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Bastien Jean

► **To cite this version:**

Bastien Jean. L2 cohomology of a polarized variation of Hodge structures on an infinite covering of a smooth open algebraic curve. Algebraic Geometry [math.AG]. Université Grenoble - Alpes, 2024. English. NNT: . tel-04858452

HAL Id: tel-04858452

<https://hal.science/tel-04858452v1>

Submitted on 29 Dec 2024

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THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES

École doctorale : MSTII - Mathématiques, Sciences et technologies de l'information, Informatique

Spécialité : Mathématiques

Unité de recherche : Institut Fourier

Cohomologie L2 des variations de structures de Hodge sur les revêtements de courbe algébrique lisse ouverte

L2 cohomology of a polarized variation of Hodge structures on an infinite covering of a smooth open algebraic curve

Présentée par :

Bastien JEAN

Direction de thèse :

Philippe EYSSIDIEUX
PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES

Directeur de thèse

Rapporteurs :

BENOÎT CLAUDON
PROFESSEUR DES UNIVERSITES, UNIVERSITE DE RENNES
JUNYAN CAO
PROFESSEUR DES UNIVERSITES, UNIVERSITE COTE D'AZUR

Thèse soutenue publiquement le **23 octobre 2024**, devant le jury composé de :

EMMANUEL PEYRE, PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES	Président
PHILIPPE EYSSIDIEUX, PROFESSEUR DES UNIVERSITES, UNIVERSITE GRENOBLE ALPES	Directeur de thèse
BENOÎT CLAUDON, PROFESSEUR DES UNIVERSITES, UNIVERSITE DE RENNES	Rapporteur
JUNYAN CAO, PROFESSEUR DES UNIVERSITES, UNIVERSITE COTE D'AZUR	Rapporteur
CATRIONA MACLEAN, MAITRESSE DE CONFERENCES HDR, UNIVERSITE GRENOBLE ALPES	Examinatrice
YA DENG, CHARGE DE RECHERCHE, CNRS DELEGATION CENTRE-EST	Examineur



Remerciements

Me voilà arrivé à la fin de ma thèse, d'où je peux contempler la fin de mon manuscrit. Ce travail n'aurait pas été possible sans l'aide de Philippe, mon directeur de thèse, qui a su me guider tout au long de ces quatre années de doctorat. Je te remercie grandement pour toutes nos discussions mathématiques. Ces nombreuses heures passées ensemble m'ont permis de progresser grandement et de gagner l'intuition nécessaire à ce travail de longue haleine.

Je remercie aussi grandement Benoit Claudon et Junyan Cao pour avoir accepté d'être mes rapporteurs. Je les remercie aussi pour leurs remarques qui ont permis l'amélioration de ce manuscrit. Je remercie également Catriona Maclean, Emmanuel Peyre et Ya Deng d'avoir accepté de faire partie de mon jury.

Un travail de recherche se fait également en discutant avec d'autres personnes. Je souhaite donc remercier tous les chercheurs de l'institut Fourier avec qui il m'a été donné l'occasion de discuter de mathématiques. Je remercie également le personnel administratif de l'institut Fourier, en particulier Christine, Laurence et Tiana pour leur aide, en particulier au moment des départs en mission.

Je remercie également tous les doctorants de l'institut Fourier qui ont rendu ces quatre années très agréables et agrémentées les pauses-café de nombreuses pâtisseries. Je remercie en particulier les doctorants en géométrie avec qui j'ai participé dans des groupes de travail : Omar, Loïs, Rufus et Matilde ce fut un plaisir de discuter de mathématiques avec vous. Dans un autre registre, je remercie aussi en particulier les (nombreuses) personnes de l'Institut Fourier qui m'ont accompagné dans les randonnées autour de Grenoble au fil des années : Alexis, Nicolas, Lucie, Arielle, Magali, Felix, Yohan, Vivek, Julien, Matthieu, Marie, Samuel, Chloë et Chloé.

J'adresse également mes remerciements aux personnes qui m'accompagnent depuis mes études à l'ENS Rennes : David, Clarence, Antoine, Maxence, Émeline et Louis. Je ne me lasserai pas de nos discussions. Je vous remercie également (avec Raoul) pour ces bons moments au festival Court Métrange. Je remercie également ma bande d'amis du lycée : Harriet, Gwenaëlle, Titouan, Raphaël, Etienne et Aurélie. C'est toujours un plaisir de vous retrouver à chaque nouvelle année qui passe. J'adresse également des remerciements à l'équipe des Scouts de Saint-Thomas qui fut également une bonne rencontre au cours de ces années grenobloise. Je remercie en particulier Cristina de m'y avoir invité.

J'adresse également mes parents pour toujours avoir été à mes côtés au cours de mon parcours.

J'adresse finalement mes derniers remerciements à Paola que j'ai rencontré à la fin de ma thèse et qui m'accompagne maintenant depuis quelques mois et me suit dans la vie après la thèse.

Introduction

Contexte général et résultats principaux.

Le sujet de cette thèse porte sur la cohomologie L^2 des variations de structure de Hodge sur des revêtements galoisiens (potentiellement infinis) de courbe lisse quasi-projective définie sur \mathbb{C} . Elle est donc à l'intersection de deux sujets : la cohomologie L^2 des revêtements et la théorie de Hodge. Avant d'énoncer nos résultats, nous donnons un rapide survol de ces théories.

Théorie de Hodge et variations de structure de Hodge. La théorie de Hodge prend ses racines en 1941 quand W. Hodge démontre que les groupes de cohomologie d'une variété projective lisse X admettent une décomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

dont la composante $H^{p,q}(X, \mathbb{C})$ s'identifie naturellement au groupe de cohomologie de Dolbeault $H^q(X, \Omega_X^p)$. Si les groupes $H^k(X, \mathbb{C})$ ne dépendent que de la topologie de X la décomposition dépend de la structure complexe de X et nous donne donc des invariants supplémentaires. Ce résultat a de nombreuses applications en géométrie algébrique, il permet par exemple de plonger une courbe projective lisse dans une variété abélienne (dite variété jacobienne) qui est définie par

$$H^{1,0}(X, \mathbb{C})^* / H_1(X, \mathbb{Z}) \cap H^{1,0}(X, \mathbb{C})^*.$$

En dimension supérieure, il permet de construire la variété d'Albanese et les variétés jacobiniennes intermédiaires. Par l'importance de ce résultat, il est naturel d'étudier les structures de Hodge en famille. Le formalisme devient celui des variations de structure de Hodge polarisée qui a été développé par P. Griffiths [Gri68a, Gri68b, Gri70] à la fin des années 60. Une variation de structure de Hodge polarisée sur une variété est alors un fibré vectoriel holomorphe plat \mathcal{V} muni d'une métrique hermitienne h satisfaisant certaines propriétés. On se donne également une décomposition orthogonale en sous fibrés lisses $\mathcal{V} = \bigoplus \mathcal{H}^{p,q}$ pour laquelle on demande que la connexion D satisfasse la condition de transversalité suivante

$$D\mathcal{H}^{p,q} \subset \mathcal{H}^{p,q} \otimes \mathcal{E}^{1,0} \oplus \mathcal{H}^{p-1,q+1} \otimes \mathcal{E}^{1,0} \oplus \mathcal{H}^{p,q} \otimes \mathcal{E}^{0,1} \oplus \mathcal{H}^{p+1,q-1} \otimes \mathcal{E}^{0,1}$$

L'exemple typique est celui du cas géométrique : on se donne $f: \mathcal{X} \rightarrow X$ un morphisme propre à fibres projectives lisses, on peut alors considérer le fibré vectoriel \mathcal{V} dont la fibre en t est la partie primitive du k -ème groupe de cohomologie de $f^{-1}(t)$ et la décomposition est alors la décomposition de Hodge de ce groupe de cohomologie. En général, si l'on se donne un morphisme propre $f: \mathcal{X} \rightarrow X$ sur une base projective, les fibres $f^{-1}(t)$ ne sont lisses qu'en dehors d'un fermé de Zariski $\Sigma \subset X$ que l'on peut supposer être un diviseur à croisements normaux. Nous obtenons donc une variation de structure de Hodge sur $M := X \setminus \Sigma$ qui dégénère sur Σ . P. Deligne conjectura dans les années 70 que les groupes de cohomologie L^2 sur M pour une métrique à singularités Poincaré devait calculer les groupes de cohomologie d'intersection du système local sous-jacent et qu'une identification avec l'espace des formes harmoniques donnerait une structure de Hodge sur ces groupes de cohomologie. La conjecture de Deligne reçut une réponse positive en dimension 1 par un article de S. Zucker [Zuc79] et en dimension supérieure par M. Kashiwara et T. Kawai [KK85] sous des hypothèses sur la monodromie à l'infini et dans le cas général par E. Cattani, A. Kaplan et W. Schmid [CKS87].

Les travaux de M. Saito [Sai88, Sai90] introduisirent la théorie des modules de Hodge, ce qui donna une généralisation de la théorie de Hodge à coefficients dégénèrescents. M. Saito montra en particulier que les groupes

de cohomologie d'intersection d'une variation de structure de Hodge à coefficients dégénérés admettaient une structure de Hodge pure de nature algébrique. Le fait que cette structure de Hodge coïncide avec la précédente est établi en dimension 1 et résulte d'un lemme de Dolbeault démontré par S. Zucker [Zuc79] le cas de la dimension supérieure est encore conjectural et présente un saut de difficulté important. Cette conjecture semble néanmoins avoir finalement reçu une démonstration correcte dans une pré-publication de T. Mochizuki [Moc22].

Cohomologie L^2 des revêtements. La théorie de Hodge sur les variétés complexes est liée à la théorie de la cohomologie L^2 . Une particularité de cette théorie est qu'elle se comporte agréablement pour des revêtements galoisiens éventuellement infinis. La théorie de la cohomologie L^2 appliquée à des revêtements galoisiens de variétés se développe à partir de la seconde moitié des années 70 avec la découverte du théorème d'indice L^2 d'Atiyah [Ati76]. Expliquons-en le principe dans le cas qui va nous intéresser. Donnons-nous M une variété fermée et E un fibré vectoriel sur M . Dans certains cas, il est alors possible, de construire des complexes de cochaînes (\mathcal{E}^\bullet, d) où \mathcal{E}^k sera un espace de k -formes différentielles lisses et d un opérateur différentiel. Les exemples les plus usuels sont :

- M est une variété complexe, E un fibré holomorphe et $d = \bar{\partial}_E$. On obtient dès lors le complexe de Dolbeault de E .
- E est un fibré plat et d est la connexion plate associée. On obtient alors le complexe de de Rham de E .

Généralement, le complexe (\mathcal{E}^\bullet, d) est naturellement quasi-isomorphe au complexe $(L^2\mathcal{E}^\bullet, d)$ où l'on a remplacé l'espace des formes lisses \mathcal{E}^k par l'espace $L^2\mathcal{E}^k$ des formes mesurables de carré intégrable à valeurs dans E , la différentielle d est alors calculée au sens des distributions et est vu comme un opérateur densément défini. On remarque que pour parler de formes de carré intégrable, il faut normalement se fixer une métrique sur M et une métrique sur E , cependant la compacité de M implique que la notion de forme L^2 ne dépend pas de ce choix de métrique. On note par \mathfrak{d} l'adjoint formel de d , alors l'adjoint d^* de d au sens de la théorie des opérateurs est donné par \mathfrak{d} calculé au sens des distributions. Sous des hypothèses d'ellipticité de l'opérateur $d^* + d$, ces complexes ont des groupes de cohomologie de dimension finie et la caractéristique d'Euler du complexe

$$\chi(M, \mathcal{E}^k) = \sum_{j \geq 0} (-1)^j \dim H^j(\mathcal{E}^k)$$

est un invariant topologique de M et de E par les résultats d'Atiyah et Singer.

On se fixe désormais un revêtement galoisien $\pi: \tilde{M} \rightarrow M$ de groupe $\Gamma := \text{Deck}(\tilde{M}/M)$. Dans ce contexte, nous pouvons considérer l'image réciproque de E pour obtenir un fibré \tilde{E} sur \tilde{M} et nous pouvons construire un complexe $(L^2\tilde{\mathcal{E}}^\bullet, d)$. C'est un complexe d'espaces de Hilbert dont nous voulons étudier la cohomologie. Si Γ est infini, il est possible de montrer que les groupes de cohomologie (réduite) sont nuls ou de dimensions infinies, on ne peut donc pas a priori parler de la caractéristique d'Euler de ce complexe. En utilisant l'action naturelle de Γ par isométries sur $L^2\tilde{\mathcal{E}}^k$, nous obtenons une action de l'algèbre de Von Neumann du groupe $\mathcal{N}(\Gamma)$, ce qui nous permet de définir la notion de dimension de Von Neumann \dim_Γ qui est une notion de dimension "renormalisée" par l'action du groupe. Cette notion de dimension satisfait alors les mêmes propriétés que la dimension usuelle, à ceci près qu'elle est à valeurs réelles au lieu d'être à valeurs entières (voir [Lüc02] pour la définition). Il est possible de montrer que les groupes de cohomologie $H^k(L^2\tilde{\mathcal{E}}^\bullet) := H_{L^2}^k(\tilde{\mathcal{E}}^\bullet)$ ont dimension de Von Neumann finie. On peut ainsi définir la caractéristique d'Euler du complexe par

$$\chi_{L^2, \Gamma}(\tilde{M}, \tilde{\mathcal{E}}^k) = \sum_{j \geq 0} (-1)^j \dim_\Gamma H_{L^2}^j(\tilde{\mathcal{E}}^k).$$

Le théorème d'indice L^2 d'Atiyah affirme alors

$$\chi(M, \mathcal{E}) = \chi_\Gamma(\tilde{M}, L^2\tilde{\mathcal{E}}^k).$$

Ce résultat remarquable a eu de nombreuses applications. Dans un cadre plus topologique, la première preuve de la simple connexité des variétés de Fano est une application du théorème d'indice L^2 . Les nombres de Betti L^2 $b_k^2(\tilde{X}) := \dim_\Gamma H_{L^2}^k(\tilde{X}, \mathbb{C}_X)$ sont alors des invariants d'homotopie. Ces nombres sont au centre de nombreuses conjectures, deux des plus célèbres étant

Conjecture (Conjecture d'Atiyah). Si l'ordre des sous groupes finis de Γ est borné, les nombres de Betti L^2 appartiennent au sous groupe de \mathbb{Q} engendré par les inverses des ordres des sous groupes finis de Γ .

Cette conjecture fut d'abord énoncée par Atiyah sans hypothèse sur les sous groupes finis de Γ . Sans cette hypothèse, de nombreux contre-exemples ont été découverts, le premier utilisant le groupe des allumeurs de réverbères [GLS00, DS02]. D'autres contre-exemples furent découverts par la suite par Austin [Aus13], toujours sans l'hypothèse de borne sur l'ordre des sous-groupes finis. Il démontra en particulier que les nombres de Betti L^2 peuvent prendre des valeurs irrationnelles et même transcendantes.

Conjecture (Conjecture de Singer). Si X est une variété réelle fermée asphérique de dimension $2n$ et $\pi: \tilde{X} \rightarrow X$ est son revêtement universel, les nombres de Betti L^2 satisfont $b_k^2(\tilde{X}) = 0$ pour $k \neq n$. Si de plus X est à courbure sectionnelle négative, nous avons de plus $b_n^2(\tilde{X}) > 0$.

Par le théorème d'indice, cette conjecture implique en particulier la conjecture de Hopf qui prédit que pour une variété X fermée asphérique de dimension $2n$ nous avons $(-1)^n \chi(X) \geq 0$.

Dans le cadre de la géométrie algébrique, nous pouvons citer le résultat suivant de Gromov sur les variétés Kähler hyperboliques : si X est une variété Kähler hyperbolique et $\pi: \tilde{X} \rightarrow X$ est son revêtement universel et si $H_{L^2, \bar{\partial}}^q(\tilde{X}, \mathcal{E}^{p, \bullet})$ dénote les groupes de cohomologie de Dolbeault L^2 alors

$$H_{L^2, \bar{\partial}}^q(\tilde{X}, \Omega_{\tilde{X}}^p) = 0 \quad \text{si } p + q \neq \dim_{\mathbb{C}} X.$$

Gromov utilise ce résultat et le théorème d'indice d'Atiyah pour montrer que X doit vérifier un certain nombre de conditions de nature algébro-géométrique. Cela démontre la conjecture de Singer dans le cadre des variétés Kähler hyperboliques. Ces résultats de Gromov ont alors été substantiellement étendus par P. Eyssidieux [Eys97] dans le cas des variations de structures de Hodge. Plusieurs applications de cette théorie en géométrie complexe ont vu le jour avec des résultats sur la cohomologie L^2 des faisceaux cohérents [CD01, Eys00]. Plus récemment, les travaux de Dingoyan [Din13] ont mis en lumière l'intérêt d'utiliser l'algèbre des opérateurs affiliés dans l'étude des groupes de cohomologie de revêtement de variétés Kähleriennes.

La théorie d'Atiyah a eu quelques généralisations depuis, notamment en remplaçant la base par une variété complète de volume fini. On peut citer entre-autre une extension de cette théorie par J. Cheeger et M. Gromov [CG85b, CG85a, CGT82] dans le cas où l'on remplace M par une variété complète de volume finie (on demande alors à ce que le revêtement soit à géométrie bornée). Celle-ci fut encore généralisée dans le cadre des fibrés de Higgs à géométrie bornée par Dingoyan et Schumacher [DS18].

Résultats principaux. Après cette discussion, il peut sembler naturel d'essayer de construire une théorie de Hodge L^2 à coefficients dégénérés sur les revêtements de variétés quasi-projectives. Le cas de la dimension ≥ 2 présentant de nombreuses difficultés supplémentaires, nous nous concentrons dans le cas de la dimension 1. Nous fixons donc X une surface de Riemann et $M := X \setminus \Sigma$ le complémentaire d'un nombre fini de points. On se donne également $\pi: \tilde{M} \rightarrow M$ un revêtement galoisien de groupe Γ et $(\mathbb{V}, F^\bullet, Q)$ une variation de structure de Hodge polarisée de poids w sur M . Nous équipons M d'une métrique à singularité Poincaré, c'est une métrique ω_{p_c} telle que tout point $p \in \Sigma$ admet un voisinage U tel que $U \cap M$ est quasi-isométrique à un disque épointé de rayon $r < 1$ muni de la métrique

$$\frac{dz \otimes d\bar{z}}{|z|^2 \ln(|z|^2)^2}.$$

Cela nous permet de construire un complexe de de Rham $L^2 DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{V}) = L^2 DR^\bullet(\tilde{M}, \omega_{p_c}, \pi^{-1}\mathbb{V}, h)$ dont les objets sont les k formes mesurables η à valeurs dans \mathbb{V} telles que η et $D\eta$ soient de carré intégrable pour les métriques $\pi^* \omega_{p_c}$ et h (la métrique de Hodge induite par la polarisation Q). Dans la suite, nous dénoterons par $H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V})$ le groupe $H^k(L^2 DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{V}))$. Nous construisons également un faisceau faiblement constructible (voir [KS90] pour la définition) de $\mathcal{N}(\Gamma)$ -modules $\ell^2 \pi^* \mathbb{V}$ défini sur X . Nous démontrons alors le résultat suivant

Théorème. Soit $\mathcal{U}(\Gamma)$ l'algèbre des opérateurs affiliés à Γ , c'est un anneau quotient au sens d'Ore de $\mathcal{N}(\Gamma)$ (voir [Lüc02, Chapitre 8]). C'est une extension plate de $\mathcal{N}(\Gamma)$, et la tensorisation par $\mathcal{U}(\Gamma)$ ne perd pas d'information sur la dimension de Von Neumann. Pour $k \in \mathbb{Z}$ le groupe de cohomologie $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$ est isomorphe à $\mathcal{U}(\Gamma) \otimes H_{L^2}^k(\tilde{M}, \pi^* \mathbb{V})$.

Ce résultat s'obtient par un lemme de Poincaré. Nous faisceautisons le complexe de de Rham L^2 pour obtenir un complexe de faisceaux fins $\mathcal{L}^2 DR^\bullet(\pi^* \mathbb{V})$. Celui-ci est défini sur X et les sections au-dessus d'un ouvert U sont les formes mesurables η sur $\pi^{-1}(U \cap M)$ telles que η et $D\eta$ soient de carré intégrable sur $\pi^{-1}(U \cap M)$ pour tout

compact K de U . Le résultat précédent revient alors à montrer que $\mathcal{U}(\Gamma) \otimes \mathcal{L}^2 DR^\bullet(\pi^*\mathbb{V})$ est une résolution fine de $\mathcal{U}(\Gamma) \otimes \ell^2(\pi^*\mathbb{V})$.

Il est à noter que si Γ est infini et \tilde{M} est connexe alors le groupe $H_{L^2}^0(\tilde{M}, \pi^*\mathbb{V})$ est nul. Par un théorème de dualité que nous allons énoncer sous peu, nous obtiendrons $\mathcal{U}(\Gamma) \otimes H_{L^2}^2(\tilde{M}, \pi^*\mathbb{V}) = 0$. Dans ce cadre, la théorie qui va être développée ne concernera que le groupe $\mathcal{U}(\Gamma) \otimes H^1(X, \ell^2 \pi^*\mathbb{V})$ qui est le seul qui n'est pas nul a priori.

On rappelle que $L^2 DR^k(\tilde{M}, \pi^*\mathbb{V})$ désigne l'espace des k -formes η à valeurs dans $\pi^*\mathbb{V}$ qui sont de carré intégrable, des résultats généraux d'analyse fonctionnelle nous apportent une identification en cohomologie

$$H_{L^2}^\bullet(\tilde{M}, \pi^*\mathbb{V}) = \text{Ker}(\square_D) \oplus \frac{\overline{\text{Ran}(D)}}{\text{Ran}(D)}$$

où D^* est l'adjoint (au sens de la théorie des opérateurs) de D et $\square_D = (D + D^*)^2$ est l'opérateur de Laplace associé à D . Pour pouvoir identifier les groupes de cohomologie de de Rham avec l'espace des formes harmoniques après tensorisation avec l'algèbre des opérateurs affiliés $\mathcal{U}(\Gamma)$, il nous faut alors démontrer

$$\mathcal{U}(\Gamma) \otimes \frac{\overline{\text{Ran}(D)}}{\text{Ran}(D)} = 0.$$

Ce résultat peut s'obtenir par une étude du bas du spectre de l'opérateur de Laplace. Une fois ce résultat obtenu, nous avons par les identités Kähleriennes le second résultat central de ce manuscrit.

Théorème. L'identification $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^*\mathbb{V})$ avec l'espace des formes harmoniques induit une structure de Hodge pure (de $\mathcal{U}(\Gamma)$ -module) de poids $w + k$.

Comme dans ce cadre, nous travaillons avec des $\mathcal{U}(\Gamma)$ -modules, il semble naturel de se demander ce que deviennent le théorème de dualité et le théorème d'indice d'Atiyah. L'énoncé d'un théorème de dualité est le suivant.

Théorème. L'étoile de Hodge $*$ induit un isomorphisme

$$\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^*\mathbb{V}) \rightarrow \mathcal{U}(\Gamma) \otimes H^{2-k}(X, \ell^2 \pi^*\mathbb{V}^*)$$

où \mathbb{V}^* est le système local sous-jacent à la variation de structure de Hodge duale.

La preuve de ce résultat est standard et repose sur le fait que l'étoile de Hodge induit un isomorphisme entre les espaces de formes harmoniques. Dans le cas de notre théorème d'indice, comme nous considérons des revêtements galoisiens arbitraires de M un terme d'erreur lié à une potentielle "ramification à l'infini" est attendu, ce qui donnerait lieu à un énoncé du type Riemann-Hurwitz. Avant de donner l'énoncé, posons

$$\chi_\Gamma(X, \ell^2 \pi^*\mathbb{V}) = \sum_k \dim_\Gamma H^k(X, \ell^2 \pi^*\mathbb{V}) \quad \chi(X, j_*\mathbb{V}) = \sum_k \dim_\Gamma H^k(X, j_*\mathbb{V}).$$

Théorème (L^2 Riemann Hurwitz). Nous avons l'égalité suivante

$$\chi_\Gamma(X, \ell^2 \pi^*\mathbb{V}) - \sum_{p \in \Sigma} \frac{\dim \text{Ker}((T_p^{n_p} - \text{Id}))}{n_p} = \chi(X, j_*\mathbb{V}) - \sum_{p \in \Sigma} \dim \text{Ker}(T_p - \text{Id}).$$

Où pour $p \in \Sigma$, T_p est l'opérateur de monodromie locale de \mathbb{V} en $p \in X$ et n_p est l'ordre de γ_p l'élément de Γ obtenu en considérant le cercle méridien autour de p . Nous adoptons la convention $\frac{\dim \text{Ker}(T_p^{n_p} - \text{Id})}{n_p} = 0$ si $n_p = +\infty$.

Comme dit précédemment, si le groupe Γ est infini, nous ne pouvons avoir de la cohomologie qu'en degré 1 par le théorème de dualité, avec le théorème d'indice cela résout les versions naturelles des conjectures de Singer et d'Atiyah que l'on pourrait se poser en dimension 1.

Il reste la question de l'algébricité de la structure de Hodge. Pour cela, nous construisons un complexe de de Rham holomorphe L^2 noté $\Omega_{L^2}^\bullet(\pi^*\mathbb{V})$ c'est un complexe de faisceaux sur X dont les sections au-dessus d'un ouvert U sont les formes holomorphes η valeurs dans $\pi^{-1}\mathbb{V}$ définies sur $\pi^{-1}(U \cap M)$ de carré intégrable. La filtration de Hodge induit une filtration de ce complexe

$$F^p \Omega_{L^2}^\bullet(\pi^*\mathbb{V}) = 0 \rightarrow F^p \rightarrow F^{p-1} \otimes \Omega^1 \rightarrow 0$$

Nous démontrons alors à l'aide d'un lemme de Dolbeault et d'un argument de suite spectrale le résultat suivant

Théorème. Il existe une suite spectrale naturelle

$$E_1^{p,q} = \mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, Gr_p^F \Omega_{L^2}^\bullet(\pi^* \mathbb{V})) \implies \mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$$

qui dégénère en page 1. La filtration induite par cette suite spectrale coïncide avec la filtration de Hodge donnée par la décomposition des formes harmoniques.

Un cas particulier de notre théorie est celui où notre revêtement est obtenu par restriction d'un revêtement galoisien $\pi: \tilde{X} \rightarrow X$. Il est connu que l'étude de la cohomologie L^2 de \mathbb{V} sur M revient à l'étude de son extension moitié sur X et que les résultats peuvent s'interpréter en termes de modules de Hodge pures polarisables. Notre théorie devrait donc s'interpréter comme une théorie de cohomologie L^2 de modules de Hodge sur \tilde{X} . Une telle théorie est conjecturée par P. Eyssidieux dans [Eys22] où pour un module de Hodge \mathbb{M} sur X il définit la cohomologie L^2 du module de Hodge $\pi^* \mathbb{M}$ sur \tilde{X} via la construction d'un foncteur

$$L^2 dR: D^b pHM(X) \rightarrow D^b E_f(\Gamma)$$

où pHM est la catégorie des modules de Hodge pures polarisables et $E_f(\Gamma)$ est la catégorie de Farber obtenue en considérant des quotients formels de $\mathcal{N}(\Gamma)$ -modules Hilbertiens de type fini. Nos résultats deviennent dans le cadre de cette théorie le théorème suivant.

Théorème. Soit \mathbb{M} un module de Hodge pur polarisable de poids w sur une surface de Riemann compact X . Soit $\pi: \tilde{X} \rightarrow X$ un revêtement galoisien de groupe Γ , alors les espaces

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathbb{H}^k(L^2 dR(\mathbb{M}))$$

admettent une structure de Hodge pure de poids $w+k$ et la filtration de Hodge est induite par la filtration de Hodge de Saito sur le complexe de de Rham pervers ${}^p DR(\mathcal{M})$ où \mathcal{M} est le \mathcal{D}_X -module sous-jacent à \mathbb{M} .

Organisation de la thèse

Le manuscrit de thèse est organisé de la manière suivante.

Premier chapitre. C'est un chapitre préliminaire sur les variations de structure de Hodge et dont le but est principalement de fixer les notations. On y rappelle les définitions de structures de Hodge, variation de structure de Hodge et les domaines de périodes associés. On y rappelle également différents résultats sur les variations de structure de Hodge sur le disque épointé et sur le demi-plan de Poincaré qui nous serviront par la suite.

Deuxième chapitre. Nous souhaitons utiliser le formalisme des complexes Hilbertiens de [BL92], c'est un type de complexe de chaînes dont les objets sont des espaces de Hilbert et les différentielles sont des opérateurs fermés densément définis. Comme nous travaillons avec des $\mathcal{N}(\Gamma)$ -modules Hilbertiens, la théorie a besoin d'être légèrement modifiée. De nombreux résultats de ce chapitre existent déjà et sont présents dans [Lüc02] sous l'hypothèse supplémentaire que la différentielle est bornée, les preuves nécessitent alors de légères modifications pour rester valables dans notre contexte. Nous y introduisons aussi la notion de double complexe $\mathcal{N}(\Gamma)$ -Hilbertien qui n'existait pas à notre connaissance dans le cadre des complexes Hilbertiens. Une application de cette théorie permet d'obtenir le caractère $\mathcal{N}(\Gamma)$ -Fredholm des complexes de de Rham de revêtements de variétés à coins ainsi qu'un théorème d'indice par des méthodes combinatoires. Ces résultats sont déjà connus et des cas particuliers d'un article de Schick [Sch01] qui les a obtenus par des méthodes d'analyse reposant sur l'algèbre des opérateurs de Boutet de Monvel. Bien que la $\mathcal{N}(\Gamma)$ -Fredholmité soit une conséquence des méthodes de [Sch01], ce résultat n'y est pas donné. Par conséquent, nous avons inclus une preuve alternative de ce résultat dans ce manuscrit par souci de complétude. Finalement, nous donnons quelques propriétés des complexes de Dolbeault des revêtements de courbes quasi-projectives.

Troisième chapitre. Comme expliqué précédemment, nous allons faisceautiser le complexe de de Rham, ce qui nous ramène à des méthodes locales. Le voisinage de nos singularités $p \in \Sigma$ étant quasi isométriquement biholomorphe à un disque épointé de rayon inférieur à 1 nous étudions d'abord ce cas-ci. Nous étudions alors le spectre du Laplacien sur des revêtements du disque épointé pour lesquels il y a deux cas possibles $\Gamma = \mathbb{Z}/n\mathbb{Z}$ ou $\Gamma = \mathbb{Z}$.

Nous devons étudier des complexes de de Rham sur des variétés non complètes, ce qui nous force à considérer des problèmes à bord pour l'étude du Laplacien. En suivant la méthode de [Zuc79], nous utilisons la filtration associée à la monodromie pour nous ramener au cas de fibrés en droite. Le cas où Γ est fini repose sur les séries de Fourier et un argument de suite spectrale; c'est une petite modification de l'argument de [Zuc79]. Le cas où $\Gamma = \mathbb{Z}$ présente une difficulté supplémentaire, car 0 peut appartenir au spectre continu du Laplacien, ce qui nous empêche d'utiliser des estimées a priori. Nous contournons ce problème avec une transformée de Fourier et nous utilisons la théorie des intégrales de champs mesurables d'espace de Hilbert pour étudier le spectre $\mathcal{N}(\Gamma)$ -essentiel du Laplacien. Cette théorie (qui est résumée en appendice) nous permet de nous ramener au cas d'un opérateur différentiel d'ordre deux sur la demi-droite. Il restera ensuite à adapter l'argument de Zucker [Zuc79] pour conclure.

Quatrième chapitre. Ce chapitre s'intéresse au cas global, nous avons à ce stade essentiellement déjà démontré le lemme de Poincaré dans le chapitre précédent ce qui nous donnera l'isomorphisme

$$\mathcal{U}(\Gamma) \otimes H^\bullet(X, \ell^2 \pi^* \mathbb{V}) \simeq \mathcal{U}(\Gamma) \otimes H_{L^2}^\bullet(\tilde{M}, \pi^{-1} \mathbb{V}).$$

Il restera à appliquer les résultats du chapitre 2 pour conclure sur le caractère $\mathcal{N}(\Gamma)$ -Fredholm du Laplacien et obtenir la représentation par des formes harmoniques des éléments de $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$. Une fois cette identification faite, nous obtenons une structure de Hodge de nature analytique sur nos groupes de cohomologie $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$. Le théorème d'indice et le théorème de dualité suivent alors simplement des résultats exposés dans le chapitre 2. Il reste ensuite à étudier la structure de Hodge algébrique et vérifier qu'elle coïncide avec la structure de Hodge analytique, ce qui sera une conséquence de notre lemme de Dolbeault.

Cinquième chapitre. Ce dernier chapitre est dédié à l'exposition du cas où le revêtement $\pi: \tilde{M} \rightarrow M$ est obtenu par restriction d'un revêtement $\pi: \tilde{X} \rightarrow X$. Dans ce cas, il est possible de reformuler nos résultats dans le cadre de la théorie conjecturale de cohomologie L^2 des modules de Hodge polarisables sur les revêtements de variétés projectives développée par P. Eyssidieux dans [Eys22]. Nous rappelons le formalisme de [Eys22] avant de montrer que nos résultats dans ce cadre sont des cas particuliers d'une conjecture énoncée dans [Eys22].

Pour ce manuscrit, nous aurons besoin de nombreux résultats d'analyse portant notamment sur l'analyse Hilbertienne, les algèbres de Von Neumann, la théorie des intégrales de champs mesurables d'espaces de Hilbert et la théorie des équations différentielles ordinaires. Tous les résultats utilisés seront énoncés dans les annexes et nous nous y référerons à chaque fois que cela sera nécessaire.

Introduction

Context and principal results.

This thesis is about the L^2 -cohomology of polarised variations of Hodge structures on Galois covering (non necessarily finite) of smooth quasi-projective curve defined over the field \mathbb{C} . Before stating our results, we begin with providing a quick review of these theories.

Hodge theory and variations of Hodge structures Hodge theory takes its roots in 1941 when W. Hodge proved that the cohomology groups of a smooth projective variety X admit a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

in which $H^{p,q}(X, \mathbb{C})$ is naturally identified with the Dolbeault cohomology group $H^q(X, \Omega_X^p)$. If the groups $H^k(X, \mathbb{C})$ only depend on the topology of X , the Hodge decomposition depends on the complex structure on X , thus giving us finer invariants. This result showed a remarkable amount of application in algebraic geometry; to name a few, it allows to embed smooth projective curves in an Abelian variety (called the Jacobian variety) defined by

$$H^{1,0}(X, \mathbb{C})^* / H_1(X, \mathbb{Z}) \cap H^{1,0}(X, \mathbb{C})^*$$

In greater dimensions, the decomposition permits the construction of the Albanese variety and intermediate Jacobians. Due to the importance of this result, it is natural to study Hodge structures in families. The formalism becomes the one of (polarised) variations of Hodge structures, which was developed by P. Griffiths at the end of the 1960s [Gri68a, Gri68b, Gri70]. A polarised variation of Hodge structure on a manifold is a flat holomorphic bundle (\mathcal{V}, D) endowed with a Hermitian metric h that must satisfy some properties. One has a smooth orthogonal decomposition $\mathcal{V} = \bigoplus \mathcal{H}^{p,q}$ in which we ask that D satisfies the following transversality condition

$$D\mathcal{H}^{p,q} \subset \mathcal{H}^{p,q} \otimes \mathcal{E}^{1,0} \oplus H^{p-1,q+1} \otimes \mathcal{E}^{1,0} \oplus H^{p,q} \otimes \mathcal{E}^{0,1} \oplus H^{p+1,q-1} \otimes \mathcal{E}^{0,1}.$$

The typical example is the geometric case : we consider $f: \mathcal{X} \rightarrow X$ a proper morphism with smooth projective fibres and the flat vector bundle \mathcal{V} whose fibre at t is given by the primitive part of the k -th cohomology group of $f^{-1}(t)$ and in which our decomposition is the Hodge decomposition of the fibre. In general, given a proper morphism $f: \mathcal{X} \rightarrow X$ on a projective basis X the fibres are only smooth outside of a nowhere dense Zariski closed subset $\Sigma \subset X$, which we can assume to be a normal crossing divisor. Thus, we only obtain a variation of Hodge structure on $M := X \setminus \Sigma$ degenerating on Σ . In the seventies, P. Deligne made the conjecture that the L^2 -cohomology groups on M endowed with a Poincaré metric should compute the intersection cohomology groups of the underlying local system and that the identification with the space of harmonic forms should give us a Hodge structure on those cohomology groups. This conjecture received a positive answer in the one-dimensional case by a work of S. Zucker [Zuc79], and in higher dimension by the work of M. Kashiwara and T. Kawai under some assumptions on the monodromy at infinity, the general case was settled by E. Cattani, A. Kaplan et W. Schmid [CKS87].

The papers of M. Saito [Sai88, Sai90] introduced the theory of Hodge modules, which gave a natural generalisation of Hodge theory with degenerating coefficients of Deligne. In particular, M. Saito proved in particular that the Hodge intersection cohomology groups of a variation of Hodge structure admit a pure Hodge structure of an algebraic nature. The fact that Saito's decomposition coincides with the one exposed previously is established in dimension one and is a consequence of a Dolbeault lemma of S. Zucker [Zuc79]. The higher dimensional case is still conjectural

and presents a huge gap in difficulty; it seems, however, to have finally obtained a positive answer in a preprint of T. Mochizuki [Moc22].

L^2 -cohomology of Galois coverings. Hodge theory on complex manifolds is linked to L^2 -cohomology. One feature of this theory is that it behaves well with arbitrary Galois coverings. The theory of L^2 -cohomology applied to Galois covers was developed in the second half of the 1970s due to the discovery of the Atiyah L^2 -index theorem. We shall explain its principle in the case that will interest us. Let us take M a closed manifold and E a vector bundle on M . In some cases, it is possible to construct some natural cochain complexes (\mathcal{E}^\bullet, d) , in which \mathcal{E}^k will be a space of smooth E -valued k -forms and d is a differential operator. The most usual examples are

- M is a complex manifold, E is a holomorphic bundle and $d = \bar{\partial}_E$. We obtain the Dolbeault complex of E .
- E is a flat bundle and d is the associated flat connection. We obtain the de Rham complex of E .

The smooth complex (\mathcal{E}^\bullet, d) is quasi-isometric to the complex $(L^2\mathcal{E}^\bullet, d)$ in which we replaced the space of smooth forms by the space $L^2\mathcal{E}^k$ of square-integrable measurable forms, the differential d is then computed in the sense of distributions and is viewed as a densely defined closed operator. To be able to talk about square-integrable forms, one needs to fix a metric on M and on E . Thankfully, the compactness of M implies that those spaces do not depend on those metrics. We denote by \mathfrak{d} the formal adjoint of d and the adjoint d^* of d in the sense of operator theory is just \mathfrak{d} computed in the distribution sense. Under some ellipticity assumption on $\mathfrak{d} + d$, these complexes have finite-dimensional cohomology groups, and the Euler characteristic of the complex

$$\chi(M, \mathcal{E}^k) = \sum_{j \geq 0} (-1)^j \dim H^j(\mathcal{E}^k)$$

is a topological invariant by results of the index theory of Atiyah and Singer.

Now, we fix a Galois cover $\pi: \tilde{M} \rightarrow M$ with covering group $\Gamma := \text{Deck}(\tilde{M}/M)$. We can pull back E to obtain a bundle \tilde{E} on \tilde{M} and construct a complex $(L^2\tilde{\mathcal{E}}^\bullet, d)$ in a natural way. This is a Hilbert complex in which we wish to study the cohomology. If Γ is infinite, it is possible to show that the (reduced) cohomology groups of this complex either vanish or are infinite-dimensional. By using the natural action of Γ by isometry on $L^2\tilde{\mathcal{E}}^k$ we obtain an action of the group Von Neumann algebra $\mathcal{N}(\Gamma)$, which allows us to define the notion of the Von Neumann dimension \dim_Γ which can be thought of as a dimension theory "renormalised" by the group action. This notion of dimension satisfies properties similar to those of the usual dimension, but it can take real values and not just integral ones. Then it is possible to show that the cohomology groups $H^k(L^2\tilde{\mathcal{E}}^\bullet) := H_{L^2}^k(\tilde{\mathcal{E}}^\bullet)$ have a finite Von Neumann dimension. Therefore, we can define the Euler characteristic to be

$$\chi_{L^2, \Gamma}(\tilde{M}, \tilde{\mathcal{E}}^k) = \sum_{j \geq 0} (-1)^j \dim_\Gamma H_{L^2}^j(\tilde{\mathcal{E}}^k).$$

Atiyah's L^2 -index theorem gives the equality

$$\chi(M, \mathcal{E}) = \chi_\Gamma(\tilde{M}, L^2\tilde{\mathcal{E}}^k).$$

This remarkable result has numerous applications. It was used for the first proof of the simple connectedness of Fano varieties.

Without Fano assumption, the L^2 -Betti numbers are $b_k^2(\tilde{X}) := \dim_\Gamma H_{L^2}^k(\tilde{X}, \mathbb{C}_X)$ are always homotopy invariants. These numbers are the subject of many conjectures. Two of the most famous one are

Conjecture (Atiyah's conjecture). If the order of finite subgroup of Γ is bounded, the L^2 -Betti numbers belongs to the subgroup of \mathbb{Q} generated by the inverse of the orders of finite subgroups of Γ .

This conjecture was first stated by Atiyah without the hypothesis on the bound on the finite subgroup of Γ . Without this assumption, numerous counterexamples have been discovered, the first using the lamplighter group [GLS00, DS02]. Austin [Aus13] discovered other counterexamples, still without the assumption on the bound of the order of finite subgroups. He proved in particular that the L^2 -Betti numbers could take irrational and even transcendental values.

Conjecture (Singer's conjecture). If X is a real aspherical closed manifold of dimension $2n$ and $\pi: \tilde{X} \rightarrow X$ is its universal cover, then the L^2 -Betti numbers $b_k^2(\tilde{X})$ vanish for $k \neq n$. If, moreover, X has negative sectional curvature, then $b_n^2(\tilde{X})$ is positive.

By the index theorem, this conjecture implies the Hopf conjecture, which predicts that if X is closed aspherical of dimension $2n$, then $(-1)^n \chi(X) \geq 0$.

In the algebraic geometry setting, we can state the following result of M. Gromov about Kähler hyperbolic manifolds : if X is Kähler hyperbolic and $\pi: \tilde{X} \rightarrow X$ is its universal cover, and if we denote by $H_{L^2, \bar{\partial}}^q(\tilde{X}, \mathcal{E}^{p, \bullet})$ the Dolbeault L^2 cohomology groups of X then

$$H_{L^2, \bar{\partial}}^q(\tilde{X}, \Omega_{\tilde{X}}^p) = 0 \quad \text{si } p + q \neq \dim_{\mathbb{C}} X.$$

M. Gromov then used this result and Atiyah's index theorem to show that X must satisfy some strong algebro-geometric conditions. In particular, he proved the Singer conjecture in the case of Kähler hyperbolic manifolds. These results have been substantially generalised by P. Eyssidieux [Eys97] in the setting of polarised variations of Hodge structure. Many applications of this theory in complex geometry have been discovered, with results on the L^2 -cohomology of coherent sheaves [CD01, Eys00]. More recently, the work of P. Dingoyan [Din13] have shown interest in the use of the affiliated operators algebra $\mathcal{U}(\Gamma)$ to view this theory from a more algebraic point of view.

Atiyah's theory had a few generalisations, in particular by replacing the compacity assumption on the basis and only assuming that it is complete with finite volume. One can illustrate the extension of this theory by J. Cheeger and M. Gromov [CG85b, CG85a, CGT82], where they work under the additional assumption that the covering space has bounded geometry. This theory admits further generalisations due to Dingoyan and Schumacher in the context of the L^2 -cohomology of Higgs bundles.

Principal results. After this discussion, it seems natural to construct a Hodge theory with degenerating coefficients on Galois covers of quasi-projective smooth varieties. The case of varieties with complex dimensions greater than 2 presents numerous difficulties, and we focus on the one-dimensional case. We fix X a compact Riemann surface and $M := X \setminus \Sigma$ obtained by removing a finite number of points from X . We also fix a Galois cover $\pi: \tilde{M} \rightarrow M$ of the covering group Γ and $(\mathbb{V}, F^\bullet, Q)$ a polarised variation of Hodge structure on M of weight w . We endow M with a metric with Poincaré singularities; it is a metric for which any point $p \in \Sigma$ admits a neighbourhood U for which $U \cap M$ is quasi-isometric to a punctured disk Δ_r^* of radius $r < 1$ endowed with the Poincaré metric

$$\frac{dz \otimes d\bar{z}}{|z|^2 \ln(|z|^2)^2}.$$

This allows us to define a de Rham complex $L^2 DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{V}) = L^2 DR^\bullet(\tilde{M}, \omega_{p_c}, \pi^{-1}\mathbb{V}, h)$ in which objects are spaces of measurable \mathbb{V} -valued k -forms η that are square-integrable for the pullback metric $\pi^* \omega_{p_c}$ and the Hodge metric h induced by flat polarisation Q . In the following, we denote by $H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V})$ the group $H^k(L^2 DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{V}))$. We also construct a weakly constructible sheaf of $\mathcal{N}(\Gamma)$ -modules $\ell^2 \pi^* \mathbb{V}$ defined on X . We then prove the result below.

Theorem. Let $\mathcal{U}(\Gamma)$ be the algebra of affiliated operators of Γ , it is an Ore quotient rings of $\mathcal{N}(\Gamma)$. It is a flat extension of $\mathcal{N}(\Gamma)$ and the tensorization by $\mathcal{U}(\Gamma)$ does not make us loose information on the Von Neumann dimension. For all $k \in \mathbb{Z}$, the cohomology group $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$ is isomorphic to $\mathcal{U}(\Gamma) \otimes H_{L^2}^k(\tilde{M}, \pi^* \mathbb{V})$.

This result will be seen as a consequence of a Poincaré lemma. For this lemma, we introduce a sheaf version of the L^2 -de Rham complex to obtain a complex of fine sheaves $\mathcal{L}^2 DR^\bullet(\pi^* \mathbb{V})$. It will be defined on X and the section above an open U is measurable in k forms η defined in $\pi^{-1}(U \cap M)$ such that both η and $D\eta$ are square-integrable on $\pi^{-1}(K \cap M)$ for all $K \Subset U$. The Poincaré lemma will state that $\mathcal{U}(\Gamma) \otimes \mathcal{L}^2 DR^\bullet(\pi^* \mathbb{V})$ is a fine resolution of $\mathcal{U}(\Gamma) \otimes \ell^2(\pi^* \mathbb{V})$.

It should be noted that if Γ is infinite and \tilde{M} is connected, the group $H_{L^2}^0(\tilde{M}, \pi^* \mathbb{V})$ vanishes. By a duality theorem that we will state below we will obtain the vanishing of $\mathcal{U}(\Gamma) \otimes H^2(X, \ell^2 \pi^* \mathbb{V})$, in this case the group $\mathcal{U}(\Gamma) \otimes H^1(X, \ell^2 \pi^* \mathbb{V})$ is the only one that may not vanish.

We recall that the space $L^2 DR^k(\tilde{M}, \pi^* \mathbb{V})$ is the space of k forms η that are $\pi^* \mathbb{V}$ -valued and that are square-integrable. General results on Hilbert theory give us a natural identification

$$H_{L^2}^\bullet(\tilde{M}, \pi^* \mathbb{V}) = \text{Ker}(\square_D) \oplus \overline{\frac{\text{Ran}(D)}{\text{Ran}(D)}}$$

where D^* is the adjoint (in the sense of operators theory) of D and $\square_D = (D + D^*)^2$ is the Laplace operator of D . To be able to identify the de Rham cohomology groups with the space of harmonic forms after tensorization with the

algebra of affiliated operators, one needs to show

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{\frac{\text{Ran}(D)}{\text{Ran}(D)}} = 0.$$

This result can be obtained by studying the bottom of the spectrum of the Laplace operator. After this result, this calls for an application of Kähler identities to prove the second central result of this manuscript.

Theorem. The identification of $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$ with the space of harmonic forms induces a pure Hodge structure (of $\mathcal{U}(\Gamma)$ -module) of weight $w + k$.

Since we work with $\mathcal{U}(\Gamma)$ -modules, it seems natural to wonder what happens to the duality theorem and to the L^2 -index theorem. The statement of the duality theorem becomes the following.

Theorem. The Hodge star operator $*$ induces an isomorphism

$$\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V}) \rightarrow \mathcal{U}(\Gamma) \otimes H^{2-k}(X, \ell^2 \pi^* \mathbb{V}^*)$$

where \mathbb{V}^* is the local system that is underlying the dual variation of Hodge structure.

The proof of this result is standard and relies on the fact that the Hodge star induced an isomorphism between the spaces of harmonic forms. In the case of the L^2 -index theorem, since we consider arbitrary Galois covering of M , an error term linked to a potential "ramification at infinity" is to be expected, in which case we would obtain a Riemann-Hurwitz type theorem. Before giving the statement, we set

$$\chi_\Gamma(X, \ell^2 \pi^* \mathbb{V}) = \sum_k \dim_\Gamma H^k(X, \ell^2 \pi^* \mathbb{V}) \quad \chi(X, j_* \mathbb{V}) = \sum_k \dim_\Gamma H^k(X, j_* \mathbb{V}).$$

Theorem (L^2 Riemann-Hurwitz). We have the equality

$$\chi_\Gamma(X, \ell^2 \pi^* \mathbb{V}) - \sum_{p \in \Sigma} \frac{\dim \text{Ker}((T_p^{n_p} - \text{Id}))}{n_p} = \chi(X, j_* \mathbb{V}) - \sum_{p \in \Sigma} \dim \text{Ker}(T_p - \text{Id}).$$

Where for $p \in \Sigma$, T_p is the local monodromy of \mathbb{V} at $p \in X$ and n_p is the order of γ_p the element of Γ obtained by considering a meridian circle at p . We take the convention $\frac{\dim \text{Ker}(T_p^{n_p} - \text{Id})}{n_p} = 0$ if $n_p = +\infty$.

As explained previously, if the covering group Γ is infinite, we can only have cohomology in degree 1 by the duality theorem, coupled with the index theorem, we solve the natural question of what happens to Singer and Atiyah's conjecture in the one-dimensional case. The question of the algebraic nature of the Hodge structure remains. To solve this, we construct a holomorphic L^2 -de Rham complex denoted by $\Omega_{L^2}^\bullet(\pi^* \mathbb{V})$. It is a complex of sheaves on X whose sections above an open U are $\pi^{-1} \mathbb{V}$ -valued holomorphic forms η defined on $\pi^{-1}(U \cap M)$ that are square-integrable on $\pi^{-1}(K \cap M)$ for all $K \Subset U$. The Hodge filtration induces a filtration of this complex given by

$$F^p \Omega_{L^2}^\bullet(\pi^* \mathbb{V}) = 0 \rightarrow F^p \rightarrow F^{p-1} \otimes \Omega^1 \rightarrow 0.$$

We prove, using a Dolbeault lemma and a spectral sequence argument.

Theorem. There exists a natural spectral sequence

$$E_1^{p,q} = \mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, Gr_p^F \Omega_{L^2}^\bullet(\pi^* \mathbb{V})) \implies \mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$$

that degenerates on page 1. The filtration induced by this spectral sequence coincides with the Hodge filtration given by the decomposition of harmonic forms.

A particular case of this theory is the one in which our covering is obtained by the restriction of a Galois covering $\pi: \tilde{X} \rightarrow X$. It is well known that the study of L^2 of \mathbb{V} is the study of its middle extension on X and that these results can be interpreted in terms of polarisable pure Hodge modules. We should interpret our theory as an L^2 theory of Hodge modules on \tilde{X} . This theory is conjectured by P. Eyssidieux in [Eys22] where for a Hodge module \mathbb{M} on X , he defines the L^2 -cohomology of the Hodge module $\pi^* \mathbb{M}$ on \tilde{X} via the construction of a functor

$$L^2 dR: D^b pHM(X) \rightarrow D^b E_f(\Gamma)$$

where pHM is the category of polarisable pure Hodge modules and $E_f(\Gamma)$ is the Farber category obtained by considering the formal quotient of finitely generated $\mathcal{N}(\Gamma)$ -Hilbert modules. Our results become in this setup.

Theorem. Let \mathbb{M} be a polarisable Hodge module of weight w on a compact Riemann surface X . Let $\pi: \tilde{X} \rightarrow X$ be a Galois covering of covering group Γ then the spaces

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathbb{H}^k(L^2 dR(\mathbb{M}))$$

admit a pure Hodge structure of weight $w + k$ and the Hodge filtration is induced by Saito's Hodge filtration on the perverse de Rham complex ${}^pDR(\mathcal{M})$ where \mathcal{M} is the underlying \mathcal{D}_X -module to \mathbb{M} .

Organisation of the manuscript.

This manuscript is organised in the following way.

First chapter. This is a preliminary chapter on variations of Hodge structures; its purpose is to fix notation. We recall the definitions of Hodge structures, variations of Hodge structures, and their associated periods domains. We also recall results about variations of Hodge structures on the punctured disk and on the Poincaré upper-half plane that we will use.

Second chapter. We wish to use the formalism of Hilbert complexes from [BL92], they are a kind of complexes in which objects are Hilbert spaces and differentials are densely defined closed operators. Since we work with $\mathcal{N}(\Gamma)$ -Hilbert modules, the formalism needs to be slightly changed. Numerous results of the chapter already exist in [Lüc02] with the additional assumption that the differentials are bounded operators, and the proofs only need to be slightly adapted in our settings. We also introduced the notion of double $\mathcal{N}(\Gamma)$ -Hilbert complexes, which did not exist to the best of our knowledge. An application of this theory allows us to obtain the $\mathcal{N}(\Gamma)$ -Fredholm property of the de Rham complexes of coverings of manifolds with corners, as well as an index theorem by combinatorial means. These results were already known and are particular cases of a result of Schick [Sch01] who obtained them using a method based on the use of the Boutet-de Monvel algebra. Although the $\mathcal{N}(\Gamma)$ -Fredholm property is a consequence of the proof in [Sch01], this result was not stated there and our alternative proof allows the present manuscript to be self-contained.

Third chapter. As explained above, we will sheafify the L^2 de Rham complex, which brings us to some local methods. The neighbourhood of our singularities $p \in \Sigma$ being quasi-isometric to a punctured disk of radius less than 1, we study first this case. We have to study the bottom of the spectrum of the Laplace operator on coverings of the punctured disks, for which there are two possible cases : $\Gamma = \mathbb{Z}/n\mathbb{Z}$ or $\Gamma = \mathbb{Z}$. We thus have to consider non-complete manifolds, and we have to study some boundary problems for our Laplace operators. Following the method of Zucker [Zuc79], we use monodromy filtration to reduce the study to the case of line bundles. The case where Γ is finite relies on Fourier series and a spectral sequence argument. The case where $\Gamma = \mathbb{Z}$ presents more difficulty as 0 may belong to the essential spectrum of the Laplacian, which prevents us from using a priori estimates. We circumvent this problem by using the theory of measurable fields of Hilbert spaces, thanks to a Fourier transform, to study the bottom of the spectrum of the Laplace operator. This theory (given in the appendices) will reduce the problem to a study of a Schrödinger operator on the half-line. It will then only remain to adapt the Zucker spectral sequence argument to conclude.

Fourth chapter. This chapter is dedicated to the global case. At this point we have essentially proved our Poincaré lemma which will give us the isomorphism

$$\mathcal{U}(\Gamma) \otimes H^\bullet(X, \ell^2 \pi^* \mathbb{V}) \simeq \mathcal{U}(\Gamma) \otimes H_{L^2}^\bullet(\tilde{M}, \pi^{-1} \mathbb{V}).$$

It will remain to apply the result of Chapter 2 to deduce the $\mathcal{N}(\Gamma)$ -Fredholmness of the Laplace operator and to obtain the representation by harmonic forms of elements of $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$. Once this identification is done, we obtain an analytic Hodge filtration on the cohomology groups $\mathcal{U}(\Gamma) \otimes H^k(X, \ell^2 \pi^* \mathbb{V})$. The index theorem and the duality theorem follow then from the general results given in Chapter 2. It remains then to study the algebraic

Hodge structure and verify that it coincides with the analytic Hodge structure, which will follow from a Dolbeault lemma.

Fifth chapter. This last chapter is dedicated to the case where the covering $\pi: \tilde{M} \rightarrow M$ is obtained by restricting a covering $\pi: \tilde{X} \rightarrow X$. In this case, it is possible to reformulate our results in the context of the conjectural theory of L^2 cohomology of Hodge modules on coverings of projective varieties developed by P. Eyssidieux in [Eys22]. We mostly recall the formalism of [Eys22] before showing that our results in this setting are particular cases of a conjecture stated in [Eys22].

For this manuscript, we will need numerous results of functional analysis, more precisely results on Hilbert analysis, Von Neumann algebras, measurable fields of Hilbert spaces theory, and ordinary differential equations. All the results will be given in the appendices, and we will refer to them when necessary.

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Chapter 1

Generalities on polarised variations of Hodge structures

This is a preliminary chapter on Hodge structures; its purpose is to fix the different notations that will be used through this manuscript.

1 Polarised complex Hodge Structures

In this section, we survey the notion of polarised complex Hodge structures and will mainly follow the conventions and definitions given in the MHM project [SS22b, Part 0, Chapter 2]. As in [SS22b] we will not discuss integral nor rational Hodge structure, we refer the interested reader to [PS08]. We will compare this notion to the case of Hodge structure defined over \mathbb{R} in some remarks. These are more common in the literature, and the main reference in this case is [Del71].

Complex Hodge structures

Definition 1.1.1. Let V be a complex vector space (not necessary of finite dimension), and $w \in \mathbb{Z}$. A complex Hodge structure on V of weight w is the data of two bounded decreasing filtrations $(F^\bullet, \bar{F}^\bullet)$ satisfying one of the equivalent following conditions

- If we set for all p, q , $V^{p,q} = F^p \cap \bar{F}^q$ one has $V = \bigoplus_{p+q=w} V^{p,q}$.
- For all p, q with $p + q \neq w$, one has $Gr_p^F Gr_q^{\bar{F}} V = 0$.

The numbers $h^{p,q} = \dim V^{p,q}$ are called the Hodge numbers, and the tuple $(h^{p,q})_{p+q=n}$ is called the Hodge tuple of the Hodge structure.

In the following, we will denote the Hodge structure by the data $(V, F^\bullet, \bar{F}^\bullet)$. When the filtrations are clear from the context, we will simply denote the Hodge structure by V .

Remark 1.1.2.

- The filtrations F, \bar{F} can be recovered from the decomposition $V = \bigoplus_{p+q=w} V^{p,q}$ with the relations

$$F^p = \bigoplus_{r \geq p} V^{r, w-r}, \quad \text{and} \quad \bar{F}^q = \bigoplus_{s \geq q} V^{w-s, s}.$$

Therefore, a Hodge structure of weight w on V can also be given by a decomposition $V = \bigoplus_{p+q=w} V^{p,q}$.

- The assumption that $p, q \in \mathbb{N}$ is common but unnecessary.

Example 1.1.3.

1. If (X, ω) is a compact Kähler manifold, its k -th cohomology group $H^k(X, \mathbb{C})$ admits a Hodge structure of weight k . In the decomposition

$$H^k(X, \mathbb{C}) = \bigoplus H^{p,q}(X, \mathbb{C})$$

the space $H^{p,q}(X, \mathbb{C})$ is canonically identified with the space of harmonic (p, q) -forms which is also identified with the cohomology group $H^q(X, \Omega_X^p)$.

2. For p, q integers, one can define a Hodge structure $\mathbb{C}_{p,q}$ of weight $p + q$ on \mathbb{C} defined by $\mathbb{C}^{r,s} = 0$ unless $(r, s) = (p, q)$ in which case $\mathbb{C}^{p,q} = \mathbb{C}$.

Remark 1.1.4. In a geometrical context, complex Hodge structures are often real Hodge structures. Those are given by the additional data of a real structure on $V = V_{\mathbb{R}} \otimes \mathbb{C}$ and by asking that the filtration \bar{F}^\bullet is defined by taking \bar{F}^p to be the space conjugated to the space F^p . It is equivalent to ask that $\bar{V}^{p,q} = V^{q,p}$ where $\bar{\cdot}$ denotes the complex conjugation. It is straightforward to check that the complex Hodge structure $\mathbb{C}_{p,q}$ cannot be underlying a real Hodge structure unless $p = q$.

Definition 1.1.5. Let $(V, F^\bullet, \bar{F}^\bullet), (W, F^\bullet, \bar{F}^\bullet)$ be two Hodge structures of weight w . A morphism of Hodge structure is a linear morphism $f: V \rightarrow W$ that respects both filtrations, that is,

$$f(F^p V) \subset F^p W \quad f(\bar{F}^q V) \subset \bar{F}^q W$$

or equivalently if and only if

$$f(V^{p,q}) \subset W^{p,q} \quad \text{for all } p, q \in \mathbb{Z}$$

For W a vector subspace of V , we will denote by W° the subspace of V^* consisting of linear forms that vanish on W .

