15. Only lower bounds are required, as in [Liu and Wu, 2020, Assumption (A1)]. Another advantage of Proposition 4.B.1 is that its proof is elementary, in the sense that it does not rely on the use of the Hahn-Banach theorem.

*Proof of Proposition 4.B.1.* Since  $\mathcal{F}$  is not assumed to be convex,  $\bar{\mu}$  may not be unique.

**Step 1. Linearisation.** Let us consider  $\mu$  in  $\mathcal{A}_\Psi$ . Then  $\mathcal{F}(\mu)$  is finite by assumption. We first assume that  $H(\mu|\nu) < +\infty$ . For  $\varepsilon \in (0, 1]$ , we set  $\bar{\mu}_\varepsilon := (1 - \varepsilon)\bar{\mu} + \varepsilon\mu$ . Since  $\mathcal{P}_\phi(E)$  is convex, we write for any  $\varepsilon \in (0, 1]$ ,

$$H(\bar{\mu}|\nu) + \mathcal{F}(\bar{\mu}) \leq H(\bar{\mu}_\varepsilon|\nu) + \mathcal{F}(\bar{\mu}_\varepsilon) \leq (1 - \varepsilon)H(\bar{\mu}|\nu) + \varepsilon H(\mu|\nu) + \mathcal{F}(\bar{\mu}_\varepsilon),$$

where we used the convexity of  $H$  in the second inequality. This rewrites

$$H(\bar{\mu}|\nu) - H(\mu|\nu) \leq \varepsilon^{-1} [\mathcal{F}(\bar{\mu}_\varepsilon) - \mathcal{F}(\bar{\mu})],$$

which yields, thanks to Definition 4.A.1,

$$H(\bar{\mu}|\nu) + \left\langle \bar{\mu}, \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle \leq H(\mu|\nu) + \left\langle \mu, \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle.$$

This inequality still holds if  $H(\mu|\nu) = +\infty$ . We thus proved that  $\bar{\mu}$  is a minimiser for

$$\inf_{\mu \in \mathcal{A}_\Psi} H(\mu|\nu) + \left\langle \mu, \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle. \quad (4.57)$$

By strict convexity of  $H$ ,  $\bar{\mu}$  is the unique minimiser for (4.57).

**Step 2. Dual problem.** For any  $\lambda$  in  $\mathcal{M}_+(\mathcal{T})$  with  $C_{\mathcal{F}} + C_\psi \lambda(\mathcal{T}) < \bar{\alpha}$ , we define the Gibbs free energy

$$G(\lambda) := -\log \left\langle \nu, \exp \left[ -\frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) - \int_{\mathcal{T}} \psi_t \lambda(dt) \right] \right\rangle,$$

together with the Gibbs measure  $\mu_\lambda$  such that

$$\frac{d\mu_\lambda}{d\nu}(x) = e^{G(\lambda)} \exp \left[ -\frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}, x) - \int_{\mathcal{T}} \psi_t(x) \lambda(dt) \right].$$

From (4.54),  $\mu_\lambda$  is well-defined and satisfies  $H(\mu_\lambda|\nu) < +\infty$ ; thus,  $\mu_\lambda$  belongs to  $\mathcal{P}_\phi(E)$  thanks to Lemma 4.2.2. For every  $\mu$  in  $\mathcal{P}_\phi(E)$ , we get as in the proof of Theorem 4.2.17 that

$$H(\mu|\nu) + \left\langle \mu, \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}) \right\rangle + \int_{\mathcal{T}} \langle \mu, \psi_t \rangle \lambda(dt) = H(\mu|\mu_\lambda) + G(\lambda)$$

this equality being in  $\mathbb{R} \cup \{+\infty\}$ . Since  $H(\mu|\mu_\lambda) \geq 0$ , this proves that

$$G(\lambda) = \inf_{\mu \in \mathcal{P}_\phi(E)} H(\mu|\nu) + \left\langle \mu, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle + \int_{\mathcal{T}} \langle \mu, \psi_t \rangle \lambda(dt), \quad (4.58)$$

the unique minimiser being  $\mu_\lambda$ . We now study the Lagrange dual problem

$$\sup_{\substack{\lambda \in \mathcal{M}_+(\mathcal{T}) \\ C_{\mathcal{F}} + C_\psi \lambda(\mathcal{T}) < \bar{\alpha}}} G(\lambda). \quad (4.59)$$

The following weak duality relation is verified

$$\sup_{\substack{\lambda \in \mathcal{M}_+(\mathcal{T}) \\ C_{\mathcal{F}} + C_\psi \lambda(\mathcal{T}) < \bar{\alpha}}} G(\lambda) \leq \inf_{\mu \in \mathcal{A}_\psi} H(\mu|\nu) + \left\langle \mu, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle, \quad (4.60)$$

guarantying the finiteness of the supremum (4.59).

**Step 3. Existence for  $\bar{\lambda}$ .** Let  $(\lambda_k)_{k \in \mathbb{N}}$  be a maximising sequence for (4.59). Noticing that

$$\sup_{\substack{\lambda \in \mathcal{M}_+(\mathcal{T}) \\ C_{\mathcal{F}} + C_\psi \lambda(\mathcal{T}) < \bar{\alpha}}} G(\lambda) \geq G(0) = -\log \left\langle \nu, \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right] \right\rangle,$$

we get that for every  $\varepsilon > 0$ ,  $G(\lambda_k) \geq G(0) - \eta\varepsilon$  for  $k$  large enough. Using (4.58),

$$G(\lambda_k) \leq H(\tilde{\mu}|\nu) + \left\langle \tilde{\mu}, \frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) \right\rangle + \int_{\mathcal{T}} \langle \tilde{\mu}, \psi_t \rangle \lambda_k(dt),$$

with  $\langle \tilde{\mu}, \psi_t \rangle \leq -\eta$ , hence

$$\lambda_k(\mathcal{T}) \leq \varepsilon + C(\nu, \mathcal{F}, \eta, \tilde{\mu}, \bar{\mu}). \quad (4.61)$$

Using the Prokhorov theorem as in the proof of Theorem 4.2.13, the bound on  $(\lambda_k(\mathcal{T}))_{k \geq 1}$  implies that  $(\lambda_k)_{k \geq 1}$  is relatively compact for the weak convergence of measures. Up to re-indexing, we can thus assume that  $(\lambda_k)_{k \in \mathbb{N}}$  weakly converges towards some  $\bar{\lambda} \in \mathcal{M}_+(\mathcal{T})$ . By assumption,  $t \mapsto \psi_t(x)$  is  $\nu$ -a.e. continuous, and  $\nu$ -a.e. bounded because  $\mathcal{T}$  is compact. As a consequence,

$$\int_{\mathcal{T}} \psi_t(x) \lambda_k(dt) \xrightarrow[k \rightarrow +\infty]{} \int_{\mathcal{T}} \psi_t(x) \bar{\lambda}(dt) \text{ for } \nu\text{-a.e. } x \in E.$$

Fatou's lemma then yields

$$\left\langle \nu, \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) - \int_{\mathcal{T}} \psi_t \bar{\lambda}(dt) \right] \right\rangle \leq \liminf_{k \rightarrow +\infty} \left\langle \nu, \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) - \int_{\mathcal{T}} \psi_t \lambda_k(dt) \right] \right\rangle,$$

proving that  $\bar{\lambda}$  realises the supremum (4.59). Since (4.61) was true for every  $\varepsilon > 0$  provided that  $k$  was large enough, we get that  $\bar{\lambda}(\mathcal{T}) \leq C(\nu, \mathcal{F}, \eta, \tilde{\mu}, \bar{\mu})$ . In particular,  $C_{\mathcal{F}} + C_\psi \bar{\lambda}(\mathcal{T}) < \bar{\alpha}$  from our assumption on  $\bar{\alpha}$ .

**Step 4. Admissibility for the Gibbs measure  $\mu_{\bar{\lambda}}$ .** Let  $\tilde{\varepsilon} > 0$  be such that  $C_{\mathcal{F}} + C_\psi [\bar{\lambda}(\mathcal{T}) + \tilde{\varepsilon}] < \bar{\alpha}$ . Given any  $(t_0, \varepsilon)$  in  $\mathcal{T} \times (0, \tilde{\varepsilon}]$ , the perturbation  $\bar{\lambda} + \varepsilon \delta_{t_0}$  is admissible for (4.59),  $\delta_{t_0}$  being the Dirac mass at  $t_0$ . The optimality of  $\bar{\lambda}$  yields

$$\log \left\langle \nu, \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) - \int_{\mathcal{T}} \psi_t \bar{\lambda}(dt) \right] \right\rangle \leq \log \left\langle \nu, \exp \left[ -\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) - \int_{\mathcal{T}} \psi_t [\bar{\lambda}(dt) + \varepsilon \delta_{t_0}(dt)] \right] \right\rangle,$$

so that subtracting the r.h.s., and dividing by  $\varepsilon$ ,

$$\varepsilon^{-1} \langle \mu_{\bar{\lambda}}, 1 - e^{-\varepsilon \psi_{t_0}} \rangle \leq 0,$$

that we rewrite as

$$\varepsilon^{-1} \langle \mu_{\bar{\lambda}}, 1 - e^{-\varepsilon |\psi_{t_0}|_+} \rangle \leq \varepsilon^{-1} \langle \mu_{\bar{\lambda}}, e^{\varepsilon |\psi_{t_0}|_-} - 1 \rangle. \quad (4.62)$$

By the mean value theorem,

$$\forall \varepsilon \in (0, \frac{\tilde{\varepsilon}}{2}], \quad \varepsilon^{-1} |e^{\varepsilon |\psi_{t_0}|_-} - 1| \leq |\psi_{t_0}|_- e^{\varepsilon |\psi_{t_0}|_-} \leq C e^{\tilde{\varepsilon} |\psi_{t_0}|_-},$$

for some constant  $C > 0$  that does not depend on  $\varepsilon$ . The integrability condition (4.54) and our choice of  $\tilde{\varepsilon}$  guarantee that  $e^{\tilde{\varepsilon} |\psi_{t_0}|_-}$  is  $\mu_{\bar{\lambda}}$ -integrable, so that

$$\varepsilon^{-1} \langle \mu_{\bar{\lambda}}, e^{\varepsilon |\psi_{t_0}|_-} - 1 \rangle \xrightarrow{\varepsilon \rightarrow 0} \langle \mu_{\bar{\lambda}}, |\psi_{t_0}|_- \rangle.$$

To send  $\varepsilon$  to 0 in the l.h.s. of (4.62) which is non-negative, we apply Fatou's lemma. At the end of the day, gathering everything,

$$\langle \mu_{\bar{\lambda}}, \psi_{t_0} \rangle \leq 0.$$

Since the above inequality holds for every  $t_0$  in  $\mathcal{T}$ ,  $\mu_{\bar{\lambda}}$  belongs to  $\mathcal{A}_{\Psi}$ .

**Step 5.** *Complementary slackness condition.* For any  $\varepsilon$  in  $(0, 1)$ , the perturbation  $\bar{\lambda} - \varepsilon \bar{\lambda}$  is admissible for (4.59). As above, the optimality of  $\bar{\lambda}$  implies

$$\left\langle \mu_{\bar{\lambda}}, \exp \left[ \varepsilon \int_{\mathcal{T}} \psi_t \bar{\lambda}(dt) \right] - 1 \right\rangle \geq 0.$$

Dividing by  $\varepsilon$  and sending  $\varepsilon$  to 0, the same splitting and domination arguments as above provide that

$$\int_{\mathcal{T}} \langle \mu_{\bar{\lambda}}, \psi_t \rangle \bar{\lambda}(dt) \geq 0.$$

From the previous step,  $\langle \mu_{\bar{\lambda}}, \psi_t \rangle$  is always non-positive. This proves that  $\langle \mu_{\bar{\lambda}}, \psi_t \rangle = 0$  for  $\bar{\lambda}$ -a.e.  $t$  in  $\mathcal{T}$ .

**Step 6.** *Conclusion.* Since  $\mu_{\bar{\lambda}}$  is the unique minimiser for (4.58), (4.60) and the previous step show that  $\mu_{\bar{\lambda}}$  is a minimiser for (4.57). From the uniqueness proved in **Step 1.**, we eventually get that  $\mu_{\bar{\lambda}} = \bar{\mu}$ .  $\square$

The same method could be adapted when the  $\Psi_t$  are non-linear convex functions.

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## CHAPTER 5

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# Regularity and stability for the Gibbs conditioning principle on path space via McKean-Vlasov control

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*Attention, les calculs sont un peu lourds.*

— Giovanni Conforti, private communication

This work is a collaboration with Giovanni Conforti.

We consider a system of diffusion processes interacting through their empirical distribution. Assuming that the empirical average of a given observable can be observed at any time, we derive regularity and quantitative stability results for the optimal solutions in the associated version of the Gibbs conditioning principle. The proofs rely on the analysis of a McKean-Vlasov control problem with distributional constraints. Some new estimates are derived for Hamilton-Jacobi-Bellman equations and the Hessian of the log-density of diffusion processes, which are of independent interest.

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## 5.1 Introduction

Consider a system of identically distributed random variables

$$X^1, X^2, \dots, X^N,$$

taking values in a topological space  $E$ . For example, as it is the case in this article,  $E$  may be the space of continuous trajectories over  $\mathbb{R}^d$  and the  $X^i$  are sample paths from a system of interacting diffusion processes. Furthermore, assume that the variables  $X^i$  are either independent or interacting through their empirical distribution (or configuration)

$$\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{X^i} \in \mathcal{P}(E).$$

Next, imagine that some measurement is made on the system configuration revealing that  $\mu^N$  belongs to a subset  $\mathcal{O}$ . Then, one is naturally led to the question of computing either the most likely configuration of the system or the law of  $X^1$  conditionally on  $\{\mu^N \in \mathcal{O}\}$ . A detailed answer to both questions can be given when the empirical distribution satisfies a large deviation principle (henceforth, LDP) with rate function  $I$ . This answer is part of a meta-principle, called *Gibbs conditioning principle*, which we now briefly describe. To this aim, recall that the LDP holds if for any closed set  $A \subseteq \mathcal{P}(E)$ ,

$$\limsup_{N \rightarrow +\infty} N^{-1} \log \mathbb{P}(\mu^N \in A) \leq - \inf_A I,$$

and the converse inequality holds for open sets. Then, an informal statement of the Gibbs conditional principle is

$$\lim_{N \rightarrow +\infty} \mathcal{L}(X^1 | \mu^N \in \mathcal{O}) = \bar{\mu}, \quad \text{with } \bar{\mu} := \operatorname{argmin}_{\mathcal{O}} I.$$

In words, the conditional distribution converges to the minimiser of the rate function among all distributions compatible with the observation.

Equivalently, the distribution  $\bar{\mu}$  may be regarded as the most likely configuration of the system conditionally on the observation: this viewpoint is the one adopted by E. Schrödinger in a celebrated thought experiment Schrödinger [1932] motivating the formulation of the Schrödinger problem, which is the prototype of a stochastic mass transport problem. In all physical experiments, measurements are affected by uncertainties of different nature. Thus, it is of paramount importance to quantify how much the sought most likely evolution or conditional law  $\bar{\mu}$  is stable with respect to variations in the observed values. The scope of this article is to address this problem and the tightly related question of quantifying the regularity properties of  $\bar{\mu}$  when the  $X^i$  are  $\mathbb{R}^d$ -valued interacting diffusion processes such as, e.g.

$$dX_t^i = [b(X_t^i) - \frac{1}{N} \sum_{j \neq i} \nabla W(X_t^i - X_t^j)] dt + \sigma(X_t^i) dB_t^i, \quad t \in [0, T], \quad i = 1, \dots, N,$$

and what is observed is whether or not at any time  $t \in [0, T]$  the value of a certain observable  $\Psi(\mu_t^N)$  is below a threshold  $\varepsilon$ , where  $\mu_t^N := \frac{1}{N} \sum_i \delta_{X_t^i}$  is the particle configuration at time  $t$ . To fix ideas, the observation may be the empirical average of a given observable  $\psi$ ,

$$\Psi(\mu_t^N) = \frac{1}{N} \sum_{i=1}^N \psi(X_t^i),$$

which can be thought of as a mean energy in a physical setting. However, our framework is more general as we shall see below, encompassing non-linear convex constraints. In this setting, the LDP is known to hold with rate function in relative entropy form.

For clarity of exposition, let us now describe our results in the simplified setting  $W \equiv 0$ . The LDP is now directly given by Sanov's Theorem. The most likely evolution  $\bar{\mu}$  then reduces to the problem of computing the entropic projection in the sense of Csiszár [1975] of the path measure  $\nu_{[0,T]}$  describing the law of  $X^1$  onto the admissible set (clearly,  $\nu_{[0,T]}$  does not depend on  $N$  when  $W \equiv 0$ ). We are therefore looking at the problem

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \Psi(\mu_t) \leq \varepsilon}} H(\mu_{[0,T]} | \nu_{[0,T]}), \quad (5.1)$$

where  $H$  is the relative entropy on path space, defined in Section 5.1.2 below. Our approach takes benefit from classical duality results for entropic projections. In particular, these results suggest that there exists a non-negative measure  $\bar{\lambda}^\varepsilon$  with finite mass on  $[0, T]$  playing the role of a Lagrange multiplier and such that the density of the minimiser  $\bar{\mu}_{[0,T]}^\varepsilon$  for (5.1) reads

$$\frac{d\bar{\mu}_{[0,T]}^\varepsilon}{d\nu_{[0,T]}}(x_{[0,T]}) = Z_\varepsilon^{-1} \exp \left[ - \int_0^T \psi(x_t) \bar{\lambda}^\varepsilon(dt) \right],$$

where  $Z_\varepsilon$  is a normalising constant. We then rely on the classical connection between entropy minimisation and stochastic control Föllmer [1988]; Dawson and Gärtner [1987]; Dai Pra [1991]; Budhiraja et al. [2012]; Léonard [2012]. This connection allows us to view the optimal measure  $\bar{\mu}_{[0,T]}^\varepsilon$  as an optimal state in a stochastic control problem (of McKean-Vlasov type) with distributional constraints. The optimal control policy  $(t, x) \mapsto \nabla \bar{\varphi}_t^\varepsilon(x)$  can be computed from the path-density through the Feynman-Kac formula, see e.g. Léonard [2022]. Leaving all general and rigorous statements to the next section, let us informally showcase our contributions in the simple though relevant setting we have just described:

- *Regularity*: we show at Theorem 5.2.2 and Theorem 5.2.3 that under suitable regularity assumptions on the coefficients:
  - The optimal marginal flow  $(\bar{\mu}_t^\varepsilon)_{t \in [0,T]}$ , together with the Lagrange multiplier  $\bar{\lambda}^\varepsilon$  and the optimal control policy  $(\nabla \bar{\varphi}_t^\varepsilon)_{t \in [0,T]}$  form the unique solution of a coupled forward-backward PDE system, whose precise form is given at (5.3) below. As it is customary in mean-field control, the forward equation is a Fokker-Planck equation and the backward equation is a Hamilton-Jacobi-Bellman equation (henceforth, HJB). The non-standard nature of the equations we study here lies in the presence of the Lagrange multiplier  $\bar{\lambda}^\varepsilon$ , accounting for the distributional constraints in the underlying control problem.
  - The restriction of  $\bar{\lambda}^\varepsilon$  to the open interval  $(0, T)$  is an absolutely continuous measure with globally bounded density. Atoms may exist at the endpoints of the interval.
- *Stability*: strengthening the regularity assumptions of Theorem 5.2.3 and adding a convexity assumption on  $\Psi$  we show at Theorem 5.2.6:
  - *Entropic stability*: the relative entropy  $H(\bar{\mu}_{[0,T]}^0 | \bar{\mu}_{[0,T]}^\varepsilon)$  is  $O(\varepsilon)$ . As a corollary of Pinsker's inequality, the total variation distance is of order  $O(\varepsilon^{1/2})$ . Furthermore, as a consequence of our assumptions, we prove that the 1-Wasserstein distance is also  $O(\varepsilon^{1/2})$ .

- *Multiplier stability*: we show that the total variation distance between the non-negative measures  $\bar{\lambda}^\varepsilon$  and  $\bar{\lambda}^0$  is  $O(\varepsilon^{1/4})$ .
- *Control stability*: we establish that, uniformly in  $t \in [0, T]$ , the  $L^\infty$ -difference between the optimal Markov policies  $\nabla \bar{\varphi}_t^\varepsilon(\cdot)$  and  $\nabla \bar{\varphi}_t^0(\cdot)$  is  $O(\varepsilon^{1/4})$ .

Both the Schrödinger problem and (5.1) aim at computing some kind of entropic projection and originate from the Gibbs conditioning principle. However, they fundamentally differ in the type of available observations. Whereas the former problem provides the full configuration of the particle system but only at initial and final times, (5.1) only has very partial information on the particle configuration, namely the sign of the empirical average of  $\psi$ , but at all times  $t \in [0, T]$ . Thus, passing from one problem to the other, there is a trade-off between how rich is the observation we have and the set of times at which observations are made. This difference has deep implications in terms of how to approach the stability problem. For example, most stability results for the Schrödinger problem (see Section 5.1.1 below for references) rely on a static equivalent formulation in terms of an optimization problem over couplings on  $\mathbb{R}^d \times \mathbb{R}^d$ . Because of the dynamic nature of the constraints we deal with in (5.1), such reduction to a static problem is impossible and the proof approach has to be different.

The backbone of our proof strategy consists of two main ingredients. The first one is a careful analysis of the regularity properties of the HJB equations that arise in connection with a relaxed version of (5.1). This relaxation employs a Lagrange multiplier to get rid of the distributional constraints. The analysis of these PDEs is rather delicate due the low regularity of the involved cost functionals, which are typically only  $L^1$  in time. Moreover, in order to obtain quantitative stability estimates, we have to push the analysis of these HJB equations up to the third derivative in Section 5.3.1, thus obtaining novel bounds (up to our knowledge) that are of independent interest. We combine these regularity bounds with new estimates (up to our knowledge) for the Hessian of the log-density of diffusion processes. These estimates are also of independent interest; they are proved in Appendix 5.A using time-reversal in the spirit of Haussmann and Pardoux [1986], extending results from Fontbona and Jourdain [2016].

The second key-ingredient in Section 5.3.1 is a procedure to construct good competitors for the regularised stochastic control problems, by slightly pushing optimal processes along the gradient flow generated by the observable  $\psi$  and reinterpreting the result as an admissible control. This construction is crucial for showing that the optimal multiplier has a bounded density.

We conclude this introduction by highlighting that the results of this paper open the door for obtaining quantitative versions of the Gibbs conditioning principle in which one computes the rate of convergence of the conditional distribution of the  $N$ -particle system towards the law of the optimal solution in (5.1).

### 5.1.1 Literature review

Some reference textbooks about the Gibbs conditioning principle are Lanford [1973]; Ruelle [1965]; Dembo and Zeitouni [2009]; Dupuis and Ellis [2011]; Ellis [2006]. Some prominent contributions are Borel [1906]; Diaconis and Freedman [1987]; Stroock and Zeitouni [1991], which are summarised and deepened in the aforementioned textbooks. The related notion of entropic projection was introduced and studied in Csiszár [1975, 1984]. For quantitative versions of the Gibbs principle in convex settings, we refer to Dembo and Zeitouni [1996]; Dembo and Kuelbs [1998]; Cattiaux and Gozlan [2007]. All these settings study the case of a finite number of constraints. Existence of a Lagrange multiplier for abstract equality

constraints is proved in Léonard [2000]. The case of an infinite number of inequality constraints is treated in Chaintron et al. [2024].

There has been an abundant literature on Schrödinger bridges, some seminal works being Csiszár [1975]; Cattiaux and Léonard [1995]; Cattiaux and Léonard [1996]. We also refer to the survey article Léonard [2014], the lecture notes Nutz [2021] and references therein. The analogous problem when replacing the relative entropy by the rate function of an interacting particle system (case  $W \neq 0$  of the introduction) is known as the mean-field Schrödinger problem Backhoff et al. [2020]. Up to our knowledge, the first result on Schrödinger bridges with distributional constraints is Chaintron et al. [2024].

Recently, important progresses on the stability problem for entropic projections have been achieved Eckstein and Nutz [2022]; Nutz and Wiesel [2023]; Chiarini et al. [2023]; Divol et al. [2024], mostly if not exclusively driven by the surge of interest around the Schrödinger problem and its applications in machine learning. However, the proofs in our setting are quite different as explained in the introduction.

Stochastic control problems under distributional constraints have received some attention, mostly motivated by mathematical finance, see e.g. Föllmer and Leukert [1999]; Guo et al. [2022] and references therein. For problems with expectation constraints, we refer to Chow et al. [2020]; Guo et al. [2022]; Pfeiffer et al. [2021] and related works. In what concerns the regularity results, the closest works to ours are the recent works Daudin [2022, 2023b], which study the problem of controlling the drift of a Brownian motion under convex distributional constraints, and sparked renewed interest around constrained stochastic control problems. Their approach is purely PDE-oriented: existence and regularity are obtained for a Lagrange multiplier using a penalisation method, following some constructions put forward in the deterministic setting in Cannarsa et al. [2018]. The main challenge of our framework compared to these works is the addition of a non-constant diffusion matrix and the McKean-Vlasov setting. In particular, our results rely on novel estimates for HJB equations, whose starting points are our recent works Chaintron [2023] and Chaintron et al. [2024]. The related (non-quantitative) mean-field limit is proved in Daudin [2023a], adapting the method of Lacker [2017] to the constrained setting. When it comes to the stability results we obtained, they appear to be the first of this kind that apply to dynamically constrained McKean-Vlasov control problems, at least to the best of our knowledge and understanding.

### 5.1.2 Frequently used notations

- $\mathcal{P}(E)$  denotes the set of probability measures over a probability space  $E$ .
- $\mathcal{L}(X)$  denotes the law in  $\mathcal{P}(E)$  of a  $E$ -valued random variable  $X$ .
- $\delta_x$  denotes the Dirac measure at some point  $x$  in  $E$ .
- $H(\mu|\nu)$  is the relative entropy of  $\mu, \nu \in \mathcal{P}(E)$ , defined by  $H(\mu|\nu) := \int_E \log \frac{d\mu}{d\nu} d\mu$  if  $\mu \ll \nu$ , and  $H(\mu|\nu) := +\infty$  otherwise.
- $\mathcal{P}_1(E)$  denotes the set of measures  $\mu \in \mathcal{P}_1(E)$  such that  $\int_E d(x, x_0) \mu(dx) < +\infty$  for some distance  $d$  on  $E$  and  $x_0 \in E$ .

- $W_1$  denotes the Wasserstein distance on  $\mathcal{P}_1(E)$ , defined by

$$W_1(\mu, \mu') := \inf_{X \sim \mu, Y \sim \mu'} \mathbb{E}[d(X, Y)].$$

- $\|\cdot\|_{\text{TV}}$  denotes the total variation distance on  $\mathcal{P}(E)$ , defined by

$$\|\mu - \mu'\|_{\text{TV}} := \inf_{X \sim \mu, Y \sim \mu'} \mathbb{P}(X \neq Y).$$



- $f * \mu$  is the convolution of a measurable  $f : E \rightarrow E'$  and  $\mu \in \mathcal{P}(E)$ , defined by  $f * \mu(x) := \int_E f(x - y)\mu(dy)$ .
- $\frac{\delta F}{\delta \mu}(\mu) : x \mapsto \frac{\delta F}{\delta \mu}(\mu, x)$  denotes the linear functional derivative at  $\mu$  of a function  $F : \mathcal{P}(E) \rightarrow \mathbb{R}$ , see Definition 5.1.1 below. The convention is adopted that  $\int_E \frac{\delta F}{\delta \mu}(\mu)d\mu = 0$ .
- $T > 0$  is a given real number, and  $d \geq 1$  is an integer.
- $x_{[0,T]} \in C([0, T], \mathbb{R}^d)$  denotes a continuous function  $x_{[0,T]} : [0, T] \rightarrow \mathbb{R}^d$ .
- $\mu_{[0,T]}$  denotes a path measure in  $\mathcal{P}(C([0, T], \mathbb{R}^d))$ . The related marginal measure at time  $t$  will be denoted by  $\mu_t$ .
- $X_t$  denotes the coordinate map  $x_{[0,T]} \mapsto x_t$ . It can be seen as a random variable on the canonical space  $\Omega = C([0, T], \mathbb{R}^d)$ .
- $\Sigma = (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$  denotes a reference probability system in the terminology of [Fleming and Soner, 2006, Chapter 4]:  $(\Omega, \mathcal{F}_T, \mathbb{P})$  is a probability space,  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is a filtration satisfying the usual conditions, and  $(B_t)_{0 \leq t \leq T}$  is a  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -Brownian motion.
- $\mathcal{M}_+([0, T])$  denotes the convex cone of positive Radon measures over  $[0, T]$ . In this setting, a Radon measure is a signed finite measure that is both inner and outer regular as defined in [Rudin, 1970, Definition 2.15].
- $\cdot^\top$  and  $\text{Tr}[\cdot]$  respectively denote the transpose and the trace of matrices.
- $|a|$  denotes the Frobenius norm  $\sqrt{\text{Tr}[aa^\top]}$  of a matrix  $a$ .
- $\nabla \cdot a$  for a matrix field  $a = (a^{i,j})_{1 \leq i, j \leq d}$  is the vector field whose  $i$  entry is the divergence of the vector field  $(a^{i,j})_{1 \leq j \leq d}$ . We similarly define differential operators on matrices.
- When using coordinates, we will always use the summing convention of repeated indices.

**Definition 5.1.1** (Linear functional derivative). *Let  $\mathcal{C}$  be a convex subset of  $\mathcal{P}(E)$ , and let  $\mu \in \mathcal{C}$ . A map  $F : \mathcal{C} \rightarrow \mathbb{R}$  is differentiable at  $\mu$  if there exists a measurable map*

$$\frac{\delta F}{\delta \mu}(\mu) : \begin{cases} E \rightarrow \mathbb{R}, \\ x \mapsto \frac{\delta F}{\delta \mu}(\mu, x), \end{cases}$$

such that for every  $\mu'$  in  $\mathcal{C}$ ,  $\frac{\delta F}{\delta \mu}(\mu)$  is  $\mu'$ -integrable and satisfies

$$\varepsilon^{-1} [F((1 - \varepsilon)\mu + \varepsilon\mu') - F(\mu)] \xrightarrow{\varepsilon \rightarrow 0^+} \int_E \frac{\delta F}{\delta \mu}(\mu)d[\mu' - \mu].$$

The map  $\frac{\delta F}{\delta \mu}(\mu)$  being defined up to an additive constant, we adopt the usual convention that  $\int_E \frac{\delta F}{\delta \mu}(\mu)d\mu = 0$ . This map is called the linear functional derivative of  $F$  at  $\mu$  (w.r.t. the set of directions  $\mathcal{C}$ ). We notice that this definition does not depend on the behaviour of  $F$  outside of an arbitrary small neighbourhood of  $\mu$ .

By direct integration, the definition implies

$$\forall \mu, \mu' \in \mathcal{C}, \quad F(\mu) - F(\mu') = \int_0^1 \int_E \frac{\delta F}{\delta \mu} ((1-r)\mu' + r\mu) d[\mu - \mu'] dr,$$

provided that the integral on the r.h.s. is well-defined.

**Example 5.1.2** (Linear case). *In the particular case  $F(\mu) = \int_E f d\mu$  for some measurable  $f : E \rightarrow \mathbb{R}$  that is  $\mu$ -integrable for every  $\mu$  in  $\mathcal{C}$ , we have  $\frac{\delta F}{\delta \mu}(\mu, x) = f(x) - \int_E f d\mu$ .*

## 5.2 Statement of the main results

For  $\nu_0 \in \mathcal{P}_1(\mathbb{R}^d)$  and  $\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d))$ , let  $\Gamma(\mu_{[0,T]})$  denote the path-law of the pathwise unique strong solution to

$$dX_t = b_t(X_t, \mu_t)dt + \sigma_t(X_t)dB_t, \quad X_0 \sim \nu_0,$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion and the coefficients are continuous functions  $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ ,  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  satisfying (A.2) below. The generalisation of (5.1) we are interested in is

$$\overline{H}_\Psi := \inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0, T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]} | \Gamma(\mu_{[0,T]})), \quad (5.2)$$

where  $\Psi : \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$  is lower semi-continuous. The map  $\mu_{[0,T]} \mapsto H(\mu_{[0,T]} | \Gamma(\mu_{[0,T]}))$  is a large deviation rate function, which is studied in e.g. Fischer [2014]; Backhoff et al. [2020]. Under (A.1)-(A.2) below, this map has compact level sets in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$  from [Fischer, 2014, Remark 5.2] or [Chaintron, 2024, Corollary B.4] (it is a good rate function in the large deviation terminology). Since  $\Psi$  is lower semi-continuous, existence of a minimiser always holds for (5.2) if  $\overline{H}_\Psi$  is finite. We now detail the regularity assumptions that are needed for writing optimality conditions for (5.2).

**Assumption (A.1)** (Initial condition). *For every  $\alpha > 0$ ,  $x \mapsto e^{\alpha|x|}$  is  $\nu_0$ -integrable.*

**Assumption (A.2)** (Coefficients). *There exists  $C > 0$  such that:*

- (i)  $\forall (t, x, y, \mu, \mu') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{P}_1(\mathbb{R}^d)$ ,  
 $|b_t(x, \mu) - b_t(y, \mu')| + |\sigma_t(x) - \sigma_t(y)| \leq C[|x - y| + W_1(\mu, \mu')].$
- (ii)  $\forall (t, x) \in [0, T] \times \mathbb{R}^d$ ,  $|\sigma_t(x)| \leq C$ , and  $t \mapsto \sigma_t(x)$  is locally Hölder-continuous.
- (iii)  $\forall (t, x, \xi) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ,  $\xi^\top \sigma_t \sigma_t^\top(x) \xi \geq C^{-1}|\xi|^2$ .

**Assumption (A.3)** (Linear derivatives). *The functions  $\mu \mapsto \Psi(\mu)$  and  $\mu \mapsto b_t(x, \mu)$  have a linear functional derivative in the sense of Definition 5.1.1. Moreover,  $(x, \mu) \mapsto \frac{\delta \Psi}{\delta \mu}(\mu, x)$  and  $(t, x, \mu, y) \mapsto \frac{\delta b_t}{\delta \mu}(x, \mu, y)$  are jointly continuous, and for every compact set  $K \subset \mathcal{P}_1(\mathbb{R}^d)$ ,*

$$\forall (t, \mu, x, y, z) \in [0, T] \times K \times (\mathbb{R}^d)^3, \quad \left| \frac{\delta b_t}{\delta \mu}(x, \mu, y) - \frac{\delta b_t}{\delta \mu}(x, \mu, z) \right| + \left| \frac{\delta \Psi}{\delta \mu}(\mu, y) - \frac{\delta \Psi}{\delta \mu}(\mu, z) \right| \leq C_K |y - z|,$$

for  $C_K > 0$  that only depends on  $K$ . Finally,  $x \mapsto \nabla \frac{\delta \Psi}{\delta \mu}(\mu, x)$  is  $C_K$ -Lipschitz for any  $\mu \in K$ .

In the statement of our results, we shall require the following assumption, called *constraint qualification*, to be satisfied at a given  $\bar{\mu}_{[0,T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d))$  such that  $\Psi(\bar{\mu}_t) \leq 0$  for every  $t \in [0, T]$ .

**Assumption (A.4)** (Qualification). *There exists  $\tilde{\varepsilon} > 0$ ,  $\tilde{\mu}_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d))$  such that  $H(\tilde{\mu}_{[0,T]}|\Gamma(\tilde{\mu}_{[0,T]})) < +\infty$  and*

$$\forall t \in [0, T], \quad \Psi(\bar{\mu}_t) + \tilde{\varepsilon} \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t) d\tilde{\mu}_t < 0.$$

An analytical equivalent of this assumption is given by Remark 5.2.5 below.

**Remark 5.2.1.** *When  $\Psi$  is convex, a condition equivalent to (A.4) is the existence of  $\tilde{\mu}_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d))$  such that  $H(\tilde{\mu}_{[0,T]}|\Gamma(\tilde{\mu}_{[0,T]})) < +\infty$  and*

$$\forall t \in [0, T], \quad \Psi(\tilde{\mu}_t) < 0,$$

see [Chaintron et al., 2024, Lemma 2.17]. Another sufficient condition is given by [Chaintron et al., 2024, Remark 2.15].

We now provide a control interpretation of (5.2) for which we write optimality conditions.

**Theorem 5.2.2** (Structure of optimisers). *Under (A.1)-(A.2)-(A.3), let us assume that  $\bar{\mu}_{[0,T]}$  is an optimal measure for (5.2) that satisfies (A.4).*

(i) McKean-Vlasov control: *consider the control problem with law constraints*

$$\bar{V}_\Psi := \inf_{\substack{(X_t^\alpha)_{0 \leq t \leq T}, \alpha \\ \forall t \in [0, T], \Psi(\mathcal{L}(X_t^\alpha)) \leq 0}} H(\mathcal{L}(X_0^\alpha)|\nu_0) + \mathbb{E} \int_0^T \frac{1}{2} |\alpha_t|^2 dt,$$

on a filtered probability space  $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  satisfying the usual conditions, where  $\alpha$  is progressively measurable,  $(X_t^\alpha)_{0 \leq t \leq T}$  is adapted, and  $\mathbb{P}$ -a.s.

$$dX_t^\alpha = b_t(X_t^\alpha, \mathcal{L}(X_t^\alpha))dt + \sigma_t(X_t^\alpha)\alpha_t dt + \sigma_t(X_t^\alpha)dB_t, \quad 0 \leq t \leq T,$$

for a given  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -Brownian motion  $(B_t)_{0 \leq t \leq T}$  in  $\mathbb{R}^d$ . We have the correspondence:

$$\bar{H}_\Psi = \bar{V}_\Psi,$$

and  $\bar{\mu}_{[0,T]}$  is the path-law of an optimally controlled process.

(ii) Optimality conditions: *there exists  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  and  $\bar{\varphi} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\bar{\mu}_0(dx) = \bar{Z}^{-1} e^{-\bar{\varphi}_0(x)} \nu_0(dx)$  and*

$$\begin{cases} \partial_t \bar{\mu}_t - \nabla \cdot [-\bar{\mu}_t b_t(\cdot, \bar{\mu}_t) + \bar{\mu}_t \sigma_t \sigma_t^\top \nabla \bar{\varphi}_t + \frac{1}{2} \nabla \cdot [\bar{\mu}_t \sigma_t \sigma_t^\top]] = 0, \\ -\bar{\varphi}_t + \int_t^T b_s(\cdot, \bar{\mu}_s) \cdot \nabla \bar{\varphi}_s - \frac{1}{2} |\sigma_s^\top \nabla \bar{\varphi}_s|^2 + \frac{1}{2} \text{Tr}[\sigma_s \sigma_s^\top \nabla^2 \bar{\varphi}_s] + \bar{c}_s ds \\ \quad + \int_{[t, T]} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_s) \bar{\lambda}(ds) = 0, \end{cases} \quad (5.3)$$

where

$$\bar{c}_s(x) := \int_{\mathbb{R}^d} \frac{\delta b}{\delta \mu}(y, \bar{\mu}_s, x) \cdot \nabla \bar{\varphi}_s(y) \bar{\mu}_s(dy).$$

Moreover,  $\bar{V}_\Psi$  is realised by the optimal control in feed-back form  $\bar{\alpha}_t = -\sigma_t \sigma_t^\top \nabla \bar{\varphi}_t(X_t^\alpha)$ . In the above, solutions to the Fokker-Planck equation have to be understood in the weak sense, and solutions to the Hamilton-Jacobi-Bellman (HJB) equation have to

be understood in the sense of Definition 5.C.1. Furthermore, the complementary slackness condition holds,

$$\Psi(\bar{\mu}_t) = 0, \quad \text{for } \bar{\lambda}\text{-a.e. } t \in [0, T], \quad (5.4)$$

and we have the pathwise representation

$$\frac{d\bar{\mu}_{[0,T]}}{d\Gamma(\bar{\mu}_{[0,T]})}(x_{[0,T]}) = \bar{Z}^{-1} \exp \left[ - \int_0^T \bar{c}_t(x_t) dt - \int_0^T \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t, x_t) \bar{\lambda}(dt) \right].$$

Let us further study the Lagrange multiplier  $\bar{\lambda}$ . To do so, we need to strengthen our regularity assumptions.

