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Essays in Dynamic General Equilibrium and Economic Growth

Ngoc-Sang Pham

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UNIVERSITÉ DE PARIS 1 PANTHÉON-SORBONNE

THÈSE

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l'Habilitation à Diriger des Recherches

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Essays in Dynamic General Equilibrium and Economic Growth

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Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 5 |
| 1.1 | General equilibrium theory | 5 |
| 1.2 | Asset valuation in general equilibrium | 8 |
| 1.3 | Financial market and production sector | 11 |
| 1.4 | Economic growth: role of foreign aid | 12 |
| 1.5 | Economic growth: role of FDI | 14 |
| 2 | General equilibrium theory | 15 |
| 2.1 | Existence of intertemporal equilibrium with borrowing constraints | 15 |
| 2.1.1 | Intertemporal equilibrium in a deterministic model with heterogeneous households, production and borrowing constraints | 15 |
| 2.1.2 | Intertemporal equilibrium in a stochastic model with heterogeneous producers, borrowing constraints and incomplete financial markets | 16 |
| 2.2 | Existence of general equilibrium: role of Sperner's lemma | 19 |
| 2.2.1 | Sperner's lemma and the Gale-Nikaido-Debreu lemma | 19 |
| 2.2.2 | Sperner's lemma and the existence of competitive equilibrium with incomplete markets | 20 |
| 2.3 | General equilibrium with non-normality and non-convexity | 22 |
| 3 | Financial frictions and macroeconomics: A general equilibrium approach | 26 |
| 3.1 | Asset valuation and asset price bubbles in general equilibrium | 26 |
| 3.1.1 | Asset price bubbles in general equilibrium models with infinitely-lived heterogeneous agents | 26 |
| 3.1.2 | Asset price bubbles in OLG models | 33 |
| 3.1.3 | Extension 1: Valuation and bubbles of productive assets | 35 |
| 3.1.4 | Extension 2: Valuation and bubbles of land and house | 39 |
| 3.2 | Equilibrium efficiency under borrowing constraints | 42 |
| 3.2.1 | Efficiency of intertemporal equilibrium in a one-sector model | 42 |
| 3.2.2 | Efficiency of intertemporal equilibrium in a two-sector model | 44 |
| 3.2.3 | Discussion: efficiency and existence of bubbles | 45 |
| 3.3 | (Non-Monotonic) impacts of productivity and credit changes on equilibrium aggregate production | 46 |
| 4 | Economic growth: roles of financial markets and globalization | 49 |
| 4.1 | Financial market, recession and economic growth | 49 |
| 4.1.1 | Financial market versus productive sector | 49 |
| 4.1.2 | Dividend taxation, public spending and economic growth | 50 |

| | | |
|----------|--|-----------|
| 4.1.3 | Bubble asset taxation, public spending and economic growth | 51 |
| 4.2 | Foreign aid and economic growth | 51 |
| 4.2.1 | Foreign grants and economic growth | 51 |
| 4.2.2 | Development loans and economic dynamics | 52 |
| 4.3 | FDI and economic growth | 53 |
| 5 | Research program | 57 |
| 5.1 | Dynamics of economic distributions | 57 |
| 5.2 | Corruption, inequality, and economic development | 58 |
| 5.3 | FDI and endogenous growth | 61 |
| 5.4 | Wariness | 62 |
| 5.4.1 | Wariness and economic growth | 63 |
| 5.4.2 | Wariness, altruism, and their macroeconomic impacts | 63 |
| A | Curriculum Vitae | 78 |
| A.1 | Positions and educations | 78 |
| A.2 | Teachings and supervisions | 79 |
| A.3 | Visits and professional activities | 80 |
| A.4 | Talks | 81 |
| A.5 | Others | 82 |
| A.6 | Publications | 82 |
| | A.6.1 Journal articles | 82 |
| | A.6.2 Book chapters | 83 |
| A.7 | Lectures | 84 |
| A.8 | Working papers | 84 |
| A.9 | Work in progress | 84 |

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Résumé

Depuis la publication de "the Wealth of Nations" d'Adam Smith en 1776, deux questions centrales en économie se posent : Comment fonctionne la main invisible et quel est son rôle dans l'économie ? Comment pouvons-nous atteindre la prospérité économique ? Cette thèse se concentre sur ces deux questions. De manière formelle, nous allons étudier la notion d'équilibre général dans divers modèles et ses applications en macroéconomie, particulièrement dans la croissance économique.

Dans le chapitre 1, nous offrons un aperçu de nos contributions.

Le chapitre 2 présente nos contributions à la théorie de l'équilibre général dynamique. Nous établissons l'existence d'un équilibre général dans deux modèles : (1) un modèle déterministe à horizon infini avec une entreprise représentative et un nombre fini de ménages hétérogènes (qui sont confrontés à une contrainte d'emprunt) et (2) un modèle stochastique à horizon infini avec des ménages hétérogènes, producteurs, les contraintes d'emprunt et les marchés financiers incomplets. Ensuite, nous utilisons le lemme de Sperner pour prouver l'existence d'un équilibre général dans l'économie de production (Debreu, 1959) et d'un équilibre concurrentiel avec des marchés financiers incomplets (Cass, 2006). Enfin, nous apportons de nouvelles visions concernant l'analyse de l'équilibre général en l'absence de normalité des biens et de convexité des préférences.

Le chapitre 3 utilise l'approche d'équilibre général pour aborder les questions de valorisation des actifs, de bulles de prix d'actifs et d'impacts des contraintes financières. Nous donnons les conditions dans lesquelles des bulles de divers actifs (arbre de Lucas, bien de capital physique, terrain, maison) peuvent apparaître ou être exclues à l'équilibre et leurs impacts macroéconomiques. Nous montrons le rôle de l'hétérogénéité, des contraintes financières, du taux d'intérêt, de la fiscalité, de l'altruisme. Ensuite, nous étudions l'efficacité à l'équilibre et discutons du lien entre les bulles et l'efficacité. Enfin, nous étudions les effets macroéconomiques des changements dans la limite de crédit et la productivité.