Proposition 1.1.6. Let $(V, F^\bullet, \bar{F}^\bullet)$ be a Hodge structure of weight w . The following filtrations on V^*

$$F^p V^* = (F^{-p+1} V)^\circ \quad \bar{F}^q V^* = (F^{-q+1} V)^\circ$$

define a complex Hodge structure on V^* of weight $-w$. Then we have

$$(V^*)^{-p,-q} := \{f: V \rightarrow \mathbb{C} \mid f \text{ vanishes outside of } V^{p,q}\}.$$

Proposition 1.1.7. Let $(V, F^\bullet, \bar{F}^\bullet)$ and $(W, F^\bullet, \bar{F}^\bullet)$ be two Hodge structures of weight w_1, w_2 respectively. We can define a Hodge structure $(V \otimes W, F^\bullet, \bar{F}^\bullet)$ of weight $w_1 + w_2$ where the filtrations are defined by

$$F^p(V \otimes W) = \bigoplus_{i+j=p} F^i V \otimes F^j W, \quad \bar{F}^q(V \otimes W) = \bigoplus_{i+j=q} \bar{F}^i V \otimes \bar{F}^j W$$

In this case, one has

$$(V \otimes W)^{p,q} = \bigoplus_{\substack{r+r'=p \\ s+s'=q}} V^{r,s} \otimes W^{r',s'}.$$

It is also possible to define a Hodge structure on the space of morphisms (of vector spaces) between two Hodge structures.

Proposition 1.1.8. Let $(V, F^\bullet, \bar{F}^\bullet)$ and $(W, F^\bullet, \bar{F}^\bullet)$ be two Hodge structures of weight w_1, w_2 respectively. Then the following filtrations on $\text{End}(V, W)$

$$F^p \text{End}(V, W) = \left\{ f: V \rightarrow W \mid f(F^k V) \subset F^{k+p} V \quad \forall k \right\} \quad \bar{F}^q \text{End}(V, W) = \left\{ f: V \rightarrow W \mid f(\bar{F}^k V) \subset \bar{F}^{k+q} V \quad \forall k \right\}$$

define a Hodge structure of weight $w_2 - w_1$ on $\text{End}(V, W)$. In this case, one has

$$\text{End}(V, W)^{p,q} = \left\{ f: V \rightarrow W \mid f(V^{r,s}) \subset W^{r+p,s+q} \quad \forall r, s \right\}$$

and the natural injection

$$V^* \otimes W \rightarrow \text{End}(V, W)$$

is a morphism of Hodge structures.

An important notion in the theory of Hodge structure is the notion of shift.

Definition 1.1.9. Let $(V, F^\bullet, \bar{F}^\bullet)$ be a Hodge structure. We define its (r, s) shift $(V, F^\bullet, \bar{F}^\bullet)[r, s]$ by

$$(V, F^\bullet, \bar{F}^\bullet)[r, s] = (V, F^{\bullet+r}, \bar{F}^{\bullet+s}).$$

We will also denote $(V, F^\bullet, \bar{F}^\bullet)[r] := (V, F^\bullet, \bar{F}^\bullet)[r, r]$.

When the filtrations are clear from the context, we will simply denote it by $V[r, s]$ or $V[r]$.

If V is a Hodge structure of weight w , then $V[r, s]$ is a Hodge structure of weight $w - r - s$ and

$$(V[r, s])^{p, q} = V^{p+r, q+s} \quad \forall p, q.$$

Polarisations

For the rest of this section, all the vector spaces will be assumed to be of finite dimension.

Definition 1.1.10. Let V be a Hodge structure. A polarisation of V is a non-degenerate sesquilinear form $Q: V \otimes \bar{V} \rightarrow \mathbb{C}$ such that the decomposition $V = \bigoplus V^{p, q}$ is orthogonal with respect to Q and such that $(-1)^q Q|_{V^{p, q}}$ is definite positive. A Hodge structure that admits a polarisation is said to be polarisable. A polarised Hodge structure is the data $(V, F^\bullet, \bar{F}^\bullet, Q)$ where Q is a polarisation of the Hodge structure $(V, F^\bullet, \bar{F}^\bullet)$. A polarisation Q induces a Hermitian product h on V defined by

$$h = \bigoplus_{p+q=w} (-1)^q Q|_{V^{p, q}}.$$

Remark 1.1.11. If $(V, F^\bullet, \bar{F}^\bullet, Q)$ is a polarised Hodge structure of weight w , one has the relation

$$\bar{F}^{w-p+1} = (F^p)^\perp$$

where the orthogonal is taken with respect to Q (which in this case coincides with the orthogonal with respect to h). For this reason, we will often denote a polarised Hodge structure by the data (V, F^\bullet, Q) .

Example 1.1.12.

- Let (X, ω) be a Kähler manifold. Denote by

$$P^k(X, \omega) := \text{Ker}(\omega^{n-k+1} \wedge : H^k(X, \mathbb{C}) \rightarrow H^{2n-k+2}(X, \mathbb{C}))$$

the space of primitive cycles. Then $P^k(X, \omega)$ admits a Hodge structure of weight k polarised by

$$Q(\alpha, \beta) = i^{k^2} (-1)^{\frac{k(k+1)}{2}} \int_X \omega^{n-k} \wedge \alpha \wedge \bar{\beta}.$$

- The Hodge structure $\mathbb{C}_{p, q}$ is polarised by $(-1)^{| \cdot |^2}$ where $| \cdot |^2$ is the Euclidean metric on \mathbb{C} .

Recall that giving a non-degenerate sesquilinear form on a finite-dimensional vector space $Q: V \otimes \bar{V} \rightarrow \mathbb{C}$ amounts to fixing a \mathbb{C} -linear isomorphism $Q: \bar{V} \rightarrow V^*$, in particular, it induces an isomorphism $Q^*: \bar{V}^* \rightarrow V \simeq V^{**}$ and thus one obtains a non-degenerate sesquilinear form on V^* .

Proposition 1.1.13. Let $(V, F^\bullet, \bar{F}^\bullet)$ be a complex Hodge structure of weight w . Then if Q polarises V , then Q^* polarises V^* .

Also, if Q_1 and Q_2 are sesquilinear forms on V_1 and V_2 respectively, the sesquilinear form $Q_1 \otimes Q_2$ by $Q_1 \otimes Q_2(u_1 \otimes u_2, v_1 \otimes v_2) = Q_1(u_1, v_1)Q_2(u_2, v_2)$. As one can expect, we have the following.

Proposition 1.1.14. If V_1 and V_2 are complex Hodge structures polarised by Q_1 and Q_2 respectively, then $Q_1 \otimes Q_2$ polarises $V_1 \otimes V_2$.

Recall that in the case where V_1 and V_2 are finite-dimensional, the natural morphism $V_1^* \otimes V_2 \rightarrow \text{End}(V_1, V_2)$ is an isomorphism. Thus, the polarisations of V_1 and V_2 induce a polarisation of $\text{End}(V_1, V_2)$.

Remark 1.1.15. A polarisation of a real Hodge structure of weight w $(V_{\mathbb{R}}, F^\bullet)$ is defined to be a bilinear form S defined over \mathbb{R} , $S_{\mathbb{R}}: V_{\mathbb{R}} \otimes V_{\mathbb{R}} \rightarrow \mathbb{R}$ satisfying the three following properties

1. S is $(-1)^w$ -symmetric.

2. For all p , F^p is S -orthogonal to F^{w-p+1} .

3. The sesquilinear form $S(\cdot, \bar{\cdot})$ is positive definite.

Taking $Q = i^w S(\cdot, \bar{\cdot})$, we recover a polarisation of the complex Hodge structure obtained by forgetting the real structure of $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

2 Griffiths domain of polarised Hodge structures

This section is dedicated to the construction of periods domains, which are locally homogeneous complex manifolds, primarily introduced by Griffiths [Gri68b, Gri70]. We will follow the construction given in [SS22a], and we also refer the reader to [CMSP03] for the construction of period domains of polarised real Hodge structures.

Settings

Let V be a complex vector space and $Q: V \otimes \bar{V} \rightarrow \mathbb{C}$ be a non-degenerate Hermitian form of signature (h^+, h^-) on V . Consider a family $(h^{p,q})_{p+q=w}$ of integers such that

$$\sum_{p+q=w} h^{p,q} = \dim V \quad \sum_{q \text{ even}} h^{p,q} = h^+ \quad \sum_{q \text{ odd}} h^{p,q} = h^-$$

We want to construct a space that classifies Hodge structures $V = \bigoplus V^{p,q}$ of weight w on V that are polarised by Q and that satisfies the relation $\dim V^{p,q} = h^{p,q}$. The above conditions are necessary and sufficient for this space to be non-empty.

It is included in the classifying space of filtration satisfying

$$\dim Gr_F^p = h^{p,q} \tag{1.1}$$

This space is well known, as it is a flag manifold and will be denoted by $\check{\mathcal{D}}$. We briefly recall its construction in the following.

The complex manifold structure on $\check{\mathcal{D}}$

The complex Lie group $G := SL(V)$ acts transitively on the space of filtrations that satisfies the condition 1.1. Set F_0^\bullet such a filtration and $B = \text{Stab}_G(F_0^\bullet)$, it acts properly on G by right multiplication. One has then an identification of $\check{\mathcal{D}}$ with the homogeneous space G/B given by

$$\begin{aligned} G/B &\rightarrow \check{\mathcal{D}} \\ gB &\mapsto gF_0^\bullet \end{aligned}$$

This endows $\check{\mathcal{D}}$ with the structure of a complex manifold, and this structure does not depend on the choice of the base point F_0^\bullet . It is a projective manifold, since it can be embedded in a suitable product of Grassmanian manifolds.

Recall that the Lie algebra \mathfrak{g} is the space of operators with vanishing trace and that the filtration F_0^\bullet induces a filtration F^\bullet on $\mathfrak{gl}(V) = \text{End}(V)$ and thus on \mathfrak{g} given by

$$F^k \mathfrak{g} := \left\{ g \in \mathfrak{g} \mid gF_0^\bullet \subset F_0^{\bullet+k} \right\}.$$

The Lie algebra \mathfrak{b} of B is just $F^0 \mathfrak{g}$ by definition. The tangent space at the point F_0^\bullet is given by $\mathfrak{g}/\mathfrak{b}$.

The complex manifold structure on \mathcal{D}

We construct \mathcal{D} as a homogeneous space similarly. The group $G_{\mathbb{R}} = \text{SU}(Q)$ acts transitively on \mathcal{D} , we set $F_0 \in \mathcal{D}$ and $V = \text{Stab}_{G_{\mathbb{R}}} F_0 = B \cap G_{\mathbb{R}}$ we have an identification of \mathcal{D} with the homogeneous space $G_{\mathbb{R}}/V$. This identification endows \mathcal{D} with the structure of a real analytic manifold.

If \dagger denotes the adjunction relation with respect to the sesquilinear form Q , it induces a complex conjugation c on G given by

$$c: g \mapsto (g^\dagger)^{-1}.$$

By definition, the space of fixed points is the group $G_{\mathbb{R}}$. On the Lie algebra \mathfrak{g} the conjugation is given by

$$X \mapsto -X^\dagger.$$

The space of real points is then just the Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Taking the decomposition

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^{p, -p}$$

it is straightforward to see that if $f \in \mathfrak{g}^{p, -p}$ then $-f^\dagger \in \mathfrak{g}^{-p, p}$. It follows that if $f = \bigoplus f^{p, -p} \in \mathfrak{g}$ then $f \in \mathfrak{g}_{\mathbb{R}}$ if and only if $f^{-p, p} = -(f^{p, -p})^\dagger$. This gives

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}} \oplus \bigoplus_{p>0} (\mathfrak{g}^{-p, p} \oplus \mathfrak{g}^{p, -p}) \cap \mathfrak{g}_{\mathbb{R}}.$$

The Lie algebra \mathfrak{b} and \mathfrak{v} are given by

$$\mathfrak{b} = \bigoplus_{p \geq 0} \mathfrak{g}^{p, -p}, \quad \mathfrak{v} = \mathfrak{b} \cap \mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^{0,0} \cap \mathfrak{g}_{\mathbb{R}}$$

one has for any $f = \bigoplus f^{p, -p} \in \mathfrak{g}$

$$f = \bigoplus_{p>0} (f^{-p, p} - (f^{-p, p})^\dagger) \pmod{\mathfrak{b}}.$$

Since the right-hand side is an element of $\mathfrak{g}_{\mathbb{R}}$ we obtain the following lemma.

Lemma 1.2.16. The natural morphism $\mathfrak{g}_{\mathbb{R}}/\mathfrak{v} \rightarrow \mathfrak{g}/\mathfrak{b}$ is an isomorphism.

Thanks to this lemma, the inverse function theorem gives us the following.

Theorem 1.2.17. The natural inclusion $\mathcal{D} \rightarrow \check{\mathcal{D}}$ is an open embedding. In particular, \mathcal{D} inherits a complex manifold structure when viewed as a Euclidean open subset of $\check{\mathcal{D}}$.

The horizontal tangent space. We have seen that the tangent spaces of $\check{\mathcal{D}}$ and \mathcal{D} at the base point are identified with $\text{End}(V)/_{F^0} \text{End}(V)$. It admits a distinguished holomorphic subbundle.

Definition 1.2.18. The horizontal tangent space $T_h \check{\mathcal{D}}$ of $\check{\mathcal{D}}$ is the holomorphic subbundle of the tangent space whose fibre at the point F^\bullet is given by

$$F^{-1} \text{End}(V)/_{F^0} \text{End} V \subset \text{End}(V)/_{F^0} \text{End} V.$$

A holomorphic map $\phi: X \rightarrow \check{\mathcal{D}}$ is said to be horizontal if its differential takes values in $T_h(\check{\mathcal{D}})$.

3 Polarised Variation of Hodge structures

In the following section, M will denote a complex manifold and n will be its dimension. We will define polarised variations of Hodge structures (pVHS) and recall basic properties about the one defined on a punctured disk. The first definition of polarised variations of Hodge structures is due to Griffiths [Gri68b, Gri70] in the real setting. We will follow again the definitions used in the MHM project [SS22b, Part 0, Chapter 4]. A correspondence due to Simpson exists between polarised variation of Hodge structure and a certain class of Higgs bundle, it motivates some of the terminology that will be used.

Definition of polarised variation of Hodge structures

Definition 1.3.19. A complex polarised variation of Hodge structure of weight w on M is a tuple $(\mathbb{V}, F^\bullet, \bar{F}^\bullet, Q)$ where \mathbb{V} is a local system of \mathbb{C} -vector space, F^\bullet (resp. \bar{F}^\bullet) is a decreasing filtration of the holomorphic vector bundle $\mathcal{V} := \mathcal{O}_M \otimes \mathbb{V}$ by holomorphic subbundles (resp. by antiholomorphic subbundles) and $Q: \mathbb{V} \otimes \bar{\mathbb{V}} \rightarrow \mathbb{C}_M$ is a flat non-degenerate Hermitian pairing. We ask that those data satisfy the following properties

- If $H^{p,q} := F^p \cap \bar{F}^q$ then $\mathcal{E}(\mathcal{V}) = \bigoplus_{p+q=w} \mathcal{E}(H^{p,q})$ where $\mathcal{E}(\mathcal{V}) = C_X^\infty \otimes_{\mathcal{O}_X} \mathcal{V}$ is the sheaf of smooth sections of \mathcal{V} .
- If $p \neq r$ then $Q(H^{p,q}, H^{r,s}) = 0$, and $(-1)^q Q$ is positive definite on $H^{p,q}$.
- If D denotes the flat connection $d \otimes 1$ on the smooth bundle $\mathcal{E}(\mathcal{V}) := C_X^\infty \otimes \mathbb{V}$ then $D^{1,0} \mathcal{E}(F^p) \subset F^{p-1} \otimes \mathcal{E}^{1,0}$ and $D^{0,1} \mathcal{E}(\bar{F}^q) \subset \bar{F}^{q-1} \otimes \mathcal{E}^{0,1}$ where $D^{1,0}$ (resp. $D^{0,1}$) is the $(1,0)$ -component (resp. $(0,1)$ -component) of D .

Remark 1.3.20. The bundles $H^{p,q}$ have a structure of holomorphic vector bundles given by the isomorphisms

$$H^{p,q} \simeq \text{Gr}_p^F \mathcal{V} := F^p / F^{p+1}.$$

The polarisation also allows us to recover the filtration \bar{F}^\bullet from the filtration F^\bullet thanks to the relation

$$\bar{F}^{w-p+1} = (F^p)^\perp.$$

In the rest of this manuscript, we will denote a complex polarised variation of Hodge structure by $(\mathbb{V}, F^\bullet, Q)$.

The Hermitian metric $h = \bigoplus (-1)^q Q|_{H^{p,q}}$ is called the Hodge metric. The connection D induces maps

- $\partial: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p,q} \otimes \mathcal{E}^{r+1,s}$
- $\bar{\partial}: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p,q} \otimes \mathcal{E}^{r,s+1}$
- $\theta: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p-1,q+1} \otimes \mathcal{E}^{r+1,s}$
- $\bar{\theta}: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p+1,q-1} \otimes \mathcal{E}^{r,s+1}$

It should be noted that the operator $\bar{\partial}$ is the one that induces the holomorphic structure on the bundles $H^{p,q}$. The map θ is called the Gauss-Manin connection or the Higgs field. We set

$$\begin{aligned} D' &= \partial + \bar{\theta} & D'' &= \bar{\partial} + \theta \\ \mathcal{E}(\mathbb{V})^{P,Q} &= \bigoplus_{\substack{p+r=P \\ q+s=Q}} H^{p,q} \otimes \mathcal{E}^{r,s} \end{aligned}$$

One has $D = D' + D''$, $D' \mathcal{E}(\mathbb{V})^{P,Q} \subset \mathcal{E}(\mathbb{V})^{P+1,Q}$ and $D'' \mathcal{E}(\mathbb{V})^{P,Q} \subset \mathcal{E}(\mathbb{V})^{P,Q+1}$. One also has a decomposition of $\mathcal{E}^\bullet(\mathbb{V})$, which is the smooth de Rham complex of \mathcal{V} , the decomposition is given by

$$\mathcal{E}^\bullet(\mathbb{V}) = \bigoplus_{P,Q} \mathcal{E}(\mathbb{V})^{P,Q}.$$

This decomposition allows us to define the Hodge filtration on the complex $\mathcal{E}^\bullet(\mathbb{V})$ by setting

$$F^p \mathcal{E}^\bullet(\mathbb{V}) = \bigoplus_{P \geq p} \mathcal{E}(\mathbb{V})^{P,Q}.$$

Induced mapping to a suitable period domain

Consider $(\mathbb{V}, F^\bullet, Q)$ a variation of Hodge structure on a complex manifold M . We denote by $\pi: \tilde{M} \rightarrow M$ its universal cover, by pullback we obtain a variation of Hodge structure $(\pi^{-1}\mathbb{V}, \pi^*F^\bullet, \pi^*Q)$ on \tilde{M} . Denote by V the space of global sections of $\pi^*\mathbb{V}$ (i.e. the space of multivalued flat sections on M), since \tilde{M} is simply connected, one has an isomorphism of flat bundle $\mathcal{V} \simeq \tilde{M} \times V$. By letting \mathcal{D} be the period domain parametrizing Hodge structures on V polarised by Q with Hodge numbers $h^{p,q} = \text{rk}(H^{p,q})$ one obtains the period mapping

$$\tilde{\phi}: \tilde{M} \rightarrow \mathcal{D}$$

that sends a point $p \in \tilde{M}$ to the Hodge structure induced by π^*F^\bullet on the fibre above p . The composition

$$\tilde{M} \xrightarrow{\tilde{\phi}} \mathcal{D} \hookrightarrow \check{\mathcal{D}}$$

is clearly holomorphic : it is the natural map associated to a holomorphic filtration of a trivial bundle taking values in a flag manifold. Hence, we obtain

Proposition 1.3.21. The period mapping $\tilde{\phi}$ is holomorphic.

It should also be noted that the fundamental group of M acts on \tilde{M} and also on \mathcal{D} via the monodromy representation and that the period mapping $\tilde{\phi}$ is $\pi_1(M)$ -equivariant.

The Higgs field $\theta: \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\tilde{M}}^1$ being $\mathcal{O}_{\tilde{M}}$ linear, one can view it as a morphism

$$\theta: T\tilde{M} \rightarrow \text{End}_{\mathcal{O}_{\tilde{M}}}(\mathcal{V}).$$

The tangent space of \mathcal{D} being a quotient of $\mathcal{O}_{\mathcal{D}} \otimes \text{End}(V)_{\mathcal{D}}$, θ induces a morphism

$$\theta: T\tilde{M} \rightarrow T\mathcal{D}$$

and almost by definition, one has

Proposition 1.3.22. The differential of the period mapping $\phi: \tilde{M} \rightarrow \mathcal{D}$ is the morphism induced by the Higgs field θ .

The Griffiths transversality $\theta(H^{p,q}) \subset H^{p,q} \otimes \Omega_{\tilde{M}}^1$ implies

Proposition 1.3.23. The period mapping ϕ is horizontal, i.e.

$$d\phi(T\tilde{M}) = \theta(T\tilde{M}) \subset T_h(\mathcal{D}).$$

Remark 1.3.24. This horizontality theorem has been first proven by Griffiths [Gri68b, Gri70] in the case of a polarised variation of Hodge structures on a manifold M coming from the study of a family $\mathcal{X} \rightarrow M$ of smooth projective varieties above M and was known as the Griffiths transversality theorem. It is the motivation for the Griffiths transversality condition in the definition of polarised variation of Hodge structures.

Variation of Hodge structure on a punctured disk

In this section, we review some characterisation of square integrability of sections of a polarised variation of Hodge structure on a punctured disk in terms of meromorphic extension and monodromy filtration. These results are due to Schmid [Sch73] and Zucker [Zuc79] for the case of real variations of Hodge structure, the reader can refer to [SS22b, Chapter 6] for the case of complex variation of Hodge structure, as before we use the notation of [SS22b] as much as possible.

Proposition 1.3.25. [[Mal87, Theorem 4.4] and [Del70, Proposition 5.4]] Let Σ be a discrete set of points of a Riemann surface X and $M := X \setminus \Sigma$. Let $(\mathcal{V}, D^{1,0})$ be a flat holomorphic vector bundle on M , then for all $\beta \in \mathbb{R}$ there exists a unique extension \mathcal{V}_*^β of \mathcal{V} to X such that

1. \mathcal{V}_*^β is a vector bundle on X .
2. The connection $D^{1,0}$ has at most a logarithmic pole at points $p \in \Sigma$ with respect to the meromorphic extension associated to $\mathcal{V}_* = \mathcal{O}_X(*\Sigma) \otimes_{\mathcal{O}_X} \mathcal{V}_*^\beta$.
3. The real part of the eigenvalues of the residue of the connection lie in $[\beta, \beta + 1[$.

We recall the basics of the construction : first, we localise around a puncture $p \in \Sigma$. We set $\pi: \mathbb{H} \rightarrow \Delta^*$, $z \mapsto \exp(2i\pi z)$ the universal covering, the pullback $\pi^*\mathcal{V}$ is isomorphic as a flat vector bundle to $\mathbb{H} \times V$ where V is a n -dimensional vector space. We have $\pi_1(\Delta^*) \simeq \mathbb{Z}$, and the monodromy is given by the image $T^{-1} \in GL(V)$ of γ by the monodromy representation. One has

$$\mathcal{V} = (\mathbb{H} \times V) / \mathbb{Z}.$$

Set e_1, \dots, e_n a basis of horizontal sections of $\pi^*\mathcal{V}$ flagged according to the decomposition into generalised eigenspaces $V = \bigoplus_{\lambda \in \mathbb{C}} \text{Ker}(T - \lambda)^n$. If $e_j \in \text{Ker}(T - \lambda)^n$, there exists a unique α with $\beta \leq \alpha < \beta + 1$ with $\lambda = e^{2i\pi\alpha}$ and on this space $T = e^{2i\pi\alpha} e^{N_\alpha}$ where N_α is nilpotent, one can then set

$$\xi_j = \exp((2i\pi\alpha + N_\alpha)\tau) e_j$$

it is a holomorphic basis of sections satisfying $\xi_j(\tau + 1) = T\xi_j(\tau)$, it is actually defined on Δ^* and it generates the sections of \mathcal{V}_*^β at p . An isomorphism of vector bundle is given by

$$\begin{aligned} V \times \Delta &\rightarrow \mathcal{V}_*^\beta \\ e_j &\mapsto \xi_j \end{aligned}$$

In the following, we consider the case where $M = \Delta^*$ is a punctured disk of radius 1 and $X = \Delta$ is the disk of radius 1. We will also work with a flat bundle \mathbb{V} underlying a polarised variation of Hodge structure of weight w . This imposes some conditions on the eigenvalues of the monodromy as stated by the monodromy theorem.

Theorem 1.3.26 (Monodromy theorem). [SS22b, Theorem 6.3.2] In the case where \mathcal{V} is underlying a polarised complex variation of Hodge structure on $M = \Delta^*$, the monodromy has eigenvalues in \mathbb{S}^1 . In particular, the eigenvalues of the residue on \mathcal{V}_*^β will be real numbers.

Remark 1.3.27. The morphism of \mathcal{O}_Δ -module $\mathcal{V}_* \rightarrow \mathcal{V}_*$ given by the multiplication by z induces an isomorphism between \mathcal{V}_*^β and $\mathcal{V}_*^{\beta+1}$.

Notation. If we set $\mathcal{V}^{>\beta} = \bigcup_{\beta' > \beta} \mathcal{V}_*^{\beta'}$ there exists $\beta' > \beta$ such that $\mathcal{V}_*^{\beta'} = \mathcal{V}_*^\beta$. We will set $Gr^\beta \mathcal{V}_* = \mathcal{V}_*^\beta / \mathcal{V}_*^{>\beta}$.

If N is a nilpotent endomorphism of a vector space V , there exists a unique increasing filtration $W_\bullet(N)$ of V satisfying

$$\begin{aligned} NW_k &\subset W_{k-2} \\ N^k : Gr_k^W &\rightarrow Gr_{-k}^W \text{ is an isomorphism.} \end{aligned}$$

The construction of this filtration is an application of the Jordan decomposition. Its introduction for the study of period mappings is due to Schmid [Sch73, §4].

Remark 1.3.28. In the papers of Schmid [Sch73], Zucker [Zuc79] and Cattani-Kaplan-Schmid [CKS86], this filtration is shifted by $-w$ where w is the weight of the Hodge structures considered on V . The reason for this shift in their work is to have Hodge structures of weight k on the graded $Gr_k^W V$ coming from the \mathfrak{sl}_2 orbit theorem, while for our convention those Hodge structures would have weight $w + k$. Our convention for the filtration W_\bullet is the one used in [KK85] and [SS22b].

Using the same notation as before, each N_α induces a filtration $W_\bullet(N_\alpha)$ of $\text{Ker}((T - e^{2i\pi\alpha})^{\text{rk}(\mathbb{V})})$, we denote by W_k the direct sum of the $W_\bullet(N_\alpha)$. The filtration W_\bullet can be characterised by its asymptotic behaviour as stated in the lemma below (see [Sch73, Theorem 6.6] for the case of integral variations or [SS22b, Chapter 6] for the general case).

Lemma 1.3.29. A horizontal multivalued section $e \in V$ is in $W_k \setminus W_{k-1}$ if and only if its Hodge norm $\|e\|_h^2 \simeq \text{Im}(\tau)^k$ uniformly on any vertical strip.

The operator N_β induces a nilpotent endomorphism on $Gr^\beta \mathcal{V}_*$ and we will also denote by $W_k(N_\beta)$ the induced filtration, we take $p : \mathcal{V}_*^\beta \rightarrow Gr^\beta \mathcal{V}_*$ the projection and set as in [SS22b] $M_k \mathcal{V}_*^\beta = p^{-1}(W_k(N_\beta))$.

Remark 1.3.30. The filtration $W_k(N_\beta) \subset \text{Ker}(T - e^{2i\pi\beta})^n \subset V$ defines locally near the punctures, a flat vector sub-bundle $\mathcal{W}_k(N_\beta)$ of \mathcal{V} , we take $\overline{\mathcal{W}_k(N_\beta)}$ its extension to X in the sense of Proposition 1.3.25, such that the eigenvalues of the residue is equal to β . Then we have an isomorphism

$$M_k \mathcal{V}_*^\beta \simeq \overline{\mathcal{W}_k(N_\beta)} + \mathcal{V}_*^{>\beta}.$$

The asymptotic behaviour of the monodromy filtration gives us the following result.

Theorem 1.3.31. [SS22b, Theorem 6.3.5] If $\zeta \in \mathcal{V}_*^\beta$ extends to a section of $M_k \mathcal{V}_*^\beta$ and projects non-trivially in $Gr_M^k Gr^\beta \mathcal{V}_*$ if and only if

$$\|\zeta\|_h^2 \sim |z|^{2\beta} |\ln(z)|^k$$

on any angular sector.

With this norm approximation we can express locally square-integrable forms in the neighbourhood of a puncture in terms of the minimal extension ([Zuc79, Proposition 4.4, Proposition 6.9], [SS22b, Lemma 6.10.16]).

Proposition 1.3.32. [Zuc79, Proposition 4.4] Let $(\Omega^\bullet(\mathbb{V})_{(2)}, D^{1,0})$ be the complex of sheaves defined on X , such that $\Omega^k(\mathbb{V})_{(2)}(U)$ is the space of holomorphic k -forms on $M \cap U$ that are square integrable on $M \cap K$ for all K compact of

U. Then it is equal to the following complex

$$0 \longrightarrow M_0 \mathcal{V}_*^0 \xrightarrow{D^{1,0}} \Omega_{\mathbb{X}}^1(\log \Sigma) \otimes_{\mathcal{O}_{\mathbb{X}}} M_{-2} \mathcal{V}_*^{-1} \longrightarrow 0$$

The proposition below is a consequence of the result below [Zuc79, p.433], it is also given in [SS22b, Proposition 6.11.2].

Lemma 1.3.33. If (e_1, \dots, e_n) is a flat multivalued basis of \mathbb{V} flagged according to the filtration W_\bullet then the induced frame $(\xi_1, \dots, \xi_n) \in \mathcal{V}_*^0$ is L^2 adapted, i.e. there exists a constant $C > 1$ if $(f_1, \dots, f_n) \in \mathbb{C}^n$ then

$$C^{-1} \sum \|f_i \xi_i\|^2 \leq \left\| \sum f_i \xi_i \right\|^2 \leq C \sum \|f_i \xi_i\|^2.$$

Asymptotic behaviour of the metrics. As we have said before, one can associate a period mapping to a variation of Hodge structure on Δ^*

$$\phi: \mathbb{H} \rightarrow \mathcal{D}$$

where \mathbb{H} is the upper half-plane, which we see as the universal cover of the punctured disk via the map $z \mapsto \exp(2i\pi z)$. We have seen that the differential of this mapping is induced by the Higgs field θ . It will be useful to have some estimates on θ . The most fundamental one is Simpson's basic estimate, which can be found in [SS22a, Corollary §57].

Theorem 1.3.34 (Simpson's basic estimate). Let $(\mathbb{V}, F^\bullet, Q)$ be a polarised variation of Hodge structure of rank r on the upper half-plane \mathbb{H} . Then the Higgs field θ satisfies the inequality

$$h_{\text{End}(E)}(\theta_{\partial_z}, \theta_{\partial_z}) \leq \frac{\binom{r+1}{3}}{4\text{Im}(z)^2}.$$

We can see by the result above that we can bound the Hodge metric by a constant that depends only on the rank of the local system. In fact, the behaviour of the Hodge metric does not depend on the variation of Hodge structure but only of the underlying local system, this result is due to Schnell and is stated as the comparison theorem [SS22a, §18].

Theorem 1.3.35 (Comparison theorem). Let $(\mathbb{V}_1, F_1^\bullet, Q_1)$ and $(\mathbb{V}_2, F_2^\bullet, Q_2)$ be two polarised variations of Hodge structures on the punctured disk Δ^* and denote by h_1 and h_2 their Hodge norm. Then, if \mathbb{V}_1 and \mathbb{V}_2 are isomorphic as local systems, h_1 and h_2 must be mutually bounded.

Chapter 2

$\mathcal{N}(\Gamma)$ -Hilbert complexes

In [BL92] a notion of Hilbert complexes has been introduced, they are complexes whose objects are Hilbert spaces and differentials are closed densely defined operators. Such complexes were introduced to compute L^2 cohomology groups of a Hermitian flat bundle on a Riemannian manifold M . Now, if one considers a Galois covering $\pi: \tilde{M} \rightarrow M$ of our Riemannian manifold with covering group $\text{Deck}(\tilde{M}/M) = \Gamma$, one can pull back our Hermitian bundle and compute the L^2 cohomology groups of our pullback bundle. If one wishes to study the action of the group Γ on the cohomology group, a notion of $\mathcal{N}(\Gamma)$ -Hilbert complex would be needed; those will be complexes whose objects are $\mathcal{N}(\Gamma)$ -Hilbert modules and differentials are closed densely defined operators.

In the literature, to the best of our knowledge, if one studies complexes of $\mathcal{N}(\Gamma)$ -Hilbert modules, the differentials are always required to be bounded (see [Lüc02, Shu95, Din13, DS18, Eys22]...). The goal of this chapter is to develop such a notion; thankfully, the existing results in [BL92, Lüc02, Shu95] generalise without any trouble. We will focus in particular on the notion of $\mathcal{N}(\Gamma)$ -Fredholm complexes. One thing that will be missing is the notion of Poincaré duality developed in [BL92].

This notion of double $\mathcal{N}(\Gamma)$ -Hilbert complexes will allow us to recover some particular case of L^2 -index theorems on a covering $\pi: \tilde{M} \rightarrow M$ compact manifold similar to the one due to Atiyah [Ati76] and Schick [Sch01] by combinatorial means; we also give a topological interpretation of these cohomology groups. These theorems relate the L^2 Euler characteristic of \tilde{M} to the L^2 Euler characteristic of M when M is compact, it is a topological invariant equal to its usual Euler characteristic.

This theory was extended by J.Cheeger and M. Gromov in a series of articles [CG85b, CG85a] to the coverage of the complete manifold M with finite volume under the assumption that the covering space \tilde{M} has a bounded geometry. In this context, the L^2 Euler characteristic of M has to be replaced by the integral of the Chern-Gauss-Bonnet form, it differs from the L^2 -Euler characteristic by a term that depends on a compactification of M (see, for instance, [CG91]).

1 $\mathcal{N}(\Gamma)$ -Hilbert complexes

We begin with the definition of the main object of this chapter. In the whole chapter, Γ will denote a discrete group and $\mathcal{N}(\Gamma)$ its Von Neumann algebra. The standard reference on groups Von Neumann algebra is the book of Lück [Lüc02]. Basic properties of group Von Neumann algebras as well as slightly advanced results of functional analysis needed for this chapter are given in the first and third sections of the appendices of the present manuscript. In particular, we refer to Definition A.3.48 for the definition of $\mathcal{N}(\Gamma)$ -Hilbert modules.

Definitions of $\mathcal{N}(\Gamma)$ -Hilbert complexes.

Definition 2.1.1. A $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) is the data of a family of $\mathcal{N}(\Gamma)$ -Hilbert module $(C^k)_{k \in \mathbb{Z}}$ together with closed densely defined operators $d^k: C^k \dashrightarrow C^{k+1}$ such that

- For all $k \in \mathbb{Z}$, $\text{Dom}(d^k)$ is $\mathcal{N}(\Gamma)$ -invariant.
- For all $k \in \mathbb{Z}$, d^k is $\mathcal{N}(\Gamma)$ -equivariant.
- For all $k \in \mathbb{Z}$, $\text{Ran}(d^k) \subset \text{Ker}(d^{k+1})$.

A morphism of $\mathcal{N}(\Gamma)$ -Hilbert complexes $f: (C^\bullet, d_C) \rightarrow (K^\bullet, d_K)$ is a family of bounded $\mathcal{N}(\Gamma)$ -equivariant morphisms $(f_k: C^k \rightarrow K^k)_{k \in \mathbb{Z}}$ such that for all k one has $f_k(\text{Dom}(d_C^k)) \subset \text{Dom}(d_K^k)$ and such that for all $x \in \text{Dom}(d_C^k)$

$$f_{k+1}(d_C^k(x)) = d_K^k(f_k(x))$$

When the operator d is clear from the context, we will often denote the $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) by C^\bullet . Moreover, we will often forget the subscript k and write d, f instead of d^k, f_k . The forgetful functor from the category of $\mathcal{N}(\Gamma)$ -Hilbert complexes to the category of complexes of $\mathcal{N}(\Gamma)$ -modules is defined by

$$(C^\bullet, d) \rightarrow (\cdots \rightarrow \text{Dom}(d^{k-1}) \rightarrow \text{Dom}(d^k) \rightarrow \text{Dom}(d^{k+1}) \rightarrow \cdots).$$

We will say that a complex $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) is bounded if the underlying complex is bounded. And we define the translation operator in a natural way, namely we set

$$(C^\bullet[n])^k = C^{n+k} \quad d[n] = (-1)^n d$$

Definition 2.1.2. The cohomology groups $H^k(C^\bullet)$ of a $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) will then be defined as the cohomology groups of the underlying complex. Reduced cohomology groups $H_{\text{red}}^k(C^\bullet)$ of a $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) are defined as

$$H_{\text{red}}^k(C^\bullet) = \frac{\text{Ker}(d^k)}{\text{Ran}(d^{k-1})}.$$

Weak Hodge decomposition. One of the key features of these complexes is the existence of a Laplace operator and a weak Hodge decomposition (sometimes called the weak Kodaira decomposition). Set d^* to be the adjoint of d , it is a closed densely defined $\mathcal{N}(\Gamma)$ -equivariant operator, and we define the Laplace operator by

$$\square_d = dd^* + d^*d.$$

It is a closed self-adjoint (in particular, densely defined) non-negative $\mathcal{N}(\Gamma)$ -equivariant operator. Its domain is given by

$$\text{Dom}(\square_d) = \{x \in C^\bullet \mid x \in \text{Dom}(d) \cap \text{Dom}(d^*), dx \in \text{Dom}(d^*) \text{ and } d^*x \in \text{Dom}(d)\}.$$

We set $\text{Harm}_d = \text{Ker}(\square_d)$, its elements will be called harmonic forms. We will denote $\text{Harm}_d^k = \text{Harm}_d \cap C^k$.

Theorem 2.1.3 (Weak Hodge decomposition). Let (C^\bullet, d^\bullet) be a $\mathcal{N}(\Gamma)$ -Hilbert complex, then we have $\text{Harm}_d^k = \text{Ker}(d) \cap \text{Ker}(d^*)$. Moreover, for any $k \in \mathbb{Z}$, we have the following orthogonal decomposition called the weak Hodge decomposition

$$C^k = \text{Harm}_d^k \oplus \overline{\text{Ran}(d)} \oplus \overline{\text{Ran}(d^*)}.$$

The action of a group is not used to obtain this decomposition (see [BL92, Section 2]), it is straightforward to check that it is a decomposition of $\mathcal{N}(\Gamma)$ -Hilbert modules.

Remark 2.1.4. The reader can note that this decomposition is compatible with the domain of d, d^* and \square_d^n for any $n \geq 0$ in the sense that if $x = x_h + x_d + x_{d^*}$ with $x_h \in \text{Harm}_d, x_d \in \overline{\text{Ran}(d)}$ and $x_{d^*} \in \overline{\text{Ran}(d^*)}$ then $x \in \text{Dom}(d)$ (resp. $x \in \text{Dom}(d^*),$ resp. $\text{Dom}(\square_d^n)$) if and only if the same is true x_h, x_d and x_{d^*} . This property is false for an arbitrary orthogonal decomposition.

From the weak Hodge decomposition, one easily deduces the

Corollary 2.1.5. There exist canonical isomorphisms

$$H^k(C^\bullet) = \text{Harm}^k(C^\bullet) \oplus \frac{\overline{\text{Ran}(d)}}{\text{Ran}(d)} \quad H_{\text{red}}^k(C^\bullet) = \text{Harm}^k(C^\bullet)$$

Homotopies and quasi-isomorphisms.**Definition 2.1.6.**

- A morphism $f: C^\bullet \rightarrow K^\bullet$ of $\mathcal{N}(\Gamma)$ -Hilbert complexes is said to be a quasi-isomorphism if it induces an isomorphism of $\mathcal{N}(\Gamma)$ -module between the cohomology groups.
- Let $f, g: C^\bullet \rightarrow K^\bullet$ be two morphisms of $\mathcal{N}(\Gamma)$ -Hilbert complexes. A homotopy H between f and g is given by the data of bounded morphisms $H^k: K^k \rightarrow C^{k-1}$ indexed by $k \in \mathbb{Z}$ such that

$$f - g = d_C H - H d_K$$

Two morphisms f, g are said to be homotopic if there exists a homotopy between f and g . A complex is said to be contractible if the identity is homotopic to the zero morphism.

A morphism $f: C^\bullet \rightarrow K^\bullet$ of $\mathcal{N}(\Gamma)$ -Hilbert complexes is a homotopy equivalence if and only if there exists $g: K^\bullet \rightarrow C^\bullet$ such that both gf and fg are homotopic to the identity.

A homotopy between two morphisms f and g of $\mathcal{N}(\Gamma)$ -Hilbert complexes induces a homotopy between f and g as morphisms of complexes of $\mathcal{N}(\Gamma)$ -module. In particular, one has

Lemma 2.1.7. If f and g are two homotopic morphisms of $\mathcal{N}(\Gamma)$ -Hilbert complexes, then they induce the same morphisms on cohomology.

The reader can note that the morphism induced in cohomology is continuous for the quotient topology, however, in the definition we do not require the inverse to be continuous.

Remark 2.1.8. Our definition of homotopy equivalence of $\mathcal{N}(\Gamma)$ -Hilbert complexes is a priori stronger than a homotopy equivalence of the underlying $\mathcal{N}(\Gamma)$ -module, since we require the homotopy H to be bounded. In some cases, however, these notions coincide. For instance, consider C^\bullet and K^\bullet , two $\mathcal{N}(\Gamma)$ -Hilbert complexes whose objects are finitely generated $\mathcal{N}(\Gamma)$ -Hilbert module and whose differential d is bounded, take $f: C^\bullet \rightarrow K^\bullet$ to be a homotopy equivalence of the underlying complexes of $\mathcal{N}(\Gamma)$ -modules then since any $\mathcal{N}(\Gamma)$ -equivariant map between finitely generated $\mathcal{N}(\Gamma)$ -Hilbert module is bounded by Theorem A.3.61 it follows that f is bounded and is a homotopy equivalence of $\mathcal{N}(\Gamma)$ -Hilbert complexes.

Smooth subcomplex. We give now an important example of homotopy retract. Let (C^\bullet, d) be a $\mathcal{N}(\Gamma)$ -Hilbert complex and $E_{\square_d}^k$ denotes the projection-valued measure associated with the Laplace operator \square_d acting on C^k and for $\varepsilon > 0$ we denote by $E_{\square_d, \varepsilon}^k$ the orthogonal projector $E_{\square_d}^k([0, \varepsilon[)$. Set

$$C_\infty^k = \bigcap_{n \in \mathbb{N}} \text{Dom}(\square_d^n).$$

This is a complex of $\mathcal{N}(\Gamma)$ -modules. However, it is not a $\mathcal{N}(\Gamma)$ -Hilbert complex, as C_∞^k is not a $\mathcal{N}(\Gamma)$ -Hilbert module in general.

Definition 2.1.9. The complex C_∞^\bullet is called the smooth subcomplex of (C^\bullet, d) .

Proposition 2.1.10. For all $\varepsilon > 0$, $(\text{Ran}(E_{\square_d, \varepsilon}^\bullet), d)$ defines a $\mathcal{N}(\Gamma)$ -Hilbert subcomplex of the complex (C^\bullet, d) . Furthermore, injection $(E_{\square_d, \varepsilon}^\bullet, d) \rightarrow (C^\bullet, d)$ is a homotopy equivalence of $\mathcal{N}(\Gamma)$ -Hilbert complexes, and the following inclusions

$$(E_{\square_d, \varepsilon}^\bullet, d) \rightarrow (C_\infty^\bullet, d) \rightarrow (C^\bullet, d)$$

are homotopy equivalences of complexes of $\mathcal{N}(\Gamma)$ -modules.

Proof. We begin to show that the inclusion

$$\iota: (E_{\square_d, \varepsilon}^\bullet, d) \rightarrow (C^\bullet, d)$$

is a homotopy equivalence of $\mathcal{N}(\Gamma)$ -Hilbert complexes. Set p as the projector $E_{\square_d, \varepsilon}$. One has $p\iota = \text{Id}$, so it suffices to prove that ιp is homotopic to the identity. By definition, one has

$$\text{Id} - \iota p = E_{\square_d}(\] \varepsilon, +\infty])$$

If $x \in \text{Ran}(E_{\square_d}(\cdot]_{\varepsilon, +\infty}[\cdot))$ one has the inequality

$$\langle dx, dx \rangle + \langle d^*x, d^*x \rangle \geq \varepsilon \|x\|^2.$$

We denote, as before, $x = x_h + x_d + x_{d^*}$ the weak Hodge decomposition of x . One has $x_h = 0$ and the above inequality shows that the restriction of $d + d^*$ to $\text{Ran}(E_{\square_d}(\cdot]_{\varepsilon, +\infty}[\cdot))$ is surjective, and we can find two bounded operators h_1, h_2 such that for all $x \in \text{Ran}(E_{\square_d}(\cdot]_{\varepsilon, +\infty}[\cdot))$ one has

$$x = dh_1(x) + d^*h_2(x).$$

It follows that if we take $H = h_1 E_{\square_d}(\cdot]_{\varepsilon, +\infty}[\cdot)$, it defines the desired homotopy.

It remains to prove that the inclusion $(C_{\infty}^{\bullet}, d) \rightarrow (C^{\bullet}, d)$ is a homotopy equivalence of complexes of $\mathcal{N}(\Gamma)$ -modules. The proof of [BL92, Theorem 2.12] works verbatim : we take $\chi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a decreasing function such that $\chi = 1$ in a neighbourhood of 0 and $\chi = 0$ in a neighbourhood of $+\infty$, and we set A as the bounded inverse of $\square_d + \text{Id}$. Then the operator

$$H := -d^*A \left(\text{Id} + \int_0^{\infty} \chi(t) e^{-t\square_d} dt \right)$$

defines the homotopy. □

Short exact sequences, cones, and cylinders. As in regular homological algebra, one has the notion of short exact sequences.

Definition 2.1.11. Let $A^{\bullet}, B^{\bullet}, C^{\bullet}$ be three $\mathcal{N}(\Gamma)$ -Hilbert complexes, $\iota: A^{\bullet} \rightarrow B^{\bullet}$ and $p: B^{\bullet} \rightarrow C^{\bullet}$ be morphisms of $\mathcal{N}(\Gamma)$ -Hilbert complexes. We say that we have a short exact sequence of $\mathcal{N}(\Gamma)$ -Hilbert complexes

$$0 \longrightarrow A^{\bullet} \xrightarrow{\iota} B^{\bullet} \xrightarrow{p} C^{\bullet} \longrightarrow 0$$

if the two following properties are satisfied.

1. This sequence is exact in the category of Hilbert spaces.
2. The sequence

$$0 \longrightarrow \text{Dom}(d_A)^{\bullet} \xrightarrow{\iota} \text{Dom}(d_B)^{\bullet} \xrightarrow{p} \text{Dom}(d_C)^{\bullet} \longrightarrow 0$$

is exact in the category of $\mathcal{N}(\Gamma)$ -modules.

A short exact sequence of $\mathcal{N}(\Gamma)$ -Hilbert complexes induces a long exact sequence of $\mathcal{N}(\Gamma)$ -modules

$$\dots \longrightarrow H^k(A^{\bullet}) \longrightarrow H^k(B^{\bullet}) \longrightarrow H^k(C^{\bullet}) \longrightarrow H^{k+1}(A^{\bullet}) \longrightarrow \dots$$

We define the cone and the cylinder of a morphism in a natural fashion.

Definition 2.1.12. Let (C_0^{\bullet}, d_0) and (C_1^{\bullet}, d_1) be two $\mathcal{N}(\Gamma)$ -Hilbert complexes, and $f: C_0^{\bullet} \rightarrow C_1^{\bullet}$ be a morphism of complexes of Hilbert $\mathcal{N}(\Gamma)$ -modules. We define the cone of f as the $\mathcal{N}(\Gamma)$ -Hilbert complex $\text{cone}(f)^{\bullet}$ defined by

$$\text{cone}(f)^n = C_0^{n+1} \oplus C_1^n \quad d_{\text{cone}} = \begin{pmatrix} -d_0 & 0 \\ f & d_1 \end{pmatrix}$$

where the domain of the differential of the cone is $\text{Dom}(d_0) \oplus \text{Dom}(d_1)$.

The cylinder of f is the complex $\text{cyl}(f)^{\bullet} := \text{cone}((- \text{Id}, f): C_0^{\bullet} \rightarrow C_0^{\bullet} \oplus C_1^{\bullet})$.

Remark 2.1.13. The usual proof shows that if $f: C_0^{\bullet} \rightarrow C_1^{\bullet}$ is a quasi-isomorphism of $\mathcal{N}(\Gamma)$ -Hilbert complexes, then $\text{cone}(f)^{\bullet}$ is acyclic (i.e. its cohomology groups vanish). One has canonical isomorphisms of $\mathcal{N}(\Gamma)$ -Hilbert complexes

$$\text{cyl}(f)^{\bullet} \simeq \text{cone}(\text{cone}(f)^{\bullet}[-1] \rightarrow C_0^{\bullet}) \simeq \text{cone}(f \oplus 0: \text{cone}(\text{Id})[-1] \rightarrow C_1^{\bullet})$$

An important property of $\mathcal{N}(\Gamma)$ -Hilbert complexes is that any acyclic complex is contractible, which allows us to see cones of morphism as trivial extensions of our complexes. More precisely, one has

Lemma 2.1.14. We have the following properties

1. An acyclic $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) is contractible.
2. If $f: C_0^\bullet \rightarrow C_1^\bullet$ is a morphism of complexes and C_0^\bullet is acyclic, then there exists an isomorphism of $\mathcal{N}(\Gamma)$ -Hilbert complexes $\text{cone}(f)^\bullet \simeq C_0^\bullet[1] \oplus C_1^\bullet$.

These results are in [Lüc02, Theorem 2.20] for complexes with bounded differentials.

Proof. For the first point, consider an acyclic complex (C^\bullet, d) , the closed operator $d: \text{Ran}(d)^\perp \dashrightarrow \text{Ker}(d)$ is both surjective and injective, as the complex is acyclic. It follows that there exists $c > 0$ such that for all $x \in \text{Ran}(d)^\perp \cap \text{Dom}(d)$

$$c\|x\| \leq \|dx\|.$$

Set $h: \text{Ker}(d) \rightarrow \text{Ker}(d)^\perp \cap \text{Dom}(d)$ to be equal to the inverse of the operator d , h is bounded, since it is a closed operator defined on the whole Banach space $\text{Ker}(d)$ by setting

$$H = h \oplus 0: C^\bullet = \text{Ker}(d) \oplus \text{Ker}(d)^\perp \rightarrow C^\bullet$$

we obtain a bounded operator such that the minimal closure of $Hd + dH$ is equal to the identity.

For the second point, one has a short exact sequence of complexes

$$0 \longrightarrow C_1^\bullet \longrightarrow \text{cone}(f)^\bullet \longrightarrow C_0^\bullet[1] \longrightarrow 0.$$

If $\iota: C_0^\bullet[1] \rightarrow \text{cone}(f)^\bullet$ is the canonical injection (which is not a morphism of complexes), take H the contraction of C_0^\bullet defined earlier, then

$$s = d_{\text{cone}} \iota H - \iota H d_0$$

will be a section of the previous short exact sequence and a morphism of complexes, provided it is bounded. For the short exact sequence to split, one only needs to check that (the minimal closure of) s is bounded, which is straightforward, since Hd_0 is equal to the orthogonal projection on $\text{Ker}(d)^\perp$, and $d_{\text{cone}} \iota H$ is equal to $-\iota \oplus fh$ which is bounded. \square

Induction of $\mathcal{N}(\Gamma)$ -Hilbert complex. We now turn to the induction of complex. Recall by Proposition A.3.77 that if $G < \Gamma$ is a subgroup, we have a faithfully flat additive functor ι_* from the category of $\mathcal{N}(G)$ -Hilbert module to the category of $\mathcal{N}(\Gamma)$ -Hilbert module. Since the functor ι_* extends to densely defined closed operators, it induces a functor from the category of $\mathcal{N}(G)$ -Hilbert complexes to the category of $\mathcal{N}(\Gamma)$ -Hilbert complexes.

Let (C^\bullet, d) be a $\mathcal{N}(G)$ -complex. The weak Hodge decomposition

$$C^k = \text{Harm}^k(C^\bullet) \oplus \overline{\text{Ran}(d)} \oplus \overline{\text{Ran}(d^*)}$$

gives a decomposition

$$\iota_* C^k = \iota_* \text{Harm}^k(C^\bullet) \oplus \iota_* \overline{\text{Ran}(d)} \oplus \iota_* \overline{\text{Ran}(d^*)}$$

and this is the weak Hodge decomposition of the $\mathcal{N}(\Gamma)$ -complex $\iota_* C^\bullet$.

By Proposition A.3.78 the Laplace operator of $\iota_* d$ is just $\iota_* \square_d$ and the projection-valued measure of $\iota_* \square_d$ is given by $\iota_* E_{\square_d}$.

Tensor product of $\mathcal{N}(\Gamma)$ -Hilbert complexes. In this paragraph, we consider two discrete groups Γ_1 and Γ_2 . We refer the reader to Definitions A.1.26 and A.3.50 for the notion of tensor product of Hilbert spaces and of tensor product of Von Neumann algebras.

Definition 2.1.15. Let C_1^\bullet (resp. C_2^\bullet) be a $\mathcal{N}(\Gamma_1)$ -Hilbert complex (resp. $\mathcal{N}(\Gamma_2)$ -Hilbert complex). The tensor product $C_1^\bullet \hat{\otimes} C_2^\bullet$ of $\mathcal{N}(\Gamma_1 \times \Gamma_2)$ -Hilbert complexes is defined by

$$(C_1^\bullet \otimes C_2^\bullet)^k = \bigoplus_{p+q=k} C_1^p \hat{\otimes} C_2^q$$

and whose differential is given by the minimal closure of

$$d_{C_1 \otimes C_2}^k = \bigoplus_{p+q=k} d_{C_1}^p \otimes id + (-1)^p id \otimes d_{C_2}^q$$

The main proposition, which is stated below, is due to Berger [Ber16, Proposition 3.2].

Proposition 2.1.16. The Laplace operator on $(C_1^\bullet \otimes C_2^\bullet)^k$ is given by (the minimal closure of)

$$\bigoplus_{p+q=k} \square_{C_1} \otimes id + id \otimes \square_{C_2}.$$

2 $\mathcal{N}(\Gamma)$ -Fredholm complexes

This section is dedicated to the central notion of $\mathcal{N}(\Gamma)$ -Fredholm complexes, whose definition is given below.

Definition of Fredholm complexes.

Definition 2.2.17. A $\mathcal{N}(\Gamma)$ -Hilbert complex (C^\bullet, d) is said to be $\mathcal{N}(\Gamma)$ -Fredholm if it satisfies one of the equivalent properties below

1. The restriction of the differential d to $\text{Ran}(d)^\perp$ is $\mathcal{N}(\Gamma)$ -Fredholm.
2. If \square_d denotes the Laplace operator $\square_d := (d + d^*)^2$ and $(E_\lambda^{\square_d})_{\lambda > 0}$ is its associated projection-valued measure, then there exists $\varepsilon > 0$ such that $\text{tr}_{\mathcal{N}(\Gamma)} E_\varepsilon^{\square_d} < +\infty$.
3. 0 does not belong to the $\mathcal{N}(\Gamma)$ -essential spectrum of \square_d .

Remark 2.2.18. If C_0^\bullet and C_1^\bullet are two $\mathcal{N}(\Gamma)$ -Hilbert complexes, then $C_0^\bullet \oplus C_1^\bullet$ is $\mathcal{N}(\Gamma)$ -Fredholm if and only if both C_0^\bullet and C_1^\bullet are $\mathcal{N}(\Gamma)$ -Fredholm.

The equivalence between the last two points is just the definition of the $\mathcal{N}(\Gamma)$ -essential spectrum (see Definition A.3.51). For the equivalence between the first two properties, by [Lüc02, Lemma 2.3], a closed operator $f: H_1 \rightarrow H_2$ is $\mathcal{N}(\Gamma)$ -Fredholm if and only if $\text{tr}_{\mathcal{N}(\Gamma)} E_{f^*f, \varepsilon} < +\infty$ for $\varepsilon > 0$ small enough, we use the weak Kodaira decomposition $C^p = \text{Ker}(\square_d) \oplus \overline{\text{Ran}(d)} \oplus \overline{\text{Ran}(d^*)}$ to obtain the decomposition of the Laplacian

$$\square_d = d^*d \Big|_{\overline{\text{Ran}(d)}^\perp} \oplus dd^* \Big|_{\overline{\text{Ran}(d)}}$$

which gives 2. \implies 1. Conversely, if d is $\mathcal{N}(\Gamma)$ -Fredholm then $\text{Ker}(\square_d) = \text{Ker}(d) \cap \text{Ker}(d^*)$ has finite dimension, so $\text{tr}_{\mathcal{N}(\Gamma)} E_{\square_d, 0} < +\infty$ and

$$\text{tr}_{\mathcal{N}(\Gamma)} E_{\square_d, \varepsilon} - \text{tr}_{\mathcal{N}(\Gamma)} E_{\square_d, 0} = \text{tr}_{\mathcal{N}(\Gamma)} E_{d^*d \Big|_{\text{Ker}(d)^\perp}, \varepsilon} + \text{tr}_{\mathcal{N}(\Gamma)} E_{dd^* \Big|_{\text{Ker}(d^*)^\perp}, \varepsilon} < +\infty.$$

Recall that if $f: H_0 \rightarrow H_1$, $g: H_1 \rightarrow H_2$ are two closed densely defined Γ -equivariant operators with g bounded and gf is $\mathcal{N}(\Gamma)$ -Fredholm, then f is $\mathcal{N}(\Gamma)$ -Fredholm. In particular, $\mathcal{N}(\Gamma)$ -Fredholmness is a property invariant by isomorphism. In fact, it is even invariant by quasi-isomorphism, as will be seen in Lemma 2.2.20.

In the case where $\Gamma = \{e\}$ (or more generally, if Γ is finite), as noted in [BL92, Corollary 2.5], the differential of a $\mathcal{N}(\Gamma)$ -Fredholm complex has a closed range. This is not true for a general discrete group Γ , however, one might expect the part $\overline{\text{Ran}(d)} / \text{Ran}(d)$ to vanish in a suitable torsion theory. There exist two theories of natural torsion in the category of $\mathcal{N}(\Gamma)$ -modules, they are discussed in [Lüc02, Section 8.4] and in the third section of the appendices, in our case we would like to have

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \frac{\overline{\text{Ran}(d)}}{\text{Ran}(d)} = 0 \quad \text{or} \quad \dim_\Gamma \frac{\overline{\text{Ran}(d)}}{\text{Ran}(d)}.$$

Where $\mathcal{U}(\Gamma)$ is the algebra of operators affiliated with $\mathcal{N}(\Gamma)$, for details about the algebra of affiliated operators, we refer the reader to [Lüc02, Chapter 8] or the appendices of the present manuscript. Note that the first condition implies the second by [Lüc02, Lemma 8.33] (or see Proposition A.3.66). This is true for $\mathcal{N}(\Gamma)$ -Fredholm complexes, and this key property was first observed by Dingoyan [Din13], the result is stated below and is a consequence of Proposition A.3.67.

Proposition 2.2.19. Let (C^\bullet, d^\bullet) be a $\mathcal{N}(\Gamma)$ -Fredholm complex. Then we have

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \frac{\overline{\text{Ran}(d)}}{\text{Ran}(d)} = 0 \quad \dim_{\mathcal{N}(\Gamma)} \frac{\overline{\text{Ran}(d)}}{\text{Ran}(d)} = 0.$$

In particular

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^k(\mathbf{C}^\bullet) \simeq \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_{\text{red}}^k(\mathbf{C}^\bullet) \simeq \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \text{Harm}^k(\mathbf{C}^\bullet).$$

Stability by quasi-isomorphism.

Lemma 2.2.20. We have the following properties

1. An acyclic complex (\mathbf{C}^\bullet, d) is $\mathcal{N}(\Gamma)$ -Fredholm.
2. If two $\mathcal{N}(\Gamma)$ -Hilbert complexes $(\mathbf{C}_0^\bullet, d_0)$ and $(\mathbf{C}_1^\bullet, d_1)$ are quasi-isomorphic then if one is $\mathcal{N}(\Gamma)$ -Fredholm, the other is also $\mathcal{N}(\Gamma)$ -Fredholm.

These results are in [Lüc02] in the case of complexes with bounded differentials.

Proof. For the first point, if a complex is acyclic, it has a closed range and $\text{Ker}(d)^\perp = \text{Ran}(d)^\perp$. It follows that there exists a constant $c > 0$ such that for all $x \in \text{Ran}(d)^\perp \cap \text{Dom}(d)$

$$c\|x\| \leq \|dx\|.$$

For $f: H_1 \dashrightarrow H_2$ a densely defined morphism between $\mathcal{N}(\Gamma)$ -Hilbert modules, and $\lambda > 0$ we set $\mathcal{L}(f, \lambda)$ as the set of Hilbert submodules $L \subset H_1$ that satisfy $L \subset \text{Dom}(f)$ and $\|f(x)\| \leq \lambda\|x\|$ for all $x \in L$. We recall that by [Lüc02, Lemma 2.2] f is $\mathcal{N}(\Gamma)$ -Fredholm if and only if there exist two positive constants $c, \varepsilon > 0$ such that for all $L \in \mathcal{L}(f, \varepsilon)$ one has $\dim_\Gamma L < C$. Since $\text{Ran}(d)^\perp = \text{Ker}(d)^\perp$ it follows that for all $\varepsilon < c$ one has $\mathcal{L}(d_{\text{Ran}(d)^\perp}, \varepsilon) = \{0\}$ and the complex is Γ -Fredholm.