**Assumption (A.5)** (Strengthening of (A.1)). *Assumption (A.1) holds and  $\nu_0$  has a positive density w.r.t. the Lebesgue measure, still denoted by  $\nu_0$ . Moreover,  $C, \varepsilon_0 > 0$  exist such that*

- (i)  $\log \nu_0$  is (a.e.) differentiable and  $\int_{\mathbb{R}^d} [(\log \nu_0)^{1+\varepsilon_0} + |\nabla \log \nu_0|^{2+\varepsilon_0}] d\nu_0 \leq C$ .
- (ii)  $\nabla \log \nu_0$  is differentiable and  $\int_{\mathbb{R}^d} [(\log \nu_0)^{1+\varepsilon_0} + |\nabla \log \nu_0|^{4+\varepsilon_0} + |\nabla^2 \log \nu_0|^{2+\varepsilon_0}] d\nu_0 \leq C$ .
- (iii) We further have  $\int_{\mathbb{R}^d} e^{\varepsilon_0|x|^2} \nu_0(dx) \leq C$ .

**Assumption (A.6)** (Strengthening of (A.2)-(A.3)). *Assumptions (A.2)-(A.3) hold, and  $\mu \mapsto \frac{\delta\Psi}{\delta\mu}(\mu, x)$  has a linear derivative  $y \mapsto \frac{\delta^2\Psi}{\delta\mu^2}(\mu, x, y)$ . Moreover, the functions*

$$t \mapsto b_t(x, \mu), \quad t \mapsto \sigma_t(x), \quad x \mapsto \nabla \sigma_t(x),$$

$$(\mu, x, y) \mapsto \frac{\delta b_t}{\delta\mu}(x, \mu, y), \quad (\mu, x) \mapsto \frac{\delta\Psi}{\delta\mu}(\mu, x), \quad x \mapsto \nabla \frac{\delta\Psi}{\delta\mu}(\mu, x), \quad y \mapsto \frac{\delta^2\Psi}{\delta\mu^2}(\mu, x, y),$$

are well-defined and Lipschitz-continuous uniformly in  $(t, x, \mu)$ .

**Assumption (A.7)** (Strengthening of (A.6)). *Assumption (A.6) holds and there exists  $C > 0$  such that for every  $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ ,*

- (i)  $x \mapsto \nabla_x b_t(x, \mu)$ ,  $x \mapsto \nabla^2 \sigma_t(x)$  and  $y \mapsto \nabla_y \frac{\delta b_t}{\delta\mu}(x, \mu, y)$  are well-defined and  $C$ -Lipschitz.
- (ii)  $x \mapsto \frac{\delta\Psi}{\delta\mu}(x, \mu)$  is  $C^2$  with  $C$ -Lipschitz derivatives which are continuous in  $(x, \mu)$ .
- (iii)  $x \mapsto \frac{\delta\Psi}{\delta\mu}(\mu, x)$  is  $C^4$  and  $y \mapsto \frac{\delta^2\Psi}{\delta\mu^2}(\mu, x, y)$  is  $C^2$ , both with derivatives bounded by  $C$  and jointly Lipschitz-continuous in  $(\mu, x, y)$ .

We now state sufficient conditions for  $\bar{\lambda}$  having a density.

**Theorem 5.2.3** (Density for the multiplier). *Under (A.4) at  $\bar{\mu}_{[0,T]}$ , there exists  $(\bar{\lambda}, \bar{\varphi})$  satisfying Theorem 5.2.2-(ii) such that  $x \mapsto \nabla \bar{\varphi}_t(x)$  is Lipschitz-continuous and*

$$\bar{\lambda}(dt) = \bar{\lambda}_0 \delta_0(dt) + \bar{\lambda}_t dt + \bar{\lambda}_T \delta_T(dt),$$

for  $t \mapsto \bar{\lambda}_t$  in  $L^\infty(0, T)$ , if we assume that either (A.5)-(i) and (A.7)-(ii) hold, or that (A.5)-(ii) and (A.7)-(i) hold. In the second case,  $x \mapsto \nabla^2 \bar{\varphi}_t(x)$  is Lipschitz-continuous and the densities of the time-marginals of  $\bar{\mu}_{[0,T]}$  satisfy

$$\left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^d} [\log \bar{\mu}_t + |\nabla \log \bar{\mu}_t|^4 + |\nabla^2 \log \bar{\mu}_t|^2] d\bar{\mu}_t \right] + \int_0^T \int_{\mathbb{R}^d} |\nabla^3 \log \bar{\mu}_t|^2 d\bar{\mu}_t dt < +\infty. \quad (5.5)$$

**Remark 5.2.4.** *As highlighted in the introduction, constrained mean field control problems have recently been studied by Daudin in a series of articles. More specifically, optimality conditions of the form (5.3) and the existence of a bounded Radon-Nikodym density for Lagrange multipliers are established in [Daudin, 2023b, Theorem 2.2]. These results are very similar in spirit to Theorem 5.2.2 and Theorem 5.2.3 and have partially inspired our work. One should note that the control problems considered here are slightly different: in Daudin [2023b], the objective function includes an additional mean field term in both the running and terminal costs, the reference dynamics has zero drift and a constant diffusion matrix, and the initial distribution is fixed. Here, the initial distribution is a variable over which we optimize, and the controlled dynamics corresponds to a stochastic differential equation of McKean-Vlasov-type with non constant diffusions coefficient. Regarding the constraint function  $\Psi$ , we are able to work under weaker regularity assumptions and do not require convexity. We also rephrase the transversality conditions from Daudin's work in terms of a constraint qualification condition, see Remark 5.2.5. Our proof approaches are rather different too. We study regularity properties by pushing the optimal process along the gradient flow of the constraint to build suitable competitors, whereas Daudin [2023b] approximates the constrained problem by relaxing the constraint and penalising the flows that violate it. Then, he shows that if the penalisation strength is sufficiently large, the penalised problem and the constrained problem are equivalent.*

The results on the density of  $\bar{\lambda}$  can be seen as a regularity trade-off between the constraint and the coefficients. They rely on the new technical estimate (5.5) for diffusions, which is proved in Proposition 5.A.4. The proof of Theorem 5.2.3 and Corollary 5.4.2.1 further show that  $t \in (0, T] \mapsto \nabla \varphi_t(x)$  is continuous and  $t \in [0, T] \mapsto \Psi(\bar{\mu}_t)$  is  $C^1$  under (A.7)-(ii), whereas  $t \in (0, T] \mapsto \nabla^2 \varphi_t(x)$  is continuous under (A.7)-(i). Depending on where the regularity is assumed (constraints or coefficients), we thus get more smoothness on either the constrained curve or the optimal control. Requiring more derivatives on coefficients would still improve the regularity in  $(t, x)$ .

**Remark 5.2.5** (Equivalent qualification condition). *A consequence of (A.4) is that*

$$\forall t \in [0, T], \quad \Psi(\bar{\mu}_t) = 0 \Rightarrow \int_{\mathbb{R}^d} \left| \nabla \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t) \right|^2 d\bar{\mu}_t \neq 0.$$

*In the setting of Theorem 5.2.3, this condition is equivalent to (A.4) from Corollary 5.3.5.1.*

We now prove quantitative stability with respect to the constraint  $\Psi$ . To guarantee that uniqueness holds for (5.2), we restrict ourselves to  $b_t(x, \mu) = b_t(x)$  and  $\Psi$  convex. This setting still includes the entropic projection (5.1) of the introduction. Let  $\nu_{[0, T]}$  denote the path-law of the solution to

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 \sim \nu_0,$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion. To alleviate notations, optimal measures will be denoted by  $\mu_{[0, T]}^\varepsilon$  in the following statement, instead of  $\bar{\mu}_{[0, T]}$ . The same goes for  $\varphi^\varepsilon$  and  $\lambda^\varepsilon$ .

**Theorem 5.2.6** (Quantitative stability). *We assume (A.4)-(A.5)-(A.7) and that  $\Psi$  is convex. For  $\varepsilon \geq 0$ , let  $\mu_{[0, T]}^\varepsilon$  denote the unique minimiser for*

$$\inf_{\substack{\mu_{[0, T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)) \\ \forall t \in [0, T], \Psi(\mu_t) \leq \varepsilon}} H(\mu_{[0, T]} | \nu_{[0, T]}).$$

Let  $(\varphi_t^\varepsilon, \lambda_t^\varepsilon, \lambda_0^\varepsilon, \lambda_T^\varepsilon)$  be related to  $\mu_{[0,T]}^\varepsilon$  by Theorems 5.2.2-5.2.3. We assume that the number of non-trivial intervals where  $\Psi(\mu_t^0)$  is identically zero is finite. Then, there exists  $C > 0$  independent of  $\varepsilon$  such that uniformly in  $\varepsilon \in [0, 1]$ ,

(i) Entropic stability:  $H(\mu_{[0,T]}^0 | \mu_{[0,T]}^\varepsilon) \leq C\varepsilon$ .

(ii) Multiplier stability:  $|\lambda_0^\varepsilon - \lambda_0^0| + \|\lambda^\varepsilon - \lambda^0\|_{L^1(0,T)} + |\lambda_T^\varepsilon - \lambda_T^0| \leq C\varepsilon^{1/4}$ .

(iii) Control stability:  $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |\nabla \varphi_t^\varepsilon(x) - \nabla \varphi_t^0(x)| \leq C\varepsilon^{1/4}$ .

We notice that the l.h.s. of (ii) corresponds to the total variation distance between the multipliers, seen as measures over  $[0, T]$ . The Gaussian bound (A.5)-(iii) is only used to simplify the convergence rate (otherwise, additional  $\log \varepsilon$  factors would have been needed). Similarly, the assumption on the finite number of intervals can be removed if we deteriorate the rate, see Remark 5.4.9 below.

**Remark 5.2.7** ( $L^\infty$ -stability). *From the proof of Proposition 5.4.8, the  $L^1$ -estimate on  $\lambda^\varepsilon$  is actually a  $L^\infty$ -one on intervals where  $\Psi(\mu_t^0) = 0$  and  $\Psi(\mu_t^\varepsilon) = \varepsilon$  simultaneously. We cannot expect a global  $L^\infty$ -estimate, since moving the constraint can produce strong discontinuities for  $\lambda_t^\varepsilon$  at the boundary of  $\{t \in [0, T], \Psi(\mu_t^\varepsilon) = \varepsilon\}$ . However, under stronger assumptions on coefficients, Lemma 5.4.6 can provide continuity for  $\lambda_t^\varepsilon$  within intervals where  $\Psi(\mu_t^\varepsilon) = \varepsilon$  identically.*

The proofs are organised as follows. Section 5.3 proves bounds on the multiplier in a linearised setting with smooth coefficients. However, we precisely keep track of regularity constants along the estimates. The main results are then proved in Section 5.4. The proof of Theorem 5.2.2 essentially relies on results from Chaintron et al. [2024]. To cope with the delicate regularity assumptions of Theorem 5.2.3, a careful smoothing procedure is detailed in Section 5.4.2. The non-linear problem (5.2) is first approximated by smooth non-linear problems. These smooth problems are then linearised to enter the scope of Section 5.3. Theorem 5.2.3 is obtained by taking the limit in the bounds therein. Starting from Theorems 5.2.2-5.2.3, Theorem 5.2.6 is proved in an independent way in Section 5.4.3.

### 5.3 Regularity bounds on the multiplier

This section is concerned with the strictly convex minimisation problem

$$\inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \int_{\mathbb{R}^d} \psi_t d\mu_t \leq 0}} H(\mu_{[0,T]} | \nu_{[0,T]}) + \int_0^T \int_{\mathbb{R}^d} c_t d\mu_t dt, \quad (5.6)$$

where  $\nu_{[0,T]}$  is the path-law of the pathwise unique strong solution to

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t, \quad X_0 \sim \nu_0,$$

under the following assumption.

**Assumption (A.8)** (Smooth setting).

(i)  $\nu_0$  satisfies (A.1), and  $\sigma$  satisfies (A.2)-(ii)-(iii).

(ii)  $b_t, \sigma_t, c_t, \psi_t$  are  $C^\infty$  function of  $x$ , with bounded derivatives of all orders independently of  $t$ . Moreover, all derivatives of  $b_t, \sigma_t, \psi_t$  are locally Hölder-continuous in  $(t, x)$ .

(iii)  $\partial_t b_t, \partial_t \sigma_t$  are well-defined,  $C^\infty$  and bounded independently of  $t$ , as well as of all their derivatives.

(iv) There exists a minimiser  $\bar{\mu}_{[0,T]}$  for (5.6), which is necessarily unique by strict convexity of the relative entropy. Moreover,  $\psi$  satisfies

$$\forall t \in [0, T], \quad \int_{\mathbb{R}^d} \partial_t \psi_t d\bar{\mu}_t = 0. \quad (5.7)$$

and constraint qualification holds at  $\bar{\mu}_{[0,T]}$ : there exists  $\tilde{\varepsilon} > 0, \tilde{\mu}_{[0,T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d))$  with  $H(\tilde{\mu}_{[0,T]} | \nu_{[0,T]}) < +\infty$  such that

$$\forall t \in [0, T] \quad \int_{\mathbb{R}^d} \psi_t d\bar{\mu}_t + \tilde{\varepsilon} \int_{\mathbb{R}^d} \psi_t d(\tilde{\mu}_t - \bar{\mu}_t) < 0. \quad (5.8)$$

This setting satisfies the assumptions of Theorem [Chaintron et al., 2024, Theorem 2.12] with  $E = C([0, T], \mathbb{R}^d)$ ,  $\phi(x_{[0,T]}) = \sup_{t \in [0, T]} |x_t|$ ,  $\mathcal{F}(\mu_{[0,T]}) = \int_0^T \int_{\mathbb{R}^d} c_t d\mu_t dt$ , the inequality constraints  $\Psi_t : \mu_{[0,T]} \mapsto \int_{\mathbb{R}^d} \psi_t d\mu_t$  with  $\mathcal{T} = [0, T]$ , and no equality constraint. Indeed,  $c_t$  has linear growth uniformly in  $t$ , implying that  $x_{[0,T]} \mapsto \int_0^T c_t(x_t)$  has linear growth, and thus [Chaintron et al., 2024, Assumptions (A6)-(A8)] are satisfied. Moreover, [Chaintron et al., 2024, Assumptions (A7)] is satisfied using (A.8) and [Chaintron et al., 2024, Remark 2.14]. Since (5.8) provides the qualification condition, we obtain a Lagrange multiplier  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  such that

$$\frac{d\bar{\mu}_{[0,T]}}{d\nu_{[0,T]}}(x_{[0,T]}) = \bar{Z}^{-1} \exp \left[ - \int_0^T c_t(x_t) dt - \int_0^T \psi_t(x_t) \bar{\lambda}(dt) \right], \quad (5.9)$$

with the complementary slackness condition

$$\int_{\mathbb{R}^d} \psi_t d\mu_t = 0, \quad \text{for } \bar{\lambda}\text{-a.e. } t \in [0, T]. \quad (5.10)$$

This section aims to show that the restriction of  $\bar{\lambda}$  to  $(0, T)$  has a bounded density w.r.t. the Lebesgue measure. Up to changing  $\nu_0(dx)$  into  $Z_0^{-1} e^{-\bar{\lambda}(\{0\})\psi_0(x)} \nu_0(dx)$ , we can assume that  $\bar{\lambda}(\{0\}) = 0$ . Indeed, under (A.6),  $\nu_0$  still satisfies (A.5) after this change. In the following, we set  $\bar{\lambda}_T := \bar{\lambda}(\{T\})$  and we will use the convenient notation  $a := \sigma\sigma^\top$ .

### 5.3.1 Smoothed multiplier and stochastic control

Let  $\Sigma = (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$  be a reference probability system as defined in Section 5.1.2. For  $(t, x) \in [0, T] \times \mathbb{R}^d$ , we consider the controlled dynamics

$$\begin{cases} dX_s^{t,x,\alpha} = b_s(X_s^{t,x,\alpha}) ds + \sigma_s(X_s^{t,x,\alpha}) \alpha_s ds + \sigma_s(X_s^{t,x,\alpha}) dB_s, & t \leq s \leq T, \\ X_t^{t,x,\alpha} = x, \end{cases} \quad (5.11)$$

where  $\alpha = (\alpha_s)_{t \leq s \leq T}$  is any progressively measurable square-integrable process on  $\Sigma$ . Such a process will be called an *open-loop control*. Following [Budhiraja et al., 2012, Section 3, page 9], strong existence and pathwise uniqueness for (5.11) is given by the Girsanov transform if  $\int_0^T |\alpha_t|^2 dt \leq M$  a.s. for some  $M > 0$ , and still holds otherwise using a localisation argument. In the following, we will often consider *feedback controls*  $\alpha_s = \alpha(s, X_s^\alpha)$ , as it is customary in control theory. In this case, regularity properties will

be needed on the function  $\alpha$  to prove strong existence and pathwise uniqueness for the SDE

$$dY_s^{t,x,\alpha} = b_s(Y_s^{t,x,\alpha})ds + \sigma_s(Y_s^{t,x,\alpha})\alpha(s, Y_s^{t,x,\alpha})ds + \sigma_s(Y_s^{t,x,\alpha})dB_s, \quad t \leq s \leq T, \quad (5.12)$$

with  $Y_t^{t,x,\alpha} = x$ . If well-posedness is granted for this SDE, we will still denote its solution by  $X^{t,x,\alpha}$ , with a slight abuse of notations, and we may write  $\alpha_s(X_s^{t,x,\alpha})$  instead of  $\alpha(s, X_s^{t,x,\alpha})$  when it is non-ambiguous. Furthermore, we will use the infinitesimal generator  $L_t^\alpha$  defined on  $C^2$  functions  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$L_t^\alpha \phi := b_t \cdot \nabla \phi + \sigma_t \alpha_t \cdot \nabla \phi + \frac{1}{2} \text{Tr}[a_t \nabla^2 \phi].$$

Given any  $(\mu_0, \lambda) \in \mathcal{P}_1(\mathbb{R}^d) \times \mathcal{M}_+([0, T])$ , we define the value functions:

$$V_\Sigma^\lambda(t, x) := \inf_\alpha \mathbb{E} \int_t^T \frac{1}{2} |\alpha_s|^2 + c_s(X_s^{t,x,\alpha}) ds + \int_{[t, T]} \psi_s(X_s^{t,x,\alpha}) \lambda(ds),$$

$$V_\Sigma^\lambda(\mu_0) := \inf_{\substack{X_0, \alpha \\ X_0 \sim \mu_0}} \mathbb{E} \int_0^T \frac{1}{2} |\alpha_s|^2 + c_s(X_s^{0, X_0, \alpha}) ds + \int_{[0, T]} \psi_s(X_s^{0, X_0, \alpha}) \lambda(ds),$$

where we minimise over open-loop controls  $\alpha$ , and  $\mathcal{F}_0$ -adapted  $X_0$  with law  $\mu_0$  for  $V_\Sigma^\lambda$ . We have to keep track of  $\Sigma$  here, because the proof of Proposition 5.3.5 below will require a change of reference system. Throughout this subsection, we fix  $(\mu_0, \lambda)$  and we make the following assumption in addition to (A.8).

**Assumption (A.9).**

- (i) There exists a  $C^1$  non-negative function  $(\lambda_s)_{0 \leq s \leq T}$  such that  $\lambda(ds) = \lambda_s ds + \bar{\lambda}_T \delta_T(ds)$ .
- (ii)  $(t, x) \mapsto c_t(x)$  is locally Hölder-continuous, as well as all its derivatives w.r.t.  $x$ .

Under (A.9) the above control problem is standard. To solve it, we introduce the HJB equation in  $[0, T] \times \mathbb{R}^d$ :

$$\begin{cases} \partial_t \varphi_t + L_t^0 \varphi_t - \frac{1}{2} |\sigma_t^\top \nabla \varphi_t|^2 + c_t = -\lambda_t \psi_t, & 0 \leq t < T, \\ \varphi_T = \bar{\lambda}_T \psi_T. \end{cases} \quad (5.13)$$

From [Chaintron, 2023, Theorem 2.2], (5.13) has a unique  $C^{1,2}$  solution  $\varphi$ , and

$$\sup_{t \in [0, T]} \|\nabla \varphi_t\|_\infty \leq C, \quad (5.14)$$

for  $C > 0$  that only depends on  $\lambda([0, T])$  and the uniform norms of  $\sigma$ ,  $\sigma^{-1}$ ,  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla c$ , and  $\nabla \psi$ . Moreover,  $\varphi$  and its derivatives are Hölder-continuous, and

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |\varphi_t(x)| \leq C[1 + |x|], \quad (5.15)$$

using [Chaintron, 2023, Lemma 3.1]. Since the coefficients are  $C^\infty$ ,  $\varphi_t$  is in fact  $C^\infty$  from [Chaintron et al., 2024, Corollary 5.6]. We can now introduce the feed-back control  $\alpha_s^0 := -\sigma_s^\top \nabla \varphi_s$ , because strong existence and pathwise-uniqueness holds for the related SDE (5.12) using (A.8), (5.14) and the fact that  $\varphi$  is  $C^2$ .

**Lemma 5.3.1** (Verification). *For every  $(t, x)$  in  $[0, T] \times \mathbb{R}^d$ ,  $V_\Sigma^\lambda(t, x)$  is uniquely realised by the feed-back control  $\alpha^0$ . As a consequence,  $V_\Sigma^\lambda(\mu_0) = \int_{\mathbb{R}^d} V_\Sigma^\lambda(0, x) d\mu_0(x)$  does not depend on  $\Sigma$ .*



This result is classical in control theory, see e.g. [Fleming and Soner, 2006, Chapter III.8, Theorem 8.1], but we still provide the proof for the sake of completeness.

*Proof.* Let  $\alpha = (\alpha_s)_{t \leq s \leq T}$  be an open-loop control. The regularity of  $\varphi$  allows us to apply Ito's formula:

$$\begin{aligned} \varphi_T(X_T^{t,x,\alpha}) - \varphi_t(x) &= \int_t^T (\partial_s + L_s^0) \varphi_s(X_s^{t,x,\alpha}) + \sigma_s(X_s^{t,x,\alpha}) \alpha_s \cdot \nabla \varphi_s(X_s^{t,x,\alpha}) ds \\ &\quad + \int_t^T \nabla \varphi_s \cdot \sigma_s(X_s^{t,x,\alpha}) dB_s. \end{aligned}$$

The stochastic integral is a true martingale because of the uniform bounds on  $\sigma$  and  $\nabla \varphi$ . The HJB equation (5.13) yields

$$\begin{aligned} \varphi_t(x) &= \int_t^T -\alpha_s \cdot \sigma_s^\top \nabla \varphi_s(X_s^{t,x,\alpha}) - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s(X_s^{t,x,\alpha})|^2 + c_s(X_s^{t,x,\alpha}) + \psi_s(X_s^{t,x,\alpha}) \lambda_s ds \\ &\quad - \int_t^T \nabla \varphi_s \cdot \sigma_s(X_s^{t,x,\alpha}) dB_s + \bar{\lambda}_T \psi_T(X_T^{t,x,\alpha}). \end{aligned}$$

Moreover,

$$\int_t^T -\alpha_s \cdot \sigma_s^\top \nabla \varphi_s(X_s^{t,x,\alpha}) - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s(X_s^{t,x,\alpha})|^2 ds \leq \int_t^T \frac{1}{2} |\alpha_s|^2 ds \quad \text{a.s.},$$

with equality if and only if  $\alpha_s = -a_s \nabla \varphi_s(X_s^{t,x,\alpha})$  for a.e.  $s \in [t, T]$ , so that taking expectations,

$$\varphi_t(x) \leq \mathbb{E} \int_t^T \frac{1}{2} |\alpha_s|^2 + c_s(X_s^{t,x,\alpha}) + \psi_s(X_s^{t,x,\alpha}) \lambda_s ds + \bar{\lambda}_T \psi_T(X_T^{t,x,\alpha}),$$

with equality if and only if  $\alpha_s = -a_s \nabla \varphi_s(X_s^{t,x,\alpha})$ ,  $ds \otimes \mathbb{P}$ -a.s.  $\square$

We now prove further regularity bounds. For  $R \geq 0$ , let  $B(0, R)$  denote the centred ball of  $\mathbb{R}^d$  with radius  $R$ . For technical reasons, we need to introduce the matrix

$$a_t^R(x) := \rho^R(x) a_t(x) + [1 - \rho^R(x)] \text{Id},$$

where  $(\rho^R)_{R \geq 0}$  is any family of smooth functions  $\mathbb{R}^d \rightarrow [0, 1]$  such that  $\rho^R|_{B(0, R)} \equiv 1$  and  $\rho^R|_{B^c(0, R+1)} \equiv 0$ , the derivatives of  $\rho^R$  being globally bounded uniformly in  $R$ . Let  $\sigma_t^R$  denote the positive definite square-root of  $a_t^R$ . Let  $\varphi_t^R$  denote the solution of (5.13) when replacing  $\sigma$  by  $\sigma^R$ . We notice that  $\sigma^R$  and  $(\sigma^R)^{-1}$  satisfy the same bounds as  $\sigma$  and  $\sigma^{-1}$ , for regularity constants that are independent of  $R$ . As a consequence,  $\nabla \varphi^R$  satisfies the bound (5.14)-(5.15) independently of  $R$ . Let us consider the feed-back control  $\alpha_t^R := -(\sigma_t^R)^\top \nabla \varphi_t^R$ , the related SDE (5.12) having a pathwise unique strong solution because  $\varphi^R$  is  $C^2$  with bounded  $\nabla \varphi^R$ . Since  $\varphi$  is the classical solution of (5.13), it is also the mild solution of (5.13) in the sense of Definition 5.C.1, which is given by [Chaintron et al., 2024, Theorem 3.6]. As a consequence of [Chaintron et al., 2024, Lemma 5.3], for Lebesgue-a.e.  $t \in [0, T]$ ,

$$\varphi_t^R \xrightarrow{R \rightarrow +\infty} \varphi_t, \quad \nabla \varphi_t^R \xrightarrow{R \rightarrow +\infty} \nabla \varphi_t, \quad (5.16)$$

uniformly on every compact set of  $\mathbb{R}^d$ , if we show that  $(\nabla \varphi_t^R)_{R \geq 0}$  is pre-compact on any compact set. From the uniform bound on  $(\nabla \varphi_t^R)_{R \geq 0}$  and the Arzelà-Ascoli theorem, a sufficient condition for this is  $(\nabla^2 \varphi_t^R)_{R \geq 0}$  being uniformly bounded.

To the best of our knowledge the following estimate is new in the stochastic control literature. Based on [Conforti, 2023, Proposition 3.2], its key element is a gradient estimate that is proved in Priola and Wang [2006] using reflection coupling.

**Proposition 5.3.2** (Hessian bound). *The following estimate holds*

$$\sup_{t \in [0, T]} \|\nabla \varphi_t\|_\infty + \sup_{t \in [0, T]} \|\nabla^2 \varphi_t\|_\infty \leq C,$$

for a constant  $C$  that only depends on coefficients through  $\lambda([0, T])$  and the uniform norms of  $\sigma$ ,  $\sigma^{-1}$ ,  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla c$ ,  $\nabla \psi_s$  and  $\nabla^2 \psi_s$ .

*Proof.* Starting from (5.14), we have to prove the Hessian bound. Since we do not know that  $\nabla^2 \varphi_t$  is bounded beforehand, we first reason on  $\varphi_t^R$ . Differentiating the HJB equation, for  $1 \leq i \leq d$ ,

$$(\partial_t + L_t^{\alpha^R}) \partial_i \varphi_t^R = -\partial_i b_t \cdot \nabla \varphi_t^R + \frac{1}{2} \nabla \varphi_t^R \cdot \partial_i a_t^R \nabla \varphi_t^R - \frac{1}{2} \text{Tr}[\partial_i a_t^R \cdot \nabla^2 \varphi_t^R] - \partial_i c_t - \lambda_t \partial_i \psi_t. \quad (5.17)$$

With a slight abuse, we still write  $L_t^{\alpha^R}$  for the operator where  $\sigma_t$  has been replaced by  $\sigma_t^R$ . Similarly, we write  $X^{t,x,\alpha^R}$  for the process (5.12) controlled by the feed-back  $\alpha^R$ , where the diffusion matrix has been replaced by  $\sigma_t^R$ . From [Chaintron, 2023, Theorem 2.2],  $\varphi^R$  is  $C^{1,2}$  with Hölder-continuous derivatives. Since  $\nabla b$ ,  $\nabla c$ ,  $\nabla \sigma^R$  and  $\nabla \psi$  are Hölder-continuous, [Friedman, 2008, Chapter 3, Theorem 13] guarantees that  $\nabla \varphi^R$  is  $C^{1,2}$  with Hölder-continuous derivatives. Ito's formula applied to  $\partial_i \varphi_s^R(X_s^{t,x,\alpha^R})$  then yields

$$\begin{aligned} \partial_i \varphi_t^R(x) &= \bar{\lambda}_T \partial_i \psi_T(X_T^{t,x,\alpha^R}) + \int_t^T \left\{ f_s + \frac{1}{2} \text{Tr}[\partial_i a_s^R \nabla^2 \varphi_s^R] + \lambda_s \partial_i \psi_s \right\} (X_s^{t,x,\alpha^R}) ds \\ &\quad + \int_t^T (\sigma_s^R)^\top \nabla \partial_i \varphi_s^R(X_s^{t,x,\alpha^R}) dB_s, \end{aligned} \quad (5.18)$$

where

$$f_s := \partial_i b_s \cdot \nabla \varphi_s^R - \frac{1}{2} \nabla \varphi_s^R \cdot \partial_i a_s^R \nabla \varphi_s^R + \partial_i c_s.$$

Using the uniform bound on  $\nabla \varphi_s^R$ ,  $f_s$  is bounded by a constant  $C > 0$  independent of  $(s, R)$ , which satisfies the requirements of Proposition 5.3.2. In the following,  $C$  may change from line to line, but still satisfying these requirements. For  $y \in \mathbb{R}^d$ , we similarly decompose  $\partial_i \varphi_t^R(y)$  and we subtract this expression to (5.18), before taking expectations to get rid of the martingale. Since  $\alpha^R$  is bounded continuous, we can use the estimate from [Priola and Wang, 2006, Theorem 3.4 (a)]. Strictly speaking, [Priola and Wang, 2006, Theorem 3.4 (a)] only allows for time-homogeneous coefficients, but the adaptation of these arguments to the time-dependent setting is straight-forward. This yields

$$\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |\mathbb{E}[f_s(X_s^{t,x,\alpha^R})] - \mathbb{E}[f_s(X_s^{t,y,\alpha^R})]| \leq \frac{C|x-y|}{1 \wedge \sqrt{t-s}}. \quad (5.19)$$

We similarly handle the other terms to obtain that

$$|\partial_i \varphi_t^R(x) - \partial_i \varphi_t^R(y)| \leq C \bar{\lambda}_T |x-y| + \int_t^T \frac{C|x-y|}{1 \wedge \sqrt{s-t}} [1 + \|\partial_i a_s^R \nabla^2 \varphi_s^R\|_\infty + \lambda_s] ds,$$

where  $\|\partial_i a_s^R \nabla^2 \varphi_s^R\|_\infty$  is finite because  $\nabla^2 \varphi^R$  is continuous and  $\partial_i a_s^R \equiv 0$  out of  $B(0, R+1)$ . Dividing by  $|x-y|$  and taking the supremum over  $(x, y)$  and  $i$ , this proves that  $\sup_{0 \leq t \leq T} \|\nabla^2 \varphi_t^R\|_\infty$  is finite.

We now perform the same reasoning, but applying Ito's formula to  $\partial_i \varphi_s^R(X_s^{t,x,0})$  instead of  $\partial_i \varphi_s^R(X_s^{t,x,\alpha^R})$ , still with  $\sigma^R$  instead of  $\sigma$  in the definition of  $X^{t,x,0}$ . Using that  $L_t^{\alpha^R} = L_t^0 + \sigma_t^R \alpha_t^R \cdot \nabla$  in (5.17), this yields

$$\begin{aligned} \partial_i \varphi_t^R(x) &= \int_t^T \left\{ f_s + \sigma_s^R \alpha_s^R \cdot \nabla \partial_i \varphi_s^R + \frac{1}{2} \text{Tr}[\partial_i a_s^R \nabla^2 \varphi_s^R] + \lambda_s \partial_i \psi_s \right\} (X_s^{t,x,0}) ds \\ &\quad + \bar{\lambda}_T \partial_i \psi_T(X_T^{t,x,0}) + \int_t^T (\sigma_s^R)^\top \nabla \partial_i \varphi_s^R(X_s^{t,x,0}) dB_s, \end{aligned}$$

Using (5.19) as previously, with  $X^{t,x,0}$  instead of  $X^{t,x,\alpha^R}$ , we obtain

$$\begin{aligned} |\partial_i \varphi_t^R(x) - \partial_i \varphi_t^R(y)| &\leq C \bar{\lambda}_T |x - y| + \int_t^T \frac{C|x-y|}{1 \wedge \sqrt{s-t}} [1 + \|\nabla^2 \varphi_s^R\|_\infty] ds \\ &\quad + \int_t^T \lambda_s |\mathbb{E}[\partial_i \psi_s(X_s^{t,x,0})] - \mathbb{E}[\partial_i \psi_s(X_s^{t,y,0})]| ds. \end{aligned} \quad (5.20)$$

Using e.g. synchronous coupling, it is standard that

$$|\mathbb{E}[\partial_i \psi_s(X_s^{t,x,0})] - \mathbb{E}[\partial_i \psi_s(X_s^{t,y,0})]| \leq C \|\nabla \partial_i \psi_s\|_\infty |x - y|.$$

The last term in (5.20) can then be bounded using that  $\lambda([0, T]) \leq C$ . We now divide by  $|x - y|$ , and we take the supremum over  $(x, y)$  and  $i$  to get that

$$\forall t \in [0, T], \quad \|\nabla^2 \varphi_t^R\|_\infty \leq C + \int_t^T \frac{C}{1 \wedge \sqrt{s-t}} [1 + \|\nabla^2 \varphi_s^R\|_\infty] ds.$$

Using the Gronwall lemma, we eventually obtain

$$\sup_{0 \leq t \leq T} \|\nabla^2 \varphi_t^R\|_\infty \leq C \left[ 1 + \int_0^T \frac{ds}{1 \wedge \sqrt{s}} \right] \exp \left[ \int_0^T \frac{C}{1 \wedge \sqrt{s}} ds \right],$$

where the constant  $C$  is independent of  $R$  and satisfies the requirements of Proposition 5.3.2. This uniform bound on  $(\nabla^2 \varphi_t^R)_{R \geq 0}$  implies the convergence (5.16), for Lebesgue-a.e.  $t \in [0, T]$ . From the Banach-Alaoglu theorem,  $(\nabla^2 \varphi_t^R)_{R \geq 0}$  has a weakly- $\star$  converging sub-sequence as  $R \rightarrow +\infty$  for the  $\sigma(L^\infty, L^1)$ -topology, whose limit still satisfies the above bound. Since  $\nabla^2 \varphi_t$  was already well-defined, this proves that  $\nabla^2 \varphi_t^R \rightarrow \nabla^2 \varphi_t$  weakly- $\star$ . In particular,  $\nabla^2 \varphi_t$  satisfies the above bound, for Lebesgue-a.e.  $t \in [0, T]$ . Since  $t \mapsto \nabla^2 \varphi_t(x)$  is continuous, this concludes the proof.  $\square$

In the above proof, the dependence on  $\nabla^2 \psi_s$  for  $C$  is needed because we want an estimate that only depends on  $\lambda$  through  $\lambda([0, T])$ . Otherwise, instead of synchronous coupling, we would use reflection coupling to handle the last term in (5.20) as the other ones. Still relying on the estimate from Priola and Wang [2006], the next result is a new third-order bound for HJB equations. In contrast with the previous bound, which only depends on  $\lambda([0, T])$ , the next bound depends on  $\sup_{0 \leq t \leq T} |\lambda_t|$ . However, it is crucial for the sequel that this dependence comes through a multiplicative constant that can be taken as small as desired.

**Lemma 5.3.3** (Third-order bound). *For every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that*

$$\sup_{t \in [0, T]} \|\nabla^3 \varphi_t\|_\infty \leq C_\varepsilon + \varepsilon \sup_{0 \leq t \leq T} |\lambda_t|,$$

for a constant  $C_\varepsilon$  that only depends on coefficients through  $\lambda([0, T])$  and the uniform norms of  $\sigma$ ,  $\sigma^{-1}$ ,  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla c$ ,  $\nabla \psi_s$ ,  $\nabla^2 b$ ,  $\nabla^2 \sigma$ ,  $\nabla^2 c$  and  $\nabla^2 \psi_s$ .

*Proof.* As in the proof of Proposition 5.3.2, we first reason on  $\varphi^R$ . For  $1 \leq i, j \leq d$ , we differentiate (5.17) and we use the bounds that were proved on  $\varphi^R$  to get as previously that

$$(\partial_t + L_t^{\alpha^R})\partial_{i,j}\varphi_t^R = -f_t - \frac{1}{2}\text{Tr}[\partial_i a_t^R \partial_j \nabla^2 \varphi_t^R] - \lambda_t \partial_{i,j}\psi_t,$$

for some measurable  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  that is bounded by a constant  $C > 0$  independent of  $R$ , which satisfies the requirements of Lemma 5.3.3. In the following,  $C$  may change from line to line, but still satisfying these requirements. Since  $\nabla^2 b$ ,  $\nabla^2 c$ ,  $\nabla^2 \sigma^R$  and  $\nabla^2 \psi$  are Hölder-continuous, [Friedman, 2008, Chapter 3, Theorem 13] guarantees as previously that  $\nabla^2 \varphi^R$  is  $C^{1,2}$  with Hölder-continuous derivatives. Ito's formula applied to  $\partial_{i,j}\varphi_s^R(X_s^{t,x,\alpha^R})$  then yields

$$\begin{aligned} \partial_{i,j}\varphi_t^R(x) &= \bar{\lambda}_T \partial_{i,j}\psi_T(X_T^{t,x,\alpha^R}) + \int_t^T \left\{ f_s + \frac{1}{2}\text{Tr}[\partial_i a_s^R \partial_j \nabla^2 \varphi_s^R] + \lambda_s \partial_{i,j}\psi_s \right\} (X_s^{t,x,\alpha^R}) ds \\ &\quad + \int_t^T (\sigma_s^R)^\top \nabla \partial_{i,j}\varphi_s^R(X_s^{t,x,\alpha^R}) dB_s. \end{aligned}$$

As previously, we subtract  $\partial_{i,j}\varphi_t^R(y)$ , we take expectations, and we use (5.19) to get that

$$|\partial_{i,j}\varphi_t^R(x) - \partial_{i,j}\varphi_t^R(y)| \leq C|x - y| + \int_t^T \frac{C|x - y|}{1 \wedge \sqrt{s - t}} [1 + \|\partial_i a_s^R \nabla^3 \varphi_s^R\|_\infty + \lambda_s] ds,$$

where  $\|\partial_i a_s^R \nabla^3 \varphi_s^R\|_\infty$  is finite because  $\nabla^3 \varphi^R$  is continuous and  $\partial_i a_s^R \equiv 0$  out of  $B(0, R + 1)$ . Dividing by  $|x - y|$  and taking the supremum over  $(x, y)$  and  $(i, j)$ , this proves that  $\sup_{0 \leq t \leq T} \|\nabla^3 \varphi_t^R\|_\infty$  is finite. For any  $\varepsilon \in (0, 1)$ , we split the last integral between  $t$  and  $t + \varepsilon^2$ :

$$\int_t^T \frac{\lambda_s}{\sqrt{t - s}} ds \leq 2\varepsilon \sup_{t \leq s \leq t + \varepsilon^2} |\lambda_s| + \frac{1}{\varepsilon} \int_{t + \varepsilon^2}^T \lambda_s ds,$$

so that

$$\forall t \in [0, T], \quad \|\nabla^3 \varphi_t^R\|_\infty \leq C + \int_t^T \frac{C}{1 \wedge \sqrt{s - t}} [\varepsilon^{-1} + \|\nabla^3 \varphi_s^R\|_\infty + \varepsilon \sup_{0 \leq r \leq T} |\lambda_r|] ds.$$

Using the Gronwall lemma, we eventually obtain

$$\sup_{0 \leq t \leq T} \|\nabla^3 \varphi_t^R\|_\infty \leq C[\varepsilon^{-1} + \varepsilon \sup_{0 \leq t \leq T} |\lambda_t|] \left[ 1 + \int_0^T \frac{ds}{1 \wedge \sqrt{s}} \right] \exp \left[ \int_0^T \frac{C}{1 \wedge \sqrt{s}} ds \right],$$

where the constant  $C$  is independent of  $R$  and satisfies the requirements of Lemma 5.3.3. We conclude the proof using the convergence (5.16) as previously.  $\square$

We fix  $X_0 \sim \bar{\mu}_0$ , and we shall write  $X^0 := X^{0, X_0, \alpha^0}$ . Since  $\bar{\mu}_0(dx) = \bar{Z}^{-1} e^{-\bar{\varphi}_0(x)} \nu_0(dx)$  from [Chaintron et al., 2024, Theorem 3.6], and  $\bar{\varphi}_0$  has linear growth from (5.15), (A.1) implies the following useful result.