Le chapitre 4 expose nos contributions à la littérature sur la croissance économique, en examinant le rôle des marchés financiers, de l'aide internationale et des investissements directs étrangers (IDE). Nous utilisons principalement la théorie de l'équilibre général et la théorie de la croissance optimale.

Le programme de recherche est décrit dans le chapitre 5.

Le CV du candidat est présenté dans l'annexe A.

Chapter 1

Introduction

Since the publication of Adam Smith's *The Wealth of Nations* in 1776, two central questions in economics have been: What is the invisible hand, and what role does it play in the economy? How can we achieve economic prosperity? This thesis focuses on these two questions. Formally, we study the notion of general equilibrium in various models and its applications in macroeconomics, particularly in economic growth.

In this chapter, we provide an overview of our contributions. Chapter 2 presents our contributions to the theory of (dynamic) general equilibrium. Chapter 3 uses the general equilibrium approach to address the issues of asset valuation, asset price bubbles and impacts of financial constraints. It also investigates equilibrium efficiency and discusses the connection between bubbles and efficiency. Chapter 4 focuses on economic growth. We explore the roles of financial markets, foreign aid, and foreign direct investment (FDI). The primary tools used are the general equilibrium theory and the optimal growth theory. The research program is outlined in Chapter 5.

1.1 General equilibrium theory

Existence of intertemporal equilibrium with borrowing constraints

[Becker, Boyd III and Foias \(1991\)](#) demonstrated the existence of intertemporal equilibrium under borrowing constraints with inelastic labor supply. [Becker, Bosi, Le Van and Seegmuller \(2015\)](#) proved the existence of a Ramsey equilibrium with endogenous labor supply and borrowing constraint on physical capital; however, they only considered an implicit financial market and assumed that no one can borrow. In these papers, they needed some assumptions about production function to ensure that aggregate capital and consumption stocks are uniformly bounded.

In [Le Van and Pham \(2016\)](#), we consider an infinite-horizon deterministic general equilibrium model with a finite number of heterogeneous households (who face a borrowing constraint), a representative firm. We prove the existence of equilibrium in this model. In our proof of the existence of an intertemporal equilibrium, we allow non-stationary and even linear production functions and do not need that aggregate capital and consumption stocks be uniformly bounded. We firstly prove that there exists an equilibrium for each T -truncated economy. We then obtain a sequence of equilibria (indexed by T) which will be proved to have a limit for the product topology. Last, we prove that such limit is indeed an intertemporal equilibrium.

Existence of intertemporal equilibrium in a stochastic model with heterogeneous producers, borrowing constraints and incomplete financial markets

There is a vast literature on the existence of equilibrium with incomplete financial markets;¹ see [Magill and Quinzii \(1994\)](#), [Florenzano \(1999\)](#), [Magill and Quinzii \(1996\)](#), [Kubler and Schmedders \(2003\)](#), [Magill and Quinzii \(2008\)](#) and references therein, or more recently [Araujo, Pascoa and Torres-Martinez \(2011\)](#), [Pascoa et al. \(2011\)](#). We only mention here some papers that are very closed to ours. [Geanakoplos and Zame \(2002\)](#) prove the existence of collateral equilibrium in a two-period models with durable goods and collateralized securities. By extending [Geanakoplos and Zame \(2002\)](#), [Araujo, Pascoa and Torres-Martinez \(2002\)](#) prove the existence of equilibrium in an infinite-horizon model with a collateral requirement on the sale of financial assets. [Kubler and Schmedders \(2003\)](#) construct and prove, with additional conditions on the exogenous variables, the existence of a Markov equilibrium in an infinite-horizon asset pricing model with incomplete markets and collateral constraints; such a Markov equilibrium is also proved to be competitive equilibrium. However, these papers did not take into account the production sector.

In macroeconomics, since [Kiyotaki and Moore \(1997\)](#), there has been a growing literature focusing on general equilibrium models with heterogeneous firms ([Khan and Thomas, 2013](#)). However, in general, this literature does not pay much attention on the equilibrium existence.

In [Bosi, Le Van and Pham \(2018\)](#), we try to fill the gap between these two literature by building a stochastic model with heterogeneous producers, borrowing constraints and incomplete markets.

A proof à la [Becker, Bosi, Le Van and Seegmuller \(2015\)](#) or [Le Van and Pham \(2016\)](#) no longer applies in our model because agents trade short-lived financial assets with zero supply instead of long-lived assets. The difficulty is to prove that individual asset volumes are bounded. To overcome the difficulty, we introduce an intermediate economy where the real asset is replaced by a nominal one. In this intermediate economy, we can bound the volume of the financial asset, and so can prove the existence of equilibrium by adapting the method of [Becker, Bosi, Le Van and Seegmuller \(2015\)](#) and [Le Van and Pham \(2016\)](#): (1) we prove the existence of equilibrium for each T -truncated economy \mathcal{E}^T ; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy \mathcal{E} . Last, we construct an equilibrium for the original economy from an equilibrium of the intermediate economy.

Notice that our proof can apply to a large class of general equilibrium models used in macroeconomics.

Sperner's lemma and general equilibrium

The Gale-Nikaido-Debreu lemma ([Debreu, 1959](#); [Gale, 1955](#); [Nikaido, 1956](#)) plays an important role in establishing the existence of general equilibrium.² Let us recall the following version (Theorem 1 in [Debreu \(1982\)](#), page 82) of the GND lemma.

Lemma 1 (Gale-Nikaido-Debreu lemma). *Let Δ be the unit-simplex of \mathbb{R}^N . Let ζ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ into*

¹An excellent introduction to asset pricing models with incomplete markets and infinite-horizon can be found in [Magill and Quinzii \(2008\)](#)

²Another important lemma in the general equilibrium theory is Gale and Mas-Colell's lemma introduced and proved by ([Gale and Mas-Colell, 1975, 1979](#)). Their proofs are based on the Kakutani fixed point theorem and Michael selection theorem ([Michael, 1956](#)). See [Florenzano \(2009\)](#) for the role of these two lemmas in the general equilibrium theory.