For the second point, fix a quasi-isomorphism $f: \mathbf{C}_0^\bullet \rightarrow \mathbf{C}_1^\bullet$, then the mapping cones $\text{cone}(f)^\bullet$ and $\text{cone}(\text{Id}_{\mathbf{C}_0^\bullet})$ are acyclic and thus $\mathcal{N}(\Gamma)$ -Fredholm. By using the last property, we have isomorphisms of $\mathcal{N}(\Gamma)$ -Hilbert complexes

$$\begin{aligned} \text{cyl}(f)^\bullet &\simeq \text{cone}(\text{cone}(f)^\bullet[-1] \rightarrow \mathbf{C}_0^\bullet) && \simeq \text{cone}(f)^\bullet \oplus \mathbf{C}_0^\bullet \\ &\simeq \text{cone}(f \oplus 0: \text{cone}(\text{Id})[-1] \rightarrow \mathbf{C}_1^\bullet) && \simeq \text{cone}(\text{Id})^\bullet \oplus \mathbf{C}_1^\bullet. \end{aligned}$$

The isomorphisms in the first line show that $\text{cyl}(f)^\bullet$ is (Γ) -Fredholm if and only if \mathbf{C}_0^\bullet is (Γ) -Fredholm, and the isomorphisms in the second line show that $\text{cyl}(f)^\bullet$ is $\mathcal{N}(\Gamma)$ -Fredholm if and only if \mathbf{C}_1^\bullet is $\mathcal{N}(\Gamma)$ -Fredholm. \square

Characterisation in terms of spectra of Laplace operators. To study the $\mathcal{N}(\Gamma)$ -Fredholmness, one should study the spectrum of the Laplace operators. The following lemma allows us to study the spectrum of only half of the Laplace operators.

Lemma 2.2.21. Let (\mathbf{C}^\bullet, d) be a bounded $\mathcal{N}(\Gamma)$ -Hilbert complex, for $k \in \mathbb{Z}$ denote by $\square_{d,k}$ the Laplace operator acting on \mathbf{C}^k and assume that there exists $\varepsilon > 0$ such that one has for all $k \in \mathbb{Z}$

$$\text{tr}_{\mathcal{N}(\Gamma)} E_{\square_{d,2k}, \varepsilon} < +\infty \quad \text{and} \quad \dim_\Gamma \text{Harm}^k < +\infty.$$

Then (\mathbf{C}^\bullet, d) is $\mathcal{N}(\Gamma)$ -Fredholm.

Proof. Recall that $\text{Ran}(E_{\square_{d,\varepsilon}})^\bullet$ is a subcomplex of \mathbf{C}^\bullet , therefore, one has a decomposition

$$\text{Ran}(E_{\square_{d,\varepsilon}})^k = \text{Harm}^k \oplus \overline{d(\text{Ran}(E_{\square_{d,\varepsilon}})^{k-1})} \oplus \overline{d^*(\text{Ran}(E_{\square_{d,\varepsilon}})^{k-1})}.$$

We already know that $\text{Ran}(E_{\square_{d,\varepsilon}})^k$ has a finite dimension for k even, and if k is odd then both

$$\overline{d(\text{Ran}(E_{\square_{d,\varepsilon}})^{k-1})} \quad \text{and} \quad \overline{d^*(\text{Ran}(E_{\square_{d,\varepsilon}})^{k-1})}$$

are finite-dimensional, and it follows that $\text{Ran}(E_{\square_{d,\varepsilon}})^k$ must also be, therefore the complex is $\mathcal{N}(\Gamma)$ -Fredholm. \square

Remark 2.2.22. As it can be seen in the proof, the hypothesis $\dim_\Gamma(\text{Harm}^k) < +\infty$ is only here to prevent some rather simple case of complexes of the form

$$\dots \longrightarrow 0 \longrightarrow H \longrightarrow 0 \longrightarrow \dots$$

where H is a $\mathcal{N}(\Gamma)$ -Hilbert module of infinite dimension.

Corollary 2.2.23. Consider a short exact sequence of $\mathcal{N}(\Gamma)$ -Hilbert complexes

$$0 \longrightarrow C_1^\bullet \longrightarrow C_2^\bullet \longrightarrow C_3^\bullet \longrightarrow 0.$$

Then if two of the complexes are $\mathcal{N}(\Gamma)$ -Fredholm, the third one is also $\mathcal{N}(\Gamma)$ -Fredholm.

Proof. For $j \in \{1, 2, 3\}$, let $G(C_j^\bullet)$ be the $\mathcal{N}(\Gamma)$ -Hilbert complex whose underlying Hilbert spaces are given by $\text{Dom}(d_j)$ endowed with the graph hermitian product

$$\langle x, y \rangle_{G(C_j)} = \langle x, y \rangle_{C_j} + \langle dx, dy \rangle_{C_j}.$$

and the differential is induced by d . Then $G(C_j^\bullet)$ is a $\mathcal{N}(\Gamma)$ -Hilbert complex whose differential d is bounded, and the inclusion $G(C_j^\bullet) \rightarrow C_j^\bullet$ is a quasi-isomorphism, hence C_j^\bullet is $\mathcal{N}(\Gamma)$ -Fredholm if and only if $G(C_j^\bullet)$ is $\mathcal{N}(\Gamma)$ -Fredholm. Our short exact sequence induces a short exact sequence

$$0 \longrightarrow G(C_1^\bullet) \longrightarrow G(C_2^\bullet) \longrightarrow G(C_3^\bullet) \longrightarrow 0.$$

And it suffices to prove our result in the case where we have complexes with bounded differentials. This is done in the book of Lück [Lüc02, Theorem 2.20], which concludes the proof. \square

Weakly exact sequences in cohomology. Another interesting feature of the $\mathcal{N}(\Gamma)$ -Fredholm complex is the following result due to Shubin [Shu95, Theorem 2.11]

Theorem 2.2.24. Consider a short exact sequence of $\mathcal{N}(\Gamma)$ -Hilbert complexes that are $\mathcal{N}(\Gamma)$ -Fredholm

$$0 \longrightarrow A^\bullet \xrightarrow{\iota} B^\bullet \xrightarrow{p} C^\bullet \longrightarrow 0.$$

Then it induces a long sequence in reduced cohomology

$$\dots \longrightarrow H_{\text{red}}^k(A^\bullet) \longrightarrow H_{\text{red}}^k(B^\bullet) \longrightarrow H_{\text{red}}^k(C^\bullet) \longrightarrow H_{\text{red}}^{k+1}(A^\bullet) \longrightarrow \dots$$

that is weakly exact, in the sense that the range of each map is essentially dense in the kernel of the next one.

We recall the reader that the notion of essential density is discussed in the appendices; we also refer to [Lüc02, Section 8.1] or [Shu95, Definition 1.7] for more details.

Stability by induction. Let $G < \Gamma$ be a subgroup. We have seen that we have a functor ι_* from the category of $\mathcal{N}(G)$ -Hilbert complexes to the category of $\mathcal{N}(\Gamma)$ -Hilbert complexes. Proposition A.3.78 yields $\iota_* \square_{C^\bullet} = \square_{\iota_* C^\bullet}$ and then one sees that at the level of the spectral projection one has

$$\iota_* E_{\square_{C^\bullet}} = E_{\square_{\iota_* C^\bullet}}$$

The proposition below follows from the above discussion.

Proposition 2.2.25. Let (C^\bullet, d) be a $\mathcal{N}(G)$ -Fredholm $\mathcal{N}(G)$ -Hilbert complex. Then the $\mathcal{N}(\Gamma)$ -Hilbert complex $\iota_* C^\bullet$ is $\mathcal{N}(\Gamma)$ -Fredholm.

Stability by tensor product. We have seen that given C_1^\bullet (resp. C_2^\bullet) a bounded $\mathcal{N}(\Gamma_1)$ -Hilbert complex (resp. $\mathcal{N}(\Gamma_2)$ -Hilbert complex) one can form the $\mathcal{N}(\Gamma_1 \times \Gamma_2)$ -Hilbert complex $C_1^\bullet \hat{\otimes} C_2^\bullet$. One has the following proposition, which is given in Lück [Lüc02, Lemma 2.35].

Proposition 2.2.26. Let C_1^\bullet and C_2^\bullet be as above then if C_j is $\mathcal{N}(\Gamma_j)$ -Fredholm for $j = 1, 2$ then $C_1^\bullet \hat{\otimes} C_2^\bullet$ is $\mathcal{N}(\Gamma_1 \times \Gamma_2)$ -Fredholm.

Proof. We have seen that the Laplace operator on $(C_1^\bullet \hat{\otimes} C_2^\bullet)^k$ is given by

$$\bigoplus_{p+q=k} \square_{C_1^p} \otimes id + id \otimes \square_{C_2^q}$$

we infer using Theorem A.1.30 and, setting $E_{1,p}$ (resp $E_{2,p}$) the spectral projectors of the Laplace operators on C_1^p (resp C_2^q) that it is equal to

$$\bigoplus_{p+q=k} \int_{\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}} \lambda_1 + \lambda_2 dE_{1,p} \otimes E_{2,q}.$$

Let E denote the spectral measure for the Laplace operator on $C_1 \hat{\otimes} C_2$ and take $\varepsilon > 0$ that satisfies $\text{tr}_{\mathcal{N}(\Gamma_1)} E_{1,p}([0, \varepsilon]) < +\infty$ and $\text{tr}_{\mathcal{N}(\Gamma_2)} E_{2,q}([0, \varepsilon]) < +\infty$ for all p, q . One obtains

$$\text{tr}_{\mathcal{N}(\Gamma_1 \times \Gamma_2)} E([0, \varepsilon]) \leq \sum_{p+q=k} \text{tr}_{\mathcal{N}(\Gamma_1)} E_{1,p}([0, \varepsilon]) \cdot \text{tr}_{\mathcal{N}(\Gamma_2)} E_{2,p}([0, \varepsilon]) < +\infty.$$

From this follows the $\mathcal{N}(\Gamma_1 \times \Gamma_2)$ -Fredholmness. \square

3 Double $\mathcal{N}(\Gamma)$ -complexes

Definition 2.3.27. A $\mathcal{N}(\Gamma)$ -Hilbert double complex $(C^{\bullet, \bullet}, d_1, d_2)$ is the data of $\mathcal{N}(\Gamma)$ -Hilbert modules $C^{p,q}$ for $(p, q) \in \mathbb{Z}^2$ and two closed densely defined operators $d_1: C^{p,q} \dashrightarrow C^{p+1,q}$ and $d_2: C^{p,q} \dashrightarrow C^{p,q+1}$ such that either d_1 or d_2 is bounded, and we have for all $i, j \in \{1, 2\}$ $\text{Dom}(d_j) \cap \text{Ran}(d_i) = d_i(\text{Dom}(d_j))$ and relations

$$d_1^2 = d_2^2 = 0 \quad d_1 d_2 = d_2 d_1.$$

The simple complex (C^\bullet, d) associated with the double complex $(C^{\bullet, \bullet}, d_1, d_2)$ is the complex defined by

$$C^n = \bigoplus_{p+q=n} C^{p,q} \quad d|_{C^{p,q}} = d_1 + (-1)^p d_2 \quad \text{Dom}(d) = \text{Dom}(d_1) \cap \text{Dom}(d_2).$$

The definition is pretty much similar to the one we have when working with complexes in Abelian categories. We bring the reader attention to the hypothesis regarding the domain of the differentials. The assumption that one of them is bounded ensures that in the associated simple complex, the differential d is closed. In particular, it is a $\mathcal{N}(\Gamma)$ -Hilbert complex.

For the other condition on the domains, the one we impose is stronger than the naive one, namely $d_i(\text{Dom}(d_j)) \subset \text{Dom}(d_j)$. The reason is to recover the following lemma, which is classical in the case of standard homological algebra.

Lemma 2.3.28. Let (C^\bullet, d) be a bounded $\mathcal{N}(\Gamma)$ -Hilbert complex and $(K^{\bullet, \bullet}, d_1, d_2)$ a double $\mathcal{N}(\Gamma)$ -Hilbert complex. Assume that we have the following

1. There exists $i \in \mathbb{Z}$ such that $K^{p,q} = 0$ if $q < 0$ or $p < i$.
2. We have an injective morphism of $\mathcal{N}(\Gamma)$ -Hilbert complexes $\iota: C^\bullet \rightarrow K^{\bullet, 0}$, which induces an isomorphism $C^n \rightarrow \text{Ker}(d_2) \cap K^{n,0}$ for each $n \in \mathbb{Z}$ and $\text{Ker}(d_2) \cap \text{Dom} d_1 = \iota(\text{Dom}(d))$.
3. For all p the morphism $\iota: C^p \rightarrow (K^{p, \bullet}, d_2)$ is a resolution of C^p .

Then ι induces a quasi-isomorphism $\iota: C^\bullet \rightarrow K^\bullet$.

Proof. The condition we impose on the domains allows us to use the usual proof in the case of Abelian group almost verbatim (proof given, for example, in [Voi02, Lemma 8.5]) by being careful with the domain of our operators. Using a shift, one can assume that $C^{p,q} = 0$ if $p < 0$ or $q < 0$. Let us begin to show that ι is injective in cohomology, take $y \in \text{Ran}(\iota) \cap K^n \subset K^{n,0}$ which is d -exact, so there exists $x = (x_{p,q})_{p+q=n-1}$ with $dx = y$ in particular $d_2 x_{0,n-1} = 0$ so if $n-1 = 0$, $x = x_{0,0} \in \text{Ker}(d_2) = \text{Ran}(\iota)$. If $n-1 > 0$, we can take $z \in K^{0,n-2} \cap \text{Dom}(d_2)$ such that $d_2 z = x_{0,n-2}$, since $x_{0,n-1} \in \text{Dom}(d_1)$, we can take $z \in \text{Dom}(d_2) \cap \text{Dom}(d_1)$, so dz is well-defined (this is where we need the assumption about the domains in the definitions of double $\mathcal{N}(\Gamma)$ -Hilbert complexes) and $d(x - dz) = Dx = y$ so we can assume $x \in \bigoplus_{q < n-1} K^{p,q}$. By induction, we see that we can assume that $x \in K^{n-1,0}$, but in this case $d_2 x = 0$ since $y \in K^{n,0}$, and $x \in \text{Ran}(\iota)$ which gives us injectivity in cohomology since $x \in \text{Dom}(d_1)$ is well-defined $\iota^{-1}x \in \text{Dom}(d)$ since $\text{Ker}(d_2) \cap \text{Dom} d_1 = \iota(\text{Dom}(d))$ which gives injectivity.

For the surjectivity, if $y \in K^n$ is a closed form, by a similar induction one can assume $y \in K^{n,0}$, in which case y is both d_2 and d_1 closed so it is the image of a d -closed form. \square

4 The L^2 -de Rham complex of a covering of a compact manifold with corners.

Generalities on L^2 de Rham complex. In this section we consider (M, g) a Riemannian manifold and a Galois covering $\pi: \tilde{M} \rightarrow M$ with transformation group $\text{Deck}(\tilde{M}/M) = \Gamma$. We consider \mathbb{V} a local system on M , and if we denote by $\mathcal{A}_M(\mathbb{V})$ the sheaf on M of smooth sections on the underlying flat vector bundle with flat connection D . We denote by $L^2DR^\bullet(\tilde{M}, \pi^*g, \pi^*\mathbb{V}, \pi^*h)$ the space of measurable k -forms ω on \tilde{M} with values on $\pi^*\mathbb{V}$ satisfying

$$\int_{\tilde{M}} \langle \omega, \omega \rangle d\text{Vol} < +\infty.$$

When the metrics are clear from the context, we will simply denote them by $L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V})$. The differential $D: \mathcal{A}_c^k(\tilde{M}, \pi^*\mathbb{V}) \rightarrow \mathcal{A}_c^{k+1}(\tilde{M}, \pi^*\mathbb{V})$ can be seen as a Γ -equivariant unbounded operator on $L^2DR^k(\tilde{M}, \pi^*\mathbb{V})$, and the choice of a Γ -equivariant closure of D should give a $\mathcal{N}(\Gamma)$ -Hilbert complex. It is not true that any closure of D satisfies $\text{Ran}(D) \subset \text{Dom}(D)$, this brings the following definition.

Definition 2.4.29. An ideal boundary condition D_{bc} is a Γ -equivariant closure of D that satisfies the condition $\text{Ran}(D_{bc}) \subset \text{Dom}(D_{bc})$.

Given an ideal boundary condition, one can consider the $\mathcal{N}(\Gamma)$ -Hilbert complex

$$L^2DR_{bc}^\bullet(\tilde{M}, \pi^*\mathbb{V}) = (L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V}), D_{bc}).$$

We will denote its cohomology group by $H_{L^2, bc}^k(\tilde{M}, \pi^*\mathbb{V})$, we recall that by the weak Hodge decomposition if we set $\text{Harm}_{bc}^k(\tilde{M}, \pi^*\mathbb{V}) := \text{Harm}_{D_{bc}}^k = \text{Ker}(\square_{D_{bc}})$ one has a canonical isomorphism

$$H_{L^2, bc}^k(\tilde{M}, \pi^*\mathbb{V}) \simeq \text{Harm}_{bc}^k(\tilde{M}, \pi^*\mathbb{V}) \oplus \overline{\text{Ran}(D_{bc})} / \text{Ran}(D_{bc}).$$

Example 2.4.30. The minimal closure D_{\min} of D is an ideal boundary condition. We recall that its domain consists of measurable forms ω such that there exists a sequence $(\omega_n)_{n \in \mathbb{N}} \in (\mathcal{A}_c^k(\tilde{M}, \pi^*\mathbb{V}))^{\mathbb{N}}$ such that

$$\omega_n \xrightarrow{L^2} \omega \quad D\omega_n \xrightarrow{L^2} D\omega.$$

The maximal closure D_{\max} is an ideal boundary condition. We recall that the domain of the maximal closure consists of square-integrable measurable forms ω such that $D\omega$ is square-integrable ($D\omega$ is computed here in the sense of distributions).

These are the main examples of ideal boundary conditions. An important fact is that given any closure \bar{D} of D satisfies $D_{\min} \subset \bar{D} \subset D_{\max}$. We denote by \mathfrak{d} the formal adjoint of D , it is the differential operator defined on compactly supported \mathbb{V} -valued smooth forms by the identity

$$\langle \mathfrak{d}\phi, \psi \rangle = \langle \phi, D\psi \rangle$$

when ϕ, ψ are compactly supported smooth forms. The Laplace operator is the differential operator

$$\square_D = (D + \mathfrak{d})^2.$$

It is formally self-adjoint, which means that if ϕ and ψ are compactly supported smooth forms one has

$$\langle \square_D \phi, \psi \rangle = \langle \phi, \square_D \psi \rangle.$$

The operator \mathfrak{d} also admits two closures \mathfrak{d}_{\min} and \mathfrak{d}_{\max} defined similarly, and we have the adjunction relations

$$D_{\min}^* = \mathfrak{d}_{\max} \quad D_{\max}^* = \mathfrak{d}_{\min}$$

where D_{\min}^* (resp. D_{\max}^*) denotes the adjoint of D_{\min} (resp. D_{\max}) taken in the sense of operator theory.

We recall that one has a morphism of smooth vector bundles given by the Hodge star operator $*$:

$$*: \mathcal{A}^k(M, \mathbb{V}) \rightarrow \mathcal{A}^{n-k}(M, \mathbb{V}^*)$$

where \mathbb{V}^* is the dual local system of \mathbb{V} and n is the real dimension of M . This operator is characterised by the property

$$\|s\|^2 d\text{Vol} = \{*\nu, \nu\}$$

where $\{\cdot, \cdot\}: \mathbb{V}^* \otimes \Omega^k \otimes \mathbb{V} \otimes \Omega^{n-k} \rightarrow \Omega^n$ is the natural contraction. If we equip $\mathcal{A}(\mathbb{V}^*)$ with the metric induced by the metric h in \mathbb{V} , we obtain an isometry

$$*: L^2DR^k(M, \mathbb{V}) \rightarrow L^2DR^{n-k}(M, \mathbb{V}^*).$$

We recall the well known result

Proposition 2.4.31. As a differential operator, the formal adjoint \mathfrak{d} is given by the formula

$$\mathfrak{d} = *^{-1}d*.$$

From this one can obtain the

Theorem 2.4.32. The Hodge star operator $*$ induces an isomorphism

$$*: \text{Harm}_{\min}^k(M, \mathbb{V}) \rightarrow \text{Harm}_{\max}^{n-k}(M, \mathbb{V}^*).$$

Proof. The relations $D_{\min}^* = \mathfrak{d}_{\max}$ and $D_{\max}^* = \mathfrak{d}_{\min}$ imply that $*$ maps $\text{Dom}(D_{\min}^*)$ to $\text{Dom}(D_{\max})$ and $\text{Dom}(D_{\min})$ to $\text{Dom}(D_{\max}^*)$. Furthermore, a form ω is closed if and only if $*\omega$ is co-closed, and conversely ω is co-closed if and only if $*\omega$ is closed. Therefore, $*$ induces an isometry between spaces of harmonic forms. \square

Corollary 2.4.33. Assume that both $L^2DR_{\min}^\bullet(M, \mathbb{V})$ and $L^2DR_{\max}^\bullet(M, \mathbb{V}^*)$ are $\mathcal{N}(\Gamma)$ -Fredholm, then the Hodge star operator induces an isomorphism of $\mathcal{U}(\Gamma)$ -module

$$*: \mathcal{U}(\Gamma) \otimes H_{L^2, \min}^k(\tilde{M}, \pi^*\mathbb{V}) \rightarrow \mathcal{U}(\Gamma) \otimes H_{L^2, \max}^{n-k}(\tilde{M}, \pi^*\mathbb{V}^*).$$

Case of a non-connected covering. We still consider a Galois covering $\pi: \tilde{M} \rightarrow M$ with transformation group $\text{Deck}\left(\frac{\tilde{M}}{M}\right) = \Gamma$. We assume that \tilde{M} is not connected, then Γ acts transitively on the connected component of \tilde{M} . Set \tilde{U} to be a connected component of \tilde{M} and G as the isotropy subgroup of \tilde{U} . Then we can consider the $\mathcal{N}(G)$ -Hilbert complexes $L^2DR_{\min}^\bullet(\tilde{U}, \pi^*\mathbb{V})$ and $L^2DR_{\max}^\bullet(\tilde{U}, \pi^*\mathbb{V})$.

Recall that we have defined the induction functor ι_* from the category of $\mathcal{N}(G)$ -Hilbert complexes to the category of $\mathcal{N}(\Gamma)$ -Hilbert complexes when G is a subgroup of Γ . We recall that ι_*C^k is given by the completion of

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[G]} C^k \simeq \bigoplus_{\Gamma/G} C^k.$$

In our case, one has $C^k = L^2DR_{bc}^k(\tilde{U}, \pi^*\mathbb{V})$, and we have a natural Γ -equivariant morphism

$$\bigoplus_{\Gamma/G} L^2DR_{bc}^k(\tilde{U}, \pi^*\mathbb{V}) \rightarrow L^2DR_{bc}^k(\tilde{M}, \pi^*\mathbb{V})$$

that induces an isomorphism after completion. In summary, we obtain the following.

Proposition 2.4.34. If ι_* denotes the induction functor from the category of $\mathcal{N}(G)$ -Hilbert complexes to the category of $\mathcal{N}(\Gamma)$ -Hilbert complexes, one has

$$\iota_*L^2DR_{\min}^\bullet(\tilde{U}, \pi^*\mathbb{V}) = L^2DR_{\min}^\bullet(\tilde{M}, \pi^*\mathbb{V}) \quad \text{and} \quad \iota_*L^2DR_{\max}^\bullet(\tilde{U}, \pi^*\mathbb{V}) = L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})$$

In particular, if one wants to check the $\mathcal{N}(\Gamma)$ -Fredholmness, it is possible to reduce to connected coverings by Proposition 2.2.25.

The case of compact manifold with corners. In the rest of this section, we treat the case where M is the interior of a compact Riemannian manifold with corners X and under the assumption that \mathbb{V}, g and h are defined on the whole space X , and we consider coverings of the form $\pi: \tilde{X} \rightarrow X$. There exist several definitions of smooth manifolds with corners; we will follow the one given in [MROD92] (and restrict ourselves to the finite-dimensional case). It is an object that is locally diffeomorphic to an open of the quadrant

$$Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 1 \leq j \leq n, \quad x_j \geq 0\}.$$

The definition of the tangent space and of Riemannian metrics is similar to the one developed for manifolds with boundary. Any metrics g, h that satisfy our assumption are mutually bounded; therefore, the spaces of square-integrable forms $L^2DR^k(\tilde{M}, \pi^*g, \pi^*\mathbb{V}, \pi^*h)$ as well as the de Rham complexes $L^2DR_{\min}(\tilde{M}, \pi^*g, \pi^*\mathbb{V}, \pi^*h)$ and $L^2DR_{\min}(\tilde{M}, \pi^*g, \pi^*\mathbb{V}, \pi^*h)$ do not depend on the metrics. We wish to answer the following questions, where $bc \in \{\min, \max\}$.

1. Is there a topological interpretation to the cohomology groups $H_{L^2, bc}^k(\tilde{M}, \pi^*\mathbb{V})$?
2. Are the complexes $L^2DR_{bc}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ $\mathcal{N}(\Gamma)$ -Fredholm?
3. Is there an index theorem giving a relation between the cohomology groups $H_{L^2, bc}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ and $H_{L^2, bc}^\bullet(M, \mathbb{V})$?

The answers to all these questions are affirmative and are already known in a more general setting thanks to the work of Atiyah [Ati76] and Schick [Sch01] among many others (see also [Shu95, Lüc02, DS18]). Those proofs often involve at some point the use of pseudo-differential operators or Boutet-de Monvel calculus, and we will use our simpler setting to give alternative proofs of a combinatorial nature that rely on sheaf cohomology. We begin with the following lemma that can be found in [IL93] that settles the case where $\tilde{X} = X$ is a bounded convex subset in \mathbb{R}^n .

Lemma 2.4.35 (L^2 -Poincaré lemma for bounded convex domains). [IL93, Lemma 4.2] Set $K \subset \mathbb{R}^n$ an open bounded convex subset for $k \geq 1$, there exists a compact operator $T: L^2DR^k(K, \mathbb{C}) \rightarrow L^2DR^{k-1}(K, \mathbb{C})$ such that for any k -form ω with $\omega \in \text{Dom}(D_{\max})$ one has

$$\omega = TD\omega + DT\omega.$$

Since any local system \mathbb{V} on a convex $K \subset \mathbb{R}^n$ is isomorphic to a constant sheaf \mathbb{C}_K^l , it follows from this lemma that

$$H_{L^2, \max}^k(K, \mathbb{V}) = H^k(K, \mathbb{V})$$

where the right-hand side is just the sheaf cohomology of \mathbb{V} which vanishes when $k \geq 1$ since K is contractible. When we have a covering $\pi: \tilde{X} \rightarrow X$ one should consider the sheaf $\ell^2\pi^*\mathbb{V} = \ell^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \pi_!\pi^*\mathbb{V}$. This is a weakly constructible sheaf of $\mathcal{N}(\Gamma)$ -modules on X and one has the following result.

Theorem 2.4.36. Set M the interior of a compact manifold with corners X , \mathbb{V} to be a local system on X and $\pi: \tilde{X} \rightarrow X$ a Galois covering with group $\text{Deck}\left(\frac{\tilde{X}}{X}\right) = \Gamma$. Set $\tilde{M} = \pi^{-1}(M)$ then for $bc \in \{\min, \max\}$, the $\mathcal{N}(\Gamma)$ -Hilbert complex $L^2DR_{bc}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm, and we have

$$H_{L^2, \max}^k(\tilde{M}, \pi^*\mathbb{V}) = H^k(X, \ell^2\pi^*\mathbb{V}) \text{ and } H_{L^2, \min}^k(\tilde{M}, \pi^*\mathbb{V}) = H_\Phi^k(X, \ell^2\pi^*\mathbb{V})$$

where Φ is the space of compact subsets of M and $H_\Phi^k(X, \ell^2\pi^*\mathbb{V})$ denotes cohomology groups with support in Φ .

As we have said before, the proof of this theorem will require the use of sheaf cohomology. We define the complex of sheaves $\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})$ on X where the space of sections above an open subset set $U \subset X$ of $\mathcal{L}^2DR^k(\pi^*\mathbb{V})$ is given by the space of $\pi^*\mathbb{V}$ measurable k -forms ω on $\pi^{-1}(U)$ such that $D\omega$ is a measurable $k+1$ -form and one has for all $K \Subset U$

$$\int_{\pi^{-1}(K)} \|\omega\|^2 + \|D\omega\|^2 < +\infty.$$

and the differential is given by D . The only result we need about this complex of sheaves is the following.

Lemma 2.4.37. The complex of sheaves $\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})$ is a soft resolution of the sheaf $\ell^2\pi^*\mathbb{V}$.

It is straightforward to see that the sheaves $\mathcal{L}^2DR^k(\pi^*\mathbb{V})$ have a structure of C_X^∞ -modules given by

$$f \cdot \omega = (f \circ \pi)\omega \quad \forall f \in C_X^\infty, \omega \in \mathcal{L}^2DR^k(\pi^*\mathbb{V})$$

in particular, they are soft since any C_X^∞ -module on a manifold is soft (see, for instance, [God60, Theorem 3.7.1]).

We recall that soft sheaves are acyclic for the functors Γ and Γ_Φ for any family of support Φ (see [God60, Theorem 4.6.2] or [KS90, Proposition 2.5.10]), and it follows that this lemma gives that the cohomology groups $H^k(X, \ell^2 \pi^* \mathbb{V})$ can be computed as the cohomology of the complex of global section of $\mathcal{L}^2 DR^\bullet(\pi^* \mathbb{V})$ which is nothing but the complex $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})$.

Proof of the lemma. Let us begin with the fineness property. The considered sheaves have a structure of C_X^∞ -module given by

$$(\chi \cdot \omega)(p) = \chi(\pi(p))\omega(p) \quad \text{for } \chi \in C_X^\infty \text{ and } \omega \in \mathcal{L}^2 DR^\bullet(\pi^* \mathbb{V}).$$

Now we have to show that it is a resolution of $\ell^2 \pi^* \mathbb{V}$. It is clear that $\ell^2 \pi^* \mathbb{V} = \text{Ker}(D: \mathcal{L}^2 DR^0(\pi^* \mathbb{V}) \rightarrow \mathcal{L}^2 DR^1(\pi^* \mathbb{V}))$ and we have to prove that we have no cohomology in positive degree. Each point $p \in X$ admits a basis of neighbourhood U quasi-isometric to a bounded convex subset of \mathbb{R}^n and $\pi^{-1}(U)$ is isometric to $\Gamma \times U$ and this induces a unitary isomorphism

$$L^2 DR^k(\Gamma \times U, \pi^* \mathbb{V}) \simeq \ell^2(\Gamma) \hat{\otimes} L^2 DR^k(U, \mathbb{V}).$$

The differential on the right side is given by $\text{Id} \otimes D_{\max}$. Taking $\omega \in \mathcal{L}^2 DR^k(\pi^* \mathbb{V})(U)$, restricting U one can assume that ω is L^2 and if T is the contraction given by the lemma 2.4.35, one has

$$\omega = (\text{Id} \otimes T)(\text{Id} \otimes D)\omega + (\text{Id} \otimes D)(\text{Id} \otimes T)\omega.$$

It follows that ω is exact, provided that it is closed. Hence, one has no cohomology in positive degree. \square

With this, the theorem follows easily in the case of the maximal ideal boundary conditions.

Proof of the theorem for the maximal ideal boundary condition. As we have said before, the lemma implies directly that

$$H_{L^2, \max}^k(\tilde{M}, \pi^* \mathbb{V}) = H^k(X, \ell^2 \pi^* \mathbb{V}).$$

The only thing we have to check is that the complex $L^2 DR^\bullet(\tilde{M}, \pi^* \mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm. To see this quickly, we will use the results on double $\mathcal{N}(\Gamma)$ -Hilbert complexes, and a method similar to the one in [BT82, §8]. Consider $\mathfrak{U} = (U_i)_{i \in I}$ a finite covering of X by an open subset such that any intersection of open subsets of \mathfrak{U} is quasi-isometric to a convex subset. We endow I with a total order and define the complex $(K^{\bullet, \bullet}, D_1, D_2)$ where for $p, q \in \mathbb{Z}^2$

$$K^{p, q} = \begin{cases} \bigoplus_{\alpha_0 < \dots < \alpha_q \in I} L^2 DR_{\max}^p(\pi^{-1}(U_{\alpha_0, \dots, \alpha_q}), \pi^* \mathbb{V}) & \text{if } (p, q) \in \mathbb{N}^2 \\ 0 & \text{if } p < 0 \text{ or } q < 0 \end{cases}$$

Here $U_{\alpha_0, \dots, \alpha_q} = \bigcap_{j=0}^q U_{\alpha_j}$. The differential D_1 is given by D_{\max} on each $U_{\alpha_0, \dots, \alpha_q}$. If $\alpha_0 < \dots < \alpha_q$ are elements of I , fix $0 \leq i \leq q$, and $\omega \in K^{p, q}$, with $\omega = \bigoplus \omega_{\alpha_0, \dots, \alpha_q}$ where $\omega_{\alpha_0, \dots, \alpha_q} \in L^2 DR^p(\pi^{-1}(U_{\alpha_0, \dots, \alpha_q}), \pi^* \mathbb{V})$, we define $D_2 \omega$ by

$$(D_2 \omega)_{\alpha_0, \dots, \alpha_{q+1}} = \sum_{k=0}^{q+1} (-1)^k \omega_{\alpha_0, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_{q+1}} \in L^2 DR^p(\pi^{-1}(U_{\alpha_0, \dots, \alpha_{q+1}}), \pi^* \mathbb{V}).$$

The finiteness of \mathfrak{U} implies that D_2 is bounded, and we can check that we have then a proper double $\mathcal{N}(\Gamma)$ -complex. Moreover, by the previous lemma one has that for all q , $(K^{\bullet, q}, D_1)$ has no cohomology in positive degree. The kernel of $D_1: K^{0, q} \rightarrow K^{1, q}$ is given by

$$C^q = \bigoplus_{\alpha_0 < \dots < \alpha_q} \ell^2 \pi^* \mathbb{V}(U_{\alpha_0, \dots, \alpha_q})$$

and the complex (C^\bullet, D_2) is a $\mathcal{N}(\Gamma)$ -Hilbert complex whose objects are finitely generated; in particular, it is $\mathcal{N}(\Gamma)$ -Fredholm. This implies that the simple complex (K^\bullet, D^\bullet) associated to $(K^{\bullet, \bullet}, D_1, D_2)$ is $\mathcal{N}(\Gamma)$ -Fredholm as it is quasi-isomorphic to C^\bullet .

Now we wish to see that the complexes $(K^{p, \bullet}, D_2)$ have no cohomology in positive degree. We fix a partition of unity $(\rho_\alpha)_{\alpha \in I}$ subordinate to the covering \mathfrak{U} , we define $H: K^{p, q+1} \rightarrow K^{p, q}$ by

$$(H\omega)_{\alpha_0, \dots, \alpha_q} = \sum_{\alpha \in I \setminus \{\alpha_0, \dots, \alpha_q\}} (\rho_\alpha \circ \pi) \omega_{\alpha, \alpha_0, \dots, \alpha_q}$$

where we extend the forms $\rho_\alpha \omega_{\alpha, \alpha_0, \dots, \alpha_q}$ to $U_{\alpha_0, \dots, \alpha_q}$ by 0. The operator H is bounded and for $p, q \geq 0$, one has on $K^{p, q+1}$ the relation

$$\text{Id} = HD_2 + D_2H.$$

It follows that the complexes $(K^{p, \bullet}, D_2)$ have no cohomology in positive degree and by looking at the kernel of $D_2: K^{p, 0} \rightarrow K^{p, 1}$ we find that the complex (K^\bullet, D^\bullet) is quasi-isomorphic to $L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})$. Since K^\bullet is $\mathcal{N}(\Gamma)$ -Fredholm, so is $L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})$. \square

The case of the minimal ideal boundary condition is a little more problematic, as we do not have that

$$\Gamma_\Phi(\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})) = L^2DR_{\min}^\bullet(\tilde{M}, \pi^*\mathbb{V}).$$

The equality

$$H_{L^2, \min}^k(\tilde{M}, \pi^*\mathbb{V}) = H_\Phi^k(X, \ell^2\pi^*\mathbb{V})$$

follows from the following lemma.

Lemma 2.4.38. The inclusion

$$\Gamma_\Phi(\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})) \hookrightarrow L^2DR_{\min}^\bullet(\tilde{M}, \pi^*\mathbb{V})$$

is a quasi-isomorphism.

Proof. Let us begin to show that it induces a surjection in cohomology. Take U a neighbourhood of ∂M such that there exists a quasi-isometry

$$i: U \rightarrow [0, 1] \times \partial M$$

that satisfies $i(\partial M) = \{1\} \times \partial M$. If we denote by $\partial\tilde{M}$ the boundary of \tilde{M} , the space $\pi^{-1}(U)$ is quasi-isometric to $[0, 1] \times \partial\tilde{M}$, we will also denote by i this quasi-isometry.

Let $\chi: [0, 1] \rightarrow [0, 1]$ be a smooth non-decreasing function satisfying $\chi(t) = t$ in a neighbourhood of 0 and $\chi = 1$ in a neighbourhood of 1. Let $\omega \in L^2DR_{\min}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ be a closed form. Consider the form ψ defined by $\psi = \omega$ on $\tilde{M} \setminus \pi^{-1}(U)$ and defined on $\pi^{-1}(U)$ by $\psi(t, x) = (i^{-1} \circ (\chi \times \text{Id}_{\partial M}) \circ i)^*\omega$. Recall that on U one can write $\omega(t, x) = \omega_N(t, x) \wedge dt + \omega_T(t, x)$ with $t \in [0, 1]$ and $x \in \partial\tilde{M}$, with ω_N and ω_T that do not contain dt . Recall by [BL92, Theorem 4.1] that the hypothesis $\omega \in \text{Dom}(D_{\min})$ imposes $\omega_T(1, \cdot) = 0$. From this we find that ψ is a closed form belonging to $\Gamma_\Phi(\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V}))$ and we wish to see that $\omega - \psi$ is exact. To do this, it is sufficient to prove that a closed form belonging to $\text{Dom}(D_{\min})$ with support in $\pi^{-1}(U) \simeq [0, 1] \times \partial\tilde{M}$ is exact. Taking ω such a form, the closeness of the form imposes

$$\partial_t \omega_T = \pm d_x \omega_N$$

where d_x is the connection induced on $\partial\tilde{M}$. In particular, we find that

$$\omega_T(t, x) = \pm d_x \int_t^1 \omega_N(\tau, x) d\tau \quad \omega = D \pm \left(\int_t^1 \omega_N(\tau, x) d\tau \right)$$

hence ω is exact, and we have proved the surjectivity.

Now, we have to prove the injectivity. Take $\omega \in \Gamma_\Phi(\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V}))$ and assume that there exists $\psi \in \text{Dom}(D_{\min})$ such that $D\psi = \omega$, we want to show that there exists $\phi \in \Gamma_\Phi(\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V}))$ such that $D\phi = \omega$. We can set U as before and, shrinking U if necessary, we can assume $\omega|_{\pi^{-1}(U)} = 0$. In this case, we can take $\phi = \psi$ on $\tilde{M} \setminus \pi^{-1}(U)$ and $\phi = (\chi \times \text{Id})^*\psi$ on $\pi^{-1}(U)$. \square

It remains to prove

Proposition 2.4.39. The complex $L^2DR_{\min}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm.

Proof. We wish to use a proof similar to the one made for the maximal complex $L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})$. However, the restriction behaves poorly with the minimal ideal boundary condition, indeed if $\omega \in \text{Dom}(D_{\min})$ one does not have $\omega|_U \in \text{Dom}(D_{\min})$. Denote by \mathfrak{d} the differential operator obtained as the formal adjoint of D . Recall that $D_{\min}^* = \mathfrak{d}_{\max}$, it is immediate that the $\mathcal{N}(\Gamma)$ -Hilbert complex $L^2DR_{\min}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm if and only if the complex $(L^2DR^{n-\bullet}, \mathfrak{d}_{\max})$ is $\mathcal{N}(\Gamma)$ -Fredholm. Now that we have a maximal boundary condition, we can reason as we did for the maximal de Rham complex. Take $(U_i)_{i \in I}$ to be a finite open covering of X such that all the U_i and their intersection are quasi-isometric to a convex subset of \mathbb{R}^n , as before if $\alpha_1, \dots, \alpha_l \in I$ the open $U_{\alpha_1, \dots, \alpha_l}$ denotes

the intersection $\bigcap_{i=1}^k U_{\alpha_i}$. We endow I with a total order and define the double $\mathcal{N}(\Gamma)$ -Hilbert complex $(K^{\bullet,\bullet}, D_1, D_2)$ where for $p, q \in \mathbb{Z}^2$

$$K^{p,q} = K^{p,q} = \begin{cases} \bigoplus_{\alpha_0 < \dots < \alpha_q \in I} L^2 DR^{n-p}((\pi^{-1}(U_{\alpha_0, \dots, \alpha_q}), \pi^* \mathbb{V}^*)) & \text{if } (p, q) \in \mathbb{N}^2 \\ 0 & \text{if } p < 0 \text{ or } q < 0 \end{cases}$$

and the differential are given by

$$D_1 = \mathfrak{d}_{\max} \quad \text{and} \quad (D_2 \omega)_{\alpha_0, \dots, \alpha_{q+1}} = \sum_{k=0}^{q+1} (-1)^k \omega_{\alpha_0, \dots, \alpha_{k-1}, \alpha_{k+1}, \dots, \alpha_{q+1}}$$

As before for each p the complex $(K^{p,\bullet}, D_2)$ does not have cohomology in positive degree and

$$(\text{Ker}(D_2|_{K^{\bullet,0}}), D_1) = H^0(K^{\bullet,0}).$$

Similarly, the complexes $(K^{\bullet,q}, D_1)$ have no cohomology in positive degree and the complex $(\text{Ker}(D_1|_{K^{0,\bullet}}), D_2)$ is given by a Čech complex of $\pi^* \mathbb{V}^*$ -valued harmonic n -forms associated to our covering. Since the space of harmonic forms on a bounded convex subset of \mathbb{R}^n has a finite dimension, it follows that the object appearing in our Čech complex has finite $\mathcal{N}(\Gamma)$ -dimension. As before, this brings the $\mathcal{N}(\Gamma)$ -Fredholmness of the simple complex associated with the double complex $K^{\bullet,\bullet}$ and hence the $\mathcal{N}(\Gamma)$ -Fredholmness of the complex $L^2 DR_{\min}(\tilde{M}, \pi^* \mathbb{V})$. \square

In the proof, we have seen that the cohomology of the complex $L^2 DR_{bc}^{\bullet}(\tilde{M}, \pi^* \mathbb{V})$ is computed by some Čech complex C^{\bullet} associated with some finite covering of X for $bc \in \{\min, \max\}$. All the C^k are finite-dimensional and are sums of spaces of the form $\ell^2 \pi^* \mathbb{V}(U)$ with U quasi-isometric to some bounded convex space of \mathbb{R}^n . In particular, one has a natural isomorphism of $\mathcal{N}(\Gamma)$ -Hilbert modules

$$\ell^2 \pi^* \mathbb{V}(U) \simeq \ell^2(\Gamma) \otimes \mathbb{V}(U)$$

and one has $\dim_{\Gamma}(\ell^2 \pi^* \mathbb{V}(U)) = \dim(\mathbb{V}(U)) = \text{rk}(\mathbb{V})$. Similarly, one can compute the L^2 cohomology of M (with the min or max ideal boundary condition) using a Čech complex K^{\bullet} associated with the *same* open covering of X . One then obviously has

$$\dim_{\Gamma} C^k = \dim K^k. \quad (2.1)$$

Now one can define for $bc \in \{\min, \max\}$ the L^2 Euler characteristic

$$\begin{aligned} \chi_{\Gamma, L^2, bc}(\tilde{M}, \pi^* \mathbb{V}) &= \sum_{k \in \mathbb{N}} (-1)^k \dim_{\Gamma} H_{L^2, bc}^k(\tilde{M}, \pi^* \mathbb{V}) \\ \chi_{L^2, bc}(M, \mathbb{V}) &= \sum_{k \in \mathbb{N}} (-1)^k \dim H_{L^2, bc}^k(M, \mathbb{V}) \end{aligned}$$

Since all the C^k and K^k have finite $\mathcal{N}(\Gamma)$ -dimension, one also has

$$\begin{aligned} \chi_{\Gamma, L^2, bc}(\tilde{M}, \pi^* \mathbb{V}) &= \sum_{k \in \mathbb{N}} (-1)^k \dim_{\Gamma} C^k \\ \chi_{L^2, bc}(M, \mathbb{V}) &= \sum_{k \in \mathbb{N}} (-1)^k \dim K^k \end{aligned}$$

And by (2.1) one obtains the

Proposition 2.4.40. For $bc \in \{\min, \max\}$ one has

$$\chi_{\Gamma, L^2, bc}(\tilde{M}, \pi^* \mathbb{V}) = \chi_{L^2, bc}(M, \mathbb{V}).$$

This result is a special case of the Schick result [Sch01], but we have retrieved it by combinatorial means and did not have to use the Boutet de Monvel calculus.

5 The L^2 complexes associated to a polarized variation of Hodge structure

Now we present some other examples of $\mathcal{N}(\Gamma)$ -Hilbert complexes associated to some object appearing in geometry. Consider (M, ω_M) a complex Hermitian manifold of dimension n and $(\mathbb{V}, F^\bullet, Q)$ a polarised variation of Hodge structures on M . Again, consider $\pi: \tilde{M} \rightarrow M$ a Galois covering of M . Since \mathbb{V} is a local system on M endowed with a Hermitian metric (in this case, the Hodge metric h), the above discussion applies, and we can consider the L^2 de Rham complexes

$$L^2DR_{\min}^\bullet(\tilde{M}, h_M, \pi^*\mathbb{V}, h) \quad \text{and} \quad L^2DR_{\max}^\bullet(\tilde{M}, h_M, \pi^*\mathbb{V}, h).$$

We will not write the metric ω_M and h when they are clear from the context. Since one has additional structures, it is possible to construct other natural L^2 -Hilbert complexes. The main ones are the Dolbeault complexes. Before introducing it, we recall that one has maps

- $\partial: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p,q} \otimes \mathcal{E}^{r+1,s}$
- $\bar{\partial}: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p,q} \otimes \mathcal{E}^{r,s+1}$
- $\theta: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p-1,q+1} \otimes \mathcal{E}^{r+1,s}$
- $\bar{\theta}: H^{p,q} \otimes \mathcal{E}^{r,s} \rightarrow H^{p+1,q-1} \otimes \mathcal{E}^{r,s+1}$

with $D = \partial + \theta + \bar{\partial} + \bar{\theta}$. Also recall that we set

$$\begin{aligned} D' &= \partial + \bar{\theta}, & D'' &= \bar{\partial} + \theta \\ \mathcal{E}(\mathbb{V})^{P,Q} &= \bigoplus_{\substack{p+r=P \\ q+s=Q}} H^{p,q} \otimes \mathcal{E}^{r,s}. \end{aligned}$$

One has $D = D' + D''$, $D'\mathcal{E}(\mathbb{V})^{P,Q} \subset \mathcal{E}(\mathbb{V})^{P+1,Q}$ and $D''\mathcal{E}(\mathbb{V})^{P,Q} \subset \mathcal{E}(\mathbb{V})^{P,Q+1}$. One also has a decomposition of $\mathcal{E}^\bullet(\mathbb{V})$, which is the smooth de Rham complex of \mathcal{V} , the decomposition is given by

$$\mathcal{E}^\bullet(\mathbb{V}) = \bigoplus_{P,Q} \mathcal{E}(\mathbb{V})^{P,Q}.$$

This decomposition allows us to define the Hodge filtration on the complex $\mathcal{E}^\bullet(\mathbb{V})$ by setting

$$F^p \mathcal{E}^\bullet(\mathbb{V}) = \bigoplus_{P \geq p} \mathcal{E}(\mathbb{V})^{P,Q}.$$

Definition 2.5.41. The Dolbeault complex $L^2 \text{Dolb}^{P,\bullet}(\tilde{M}, \pi^{-1}\mathbb{V})$ is the elliptic complex

$$0 \longrightarrow L^2DR^{0,0}(Gr_F^P) \xrightarrow{D''} L^2DR^{1,0}(Gr_F^{P-1}) \oplus L^2DR^{0,1}(Gr_F^P) \xrightarrow{D''} \dots$$

Similarly to the case of the de Rham complex, one has two closed extensions to consider D''_{\min} and D''_{\max} . We will denote by \mathfrak{d}'' the formal adjoints of D'' , and consider the Laplace operator $\square_{D''} := (D'' + \mathfrak{d}'')$, which is formally self-adjoint. If the metric ω_M is Kähler, the following equality (see [Zuc79, Theorem 2.7]) over the space of compactly supported smooth forms holds

$$\square_D = 2\square_{D''}.$$

One wishes to have this equality as an equality between closed operators, however, trouble arises from the ideal boundary conditions. It becomes important to know if one has a unique ideal boundary condition, as it might solve those issues. One has $D'' = \bar{\partial} + \theta$ and since the $\bar{\partial}$ operator defines the holomorphic structure of the vector bundle $\mathcal{H}^{p,q}$, by [AV65, p.92] we have that if the metric ω_M is complete Kähler, one has $\bar{\partial}_{\min} = \bar{\partial}_{\max}$. Some other trouble might arise from the different closures of the operator θ . Fortunately, θ will often be a bounded operator, in which case one has the following.

Proposition 2.5.42. [Zuc79, Theorem 7.1] Assume $\theta: L^2DR^k(\tilde{M}, \pi^*\mathbb{V}) \rightarrow L^2DR^{k+1}(\tilde{M}, \pi^*\mathbb{V})$ is a bounded operator and assume furthermore that the Kähler metric ω_M is complete, then one has a unique ideal boundary condition for the operator D'' , i.e. one has

$$D''_{\min} = D''_{\max}.$$

One could do the same thing with the operator $D' = \partial + \bar{\theta}$ and since the boundedness of θ implies the boundedness of $\bar{\theta}$, one obtains a similar result for the operator D' . This also implies the uniqueness of the ideal boundary condition for the connection D , that is,

$$D_{\min} = D_{\max}.$$

Proposition 2.5.43. Under the same hypothesis of the above proposition, we have the equality of closed operators of $L^2DR^\bullet(M, \mathbb{V})$

$$\square_D = \square_{D'} + \square_{D''} = 2\square_{D''}.$$

Moreover, we also have the following orthogonal decomposition of \square_D

$$\square_D|_{L^2_\bullet(M, \mathbb{V})} = \bigoplus_{P+Q=w+k} 2\square_{D''}|_{L^2_\bullet(M, \mathbb{V})^{P,Q}}.$$

Remark 2.5.44. In the above proposition, we have used the notation \square_D instead of $\square_{D_{\min}}$ or $\square_{D_{\max}}$, since the subscript min, max is unnecessary when one has a unique ideal boundary condition.

If we have $P + Q = k + w$ we set $\text{Harm}^{P,Q}(\tilde{M}, \pi^*\mathbb{V}) := \text{Harm}^k(\tilde{M}, \pi^*\mathbb{V}) \cap \mathcal{E}^{P,Q}(\pi^*\mathbb{V})$ the space of harmonic forms of type (P, Q) , and one has

$$\text{Harm}^k(\tilde{M}, \pi^*\mathbb{V}) = \bigoplus_{P+Q=k+w} \text{Harm}^{P,Q}(\tilde{M}, \pi^*\mathbb{V}).$$

The space $\text{Harm}^{P,Q}(\tilde{M}, \pi^*\mathbb{V})$ can be identified with the reduced L^2 Dolbeault cohomology. This gives the following proposition.

Proposition 2.5.45. If \mathbb{V} is a local system underlying a polarized variation of Hodge structure of weight w on a complete Kähler manifold M , the reduced L^2 cohomology groups $H_{2,\text{red}}^k(M, \mathbb{V})$ admit a pure Hodge structure of weight $k + w$. The component of type (P, Q) is canonically isomorphic to $\text{Harm}^{P,Q}(M, \mathbb{V})$.

The case of unreduced cohomology is more difficult to deal with. In [Eys22] it was observed that the smooth subcomplex is more convenient $L^2DR^\bullet_\infty(M, \mathbb{V})$ to deal with our filtrations. The reason is that we have a filtration of the smooth complex $F^p\mathcal{E}^\bullet(\pi^*\mathbb{V})$, we can define the filtration on the L^2 -de Rham complex by taking $F^pL^2DR^k(\tilde{M}, \pi^*\mathbb{V})$ as the space of square-integrable measurable sections with value in F^p . One has a decomposition of Hilbert spaces

$$L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V}) = \bigoplus L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V})^{P,Q}.$$

However, it is not true that

$$\text{Dom}(D) = \bigoplus \text{Dom}(D) \cap L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V})^{P,Q}.$$

This makes it difficult to talk about the graded complex $\text{Gr}_pL^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V})$. However, this filtration induces a filtration on the smooth subcomplex $L^2DR^\bullet_\infty(M, \mathbb{V})$, and in this case we have compatibility with the Hodge decomposition in the following sense

$$L^2DR^\bullet_\infty(\tilde{M}, \pi^*\mathbb{V}) = \bigoplus L^2DR^\bullet_\infty(\tilde{M}, \pi^*\mathbb{V}) \cap L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V})^{P,Q}$$

thanks to the equality between the Laplace operators $\square_D = 2\square_{D''}$. Since all the (P, Q) -components are in the domain of all the power of the Laplace operators $\square_{D''}$ one obtains the

Proposition 2.5.46. We have a canonical isomorphism

$$\text{Gr}_p^F L^2DR^\bullet_\infty(\tilde{M}, \pi^*\mathbb{V}) \rightarrow L^2\text{Dolb}^{p,\bullet}_\infty(\tilde{M}, \pi^*\mathbb{V}).$$

Chapter 3

The L^2 De Rham of a pVHS on covering of a punctured disk.

1 Main results of the chapter

In this chapter, we consider a polarised variation of Hodge structure $(\mathbb{V}, F^\bullet, Q)$ on a punctured disk $M = \Delta_R^* := \{t \in \mathbb{C}^* \mid |z| < R\}$ for $R < 1$. We denote by h the Hermitian metric induced by the polarisation. We endow our punctured disk Δ_R^* with the restriction of the Poincaré metric on Δ , i.e.

$$\omega_{Pc} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2 (\ln(|z|^2))^2}.$$

As seen in the previous chapter, if we give a covering $\pi: \tilde{M} \rightarrow M$ with Deck group $\Gamma := \text{Deck}(\tilde{M}/M)$ one can associate the $\mathcal{N}(\Gamma)$ -Hilbert complex $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V}) := L^2 DR_{\max}^\bullet(\tilde{M}, \omega_{Pc}, \pi^*\mathbb{V}, h)$. This chapter is dedicated to prove the following result.

Theorem 3.1.1. For $\pi: \tilde{M} \rightarrow M$ given above the complex $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm. Moreover, for $k \geq 1$, one has

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_{L^2, \max}^k(\tilde{M}, \pi^*\mathbb{V}) = 0.$$

By Proposition 2.4.34 it is sufficient to consider only a connected covering. Let $\gamma \in \Gamma$ be the element generated by the meridian circle around the puncture 0. There are two cases : γ has finite order, in which case the covering is finite, and we are reduced to study the L^2 -de Rham complex on a punctured disk or γ has infinite order and \tilde{M} is a horodisk $\mathbb{H}_A := \{z = x + iy \in \mathbb{C} \mid y > A\}$ with $A = \frac{-\ln(R)}{2\pi}$ for the covering is given by $\pi(z) = \exp(2i\pi z)$. Before diving into the proof let us recall some notation used in the first chapter, we denote by T the monodromy operator of \mathbb{V} , and $T = T_s T_u$ its Jordan decomposition with T_u unipotent and T_s semi-simple. We take N the nilpotent element that satisfies $T_u = \exp(N)$ and by choosing a branch of the logarithm we can choose S semi-simple such that $T_s = \exp(S)$. Then the local system \mathbb{V} admits an increasing filtration by local systems \mathbb{W}_\bullet such that

- $N\mathbb{W}_k \subset \mathbb{W}_{k-2}$
- For all $k \in \mathbb{N}_{>0}$, $N^k: \text{Gr}_W^k \rightarrow \text{Gr}_W^{-k}$ is an isomorphism.

For a multivalued horizontal section e , we will denote by \tilde{e} the single-valued section of \mathbb{V} defined on the universal cover by

$$\tilde{e} = \exp((S + T)z)e.$$

We have the following result.

Lemma 3.1.2.

- [Sch73, Theorem 6.6], [SS22a, Theorem §15] A multivalued horizontal section e belongs to $\mathbb{W}_k \setminus \mathbb{W}_{k-1}$ if and only if on any vertical strips one has

$$h(e, e) \sim y^k.$$

- [Zuc79, p. 433][SS22b, Proposition 6.11.2] If (e_1, \dots, e_n) is a frame of multivalued horizontal sections flagged according to the filtration W_\bullet then the frame $(\tilde{e}_1, \dots, \tilde{e}_n)$ is L^2 adapted, i.e. there exists a positive constant C such that for any measurable functions $(f_j)_{1 \leq j \leq n}$ one has

$$C^{-1} \sum_{j=1}^n h(f_j \tilde{e}_j, f_j \tilde{e}_j) \leq h \left(\sum_{j=1}^n f_j \tilde{e}_j, \sum_{j=1}^n f_j \tilde{e}_j \right) \leq C \sum_{j=1}^n h(f_j \tilde{e}_j, f_j \tilde{e}_j)$$

This lemma is fundamental for us to be able to make the computations. In particular, it implies the

Corollary 3.1.3. For all $k \in \mathbb{Z}$ the following short sequence of $\mathcal{N}(\Gamma)$ -Hilbert module is exact

$$0 \longrightarrow L^2 DR_{\max}^\bullet(\tilde{M}, \mathbb{W}_{k-1}) \longrightarrow L^2 DR_{\max}^\bullet(\tilde{M}, \mathbb{W}_k) \longrightarrow L^2 DR_{\max}^\bullet(\tilde{M}, Gr_W^k) \longrightarrow 0.$$

Now that the general result has been stated, we are able to begin the proof. The proof will be separated into two parts depending on whether the covering is finite or not. The case where the covering is finite could follow from the work of Zucker [Zuc79], we will give an alternative proof using the study of the essential spectra of the Laplace operator, a method which will be easier to generalise in the case of an infinite covering.

2 The L^2 -de Rham complex for a finite covering

In this section, we consider a finite covering of $M = \Delta_R^*$. In this case one has $\tilde{M} = \Delta_{R'}^*$ with $R' = R^{1/n}$ where n is the degree of the covering. In this case one has $\Gamma = \mathbb{Z}/n\mathbb{Z}$ and the notion of being $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Fredholm is the same as being Fredholm as the Γ -dimension and the dimension only differs from a multiplicative constant.

2.1 Fredholmness of the complexes $L^2 DR_{\max}^\bullet(\tilde{M}, Gr_W^k)$.

Corollary 3.1.3 tells us that we should study the complexes $L^2 DR_{\max}^\bullet(\tilde{M}, Gr_W^k)$ first, and by Lemma 3.1.2 we can reduce to the case where the bundles Gr_W^k are trivial flat bundles $\mathcal{L}_{\alpha,k}, \alpha \in \mathbb{R}$ with generating section e and the differential D is given by

$$D(\omega \otimes e) = (d\omega + \alpha \frac{dt}{t} \wedge \omega) \otimes e$$

where $t = re^{i\theta}$ is our complex variable and the Hermitian norm of e is given by

$$h(e, e) = r^{2\alpha} |\ln(r)|^2.$$

By Lemma 2.2.21, we will begin by studying the Laplace operators on 0 and 2-forms. We will prove the

Proposition 3.2.4. On the complex $L^2 DR_{\max}^\bullet(\tilde{M}, \mathcal{L}_{\alpha,k})$ the Laplace operators $\square_{D_{\max},0}$ and $\square_{D_{\max},2}$ are $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Fredholm unless we have both $k \in \{-1, 1\}$ and $\alpha \in \mathbb{Z}$.

Proof. We fix a generating section e satisfying the above properties. We consider the natural action of the circle S^1 on the punctured disk and make it act trivially on our line bundle $\mathcal{L}_{\alpha,k}$ for the trivialisation given by e . This allows us to decompose our forms using Fourier series. For f, g measurable functions one has

$$\begin{aligned} f(r, \theta) \otimes e &= \left(\sum_{v \in \mathbb{Z}} f_v(r) e^{iv\theta} \right) \otimes e \\ f(r, \theta) dr \otimes e + g(r, \theta) d\theta \otimes e &= \left(\sum_{v \in \mathbb{Z}} f_v(r) e^{iv\theta} dr + g_v(r) e^{in\theta} d\theta \right) \otimes e \\ f(r, \theta) dr \wedge d\theta \otimes e &= \left(\sum_{v \in \mathbb{Z}} f_v(r) e^{iv\theta} \right) dr \wedge d\theta \otimes e \end{aligned} \tag{3.1}$$

This orthogonal decomposition is invariant by D_{\max} hence it is also left invariant by the Laplace operator $\square_{D_{\max}}$. Note that to obtain this decomposition, we had to choose a frame of the bundles $\mathcal{L}_{\alpha,k}, \mathcal{A}^1 \otimes \mathcal{L}_{\alpha,k}$ and $\mathcal{A}^2 \otimes \mathcal{L}_{\alpha,k}$ given by the sections

$$e, \quad dr \otimes e, \quad d\theta \otimes e, \quad dr \wedge d\theta \otimes e.$$

Since the decomposition is orthogonal it is enough to show the Laplace operators $\square_{D_{\max,j}}$ are Fredholm while restricted to sections of the form

$$f_\nu e^{i\nu\theta} \otimes e, \quad f_\nu e^{i\nu\theta} dr \otimes e, \quad f_\nu e^{i\nu\theta} d\theta \otimes e, \quad \text{and} \quad f_\nu e^{i\nu\theta} dr \wedge d\theta \otimes e.$$

with $\nu \in \mathbb{Z}$. We will denote by $D_{\max,j,\nu}$ ($j = 0, 1, 2$) (resp. $\square_{\nu,j}$) the restriction of the operator $D_{\max,j}$ (resp. $\square_{D_{\max,j}}$) on the space of sections of the above form. To prove the Fredholmness of our Laplace operators we have to prove that all $\square_{\nu,j}$ are Fredholm and find a positive constant $c > 0$ such that the essential spectrum of all the $\square_{\nu,j}$ is bounded below by c and only admits a finite number of eigenvalues below c .