**Lemma 5.3.4** (Initial bound). *If any of the bounds (A.5)-(i)-(ii)-(iii) is satisfied by  $\nu_0$  for some  $\varepsilon_0 > 0$ , then the same property holds for  $\bar{\mu}_0$  with  $\varepsilon_0 = 0$ , for a constant  $C > 0$  that satisfies the same requirements as the one in (5.15).*

The following result is crucial to obtain density estimates on the multiplier.

**Proposition 5.3.5.** *There exist  $\delta \in (0, 1)$  and  $C > 0$  such that for any  $C^1$  function  $h : [t, t'] \rightarrow [0, 1]$  with  $0 \leq t < t' \leq T$ ,  $h_t = h_{t'} = 0$  and  $\|h\|_\infty + \|\frac{d}{ds}h\|_\infty^2 \leq \delta$ ,*

$$\int_t^{t'} h_s \mathbb{E}[|\sigma_s \nabla \psi_s(X_s^0)|^2] \lambda_s ds \leq \int_t^{t'} C(h_s + |\frac{d}{ds}h_s|^2) - 2\left\{\frac{d}{ds}\mathbb{E}[\psi_s(X_s^0)] - \mathbb{E}[\partial_s \psi_s(X_s^0)]\right\} \frac{d}{ds}h_s ds.$$

Moreover, under (A.5)-(i),  $\delta$  and  $C$  only depend on  $\lambda$  through  $\lambda([0, T])$ , and  $C$  only depends on the other coefficients through the uniform norms of  $\sigma$ ,  $\sigma^{-1}$ ,  $\partial_s b$ ,  $\partial_s \sigma$ ,  $\partial_s \psi$ ,  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla c$ ,  $\nabla \psi$ ,  $\nabla \partial_s \psi$ ,  $\nabla^2 \sigma$ ,  $\nabla^2 \psi$  and  $\nabla^3 \psi$ . Under (A.5)-(ii), for every  $\varepsilon > 0$ , there further exists  $C_\varepsilon > 0$  satisfying the same requirements except that  $C_\varepsilon$  depends on  $\nabla^2 b$  and  $\nabla^3 \sigma$  but not on  $\nabla^3 \psi$ , such that the bound holds for the constant  $C := C_\varepsilon + \varepsilon \sup_{0 \leq s \leq T} |\lambda_s|$ .

In order to prove the above upper bound, we are going to construct a competitor in the stochastic control problem by pushing the marginal flow of the optimal solution along the gradient flow generated by  $\psi$ . Such construction shares some analogies with earlier results obtained e.g. in [Erbar et al., 2015, Section 4] or Monsaingeon et al. [2023] in the context of the study of (an abstract version of) the Schrödinger problem.

*Proof.* Let  $(\mu_s^0)_{0 \leq s \leq T}$  be the flow of marginal laws of  $(X_s^0)_{0 \leq s \leq T}$ . To prove the bound, we build a competitor  $\alpha^h$  for  $V_\Sigma^\lambda(\bar{\mu}_0)$  in **Step 1.**, by perturbing the optimal flow on  $[t, t']$  only. The related controlled process  $X^h$  is such that  $\mathbb{E}[\psi_s(X_s^h)] < \mathbb{E}[\psi_s(X_s^0)]$  for  $s \in (t, t')$ . The related cost  $\mathbb{E} \int_0^T \frac{1}{2} |\alpha_s^h(X_s^h)|^2 ds$  is computed in **Step 2.** and **Step 3.** The comparison between  $X^0$  and  $X^h$  is done in **Step 4.**, yielding the desired estimate. Final computations are done in **Step 5.** to obtain the wanted dependence on the coefficients.

**Step 1. Perturbation.** For  $s \in [0, T]$ , let  $(S_{s,t})_{t \geq 0}$  be the flow defined by

$$\forall x \in \mathbb{R}^d, \forall t \geq 0, \quad \frac{d}{dt} S_{s,t}(x) = -a_s \nabla \psi_s(S_{s,t}(x)), \quad S_{s,0}(x) = x,$$

where  $a_s := \sigma_s \sigma_s^\top$ . Since  $\psi_s$  is  $C^4$  and  $a_s$  is bounded and  $C^3$ , both with uniformly bounded derivatives, the Cauchy-Lipschitz theory guarantees that  $x \mapsto S_{s,t}(x)$  is a  $C^3$ -diffeomorphism. We define the perturbed flow as

$$\mu_s^h := \mu_s^0 \circ S_{s,h_s}^{-1}, \quad 0 \leq s \leq T.$$

Writing  $h'_s := \frac{d}{ds}h_s$ , Ito's formula yields

$$dS_{s,h_s}(X_s^0) = [-h'_s a_s \nabla \psi_s(S_{s,h_s}(X_s^0)) + (\partial_s + L_s^{\alpha^0})S_{s,h_s}(X_s^0)] ds + \nabla S_{s,h_s}(X_s^0) \cdot \sigma_s(X_s^0) dB_s.$$

Since  $a$  is uniformly elliptic and the coefficients are smooth from (A.8), it is standard that  $\mu_s^0$  has a positive  $C^2$  density for  $s > 0$ . Since  $x \mapsto S_{s,t}(x)$  is a  $C^3$ -diffeomorphism,  $\mu_s^h$  also has a positive  $C^2$  density that we still write  $x \mapsto \mu_s^h(x)$  with a slight abuse. We then define

$$\begin{aligned} A_s^h &:= -h'_s a_s \nabla \psi_s, & B_s^h &:= (\partial_s + L_s^{\alpha^0})[S_{s,h_s}] \circ S_{s,h_s}^{-1} - b_s, \\ C_s^h &:= -\frac{1}{2}(\mu_s^h)^{-1} \nabla \cdot [\mu_s^h (\nabla S_{s,h_s} a_s \nabla S_{s,h_s}^\top \circ S_{s,h_s}^{-1} - a_s)], & \beta_s^h &:= A_s^h + B_s^h + C_s^h. \end{aligned}$$

We refer to the notation section 5.1.2 for the meaning of operators applied to vector fields or matrices. Let us now define the feed-back  $\alpha_s^h := \sigma_s^{-1} \beta_s^h$ . Since the law of  $S_{s,h_s}(X_s^0)$  is precisely  $\mu_s^h$ , we get that  $\partial_s \mu_s^h = L_s^{\alpha^h} \mu_s^h$  in the sense of distributions. To obtain an admissible control from  $\alpha^h$ , we need to show that  $\alpha^h$  has a finite  $L^2(ds \otimes \mu_s^h)$ -norm by computing  $\mathbb{E}[|\alpha_s^h(S_{s,h_s}(X_s^0))|^2]$ . This computation is done in **Step 3.**, after computing  $\mathbb{E}[|\beta_s^h(S_{s,h_s}(X_s^0))|^2]$  in **Step 2.**

**Step 2. Expansion.** In the following, we write  $O(h_s^i)$  for quantities that are bounded by  $C|h_s|^i$  where the constant  $C$  is allowed to depend only on the uniform norm of the coefficients and their derivatives (of any order). For  $(i, j) \in \{0, 1\} \times \{0, 1, 2\}$ , we will repeatedly use the (uniform in  $x$ ) expansion

$$\partial_s^i \nabla^j S_{s, h_s} = \partial_s^i \nabla^j \text{Id} - h_s \nabla^j \partial_s^i (a_s \nabla \psi_s) + O(h_s^2),$$

which stems from a uniform bound on  $\nabla^{j+1} \partial_s^i (a \nabla \psi)$  (we will always have  $i = 0$  when  $j = 2$ ). Let us start the expansion of  $\beta_s^h \circ S_{s, h_s}$ : for  $x$  in  $\mathbb{R}^d$ ,

$$\begin{aligned} A_s^h(S_{s, h_s}(x)) &= -h'_s a_s \nabla \psi_s(x) + O(h'_s h_s), \\ B_s^h(S_{s, h_s}(x)) &= \sigma_s \alpha_s^0(x) - h_s [(\partial_s + L_s^{\alpha^0})[a_s \nabla \psi_s] - \nabla b_s a_s \nabla \psi_s](x) + O((1 + |x|)h_s^2), \end{aligned}$$

using that  $b$  is globally Lipschitz. Similarly,

$$\nabla S_{s, h_s} a_s \nabla S_{s, h_s}^\top \circ S_{s, h_s}^{-1} - a_s = -h_s [a_s [\nabla(a_s \nabla \psi_s)]^\top + \nabla(a_s \nabla \psi_s) a_s - \nabla a_s \cdot a_s \nabla \psi_s] + O(h_s^2),$$

using that  $\nabla^2(a \nabla \psi)$  is uniformly bounded. Using the bound on  $\nabla^3(a \nabla \psi)$ , we turn this into

$$\begin{aligned} C_s^h(S_{s, h_s}(x)) &= \frac{h_s}{2} [a_s [\nabla(a_s \nabla \psi_s)]^\top + \nabla(a_s \nabla \psi_s) a_s - \nabla a_s \cdot a_s \nabla \psi_s][(\nabla \log \mu_s^h) \circ S_{s, h_s}] \\ &\quad + \frac{h_s}{2} \nabla \cdot [a_s [\nabla(a_s \nabla \psi_s)]^\top + \nabla(a_s \nabla \psi_s) a_s - \nabla a_s \cdot a_s \nabla \psi_s] + O(h_s^2) \\ &=: D_s^h[(\nabla \log \mu_s^h) \circ S_{s, h_s}] + E_s^h + O(h_s^2). \end{aligned}$$

Gathering everything:

$$\begin{aligned} \beta_s^h(S_{s, h_s}(X_s^0)) &= \beta_s^0(X_s^0) - h'_s a_s \nabla \psi_s(X_s^0) - h_s [(\partial_s + L_s^{\alpha^0})[a_s \nabla \psi_s] - \nabla b_s a_s \nabla \psi_s](X_s^0) \\ &\quad + D_s^h(X_s^0) \nabla \log \mu_s^h(S_{s, h_s}(X_s^0)) + E_s^h(X_s^0) + O((1 + |X_s^0|)h_s^2 + h'_s h_s). \end{aligned} \quad (5.21)$$

Since  $\mu_s^h = \mu_s^0 \circ S_{s, h_s}^{-1}$ :

$$\begin{aligned} \nabla \log \mu_s^h(S_{s, h_s}(X_s^0)) &= \nabla S_{s, h_s}^{-1}(S_{s, h_s}(X_s^0)) \nabla \log \mu_s^0(X_s^0) + \nabla \log |\det \nabla S_{s, h_s}^{-1}|(S_{s, h_s}(X_s^0)) \\ &= \nabla \log \mu_s^0(X_s^0) + O((1 + |\nabla \log \mu_s^0(X_s^0)|)h), \end{aligned}$$

where we used that  $\nabla^2(a \nabla \psi)$  is uniformly bounded to get the  $O(h)$  term. Using Lemma 5.3.4 and the bound on  $\nabla \alpha^0$  given by Proposition 5.3.2, we can apply Lemma 5.A.2 to  $(\mu_s^0)_{0 \leq s \leq T}$ . This provides a bound on  $\mathbb{E}[|\nabla \log \mu_s^0(X_s^0)|]$  involving  $\int_{\mathbb{R}^d} |\nabla \log \nu_0|^{2+\varepsilon_0} d\nu_0$ . Bounding  $\mathbb{E}[|X_s^0|]$  is then standard from (A.2) and the bound on  $\nabla \alpha^0$ , so that  $\mathbb{E}[|\beta_s^h(S_{s, h_s}(X_s^0))|^2]$  is finite.

**Step 3. Integrations by parts.** Since  $\mathbb{E}[|\nabla \log \mu_s^0(X_s^0)|]$  is bounded, we can integrate by parts the density  $\mu_s^0$  to obtain that

$$\mathbb{E}[D_s^h(X_s^0) \nabla \log \mu_s^0(X_s^0)] = -\mathbb{E}[E_s^h(X_s^0)].$$

Plugging this cancellation within (5.21) yields

$$\begin{aligned} \mathbb{E}[|\beta_s^h(S_{s, h_s}(X_s^0))|^2 - |\beta_s^0(X_s^0)|^2] &= -2h'_s \mathbb{E}[\sigma_s \alpha_s^0 \cdot a_s \nabla \psi_s(X_s^0)] \\ &\quad - 2h_s \mathbb{E}\{\sigma_s \alpha_s^0 \cdot [\partial_s(a_s \nabla \psi_s) + \frac{1}{2} \text{Tr}[a_s \nabla^2(a_s \nabla \psi_s)]](X_s^0)\} + C' h_s + O(h_s^2 + (h'_s)^2), \end{aligned}$$

where  $C'$  is a constant that satisfies the requirements of Proposition 5.3.5. In the following,  $C'$  may change from line to line but still satisfying these requirements. The quantities

$\nabla\partial_s\psi_s$  and  $\nabla^3\psi_s$  appear in the first term of the second line. To get rid of them, we use coordinates (and the summing convention of repeated indices) to rewrite this term as

$$-2h_s\mathbb{E}\{\sigma_s^{i,j}(\alpha_s^0)^j[\partial_s a_s^{i,k}\partial_k\psi_s + a_s^{i,k}\partial_k\partial_s\psi_s + \frac{1}{2}a_s^{l,m}\partial_{l,m}(a_s^{i,k}\partial_k\psi_s)](X_s^0)\}.$$

As above, we integrate by parts the  $\partial_k$ -derivative to get that

$$\mathbb{E}[|\beta_s^h(S_{s,h_s}(X_s^0))|^2 - |\beta_s^0(X_s^0)|^2] = -2h'_s\mathbb{E}[\sigma_s\alpha_s^0 \cdot a_s\nabla\psi_s(X_s^0)] + C'h_s + O(h_s^2 + (h'_s)^2).$$

We can now write that

$$\begin{aligned} \alpha_s^h(S_{s,h_s}(X_s^0)) &= \sigma_s^{-1}(X_s^0)\beta_s^h(S_{s,h_s}(X_s^0)) - h_s\beta_s^h(S_{s,h_s}(X_s^0)) \cdot \nabla\sigma_s^{-1}a_s\nabla\psi_s(X_s^0) \\ &\quad + O(h_s^2\beta_s^h(S_{s,h_s}(X_s^0))), \end{aligned}$$

using that  $\nabla\sigma^{-1}$  is uniformly bounded, and the same computations as above yield

$$\mathbb{E}[|\alpha_s^h(S_{s,h_s}(X_s^0))|^2] - \mathbb{E}[|\alpha_s^0(X_s^0)|^2] = -2h'_s\mathbb{E}[\sigma_s\alpha_s^0 \cdot \nabla\psi_s(X_s^0)] + C'h_s + O(h_s^2 + (h'_s)^2).$$

In particular, we obtained that  $\int_0^T \mathbb{E}[|\alpha_s^h(S_{s,h_s}(X_s^0))|^2]ds$  is finite.

**Step 4. Comparison.** Using **Step 3.** and [Trevisan, 2015, Theorem 2.5], there exists a weak solution for the SDE (5.12) with the feed-back control  $\alpha = \alpha^h$  and the initial law  $\bar{\mu}_0$  ([Trevisan, 2015, Theorem 2.5] actually shows existence of a solution for the related martingale problem, but this is equivalent, see e.g. [Daudin, 2022, Proposition 2.1]). As a consequence, there exists a reference probability system  $\Sigma'$  and an adapted process  $(X_s^h)_{0 \leq s \leq T}$  on it satisfying (5.12) with  $\alpha = \alpha^h$  and  $\mathcal{L}(X_0^h) = \bar{\mu}_0$ . In particular,  $(\alpha_s^h(X_s^h))_{0 \leq s \leq T}$  can be seen as a specific open-loop control for (5.11). Thus,

$$\begin{aligned} V_\Sigma^\lambda(\bar{\mu}_0) &= V_{\Sigma'}^\lambda(\bar{\mu}_0) \leq \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_s^h(X_s^h)|^2 + c_s(X_s^h) + \psi_s(X_s^h)\lambda_s ds + \bar{\lambda}_T\psi_T(X_T^h)\right] \\ &= \int_0^T \mathbb{E}\left[\frac{1}{2}|\alpha_s^h(S_{s,h_s}(X_s^0))|^2 + c_s(S_{s,h_s}(X_s^0)) + \psi_s(S_{s,h_s}(X_s^0))\lambda_s\right] ds + \bar{\lambda}_T\mathbb{E}[\psi_T(X_T^0)], \end{aligned}$$

using that  $X_s^h$  and  $S_{s,h(s)}(X_s^0)$  have the same law  $\mu_s^h$ , and  $X_T^h = X_T^0$ . We further expand

$$\begin{aligned} c_s(S_{s,h_s}(X_s^0)) &= c_s(X_s^0) - h_s\nabla c_s \cdot a_s\nabla\psi_s(X_s^0) + O(h_s^2) = C'h_s + O(h_s^2), \\ \psi_s(S_{s,h_s}(X_s^0)) &= \psi_s(X_s^0) - h_s\nabla\psi_s \cdot a_s\nabla\psi_s(X_s^0) + O(h_s^2), \end{aligned}$$

using that  $\nabla^2 c$  and  $\nabla^2\psi$  are uniformly bounded. Using **Step 3.** to expand  $\alpha_s^h$ , we obtain  $\delta > 0$  that only depends on  $\lambda$  through  $\lambda([0, T])$  such that  $\|h\|_\infty + \|h'\|_\infty^2 \leq \delta$  implies

$$\int_t^{t'} h_s\mathbb{E}[|\sigma_s^\top\nabla\psi_s(X_s^0)|^2]\lambda_s ds \leq \int_t^{t'} C'(h_s + |h'_s|^2) - 2h'_s\mathbb{E}[\sigma_s\alpha_s^0 \cdot \nabla\psi_s(X_s^0)] ds.$$

The first term in the integral on the r.h.s. has the desired shape, but the second one still involves the leading order term  $h'_s$ , so we must integrate it.

**Step 5. Last integrations by parts.** Ito's formula gives that

$$\mathbb{E}[\sigma_s\alpha_s^0 \cdot \nabla\psi_s(X_s^0)] = \frac{d}{ds}\mathbb{E}[\psi_s(X_s)] - \mathbb{E}[(\partial_s + L_s^0)\psi_s(X_s^0)],$$

and we integrate by parts in time using that  $h_t = h_{t'} = 0$ :

$$-\int_t^{t'} h'_s\mathbb{E}[L_s^0\psi_s(X_s^0)] ds = \int_t^{t'} h_s \frac{d}{ds}\mathbb{E}[L_s^0\psi_s(X_s^0)] ds = \int_t^{t'} h_s\mathbb{E}[(\partial_s + L_s^0)L_s^0\psi_s(X_s^0)] ds.$$

As previously, we integrate by parts the density to get rid of the  $\nabla^2 \partial_s \psi_s$  term:

$$\mathbb{E} \operatorname{Tr}[a_s \nabla^2 \partial_s \psi_s(X_s^0)] = C' - \mathbb{E} [a_s^{i,j} \partial_i \partial_s \psi_s \partial_j \log \mu_s^0(X_s^0)].$$

Similarly, we get rid of the  $\nabla^2 b_s$  and  $\nabla^4 \psi_s$  terms. The resulting  $C'$  involves  $\int |\nabla \log \nu_0|^{2+\varepsilon_0} d\nu_0$  and a  $\nabla^3 \psi_s$  term. To get rid of this last term, we can integrate by parts one more time:

$$\mathbb{E} \operatorname{Tr}\{a_s \nabla^2 \operatorname{Tr}[a_s \nabla^2 \psi_s]\}(X_s^0) \leq C' - \mathbb{E} \operatorname{Tr}[a_s \nabla^2 \psi_s] \operatorname{Tr}[a_s \nabla^2 \log \mu_s^0](X_s^0).$$

We deduce from Proposition 5.A.4 applied to  $(\mu_s^0)_{0 \leq s \leq T}$  that

$$\sup_{0 \leq s \leq T} \mathbb{E}[|\nabla^2 \log \mu_s^0(X_s^0)|] \leq C[1 + \sup_{0 \leq s \leq T} \|\nabla^3 \varphi_s\|_\infty],$$

where the constant  $C$  now involves  $\int |\nabla \log \nu_0|^{4+\varepsilon_0} d\nu_0$  and  $\int |\nabla^2 \log \nu_0|^{2+\varepsilon_0}$  too. The desired dependence on  $\varepsilon \sup_{0 \leq s \leq T} |\lambda_s|$  eventually results from Lemma 5.3.3.  $\square$

**Corollary 5.3.5.1** (Qualification condition). *A sufficient condition for (5.8) is that*

$$\forall s \in [0, T], \quad \int_{\mathbb{R}^d} \psi_s d\bar{\mu}_s = 0 \Rightarrow \int_{\mathbb{R}^d} |\nabla \psi_s|^2 d\bar{\mu}_s \neq 0. \quad (5.22)$$

*This condition is also necessary from Remark 5.2.5.*

*Proof.* We follow the proof of Proposition 5.3.5, defining the perturbed flow

$$\mu_s^h := \bar{\mu}_s \circ S_{s,h}^{-1}, \quad 0 \leq s \leq T,$$

for  $h > 0$ . As previously, we can show that for  $h$  small enough,  $(\mu_s^h)_{0 \leq s \leq T}$  are the marginal laws of a process of type (5.11), for a square-integrable control process. Moreover,

$$\frac{d}{dh} \int_{\mathbb{R}^d} \psi_s d\mu_s^h = - \int_{\mathbb{R}^d} |\sigma_s^\top \nabla \psi_s|^2 d\mu_s^h \leq 0.$$

Since  $\sigma^{-1}$  is bounded from (A.2), (5.22) guarantees that the derivative is negative at points where  $\int_{\mathbb{R}^d} \psi_s d\bar{\mu}_s$  (which is always non-positive) is not already negative. As a consequence (5.8) is satisfied, choosing  $\tilde{\nu}_{[0,T]}$  as being the path-law of a controlled process with marginal laws  $(\mu_s^h)_{0 \leq s \leq T}$ , for  $h$  small enough.  $\square$

### 5.3.2 Existence of a density for the Lagrange multiplier

In our smooth setting (A.8), [Chaintron et al., 2024, Theorem 3.6] gives that the measure  $\bar{\mu}_{[0,T]}$  satisfying (5.9) is the path-law of the solution to

$$d\bar{X}_t = b_t(\bar{X}_t)dt - a_t \nabla \bar{\varphi}_t(\bar{X}_t)dt + \sigma_t(\bar{X}_t)dB_t, \quad \bar{X}_0 \sim \bar{Z}^{-1} e^{-\bar{\varphi}_0(x)} \nu_0(dx),$$

in a reference system  $\Sigma = (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$ , where  $\bar{\varphi}$  is the solution to

$$-\bar{\varphi}_t + \int_t^T b_s \cdot \nabla \bar{\varphi}_s - \frac{1}{2} |\sigma_s^\top \nabla \bar{\varphi}_s|^2 + c_s + \frac{1}{2} \operatorname{Tr}[a_s \nabla^2 \bar{\varphi}_s] ds + \int_{[t,T]} \psi_s \bar{\lambda}(ds) = 0,$$

in the sense of Definition 5.C.1.

Using convolution with a smooth kernel, let us consider approximations of  $\bar{\lambda} - \bar{\lambda}_T \delta_T$  given by measures that have  $C^1$  densities  $(\lambda^k)_{k \geq 1}$  w.r.t. the Lebesgue measure. We similarly regularise  $t \mapsto c_t(x)$  into  $t \mapsto c_t^k(x)$ . Since convolution in time does not affect  $x$ -regularity,  $c^k$  still satisfies (A.8), with regularity constants independent of  $k$ .  $(\lambda^k, c^k)$  further satisfy (A.9). Let  $(X_s^k)_{0 \leq s \leq T}$  be the optimal process for  $V_\Sigma^{\lambda^k}(\bar{\mu}_0)$  obtained from the solution  $\varphi^k$  of (5.13), and let us consider the feed-back control  $\alpha^k := -\sigma^\top \nabla \varphi^k$ . We recall that the number of atoms of the measure  $\bar{\lambda}$  is at most countable.



**Lemma 5.3.6.** For  $t = 0$  and every  $t \in (0, T]$  with  $\bar{\lambda}(\{t\}) = 0$ ,

$$\varphi_t^k \xrightarrow[k \rightarrow +\infty]{} \bar{\varphi}_t, \quad \nabla \varphi_t^k \xrightarrow[k \rightarrow +\infty]{} \nabla \bar{\varphi}_t,$$

uniformly on every compact set of  $\mathbb{R}^d$ . Moreover, the bounds in Proposition 5.3.2 extend to  $\nabla \bar{\varphi}$  and  $\nabla^2 \bar{\varphi}$ ,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^k - \bar{X}_t| \right] \xrightarrow[k \rightarrow +\infty]{} 0,$$

and for every  $1 \leq i \leq d$ ,  $x \in \mathbb{R}^d$ , and  $t \in [0, T]$  with  $\bar{\lambda}(\{t\}) = 0$ ,

$$\begin{aligned} \partial_i \bar{\varphi}_t(x) = & \int_t^T \mathbb{E} \left\{ \partial_i b_s \cdot \nabla \bar{\varphi}_s - \frac{1}{2} \nabla \bar{\varphi}_s \cdot \partial_i a_s \nabla \bar{\varphi}_s + \frac{1}{2} \text{Tr}[\partial_i a_s \nabla^2 \bar{\varphi}_s] + \partial_i c_s \right\} (X_s^{t,x,\bar{\alpha}}) ds \\ & + \int_{[t,T]} \mathbb{E}[\partial_i \psi_s^k(X_s^{t,x,\bar{\alpha}})] \bar{\lambda}(ds) + \bar{\lambda}_T \mathbb{E}[\partial_i \psi_T(X_T^{t,x,\bar{\alpha}})]. \end{aligned}$$

*Proof.* For every  $t \in [0, T]$ , Proposition 5.3.2 provides equi-continuity for  $(\nabla \varphi_t^k)_{k \geq 1}$ , and  $\varphi^k$  has linear growth uniformly in  $k$  from (5.15). The stability result [Chaintron et al., 2024, Lemma 5.3] then provides the convergence of  $\bar{\varphi}$  and  $\nabla \bar{\varphi}$ , and the bounds are inherited from Proposition 5.3.2. Using this and our Lipschitz assumptions, a standard coupling argument (a variation of the proof of [Friedman, 1975, Chapter 5, Theorem 5.2]) shows that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^k - \bar{X}_t|] \rightarrow 0$ .

From [Billingsley, 2013, Theorem 2.7], the measures  $\lambda_s^k ds + \bar{\lambda}_T \delta_T(ds)$  restricted to  $[t, T]$  weakly converges towards the restriction of  $\bar{\lambda}$  to  $[t, T]$ , for every  $t$  with  $\bar{\lambda}(\{t\}) = 0$ . The equation on  $\nabla \bar{\varphi}_t$  then results from taking the  $k \rightarrow +\infty$  limit within (5.18).  $\square$

Let us now show that  $\bar{\lambda}$  can only have atoms at 0 and  $T$ .

**Lemma 5.3.7.**  $\bar{\lambda}$  has no atom in  $(0, T)$ .

*Proof.* For every  $t \in (0, T)$ , we can write that  $\bar{\lambda} = \bar{\lambda}_t \delta_t + \lambda'$ , for  $\bar{\lambda}_t \in \mathbb{R}$  and  $\lambda' \in \mathcal{M}_+([0, T])$  with  $\lambda'(\{t\}) = 0$ . From the complementary slackness condition (5.10), we can assume that  $\mathbb{E}[\psi_t(\bar{X}_t)] = 0$ . Let  $(t_0, t'_0)$  be in  $(0, T)^2$  with  $t_0 < t < t'_0$ . By induction, we define a sequence  $(t_l, t'_l)_{l \geq 0}$  in  $(0, T)^2$  such that  $t_l < t < t'_l$  and

$$t_{l+1} = \inf \left\{ s \in \left[ \frac{t_0+t}{2}, t \right], \forall r \in [s, t], \mathbb{E}[\psi_s(\bar{X}_s)] \leq \mathbb{E}[\psi_r(\bar{X}_r)] \right\},$$

$$t'_{l+1} = \sup \left\{ s' \in \left[ t, \frac{t+t'_0}{2} \right], \forall r \in [t, s'], \mathbb{E}[\psi_{s'}(\bar{X}_{s'})] \leq \mathbb{E}[\psi_r(\bar{X}_r)] \right\}.$$

The iterations are well-defined because  $s \mapsto \mathbb{E}[\psi_s(\bar{X}_s)]$  is a non-positive continuous function and  $\mathbb{E}[\psi_t(\bar{X}_t)] = 0$ . Let now  $h^l : [t_l, t'_l] \rightarrow [0, 1]$  be a  $C^2$  function such that

$$h^l_{t_l} = h^l_{t'_l} = 0, \quad \frac{d}{ds} \Big|_{s=t} h^l_s = 0, \quad \text{and} \quad \forall s \in [t_l, t'_l], \quad \frac{d^2}{ds^2} h^l_s \leq 0.$$

Thus,  $h^l$  has a global maximum at  $t$ . Let us choose  $h^l_t = \|h^l\|_\infty \leq 2 \min[(t - t_l)^2, (t'_l - t)^2]$  and such that  $\|\frac{d}{ds} h^l\|_\infty \leq 2 \|h^l\|_\infty$ .

Let now  $\delta, C > 0$  be given by Proposition 5.3.5 for  $\lambda(ds) = \lambda_s^k ds + \bar{\lambda}_T \delta_T(ds)$ : these constants do not depend on  $(k, l)$  because they only depend on  $\lambda$  through  $\lambda([0, T])$ , which is unaffected by the regularisation procedure. From (5.7),  $\mathbb{E}[\partial_s \psi_s(\bar{X}_s)] = 0$  for  $s \in [0, T]$ , hence

$$\int_{t_l}^{t'_l} \mathbb{E}[\partial_s \psi_s(X_s^k)] \frac{d}{ds} h^l_s ds \xrightarrow[k \rightarrow +\infty]{} 0. \quad (5.23)$$

Integrating by parts,

$$\begin{aligned}
 & - \int_{t_l}^{t'_l} \frac{d}{ds} \mathbb{E}[\psi_s(X_s^k)] \frac{d}{ds} h_s^l ds \\
 & = \int_{t_l}^t \{ \mathbb{E}[\psi_s(X_s^k)] - \mathbb{E}[\psi_{t_l}(X_{t_l}^k)] \} \frac{d^2}{ds^2} h_s^l ds + \int_t^{t'_l} \{ \mathbb{E}[\psi_s(X_s^k)] - \mathbb{E}[\psi_{t'_l}(X_{t'_l}^k)] \} \frac{d^2}{ds^2} h_s^l ds,
 \end{aligned} \tag{5.24}$$

where we used that  $\frac{d}{ds} \Big|_{s=t} h_s^l = 0$ . Since  $\frac{d^2}{ds^2} h_s^l \leq 0$ , and  $\mathbb{E}[\psi_s(\bar{X}_s)] - \mathbb{E}[\psi_{t_l}(\bar{X}_{t_l})] \leq 0$ ,  $\mathbb{E}[\psi_s(\bar{X}_s)] - \mathbb{E}[\psi_{t'_l}(\bar{X}_{t'_l})] \leq 0$  by definition of  $(t_l, t'_l)$ , the above integrals are non-positive as  $k \rightarrow +\infty$ . Plugging (5.23)-(5.24) into Proposition 5.3.5 yields

$$\int_{(t_l, t'_l)} h_s^l \mathbb{E}[|\sigma_s^\top \nabla \psi_s(\bar{X}_s)|^2] \bar{\lambda}(ds) \leq \limsup_{k \rightarrow +\infty} \int_{t_l}^{t'_l} h_s^l \mathbb{E}[|\sigma_s \nabla \psi_s(X_s^k)|^2] \lambda_s^k ds \leq \int_{t_l}^{t'_l} C |h_s^l| ds,$$

where we used that  $\|\lambda^k\|_{L^1(0, T)}$  is bounded uniformly in  $k$ . Moreover, the constant  $C$  does not depend on  $l$ . From the decomposition  $\bar{\lambda} = \bar{\lambda}_t \delta_t + \lambda'$ , the l.h.s. is lower-bounded by  $h_t^l \bar{\lambda}_t \mathbb{E}[|\sigma_t \nabla \psi_t(\bar{X}_t)|^2]$ . From the qualification condition (5.8) and Remark 5.2.5,  $\mathbb{E}[|\sigma_t \nabla \psi_t(\bar{X}_t)|^2] \neq 0$ . Since  $h_t^l = \|h^l\|_\infty$ , dividing by  $h_t^l$  and sending  $l \rightarrow +\infty$  shows that  $\bar{\lambda}_t = 0$ . We thus proved that  $\bar{\lambda}(\{t\}) = 0$ , for any  $t$  in  $(0, T)$ .  $\square$

**Proposition 5.3.8** (Density). *The multiplier has the following decomposition*

$$\bar{\lambda}(dt) = \bar{\lambda}_0 \delta_0(dt) + \bar{\lambda}_t dt + \bar{\lambda}_T \delta_T(dt).$$

Moreover,  $\|\bar{\lambda}\|_{L^\infty(0, T)} \leq C$ , for  $C > 0$  independent of  $\bar{\lambda}$  that only depends on the coefficients through the uniform norms of  $\sigma$ ,  $\sigma^{-1}$ ,  $\partial_s b$ ,  $\partial_s \sigma$ ,  $\partial_s \psi$ ,  $\nabla b$ ,  $\nabla \sigma$ ,  $\nabla c$ ,  $\nabla \psi$ ,  $\nabla \partial_s \psi$ ,  $\nabla^2 \sigma$ ,  $\nabla^2 \psi$ , and either  $\nabla^3 \psi$  or  $\nabla^2 b$  and  $\nabla^3 \sigma$ .

We recall that we restricted ourselves to  $\bar{\lambda}(\{0\}) = 0$  at the beginning of Section 5.3, without loss of generality.

*Proof.* Let us define the set

$$F := \{t \in [0, T], \mathbb{E}[\psi_t(\bar{X}_t)] = 0\}.$$

From the complementary slackness condition (5.10), we only need to study the restriction of  $\bar{\lambda}$  to  $F$ . Since  $t \mapsto \mathbb{E}[\psi_t(\bar{X}_t)]$  is continuous,  $F$  is a closed set, and  $F$  is a (at most) countable union of disjoint intervals. Since  $\bar{\lambda}$  has no atom in  $(0, T)$  from Lemma 5.3.7, we only need to look at intervals  $[t_0, t'_0] \subset F$  with  $t_0 < t'_0$ . Let us fix such a  $[t_0, t'_0]$ , and let  $\delta_0, C > 0$  be given by Proposition 5.3.5, so that

$$\int_{t_1}^{t'_1} h_s \mathbb{E}[|\sigma_s^\top \nabla \psi_s^k(X_s^k)|^2] \lambda_s^k ds \leq \int_{t_1}^{t'_1} C [h_s + |\frac{d}{ds} h_s|^2] - 2 \frac{d}{ds} h_s \{ \frac{d}{ds} \mathbb{E}[\psi_s(X_s^k)] - \mathbb{E}[\partial_s \psi_s(X_s^k)] \} ds,$$

for every  $C^2$  function  $h : [t_1, t'_1] \rightarrow [0, 1]$  with  $0 \leq t_1 \leq t'_1 \leq T$ ,  $h_{t_1} = h_{t'_1} = 0$  and  $\|h\|_\infty + \|\frac{d}{ds} h\|_\infty^2 \leq \delta_0$ .

We first allow  $C$  to depend on  $\nabla^3 \psi$  in the setting of Proposition 5.3.5, so that  $(C, \delta_0)$  does not depend on  $k$ , because  $(\|\lambda^k\|_{L^1(0, T)})_{k \geq 1}$  is bounded. Similarly to (5.24),

$$\begin{aligned}
 & - \int_{t_1}^{t'_1} \frac{d}{ds} \mathbb{E}[\psi_s(X_s^k)] \frac{d}{ds} h_s ds = \mathbb{E}[\psi_s(X_{t_1}^k)] \frac{d}{ds} \Big|_{s=t_1} h_s - \mathbb{E}[\psi_s(X_{t'_1}^k)] \frac{d}{ds} \Big|_{s=t'_1} h_s \\
 & \qquad \qquad \qquad + \int_{t_1}^{t'_1} \mathbb{E}[\psi_s(X_s^k)] \frac{d^2}{ds^2} h_s ds \xrightarrow[k \rightarrow +\infty]{} 0,
 \end{aligned}$$

if we assume that  $[t'_1, t'_1] \subset F$ . From the qualification condition (5.8) and Remark 5.2.5, the continuous function  $s \mapsto \mathbb{E}[|\sigma_s^\top \nabla \psi_s(\bar{X}_s)|^2]$  is positive on the compact set  $F$  ( $F$  is a closed subset of the compact  $[0, T]$ ), so that there exists  $\eta > 0$  such that

$$\forall s \in F, \quad 0 < \eta \leq \mathbb{E}[|\sigma_s^\top \nabla \psi_s(\bar{X}_s)|^2].$$

Taking the  $k \rightarrow +\infty$  limit and using (5.23), as in the proof of Lemma 5.3.7,

$$\eta \int_{[t_1, t'_1]} h_s \bar{\lambda}(ds) \leq C \int_{t_1}^{t'_1} [h_s + |\frac{d}{ds} h_s|^2] ds,$$

for every  $h$  as above with  $[t_1, t'_1] \subset F$ . For  $\delta \in (0, (t'_0 - t_0)/2)$  and  $[t, t'] \subset [t_0 - \delta, t'_0 + \delta]$  with  $t' - t < \delta$ , we apply this estimate to  $t_1 = t - (t' - t)$  and  $t'_1 = t + (t' - t)$ , so that  $[t, t'] \subset [t_1, t'_1] \subset [t_0, t'_0]$ . We further choose  $h_s = -\varepsilon_0(s - t_1)(s - t'_1)$ , for  $\varepsilon_0 \in (0, 1)$  small enough so that  $\|h\|_\infty + \|\frac{d}{ds} h\|_\infty^2 \leq \delta_0$ . In particular,  $h_s \leq \varepsilon_0[2(t' - t)]^2$ , and  $h_s \geq \varepsilon_0|t' - t|^2$  for  $s \in [t, t']$ . As a consequence,

$$\varepsilon_0 \eta |t' - t|^2 \bar{\lambda}([t, t']) \leq \eta \int_{[t, t']} h_s \bar{\lambda}(ds) \leq C \int_{t_1}^{t'_1} [h_s + |\frac{d}{ds} h_s|^2] ds \leq 5\varepsilon_0 C [2(t' - t)]^3.$$

We deduce that

$$\forall 0 < \delta < \frac{t'_0 - t_0}{2}, \quad \forall [t, t'] \subset [t_0 - \delta, t'_0 + \delta], \quad t' - t < \delta \Rightarrow \bar{\lambda}([t, t']) \leq 40\eta^{-1} C |t' - t|.$$

The cumulative distribution function of  $\bar{\lambda}$  is thus absolutely continuous on every compact subset of  $(t_0, t'_0)$ , with a.s. derivative bounded by  $40\eta^{-1} C$ . This shows that  $\bar{\lambda}$  has a density bounded by  $40\eta^{-1} C$  on  $(t_0, t_0)$ . Since  $\bar{\lambda}(\{0\}) = 0$ , Lemma 5.3.7 extends the result to  $[t_0, t'_0)$ , and then to  $[t_0, t'_0]$  if  $t'_0 \neq T$ . Since  $(\eta, C)$  does not depend on  $[t_0, t'_0]$ , the result holds on the whole set  $F \cap (0, T)$ , and thus on  $(0, T)$ .