\mathbb{R}^N . Suppose ζ satisfies the following condition:

$$\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z \leq 0. \quad (1.1)$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}_-^N \neq \emptyset$.

In many setups, the existence of equilibrium is a direct consequence of this lemma. To see the intuition, consider a particular case: an exchange economy with N goods and m agents whose utility functions are continuous, strictly increasing, strictly concave. Given the price $p \in \Delta$, the exceed demand is a function $Z(p) \equiv \sum_i (x_i(p) - e_i)$, where $e_i \in \mathbb{R}_+^N$ is the endowment of agent i , $x_i(p)$ is the demand function of agent i . We have the Walras law: $pZ(p) \leq 0, \forall p \in \Delta$. So, by the Gale-Nikaido-Debreu lemma, there exists $p \in \Delta$ satisfying $Z_l(p) \leq 0, \forall l = 1, \dots, N$. It means that p is an equilibrium price.

The classical proofs of the Gale-Nikaido-Debreu lemma require the use of the fixed point theorems. Indeed, (Debreu, 1952, 1956) and Nikaido (1956) used the Kakutani fixed point theorem while Gale (1955) used lemma Knaster, Kuratowski, and Mazurkiewicz (1929)'s lemma to prove the Gale-Nikaido-Debreu lemma. Kuhn (1956) provided another proof of the Gale-Nikaido-Debreu lemma, which is based on the Eilenberg-Montgomery fixed-point theorem.

As mentioned by Duppe and Weintraub (2014) and Khan (2021), Debreu wanted to discuss the question whether one could dispense with a fixed point theorem in proving the Gale-Nikaido-Debreu lemma.

In Le, Le Van, Pham and Saglam (2022), we address Debreu's question by providing a new proof of the Gale-Nikaido-Debreu lemma directly from Sperner's lemma which is a combinatorial result on colorings of triangulations.³ More precisely, our proof relies on Sperner's lemma, Carathéodory (1907)'s convexity theorem, and basic properties of topology such as the finite covering of a compact set. It should be noticed that the Sperner lemma and the mathematical tools that we use to prove the Gale-Nikaido-Debreu lemma date back to 1928.

In a companion paper (Le, Le Van, Pham and Saglam, 2024), we use Sperner's lemma to prove that if the rank of the payoff matrix equals the number of financial assets and other mild conditions hold, there exists an existence of competitive equilibrium with incomplete markets, whose asset price is a convex combination of the return matrix where the weights are exogenously given. Interestingly, our result recovers the existence of competitive equilibrium in economies with nominal assets as in Cass (2006) and Florenzano (1999).

General equilibrium with non-normality and non-convexity

In the general equilibrium theory and comparative statics, the normality and preferences' convexity play an important role. In Le Van and Pham (2020) and Le Van and Pham (2023), we revisit this issue.

We are firstly interested in inferior and Giffen goods. These notions have been mentioned in most microeconomics textbooks (see Mas-Colell et al. (1995), Jehle and Reny (2011), Varian (2014) for instance).⁴ However, these goods are usually illustrated by pictures.

³Sperner (1928)'s lemma can be viewed as a combinatorial variant of the Brouwer fixed point theorem and actually equivalent to it. For instance, Knaster, Kuratowski, and Mazurkiewicz (1929) used the Sperner lemma to prove the Knaster-Kuratowski-Mazurkiewicz theorem which implies the Brouwer theorem. Meanwhile, Yoseloff (1974), Park and Jeong (2003) proved the Sperner lemma by using the Brouwer theorem. The reader is referred to Park (1999) for a more complete survey of fixed point theorems and Ben-El-Mechaiekh et al. (2009) for a survey of general equilibrium and fixed point theory.

⁴Jensen and Miller (2008) provide real evidences (in two provinces of China: Hunan and Gansu) of Giffen behavior.

In [Le Van and Pham \(2020\)](#), we present a class of differentiable, strictly increasing, concave utility functions exhibiting an explicit demand of a good which may have Giffen behavior. Then, we study the equilibrium property under this utility function.

We now discuss the preferences' convexity. When proving the existence of general equilibrium, the standard approach requires the convexity of the preferences (or the (quasi)concavity of the utility function).⁵ The existence of equilibrium with non-convex preferences is an important issue not only in microeconomics but also in finance because investors' preferences may not be convex. However, this question remains open. Although [Aumann \(1966\)](#) proves the existence of general equilibrium in an exchange economy consisting of a continuum of agents with non-convex preferences,⁶ he also recognizes that such a result may not hold when there are finitely many agents.

Recently, [Araujo, Chateauneuf, Gama and Novinski \(2018\)](#) study the equilibrium existence in a model with two kinds of agents: a risk averse agent (having strictly concave utility function) and a risk loving agent (having strictly convex utility function). A key result in [Araujo, Chateauneuf, Gama and Novinski \(2018\)](#) is that there exists an equilibrium if the aggregate endowment of risk averse agents is sufficiently large in some state of nature compared to the aggregate endowment in other states. Moreover, such an equilibrium is a corner equilibrium.

In [Le Van and Pham \(2023\)](#), we study the issue of the existence of equilibrium when agents may be neither risk averse nor risk loving (i.e., the agents' utility functions are neither concave nor convex). Under specific utility functions, we provide a necessary and sufficient condition for the existence of equilibrium. This condition allows us to understand the role of endowments and agents' preferences.

1.2 Asset valuation in general equilibrium

Asset valuation is one of the fundamental issues in economics and finance. When evaluating the asset value, investigating the existence and dynamics of asset price bubbles are crucial. According to the classical paper by [Santos and Woodford \(1997\)](#), conditions under which bubbles exist are relatively fragile. After the global financial crisis of 2007-2009, this topic has regained momentum and different new mechanisms of bubbles have been proposed.⁷ In the following, we present our contribution to the literature of asset price bubbles.

Asset pricing and asset price bubble: a basic idea

We start with a deterministic setup. Consider a long-lived financial asset with the following structure: at period t , if one economic agent buys 1 unit of financial asset with price q_t , she will receive d_{t+1} units of consumption good as dividend and she will be able to resell 1 units of financial asset with price q_{t+1} . This asset can be interpreted as Lucas' tree or security ([Santos and Woodford, 1997](#)), or stock ([Kocherlakota, 1992](#)). When there is no dividend, this asset can be viewed as fiat money.