In those frames, the differential of 0 and 1 forms are given by the matrix of differential operators.

$$D_{\max,0,\nu} = \begin{pmatrix} \partial_{r,\max} + \frac{\alpha}{r} \\ i(\alpha + \nu) \end{pmatrix} \quad D_{\max,1,\nu} = \begin{pmatrix} -i(\alpha + \nu) & \partial_{r,\max} + \frac{\alpha}{r} \end{pmatrix}.$$

Its adjoint is given by \mathfrak{d}_{\min} where \mathfrak{d} is the formal adjoint of $D_{\max,0,\nu}$ and the subscript min denotes the minimal closure. Hence, one has

$$\begin{aligned} D_{\max,0,\nu}^* &= (-r^{1-2\alpha} \ln(r)^{2-k} \partial_{r,\min} r^{2\alpha+1} \ln(r)^k + \alpha r \ln(r)^2 \quad -i(\alpha + \nu) \ln(r)^2) \\ D_{\max,1,\nu}^* &= \begin{pmatrix} i(\alpha + \nu) \ln(r)^2 \\ -r^{1-2\alpha} \ln(r)^{-k} \partial_{r,\min} r^{1+2\alpha} \ln(r)^{k+2} + \alpha r \ln(r)^2 \end{pmatrix}. \end{aligned} \quad (3.2)$$

And the Laplace operator on 0-forms is given by

$$\square_{\nu,0} = (-r^{1-2\alpha} |\ln(r)|^{2-k} \partial_{r,\min} r^{2\alpha+1} |\ln(r)|^k + \alpha r |\ln(r)|^2) (\partial_{r,\max} + \frac{\alpha}{r}) + (\alpha + \nu)^2 |\ln(r)|^2$$

We will drop the subscript min, max for the differential operator ∂_r for the reader's convenience and will explicit the boundary conditions. As a differential operator, it is given by

$$\begin{aligned} \square_{\nu,0} &= -r^{1-2\alpha} |\ln(r)|^{2-k} \partial_r r^{2\alpha+1} |\ln(r)|^k \partial_r - r^{1-2\alpha} |\ln(r)|^{2-k} \partial_r r^{2\alpha+1} |\ln(r)|^k \frac{\alpha}{r} + \alpha r |\ln(r)|^2 \partial_r \\ &\quad + \alpha \ln(r)^2 + (\alpha + \nu)^2 |\ln(r)|^2 \\ &= -r^{1-2\alpha} \ln(r)^{2-k} \partial_r r^{2\alpha+1} \ln(r)^k \partial_r - \alpha^2 \ln(r)^2 - \alpha k \ln(r) + (\alpha + \nu)^2 \ln(r)^2 \end{aligned}$$

A measurable function $f_\nu(r) e^{i\nu\theta}$ is in the domain of $\square_{\nu,0}$ if and only if it satisfies the Neumann boundary condition given by

$$(f'(R') + \frac{\alpha}{R'} f(R')) = 0 \quad (3.3)$$

Indeed, a form $gdr + h d\theta$ is in $\text{Dom}(d_{\max}^*) = \text{Dom}(\mathfrak{d}_{\min})$ if and only if $g \in \text{Dom}(\partial_{r,\min})$ i.e. $g(R') = 0$, and we only have to apply this result to Df .

We have to study the spectra of a non-negative self-adjoint ordinary differential operator of order 2. The general method is to do a Liouville transform (Theorem A.4.81) to reduce to the well-known case of Schrödinger's operators. For the Liouville transform, we set

$$u := u(r) = - \int_r^{R'} \sqrt{\frac{\rho^{2\alpha-1} |\ln(\rho)|^{k-2}}{\rho^{2\alpha+1} |\ln(\rho)|^k}} d\rho = -\ln(|\ln(r)|) + \ln(|\ln(R')|)$$

Then we conjugate our Laplace operator by the unitary transform

$$\begin{aligned} \mathbb{C}: L^2([0, R'], \frac{dr}{r |\ln(r)|^{2-k}}) &\longrightarrow L^2(\mathbb{R}, du) \\ f &\longmapsto ((r^\alpha |\ln(r)|^{(k-1)/2}) f) \circ u^{-1} \end{aligned}$$

which conjugate our Laplace operator to (a self-adjoint realization of) the differential operator

$$L_{\alpha,k,\nu} := -\partial_u^2 + \frac{(k-1)^2}{4} + \ln(R')^2(\alpha + \nu)^2 e^{2u}. \quad (3.4)$$

and the boundary condition becomes $f'(0) = \frac{k-1}{2}f(0)$. This operator acts on the positive half line and its spectra is bounded below by $\frac{(k-1)^2}{4} + \ln(R')^2(\alpha + \nu)^2$. And since the operator $-\partial_u^2 + \frac{(k-1)^2}{4}$ admits at most one eigenvalue for our boundary condition given by $\exp(\frac{k-1}{2}u)$, it follows that the essential spectrum of the Laplace operators \square_0 is clearly bounded below by

$$\frac{(k-1)^2}{4} + \ln(R')^2 \inf_{\nu \in \mathbb{Z}} (\alpha + \nu)^2.$$

Hence, if either $k \neq 1$ or $\alpha \notin \mathbb{Z}$ the spectrum is bounded below by a constant that does not depend on ν , hence the Laplace operator is $\square_{D_{\max,0}}$ is Fredholm. Otherwise, if $k = 1$ and $\alpha \in \mathbb{Z}$ then by taking $\nu = -\alpha$ one has

$$L_{\alpha,1,-\alpha} = -\partial_u^2$$

whose essential spectrum is \mathbb{R}^+ , in particular it is not Fredholm and the Laplace operator $\square_{D_{\max,0}}$ cannot be Fredholm. In particular, if $\alpha \in \mathbb{Z}$ one should have a cohomology group $H_{L^2}^1(\tilde{M}, \mathcal{L}_{\alpha,1})$ that is not fully reduced, thus infinite-dimensional (compare [Zuc79, Proposition 6.6] for the case $\alpha = 0$).

We make a similar computation for the Laplace operator on 2-forms given by the differential operator

$$\square_{D_{\max,2}} = (\partial_r + \frac{\alpha}{r})(-r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} + \alpha r \ln(r)^2) + (\alpha + \nu)^2 \ln(r)^2$$

with the Dirichlet boundary condition $f_\nu(R') = 0$. By expanding our expression we find

$$\begin{aligned} \square_{D_{\max,2}} &= (\partial_r + \frac{\alpha}{r})(-r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} + \alpha r \ln(r)^2) + (\alpha + \nu)^2 \ln(r)^2 \\ &= -\partial_r r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} - \alpha r^{-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} + \alpha \partial_r r \ln(r)^2 + (\alpha + \nu)^2 \ln(r)^2 \\ &= -\partial_r r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} - \alpha(1+2\alpha) \ln(r)^2 - \alpha(k+2) \ln(r) + \alpha \ln(r)^2 + 2\alpha \ln(r) + (\alpha + \nu)^2 \ln(r)^2 \\ \square_{D_{\max,2}} &= -\partial_r r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} - 2\alpha^2 \ln(r)^2 - \alpha k \ln(r) + (\alpha + \nu)^2 \ln(r)^2 \end{aligned} \quad (3.5)$$

To apply a Liouville Transform we have to put the term $-\partial_r r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2}$ into Sturm–Liouville form $-\frac{1}{w} \partial_r p \partial_r + Q$, by looking at our metric one ought to have $w = r^{1+2\alpha} \ln(r)^{k+2}$, and $p/w = \|dr\|^2 = r^2 \ln(r)^2$. It remains to compute the function Q , which will appear naturally by using the relations between the operator ∂_r and the multiplication by some functions. One has

$$\begin{aligned} -\partial_r r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} &= -\left[\partial_r r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \right] r^2 \ln(r)^2 \left[\partial_r r^{1+2\alpha} \ln(r)^{k+2} \right] \\ &= -\left[r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r - r^{-(2+2\alpha)} \left((1+2\alpha) \ln(r)^{-(k+2)} - (k+2) \ln(r)^{-(k+3)} \right) \right] \\ &\quad \times r^2 \ln(r)^2 \left[r^{1+2\alpha} \ln(r)^{k+2} \partial_r + (1+2\alpha) r^{2\alpha} \ln(r)^{k+2} + (k+2) r^{2\alpha} \ln(r)^{k+1} \right] \\ &= -\left[r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r - r^{-(2+2\alpha)} \left((1+2\alpha) \ln(r)^{-(k+2)} - (k+2) \ln(r)^{-(k+3)} \right) \right] \\ &\quad \times \left[r^{3+2\alpha} \ln(r)^{k+4} \partial_r + (1+2\alpha) r^{2+2\alpha} \ln(r)^{k+4} + (k+2) r^{2+2\alpha} \ln(r)^{k+3} \right] \\ &= -r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r r^{3+2\alpha} \ln(r)^{k+4} \partial_r \\ &\quad - r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r \left[(1+2\alpha) r^{2+2\alpha} \ln(r)^{k+4} + (k+2) r^{2+2\alpha} \ln(r)^{k+3} \right] \\ &\quad + \left[(1+2\alpha) r^{-(2+2\alpha)} \ln(r)^{-(k+2)} + (k+2) r^{-(2+2\alpha)} \ln(r)^{-(k+3)} \right] r^{3+2\alpha} \ln(r)^{k+4} \partial_r \\ &\quad + (1+2\alpha)^2 \ln(r)^2 + 2(1+2\alpha)(k+2) \ln(r) + (k+2)^2. \end{aligned}$$

We have a simplification

$$\begin{aligned} -r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r \left[(1+2\alpha) r^{2+2\alpha} \ln(r)^{k+4} + (k+2) r^{2+2\alpha} \ln(r)^{k+3} \right] &+ \left[(1+2\alpha) r \ln(r)^2 + (k+2) r \ln(r) \right] \partial_r \\ &= -(2+2\alpha)(1+2\alpha) \ln(r)^2 - (4\alpha(k+3) + 3k+8) \ln(r) - (k+2)(k+3). \end{aligned}$$

From this one obtains

$$-\partial_r r^{1-2\alpha} \ln(r)^{-k} \partial_r r^{1+2\alpha} \ln(r)^{k+2} = -r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r r^{3+2\alpha} \ln(r)^{k+4} \partial_r - (1+2\alpha) \ln(r)^2 - (4\alpha + k + 4) \ln(r) - (k+2) \quad (3.6)$$

And finally, by using the expression of $\square_{D_{\max,2}}$ (3.5) one obtains

$$\begin{aligned} \square_{D_{\max,2}} &= -r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r r^{3+2\alpha} \ln(r)^{k+4} \partial_r - (\alpha^2 + 2\alpha + 1) \ln(r)^2 - (\alpha k + 4\alpha + k + 4) \ln(r) \\ &\quad - (k+2) + (\alpha + \nu)^2 \ln(r)^2 \\ \square_{D_{\max,2}} &= -r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r r^{3+2\alpha} \ln(r)^{k+4} \partial_r - (\alpha + 1)^2 \ln(r)^2 - (\alpha + 1)(k + 4) \ln(r) - (k+2) + (\alpha + \nu)^2 \ln(r)^2 \end{aligned} \quad (3.7)$$

It remains to apply the Liouville transform by setting again $u(r) = \ln(|\ln(r)|) - \ln(|\ln(R')|)$ and using the unitary operator

$$\begin{aligned} C: L^2([0, R'][, r^{2\alpha+1} |\ln(r)|^{k+2} dr d\theta]) &\longrightarrow L^2(\mathbb{R}_{>0}, du) \\ f &\longmapsto \left((r^{\alpha+1} \ln(r)^{\frac{k+3}{2}}) f \right) \circ u^{-1} \end{aligned}$$

This operator conjugates $-r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r r^{3+2\alpha} \ln(r)^{k+4} \partial_r$ to the differential operator

$$-\partial_u^2 + \frac{(k+3)^2}{4} + (\alpha+1)^2 \ln(r)^2 + (\alpha+1)(k+4) \ln(r)$$

with $r(u) = \exp(\ln(R')e^u)$. In the end, we find that the operator $\square_{D_{\max,2}}$ is conjugated to

$$L_{2,\alpha,k,\nu} = -\partial_u^2 + \frac{(k+1)^2}{4} + (\alpha+\nu)^2 \ln(r)^2 = -\partial_u^2 + \frac{(k+1)^2}{4} + \ln(R')^2 (\alpha+\nu)^2 e^{2u} \quad (3.8)$$

and this operator is Fredholm unless $k = -1$ and $\alpha = -\nu$ by the same arguments as before. \square

Remark 3.2.5. In the above proof we have seen that the Laplace operator $\square_{\nu,j}$ is conjugated to some self-adjoint realization of the differential operator on the half line defined by

$$L_{0,\alpha,k,\nu} := -\partial_u^2 + \frac{(k-1)^2}{4} + \ln(R')^2 (\alpha+\nu)^2 e^{2u} \quad \text{and} \quad L_{2,\alpha,k,\nu} = -\partial_u^2 + \frac{(k+3)^2}{4} + (\alpha+1)^2 \ln(r)^2 + (\alpha+1)(k+4) \ln(r).$$

Fixing such an adjoint self-realisation amounts to fixing some boundary conditions at $u = 0$. Those boundaries conditions have not been given in an explicit way in the proof, as the Fredholmness does not depend on them (see Theorem A.4.83). It will be useful to know them. Let us start with the Laplace operator on 0-form. The boundary condition is forced by the fact that we ask $D_{\max,0} f \otimes e \in \text{Dom}(\mathfrak{d}_{\min})$. Take a 1-form $\omega := f dr \otimes e + g d\theta \otimes e$ in the domain of \mathfrak{d}_{\min} . By definition there must exist a sequence $(f_n, g_n)_{n \in \mathbb{N}}$ of smooth function with compact support in $\Delta_{R'}$ such that if $\omega_n := f_n dr \otimes e + g_n d\theta \otimes e$ then

$$\omega_n \xrightarrow{L^2} \omega \quad \text{and} \quad \mathfrak{d}\omega_n \xrightarrow{L^2} \mathfrak{d}\omega$$

As before, since \mathfrak{d} preserves our Fourier decomposition we can reduce the study to the case where $f = f_\nu(r)e^{i\nu\theta}$ and $g = g_\nu(r)e^{i\nu\theta}$. Let us fix such a form in the domain $\text{Dom}(\mathfrak{d}_{\min})$ and $h := h_\nu(r)e^{i\nu\theta}$ an L^2 function. Then one has

$$\langle \mathfrak{d}_{\min} \omega_n, h \rangle \longrightarrow \langle \mathfrak{d}_{\min} \omega, h \rangle.$$

The left-hand side is equal to

$$\begin{aligned} \langle \mathfrak{d}\omega_n, h \rangle &= \int_0^{R'} \left(-r^{1-2\alpha} |\ln(r)|^{2-k} \partial_r \left(r^{2\alpha+1} |\ln(r)|^k f_n \right) + \alpha r |\ln(r)|^2 f_n - i(\nu + \alpha) \ln(r)^2 g_n \right) \bar{h} \frac{r^{2\alpha} |\ln(r)|^k dr}{r \ln(r)^2} \\ &= \int_0^{R'} r^{2\alpha+1} |\ln(r)|^k f_n \partial_r \bar{h} + \left(\alpha r |\ln(r)|^2 f_n - i(\nu + \alpha) \ln(r)^2 g_n \right) \bar{h} \frac{r^{2\alpha} |\ln(r)|^k dr}{r \ln(r)^2} \end{aligned}$$

And the right-hand side is equal to

$$\begin{aligned} \langle \partial\omega, h \rangle &= \int_0^{R'} r^{2\alpha+1} |\ln(r)|^k f \partial_r h_n + \left(\alpha r |\ln(r)|^2 f - i(\nu + \alpha) \ln(r)^2 g \right) h \frac{r^{2\alpha} \ln(r)^k dr}{r \ln(r)^2} \\ &\quad + R'^{2\alpha+1} \ln(R')^k f(R') \bar{h}(R') \end{aligned}$$

This imposes $f(R') = 0$. Therefore, a 0-form $f \otimes e$ belongs to the domain of $\square_{D_{\max}, 0}$ if one has $\frac{d}{dr} f(R') + \frac{\alpha}{R'} f(R') = 0$. Once we conjugated with the Liouville transform C the Laplace operator was conjugated to

$$L_{0,\alpha,k,\nu} := -\partial_u^2 + \frac{(k-1)^2}{4} + \ln(R')^2 (\alpha + \nu)^2 e^{2u}$$

and if $h = Cf$ it satisfies the boundary conditions

$$h'(0) = \frac{k-1}{2} h(0). \quad (3.9)$$

Indeed, one has

$$\begin{aligned} \frac{d}{du} h &= r |\ln(r)| \frac{d}{dr} (r^\alpha |\ln(r)|^{(k-1)/2} f) \\ &= r^{\alpha+1} |\ln(r)|^{(k+1)/2} \frac{d}{dr} f + (\alpha r^\alpha |\ln(r)|^{(k+1)/2}) f + \frac{(k-1)}{2} (r^\alpha |\ln(r)|^{(k-1)/2}) f \\ &= r^{\alpha+1} |\ln(r)|^{(k+1)/2} \frac{d}{dr} f + (\alpha r^\alpha |\ln(r)|^{(k+1)/2}) f + \frac{(k-1)}{2} h \end{aligned}$$

and it remains to evaluate in $u = 0$ (i.e. $r = R'$) to find the sought boundary condition.

For the boundary condition on 2-form, a form $f_\nu(r) e^{i\nu\theta} dr \wedge d\theta \otimes e$ belongs to $\text{Dom}(\partial_{\min})$ if $h_\nu(R') = 0$. It follows that one has the Dirichlet boundary condition on the Laplace operator $\square_{D_{\max}, 2}$. Hence, when we conjugate it with the Liouville transform we find the differential operator

$$L_{2,\alpha,k,\nu} - \partial_u^2 + \frac{(k+3)^2}{4} + (\alpha+1)^2 \ln(r)^2 + (\alpha+1)(k+4) \ln(r)$$

with Dirichlet boundary condition.

Proposition 3.2.6. Let $\text{Harm}^\bullet(\tilde{M}, \mathcal{L}_{\alpha,k})$ be the space of L^2 -harmonic forms with values in the line bundle $\mathcal{L}_{\alpha,k}$ that satisfy the boundary conditions associated with D_{\max} . Then one has

$$\begin{aligned} \text{Harm}^0(\tilde{M}, \mathcal{L}_{\alpha,k}) &= \begin{cases} \mathbb{C} z^{-\alpha} \otimes e & \text{if } \alpha \in \mathbb{Z} \text{ and } k \leq 0 \\ 0 & \text{otherwise} \end{cases} \\ \text{Harm}^1(\tilde{M}, \mathcal{L}_{\alpha,k}) &= \begin{cases} \mathbb{C} z^{-\alpha} d\theta \otimes e & \text{if } \alpha \in \mathbb{Z} \text{ and } k \leq -2 \\ 0 & \text{otherwise} \end{cases} \\ \text{Harm}^2(\tilde{M}, \mathcal{L}_{\alpha,k}) &= 0 \end{aligned}$$

Proof. The space $\text{Harm}^0(\tilde{M}, \mathcal{L}_{\alpha,k})$ is given by the space of L^2 closed 0-form. Since the differential is given by $D = d + \alpha \frac{dz}{z}$ such forms exist if and only if $\alpha \in \mathbb{Z}$, and they belong to $L^2(\Delta_{R'}, \frac{r^{2\alpha} |\ln(r)|^k dr \wedge d\theta}{r \ln(r)^2})$ if and only if $k \leq 0$.

For the space of harmonic 2-forms, we have seen that the Laplace operator is conjugated to the direct sum of the operators $-\partial_u^2 + \frac{(k+1)^2}{4} + \ln(R')^2 (\alpha + \nu)^2 e^{2u}$ for $\nu \in \mathbb{Z}$ with the Dirichlet boundary condition. We have seen that these operators can have a kernel if and only if $\alpha = -\nu$, and it is straightforward to check that if $f \in L^2(\mathbb{R}^+, du)$ satisfies $-f''(u) + \frac{(k-1)^2}{4} f = 0$ and $f(0) = 0$, then one has $f = 0$. This gives the result for harmonic two forms.

It remains to do the computation for the harmonic 1-forms. A harmonic 1-form $\omega = f(r) e^{i\nu\theta} dr + g(r) e^{i\nu\theta} d\theta$ must satisfy $D\omega = \partial\omega = 0$ which gives

$$\begin{aligned} i(\alpha + \nu) f &= \partial_r g + \frac{\alpha}{r} g \\ i(\alpha + \nu) \ln(r)^2 g &= -r^{1-2\alpha} \ln(r)^{2-k} \partial_r r^{1+2\alpha} \ln(r)^k f + \alpha r \ln(r)^2 f \end{aligned}$$

We set $w(r) = r^{2\alpha-1} \ln(r)^k$ and $p(r) = r^{1+2\alpha} \ln(r)^k$ to simplify the second line into

$$i(\alpha + \nu) \ln(r)^2 g = -\frac{\ln(r)^2}{w} \partial_r p f + \alpha r \ln(r)^2 f$$

we multiply the second line by $i(\alpha + \nu)$ and replace the terms $i(\alpha + \nu)f$ by their expression in terms of g to find

$$\begin{aligned} -(\alpha + \nu)^2 r \ln(r)^2 g &= -\frac{\ln(r)^2}{w} \partial_r p [\partial_r g + \frac{\alpha}{r} g] + \alpha r \ln(r)^2 [\partial_r g + \frac{\alpha}{r} g] \\ -(\alpha + \nu)^2 g &= -\frac{1}{w} \partial_r p [\partial_r g + \frac{\alpha}{r} g] + \alpha r \partial_r g + \alpha^2 g \end{aligned}$$

One can then use the relation

$$\partial_r \frac{\alpha p}{r} = \alpha \partial_r r^{2\alpha} \ln(r)^k = \alpha r w \partial_r + 2\alpha^2 w + \alpha \frac{k w}{\ln r}$$

to find

$$-(\alpha + \nu)^2 g = -\frac{1}{w} \partial_r p \partial_r g - \frac{\alpha k}{\ln(r)} g - \alpha^2 g.$$

It amounts to compute the kernel of the Sturm-Liouville operator

$$C = -\frac{1}{w} \partial_r p \partial_r - \frac{\alpha k}{\ln(r)} - \alpha^2 + (\alpha + \nu)^2.$$

With boundary condition $g'(R') = -\frac{\alpha}{R'} g$ We have to use the Liouville transform one more time to conjugate C with the operator L on the half line $\mathbb{R}_{\geq 0}$ defined by

$$L = -\partial_u^2 + (\alpha + \nu)^2 + \frac{k(k-2)}{4(u - \ln(R'))^2}$$

with boundary condition $h'(0) = \frac{k}{2 \ln(R')} h(0)$. It has no kernel unless $\alpha = -\nu$ as $-\partial_u^2 + \frac{k(k-2)}{4(u - \ln(R'))^2}$ is non-negative. Moreover, it has a one-dimensional kernel given by $(u - \ln(R'))^{k/2}$ that is square-integrable if and only if $k < -1$. By conjugating, one finds $g(r) = \frac{1}{r^\alpha}$ and $f = 0$. Finally, the space of harmonic forms is generated by $\omega = r^{-\alpha} e^{-i\alpha\nu} d\theta = z^{-\alpha} d\theta$. \square

Remark 3.2.7. The reader can compare our results with those of Zucker [Zuc79, Proposition 6.6]. The case where Zucker found cohomology groups that are finite-dimensional coincide with the case where we have harmonic one forms. The only infinite-dimensional cohomology groups appearing in [Zuc79] are fully unreduced, and this happens for the weight k for which the complex is not Fredholm.

2.2 Fredholmness of the complex $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})$

Now one wishes to prove the

Theorem 3.2.8. The $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Hilbert complex $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})$ is $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Fredholm.

By Corollary 3.1.3, if one had Fredholmness of all $L^2 DR_{\max}^\bullet(\tilde{M}, Gr_W^k)$ one would obtain Fredholmness of the de Rham complexes $L^2 DR_{\max}^\bullet(\tilde{M}, W_k)$ by induction on k using Corollary 2.2.23. The problem is that the complexes $L^2 DR_{\max}^\bullet(\tilde{M}, Gr_W^k)$ are not $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Fredholm for $k = \pm 1$. The induction argument only allows us to conclude for the complex $L^2 DR_{\max}^\bullet(\tilde{M}, W_{-2})$ and one will need to prove the Fredholmness of $L^2 DR_{\max}^\bullet(\tilde{M}, W_1)$ to obtain Theorem 3.2.8. Fortunately, the existence of an L^2 -adapted basis given by Lemma 3.1.2 gives us the following proposition.

Proposition 3.2.9. One has a short exact sequence of $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Hilbert complexes

$$0 \longrightarrow L^2 DR_{\max}^\bullet(\tilde{M}, W_{-2}) \longrightarrow L^2 DR_{\max}^\bullet(\tilde{M}, W_1) \longrightarrow L^2 DR_{\max}^\bullet(\tilde{M}, W_1/W_{-2}) \longrightarrow 0$$

This proposition tells us that to obtain the Fredholmness of $L^2 DR_{\max}^\bullet(\tilde{M}, W_1)$ (and hence of $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})$) it is then sufficient to prove the following lemma.

Lemma 3.2.10. The complex $L^2DR_{\max}^{\bullet}(\tilde{M}, W_1/W_{-2})$ is $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Fredholm.

Proof. Choose a family (e_1, \dots, e_l) of multivalued horizontal sections such that it induces a horizontal base of Gr_W^1 and (f_1, \dots, f_k) a family of multivalued horizontal sections that induces a basis of Gr_W^0 . Then the family $(\tilde{e}_1, \dots, \tilde{e}_l, N\tilde{e}_1, \dots, N\tilde{e}_l, \tilde{f}_1, \dots, \tilde{f}_k)$ defines an L^2 -adapted frame. Recall that for a flat multivalued section e we have denoted by \tilde{e} the single-valued section whose expression on the universal covering is given by

$$\tilde{e} = \exp(z(S + N))e.$$

Using L^2 -adaptedness, up to a quasi-isometry, we can assume that our frame is orthogonal. This allows us to reduce to the case where $l = k = 1$. We can assume that e_1 and f_1 are in an eigenspace of S . We will denote by α (resp. β) the eigenvalue associated with e_1 (resp. f_1). In the frame $(\tilde{e}_1, N\tilde{e}_1, \tilde{f}_1)$ the connection D is given by the matrix of differential operator

$$D = \begin{pmatrix} d + \alpha\left(\frac{dr}{r} + id\theta\right)\wedge & 0 & 0 \\ \left(\frac{dr}{r} + id\theta\right)\wedge & d + \alpha\left(\frac{dr}{r} + id\theta\right)\wedge & 0 \\ 0 & 0 & d + \beta\left(\frac{dr}{r} + id\theta\right)\wedge \end{pmatrix} \quad (3.10)$$

Recall that this choice of frame also induces an isomorphism

$$\begin{aligned} L^2DR^0(\Delta_{R'}^*, W_1/W_{-2}) &\rightarrow L^2\left(\Delta_{R'}^*, \frac{r^{2\alpha}|\ln(r)|drd\theta}{r\ln(r)^2}\right) \oplus L^2\left(\Delta_{R'}^*, \frac{r^{2\alpha}|\ln(r)|^{-1}drd\theta}{r\ln(r)^2}\right) \oplus L^2\left(\Delta_{R'}^*, \frac{r^{2\beta}drd\theta}{r\ln(r)^2}\right) \\ u \otimes \tilde{e}_1 + v \otimes \frac{N}{2i\pi}\tilde{e}_1 + w \otimes \tilde{f}_1 &\mapsto \begin{matrix} u & v & w \end{matrix} \end{aligned} \quad (3.11)$$

In the same fashion, we have an isomorphism φ that sends $L^2DR^1(\Delta_{R'}^*, W_1/W_{-1})$ to a direct sum of L^2 -spaces by sending $fdr \otimes e_1 \mapsto f$, $fd\theta \otimes \tilde{e}_1 \mapsto f\dots$ The different weights depend on the asymptotic behaviour of the norms of \tilde{e}_1 , $N\tilde{e}_1$ and \tilde{f}_1 and on the asymptotic behaviour of the norms of dr and $d\theta$. We give them explicitly below.

$$\begin{aligned} \varphi(fdr \otimes \tilde{e}_1) &\in L^2(\Delta_{R'}^*, r^{2\alpha+1}|\ln(r)|drd\theta) & \varphi(fd\theta \otimes \tilde{e}_1) &\in L^2(\Delta_{R'}^*, \frac{r^{2\alpha}|\ln(r)|}{r}drd\theta) \\ \varphi(fdr \otimes \frac{N}{2i\pi}\tilde{e}_1) &\in L^2(\Delta_{R'}^*, r^{2\alpha+1}|\ln(r)|^{-1}drd\theta) & \varphi(fdr \otimes \frac{N}{2i\pi}\tilde{e}_1) &\in L^2(\Delta_{R'}^*, \frac{r^{2\alpha}|\ln(r)|^{-1}}{r}drd\theta) \\ \varphi(fdr \otimes f_1) &\in L^2(\Delta_{R'}^*, r^{2\alpha+1}drd\theta) & \varphi(fd\theta \otimes f_1) &\in L^2(\Delta_{R'}^*, r^{R\alpha}drd\theta) \end{aligned}$$

Recall that we have a unitary action of the group \mathbb{Z} on $\Delta^* \simeq S^1 \otimes]0, 1[$ and of \mathbb{Z} on the bundle W_1/W_{-2} that is trivial in our frame. As before we have a Fourier decomposition which allows us to reduce to the cases where the functions are of the form $f(r)e^{iv\theta}$ and where the functions depend only on one variable r .

Computation of the Laplace operator on 0-forms. We consider the restriction of the connection D to 0-form of this form we obtain the matrix of differential operators given by

$$D_{v,0} = \begin{pmatrix} d_{v,0,1} & 0 & 0 \\ \begin{pmatrix} \frac{1}{r} \\ i \end{pmatrix} & d_{v,0,-1} & 0 \\ 0 & 0 & d_{v,0,0} \end{pmatrix}$$

with

$$d_{v,0,1} = \begin{pmatrix} \partial_r + \frac{\alpha}{r} \\ i(v + \alpha) \end{pmatrix} \quad d_{v,0,-1} = \begin{pmatrix} \partial_r + \frac{\alpha}{r} \\ i(v + \alpha) \end{pmatrix} \quad d_{v,0,0} = \begin{pmatrix} \partial_r + \frac{\beta}{r} \\ i(v + \beta) \end{pmatrix}.$$

Here the matrix of differential operators acts on a direct sum of spaces of the form $L^2((0, R'), w(r)dr)$ where the weights w are given by the isomorphisms (3.11). By taking the weights on our L^2 -spaces into account, the adjoint operator is given by the matrix

$$D_{v,0}^* = \begin{pmatrix} d_{v,0,1}^* & (r \ -i) & 0 \\ 0 & d_{v,0,-1}^* & 0 \\ 0 & 0 & d_{v,0,0}^* \end{pmatrix}$$

where the terms $d_{v,0,k}^*$ were already computed in the previous section in the equation (3.2)

$$\begin{aligned} d_{v,0,1}^* &= (-r^{1-2\alpha} \ln(r) \partial_r r^{1+2\alpha} \ln(r) + \alpha r \ln(r)^2 - i(v + \alpha) \ln(r)^2). \\ d_{v,0,-1}^* &= (-r^{1-2\alpha} \ln(r)^3 \partial_r r^{1+2\alpha} \ln(r)^{-1} + \alpha r \ln(r)^2 - i(v + \alpha) \ln(r)^2). \\ d_{v,0,0}^* &= (-r^{1+2\beta} \ln(r)^2 \partial_r r^{1+2\beta} + \beta r \ln(r)^2 - i(v + \beta) \ln(r)^2). \end{aligned}$$

And the Laplace operator on 0-form is thus given by the matrix of differential operators

$$D_{v,0}^* D_{v,0} = \begin{pmatrix} d_{v,0,1}^* d_{v,0,1} + 2 & r \partial_r + v + 2\alpha & 0 \\ -r^{1+2\alpha} \ln(r)^3 \partial_r r^{2\alpha} \ln(r)^{-1} + (v + 2\alpha) \ln(r)^2 & d_{v,0,-1}^* d_{v,0,-1} & 0 \\ 0 & 0 & d_{v,0,0}^* d_{v,0,0} \end{pmatrix} \quad (3.12)$$

Recall that for the differential operator $\tau = -\frac{1}{w} \partial_x p \partial x$ with $p, w > 0$ acting on $L^2([a, b[, w dx)$, by setting $u(x) = -\int_x^b \sqrt{w/p}$, the Liouville transform is given by the unitary isomorphism

$$\begin{aligned} C: L^2([a, b[, w dx) &\longrightarrow L^2([0, -u(a)[, du) \\ f &\longmapsto (pw)^{1/4} f \circ u^{-1}. \end{aligned}$$

It is a unitary operator and one has $C\tau C^* = -\partial_u^2 + Q(u)$. Here, the function Q can be made explicit in terms of the derivative of w and p . For $\eta \in \mathbb{R}$ and $k \in \mathbb{Z}$ let $C_{\eta,k}$ be the unitary isomorphism associated with the differential operator $\tau_{\eta,k} = -r^{1-2\eta} \ln(r)^{2-k} \partial_r r^{1+2\eta} \ln(r)^k \partial_r$. We want to conjugate the differential operator given by the expression (3.12) by the unitary transform given by the matrix

$$\begin{pmatrix} C_{\alpha,1} & 0 & 0 \\ 0 & C_{\alpha,-1} & 0 \\ 0 & 0 & C_{\beta,0} \end{pmatrix}$$

We recall the reader that we have already computed the terms $C_{\alpha,k} d_{v,0,k}^* d_{v,0,k} C_{\alpha,k}^*$ in the previous section when we computed the Laplace operators on the line bundles $\mathcal{L}_{\alpha,k}$ and that their expression is given by (3.4). Hence, we only need to compute the terms

$$C_{\alpha,1} r \partial_r + v + 2\alpha C_{\alpha,-1}^* \quad \text{and} \quad -C_{\alpha,-1} r^{1+2\alpha} \ln(r)^3 \partial_r r^{2\alpha} \ln(r)^{-1} C_{\alpha,1}^*$$

to compute the product

$$\begin{aligned} &\begin{pmatrix} C_{\alpha,1} & 0 & 0 \\ 0 & C_{\alpha,-1} & 0 \\ 0 & 0 & C_{\beta,0} \end{pmatrix} D_{v,0}^* D_{v,0} \begin{pmatrix} C_{\alpha,1}^* & 0 & 0 \\ 0 & C_{\alpha,-1}^* & 0 \\ 0 & 0 & C_{\beta,0}^* \end{pmatrix} \\ &= \begin{pmatrix} -\partial_u^2 + 2 + (v + \alpha)^2 \ln(R')^2 e^{2u} & \partial_u + 1 + (v + \alpha) \ln(R') e^u & 0 \\ -\partial_u + 1 + (v + \alpha) \ln(R') e^u & -\partial_u^2 + 1 + (v + \alpha)^2 \ln(R')^2 e^{2u} & 0 \\ 0 & 0 & -\partial_u^2 + \frac{1}{4} + (v + \beta)^2 \ln(R')^2 e^{2u} \end{pmatrix} \quad (3.13) \end{aligned}$$

where this matrix of differential operators acts on a direct sum of $L^2(\mathbb{R}^+, du)$. The term $-\partial_u^2 + \frac{1}{4} + (v + \beta)^2 \ln(R')^2 e^{2u}$ gives us no trouble as it was already proven to be Fredholm, and we only have to focus on the term

$$\begin{aligned} &\begin{pmatrix} -\partial_u^2 + 2 + (v + \alpha)^2 \ln(R')^2 e^{2u} & \partial_u + 1 + (v + \alpha) \ln(R') e^u \\ -\partial_u + 1 + (v + \alpha) \ln(R') e^u & -\partial_u^2 + 1 + (v + \alpha)^2 \ln(R')^2 e^{2u} \end{pmatrix} = \begin{pmatrix} -\partial_u^2 + 2 & \partial_u + 1 \\ -\partial_u + 1 & -\partial_u^2 + 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} (v + \alpha)^2 \ln(R')^2 e^{2u} & (v + \alpha) \ln(R') e^u \\ (v + \alpha) \ln(R') e^u & (v + \alpha)^2 \ln(R')^2 e^{2u} \end{pmatrix} \end{aligned}$$

We recall that by construction the considered self-adjoint resolution of the operator

$$\begin{pmatrix} -\partial_u^2 + 2 & \partial_u + 1 \\ -\partial_u + 1 & -\partial_u^2 + 1 \end{pmatrix}$$

must be positive. Since $u \in \mathbb{R}^+$, one can check that the eigenvalues of the matrix

$$\begin{pmatrix} (\nu + \alpha)^2 \ln(R')^2 e^{2u} & (\nu + \alpha) \ln(R') e^u \\ (\nu + \alpha) \ln(R') e^u & (\nu + \alpha)^2 \ln(R')^2 e^{2u} \end{pmatrix}$$

are equal to $\lambda^2 \pm \lambda$ with $\lambda = (\nu + \alpha)^2 \ln(R') e^u$, which bounded below by a positive constant, except for at most one value ν_0 since $\nu \in \mathbb{Z}$, and it is always bounded below by $-1/4$. It remains to bound below the essential spectrum of the differential operators matrix

$$\begin{pmatrix} -\partial_u^2 + 2 & \partial_u + 1 \\ -\partial_u + 1 & -\partial_u^2 + 1 \end{pmatrix}$$

by a constant c strictly greater than $1/4$. According to the usual results on the essential spectrum of a differential operator with constant coefficient, its essential spectra is the same as the essential spectra of the differential operator acting on $L^2(\mathbb{R}, du)$ and using the Fourier transform it is the set

$$\left\{ \lambda(\xi) \mid \lambda(\xi) \text{ eigenvalue of } A(\xi) = \begin{pmatrix} \xi^2 + 2 & i\xi + 1 \\ -i\xi + 1 & \xi^2 + 1 \end{pmatrix} \quad \xi \in \mathbb{R} \right\}$$

The characteristic polynomial of the matrix $A(\xi)$ is given by

$$P_\xi(X) = X^2 - (2\xi^2 + 3)X + \xi^4 + 2\xi^2 + 1$$

and its smallest eigenvalue is given by

$$r^-(\xi^2) = \frac{1}{2} \left((2\xi^2 + 3) - \sqrt{(2\xi^2 + 3)^2 - 4(\xi^4 + 2\xi^2 + 1)} \right) = \frac{1}{2} \left((2\xi^2 + 3) - \sqrt{4\xi^2 + 5} \right)$$

and a straightforward study of the function r^- shows that $r^-(\xi^2)$ is bounded below by $r^-(0) = \frac{3-\sqrt{5}}{2} > 1/4$. This concludes the proof of the Fredholmness of the Laplace operator on 0-forms on W_1/W_{-2} .

Study of the Laplace operator on 2-forms. The study of the Laplace operator on 2-forms is quite similar. For $\nu \in \mathbb{Z}$ we denote by $D_{\nu,1}$ the restriction of our connection to 1-form of the form $f(r)e^{i\nu\theta} dr \otimes \tilde{e} + g(r)e^{i\nu\theta} d\theta \otimes \tilde{e}$. On the frame

$$(dr \otimes \tilde{e}_1, d\theta \otimes \tilde{e}_1, dr \otimes N\tilde{e}_1, d\theta \otimes N\tilde{e}_1, dr \otimes \tilde{f}_1, d\theta \otimes \tilde{f}_1)$$

it is given by the matrix of differential operators

$$D_{\nu,1} = \begin{pmatrix} d_{\nu,1,1} & 0 & 0 \\ (-i \quad \frac{1}{r}) & d_{\nu,1,-1} & 0 \\ 0 & 0 & d_{\nu,1,0} \end{pmatrix}$$

with

$$\begin{aligned} d_{\nu,1,1} &= (-i(\nu + \alpha) \quad \partial_r + \frac{\alpha}{r}) . \\ d_{\nu,1,-1} &= (-i(\nu + \alpha) \quad \partial_r + \frac{\alpha}{r}) . \\ d_{\nu,1,0} &= (-i(\nu + \beta) \quad \partial_r + \frac{\beta}{r}) . \end{aligned}$$

Its adjoint is then given by

$$D_{\nu,1}^* = \begin{pmatrix} d_{\nu,1,1}^* & \begin{pmatrix} i \\ r \end{pmatrix} & 0 \\ 0 & d_{\nu,1,-1}^* & 0 \\ 0 & 0 & d_{\nu,1,0}^* \end{pmatrix}$$

Where the terms $d_{\nu,1,k}^*$ were already computed in the previous section in the equation (3.2)

$$\begin{aligned} d_{\nu,1,1}^* &= \begin{pmatrix} -i(\alpha + \nu) \ln(r)^2 \\ -r^{1-2\alpha} \ln(r)^{-1} \partial_r r^{1+2\alpha} \ln(r)^3 + \alpha r \ln(r)^2 \end{pmatrix} . \\ d_{\nu,1,-1}^* &= \begin{pmatrix} -i(\alpha + \nu) \ln(r)^2 \\ -r^{1-2\alpha} \ln(r) \partial_r r^{1+2\alpha} \ln(r) + \alpha r \ln(r)^2 \end{pmatrix} . \\ d_{\nu,1,0}^* &= \begin{pmatrix} -i(\beta + \nu) \ln(r)^2 \\ -r^{1-2\beta} \partial_r r^{1+2\beta} \ln(r)^2 + \alpha r \ln(r)^2 \end{pmatrix} . \end{aligned}$$

The Laplace operator is then given by

$$D_{v,1}D_{v,1}^* = \begin{pmatrix} d_{v,1,1}d_{v,1,1}^* + 2 & r\partial_r + v + 2\alpha & 0 \\ -r^{1+2\alpha} \ln(r)^3 \partial_r r^{2\alpha} \ln(r)^{-1} + (v + 2\alpha) \ln(r)^2 & d_{v,0,-1}d_{v,0,-1}^* & 0 \\ 0 & 0 & d_{v,0,0}d_{v,0,0}^* \end{pmatrix} \quad (3.14)$$

where a computation of the $d_{v,1,1}d_{v,1,1}^*$ in Sturm-Liouville form has already been achieved in the last section. We recall that we have by (3.7)

$$\begin{aligned} d_{v,1,1}^*d_{v,1,1} &= -r^{-(1+2\alpha)} \ln(r)^{-3} \partial_r r^{3+2\alpha} \ln(r)^5 \partial_r - (\alpha + 1)^2 \ln(r)^2 - 5(\alpha + 1) \ln(r) - 3 + (\alpha + v)^2 \ln(r)^2 \\ d_{v,1,-1}^*d_{v,1,-1} &= -r^{-(1+2\alpha)} \ln(r)^{-1} \partial_r r^{3+2\alpha} \ln(r)^3 \partial_r - (\alpha + 1)^2 \ln(r)^2 - 3(\alpha + 1) \ln(r) - 1 + (\alpha + v)^2 \ln(r)^2 \\ d_{v,1,0}^*d_{v,1,0} &= -r^{-(1+2\alpha)} \ln(r)^{-2} \partial_r r^{3+2\alpha} \ln(r)^4 \partial_r - (\alpha + 1)^2 \ln(r)^2 - 4(\alpha + 1) \ln(r) - 2 + (\alpha + v)^2 \ln(r)^2 \end{aligned}$$

This time we denote by $C_{\alpha,k}$ the isometry given by the Liouville transform A.4.81 of the differential operator $\tau_{\alpha,k} = -r^{-(1+2\alpha)} \ln(r)^{-(k+2)} \partial_r r^{3+2\alpha} \ln(r)^{k+4} \partial_r$. And we want to conjugate the expression (3.14) by the unitary transform

$$\begin{pmatrix} C_{\alpha,1} & 0 & 0 \\ 0 & C_{\alpha,-1} & 0 \\ 0 & 0 & C_{\beta,k} \end{pmatrix}$$

As before we recall the reader that we have already computed the terms we have already computed the terms $C_{\alpha,k}d_{v,0,k}^*d_{v,0,k}C_{\alpha,k}^*$ in the previous section when we computed the Laplace operators on the line bundles $\mathcal{L}_{\alpha,k}$ and that their expressions are given by (3.8). Finally, we find

$$\begin{aligned} &\begin{pmatrix} C_{\alpha,1} & 0 & 0 \\ 0 & C_{\alpha,-1} & 0 \\ 0 & 0 & C_{\beta,0} \end{pmatrix} D_{v,1}D_{v,1}^* \begin{pmatrix} C_{\alpha,1}^* & 0 & 0 \\ 0 & C_{\alpha,-1}^* & 0 \\ 0 & 0 & C_{\beta,0}^* \end{pmatrix} \\ &= \begin{pmatrix} -\partial_u^2 + 2 + (v + \alpha)^2 \ln(R')^2 e^{2u} & \partial_u + 1 + (v + \alpha) \ln(R') e^u & 0 \\ -\partial_u + 1 + (v + \alpha) \ln(R') e^u & -\partial_u^2 + 1 + (v + \alpha)^2 \ln(R')^2 e^{2u} & 0 \\ 0 & 0 & -\partial_u^2 + \frac{1}{4} + (v + \beta)^2 \ln(R')^2 e^{2u} \end{pmatrix} \quad (3.15) \end{aligned}$$

and we conclude that \square_2 is $\mathcal{N}(\mathbb{Z}/n\mathbb{Z})$ -Fredholm with the same argument as the one used in the case of the Laplace operator on functions as it is the same operator as the one given in (3.13) up to boundary conditions. \square

2.3 Computation of the cohomology groups.

In this subsection, we prove the second part of Theorem 3.1.1. Recall that we denote by M the punctured disk Δ_R with $R < 1$, endowed with a Poincaré metric ω_{Pc} . We consider the finite covering $\pi: \tilde{M} \rightarrow M$, with $\tilde{M} = \Delta_{R'} = \Delta_{R^{1/n}}$ and $\pi(z) = z^n$. We consider a polarised variation of Hodge structure $(\mathbb{V}, F^\bullet, Q)$ on M and assume that it extends to a slightly bigger punctured disk $\Delta_{R+\varepsilon}$.

Theorem 3.2.11. Let $\pi: \tilde{M} \rightarrow M$ be the finite covering defined above; then we have

$$H_{L^2, \max}^j(\tilde{M}, \pi^*\mathbb{V}) = 0 \quad \text{if } j > 0$$

Proof. We use the argument of Zucker [Zuc79, p.441]. We consider the spectral sequence with respect to the filtration W_\bullet on the complex $L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})$. We have

$$E_1^{p,q} = H^{p+q}(\text{Gr}_W^{-p} L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})) \implies H^{p+q}(L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})).$$

The differential

$$d_1: H^{p+q}(\text{Gr}_W^{-p} L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V})) \rightarrow H^{p+q+1}(\text{Gr}_W^{-(p+1)} L^2DR_{\max}^\bullet(\tilde{M}, \pi^*\mathbb{V}))$$

of the spectral sequence vanishes, since it is given by $\frac{dz}{2i\pi z} \otimes \tilde{N}_0$ and

$$\tilde{N}_0 W_k((\pi^*\mathbb{V})_0) \subset W_{k-2}((\pi^*\mathbb{V})_0) \quad \text{for all } k$$

One has then

$$E_2^{p,q} = H^{p+q}(\mathrm{Gr}_W^{-p} L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})).$$

with differential

$$d_2: H^{p+q}(\mathrm{Gr}_W^{-p} L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})) \rightarrow H^{p+q+1}(\mathrm{Gr}_W^{-(p+2)} L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V}))$$

that is also induced by $\frac{dz}{2i\pi z} \otimes \tilde{N}_0$. Noting that $H^{p+q}(\mathrm{Gr}_W^{-p} L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})) = 0$ unless $0 \leq p+q \leq 2$ and $p \geq -1$ by Propositions 3.2.4 and 3.2.6. We begin by treating the case where $p+q = 0$ and $p \geq 0$, where we need to see that d_2 is surjective, since N_0 induces a surjection $W_l \rightarrow W_{l-2}$ for any $l \leq 0$. We are left to show that

$$d_2: H^1(\mathrm{Gr}_W^1 L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})) \rightarrow H^2(\mathrm{Gr}_W^{-1} L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V}))$$

is an isomorphism derived from the fact that \tilde{N}_0 induces an isomorphism

$$\tilde{N}_0: \mathrm{Gr}_1^W((\pi^* \mathbb{V})_0) \rightarrow \mathrm{Gr}_{-1}^W((\pi^* \mathbb{V})_0).$$

It follows that all $E_3^{p,q}$ vanishes unless $p = -q$ with $p \geq 1$. Therefore, the cohomology groups $H_2^k(\tilde{M}, \pi^* \mathbb{V})$ vanish for $k \geq 1$. \square

3 The L^2 -de Rham complex for an infinite covering

3.1 On the $\mathcal{N}(\mathbb{Z})$ -Fredholmness of $L^2 DR_{\max}^\bullet(\tilde{M}, \mathrm{Gr}_W^k)$

In this section, we set $\tilde{M} = \mathbb{H}_A$ and $\pi(z) = \exp(2i\pi z)$. We recall that the monodromy T can be decomposed into $T = \exp(S + N)$ with N nilpotent and S semisimple with imaginary eigenvalues. We begin with the case $S = 0$ since it is the easiest case. Corollary 3.1.3 suggests that we first study the cohomology of flat line bundles \mathcal{L}_k on M which is a trivial flat line bundle endowed with a Hermitian metric h such that if e is a horizontal generating section of $\pi^* \mathcal{L}_k$ one has

$$h(e, e) \sim y^k.$$

In the case $S \neq 0$, the term $h(e, e)$ will be slightly more complicated, and the section e won't be flat. This case will be treated at the end of the subsection.

The main ingredient of the proof is to realise the L^2 -de Rham complex as a complex of integrals of a measurable field of Hilbert spaces and to apply the Plancherel theorem. The necessary results on measurable fields of Hilbert spaces are collected in the second section of the appendices.

$L^2 DR_{\max}^\bullet(\mathbb{H}_A, \mathcal{L}_k)$ as a complex of measurable fields of Hilbert spaces. We denote by $\mathcal{A}_2^j(\mathbb{H}_A, \mathcal{L}_k)$ the space of square-integrable \mathcal{L}_k -valued j -forms, we set e a horizontal section generating \mathcal{L}_k , we recall that we assume $\|e\|^2 = y^k$. We denote by \mathcal{F}_x the Fourier transform in the variable x , it gives us isomorphisms :

$$\begin{aligned} \mathcal{A}_2^0(\mathbb{H}_A, \mathcal{L}_k) &\rightarrow L^2(\mathbb{R} \times \mathbb{R}_{>A}, y^{k-2} d\tilde{\xi} dy) \\ f \otimes e &\mapsto \mathcal{F}_x f \\ \mathcal{A}_2^1(\mathbb{H}_A, \mathcal{L}_k) &\rightarrow L^2(\mathbb{R} \times \mathbb{R}_{>A}, y^k d\tilde{\xi} dy) \oplus L^2(\mathbb{R} \times \mathbb{R}_{>A}, y^k d\tilde{\xi} dy) \\ f dx \otimes e + g dy \otimes e &\mapsto \begin{pmatrix} \mathcal{F}_x f \\ \mathcal{F}_x g \end{pmatrix} \\ \mathcal{A}_2^2(\mathbb{H}_A, \mathcal{L}_k) &\rightarrow L^2(\mathbb{R} \times \mathbb{R}_{>A}, y^{k+2} d\tilde{\xi} dy) \\ f dx \wedge dy \otimes e &\mapsto \mathcal{F}_x f. \end{aligned}$$

One has the natural isomorphism (see Appendix 2 for integrals of measurable fields of Hilbert spaces)

$$\begin{aligned} L^2(\mathbb{R} \times \mathbb{R}_{>A}, y^k d\tilde{\xi} dy) &\rightarrow L^2(\mathbb{R}, d\tilde{\xi}, L^2(\mathbb{R}_{>A}, y^k dy)) = \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_{>A}, y^k dy) d\tilde{\xi} \\ f &\mapsto (\tilde{\xi} \mapsto f(\tilde{\xi}, \cdot)). \end{aligned}$$

Composing the above isomorphisms, one obtains the following results.

$$\begin{aligned}\mathcal{A}_2^0(\mathbb{H}_A, \mathcal{L}_k) &\simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_{>A}, y^{k-2} dy) d\zeta \\ \mathcal{A}_2^1(\mathbb{H}_A, \mathcal{L}_k) &\simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_{>A}, y^k dy) \oplus L^2(\mathbb{R}, y^k dy) d\zeta \\ \mathcal{A}_2^2(\mathbb{H}_A, \mathcal{L}_k) &\simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_{>A}, y^{k+2} dy) d\zeta\end{aligned}\tag{3.16}$$

where the isomorphism is an isomorphism of Hilbert $\mathcal{N}(\mathbb{Z})$ -modules.

Note that the closed operator $D_{0,\max}$ is conjugated by the Fourier transform into the maximal operator

$$S_{0,\max}: L^2(\mathbb{R}, d\zeta, L^2(\mathbb{R}_{>A}, y^{k-2} dy)) \rightarrow L^2(\mathbb{R}, d\zeta, L^2(\mathbb{R}_{>A}, y^k dy))$$

whose domain consists of square-integrable functions f such that for all ζ , $f(\zeta) \in L^2(\mathbb{R}_{>A}, y^{k-2} dy)$ and $\partial_y f(\zeta)$ are measurable and satisfy

$$\int_{\mathbb{R}} \left(\|\partial_y f(\zeta)\|_{L^2(\mathbb{R}_{>A}, y^k dy)}^2 + \zeta^2 \|f(\zeta)\|_{L^2(\mathbb{R}_{>A}, y^k dy)}^2 \right) d\zeta < +\infty.$$

By setting $S_{0,\max}(\zeta)$ to be the maximal operator associated with the differential operator $\begin{pmatrix} i\zeta \\ \partial_y \end{pmatrix}$, it follows from the definition of integral of a measurable field of closed operator Definition A.2.37 that when viewed as an operator on $\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_{>A}, y^{k-2} dy)$ one has

$$S_{0,\max} = \int_{\mathbb{R}}^{\oplus} S_{0,\max}(\zeta) d\zeta.$$

Similarly, for $D_{1,\max}$, if one defines $S_{1,\max}(\zeta)$ as the maximal operator associated with the differential operator $\begin{pmatrix} -\partial_y & i\zeta \end{pmatrix}$ one obtain that $D_{1,\max}$ is transformed into

$$S_{1,\max} := \int_{\mathbb{R}}^{\oplus} S_{1,\max}(\zeta) d\zeta$$

The adjoint of $S_{0,\max}(\zeta)$ is $T_{0,\min}(\zeta)$ the minimal operator associated with the differential operator

$$\begin{pmatrix} -i\zeta y^2 & -y^{2-k} \partial_y y^k \end{pmatrix}$$

(i.e., the minimal closure of the differential operator when restricted to the space of smooth functions with compact support). And the adjoint of $S_{1,\max}(\zeta)$ is given by $T_{1,\min}(\zeta)$ the minimal operator associated with the differential operator $\begin{pmatrix} y^{-k} \partial_y \min y^{2+k} \\ -i\zeta y^2 \end{pmatrix}$. By the remark following Definition A.2.37 we see that the operator $D_{0,\max}^*$ and $D_{1,\max}^*$ are transformed into

$$T_{0,\min} = \int_{\mathbb{R}}^{\oplus} T_{0,\min}(\zeta) d\zeta$$

$$T_{1,\min} = \int_{\mathbb{R}}^{\oplus} T_{1,\min}(\zeta) d\zeta$$

It follows that the Laplacian on forms is also a closed decomposable operator. The Laplacian on 0-forms is given by

$$\square_0 := \int_{\mathbb{R}}^{\oplus} \square_0(\zeta) d\zeta = \int_{\mathbb{R}}^{\oplus} T_{0,\min}(\zeta) S_{0,\max}(\zeta) d\zeta.$$

where $T_{0,\min}(\zeta) S_{0,\max}(\zeta)$ is a self-adjoint realisation of the differential operator $\zeta^2 y^2 - y^{2-k} \partial_y y^k \partial_y$. It is possible to give the explicit self-adjoint extension, since $T_{0,\min}(\zeta)$ is the minimal closure of a differential operator of degree one, it is defined on the space of function f satisfying $f(A) = 0$, hence the domain of $\square_0(\zeta)$ is given by the Neumann boundary conditions

$$\text{Dom}(\square_0(\zeta)) = \{f \in L^2(\mathbb{R}_{>A}, y^{k-2} dy) \mid (-y^{2-k} \partial_y y^k \partial_y + \zeta^2 y^2) f \in L^2 \text{ and } f'(A) = 0\}.$$

Similarly, the Laplacian on two forms is given by

$$\square_2 := \int_{\mathbb{R}}^{\oplus} \square_2(\zeta) d\zeta = \int_{\mathbb{R}}^{\oplus} S_{1,\max} T_{1,\min} d\zeta.$$

where $S_{1,\max}(\xi)T_{1,\min}(\xi)$ is a self-adjoint realisation of the differential operator $\xi^2 y^2 - \partial_y y^{-k} \partial_y y^{2+k}$. Once again we can explicitly give the boundary conditions since $T_{1,\min}$ is defined on the space of functions vanishing at A one finds that the self-adjoint realisation is obtained by considering the Dirichlet boundary condition, namely

$$\text{Dom}(\square_0(\xi)) = \{f \in L^2(\mathbb{R}_{>A}, y^{k+2} dy) \mid (\xi^2 y^2 - \partial_y y^{-k} \partial_y y^{2+k})f \in L^2 \text{ and } f(A) = 0\}.$$

Note that by the Theorem of Chow (see Theorem A.2.39) $E_\lambda(\square_j)$ is conjugated by the isomorphisms (3.16) to $\int_{\mathbb{R}}^\oplus E_\lambda(\square_j(\xi)) d\xi$, and that under the Fourier transform the action of \mathbb{Z} is given by $n \mapsto \int_{\mathbb{R}}^\oplus e^{in\xi} d\xi$, hence by Proposition A.3.75 one has the following result which will help us to compute the trace of the spectral projectors.

Proposition 3.3.12. For $\lambda \geq 0$ one has

$$\text{tr}_{\mathcal{N}(\mathbb{Z})} E_\lambda(\square_j) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr} E_\lambda(\square_j(\xi)) d\xi.$$

Hence, we are reduced to study the spectral property of each $\square_j(\xi)$ for $\xi \in \mathbb{R}$. Before jumping to this computation, however, let us explicitly give the spaces $\text{Ker}(\square_0)$ and $\text{Ker}(\square_2)$. Recall that one has $\text{Ker}(\square_0) = \text{Ker} D_{0,\max}$ and $\text{Ker}(\square_2) = \text{Ker}(D_{1,\max}^*)$. However, d_0 and d_1 are conjugated to the operators

$$\int_{\mathbb{R}}^\oplus \begin{pmatrix} i\xi \\ \partial_{y,\max} \end{pmatrix} d\xi \quad \int_{\mathbb{R}}^\oplus \begin{pmatrix} y^{-k} \partial_{y,\min} y^{2+k} \\ -i\xi y^2 \end{pmatrix} d\xi$$

which are clearly injective. Hence, $\text{Ker}(\square_0) = \text{Ker}(\square_2) = 0$.

$\mathcal{N}(\mathbb{Z})$ -Fredholmness of the Laplace operators on the graded Gr_W^k . In this paragraph, we show that the Laplace operators \square_j are $\mathcal{N}(\mathbb{Z})$ -Fredholm. Recall that by the previous section, one has

$$\dim_{\mathcal{N}(\mathbb{Z})} E_\lambda(\square_j) = \frac{1}{2\pi} \int_{\mathbb{R}} \dim E_\lambda(\square_j(\xi)) d\xi.$$

Proposition 3.3.13. For $k \neq 1$, the Laplace operator \square_0 is $\mathcal{N}(\mathbb{Z})$ -Fredholm. More precisely, for all $k \in \mathbb{Z}$, and all $0 < \lambda < \frac{(k-1)^2}{4}$ the $\mathcal{N}(\mathbb{Z})$ -dimension of the image of $E_\lambda(\square_0)$ is bounded above by the constant $\frac{\sqrt{\lambda}}{2\pi A}$.