To refine the bound, we now allow for  $C = C_\varepsilon + \varepsilon \sup_{0 \leq t \leq T} \lambda_s^k$  when applying Proposition 5.3.5. Since we now know that  $\bar{\lambda}$  restricted to  $(0, T)$  has a bounded density  $\bar{\lambda}|_{(0, T)} \in L^\infty(0, T)$ , and the continuous function  $\lambda^k$  was obtained from  $\bar{\lambda} - \bar{\lambda}_T \delta_T$  by convolution, we can bound  $\sup_{0 \leq s \leq T} \lambda_s^k$  by  $\|\bar{\lambda}|_{(0, T)}\|_{L^\infty(0, T)}$ . Since there is no more dependence on  $k$  in the bounds, we then perform the same computations as above to obtain that

$$\|\bar{\lambda}|_{(0, T)}\|_{L^\infty(0, T)} \leq 40\eta^{-1} [C_\varepsilon + \varepsilon \|\bar{\lambda}|_{(0, T)}\|_{L^\infty(0, T)}].$$

By choosing  $\varepsilon$  small enough beforehand, this yields the desired bound on  $\|\bar{\lambda}|_{(0, T)}\|_{L^\infty(0, T)}$ .  $\square$

## 5.4 Proof of the main results

We now turn to the proofs of the results stated in Section 5.2. The proof of Theorem 5.2.2 essentially relies on results from Chaintron et al. [2024], see Section 5.4.1 below. For Theorem 5.2.3, a smoothing procedure is detailed in Section 5.4.2 to enter the framework of Section 5.3. Theorem 5.2.6 is proved independently in Section 5.4.3. We recall the convenient notation  $a := \sigma \sigma^\top$ .

### 5.4.1 Optimality conditions

This section is devoted to the proof of Theorem 5.2.2. Let  $\bar{\mu}_{[0, T]}$  be an optimiser for (5.2) that satisfies (A.4). We first rewrite (5.2) to make it enter the framework of Chaintron et al. [2024].

**Lemma 5.4.1.** *For any  $\mu_{[0,T]}$  in  $\mathcal{P}_1(C([0,T], \mathbb{R}^d))$  such that  $H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]}))$ ,  $H(\mu_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]}))$  are finite, we define*

$$\mathcal{F}(\mu_{[0,T]}) := H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]})) - H(\mu_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]})).$$

$\mathcal{F}$  is differentiable at  $\bar{\mu}_{[0,T]}$ , its linear functional derivative being given by

$$\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_{[0,T]}, x_{[0,T]}) = -\mathbb{E}_{\bar{\mu}_{[0,T]}} \int_0^T a_t^{-1}(\mathbf{X}_t) \frac{\delta b_t}{\delta \mu}(\mathbf{X}_t, \bar{\mu}_t, x_t) [d\mathbf{X}_t - b_t(\mathbf{X}_t, \bar{\mu}_t) dt]. \quad (5.25)$$

Moreover, there exists a measurable  $\bar{c} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}_{[0,T]}, x_{[0,T]}) = \int_0^T \int_{\mathbb{R}^d} \bar{c}_t(x_t) \bar{\mu}_t(dx) dt,$$

and  $\bar{c}_t$  is globally Lipschitz uniformly in  $t$ .

We recall that  $(\mathbf{X}_t)_{0 \leq t \leq T}$  denotes the canonical process on  $C([0, T], \mathbb{R}^d)$ , see Section 5.1.2.

*Proof.* The finiteness of the entropy allows us to write

$$H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]})) = H(\mu_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]})) - \int_{C([0,T], \mathbb{R}^d)} \log \frac{d\Gamma(\mu_{[0,T]})}{d\Gamma(\bar{\mu}_{[0,T]})} d\mu_{[0,T]}.$$

Since  $\sigma^{-1}$  is bounded and  $b$  is globally Lipschitz in  $\mu$  from (A.2), the Novikov condition

$$\mathbb{E}_{\Gamma(\bar{\mu}_{[0,T]})} \exp \left[ \frac{1}{2} \int_0^T |\sigma_t^{-1}(\mathbf{X}_t) [b_t(\mathbf{X}_t, \mu_t) - b_t(\mathbf{X}_t, \bar{\mu}_t)]|^2 dt \right] < +\infty,$$

is satisfied, and the Girsanov transform [Léonard, 2012, Theorem 2.3] gives that

$$\begin{aligned} \frac{d\Gamma(\mu_{[0,T]})}{d\Gamma(\bar{\mu}_{[0,T]})} &= \exp \left[ \int_0^T \sigma_t^{-1}(\mathbf{X}_t) [b_t(\mathbf{X}_t, \mu_t) - b_t(\mathbf{X}_t, \bar{\mu}_t)] d\mathbf{B}_t \right. \\ &\quad \left. - \frac{1}{2} \int_0^T |\sigma_t^{-1}(\mathbf{X}_t) [b_t(\mathbf{X}_t, \mu_t) - b_t(\mathbf{X}_t, \bar{\mu}_t)]|^2 dt \right], \end{aligned}$$

where  $(\mathbf{B}_t)_{0 \leq t \leq T}$  is a Brownian motion under  $\Gamma(\bar{\mu}_{[0,T]})$ , and  $\Gamma(\bar{\mu}_{[0,T]})$ -almost surely,

$$d\mathbf{X}_t = b_t(\mathbf{X}_t, \bar{\mu}_t) dt + \sigma_t(\mathbf{X}_t) d\mathbf{B}_t.$$

Using this to rewrite  $d\mathbf{B}_t$ ,

$$\begin{aligned} \mathcal{F}(\mu_{[0,T]}) &= \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |\sigma_t^{-1}(x) [b_t(x, \mu_t) - b_t(x, \bar{\mu}_t)]|^2 \mu_t(dx) dt \\ &\quad - \mathbb{E}_{\mu_{[0,T]}} \int_0^T a_t^{-1}(\mathbf{X}_t) [b_t(\mathbf{X}_t, \mu_t) - b_t(\mathbf{X}_t, \bar{\mu}_t)] [d\mathbf{X}_t - b_t(\mathbf{X}_t, \bar{\mu}_t) dt], \end{aligned}$$

and it is direct that  $\mathcal{F}(\bar{\mu}_{[0,T]}) = 0$ . For any  $\mu_{[0,T]}$  with  $H(\mu_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]})) < +\infty$ , the differentiability of  $b$  yields

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{F}((1-\varepsilon)\bar{\mu}_{[0,T]} + \varepsilon\mu_{[0,T]}) &= \\ &= -\mathbb{E}_{\bar{\mu}_{[0,T]}} \int_0^T a_t^{-1}(\mathbf{X}_t) \int_{\mathbb{R}^d} \frac{\delta b_t}{\delta \mu}(\mathbf{X}_t, \bar{\mu}_t, x) [\mu_t(dx) - \bar{\mu}_t(dx)] [d\mathbf{X}_t - b_t(\mathbf{X}_t, \bar{\mu}_t) dt]. \end{aligned}$$

Since  $\int_{\mathbb{R}^d} \frac{\delta b_t}{\delta \mu}(\mathbf{X}_t, \bar{\mu}_t, x) \bar{\mu}_t(dx) = 0$  by definition of  $\frac{\delta b}{\delta \mu}$ , this proves that  $\mathcal{F}$  admits a linear functional derivative at  $\bar{\mu}_{[0,T]}$  given by (5.25), in the sense of Definition 5.1.1. Since  $H(\bar{\mu}_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]}))$  is finite, [Léonard, 2012, Theorem 2.1] provides an adapted process  $(c_t)_{0 \leq t \leq T}$  on the canonical space  $\Omega = C([0, T], \mathbb{R}^d)$  such that

$$\mathbb{E}_{\bar{\mu}_{[0,T]}} \int_0^T |\sigma_t^\top(\mathbf{X}_t) c_t|^2 dt < +\infty,$$

where

$$d\mathbf{X}_t = b_t(\mathbf{X}_t, \bar{\mu}_t) dt + a_t(\mathbf{X}_t) c_t dt + \sigma_t(\mathbf{X}_t) dB_t, \quad \bar{\mu}_{[0,T]}\text{-a.s.}, \quad (5.26)$$

the process  $(\mathbf{B}_t)_{0 \leq t \leq T}$  being a Brownian motion under  $\bar{\mu}_{[0,T]}$ . For  $x$  in  $\mathbb{R}^d$ , we then define

$$\bar{c}_t(x) := -\mathbb{E}_{\bar{\mu}_{[0,T]}} c_t \cdot \frac{\delta b_t}{\delta \mu}(\mathbf{X}_t, \bar{\mu}_t, x). \quad (5.27)$$

From (A.3),  $\bar{c}_t$  is globally Lipschitz in  $x$  uniformly in  $t$ . Plugging this into (5.25) concludes.  $\square$

*Proof of Theorem 5.2.2-(ii).* Using Lemma 5.4.1,

$$\bar{H}_\Psi = \inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]})) + \mathcal{F}(\mu_{[0,T]}).$$

This setting satisfies the assumptions of Theorem [Chaintron et al., 2024, Theorem 2.12] with  $E = C([0, T], \mathbb{R}^d)$ ,  $\phi(x_{[0,T]}) = \sup_{t \in [0,T]} |x_t|$ ,  $\mathcal{F}$  as above, the inequality constraints  $\Psi_t : \mu_{[0,T]} \mapsto \Psi(\mu_t)$  with  $\mathcal{T} = [0, T]$ , and no equality constraint. Indeed,  $\mu_{[0,T]} \mapsto H(\mu_{[0,T]}|\Gamma(\bar{\mu}_{[0,T]}))$  is lower semi-continuous, so that [Chaintron et al., 2024, Assumptions (A1)-(A2)-(A3)] are satisfied. From Corollary 5.B.1.1, the domain of  $\mathcal{F}$  is convex and  $\mathcal{F}(\mu_{[0,T]}) < +\infty$  implies  $\mathcal{F}(\mu'_{[0,T]}) < +\infty$  for any  $\mu'_{[0,T]}$  that has a bounded density w.r.t.  $\mu_{[0,T]}$ . Using Lemma 5.4.1, this implies [Chaintron et al., 2024, Assumptions (A6)-(A8)]. The regularity assumptions (A.2)-(A.3) and the qualification condition (A.4) eventually imply [Chaintron et al., 2024, Assumption (B3)], and thus [Chaintron et al., 2024, Assumptions (A7)]. As a consequence, we obtain a Lagrange multiplier  $\bar{\lambda} \in \mathcal{M}_+([0, T])$  such that

$$\frac{d\bar{\mu}_{[0,T]}}{d\Gamma(\bar{\mu}_{[0,T]})}(x_{[0,T]}) = \bar{Z}^{-1} \exp \left[ - \int_0^T \bar{c}_s(x_s) ds - \int_0^T \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_s, x_s) \bar{\lambda}(ds) \right],$$

together with the complementary slackness condition (5.4).

By convolution with a smooth kernel with compact support, we regularise  $(t, x) \mapsto b_t(x, \bar{\mu}_t)$ ,  $\bar{c}_t(x)$ ,  $\sigma_t(x)$ ,  $\frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t, x)$  into  $C^\infty$  functions  $b^k$ ,  $c^k$ ,  $\sigma^k$ ,  $\psi^k$ . We similarly obtain a weak approximation of  $\bar{\lambda}$  given by a sequence  $(\lambda^k)_{k \geq 1}$  of functions in  $C^1([0, T], \mathbb{R}_+)$ . Since we assumed (A.2)-(A.3) and convolution preserves already existing regularity properties,  $b^k$ ,  $c^k$ ,  $\sigma^k$ ,  $\psi^k$  satisfy (A.2)-(A.3) for regularity constants independent of  $k$ . We thus enter the setting of Section 5.3, and let  $\varphi^k$  denote the solution of the corresponding HJB equation (5.13) with regularised coefficients. For every  $t \in [0, T]$ , Proposition 5.3.2 shows that  $(\nabla \varphi_t^k)_{k \geq 1}$  and  $(\nabla^2 \varphi_t^k)_{k \geq 1}$  are uniformly bounded. Hence, from the Arzelà-Ascoli theorem,  $(\nabla \varphi_t^k)_{k \geq 1}$  is pre-compact on any compact set. Using [Chaintron et al., 2024, Lemma 5.4], [Chaintron et al., 2024, Lemma 5.3] now shows that for Lebesgue-a.e.  $t \in [0, T]$ ,  $\varphi_t^k$  converges towards  $\bar{\varphi}_t$  on any compact set, where  $\bar{\varphi}$  is the mild solution of (5.3) in the sense of Definition 5.C.1. Moreover,  $\bar{\mu}_{[0,T]}$  is the law of the solution to

$$d\bar{X}_t = b_t(\bar{X}_t) dt - a_t \nabla \varphi_t(\bar{X}_t) dt + \sigma_t(\bar{X}_t) dB_t, \quad \bar{X}_0 \sim \bar{Z}^{-1} e^{-\bar{\varphi}_0(x)} \nu_0(dx),$$

which we now identify to (5.26). This yields  $c_t = -\nabla\bar{\varphi}_t(\bar{X}_t)$  a.s., giving the desired expression for  $\bar{c}_t$  using (5.27).  $\square$

From [Chaintron et al., 2024, Remark 4.4], we further have

$$\bar{\lambda}([0, T]) \leq \tilde{\varepsilon}\eta^{-1} \int_{\mathcal{P}_1(C([0, T], \mathbb{R}^d))} \left[ \log \frac{d\bar{\mu}_{[0, T]}}{d\Gamma(\bar{\mu}_{[0, T]})} + \frac{\delta\mathcal{F}}{\delta\mu}(\bar{\mu}_{[0, T]}) \right] d(\tilde{\mu}_{[0, T]} - \bar{\mu}_{[0, T]}), \quad (5.28)$$

where  $\eta := -\sup_{0 \leq t \leq T} \Psi(\bar{\mu}_t) + \int \frac{\delta\Psi}{\delta\mu}(\bar{\mu}_t) d\tilde{\mu}_t$ .

*Proof of Theorem 5.2.2-(i).* From Lemma 5.B.1,

$$\bar{H}_\Psi = \inf_{\Sigma} \bar{V}_\Psi.$$

From the above proof of Theorem 5.2.2-(ii), this infimum is realised on any reference system  $\Sigma$  by the initial law  $\bar{\mu}_0$  and the feed-back control  $\bar{\alpha}_t = -a_t \nabla\bar{\varphi}_t$ . Consequently,  $\bar{H}_\Psi = \bar{V}_\Psi = H(\bar{\mu}_0|\nu_0) + V_\Sigma(\bar{\mu}_0)$ .  $\square$

### 5.4.2 Existence of a density for the Lagrange multiplier

Under (A.6) and (A.5)-(i), let  $\bar{\mu}_{[0, T]}$  be an optimiser for (5.2) that satisfies (A.4). We are going to smooth Problem (5.2) to make it enter the framework of Section 5.3. Let  $\rho : \mathbb{R}^d \rightarrow [0, 1]$  be a  $C^\infty$  symmetric function with compact support and  $\int_{\mathbb{R}^d} \rho(x) dx = 1$ . For  $k \geq 1$ , we define  $\rho^k(x) := k^d \rho(kx)$ , and the regularised coefficients

$$b_t^k(x, \mu) := [b_t^k(\cdot, \rho^k * \mu)] * \rho^k(x), \quad \sigma_t^k(x) := \sigma_t^k * \rho^k(x), \quad \Psi^k(\mu) := \Psi(\rho^k * \mu).$$

The convolution against vectors or matrices is done component-wise. This mollifying procedure of the measure argument well behaves w.r.t. linear derivatives, as recalled in Lemma 5.B.2. Since the coefficients were already Lipschitz-continuous,  $b^k$ ,  $\sigma^k$ ,  $\Psi^k$ ,  $\frac{\delta}{\delta\mu} b^k$  and  $\frac{\delta}{\delta\mu} \Psi^k$  converge towards  $b$ ,  $\sigma$ ,  $\Psi$ ,  $\frac{\delta}{\delta\mu} b$  and  $\frac{\delta}{\delta\mu} \Psi$  uniformly on  $[0, T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ . Moreover, since convolution preserves already existing regularity properties, (A.6) holds for  $b^k$ ,  $\sigma^k$ ,  $\frac{\delta}{\delta\mu} b^k$  and  $\frac{\delta}{\delta\mu} \Psi^k$  with regularity bounds independent of  $k$ . For  $\mu_{[0, T]}$  in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$ , let  $\Gamma^k(\mu_{[0, T]})$  denote the path-law of the pathwise unique strong solution to

$$dY_t^k = b_t^k(Y_t^k, \mu_t) dt + \sigma_t^k(Y_t^k) dB_t, \quad Y_0^k \sim \nu_0,$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion. From Theorem 5.2.2, on the canonical space  $\Omega = C([0, T], \mathbb{R}^d)$ , there exists an essentially bounded process  $(\bar{\alpha}_t)_{0 \leq t \leq T}$  such that the canonical process satisfies

$$dX_t = b_t(X_t, \bar{\mu}_t) dt + \sigma_t(X_t) \bar{\alpha}_t dt + \sigma_t(X_t) dB_t, \quad \bar{\mu}_{[0, T]} \text{-a.s.},$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion under  $\bar{\mu}_{[0, T]}$ . In our Lipschitz setting,  $\bar{\alpha}$  being essentially bounded, the following McKean-Vlasov equation

$$dX_t^k = b_t^k(X_t^k, \mathcal{L}(X_t^k)) dt + \sigma_t^k(X_t^k) \bar{\alpha}_t dt + \sigma_t^k(X_t^k) dB_t, \quad X_0^k = X_0,$$

has a pathwise-unique strong solution, and let  $\mu_{[0, T]}^k$  denote its law. Using the Girsanov transform,

$$H(\mu_{[0, T]}^k | \Gamma(\mu_{[0, T]}^k)) = H(\bar{\mu}_{[0, T]} | \Gamma(\bar{\mu}_{[0, T]})) = H(\bar{\mu}_0 | \nu_0) + \mathbb{E}_{\bar{\mu}_{[0, T]}} \int_0^T \frac{1}{2} |\bar{\alpha}_t|^2 dt. \quad (5.29)$$

Using the uniform convergence of coefficients, a standard coupling argument yields

$$\mathbb{E}_{\bar{\mu}_{[0,T]}} \left[ \sup_{0 \leq t \leq T} |X_t - X_t^k| \right] \xrightarrow{k \rightarrow +\infty} 0. \quad (5.30)$$

As a consequence,  $\Psi^k(\mu_t^k)$  converges towards  $\Psi(\bar{\mu}_t)$ , and we can assume that

$$\forall k \geq 1, \forall t \in [0, T], \quad \Psi^k(\mu_t^k) \leq \Psi(\bar{\mu}_t) \leq 0, \quad (5.31)$$

up to subtracting a small constant to  $\Psi^k$  that vanishes as  $k \rightarrow +\infty$ . Let now  $(\phi_i)_{i \geq 1}$  be a countable set of  $C^\infty$  functions  $\mathbb{R}^d \rightarrow \mathbb{R}$  that separates measures, with  $\|\nabla^j \phi_i\|_\infty \leq 1$  for any  $j \geq 0$ . We further define

$$\forall \mu_{[0,T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)), \quad \Phi(\mu_{[0,T]}) := \sum_{i \geq 1} 2^{-i-1} \int_0^T \left( \int_{\mathbb{R}^d} \phi_i d(\mu_t - \bar{\mu}_t) \right)^2 dt.$$

This positive term only vanishes when  $(\mu_t)_{0 \leq t \leq T} = (\bar{\mu}_t)_{0 \leq t \leq T}$ , hence

$$\bar{H}_\Psi = \inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \Psi(\mu_t) \leq 0}} H(\mu_{[0,T]} | \Gamma(\mu_{[0,T]})) + \Phi(\mu_{[0,T]}),$$

and the r.h.s. is uniquely realised by  $\bar{\mu}_{[0,T]}$ . Indeed,  $\Gamma(\bar{\mu}_{[0,T]})$  only depends on  $(\bar{\mu}_t)_{0 \leq t \leq T}$ , and there is only one path-measure with given time-marginals and minimal entropy (because  $H$  is strictly convex). We now approximate this problem.

**Lemma 5.4.2** (Non-linear approximation). *For large enough  $k \geq 1$ ,*

$$\bar{H}_\Psi^k := \inf_{\substack{\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d)) \\ \forall t \in [0,T], \Psi^k(\mu_t) \leq 0}} H(\mu_{[0,T]} | \Gamma^k(\mu_{[0,T]})) + \Phi(\mu_{[0,T]}),$$

is realised by at least one measure  $\bar{\mu}_{[0,T]}^k$  at which the qualification constraint (A.4) holds. Moreover,  $(\bar{\mu}_{[0,T]}^k)_{k \geq 1}$  converges towards  $\bar{\mu}_{[0,T]}$  in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$ .

*Proof.* We first prove existence and convergence, before to show constraint qualification.

**Step 1. Convergence of minimisers.** From (5.29)-(5.31), the competitor  $\mu_{[0,T]}^k$  that we previously built shows that  $\bar{H}_\Psi^k$  is finite and bounded uniformly in  $k$ . Since  $\Phi$  is non-negative continuous and  $\mu_{[0,T]} \mapsto H(\mu_{[0,T]} | \Gamma^k(\mu_{[0,T]}))$  has compact level sets in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$  (see e.g. [Chaintron, 2024, Corollary B.4]), existence holds for at least one minimiser  $\bar{\mu}_{[0,T]}^k$ . Since  $H(\bar{\mu}_{[0,T]}^k | \Gamma^k(\bar{\mu}_{[0,T]}^k))$  is finite, there exists a square-integrable process  $(\bar{\alpha}_t^k)_{0 \leq t \leq T}$  on the canonical space such that

$$dX_t = b_t^k(X_t, \bar{\mu}_t^k) dt + \sigma_t^k(X_t) \bar{\alpha}_t^k dt + \sigma_t^k(X_t) d\mathbf{B}_t^k, \quad \bar{\mu}_{[0,T]}^k\text{-a.s.},$$

where  $(\mathbf{B}_t^k)_{0 \leq t \leq T}$  is a Brownian motion under  $\bar{\mu}_{[0,T]}^k$  (this argument was detailed in the proof of Lemma 5.4.1). This is a McKean-Vlasov equation whose coefficients are Lipschitz uniformly in  $k$ . Since  $\mathbb{E}[\int_0^T |\bar{\alpha}_t^k|^2 dt]$  and  $\sigma_t^k$  are furthermore uniformly bounded, a standard Gronwall argument yields

$$\sup_{k \geq 1} \mathbb{E}_{\bar{\mu}_{[0,T]}^k} \left[ \sup_{0 \leq t \leq T} |X_t| \right] < +\infty,$$

hence the Lipschitz function  $x \mapsto b_t^k(x, \bar{\mu}_t^k)$  has linear growth uniformly in  $k$ . Using [Chaintron, 2024, Lemma B.2 and Remark B.3], we deduce that  $(\bar{\mu}_{[0,T]}^k)_{k \geq 1}$  is pre-compact in

$\mathcal{P}_1(C([0, T], \mathbb{R}^d))$ . Up to re-indexing, we can assume that it converges towards some  $\bar{\mu}_{[0, T]}^\infty$ . Using a coupling argument as for (5.30), we get that  $\Gamma^k(\bar{\mu}_{[0, T]}^k)$  converges towards  $\Gamma(\bar{\mu}_{[0, T]}^\infty)$  in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$ . Moreover, by optimality of  $\bar{\mu}_{[0, T]}^k$ ,

$$\forall k \geq 1, \quad H(\bar{\mu}_{[0, T]}^k | \Gamma^k(\bar{\mu}_{[0, T]}^k)) + \Phi(\bar{\mu}_{[0, T]}^k) \leq H(\mu_{[0, T]}^k | \Gamma^k(\mu_{[0, T]}^k)) + \Phi(\mu_{[0, T]}^k).$$

From (5.29)-(5.30), the r.h.s. converges to  $\bar{H}_\Psi$  as  $k \rightarrow +\infty$ . Since  $(\mu_{[0, T]}, \nu_{[0, T]}) \mapsto H(\mu_{[0, T]} | \nu_{[0, T]})$  is lower semi-continuous, we get

$$H(\bar{\mu}_{[0, T]}^\infty | \Gamma(\bar{\mu}_{[0, T]}^\infty)) + \Phi(\bar{\mu}_{[0, T]}^\infty) \leq \bar{H}_\Psi,$$

and using the uniform convergence of  $\Psi^k$ :

$$\forall t \in [0, T], \quad \Psi^k(\bar{\mu}_t^\infty) \leq 0.$$

By uniqueness for the problem defining  $\bar{H}_\Psi$ , we get  $\bar{\mu}_{[0, T]}^\infty = \bar{\mu}_{[0, T]}$ . By uniqueness of limit points, the full sequence  $(\bar{\mu}_{[0, T]}^k)_{k \geq 1}$  thus converges towards  $\bar{\mu}_{[0, T]}$ .

**Step 2. Constraint qualification.** From (A.4), we have existence of  $\tilde{\mu}_{[0, T]}$  with finite  $H(\tilde{\mu}_{[0, T]} | \Gamma(\bar{\mu}_{[0, T]}))$  such that

$$\forall t \in [0, T], \quad \Psi(\bar{\mu}_t) + \tilde{\varepsilon} \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t) d[\tilde{\mu}_t - \bar{\mu}_t] < 0.$$

Reasoning as we did to build  $\mu_{[0, T]}^k$ , we can produce  $\tilde{\mu}_{[0, T]}^k$  with  $\sup_{k \geq 1} H(\tilde{\mu}_{[0, T]}^k | \Gamma^k(\bar{\mu}_{[0, T]}^k)) < +\infty$  such that  $(\tilde{\mu}_{[0, T]}^k)_{k \geq 1}$  converges towards  $\tilde{\mu}_{[0, T]}$  in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$ . The uniform convergence of  $\Psi^k$  and  $\frac{\delta}{\delta \mu} \Psi^k$  then implies that

$$\sup_{k \geq 1} H(\tilde{\mu}_{[0, T]}^k | \Gamma^k(\bar{\mu}_{[0, T]}^k)) < +\infty, \quad \sup_{k \geq k_0} \sup_{t \in [0, T]} \Psi^k(\bar{\mu}_t^k) + \tilde{\varepsilon} \int_{\mathbb{R}^d} \frac{\delta \Psi^k}{\delta \mu}(\bar{\mu}_t^k) d\bar{\mu}_t^k < 0, \quad (5.32)$$

for a large enough  $k_0$ , hence constraint qualification holds at  $\bar{\mu}_{[0, T]}^k$ .  $\square$

We now set  $\psi_t^k(x) := \Psi^k(\bar{\mu}_t^k) + \frac{\delta \Psi^k}{\delta \mu}(\bar{\mu}_t^k, x)$  and  $\bar{\psi}_t(x) := \Psi(\bar{\mu}_t) + \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t, x)$ . We then linearise the problem defining  $\bar{H}_\Psi$  at  $\bar{\mu}_{[0, T]}^k$  using Lemma 5.4.1 and [Chaintron et al., 2024, Lemma 4.3]. We obtain that  $\bar{\mu}_{[0, T]}^k$  is the unique minimiser of the strictly convex problem

$$\inf_{\substack{\mu_{[0, T]} \in \mathcal{P}_1(C([0, T], \mathbb{R}^d)) \\ \forall t \in [0, T], \int_{\mathbb{R}^d} \psi_t^k d\mu_t \leq 0}} H(\mu_{[0, T]} | \Gamma^k(\bar{\mu}_{[0, T]}^k)) + \int_0^T \int_{\mathbb{R}^d} c_t^k d\mu_t dt, \quad (5.33)$$

where

$$\bar{c}_t^k(x) := \bar{c}_t^k(x) + \Phi_t^k(x), \quad \Phi_t^k(x) := \sum_{i \geq 1} 2^{-i} \left( \int_{\mathbb{R}^d} \phi_i d(\bar{\mu}_t^k - \bar{\mu}_t) \right) \phi_i(x).$$

The  $\bar{c}_t^k$  term is produced using Lemma 5.4.1 applied to  $\bar{\mu}_{[0, T]}^k$ . The second term results from computing the linear derivative of  $\Phi$ . Noticing that  $\Psi^k(\bar{\mu}_t^k) = \int_{\mathbb{R}^d} \psi_t^k d\bar{\mu}_t^k$ , differentiating yields that  $\int_{\mathbb{R}^d} \partial_t \psi_t^k d\bar{\mu}_t^k = 0$ , as required by (5.7). Since our regularised coefficients satisfy



(A.8), Problem (5.33) now enters the scope of Section 5.3 for the drift coefficient  $b_t^k(x) := b_t^k(x, \bar{\mu}_t^k)$ . Let  $(\bar{\lambda}^k, \bar{\varphi}^k)$  be related to  $\bar{\mu}_{[0,T]}^k$  as in (5.9): the path-density reads

$$\frac{d\bar{\mu}_{[0,T]}^k}{d\Gamma^k(\bar{\mu}_{[0,T]}^k)}(x_{[0,T]}) = \bar{Z}_k^{-1} \exp \left[ - \int_0^T c_t^k(x_t) dt - \int_0^T \psi_t^k(x_t) \bar{\lambda}^k(dt) \right], \quad (5.34)$$

and  $\bar{\varphi}^k$  is the solution to

$$-\bar{\varphi}_t^k + \int_t^T b_s^k \cdot \nabla \bar{\varphi}_s^k - \frac{1}{2} |(\sigma_s^k)^\top \nabla \bar{\varphi}_s^k|^2 + c_s^k + \frac{1}{2} \text{Tr}[\sigma_s^k (\sigma_s^k)^\top \nabla^2 \bar{\varphi}_s^k] ds + \int_{[t,T]} \psi_s^k \bar{\lambda}^k(ds) = 0,$$

in the sense of Definition 5.C.1. Moreover, reasoning as in the proof of Theorem 5.2.2-(ii),

$$\bar{c}_s^k(x) = \int_{\mathbb{R}^d} \frac{\delta b_s^k}{\delta \mu}(y, \bar{\mu}_s^k, x) \cdot \nabla \bar{\varphi}_s^k \bar{\mu}_s^k(dy). \quad (5.35)$$

Let us further define  $\alpha_t^k := -(\sigma_t^k)^\top \nabla \bar{\varphi}_t^k$ ,  $a_t^k := \sigma_t^k (\sigma_t^k)^\top$  and

$$\forall \phi \in C^2(\mathbb{R}^d), \quad L_t^{\alpha^k} \phi := b_t^k \cdot \nabla \phi + \sigma_t^k \alpha_t^k \cdot \nabla \phi + \frac{1}{2} \text{Tr}[a_t^k \nabla^2 \phi], \quad (5.36)$$

the operator  $L_t^{\alpha^k}$  being the infinitesimal generator of the process with path-law  $\bar{\mu}_{[0,T]}^k$ .

*Proof of Theorem 5.2.3.* Since  $\bar{\mu}_{[0,T]}^k$  converges to  $\bar{\mu}_{[0,T]}$  in  $\mathcal{P}_1(C([0,T], \mathbb{R}^d))$ ,  $\Phi_t^k(x)$  converges to 0 uniformly in  $(t, x)$ . Similarly, using our Lipschitz assumptions,  $\psi^k$  uniformly converges towards  $\bar{\psi}$ . Since convolution preserves already existing regularity properties, the regularity properties w.r.t.  $x$  assumed by Theorem 5.2.3 hold for  $b^k$ ,  $\sigma^k$ ,  $\frac{\delta}{\delta \mu} b^k$  and  $\frac{\delta}{\delta \mu} \Psi^k$ , with regularity bounds independent of  $k$ . To take the  $k \rightarrow +\infty$  limit within Proposition 5.3.8, we still have to check that the time-derivatives of the coefficients are bounded uniformly in  $k$ . This is clear for  $\sigma_t^k$  because  $x$ -convolution does not affect time-regularity. Moreover, from e.g. [Daudin, 2023b, Proposition 1.3],

$$\begin{aligned} \frac{d}{dt} b_t^k(x, \bar{\mu}_t^k) &= \partial_t b_t^k(x, \bar{\mu}_t^k) + \int_{\mathbb{R}^d} L_t^{\alpha^k} [y \mapsto \frac{\delta b_t}{\delta \mu}(x, \mu_t, y)] d\bar{\mu}_t^k, \\ \partial_t \psi_t^k(x) &= \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t^k) + L_t^{\alpha^k} [y \mapsto \frac{\delta^2 \Psi}{\delta \mu}(\mu_t, x, y)] d\bar{\mu}_t^k. \end{aligned}$$

From Lemma 5.3.6,  $\alpha^k$  is bounded and Lipschitz uniformly in  $k$ . Using the bounds on  $\varphi^k$ , we similarly deduce that  $\bar{\mu}_0^k$  satisfies Lemma 5.3.4 with bounds independent of  $k$ . Applying Lemma 5.A.2 to  $(\bar{\mu}_t^k)_{0 \leq t \leq T}$  thus provides a bound on  $\mathbb{E}[|\nabla \log \bar{\mu}_t^k(X_t^0)|]$  that is uniform in  $k$ . Integrating by parts, we get rid of second order derivatives in  $y$ , so that  $\partial_t b_t^k$  and  $\partial_t \psi_t^k$  are bounded uniformly in  $k$  using (A.6). From Proposition 5.3.8, we can now decompose

$$\bar{\lambda}^k(dt) = \bar{\lambda}_0^k \delta_0(dt) + \bar{\lambda}_t^k dt + \bar{\lambda}_T^k \delta_T(dt),$$

for some bounded measurable  $t \mapsto \bar{\lambda}^k$ , whose  $L^\infty$ -norm only depends on  $\bar{\lambda}^k([0, T])$  and the regularity constants allowed by Theorem 5.2.3.

In the proof of Lemma 5.4.2, qualification was obtained using  $\tilde{\mu}_{[0,T]}^k$  converging to  $\tilde{\mu}_{[0,T]}$  with bounded  $H(\tilde{\mu}_{[0,T]}^k | \Gamma^k(\bar{\mu}_{[0,T]}^k))$  uniformly in  $k$ . Using (5.28) and (5.32), there exists  $\eta > 0$  independent of  $k$  such that

$$\bar{\lambda}^k([0, T]) \leq \tilde{\varepsilon} \eta^{-1} \int_{\mathcal{P}_1(C([0,T], \mathbb{R}^d))} \left[ \log \frac{d\bar{\mu}_{[0,T]}^k}{d\Gamma^k(\bar{\mu}_{[0,T]}^k)} + \frac{\delta \mathcal{F}^k}{\delta \mu}(\bar{\mu}_{[0,T]}^k) \right] d\tilde{\mu}_{[0,T]}^k.$$

Since  $\tilde{\mu}_{[0,T]}^k$  converges in  $\mathcal{P}_1(C([0,T], \mathbb{R}^d))$  and  $\frac{\delta \mathcal{F}^k}{\delta \mu}$  has linear growth uniformly in  $k$  from Lemma 5.4.1 and (5.35),  $\int \frac{\delta \mathcal{F}^k}{\delta \mu}(\tilde{\mu}_{[0,T]}^k) d\tilde{\mu}^k$  is bounded uniformly in  $k$ . From [Nutz, 2021, Lemma 1.4-(b)],

$$\int_{\mathcal{P}_1(C([0,T], \mathbb{R}^d))} \log \frac{d\tilde{\mu}_{[0,T]}^k}{d\Gamma^k(\tilde{\mu}_{[0,T]}^k)} d\tilde{\mu}_{[0,T]}^k \leq H(\tilde{\mu}_{[0,T]}^k | \Gamma^k(\tilde{\mu}_{[0,T]}^k)),$$

which is bounded uniformly in  $k$ . This implies a bound on  $(\bar{\lambda}^k([0,T]))_{k \geq 1}$ . From (a variant of) the Prokhorov theorem, up to re-indexing, we can thus assume that  $\bar{\lambda}^k$  converges towards some  $\bar{\lambda}$  in  $\mathcal{M}_+([0,T])$ . In particular, we assume that  $(\bar{\lambda}_0^k)_{k \geq 1}$  and  $(\bar{\lambda}_T^k)_{k \geq 1}$  respectively converge towards  $\bar{\lambda}(\{0\})$  and  $\bar{\lambda}(\{T\})$ . This shows that

$$\bar{\lambda}(dt) = \bar{\lambda}_0 \delta_0(dt) + \lambda_t dt + \bar{\lambda}_T \delta_T(dt),$$

for some bounded measurable function  $t \mapsto \bar{\lambda}_t$ , which is the weak- $\star$  limit of  $(\bar{\lambda}^k)_{k \geq 1}$  (for the  $\sigma(L^\infty, L^1)$ -topology).

From Lemma 5.3.6,  $\nabla \bar{\varphi}_t^k$  and  $\nabla^2 \bar{\varphi}_t^k$  are bounded uniformly in  $(t, k)$ . Moreover, for every  $1 \leq i \leq d$  and  $0 < t \leq T$ ,

$$\begin{aligned} \partial_i \bar{\varphi}_t^k(x) &= \bar{\lambda}_T^k \mathbb{E}[\partial_i \psi_T^k(X_T^{t,x,\alpha^k})] + \int_t^T \mathbb{E}\left\{ \partial_i b_s^k \cdot \nabla \bar{\varphi}_s^k - \frac{1}{2} \nabla \bar{\varphi}_s^k \cdot \partial_i a_s^k \nabla \bar{\varphi}_s^k + \frac{1}{2} \text{Tr}[\partial_i a_s^k \nabla \bar{\varphi}_s^k] \right. \\ &\quad \left. + \partial_i \bar{c}_s^k + \bar{\lambda}_s^k \partial_i \frac{\delta \Psi^k}{\delta \mu}(\bar{\mu}_s^k) \right\} (X_s^{t,x,\alpha^k}) ds. \end{aligned} \quad (5.37)$$

Since  $(\bar{\lambda}^k)_{k \geq 1}$  is now bounded in  $L^\infty(0, T)$ , we deduce that  $(\nabla \bar{\varphi}^k)_{k \geq 1}$  is equi-continuous on every compact set of  $(0, T] \times \mathbb{R}^d$ . Up to re-indexing, the Arzelà-Ascoli allows us to assume that  $\bar{\varphi}^k$  converges towards  $\bar{\varphi} \in C^1((0, T] \times \mathbb{R}^d)$  uniformly on every compact set. Similarly, we can define  $\bar{\varphi}_0$  so that  $\nabla \bar{\varphi}_0^k$  uniformly converges towards  $\nabla \bar{\varphi}_0$  on every compact set of  $\mathbb{R}^d$ . Going back to (5.35),  $\bar{c}^k$  uniformly converges towards  $\bar{c}$  defined by

$$\bar{c}_t(x) := \int_{\mathbb{R}^d} \frac{\delta b_s}{\delta \mu}(y, \bar{\mu}_s, x) \cdot \nabla \bar{\varphi}_s(y) \bar{\mu}_s(dy).$$

From the stability result [Chaintron et al., 2024, Proposition 5.2], we get that  $\bar{\varphi}$  is a solution of (5.3), so that  $(\bar{\mu}_{[0,T]}, \bar{\lambda}, \bar{\varphi})$  satisfies Theorem 5.2.2-(ii).

Taking the  $k \rightarrow +\infty$  limit in Lemma 5.3.6 applied to  $\bar{\varphi}^k$ , we get that  $\nabla \bar{\varphi}_t$  is bounded and Lipschitz uniformly in  $t$ . Under (A.5)-(ii) and (A.7)-(i), we can further take the limit in Lemma 5.3.3 and get that  $\nabla^2 \bar{\varphi}_t$  is globally Lipschitz-continuous. Similarly, the analogous of (5.37) yields that  $t \mapsto \nabla^2 \bar{\varphi}_t(x)$  is continuous on  $(0, T]$ . The bound (5.5) results from Proposition 5.A.4.  $\square$

As a consequence we get the following result,  $(\bar{X}_t)_{0 \leq t \leq T}$  denoting the optimally controlled process for the feed-back  $\bar{\alpha} := -\sigma^\top \nabla \bar{\varphi}$ .

**Corollary 5.4.2.1** (First order regularity). *Under (A.7)-(ii),*

(i)  $t \mapsto \varphi_t(x)$  is continuous on  $(0, T]$ . Moreover,  $(\nabla \bar{\varphi}_t(\bar{X}_t))_{0 < t \leq T}$  is almost surely continuous, and for  $1 \leq i \leq d$ ,

$$\begin{aligned} d\partial_i \bar{\varphi}_t(\bar{X}_t) &= \{-\partial_i b_t \cdot \nabla \bar{\varphi}_t + \frac{1}{2} \nabla \bar{\varphi}_t \cdot \partial_i a_t \nabla \bar{\varphi}_t - \frac{1}{2} \text{Tr}[\partial_i a_t \nabla^2 \bar{\varphi}_t] \\ &\quad - \partial_i \bar{c}_t - \bar{\lambda}_t \partial_i \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t)\}(\bar{X}_t) dt - \sigma_t \nabla \partial_i \bar{\varphi}_t(\bar{X}_t) dB_t. \end{aligned}$$

(ii)  $t \mapsto \Psi(\bar{\mu}_t)$  is  $C^1$  on  $[0, T]$ , with bounded derivative given by

$$\frac{d}{dt}\Psi(\bar{\mu}_t) = \mathbb{E} \left[ L_t^{\bar{\alpha}} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_t, \bar{X}_t) \right], \quad t \in (0, T).$$

*Proof.* We follow the regularisation procedure in the above proof of Theorem 5.2.3. Since  $\bar{\lambda}$  has a bounded density, the continuity of  $t \mapsto \bar{\varphi}_t(x)$  can be read directly on (5.3). To get (i), we simply take the  $k \rightarrow +\infty$  limit within (5.37) initialised at  $x = \bar{X}_t$ .