⁵[Debreu \(1982\)](#) and [Florenzano \(2003\)](#) offer excellent treatments of the existence of equilibrium.

⁶A key point in [Aumann \(1966\)](#) is that the aggregate preferred set is convex. He proves this by using a mathematical result which states that the integral of any set-valued function over a non-atomic measure space is convex ([Aumann, 1965](#); [Richter, 1963](#)).

⁷See [Farhi and Tirole \(2012\)](#), [Martin and Ventura \(2012\)](#), [Gali \(2014, 2021\)](#), [Hirano and Yanagawa \(2017\)](#), [Miao and Wang \(2012, 2018\)](#), [Barbie and Hillebrand \(2018\)](#) among others. The reader can also find detailed surveys in [Brunnermeier and Oehmke \(2012\)](#), [Miao \(2014\)](#), [Martin and Ventura \(2018\)](#), [Hirano and Toda \(2024\)](#).

Denote its price at date t is q_t . Assume that we have the following asset pricing (later, we will provide conditions under which this happens):

$$q_t = \gamma_{t+1}(q_{t+1} + d_{t+1}). \quad (1.2)$$

where γ_t is the discount factor of the economy from date t to date $t + 1$. In equilibrium, this is endogenous.

Following the standard literature (Tirole, 1982, 1985; Kocherlakota, 1992; Santos and Woodford, 1997), we introduce the notion of rational asset price bubbles.

We define the discount factor, Q_t , of the economy from date 1 to date t

$$Q_t := \gamma_1 \cdots \gamma_t. \quad (1.3)$$

We then have

$$Q_t q_t = Q_{t+1} q_{t+1} + Q_{t+1} d_{t+1}, \quad q_0 = \sum_{t=0}^T Q_t d_t + Q_T q_T. \quad (1.4)$$

1. At date 1, one unit (from date 0) of this asset will give back 1 units of the same asset and ξ_1 units of consumption good as its dividend. This is represented by $q_0 = Q_1 \xi_1 + Q_1 q_1$
2. At date 2, one unit of long lived asset will give one unit of the same asset and ξ_2 units of consumption good. This is represented by $Q_1 q_1 = Q_2 \xi_2 + Q_2 q_2$, and so on.

This leads us to define the fundamental value of financial asset

$$FV_0 := \sum_{t=1}^{+\infty} Q_t \xi_t. \quad (1.5)$$

Definition 1. (1) Consider the asset pricing (1.2), there is an asset price bubble (or a bubble, for short) if and only if the price of the asset is strictly higher than its fundamental value: $q_0 > FV_0$.

(2) Consider a model economy where there exists an equilibrium satisfying the asset pricing (1.2). We say that this equilibrium is bubbly (bubbleless, respectively) if there exists (does not exist, respectively) a bubble.

Consider the asset pricing (1.2). There is a bubble if and only if $\lim_{t \rightarrow \infty} Q_t q_t > 0$. If we assume, in addition, that $q_t > 0, \forall t$, then there is a bubble if and only if $\sum_{t \geq 0} \frac{\xi_t}{q_t} < \infty$.⁸

We are interested in the following questions:

1. Why does there exist a bubble in general equilibrium? What are the key factors determining the existence of bubble? What are the characteristics of a bubbly equilibrium?
2. Does there exist an equilibrium indeterminacy? If yes, what is the different between the bubbly and bubbleless equilibria?

These questions are not easy because not only the asset prices (q_t) but also the discount factor (γ_t) are endogenous. The literature on rational asset price bubbles has focused on two frameworks: (1) infinite-horizon general equilibrium models with infinitely lived agents and (2) overlapping generations models (OLG).

⁸This is proved in Montrucchio (2004), Bosi, Le Van and Pham (2018). The proof is simple. Indeed, observe that $q_0 = Q_T q_T \prod_{t=1}^T (1 + \frac{d_t}{q_t})$. Since $q_0 > 0$, we see that $\lim_{t \rightarrow +\infty} Q_t q_t > 0$ if and only if $\lim_{t \rightarrow \infty} \prod_{t=1}^T (1 + \frac{\xi_t}{q_t}) < \infty$. It is easy to prove that this condition is equivalent to $\sum_{t=1}^{\infty} \frac{\xi_t}{q_t} < +\infty$.

Bubbles in overlapping generations (OLG) models

Since the influential paper of [Tirole \(1985\)](#), numerous studies have privileged OLG models to study the existence of bubbles and their macroeconomic implications.

In [Bosi and Pham \(2016\)](#), we study the interplay between taxation, bubble formation and economic growth. We prove that a pure bubble asset may be beneficial when growth is fueled by R&D externalities and the government levies taxes on bubble returns to finance this R&D. We provide a bigger picture concerning the effect of bubbles in endogenous growth, which complements the conventional view about the negative effect of bubbles in endogenous growth ([Grossman and Yanagawa, 1993](#)).

In [Bosi, Ha-Huy, Le Van, Pham and Pham \(2018a\)](#), [Bosi, Ha-Huy, Pham and Pham \(2022\)](#), we explore the role of (ascendant and descendant) altruism on the existence and dynamics of asset price bubbles. [Bosi, Ha-Huy, Pham and Pham \(2019\)](#) investigates bubbles in an OLG model where altruism à la Barro (1974) is introduced through a recursive utility.

Asset valuation and bubbles in infinite-horizon general equilibrium models with infinitely lived agents

Although it is also important to study infinite-horizon models of bubbles,⁹ this kind of framework has received relatively less attention.¹⁰ As recognized by [Kocherlakota \(2008\)](#), [Miao \(2014\)](#) and [Martin and Ventura \(2018\)](#), our understanding of bubbles in infinite-horizon models is far from complete.