Proof. First we consider the isometry

$$C: \begin{array}{ccc} L^2(\mathbb{R}_{>A}, y^{k-2} dy) & \rightarrow & L^2(\mathbb{R}^+, du) \\ f & \mapsto & A^{\frac{k-1}{2}} e^{(k-1)x/2} f(Ae^x) \end{array}$$

The conjugation by C transforms the differential operator $\square_0(\xi)$ into the Schrödinger operator (cf the Liouville transform explained in Appendix, Proposition A.4.81)

$$L(\xi) := -\partial_x^2 + Q(x) + \xi^2 A^2 e^{2x}.$$

We have

$$Q(x) = \left(A^k e^{kx} A^{k-2} e^{(k-2)x} \right)^{-1/4} \frac{d^2}{dx^2} \left(A^k e^{kx} A^{k-2} e^{(k-2)x} \right)^{1/4} = \frac{(k-1)^2}{4}$$

Hence, one has

$$L(\xi) := -\partial_x^2 + \frac{(k-1)^2}{4} + \xi^2 A^2 e^{2x}.$$

To find the boundary conditions satisfied by elements in the domain of L , set $g = Cf$ one has

$$\begin{aligned} \frac{d}{du} g &= \frac{dy}{du} \cdot \frac{d}{dy} \left(y^{(k-1)/2} f \right) \\ &= \frac{k-1}{2} y^{(k-1)/2} f + y^{(k+1)/2} \frac{d}{dy} f \end{aligned}$$

Since $\frac{d}{dy}f(A) = 0$ if f is in the domain of $\square_0(\xi)$, it follows that if g is in the domain of $L(\xi)$ it must satisfy the boundary condition

$$g'(0) = \frac{k-1}{2}g(0).$$

The operator $\square_0(0)$ is just a self-adjoint realisation of the differential operator $-\partial_u^2 + \frac{(k-1)^2}{4}$ and it is well known that its essential spectra is $[\frac{(k-1)^2}{4}, +\infty[$ in the case of the line \mathbb{R} , and the essential spectra on a half line is the same by Proposition A.4.83. We still need to compute the number of eigenvalues below the essential spectrum. A function g is an eigenfunction associated with the eigenvalue λ if and only if g satisfies the differential equation

$$\partial_u^2 g = \left(\frac{(k-1)^2}{4} - \lambda\right)g$$

We have one square-integrable solution if $\lambda \leq \frac{(k-1)^2}{4}$ that's given by $\exp\left(-\sqrt{\frac{(k-1)^2}{4} - \lambda}u\right)$. It satisfies the boundary condition if and only if $\lambda = 0$ and $k-1 < 0$. Therefore, we have at most one eigenvalue below the essential spectrum. By the Min-Max principle, it follows that for all ξ , the operator $\square_0(\xi)$ admits at most one eigenvalue less than $\frac{(k-1)^2}{4}$. In particular, for $\lambda < \frac{(k-1)^2}{4}$, one has $\dim(E_\lambda(L(\xi))) \leq 1$. Moreover, since $L(\xi)$ is bounded below by $A^2\xi^2$, it follows that for $\lambda < \frac{(k-1)^2}{4}$ one has :

$$\dim_{\mathbb{Z}} E_\lambda(\square_0) := \frac{1}{2\pi} \int_{\mathbb{R}} \dim(E_\lambda(L(\xi)))d\xi = \frac{1}{2\pi} \int_{\sqrt{\lambda}/2A}^{\sqrt{\lambda}/2A} \dim(E_\lambda(L(\xi)))d\xi \leq \frac{\sqrt{\lambda}}{2\pi A}.$$

□

Remark 3.3.14. In fact, one obtains more than just the finite $\mathcal{N}(\mathbb{Z})$ -dimensionality of the $E_\lambda(\square_0)$, since $E_\lambda(\square_0) = \int_{-\sqrt{\lambda}/2A}^{\sqrt{\lambda}/2A} E_\varepsilon(\square_0(\xi))d\xi$ and the fact that the $E_\varepsilon(\square_0(\xi))$ are finite dimensional one obtains that the range of $E_\varepsilon(\square_0)$ is a finitely generated Hilbert $\mathcal{N}(\mathbb{Z})$ -module.

Proposition 3.3.15. For $k \neq -1$, the Laplace operator \square_2 is $\mathcal{N}(\mathbb{Z})$ -Fredholm. More precisely, for all $k \in \mathbb{Z}$, and all $0 < \lambda < \frac{(k+1)^2}{4}$ the $\mathcal{N}(\mathbb{Z})$ -dimension of the image of $E_\lambda(\square_0)$ is bounded above by $\frac{\sqrt{\lambda}}{2\pi A}$.

Proof. We begin by writing $\square_2(\xi)$ in Sturm-Liouville form

$$\begin{aligned} \square_2(\xi) &= -\partial_y y^{-k} \partial_y y^{2+k} + y^2 \xi^2 \\ &= -\left[\partial_y y^{-(2+k)}\right] y^2 \left[\partial_y y^{2+k}\right] + \xi^2 y^2 \\ &= -\left[y^{-(2+k)} \partial_y - (2+k)y^{-(3+k)}\right] y^2 \left[y^{2+k} \partial_y + (2+k)y^{1+k}\right] + y^2 \xi^2 \\ &= -\left[y^{-(2+k)} \partial_y - (2+k)y^{-(3+k)}\right] \left[y^{4+k} \partial_y + (2+k)y^{3+k}\right] + y^2 \xi^2 \\ &= -\left(y^{-(2+k)} \partial_y y^{4+k} \partial_y - (2+k)y \partial_y + y^{-(2+k)} \partial_y (2+k)y^{3+k} - (2+k)^2\right) + y^2 \xi^2 \\ &= -y^{-(2+k)} \partial_y y^{4+k} \partial_y - (2+k) + \xi^2 y^2. \end{aligned}$$

This time, the Liouville transform reduces the study of $\square_2(\xi)$ to the study of the operator

$$L(\xi) := -\partial_u^2 + Q(u) - (k+2) + \xi^2 A^2 e^{2u}$$

with Dirichlet boundary conditions. With

$$Q(u) = \left(A^{k+4} e^{(k+4)u} A^{k+2} e^{(k+2)u}\right)^{-1/4} \frac{d^2}{du^2} \left(A^{k+4} e^{(k+4)u} A^{k+2} e^{(k+2)u}\right)^{1/4} = \frac{(k+3)^2}{4}$$

In the end, we find

$$L(\xi) := -\partial_u^2 + \frac{(k+1)^2}{4} + \xi^2 A^2 e^{2u}$$

The reasoning is then exactly the same as we did before by noting the change of $\frac{(k+1)^2}{4}$ into $\frac{(k-1)^2}{4}$. □

Those results give us the proposition.

Proposition 3.3.16. For $k \neq 1$ one has $\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\mathbb{H}_A, \mathcal{L}_k) = \mathcal{U}(\mathbb{Z}) \otimes E_0(\square_1)$ and for $k \neq -1$ one has $\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^2(\mathbb{H}_A, \mathcal{L}_k) = 0$.

Proof. For any $\lambda > 0$, the complex

$$0 \longrightarrow E_\lambda(\square_0) \longrightarrow E_\lambda(\square_1) \longrightarrow E_\lambda(\square_2) \longrightarrow 0$$

is homotopic to the complex $L^2 DR_{\max}^\bullet(\mathbb{H}_A, \mathcal{L}_k)$. For $k \neq 1$ and λ small enough, the Hilbert $\mathcal{N}(\mathbb{Z})$ -module $E_\lambda(\square_0)$ is finitely generated, hence $\overline{\text{Ran}(D|_{E_\lambda(\square_0)})}$ is finitely generated and the Fredholmness implies

$$\dim_{\mathcal{N}(\mathbb{Z})} \frac{\overline{\text{Ran}(D)}}{\text{Ran}(D)} = 0.$$

Hence, $\frac{\overline{\text{Ran}(D)}}{\text{Ran}(D)}$ is of $\mathcal{U}(\mathbb{Z})$ -torsion, since a finitely generated $\mathcal{N}(\mathbb{Z})$ -module of dimension 0 is of $\mathcal{U}(\mathbb{Z})$ -torsion (see [Lüc02, Theorem 8.22]) this gives the isomorphism $\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\mathbb{H}_A, \mathcal{L}_k) = \mathcal{U}(\mathbb{Z}) \otimes E_0(\square_1)$, since $E_0(\square_1)$ is just the space of harmonic forms.

For $k \neq -1$, $H_{L^2}^2(\mathbb{H}_A, \mathcal{L}_k)$ is finitely generated since $E_\lambda(\square_2)$ is finitely generated and is of dimension 0 since $\lim_{\lambda \rightarrow 0} \dim_{\mathcal{N}(\mathbb{Z})} E_\lambda(\square_2) = 0$ hence $\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^2(\mathbb{H}_A, \mathcal{L}_k) = 0$. \square

The vanishing of $\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\mathbb{H}_A, \mathcal{L}_k)$ is given by the following lemma.

Proposition 3.3.17. For any $k \in \mathbb{Z}$, one has $\text{Ker}(\square_1) = 0$. In particular, for $k \neq 1$ one has $\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\mathbb{H}_A, \mathcal{L}_k) = 0$.

Proof. One has $\text{Ker}(\square_1(\xi)) = \text{Ker}(d_1(\xi)) \cap \text{Ker}(d_0(\xi)^*)$, so if $\begin{pmatrix} f \\ g \end{pmatrix} \in \text{Ker}(\square_1(\xi))$ one has

$$i\xi g = \partial_y f, \text{ and } -\xi y^2 f = -y^{2-k} \partial_y (y^k g).$$

It follows that

$$\xi^2 y^2 f = y^{2-k} \partial_y y^k \partial_y f.$$

We then have

$$\langle -y^{2-k} \partial_y y^k \partial_y f + \xi^2 y^2 f, f \rangle = \int_A^{+\infty} y^k (-\bar{f} y^{-k} \partial_y y^k \partial_y f + \xi^2 |f|^2) dy = 0.$$

One must also satisfy the boundary condition $g(A) = f'(A) = 0$ that allows us to integrate by part the term $-\bar{f} \partial_y y^k \partial_y f$ to find

$$\int_A^{+\infty} y^k (|\partial_y f|^2 + \xi^2 |f|^2) dy = 0.$$

From this equality one obtains that $\text{Ker}(\square_1(\xi)) = 0$ for all $\xi \neq 0$ hence $\text{Ker}(\square_1) = 0$. The second part of the lemma follows from the above proposition. \square

$\mathcal{N}(\mathbb{Z})$ -Fredholmness with generic monodromy. In the last paragraph, we proved the $\mathcal{N}(\mathbb{Z})$ -Fredholmness of the maximal L^2 -de Rham complex under the additional assumption that the monodromy was unipotent, i.e. $S = 0$. We will show how the proof can be generalised under arbitrary monodromy. In this case, if we take $T = \exp(S + N)$, and e a multivalued horizontal section in $W_k \setminus W_{k-1}$ and associated with the eigenvalue $2i\pi\beta$ of the operator S with $\beta \in \mathbb{R}$ then we have

$$h(\tilde{e}, \tilde{e}) \sim e^{-2\pi\beta y} y^k,$$

for a form ω the differential is given by

$$D_{\max}(\omega \otimes \tilde{e}) = d\omega \otimes \tilde{e} + 2i\pi\beta dz \wedge \omega \otimes \tilde{e}.$$

As before, by Lemma 3.1.2, we can reduce to the case of a trivial line bundle, and we will obtain the following.

Proposition 3.3.18. Let $\mathcal{L}_{\beta,k}$ be a line bundle on a horodisk \mathbb{H}_A with generating section e endowed with a Hermitian metric such that $h(e, e) \simeq e^{-4\pi\beta y} y^k$ and a flat connection D given by the above relation. Then the complex $L^2 DR_{\max}^\bullet(\mathbb{H}_A, \mathcal{L}_{\beta,k})$ is $\mathcal{N}(\mathbb{Z})$ -Fredholm for $k \neq -1, 1$.

Proof. We have isomorphisms given by Fourier transform

$$\begin{aligned} \mathcal{A}_2^0(\mathbb{H}_A, \mathcal{L}_{\beta,k}) &\simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}, e^{-4\pi\beta y} y^{k-2} dy) dx \\ \mathcal{A}_2^1(\mathbb{H}_A, \mathcal{L}_{\beta,k}) &\simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}, e^{-4\pi\beta y} y^k dy) \oplus L^2(\mathbb{R}, e^{-4\pi\beta y} y^k dy) dx \\ \mathcal{A}_2^2(\mathbb{H}_A, \mathcal{L}_{\beta,k}) &\simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}, e^{-4\pi\beta y} y^{k+2} dy) dx \end{aligned}$$

and the connection becomes

$$\begin{aligned} d_0 &= \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} i\xi + 2i\pi\beta \\ \partial_y - 2\pi\beta \end{pmatrix} d\xi \text{ on 0-forms} \\ d_1 &= \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} -\partial_y + 2\pi\beta & i\xi + 2i\pi\beta \end{pmatrix} d\xi \text{ on 1-forms} \end{aligned}$$

The adjoint of the operators d_0 and d_1 are given by

$$\begin{aligned} d_0^* &= \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} -i(\xi + 2\pi\beta)y^2 & -y^{2-k} e^{4\pi\beta y} \partial_y e^{-4\pi\beta y} y^k - 2\pi\beta y^2 \end{pmatrix} d\xi \\ d_1^* &= \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} + 2\pi\beta y^2 \\ -i(\xi + 2\pi\beta)y^2 \end{pmatrix} d\xi \end{aligned}$$

Hence, the Laplace operators $\square_0(\xi)$ is given by

$$\begin{aligned} \square_0(\xi) &= \begin{pmatrix} -i(\xi + 2\pi\beta)y^2 & -y^{2-k} e^{4\pi\beta y} \partial_y e^{-4\pi\beta y} y^k - 2\pi\beta y^2 \end{pmatrix} \begin{pmatrix} i\xi + 2i\pi\beta \\ \partial_y - 2\pi\beta \end{pmatrix} \\ &= (-y^{2-k} e^{4\pi\beta y} \partial_y e^{-4\pi\beta y} y^k - 2\pi\beta y^2)(\partial_y - 2\pi\beta) + (\xi + 2\pi\beta)^2 \\ &= -y^{2-k} e^{4\pi\beta y} \partial_y y^k e^{-4\pi\beta y} \partial_y + 4\pi\beta y^2 + (\xi + 2\pi\beta)^2 \\ &\quad + 2\pi\beta y^{2-k} e^{4\pi\beta y} \partial_y e^{-4\pi\beta y} y^k - 2\pi\beta y^2 \partial_y \end{aligned}$$

We can note that the penultimate term is the $2\pi\beta y^{2-k} e^{4\pi\beta y}$ times the commutator $[\partial_y, e^{-4\pi\beta y} y^k]$ so it is equal to

$$2\pi\beta y^{2-k} (-4\pi\beta y^k + ky^{k-1}) = -8\pi\beta y^2 + 2\pi\beta ky.$$

This gives us

$$\begin{aligned} \square_0(\xi) &= -e^{4\pi\beta y} y^{2-k} \partial_y e^{-4\pi\beta y} y^k \partial_y + (\xi + 2\pi\beta)^2 y^2 - 4\pi^2 \beta^2 y^2 + 2k\pi\beta y \\ \square_2 &:= \int_{\mathbb{R}}^{\oplus} \square_2(\xi) d\xi = \int_{\mathbb{R}}^{\oplus} \left(-\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{k+2} + (\xi + 2\pi\beta)^2 y^2 - 4\pi^2 \beta^2 y^2 + 2(k+2)\pi\beta y \right) d\xi. \end{aligned}$$

We set again $y = Ae^u$, the Liouville transform is then given by

$$\begin{aligned} \mathcal{C}: \quad L^2(\mathbb{R}_{A>}, e^{-4\pi\beta y} y^{k-2}) &\rightarrow L^2(\mathbb{R}^+, du) \\ f &\mapsto A^{1/2} e^{-2\pi\beta u} y^{(k-1)/2} f(Ae^u) \end{aligned}$$

The Liouville transformation conjugates $\square_0(\xi)$ to the operator

$$L_0(\xi) = -\partial_u^2 + (\xi + 2\pi\beta)^2 y^2 - 4\pi^2 \beta^2 y^2 + 2k\pi\beta y + Q(u)$$

with

$$\begin{aligned} Q(u) &= (e^{-8\pi\beta y} y^{2k-2})^{-1/4} \frac{d^2}{du^2} \left(e^{-8\pi\beta y} y^{2k-2} \right)^{1/4} \\ &= e^{2\pi\beta y} y^{(1-k)/2} \frac{d^2}{du^2} \left(e^{-2\pi\beta y} y^{(k-1)/2} \right). \end{aligned}$$

Noting that $\frac{d}{du} = y \frac{d}{dy}$ we obtain

$$\begin{aligned} Q(u) &= e^{2\pi\beta y} y^{(1-k)/2} \frac{d^2}{du^2} \left(e^{-2\pi\beta y} y^{(k-1)/2} \right) \\ &= e^{2\pi\beta y} y^{(1-k)/2} \frac{d}{du} \left(-2\pi\beta e^{-2\pi\beta y} y^{(k+1)/2} + \frac{k-1}{2} e^{-2\pi\beta y} y^{(k-1)/2} \right) \\ &= y^{(1-k)/2} \left(4\pi^2 \beta^2 y^{(k+3)/2} - \pi\beta(k+1) y^{(k+1)/2} - \pi\beta(k-1) y^{(k+1)/2} + \frac{(k-1)^2}{4} y^{(k-1)/2} \right) \\ &= 4\pi^2 \beta^2 y^2 - 2\pi\beta k y + \frac{(k-1)^2}{4}. \end{aligned}$$

In the end, we find

$$L_0(\xi) = L_0(\xi) = -\partial_u^2 + (\xi + 2\pi\beta)^2 y^2 + \frac{(k-1)^2}{4}.$$

Then we can do the exact same proof that we used for the unipotent case up to the shift of $2\pi\beta$ on the ξ variable.

The case of the Laplace operator on the 2-forms is also similar, but the computation of $\square_2(\xi)$ is a little more complicated. We give the details below.

$$\begin{aligned} \square_2(\xi) &= (-\partial_y + 2\pi\beta \quad i\xi + 2i\pi\beta) \begin{pmatrix} e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} + 2\pi\beta y^2 \\ -i(\xi + 2\pi\beta) y^2 \end{pmatrix} \\ &= -\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} + 4\pi^2 \beta^2 y^2 + (\xi + 2\pi\beta)^2 y^2 \\ &\quad - 2\pi\beta \partial_y y^2 + 2\pi\beta e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} \end{aligned}$$

The terms in the last line are equal to $2\pi\beta e^{4\pi\beta y} y^{-k} [\partial_y, e^{-4\pi\beta y} y^k] y^2$. So we have

$$\begin{aligned} 2\pi\beta e^{4\pi\beta y} y^{-k} [\partial_y, e^{-4\pi\beta y} y^k] y^2 &= 2\pi\beta e^{4\pi\beta y} y^{-k} (-4\pi\beta e^{-4\pi\beta y} y^k + k e^{-4\pi\beta y} y^{k-1}) y^2 \\ &= -8\pi^2 \beta^2 y^2 + 2k\pi\beta y \end{aligned}$$

Summing up, we find

$$\square_2(\xi) = -\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} - 4\pi^2 \beta^2 y^2 + (\xi + 2\pi\beta)^2 y^2 + 2k\pi\beta y. \quad (3.17)$$

It is not yet in a Sturm-Liouville form to apply the Liouville transform, so we still have to work with the term $-\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k}$. It gives

$$\begin{aligned} -\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} &= -[\partial_y e^{4\pi\beta y} y^{-(k+2)}] y^2 [\partial_y e^{-4\pi\beta y} y^{2+k}] \\ &= - \left[e^{4\pi\beta y} y^{-(k+2)} \partial_y + \left(4\pi\beta y^{-(k+2)} - (k+2) y^{-(k+3)} \right) e^{4\pi\beta y} \right] \\ &\quad \times y^2 \left[e^{-4\pi\beta y} y^{2+k} \partial_y + \left(-4\pi\beta y^{2+k} + (2+k) y^{1+k} \right) e^{-4\pi\beta y} \right] \\ &= - \left\{ e^{4\pi\beta y} y^{-(k+2)} \partial_y e^{-4\pi\beta y} y^{4+k} \partial_y \right. \\ &\quad + e^{4\pi\beta y} y^{-(k+2)} \partial_y (-4\pi\beta y^{4+k} + (2+k) y^{3+k}) e^{-4\pi\beta y} \\ &\quad + (4\pi\beta y^2 - (k+2) y) \partial_y \\ &\quad \left. + (4\pi\beta y^{-k+2} - (k+2) y^{-(k+3)}) (-4\pi\beta y^{4+k} + (2+k) y^{3+k}) \right\} \end{aligned}$$

We can simplify the term $e^{4\pi\beta y} y^{-(k+2)} \partial_y (-4\pi\beta y^{2+k} + (2+k) y^{1+k}) e^{-4\pi\beta y}$ by

$$\begin{aligned} e^{4\pi\beta y} y^{-(k+2)} \partial_y (-4\pi\beta y^{2+k} + (2+k) y^{1+k}) e^{-4\pi\beta y} &= -(4\pi\beta y^2 - (k+2) y) \partial_y \\ &\quad + 16\pi^2 \beta^2 y^2 - 8\pi\beta(k+3) y + (k+2)(k+3). \end{aligned}$$

This reduces to

$$\begin{aligned} -\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} &= - \left\{ [e^{4\pi\beta y} y^{-(k+2)} \partial_y e^{-4\pi\beta y} y^{4+k} \partial_y \right. \\ &\quad + 16\pi^2 \beta^2 y^2 - 8\pi\beta(k+3) y + (k+2)(k+3) \\ &\quad \left. + (4\pi\beta y^{-k+2} - (k+2) y^{-(k+3)}) (-4\pi\beta y^{4+k} + (2+k) y^{3+k}) \right\}. \end{aligned}$$

And we end up with

$$-\partial_y e^{4\pi\beta y} y^{-k} \partial_y e^{-4\pi\beta y} y^{2+k} = -e^{4\pi\beta y} y^{-(k+2)} \partial_y e^{-4\pi\beta y} y^{4+k} \partial_y + 8\pi\beta y - (k+2).$$

By replacing the appropriate term in (3.17) we find

$$\square_2(\xi) = -e^{4\pi\beta y} y^{-(k+2)} \partial_y e^{-4\pi\beta y} y^{4+k} \partial_y - 4\pi^2 \beta^2 y^2 + (\xi + 2\pi\beta)^2 y^2 + 2(k+4)\pi\beta y - (k+2).$$

It remains to apply the Liouville transform (again with $y = Ae^u$). This time the additional term Q is given by

$$\begin{aligned} Q(u) &= (e^{-8\pi\beta y} y^{2k+6})^{-1/4} \frac{d^2}{du^2} (e^{-8\pi\beta y} y^{2k+6})^{1/4} \\ &= 4\pi^2 \beta^2 y^2 - 2\pi(k+4)\beta y + \frac{(k+3)^2}{4}. \end{aligned}$$

So we reduce to the study of $\square_2(\xi)$ to

$$L_2(\xi) = -\partial_u^2 + (\xi + 2\pi\beta)^2 e^{2u} + \frac{(k+1)^2}{4}.$$

Then the reasoning is the same as before. \square

3.2 The $\mathcal{N}(\mathbb{Z})$ -Fredholmness of $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})$.

Recall that $\tilde{M} \rightarrow M$ is an infinite-connected cover of a punctured disk. In particular, \tilde{M} is quasi-isometric to some horodisk \mathbb{H}_A .

Theorem 3.3.19. Let $(\mathbb{V}, F^\bullet, Q)$ be a variation of Hodge structure on M . Then the complex $L^2 DR_{\max}^\bullet(\tilde{M}, \pi^* \mathbb{V})$ is $\mathcal{N}(\mathbb{Z})$ -Fredholm.

Recall that if T is the monodromy operator, one has a decomposition $T = e^{S+N}$, where N is nilpotent and S is semisimple and that we denote by W_\bullet the nilpotent filtration of N . Recall that by the existence of L^2 -adapted basis flagged according to the filtration, we have short exact sequences for all $k \in \mathbb{Z}$

$$0 \rightarrow L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_k) \rightarrow L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_{k+1}) \rightarrow L^2 DR_{\max}^\bullet(\mathbb{H}_A, \text{Gr}_W^{k+1}) \rightarrow 0.$$

In the proof of the L^2 Poincaré lemma, we have shown that $L^2 DR_{\max}^\bullet(\mathbb{H}_A, \text{Gr}_W^k)$ is $\mathcal{N}(\mathbb{Z})$ -Fredholm for $k \neq -1, 1$. It follows by a straightforward induction that $L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_{-2})$ is $\mathcal{N}(\mathbb{Z})$ -Fredholm. And by induction on k , if we have that the complex $L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_1)$ is $\mathcal{N}(\mathbb{Z})$ -Fredholm, the whole complex will be $\mathcal{N}(\mathbb{Z})$ -Fredholm. Since one has also a short exact sequence

$$0 \rightarrow L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_{-2}) \rightarrow L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_1) \rightarrow L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_1/W_{-2}) \rightarrow 0$$

it suffices to prove the

Lemma 3.3.20. The $\mathcal{N}(\mathbb{Z})$ -Hilbert complex $L^2 DR_{\max}^\bullet(\mathbb{H}_A, W_1/W_{-2})$ is $\mathcal{N}(\mathbb{Z})$ -Fredholm.

Proof. Choose a family (e_1, \dots, e_l) of horizontal sections such that it induces a horizontal base of Gr_W^1 and $(\epsilon_1, \dots, \epsilon_k)$ a family of horizontal sections that induces a basis of Gr_W^0 . Since S preserves the filtration W_\bullet , we can choose them so that they are eigenvalues of S . As before, if s is a multivalued horizontal section, we denote by \tilde{s} the single-valued section $\exp((S+N)z)s$. Then the family

$$(\tilde{e}_1, \dots, \tilde{e}_l, N\tilde{e}_1, \dots, N\tilde{e}_l, \tilde{\epsilon}_1, \dots, \tilde{\epsilon}_k)$$

defines an L^2 -adapted frame of W_1/W_{-2} . The connection D expressed with respect to this frame is given by the connection matrix

$$D = \begin{pmatrix} d + Sdz \wedge & 0 & 0 \\ dz \wedge & d + Sdz \wedge & 0 \\ 0 & 0 & d + Sdz \wedge \end{pmatrix}. \quad (3.18)$$

Using the L^2 -adaptedness, up to a quasi-isometry, we can assume that our frame is orthogonal. This allows us to reduce to the case where $l = k = 1$. We will denote by $2i\pi\alpha$ (resp. $2i\pi\beta$) the eigenvalues of S associated to the eigenvectors e_1 (resp. ε_1)

The choice of frame induces isomorphisms of the spaces $L^2DR^k(\mathbb{H}_A, \pi^*\mathbb{V})$ into sums of L^2 spaces given by the maps

$$\begin{aligned} f \otimes \tilde{e}_1 &\mapsto f & f \otimes \tilde{\varepsilon}_1 &\mapsto f \\ f dx \otimes \tilde{e}_1 &\mapsto f & f dx \otimes \tilde{\varepsilon}_1 &\mapsto f \\ f dy \otimes \tilde{e}_1 &\mapsto f & f dy \otimes \tilde{\varepsilon}_1 &\mapsto f \\ f dx \wedge dy \otimes \tilde{e}_1 &\mapsto f & f dx \wedge dy \otimes \tilde{\varepsilon}_1 &\mapsto f \end{aligned}$$

Using the Fourier transform, we can express them as direct integrals of measurable fields of Hilbert spaces, and the connection D becomes a closed decomposable operator.

$$\begin{aligned} L^2DR^0(\mathbb{H}_A, W_1/W_{-2}) &\simeq \int_{\mathbb{R}}^{\oplus} H_0(\xi) d\xi \\ L^2DR^1(\mathbb{H}_A, W_1/W_{-2}) &\simeq \int_{\mathbb{R}}^{\oplus} H_1(\xi) d\xi \\ L^2DR^2(\mathbb{H}_A, W_1/W_{-2}) &\simeq \int_{\mathbb{R}}^{\oplus} H_2(\xi) d\xi \end{aligned} \tag{3.19}$$

with $H_0(\xi)$ being a constant field of Hilbert spaces defined by

$$\begin{aligned} H_0(\xi) &= L^2(\mathbb{R}_{>A}, e^{-4\pi\alpha y} y^{-1} dy) \oplus L^2(\mathbb{R}_{>A}, e^{-4\pi\alpha y} y^{-3} dy) \oplus L^2(\mathbb{R}_{>A}, e^{-4\pi\beta y} y^{-2} dy) \\ H_1(\xi) &= L^2(\mathbb{R}_{>A}, e^{-4\pi\alpha y} y dy)^{\oplus 2} \oplus L^2(\mathbb{R}_{>A}, e^{-4\pi\alpha y} y^{-1} dy)^{\oplus 2} \oplus L^2(\mathbb{R}_{>A}, e^{-4\pi\beta y} dy)^{\oplus 2} \\ H_2(\xi) &= L^2(\mathbb{R}_{>A}, e^{-4\pi\alpha y} y^3 dy) \oplus L^2(\mathbb{R}_{>A}, e^{-4\pi\alpha y} y^1 dy) \oplus L^2(\mathbb{R}_{>A}, e^{-4\pi\beta y} y^2 dy) \end{aligned}$$

The differential $D_k : L^2DR^k(\mathbb{H}_A, W_1/W_{-2})$ are then given by the direct integrals of differential operators

$$\begin{aligned} D_0 &\simeq \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} d_{0,1}(\xi) & 0 & 0 \\ \begin{pmatrix} 1 \\ i \end{pmatrix} & d_{0,-1}(\xi) & 0 \\ 0 & 0 & d_{0,0}(\xi) \end{pmatrix} d\xi \\ D_1 &\simeq \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} d_{1,1}(\xi) & 0 & 0 \\ \begin{pmatrix} -i & 1 \end{pmatrix} & d_{1,-1}(\xi) & 0 \\ 0 & 0 & d_{1,0}(\xi) \end{pmatrix} d\xi \end{aligned}$$

with

$$\begin{aligned} d_{0,k}(\xi) &= \begin{pmatrix} i\xi + 2i\pi\alpha \\ \partial_y - 2\pi\alpha \end{pmatrix} && \text{for } k = -1, 1 \\ d_{0,0}(\xi) &= \begin{pmatrix} i\xi \\ \partial_y - 2\pi\beta \end{pmatrix} && \text{for } k = 0 \\ d_{1,k}(\xi) &= \begin{pmatrix} -\partial_y + 2\pi\alpha & i\xi + 2i\pi\alpha \end{pmatrix} && \text{for } k = -1, 1 \\ d_{1,0}(\xi) &= \begin{pmatrix} -\partial_y + 2\pi\beta & i\xi + 2i\pi\beta \end{pmatrix} && \text{for } k = 0. \end{aligned}$$

By taking into account the different weights on our L^2 spaces, the adjoints maps are given by

$$\begin{aligned} D_0^* &\simeq \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} d_{0,1}^*(\xi) & \begin{pmatrix} 1 & -i \end{pmatrix} & 0 \\ 0 & d_{0,-1}^*(\xi) & 0 \\ 0 & 0 & d_{0,0}^*(\xi) \end{pmatrix} d\xi \\ D_1^* &\simeq \int_{\mathbb{R}}^{\oplus} \begin{pmatrix} d_{1,1}^*(\xi) & \begin{pmatrix} i \\ 1 \end{pmatrix} & 0 \\ 0 & d_{1,-1}^*(\xi) & 0 \\ 0 & 0 & d_{1,0}^*(\xi) \end{pmatrix} d\xi \end{aligned}$$

where the $d_{0,k}^*(\xi), d_{1,k}^*(\xi)$ are formally given by the adjoints of the differential operators $d_0(\xi)$ and $d_1(\xi)$ we considered while computing the spectrum of the Laplace operator on Gr_W^k , whose expressions we recall below

$$\begin{aligned} d_{0,k}^*(\xi) &= (-i(\xi + 2\pi\alpha)y^2 - y^{k-2}e^{4\pi\alpha y}\partial_y e^{-4\pi\alpha y}y^k - 2\pi\alpha y^2) & \text{if } k \neq 0 \\ d_{0,0}^*(\xi) &= (-i(\xi + 2\pi\beta)y^2 - y^2e^{4\pi\beta y}\partial_y e^{-4\pi\beta y} - 2\pi\beta y^2) \\ d_{1,k}^*(\xi) &= \begin{pmatrix} y^{-k}e^{4\pi\alpha y}\partial_y e^{-4\pi\alpha y}y^{2+k} + 2\pi\alpha y^2 \\ -i(\xi + 2\pi\alpha)y^2 \end{pmatrix} & \text{if } k \neq 0 \\ d_{1,0}^*(\xi) &= \begin{pmatrix} e^{4\pi\beta y}\partial_y e^{-4\pi\beta y}y^2 + 2\pi\beta y^2 \\ -i(\xi + 2\pi\beta)y^2 \end{pmatrix} \end{aligned}$$

The Laplace operator on the 0-forms and 2-forms are thus given by

$$\square_0 = \int_{\mathbb{R}}^{\oplus} \square_0(\xi)d\xi \quad (3.20)$$

$$\square_2 = \int_{\mathbb{R}}^{\oplus} \square_2(\xi)d\xi \quad (3.21)$$

with

$$\begin{aligned} \square_0(\xi) &= \begin{pmatrix} \square_{0,1}(\xi) + 2 & i(\xi + 4\pi\alpha) - i\partial_y & 0 \\ -i(\xi + 4\pi\alpha)y^2 - iy^3e^{4\pi\alpha y}\partial_y e^{-4\pi\alpha y}y^{-1} & \square_{0,-1}(\xi) & 0 \\ 0 & 0 & \square_{0,0}(\xi) \end{pmatrix} \\ \square_2(\xi) &= \begin{pmatrix} \square_{2,1}(\xi) & i(\xi + 4\pi\alpha) - i\partial_y & 0 \\ -i(\xi + 4\pi\alpha)y^2 - iy^{-1}e^{4\pi\alpha y}\partial_y e^{-4\pi\alpha y}y^3 & \square_{2,-1}(\xi) + 2 & 0 \\ 0 & 0 & \square_{2,0}(\xi) \end{pmatrix} \end{aligned}$$

Where $\square_{0,k}(\xi), \square_{2,k}(\xi)$ are given by the expression of the corresponding Laplace operators on Gr_W^k which we computed while proving the Poincaré lemma. We recall their expressions below.

$$\begin{aligned} \square_{0,k}(\xi) &= -e^{4\pi\alpha y}y^{2-k}\partial_y y^k e^{-4\pi\alpha y}\partial_y + (\xi + 2\pi\alpha)^2 - 4\pi^2\alpha^2 + 2k\pi\alpha y \quad \text{for } k = -1, 1 \\ \square_{0,0}(\xi) &= -e^{4\pi\beta y}y^{2-k}\partial_y y^k e^{-4\pi\beta y}\partial_y + (\xi + 2\pi\beta)^2 - 4\pi^2\beta^2 + 2k\pi\beta y \\ \square_{2,k}(\xi) &= -e^{4\pi\alpha y}y^{-(k+2)}\partial_y y^{4+k} e^{-4\pi\alpha y}\partial_y + (\xi + 2\pi\alpha)^2 - 4\pi^2\alpha^2 + 2(k+4)\pi\alpha y - (k+2) \quad \text{for } k = -1, 1 \\ \square_{2,0}(\xi) &= -e^{4\pi\beta y}y^{-2}\partial_y y^4 e^{-4\pi\beta y}\partial_y + (\xi + 2\pi\beta)^2 - 4\pi^2\beta^2 + 8\pi\beta y - 2 \end{aligned}$$

We already computed the spectrum of $\square_{0,0}(\xi)$ and $\square_{2,0}(\xi)$ and showed that $\int_{\mathbb{R}}^{\oplus} \square_{0,0}(\xi)d\xi$ and $\int_{\mathbb{R}}^{\oplus} \square_{2,0}(\xi)d\xi$ are $\mathcal{N}(\mathbb{Z})$ -Fredholm. We are thus reduced to study the Fredholmness of the operators

$$P_0(\xi) = \begin{pmatrix} \square_{0,1}(\xi) + 2 & i(\xi + 2\pi\alpha) - i\partial_y \\ -i(\xi + 2\pi\alpha)y^2 - iy^3e^{4\pi\alpha y}\partial_y e^{-4\pi\alpha y}y^{-1} & \square_{0,-1}(\xi) \end{pmatrix} \quad (3.22)$$

$$P_2(\xi) = \begin{pmatrix} \square_{2,1}(\xi) & i(\xi + 4\pi\alpha) - i\partial_y \\ -i(\xi + 4\pi\alpha)y^2 - iy^{-1}e^{4\pi\alpha y}\partial_y e^{-4\pi\alpha y}y^3 & \square_{2,-1}(\xi) + 2 \end{pmatrix}. \quad (3.23)$$

As in the proof of the Poincaré lemma, we consider the Liouville transforms, i.e. the unitary transformation $C_{0,1}, C_{0,-1}$ that conjugate $\square_{0,k}(\xi)$ into a Schrödinger operator. Namely, we set $\ell(y) = \ln(\frac{y}{A})$ and set

$$\begin{aligned} C_{0,k}: L^2(\mathbb{R}_{>A}, y^{k-2}e^{4\pi\alpha y}dy) &\rightarrow L^2(\mathbb{R}^+, du) \\ f &\mapsto (y^{\frac{k-1}{2}}e^{-2\pi\alpha y}f) \circ \ell^{-1} \end{aligned}$$

We also set

$$C_0 = \begin{pmatrix} C_{0,1} & 0 \\ 0 & C_{0,-1} \end{pmatrix}$$

and we compute

$$C_0 P_0(\xi) C_0^* = \begin{pmatrix} -\partial_u^2 + A(\xi + 2\pi\alpha)^2 e^{2u} + 2 & -i\partial_u + iA(\xi + 2\pi\alpha)e^u - i \\ -i\partial_u - iA(\xi + 2\pi\alpha)e^u + i & -\partial_u^2 + A(\xi + 2\pi\alpha)^2 e^{2u} + 1 \end{pmatrix} \quad (3.24)$$

which is a differential operator acting on $L^2(\mathbb{R}^+, du)^{\oplus 2}$. Recall that we have Neumann boundary condition for the operator $A_0(\xi)$, and the unitary transformation changes those boundary conditions into

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \text{Dom}(CA(\xi)C^*) \text{ then } \begin{cases} f'(0) & = -2\pi\alpha Af(0) \\ g'(0) & = (-1 - 2\pi\alpha A)g(0) \end{cases}$$

We split the right-hand side into a sum

$$\begin{pmatrix} -\partial_u^2 + 2 & -i\partial_u - i \\ -i\partial_u + i & -\partial_u^2 + 1 \end{pmatrix} + \begin{pmatrix} A^2(\xi + 2\pi\alpha)^2 e^{2u} & iA(\xi + 2\pi\alpha)e^u \\ -iA(\xi + 2\pi\alpha)e^u & A^2(\xi + 2\pi\alpha)^2 e^{2u} \end{pmatrix}$$

A quick study of the eigenvalues of the matrix

$$\begin{pmatrix} A^2(\xi + 2\pi\alpha)^2 e^{2u} & iA(\xi + 2\pi\alpha)e^u \\ -iA(\xi + 2\pi\alpha)e^u & A^2(\xi + 2\pi\alpha)^2 e^{2u} \end{pmatrix}$$

shows that the operator is bounded below by $-1/4$. Hence, it suffices to bound below the essential spectra of the differential operator

$$\begin{pmatrix} -\partial_u^2 + 2 & -i\partial_u - i \\ -i\partial_u + i & -\partial_u^2 + 1 \end{pmatrix}$$

by a constant greater than $1/4$. As we did for the Poincaré lemma, the essential spectra of this operator does not depend on the boundary conditions and Fourier theory tells us it is equal to the set of eigenvalues of the matrices

$$B(\zeta) = \begin{pmatrix} \zeta^2 + 2 & \zeta - i \\ \zeta + i & \zeta^2 + 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}.$$

Its characteristic polynomial is given by

$$X^2 - (2\zeta^2 + 3)X + (\zeta^2 + 1)^2.$$

We deduce that its smallest eigenvalue of $B(\zeta)$ is given by

$$r_-(\zeta) = \frac{2\zeta^2 + 3 - \sqrt{4\zeta^2 + 5}}{2}$$

one has

$$r'_-(\zeta) = 2\zeta - \frac{4\zeta}{4\sqrt{4\zeta^2 + 5}} = \zeta \left(2 - \frac{1}{\sqrt{4\zeta^2 + 5}} \right).$$

Hence the function r_- is bounded below by $(3 - \sqrt{5})/2 > 1/4$, which concludes the proof of the Fredholmness of the Laplace operator on 0-forms.

The case of the Laplace operator for 2-forms is similar. We have to study the operators $P_2(\xi)$ given by (3.23). This time we use the Liouville transform $C_{2,k}$ that conjugates $\square_{2,k}(\xi)$ to Schrödinger's operators. This time we set

$$C_2 = \begin{pmatrix} C_{2,1} & 0 \\ 0 & C_{2,-1} \end{pmatrix}$$

and we compute

$$C_2 P_2(\xi) C_2^* = \begin{pmatrix} -\partial_u^2 + (\xi + 2\pi\alpha)^2 e^u + 1 & i(\xi + 2\pi\alpha) - i\partial_u + i \\ -i(\xi + 2\pi\alpha) + i\partial_u + i & -\partial_u^2 + (\xi + 2\pi\alpha)^2 e^u + 1 \end{pmatrix} \quad (3.25)$$

with Dirichlet boundary conditions. It follows that up to conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and boundary conditions we have the same differential operator as the one we just studied, and the rest is thus similar. \square

The lemma being proven, the Theorem follows immediately.

3.3 Computation of the group $\mathcal{U}(\mathbb{Z}) \otimes_{\mathcal{N}(\mathbb{Z})} H_{L^2, \max}^k(\tilde{M}, Gr_W^k)$

This section is dedicated to the proof of the second part of the theorem 3.1.1, namely,

Theorem 3.3.21. Let $(\mathbb{V}, F^\bullet, Q)$ be a polarised Hodge structure on $M := \Delta_R$ and $\pi: \tilde{M} = \mathbb{H}_A \rightarrow \Delta_R$ be a Galois covering of Deck group $\text{Deck}(\tilde{M}/M) = \mathbb{Z}$. Then we have

$$\mathcal{U}(\mathbb{Z}) \otimes_{\mathcal{N}(\mathbb{Z})} H_{L^2, \max}^k(\tilde{M}, \pi^*\mathbb{V}) = 0. \quad \text{for all } k$$

The proof is an adaptation of the spectral sequence argument used in [Zuc79] in the case where the covering is trivial. Recall that while studying $\mathcal{N}(\mathbb{Z})$ -Fredholmness we have also shown the following, which we will use.

Lemma 3.3.22. For $\beta, k \in \mathbb{R}$, let $(\mathcal{L}_{\beta, k}, D)$ be a flat line bundle on \tilde{M} with generating section e , endowed with a Hermitian metric h satisfying $h(e, e) = e^{-4\pi\beta y^k}$. Then the complex $L^2 DR_{\max}^\bullet(\tilde{M}, \mathcal{L}_{\beta, k})$ has no reduced cohomology and

$$\begin{aligned} \mathcal{U}(\mathbb{Z}) \otimes H_{L^2, \max}^0(\tilde{M}, \mathcal{L}_{\beta, k}) &= 0 \\ \mathcal{U}(\mathbb{Z}) \otimes H_{L^2, \max}^1(\tilde{M}, \mathcal{L}_{\beta, k}) &= 0 \quad \text{for } k \neq 1 \\ \mathcal{U}(\mathbb{Z}) \otimes H_{L^2, \max}^2(\tilde{M}, \mathcal{L}_{\beta, k}) &= 0 \quad \text{for } k \neq -1. \end{aligned}$$

The main ingredient of this proof is the following.

Lemma 3.3.23. Set $N \in \text{End}(\mathcal{V})$ to be the morphism obtained by taking the logarithm of the monodromy. The morphism $(dz \wedge \cdot) \otimes N$ induces an isomorphism

$$\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\pi^{-1}(U), \omega_{P_c}, Gr_1^W, h) \rightarrow \mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^2(\pi^{-1}(U), \omega_{P_c}, Gr_{-1}^W, h).$$

Proof. Again by Lemma 3.1.2 we are reduced to the case where $Gr_W^k = \mathcal{L}_k$ is a trivial flat line bundle. We consider the following morphism of complexes induced by $dz \wedge$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{U}(\mathbb{Z}) \otimes L^2 DR^0(\mathcal{L}_1) & \xrightarrow{D} & \mathcal{U}(\mathbb{Z}) \otimes Z^1(\mathcal{L}_1) & \longrightarrow & 0 \\ & & \downarrow dz \wedge & & \downarrow dz \wedge & & \\ 0 & \longrightarrow & \mathcal{U}(\mathbb{Z}) \otimes L^2 DR^1(\mathcal{L}_{-1})/Z^1 & \xrightarrow{D} & \mathcal{U}(\mathbb{Z}) \otimes Z^2(\mathcal{L}_{-1}) & \longrightarrow & 0 \end{array}$$

where Z^1 is the space of closed 1-forms. We will show that it is a surjective morphism of complexes and that the first cohomology group of the kernel vanishes. The exact long sequence in cohomology will then show that $dz \wedge: \mathcal{U}(\mathbb{Z}) \otimes H^1(\mathcal{L}_1) \rightarrow \mathcal{U}(\mathbb{Z}) \otimes H^2(\mathcal{L}_{-1})$ is an isomorphism.

Step 1: Surjectivity of $\mathcal{U}(\mathbb{Z}) \otimes Z^1(\mathcal{L}_1) \rightarrow \mathcal{U}(\mathbb{Z}) \otimes Z^2(\mathcal{L}_{-1})$. Using the decomposition into integral of measurable fields of Hilbert spaces of $L^2 DR^1(\mathbb{H}_A, \mathcal{L}_1)$ given by the isomorphism (3.16) the morphism $dz \wedge$ is given by

$$\int_{\mathbb{R}}^{\oplus} \begin{pmatrix} f_{\xi} \\ g_{\xi} \end{pmatrix} d\xi \rightarrow \int_{\mathbb{R}}^{\oplus} -if_{\xi} + g_{\xi} d\xi.$$

Take $\int_{\mathbb{R}}^{\oplus} h_{\xi} d\xi \in Z^2(\mathcal{L}_{-1})$. We want to find $f_{\xi}, g_{\xi} \in L^2(\mathbb{R}_{>A}, ydy)$ such that

$$\partial_y f_{\xi} = i\xi g_{\xi} \tag{3.26}$$

$$g_{\xi} - if_{\xi} = h_{\xi} \tag{3.27}$$

The condition (3.26) comes from the fact that we are looking at closed forms, while the condition (3.27) is just the condition for the morphism to be surjective.

According to Lemma A.3.76, if we can find such f_{ξ} that also satisfies $\|f_{\xi}\|_{L^2}^2 + \|g_{\xi}\|_{L^2}^2 \leq C(\xi)\|h(\xi)\|^2$ with $C(\xi)$ being a 2π -periodic functions, finite almost everywhere, we will obtain the surjectivity after tensorization with $\mathcal{U}(\mathbb{Z})$. In the following the function f_{ξ} will be measurable in ξ , however, the reader can note that Lemma A.3.76 also tells us

that it is not necessary to check the measurability. From now on, we will drop the subscript ξ for f, g, h to simplify the notation.

Multiplying (3.27) by $i\bar{\xi}$, we are reduced to solving the differential equation $\partial_y f + \bar{\xi}f = i\bar{\xi}h$, and we need to control the L^2 -norm of the solution f in terms of the L^2 -norm of h . Using the density of smooth functions with compact support on Z^2 we can assume that h is smooth, in this case we take

$$f(y) = \begin{cases} i\bar{\xi}e^{-\bar{\xi}y} \int_A^y e^{\bar{\xi}t} h(t) dt & \text{if } \bar{\xi} > 0 \\ -i\bar{\xi}e^{-\bar{\xi}y} \int_y^{+\infty} e^{\bar{\xi}t} h(t) dt & \text{if } \bar{\xi} < 0. \end{cases}$$

We can bound the L^2 norm of f by the L^2 norm of h , we treat here the case $\bar{\xi} > 0$. By using Theorem A.4.84, with $U = \bar{\xi}e^{-\bar{\xi}y} \sqrt{y}$ and $V = e^{-\bar{\xi}y} \sqrt{y}$, we know that we have a bound

$$\|f\|^2 \leq 4C(\bar{\xi}) \|h\|^2$$

with

$$C(\bar{\xi}) = \sup_{r>A} \int_r^{+\infty} \left| i\bar{\xi}e^{-\bar{\xi}y} \sqrt{y} \right|^2 dy \int_A^r \left| e^{-\bar{\xi}y} \sqrt{y} \right|^{-2} dy$$

The first integral is equal to $\frac{e^{-2\bar{\xi}r}(2\bar{\xi}r+1)}{4}$. If we set $F(y) = \int_1^y \frac{e^t}{t} dt$ the second integral is equal to $F(2\bar{\xi}r) - F(2\bar{\xi}A)$. Hence, one has

$$4C(\bar{\xi}) = \sup_{r>A} e^{-2\bar{\xi}r}(2\bar{\xi}r+1)(F(2\bar{\xi}r) - F(2\bar{\xi}A))$$

Since we have the asymptotic behaviour

$$\frac{e^t}{t} \underset{0}{\sim} \frac{1}{t} \quad \frac{e^t}{t} \underset{\infty}{\sim} \frac{d}{dt} \frac{e^t}{t}$$

and the integral defining F is divergent one has

$$F(y) \underset{0}{\sim} \ln(y) \quad F(y) \underset{\infty}{\sim} \frac{e^y}{y}$$

and there exists a constant $C_0 > 0$ such that

$$|F(y)| \leq C_0 \frac{e^y}{y} \text{ for } y \geq 1 \quad |F(y)| \leq C_0(|\ln(y)| + 1) \text{ for } y \leq 1. \quad (3.28)$$

For $\bar{\xi} \geq 1/A$, using that the function $t \mapsto e^{-t}(t+1)$ is decreasing on $\mathbb{R}_{>0}$, we have

$$\begin{aligned} 4C(\bar{\xi}) &\leq C_0 \sup_{r>A} \left(\frac{(2\bar{\xi}r+1)}{2\bar{\xi}r} + e^{2\bar{\xi}(A-r)} \frac{2\bar{\xi}r+1}{2\bar{\xi}A} \right) \\ &\leq C_0 \sup_{r>A} \frac{3}{2} + e^{2\bar{\xi}(A-r)} \frac{2\bar{\xi}r+1}{2\bar{\xi}A} \\ &\leq C_0 \left(3/2 + \frac{2\bar{\xi}A+1}{2\bar{\xi}A} \right) \\ &\leq 3C_0 \end{aligned}$$

which is bounded uniformly in $\bar{\xi} \geq 1/A$. For $\bar{\xi} < 1/A$, one has the bound

$$4C(\bar{\xi}) \leq C_0 \sup_{r>A} \left(\frac{(2\bar{\xi}r+1)}{2\bar{\xi}r} + e^{-2\bar{\xi}r} (2\bar{\xi}r+1) (|\ln(2\bar{\xi}A)| + 1) \right) \quad (3.29)$$

$$\leq C_0 \left(1 + \frac{1}{2\bar{\xi}A} + (|\ln(2\bar{\xi}A)| + 1) \right). \quad (3.30)$$

Since $C(\bar{\xi})$ is measurable and finite for all $\bar{\xi} \neq 0$, it is straightforward that one can bound $4C(\bar{\xi})$ by a measurable 2π -periodic function that is finite almost-everywhere.

Step 2 : Surjectivity of $dz: L^2DR^0(\mathcal{L}_1) \rightarrow L^2DR^1(\mathcal{L}_{-1})/Z^1$. The method will be similar, albeit a little more complicated. We use the fact that an element of $L^2DR^1(\mathcal{L}_{-1})/Z^1$ admits a unique representative in $\overline{\text{Ran}(d_1^*)}$. Again, we will work with a fixed ξ . An element of $\text{Ran}(d_1(\xi)^*)$ is of the form $d_1(\xi)^*h = \begin{pmatrix} y\partial_y(yh) \\ -i\xi y^2h \end{pmatrix} \in L^2(\mathbb{R}_{>A}, y^{-1}dy)^{\oplus 2}$ with $h \in L^2(\mathbb{R}_{>A}, ydy)$. We wish to find $r \in L^2(\mathbb{R}_{>A}, y^{-1}dy)$, C a 2π -periodic measurable function that is finite almost everywhere such that

$$\begin{pmatrix} y\partial_y(yh) \\ -i\xi y^2h \end{pmatrix} + \omega = \begin{pmatrix} r \\ ir \end{pmatrix} \text{ with } \omega \text{ closed}$$

$$\|r\|_{L^2} + \|d_0(\xi)r\|_{L^2} \leq C(\xi)(\|d_1^*h\|_{L^2} + \|d_1d_1^*h\|_{L^2})$$

This amounts to solving the following differential equation.

$$-\partial_y y \partial_y y h + \xi^2 y^2 h = -\partial_y r - \xi r$$

We set

$$r(y) = \begin{cases} -e^{\xi y} \int_y^{+\infty} e^{-\xi t} (\partial_t t \partial_t t h(t) - \xi^2 y^2 h(t)) dt & \text{if } \xi > 0 \\ e^{\xi y} \int_A^y e^{-\xi t} (\partial_t t \partial_t t h(t) - \xi^2 y^2 h(t)) dt & \text{if } \xi < 0 \end{cases}.$$

Since smooth functions are dense for the graph norm of \square_2 , one can assume h smooth and by integrating by part the term $e^{-\xi t} \partial_t t \partial_t t h(t)$ we obtain

$$r(y) = \begin{cases} -e^{\xi y} \int_y^{+\infty} \xi e^{-\xi t} (t \partial_t t h(t) + \xi y^2 h(t)) dt - y \partial_y(yh) & \text{if } \xi > 0 \\ e^{\xi y} \int_A^y \xi e^{-\xi t} (t \partial_t t h(t) + \xi y^2 h(t)) dt + y \partial_y(yh) & \text{if } \xi < 0 \end{cases}$$

We assume $\xi > 0$, the case $\xi < 0$ being similar. We have three terms whose norm must be bounded in terms of the norm of d_1^*h namely

$$\xi e^{\xi y} \int_y^{+\infty} e^{-\xi t} t \partial_t t h(t) dt \tag{3.31}$$

$$\xi e^{\xi y} \int_y^{+\infty} e^{-\xi t} (\xi y^2 h(t)) dt \tag{3.32}$$

$$y \partial_y(yh) \tag{3.33}$$

The estimate for the third term is immediate, as it is one of the components of d_1^*h . We begin with the first term. We use Muckenhoupt inequalities (Theorem A.4.84) with $U(y) = V(y) = \frac{e^{\xi y}}{\sqrt{y}}$ to obtain

$$\begin{aligned} \int_A^{+\infty} \xi^2 \frac{e^{2\xi y}}{y} \left| \int_y^{+\infty} e^{-\xi t} (y \partial_y y h)(t) dt \right|^2 dy &\leq 4\xi^2 C(\xi) \int_A^{+\infty} |e^{-\xi y} ((y \partial_y y h)(t))|^2 \frac{e^{2\xi y}}{y} dy \\ &\leq 4C(\xi) \int_A^{+\infty} |y \partial_y y h|^2 \frac{dy}{y} \\ &\leq 4\xi^2 C(\xi) \|d_1^*h\|_{L^2}^2 \end{aligned}$$

with $C(\xi) = \xi^2 \sup_{r>A} \int_A^r \frac{e^{2\xi y}}{y} dy \int_r^{+\infty} y e^{-2\xi y} dy = \sup(F(2\xi r) - F(2\xi A)) \frac{e^{-2\xi r} (2\xi r + 1)}{4\xi^2}$, with $F(y) := \int_1^y \frac{e^t}{t} dt$ the primitive of e^t/t considered previously. Similar computations for the second term gives

$$\int_A^{+\infty} \xi^2 \frac{e^{2\xi y}}{y} \left| \int_y^{+\infty} e^{-\xi t} (\xi t^2) h(t) dt \right|^2 dy \leq 4C(\xi) \|d_1^*h\|_{L^2}^2.$$

The function $C(\xi)$ is also the one encountered in the first step, and we already know that we can bound it by a measurable function 2π -periodic that is finite almost everywhere by (3.30).

One still needs to control the L^2 norm of $d_0 r$ in terms of the L^2 norm of $\square_2(h)$. Since $-\partial_y r - \xi r = \square_2(h)$, it suffices to find a suitable $C(\xi)$ such that

$$\|\xi r\|_{L^2(\mathbb{R}_{>A}, ydy)} \leq C(\xi) \|\square_2(\xi)h(\xi)\|_{L^2(\mathbb{R}_{>A}, ydy)}$$

One has for $\xi > 0$

$$\xi r(y) = \xi e^{\xi y} \int_y^{+\infty} e^{-\xi t} \square_2(\xi) h(t) dt.$$

Applying Theorem A.4.84 again with $U(y) = V(y) = e^{\xi y} \sqrt{y}$ we find

$$\|\xi r\|_{L^2(\mathbb{R}_{>A}, y dy)}^2 \leq 4\xi^2 C(\xi) \|\square_2(\xi) h\|_{L^2(\mathbb{R}_{>A}, y dy)}^2$$

with $C(\xi) = \sup \int_A^y e^{2\xi t} t dt \int_y^{+\infty} \frac{e^{-2\xi t}}{t} dt$, and we have already shown that we can bound $4\xi^2 C(\xi)$ by a 2π -periodic measurable function, finite almost everywhere, which gives the desired estimate for $\|d_0 r\|$, and we only have to use Lemma A.3.76 to conclude the proof of the surjectivity of $dz \wedge : \mathcal{U}(\mathbb{Z}) \otimes L^2 DR^0(L_1) \rightarrow L^2 DR^1(L_{-1})/Z^1$.

Step 3 : The vanishing of the cohomology of the kernel. The kernel of $dz \wedge$ is given by

$$0 \longrightarrow \mathcal{U}(\mathbb{Z}) \otimes \int_{\mathbb{R}_{>0}}^{\oplus} \mathbb{C} e^{-\xi y} d\xi \xrightarrow{d_0} \mathcal{U}(\mathbb{Z}) \otimes \int_{\mathbb{R}_{>0}}^{\oplus} \mathbb{C} e^{-\xi y} \begin{pmatrix} 1 \\ i \end{pmatrix} d\xi \longrightarrow 0$$

By computing $d_0(\xi) e^{-\xi y}$ we find that $d_0(\xi)$ is given by $i\xi \begin{pmatrix} 1 \\ i \end{pmatrix}$, which is an isomorphism after tensorization with $\mathcal{U}(\mathbb{Z})$ so this complex is acyclic. By considering the long exact sequence in cohomology associated with the following short exact sequence of complexes and using the flatness of $\mathcal{U}(\mathbb{Z})$

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U}(\mathbb{Z}) \otimes \int_{\mathbb{R}_{>0}}^{\oplus} \mathbb{C} e^{-\xi y} d\xi & \xrightarrow{d_0} & \mathcal{U}(\mathbb{Z}) \otimes \int_{\mathbb{R}_{>0}}^{\oplus} \mathbb{C} e^{-\xi y} \begin{pmatrix} 1 \\ i \end{pmatrix} d\xi & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U}(\mathbb{Z}) \otimes L^2 DR^0(L_1) & \xrightarrow{d_0} & \mathcal{U}(\mathbb{Z}) \otimes Z^1(L_1) & \longrightarrow & 0 \\ & & \downarrow dz \wedge & & \downarrow dz \wedge & & \\ 0 & \longrightarrow & \mathcal{U}(\mathbb{Z}) \otimes L^2 DR^1(L_{-1})/Z^1 & \xrightarrow{d_1} & \mathcal{U}(\mathbb{Z}) \otimes Z^2(L_{-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

we find that $dz \wedge : \mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\mathbb{H}_A, L_1) \rightarrow \mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^2(\mathbb{H}_A, L_{-1})$ is an isomorphism which concludes the proof of the lemma. \square

With this lemma, the proof of Theorem 3.3.21 follows from a spectral sequence argument.

Proof of Theorem 3.3.21. As in the Zucker's paper [Zuc79] we compute the cohomology germs using the spectral sequence induced by the monodromy filtration W_\bullet . Thanks to the existence of L^2 adapted basis flagged according to the filtration W_\bullet we have

$$\mathrm{Gr}_W^{-p} L^2 DR_{\max}^\bullet(\pi^{-1}(U), \pi^* \mathbb{V}, \omega_{pc}, h) = L^2 DR_{\max}^\bullet(\pi^{-1}(U), \mathrm{Gr}_W^{-p}, \omega_{pc}, h)$$

so the spectral sequence becomes

$$E_1^{p,q} = \mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^{p+q}(\mathbb{H}_A, \mathrm{Gr}_W^{-p}) \rightarrow \mathcal{U}(\mathbb{Z}) \otimes \mathrm{Gr}_W^{-p} H^{p+q}(\mathbb{H}_A, \pi^* \mathbb{V},).$$

Thanks to Lemma 3.3.22, we find that the $E_1^{p,q}$ vanishes unless $p = -1$ and $p + q = 2$ or $p = 1$ and $p + q = 2$.