For (ii), we differentiate  $\Psi^k(\bar{\mu}_s^k)$  to get that for every  $0 < t < t' < T$ ,

$$\Psi_{t'}^k(\bar{\mu}_{t'}^k) - \Psi_t^k(\bar{\mu}_t^k) = \mathbb{E} \int_t^{t'} L_s^{\alpha^k} \frac{\delta \Psi^k}{\delta \mu}(\bar{\mu}_s^k, X_s^{\alpha^k}) ds.$$

Taking the  $k \rightarrow +\infty$  limit and using that  $(\nabla \bar{\varphi}_s(\bar{X}_s))_{0 < s \leq T}$  is continuous on  $(0, T]$ ,  $t \mapsto \Psi(\bar{\mu}_t)$  is  $C^1$  on  $(0, T]$  with derivative given by (ii). From (i), we see that  $s \mapsto \mathbb{E}[L_s^{\bar{\alpha}} \frac{\delta \Psi}{\delta \mu}(\bar{\mu}_s, \bar{X}_s)]$  has a finite limit as  $s \rightarrow 0$ . By continuous prolongation,  $t \mapsto \Psi(\bar{\mu}_t)$  is thus differentiable on the full  $[0, T]$  with continuous derivative.  $\square$

**Remark 5.4.3** (Linearising before regularising). *In the proof of Theorem 5.2.3, we regularised the non-linear problem before to linearise it and to take the limit in the optimality conditions. An alternative (and easier) way is to linearise before to regularise. In particular, this avoids using Lemma 5.B.2. However, this approach requires additional derivatives w.r.t.  $\mu$  to bound the time-derivatives  $\partial_s b_s^k$  and  $\partial_s \psi_s^k$  uniformly in  $k$ .*

### 5.4.3 Quantitative stability

We can apply Theorem 5.2.2 to  $\mu_{[0, T]}^\varepsilon$  and the constraints  $\Psi - \varepsilon$ , because  $\mu_{[0, T]}^0$  provides constraint qualification using Remark 5.2.1. Let us fix a reference probability system  $\Sigma = (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$ . From Theorem 5.2.2,  $\mu_{[0, T]}^\varepsilon$  is the law of the solution to the SDE

$$dX_t^\varepsilon = b_t(X_t^\varepsilon)dt + \sigma_t \alpha_t^\varepsilon(X_t^\varepsilon)dt + \sigma_t(X_t^\varepsilon)dB_t,$$

where  $\alpha_t^\varepsilon = -\sigma_t^\top \nabla \varphi_t^\varepsilon$  is the optimal feed-back control. In the following,  $C$  is a positive constant that may change from line to line, but always independent of  $\varepsilon$ . From (5.28), we thus have  $\lambda^\varepsilon([0, T]) \leq C$ . Using Theorem 5.2.3, we further get

$$\sup_{0 \leq t \leq T} \|\nabla \varphi_t^\varepsilon\|_\infty + \|\nabla^2 \varphi_t^\varepsilon\|_\infty \leq C, \quad \sup_{0 \leq t \leq T} \|\alpha_t^\varepsilon\|_\infty + \|\nabla \alpha_t^\varepsilon\|_\infty \leq C, \quad (5.38)$$

and a straightforward consequence is that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^\varepsilon|^2] \leq C$ .

**Lemma 5.4.4** ( $L^2$ -stability).

$$(i) \quad H(\mu_{[0, T]}^0 | \mu_{[0, T]}^\varepsilon) = H(\mu_0^0 | \mu_0^\varepsilon) + \mathbb{E} \int_0^T \frac{1}{2} |\alpha_t^\varepsilon(X_t^0) - \alpha_t^0(X_t^0)|^2 dt \leq C\varepsilon.$$

$$(ii) \quad \sup_{0 \leq t \leq T} W_1(\mu_t^\varepsilon, \mu_t^0) \leq C\varepsilon^{1/2}.$$

(iii) Let  $r_t^\varepsilon(x) := f_t(x)g_t(x)h_t(\alpha_t^\varepsilon(x))$ , for measurable  $f, g, h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ . If  $f$  is bounded,  $g$  is globally Lipschitz in  $x$  and  $h$  is locally-Lipschitz (uniformly in  $t$ ),

$$\mathbb{E} \int_0^T |r_t^\varepsilon(X_t^\varepsilon) - r_t^0(X_t^0)| dt \leq C\varepsilon^{1/4}.$$

*Proof.* First, (i) is a direct application of the stability result proved [Chaintron et al., 2024, Proposition 2.18 and Remark 2.19], the relative entropy being computed using the Girsanov transform. Let us bound  $W_1(\mu_0^\varepsilon, \mu_0^0)$ . Since  $\varphi_0^\varepsilon$  has linear growth independently of  $\varepsilon$ , Lemma 5.3.4 implies that the bound (A.5)-(iii) holds for  $\bar{\mu}^\varepsilon$  with a constant independent of  $\varepsilon$ . This Gaussian bound is known to imply a transport-entropy inequality:  $C > 0$  independent of  $\varepsilon$  exists such that

$$\forall \mu' \in \mathcal{P}_1(\mathbb{R}^d), \quad W_1^2(\mu', \bar{\mu}_0^\varepsilon) \leq CH(\mu' | \bar{\mu}_0^\varepsilon).$$

Together with (i), this implies  $W_1(\mu_0^\varepsilon, \mu_0^0) \leq C\varepsilon^{1/2}$ . Using Ito's formula, our Lipschitz assumptions on the coefficients and the uniform bound on  $\alpha_t^\varepsilon$ :

$$\mathbb{E}[|X_t^\varepsilon - X_t^0|] \leq \mathbb{E}[|X_0^\varepsilon - X_0^0|] + C \int_0^t \mathbb{E}[|X_s^\varepsilon - X_s^0|] ds + C \int_0^t \mathbb{E}[|\alpha_s^\varepsilon - \alpha_s^0|] ds.$$

Using (i) and Gronwall's Lemma, we establish (ii). To prove (iii), we decompose:

$$\begin{aligned} & \mathbb{E} \int_0^T |r_t^\varepsilon(X_t^\varepsilon) - r_t^0(X_t^0)| dt \leq \mathbb{E} \int_0^T |g_t(X_t^\varepsilon)[f_t(X_t^\varepsilon)h_t(\alpha_t^\varepsilon(X_t^\varepsilon)) - f_t(X_t^0)h_t(\alpha_t^\varepsilon(X_t^0))] \\ & + \mathbb{E} \int_0^T |f_t(X_t^0)g_t(X_t^\varepsilon)[h_t(\alpha_t^\varepsilon(X_t^0)) - h_t(\alpha_t^0(X_t^0))] + |f_t(X_t^0)h_t(\alpha_t^0(X_t^0))[g_t(X_t^\varepsilon) - g_t(X_t^0)]| dt. \end{aligned}$$

Using the Cauchy-Schwarz inequality, the first term is bounded by

$$C \left[ \mathbb{E} \int_0^T |g_t(X_t^\varepsilon)|^2 dt \right]^{1/2} \sup_{0 \leq t \leq T} \|(h_t \circ \alpha_t^\varepsilon) f_t\|_\infty \|\mu_t^\varepsilon - \mu_t^0\|_{\text{TV}}^{1/2}.$$

This term is bounded by  $C\varepsilon^{1/4}$  using (5.38), (i) and Pinsker's inequality. Similarly, we bound the second term by

$$C \left[ \mathbb{E} \int_0^T |g_t(X_t^\varepsilon)|^2 dt \right]^{1/2} \left[ \mathbb{E} \int_0^T |\alpha_t^\varepsilon(X_t^0) - \alpha_t^0(X_t^0)|^2 dt \right]^{1/2},$$

using that  $h_t$  is locally Lipschitz and  $\alpha^\varepsilon$  is bounded. This quantity is smaller than  $C\varepsilon^{1/2}$  using (i) and the fact that  $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t^\varepsilon|^2] \leq C$ . Since  $g_t$  is Lipschitz and  $(h_t \circ \alpha_t^\varepsilon) f_t$  is bounded, the third term is bounded by  $C \sup_{0 \leq t \leq T} W_1(\mu_t^\varepsilon, \mu_t^0)$ , which we bound using (ii).  $\square$

In the following, we use the notation  $\psi_t^\varepsilon := \Psi(\mu_t^\varepsilon) + \varepsilon + \frac{\delta\Psi}{\delta\mu}(\mu_t^\varepsilon)$ . We recall that  $\frac{\delta\Psi}{\delta\mu}(\mu_t^\varepsilon) : x \mapsto \frac{\delta\Psi}{\delta\mu}(\mu_t^\varepsilon, x)$  is defined in Definition 5.1.1. The Lipschitz assumption (A.7)-(iii) and Lemma 5.4.4-(iii) imply that  $\Psi$  is Lipschitz in  $W_1$  (see e.g. [Cardaliaguet et al., 2019, Section 2.2.1]). From Lemma 5.4.4-(ii), we get that

$$\sup_{0 \leq s \leq T} |\Psi(\mu_s^\varepsilon) - \Psi(\mu_s^0)| \leq C\varepsilon^{1/2}. \quad (5.39)$$

Similarly,

$$\max_{0 \leq i \leq 4} \sup_{0 \leq s \leq T} \|\nabla^i \psi_s^\varepsilon - \nabla^i \psi_s^0\|_\infty \leq C\varepsilon^{1/2}. \quad (5.40)$$

As a consequence, we can readily adapt the proof of Lemma 5.4.4-(iii) to include the derivatives of  $\psi_t^\varepsilon$ .

**Lemma 5.4.5.** *Lemma 5.4.4-(iii) still holds if*

$$r_t^\varepsilon(x) = f_t(x)\psi_t^\varepsilon(x)h_t(\alpha_t^\varepsilon(x)) \quad \text{or} \quad r_t^\varepsilon(x) = f_t(x)g_t(x)h_t(\alpha_t^\varepsilon(x))k_t^\varepsilon(x),$$

where  $k_t^\varepsilon$  is any of the four derivatives of  $\psi_t^\varepsilon$ .

To make  $\lambda_t^\varepsilon$  explicit, we differentiate twice  $\Psi(\mu_t^\varepsilon)$ . We recall the definition (5.36) of the infinitesimal generator  $L_t^{\alpha^\varepsilon}$ .

**Lemma 5.4.6** (Second order regularity). *Under (A.7) there exists (a linear combination of terms)  $r^\varepsilon$  as in Lemma 5.4.5 such that for every  $0 \leq t < t' \leq T$ ,*

$$\begin{aligned} \mathbb{E}[L_{t'}^{\alpha^\varepsilon} \psi_{t'}^\varepsilon(X_{t'}^\varepsilon)] - \mathbb{E}[L_t^{\alpha^\varepsilon} \psi_t^\varepsilon(X_t^\varepsilon)] &= \mathbb{1}_{t=0} \lambda_0^\varepsilon \mathbb{E}[|\sigma_0^\top \nabla \psi_0^\varepsilon|^2(X_0^\varepsilon)] + \mathbb{1}_{t'=T} \lambda_T^\varepsilon \mathbb{E}[|\sigma_T^\top \nabla \psi_T^\varepsilon|^2(X_T^\varepsilon)] \\ &\quad + \int_t^{t'} \mathbb{E}[r_s^\varepsilon(X_s^\varepsilon) - s_s^\varepsilon(X_s^\varepsilon) + \lambda_s^\varepsilon |\sigma_s^\top \nabla \psi_s^\varepsilon|^2(X_s^\varepsilon)] ds, \end{aligned}$$

where

$$s_s^\varepsilon = a_s^{i,j} \nabla \partial_i \psi_s^\varepsilon \cdot a_s \nabla \partial_j \varphi_s^\varepsilon + \partial_i \psi_s^\varepsilon \nabla a_s^{i,j} \cdot a_s \nabla \partial_j \varphi_s^\varepsilon - \frac{1}{2} a_s^{i,j} \partial_j a_s^{k,l} \partial_i \psi_s^\varepsilon \partial_{k,l}^2 \varphi_s^\varepsilon.$$

As a consequence of Corollary 5.4.2.1-(ii),  $\mathbb{E}[L_t^{\alpha^\varepsilon} \psi_t^\varepsilon(X_t^\varepsilon)] = \frac{d}{dt} \Psi(\mu_t^\varepsilon)$  when  $t > 0$ , hence  $t \mapsto \Psi(\mu_t^\varepsilon)$  is a.e. twice differentiable, and  $C^2$  on intervals where  $\lambda_t^\varepsilon = 0$  identically.

*Proof.* Using Lemma 5.3.6, we can compute on a regularised version of the multiplier before to take the limit (and then recover the atoms at  $t \in \{0, T\}$ ). We thus assume that  $\lambda^\varepsilon$  is  $C^1$ , and we use Ito's formula to write that

$$\mathbb{E}[L_{t'}^{\alpha^\varepsilon} \psi_{t'}^\varepsilon(X_{t'}^\varepsilon)] - \mathbb{E}[L_t^{\alpha^\varepsilon} \psi_t^\varepsilon(X_t^\varepsilon)] = \int_t^{t'} \mathbb{E}[(\partial_s + L_s^{\alpha^\varepsilon}) L_s^{\alpha^\varepsilon} \psi_s^\varepsilon(X_s^\varepsilon)] ds.$$

In the following,  $r_s^\varepsilon$  is a remainder of the desired shape which may change from line one to another. Decomposing  $L_s^{\alpha^\varepsilon} = L_s^0 + \sigma_s \alpha_s^\varepsilon \cdot \nabla$ ,

$$\begin{aligned} (\partial_s + L_s^{\alpha^\varepsilon}) L_s^{\alpha^\varepsilon} \psi_s^\varepsilon &= (\partial_s + L_s^{\alpha^\varepsilon}) \sigma_s \alpha_s^\varepsilon \cdot \nabla \psi_s^\varepsilon + (\partial_s + L_s^{\alpha^\varepsilon}) L_s^0 \psi_s^\varepsilon \\ &= (\partial_s + L_s^{\alpha^\varepsilon}) \sigma_s \alpha_s^\varepsilon \cdot \nabla \psi_s^\varepsilon + r_s^\varepsilon. \end{aligned}$$

We now use coordinates:

$$\begin{aligned} (\partial_s + L_s^{\alpha^\varepsilon}) \sigma_s \alpha_s^\varepsilon \cdot \nabla \psi_s^\varepsilon &= (\partial_s + L_s^{\alpha^\varepsilon}) (\sigma_s \alpha_s^\varepsilon)^i \partial_i \psi_s^\varepsilon \\ &= \partial_i \psi_s^\varepsilon (\partial_s + L_s^{\alpha^\varepsilon}) (\sigma_s \alpha_s^\varepsilon)^i + (\sigma_s \alpha_s^\varepsilon)^i (\partial_s + L_s^{\alpha^\varepsilon}) \partial_i \psi_s^\varepsilon + \nabla \partial_i \psi_s^\varepsilon \cdot a_s \nabla (\sigma_s \alpha_s^\varepsilon)^i \\ &= (\partial_s + L_s^{\alpha^\varepsilon}) (\sigma_s \alpha_s^\varepsilon)^i + r_s^\varepsilon. \end{aligned}$$

Since  $(\sigma_s \alpha_s^\varepsilon)^i = -(a_s \nabla \varphi_s^\varepsilon)^i = -a_s^{i,j} \partial_j \varphi_s^\varepsilon$ , we further get

$$\begin{aligned} (\partial_s + L_s^{\alpha^\varepsilon}) (\sigma_s \alpha_s^\varepsilon)^i &= -\partial_j \varphi_s^\varepsilon (\partial_s + L_s^{\alpha^\varepsilon}) a_s^{i,j} - a_s^{i,j} (\partial_s + L_s^{\alpha^\varepsilon}) \partial_j \varphi_s^\varepsilon - \nabla a_s^{i,j} \cdot a_s \nabla \partial_j \varphi_s^\varepsilon \\ &= -a_s^{i,j} (\partial_s + L_s^{\alpha^\varepsilon}) \partial_j \varphi_s^\varepsilon + r_s^\varepsilon. \end{aligned}$$

We eventually differentiate the HJB equation satisfied by  $\varphi^\varepsilon$ :

$$(\partial_s + L_s^{\alpha^\varepsilon}) \partial_j \varphi_s^\varepsilon = -\partial_j b_s \cdot \nabla \varphi_s^\varepsilon + \frac{1}{2} \nabla \varphi_s^\varepsilon \partial_j a_s \nabla \varphi_s^\varepsilon - \frac{1}{2} \text{Tr}[\partial_j a_s \nabla^2 \varphi_s^\varepsilon] - \lambda_s^\varepsilon \partial_j \psi_s^\varepsilon.$$

Gathering everything, we get  $r^\varepsilon$  such that

$$\begin{aligned} (\partial_s + L_s^{\alpha^\varepsilon}) L_s^{\alpha^\varepsilon} \psi_s^\varepsilon &= r_s^\varepsilon - \nabla \partial_i \psi_s^\varepsilon \cdot a_s \nabla (a_s \nabla \varphi_s^\varepsilon)^i - \partial_i \psi_s^\varepsilon \nabla a_s^{i,j} \cdot a_s \nabla \partial_j \varphi_s^\varepsilon \\ &\quad + \frac{1}{2} a_s^{i,j} \partial_i \psi_s^\varepsilon \text{Tr}[\partial_j a_s \cdot \nabla^2 \varphi_s^\varepsilon] + \lambda_s^\varepsilon a_s^{i,j} \partial_i \psi_s^\varepsilon \partial_j \psi_s^\varepsilon. \end{aligned}$$

We eventually obtain the desired  $s^\varepsilon$  after a slight modification of  $r^\varepsilon$ .  $\square$

We will repeatedly use the following result.

**Lemma 5.4.7.** *For any global maximum or minimum  $t_0 \in (0, T)$  of  $s \mapsto \Psi(\mu_s^0)$ , we have*

$$\left| \frac{d}{ds} \Big|_{s=t_0} \Psi(\mu_s^\varepsilon) \right| \leq C\varepsilon^{1/4},$$

for  $C > 0$  that does not depend on  $\varepsilon$  nor  $t_0$ .

The result still holds if we exchange the roles of  $\Psi(\mu_s^\varepsilon)$  and  $\Psi(\mu_s^0)$ .

*Proof.* We prove the result for a maximum, the case of a minimum being very similar. For any  $\beta, \gamma > 0$ ,

$$\forall s \in [0, T] \setminus [t_0 - \beta, t_0 + \beta], \quad \Psi(\mu_{t_0}^0) \geq \Psi(\mu_s^0) - \gamma(s - t_0)^2 + \gamma\beta^2.$$

Using (5.39), this implies

$$\forall s \in [0, T] \setminus [t_0 - \beta, t_0 + \beta], \quad \Psi(\mu_{t_0}^\varepsilon) \geq \Psi(\mu_s^\varepsilon) - \gamma(s - t_0)^2 + \gamma\beta^2 - 2C\varepsilon^{1/2}.$$

If  $\gamma\beta^2 \geq 2C\varepsilon^{1/2}$ , this proves that  $s \mapsto \Psi(\mu_s^\varepsilon) - \gamma(s - t_0)^2$  has a global maximum at some point  $t'_0 \in [t_0 - \beta, t_0 + \beta]$ . As a consequence  $\frac{d}{ds} \Big|_{s=t'_0} \Psi(\mu_s^\varepsilon) = 2\gamma(t'_0 - t_0)$ . From Lemma 5.4.6,  $\frac{d^2}{ds^2} \Psi(\mu_s^\varepsilon)$  is bounded on  $[0, T]$  by some  $K > 0$  independent of  $\varepsilon$ , hence

$$\left| \frac{d}{ds} \Big|_{s=t_0} \Psi(\mu_s^\varepsilon) \right| \leq 2\beta\gamma + K\beta,$$

for any  $\beta, \gamma > 0$  such that  $\gamma\beta^2 \geq 2C\varepsilon^{1/2}$ . Optimising over  $(\beta, \gamma)$  concludes.  $\square$

By continuity of  $t \mapsto \Psi(\mu_t^\varepsilon)$ , the sets  $F_\varepsilon := \{t \in [0, T], \Psi(\mu_t^\varepsilon) = \varepsilon\}$  are closed. Moreover,  $F_\varepsilon$  is a countable union of singletons and non-trivial intervals, and we know that  $F_\varepsilon$  contains the support of  $\lambda^\varepsilon$  from the complementary slackness condition (5.4) with  $\Psi - \varepsilon$  in place of  $\Psi$ . From (A.4) and Remark 5.2.5, the continuous function  $s \mapsto \mathbb{E}[|\sigma_s^\top \nabla \psi_s^0(\bar{X}_s)|^2]$  is positive on the closed set  $F_0$ , hence lower-bounded by a positive number. Using this and Lemma 5.4.4-(ii), there exists  $\eta > 0$  such that for every small enough  $\varepsilon$ ,

$$\forall t \in F_\varepsilon, \quad \mathbb{E}[|\sigma_t^\top \nabla \psi_t^\varepsilon|^2(X_t^\varepsilon)] \geq \eta > 0. \quad (5.41)$$

We now prove the main stability estimate on  $\lambda^\varepsilon$ .

**Proposition 5.4.8.** *There exists  $C > 0$  independent of  $\varepsilon$  such that  $\|\lambda^\varepsilon - \lambda^0\|_{L^1(0, T)} \leq C\varepsilon^{1/4}$ .*

*Proof.* We decompose the norm using  $F_\varepsilon$  and  $F_0$ .

**Step 1.** *On  $F_\varepsilon \cap F_0$ .* We first write that

$$\begin{aligned} |\mathbb{E}[|\lambda_t^\varepsilon| \sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2] - \lambda_t^0 | \sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2| &\leq |\mathbb{E}[|\lambda_t^\varepsilon| \sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2] - \lambda_t^0 | \sigma_t^\top \nabla \psi_t^0(X_t^0)|^2| \\ &\quad + \lambda_t^0 |\mathbb{E}[|\sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2] - |\sigma_t^\top \nabla \psi_t^0(X_t^0)|^2|. \end{aligned}$$

Moreover,

$$\begin{aligned} |\mathbb{E}[|\sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2] - |\sigma_t^\top \nabla \psi_t^0(X_t^0)|^2| &\leq |\mathbb{E}[|\sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2] - |\sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2| \\ &\quad + |\mathbb{E}[|\sigma_t^\top \nabla \psi_t^\varepsilon(X_t^0)|^2] - |\sigma_t^\top \nabla \psi_t^0(X_t^0)|^2| \leq C\varepsilon^{1/2}, \end{aligned}$$

combining Lemma 5.4.4-(ii) and (5.40) for the second inequality. Using (5.41) and the bound on  $\lambda_t^0$  given by Theorem 5.2.3, this implies

$$\eta|\lambda_t^\varepsilon - \lambda_t^0| \leq |\mathbb{E}[\lambda_t^\varepsilon |\sigma_t^\top \nabla \psi_t^\varepsilon(X_t^\varepsilon)|^2 - \lambda_t^0 |\sigma_t^\top \nabla \psi_t^0(X_t^0)|^2]| + C\varepsilon^{1/2}.$$

By continuity of  $\Psi$ ,  $F_\varepsilon \cap F_0$  is a (at most) countable union of closed disjoint intervals.  $\Psi(\mu_s^\varepsilon)$  and  $\Psi(\mu_s^0)$  are identically 0 on  $F_\varepsilon \cap F_0$ , hence their second derivative is a.e. 0 on  $F_\varepsilon \cap F_0$ . From Lemmata 5.4.2.1 and 5.4.6,

$$0 = \frac{d^2}{ds^2} \Psi(\mu_s^\varepsilon) = \mathbb{E}[r_s^\varepsilon(X_s^\varepsilon) - s_s^\varepsilon(X_s^\varepsilon) + \lambda_s^\varepsilon |\sigma_s^\top \nabla \psi_s^\varepsilon|^2(X_s^\varepsilon)] \quad \text{for a.e. } s \in F_0 \cap F_\varepsilon,$$

hence

$$\eta|\lambda_t^\varepsilon - \lambda_t^0| \leq |\mathbb{E}[r_t^\varepsilon(X_t^\varepsilon) - r_t^0(X_t^0)]| + |\mathbb{E}[s_t^\varepsilon(X_t^\varepsilon) - s_t^0(X_t^0)]|. \quad (5.42)$$

The term  $s_s^\varepsilon$  can be written as a sum of terms of shape  $m_s^\varepsilon \cdot \nabla \partial_j \varphi_s^\varepsilon$ , where  $m^\varepsilon$  is uniformly bounded with bounded  $\partial_j$ -derivative. However,  $\mathbb{E}[\nabla \partial_j \varphi_s^\varepsilon(X_s^\varepsilon) - \nabla \partial_j \varphi_s^0(X_s^0)]$  cannot be controlled using Lemma 5.4.5. To circumvent this, we integrate by parts:

$$\begin{aligned} \mathbb{E}[m_s^0 \cdot \nabla \partial_j \varphi_s^0(X_s^0)] &= -\mathbb{E}[\partial_j m_s^0 \cdot \nabla \varphi_s^0(X_s^0)] - \mathbb{E}[(m_s^0 \cdot \nabla \varphi_s^0) \partial_j \log \mu_s^0(X_s^0)] \\ &= -\mathbb{E}[\partial_j m_s^0 \cdot \nabla \varphi_s^0(X_s^0)] - \mathbb{E}[(m_s^0 \cdot \nabla \varphi_s^\varepsilon) \partial_j \log \mu_s^0(X_s^0)] + \mathbb{E}[m_s^0 \cdot (\nabla \varphi_s^\varepsilon - \nabla \varphi_s^0) \partial_j \log \mu_s^0(X_s^0)] \\ &= \mathbb{E}[m_s^0 \cdot \nabla \partial_j \varphi_s^\varepsilon(X_s^0)] + \mathbb{E}[\partial_j m_s^0 \cdot (\nabla \varphi_s^\varepsilon - \nabla \varphi_s^0)(X_s^0)] + \mathbb{E}[m_s^0 \cdot (\nabla \varphi_s^\varepsilon - \nabla \varphi_s^0) \partial_j \log \mu_s^0(X_s^0)]. \end{aligned}$$

Using the Cauchy-Schwarz inequality and Lemmata 5.3.4-5.A.2 to bound  $\mathbb{E}[|\nabla \log \mu_s^0(X_s^0)|^2]$ ,

$$\begin{aligned} &\int_0^T |\mathbb{E}[m_s^0 \cdot (\nabla \varphi_s^\varepsilon - \nabla \varphi_s^0) \partial_j \log \mu_s^0(X_s^0)]| ds \\ &\leq \|m^0\|_\infty \left\{ \int_0^T \mathbb{E}[|\nabla \log \mu_s^0(X_s^0)|^2] ds \right\}^{1/2} \left\{ \int_0^T \mathbb{E}[|\nabla \varphi_s^\varepsilon(X_s^0) - \nabla \varphi_s^0(X_s^0)|^2] ds \right\}^{1/2} \leq C\varepsilon^{1/2}, \end{aligned}$$

Lemma 5.4.4-(i) giving the last bound. We similarly show that

$$\int_0^T |\mathbb{E}[\partial_j m_s^0 \cdot (\nabla \varphi_s^\varepsilon - \nabla \varphi_s^0)(X_s^0)]| ds \leq C\varepsilon^{1/2}.$$

We can now use Lemma 5.4.5 to write

$$\int_0^T |\mathbb{E}[m_s^\varepsilon \cdot \nabla \partial_j \varphi_s^\varepsilon(X_s^\varepsilon) - m_s^0 \cdot \nabla \partial_j \varphi_s^\varepsilon(X_s^0)]| ds \leq C\varepsilon^{1/4}.$$

Gathering everything, we showed that

$$\int_0^T |\mathbb{E}[m_s^\varepsilon \cdot \nabla \partial_j \varphi_s^\varepsilon(X_s^\varepsilon) - m_s^0 \cdot \nabla \partial_j \varphi_s^0(X_s^0)]| ds \leq C\varepsilon^{1/4},$$

hence the same holds when replacing  $m_t^\varepsilon$  by  $s_t^\varepsilon$ . Going back to (5.42), Lemma 5.4.5 now gives that  $\int_{F_\varepsilon \cap F_0} |\lambda_s^\varepsilon - \lambda_s^0| ds \leq C\varepsilon^{1/4}$ .

**Step 2.** On  $F_\varepsilon^c \cap F_0$ . Using the complementary slackness condition

$$\int_{(F_\varepsilon \cap F_0)^c} |\lambda_s^\varepsilon - \lambda_s^0| ds = \int_{F_\varepsilon^c \cap F_0} \lambda_s^0 ds + \int_{F_\varepsilon \cap F_0^c} \lambda_s^\varepsilon ds. \quad (5.43)$$

Since  $t \mapsto \Psi(\mu_t^\varepsilon)$  and  $t \mapsto \Psi(\mu_t^0)$  are continuous,  $[0, T] \setminus (F_\varepsilon \cap F_0)$  is a (at most) countable union of disjoint intervals. Let  $I = (t, t')$  be one them that is not trivial.

If  $0 < t < t' < T$ , then  $\Psi(\mu_t^\varepsilon) = \Psi(\mu_{t'}^\varepsilon) = \varepsilon$  and  $\Psi(\mu_t^0) = \Psi(\mu_{t'}^0) = 0$  by definition of  $(F_\varepsilon \cap F_0)^c$ . From Lemma 5.4.2.1,  $s \mapsto \frac{d}{ds}\Psi(\mu_s^\varepsilon)$  and  $s \mapsto \frac{d}{ds}\Psi(\mu_s^0)$  are  $C^1$  at  $t$  and  $t'$ , hence they vanish there ( $t$  and  $t'$  are local maxima). We then set  $t_0 := t$  and  $t'_0 := t'$ . Otherwise, if  $t' = 0$  or  $t = T$ , we consider  $t_0, t'_0 \in I$  such that  $0 < t_0 < t'_0 < T$  and  $\Psi(\mu_{t_0}^0) = \Psi(\mu_{t'_0}^0) = 0$ .

Let us apply Lemma 5.4.7 at  $t_0$  and  $t'_0$ . Using Lemma 5.4.6 on  $[t_0, t'_0]$ , reasoning as in **Step 1.** yields

$$\int_{t_0}^{t'_0} \lambda_s^0 ds - \int_{t_0}^{t'_0} \lambda_s^\varepsilon ds \leq C\varepsilon^{1/4},$$

hence

$$\int_{t_0}^{t'_0} \lambda_s^0 ds \leq C\varepsilon^{1/4} + \int_I \lambda_s^\varepsilon ds.$$

Since this holds for every  $[t_0, t'_0] \subset I$  with  $\Psi(\mu_{t_0}^0) = \Psi(\mu_{t'_0}^0) = 0$ , we deduce that

$$\int_I \lambda_s^0 ds \leq C\varepsilon^{1/4} + \int_I \lambda_s^\varepsilon ds,$$

recalling the complementary slackness condition (5.4). To bound  $\int_{F_\varepsilon^c \cap F_0} \lambda_s^0 ds$ , it is thus sufficient to control  $\int_{F_\varepsilon \cap F_0^c} \lambda_s^\varepsilon ds$ . This is the content of the next step.

**Step 3.** On  $F_\varepsilon \cap F_0^c$ . We now use the assumption that there are a finite number of non-trivial intervals where  $\Psi(\mu_s^0) = 0$  identically. Let  $[a_k, b_k]$  be these intervals in an increasing order, with  $a_k < b_k$  and  $1 \leq k \leq m$ . We further set  $b_0 := 0$  and  $a_{m+1} := T$  so that

$$\int_{F_\varepsilon \cap F_0^c} \lambda_s^\varepsilon ds = \int_{F_0^c} \lambda_s^\varepsilon ds = \sum_{k=0}^m \int_{b_k}^{a_{k+1}} \lambda_s^\varepsilon ds.$$

For  $0 \leq k \leq m$ , let us consider  $b_k < t_k < t'_k < a_{k+1}$  with  $\Psi(\mu_{t_k}^\varepsilon) = \Psi(\mu_{t'_k}^\varepsilon) = \varepsilon$ . Indeed, if such a  $(t_k, t'_k)$  does not exist,  $\int_{b_k}^{a_{k+1}} \lambda_s^\varepsilon ds = 0$  from the complementary slackness condition (5.4) with  $\Psi - \varepsilon$  in place of  $\Psi$ . We then write

$$\int_{t_k}^{t'_k} \lambda_s^\varepsilon ds = \int_{t_k}^{t'_k} \lambda_s^\varepsilon ds - \int_{t_k}^{t'_k} \lambda_s^0 ds,$$

and we apply Lemma 5.4.7 at  $t_k$  and  $t'_k$ , as in **Step 2.** Using Lemma 5.4.6 on  $[t_k, t'_k]$ , reasoning as in **Step 1.** shows that the r.h.s. is bounded by  $C\varepsilon^{1/4}$ . Since this holds for every  $[t_k, t'_k] \subset [b_k, a_{k+1}]$  with  $\Psi(\mu_{t_k}^\varepsilon) = \Psi(\mu_{t'_k}^\varepsilon) = \varepsilon$ , we deduce that

$$\int_{b_k}^{a_{k+1}} \lambda_s^\varepsilon ds \leq C\varepsilon^{1/4},$$

recalling the complementary slackness condition. Gluing the steps together in (5.43) concludes.  $\square$

**Remark 5.4.9** (Infinite number of intervals). *Without assuming that the number of  $[a_k, b_k]$  is finite, we can still write that*

$$\int_{F_\varepsilon \cap F_0^c} \lambda_s^\varepsilon ds = \sum_{k \geq 0} \int_{I_k} \lambda_s^\varepsilon ds,$$



the sum being possibly countable with no more required ordering on the  $I_k$ . For each  $k$ , let us consider a maximal  $[t_k, t'_k] \subset I_k$  such that  $\Psi(\mu_{t_k}^\varepsilon) = \Psi(\mu_{t'_k}^\varepsilon) = \varepsilon$ . Using Lemma 5.4.6 and **Step 1.**, the problem boils down to controlling

$$\sum_{k \geq 0} \frac{d}{ds} \Big|_{s=t_k} \Psi(\mu_s^0) - \frac{d}{ds} \Big|_{s=t'_k} \Psi(\mu_s^0). \quad (5.44)$$

From Lemma 5.4.6,  $s \mapsto \Psi(\mu_s^0)$  is  $C^2$  on each  $I_k$ . Let us assume that  $\frac{d^2}{ds^2} \Psi(\mu_s^0)$  has a uniform continuity modulus  $m_\Psi$  on the union of these intervals: for any  $\eta > 0$ ,

$$\forall t, t' \in \cup_k I_k, \quad |t - t'| \leq m_\Psi(\eta) \Rightarrow \left| \frac{d^2}{ds^2} \Big|_{s=t} \Psi(\mu_s^0) - \frac{d^2}{ds^2} \Big|_{s=t'} \Psi(\mu_s^0) \right| \leq \eta.$$

We can then split the sum (5.44) between intervals of size smaller and bigger than  $m_\Psi(\eta)$ . On intervals  $I_k$  smaller than  $m_\Psi(\eta)$ ,  $\left| \frac{d^2}{ds^2} \Psi(\mu_s^0) \right|$  is smaller than  $\eta$  because  $\frac{d^2}{ds^2} \Psi(\mu_s^0)$  vanishes within  $I_k$ ; this is indeed a consequence of Rolle's theorem because  $\frac{d}{ds} \Psi(\mu_s^0)$  vanishes at the boundary of  $I_k$  (except possibly for the first and the last  $I_k$ ). On intervals bigger than  $m_\Psi(\eta)$ , we apply the error bound of **Step 3.** Since there are at most  $\lfloor T m_\Psi^{-1}(\eta) \rfloor + 1$  such intervals within  $[0, T]$ , we can bound (5.44) by

$$\frac{C}{m_\Psi(\eta)} \varepsilon^{1/4} + \sum_{a_{k+1} - b_k \leq m_\Psi(\eta)} \eta [t'_k - t_k] \leq C [m_\Psi^{-1}(\eta) \varepsilon^{1/4} + \eta]. \quad (5.45)$$

Such a continuity modulus  $m_\Psi$  can be obtained from Lemma 5.4.6 by differentiating once again to get third-order time-regularity. Strengthening a bit the regularity assumptions on coefficients, we could thus bound  $\frac{d^3}{ds^3} \Psi(\mu_s^0)$  and optimise over  $\eta$  in (5.45), to get that (5.44) is of order  $\varepsilon^{1/8}$ .

*Proof of Theorem 5.2.6.* Let us first handle the convergence of the atom at 0. If  $\Psi(\mu_0^0) < 0$ , then the same holds for  $\Psi(\mu_0^\varepsilon) < 0$  for  $\varepsilon$  small enough, hence  $\lambda_0^\varepsilon = \lambda_0^0 = 0$  and there is nothing to do. We thus assume  $\Psi(\mu_0^0) = 0$ . Reasoning as in **Step 1.** in the proof of Proposition 5.4.8,

$$\eta |\lambda_0^\varepsilon - \lambda_0^0| \leq |\mathbb{E}[\lambda_0^\varepsilon |\sigma_0^\top \nabla \psi_0^\varepsilon(X_0^\varepsilon)|^2 - \lambda_0^0 |\sigma_0^\top \nabla \psi_0^0(X_0^0)|^2]| + C \varepsilon^{1/2}.$$

The first term on the r.h.s can be computed using Lemma 5.4.6, so that

$$\begin{aligned} \eta |\lambda_0^\varepsilon - \lambda_0^0| &\leq |\mathbb{E}[L_0^{\alpha^\varepsilon} \psi_0^\varepsilon(X_0^\varepsilon) - L_0^{\alpha^0} \psi_0^0(X_0^0)]| + \left| \frac{d}{dt} \Big|_{t=t'} \Psi(\mu_t^\varepsilon) \right| + \int_0^{t'} |\mathbb{E}[r_s^\varepsilon(X_s^\varepsilon) - r_s^0(X_s^0)]| ds \\ &+ C \varepsilon^{1/2} + \int_0^{t'} |\mathbb{E}[s_s^\varepsilon(X_s^\varepsilon) - s_s^0(X_s^0)]| + |\mathbb{E}[\lambda_s^\varepsilon |\sigma_s^\top \nabla \psi_s^\varepsilon|^2(X_s^\varepsilon) - \lambda_s^0 |\sigma_s^\top \nabla \psi_s^0|^2(X_s^0)]| ds, \end{aligned} \quad (5.46)$$

for any  $t' \in (0, T)$  with  $\frac{d}{dt} \Big|_{t=t'} \Psi(\mu_t^\varepsilon) = 0$ . The  $|\mathbb{E}[r_s^\varepsilon(X_s^\varepsilon) - r_s^0(X_s^0)]|$  term can be bounded using Lemma 5.3.6-(iii). We deal with the  $|\mathbb{E}[s_s^\varepsilon(X_s^\varepsilon) - s_s^0(X_s^0)]|$  as previously in **Step 1.** For the remaining terms, we distinguish between two cases.