In the existing literature, there are some examples of bubbles in general equilibrium models with infinitely lived agents.¹¹ Concerning the asset having zero dividends and positive supply (i.e., fiat money), [Bewley \(1980\)](#) (Section 13), [Townsend \(1980\)](#), [Kocherlakota \(1992\)](#) (Example 1) and [Scheinkman and Weiss \(1986\)](#) show that, when borrowing is not allowed, fiat money may have positive value in infinite-horizon general equilibrium models.¹² [Kocherlakota \(1992\)](#) considers an asset with positive supply and zero dividends (i.e., fiat money) in a deterministic model without short-sale and shows that money may have a positive value. [Santos and Woodford \(1997\)](#) present several examples of this kind of bubbles: their examples 4.1, 4.2 study fiat money in deterministic models while and their example 4.4 investigates fiat money in a stochastic model. [Hirano and Yanagawa \(2017\)](#) also give sufficient conditions for the existence of stochastic bubbles of an asset without dividend.¹³

There are a few examples of bubbles of assets with positive dividends. In a deterministic set-up, Example 4.3 in [Santos and Woodford \(1997\)](#) studies bubbles of an asset with positive dividends but with zero net supply. Example 4.5 in [Santos and Woodford \(1997\)](#) also investigates bubbles of the Lucas' tree, although they use a stochastic model with a single representative household.¹⁴ [Bloise and Citanna \(2019\)](#) provide a sufficient condition based on trade and punishment for default for the existence of the bubble of an asset with vanishing dividends of an equilibrium whose sequence of allocations converges.

⁹[Miao \(2014\)](#) explains why we need to study infinite-horizon models of bubbles.

¹⁰In such models, it is difficult to characterize or compute the equilibrium. It is also not easy to provide non-trivial examples of intertemporal equilibrium.

¹¹[Brunnermeier and Oehmke \(2012\)](#), [Miao \(2014\)](#) and [Martin and Ventura \(2018\)](#) provide excellent surveys on bubbles. Here, we focus on bubbles in general equilibrium models with infinitely lived agents.

¹²[Bewley \(1980\)](#) (Section 13), [Townsend \(1980\)](#), [Kocherlakota \(1992\)](#) consider discrete-time, exchange deterministic models while [Scheinkman and Weiss \(1986\)](#) study a continuous-time model with labor.

¹³They also study how the existence of bubbles, economic growth, welfare depend on the degree of pledgeability.

¹⁴In this example, they introduce a sequence of non-stationary stochastic discount factors and show that bubbles may exist under a state-price process but not under another state-price process.

Our contributions. [Le Van and Pham \(2016\)](#) show that there may exist a price bubble of an asset (having positive dividends and positive net supply) in infinite-horizon general equilibrium models. Then, [Bosi, Le Van and Pham \(2017\)](#), [Bosi, Le Van and Pham \(2017c\)](#), [Bosi, Le Van and Pham \(2018\)](#) extend the asset pricing to the case of other assets (land, house, capital good, ...) in both deterministic and stochastic models. We also construct models where there may exist a price bubble of these assets.

Recently, in [Bosi, Le Van and Pham \(2022\)](#), we show that both real indeterminacy and asset price bubble may appear in an infinite-horizon exchange economy with infinitely lived agents and an imperfect financial market. We explain how the asset structure and heterogeneity (in terms of preferences and endowments) affect the existence and the dynamics of asset price bubbles as well as the equilibrium indeterminacy.

1.3 Financial market and production sector

The financial market plays an important role on the functioning of the economy. However, it has also been considered as one of main causes of economic recession or/and fluctuation. We revisit this issue by investigating several questions: does financial market always cause an economic recession? How is the interaction between financial market and the productive sector? What is the role of borrowing constraints? What should we do with the financial market to enhance the economic growth?

Recession, asset taxation and growth

[Le Van and Pham \(2016\)](#) consider the interaction between financial market and production sector in an infinite-horizon general equilibrium. We point out that when the productivity is high enough, the productive sector never falls in recession. When the productivity is low, the productive sector will fall in recession at infinitely many dates (not necessary at all dates) because the agents prefer financial assets to physical capital. However, at some dates, even when the productivity is low, financial assets may be beneficial to the productive sector by providing financial support for the purchase of the physical capital. Thanks to that, a recession may be avoided.

In [Pham \(2023\)](#), we extend [Le Van and Pham \(2016\)](#) by introducing a dividend taxation. Our point is that imposing a dividend tax and using the revenue from this tax to finance productive government spending may prevent economic recession and promote economic growth.

In [Bosi and Pham \(2016\)](#), we propose a tax on the pure bubble asset and use this tax again to finance the public investment which can in turn enhance productivity. Applying this strategy may be beneficial to the economic growth. In some cases, ruling out a bubble may harm the economy because the government cannot get tax revenue from this asset and by consequence does not have enough financing for productivity investment.

(Non-Monotonic) impacts of productivity and credit changes on equilibrium aggregate production

In [Pham and Pham \(2021\)](#) and [Pham \(2023\)](#), we build general equilibrium models with borrowing constraint and heterogeneous producers in order to analyze the effects of changes in credit limit and productivity on the equilibrium outcome. We point out that, due to financial frictions, the equilibrium aggregate production may be non-monotonic in both individual productivity and credit limit. We provide conditions under which this phenomenon happens. By consequence, the emergence of some firms (for example, improving productivity

or relaxing credit limit) may not necessarily be beneficial to economic development. Our findings complement the main point in [Buera and Shin \(2013\)](#), [Khan and Thomas \(2013\)](#), [Midrigan and Xu \(2014\)](#), [Moll \(2014\)](#), [Catherine, Chaney, Huang, Sraer and Thesmar \(2017, 2022\)](#). Recall that these studies provide conditions under which relaxing credit limits has positive impact on the aggregate output.

Then, we prove that the non-monotonic effect of productivity and credit limit on the aggregate output cannot appear at the steady state in infinite-horizon models à la Ramsey. The reason is that the steady state interest rate only depends on the rate of time preferences of agents.

Therefore, we should focus on the global dynamics of intertemporal equilibrium in order to fully understand the effects of productivity changes. First, our findings suggest that a permanent increase of productivity of less productive agents improves the aggregate output in the long run. However, when this productivity change is quite small and credit constraints are tight, the aggregate output may decrease in the short-run and then increase in the long-run.