The first differential of the spectral complex d_1 vanishes as the logarithm of the monodromy N lowers the weight of the filtration W_\bullet by two. The second differential d_2 is given by $dz \wedge \otimes N$ and induces an isomorphism

$$\mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^1(\mathbb{H}_A, \mathrm{Gr}_W^1) \longrightarrow \mathcal{U}(\mathbb{Z}) \otimes H_{L^2}^2(\mathbb{H}_A, \mathrm{Gr}_W^{-1})$$

by Lemma 3.3.23. It implies the vanishing of all cohomology groups. \square

Chapter 4

The L^2 complexes of a pVHS on the covering of an open algebraic curve

In this chapter, we want to study the L^2 -cohomology of a polarised variation of Hodge structure $(\mathbb{V}, F^\bullet, Q)$ on a covering of a smooth open algebraic curve M that we assume to be embedded in a compact Riemann surface X . The metric considered on M will again be a metric ω_{p_c} having Poincaré singularities. Consider $\pi: \tilde{M} \rightarrow M$ a Galois covering of M of Deck group Γ . As in Chapter 2, Section 3, we begin by providing a sheaf-theoretic interpretation of L^2 -cohomology in the first section and prove a Poincaré lemma. The second section will be dedicated to the proof of a Dolbeault lemma, and we will see how it implies a holomorphic version of the L^2 -Poincaré lemma.

1 The global L^2 complexes on a smooth open curve.

We consider X a compact Riemann surface, $\Sigma \subset X$ a finite set of points, and set $M := X \setminus \Sigma$. We endow M with a metric ω_{p_c} with Poincaré singularities. We recall that it is a metric ω_{p_c} for which any puncture $p \in \Sigma$ admits a neighbourhood U such that one has a holomorphic quasi-isometry

$$(U \cap M, \omega_{p_c}) \rightarrow (\Delta_r, \frac{dz \wedge d\bar{z}}{|z|^2 (\ln|z|)^2}).$$

Such a metric always exists (see [Zuc79, Proposition 3.2]), and in the case of a hyperbolic Riemann surface, one can take the standard hyperbolic metric on M . It is straightforward, using the compacity of X and the asymptotic behaviour of the metric, to verify the following.

Proposition 4.1.1. The Riemann surface (M, ω_{p_c}) is complete and has finite volume.

We will now consider a Galois covering $\pi: \tilde{M} \rightarrow M$ (unlike Chapter 2, Section 3, we do not require this covering to come from a covering of X) and we denote by Γ its covering group. Recall that in Chapter 2, we have defined the L^2 de Rham complex $L^2 DR^\bullet(\tilde{M}, \pi^* \mathbb{V})$ and the L^2 Dolbeault complex $L^2 \text{Dolb}^{p, \bullet}(\tilde{M}, \pi^* \mathbb{V})$, which are defined as $\mathcal{N}(\Gamma)$ -Hilbert complexes after choosing an ideal boundary condition. Fortunately, by Proposition 2.5.42, one has the following.

Proposition 4.1.2. The complex $L^2 DR^\bullet(\tilde{M}, \pi^* \mathbb{V})$ has a unique ideal boundary condition.

Also recall that by Simpson's basic estimate Theorem 1.3.34 one has the following.

Proposition 4.1.3. The Higgs field $\theta: L^2 DR^\bullet(M, \mathbb{V}) \rightarrow L^2 DR^\bullet(M, \mathbb{V})$ is a bounded operator.

In particular, this implies

Proposition 4.1.4. For $p \in \mathbb{Z}$, the complex $L^2 \text{Dolb}^{p, \bullet}(\tilde{M}, \pi^* \mathbb{V})$ admits a unique ideal boundary condition.

We are interested in the $\mathcal{N}(\Gamma)$ -Fredholmness of these complexes. The uniqueness of the ideal boundary condition implies the equality of the Laplace operator $\square_D = \square_{D'}$ and is sufficient to check the $\mathcal{N}(\Gamma)$ -Fredholmness of the de Rham complex $L^2 DR^\bullet(\tilde{M}, \pi^* \mathbb{V})$.

A generic lemma. Take M a Riemannian manifold and \mathbb{V} a local system on M endowed with a Hermitian metric that is not necessarily flat. Set V to be an open subset of M , and we set $U = M \setminus \bar{V}$. The restriction of a \mathbb{V} -valued form on M to V defines a morphism of Hilbert complex

$$r: L^2DR_{\max}^\bullet(M, \mathbb{V}) \rightarrow L^2DR_{\max}^\bullet(V, \mathbb{V}).$$

If ϕ is a smooth \mathbb{V} -valued form with compact support on U , we set $e(\phi)$ to be the form on M defined by the extension by 0 of ϕ . The morphism e is bounded for both the L^2 -norm and the graph norm of D_{\min} , it also commutes with the differential hence it defines a morphism of complexes

$$e: L^2DR_{\min}^\bullet(U, \pi^*\mathbb{V}) \rightarrow L^2DR_{\max}^\bullet(M, \pi^*\mathbb{V}).$$

In the case where M is complete, and the boundary is collared, one has the following.

Lemma 4.1.5. Let M be a complete manifold and \mathcal{V} be a flat Hermitian vector bundle on M . Consider $U \subset M$ to be an open of M and $V = M \setminus \bar{U}$. Assume that the boundary ∂U is collared, that is, it admits a neighbourhood W quasi-isometric to $(-1, 1) \times \partial U$ such that $(t, x) \in U$ if and only if $t < 0$. We also require that on the collar, the Hermitian metric is quasi-isometric to a metric h that depends only on ∂U . Then the following sequence of $\mathcal{N}(\Gamma)$ -Hilbert complexes is exact

$$0 \rightarrow L^2DR_{\min}(U, \mathbb{V}) \xrightarrow{e} L^2DR^\bullet(M, \mathbb{V}) \xrightarrow{r} L^2DR_{\max}^\bullet(V, \mathbb{V}) \rightarrow 0.$$

Moreover, if e^*, r^* denote the induced morphisms in cohomology, then e^* and r^* are continuous for the quotient topology on the cohomology groups and e^* sends harmonic forms to harmonic forms.

Proof. The continuity of e and r is immediate, as is the fact that e sends harmonic forms to harmonic forms. We begin by proving that the kernel of the restriction morphism r coincides with the range of the morphism e . We take a form $\omega \in \text{Ker}(r)$, we want to approximate it in the graph norm of D by smooth sections with compact support. Take a collared neighbourhood $W \simeq (-1, 1) \times \partial U$ of the boundary of U .

For $\varepsilon > 0$, set $\varphi_\varepsilon: [-1, 1] \rightarrow [-1, 1]$ to be a smooth function such that

- φ_ε induces a diffeomorphism between $(-1 + \varepsilon, -\varepsilon)$ and $(-1, 0)$
- $\varphi_\varepsilon = -1$ on $[-1, -1 + \varepsilon]$ and $\varphi_\varepsilon = 0$ on $[-\varepsilon, 0]$
- φ_ε is the identity on $[0, 1]$

Define the function $\psi_\varepsilon: M \rightarrow M$ by

$$\psi_\varepsilon(p) = \begin{cases} p & \text{if } p \in M \setminus W \\ (\varphi_\varepsilon(t), x) & \text{if } p = (t, x) \in W \end{cases}$$

One of the remaining problems is for $\psi_\varepsilon^*\omega$ to be a \mathbb{V} -valued section : for $p = (t, x) \in W$, $\psi_\varepsilon^*\omega(p)$ lies in the fibre $(\varphi_\varepsilon(t), x) \neq (t, x)$ and one has to consider an isometry between these two fibres. Using the collared neighbourhood of ∂U one has a natural candidate for such an isometry, one can consider the morphism induced by the straight path $\gamma_{\varepsilon, p}$ from (t, x) to $(\varphi_\varepsilon(t), x)$. The convergence of the form $p \mapsto \gamma_\varepsilon(p)(\psi_\varepsilon^*\omega)(p)$ to ω in the graph norm of D_{\min} is then straightforward. The form $\psi_\varepsilon^*\omega$ may not have a compact support; fortunately, the completeness of M implies that the space of smooth forms with compact support is dense for the graph norm of d , so we can approximate each $\psi_\varepsilon^*\omega$ by smooth forms ω_n with a support that is compact in M . We can assume that the support of ω_n is contained in a neighbourhood of the support of $\psi_\varepsilon^*\omega$ and, since the support of ψ_ε is strictly contained in U , $\psi_\varepsilon^*\omega$ is approximated by smooth forms whose support is compact in U . Hence $\omega \in L^2DR_{\min}^\bullet(U, \mathbb{V})$.

We still have to prove the surjectivity. It is proven in [Che80, Lemma 4.2] for the case of the trivial bundle with trivial metric. The general case is similar; we recall the principle of the proof below. We will construct a section s of the restriction morphism r . Set $\chi: [0, 1] \rightarrow [0, 1]$ to be a smooth function such that $\chi = 1$ in a neighbourhood of 0 and $\chi = 0$ in a neighbourhood of 1. We can then define $s(\omega) = \omega$ on V , $s(\omega) = 0$ on $U \setminus W$ and $s(\omega)(t, x) = \chi(t)\gamma_{t, x}(\omega(-t, x))$ for $(t, x) \in W \setminus V$ where γ_p is the isomorphism between $\mathbb{V}_{(t, x)}$ and $\mathbb{V}_{(-t, x)}$ induced by the straight path. The section s is then continuous with respect to the L^2 norm and the graph norm of D . \square

On the $\mathcal{N}(\Gamma)$ -Fredholmness. We now return to our setting with M being a smooth open curve embedded in a compact Riemann surface X . We wish to apply Lemma 4.1.5. For this we take U to be an open such that

- U is relatively compact in M and its boundary is smooth.
- $M \setminus U$ is quasi-isometric to a finite union of punctured disks $(\Delta_r^*, \omega_{P_C})$.

In this case, $L^2DR_{\min}^\bullet(\pi^{-1}(U), \pi^*\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm due to Proposition 2.4.39 and $L^2DR_{\max}^\bullet(\pi^{-1}(M \setminus \bar{U}), \pi^*\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm according to Theorem 3.1.1. Then from Corollary 2.2.23 one has

Theorem 4.1.6. The $\mathcal{N}(\Gamma)$ -Hilbert complexes $L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V})$ and $L^2Dolb^{P,\bullet}(\tilde{M}, \pi^*\mathbb{V})$ are $\mathcal{N}(\Gamma)$ -Fredholm.

From this result, it follows that

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}) \simeq \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \text{Harm}_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}).$$

Recall that one has a complex Hodge structure of weight $k + w$ on the space of harmonic forms $\text{Harm}_{L^2}^k(\tilde{M}, \pi^*\mathbb{V})$, obtained by setting

$$\text{Harm}^k(\tilde{M}, \pi^*\mathbb{V})^{P,Q} = \text{Harm}^k(\tilde{M}, \pi^*\mathbb{V}) \cap \mathcal{E}^k(\pi^*\mathbb{V})^{P,Q} \quad \text{for } P + Q = k + w.$$

From this one obtains the

Theorem 4.1.7. The spaces $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V})$ admits a pure Hodge structure of weight $k + w$ induced by the identification with the spaces of harmonic forms.

This Hodge structure obtained from the identification with the space of harmonic forms is of analytic nature, and we will denote it by F_{an}^\bullet . We will show later that it is also of algebraic nature.

2 The L^2 -Poincaré lemma.

The sheaf $\ell^2\pi^*\mathbb{V}$. In this paragraph, we construct the sheaf $\ell^2\pi^*\mathbb{V}$, it will be a sheaf defined on X that "parametrises" the flat L^2 section of $\pi^*\mathbb{V}$ on \tilde{M} .

Definition 4.2.8. We define the sheaf $\ell^2\pi^*\mathbb{V}$ by

$$\ell^2\pi^*\mathbb{V}(U) = \{s \in \pi^*\mathbb{V}(\pi^{-1}(U \cap M)) \mid \int_{\pi^{-1}(K \cap M)} h(s, s) d\text{Vol} < +\infty \forall K \Subset U\}$$

Remark 4.2.9. This sheaf is a weakly constructible sheaf of $\mathcal{N}(\Gamma)$ -modules in the sense of [KS90, Definition 8.1.3], i.e. there exists a realisation of X as a simplicial complex \mathbf{S} such that $\ell^2\pi^*\mathbb{V}$ is constant on each simplex of \mathbf{S} .

The complex $\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})$. For an open $U \subset X$, the complex $(\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})(U), D)$ is the complex of $\pi^*\mathbb{V}$ -valued measurable k -forms ϕ over $\pi^{-1}(U \cap M)$ with ϕ and $D\phi$ square-integrable on $\pi^{-1}(K \cap M)$ for all compact $K \subset U$ (here D is computed in the sense of distribution). The presheaves

$$U \mapsto \mathcal{L}^2DR^k(\pi^*\mathbb{V})(U)$$

are actually sheaves. Since a smooth form on X has a bounded norm for the metric ω_{P_C} , if χ is a smooth function and $\phi \in \mathcal{L}^2DR^k(\pi^*\mathbb{V})$ the form

$$\chi \cdot \phi = (\chi \circ \pi)\phi$$

is also in $\mathcal{L}^2DR^k(\pi^*\mathbb{V})$ and there is a natural structure of C_X^∞ -modules on the sheaves $\mathcal{L}^2DR^k(\pi^*\mathbb{V})$. It implies the following.

Lemma 4.2.10. The complex $\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})$ is a complex of fine sheaves, and the following equality holds.

$$\mathbb{H}^\bullet(X, \mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})) = H_2^\bullet(\tilde{M}, \pi^*\mathbb{V}, \omega_{P_C}, \pi^*h).$$

The smooth subcomplex $\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})$. For each open $U \subset X$, we can consider the Laplace operator $\square_D = (D + \mathfrak{d})^2$ computed in the sense of distribution. We set $\mathcal{L}^2DR_0^k(\pi^*\mathbb{V}) = \mathcal{L}^2DR^k(\pi^*\mathbb{V})$, we can define the sheaves $\mathcal{L}^2DR_j^k(\pi^*\mathbb{V})$, where $\phi \in \mathcal{L}^2DR_{j+1}^k(\pi^*\mathbb{V})(U)$ if and only if $\phi \in \mathcal{L}^2DR_j^k(\pi^*\mathbb{V})(U)$ and $D\square_D^j\phi$ and $\mathfrak{d}\square_D^j\phi$ are square-integrable on $\pi^{-1}(K \cap U)$ for any compact $K \subset U$. We define

$$\mathcal{L}^2DR_\infty^k(\pi^*\mathbb{V}) := \bigcap_{j \in \mathbb{N}} \mathcal{L}^2DR_j^k(\pi^*\mathbb{V})$$

which gives a subcomplex of sheaves $\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})$. It is not clear whether it is a sheaf of C_X^∞ -modules in general, since the covariant derivatives of the metric ω_{p_c} are involved in the computation of the adjoint \mathfrak{d} . However, one still has the following result.

Lemma 4.2.11. The complex $\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})$ is a complex of soft sheaves. And we have

$$H^\bullet(X, \mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})) = H_{(2)}^\bullet(\tilde{M}, \pi^*\omega_{p_c}, \pi^*\mathbb{V}, \pi^*h).$$

Proof. The last assertion of the lemma comes from the fact that soft sheaves are acyclic, so its hypercohomology groups coincide with the cohomology groups of the complex $\Gamma\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V}) = L^2DR_\infty^\bullet(\tilde{M}, \pi^*\mathbb{V})$ and the conclusion follows from Lemma 2.1.10. To prove the softness by [God60, Theorem 3.4.1] we need to check that every point $p \in X$ admits an open neighbourhood U such that if $K \subset U$, with K closed in X , then the restriction of global sections to K is surjective. Take U a neighbourhood of a puncture, with $U \cap M \simeq \Delta_r^*$, and $K \subset U$ a closed subset of X , and $\phi \in \mathcal{L}^2DR^k(\pi^*\mathbb{V})(K)$. The form ϕ is defined on an open neighbourhood W of K that we can assume to be relatively compact on U . One can take a smooth function χ , where $\chi = 1$ on a neighbourhood of K and $\text{supp}(\chi) \subset W$, then $(\chi \circ \pi)\phi$ is a form that extends ϕ to $\pi^{-1}(U)$ and is also an element of $\mathcal{L}^2DR_\infty^k(\pi^*\mathbb{V})(U)$. Thus, the sheaf $\mathcal{L}^2DR_\infty^k(\pi^*\mathbb{V})$ is soft. \square

Lemma 4.2.12. The inclusion

$$\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V}) \rightarrow \mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})$$

is a quasi-isomorphism.

Proof. For a point $p \in X$ we will denote it by \mathcal{H}_p^k (resp. $\mathcal{H}_{\infty,p}^k$) the cohomology germs of the complex $\mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})$ (resp. $\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})$). If $p \in M$, it is easy to see that both \mathcal{H}_p^k and $\mathcal{H}_{\infty,p}^k$ vanish if $k \neq 0$ and are isomorphic to \mathbb{C} if $k = 0$. Thus, we focus on the case of $p \in X \setminus M$.

Let ϕ be the germ of a closed form in $\mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})_p$, if there exists $\psi \in \mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})_p$ such that $D\psi = \phi$, one can take a neighbourhood U of p , such that ϕ and ψ are square-integrable in $\tilde{U} := \pi^{-1}(U \cap M)$. By restricting the open set U , if necessary, we can view ϕ and ψ as elements of the Hilbert space $L^2DR^\bullet(\pi^{-1}(U \cap M), \pi^*\mathbb{V})$, satisfying $\phi = D_{\max}\psi$. By the weak Hodge decomposition, we have

$$L^2DR^\bullet(\pi^{-1}(U \cap M), \pi^*\mathbb{V}) = \text{Harm}^\bullet(\pi^{-1}(U \cap M), \pi^*\mathbb{V}) \oplus \overline{\text{Ran}(D_{\max})} \oplus \overline{\text{Ran}(\mathfrak{d}_{\min})}$$

where $\text{Harm}^\bullet(\pi^{-1}(U \cap M), \pi^*\mathbb{V})$ is the space of forms that are D_{\max} -closed and \mathfrak{d}_{\min} -closed. One can set $\psi_0 = \text{pr}_{\overline{\text{Ran}(\mathfrak{d}_{\min})}}\psi$ which is square-integrable on $\pi^{-1}(U \cap M)$ and satisfies $D_{\max}\psi_0 = D_{\max}\psi = \phi$. We set the operator $\square_{D_{\max}} = (D_{\max} + \mathfrak{d}_{\min})^2$ and for all $j \in \mathbb{N}$ one has

$$\square_{D_{\max}}^{j+1}\psi_0 = \mathfrak{d}_{\min}\square_{D_{\max}}^j D_{\max}\psi_0 = \mathfrak{d}_{\min}\square_D^j\phi.$$

We infer that $\square_{D_{\max}}^j\psi_0$ is defined for all $j \in \mathbb{N}$ and we obtain $\psi_0 \in \mathcal{L}^2DR_\infty^\bullet(\pi^*\mathbb{V})_p$ and $D\psi_0 = \phi$. Thus, the inclusion induces an injection in cohomology.

Regarding the surjectivity, if $\phi \in \mathcal{L}^2DR^\bullet(\pi^*\mathbb{V})_p$ is a closed form, we fix U a neighbourhood of p such that ϕ is a closed square-integrable form on $\tilde{U} := \pi^{-1}(U \cap M)$. By [BL92, Theorem 2.12], there exists a closed form ψ also square-integrable on $\pi^{-1}(U \cap M)$ in the cohomology class of ϕ , that satisfies

$$\psi \in \bigcap_{j \in \mathbb{N}} \text{Dom}(\square_{D_{\max}}^j)$$

and one obtains the surjectivity. \square

The Poincaré lemma. This paragraph is dedicated to the main result of this section, namely,

Theorem 4.2.13 (L^2 Poincaré lemma). The complexes of sheaves $\mathcal{L}^2 DR^\bullet(\pi^*\mathbb{V})$ and $\mathcal{L}^2 DR_\infty^\bullet(\pi^*\mathbb{V})$ are fine resolutions of $\ell^2 \pi^*\mathbb{V}$. In particular

$$H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}) = H^k(X, \ell^2 \pi^*\mathbb{V}).$$

Proof. We have inclusions

$$\ell^2 \pi^*\mathbb{V} \rightarrow \mathcal{L}^2 DR_\infty^\bullet(\pi^*\mathbb{V}) \rightarrow \mathcal{L}^2 DR^\bullet(\pi^*\mathbb{V})$$

the last one being a quasi-isomorphism; we only need to prove the result for $\mathcal{L}^2 DR^\bullet(\pi^*\mathbb{V})$. We want to show that at each point $p \in X$ the cohomology germs at p vanish. In the case $p \in M$, p admits a basis of neighbourhood quasi-isometric to Euclidean disks and the result follows from Lemma 2.4.35. In the case of $p \in X \setminus M$, it admits a basis of neighbourhood quasi-isometric to punctured disks $(\Delta_R, \frac{dz \wedge d\bar{z}}{|z|^2 |\ln(|z|^2)|^2})$ and the result comes from Theorem 3.1.1. \square

3 The Dolbeault complexes and the Dolbeault lemma

The holomorphic de Rham complex $\Omega_{L^2}^\bullet(\pi^*\mathbb{V})$ and its filtration.

Definition 4.3.14. The holomorphic de Rham complex $\Omega_{L^2}^\bullet(\pi^*\mathbb{V})$ is the subcomplex of the L^2 de Rham complex $\mathcal{L}^2 DR^\bullet(\pi^*\mathbb{V})$ consisting of holomorphic forms.

Here, we want to use the Hodge filtration F^\bullet of our variation of Hodge structure to have a filtration of the holomorphic de Rham complex. We define this filtration by

$$F^p \Omega_{L^2}^\bullet(\pi^*\mathbb{V}) := 0 \longrightarrow F^p \cap \Omega_{L^2}^0(\pi^*\mathbb{V}) \xrightarrow{D^{1,0}} \Omega^1(\pi^* F^{p-1}) \cap \Omega_{L^2}^1(\pi^*\mathbb{V}) \longrightarrow 0$$

The grading of the Hodge filtration behaves well when taking L^2 -form, namely one has from [Zuc79, Proposition 5.2]

Proposition 4.3.15. One has the equality

$$\text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V}) = \Omega_{L^2}^\bullet(\text{Gr}_{p-\bullet})$$

where $\Omega_{L^2}^k(\text{Gr}_{p-k})$ is the space of Gr_{p-k} -valued holomorphic k -forms that are square-integrable.

The Dolbeault complexes $\mathcal{L}^2 \text{Dolb}^{p,\bullet}$ and their smooth subcomplexes. In this paragraph, we study a sheaf version of the Dolbeault complexes.

Definition 4.3.16. For $p \in \mathbb{Z}$, we define the Dolbeault complex $\mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$ by taking sections of $\mathcal{L}^2 \text{Dolb}^{p,k}(\pi^*\mathbb{V})$ over an open $U \subset X$ to $\pi^* \text{Gr}_F^{p-k}$ -valued forms ϕ on $\pi^{-1}(U \cap M)$ with ϕ and $D''\phi$ square-integrable on $\pi^{-1}(K \cap M)$ for any compact K of U . The differential of the complex is the morphism induced by D'' .

Remark 4.3.17. The square-integrability of $D''\phi$ is equivalent to the square-integrability of $\bar{\partial}\phi$ since θ is bounded in the L^2 norm.

Similarly to the de Rham complex, the Dolbeault complex $\mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$ has a natural structure of complexes of sheaves of C_X^∞ -modules; in particular, they are complexes of fine sheaves; hence, one has the following proposition.

Proposition 4.3.18.

$$H_{L^2}^k(\mathcal{L}^2 \text{Dolb}_{\max}^{p,\bullet}) = \mathbb{H}^k(\mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V}))$$

Again, we can consider the smooth subcomplex $\mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V})$ whose definition is given below. By induction on $j \in \mathbb{N}$, we set $\mathcal{L}^2 \text{Dolb}_0^{p,\bullet}(\pi^*\mathbb{V}) = \mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$ and $\mathcal{L}^2 \text{Dolb}_{j+1}^{p,\bullet}(\pi^*\mathbb{V})$ as the subcomplex of $\mathcal{L}^2 \text{Dolb}_j^{p,\bullet}$ consisting of forms ϕ such that $\square_{D''}\phi \in \mathcal{L}^2 \text{Dolb}_j^{p,\bullet}$. The smooth subcomplex is finally defined by

$$\mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V}) = \bigcap_j \mathcal{L}^2 \text{Dolb}_j^{p,\bullet}(\pi^*\mathbb{V})$$

A proof similar to the one used for the smooth de Rham complex yields the

Proposition 4.3.19. The complex $\mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V})$ is a complex of soft sheaves and the inclusion

$$\mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V}) \rightarrow \mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$$

is a quasi-isomorphism.

An interesting fact is the relationship between the smooth subcomplex $\mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V})$ and $\mathcal{L}^2 \text{DR}_\infty^\bullet(\pi^*\mathbb{V})$.

Proposition 4.3.20. One has a natural isomorphism of complex

$$\text{Gr}_{\pi^*F}^P \mathcal{L}^2 \text{DR}_\infty^\bullet(\pi^*\mathbb{V}) \rightarrow \mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V}).$$

The L^2 Dolbeault lemma. In this paragraph, we are interested in proving the following Dolbeault lemma.

Theorem 4.3.21 (L^2 Dolbeault lemma). The complexes $\mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$ and $\mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V})$ are soft resolutions of the complex $\text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V})$.

Before giving the proof, we will discuss a consequence of this Dolbeault lemma. Recall that one has a canonical isomorphism

$$\text{Gr}_{\pi^*F}^P \mathcal{L}^2 \text{DR}_\infty^\bullet(\pi^*\mathbb{V}) \rightarrow \mathcal{L}^2 \text{Dolb}_\infty^{p,\bullet}(\pi^*\mathbb{V}).$$

This implies that the complex $\mathcal{L}^2 \text{DR}_\infty^\bullet(\pi^*\mathbb{V})$ is a graded resolution of $\Omega_{L^2}^\bullet(\pi^*\mathbb{V})$, (hence a resolution). Using that $\mathcal{L}^2 \text{DR}_\infty^\bullet(\pi^*\mathbb{V})$ also is a resolution of $\ell^2 \pi^*\mathbb{V}$ one obtains the following.

Theorem 4.3.22 (Holomorphic Poincaré lemma). The natural map

$$\ell^2 \pi^*\mathbb{V} \rightarrow \Omega_{L^2}^\bullet(\pi^*\mathbb{V})$$

is a quasi-isomorphism.

Now we will proceed with the proof of Theorem 4.3.21. It is sufficient to prove the result for the complex $\mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$. The complex $\mathcal{L}^2 \text{Dolb}^{p,\bullet}(\pi^*\mathbb{V})$ is the simple complex associated to the double complex

$$\begin{array}{ccc} \mathcal{L}^2 \text{Dolb}^{0,0}(\pi^* \text{Gr}_F^p) & \xrightarrow{\theta} & \mathcal{L}^2 \text{Dolb}^{1,0}(\pi^* \text{Gr}_F^{p-1}) \\ \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\ \mathcal{L}^2 \text{Dolb}^{0,1}(\pi^* \text{Gr}_F^p) & \xrightarrow{\theta} & \mathcal{L}^2 \text{Dolb}^{1,1}(\pi^* \text{Gr}_F^{p-1}) \end{array}$$

Where $\mathcal{L}^2 \text{Dolb}^{r,s}(\pi^* \text{Gr}_F^p)$ is the subsheaf of $\mathcal{L}^2 \text{Dolb}^\bullet(\pi^*\mathbb{V})$ consisting of $\pi^* \text{Gr}_F^p$ -valued (r,s) forms. To have a quasi-isomorphism it suffices to check that the following sequences are exact

$$0 \longrightarrow \mathcal{O}(\pi^* \text{Gr}_F^p)_{(2)} \longrightarrow \mathcal{L}^2 \text{Dolb}^{0,0}(\pi^* \text{Gr}_F^p) \xrightarrow{\bar{\partial}} \mathcal{L}^2 \text{Dolb}^{0,1}(\pi^* \text{Gr}_F^p) \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow \Omega^1(\pi^* \text{Gr}_F^{p-1})_{(2)} \longrightarrow \mathcal{L}^2 \text{Dolb}^{1,0}(\pi^* \text{Gr}_F^{p-1}) \xrightarrow{\bar{\partial}} \mathcal{L}^2 \text{Dolb}^{1,1}(\pi^* \text{Gr}_F^{p-1}) \longrightarrow 0$$

It follows that in the case of the point of M is thus reduced to the problem of the $\bar{\partial}$ on the disk with the Euclidean metric, in this case it is well known that the above sequences are exact.

The only problems that arise are at points $p \in \Sigma$. In this case, p admits a neighbourhood basis open U quasi-isometric to a punctured Δ_R^* endowed with the Poincaré metric. There are two cases to consider, depending on whether the element $\gamma \in \Gamma$, defined by the meridian circle around p , has finite order or not. If γ has finite order, $\pi^{-1}(U)$ is quasi-isometric to a disjoint union of punctured disks Δ_R^* endowed with the Poincaré metric. By an induction argument, we can reduce the study to the case where $\pi^{-1}(U)$ is a punctured disk Δ_R^* . If γ has infinite order, $\pi^{-1}(U)$ is quasi-isometric to a disjoint union of horodisks \mathbb{H}_A and by an induction argument one can reduce the study to the case where $\pi^{-1}(U)$ is a horodisk.

We begin with the case where γ has finite order. Since we are only dealing with a punctured disk, it is a consequence of results from Zucker [Zuc79, Proposition 6.4, Proposition 11.5]. The proof relies on the following lemma, which is due to Zucker.

Lemma 4.3.23. [Zuc79, Proposition 11.5] If (\mathcal{V}, h) is a holomorphic Hermitian line bundle on a disk Δ_R of radius $R < 1$, admitting a generating section $\tilde{\xi}$ satisfying

$$C^{-1}r^{2\beta}|\ln r|^k \leq h(\tilde{\xi}, \tilde{\xi}) \leq Cr^{2\beta}|\ln r|^k$$

for an integer number k , $-1 < \beta \leq 0$ with $\beta, (k-1) \neq 0$ and a real constant $C > 1$. Then for any square-integrable holomorphic one form $fd\bar{z} \otimes \tilde{\xi}$, there exists a holomorphic one form $g \otimes e$ satisfying

$$\begin{aligned} (\bar{\partial}g) \otimes \tilde{\xi} &= fd\bar{z} \otimes \tilde{\xi} \\ \|g \otimes \tilde{\xi}\|_2^2 &\leq K\|fd\bar{z} \otimes \tilde{\xi}\|_2^2 \end{aligned}$$

where K is a positive constant depending only on β , R and C .

Proof of Theorem 4.3.21 in the case where γ has finite order. For a point $p \in X \setminus M$ the sequences (4.1) may not be exact, and we need to slightly adapt our proof. As usual, we take a distinguished neighbourhood U of p , \tilde{U} a connected component $\pi^{-1}(U)$ and consider an L^2 -adapted frame of $\pi^*\mathbb{V}_{\tilde{U}}$ $(\tilde{\xi}_1, \dots, \tilde{\xi}_n)$ constructed in the usual way.

Let η be a closed square-integrable one-form, and $\eta_\gamma = \eta|_{\gamma\tilde{U}}$ have the decomposition by type $\eta_\gamma = \eta_\gamma^{1,0} + \eta_\gamma^{0,1}$. One can write

$$\eta_\gamma^{0,1} = \sum_{j=1}^n u_j d\bar{z} \otimes \tilde{\gamma}^* \tilde{\xi}_j.$$

By Lemma 4.3.23, for each j such that $\|\tilde{\xi}_j\|^2 \in W_k \setminus W_{k-1}$ with $k \neq 1$ there exists v_j such that

$$\begin{aligned} \bar{\partial}v_j \otimes \tilde{\gamma}^* \tilde{\xi}_j &= u_j \otimes \tilde{\gamma}^* \tilde{\xi}_j \\ \|v_j \otimes \tilde{\gamma}^* \tilde{\xi}_j\|^2 &\leq C\|u_j \otimes \tilde{\gamma}^* \tilde{\xi}_j\|^2. \end{aligned}$$

We take v_γ the sum of such forms, then since the basis is L^2 adapted one has

$$\begin{aligned} \|v_\gamma\|^2 &\leq C\|\eta_\gamma^{0,1}\|^2 \leq C\|\eta_\gamma\|^2 \\ \|\theta(v_\gamma)\|^2 &\leq C\|\eta_\gamma^{0,1}\|^2 \leq C\|\eta_\gamma\|^2. \end{aligned}$$

Note that the constant C does not depend on γ since all connected components are isometric, so it defines a square-integrable form v with $D''(v)$ square-integrable. It follows that without loss of generality by replacing η with $\eta - D''v$, one can assume that the terms $\eta_\gamma^{0,1}$ only have terms of the form $ud\bar{z} \otimes \tilde{\xi}_j$ where $\tilde{\xi}_j$ is a section which has asymptotic behaviour $\|\tilde{\xi}_j\|^2 \simeq |\ln(r)|$. The form η being closed, one has

$$\theta(\eta^{0,1}) = d''\eta^{1,0}.$$

And $\theta(\eta^{0,1})$ is a sum of terms of the form $ud\bar{z} \wedge D^{1,0}(\tilde{\xi}_j)$, so $\eta^{1,0}$ needs to be a sum of terms of the form $v_j \otimes D^{1,0}(\tilde{\xi}_j)$ up to a holomorphic 1-form. It follows that we have up to a holomorphic one-form

$$\eta^{1,0} + \eta^{0,1} = D''(\sum v_j \otimes \tilde{\xi}_j)$$

This proves that the morphism

$$Gr_{\pi^*F}^p \Omega^\bullet(\pi^*\mathbb{V})_{(2)} \rightarrow \mathcal{L}^2 \text{Dolb}^{P,\bullet}(\pi^*\mathbb{V})$$

induces a surjection at the cohomological level.

Similarly, if we have a two form η for a fixed γ one has

$$\alpha_\gamma = \sum u_j d\bar{z} \wedge \frac{dz}{z} \otimes \tilde{\gamma}^* \tilde{\xi}_j.$$

Lemma 4.3.23 allows us to find a primitive $v_j dz \otimes \tilde{\gamma}^* \tilde{\xi}_j$ for each term such that $\|\frac{dz}{z} \otimes \tilde{e}_j\|^2 \simeq r^{2\beta} |\ln(r)|^k$ with $\beta, (k-1) \neq 0$. The case $k = 1$ occurs when $\tilde{e}_j \in W_{-1}(\pi^*(\mathbb{V})_0) \setminus W_{-2}((\pi^*\mathbb{V})_0)$ since $\|\frac{dz}{z}\|^2 = |\ln(r)|^2$. The operator θ acts as $1 \wedge \frac{1}{2i\pi} \frac{dz}{z} \otimes \tilde{N}_0$, and

$$\tilde{N}_0: W_1((\pi^*\mathbb{V})_0) \rightarrow W_{-1}((\pi^*\mathbb{V})_0)$$

is surjective, we fix \tilde{f}_j such that $\tilde{N}_0 \tilde{f} = \tilde{e}_j$. And we have

$$2i\pi\theta \left(u_j d\bar{z} \otimes \exp \left(\frac{\tilde{N}_0}{2i\pi} \ln(z) \right) \gamma^* \tilde{f}_j \right) = u_j d\bar{z} \wedge \frac{dz}{z} \otimes \gamma^* \tilde{\xi}_j$$

and we obtain the surjectivity in cohomology. The injectivity being clear, the morphism

$$\mathrm{Gr}_{\pi^*F}^P \Omega^\bullet(\pi^*\mathbb{V})_{(2)} \rightarrow \mathcal{L}^2 \mathrm{Dolb}_{\infty}^{P,\bullet}(\pi^*\mathbb{V}) \simeq \mathrm{Gr}_{\pi^*F}^P L^2 \mathrm{DR}_{\infty}^{P,\bullet}(\pi^*\mathbb{V})$$

is a quasi-isomorphism, and this concludes the proof of Theorem 4.3.21. \square

We still have to give the proof in the case where γ has infinite order.

Proof of the Dolbeault lemma when γ has infinite order. As before, the Dolbeault complex will be a resolution if the following sequences are exact

$$0 \rightarrow \mathcal{U}(\Gamma) \otimes \Omega^0(\pi^*\mathbb{V})_2 \rightarrow \mathcal{U}(\Gamma) \otimes L^2 \mathrm{DR}^{0,0}(\mathrm{Gr}_p^F) \xrightarrow{\bar{\partial}} \mathcal{U}(\Gamma) \otimes L^2 \mathrm{DR}^{0,1}(\mathrm{Gr}_p^F) \rightarrow 0$$

$$0 \rightarrow \mathcal{U}(\Gamma) \otimes \Omega^1(\pi^*\mathbb{V})_2 \rightarrow \mathcal{U}(\Gamma) \otimes L^2 \mathrm{DR}^{1,0}(\mathrm{Gr}_p^F) \xrightarrow{\bar{\partial}} \mathcal{U}(\Gamma) \otimes L^2 \mathrm{DR}^{1,1}(\mathrm{Gr}_p^F) \rightarrow 0$$

In other words, we have to solve the $\bar{\partial}$ -equation. We focus here on the case where the meridian loop has infinite order in Γ , as before, we focus on a connected component of a neighbourhood U of p which is quasi-isometric to a horodisk. For $g \in L^2(\mathbb{H}_A, e^{-4\pi\beta y} y^k dz \wedge d\bar{z})$, we wish to find $f \in L^2(\mathbb{H}_A, e^{-4\pi\beta y} y^{k-2} dz \wedge d\bar{z})$ such that $\partial_{\bar{z}} f = g$. We use the decomposition into direct integral using the Fourier transform as we did for the L^2 -Poincaré Lemma (isomorphisms (3.16))

$$L^2(\mathbb{H}_A, e^{-4\pi\beta y} y^k dz \wedge d\bar{z}) \simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}_{>A}, e^{-4\pi\beta y} y^k dy) d\tilde{\xi}.$$

The operator $\bar{\partial}$ is transformed into $\int_{\mathbb{R}}^{\oplus} \frac{i}{2}(\tilde{\xi} + \partial_y)$. For $g_{\tilde{\xi}} \in L^2(\mathbb{R}_{>A}, e^{-4\pi\beta y} y^k dy)$ we set

$$f_{\tilde{\xi}} := \begin{cases} -2ie^{-\tilde{\xi}y} \int_A^y e^{\tilde{\xi}t} g(t) dt & \text{if } \tilde{\xi} > 0 \\ 2ie^{-\tilde{\xi}y} \int_y^{+\infty} e^{\tilde{\xi}t} g(t) dt & \text{if } \tilde{\xi} < 0 \end{cases}.$$

The Muckenhoupt inequalities give us (cf. Theorem A.4.84)

$$\|f_{\tilde{\xi}}\|^2 \leq 4C(\tilde{\xi}) \|g_{\tilde{\xi}}\|^2$$

with

$$C(\tilde{\xi}) = \begin{cases} \sup_{r>A} \int_A^r e^{(4\pi\beta y + 2\tilde{\xi})y} y^{-k} dy \int_r^{+\infty} e^{-(4\pi\beta + 2\tilde{\xi})y} y^{k-2} dy & \text{if } \tilde{\xi} > -2\pi\beta \\ \sup_{r>A} \int_A^r e^{(4\pi\beta + 2\tilde{\xi})y} y^{k-2} dy \int_r^{+\infty} e^{-(4\pi\beta + 2\tilde{\xi})y} y^{-k} dy & \text{if } \tilde{\xi} < -2\pi\beta \end{cases}$$

We have to show that we can bound $C(\tilde{\xi})$ by a measurable function 2π -periodic and finite almost everywhere, to obtain the surjectivity of $\bar{\partial}$ after tensorization with the algebra of affiliated operators (this follows from Lemma A.3.76 as we did for the Poincaré lemma). We treat the case $\tilde{\xi} > -2\pi\beta$.

$$\begin{aligned} C(\tilde{\xi}) &= \sup_{r>A} \int_A^r e^{(4\pi\beta + 2\tilde{\xi})y} y^{-k} dy \int_r^{+\infty} e^{-(4\pi\beta + 2\tilde{\xi})y} y^{k-2} dy \\ &= \sup_{r>A} \int_{(4\pi\beta + 2\tilde{\xi})A}^{(4\pi\beta + 2\tilde{\xi})r} e^y y^{-k} dy \int_{(4\pi\beta + 2\tilde{\xi})r}^{+\infty} e^{-y} y^{k-2} dy \\ &= \sup_{r>A} \int_{(4\pi\beta + 2\tilde{\xi})A}^{(4\pi\beta + 2\tilde{\xi})r} e^y y^{-k} dy \int_{(4\pi\beta + 2\tilde{\xi})r}^{+\infty} e^{-y} y^{k-2} dy \end{aligned}$$

However, by usual asymptotic behaviour of improper integral for $\varepsilon > 0$, there exists a constant $K(\varepsilon) > 0$ such that for all $r > \varepsilon$

$$\int_{\varepsilon}^r e^y y^{-k} dy \int_r^{+\infty} e^{-y} y^{k-2} dy \leq K(\varepsilon)(e^r r^{-k})(e^{-r} r^{k-2}) \leq K(\varepsilon)/\varepsilon^2.$$

It follows that $C(\xi)$ is uniformly bounded above outside of an arbitrarily small neighbourhood of $2\pi\beta$, therefore we can bound $C(\xi)$ by a 2π -periodic measurable function that is finite almost everywhere and Lemma A.3.76 allows us to conclude the proof. \square

4 Algebraicity of the analytic Hodge structure.

Recall that in the first section we have found an analytic Hodge structure on the cohomology group $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V})$. However, in the previous section we have also found that the complex $\mathcal{L}^2 DR^\bullet(\pi^*\mathbb{V})$ is a fine resolution of the holomorphic de Rham complex $\Omega_{L^2}^\bullet(\pi^*\mathbb{V})$. From this one obtains a spectral sequence

$$E_1^{p,q} = \mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, \text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V})) \implies \mathcal{U}(\Gamma) \otimes H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}, \omega_{P_C}, h).$$

This spectral sequence induces a filtration on the spaces $\mathcal{U}(\Gamma) \otimes H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}, \omega_{P_C}, h)$. We will call this filtration the algebraic Hodge filtration and denote it by F_{alg}^\bullet . We are interested in the properties of this spectral sequence. The L^2 Dolbeault lemma implies that the spaces $\mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, \text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V}))$ can be computed using the L^2 -Dolbeault complex, that is

$$\mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, \text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V})) = \mathcal{U}(\Gamma) \otimes H^{p+q}(L^2 \text{Dolb}^{p,\bullet}(\tilde{M}, \pi^*\mathbb{V})).$$

The equality of the Laplace operators

$$\square_D = 2\square_{D''} = 2\square_{D'}$$

implies the $\mathcal{N}(\Gamma)$ -Fredholmness of the Dolbeault complexes, and the cohomology classes are represented by harmonic forms after tensorization with $\mathcal{U}(\Gamma)$. From this relation, one can also deduce the following $D'D''$ -lemma.

Lemma 4.4.24 (The $D'D''$ -lemma). Let α be a form that is both D and D'' closed and is orthogonal to the space of harmonic forms. Then there exists $r \in \mathcal{N}(\Gamma)$ injective with dense range (i.e., invertible in $\mathcal{U}(\Gamma)$) and β such that

$$r \cdot \alpha = D'D''\beta$$

Proof. The reader can note that if α is both D and D'' -closed, it is also D' -closed. Using the Fredholmness of α and Proposition A.3.67 we find $r_1 \in \mathcal{N}(\Gamma)$ and β_1 such that

$$r_1 \cdot \alpha = D'\beta_1.$$

And one can take β_1 orthogonal to $\text{Ker}(D')$ (in particular to harmonic forms). Since α is D'' -closed, β_1 is also D'' -closed, and we can find $r_2 \in \mathcal{N}(\Gamma)$ and β_2 such that $r_2\beta_1 = D''\beta_2$ again using Proposition A.3.67. Then it suffices to take $\beta = \beta_2$ and $r = r_1 r_2$ to obtain the result. \square

The usual proof brings us to the following.

Theorem 4.4.25. The spectral sequence

$$E_1^{p,q} = \mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, \text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V})) \implies \mathcal{U}(\Gamma) \otimes H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}, \omega_{P_C}, h).$$

degenerates on page 1, and the analytic and algebraic Hodge filtrations F_{an}^\bullet and F_{alg}^\bullet coincide.

Proof. A cohomology class in $\mathcal{U}(\Gamma) \otimes \mathbb{H}^{p+q}(X, \text{Gr}_p^F \Omega_{L^2}^\bullet(\pi^*\mathbb{V}))$ admits a representant of the form $\sum_k u_k \otimes \alpha_k$ with $u_k \in \mathcal{U}(\Gamma)$ and α_k harmonic. By the equality of the Laplace operators $\alpha_k \in \text{Ker}(D)$, and since the differential of the spectral sequence is induced by D , it must vanish identically and the spectral sequence degenerates at page 1. \square

5 L^2 -index and a duality theorem

Recall that in Chapter 2, for a Galois covering $\tilde{M} \rightarrow M$ of a complete manifold M with covering group Γ and \mathbb{V} a local system on M endowed with a Hermitian metric, we set the L^2 -Euler characteristics as

$$\chi_{\Gamma, L^2}(\tilde{M}, \pi^*\mathbb{V}) = \sum_k \dim_{\Gamma} H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}) \quad \chi_{L^2}(M, \mathbb{V}) = \sum_k \dim H_{L^2}^k(M, \mathbb{V}).$$

They are related by the

Theorem 4.5.26 (L^2 Riemann Hurwitz). Let $(\mathbb{V}, F^\bullet, Q)$ be a polarised variation of Hodge structure on a smooth open curve M and $\pi: \tilde{M} \rightarrow M$ be a Galois cover with covering group Γ . We denote by X a compactification of M such that $\Sigma := X \setminus M$ is a finite set of points, one has

$$\chi_{\Gamma, L^2}(\tilde{M}, \pi^*\mathbb{V}) - \sum_{p \in \Sigma} \frac{\dim \text{Ker}((T_p^{n_p} - \text{Id}))}{n_p} = \chi_{L^2}(M, \mathbb{V}) - \sum_{p \in \Sigma} \dim \text{Ker}(T_p - \text{Id}).$$

Where for $p \in \Sigma$, T_p denotes the local monodromy of \mathbb{V} at $p \in X$ and n_p is the order of γ_p the element of Γ represented by the meridian circle around p . We take $\frac{\dim \text{Ker}(T_p^{n_p} - \text{Id})}{n_p} = 0$ if $n_p = +\infty$.

Remark 4.5.27.

- We recall by the result of Zucker [Zuc79] the Euler characteristics $\chi_{L^2}(M, \mathbb{V})$ is the alternate sum of the dimension of the spaces $H^k(X, j_*\mathbb{V})$
- In the case where Γ is finite and $\mathbb{V} = \mathbb{C}_X$, the covering $\pi: \tilde{M} \rightarrow X$ extends to a ramified covering $\pi: \tilde{X} \rightarrow X$, where \tilde{X} is a compactification of \tilde{M} , in this case this theorem is just the standard Riemann-Hurwitz theorem.
- Still assuming \mathbb{V} to be a trivial local system, by a result of Gromov [CG91], the term

$$\chi_{L^2}(M, \mathbb{V}) - \sum_{p \in \Sigma} \dim \text{Ker}(T_p - \text{Id})$$

can be computed as the integral of an invariant polynomial on M and this gives a topological invariant of M . If we assume that all n_p have infinite order and $\mathbb{V} = \mathbb{C}_X$ then the covering space \tilde{M} has bounded geometry, and we recover the result of [CG85b] in the case of Riemann surfaces.

Proof. We take a finite covering $\mathcal{U} = (U_i)_{i \in I}$ of X that satisfies

1. For all $J \subset I$ finite the intersection $\bigcap_{j \in J} U_j$ is contractible (or empty).
2. Each $p \in \Sigma$ is contained in exactly one U_i .
3. If U_i contains a point $p \in \Sigma$, then $U_i \cap M$ is quasi-isometric to a punctured disk $(\Delta_r^*, \omega_{pc})$ with $r < 1$.

We denote by $C_{\mathcal{U}}(j_*\mathbb{V})^\bullet$ (resp. $C_{\mathcal{U}}(\mathcal{U}(\Gamma) \otimes \ell^2 \pi^*\mathbb{V})^\bullet$) the Čech complex of $j_*\mathbb{V}$ (resp. $\ell^2 \pi^*\mathbb{V}$). From the proof of the Poincaré lemma, it follows that those Čech complex compute the cohomology groups of $j_*\mathbb{V}$ and $\mathcal{U}(\Gamma) \otimes \ell^2 \pi^*\mathbb{V}$. The covering being finite, each term of those complexes has finite dimension and one has

$$\chi_{\Gamma, L^2}(\tilde{M}, \pi^*\mathbb{V}) = \sum_k \dim_{\Gamma} C_{\mathcal{U}}(\mathcal{U}(\Gamma) \otimes \ell^2 \pi^*\mathbb{V})^k \quad \chi_{L^2}(M, \mathbb{V}) = \sum_k \dim C_{\mathcal{U}}(j_*\mathbb{V})^k.$$

However, if $W = \bigcap_{j \in J} U_j$ does not intersect Σ , since it is contractible, $\pi^{-1}(W)$ is isometric to a disjoint union of copies of W indexed by Γ , from which it follows that

$$\dim_{\Gamma} \ell^2 \pi^*\mathbb{V}(W) = \dim_{\mathbb{V}}(W).$$

If W is the intersection of at least 2 of the open subset of our open covering, then it does not intersect Σ and one thus obtains

$$\dim_{\Gamma} C_{\mathcal{U}}(\mathcal{U}(\Gamma) \otimes \ell^2 \pi^*\mathbb{V})^k = \dim C_{\mathcal{U}}(j_*\mathbb{V})^k \quad \text{for } k \geq 1.$$

In the case where $k = 0$, the only terms that can cause trouble are the terms of the form $\ell^2 \pi^*\mathbb{V}(U_i)$ and $j_*\mathbb{V}(U_i)$ where U_i contains a point $p \in \Sigma$. In this case, by the definition of local monodromy around p we have

$$\dim j_*\mathbb{V}(U_i) = \dim \text{Ker}(T_p - \text{Id}).$$

It remains to compute $\dim_{\Gamma} \ell^2 \pi^* \mathbb{V}(U_i)$. Taking W_i to be a connected component of $\pi^{-1}(U_i)$, $\pi^{-1}(U_i)$ is then given by a disjoint union of copies of U_i indexed by Γ/H where H is the isotropy group of W_i and is isomorphic to $\mathbb{Z}/n_p \mathbb{Z}$. It follows that $\dim_{\Gamma} \ell^2 \pi^* \mathbb{V}(U_i) = \dim_{\mathbb{Z}/n_p \mathbb{Z}} \pi^{-1} \mathbb{V}(W_i)$ where $\pi^{-1} \mathbb{V}(W_i)$ is the space of horizontal sections on W_i that is given by $\text{Ker}(T_p^{n_p} - \text{Id})$ hence

$$\dim_{\Gamma} \ell^2 \pi^* \mathbb{V}(U_i) = \dim_{\mathbb{Z}/n_p \mathbb{Z}} \pi^{-1} \mathbb{V}(W_i) = \frac{\dim \text{Ker}(T_p^{n_p} - \text{Id})}{n_p}.$$

This concludes the proof. \square

Theorem 4.5.28 (Duality theorem). With the same assumption as in Theorem 4.5.26, the Hodge star operator $*$ induces isomorphisms of $\mathcal{U}(\Gamma)$ -modules

$$\mathcal{U}(\Gamma) \otimes H_{L^2}^{\bullet}(\tilde{M}, \pi^* \mathbb{V})^{P,Q} \rightarrow \mathcal{U}(\Gamma) \otimes H_{L^2}^{\bullet}(\tilde{M}, \pi^* \mathbb{V}^*)^{-P,-Q} \quad \text{for all } (P, Q)$$

Proof. By Corollary 2.4.33 the Hodge star operator induces an isomorphism between the spaces of harmonic forms with the correct ideal boundary conditions. We have, however, only one ideal boundary condition, so one has an isomorphism

$$*: \mathcal{U}(\Gamma) \otimes H_{L^2}^{\bullet}(\tilde{M}, \pi^* \mathbb{V}) \rightarrow \mathcal{U}(\Gamma) \otimes H_{L^2}^{\bullet}(\tilde{M}, \pi^* \mathbb{V}^*).$$

It remains to see that the Hodge star operator must send the space $\text{Harm}^k(\tilde{M}, \pi^* \mathbb{V})^{P,Q}$ to $\text{Harm}^{2-k}(\tilde{M}, \pi^* \mathbb{V}^*)^{-P,-Q}$ thanks to the relation $\|s\|^2 d\text{Vol} = \{s, *s\}$ as it must send the h -orthogonal space of $H^{p,q}$ to its polar orthogonal which is $\bigoplus_{r \neq p, s \neq q} H^{-r,-s}$. Since $*$ is an isomorphism and is compatible with the Hodge decomposition, it is an isomorphism of Hodge structures. \square

Corollary 4.5.29. If the covering space \tilde{M} has an infinite volume (i.e. if Γ is infinite) then

$$\mathcal{U}(\Gamma) \otimes H_{L^2}^0(\tilde{M}, \pi^* \mathbb{V}) = \mathcal{U}(\Gamma) \otimes H_{L^2}^2(\tilde{M}, \pi^* \mathbb{V}) = 0.$$

Proof. The infinite volume condition imposes $H_{L^2}^0(\tilde{M}, \pi^* \mathbb{V}) = 0$ and the duality theorem gives us the result for the second cohomology group. \square

6 The example of locally homogeneous variation of Hodge structures

Another important example is the case of locally homogeneous variations of Hodge structure. We take Γ to be a Fuchsian group acting freely on \mathbb{H} , we set $M = \Gamma \backslash \mathbb{H}$ and $\tilde{M} = \mathbb{H}$ to be the universal cover, it is a symmetric space which is naturally isomorphic to $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$. As before, we consider X a compactification of M .

We recall the definition of locally homogeneous variation of Hodge structure presented in [Zuc81]. For m a positive integer, we set $V_m = \text{Sym}^m(\mathbb{C}^2)$ we endow it with the natural representation of $\text{SL}_2(\mathbb{R})$, which induces a representation of Γ by restriction. This representation of $\text{SL}_2(\mathbb{R})$ will be denoted by ρ . We set \mathbb{V}_m the associated local system defined on M . We will endow it with a structure of polarised variation of Hodge structure m . Recall that since the group $\text{SO}_2(\mathbb{R})$ is isomorphic to a circle, one has a $\text{SO}_2(\mathbb{R})$ -invariant decomposition

$$V_m = \bigoplus_{k \in \mathbb{Z}} V_m \langle k \rangle$$

where on $V_m \langle k \rangle$, the action is given by

$$\rho \left(\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \right) = e^{i\theta k}.$$

We have an explicit description of $V_m \langle k \rangle$. If we set $e_+ = {}^t(1, i)$ and $e_- = {}^t(1, -i)$, one has

$$V_m \langle k \rangle = \begin{cases} 0 & \text{if } |k| > m \\ 0 & \text{if } k + m = 1 \pmod{2} \\ \text{Span}(e_+^{m-l} e_-^l) & \text{if } k = m - 2l \end{cases}$$

By setting $H_0^{p,q} = V_m \langle m - 2p \rangle = \text{Span}(e_-^{m-p} e_+^p)$, one has the basic Hodge structure

$$V_m = \bigoplus_{p+q=m} H_0^{p,q}.$$

For each $g \in \text{SL}_2(\mathbb{R})$, we set $H_g^{p,q} = \rho(g)H_0^{p,q}$ and we obtain a Hodge structure of weight m , this Hodge structure only depends on the coset $[g] \in \text{SL}_2/\text{SO}_2 \simeq \tilde{M}$, and this induces a variation of Hodge structure on \tilde{M} (see [Zuc81, p. 263]). To polarise V_m , we need to polarise V_1 , as V_m is just the m -th symmetric power of V_1 . The polarisation Q is given by the properties :

$$Q(e_+, e_+) = -2, \quad Q(e_-, e_-) = 2, \text{ and } Q(e_+, e_-) = 0.$$

The reader can note that we differ from the polarisation of [Zuc81] by a factor -1 as we took the convention that $(-1)^q Q$ to be positive-definite on $H^{p,q}$, while in [Zuc81] the convention was $(-1)^p Q$ to be positive-definite on $H^{p,q}$. From this, we obtain a polarised variation of Hodge structure on $\tilde{M} = \mathbb{H}$. It induces a polarised variation of Hodge structures on any quotient $M = \Gamma \backslash \tilde{M}$. We set $\tilde{\mathcal{V}}_m = \mathcal{O}_{\tilde{M}} \otimes V_m$ and $\mathcal{V}_m = \mathcal{O}_M \otimes \mathbb{V}_m$ and the decomposition into smooth vector bundles by

$$\begin{aligned} \tilde{\mathcal{V}}_m &= \bigoplus \tilde{\mathcal{V}}_m^{p,q} \\ \mathcal{V}_m &= \bigoplus \mathcal{V}_m^{p,q}. \end{aligned}$$

If we set $\omega_- = \begin{pmatrix} z \\ 1 \end{pmatrix}$ and $\omega_+ = \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}$ then ω_- generates $\tilde{\mathcal{V}}_1^{1,0}$ and ω_+ generates $\tilde{\mathcal{V}}_1^{0,1}$. It follows that for $p+q=m$, $\omega_-^p \omega_+^q$ generates $\mathcal{V}_m^{p,q}$. The reader can also note that \mathbb{V}_m has also a natural real structure induced by the natural embedding $\mathbb{R}^2 \rightarrow \mathbb{C}^2$. Considering this real structure, one has $\mathcal{V}^{p,q} = \overline{\mathcal{V}^{q,p}}$. Thus, one has a natural polarised variation of *real* Hodge structure.

We will compute the norm of ω_- , which is equal to the one of ω_+ . One has $\omega_- = \frac{z+i}{2}e_- + \frac{z-i}{2}e_+$. From this one obtains

$$h(\omega_-, \omega_-) = Q(\omega_-, \omega_-) = 2\text{Im}(z). \quad (4.2)$$

If D is the flat connection we recall that we have a decomposition

$$D = D' + D'', \quad D' = \partial + \bar{\theta}, \text{ and } D'' = \bar{\partial} + \theta.$$

An important result is the following

Proposition 4.6.30. [Zuc81, Proposition 3.12] One has the following equalities between differential operators.

- $\square_{D''} = \square_{\bar{\theta}} + \square_{\theta}$
- $\square_{D'} = \square_{\partial} + \square_{\bar{\theta}}$

Here we kept the convention $\square_d = \partial d + d\bar{\partial}$ for a differential operator d , where ∂ is the formal adjoint of d .

We recall that θ and $\bar{\theta}$ are bounded, and that all the operators in the above proposition are essentially self adjoint. It follows that the above equalities also yields as equalities between closed operators. Moreover, one has a decomposition of the cohomology groups

$$\begin{aligned} H^k(X, \ell^2 \pi^* \mathbb{V})^{P,Q} &= \bigoplus_{\substack{p+r=P \\ q+s=Q}} \tilde{\mathcal{H}}^{(p,q),(r,s)} \\ H^k(X, j_* \mathbb{V})^{P,Q} &= \bigoplus_{\substack{p+r=P \\ q+s=Q}} \mathcal{H}^{(p,q),(r,s)}. \end{aligned}$$

Where $\tilde{\mathcal{H}}^{(p,q),(r,s)}$ (resp. $\mathcal{H}^{(p,q),(r,s)}$) denote the space of square integrable $\tilde{\mathcal{V}}_m^{r,s}$ -valued (resp. $\mathcal{V}_m^{r,s}$) (p,q) -harmonic forms. We are interested in the Laplace operator associated to the Higgs field θ . It is sufficient to study it on \mathcal{V}_1 , since $D\omega_- = \frac{dz}{2\text{Im}(z)}(\omega_- - \omega_+)$, we have

$$\theta(\omega_-) = \frac{-dz}{2\text{Im}(z)}\omega_+, \text{ and } \theta(\omega_+) = 0.$$

This yields

$$\theta(\omega_-^p \omega_+^q) = \frac{pdz}{2\text{Im}(z)}\omega_-^{p-1} \omega_+^{q+1}.$$

Similarly, we find $\theta^*(dz \otimes \omega_+) = \frac{-2}{\text{Im}(z)}\omega_-$, where θ^* denotes the L^2 -adjoint of θ . It follows that the action of the Laplace operator \square_θ on $\mathcal{V}_m^{p,q}$ -valued $(0,s)$ -form is simply the multiplication by $\frac{p^2\|dz\|^2}{2\text{Im}(z)^2} = \frac{p^2}{4}$ which does not depend on z . Moreover, the action of \square_θ on the space of $\mathcal{V}^{\tilde{p},q}$ -valued $(1,0)$ -forms is given by the multiplication by $\frac{m-p}{4}$. With this we obtain the following lemma.

Lemma 4.6.31. For $0 < p < m + 1$, the complex of sheaves $Gr^p\Omega_{L^2}^\bullet(\pi^*\mathbb{V}_m)$ is acyclic. In particular, its hypercohomology vanishes.

Using this lemma and our Hodge decomposition, one obtains

$$H_{L^2}^1(\tilde{M}, \pi^*\mathbb{V}_m) = H^0(X, \Omega_{L^2}^1(\mathcal{V}^{m,0})) + H^1(X, \Omega_{L^2}^0(\mathcal{V}^{0,m})) \quad (4.3)$$

We recall the reader that Lemma 4.6.31, remains true in the case of the trivial covering (see [Zuc79, Lemma 12.14]). We then obtain the decomposition

$$H^1(X, j_*\mathbb{V}) = H^0(X, \Omega_{(2)}^1(\mathcal{V}_0^{m,0})) \oplus \overline{H^0(X, \Omega_{(2)}^1(\mathcal{V}_0^{0,m}))}$$

which can be viewed as an analytic interpretation of the Eichler-Shimura isomorphism (see [Zuc79, Lemma 12.15]). By our results, we have two methods of computing the Euler characteristic $\chi_\Gamma(X, \mathcal{U}(\Gamma) \otimes \ell^2\pi^*\mathbb{V}_m)$, which is equal to $-\dim_\Gamma H^1(X, \ell^2\pi^*\mathbb{V}_m)$.