1. Let us assume that  $t \mapsto \Psi(\mu_t^0)$  has a global minimum at  $t' \in (0, T)$ , so that  $\frac{d}{dt} \Big|_{t=t'} \Psi(\mu_t^0) = 0$ . Using Lemma 5.4.7, we get that  $\left| \frac{d}{dt} \Big|_{t=t'} \Psi(\mu_t^\varepsilon) \right| \leq C \varepsilon^{1/4}$ . It remains to control the difference  $\mathbb{E}[L_0^{\alpha^\varepsilon} \psi_0^\varepsilon(X_0^\varepsilon)] - \mathbb{E}[L_0^{\alpha^0} \psi_0^0(X_0^0)]$ . Since

$$L_0^{\alpha^\varepsilon} \psi_0^\varepsilon = L_0^{\alpha^0} \psi_0^0 + a_0 \nabla \varphi_0^\varepsilon \cdot \nabla \psi_0^\varepsilon, \quad X_0^\varepsilon \sim Z_\varepsilon^{-1} e^{-\varphi_0^\varepsilon} d\nu_0,$$

we notice that

$$\mathbb{E}[L_0^{\alpha^\varepsilon} \psi_0^\varepsilon(X_0^\varepsilon)] = \mathbb{E}[L_0^0 \psi_0^\varepsilon(X_0^\varepsilon)] + Z_\varepsilon^{-1} \int_{\mathbb{R}^d} -a_0 \nabla[e^{-\varphi_0^\varepsilon}] \cdot \nabla \psi_0^\varepsilon d\nu_0.$$

Integrating by parts:

$$Z_\varepsilon^{-1} \int_{\mathbb{R}^d} -a_0 \nabla[e^{-\varphi_0^\varepsilon}] \cdot \nabla \psi_0^\varepsilon d\nu_0 = \mathbb{E}[\nabla \cdot (a_0 \nabla \psi_0^\varepsilon)(X_0^\varepsilon)] + \mathbb{E}[a_0 \nabla \psi_0^\varepsilon \cdot \nabla \log \nu_0(X_0^\varepsilon)].$$

The first term on the r.h.s generates the difference  $\mathbb{E}[\nabla \cdot (a_0 \nabla \psi_0^\varepsilon)(X_0^\varepsilon) - \nabla \cdot (a_0 \nabla \psi_0^0)(X_0^0)]$  which enters the scope of Lemma 5.4.5. To handle the difference of the second second terms, we first write that

$$\mathbb{E}[|\nabla \log \nu_0(X_0^\varepsilon) - \nabla \log \nu_0(X_0^0)| \mathbb{1}_{|\nabla \log \nu_0(X_0^\varepsilon)| + |\nabla \log \nu_0(X_0^0)| \leq M}] \leq CM\varepsilon^{1/2},$$

using Lemma 5.4.4-(i) and Pinsker's inequality. We then use that

$$\begin{aligned} \mathbb{E}[|\nabla \log \nu_0(X_0^\varepsilon) - \nabla \log \nu_0(X_0^0)| \mathbb{1}_{|\nabla \log \nu_0(X_0^\varepsilon)| + |\nabla \log \nu_0(X_0^0)| > M}] \\ \leq M^{-1} \{ \mathbb{E}[|\nabla \log \nu_0(X_0^\varepsilon)|^2 + |\nabla \log \nu_0(X_0^0)|^2] \}^{3/2}, \end{aligned}$$

using the Cauchy-Schwarz and the Markov inequalities. We further bound the expectation using (A.5) and Lemma 5.3.4. Optimising in  $M$ , we get that

$$\mathbb{E}[|\nabla \log \nu_0(X_0^\varepsilon) - \nabla \log \nu_0(X_0^0)|] \leq C\varepsilon^{1/4}.$$

Gathering everything, we deduce that

$$|\mathbb{E}[L_0^{\alpha^\varepsilon} \psi_0^\varepsilon(X_0^\varepsilon) - L_0^0 \psi_0^0(X_0^0)]| \leq C\varepsilon^{1/4}.$$

Going back to (5.46) and using Lemma 5.4.5, we eventually obtain that  $|\lambda_0^\varepsilon - \lambda_0^0| \leq C\varepsilon^{1/4}$ .

2. If such a minimiser  $t'$  does not exist in  $(0, T)$ ,  $t \mapsto \Psi(\mu_t^0)$  is necessarily (strictly) decreasing hence  $\Psi(\mu_T^0) < 0$ , and  $\Psi(\mu_T^\varepsilon) < 0$  for small enough  $\varepsilon$ . We can then set  $t' = T$ , and (5.46) still holds using Lemma 5.4.6 because

$$\frac{d}{dt} \Big|_{t=T} \Psi(\mu_t^\varepsilon) = \lambda_T^\varepsilon \mathbb{E}[|\sigma_T^\top \nabla \psi_T^\varepsilon|^2(X_T^\varepsilon)] = 0,$$

from the complementary slackness. Reasoning as above then yields  $|\lambda_0^\varepsilon - \lambda_0^0| \leq C\varepsilon^{1/4}$ .

Since we now control  $|\lambda_0^\varepsilon - \lambda_0^0|$  and  $\|\lambda^\varepsilon - \lambda^0\|_{L^1(0, T)}$ , Corollary 5.4.2.1-(ii) and Lemma 5.4.6 yield

$$\forall t \in [0, T), \quad \left| \frac{d}{dt} \Psi(\mu_t^\varepsilon) - \frac{d}{dt} \Psi(\mu_t^0) \right| \leq C\varepsilon^{1/4},$$

for  $C$  independent of  $(t, \varepsilon)$ . From Corollary 5.4.2.1-(ii),  $t \mapsto \Psi(\mu_t^\varepsilon)$  is  $C^1$  hence this identity still holds for  $t = T$ . Since  $\frac{d}{dt} \Big|_{t=T} \Psi(\mu_t^\varepsilon) = \lambda_T^\varepsilon \mathbb{E}[|\sigma_T^\top \nabla \psi_T^\varepsilon|^2(X_T^\varepsilon)]$ , we obtain using (5.41) that  $|\lambda_T^\varepsilon - \lambda_0^0| \leq C\varepsilon^{1/4}$ .

Since  $\lambda^\varepsilon - \lambda^0$  is now fully estimated, we can introduce  $u^\varepsilon := \varphi^\varepsilon - \varphi^0$ , which satisfies

$$-u_t^\varepsilon + \int_t^T b_s^\varepsilon \cdot \nabla u_s^\varepsilon + \frac{1}{2} \text{Tr}[a_s \nabla^2 u_s^\varepsilon] ds + \int_{[t, T]} \psi_s^\varepsilon \lambda^\varepsilon(ds) - \int_{[t, T]} \psi_s^0 \lambda^0(ds) = 0,$$

in the sense of Definition 5.C.1, where  $b_t^\varepsilon := b_t - \frac{1}{2}a_t \nabla[\varphi_t^\varepsilon + \varphi_t^0]$ . As for Proposition 5.3.2, we can use a regularisation procedure to obtain the Feynman-Kac formula

$$u_t^\varepsilon(x) = \mathbb{E} \left[ \int_{[t,T]} \psi_s^\varepsilon(Y_s^{\varepsilon,t,x}) \lambda^\varepsilon(ds) - \int_{[t,T]} \psi_s^0(Y_s^{\varepsilon,t,x}) \lambda^0(ds) \right],$$

using the solution to the following SDE,

$$dY_s^{\varepsilon,t,x} = b_s^\varepsilon(Y_s^{\varepsilon,t,x})ds + \sigma_s(Y_s^{\varepsilon,t,x})dB_s, \quad Y_t^{\varepsilon,t,x} = x.$$

We can then estimate  $|u_t^\varepsilon(x) - u_t^\varepsilon(y)|$  using the gradient estimate from Priola and Wang [2006] as in Proposition 5.3.2. Indeed,  $\nabla b_t^\varepsilon$  is bounded uniformly in  $(\varepsilon, t)$  using (5.38), hence we get uniform in  $\varepsilon$  estimates. Using (5.40) and our estimate of  $\lambda^\varepsilon - \lambda^0$ , this yields  $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |\nabla u_t^\varepsilon(x)| \leq C\varepsilon^{1/4}$  as desired.  $\square$

## Appendix

### 5.A Time-reversal and density estimates

In this section, we prove bounds on the density  $\mu_t$  of the solution to

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t.$$

Lemmata 5.A.1 and 5.A.2 can be rephrased in terms of entropy and Fisher information: they are non-quantitative versions of results from Bakry et al. [2014]. Contrary to the classical semi-group approach, our estimates rely on semi-martingale decompositions in the spirit of Fontbona and Jourdain [2016]. They provide probabilistic counterparts to PDE results of [Bogachev et al., 2022, Chapter 7.4]. To the best of our knowledge, the third order estimate in Proposition 5.A.4 is new.

We work under global Lipschitz assumptions which are not optimal, see Haussmann and Pardoux [1986]; Fontbona and Jourdain [2016] for sharper discussions. From [Haussmann and Pardoux, 1986, Theorem 3.1], (A.2) is enough for existence of a density  $\mu_t \in H^1(\mathbb{R}^d)$ . In the forthcoming proofs, we assume that coefficients are smooth so that  $\mu_t$  is the smooth positive solution of  $\partial_t \mu_t = L_t^* \mu_t$ . To get the estimates with the desired regularity, it is classically sufficient to regularise coefficients and to take the limit.

We define  $\overleftarrow{X}_t := X_{T-t}$ . From [Haussmann and Pardoux, 1986, Theorem 2.1],  $(\overleftarrow{X}_t)_{0 \leq t \leq T}$  is a time-inhomogeneous Markov process with generator

$$\overleftarrow{L}_t := \overleftarrow{b}_t^j \partial_j + \frac{1}{2} \overleftarrow{a}_t^{j,k} \partial_{j,k}^2,$$

where

$$\overleftarrow{b}_t^j := -b_{T-t}^j + \mu_{T-t}^{-1} \partial_k (a_{T-t}^{j,k} \mu_{T-t}), \quad \overleftarrow{a}_t^{j,k} := a_{T-t}^{j,k}.$$

Let  $\overleftarrow{\mu}_t := \mu_{T-t}$  be the density of the law of  $\overleftarrow{X}_t$  at time  $t$ . We then set  $\eta_t := \log \overleftarrow{\mu}_t$ . After simplifications, the Fokker-Planck equation  $\partial_t \overleftarrow{\mu}_t = \overleftarrow{L}_t^* \overleftarrow{\mu}_t$  yields

$$(\partial_t + \overleftarrow{L}_t) \eta_t = \partial_j b_{T-t}^j - \frac{1}{2} \partial_{j,k} \overleftarrow{a}_t^{j,k} + \frac{1}{2} \overleftarrow{a}_t^{j,k} \partial_j \eta_t \partial_k \eta_t. \quad (5.47)$$

We then define  $Y_t^i := \partial_i \eta_t(\overleftarrow{X}_t)$ , for  $1 \leq i \leq d$ . In all what follows, we underly evaluation at  $\overleftarrow{X}_t$ , writing e.g.  $d\eta_t$  instead of  $d\eta_t(\overleftarrow{X}_t)$ .

**Lemma 5.A.1.** *Under (A.2), if  $\mu_0$  satisfies (A.1) then*

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^d} \log \mu_t \, d\mu_t + \int_0^T \int_{\mathbb{R}^d} |\nabla \log \mu_t|^2 \, d\mu_t \, dt \leq C,$$

where  $C > 0$  only depends on coefficients through their uniform norm or the one or their derivatives.

*Proof.* Using Ito's formula and (5.47),

$$d\eta_t = [\partial_j b_{T-t}^j + \frac{1}{2} \overleftarrow{a}_t^{j,k} Y_t^j Y_t^k - \frac{1}{2} \partial_{j,k} \overleftarrow{a}_t^{j,k}] dt + Y_t^j \overleftarrow{\sigma}_t^{j,k} dB_t^k. \quad (5.48)$$

Let  $\rho(x) := Z^{-1} \exp[-\sqrt{x^2 + 1}]$  be a probability density w.r.t. Lebesgue. A similar computation yields

$$\begin{aligned} d \log \frac{d\overleftarrow{\mu}_t}{d\rho} &= \frac{1}{2} \overleftarrow{a}_t^{j,k} Y_t^j Y_t^k dt + \partial_j \log \frac{d\overleftarrow{\mu}_t}{d\rho} \overleftarrow{\sigma}_t^{j,k} dB_t^k \\ &+ [\partial_j b_{T-t}^j - \frac{1}{2} \partial_{j,k} \overleftarrow{a}_t^{j,k} + b_{T-t}^j \partial_j \log \rho - \frac{1}{2} \partial_k \overleftarrow{a}_t^{j,k} \partial_j \log \rho - \overleftarrow{a}_t^{j,k} Y_t^k \partial_j \log \rho + \frac{1}{2} \overleftarrow{a}_t^{j,k} \partial_{j,k} \log \rho] dt. \end{aligned}$$

We then integrate in time and we take expectations. We integrate by parts to get rid of second order derivatives on  $\overleftarrow{a}_t^{j,k}$ :

$$\mathbb{E}[-\frac{1}{2} \partial_{j,k} \overleftarrow{a}_t^{j,k}] = \mathbb{E}[\frac{1}{2} \partial_j \overleftarrow{a}_t^{j,k} Y_t^k] \geq -\frac{\varepsilon}{2} \mathbb{E}[Y_t^k Y_t^k] - \frac{\varepsilon^{-1}}{2} \mathbb{E}[\partial_j \overleftarrow{a}_t^{j,k} \partial_j \overleftarrow{a}_t^{j,k}],$$

for  $\varepsilon$  small enough. Using (A.1) on  $\mu_0$ ,  $\int \log \frac{d\overleftarrow{\mu}_T}{d\rho} d\overleftarrow{\mu}_T = \int \log \frac{d\mu_0}{d\rho} d\mu_0$  is finite. Similarly,

$$\mathbb{E}[-\overleftarrow{a}_t^{j,k} Y_t^k \partial_j \rho] \geq -\frac{\varepsilon}{2} \mathbb{E}[Y_t^k Y_t^k] - \frac{\varepsilon^{-1}}{2} \mathbb{E}[\overleftarrow{a}_t^{j,k} \partial_j \rho \overleftarrow{a}_t^{j,k} \partial_j \rho]. \quad (5.49)$$

Using (A.2)-(iii),  $\overleftarrow{a}_t^{j,k} Y_t^j Y_t^k \geq C^{-1} Y_t^k Y_t^k$  for some  $C > 0$ , and we choose  $\varepsilon$  such that  $C^{-1} > 2\varepsilon$ : we thus obtain a bound on  $\mathbb{E}[\overleftarrow{a}_t^{j,k} Y_t^j Y_t^k]$ . This in turn gives a bound on  $\int \log \frac{d\overleftarrow{\mu}_0}{d\rho} d\overleftarrow{\mu}_0 = \int \log \frac{d\mu_T}{d\rho} d\mu_T$  follows, and then on  $\int \log \mu_T d\mu_T$ . Integrating and taking expectation in (5.48), the bound on  $\int \log \mu_t d\mu_t = \mathbb{E}[\eta_{T-t}]$  follows by a Gronwall argument.  $\square$

We now differentiate (5.47). After simplifications:

$$\begin{aligned} (\partial_t + \overleftarrow{L}_t) \partial_i \eta_t &= \partial_i b_{T-t}^j \partial_j \eta_t - \partial_{i,k} \overleftarrow{a}_t^{j,k} \partial_j \eta_t - \frac{1}{2} \partial_i \overleftarrow{a}_t^{j,k} \partial_k \eta_t \partial_j \eta_t - \frac{1}{2} \partial_i \overleftarrow{a}_t^{j,k} \partial_{j,k} \eta_t \\ &+ \partial_{i,j} b_{T-t}^j - \frac{1}{2} \partial_{i,j,k} \overleftarrow{a}_t^{j,k}. \end{aligned} \quad (5.50)$$

Let us define  $Z_t^{j,k} := \partial_{j,k} \eta_t(\overleftarrow{X}_t)$ .

**Lemma 5.A.2.** *Under (A.2) and (A.6)-(i), if  $\mu_0$  satisfies (A.1) then*

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^d} |\nabla \log \mu_t|^2 \, d\mu_t + \int_0^T \int_{\mathbb{R}^d} |\nabla^2 \log \mu_t|^2 \, d\mu_t \, dt \leq C,$$

where  $C > 0$  only depends on coefficients through their uniform norm or the one or their derivatives.

*Proof.* Using Ito's formula and (5.50),

$$\begin{aligned} dY_t^i &= [\partial_i b_{T-t}^j Y_t^j + \partial_{i,j} b_{T-t}^j - \frac{1}{2} \partial_{i,j,k} \overleftarrow{a}_t^{j,k} - \partial_{i,k} \overleftarrow{a}_t^{j,k} Y_t^j] dt + Z_t^{i,j} \overleftarrow{\sigma}_t^{j,k} dB_t^k \\ &- \frac{1}{2} [\partial_i \overleftarrow{a}_t^{j,k} Z_t^{j,k} + \partial_i \overleftarrow{a}_t^{j,k} Y_t^j Y_t^k] dt. \end{aligned}$$

On the other hand

$$d\overleftarrow{\sigma}_t^{j,i} = [\partial_t \overleftarrow{\sigma}_t^{j,i} - b_{T-t}^k \partial_k \overleftarrow{\sigma}_t^{j,i} + \partial_l \overleftarrow{a}_t^{k,l} \partial_k \overleftarrow{\sigma}_t^{j,i} + \overleftarrow{a}_t^{k,l} Y_t^l \partial_k \overleftarrow{\sigma}_t^{j,i} + \frac{1}{2} \overleftarrow{a}_t^{k,l} \partial_{k,l} \overleftarrow{\sigma}_t^{j,i}] dt + \partial_k \overleftarrow{\sigma}_t^{j,i} \overleftarrow{\sigma}_{k,l} dB_l.$$

We then define  $\overline{Y}_t^i := \overleftarrow{\sigma}_t^{j,i} Y_t^j$ , so that

$$\begin{aligned} d\overline{Y}_t^i &= Y_t^j d\overleftarrow{\sigma}_t^{j,i} + \overleftarrow{\sigma}_t^{j,i} dY_t^j + d[\overleftarrow{\sigma}_t^{j,i}, Y_t^j]_t \\ &= [\partial_t \overleftarrow{\sigma}_t^{j,i} Y_t^j + \mathbf{B}_t^i + \mathbf{Y}_t^i + \overline{\mathbf{Y}}_t^i + \mathbf{Z}_t^i] dt + [Y_t^j \partial_k \overleftarrow{\sigma}_t^{j,i} \overleftarrow{\sigma}_t^{k,l} + \overleftarrow{\sigma}_t^{j,i} Z_t^{j,k} \overleftarrow{\sigma}_t^{k,l}] dB_t^l, \end{aligned}$$

with

$$\mathbf{B}_t^i := -b_{T-t}^k \partial_k \overleftarrow{\sigma}_t^{j,i} Y_t^j + \overleftarrow{\sigma}_t^{j,i} [\partial_j b_{T-t}^k Y_t^k + \partial_{j,k} b_{T-t}^k],$$

$$\mathbf{Z}_t^i := \partial_k \overleftarrow{\sigma}_t^{j,i} \overleftarrow{\sigma}_{k,l} Z_t^{j,m} \overleftarrow{\sigma}_t^{m,l} - \frac{1}{2} \overleftarrow{\sigma}_t^{j,i} \partial_j \overleftarrow{a}_t^{k,l} Z_t^{k,l}, \quad \overline{\mathbf{Y}}_t^i := Y_t^j \overleftarrow{a}_t^{k,l} Y_t^l \partial_k \overleftarrow{\sigma}_t^{j,i} - \frac{1}{2} \overleftarrow{\sigma}_t^{j,i} \partial_j \overleftarrow{a}_t^{k,l} Y_t^k Y_t^l,$$

$$\mathbf{Y}_t^i := Y_t^j [\partial_l \overleftarrow{a}_t^{k,l} \partial_k \overleftarrow{\sigma}_t^{j,i} + \frac{1}{2} \overleftarrow{a}_t^{k,l} \partial_{k,l} \overleftarrow{\sigma}_t^{j,i}] - \frac{1}{2} \overleftarrow{\sigma}_t^{j,i} \partial_{j,k,l} \overleftarrow{a}_t^{k,l} - \overleftarrow{\sigma}_t^{j,i} \partial_{j,k} \overleftarrow{a}_t^{k,l} Y_t^l.$$

We now want to compute

$$d[\overline{Y}_t^i \overline{Y}_t^i] = 2\overline{Y}_t^i d\overline{Y}_t^i + \sum_{i,l=1}^d [Y_t^j \partial_k \overleftarrow{\sigma}_t^{j,i} \overleftarrow{\sigma}_t^{k,l} + \overleftarrow{\sigma}_t^{j,i} Z_t^{j,k} \overleftarrow{\sigma}_t^{k,l}]^2.$$

We first observe that

$$\overline{Y}_t^i \overline{Y}_t^i = \partial_k \overleftarrow{\sigma}_t^{j,i} \overleftarrow{\sigma}_t^{m,i} \overleftarrow{a}_t^{k,l} Y_t^j Y_t^l Y_t^m - \frac{1}{2} \overleftarrow{\sigma}_t^{m,i} \overleftarrow{\sigma}_t^{j,i} \partial_j \overleftarrow{a}_t^{k,l} Y_t^k Y_t^l Y_t^m.$$

We then write that  $\overleftarrow{\sigma}_t^{m,i} \overleftarrow{\sigma}_t^{j,i} \partial_j \overleftarrow{a}_t^{k,l} = \overleftarrow{a}_t^{j,m} (\partial_j \overleftarrow{\sigma}_t^{k,i} \overleftarrow{\sigma}_t^{l,i} + \overleftarrow{\sigma}_t^{k,i} \partial_j \overleftarrow{\sigma}_t^{l,i})$ , and we re-arrange terms to get that  $\overline{Y}_t^i \overline{Y}_t^i = 0$ . Using (A.2)-(iii), we get

$$\sum_{i,l=1}^d [Y_t^j \partial_k \overleftarrow{\sigma}_t^{j,i} \overleftarrow{\sigma}_t^{k,l} + \overleftarrow{\sigma}_t^{j,i} Z_t^{j,k} \overleftarrow{\sigma}_t^{k,l}]^2 \geq -C Y_t^j Y_t^j + D Z_t^{j,l} Z_t^{j,l},$$

for some  $C, D > 0$ . As for proving Lemma 5.A.1, we take expectation and we integrate by parts to lower derivative orders:

$$\begin{aligned} \mathbb{E}[-\overline{Y}_t^i \overleftarrow{\sigma}_t^{j,i} [\partial_j b_{T-t}^k Y_t^k + \partial_{j,k} b_{T-t}^k]] &= \mathbb{E}[\overline{Y}_t^i \partial_k \overleftarrow{\sigma}_t^{j,i} \partial_j b_{T-t}^k + \partial_k \overleftarrow{\sigma}_t^{m,i} Y_t^m \overleftarrow{\sigma}_t^{j,i} \partial_j \overleftarrow{b}_{T-t}^k \\ &\quad + \overleftarrow{\sigma}_t^{m,i} Z_t^{j,k} \overleftarrow{\sigma}_t^{j,i} \partial_j \overleftarrow{b}_{T-t}^k], \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[-\overline{Y}_t^i \overleftarrow{\sigma}_t^{j,i} \partial_{j,k,l} \overleftarrow{a}_t^{k,l}] &= \mathbb{E}[\overline{Y}_t^i \overleftarrow{\sigma}_t^{j,i} \partial_{j,k} \overleftarrow{a}_t^{k,l} Y_t^l + \overline{Y}_t^i \partial_l \overleftarrow{\sigma}_t^{j,i} \partial_{j,k} \overleftarrow{a}_t^{k,l} + \partial_l \overleftarrow{\sigma}_t^{m,i} Y_t^m \overleftarrow{\sigma}_t^{j,i} \partial_{j,k} \overleftarrow{a}_t^{k,l} \\ &\quad + \overleftarrow{\sigma}_t^{m,i} Z_t^{m,l} \overleftarrow{\sigma}_t^{j,i} \partial_{j,k} \overleftarrow{a}_t^{k,l}]. \end{aligned}$$

To handle product terms, we repeatedly use  $\overline{Y}_t^i Z_t^{j,k} \geq -\frac{\varepsilon}{2} (Z_t^{j,k})^2 + \frac{\varepsilon^{-1}}{2} (\overline{Y}_t^i)^2$  as in (5.49), for  $\varepsilon$  small enough compared to  $D$ . Using bounds (A.2) and (A.6)-(i) on coefficients, we eventually get

$$\frac{d}{dt} \mathbb{E}[\overline{Y}_t^i \overline{Y}_t^i] \geq -\alpha (1 + \mathbb{E}[Y_t^i Y_t^i]) + \beta \mathbb{E}[Z_t^{j,l} Z_t^{j,l}] \geq -\gamma (1 + \mathbb{E}[\overline{Y}_t^i \overline{Y}_t^i]) + \beta \mathbb{E}[Z_t^{j,l} Z_t^{j,l}], \quad (5.51)$$

for some  $\alpha, \beta, \gamma > 0$ . Integrating in time, we conclude as in the proof of Lemma 5.A.1.  $\square$

**Lemma 5.A.3.** *Under (A.2), (A.6)-(i), if  $\mu_0$  satisfies (A.1) and  $\int |\nabla \log \mu_0|^4 d\mu_0 < +\infty$  then*

$$(i) \sup_{t \in [0, T]} \mathbb{E}[|Y_t|^4] = \sup_{t \in [0, T]} \int |\nabla \log \mu_t|^4 d\mu_t \leq C,$$

$$(ii) \mathbb{E}[\int_0^T Y_t^j Y_t^j Z_t^{i,l} Z_t^{i,l}] \leq C,$$

where  $C > 0$  only depends on coefficients through their uniform norm or the one or their derivatives.

*Proof.* We apply the same scheme as before writing that

$$d[\overline{Y}_t^j \overline{Y}_t^j \overline{Y}_t^l \overline{Y}_t^l] = 2\overline{Y}_t^j \overline{Y}_t^j d[\overline{Y}_t^l \overline{Y}_t^l] + d[\overline{Y}_t^j \overline{Y}_t^j, \overline{Y}_t^l \overline{Y}_t^l] \geq 2\overline{Y}_t^j \overline{Y}_t^j d[\overline{Y}_t^l \overline{Y}_t^l],$$

and we follow the same computations that yield to (5.51) in the proof of Lemma 5.A.2. In particular, we lower bound product terms by writing that  $(\overline{Y}_t^j)^2 Z_t^{i,l} \geq -\frac{\varepsilon^{-1}}{2} (\overline{Y}_t^j)^4 + \frac{\varepsilon}{2} (Z_t^{i,l})^2$  for small enough  $\varepsilon > 0$ . We end up to

$$\frac{d}{dt} \mathbb{E}[\overline{Y}_t^j \overline{Y}_t^j \overline{Y}_t^l \overline{Y}_t^l] \geq -\alpha \mathbb{E}[(1 + Y_t^i Y_t^i)^2] + \beta \mathbb{E}[Y_t^j Y_t^j Z_t^{i,l} Z_t^{i,l}],$$

for some  $\alpha, \beta > 0$ . A Gronwall argument now gives the bound on  $\mathbb{E}[\overline{Y}_t^j \overline{Y}_t^j \overline{Y}_t^l \overline{Y}_t^l]$ . The bound on  $\mathbb{E}[Y_t^j Y_t^j Z_t^{i,l} Z_t^{i,l}]$  a posteriori follows.  $\square$

We now differentiate (5.50). After computations, we get that  $(\partial_t + \overleftarrow{L}_t) \partial_{i,l} \eta_t$  equals

$$\begin{aligned} & -(\partial_t \overleftarrow{L}_t) \partial_{i,l} \eta_t + \partial_{i,j,l} b_{T-t}^j + \partial_{i,l} b_{T-t}^j \partial_j \eta_t + \partial_i b_{T-t}^j \partial_{i,j} \eta_t - \frac{1}{2} \partial_{i,j,k,l} \overleftarrow{a}_t^{j,k} - \partial_{i,k} \overleftarrow{a}_t^{j,k} \partial_{j,l} \eta_t \\ & - \partial_{i,k,l} \overleftarrow{a}_t^{j,k} \partial_j \eta_t - \frac{1}{2} \partial_{i,l} \overleftarrow{a}_t^{j,k} \partial_k \eta_t \partial_j \eta_t - \partial_i \overleftarrow{a}_t^{j,k} \partial_{j,l} \eta_t \partial_k \eta_t - \frac{1}{2} \partial_i \overleftarrow{a}_t^{j,k} \partial_{j,k,l} \eta_t - \frac{1}{2} \partial_{i,l} \overleftarrow{a}_t^{j,k} \partial_{j,k} \eta_t, \end{aligned}$$

where

$$\begin{aligned} -(\partial_t \overleftarrow{L}_t) \partial_{i,l} \eta_t &= \partial_l b_{T-t}^k \partial_{i,k} \eta_t - \partial_{j,l} \overleftarrow{a}_t^{j,k} \partial_{i,k} \eta_t - \partial_l \overleftarrow{a}_t^{j,k} \partial_j \eta_t \partial_{i,k} \eta_t - \frac{1}{2} \partial_l \overleftarrow{a}_t^{j,k} \partial_{i,j,k} \eta_t \\ &\quad - \overleftarrow{a}_t^{j,k} \partial_{j,l} \eta_t \partial_{i,k} \eta_t. \end{aligned}$$

We define  $\Gamma_t^{i,j,k} := \partial_{i,j,k} \eta_t(\overleftarrow{X}_t)$ .

**Proposition 5.A.4.** *Under (A.5) and (A.6), for every  $t \in [0, T]$ ,*

$$\sup_{s \in [0, t]} \int_{\mathbb{R}^d} |\nabla^2 \log \mu_s|^2 d\mu_s + \int_0^t \int_{\mathbb{R}^d} |\nabla^3 \log \mu_s|^2 d\mu_s ds \leq C [1 + \sup_{s \in [0, t]} \|\nabla^2 b_s\|_\infty^2],$$

where  $C > 0$  does not depend on  $\|\nabla^2 b_s\|_\infty$ , and  $C$  only depends on coefficients through their uniform norm or the one or their derivatives.

*Proof.* Using Ito's formula and the above expression for  $(\partial_t + \overleftarrow{L}_t) \partial_{i,l} \eta_t$ ,

$$dZ_t^{i,l} = [\partial_{i,j,l} b_{T-t}^j - \frac{1}{2} \partial_{i,j,k,l} a_t^{j,k} + \mathbf{Y}_t^{i,l} + \mathbf{Z}_t^{i,l} + \mathbf{\Gamma}_t^{i,l} - \overleftarrow{a}_t^{j,k} Z_t^{j,l} Z_t^{k,i}] dt + \mathbf{\Gamma}_t^{i,j,l} \overleftarrow{\sigma}_t^{j,k} dB_t^k,$$

where

$$\begin{aligned} \mathbf{Y}_t^{i,l} &:= \partial_{i,l} b_{T-t}^j Y_t^j - \frac{1}{2} \partial_{i,l} \overleftarrow{a}_t^{j,k} Y_t^j Y_t^k - \partial_{i,k,l} \overleftarrow{a}_t^{j,k} Y_t^j, & \mathbf{\Gamma}_t^{i,l} &= -\frac{1}{2} \partial_i \overleftarrow{a}_t^{j,k} \Gamma_t^{j,k,l} - \frac{1}{2} \partial_l \overleftarrow{a}_t^{j,k} \Gamma_t^{i,j,k}, \\ \mathbf{Z}_t^{i,l} &:= \partial_i b_{T-t}^j Z_t^{j,l} + \partial_l b_{T-t}^k Z_t^{i,k} - \partial_{j,l} \overleftarrow{a}_t^{j,k} Z_t^{i,k} - \partial_l \overleftarrow{a}_t^{j,k} Y_t^j Z_t^{i,k} - \partial_i \overleftarrow{a}_t^{j,k} Z_t^{j,l} Y_t^k \\ &\quad - \partial_{i,k} \overleftarrow{a}_t^{j,k} Z_t^{j,l} - \frac{1}{2} \partial_{i,l} \overleftarrow{a}_t^{j,k} Z_t^{j,k}. \end{aligned}$$

Therefore,

$$\begin{aligned} dZ_t^{i,l} Z_t^{i,l} &= 2Z_t^{i,l} [\partial_{i,j,l} b_{T-t}^j - \frac{1}{2} \partial_{i,j,k,l} a_t^{j,k} + \mathbf{Y}_t^{i,l} + \mathbf{Z}_t^{i,l} + \Gamma_t^{i,l} - \overleftarrow{a}_t^{j,k} Z_t^{j,l} Z_t^{k,i}] dt \\ &\quad + \Gamma_t^{i,j,l} \overleftarrow{\sigma}_t^{j,k} \Gamma_t^{i,l,m} \overleftarrow{\sigma}_t^{m,k} dt + 2\Gamma_t^{i,j,l} \overleftarrow{\sigma}_t^{j,k} dB_t^k. \end{aligned}$$

Using (A.2)-(iii),  $\Gamma_t^{i,j,l} \overleftarrow{\sigma}_t^{j,k} \Gamma_t^{i,l,m} \overleftarrow{\sigma}_t^{m,k} \geq C\Gamma_t^{i,j,l} \Gamma_t^{i,j,l}$  for some  $C > 0$ . Then using (A.6),

$$2Z_t^{i,l} [\mathbf{Y}_t^{i,l} + \mathbf{Z}_t^{i,l}] \geq -D[1 + Y_t^j Y_t^j] Z_t^{i,l} Z_t^{i,l}, \quad \text{and} \quad 2Z_t^{i,l} \Gamma_t^{i,l} \geq -\varepsilon \Gamma_t^{i,j,k} \Gamma_t^{i,j,k} - \varepsilon^{-1} Z_t^{i,l} Z_t^{i,l},$$

for  $D > 0$  and  $\varepsilon > 0$  small enough. We now take expectation. To lower derivative orders, we integrate by parts:

$$\begin{aligned} \mathbb{E}\{Z_t^{i,l} [\partial_{i,j,l} b_{T-t}^j - \frac{1}{2} \partial_{i,j,k,l} a_t^{j,k}]\} &= -\mathbb{E}\{[\Gamma_t^{i,j,l} + Y_t^j Z_t^{i,l}] [\partial_{i,l} b_{T-t}^j - \frac{1}{2} \partial_{i,k,l} \overleftarrow{a}_t^{j,k}]\} \\ &\geq -\mathbb{E}[\varepsilon \Gamma_t^{i,j,l} \Gamma_t^{i,j,l} + \varepsilon^{-1} D \|\nabla^2 b_{T-t}\|_\infty^2 + \varepsilon^{-1} D [1 + Y_t^j Y_t^j] Z_t^{i,l} Z_t^{i,l}]. \end{aligned}$$

for some  $D > 0$  that does not depend on  $\|\nabla^2 b_{T-t}\|_\infty$ . Similarly, integrating  $\mathbb{E}[-\overleftarrow{a}_t^{j,k} Z_t^{j,l} Z_t^{i,k} Z_t^{i,l}]$  by parts gives

$$\begin{aligned} \mathbb{E}[\partial_j \overleftarrow{a}_t^{j,k} Y_t^l Z_t^{i,k} Z_t^{i,l} + \overleftarrow{a}_t^{j,k} Y_t^l \Gamma_t^{i,j,k} Z_t^{i,l} + \overleftarrow{a}_t^{j,k} Y_t^l Z_t^{i,k} \Gamma_t^{i,j,l} + \overleftarrow{a}_t^{j,k} Y_t^j Y_t^l Z_t^{i,k} Z_t^{i,l}] \\ \geq -\mathbb{E}[\varepsilon \Gamma_t^{i,j,l} \Gamma_t^{i,j,l} + \varepsilon^{-1} D [1 + Y_t^j Y_t^j] Z_t^{i,l} Z_t^{i,l}]. \end{aligned}$$

Choosing  $\varepsilon$  small enough compared to  $C$ , we obtain that

$$\frac{d}{dt} \mathbb{E}[Z_t^{i,l} Z_t^{i,l}] \geq -\alpha \sup_{s \in [t, T]} \|\nabla^2 b_{T-s}\|_\infty^2 - \alpha \mathbb{E}[[1 + Y_t^j Y_t^j] Z_t^{i,l} Z_t^{i,l}] + \beta \mathbb{E}[\Gamma_t^{i,j,l} \Gamma_t^{i,j,l}],$$

for some  $\alpha, \beta > 0$ . Using Lemma 5.A.3, a Gronwall-type argument concludes as previously.  $\square$

## 5.B Relative entropy and regularity on measures

We recall that the notion of reference system  $\Sigma = (\Omega, (\mathcal{F}_t)_{t \leq 0 \leq T}, \mathbb{P}, (B_t)_{0 \leq t \leq T})$  is defined in Section 5.1.2. For such a  $\Sigma$  and any square-integrable progressively measurable process  $\alpha = (\alpha_t)_{0 \leq t \leq T}$  on  $\Sigma$ , we say that  $X_{[0, T]}^\alpha := (X_t^\alpha)_{0 \leq t \leq T}$  is a solution of the McKean-Vlasov SDE

$$dX_s^\alpha = b_s(X^\alpha, \mathcal{L}(X^\alpha)) ds + \sigma_s(X^\alpha, \mathcal{L}(X^\alpha)) \alpha_s ds + \sigma_s(X^\alpha, \mathcal{L}(X^\alpha)) dB_s, \quad (5.52)$$

if  $(X_t^\alpha)_{0 \leq t \leq T}$  is  $(\mathcal{F}_t)_{t \leq 0 \leq T}$ -adapted and the integrated version of (5.52) holds  $\mathbb{P}$ -a.s. The following result is [Chaintron, 2024, Lemma B.1].

**Lemma 5.B.1.** *Under (A.1)-(A.2), for every measure  $\mu_{[0, T]}$  in  $\mathcal{P}_1(C([0, T], \mathbb{R}^d))$ ,*

$$H(\mu_{[0, T]} | \Gamma(\mu_{[0, T]})) = \inf_{\Sigma} \inf_{\mathcal{L}(X_{[0, T]}^\alpha) = \mu_{[0, T]}} H(\mathcal{L}(X_0^\alpha) | \nu_0) + \mathbb{E} \int_0^T \frac{1}{2} |\alpha_t|^2 dt.$$

where we minimise over  $(X_{[0, T]}^\alpha, \alpha)$  satisfying (5.52) in the reference system  $\Sigma$ , with the convention that an infimum over an empty set equals  $+\infty$ .

This results allows us to extend [Backhoff et al., 2020, Lemma 5.2] to our setting.

**Corollary 5.B.1.1** (Range of finiteness). *Under (A.1)-(A.2), let us fix any  $\nu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d))$ . Let  $W_{[0,T]}^\nu$  denote the path-law of the solution to*

$$dY_t = b_t(Y_t, \nu_t)dt + \sigma_t(Y_t)dB_t, \quad Y_0 \sim \nu_0.$$

*Then for every  $\mu_{[0,T]} \in \mathcal{P}_1(C([0,T], \mathbb{R}^d))$ ,*

$$H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]})) < +\infty \iff H(\mu_{[0,T]}|W_{[0,T]}^\nu) < +\infty.$$

*Proof.* If  $H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]})) < +\infty$ , [Léonard, 2012, Theorem 2.1] provides an adapted square-integrable process  $(c_t)_{0 \leq t \leq T}$  on the canonical space  $\Omega = C([0,T], \mathbb{R}^d)$  such that

$$dX_t = b_t(X_t, \mu_t)dt + \sigma_t(X_t)c_tdt + \sigma_t(X_t)dB_t, \quad \mu_{[0,T]}\text{-a.s.},$$

the process  $(B_t)_{0 \leq t \leq T}$  being a Brownian motion under  $\mu_{[0,T]}$ . We then apply Lemma 5.B.1 with  $x \mapsto b_t(x, \nu_t)$  instead of  $b$  and  $\alpha_t := c_t + \sigma_t^{-1}(X_t)[b_t(X_t, \mu_t) - b_t(X_t, \nu_t)]$ . Since  $b$  is Lipschitz-continuous in  $\mu$  independently of  $x$ , this ensures that  $H(\mu_{[0,T]}|W_{[0,T]}^\nu) < +\infty$ .

Reciprocally if  $H(\mu_{[0,T]}|W_{[0,T]}^\nu) < +\infty$ , we similarly obtain a square-integrable  $(c_t)_{0 \leq t \leq T}$  such that

$$dX_t = b_t(X_t, \nu_t)dt + \sigma_t(X_t)c_tdt + \sigma_t(X_t)dB_t, \quad \mu_{[0,T]}\text{-a.s.},$$

where  $(B_t)_{0 \leq t \leq T}$  is a Brownian motion under  $\mu_{[0,T]}$ . Setting  $\alpha_t := c_t + \sigma_t^{-1}(X_t)[b_t(X_t, \nu_t) - b_t(X_t, \mu_t)]$ ,  $H(\mu_{[0,T]}|\Gamma(\mu_{[0,T]})) < +\infty$  then results from Lemma 5.B.1.  $\square$

The following result is used to regularise linear derivatives of functions defined on  $\mathcal{P}_1(\mathbb{R}^d)$ . It is a slight variation of [Daudin et al., 2023, Lemma 4.2].