Concerning the effects of credit limits, along intertemporal equilibrium, we show that an increase of the most productive producer's credit may reduce the output at every period. The intuition behind is that when her(his) credit limit goes up, the equilibrium interest rate increases, and hence, her(his) repayment also increases. This in turn reduces her(his) net worth in the next period. By consequence, her(his) saving and hence the production decrease. The mechanism can be summarized by the following schema:

$$\begin{aligned} \text{Credit limit } \uparrow &\Rightarrow \text{Interest rate } \uparrow \Rightarrow \text{Agent's net worth } \downarrow \Rightarrow \\ &\Rightarrow \text{Saving } \downarrow \Rightarrow \text{Production } \downarrow \Rightarrow \dots \end{aligned} \tag{1.6}$$

As in the static model, this mechanism can happen because the credit limit of the most productive agent remains low and the productivity dispersion is high.

Equilibrium efficiency: role of borrowing constraint and asset dividend

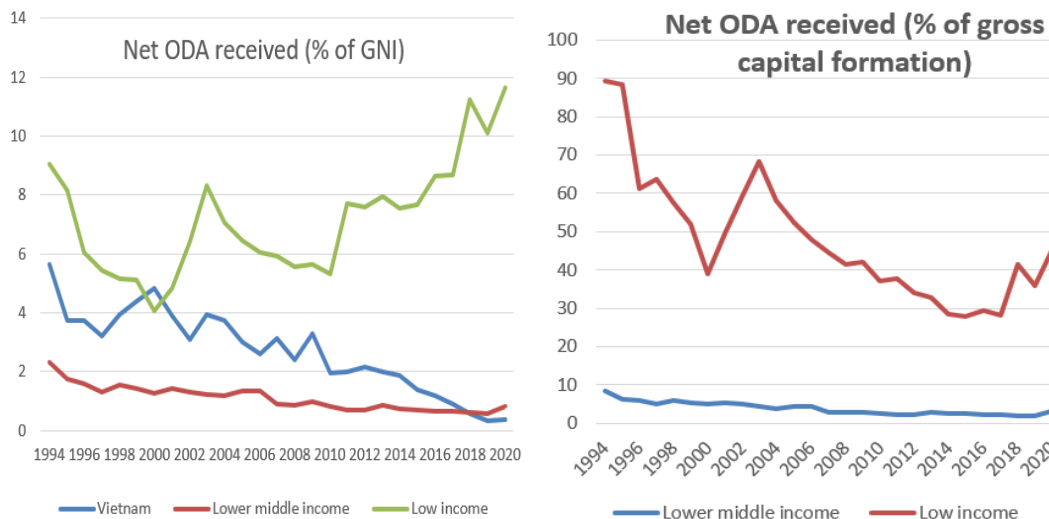
In [Le Van and Pham \(2016\)](#), [Bosi, Le Van and Pham \(2017\)](#), we investigate the efficiency of intertemporal equilibria in Ramsey models with heterogeneous households. We consider both one-sector and two-sector models. We provide sufficient conditions (either based on endogenous variables or exogenous variables) for the equilibrium efficiency. Interestingly, we prove that if the asset dividend is quite good, then every equilibrium is efficient. We also show that the borrowing constraints play an important role.

In a two-period general equilibrium model with borrowing constraint and two producers ([Pham and Pham, 2021](#); [Pham, 2022](#)), we prove that the equilibrium is efficient if and only if the credit limit of the most productive producer is high enough and the productivity dispersion is low.

1.4 Economic growth: role of foreign aid

The official development assistance (ODA) is important for low income countries as shown in the following graphics taken from the World Bank's website.

The evaluation of aid effects is necessary to determine an efficient allocation of aid to recipient countries. Many issues are under debate regarding the effectiveness of foreign aid in terms of economic growth. While empirical studies are abundant, there are relatively few theoretical analyses on this issue ([Chenery and Strout, 1966](#); [Charterjee et al., 2003](#); [Charterjee and Tursnovky, 2007](#); [Dalggaard, 2008](#); [Carter, 2014](#)), specially from a dynamic



point of view.¹⁵ We contribute to the literature from a theoretical point of view by modeling foreign aid and studying its role in general equilibrium and optimal growth models.

Pham and Pham (2020) consider discrete-time infinite-horizon frameworks where public investment, partially financed by aid, may improve the recipient economy's TFP. We model aid per capita at period t as $a_t = \max(\bar{a} - \phi k_t, 0)$ where the couple (\bar{a}, ϕ) is interpreted as the aid rule imposed by the donor and represents aid conditionalities while the physical capital per capita k_t represents the recipient country's need.¹⁶ This modeling takes into account the donor's rules and the recipient's need which is represented by a low initial capital stock. According to the above formula, aid flows are limited by an upper threshold, and the recipient country would no longer receive aid once was rich enough. This modeling is also compatible with aid allocation rules used in several bilateral and multilateral aid policies (World Bank's International Development Association, Asian Development Bank, European Development Fund, etc.). Notice that in Pham and Pham (2020) we explore the effects of foreign aid (grants) in two models: infinite-horizon general equilibrium model and an optimal growth model with externality.

Le Van, Pham and Pham (2023) investigate the nexus between foreign aid (in both forms: grant and loan), poverty trap, and economic development in a recipient country by using a Solow model with two new ingredients: a development loan and a fixed cost in the production process. The presence of this fixed cost generates a poverty trap. Then, we explain how the development loans can help the recipient country to avoid the poverty trap and eventually get economic growth.

Our results indicate that whether or not the country can overcome the poverty trap and obtain economic growth depends not only on the foreign aid (grants and loans) but also, and mainly, on its own capacity (such as the TFP, the saving rate, and the governance quality, ...). If the recipient country only waits for foreign aid, an endogenous cycle may arise.

In Pham and Pham (2020) and Le Van, Pham and Pham (2023), the rate of wasted aid is exogenous. In Pham and Pham (2019), we endogenize this rate and investigate the nexus between foreign aid, fiscal policy and economic outcomes in a small recipient country. Foreign aid may increase not only government expenditures but also private capital, and hence improve economic growth. However, it may also discourage the recipient's tax effort. The effects of foreign aid on fiscal policy and economic growth depend on the circumstances

¹⁵See Pham and Pham (2020) for a survey.