The first method is the usual one used to compute the Euler characteristics of weakly constructible sheaves on simplicial complexes (see [KS90, Chapter 8, Section 1]). We fix \mathbf{S} , a realisation of X as a simplicial complex such that $\Sigma = X \setminus M$ is included in the set of vertices of \mathbf{S} . In this case, $\ell^2\pi^*\mathbb{V}_m$ is a weakly \mathbf{S} -constructible sheaf of $\mathcal{U}(\Gamma)$ -module, i.e. it is constant on any simplex of \mathbf{S} (see [KS90, Definition 8.3.1]).

By using [KS90, Proposition 8.14], we can use this triangulation to compute the Euler characteristic. On any simplex of \mathbf{S} at the exception to the points Σ , the sheaf $\mathcal{U}(\Gamma) \otimes \ell^2\pi^*\mathbb{V}_m$ is constant to $\mathcal{U}(\Gamma)^{\text{rk}\mathbb{V}_m} = \mathcal{U}(\Gamma)^{m+1}$ and have Γ -dimension equal to $m + 1$. At the point $p \in \Sigma$ the stalks of our sheaf vanish. Hence, we obtain

$$\begin{aligned} \chi_\Gamma(X, \mathcal{U}(\Gamma) \otimes \ell^2\pi^*\mathbb{V}_m) &= (m + 1) (\#\text{faces} - \#\text{edges} + \#\text{points} - |\Sigma|) \\ &= (m + 1)\chi(M, \mathbb{C}) \end{aligned} \quad (4.4)$$

where $\chi(M, \mathbb{C})$ is the usual Euler characteristics.

We can find this result by a second method by using that $H^k(X, \mathcal{U}(\Gamma) \otimes \ell^2(\pi^*\mathbb{V}_m))$ is isomorphic to $\mathcal{U}(\Gamma) \otimes H_{L^2}^k(\tilde{M}, \pi^*\mathbb{V}_m)$. We have seen that the Laplace operator is positive definite except on $\mathcal{V}^{m,0}$ -valued $(1,0)$ -forms and on $\mathcal{V}^{0,m}$ -valued $(0,1)$ -forms. Since the space of $\mathcal{V}^{m,0}$ -valued holomorphic forms is isomorphic to the space of $\mathcal{V}^{0,m}$ -valued antiholomorphic forms we only have to compute the Γ -dimension of $\text{Harm}^{1,0}(\tilde{M}, \mathcal{V}^{m,0})$. This can be done by integrating the trace of the Bergman kernel on a fundamental domain. Since in our case, the bundle $\mathcal{V}^{m,0}$ is invariant by the whole group SL_2 and not just Γ , the trace of the Bergman kernel is constant, and we obtain

$$\chi_{\Gamma, L^2}(\tilde{M}, \pi^*\mathbb{V}_m) = c \cdot \text{Vol}(M) = -2\pi c \chi(M, \mathbb{C}).$$

Here c is a constant independent of M . It is equal to twice the trace of the Bergman operator. To compute it, we fix $p: \tilde{M} \rightarrow N$ a compact quotient of \tilde{M} , and set $G = \text{Deck}(\tilde{M}/N)$. The locally homogeneous variation of Hodge structure also lives on N and we also denote by \mathbb{V}_m the underlying local system. N being compact, we can apply the Atiyah's L^2 -index theorem to find

$$\chi_{G, L^2}(\tilde{M}, p^*\mathbb{V}_m) = \chi(N, \mathbb{V}_m) = \text{rk}(\mathbb{V}_m)\chi(N) = (m + 1)\chi(N).$$

We deduce $c = -\frac{m+1}{2\pi}$ and we find

$$\chi_{\Gamma, L^2}(\tilde{M}, \pi^*\mathbb{V}_m) = (m + 1)\chi(M, \mathbb{C}). \quad (4.5)$$

This is indeed the same result as (4.4) as predicted by Theorem 4.2.13 and the value of the Euler characteristics coincides with the one predicted by Theorem 4.5.26.

On an alternative proof of the $\mathcal{N}(\Gamma)$ -Fredholmness In this paragraph, we sketch how the study of the locally homogeneous variation of Hodge structures can prove Theorem 4.1.6 under the additional hypothesis that the monodromy of the underlying local system is unipotent at infinity. This alternative proof will not use the computations made in Chapter 3. We will need the following result that can be found in [Dem12, Theorem 4.5, p.371].

Lemma 4.6.32. Let (M, ω) be a complete Kähler manifold. Consider (E, h) to be a rank r hermitian holomorphic vector bundle on M , and set $A_{E,h} = [i\Theta(h), \Lambda_\omega]$ where $\Theta(h)$ is the curvature of the metric and Λ_ω is the adjoint of the Lefschetz map $\omega \wedge \cdot$. Under the assumption that $A_{E,h}$ is positive definite everywhere in bi-degree (p, q) with $q \geq 1$, one has that for any g E -valued square-integrable (p, q) -form, then if $\bar{\partial}g = 0$ and $\int_M \langle A_{E,h}^{-1}g, g \rangle > d\text{Vol} < +\infty$ there exists a measurable $(p, q - 1)$ -form f satisfying

$$\begin{aligned} \bar{\partial}f &= g \\ \|f\|_{L^2}^2 &\leq \int_M \langle A_{E,h}^{-1}g, g \rangle > d\text{Vol} \end{aligned}$$

From now on, M denote a hyperbolic Riemann surface of finite volume endowed with its hyperbolic metric. We begin to show the following.

Proposition 4.6.33. Let \mathbb{V}_m be the local system underlying a locally homogeneous variation of Hodge structures on M as above. Then for any Galois covering $\pi: \tilde{M} \rightarrow M$ the complex $L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V}_m)$ is $\mathcal{N}(\Gamma)$ -Fredholm.

Proof. We can without loss of generality assume that \tilde{M} is a quotient of the Poincaré half plane and endowed with its hyperbolic metric ω_{Pc} . We recall that one has an orthogonal decomposition of the Hilbert space

$$L^2DR^k(\tilde{M}, \pi^*\mathbb{V}_m) = \bigoplus_{\substack{r+s=k \\ p+q=m}} L^2DR^{r,s}(\tilde{M}, \pi^*\mathcal{V}^{p,q})$$

In the case of a locally homogeneous variation of Hodge structure, we have seen that this decomposition is preserved by the Laplace operator $\square_D = 2\square_{\bar{\partial}} + 2\square_{\partial}$. Moreover, we have also seen that the operator \square_{∂} acts by multiplication by $\frac{p^2}{4}$ (resp. $\frac{(m-p)^2}{4}$) on the spaces $L^2DR^{0,s}(\tilde{M}, \pi^*\mathcal{V}^{p,q})$ (resp. $L^2DR^{1,s}(\tilde{M}, \pi^*\mathcal{V}^{p,q})$). This implies that for $\varepsilon < 1/4$ one has

$$\begin{aligned} \text{Ran}(E_\varepsilon^0(\square_D)) &\subset L^2DR^{0,0}(\tilde{M}, \pi^*\mathcal{V}^{0,m}) \oplus L^2DR^{0,1}(\tilde{M}, \pi^*\mathcal{V}^{0,m}) \\ \text{Ran}(E_\varepsilon^1(\square_D)) &\subset L^2DR^{0,1}(\tilde{M}, \pi^*\mathcal{V}^{0,m}) \oplus L^2DR^{1,0}(\tilde{M}, \pi^*\mathcal{V}^{m,0}) \\ \text{Ran}(E_\varepsilon^2(\square_D)) &\subset L^2DR^{1,1}(\tilde{M}, \pi^*\mathcal{V}^{0,m}). \end{aligned}$$

As in Chapter 2, while talking about the complex we make the abuse of notation $E_\varepsilon^\bullet(\square_D) = \text{Ran}(E_\varepsilon^\bullet(\square_D))$. Recall that $E_\varepsilon^\bullet(\square_D)$ is homotopic to $L^2DR^\bullet(\tilde{M}, \pi^*\mathbb{V}_m)$ by Proposition 2.1.10. And for $\varepsilon < 1/4$, one has that the Laplace operator on $E_\varepsilon^\bullet(\square_D)$ is given by $2\square_{\bar{\partial}}$. It follows that the complex is $\mathcal{N}(\Gamma)$ -Fredholm, provided that the complex

$$0 \longrightarrow L^2DR^{1,0}(\tilde{M}, \pi^*\mathcal{V}^{m,0}) \xrightarrow{\bar{\partial}} L^2DR^{1,1}(\tilde{M}, \pi^*\mathcal{V}^{m,0}) \longrightarrow 0$$

is $\mathcal{N}(\Gamma)$ -Fredholm. However, in this case Lemma 4.6.32 ensures us that the $\bar{\partial}$ operator has closed range (it is even onto), hence 0 must be isolated in the spectrum and the complex is $\mathcal{N}(\Gamma)$ -Fredholm if and only if the space of harmonic form has finite $\mathcal{N}(\Gamma)$ -dimension. The $\mathcal{N}(\Gamma)$ -dimension of this space is obtained by integrating the trace of the Bergman kernel on a fundamental domain. Since a fundamental domain has finite volume, it is sufficient to prove that the trace of the Bergman kernel is bounded. Since it is invariant by the group Γ it is bounded on the preimage of any compact subset of M and one only has to study the behaviour of the Bergman kernel on the preimage of neighbourhoods of punctures.

We recall some facts about Bergman kernels. Given a hermitian line bundle \mathcal{L} on a manifold N and $p \in N$ we will denote by $B_{N,L}(p)$ the trace of the Bergman kernel at p . We recall that one has the following equality [CM15, Lemma 3.1]

$$B_{N,L}(p) = \sup \left\{ \frac{|f(p)|^2}{\|f\|_{L^2}^2} \mid f \text{ non-zero holomorphic } L^2 \text{ section of } \mathcal{L} \right\}.$$

In our settings we will take $N = \tilde{M}$ and $\mathcal{L} = \Omega^1 \otimes \mathcal{V}^{m,0}$. The above equality implies that if $U \subset \tilde{M}$ is an open subset, then $B_{U,\mathcal{L}}(p) \geq B_{\tilde{M},L}(p)$. To end the proof, we then need to prove the lemma below.

Lemma 4.6.34.

1. Let (\mathcal{L}, h) be a line bundle on a cusp $(\Delta_{R'}^*, \frac{dz \wedge d\bar{z}}{|z|^2 \ln(|z|^2)^2})$, with generating section e . Assume that $h(e, e) = |z|^2 |\ln |z|^2|^{m+2}$ with $m \geq 0$, then the trace of the Bergman kernel is bounded on any neighbourhood of 0.
2. Let (\mathcal{L}, h) be a line bundle on a horodisk $(\mathbb{H}_A, \frac{dz \wedge d\bar{z}}{y^2})$, with generating section e . Assume that $h(e, e) = y^{m+2}$ with $m \geq 0$, then the trace of the Bergman kernel is bounded on any smaller horodisk.

This lemma allows us to conclude the proof of Proposition 4.6.33 since we only need to prove the trace of the Bergman kernel of the line bundle $\mathcal{L} = \Omega^1 \otimes \pi^* \mathcal{V}_m^{m,0}$ to be bounded on the preimage neighbourhoods of the punctures. The connected component of such a neighbourhood are isometric to either

- A cusp $(\Delta_{R'}^*, \frac{dz \wedge d\bar{z}}{|z|^2 \ln(|z|^2)^2})$, in which case \mathcal{L} has generating section $e = \frac{1}{\sqrt{2}} dz \otimes \omega_-^m$ and $h(e, e) = \frac{1}{2} \|dz\|^2 h(\omega_-, \omega_-) = |z|^2 |\ln(|z|^2)|^{m+2}$ by (4.2)
- A horodisk $(\mathbb{H}_A, \frac{dz \wedge d\bar{z}}{y^2})$, in which case \mathcal{L} has generating section $e = \frac{1}{\sqrt{2}} dz \otimes \omega_-^m$ and $h(e, e) = \frac{1}{2} \|dz\|^2 h(\omega_-, \omega_-) = y^{m+2}$ by (4.2).

Proof of the lemma. We begin with the case of the cusp $(\Delta_{R'}^*, \frac{dz \wedge d\bar{z}}{|z|^2 \ln(|z|^2)^2})$. The norm of a section $s = f \otimes e$ is the given by

$$\|s\|_{L^2}^2 = \int_{\Delta_{R'}^*} |f(z)|^2 |\ln(|z|^2)|^m d\text{Leb}(z)$$

The square integrability implies that f has a removable singularity at 0 and hence the orthogonal family $(z^n)_{n \geq 0}$ is dense in the space of square integrable holomorphic sections. It follows that the trace of the Bergman kernel is given by

$$B_{\Delta_{R'}^*, \mathcal{L}}(z) = \sum_{n=0}^{\infty} \frac{1}{\|z^n\|_{L^2}^2} |z|^{2n}.$$

This is a decreasing positive function of $|z|$ and thus is bounded in a neighbourhood of 0.

The second case is the case of the horodisk \mathbb{H}_A . Set $s = f \otimes e$ to be a square-integrable holomorphic section of \mathcal{L} . Its L^2 -norm is given by

$$\|s\|_{L^2}^2 = \int_{\mathbb{H}_A} |f(z)|^2 y^m d\text{Leb}(z).$$

One has $h(s(z), s(z)) = |f(z)| y^{m+2}$. By the mean value equality, one has

$$\begin{aligned} |f(z)|^2 &= \frac{4}{\pi^2 (y-A)^4} \left| \int_{B(z, \frac{y-A}{2})} f(\zeta) d\text{Leb}(\zeta) \right|^2 \\ &\leq \frac{4}{\pi^2 (y-A)^4} \int_{B(z, \frac{y-A}{2})} |f(\zeta)|^2 v^m d\text{Leb}(\zeta) \int_{B(z, \frac{y-A}{2})} v^{-m} d\text{Leb}(\zeta) \quad \text{for } \zeta = u + iv \\ &\leq \frac{4}{\pi^2 (y-A)^4} \|s\|_{L^2}^2 \int_{x-\frac{y-A}{2}}^{x+\frac{y-A}{2}} \int_{\frac{y-A}{2}}^{\frac{3y-A}{2}} v^{-m} dv du \\ &\leq \frac{8}{(m-1)\pi^2 (y-A)^3} \left(\left(\frac{y+A}{2}\right)^{-m+1} - \left(\frac{3y-A}{2}\right)^{-m+1} \right) \|s\|_{L^2}^2 \quad \text{for } m \neq 1 \end{aligned}$$

It follows that $B_{\mathbb{H}_A, \mathcal{L}}(z) \leq \frac{8}{(m-1)\pi^2 (y-A)^3} \left(\left(\frac{y+A}{2}\right)^{-m+1} - \left(\frac{3y-A}{2}\right)^{-m+1} \right) y^{m+2}$ which is bounded on any $\mathbb{H}_{A'}$ for $A' > A$. For the case of $m = 1$, the inequality above becomes

$$|f(z)|^2 \leq \frac{8}{\pi^2 (y-A)^3} \left(\ln\left(\frac{3y-A}{2}\right) - \ln\left(\frac{y+A}{2}\right) \right) \|s\|_{L^2}^2$$

and

$$h(s(z), s(z)) \leq \frac{8}{\pi^2 (y-A)^3} \left(\ln\left(\frac{3y-A}{y+A}\right) \right) y^3 \|s\|_{L^2}^2.$$

It follows that

$$B_{\mathbb{H}_A, \mathcal{L}}(z) \leq \frac{8}{\pi^2 (y-A)^3} \left(\ln\left(\frac{3y-A}{y+A}\right) \right) y^3$$

which is also bounded on any $\mathbb{H}_{A'}$ for any $A' > A$. This concludes the proof of the lemma and hence the proof of Proposition 4.6.33. \square

Using Proposition 4.6.33, we can apply Lemma 4.1.5 and Corollary 2.2.23 to obtain the following result.

Corollary 4.6.35. If $U \subset M$ is a neighbourhood of the punctures isometric to a disjoint union of punctured disks, the complex $L^2DR_{\max}^\bullet(\pi^{-1}(U), \pi^{-1}\mathbb{V}_m)$ is $\mathcal{N}(\Gamma)$ -Fredholm.

This corollary is enough to obtain the fact the L^2 de Rham complex is $\mathcal{N}(\Gamma)$ -Fredholm when our variation of Hodge structure on M is unipotent at infinity, as explain below.

Theorem 4.6.36. Let $(\mathbb{V}, F^\bullet, Q)$ be a polarised variation of Hodge structures on M , with \mathbb{V} having a monodromy that is unipotent at infinity. Let $\pi: \tilde{M} \rightarrow M$ be a Galois covering of covering group Γ . Then the $\mathcal{N}(\Gamma)$ -Hilbert complex $L^2DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{V})$ is $\mathcal{N}(\Gamma)$ -Fredholm.

Proof. Recall that in the locally homogeneous setting \mathbb{V}_m is unipotent at infinity, it is straight-forward to check that the logarithm of the local monodromy at infinity is given by a single Jordan block of size m . If U_0 is a neighbourhood of a puncture isometric to a punctured disk, then $\mathbb{V}|_{U_0}$ is isomorphic to a local system \mathbb{L} which is a direct sum of \mathbb{V}_m given by this isomorphism is given by the Jordan canonical form.

By shifting the weight of the Hodge structure underlying \mathbb{V}_m , one can assume that \mathbb{L} is also underlying a polarised variation of Hodge structure. We then denote by h the Hodge metric on the vector bundle associated to $\mathbb{V}|_{U_0}$ and by $h_{\mathbb{L}}$ the Hodge metric on \mathbb{L} . Theorem 1.3.35 implies that h and $h_{\mathbb{L}}$ are mutually bounded. This implies that if U is a neighbourhood of the punctures isometric to a disjoint union of punctured disks the $\mathcal{N}(\Gamma)$ -Hilbert complexes $L^2DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{V})$ and $L^2DR^\bullet(\tilde{M}, \pi^{-1}\mathbb{L})$ are isomorphic. Since the latter complex is $\mathcal{N}(\Gamma)$ -Fredholm, the former is also $\mathcal{N}(\Gamma)$ -Fredholm. It remains to apply Lemma 4.1.5 and Corollary 2.2.23 to obtain the theorem. \square

Chapter 5

Interpretation in terms of Hodge modules

This chapter is dedicated to an interpretation of our results in terms of Hodge modules. We first recall some results of the theory of pure Hodge module developed by Saito in [Sai88, Sai90] in the case of complex curves, our main reference being [SS22b] about pure Hodge module and middle extension of variation of Hodge structures. Then we will recall the different notions introduced in [Eys22] that are needed to define the L^2 -cohomology of a pure Hodge module on a covering. We again consider $j: M \rightarrow X$ an embedding of Riemann surfaces, with X compact and $\Sigma := X \setminus M$ consisting of a finite set of points. We want to give an interpretation of our result in terms of the conjectural theory of L^2 -cohomology of mixed Hodge modules on infinite covering developed in [Eys22].

1 Middle extension of a variation of Hodge structure

We set \mathcal{D}_X the sheaf of differential operators on X , and by a \mathcal{D}_X -module we will always mean a left \mathcal{D}_X -module unless stated otherwise. If $(\mathcal{V}, F^\bullet, D^{1,0})$ is a holomorphic vector bundle on M with a flat connection, underlying a polarisable variation of the Hodge structure, we will denote by \mathcal{V}_* its canonical meromorphic extension.

We localise around a puncture $p \in \Sigma$, and we take $X = \Delta$ and $M = \Delta^*$. As in Chapter 1, for $\beta \in \mathbb{R}$, we denote by \mathcal{V}_*^β the lattice of \mathcal{V}_* for which the eigenvalues of the residue lie in the interval $[\beta, \beta + 1[$. This defines a decreasing filtration \mathcal{V}_*^\bullet of \mathcal{V}_* and we set

$$\mathcal{V}_*^{>\beta} = \bigcup_{\beta' > \beta} \mathcal{V}_*^{\beta'}.$$

The filtration \mathcal{V}_*^\bullet is called the parabolic filtration, it is only defined locally.

We denote by \mathcal{V}_{mid} the \mathcal{D}_X -module generated by $\mathcal{V}_*^{>-1}$, i.e.

$$\mathcal{V}_{\text{mid}} = \sum_{j \in \mathbb{N}} (D_{\partial_z}^{1,0})^j \mathcal{V}_*^{>-1}.$$

We call \mathcal{V}_{mid} the middle extension of \mathcal{V} , it is endowed with a coherent good filtration $F^\bullet \mathcal{V}_{\text{mid}}$ defined by

$$F^p \mathcal{V}_{\text{mid}} := \sum_{j \in \mathbb{N}} (D_{\partial_z}^{1,0})^j F^{p-j} \mathcal{V}_*^{>-1}.$$

It is locally endowed with the Kashiwara–Malgrange filtration $V^\bullet \mathcal{V}_{\text{mid}} := \mathcal{V}_{\text{mid}}^\bullet$ that is given by

$$\mathcal{V}_{\text{mid}}^\beta = \begin{cases} \mathcal{V}_*^\beta & \text{if } \beta > -1 \\ (D_{\partial_z}^{1,0})^{-[\beta]} \mathcal{V}_*^{\beta - [\beta]} + \mathcal{V}_*^{>\beta} & \text{if } \beta \leq -1 \end{cases}$$

Here $[\cdot]$ denote the upper integral part of a real number, so for $\beta \in \mathbb{R}$ one has $[\beta] - 1 < \beta \leq [\beta]$. For $\beta \in \mathbb{R}$, the sheaf $\mathcal{V}_{\text{mid}}^\beta$ is \mathcal{O}_X -coherent. The space $\text{Gr}^\beta \mathcal{V}_{\text{mid}}$ is finite-dimensional and $(z\partial_z - \beta)$ induces a nilpotent operator on $\text{Gr}^\beta \mathcal{V}_{\text{mid}}$, which coincides with the action of the N_β we had on \mathcal{V}_* and induces a filtration $W_\bullet(N_\beta)$ on $\text{Gr}^\beta \mathcal{V}_{\text{mid}}$, similarly to what we did for the meromorphic extension we set

$$M_k \mathcal{V}_{\text{mid}}^\beta := p^{-1} \left(W_k(N_\beta) \text{Gr}^\beta \mathcal{V}_{\text{mid}} \right)$$

where $p: \mathcal{V}_{\text{mid}}^\beta \rightarrow \text{Gr}^\beta \mathcal{V}_{\text{mid}}$ is the projection.

Proposition 5.1.1. [SS22b, Proposition 6.14.2, Corollary 6.14.4] We have the following.

- The filtration $F^\bullet \mathcal{V}_{\text{mid}}$ is exhaustive.
- For $\beta > -1$ we have $F^p \mathcal{V}_{\text{mid}}^\beta = (j_* F^p \mathcal{V}) \cap \mathcal{V}_*^\beta$ and $z(F^p \mathcal{V}_{\text{mid}}^\beta) = F^p \mathcal{V}_{\text{mid}}^{\beta+1}$.
- For $\beta \leq 0$, $\partial_z F^p \text{Gr}^\beta \mathcal{V}_{\text{mid}} = F^{p-1} \text{Gr}^{\beta-1} \mathcal{V}_{\text{mid}}$.
- The sheaves $F^p \mathcal{V}_{\text{mid}}$, $F^p \mathcal{V}_{\text{mid}}^\beta$ and $F^p M_k \mathcal{V}_{\text{mid}}$ are \mathcal{O}_X locally free and of finite rank.

The de Rham complex of \mathcal{V}_{mid} is given by

$$DR(\mathcal{V}_{\text{mid}}) := \left\{ 0 \rightarrow \mathcal{V}_{\text{mid}} \xrightarrow{D^{1,0}} \Omega_X^1 \otimes \mathcal{V}_{\text{mid}} \rightarrow 0 \right\}.$$

And its perverse de Rham complex is given by ${}^p DR(\mathcal{V}_{\text{mid}}) := DR(\mathcal{V}_{\text{mid}})[1]$. We have the Kashiwara–Malgrange filtration and the Hodge filtration on those bundles defined by

$$V^\beta DR(\mathcal{V}_{\text{mid}}) := 0 \longrightarrow \mathcal{V}_{\text{mid}}^\beta \xrightarrow{D^{1,0}} \Omega_X^1 \otimes \mathcal{V}_{\text{mid}}^{\beta-1} \longrightarrow 0$$

$$F^p DR(\mathcal{V}_{\text{mid}}) := 0 \longrightarrow F^p \mathcal{V}_{\text{mid}} \xrightarrow{D^{1,0}} \Omega_X^1 \otimes F^{p-1} \mathcal{V}_{\text{mid}} \longrightarrow 0$$

One can check that by Proposition 1.3.32, the holomorphic de Rham complex $\Omega^\bullet(\mathbb{V})_{(2)}$ of \mathcal{V} is a subcomplex of $V^0 DR(\mathcal{V}_{\text{mid}})$. We recall that we have the following result [SS22b, Proposition 6.14.8]

Proposition 5.1.2. For any $p \in \mathbb{Z}$ the inclusions

$$F^p \Omega^\bullet(\mathbb{V})_{(2)} \rightarrow F^p V^0 DR(\mathcal{V}_{\text{mid}}) \rightarrow F^p DR(\mathcal{V}_{\text{mid}})$$

are quasi-isomorphisms.

2 Polarised Hodge module on a curve

If X is a complex manifold, a polarised Hodge module on X is given by the data of (\mathcal{M}, F^\bullet) a \mathcal{D}_X -module endowed with a good filtration, a perverse sheaf $\mathbb{M}^{\text{Betti}}$ endowed with a quasi-isomorphism $\alpha: \mathbb{M}^{\text{Betti}} \rightarrow {}^p DR(\mathcal{M})$ and a sesquilinear pairing $S: \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \rightarrow \mathfrak{D}b_X$ called polarisation. We require that those data satisfy some non-trivial relations. In the case where X is a curve, the theorem of support decomposition (see [SS22b, Chapter 7]) becomes rather simple, and we will use it as a working definition.

In the following X will denote a compact Riemann surface, Σ will be a finite set of points of X , and we set $M := X \setminus \Sigma$.

Definition 5.2.3. A polarised Hodge module X of weight w with singularities at most at Σ and of pure support X is a middle extension of a polarised variation of Hodge structure on M of weight $w - 1$.

In this case, the filtered \mathcal{D}_X module is given by \mathcal{V}_{mid} endowed with the Hodge filtration, and the perverse sheaf is given by $j_* \mathbb{V}[1]$ where $j: M \rightarrow X$ is the inclusion and \mathbb{V} is the underlying local system.

This definition of a polarised Hodge module with pure support X is not the usual one, however, it is equivalent to the usual one thanks to [SS22b, Proposition 7.4.12]. The shift in the weight is there because we will consider the perverse de Rham complex ${}^p DR(\mathcal{V}_{\text{mid}})$ which is a shift of the usual de Rham complex.

We also need to define polarised Hodge modules with punctual support Σ (also called modules with strict support Σ). Taking \mathcal{H}_Σ a sheaf of polarised Hodge structure of weight w on Σ , it is equivalent to fixing a polarised Hodge structure $(H_{\Sigma,p}, F^\bullet H_{\Sigma,p}, h_p)$ for every point $p \in \Sigma$. We define ${}_{D^i} \mathcal{H}_\Sigma$ as a skyscraper sheaf supported in Σ , the stalk at $p \in \Sigma$ is given by

$${}_{D^i} \mathcal{H}_{\Sigma,p} := H_{\Sigma,p}[\partial_z].$$

We define $\partial_z \cdot (v \partial_z^k) = v \partial_z^{k+1}$, and for $n \in \mathbb{N}$ we define the action of $\mathbb{C}[z]$ on $H_{\Sigma,p}$ by

$$z^n \cdot v \partial_z^k = \begin{cases} 0 & \text{if } n \geq k \\ (-1)^n k(k-1) \dots (k-n+1) v \partial_z^{k-n} & \text{otherwise} \end{cases}$$

It extends naturally to an action of $\mathcal{O}_{X,p}$, moreover one has the relation

$$f \cdot (\partial_z \cdot v \partial_z^k) + f' \cdot v \partial_z^k = \partial_z \cdot (f \cdot v \partial_z^k),$$

so this endows ${}_{D^i}(\mathcal{H}_\Sigma)$ with the structure of a \mathcal{D}_X -module. It is endowed with the good filtration

$$F^p {}_{D^i} \mathcal{H}_{\Sigma,p} := \bigoplus_{k \in \mathbb{N}} (F^{p-k} H_{\Sigma,p}) \partial_z^k.$$

Now we define polarised Hodge modules with punctual support Σ .

Definition 5.2.4. A polarised (pure) Hodge module \mathcal{M} of weight w with punctual support Σ is a \mathcal{D}_X -module of the form ${}_{D^i} \mathcal{H}_\Sigma$ where \mathcal{H}_Σ is a sheaf of polarised Hodge structure of weight w on the finite set Σ .

Now we recall the support decomposition theorem for polarised Hodge modules on Riemann surfaces (see [SS22b, Theorem 7.4.10]).

Theorem 5.2.5. A polarised Hodge module \mathcal{M} of weight w on X with singularities at most at Σ is a \mathcal{D}_X -module \mathcal{M} of the form $\mathcal{M}_1 \oplus \mathcal{M}_2$ where \mathcal{M}_1 (resp. \mathcal{M}_2) is a polarised Hodge module of weight w with pure support X (resp. with punctual support Σ).

3 L^2 -direct image of polarised Hodge module

In this section we assume that the cover $\pi: \tilde{M} \rightarrow M$ is induced by an unramified covering $\pi: \tilde{X} \rightarrow X$, we set $\Sigma := X \setminus M$. We recall the following result of [Eys22] that we state in the case of polarisable (pure) Hodge module. We first recall the definition of the Abelian category $E_f(\Gamma)$ that was introduced by Farber and Lück.

Definition 5.3.6. The category $E_f(\Gamma)$ is the category whose objects are triple (E_1, E_2, e) where E_1 and E_2 are finitely generated $\mathcal{N}(\Gamma)$ -Hilbert modules and $e: E_1 \rightarrow E_2$ is a bounded $\mathcal{N}(\Gamma)$ -equivariant morphism.

If $E = (E_1, E_2, e)$ and $F = (F_1, F_2, f)$ are two objects of $E_f(\Gamma)$, the set $\text{Hom}_{E_f(\Gamma)}(E, F)$ is the set of pair (ϕ_1, ϕ_2) where $\phi_1: E_1 \rightarrow F_1$, $\phi_2: E_2 \rightarrow F_2$ are morphisms of $\mathcal{N}(\Gamma)$ -Hilbert modules satisfying $\phi_2 e = f \phi_1$, under the equivalence relation $(\phi_1, \phi_2) \sim (\psi_1, \psi_2)$ if there exists $T: F_2 \rightarrow E_1$ such that $\psi_2 - \phi_2 = fT$.

Proposition 5.3.7. [Far96] The category $E_f(\Gamma)$ is Abelian.

Remark 5.3.8. The category of $\mathcal{N}(\Gamma)$ -Hilbert modules is embedded in the category of Farber through the functor

$$E \rightarrow (\{0\}, E, 0).$$

Moreover, one has a forgetful functor from the category $E_f(\Gamma)$ to the category of finitely generated $\mathcal{N}(\Gamma)$ -modules given by

$$(E_1, E_2, e) \rightarrow E_2/e(E_1).$$

With this, we can state the following result of [Eys22].

Theorem 5.3.9. [Eys22, Corollary 2] Let (X, ω_X) be a compact Kähler manifold, $\pi: \tilde{X} \rightarrow X$ a Galois group of Deck group Γ . We denote by $pHM(X)$ the category of polarised Hodge modules on X . There exists a ∂ -functor satisfying the Atiyah index theorem and Poincaré–Verdier duality

$$L^2 dR: D^b pHM(X) \rightarrow D^b E_f(\Gamma)$$

such that for all Hodge modules \mathbb{M} and $q \in \mathbb{Z}$

$$H^q(L^2 dR(\mathcal{M})) \simeq H_{(2)}^q(\tilde{X}, \pi^* \mathbb{M}^{\text{Betti}}).$$

These cohomology groups are endowed with a filtration F^\bullet induced by the Saito Hodge filtration on \mathcal{M} .

We recall the definition of functor L^2dR and some of its properties. For this we need to recall the definition of the functor $\ell^2\pi_*$ of L^2 -direct image as introduced in [CD01, Eys00]. It is a functor from the category of coherent \mathcal{O}_X module to the category of $\mathcal{N}(\Gamma)_X$ -modules (here $\mathcal{N}(\Gamma)_X$ denotes the constant sheaf of algebra with germ equal to $\mathcal{N}(\Gamma)$) where if \mathcal{F} is a coherent \mathcal{O}_X -module, and U is a coordinates chart such that $\pi^{-1}(U) \simeq \Gamma \times U$, and we fix $\phi: \mathcal{O}_{X|U}^{\oplus N} \rightarrow \mathcal{F}_U$ a local presentation, and we have

$$\ell^2\pi_*\mathcal{F}(U) := \left\{ (s_\gamma)_\gamma \in \mathcal{F}(U)^\Gamma \mid \begin{array}{l} \exists (r_\gamma)_\gamma \in \mathcal{O}_X(U)^\Gamma, \quad \phi(r_\gamma) = s_\gamma \\ \sum_{\gamma \in \Gamma} \int_K |r_\gamma|^2 < +\infty \quad \forall K \subset U \text{ compact} \end{array} \right\}.$$

It can be checked that it does not depend on the local presentation and since X is compact, $\ell^2\pi_*$ does not depend on the Kähler metric ω_X , moreover, by definition $\ell^2\pi_*\mathcal{F}$ is a subsheaf of $\pi_*\pi^*\mathcal{F}$.

Lemma 5.3.10. [Eys22, Lemma 2.1.1] The functor $\ell^2\pi_*$ can be extended in an exact functor in the category $Mod(\mathcal{O}_X)$ of \mathcal{O}_X -modules by setting

$$\ell^2\pi_*\mathcal{F} := \ell^2\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}.$$

We need to extend this functor in the category of $Qcoh(\mathcal{O}_X, Diff_X)$ of quasi-coherent \mathcal{O}_X -module whose morphisms are differential operators. We recall that a differential operator $P: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is an operator in the image of

$$\nu: \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{D}_X) \rightarrow \text{Hom}_{\mathbb{C}_X}(\mathcal{F}_1, \mathcal{F}_2)$$

where the structure of \mathcal{O}_X -module on $\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is given by the right \mathcal{O}_X -module structure and ν is given by the left composition by the morphism

$$\begin{aligned} \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{D}_X &\rightarrow \mathcal{F}_2 \\ f \otimes P &\mapsto P(1)f \end{aligned}$$

Lemma 5.3.11. [Eys22, Lemma 2.2.2] Let $P := \nu(p): \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a differential operator, the morphism $\ell^2\pi_*P := \nu(\ell^2\pi_*p): \ell^2\pi_*\mathcal{F}_1$ to $\ell^2\pi_*\mathcal{F}_2$ is the restriction of $\pi_*\pi^*P: \pi^*\pi^*\mathcal{F}_1 \rightarrow \pi_*\pi^*\mathcal{F}_2$, and it is a morphism of sheaves of $\mathcal{N}(\Gamma)$ -module. Therefore, $\ell^2\pi_*$ defines an additive functor from the category $Qcoh(\mathcal{O}_X, Diff_X)$ to the category $Mod(\mathcal{N}(\Gamma)_X)$.

With this lemma, one can define L^2dR as the functor $R\Gamma \circ \ell^2\pi_*{}^pDR(\cdot)$. On the Betti side, one can understand it as the functor that sends $\mathbb{M}^{\text{Betti}}$ to $\ell^2\pi_*\mathbb{M}^{\text{Betti}} := \ell^2(\Gamma) \otimes_{\mathbb{C}[\Gamma]} \pi_!\pi^*\mathbb{M}^{\text{Betti}}$.

Proposition 5.3.12. Let \mathcal{V}_{mid} be a polarised Hodge module on X with pure support X and weight w . One has a natural filtered inclusion

$$F^\bullet\Omega^\bullet(\pi^*\mathbb{V})_{(2)} \rightarrow \ell^2\pi_*F^\bullet DR(\mathcal{V}_{\text{mid}})$$

which is filtered quasi-isomorphism.

Proof. We recall that in the case of trivial covering, the following inclusion is a quasi-isomorphism for all $p \in \mathbb{Z}$ by Proposition 5.1.2

$$F^p\Omega^\bullet(\mathbb{V})_{(2)} \rightarrow F^p DR(\mathcal{V}_{\text{mid}}).$$

We want to check that we obtain quasi-isomorphisms

$$F^p\Omega^\bullet(\pi^*\mathbb{V})_{(2)} \rightarrow \ell^2\pi_*F^p DR(\mathcal{V}_{\text{mid}}).$$

The result will follow from the functoriality of $\ell^2\pi_*$ given by Lemma 5.3.11. One has a natural isomorphism

$$\Omega^\bullet(\pi^*\mathbb{V})_{(2)} \rightarrow \ell^2\pi^*\Omega(\mathbb{V})_{(2)}$$

so one has to check that $\ell^2\pi_*D^{1,0}$ induces an isomorphism

$$[\ell^2\pi_*D^{1,0}]: \ell^2\pi_* \left(F^p V^0 \mathcal{V}_{\text{mid}} / F^p \mathcal{O}(\mathbb{V})_{(2)} \right) \rightarrow \ell^2\pi_* \left(F^{p-1} V^{-1} \mathcal{V}_{\text{mid}} / F^p \Omega^1(\mathbb{V})_{(2)} \right)$$

which is true because $[\ell^2\pi_*D^{1,0}]$ coincides with $\ell^2\pi_*[D^{1,0}]$ where

$$[D^{1,0}]: F^p V^0 \mathcal{V}_{\text{mid}} / F^p \mathcal{O}(\mathbb{V})_{(2)} \rightarrow F^{p-1} V^{-1} \mathcal{V}_{\text{mid}} / F^p \Omega^1(\mathbb{V})_{(2)}$$

is the morphism induced by $D^{1,0}$ and $[D^{1,0}]$ is an isomorphism by Proposition 5.1.2. \square

Finally, one obtains the following result, which gives a positive answer to [Eys22, Conjecture 3] for the case of pure Hodge module on Riemann surfaces.

Theorem 5.3.13. Let \mathbb{M} be a Hodge module on a compact Riemann surface X and $\pi: \tilde{X} \rightarrow X$ be a Galois covering of Deck group $\text{Deck}(\tilde{X}/X) = \Gamma$. Then the groups $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathbb{H}^k(L^2dR(\mathbb{M}))$ admit a pure Hodge structure of weight $w + k$ in the Abelian category of $\mathcal{U}(\Gamma)$ -modules. The Hodge filtration is induced by Saito's Hodge filtration on the perverse de Rham complex ${}^pDR(\mathcal{M})$.

In the case where \mathbb{M} is a polarised Hodge module with pure support X and with singularities at most at Σ and one set $M = X \setminus \Sigma$ and $\tilde{M} = \pi^{-1}(M)$ there is an isomorphism of $\mathcal{U}(\Gamma)$ -modules

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathbb{H}^k(L^2dR(\mathbb{M})) \simeq \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_{(2),\text{red}}^k(\tilde{M}, \pi^*\mathbb{V}, \pi^*\omega_{P_C}, \pi^*h)$$

which is an isomorphism of Hodge structures, where the Hodge structure on the reduced L^2 -cohomology comes from the decomposition of harmonic forms by type.

Proof. The above proposition and Theorem 5.2.5 treat the case where \mathbb{M} has pure support X , so in the following we can assume that \mathbb{M} has punctual support Σ . In this case \mathbb{M} is of the form $i\mathcal{H}_\Sigma$ where \mathcal{H}_Σ is a sheaf of polarised Hodge structure of weight w on Σ . In this case $H^k(L^2dR(\mathbb{M})) = 0$ unless $k = 0$ in which case it is equal to

$$\bigoplus_{p \in \Sigma} \ell^2(\Gamma) \otimes_{\mathbb{C}} H_{\Sigma,p}.$$

so the cohomology is fully reduced and one has a Hodge structure on $H^0(L^2dR(\mathbb{M}))$ of weight w on the category of $\mathcal{N}(\Gamma)$ -module given by the Hodge structure on each $H_{\Sigma,p}$, this gives the desired Hodge structure on $\mathcal{U}(\Gamma) \otimes H^0(L^2dR(\mathbb{M}))$. \square

Appendices A

Appendices

1 Functional analysis on Hilbert spaces

This section is devoted to giving the usual results and notions in the theory of Hilbert spaces. All Hilbert spaces considered will be defined over \mathbb{C} . We present some results in Hilbert analysis; in particular, we define the notion of closed, densely defined operator, and projection-valued measure. We state the polar decomposition and the spectral theorem for closed normal operators. The main references followed for this appendix are the books of Conway [Con85] and Reed and Simon [RS81], and some results are also taken from [Dix57].

The algebra of bounded operators. Let H_1 and H_2 be Hilbert spaces. We denote by $\mathcal{B}(H_1, H_2)$ the space of bounded operators from H_1 to H_2 , i.e. the space of continuous linear morphisms $u: H_1 \rightarrow H_2$. We recall that this space is endowed with a natural norm, given by

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

If $T: H_1 \rightarrow H_2$ is a bounded operator, its adjoint is the bounded operator $T^*: H_2 \rightarrow H_1$ defined by the property

$$\langle T(x), y \rangle_2 = \langle x, T^*(y) \rangle_1.$$

This adjoint is unique by the Riesz theorem, and the adjunction defines a conjugate linear isometry $*$: $\mathcal{B}(H_1, H_2) \rightarrow \mathcal{B}(H_2, H_1)$. The most interesting case is where $H_1 = H_2 = H$ in this case $\mathcal{B}(H, H) =: \mathcal{B}(H)$ is an algebra and the adjunction endows it with a structure of involutive algebra. More precisely, we have

Proposition A.1.1. The space $\mathcal{B}(H)$ is a stellar algebra (also called C^* -algebra), i.e. it is an algebra endowed with a norm $\|\cdot\|$ and a conjugate-linear involution $*$ such that

- $\mathcal{B}(H)$ is complete for the topology induced by the norm.
- For all $T \in \mathcal{B}(H)$ one has $\|T^*T\| = \|T\|^2$
- For all $T, S \in \mathcal{B}(H)$ one has $\|TS\| \leq \|T\| \cdot \|S\|$.
- For all $T, S \in \mathcal{B}(H)$ one has $(TS)^* = S^*T^*$.

Definition A.1.2. An operator $T \in \mathcal{B}(H)$ is said to be

- normal if $T^*T = TT^*$
- self-adjoint if $T = T^*$
- unitary if u is invertible and $T^* = T^{-1}$

It is often necessary to consider other natural topologies on $\mathcal{B}(H_1, H_2)$. In particular, we will focus on the strong and weak topologies that appear in the definition of Von Neumann algebras.

Definition A.1.3. The strong topology on $\mathcal{B}(H_1, H_2)$ is the topology induced by the family of seminorms $(T \mapsto \|T(x)\|)_{x \in H_1}$.

The weak topology on $\mathcal{B}(H_1, H_2)$ is the topology induced by the family of seminorms $(T \mapsto |\langle T(x), y \rangle|)_{x \in H_1, y \in H_2}$.

Since these topologies are defined by a family of seminorms, they are compatible with the vector space structure of $\mathcal{B}(H_1, H_2)$, however, only the weak topology is compatible with both adjunction and composition. More precisely,

Proposition A.1.4. [Dix57, Chapter 2 §3] For all $T \in \mathcal{B}(H)$, the operators given by multiplication by T

$$S \mapsto ST \quad S \mapsto TS$$

are continuous for the strong and weak topologies.

The adjunction $*$: $\mathcal{B}(H) \rightarrow \mathcal{B}(H)$ is continuous for the weak topology.

For an operator $T \in \mathcal{B}(H)$, we define its resolvent set to be

$$\rho(T) = \{ \lambda \in \mathbb{C} \mid T - \lambda \text{ is an isomorphism} \}.$$

Its spectra $\sigma(T)$ is the space $\mathbb{C} \setminus \rho(T)$. It is a closed subspace contained in $\overline{B(0, \|T\|)}$, hence compact.

Projection-valued measure and the spectral theorem for bounded operator. In this paragraph, we present the notion of projection-valued measure; this notion is central to generalise the spectral theorem from finite-dimensional Hermitian spaces to arbitrary Hilbert spaces. In this section, if H is a Hilbert space, then $\text{Proj}(H)$ denotes the space of self-adjoint projectors of H .

Definition A.1.5. Let H be a Hilbert space, and (X, \mathcal{F}) be a measurable space. A measure on X with values in the self-adjoint projectors of H is a map

$$E: \mathcal{F} \rightarrow \text{Proj}(H)$$

satisfying the following properties

1. $E(\emptyset) = 0$ and $E(X) = id_H$.
2. If $(A_i)_{i \in I}$ is a countable family of measurable sets such that $A_i \cap A_j = \emptyset$ for all $i \neq j$ then

$$E\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} E(A_i)$$

3. If $A, B \in \mathcal{F}$ then $E(A \cap B) = E(A)E(B)$.

Given a projection-valued measure E , we say that a subset $N \subset X$ is E -negligible (or negligible if no confusion can occur) if there exists A measurable such that $E(A) = 0$ and $N \subset A$. We can then define the notion of a property being true E -almost everywhere, as we do in the usual measure theory. If (X, \mathcal{F}) is a Borel space, one can define the support of E as we would do for a usual measure.

Denote by $S(X)$ the space of simple functions on X , it is the linear span of the characteristic functions of measurable subsets of X . Letting $\mathcal{M}_b(X)$ be the space of bounded measurable functions on X , it has a natural structure of an Abelian involutive algebra, where the multiplication is given by pointwise multiplication and the involution $*$ is given by $f^*(x) = \overline{f(x)}$. The norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

endows $\mathcal{M}_b(X)$ with the structure of a stellar algebra. One has a well-defined mapping

$$\begin{aligned} \int_X \cdot dE(x): \quad S(X) &\longrightarrow \mathcal{B}(H) \\ f = \sum_{j=1}^n \lambda_j \mathbb{1}_{A_j} &\longmapsto \int_X f(x) dE(x) := \sum \lambda_j E(A_j) \end{aligned}$$

where in the sum the λ_j are complex numbers and the A_j are measurable subsets of X pairwise disjoint. This mapping is a morphism of involutive algebra and one has

$$\left\| \int_X f(x) dE(x) \right\| \leq \|f\|_\infty.$$

One can then deduce

Proposition A.1.6. The application $\int_X \cdot dE(x)$ extends uniquely to a morphism of stellar algebra

$$\int_X \cdot dE(x): \mathcal{M}_b(X) \rightarrow \mathcal{B}(H).$$

Since $\mathcal{M}_b(X)$ is Abelian, one has that $\int_X f dE(x)$ is normal for all f . It is straightforward to check that

- $\int_X f(x) dE(x)$ is unitary if and only if $f(x) \in \mathbb{S}^1$ $E(x)$ -almost everywhere.
- $\int_X f(x) dE(x)$ is unitary if and only if $f(x) \in \mathbb{R}$ $E(x)$ -almost everywhere.
- $\int_X f(x) dE(x)$ is invertible if and only if $1/f(x)$ is bounded $E(x)$ -almost everywhere.

Given a projection-valued measure E and for $u, v \in H$ one can construct a finite complex-valued measure $E_{u,v}$ on X defined by

$$E_{u,v}(B) = \langle E(B)u, v \rangle \quad \forall B \in \mathcal{F}.$$

Proposition A.1.7. Let $f \in \mathcal{M}_b(X)$ and $T = \int_X f(x) dE(x)$ then T is the unique operator satisfying

$$\langle Tu, v \rangle = \int_X f(x) dE_{u,v}(x) \quad \forall u, v \in H.$$

The importance of projection-valued measures is illustrated by the spectral theorem stated below.

Theorem A.1.8 (Spectral theorem for bounded operator). [Con85, Chapter 9, Theorem 2.2] Let T be a normal bounded operator, then there is a unique projection valued measure E_T on \mathbb{C} , whose support is $\sigma(T)$ satisfying

1. $T = \int_{\mathbb{C}} z dE_T(z)$
2. If $S \in \mathcal{B}(H)$, S commutes with T and T^* if and only if S commutes with all the $E_T(A)$ for $A \subset \mathbb{C}$ measurable.

Remark A.1.9. The map $z \mapsto z$ is not bounded in \mathbb{C} , however, since $\sigma(T)$ is compact, it is bounded E -almost everywhere and the integral $\int_{\mathbb{C}} z dE_T(z)$ makes sense.

We will see in the next paragraph how this theorem extends to operators that are only densely defined.

Closed operators and the spectral theorem. In this paragraph, we recall basic definitions of unbounded operators. We refer the reader to [RS81, Chapter VIII] or [Con85, Chapter X] for more details.

Definition A.1.10. Let H_1, H_2 be Hilbert spaces. An operator $T: H_1 \dashrightarrow H_2$ is a linear morphism $T: \text{Dom}(T) \rightarrow H_2$ where $\text{Dom}(T)$ is a linear subspace of H_1 called the domain of T . An operator T is said to be

- densely defined, if $\text{Dom}(T)$ is dense in H_1 .
- closed, if $\text{Graph}(T) := \{(x, T(x)) \mid x \in \text{Dom}(T)\}$ is closed in $H_1 \oplus H_2$.
- closable, if the closure of $\text{Graph}(T)$ is the graph of an operator.
- unbounded if $T \notin \mathcal{B}(H)$.

We say that an operator T is contained in an operator S and denote $T \subset S$ if $\text{Graph}(T) \subset \text{Graph}(S)$. If T is closable, there exists a closed operator \bar{T} containing T that is minimal in the sense that if $T \subset S$ with S closed, then $\bar{T} \subset S$. The graph of the operator \bar{T} is then given by the closure of the graph of u . We set $\mathcal{C}(H_1, H_2)$, the space of densely defined closed operators from H_1 to H_2 . By the closed graph theorem, a closed operator whose domain is H_1 is bounded.

Definition A.1.11.

1. Let $T, S: H_1 \dashrightarrow H_2$ be operators. We define the operator $T + S$ as the operator with domain $\text{Dom}(T) \cap \text{Dom}(S)$ defined by $(T + S)(x) = T(x) + S(x)$.
2. Let $T: H_1 \dashrightarrow H_2$ and $S: H_2 \dashrightarrow H_3$ be operators. We define ST as the operator with domain

$$\text{Dom}(ST) = \{x \in \text{Dom}(T) \mid u(x) \in \text{Dom}(S)\}$$

defined by $ST(x) = S(T(x))$.

It is to be noted that a composition or a sum of densely defined closed operators need not be a densely defined closed operator due to some problem with the domain. However, if $T \in \mathcal{C}(H_1, H_2)$ and $S \in \mathcal{B}(H_1, H_2)$, then $T + S \in \mathcal{C}(H_1, H_2)$.

If T is a densely defined operator, the space

$$\text{Graph}(T^*) = \left\{ (y, x) \in H_2 \oplus H_1 \mid (x, -y) \in \text{Graph}(T)^\perp \right\}$$

is the graph of an operator T^* called the adjoint of T . This notion coincides with the usual notion of adjunction for bounded operators. By construction, if T is closable, one has $T^* = \bar{T}^*$. The adjoint of an operator T might not be densely defined, and hence it may not be possible to consider its adjoint. The proposition below characterises the space of operator whose adjoint is densely defined.

Proposition A.1.12. [RS81, Theorem VIII.1] Let $T: H_1 \dashrightarrow H_2$ be a densely defined operator. Then one has :

- T^* is closed
- $\text{Dom}(T^*)$ is dense if and only if T is closable.
- If T is closable, then $\bar{T} = (T^*)^*$

This proposition tells us that the adjunction $*$ defines an involution $\mathcal{C}(H_1, H_2) \rightarrow \mathcal{C}(H_2, H_1)$.

Definition A.1.13. Let $T: H \dashrightarrow H$ be a densely defined operator. Then T is said to be

- symmetric if $T \subset T^*$.
- self-adjoint if $T^* = T$.
- normal if $T^*T = TT^*$.
- essentially self-adjoint if it is closable and its minimal closure is self-adjoint.

We remind the reader that the equality between operators implies the equality of the domains.

Definition A.1.14. For $T \in \mathcal{C}(H)$ its resolvent set is the space

$$\rho(T) = \{ \lambda \in \mathbb{C} \mid T - \lambda: \text{Dom}(H) \rightarrow H \text{ is a bijection} \}.$$

The spectra of T is again defined as $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

We recall a standard result below.

Proposition A.1.15. The range $\text{Ran}(T)$ of a closed densely defined operator T is closed if and only if there exists $c > 0$ such that for all $x \in \text{Dom}(T) \cap \text{Ker}(T)^\perp$ one has

$$\|T(x)\| \geq c\|x\|^2.$$

From which one can deduce the

Corollary A.1.16. Let $T \in \mathcal{C}(H)$ is a closed densely defined operator and $\lambda \in \rho(T)$ then $T - \lambda$ admits a bounded inverse

$$(T - \lambda)^{-1}: H \rightarrow \text{Dom}(T) \subset H.$$

It is well known that the spectra of a self-adjoint operator is contained in \mathbb{R} . An important theorem of Von Neumann gives us a way to construct an example of self-adjoint operators.

Theorem A.1.17. [Con85, Proposition 4.2] Let $T \in \mathcal{C}(H_1, H_2)$ then

1. T^*T is self-adjoint.
2. $1 + T^*T$ admits a bounded inverse B satisfying $\|B\| \leq 1$ and $B \geq 0$.
3. $\text{Dom}(T^*T)$ is dense in $\text{Dom}(T)$ for the graph norm $\|\cdot\|_T$ that is, the norm given by $\|x\|_T^2 = \|x\|^2 + \|T(x)\|^2$.

As for the bounded case, one has the

Proposition A.1.18. Let $T: H \dashrightarrow H$ be a self-adjoint operator, then one has an orthogonal decomposition

$$H = \text{Ker}(T) \oplus \overline{\text{Ran}(T)}.$$

Definition A.1.19. An operator $T \in \mathcal{C}(H)$ is said to be positive if for all $x \in \text{Dom}(T)$ one has

$$\langle T(x), x \rangle \geq 0.$$

An operator $T \in \mathcal{C}(H)$ is said to be bounded below if there exists $\lambda \in \mathbb{R}$ such that $T - \lambda \text{Id}$ is positive.

A positive operator is necessarily self-adjoint. For a self-adjoint operator u to be bounded below is equivalent to asking if there exists $\lambda \in \mathbb{R}$ such that $\sigma(T) \subset [\lambda, +\infty[$.

Example A.1.20.

1. Every self-adjoint projection is positive.
2. If $T \in \mathcal{C}(H)$ then T^*T is positive.

Theorem A.1.21 (Polar decomposition). [RS81, Theorem VIII.32] Let $T \in \mathcal{C}(H_1, H_2)$ be an operator. Then there exists a unique couple $(U, |T|)$ satisfying the following

1. $U: H_1 \rightarrow H_2$ is bounded and induces an isometry $U: \text{Ker}(T)^\perp \rightarrow \overline{\text{Ran}(T)}$.
2. $|T|: H_1 \rightarrow H_1$ is a closed positive operator.
3. $\text{Ker}(U) = \text{Ker}(|T|) = \text{Ker}(T)$
4. $T = U|T|$.

We have seen how to construct a normal bounded operator by integrating a measurable bounded function with respect to a projection-valued measure E . We will now explain how to construct normal closed operator by integrating arbitrary measurable functions. This is based on the following lemma.

Lemma A.1.22. [Con85, Chapter X, 4.6] Let $f: X \rightarrow \mathbb{C}$ be a measurable function, then

$$\mathcal{D}_f = \{u \in H \mid f \in L^2(X, dE_{u,u})\}$$

is dense in \mathcal{H} .

Definition A.1.23. Let (X, \mathcal{F}) be a measurable space, and let $E: \mathcal{F} \rightarrow \text{Proj}(H)$ be a projection-valued measure. For $f: X \rightarrow \mathbb{C}$ measurable, we define the operator $\int_X f dE$ taking its domain to be \mathcal{D}_f and defined by the property that for all $u, v \in \mathcal{D}_f$ we have

$$\left\langle \left(\int_X f dE \right) u, v \right\rangle = \int_X f dE_{u,v}.$$

Theorem A.1.24. [Con85, Theorem 4.10] Let (X, \mathcal{F}) be a measurable space, and E be a projection-valued measure. For all $f: X \rightarrow \mathbb{C}$ measurable, we set $\rho(f) := \int_X f dE$. Then one has for any f, g measurable

- $\rho(f)$ is normal, in particular it is closed and densely defined.
- $\rho(f)^* = \rho(\bar{f})$.
- $\rho(f)\rho(g) \subset \rho(fg)$ and $\text{Dom}(\rho(f)\rho(g)) = \text{Dom}(\rho(g)) \cap \text{Dom}(\rho(fg))$.
- $\rho(f)^*\rho(f) = \rho(|f|^2)$.

As before, it is straightforward to verify that $\rho(f)$ is self-adjoint if and only if f is real valued almost everywhere. We can now state the spectral theorem for normal operators.

Theorem A.1.25 (The spectral theorem). [Con85, Theorem 4.11] Let $T \in \mathcal{C}(H)$ be a normal operator. There exists a unique projection valued measure E_T on \mathbb{C} such that

- $T = \int_{\mathbb{C}} z dE_T(z)$.
- $\text{supp}(E_T) = \sigma(T)$.

For the case of bounded operators, if f is measurable and T is normal, we define $f(T)$ as the operator

$$\int_{\mathbb{C}} f(z) dE_T(z).$$

Hilbert tensor product. This subsection is dedicated to basic results on the Hilbert tensor product of closed operators. We refer the reader to [RS81, Chapter VIII, Section 10] for more details.

If H_1 and H_2 are two Hilbert spaces, we will denote by $H_1 \otimes H_2$ their algebraic tensor product. If $\langle \cdot, \cdot \rangle_j$ denotes the inner product of H_j one has a canonical inner product on $H_1 \otimes H_2$ given by

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle_1 \langle x_2, y_2 \rangle_2.$$

Definition A.1.26. We define the Hilbert tensor product $H_1 \hat{\otimes} H_2$ as the Hilbert space obtained after the completion of $H_1 \otimes H_2$.

It is clear that if E_1 (resp. E_2) is a dense subspace of H_1 (resp. H_2) then $E_1 \otimes E_2$ is dense in $H_1 \hat{\otimes} H_2$.

Let $T_1: H_1 \dashrightarrow K_1$ and $T_2: H_2 \dashrightarrow K_2$ be densely defined operators. Then it defines a densely defined operator $T_1 \otimes T_2: H_1 \hat{\otimes} H_2 \dashrightarrow K_1 \hat{\otimes} K_2$ with domain $\text{Dom}(u) \otimes \text{Dom}(v)$.

Proposition A.1.27. If T_1 and T_2 are closable, then $T_1 \otimes T_2$ is closable. We denote by $T_1 \hat{\otimes} T_2$ its minimal closure.

If T_1 and T_2 are bounded, then $T_1 \hat{\otimes} T_2$ is bounded and one has

$$\|T_1 \hat{\otimes} T_2\| \leq \|T_1\| \|T_2\|.$$

If $H_j = K_j$, we will also denote by $T_1 + T_2$ the minimal closure operator $T_1 \otimes \text{Id}_{H_2} + \text{Id}_{H_1} \otimes T_2$ defined on $\text{Dom}(T_1) \otimes \text{Dom}(T_2)$.

The Hilbert tensor product of closed operators is compatible with adjunction and polar decomposition as stated in the following proposition (see [KR86, Proposition 12.2.35 & Proposition 12.2.37])

Proposition A.1.28. Let $T_1: H_1 \dashrightarrow H_1$ and $T_2: H_2 \dashrightarrow H_2$ be two closed densely defined operators then one has

- If $T_j = U_j S_j$ is the polar decomposition of T_j then the polar decomposition of the operator $T_1 \hat{\otimes} T_2$ is given by $(U_1 \hat{\otimes} U_2)(S_1 \hat{\otimes} S_2)$.
- $(T_1 \hat{\otimes} T_2)^* = T_1^* \hat{\otimes} T_2^*$.

Let H_1, \dots, H_n be Hilbert spaces and $T_k \in \mathcal{C}(H_k)$ be self-adjoint operators for all $j > 0$ the operator T_k^j is densely defined by Theorem A.1.17 and if $j_1 \leq j_2$ then $\text{Dom}(T_k^{j_2}) \subset \text{Dom}(T_k^{j_1})$. Therefore, given $P \in \mathbb{C}[X_1, \dots, X_n]$ we can make sense of the expression $P(T_1, \dots, T_n)$ as a closed operator on $H_1 \hat{\otimes} \dots \hat{\otimes} H_n$. The main result of this paragraph is

Theorem A.1.29. [RS81, Theorem VIII.33] Let T_1, \dots, T_n and P as above. Moreover, assume that P has real coefficients. Then

- $P(T_1, \dots, T_n)$ is self-adjoint.
- The spectra of $P(T_1, \dots, T_n)$ is given by $\overline{P(\sigma(T_1), \dots, \sigma(T_n))}$.