**Lemma 5.B.2** (Mollification). *Let  $\mathcal{F} : \mathcal{P}_1(\mathbb{R}^d) \rightarrow \mathbb{R}$  be Lipschitz. Let  $\rho : \mathbb{R}^d \rightarrow [0, 1]$  be a  $C^\infty$  symmetric function with compact support and  $\int_{\mathbb{R}^d} \rho(x)dx = 1$ . For  $k \geq 1$ , we set*

$$\forall x \in \mathbb{R}^d, \rho^k(x) := k^d \rho(kx), \quad \forall \mu \in \mathcal{P}_1(\mathbb{R}^d), \mathcal{F}^k(\mu) := \mathcal{F}(\rho^k * \mu).$$

*Then  $\mathcal{F}^k$  is Lipschitz-continuous uniformly in  $k$  and*

$$\sup_{\mu \in \mathcal{P}_1(\mathbb{R}^d)} |\mathcal{F}^k(\mu) - \mathcal{F}(\mu)| \leq Ck^{-1},$$

*for  $C > 0$  independent of  $k$ . Moreover, if  $\mathcal{F}$  has a jointly continuous linear derivative  $\frac{\delta \mathcal{F}}{\delta \mu}$ , then so has  $\mathcal{F}^k$  and*

$$\forall (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d), \quad \frac{\delta \mathcal{F}^k}{\delta \mu}(\mu, x) = \left[ \frac{\delta \mathcal{F}^k}{\delta \mu}(\rho^k * \mu, \cdot) * \rho^k \right](x).$$

*As a consequence,  $\frac{\delta \mathcal{F}^k}{\delta \mu}(\mu) : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $C^\infty$ .*

## 5.C PDE notion of solution

The following notion of solution is extracted from [Chaintron et al., 2024, Section 3.3].

**Assumption (A.10)** (On the reference SDE). *The functions  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ ,  $\psi, c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  are measurable and locally bounded, and in addition:*

- (i) *uniformly in  $t \in [0, T]$ ,  $x \mapsto b_t(x)$  and  $x \mapsto \sigma_t(x)$  are Lipschitz continuous;*



(ii) there exists  $M_\sigma \geq 0$  such that  $|\sigma_t(x)| \leq M_\sigma$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , and  $t \mapsto \sigma_t(x)$  is locally Hölder-continuous;

(iii) uniformly in  $t \in [0, T]$ ,  $c_t$  and  $\psi_t$  are Lipschitz-continuous, and  $\sigma_t^{-1}$  is bounded.

We now fix a positive Radon measure  $\lambda \in \mathcal{M}_+([0, T])$ , and we want to make sense of the equation

$$-\varphi_t + \int_t^T \left( b_s \cdot \nabla \varphi_s - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s|^2 + \frac{1}{2} \text{Tr}[a_s \nabla^2 \varphi_s] + c_s \right) ds + \int_{[t, T]} \psi_s \lambda(ds) = 0. \quad (5.53)$$

If  $\lambda(\{t\}) \neq 0$ , an arbitrary choice has been made when considering integrals over  $[t, T]$  rather than  $(t, T]$ . However, the set of atoms of  $\lambda$  is at most countable; hence the choice of the interval does not matter if we only require equality Lebesgue-a.e. For approximation and stability purposes, we introduce a specific notion of solution, which relies on the following heat semi-group.

Under the assumptions made on  $\sigma$  in (A.10), a consequence of [Rubio, 2011, Theorem 2.1] is that for any  $0 < s \leq T$  and any continuous  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  with linear growth, the parabolic equation

$$\begin{cases} \partial_t \varphi_t + \frac{1}{2} \text{Tr}[a_t \nabla^2 \varphi_t] = 0, & 0 \leq t \leq s, \\ \varphi_s = \varphi, \end{cases}$$

has a unique solution  $\varphi \in C([0, s] \times \mathbb{R}^d) \cap C^{1,2}((0, s) \times \mathbb{R}^d)$  with linear growth. From this, we define the evolution system  $(S_{t,s})_{0 \leq t \leq s \leq T}$  by

$$S_{t,s}[\varphi](x) = \varphi_t(x),$$

for any continuous  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  with linear growth,  $S_{t,s}[\varphi]$  being a  $C^2$  function with linear growth as soon as  $t < s$ .

**Definition 5.C.1.** We say that a measurable  $\varphi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a mild solution of (5.53) if for Lebesgue-a.e.  $t \in [0, T]$ ,  $x \mapsto \varphi_t(x)$  is  $C^1$ ,  $(t, x) \mapsto \nabla \varphi_t(x)$  is bounded measurable, and for a.e.  $t \in [0, T]$ ,

$$\varphi_t = \int_t^T S_{t,s} [b_s \cdot \nabla \varphi_s - \frac{1}{2} |\sigma_s^\top \nabla \varphi_s|^2 + c_s] ds + \int_{[t, T]} S_{t,s} [\psi_s] \lambda(ds).$$

This implies that  $x \mapsto \varphi_t(x)$  is  $C^2$  for Lebesgue-a.e.  $t$ .

Lebesgue-almost sure uniqueness always holds for (5.53) in the sense of Definition 5.C.1 because the difference of two solutions solves a classical linear parabolic equation without source term.

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## CHAPTER 6

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# Existence and global Lipschitz estimates for unbounded classical solutions of a Hamilton-Jacobi equation

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*Je ne sais pas qui a posé la question, mais c'est sûrement un matheux : il a probablement raison mais cela n'apporte rien.*

— S. F.

This work has been recently accepted for publication in *Annales de la Faculté des Sciences de Toulouse*.

The purpose of this article is to prove existence, uniqueness and uniform gradient estimates for unbounded classical solutions of a Hamilton-Jacobi-Bellman equation. Such an equation naturally arises in stochastic control problems. Contrary to the classical literature which handles the case of bounded regular coefficients, we only impose Lipschitz regularity conditions, allowing for a linear growth of coefficients. These Lipschitz assumptions are natural in a probabilistic setting. In principle, these assumptions are compatible with global Lipschitz regularity for the solution. However, to the best of our knowledge, this useful result had not been established before. Our proofs rely on the Ishii-Lions method. We combine several elements from the viscosity solution theory to obtain estimates at the edges of what seems possible.

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## 6.1 Introduction

For any integer  $d \geq 1$  and  $T > 0$ , we consider the following Hamilton-Jacobi-Bellman (henceforth, HJB) equation:

$$\begin{cases} \partial_t u(t, x) + cu(t, x) + \frac{1}{2} \text{Tr}[\sigma(t, x)\sigma^\top(t, x)D^2u(t, x)] \\ \quad + \inf_{\alpha \in \mathcal{A}} [b(t, x, \alpha) \cdot Du(t, x) + f(t, x, \alpha)] = 0, \\ u(T, x) = g(x), \quad (t, x) \in (0, T) \times \mathbb{R}^d. \end{cases} \quad (6.1)$$

for  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ ,  $b : [0, T] \times \mathbb{R}^d \times \mathcal{A} \rightarrow \mathbb{R}^d$ ,  $f : [0, T] \times \mathbb{R}^d \times \mathcal{A} \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$  is a constant. The control set  $\mathcal{A}$  is a possibly unbounded closed domain in  $\mathbb{R}^d$ . The semi-linear equation (6.1) naturally arises in finite-time horizon stochastic control problems. More specifically, on a filtered probability space  $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ , we consider the controlled dynamics in  $\mathbb{R}^d$ ,

$$\begin{cases} dX_s^{t,x,\alpha} = b(s, X_s^{t,x,\alpha}, \alpha)ds + \sigma(s, X_s^{t,x,\alpha})dB_s, \quad t \leq s \leq T, \\ X_t^{t,x,\alpha} = x, \end{cases} \quad (6.2)$$

where  $(B_s)_{0 \leq s \leq T}$  is a  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -Brownian motion, and the control process  $\alpha := (\alpha_s)_{0 \leq s \leq T}$  is any  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted  $\mathcal{A}$ -valued stochastic process. The related cost function is

$$V(t, x) := \inf_{\alpha} \mathbb{E} \int_t^T e^{c(s-t)} f(s, X_s^{t,x,\alpha}, \alpha) ds + g(X_T^{t,x,\alpha}). \quad (6.3)$$

If (6.1) has a  $C^{1,2}$  solution  $u$  which satisfies suitable bounds, then it is classical that  $u = V$  (see e.g. [Fleming and Soner, 2006, Chapter III.8, Theorem III.8.1]), and it is possible to compute optimal controls for (6.2)-(6.3) by looking for feed-back controls  $\bar{\alpha}_s := \bar{\alpha}(s, X_s^{t,x,\bar{\alpha}})$  such that

$$\bar{\alpha}(s, x) \in \operatorname{argmin}_{\alpha \in \mathcal{A}} [b(s, x, \alpha) \cdot Du(s, x) + f(s, x, \alpha)]. \quad (6.4)$$

However, having such a classical solution often requires strong assumptions on the coefficients. A standard reference on this issue is [Fleming et al., 1975, Theorem 6.2], which provides classical solutions for (6.1) when  $b$ ,  $f$  and  $\sigma$  are at least  $C_b^{1,2}$  (bounded functions with bounded continuous derivatives) and the control set  $\mathcal{A}$  is compact. In the setting of Fleming et al. [1975], the solution of (6.1) is a  $C_b^{1,2}$  function. Our main objective is the following result:

**Theorem 6.1.1.** *If  $b_2$ ,  $\sigma$  and  $f_2$  are Lipschitz-continuous functions, with  $\sigma$  bounded and  $\sigma\sigma^\top$  uniformly elliptic, then the HJB equation*

$$\partial_t u(t, x) + \frac{1}{2} \text{Tr}[\sigma\sigma^\top(t, x)D^2u(t, x)] + b_2(t, x) \cdot Du - \frac{1}{2} |\sigma^\top(t, x)Du(t, x)|^2 + f_2(t, x) = 0, \quad (6.5)$$

*with globally Lipschitz terminal data  $u(T, x) = g(x)$ , has a unique linear growth  $C^{1,2}$  solution  $u$ , which has a uniformly bounded gradient.*

These Lipschitz assumptions on the coefficients are natural in a probabilistic setting. Indeed, from the Itô theorem Stroock and Varadhan [1997], a natural sufficient assumption for (6.2) to be well-posed is that both  $b$  and  $\sigma$  are globally Lipschitz ( $\alpha$  must satisfy some moment bound too). In particular, this framework allows for a linear growth of the coefficients, which implies that the solution of (6.5) can have linear growth. Equation (6.5) is a particular case of (6.1) when specifying  $b(t, x, \alpha) = \sigma^\top(t, x)\alpha + b_2(t, x)$ ,  $f(t, x, \alpha) = \frac{1}{2}|\alpha|^2 + f_2(t, x)$  together with the unbounded control set  $\mathcal{A} = \mathbb{R}^d$ . In this case, (6.4) reduces to  $\bar{\alpha}(s, x) = -\sigma^\top(s, x)Du(s, x)$ . In particular, using the global bound on  $\sigma$ , Theorem 6.1.1

proves that optimal controls are globally bounded. Theorem 6.1.1 will be a consequence of Theorem 6.2.2 below, which is concerned with more general equations of type (6.1). We now summarise the results in the literature that are close to ours.

Well-posedness for (6.5) is related to well-posedness for the linear parabolic equation

$$\partial_t v + b_2 \cdot Dv + \frac{1}{2} \text{Tr}[\sigma \sigma^\top D^2 v] - f_2 v = 0, \quad (6.6)$$

which can be obtained from (6.5) using the Cole-Hopf transform  $v = e^{-u}$ . Standard references for (6.6) are Gilbarg et al. [1977]; Ladyzhenskaia et al. [1988]; Friedman [2008] which provide classical well-posedness and Hölder-regularity estimates when the coefficients  $b$ ,  $\sigma$  and  $f_2$  are regular enough (at least Hölder-continuous), either in bounded domains, or in  $\mathbb{R}^d$  when the coefficients  $b$ ,  $\sigma$  and  $f_2$  are also globally bounded. Their work also handles Lipschitz non-linear perturbations of (6.6). This latter extension includes (6.1) when the control set  $\mathcal{A}$  is compact. Similar results for (6.1) are proved in Krylov [1987, 2008] when  $f$  is bounded uniformly in  $\alpha$  and the coefficients are twice differentiable.

Well-posedness for linear parabolic equations with unbounded coefficients is a long-standing topic. Historical approaches Bodanko [1966]; Johnson [1971]; Besala [1975] rely on the parametrix method and provide existence and bounds on a Green function for (6.6) under a Lyapunov-type assumption, but they also require heavy differentiability conditions on the coefficients. See also Deck and Kruse [2002] for a more recent generalisation. On the other side, the solution of (6.6) admits a stochastic representation using the Feynman-Kac formula. This formula can be used in turn to build a classical solution for (6.6), but this approach Friedman [2012] also requires heavy differentiability assumptions on the coefficients. Another noticeable limitation is the assumption that  $f_2$  is bounded from below, which is necessary to guarantee that solutions do not explode. The removal of this restriction on  $f_2$  can imply the loss of the maximum principle and non-uniqueness situations.

More recently, the works Lunardi [1998]; Fornaro et al. [2004]; Bertoldi and Fornaro [2004]; Bertoldi et al. [2007]; Hieber et al. [2007]... provide classical well-posedness, existence of Green functions and regularity estimates for linear parabolic equations with unbounded coefficients. These results rely on semi-group methods, under a fairly general Lyapunov-type condition. This framework also extends to Lipschitz non-linear perturbations, see e.g. Addona et al. [2017]. However, these results mostly focus on globally bounded solutions, a situation which cannot be expected in our framework for (6.1). In Ito [2001], existence of generalised strong solutions with quadratic growth is shown for a class of Hamilton-Jacobi equations whose non-linearity in  $Du$  includes the one in (6.5). A uniform bound on  $Du$  is proved in [Ito, 2001, Theorem 2.3] in some situations where  $f_2$  is Lipschitz and  $u$  is globally bounded, and [Ito, 2001, Theorem 2.7] proves linear growth for  $Du$  when  $f_2$  has quadratic growth. However, these results require a strong confinement assumption on the drift  $b_2$  and do not provide classical solutions.

An efficient method to prove global Lipschitz estimates on the solution  $u$  of an elliptic or parabolic equation of the kind

$$\partial_t u(t, x) + H(t, x, Du(t, x), D^2 u(t, x)) = 0, \quad (6.7)$$

is the classical Bernstein method (see e.g. Gilbarg et al. [1977]). By differentiating the equation, this approach looks for a linear equation satisfied by  $w := |Du|^2$ , and estimates are obtained using standard maximum principle-type arguments. In particular, this method requires extra-regularity properties on  $u$  for the differential of (6.7) to make sense, or it needs to approximate (6.7) by a smoothed version and to use continuation methods. A successful application of this approach can be found in Daudin [2022] under assumptions on



$H$  which guarantee that  $u$  is  $C_b^{1,2}$ , namely coercivity and quadratic growth in  $Du$ , together with Lipschitz-regularity of  $D_x H$  w.r.t.  $D^2u$  and the condition (6.8) below. This method is extended to linear-growth solutions of (6.5) when  $\sigma \equiv \text{Id}$  in Daudin [2023], well-posedness resulting from a fixed-point method. However, Daudin [2022, 2023] cannot handle more general  $\sigma$ , because the Bernstein method they use requires the structural assumption that

$$\forall p \in \mathbb{R}^d, \quad \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |D_x H(t, x, p, 0)| \leq C[1 + |p|], \quad (6.8)$$

for some constant  $C > 0$ . This assumption is not satisfied by (6.5) in general if  $\sigma \neq \text{Id}$ .

Another well-posedness approach for (6.1) is the viscosity solution theory Crandall and Lions [1983]; Crandall et al. [1992]; Barles [1994]. Under suitable structural assumptions like (6.8), existence and comparison principle (hence uniqueness) for bounded uniformly continuous solutions of (6.1) are by now standard results (see e.g. [Fleming and Soner, 2006, Chapter 6]). Extensions to linear-growth (or more general growth) situations can be found in e.g. Ishii [1984, 1989]; Crandall et al. [1989]; Crandall and Lions [1990]; Barles et al. [2003]. A linear growth situation similar to ours is Giga et al. [1991], but still keeping a structural condition of the kind (6.8). A general result that covers existence and uniqueness for viscosity solutions of (6.1) in our Lipschitz setting (and far more general settings) is Da Lio and Ley [2006].

The next step is to prove Lipschitz estimates on the obtained viscosity solution. A seminal work to prove regularity estimates is Ishii and Lions [1990]: under uniformly elliptic or parabolic assumptions, the Ishii-Lions method provides Hölder or Lipschitz estimates, see Barles [2008] for a summarised presentation. When Lipschitz estimates are available, the Ishii-Lions method even implies semi-concavity estimates. For Lipschitz estimates, the main idea is to prove that

$$\sup_{(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d} u(t, x) - u(t, y) - K|x - y| \leq 0$$

for large enough  $K$ , by reasoning by contradiction and combining the PDE and the second-order optimality conditions at a maximum point. Similarly, the analogous of the Bernstein method can be developed for viscosity solutions: this is the weak Bernstein method Barles [1991]. Contrary to the classical Bernstein method, this method does not require any additional differentiability assumption on the solution, but it requires similar structure conditions on the equation. In Giga et al. [1991], Lipschitz estimates and concavity-preserving properties are shown for linear growth viscosity solutions of (6.7). However, their setting does not allow for the linear growth of  $b$  in (6.5) and the structure condition (6.8) is required. Lipschitz estimates are also obtained for globally bounded solutions of an equation similar to (6.1) in Papi [2002, 2003].

After proving existence, uniqueness, and Lipschitz estimates for viscosity solutions, it is possible to obtain semi-concavity and higher-order regularity results using e.g. Caffarelli [1989]; Wang [1989]; Cannarsa and Sinestrari [2004]; Imbert [2006]. This strategy is applied to (6.7) in Daudin [2022], under (6.8) and assumptions that guarantee the global boundedness of solutions.

On the stochastic side, the representation formula (6.3) suggests a natural way to obtain Lipschitz estimates by coupling the characteristic curves. If  $f(s, x, \alpha)$  is Lipschitz in  $x$  uniformly in  $\alpha$  and  $\mathbb{E}[|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|] \leq C|x - y|$  for some  $C > 0$ , then the estimate easily follows. This method is successfully applied in the reference textbooks Krylov [2008]; Fleming and Soner [2006]; Yong and Zhou [2012] and generalised in Buckdahn and Li [2008]. In the setting of (6.5), the lack of a priori bounds on  $\alpha$  when  $\sigma \neq \text{Id}$  does not allow for such a control of  $\mathbb{E}[|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|]$ . Nevertheless, the stochastic setting provides a further

degree of freedom because the Brownian motion driving  $X_s^{t,x,\alpha}$  needs not be the same as the one of  $X_s^{t,y,\alpha}$ . The reflection coupling Eberle [2016] exploits this fact to produce uniform in time gradient estimates Priola and Wang [2006] for (6.6). This method is adapted to HJB equations like (6.5) in Conforti [2023], but assuming  $\sigma \equiv \text{Id}$ . A correspondence between reflection coupling and doubling of variables methods is established in Porretta and Priola [2013], recovering and extending the results of Priola and Wang [2006] in a viscosity solution setting. Another coupling method is developed in Buckdahn et al. [2010, 2012], exploiting the invariance of the Brownian motion under parabolic re-scaling. This yields global Lipschitz and semi-concavity estimates in  $(t, x)$  for (6.5), the main novelty being the Lipschitz-dependence in  $t$ . However, their work requires global boundedness of coefficients and Lipschitz-regularity uniformly in  $\alpha$ . Their setting can be extended to include linear growth w.r.t.  $x$  [Buckdahn et al., 2010, Remark 2.1] but not w.r.t.  $\alpha$ : this does not cover the quadratic term in (6.5).

To the best of our knowledge, the closest results to our setting are the ones in Rubio [2011], which provide classical well-posedness for polynomial-growth solutions of (6.6) under polynomial-growth conditions on the coefficients. Their proof strategy combines both stochastic representation formulae and classical domain truncation methods. Moreover, this work provides polynomial-growth solutions for (6.1) when the control set  $\mathcal{A}$  is compact.

At this point, our strategy for proving Theorem 6.1.1 is the following one. From Da Lio and Ley [2006], comparison, hence uniqueness, already holds for viscosity solutions of (6.5). We then prove a priori Lipschitz estimates for  $C^{1,2}$  solutions of (6.5). Existence follows by truncating the non-linearity to make it enter the scope of Rubio [2011]. To obtain the a priori bounds, we rely on the Ishii-Lions method Ishii [1984], exploiting the uniform ellipticity in (6.5) by using a strictly concave test function. Since solutions are classical, there is not need for the Jensen-Ishii lemma nor doubling of variables method. No new method is developed, but several tricks from the viscosity solution theory must be combined to obtain this estimate, at the edges of what the setting allows for. Up to our knowledge, this delicate estimate has no equivalent using stochastic approaches: it would be interesting to search for its analogous using coupling methods. Another challenging issue would be to prove Hessian estimates for (6.5) under minor strengthening of our assumptions, similarly to [Conforti, 2023, Proposition 3.2].

## 6.2 Assumptions and main result

In the following, the control set  $\mathcal{A}$  is a possibly unbounded closed domain in  $\mathbb{R}^d$ . The notation  $|x| := \sqrt{x^\top x}$  refers to the usual Euclidean norm on  $\mathbb{R}^d$ . Similarly, we will use the Euclidean norm  $|\sigma| := \sqrt{\text{Tr}[\sigma\sigma^\top]}$  on matrices. We write the running cost  $f$  as

$$f(t, x, \alpha) = f_1(t, x, \alpha) + f_2(t, x).$$

In the following, we restrict ourselves to

$$b(t, x, \alpha) = b_1(t, x, \alpha) + b_2(t, x), \tag{6.9}$$

the  $b_1$  part being uniformly bounded in  $x$ . Although we can handle some more general equations, we emphasise that the main purpose of the following results is to prove Theorem 6.1.1 in the Introduction. We also handle the case of unbounded diffusion matrices  $\sigma$ . Let us now detail the needed regularity assumptions for our more general results.

**Assumption (A.1)** (Regularity and growth). *The coefficients  $b_1, f_1$  (resp.  $b_2, \sigma$  and  $f_2$ ) are continuous functions on  $[0, T] \times \mathbb{R}^d \times \mathcal{A}$  (resp.  $[0, T] \times \mathbb{R}^d$ ), which satisfy the following conditions.*

- *Regularity and growth for  $b_1$  and  $b_2$ : there exists  $L_b > 0$  such that for every  $(t, x, y, \alpha)$  in  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{A}$ ,*

$$|b_1(t, x, \alpha) - b_1(t, y, \alpha)| \leq L_b(1 + |\alpha|)|x - y|, \quad |b_2(t, x) - b_2(t, y)| \leq L_b|x - y|, \quad (6.10)$$

$$|b_1(t, x, \alpha)| \leq L_b[1 + |\alpha|], \quad |b_2(t, x)| \leq L_b[1 + |x|].$$

- *Regularity and coercivity for  $f_1$ : there exist  $L_{f_1}, c_{f_1}, c'_{f_1}, C_{f_1}, C'_{f_2} > 0$  such that for every  $(t, x, y, \alpha)$  in  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{A}$ ,*

$$|f_1(t, x, \alpha) - f_1(t, y, \alpha)| \leq L_{f_1}|x - y|, \quad (6.11)$$

together with

$$c_{f_1}|\alpha|^2 - c'_{f_1} \leq f_1(t, x, \alpha) \leq C_{f_1}|\alpha|^2 + C'_{f_1}. \quad (6.12)$$

- *Regularity and growth for  $f_2$ : there exists  $L_{f_2} \in C([0, T], \mathbb{R}_+)$  such that for every  $(t, x, y)$  in  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$|f_2(t, x) - f_2(t, y)| \leq L_{f_2}(t)|x - y|, \quad |f_2(t, x)| \leq L_{f_2}(t)[1 + |x|]. \quad (6.13)$$

- *Regularity and growth for  $\sigma$  and  $g$ : there exist  $L_\sigma, L_g > 0$  such that for every  $(t, x, y)$  in  $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$|\sigma(t, x) - \sigma(t, y)| \leq L_\sigma|x - y|^{1/2}, \quad |g(x) - g(y)| \leq L_g|x - y|.$$

As explained in Section 6.1, Lipschitz assumptions on the coefficients are natural in a probabilistic setting to guarantee well-posedness for (6.2). Assumption (A.1) is needed to prove Lipschitz estimates on the solution  $u$  of (6.1).

**Assumption (A.2)** (Further assumptions on  $\sigma\sigma^\top$ ).

- *The matrix  $\sigma\sigma^\top$  is uniformly elliptic: there exists  $\eta_\sigma > 0$  such that for every  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,*

$$\forall \xi \in \mathbb{R}^d, \quad \xi^\top \sigma\sigma^\top(t, x) \xi \geq \eta_\sigma |\xi|^2.$$

- *The matrix  $\sigma\sigma^\top$  is uniformly continuous in  $x$ : there exists a uniform continuity modulus  $m_\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for every  $(t, x, y)$  in  $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$|\sigma\sigma^\top(t, x) - \sigma\sigma^\top(t, y)| \leq m_\sigma(|x - y|).$$

If (A.1) is satisfied, uniform continuity for  $\sigma\sigma^\top$  always holds if  $\sigma$  is globally bounded (as it is the case in Theorem 6.1.1). An unbounded  $\sigma$  which satisfies (A.1)-(A.2) is e.g.  $\sigma(t, x) = \sqrt{1 + |x|}\text{Id}$ . Assumption (A.2) is needed to apply the Ishii-Lions method, and we did not manage to alleviate it. In the presence of degeneracy, existence of classical solutions to (6.1) can be lost.

**Definition 6.2.1** (Local Hölder spaces). *For integers  $n, n' \geq 1$  and  $\beta, \beta' \in (0, 1]$ , a function  $\psi$  in  $C^{n, n'}((0, T) \times \mathbb{R}^d)$  belongs to the local Hölder space  $C_{\text{loc}}^{n, n', \beta, \beta'}$  if for every compact set  $F \subset (0, T) \times \mathbb{R}^d$ , there exists  $K_F > 0$  such that  $\psi$  and all its derivatives satisfy*

$$\forall (t, x), (s, y) \in F, \quad |\psi(t, x) - \psi(s, y)| \leq K_F[|t - s|^\beta + |x - y|^{\beta'}]. \quad (6.14)$$

For every differentiable  $\psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ , we define the set

$$A[\psi](t, x) := \operatorname{argmin}_{\alpha \in \mathcal{A}} [b_1(t, x, \alpha) \cdot D\psi(t, x) + f_1(t, x, \alpha)]. \quad (6.15)$$

The following assumption is not required to prove Lipschitz estimates. This assumption is copied from [Rubio, 2011, Assumption H3-(5)] to obtain existence of classical solutions with Lipschitz non-linearity. It can be seen as a black-box to use results of Rubio [2011].

**Assumption (A.3)** (Hölder regularity). *There exists  $\beta \in (0, 1)$  such that the coefficients  $b$ ,  $\sigma$  and  $f$  belong to  $C_{\text{loc}}^{0,0,\beta,1}$ . The coefficients  $b_1$  and  $f_1$  are locally Lipschitz in  $\alpha$  in the sense of (6.14). Moreover, for every  $\beta' \in \{\beta, 1\}$  and every  $\psi$  in  $C_{\text{loc}}^{0,1,\beta,\beta'}$ ,  $A[\psi]$  is a singleton, defining a function of  $(t, x)$  which belongs to  $C_{\text{loc}}^{0,0,\beta,\beta'}$ .*

Denoting by  $B(0, R)$  the centred ball  $B$  of  $\mathbb{R}^d$  with radius  $R$ , we eventually require that the above property on  $A[\psi]$  holds when replacing  $\mathcal{A}$  by  $\mathcal{A} \cap B(0, R)$  for every  $R \geq R_{\mathcal{A}}$ , for a large enough  $R_{\mathcal{A}} > 0$ .

In practice, regularity properties for  $A[\psi]$  are easier to study when assuming convexity properties in  $\alpha$  for the coefficients. Once again, we point out that (A.1)-(A.2)-(A.3) are satisfied in the setting of Theorem 6.1.1.

**Theorem 6.2.2** (Well-posedness and uniform gradient bound). *Under (A.1)-(A.2)-(A.3), Equation (6.1) has a unique linear growth solution  $u$  in  $C([0, T] \times \mathbb{R}^d) \cap C_{\text{loc}}^{1,2,\beta,\beta}$ . Moreover, there exists a constant  $K > 0$  such that*

$$\sup_{(t,x) \in (0,T) \times \mathbb{R}^d} |Du(t, x)| \leq K. \quad (6.16)$$

The constant  $K$  only depends on the regularity constants introduced in (A.1)-(A.2)-(A.3), and  $K$  only depends on  $f_2$  through  $\|L_{f_2}\|_{L^1(0,T)}$ .

The dependence on  $f_2$  is emphasised to compare Theorem 6.2.2 to [Daudin, 2023, Theorem 1.1 and Lemma A.2], where a similar dependence is obtained under the additional assumption (6.8). In [Daudin, 2023, Lemma A.2], this dependence is obtained using the classical Bernstein method. However, this method does not work without (6.8) and Theorem 6.2.2 extends the results of Daudin [2023] to our setting. In particular, following Daudin [2023], it is possible to extend Theorem 6.2.2 to measure-valued source terms  $f_2$  that are useful when dealing with constrained stochastic control problems.

Under (A.1), for every  $(t, x, \alpha)$  in  $[0, T] \times \mathbb{R}^d \times \mathcal{A}$ , the quantity to minimise on the r.h.s. of (6.15) is bounded from below by

$$F(|\alpha|) := -L_b[1 + |\alpha|]|D\psi(t, x)| + c_{f_1}|\alpha|^2 - c'_{f_1},$$

and bounded from above by

$$C(\alpha_0) := L_b[1 + |\alpha_0|]|D\psi(t, x)| + C_{f_1}|\alpha_0|^2 + C'_{f_1},$$

for some (fixed)  $\alpha_0 \in \mathcal{A}$ . As a consequence, minimisers for (6.15) belong to  $\{\alpha \in \mathcal{A}, F(|\alpha|) \leq C(\alpha_0)\}$ . Using standard computations for second-order polynomials, there exists  $L_{\mathcal{A}} > 0$  independent of  $\psi$ , such that for every  $(t, x)$  in  $[0, T] \times \mathbb{R}^d$ ,

$$\forall \bar{\alpha} \in A[\psi](t, x), \quad |\bar{\alpha}| \leq L_{\mathcal{A}}[1 + |D\psi(t, x)|]. \quad (6.17)$$

This property will be crucial in the proofs of Lipschitz estimates. If  $Du$  is bounded, then (6.17) guarantees using (6.4) that optimal controls for (6.2)-(6.3) are globally bounded. In the following, all the estimates will only depend on  $\mathcal{A}$  through  $L_{\mathcal{A}}$ .

**Remark 6.2.3** (About the coercivity assumption). *The property (6.17) is the only reason for the decomposition (6.9) and (6.12). The lower bound in (6.12) is a coercivity assumption which is standard in stochastic control. The dependence on  $|\alpha|$  for the upper bound is not really restrictive, since this term is only meant to bound the quantity to minimise in (6.15) from above, independently of  $x$ . Any other assumption that guarantees (6.17) could replace (6.9)-(6.12).*

**Remark 6.2.4** (Adding a linear term). *It would be uneasy to handle a non-constant  $c = c(t, x, \alpha)$ , because we need  $cu$  to be globally-Lipschitz when  $u$  is only globally Lipschitz. This is very restrictive, but it would be possible to adapt the result for e.g.  $c(t, x, \alpha) = \frac{c_0|\alpha|}{(1+|\alpha|)(1+|x|)}$ .*

### 6.3 Proof of global Lipschitz estimates

As explained in Section 6.1, we are going to prove a priori global Lipschitz estimates on linear growth  $C^{1,2}$  solutions of (6.1). We first start with a short lemma about the growth of such solutions. inherited from Da Lio and Ley [2006].

**Lemma 6.3.1.** *Let  $u$  in  $C([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T) \times \mathbb{R}^d)$  be a quadratic growth solution of (6.1), in the sense that*

$$\exists C_u > 0 : \forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |u(t, x)| \leq C_u[1 + |x|^2].$$

*Then under (A.1),  $u$  has linear growth:*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |u(t, x)| \leq L[1 + |x|],$$

*for a positive constant  $L$  which does not depend on  $u$  and only depends on  $(\mathcal{A}, f_2)$  through  $(L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ .*

*Proof.* The function  $z \in \mathbb{R}^d \mapsto (1 + |z|^2)^{1/2}$  is  $C^2$  and convex, and thus has non-negative Hessian. From (A.1),  $(t, x) \mapsto \sqrt{2}(1 + |x|^2)^{1/2}[\int_0^t L_{f_2}(s)ds + Le^t]$  is a super-solution of (6.1) for large enough  $L > 0$ . Similarly,  $(t, x) \mapsto \sqrt{2}(1 + |x|^2)^{1/2}[e^{-\delta t} - \int_0^t L_{f_2}(s)ds]$  is a sub-solution of (6.1) for large enough  $\delta > 0$ . The result then follows from the comparison principle proved in [Da Lio and Ley, 2006, Theorem 2.1].  $\square$

Noticeably, the above result allows us to include quadratic growth solutions in the uniqueness result of Theorem 6.2.2. We now prove Lipschitz estimates by showing that

$$\sup_{t, x, y} u(t, x) - u(t, y) - K|x - y| \leq 0 \tag{6.18}$$

for large enough  $K$ . The supremum is taken over  $t \in [0, T]$  and  $x, y \in \mathbb{R}^d$ . We reason by contradiction and we aim at combining the PDE and the second-order optimality conditions. The Ishii-Lions method provides a way to obtain a contradiction by making the second-order terms explode at a maximum point  $(\bar{x}, \bar{y})$ . A key feature for this approach is to prove that  $|\bar{x} - \bar{y}| \rightarrow 0$  as  $K \rightarrow +\infty$ . Technical difficulties arise at this stage: there is no guarantee that the supremum in (6.18) is finite and this supremum does not need to be realised at some point. This latter difficulty is classically circumvented by adding regularising terms to the supremum and progressively removing them. However, we need to know a priori that the supremum is finite to ensure that  $|\bar{x} - \bar{y}| \rightarrow 0$  as  $K \rightarrow +\infty$ . To do so, we reason as in [Crandall et al., 1992, Theorem 5.1] and we use the linear growth to first prove a deteriorated estimate.

**Lemma 6.3.2** (Deteriorated estimate). *Let  $u$  in  $C([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T) \times \mathbb{R}^d)$  be a solution of (6.1) and  $L_u > 0$  such that for every  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,*

$$|u(t, x)| \leq L_u[1 + |x|]. \quad (6.19)$$

Under (A.1), there exist constants  $\tilde{K}, \tilde{M} > 0$  such that

$$\sup_{(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d} u(t, x) - u(t, y) - \tilde{K}|x - y| \leq \tilde{M}, \quad (6.20)$$

where  $(\tilde{K}, \tilde{M})$  only depends on  $(u, \mathcal{A}, f_2)$  through  $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ .

*Proof.* We proceed in several steps.

**Step 1. Regularising parameters.** For any  $\gamma > 0$ , we notice that  $v : (t, x) \mapsto e^{\gamma t}u(t, x)$  is  $C^{1,2}$ . Moreover,  $v(T, x) = e^{\gamma T}g(x)$  and  $v$  satisfies

$$\partial_t v(t, x) - (\gamma - c)v(t, x) + \frac{1}{2}\text{Tr}[\sigma\sigma^\top(t, x)D^2v(t, x)] + e^{\gamma t}f_2(t, x) + \inf_{\alpha \in \mathcal{A}} L_\alpha[v](t, x) = 0, \quad (6.21)$$

where for every  $\alpha$  in  $\mathcal{A}$ , we define

$$L_\alpha[v](t, x) := b(t, x, \alpha) \cdot Dv(t, x) + e^{\gamma t}f_1(t, x, \alpha). \quad (6.22)$$

It is equivalent to prove (6.20) for  $v$  instead of  $u$ . The growth condition (6.19) is satisfied by  $v$  with  $e^{\gamma T}L_u$  instead of  $L_u$ . As a consequence,

$$\begin{aligned} \bar{M} := \sup_{t,x,y} v(t, x) - v(t, y) - \varepsilon t^{-1} \\ - \left[ \tilde{K} - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] [1 + |x - y|^2]^{1/2} - \varepsilon e^{-\delta t} [|x|^2 + |y|^2] \end{aligned} \quad (6.23)$$

is finite for every  $\varepsilon, \delta, \gamma, \tilde{K} > 0$ , because of the quadratic term. To alleviate notations, we do not emphasise the dependence on  $\gamma$  for  $v$ , nor the dependence on  $(\varepsilon, \delta, \gamma, \tilde{K})$  for  $\bar{M}$ . The  $L_{f_2}$ -term within the supremum will allow us to control the  $f_2$ -term in (6.21) using only  $\|L_{f_2}\|_{L^1}$ . In the following,  $\delta, \gamma$  and  $\tilde{K}$  are fixed parameters whose values will be chosen later on. On the contrary,  $\varepsilon$  will be sent to 0, and we restrict ourselves to  $\varepsilon \leq 1$ . Using the continuity and the linear growth of  $v$ , the quadratic term ensures that the supremum (6.23) is realised for some  $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ . Since  $\bar{M}$  is a supremum, it is necessary that  $\bar{t} \neq 0$ . In the following, we restrict ourselves to  $\tilde{K} \geq e^{\gamma T} \|L_{f_2}\|_{L^1}$ .

**Step 2. Bounds on the optimiser.** If there existed a sequence  $(\varepsilon_k)_{k \geq 1}$  of positive parameters converging to 0 and such that the related sequence  $(\bar{t}_k)_{k \geq 1}$  had a constant sub-sequence equal to  $T$ , then for every  $t \in [0, T]$  and  $x, y \in \mathbb{R}^d$ , we would have (up to re-labelling the sequence)

$$\begin{aligned} v(t, x) - v(t, y) - \varepsilon_k t^{-1} \\ - \left[ \tilde{K} - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] [1 + |x - y|^2]^{1/2} - \varepsilon_k e^{-\delta t} [|x|^2 + |y|^2] \\ \leq e^{\gamma T} g(\bar{x}) - e^{\gamma T} g(\bar{y}) - \varepsilon_k T^{-1} \\ - \left[ \tilde{K} - e^{\gamma T} \int_0^T L_{f_2}(s) ds \right] [1 + |\bar{x} - \bar{y}|^2]^{1/2} - \varepsilon_k e^{-\delta T} [|\bar{x}|^2 + |\bar{y}|^2], \end{aligned}$$

and using the assumption on  $g$  in (A.1), the r.h.s. is negative as soon as  $\tilde{K} \geq e^{\gamma T}(L_g + \|L_{f_2}\|_{L^1})$ . Similarly, if there existed  $(\varepsilon_k)_{k \geq 1}$  that converges to 0 while  $\bar{M} \leq 0$  for every  $k$ , then for every  $t \in [0, T]$  and  $x, y \in \mathbb{R}^d$ , we would have

$$v(t, x) - v(t, y) - \varepsilon_k t^{-1} - \varepsilon_k e^{-\delta t} [|x|^2 + |y|^2] - \left[ \tilde{K} - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] [1 + |x - y|^2]^{1/2} \leq 0.$$

In both cases, taking the limit would yield

$$\sup_{t, x, y} v(t, x) - v(t, y) - \left[ \tilde{K} - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] [1 + |x - y|^2]^{1/2} \leq 0.$$

In particular, this would imply the desired result.

We thus restrict ourselves to  $\tilde{K} \geq e^{\gamma T}(L_g + \|L_{f_2}\|_{L^1})$ , and we can assume without loss of generality that  $\bar{t} \neq T$  and  $\bar{M} > 0$  for small enough  $\varepsilon$ . From  $\bar{M} > 0$  and the linear growth of  $v$ ,

$$\begin{aligned} [\tilde{K} - e^{\gamma T} \|L_{f_2}\|_{L^1}] |\bar{x} - \bar{y}| &\leq v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}), \\ \varepsilon e^{-\delta \bar{t}} [|\bar{x}|^2 + |\bar{y}|^2] &\leq v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq 2L_u e^{\gamma T} [2 + |\bar{x}| + |\bar{y}|]. \end{aligned} \quad (6.24)$$

Up to changing  $L_u$  in  $\max(1, L_u)$ , the second equation implies the bound

$$\varepsilon |\bar{x}| + \varepsilon |\bar{y}| \leq 4e^{(\delta + \gamma)T} L_u, \quad (6.25)$$

which prevents  $\varepsilon |\bar{x}|$  and  $\varepsilon |\bar{y}|$  from exploding.