¹⁶Dalgaard (2008) considers that $a_t = \phi y_{t-1}^\lambda$, $\phi > 0$, $\lambda < 0$.

of the recipient country, including its development level, the TFP, the efficiency of public investment, and in particular the government's concern for the population's welfare.

1.5 Economic growth: role of FDI

Over the past few decades, opening up to the global economy and attracting foreign direct investment (FDI) have been policy priorities in developing countries for promoting their economic development. One of the main arguments is that multinational enterprises (MNEs) could boost investment, bring new technologies/management skills, and generate FDI spillovers to domestic firms. However, empirical studies show ambiguous FDI effects on the host country's development.

At the micro-level, MNEs could generate spillovers to domestic competitors in the same industry (horizontal spillovers) or upstream and downstream local firms (vertical spillovers).¹⁷ Empirical evidence shows positive spillovers from downstream FDI firms (mainly joint venture FDI firms) to domestic suppliers but negative spillovers from upstream FDI firms to downstream domestic producers.¹⁸ Besides, the literature provides evidence of mixed results regarding FDI horizontal spillovers.¹⁹

At the macro-level, the empirical literature finds that the effect of FDI on the host country's economic growth is relatively weak (Carkovic and Levine, 2005). More precisely, whether this effect is significant depends on local conditions such as the host country's human capital (Borensztein, De Gregorio and Lee, 1998; Li and Liu, 2005) and the development of local financial markets (Alfaro, Chanda, Kalemli-Ozcan and Sayek, 2004, 2010).

The previous conflicting results on the effects of FDI raise a fundamental question on how a host country can benefit from FDI spillovers. We try to address this issue by introducing FDI in growth models to study the optimal allocation of a host country.

In Nguyen-Huu and Pham (2018), we construct a two-period small open economy with two sectors (an old sector producing a consumption good and a new sector producing a new good), two production factors (physical capital and specific labor), and two heterogeneous firms in the new sector (a multinational firm and a domestic firm). This simple framework allows us to analyze the effects of several macroeconomic variables on the country's optimal strategy which as we prove vary across its development level.

Nguyen-Huu and Pham (2024) extend Nguyen-Huu and Pham (2018) and investigate the optimal strategy of a small open economy receiving FDI in an optimal growth framework. We prove that no domestic firm can enter the new industry when the multinational enterprise's productivity or the fixed entry cost is high. Nevertheless, the host country's investment stock converges to a higher steady state than an economy without FDI. A domestic firm enters the new industry if its productivity is high enough. Moreover, the domestic firm can dominate or even eliminate its foreign counterpart.

¹⁷See Blomstrom and Kokko (1998), Greenaway and Gorg (2004), Crespo and Fontoura (2007) for more complete reviews of FDI spillovers, and Meyer, Klaus and Sinani (2009), Irsova and Havranek (2013) for meta-analyses.

¹⁸For more discussions on vertical FDI spillover, see Javorcik (2004), Newman, Rand, Talbot and Tarp (2015), Lu, Tao and Zhu (2017) for the case of Lithuania, Vietnam, China, respectively, and Gorodnichenko, Svejnar and Terrell (2014) for 17 transition countries.

¹⁹Indeed, there are negative or nil impacts of horizontal FDI on domestic firms in developing countries as, for example, Morocco (Haddad and Harrison, 1993), Uruguay (Kokko, Tansini and Zejan, 1996), Eastern Europe countries (Jude, 2012), Vietnam (Newman, Rand, Talbot and Tarp, 2015). By contrast, evidence of positive horizontal spillovers from FDI in developed countries is reported in Ruane and Ugur (2005) for Ireland, Haskel, Pereira and Slaughter (2007) for the UK, or Keller and Yeaple (2009) for the US.

Chapter 2

General equilibrium theory

2.1 Existence of intertemporal equilibrium with borrowing constraints

The issue of existence of equilibrium is one of the fundamental questions in economics.¹ We contribute to the literature by establishing the existence of intertemporal equilibrium in infinite-horizon general equilibrium models with imperfect financial markets.

2.1.1 Intertemporal equilibrium in a deterministic model with heterogeneous households, production and borrowing constraints

In [Le Van and Pham \(2016\)](#), we consider a deterministic model with imperfect financial market (characterized by borrowing constraints). There are a representative firm and a finite number m of heterogeneous households. Each household i takes the sequence of prices $(p, q, r) = (p_t, q_t, r_t)_{t=0}^{\infty}$ as given and solves the following problem

$$(P_i(p, q, r)) : \max_{(c_{i,t}, k_{i,t+1}, a_{i,t})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \right] \quad (2.1)$$

$$\text{subject to: } c_t \geq 0, k_{i,t+1} \geq 0 \quad (2.2)$$

$$\begin{aligned} \text{(budget constraint): } p_t(c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}) + q_t a_{i,t} \\ \leq r_t k_{i,t} + (q_t + p_t \xi_t) a_{i,t-1} + \theta_t^i \pi_t \end{aligned} \quad (2.3)$$

$$\text{(borrowing constraint): } (q_{t+1} + p_{t+1} d_{t+1}) a_{i,t} \geq -f^i [p_{t+1}(1 - \delta) + r_{t+1}] k_{i,t+1}, \quad (2.4)$$

where $f^i \in [0, 1]$ is borrowing limit of agent i . f^i is an exogenous parameter and set by law. This parameter can be viewed as an index of the financial development of the economy.

For each period, there is a representative firm which takes prices (p_t, r_t) as given and maximizes its profit by choosing physical capital amount K_t .

$$(P(p_t, r_t)) : \max_{K_t \geq 0} [p_t F_t(K_t) - r_t K_t] \quad (2.5)$$

$(\theta_t^i)_{i=1}^m$ is the share of profit at date t . $\theta_i := (\theta_t^i)_t$ is exogenous, $\theta_t^i \geq 0$ for all i and $\sum_{i=1}^m \theta_t^i = 1$.