Moreover, if for all $j \in \{1, \dots, n\}$ we fix K_j a subspace of H_j dense for the graph norm of T_j then $K_1 \otimes \dots \otimes K_n$ is dense for the graph norm of $P(T_1, \dots, T_n)$

This theorem is useful, and we also give a statement involving the spectral measure. As before, we take T_1, \dots, T_n to be self-adjoint operators, where T_j acts on a Hilbert space H_j . We denote by $E_j: \text{Bor}(\mathbb{R}) \rightarrow \text{Proj}(H_j)$ the projection-valued measure associated with T_j . We can then consider the projection-valued measure

$$E_1 \otimes \dots \otimes E_n: \text{Bor}(\mathbb{R}^n) \rightarrow \text{Proj}(H_1 \hat{\otimes} \dots \hat{\otimes} H_n)$$

which is defined by the property that for all Borel subsets B of the form $B = B_1 \times \dots \times B_n$ one has

$$E_1 \otimes \dots \otimes E_n(B) = E_1(B_1) \hat{\otimes} \dots \hat{\otimes} E_n(B_n).$$

The existence of such a measure is given in [Sti59, Section 8].

Theorem A.1.30. Let T_1, \dots, T_n and P as in the previous theorem and E_1, \dots, E_n as above. Then we have

$$P(T_1, \dots, T_n) = \int_{\mathbb{R}^n} P(\lambda_1, \dots, \lambda_n) dE_1 \otimes \dots \otimes dE_n$$

Proof. The proof is given in [Sti59, Theorem 8.2] for the case of the operator $T_1 \otimes T_2$, although the proof can be easily generalised, we present it here for the sake of completeness. By the previous theorem, both $P(T_1, \dots, T_n)$ and the operator defined by the integral are self-adjoint, so it is sufficient to check that $P(T_1, \dots, T_n)$ is contained in the operator defined by the integral. Set $d = \deg(P)$ and A as the operator defined by the integral. We recall that $\text{Dom}(T_1^d) \otimes \dots \otimes \text{Dom}(T_n^d)$ is dense for the graph norm of $P(T_1, \dots, T_n)$, therefore we need to check that for all $u \in \text{Dom}(T_1^d) \otimes \dots \otimes \text{Dom}(T_n^d)$ we have $u \in \text{Dom}(A)$ and $Au = P(T_1, \dots, T_n)u$. For the first point, we set such a $u = u_1 \otimes \dots \otimes u_n$, there exists a constant $c > 0$ such that for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ one has

$$|P(\lambda_1, \dots, \lambda_n)| \leq c(|\lambda_1|^d + \dots + |\lambda_n|^d + 1)$$

By the definition the measure $dE = dE_1 \otimes \cdots \otimes dE_n$, one has $dE_{u,u} = dE_{1,u_1,u_1} \otimes \cdots \otimes dE_{n,u_n,u_n}$ the last tensor product being the usual product of finite measures. Since λ_j^d is integrable for dE_{j,u_j,u_j} , it follows that $P(\lambda_1, \dots, \lambda_n)$ is integrable for the measure dE by Fubini–Tonneli and $u \in \text{Dom}(A)$.

To show that $Au = P(T_1 \otimes \cdots \otimes T_n)u$, it is sufficient to check that for all $v \in H_1 \otimes \cdots \otimes H_n$ with $v = v_1 \otimes \cdots \otimes v_n$ one has

$$\langle P(T_1 \otimes \cdots \otimes T_n)u, v \rangle = \langle Au, v \rangle = \int_{\mathbb{R}^n} P(\lambda_1, \dots, \lambda_n) dE_{1,u_1,v_1} \cdots dE_{n,u_n,v_n}.$$

By linearity, it is possible to reduce to the case where P is a monomial. In this scenario, the result can be derived from the Fubini theorem. \square

2 Measurable fields of Hilbert spaces

In this appendix, we recall some results about direct integral of measurable field of Hilbert space. More details are available in [Dix57, Cho70]. In this section X will be a locally compact Hausdorff topological space and μ a regular positive Borel measure on X . A Hilbert field on X is a family $(H(x))_{x \in X}$ of Hilbert spaces indexed by X .

Definition of direct integral of a measurable field of Hilbert spaces

Definition A.2.31. A measurable field of Hilbert spaces on X is the data of a Hilbert field $(H(x))_{x \in X}$ on X and a linear subspace \mathcal{M} of $\prod_{x \in X} H(x)$ satisfying the following conditions.

1. An element $f \in \prod_{x \in X} H(x)$ is in \mathcal{M} if and only if, for any $g \in \mathcal{M}$, $x \mapsto \langle f(x), g(x) \rangle$ is measurable.
2. There exists a countable family $(f_n)_{n \in \mathbb{N}}$ of elements of \mathcal{M} such that for all $x \in X$ one has $\text{Span}((f_n(x))_{n \in \mathbb{N}})$ is dense in the Hilbert space $H(x)$.

In this case, an element of \mathcal{M} is called a measurable vector field.

Definition A.2.32. We say that a measurable vector field f is square integrable if and only if

$$\int_X \|f(x)\|^2 d\mu(x) < +\infty.$$

We set $\int_X^\oplus H(x) d\mu(x)$ the space of square integrable vector fields on X , where we identify two vector fields if they are equal μ -almost everywhere. We endow it with the Hermitian product defined by

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle d\mu(x).$$

This Hermitian product endows $\int_X^\oplus H(x) d\mu(x)$ with the structure of a Hilbert space, and we call this Hilbert space the direct integral of the measurable field of Hilbert spaces \mathcal{M} .

Example A.2.33. Let H be a separable Hilbert space, set for all $x \in X$ $H(x) = H$ and define the measurable vector fields to be vector fields f such that for any $e \in H$ the application $x \mapsto (f(x), e)$ is measurable. This is called the constant Hilbert field on X , the direct integral $\int_X^\oplus H d\mu(x)$ is then canonically isomorphic to the space $L^2(X, \mu, H)$ of H -valued square-integrable functions, since elements of $\int_X^\oplus H d\mu(x)$ are by definition measurable functions $f: X \rightarrow H$ that are square integrable.

Many direct integrals of Hilbert fields can be reduced to this example thanks to the proposition below [Dix57, Proposition 1 p.143].

Proposition A.2.34. Consider $((H(x))_{x \in X}, \mathcal{M})$ be a measurable vector field on the measured space (X, μ) then there exists a countable family $(f_n)_{n \in \mathbb{N}}$ of measurable vector fields such that for all $x \in X$

1. If $d = \dim H(x) < \infty$ then $(f_1(x), \dots, f_d(x))$ is an orthonormal basis of H_x and $f_n(x) = 0$ if $n > d$.
2. If $H(x)$ is infinite-dimensional, then $((f_n(x))_{n \in \mathbb{N}})$ is a Hilbert basis of $H(x)$.

In particular, for any $d \in \mathbb{N} \cup \{\infty\}$ the subset $\{x \in X \mid \dim H(x) = d\}$ is measurable.

Measurable field of bounded operators

Definition A.2.35. Let $((H_1(x))_{x \in X}, \mathcal{M}_1)$ and $((H_2(x))_{x \in X}, \mathcal{M}_2)$ be two measurable Hilbert fields on X . A measurable field of bounded operators is a family $T_x: H_1(x) \rightarrow H_2(x)$ of bounded operators such that for all $e_1 \in \mathcal{M}_1$ then $x \mapsto T(x)e_1(x) \in \mathcal{M}_2$.

A measurable field of bounded operators $(T(x))_x$ naturally defines a linear map between \mathcal{M}_1 and \mathcal{M}_2 if, moreover, one has $\text{ess. sup } \|T(x)\| < +\infty$, it induces a bounded operator $\int_X^\oplus T(x)d\mu(x): \int_X^\oplus H_1(x)d\mu(x) \rightarrow \int_X^\oplus H_2(x)d\mu(x)$ and one has

$$\left\| \int_X^\oplus T(x)d\mu(x) \right\| = \text{ess. sup } \|T(x)\| < +\infty.$$

Such operators are said to be decomposable. Note that any function $f \in L^\infty(X, \mu)$ induces a decomposable operator

$$\int_X^\oplus f(x)d\mu(x)$$

on any direct integral of the form $\int_X^\oplus H(x)d\mu(x)$, those operators are called diagonalisable operators.

Definition A.2.36. Let $((H(x))_x, \mathcal{M})$ be a measurable field of Hilbert spaces on X . A measurable field of subspaces is the data of a closed linear subspace $K(x) \subset H(x)$ for all $x \in X$ such that if $P(x)$ is the projector onto $K(x)$, then $(P(x))_{x \in X}$ is a measurable field of bounded operator. In this case, if $\mathcal{M}' \subset \mathcal{M}$ is the set of measurable vector fields f such that $f(x) \in K(x)_{x \in X}$ for all x , then $((K(x)), \mathcal{M}')$ is a measurable field of Hilbert space.

Measurable fields of closed operators. We will consider the notion of measurable fields of closed operator. The definition is taken from [Cho70].

Definition A.2.37. A closed field of operators between the measurable fields of the Hilbert spaces $((H_1(x))_{x \in X}, \mathcal{M}_1)$ and $((H_2(x))_{x \in X}, \mathcal{M}_2)$ is the data of $(T(x))_x$ where $T(x): H_1(x) \dashrightarrow H_2(x)$ is a closed operator such that the field of Hilbert spaces $(\text{Graph}(T(x)))_{x \in X}$ is a measurable field of the subspace of $(H_1(x) \times H_2(x), \mathcal{M}_1 \times \mathcal{M}_2)$.

A measurable field of closed operator defines an operator

$$\int_X^\oplus T(x)d\mu(x): \int_X^\oplus H_1(x)d\mu(x) \dashrightarrow \int_X^\oplus H_2(x)d\mu(x)$$

whose graph is given by $\int_X^\oplus \text{Graph}(T(x))d\mu(x)$.

Remark A.2.38. It is straightforward to show that if all $T(x)$ are densely defined then T is also densely defined and that for a measurable field of closed operators $(T(x))_x$ has

$$\left(\int_X^\oplus T(x)d\mu(x) \right)^* := \int_X^\oplus T(x)^*d\mu(x).$$

The following result gives the decomposability of the projection-valued measure of a self-adjoint decomposable operator. It is a particular case of [Cho70, Theorem 4.2].

Proposition A.2.39. Let $x \mapsto T(x)$ be a measurable field of self-adjoint operators, and let $(E_\lambda(x))_{\lambda \in \mathbb{R}}$ be the projector-valued measure associated with $T(x)$. Then $\int_X^\oplus T(x)d\mu(x)$ is self-adjoint and if $(E_\lambda)_{\lambda \geq 0}$ is the associated projection-valued measure, one has

$$E_\lambda = \int_X^\oplus E_\lambda(x)d\mu(x).$$

An application to the study of some differential operators. Here is an example of how the Proposition A.2.39 can be useful for studying the spectra of some differential operators. Let U be an open subset of \mathbb{R}^n and let $w_0, w_1: U \rightarrow \mathbb{R}_{>0}$ be smooth positive functions. By the example A.2.33 one has natural unitary isomorphisms for $j = 0, 1$

$$\begin{aligned} T_j: L^2(\mathbb{R} \times U, w_j(u)dxdu) &\longrightarrow \int_{\mathbb{R}}^\oplus L^2(U, w_j(u)du)dx \\ f &\longmapsto \int_{\mathbb{R}}^\oplus f(x, \cdot)dx \end{aligned}$$

Now consider P a differential operator on U . This defines a differential operator $Q = P + f$ on $\mathbb{R} \times U$, also for each $x \in \mathbb{R}$, it defines a differential operator $Q(x) = P + f(x)$ on U . One has the following.

Lemma A.2.40. Let $Q_{\max} : L^2(\mathbb{R} \times U, w_0(u)dxdu) \dashrightarrow L^2(\mathbb{R} \times U, w_1(u)dxdu)$ be the maximal closure of the differential operator Q . Also define the closed operator $Q(x)_{\max} : L^2(U, w_0(u)du) \dashrightarrow L^2(U, w_1(u)du)$ to be the maximal closure of $Q(x)$. Then

$$T_1 Q_{\max} T_0^{-1} = \int_{\mathbb{R}}^{\oplus} Q(x)_{\max} dx$$

Proof. If A is a closed operator containing Q_{\max} we have $A = Q_{\max}$, hence we only need to prove

$$T_1 Q_{\max} T_0^{-1} \subset \int_{\mathbb{R}}^{\oplus} Q(x)_{\max} dx$$

Set $g \in \text{Dom}(Q_{\max})$, so $Qg \in L^2(\mathbb{R} \times U, w_1(u)dxdu)$, in particular $Qg(x, \cdot) + f(x)g(x, \cdot) \in L^2(\mathbb{R} \times U, w_1(u)dxdu)$ for almost all x by Fubini–Tonelli which means that

$$g(x, \cdot) \in \text{Dom}(Q(x)_{\max}) \quad \text{for almost every } x \in \mathbb{R}$$

so the field $x \mapsto Q_{\max}(x)g(x, \cdot)$ is measurable, and it is also square integrable, so by definition

$$\int_{\mathbb{R}}^{\oplus} g(x, \cdot) dx \in \text{Dom} \left(\int_{\mathbb{R}}^{\oplus} Q(x)_{\max} dx \right).$$

Once this assumption on the domain is satisfied, it is clear that

$$T_1 Q_{\max} T_0^{-1} g = \int_{\mathbb{R}}^{\oplus} Q(x)_{\max} g(x, \cdot) dx$$

which concludes the proof. \square

3 Von Neumann algebras

This section is dedicated to survey some results on Von Neumann algebras that are needed for the present manuscript. Through this section H will be a Hilbert space and for a subset $A \subset \mathcal{L}(H)$, we denote its commutant by A' , it is the set of bounded operator commuting with all the elements of A .

Definition of Von Neumann Algebras. We begin by recalling the Von Neumann bicommutant theorem.

Theorem A.3.41. [Dix57, Chapter 1 §3 Theorem 2] If \mathcal{A} is an $*$ -subalgebra of $\mathcal{L}(H)$, which contains the identity, then its strong closure and its weak closure coincide and are equal to the bicommutant \mathcal{A}'' of \mathcal{A} .

We recall the definition of Von Neumann algebras.

Definition A.3.42. Let H be a Hilbert space a Von Neumann algebra \mathcal{A} on H is a sub $*$ -algebra of $\mathcal{L}(H)$ which satisfies one of the following equivalent properties

- $\mathcal{A} = \mathcal{A}''$.
- \mathcal{A} is strongly closed.
- \mathcal{A} is weakly closed.

An important notion is the notion of trace on Von Neumann algebras.

Definition A.3.43. Let \mathcal{A} be a Von Neumann algebra; we set \mathcal{A}^+ the set of its positive element. A trace on \mathcal{A} is an application $t : \mathcal{A}^+ \rightarrow \mathbb{R}^+ \cup \{\infty\}$ satisfying

1. $\forall x, y \in \mathcal{A}^+, \lambda \geq 0$ one has $t(\lambda x + y) = \lambda t(x) + t(y)$ with the convention $0 \cdot \infty = 0$
2. $t(x^*x) = t(xx^*)$ for all $x \in \mathcal{A}$

A trace is said to be finite if $t(x) < +\infty$ for all $x \neq 0$ and to be semi-finite if for all $x \in \mathcal{A}^+$, there exists $0 < y \leq x$ such that $t(y) < \infty$. It is faithful if $t(x) = 0$ if and only if $x = 0$, and it is said to be normal if it is lower semi-continuous for the weak topology on \mathcal{A} .

If tr denotes the usual trace on the Hilbert space H , it defines a faithful normal semi-finite trace on the Von Neumann algebra $\mathcal{L}(H)$.

Let H_1, H_2 be two Hilbert spaces, we denote by $H_1 \hat{\otimes} H_2$ their Hilbert tensor product. If \mathcal{A}_1 (resp. \mathcal{A}_2) is a Von Neumann algebra on H_1 (resp. H_2), we denote by $\mathcal{A}_1 \otimes \mathcal{A}_2$ then the Von Neumann algebra on $H_1 \hat{\otimes} H_2$ generated by elements of the form $x \otimes y$ with $x \in \mathcal{A}_1, y \in \mathcal{A}_2$. We have the following proposition

Proposition A.3.44. [Tak79, Theorem 5.9] Let H_1, H_2 be two Hilbert spaces and \mathcal{A} be a Von Neumann algebra on H_1 . Then we have

$$(\mathcal{A} \otimes \mathcal{L}(H_2))' = \mathcal{A}' \otimes \mathbb{C} \text{Id}_{H_2}.$$

If \mathcal{A} is endowed with a normal semi-finite trace t , it induces a semi-finite normal trace (see [Tak79, Proposition 5.13]) also denoted by t on the Von Neumann algebra $\mathcal{A} \otimes \mathcal{L}(H_2)$ that satisfies

$$t(x \otimes y) = t(x) \text{tr}(y).$$

In the following, we will mainly focus on group Von Neumann algebras that are defined in the next paragraph. Many results and notions that we will state for group Von Neumann algebras will hold for finite Von Neumann algebras.

Group Von Neumann algebras. If G is a locally compact group, it acts on the space $L^2(G, \mu)$, where μ is the Haar measure, by the left regular representation $\lambda: G \rightarrow \mathcal{L}(L^2(G, \mu))$. We wish to study the Von Neumann algebras associated to discrete groups; the main reference in this case is the book of Lück [Lüc02]. In the rest of this section, our group will be assumed to be discrete unless stated otherwise, in this case, the Haar measure considered will just be the counting measure. The smallest Von Neumann algebra containing $\lambda(G)$ is denoted by $\mathcal{N}(G)$. An other description of this algebra can be given by a result of Dixmier. Denote by $*$ the convolution product on G .

Definition A.3.45. A function $f \in L^2(G, \mu)$ is said to be moderate if and only if there exists a constant $M > 0$ such that for all g with compact support, one has

$$\|f * g\|^2 \leq M \|g\|^2$$

It is straightforward to see that a moderate function f defines a bounded operator $U_f \in \mathcal{N}(G)$ defined by $U_f(h) = f * h$ and that the space of moderate functions is invariant by convolution. If $f \in L^2(G)$, one can define f^* by $f^*(\gamma) = \bar{f}(\gamma^{-1})$, this operation endows the space of moderate functions with an involutive algebra structure. And one has

Proposition A.3.46. [Tak79, Theorem 2.22] The map

$$\begin{aligned} \{f \in L^2(G) \text{ moderate}\} &\longrightarrow \mathcal{N}(G) \\ f &\longmapsto U_f \end{aligned}$$

is an isomorphism of involutive algebras.

Example A.3.47. In the case, where $G = \mathbb{Z}$, the Fourier series gives us a unitary isomorphism

$$\ell^2(\mathbb{Z}) \rightarrow L^2(\mathbb{S}^1).$$

This isomorphism sends $\mathcal{N}(\mathbb{Z})$ to $L^\infty(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$.

If G is a discrete group, we have a faithful normal finite trace tr_G on the Von Neumann algebras $\mathcal{N}(G)$ and $\mathcal{N}(G)'$, given by

$$\text{tr}_G(x) = \langle x \delta_e, \delta_e \rangle.$$

In particular, both $\mathcal{N}(G)$ and $\mathcal{N}(G)'$ are finite.

$\mathcal{N}(G)$ -Hilbert modules. If H is a Hilbert space and G is a discrete group, we consider the action $\lambda \otimes 1_H$ on $\ell^2(G) \otimes H$ where 1_H is the trivial action on H . The Von Neumann algebra $\lambda \otimes 1_H(G)''$ is the algebra $\mathcal{N}(G) \otimes \mathbb{C} \text{Id}_H$ and its commutant is given by $\mathcal{N}(G)' \otimes \mathcal{L}(H)$. From what we have just said, it follows that the trace tr_G on $\mathcal{N}(G)'$ induces a trace tr_G on the Von Neumann algebra $\mathcal{N}(G)' \otimes \mathcal{L}(H)$ of G -equivariant operators, and this trace is faithful

normal and semifinite. If $(e_i)_{i \in I}$ is a Hilbert basis of H and $T: \ell^2(G) \hat{\otimes} H \rightarrow \ell^2(G) \hat{\otimes} H$ is a positive G -equivariant bounded operator one has

$$\mathrm{tr}_G(T) = \sum_{i \in I} \langle T \delta_e \otimes e_i, \delta_e \otimes e_i \rangle.$$

We now define the notion of $\mathcal{N}(G)$ -Hilbert modules.

Definition A.3.48. A $\mathcal{N}(G)$ -Hilbert module V is a Hilbert space endowed with an action of G that is isometric to a closed G -invariant subspace of $\ell^2(G) \hat{\otimes} H$, for some Hilbert space H . A morphism between two Hilbert modules is a G -equivariant bounded linear morphism.

We say that V is finitely generated if there exists a surjective morphism $\ell^2(G)^{\oplus n} \twoheadrightarrow V$.

This trace allows us to consider a Von Neumann dimension \dim_G of a $\mathcal{N}(G)$ -Hilbert module $V \subset \ell^2(G) \hat{\otimes} H$ by taking the trace of the projection on V , this does not depend on the isometric embedding $\iota: V \rightarrow \ell^2(G) \hat{\otimes} H$.

Proposition A.3.49. The dimension function \dim_G satisfies the following properties.

- If $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ is a weakly exact sequence of $\mathcal{N}(G)$ -Hilbert modules, then $\dim_G(M_1) = \dim_G(M_2) + \dim_G(M_0)$.
- If M is a $\mathcal{N}(G)$ -Hilbert module then $M = 0$ if and only if $\dim_G M = 0$.

If G_1 and G_2 are two discrete groups, then one has a natural isomorphism $\ell^2(G_1) \hat{\otimes} \ell^2(G_2) \simeq \ell^2(G_1 \times G_2)$. It can be used to obtain the following result [Lüc02, Theorem 1.12].

Proposition A.3.50. Let H_1 (resp. H_2) be a $\mathcal{N}(G_1)$ -Hilbert module (resp. $\mathcal{N}(G_2)$ -Hilbert module) then the natural action of $G_1 \times G_2$ on $H_1 \hat{\otimes} H_2$ endows it with the structure of a $\mathcal{N}(G_1 \times G_2)$ -Hilbert module and one has

$$\dim_{G_1 \times G_2} H_1 \hat{\otimes} H_2 = \dim_{G_1} H_1 \cdot \dim_{G_2} H_2.$$

$\mathcal{N}(G)$ -Fredholm operators. The notion of $\mathcal{N}(G)$ -Fredholm operators is defined as follows.

Definition A.3.51. Let H be a $\mathcal{N}(G)$ -Hilbert module and A a self-adjoint G -equivariant closed operator (we do not require A to be bounded). We denote by $E_A: \mathrm{Bor}(\mathbb{R}^+) \rightarrow \mathrm{Proj}(H)$ the associated projection-valued measure. Then for each Borelian U , $E_A(U)$ is G -equivariant. An element $\lambda \in \mathbb{R}^+$ is said to be in the $\mathcal{N}(G)$ essential spectrum of A if and only if for any neighbourhood V of λ one has $\mathrm{tr}_G(E_A(V)) = +\infty$.

A self-adjoint operator A is said to be $\mathcal{N}(G)$ -Fredholm if 0 does not belong to the $\mathcal{N}(G)$ essential spectrum of A . It is strongly $\mathcal{N}(G)$ -Fredholm if 0 admits a neighbourhood V such that $\mathrm{Ran}(E_A(V))$ is finitely generated.

A G -equivariant closed operator (not necessarily self-adjoint) $A: V \rightarrow W$ between two $\mathcal{N}(G)$ -Hilbert modules is said to be Fredholm if and only if A^*A is $\mathcal{N}(G)$ -Fredholm¹.

Note that for the $\mathcal{N}(G)$ -essential spectrum to be non-empty, one needs H to have infinite $\mathcal{N}(G)$ -dimension.

Remark A.3.52. A G -equivariant closed operator $f: H_0 \dashrightarrow H_1$ between $\mathcal{N}(G)$ -Hilbert module is $\mathcal{N}(G)$ -Fredholm if and only if there exist $\varepsilon > 0$ and $C > 0$ such that for all closed $\mathcal{N}(G)$ -submodules L contained in

$$\mathcal{L}(f, \varepsilon) = \{x \in \mathrm{Dom}(f) \mid \|fx\| \leq \varepsilon \|x\|\}$$

has dimension $\dim_{\mathcal{N}(G)} L < C$. The equivalence of these notions of $\mathcal{N}(G)$ -Fredholmness is proved in [Lüc02, Lemma 2.3]

Lemma A.3.53. [Lüc02, Lemma 2.14] Let H_0, H_1 and H_2 be $\mathcal{N}(G)$ -Hilbert module, $f: H_0 \rightarrow H_1$ and $g: H_1 \rightarrow H_2$ be two closed densely defined G -equivariant operators with $f(\mathrm{Dom}(f)) \subset \mathrm{Dom}(g)$ then

1. If f and g are $\mathcal{N}(G)$ -Fredholm, gf is $\mathcal{N}(G)$ -Fredholm.
2. If gf is $\mathcal{N}(G)$ -Fredholm and g is bounded, then f is $\mathcal{N}(G)$ -Fredholm.

1. This definition is equivalent to the one given by Lück [Lüc02, Definition 2.1], however in the case of $G = \{e\}$ this definition is the one of semi-Fredholm operator. A more standard definition would be to say that A is $\mathcal{N}(G)$ -Fredholm if both A^*A and AA^* are $\mathcal{N}(G)$ -Fredholm, in which case one recovers the definition of $\mathcal{N}(G)$ -Fredholmness given in [Din13, Definition 2.12].

Proof. The proof in [Lüc02] is done under the assumption that both f and g are bounded, but generalise immediately to our case. We give the proof here for the sake of completeness. The second point follows from the remark and the fact that $\|f(x)\| \leq \varepsilon\|x\|$ implies $\|gf(x)\| \leq \varepsilon\|g\|\|x\|$.

For the first point, let us fix $\varepsilon > 0$ so that for all $L \subset \mathcal{L}(g, \varepsilon)$ or $L \subset \mathcal{L}(f, \varepsilon)$ one has $\dim_G L < C$ for some finite constant $C < +\infty$. Now let us fix $L \subset \mathcal{L}(gf, \varepsilon^2)$ to be a $\mathcal{N}(G)$ -Hilbert module, we will show $\dim_G L \leq 2C$. Set $L_0 = \text{Ker}(E_{f^*f, \varepsilon}) \cap L$. One has a weakly exact sequence

$$0 \longrightarrow L_0 \longrightarrow L \longrightarrow \overline{E_{f^*f, \varepsilon}(L)} \longrightarrow 0$$

And $\overline{E_{f^*f, \varepsilon}(L)} \subset E_{f^*f, \varepsilon}(H)$ which has finite dimension bounded by C . It remains to prove that L_0 has a finite dimension. Note that $f|_{L_0}$ is an isomorphism on its range $f(L_0)$ which is closed, however, since

$$\|gf(x)\| \leq \varepsilon^2\|x\| \leq \varepsilon\|x\|$$

one has $f(L_0) \subset \mathcal{L}(g, \varepsilon)$ so it has dimension bounded by C , and it follows that $\dim_G L \leq 2C$ hence gf must be $\mathcal{N}(G)$ -Fredholm. \square

Essential density.

Definition A.3.54. Let H be a $\mathcal{N}(G)$ -Hilbert module, a (not necessarily closed) G -invariant linear subspace $L \subset H$ is said to be essentially dense if for all $\varepsilon > 0$ there exists a closed submodule M of H with $M \subset L$ and $\dim_G(M^\perp) < \varepsilon$.

Remark A.3.55. An essentially dense subspace is dense, the converse is false.

Lemma A.3.56.

1. A countable intersection of essentially dense linear subspaces is essentially dense (see [Lüc02, Lemma 8.3, (1)]).
2. If $M \subset H$ is G invariant and essentially dense in \overline{M} , then for all $L \subset H$, a $\mathcal{N}(G)$ -Hilbert submodule, we find that $M \cap L$ is essentially dense in $\overline{M} \cap L$ (see [Shu95, Lemma 1.17]).

Some algebraic properties of group Von Neumann algebras. We discuss in this paragraph few properties of the algebra $\mathcal{N}(G)$ viewed as a ring. We recall the following definitions that hold for any non-commutative rings.

Definition A.3.57. Let R be a ring (not necessarily commutative). Then R is said to be

- [Lam12, Definition 2.28] Right (resp. left) semi-hereditary if every finitely generated right (resp. left) ideal of R is projective as a right (resp. left) R -module.
- [Lam12, Definition 4.51] Right (resp. left) coherent if every finitely generated right (resp. left) ideal of R is finitely presented.
- [GWJ04, Chapter 1] Right (resp. left) Noetherian if every right (resp. left) ideal of R is finitely generated.

We say that R is semi-hereditary (resp. coherent, resp. Noetherian) if it is both right and left semihereditary (resp. coherent, resp. Noetherian).

A theorem due to Chase characterises which coherent rings are semihereditary.

Theorem A.3.58. [Lam12, Theorem 4.67] A ring R is semi-hereditary if and only if it is coherent and if any submodule of a flat module is flat.

While finite Von Neumann algebras are not Noetherian in general, they are semihereditary (hence coherent). We state this result which seems to be due to Lück [Lüc98] in the case of group Von Neumann algebra

Theorem A.3.59. [Lüc02, Theorem 6.7] Let G be a discrete group, then the Von Neumann algebra $\mathcal{N}(G)$ is semihereditary.

Lück proved this result by extending the dimension function \dim_G to arbitrary $\mathcal{N}(G)$ -modules, and not just Hilbert ones. We recall the principle of this extension, which is given in [Lüc02, Chapter 6 section 1]. If P is a finitely generated projective $\mathcal{N}(G)$ -module then there exists $n \in \mathbb{N}$ and $A \in M_n(\mathcal{N}(G))$ such that

- $A^2 = A$

- If $r_A: \mathcal{N}(G)^n \rightarrow \mathcal{N}(G)^n$ is the morphism induced by the right multiplication by A then $\text{Ran}(r_A)$ is isomorphic to P .

Then one can simply set the G -dimension of P to be

$$\dim_G(P) = \text{tr}_G(A)$$

where the trace on the right-hand side is the sum of the trace of the diagonal element. Once the dimension of finely generated projective $\mathcal{N}(G)$ -modules has been defined, one can take the dimension of an arbitrary module M to be

$$\dim_G(M) = \sup\{\dim_G(P) \mid P \subset M \text{ finitely generated projective}\}.$$

Remark A.3.60. Contrary to the case of the $\mathcal{N}(G)$ -Hilbert module, it may happen that a $\mathcal{N}(G)$ -module M is non-zero and satisfies $\dim_G(M) = 0$. For instance, if H is any $\mathcal{N}(G)$ -Hilbert module and $L \subset H$ is essentially dense, one has $\dim_G(H/L) = 0$.

The structure of $\mathcal{N}(G)$ -module is sufficient to study finitely generated $\mathcal{N}(G)$ -Hilbert module thanks to the theorem below, which is a corollary of [Gri66, Theorem 7].

Theorem A.3.61. Let $T: H_1 \rightarrow H_2$ be a linear $\mathcal{N}(G)$ -equivariant map between finitely generated $\mathcal{N}(G)$ -Hilbert modules. Then T is bounded.

The algebra of affiliated operators. In this paragraph, G still denotes a discrete group. We review the notion of affiliated operators presented in [Lüc02, Chapter 8].

Definition A.3.62. [Lüc02, Chapter 8] Let $\mathcal{N}(G)$ be the Von Neumann algebra of a discrete group G . Its algebra of the affiliated operators $\mathcal{U}(G)$ is the space of closed operators $f: \ell^2(G) \dashrightarrow \ell^2(G)$ that commutes with every element of $\mathcal{N}(G)'$ (for this to make sense, we ask the domain of f to be $\mathcal{N}(G)'$ -invariant).

Remark A.3.63. If $f \in \ell^2(G)$ one has defined the closed operator $U_f: \ell^2(G) \dashrightarrow \ell^2(G)$, which is densely defined (since it is defined on $\mathbb{C}[G]$). In particular, U_f is affiliated and one can then make sense of the following inclusions

$$\mathbb{C}[G] \subset \mathcal{N}(G) \subset \ell^2(G) \subset \mathcal{U}(G).$$

The following proposition review the basic properties of affiliated operators

Proposition A.3.64. [Lüc02, Lemma 8.3] Let $f: \ell^2(G) \dashrightarrow \ell^2(G)$ be an affiliated operator. Then if $V \subset \ell^2(G)$ is an essentially dense subspace, then $f^{-1}(V)$ is essentially dense. In particular, $\text{Dom}(f)$ is essentially dense.

Let f, g be affiliated operators then if $f \subset g$ one has $f = g$.

Since the intersection of essentially dense subspaces is essentially dense, it follows that if $f, g: \ell^2(G) \dashrightarrow \ell^2(G)$ are affiliated then both $f + g$ and $g \circ f$ are densely defined. In particular, the minimal closure of $f + g$ and gf are affiliated operators. Using the second point, one can show

Proposition A.3.65. [Lüc02, Lemma 8.8] The set of affiliated operators has a natural structure of involutive algebra, and the inclusion $\mathcal{N}(G) \rightarrow \mathcal{U}(G)$ is a morphism of involutive algebra. Moreover, given $f \in \mathcal{U}(G)$ the following are equivalents

- f is invertible in $\mathcal{U}(G)$
- f is injective.
- f has dense range.

Proposition A.3.66. [Lüc02, Lemma 8.33] If M is a $\mathcal{N}(G)$ -module (M need not be a Hilbert module) such that $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M = 0$ one has $\dim_G M = 0$. The converse is true if M is finitely presented.

The following proposition is an important result on the torsion theory for the algebra of affiliated operators which was shown by Dingoyan in [Din13]. In [Din13], it is stated in terms of abstract finite Von Neumann algebra, we state the result only in the case of Von Neumann algebra associated to some discrete group.

Proposition A.3.67. [Din13, Lemma 2.15] Let H_1, H_2 be two $\mathcal{N}(G)$ -Hilbert modules and $f: H_1 \rightarrow H_2$ be a closed G -equivariant operator. Then if $\text{Ran}(f)$ is essentially dense in its closure and if $\mathcal{U}(G)$ denotes the ring of operators affiliated to $\mathcal{N}(G)$ then one has

$$\dim_G \frac{\overline{\text{Ran}(f)}}{\text{Ran}(f)} = 0 \quad \text{and} \quad \mathcal{U}(G) \otimes_{\mathcal{N}(G)} \frac{\overline{\text{Ran}(f)}}{\text{Ran}(f)} = 0$$

Proof. As said before the proof is due to Dingoyan [Din13, Lemma 2.15], and holds when considering a Hilbert module over a general finite Von Neumann algebra. The reader can note that

$$\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \frac{\overline{\text{Ran}(f)}}{\text{Ran}(f)} = 0$$

implies that $\dim_G \frac{\overline{\text{Ran}(f)}}{\text{Ran}(f)} = 0$. Hence, we only need to check the second point.

The case where $H_1 = H_2 = \ell^2(G)$ and $\text{Ran}(f)$ is essentially dense in H_2 . For $y \in \ell^2(G)$, let $\rho(y): \ell^2(G) \dashrightarrow \ell^2(G)$ be the densely defined operator defined by $\rho(y)(z) = z * y$ (where $*$ denotes the convolution product). We recall that the map $\rho: y \mapsto \rho(y)$ is an injection from $\ell^2(G)$ into the space of closed, densely defined operators on $\ell^2(G)$.

We fix $y \in H_2 = \ell^2(G)$, we wish to find $r \in \mathcal{N}(G)$ injective (hence with dense range) such that $r * y = f(x)$ for some $x \in \text{Dom}(f)$. For this, it suffices to find x_0 and x_1 with $x_0 \in \mathcal{N}(G)$ and $x_1 \in \text{Dom}(f)$ satisfying $\rho(x_0 * y) = \rho(f(x_1))$ by the injectivity of ρ .

Set $g: \text{Ran}(f) \rightarrow \text{Ker}(f)^\perp$ to be equal to the inverse of f on $\text{Ran}(f)$. It is a closed, densely defined operator. Since both $\rho(x)$ and g commute with the left regular representation, the (minimal closure of the) composition $g \circ \rho(x)$ defines a closed densely defined operator, and we can consider its polar decomposition us , with u partial isometry and s self-adjoint positive. The positivity of s implies that $(1 + s)$ admits a bounded inverse $(1 + s)^{-1}$ and $s(1 + s)^{-1}$ is bounded, in particular $us(1 + s)^{-1}$ is also bounded. It follows that there exist x_0, x_1 such that $(1 + s)^{-1} = \rho(x_0)$, and $us(1 + s)^{-1} = \rho(x_1)$ by [Dix57, 13.8.3]. Since $\rho(x_0)$ is bounded, one must have $x_0 \in \mathcal{N}(G)$, moreover, by construction $\rho(x_0)$ is injective and with a dense range. By the associativity of the convolution product, one obtains that on $(1 + s)\rho(x)^{-1}(\text{Ran}(f))$, which is an essentially dense subset of $\ell^2(G)$, the equality

$$\rho(x_0 * x) = \rho(y)\rho(x_0) = \rho(x)(1 + s)^{-1} = fg\rho(x)(1 + s)^{-1} = f\rho(x_1) = \rho(f(x_1)).$$

It follows that $\rho(x_0 * y) = f(x_1) \in \text{Ran}(f)$ which is what we wanted to prove in the case $H = \ell^2(G)$.

The general case. One consider the injective operator

$$h: \begin{array}{ccc} \text{Ker}(f)^\perp \oplus \ell^2(G) & \rightarrow & \overline{\text{Ran}(f)} \oplus \ell^2(G) \\ (x, g) & \mapsto & (f(x), g) \end{array}$$

Let $x \in \overline{\text{Ran}(f)}$ and set $F_1 := \overline{\mathbb{C}[G] \cdot (x, e_G)}$, it is a Hilbert submodule of $\overline{\text{Ran}(f)} \oplus \ell^2(G)$, the operator h has an essentially dense range since f has an essentially dense image it follows that $\text{Ran}(h) \cap F_1$ is essentially dense in F_1 , so if we set $F_0 := h^{-1}(F_1)$, the induced morphism $h: F_0 \rightarrow F_1$ is injective and has an essentially dense image. The morphisms $p_0: F_0 \rightarrow \ell^2(G)$ and $p_1: F_1 \rightarrow \ell^2(G)$ induced by the projection on $\ell^2(G)$ are morphisms of $\mathcal{N}(G)$ -Hilbert modules that are injective with a dense range. If we set $p_j = u_j s_j$ to be its polar decomposition, the $\mathcal{N}(G)$ -equivariant operator $u_j: F_j \rightarrow \ell^2(G)$ is an isomorphism of $\mathcal{N}(G)$ -modules.

So $u_1 h u_0^{-1}: \ell^2(G) \rightarrow \ell^2(G)$ is a morphism with an essentially dense image, thanks to the previous case there exists $r \in \mathcal{N}(G)$ such that $r \cdot u_1(x, e_G) = u_1(r \cdot x, r \cdot e_G) \in \text{Ran}(p_1 h p_0)$ and so $(r \cdot x, r) \in \text{Ran}(h)$ and $r \cdot x \in \text{Ran}(f)$ which conclude the proof. \square

Algebraic properties of $\mathcal{U}(G)$. In this paragraph, we survey some algebraic properties of the algebra $\mathcal{U}(G)$. First, we begin with explaining how it can be viewed as a localisation of $\mathcal{N}(G)$. For generic results on the location of non-commutative rings, we refer to [GWJ04, Chapter 6].

Definition A.3.68. Let R be a ring and S be a multiplicative set. A right ring of fraction R' with respect to S is a homomorphism $\phi: R \rightarrow R'$ that satisfies

1. $\phi(s)$ is invertible for all $s \in S$
2. Each element of R' is of the form $\phi(x)\phi(s)^{-1}$ for $x \in R$ and $s \in S$

$$3. \text{Ker}(\phi) = \{r \in R \mid \exists s \in S \text{ } rs = 0\}$$

For a right ring of fraction to exist, S must satisfy the right Ore conditions : for all $r \in R$, $s \in S$ one has $rS \cap sR \neq \emptyset$. Conversely, if S satisfies the right Ore condition, for a right ring of fraction to exist with respect to S it is sufficient that S contains no zero-divisor [GWJ04, Theorem 6.2]². Similarly to localisation in commutative algebra, Ore localisation can be expressed in terms of a universal property.

Proposition A.3.69. [GWJ04, Proposition 6.3] Let R be a ring and S be a multiplicative set that does not contain a zero divisor. Let $\phi: R \rightarrow R'$ be a right ring of fraction with respect to S , then for any morphism $\psi: R \rightarrow T$ such that for all $s \in S$ $\psi(s)$ is a unit, then there exists a unique $\tilde{\psi}: R' \rightarrow T$ such that the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\psi} & T \\ \downarrow \phi & \searrow \tilde{\psi} & \\ R' & & \end{array}$$

This implies that a right ring of the fraction is necessarily unique and will be denoted by RS^{-1} . One defines the left ring of fraction and the left Ore condition similarly. If S is a multiplicative set satisfying the right and left Ore conditions, then one obtains a canonical isomorphism between RS^{-1} and $S^{-1}R$ given by the universal property above.

One also defines the notion of right modules of the fraction. Before giving the definition, we introduce a notation.

Notation. For $S \subset R$ and M a right R -module we denote by $t_S(M)$ the set of elements that are of S -torsion, i.e.

$$t_S(M) = \{m \in M \mid \exists s \in S, ms = 0\}$$

Definition A.3.70. Let M be a right R -module and $S \subset R$ be a multiplicative subset that satisfies the Ore condition and does not contain zero divisors. A module of fraction of M with respect to S is a right RS^{-1} -module N together with a morphism of R -modules $\phi: M \rightarrow N$ satisfying

1. Each element of N is of the form $\phi(m)s^{-1}$ for $m \in M$ and $s \in S$
2. $\text{Ker}(\phi) = t_S(M)$.

The following theorem is a summary of [GWJ04, Theorem 10.8, Proposition 10.9, Proposition 10.12].

Theorem A.3.71. If M is a right R -module and S a multiplicative set that satisfies the right Ore condition and contains no zero divisors, then there is a right module of fraction $\phi: M \rightarrow N$.

Moreover, given such a module of fraction, it satisfies the following universal property : for any right RS^{-1} -module M' and morphism of R -module $\psi: M \rightarrow M'$ there exists a unique morphism of RS^{-1} -module $\tilde{\psi}: N \rightarrow M'$ such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M' \\ \downarrow \phi & \searrow \tilde{\psi} & \\ N & & \end{array}$$

The natural morphism $M \otimes_R RS^{-1} \rightarrow M'$ given by the multiplication is an isomorphism of right RS^{-1} -modules.

In particular, this theorem gives the unicity of the right module of fraction which will be denoted by MS^{-1} and this shows that RS^{-1} is a flat left R -module.

Proposition A.3.72. [Lüc02, Theorem 8.22] Let G be a discrete group. The space S of non-zero divisors of $\mathcal{N}(G)$, i.e., the space of injective operators with dense range, satisfies the right and left Ore conditions, and there is a canonical isomorphism $\mathcal{N}(G)S^{-1} \rightarrow \mathcal{U}(G)$ induced by inclusion $\mathcal{N}(G) \rightarrow \mathcal{U}(G)$.

The algebra $\mathcal{U}(G)$ is Von Neumann regular, that is, every (left or right) $\mathcal{U}(G)$ -module is flat over $\mathcal{U}(G)$. In particular, $\mathcal{U}(G)$ is a coherent ring.

² The minimal assumption would be that for all $r \in R$, $s \in S$ with $sr = 0$ there exists s' satisfying $rs' = 0$ by [GWJ04, Theorem 10.3], however the assumption that S contains no zero divisor will be sufficient for our needs.

Remark A.3.73. In [Lüc02, Chapter 8] only the fact that S satisfies the right Ore condition is proved. The fact that it follows from the left Ore condition is a simple consequence of the fact that S is closed under adjunction : For $r \in \mathcal{N}(G)$ and $s \in S$ one has

$$(\mathcal{N}(G)r \cap Ss)^* = r^* \mathcal{N}(G) \cap s^* S$$

which is non-empty since S satisfies the right Ore condition, and hence $\mathcal{N}(G)r \cap Ss$ is non-empty and S satisfies the left Ore condition. This implies that $\mathcal{U}(G)$ is flat as both a right and a left $\mathcal{N}(G)$ -module.

Theorem A.3.71 implies that the inclusion $\mathcal{N}(G) \subset \ell^2(G) \subset \mathcal{U}(G)$ induces isomorphisms

$$\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \mathcal{N}(G) \rightarrow \mathcal{U}(G) \otimes_{\mathcal{N}(G)} \ell^2(G) \rightarrow \mathcal{U}(G) \otimes_{\mathcal{N}(G)} \mathcal{U}(G) \rightarrow \mathcal{U}(G)$$

where the last map is the multiplication.

Von Neumann algebras and direct integral of measurable fields of Hilbert spaces. The theory of measurable Hilbert fields can be useful in computing the $\mathcal{N}(G)$ -trace of an operator thanks to Plancherel theorem. We recall in the well-known case of discrete Abelian groups (for a more general statement on locally compact unimodular postliminal group, we refer to [Dix64, 18.8.1], a version of the Plancherel theorem for countable discrete groups is given in [BdlH20]).

Theorem A.3.74 (Plancherel theorem). Let G be a countable discrete Abelian group and $\hat{G} = \text{Hom}(G, S^1)$ be the space of characters of G , which is a compact Abelian group. We set μ the Haar measure on \hat{G} so that $\mu(\hat{G}) = 1$. Then we have an isometric isomorphism

$$\ell^2(G) \rightarrow L^2(\hat{G}, d\mu) = \int_{\hat{G}}^{\oplus} \mathbb{C} d\mu$$

which transform the left regular representation into the representation

$$\pi = \int_{\hat{G}}^{\oplus} \chi d\mu(\chi) : \gamma \mapsto \int_{\hat{G}}^{\oplus} \chi(\gamma) d\mu(\chi).$$

Moreover, the Von Neumann algebra $\pi(G)'$ consists of the algebra of decomposable bounded operators, and if $T = \int_{\hat{G}}^{\oplus} T(\chi) d\mu(\chi)$ is such an operator its G -trace is given by

$$\text{tr}_G T = \int_{\hat{G}} \text{tr}(T(\chi)) d\mu(\chi).$$

Remark that $T(\chi)$ is a homothety on \mathbb{C} , hence it is given by a number also denoted by $T(\chi)$ and $\text{tr}(T(\chi)) = T(\chi)$. Also note that since we consider here the case where G is Abelian, we have $\pi(G)' = \pi(G)''$.

Let H be a Hilbert space and consider $\ell^2(G) \hat{\otimes} H$ endowed with the representation $\pi = \lambda \otimes 1_H$, where λ is the left regular representation and 1_H is the trivial representation on H . The isomorphism given by Plancherel gives us an isomorphism between $\ell^2(G) \hat{\otimes} H$ and $L^2(\hat{G}, d\mu) \hat{\otimes} H \simeq \int_{\hat{G}}^{\oplus} H d\mu$, and the representation π is conjugate to $\int_{\hat{G}}^{\oplus} \chi d\mu(\chi)$. The set of G -equivariant operators is given by the Von Neumann algebra $\rho(G)' \otimes \mathcal{L}(H)$ hence after conjugation by the previous isomorphism we find that G -equivariant operators are given by the algebra decomposable operator of $\int_{\hat{G}}^{\oplus} H d\mu$. The G -trace on $\ell^2(G) \hat{\otimes} H$ is defined as $\text{tr}_G \otimes \text{tr}$ where tr is the usual trace on H , we will denote this trace by tr_G . It follows that if $T := \int_{\hat{G}}^{\oplus} T(\chi) d\mu(\chi)$ is a decomposable operator, its trace is given by

$$\text{tr}_G T = \int_{\hat{G}} \text{tr}(T(\chi)) d\mu(\chi).$$

Recall that a $\mathcal{N}(G)$ -Hilbert module L is a Hilbert space endowed with G -action, such that it is isomorphic to a G -invariant closed subspace of $\ell^2(G) \hat{\otimes} H$ for some Hilbert H endowed with the representation described above. The projection on L is G -equivariant, hence it is decomposable, and by considering its trace which is equal to the G -dimension of L we obtain the following proposition.

Proposition A.3.75. Let G be an Abelian discrete group. Any $\mathcal{N}(G)$ -Hilbert module L is isomorphic to a direct integral of Hilbert spaces $\int_{\hat{G}}^{\oplus} H(\chi) d\mu(\chi)$ endowed with the representation $\int_{\hat{G}}^{\oplus} \chi d\mu(\chi)$. Moreover, its G -dimension is given by

$$\dim_G L = \int_{\hat{G}} \dim(H(\chi)) d\mu(\chi).$$

Lemma A.3.76. Let G be a discrete group. Let

$$\mathcal{H}_1 = \int_X^\oplus H_1(x) d\mu(x) \quad \text{and} \quad \mathcal{H}_2 = \int_X^\oplus H_2(x) d\mu(x)$$

be two direct integrals of measurable fields of Hilbert spaces. We assume that they have a structure of $\mathcal{N}(G)$ -Hilbert modules given by a representation $\rho_j: G \rightarrow \mathcal{L}(\mathcal{H}_j)$ where ρ_j take values in the space of decomposable operators.

Consider $T = \int_X^\oplus T(x): \mathcal{H}_1 \rightarrow \mathcal{H}_2$ a closed decomposable morphism of $\mathcal{N}(G)$ -modules, assume that for every measurable field $g: x \mapsto g(x)$, there exist a morphism $C = \int_X^\oplus C(x) \in \mathcal{N}(G)$ and a measurable field f satisfying

1. $C: \ell^2(G) \rightarrow \ell^2(G)$ is injective with dense range.
2. $T(x)f(x) = g(x)$ $\mu(x)$ -almost everywhere, in particular, we require $f(x)$ to be in the domain of $T(x)$ $\mu(x)$ -almost everywhere.
3. $\|C(x)f(x)\| \leq \|g(x)\|$, $\mu(x)$ -almost everywhere

Then T induces a surjection $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \mathcal{H}_1 \rightarrow \mathcal{U}(G) \otimes_{\mathcal{N}(G)} \mathcal{H}_2$ of $\mathcal{U}(G)$ -modules. If, moreover, $\mathcal{N}(G)$ acts by diagonalisable operators, then we can abandon the measurability assumption on f .

Proof. Let $1_{\mathcal{U}(G)} \otimes \int_X^\oplus g(x) d\mu(x)$ be an element of \mathcal{H}_2 . Set C and f satisfying the assumptions of the lemma, f might not be a square-integrable field, but $Cf: x \mapsto C(x)f(x)$ is square-integrable, so $Cf \in \text{Dom}(T)$ and since T is a morphism of $\mathcal{N}(G)$ -module we note that C is invertible in $\mathcal{U}(G)$:

$$\begin{aligned} T(C^{-1} \otimes \int_X^\oplus C(x)f(x) d\mu(x)) &= C^{-1} \otimes \int_X^\oplus T(x)C(x)f(x) d\mu(x) \\ &= C^{-1} \otimes \int_X^\oplus C(x)T(x)f(x) d\mu(x) \\ &= C^{-1} \otimes C \int_X^\oplus g(x) d\mu(x) \\ &= 1 \otimes \int_X^\oplus g(x) d\mu(x). \end{aligned}$$

In the case where G acts by diagonalisable operators, we have to show that we can drop the measurability condition on f . Set C and f to satisfy the three points of the theorem with f not necessarily measurable. The field of operators $x \mapsto T(x)$ is measurable; therefore we have a measurable field of subspaces given by $x \mapsto \text{Ker}(T(x))^\perp$, by hypothesis for almost every x the operator $T(x)|_{(\text{Ker } T)^\perp}$ is surjective, hence $S: x \mapsto (T(x)|_{(\text{Ker } T)^\perp})^{-1}$ is a measurable field of bounded operators, and we can set $h(x) = S(x)g(x)$. In this case, h is a measurable vector field and the couple (C, h) verify the first two hypotheses of the lemma. For the third one, we only have to note that

$$\|C(x)h(x)\| = |C(x)| \|h(x)\| \leq |C(x)| \|f(x)\| = \|C(x)f(x)\|$$

which concludes the proof. □

Induction of group Von Neumann algebras. In this paragraph, G still denotes a discrete group and $\Gamma < G$ will be a subgroup of G . Let H be a $\mathcal{N}(\Gamma)$ -Hilbert module. We consider the natural action of G on the space $\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} H$. The space $\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} H$ can be naturally identified with the vector space $\bigoplus_{G/\Gamma} H$ and if we endow it with the natural pre-

Hilbert structure, with respect to this structure, G acts on $\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} H$ by isometry. We denote by $\iota_* H$ the completion of $\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} H$ for this pre-Hilbert structure.

Proposition A.3.77. [Lüc02, Lemma 1.24] The space $\iota_* H$ is a $\mathcal{N}(G)$ -Hilbert module. The functor ι_* defined from the category of the $\mathcal{N}(\Gamma)$ -Hilbert modules to the category of the $\mathcal{N}(G)$ -Hilbert modules is an exact additive functor. Moreover, if $f: H \rightarrow H$ is a positive morphism of $\mathcal{N}(\Gamma)$ -Hilbert modules, then

$$\text{tr}_\Gamma(f) = \text{tr}_G(\iota_* f).$$

In particular, this proposition implies that for a $\mathcal{N}(\Gamma)$ -Hilbert module H one has

$$\dim_\Gamma H = \dim_G H.$$

Now, we consider $f: H_1 \dashrightarrow H_1$ to be a closed densely defined $\mathcal{N}(\Gamma)$ -equivariant operator between $\mathcal{N}(\Gamma)$ -Hilbert modules. This defines a densely defined operator $\bigoplus_{G/\Gamma} H_1 \dashrightarrow \bigoplus_{G/\Gamma} H_2$. We denote by $\iota_* f: \iota_* H_1 \dashrightarrow \iota_* H_2$ the closure of this operator. It is straightforward to check $(\iota_* f)^* = \iota_*(f^*)$. By applying the previous Proposition to the bounded operators $E_{f^* f, \varepsilon}$ given by the spectral projection, one obtains the following.

Proposition A.3.78. Let $f: H_1 \dashrightarrow H_2$ be a closed densely defined $\mathcal{N}(\Gamma)$ -equivariant operator between the $\mathcal{N}(\Gamma)$ -Hilbert modules. Then f is $\mathcal{N}(\Gamma)$ -Fredholm if and only if $\iota_* f$ is $\mathcal{N}(G)$ -Fredholm.

It is also possible to define the induction over a general $\mathcal{N}(\Gamma)$ -module. If $\Gamma < G$ is a subgroup of G and M is a $\mathcal{N}(\Gamma)$ -module, the induction is then given by

$$\iota_* M = \mathcal{N}(G)_{\otimes \mathcal{N}(\Gamma)} M.$$

This again defines a faithfully flat additive functor ι_* and is compatible with the torsion theories associated with the G -dimension and the algebra of affiliated operators, as stated by the proposition below.

Proposition A.3.79. Let Γ be a subgroup of a discrete group G and M be a $\mathcal{N}(\Gamma)$ -module. Then

- [Lüc98, Theorem 3.3] We have $\dim_{\Gamma} M = 0$ if and only if $\dim_G \iota_* M = 0$
- [Vaš05, Proposition 7.3] We have $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} M = 0$ if and only if $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \iota_* M = 0$.

4 Spectra of ordinary differential operators

In this appendix, we recall some result on the spectrum of ordinary differential operator that we have used in this paper. We consider a formally self-adjoint differential operator τ on an interval $]a, b[$ (a or b may not be finite). We consider a self-adjoint realisation of τ on $L^2(]a, b[, w(x)dx)$, that is, a closure of $\tau: C_c^\infty(]a, b[) \rightarrow L^2(]a, b[, w(x)dx)$ that is self-adjoint. It is well known that the essential spectra do not depend on the self-adjoint realisation (see, for instance, [Nai67, §14, Theorem 9]). We give here a few results on essential spectra of differential operator. For such a differential operator, we denote by $\sigma_0(\tau)$ the infimum of its essential spectra, with the convention $\sigma_0(\tau) = +\infty$ if the essential spectra is empty.

Proposition A.4.80. [Zet12, Theorem 10.12.1] Let τ be a Sturm-Liouville operator on an interval I , that is, an operator of the form

$$\tau = \frac{1}{w}(-\partial_x p \partial_x + q)$$

where $w, p \in C^2$ are positive-valued functions and q is a continuous real-valued function, with $w, \frac{1}{p}, q$ locally integrable. Let L be a self-adjoint realisation of τ operating on $L^2(I, w dx)$ and assume that $\sigma_0(\tau) > -\infty$. Then the spectra of L below $\sigma_0(\tau)$ consist only of eigenvalues and possess at most one accumulation point in this case, which must be $\sigma_0(\tau)$ if this accumulation point exists.

An important class of Sturm-Liouville operators is given by Schrödinger operators, i.e., operators of the form $-\partial_u^2 + Q$. In fact, all the Sturm-Liouville operators defined above are conjugated to a Schrödinger operator, thanks to the Liouville transform that we describe below (see [BR82, Chapter X, Theorem 6]).

We fix $x_0 \in I$, and we set

$$u = \ell(x) := \int_{x_0}^x \sqrt{\frac{w(t)}{p(t)}} dt.$$

And we define the isometry

$$\begin{aligned} C: L^2(I, w(x)dx) &\longrightarrow L^2(\ell(I), du) \\ f &\longmapsto (p(\ell^{-1}(u)w(\ell^{-1}(u)))^{1/4} \cdot f(\ell^{-1}(u))). \end{aligned}$$

Proposition A.4.81. The Liouville transform $\tau \mapsto C\tau C^*$ transform τ on the differential operator $-\partial_u^2 + Q$ with $Q(u) := qw^{-1} - (pw)^{-1/4} \frac{d^2}{du^2} (pw)^{1/4}$.

Proof. The isometry C conjugates the multiplication by a function f to the multiplication by the function $f \circ \ell^{-1}$, so it suffices to do the computation in the case $q = 0$. In the following computation, the reader will be careful not to confuse $\partial_u f$ which is the operator ∂_u composed by the operator f and the operator $\frac{d}{du}f$ which is the multiplication operator by the function $\frac{d}{du}f$. Moreover, we recall that since $u = \ell(x)$, one has $\partial_x = \sqrt{\frac{w}{p}}\partial_u$.

$$\begin{aligned}
C\left(-\frac{1}{w}\partial_x p \partial_x\right) &= -(wp)^{1/4} \frac{\sqrt{w}}{w\sqrt{p}} \partial_u p \sqrt{\frac{w}{p}} \partial_u \\
&= -(wp)^{-1/4} \partial_u (pw)^{1/4} (pw)^{1/4} \partial_u \\
&= -(wp)^{-1/4} \partial_u (pw)^{1/4} \left(\partial_u (pw)^{1/4} - \frac{d}{du} (pw)^{1/4} \right) \\
&= -(wp)^{-1/4} \left((pw)^{1/4} \partial_u + \frac{d}{du} (pw)^{1/4} \right) \left(\partial_u (pw)^{1/4} - \frac{d}{du} (pw)^{1/4} \right) \\
&= -(wp)^{-1/4} \left((pw)^{1/4} \partial_u^2 (pw)^{1/4} + (pw)^{1/4} \partial_u \frac{d}{du} (pw)^{1/4} \right. \\
&\quad \left. + \frac{d}{du} (pw)^{1/4} \partial_u (pw)^{1/4} - \left(\frac{d}{du} (pw)^{1/4} \right)^2 \right) \\
&= \left(-\partial_u^2 + (pw)^{-1/4} \frac{d^2}{du^2} (pw) \right) (pw)^{1/4} \\
&= \left(-\partial_u^2 + (pw)^{-1/4} \frac{d^2}{du^2} (pw) \right) C
\end{aligned}$$

□

For this reason, it is useful to understand the spectra of Schrödinger operators on the line.

Theorem A.4.82. [Sim09, Theorem 1] Let $L := -\partial_x^2 + Q$ be a Schrödinger operator defined on $C_c^\infty(\mathbb{R})$. Assume that Q is bounded below, then L is essentially self-adjoint. Moreover, if $Q(x) \rightarrow +\infty$ when $|x|$ tends to $+\infty$ then its spectra is a discrete countable number of eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ with multiplicities equal to 1.

This criterion allows us to treat the case on the half-line thanks to the following result.

Theorem A.4.83. [Nai67, §24 Theorem 1] Let p be a differential operator on a (potentially infinite) interval $I =]a, b[$, let $c \in I$ and set $I_0 :=]a, c[$ and $I_1 =]c, b[$, denote by p_j the differential operator induced by p on the interval I_j . Then if P (resp. P_0, P_1) is a self-adjoint realisation of p (resp. p_0, p_1) one has

$$\sigma_{\text{ess}}(P) = \sigma_{\text{ess}}(P_0) \cup \sigma_{\text{ess}}(P_1)$$

We recall also the following theorem given by Hardy's inequalities with weight proven by Muckenhoupt [Muc72], which is useful to find a priori estimates.

Theorem A.4.84. [Muc72, Theorem 2] Let $U, V: \mathbb{R}_{>0} \rightarrow \mathbb{R}$ be measurable functions. If

$$B_0 = \sup_{r>0} \left(\int_0^r |U(x)|^2 dx \int_r^{+\infty} |V(x)|^{-2} dx \right) < +\infty$$

then for any measurable function f one has

$$\int_0^{+\infty} \left| U(x) \int_x^{+\infty} f(t) dt \right|^2 dx \leq 4B_0 \int_0^{+\infty} |V(x) f(x)|^2 dx.$$

And if one has

$$B_1 = \sup_{r>0} \left(\int_r^{+\infty} |U(x)|^2 dx \int_0^r |V(x)|^{-2} dx \right) < +\infty$$

then for any measurable function f one has

$$\int_0^{+\infty} \left| U(x) \int_0^x f(t) dt \right|^2 dx \leq 4B_1 \int_0^{+\infty} |V(x) f(x)|^2 dx.$$

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