**Step 3. Optimality conditions.** Let us now define

$$\bar{p} := \left[ \tilde{K} - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) ds \right] D_z [1 + |z|^2]^{1/2} \Big|_{z = \bar{x} - \bar{y}}.$$

We notice that  $|\bar{p}| \leq \tilde{K} - e^{\gamma T} \|L_{f_2}\|_{L^1} \leq \tilde{K}$ . Since  $\bar{t} \notin \{0, T\}$ , the first-order optimality conditions in (6.23) provide

$$\begin{cases} Dv(\bar{t}, \bar{x}) = \bar{p} + 2\varepsilon e^{-\delta \bar{t}} \bar{x}, \\ Dv(\bar{t}, \bar{y}) = \bar{p} - 2\varepsilon e^{-\delta \bar{t}} \bar{y}, \\ \partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y}) = -e^{\gamma T} L_{f_2}(\bar{t}) [1 + |\bar{x} - \bar{y}|^2]^{1/2} \\ \quad - \varepsilon \bar{t}^{-2} - \varepsilon \delta e^{-\delta \bar{t}} [|\bar{x}|^2 + |\bar{y}|^2]. \end{cases} \quad (6.26)$$

We now write the second-order optimality conditions in  $(x, y)$  for (6.23):

$$\begin{aligned} \begin{pmatrix} D^2 v(\bar{t}, \bar{x}) & 0 \\ 0 & -D^2 v(\bar{t}, \bar{y}) \end{pmatrix} &\leq \left[ \tilde{K} - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) ds \right] \begin{pmatrix} \bar{A} & -\bar{A} \\ -\bar{A} & \bar{A} \end{pmatrix} \\ &\quad + 2\varepsilon e^{-\delta \bar{t}} \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}, \end{aligned} \quad (6.27)$$

where  $\bar{A} := D_z^2 [1 + |z|^2]^{1/2} \Big|_{z = \bar{x} - \bar{y}}$ , the inequality being understood in the sense of quadratic forms. The matrix  $\bar{A}$  is bounded by a constant  $C_0$  independent of  $\bar{x} - \bar{y}$ . We multiply the l.h.s. of (6.27) by  $\begin{pmatrix} \sigma^\top(\bar{t}, \bar{x}) & \sigma^\top(\bar{t}, \bar{y}) \end{pmatrix}$  and the r.h.s. by the transpose of this matrix:

$$\begin{aligned} \sigma^\top(\bar{t}, \bar{x}) D^2 v(\bar{t}, \bar{x}) \sigma(\bar{t}, \bar{x}) - \sigma^\top(\bar{t}, \bar{y}) D^2 v(\bar{t}, \bar{y}) \sigma(\bar{t}, \bar{y}) &\leq 2\varepsilon e^{-\delta \bar{t}} [\sigma \sigma^\top(\bar{t}, \bar{x}) + \sigma \sigma^\top(\bar{t}, \bar{y})] \\ &\quad + \left[ \tilde{K} - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) ds \right] [\sigma(\bar{t}, \bar{x}) - \sigma(\bar{t}, \bar{y})]^\top \bar{A} [\sigma(\bar{t}, \bar{x}) - \sigma(\bar{t}, \bar{y})]. \end{aligned} \quad (6.28)$$

We now take the trace and we use the assumption on  $\sigma$  in (A.1). Since  $L_{f_2} \geq 0$ ,  $\sigma\sigma^\top$  has linear growth and  $\varepsilon[|\bar{x}| + |\bar{y}|]$  is bounded from (6.25), the symmetry property of the trace yields

$$\frac{1}{2}\text{Tr}[\sigma\sigma^\top D^2v(\bar{t}, \bar{x})] - \frac{1}{2}\text{Tr}[\sigma\sigma^\top D^2v(\bar{t}, \bar{y})] \leq C_0L_\sigma^2\tilde{K}|\bar{x} - \bar{y}| + C_1, \quad (6.29)$$

for constants  $C_0, C_1$  which do not depend on  $(\varepsilon, \tilde{K}, \mathcal{A})$ , and which only depends on  $(u, f_2)$  through  $(L_u, e^{\gamma T}\|L_{f_2}\|_{L^1})$ . Moreover,  $C_0$  does not depend on  $\gamma$ .

**Step 4. Gathering the PDE.** We now write the PDE (6.21) at  $(\bar{t}, \bar{x})$  and  $(\bar{t}, \bar{y})$ . Let  $\bar{\alpha}$  be a minimiser in  $\alpha$  of  $L_\alpha[v](\bar{t}, \bar{x})$ , defined in (6.22). Subtracting both PDEs, we get using (6.29) that

$$\begin{aligned} (\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] &\leq \partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y}) + L_{f_2}(\bar{t})e^{\gamma\bar{t}}[f_2(\bar{t}, \bar{x}) - f_2(\bar{t}, \bar{y})] \\ &\quad + L_{\bar{\alpha}}[v](\bar{t}, \bar{x}) - L_{\bar{\alpha}}[v](\bar{t}, \bar{y}) + C_0L_\sigma^2\tilde{K}|\bar{x} - \bar{y}| + C_1. \end{aligned}$$

We then replace  $\partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y})$  by its computed value in (6.26). Assumption (A.1) ensures that the  $L_{f_2}$ -term in (6.26) is greater than  $L_{f_2}(\bar{t})e^{\gamma\bar{t}}[f_2(\bar{t}, \bar{x}) - f_2(\bar{t}, \bar{y})]$ , so that

$$\begin{aligned} (\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] &\leq -\varepsilon\bar{t}^{-2} - \varepsilon\delta e^{-\delta\bar{t}}[|\bar{x}|^2 + |\bar{y}|^2] \\ &\quad + L_{\bar{\alpha}}[v](\bar{t}, \bar{x}) - L_{\bar{\alpha}}[v](\bar{t}, \bar{y}) + C_0L_\sigma^2\tilde{K}|\bar{x} - \bar{y}| + C_1. \end{aligned} \quad (6.30)$$

Since  $-\varepsilon\bar{t}^{-2} \leq 0$ , we can get rid of this term. Within (6.17), changing  $f_2$  in  $e^{\gamma T}f_2$  amounts to replacing  $L_b$  by  $e^{-\gamma T}L_b$ . Since  $e^{-\gamma T} \leq 1$ , the bound  $|\bar{\alpha}| \leq L_{\mathcal{A}}[1 + |Dv(\bar{t}, \bar{y})|]$  still holds. Moreover, from (6.25)-(6.26),

$$|Dv(\bar{t}, \bar{y})| \leq \tilde{K} + 8e^{(\delta+\gamma)T}L_u \quad \text{and} \quad |Dv(\bar{t}, \bar{x}) - Dv(\bar{t}, \bar{y})| \leq 2\varepsilon e^{-\delta\bar{t}}[|\bar{x}| + |\bar{y}|].$$

We then bound each term of  $L_{\bar{\alpha}}[v](\bar{t}, \bar{x}) - L_{\bar{\alpha}}[v](\bar{t}, \bar{y})$ . For the the  $b$ -term, we use the Lipschitz assumption (6.10) on  $b_2$  and we simply bound  $b_1$ :

$$\begin{aligned} &b(\bar{t}, \bar{x}, \bar{\alpha}) \cdot Dv(\bar{t}, \bar{x}) - b(\bar{t}, \bar{y}, \bar{\alpha}) \cdot Dv(\bar{t}, \bar{y}) \\ &\leq |Dv(\bar{t}, \bar{y})||b(\bar{t}, \bar{x}, \bar{\alpha}) - b(\bar{t}, \bar{y}, \bar{\alpha})| + |b(\bar{t}, \bar{x}, \bar{\alpha})||Dv(\bar{t}, \bar{x}) - Dv(\bar{t}, \bar{y})| \\ &\leq L_b[\tilde{K} + 8e^{(\delta+\gamma)T}L_u][2 + 2|\bar{\alpha}| + |\bar{x} - \bar{y}|] + 2\varepsilon e^{-\delta\bar{t}}L_b[2 + |\bar{x}| + |\bar{\alpha}|][|\bar{x}| + |\bar{y}|], \end{aligned}$$

where  $\varepsilon[|\bar{x}| + |\bar{y}|]$  is bounded independently of  $\varepsilon$  from (6.25). For the last term:

$$|e^{\gamma\bar{t}}f_1(\bar{t}, \bar{x}, \bar{\alpha}) - e^{\gamma\bar{t}}f_1(\bar{t}, \bar{y}, \bar{\alpha})| \leq e^{\gamma T}L_{f_2}|\bar{x} - \bar{y}|.$$

We recall that  $\varepsilon \leq 1$ . Gathering everything, (6.30) becomes

$$\begin{aligned} (\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] &\leq -\varepsilon\delta e^{-\delta\bar{t}}[|\bar{x}|^2 + |\bar{y}|^2] + 2\varepsilon\delta L_b e^{-\delta\bar{t}}|\bar{x}|[|\bar{x}| + |\bar{y}|] \\ &\quad + C_2(\tilde{K} + 1)(|\bar{\alpha}| + 1) + |\bar{x} - \bar{y}|[\tilde{K}(L_b + C_0L_\sigma^2) + L_{f_1}e^{\gamma T} + C_3], \end{aligned} \quad (6.31)$$

for constants  $C_2, C_3$  which do not depend on  $(\varepsilon, \tilde{K}, \mathcal{A})$ , and which only depend on  $(u, f_2)$  through  $(L_u, \|L_{f_2}\|_{L^1})$ . Using the bound (6.17) on  $|\bar{\alpha}|$ , we eventually get that

$$(\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] \leq \varepsilon[-\delta + 2L_b]e^{-\delta\bar{t}}[|\bar{x}|^2 + |\bar{y}|^2] + C_4 + |\bar{x} - \bar{y}|[\tilde{K}(L_b + C_0L_\sigma^2) + C_5],$$

for constants  $C_4, C_5$  which do not depend on  $\varepsilon$ , and which only depend on  $(u, \mathcal{A}, f_2)$  through  $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ .



**Step 5. Choice of parameters.** Let us fix the values of parameters:

$$\delta := 2L_b, \quad \gamma := c + L_b + C_0 L_\sigma^2 + 2,$$

$$\tilde{K} := \max(e^{\gamma T} L_g + e^{\gamma T} \|L_{f_2}\|_{L^1}, C_5 + e^{\gamma T} \|L_{f_2}\|_{L^1}),$$

where we recall that  $C_0$  was not depending on anything. This yields

$$(\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] \leq [\gamma - c - 1][\tilde{K} - e^{\gamma T} \|L_{f_2}\|_{L^1}]|\bar{x} - \bar{y}| + C_4,$$

so that, using (6.24),

$$\bar{M} \leq v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq C_4.$$

This uniform bound on  $\bar{M}$  allows us to send  $\varepsilon$  to 0 to get that

$$\sup_{t,x,y} v(t, x) - v(t, y) - \left[ \tilde{K} - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] [1 + |x - y|^2]^{1/2} \leq C_4.$$

We conclude the proof by choosing  $\tilde{M} := \tilde{K} + C_4$ .  $\square$

We are now ready to prove the Lipschitz estimate using the Ishii-Lions method. To exploit the uniform ellipticity of  $\sigma\sigma^\top$ , we are going to use a strictly concave function  $\psi$  as a test function.

**Proposition 6.3.3** (Lipschitz estimate). *Let  $u$  in  $C([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T) \times \mathbb{R}^d)$  be a solution of (6.1) and  $L_u > 0$  such that for every  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,*

$$|u(t, x)| \leq L_u[1 + |x|].$$

*Under (A.1)-(A.2), there exists a positive constant  $K > 0$  such that*

$$\forall t \in [0, T], \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |u(t, x) - u(t, y)| \leq K|x - y|, \quad (6.32)$$

*where  $K$  only depends on  $(u, \mathcal{A}, f_2)$  through  $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ .*

*Proof.* As previously, we look for  $K > 0$  such that

$$\sup_{t,x,y} u(t, x) - v(t, y) - K|x - y| \leq 0.$$

We first introduce regularising parameters.

**Step 1. Regularising parameters.** As in the proof of Lemma 6.3.2, for any  $\gamma > 0$ , we introduce the  $C^{1,2}$  solution  $v : (t, x) \mapsto e^{\gamma t} u(t, x)$  of (6.21). It is equivalent to prove (6.3) for  $v$  instead of  $u$ , and  $v$  has linear growth with constant  $e^{\gamma T} L_u$ . To make use of uniform ellipticity, we consider a smooth function  $\rho : \mathbb{R}_+ \rightarrow [0, 1]$  such that

$$\begin{cases} \rho(z) = \frac{1}{3}z^{3/2} & \text{for } z \in [0, 1], \\ \rho(z) = 0 & \text{for } z \geq 2, \end{cases}$$

and we define the non-negative function  $\psi(z) := z - \rho(z)$ . As previously, we introduce regularising parameters  $\varepsilon, \delta, \varepsilon > 0$ , before considering the supremum

$$\begin{aligned} \bar{M} := \sup_{t,x,y} v(t, x) - v(t, y) - \varepsilon t^{-1} - \varepsilon e^{-\delta t} [|x|^2 + |y|^2] \\ - \left[ K - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] \psi(|x - y|). \end{aligned} \quad (6.33)$$

As previously,  $\gamma$ ,  $\delta$  and  $K$  are fixed parameters whose values will be chosen later on. On the contrary,  $\varepsilon$  will be sent to 0, and we restrict ourselves to  $\varepsilon \leq 1$ . The  $L_{f_2}$ -term within the supremum will allow us to control the  $f_2$ -term in (6.21) using only  $\|L_{f_2}\|_{L^1}$ . In the following, we restrict ourselves to  $K \geq e^{\gamma T} \|L_{f_2}\|_{L^1}$ .

The quadratic term ensures that the supremum (6.33) is realised for some  $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ . Moreover,  $\bar{t} \neq 0$  because the supremum is finite. If there existed a sequence  $(\varepsilon_k)_{k \geq 1}$  of positive parameters which converges to 0 such that the related sequence  $(\bar{t}_k)_{k \geq 1}$  had a constant sub-sequence equal to  $T$ , we would get the desired result as soon as  $K \geq e^{\gamma T} L_g + e^{\gamma T} \|L_{f_2}\|_{L^1}$ , reasoning as in **Step 2.** in the proof of Lemma 6.3.2. Similarly, the result would be direct if there existed  $(\varepsilon_k)_{k \geq 1}$  which converges to 0 while  $\bar{M} \leq 0$  for every  $k$ .

**Step 2. Bounds on the optimiser.** We thus restrict ourselves to  $K \geq e^{\gamma T} L_g + e^{\gamma T} \|L_{f_2}\|_{L^1}$  and we reason by contradiction, assuming that  $\bar{t} \neq T$  and  $\bar{M} > 0$  for  $\varepsilon$  small enough: we are going to show that this cannot happen if  $K$  is larger than a certain threshold which does not depend on  $\varepsilon$ . From Lemma 6.3.2, we deduce that there exist large enough constant  $\tilde{K}, \tilde{M} > 0$  such that

$$\sup_{t,x,y} v(t,x) - v(t,y) - \tilde{K}|x-y| \leq \tilde{M},$$

where  $(\tilde{K}, \tilde{M})$  only depends on  $(u, \mathcal{A}, f_2)$  through  $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ . From  $\bar{M} > 0$ , we get that

$$\begin{aligned} \left[ K - e^{\gamma T} \int_0^t L_{f_2}(s) ds - \tilde{K} \right] [|\bar{x} - \bar{y}| - \rho(|\bar{x} - \bar{y}|)] \\ \leq v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) - \tilde{K}[|\bar{x} - \bar{y}| - \rho(|\bar{x} - \bar{y}|)]. \end{aligned}$$

For every  $z \geq 0$ , we have  $\frac{1}{2}z - \rho(z) \geq 0$  and  $\rho(z) \leq 1$ , hence

$$\frac{1}{2} \left[ K - e^{\gamma T} \int_0^t L_{f_2}(s) ds - \tilde{K} \right] |\bar{x} - \bar{y}| \leq \tilde{K} + \sup_{t,x,y} v(t,x) - v(t,y) - \tilde{K}|x-y| \leq \tilde{K} + \tilde{M}.$$

Up to increasing the value of  $\tilde{K}$ , we can thus assume that for  $K \geq \tilde{K}$ ,

$$|\bar{x} - \bar{y}| \leq C_0 K^{-1}, \quad (6.34)$$

for some  $C_0 > 0$  which only depends on  $(e^{\gamma T}, \tilde{M}, \tilde{K}, \|L_{f_2}\|_{L^1})$ . Up to increasing  $\tilde{K}$  again, we can then assume that  $|\bar{x} - \bar{y}| \leq 1$  for  $K \geq \tilde{K}$ . Using the linear growth of  $v$  and  $\bar{M} > 0$ ,

$$\varepsilon e^{-\delta \bar{t}} [|\bar{x}|^2 + |\bar{y}|^2] \leq v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq L_u e^{\gamma T} [2 + |\bar{x}| + |\bar{y}|].$$

Up to changing  $L_u$  in  $\max(1, L_u)$ , this implies that

$$\varepsilon |\bar{x}| + \varepsilon |\bar{y}| \leq 4e^{(\delta+\gamma)T} L_u, \quad (6.35)$$

preventing  $\varepsilon |\bar{x}|$  and  $\varepsilon |\bar{y}|$  from exploding.

**Step 3. Optimality conditions.** To ease computations, we introduce the function  $\chi : z \in \mathbb{R}^d \mapsto |z|$ . From  $\bar{M} > 0$ , necessarily  $\bar{x} \neq \bar{y}$ . Since  $\bar{t} \notin \{0, T\}$ , the first-order optimality conditions in (6.33) provide

$$\begin{cases} Dv(\bar{t}, \bar{x}) = [K - e^{\gamma T} \int_0^t L_{f_2}(s) ds] \psi'(|\bar{x} - \bar{y}|) D\chi(\bar{x} - \bar{y}) + 2\varepsilon e^{-\delta \bar{t}} \bar{x}, \\ Dv(\bar{t}, \bar{y}) = [K - e^{\gamma T} \int_0^t L_{f_2}(s) ds] \psi'(|\bar{x} - \bar{y}|) D\chi(\bar{x} - \bar{y}) - 2\varepsilon e^{-\delta \bar{t}} \bar{y}, \\ \partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y}) \\ \quad = -e^{\gamma T} L_{f_2}(\bar{t}) [1 + |\bar{x} - \bar{y}|^2]^{1/2} - \varepsilon \bar{t}^{-2} - \varepsilon \delta e^{-\delta \bar{t}} [|\bar{x}|^2 + |\bar{y}|^2]. \end{cases} \quad (6.36)$$

Since  $|\bar{x} - \bar{y}| \leq 1$ , we notice that  $0 \leq \psi'(|\bar{x} - \bar{y}|) \leq 1$ . We then write the second-order optimality conditions in  $(x, y)$  for (6.33):

$$\begin{pmatrix} D^2v(\bar{t}, \bar{x}) & 0 \\ 0 & -D^2v(\bar{t}, \bar{y}) \end{pmatrix} \leq \left[ K - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) ds \right] \begin{pmatrix} \bar{A} & -\bar{A} \\ -\bar{A} & \bar{A} \end{pmatrix} + 2\varepsilon e^{-\delta \bar{t}} \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}, \quad (6.37)$$

where

$$\bar{A} := \psi''(|\bar{x} - \bar{y}|) D\chi(\bar{x} - \bar{y}) \otimes D\chi(\bar{x} - \bar{y}) + \psi'(|\bar{x} - \bar{y}|) D^2\chi(\bar{x} - \bar{y}).$$

For every  $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ , applying (6.37) to the vector  $\begin{pmatrix} p & q \end{pmatrix}$  and using  $L_{f_2} \geq 0$ , we get that

$$\begin{aligned} p^\top D^2v(\bar{t}, \bar{x})p - q^\top D^2v(\bar{t}, \bar{y})q \\ \leq \left[ K - \int_0^{\bar{t}} L_{f_2}(s) ds \right] (p - q)^\top \bar{A} (p - q) + 2\varepsilon e^{-\delta \bar{t}} [|p|^2 + |q|^2]. \end{aligned} \quad (6.38)$$

Since  $|D\chi(z)|^2 = 1$  for every  $z$ , we get that  $D^2\chi(z)D\chi(z) = 0$ . Using (6.38) with  $p = -q = D\chi(\bar{x} - \bar{y})$  thus yields

$$\begin{aligned} D\chi(\bar{x} - \bar{y})^\top [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] D\chi(\bar{x} - \bar{y}) \\ \leq 4 \left[ K - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) ds \right] \psi''(|\bar{x} - \bar{y}|) + 4\varepsilon e^{-\delta \bar{t}}. \end{aligned}$$

Similarly, taking  $p = q$  in (6.38) gives that

$$\forall p \in \mathbb{R}^d, \quad p^\top [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] p \leq 4\varepsilon e^{-\delta \bar{t}} |p|^2. \quad (6.39)$$

Completing  $D\chi(\bar{x} - \bar{y})$  in an orthonormal basis  $(D\chi(\bar{x} - \bar{y}), p_2, \dots, p_d)$  of  $\mathbb{R}^d$ , this proves that

$$\begin{aligned} \text{Tr}[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] &= D\chi(\bar{x} - \bar{y})^\top [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] D\chi(\bar{x} - \bar{y}) \\ &\quad + \sum_{i=2}^d p_i^\top [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] p_i \\ &\leq 4 \left[ K - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) ds \right] \psi''(|\bar{x} - \bar{y}|) + 4\varepsilon d e^{-\delta \bar{t}}. \end{aligned} \quad (6.40)$$

As  $K \rightarrow +\infty$ ,  $|\bar{x} - \bar{y}| \rightarrow 0$  from (6.34), and our specific choice of  $\psi$  makes the r.h.s. of (6.40) go to  $-\infty$ : this is the main trick of the Ishii-Lions method.

**Step 4.** *Control of the second-order terms.* Let us introduce the matrix  $a := \sigma \sigma^\top$ . From (A.2),  $(t, x) \mapsto a(t, x)$  is uniformly continuous with modulus  $m_\sigma$ . We decompose:

$$\begin{aligned} \text{Tr}[a(\bar{t}, \bar{x}) D^2v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y}) D^2v(\bar{t}, \bar{y})] \\ = \frac{1}{2} \text{Tr}\{a(\bar{t}, \bar{x}) [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})]\} + \frac{1}{2} \text{Tr}\{a(\bar{t}, \bar{y}) [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})]\} \\ + \frac{1}{2} \text{Tr}\{[a(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y})] [D^2v(\bar{t}, \bar{x}) + D^2v(\bar{t}, \bar{y})]\}. \end{aligned} \quad (6.41)$$

We now apply Lemma 6.A.1 in Appendix with  $A = a(\bar{t}, \bar{x})$ ,  $B = D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})$ ,  $m = \eta_\sigma/2$  and  $M = 4\epsilon e^{-\delta\bar{t}}$ . The inequality  $A \geq m\text{Id}$  stems from (A.2), while  $B \leq M\text{Id}$  results from (6.39). This gives that

$$\text{Tr}\{a(\bar{t}, \bar{x})[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})]\} \leq \frac{\eta_\sigma}{2} \text{Tr}[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] + 4\epsilon e^{-\delta\bar{t}} \text{Tr}[a(\bar{t}, \bar{x})],$$

and  $\text{Tr}[a(\bar{t}, \bar{x})]$  has linear growth from (A.1). Using (6.35),  $4\epsilon e^{-\delta\bar{t}} \text{Tr}[a(\bar{t}, \bar{x})]$  is bounded by a constant  $C_1$  which only depends on  $(L_\sigma, T, \delta, L_u)$ . The same bound works for  $\text{Tr}\{a(\bar{t}, \bar{y})[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})]\}$ . We then use that  $A \mapsto \sup_{|\xi|=1} \xi^\top A \xi$  is a sub-multiplicative norm on symmetric matrices and that all norms are equivalent in finite dimension. By (A.1) and the continuity of the trace, there exists a constant  $C_2 > 0$  which only depends on the dimension  $d$  such that

$$\begin{aligned} \text{Tr}\{[a(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y})][D^2v(\bar{t}, \bar{x}) + D^2v(\bar{t}, \bar{y})]\} \\ \leq C_2 m_\sigma (|\bar{x} - \bar{y}|) \sup_{|\xi|=1} \xi^\top [D^2v(\bar{t}, \bar{x}) + D^2v(\bar{t}, \bar{y})] \xi. \end{aligned} \quad (6.42)$$

Using (6.37), we now apply Lemma 6.A.2 with  $A = \bar{A}$ ,  $X = D^2v(\bar{t}, \bar{x})$ ,  $Y = -D^2v(\bar{t}, \bar{x})$ ,  $m = 2\epsilon e^{-\delta\bar{t}}$  and  $C = K - e^{\gamma T} \int_0^t L_{f_2}(s) ds$ . As a consequence, the r.h.s. of (6.42) is bounded by

$$\sqrt{2} C_2 m_\sigma (|\bar{x} - \bar{y}|) \left\{ 4\epsilon e^{-\delta\bar{t}} + 2 \left[ K - e^{\gamma T} \int_0^t L_{f_2}(s) ds \right] + \sup_{|\xi|=1} \xi^\top [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] \xi \right\}.$$

The supremum that appears within the above quantity equals the spectral radius of the symmetric matrix  $D^2u(\bar{t}, \bar{x}) - D^2u(\bar{t}, \bar{y})$ . From (6.39), any eigenvalue of  $D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})$  is bounded from above by  $4\epsilon e^{-\delta\bar{t}}$ . As a consequence, any eigenvalue of  $D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})$  is bounded from below by

$$\text{Tr}[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] - 4(d-1)\epsilon e^{-\delta\bar{t}},$$

so that

$$\sup_{|\xi|=1} \xi^\top [D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] \xi \leq |\text{Tr}[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})]| + 4d\epsilon e^{-\delta\bar{t}}.$$

At the end of the day, gathering everything from (6.41) yields

$$\begin{aligned} \text{Tr}[a(\bar{t}, \bar{x})D^2v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y})D^2v(\bar{t}, \bar{y})] &\leq \frac{\eta_\sigma}{2} \text{Tr}[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})] + C_1 \\ &+ \sqrt{2} C_2 m_\sigma (|\bar{x} - \bar{y}|) [2\epsilon e^{-\delta\bar{t}} + K + 2d\epsilon e^{-\delta\bar{t}} + \frac{1}{2} |\text{Tr}[D^2v(\bar{t}, \bar{x}) - D^2v(\bar{t}, \bar{y})]|]. \end{aligned} \quad (6.43)$$

Using (6.40), we see that the r.h.s. of (6.43) goes to  $-\infty$  as  $K \rightarrow +\infty$ .

**Step 5. Gathering the PDE.** We now write the PDE (6.21) at  $(\bar{t}, \bar{x})$  and  $(\bar{t}, \bar{y})$ . Let  $\bar{\alpha}$  be a minimiser in  $\alpha$  of  $L_\alpha[v](\bar{t}, \bar{x})$ , defined in (6.22). Subtracting both PDEs, we get that

$$\begin{aligned} (\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] &= \partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y}) + L_{f_2}(\bar{t}) e^{\gamma\bar{t}} [f_2(\bar{t}, \bar{x}) - f_2(\bar{t}, \bar{y})] \\ &+ L_{\bar{\alpha}}[v](\bar{t}, \bar{x}) - L_{\bar{\alpha}}[v](\bar{t}, \bar{y}) + \frac{1}{2} \text{Tr}[a(\bar{t}, \bar{x})D^2v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y})D^2v(\bar{t}, \bar{y})]. \end{aligned}$$

Using (6.35)-(6.36), we now reason as in **Step 4.** in the proof of Lemma 6.3.2: we replace  $\partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y})$  by its computed value in (6.36), we use it to get rid of  $L_{f_2}(\bar{t}) e^{\gamma\bar{t}} [f_2(\bar{t}, \bar{x}) -$

$f_2(\bar{t}, \bar{y})]$ , we get rid of the  $-\varepsilon\bar{t}^{-2}$  term, and then we control the  $L_{\bar{\alpha}}[v](\bar{t}, \bar{x}) - L_{\bar{\alpha}}[v](\bar{t}, \bar{y})$  term. For the  $b$ -term, we now use the Lipschitz assumption (6.10) on  $b_1$  and  $b_2$ :

$$\begin{aligned} & b(\bar{t}, \bar{x}, \bar{\alpha}) \cdot Dv(\bar{t}, \bar{x}) - b(\bar{t}, \bar{y}, \bar{\alpha}) \cdot Dv(\bar{t}, \bar{y}) \\ & \leq |Dv(\bar{t}, \bar{y})| |b(\bar{t}, \bar{x}, \bar{\alpha}) - b(\bar{t}, \bar{y}, \bar{\alpha})| + |b(\bar{t}, \bar{x}, \bar{\alpha})| |Dv(\bar{t}, \bar{x}) - Dv(\bar{t}, \bar{y})| \\ & \leq L_b[\tilde{K} + 8e^{(\delta+\gamma)T} L_u](1 + |\bar{\alpha}|) |\bar{x} - \bar{y}| + 2\varepsilon e^{-\delta\bar{t}} L_b[1 + |\bar{x}| + |\bar{\alpha}|] (|\bar{x}| + |\bar{y}|), \end{aligned}$$

where from (6.35),  $\varepsilon[|\bar{x}| + |\bar{y}|]$  is bounded independently of  $\varepsilon$ . Similarly,

$$|e^{\gamma t} f_1(\bar{t}, \bar{x}, \bar{\alpha}) - f_1(\bar{t}, \bar{y}, \bar{\alpha})| \leq e^{\gamma T} L_{f_2} |\bar{x} - \bar{y}|.$$

As in **Step 4.** in the proof of Lemma 6.3.2, we obtain using (6.17) that  $|\bar{\alpha}| \leq L_{\mathcal{A}}[1 + |Dv(\bar{t}, \bar{y})|]$ . Gathering everything, the analogous of (6.31) now reads

$$\begin{aligned} (\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] & \leq \varepsilon[2L_b - \delta] \delta e^{-\delta\bar{t}} [|\bar{x}|^2 + |\bar{y}|^2] \\ & \quad + \frac{1}{2} \text{Tr}[a(\bar{t}, \bar{x}) D^2 v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y}) D^2 v(\bar{t}, \bar{y})] \\ & \quad + L_b[\tilde{K} + 8e^{(\delta+\gamma)T} L_u](1 + K) |\bar{x} - \bar{y}| + C_3(1 + K + |\bar{x} - \bar{y}|), \end{aligned}$$

for a constant  $C_3$  which does not depend on  $(\varepsilon, K)$ , and which only depends on  $(u, \mathcal{A}, f_2)$  through  $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ .

**Step 6. Contradiction.** We now fix the values  $\gamma := c$  and  $\delta := 2L_b$ . Using (6.34) and imposing that  $K \geq 1$ , we get that

$$\begin{aligned} 0 \leq \frac{1}{2} \text{Tr}[a(\bar{t}, \bar{x}) D^2 v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y}) D^2 v(\bar{t}, \bar{y})] \\ + 2C_0 L_b [K + 8e^{(\delta+\gamma)T} L_u] + 2C_3(1 + K + C_0). \end{aligned} \quad (6.44)$$

Similarly, (6.43) becomes

$$\begin{aligned} \text{Tr}[a(\bar{t}, \bar{x}) D^2 v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y}) D^2 v(\bar{t}, \bar{y})] & \leq \frac{\eta_\sigma}{2} \text{Tr}[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] + C_1 \\ & \quad + \sqrt{2} C_2 m_\sigma(|\bar{x} - \bar{y}|) [2 + K + 2d + \frac{1}{2} |\text{Tr}[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})]|], \end{aligned} \quad (6.45)$$

and  $m_\sigma(|\bar{x} - \bar{y}|) \rightarrow 0$  as  $K \rightarrow +\infty$ . From (6.34)-(6.40),

$$\text{Tr}[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] \leq -3\sqrt{C_0} K^{3/2} + 4d,$$

proving that  $\text{Tr}[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})]$  goes to  $-\infty$  as  $K \rightarrow +\infty$  uniformly in  $\varepsilon$ . From (6.45),  $\text{Tr}[a(\bar{t}, \bar{x}) D^2 v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y}) D^2 v(\bar{t}, \bar{y})]$  goes to  $-\infty$  at least as fast as  $-K^{3/2}$  as  $K \rightarrow +\infty$ , uniformly in  $\varepsilon$ . From this, there exists a finite value  $K' \geq \max(1, \tilde{K}, e^{\gamma T}(L_g + \|L_{f_2}\|_{L^1}))$  independent of  $\varepsilon$  such that the r.h.s. of (6.44) is always negative for  $K \geq K'$ . This gives the desired contradiction. We moreover notice that  $K'$  only depends on  $(u, \mathcal{A}, f_2)$  through  $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$ .  $\square$

*Proof of Theorem 6.2.2.* From Lemma 6.3.1, there exists a growth constant  $L$  which is valid for any  $C^{1,2}$  linear growth solution of (6.1), and  $L$  only depends on  $\mathcal{A}$  through  $L_{\mathcal{A}}$ . Let  $K$  be the value given by Proposition 6.3.3 for  $L_u = L$ :  $K$  only depends on  $\mathcal{A}$  through  $L_{\mathcal{A}}$ .  $K$  can thus be chosen as an increasing function of  $L_{\mathcal{A}}$ . Up to choosing a larger  $L_{\mathcal{A}}$  in (6.17), we can assume that  $L_{\mathcal{A}}[1 + K] \geq R_{\mathcal{A}}$ , where  $R_{\mathcal{A}}$  is the constant given by (A.3). Let us define the compact sub-set of  $\mathcal{A}$ :

$$\mathcal{A}' := \mathcal{A} \cap B(0, L_{\mathcal{A}}[1 + L + K]).$$

We now introduce a truncated version of (6.1): we truncate the non-linearity by using the compact control set  $\mathcal{A}'$  instead of  $\mathcal{A}$ . From [Rubio, 2011, Theorem 3.1], the resulting truncated equation has a unique linear growth solution  $\tilde{u}$  in  $C([0, T] \times \mathbb{R}^d) \cap C_{\text{loc}}^{1,2,\beta,\beta}$ . From the definition of  $\mathcal{A}'$ , (6.17) is still valid when replacing  $\mathcal{A}$  by  $\mathcal{A}'$ , with the same constant  $L_{\mathcal{A}}$ . Since the Lipschitz estimate of Proposition 6.3.3 only depends on  $\mathcal{A}$  through  $L_{\mathcal{A}}$ ,  $\tilde{u}$  is  $K$ -Lipschitz continuous with the same constant  $K$ . This proves that  $\tilde{u}$  does not feel the truncation, hence  $\tilde{u}$  is a  $C^{1,2}$  solution of (6.1).

Reciprocally, any linear growth solution in  $C([0, T] \times \mathbb{R}^d) \cap C_{\text{loc}}^{1,2,\beta,\beta}$  of (6.1) is  $K$ -Lipschitz continuous from Lemma 6.3.1 and Proposition 6.3.3, hence it is a solution of the truncated equation. Since uniqueness holds for the truncated equation, this concludes the proof.  $\square$

**Remark 6.3.4** (Possible improvements of Lipschitz hypotheses). *The Lipschitz assumptions on the coefficients are two-fold. For instance, (6.11) provides a control of  $|f_1(t, x, \alpha) - f_1(t, y, \alpha)|$  when  $|x - y|$  is large, together with local regularity when  $|x - y|$  is small. Both these properties are needed in the proofs, but not simultaneously. The proof of Lemma 6.3.2 only uses this control when  $|x - y|$  is large, whereas the proof of Proposition 6.3.3 uses it when  $|x - y|$  is small. This suggests a weakening of the assumption by separating these properties: we could require that  $r_0 > 0$  exists such that*

$$\begin{cases} |f_1(t, x, \alpha) - f_1(t, y, \alpha)| \leq 1 + L_{f_1}|x - y| + h(|\alpha|), & \text{if } |x - y| > r_0, \\ |f_1(t, x, \alpha) - f_1(t, y, \alpha)| \leq L_{f_1}(1 + |\alpha|^2)|x - y|^\mu, & \text{if } |x - y| \leq r_0, \end{cases}$$

for any non-negative  $h : \mathcal{A} \rightarrow \mathbb{R}$  and  $\mu \in (0, 1]$ . Using this assumption instead of (6.11), the proof of Lemma 6.3.2 would still work without any change. The proof of Proposition 6.3.3 should then be modified by considering  $\rho(z) = z^{1+\nu}$  with  $\nu < \mu$ , instead of  $\rho(z) = z^{3/2}$ . We could split the condition (6.10) similarly. For the sake of clarity, we refrained ourselves from adding these subtleties in (A.1).

## Appendix

### 6.A Linear algebra results

This appendix gathers two linear algebra results that were needed for **Step 4.** in the proof of Proposition 6.3.3. These lemmas correspond to intermediary results which can be respectively found within the proofs of [Ishii and Lions, 1990, Proposition III.1] and [Ishii and Lions, 1990, Lemma III.1]. We provide the proofs for the sake of completeness.

**Lemma 6.A.1.** *Let  $A$  and  $B$  be two symmetric matrices in  $\mathbb{R}^d$  with  $A \geq m\text{Id}$  and  $B \leq M\text{Id}$  for real numbers  $m, M \geq 0$ . We have that*

$$\text{Tr}[AB] \leq m\text{Tr}[B] + M[\text{Tr}[A] - dm].$$

*Proof.* We first handle the case  $M = 0$ . Let us write  $A = P^\top DP$  for some orthogonal matrix  $P$  and  $D := \text{diag}(d_1, \dots, d_d)$ . The  $d_i$  are the eigenvalues of  $A$  and all satisfy  $d_i \geq m$ . Moreover, using the symmetry property of the trace,

$$\text{Tr}[AB] = \text{Tr}[D(PBP^\top)] = \sum_{i=1}^d d_i e_i^\top (PBP^\top) e_i,$$

where  $(e_1, \dots, e_d)$  is an orthonormal basis of  $\mathbb{R}^d$ . From the non-positiveness assumption  $M \leq 0$  on  $B$ , we have  $e_i^\top (PBP^\top) e_i \leq 0$ , so that

$$\text{Tr}[AB] \leq m \sum_{i=1}^d e_i^\top (PBP^\top) e_i = m\text{Tr}[PBP^\top] = m\text{Tr}[B].$$

In the general case  $B$  does not satisfy the non-positiveness condition, but  $B - M\text{Id}$  does. The above result applied to  $B - M\text{Id}$  then yields

$$\text{Tr}[AB] - M\text{Tr}[A] \leq m\text{Tr}[B] - mM\text{Tr}[\text{Id}] = m\text{Tr}[B] - dmM,$$

concluding the proof.  $\square$

**Lemma 6.A.2.** *Let  $A$ ,  $X$  and  $Y$  be symmetric matrices in  $\mathbb{R}^d$  such that*

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq C \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} + m \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}, \quad (6.46)$$

for real numbers  $C, m > 0$ . Then

$$\sup_{|\xi|=1} \xi^\top (X - Y)\xi \leq \sqrt{2} \left[ 2m + 2C + \sup_{|\xi|=1} \xi^\top (X + Y)\xi \right].$$

*Proof.* The inequality (6.46) is unchanged if we multiply each side on the left and on the right by the symmetric matrix  $\begin{pmatrix} \text{Id} & \text{Id} \\ \text{Id} & -\text{Id} \end{pmatrix}$ . This yields

$$\begin{pmatrix} X + Y & X - Y \\ X - Y & X + Y \end{pmatrix} \leq 4C \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} + 2m \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix}.$$

For any  $(t, \xi)$  in  $\mathbb{R} \times \mathbb{R}^d$ , we apply this to the vector  $\begin{pmatrix} t\xi & \xi \end{pmatrix}$ , so that

$$t^2 \xi^\top (X + Y)\xi + 2t \xi^\top (X - Y)\xi + \xi^\top (X + Y)\xi \leq 4C \xi^\top A \xi + 2m(t^2 + 1)|\xi|^2.$$

Since this holds for every  $t$  in  $\mathbb{R}$ , we get that

$$\begin{aligned} [\xi^\top (X - Y)\xi]^2 &\leq [\xi^\top (X + Y)\xi - 2m|\xi|^2][\xi^\top (X + Y)\xi - 4C \xi^\top A \xi - 2m|\xi|^2] \\ &\leq \frac{1}{2}[\xi^\top (X + Y)\xi - 2m|\xi|^2]^2 + \frac{1}{2}[\xi^\top (X + Y)\xi - 4C \xi^\top A \xi - 2m|\xi|^2]^2. \end{aligned}$$

To conclude, we apply this to any normalised vector  $\xi$  and we take the square-root of each side.  $\square$

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