¹Looking back at history, [Debreu \(1952\)](#) used the Eilenberg-Montgomery fixed point theorem to prove the existence of a social equilibrium. Then, by using this social equilibrium existence theorem, [Arrow and Debreu \(1954\)](#) proved the existence of a general equilibrium for a competitive economy with productions. See [Debreu \(1956\)](#) and [Florenzano \(1981, 2003\)](#) for excellent treatments of the existence of equilibrium. See also [Duppe and Weintraub \(2014\)](#), [Khan \(2021\)](#) for discussions about the history of the general equilibrium theory.

Here we allow growth for the physical quantities (consumption, capital stocks, outputs). Our framework is rich enough to cover both productive sector and imperfect financial market.²

Definition 2. Denote \mathcal{E} the economy which is characterized by a list

$$\left((u_i, \beta_i, k_{i,0}, a_{i,-1}, f^i, \theta^i)_{i=1}^m, (F_t, \xi_t)_{t=0}^\infty, \delta \right).$$

Definition 3. A sequence of prices and quantities $(\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t})_{i=1}^m, \bar{K}_t)_{t=0}^{+\infty}$ is an equilibrium of the economy \mathcal{E} if the following conditions are satisfied:

(i) Price positivity: $\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$ for $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\text{good : } \sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1 - \delta)\bar{k}_{i,t}) = F_t(\bar{K}_t) + \xi_t,$$

$$\text{capital : } \bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t},$$

$$\text{financial asset : } \sum_{i=1}^m \bar{a}_{i,t} = 1.$$

(iii) Optimal consumption plans: for each i , $(\bar{c}_{i,t}, \bar{k}_{i,t+1}, \bar{a}_{i,t})_{t=0}^\infty$ is a solution of the problem $(P_i(\bar{p}, \bar{q}, \bar{r}))$.

(iv) Optimal production plan: for each $t \geq 0$, \bar{K}_t is a solution of the problem $(P(\bar{p}_t, \bar{r}_t))$.

Proposition 1 (Le Van and Pham (2016)). There exists an equilibrium in the economy described in Definition 3 if the following assumptions hold:

Assumption (H1): u_i is in C^1 , $u_i(0) = 0$, $u_i'(0) = +\infty$, and u_i is strictly increasing, concave, continuously differentiable.

Assumption (H2): $F_t(\cdot)$ is strictly increasing, concave, continuously differentiable, $F_t(0) \geq 0$.

Assumption (H3): For every $t \geq 0$, $0 < \xi_t < \infty$.

Assumption (H4): At initial period 0, $k_{i,0}, a_{i,-1} \geq 0$, and $(k_{i,0}, a_{i,-1}) \neq (0, 0)$ for $i = 1, \dots, m$. Moreover, we assume that $\sum_{i=1}^m a_{i,-1} = 1$ and $K_0 := \sum_{i=1}^m k_{i,0} > 0$.

Assumption (H5): For each agent i , her utility is finite

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)) < \infty. \quad (2.6)$$

2.1.2 Intertemporal equilibrium in a stochastic model with heterogeneous producers, borrowing constraints and incomplete financial markets

We describe the model in Bosi, Le Van and Pham (2018). This is an infinite horizon discrete time economy where the set of dates is $0, 1, \dots$ and there is no uncertainty at initial date ($t = 0$). Given a history of realizations of the states of nature for the first $t - 1$ dates, with $t \geq 1$, $\bar{s}_t = (s_0, \dots, s_{t-1})$, there is a finite set $\mathcal{S}(\bar{s}_t)$ of states that may occur at date t . A

²However, for simplicity, we assume exogenous supply of labor.

vector $\xi = (t, \bar{s}_t, s)$, where $t \geq 1$ and $s \in \mathcal{S}(\bar{s}_t)$, is called a *node*. The only node at $t = 0$ is denoted by ξ_0 . Let \mathcal{D} be the (countable) event-tree, i.e., the set of all nodes. We denote by $t(\xi)$ the date associated with a node ξ .

Given $\xi := (t, \bar{s}_t, s)$ and $\mu := (t', \bar{s}_{t'}, s')$, we say that μ is a successor of ξ , and we write $\mu > \xi$, if $t' > t$ and the first $t + 1$ coordinates of $\bar{s}_{t'}$ are (\bar{s}_t, s) . We write $\mu \geq \xi$ to say that either $\mu > \xi$ or $\mu = \xi$.

For each T and ξ , we denote $D(\xi) := \{\mu : \mu \geq \xi\}$ the sub-tree with root ξ ; $D_T := \{\xi : t(\xi) = T\}$ the family of nodes with date T ; $D^T(\xi) := \bigcup_{t=t(\xi)}^T D_t(\xi)$, where $D_T(\xi) := D_T \cap D(\xi)$; $D^T := D^T(\xi_0)$; $\xi^+ := \{\mu \geq \xi : t(\mu) = t(\xi) + 1\}$ the set of immediate successors of ξ ; ξ^- the unique predecessor of ξ .

There is a single consumption good at each node. The number m of agents is finite. I denotes the set of agents. At each node ξ , each agent i is endowed $e_{i,\xi} > 0$ units of consumption good.

Each household i takes the sequence of prices $(p, q, R) := (p_\xi, q_\xi, R_\xi)_{\xi \in \mathcal{D}}$ as given and chooses sequences of consumption, land, and asset volume $(c_i, l_i, a_i) := (c_{i,\xi}, l_{i,\xi}, a_{i,\xi})_{\xi \in \mathcal{D}}$ in order to maximize her intertemporal utility

$$P_i(p, q, R) : \max_{(c_i, l_i, a_i)} \left[U_i(c_i) := \sum_{\xi \in \mathcal{D}} u_{i,\xi}(c_{i,\xi}) \right]$$

subject to, for each $\xi \geq \xi_0$,

$$l_{i,\xi} \geq 0 \tag{2.7}$$

$$p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + p_\xi a_{i,\xi} \leq p_\xi e_{i,\xi} + q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + R_\xi a_{i,\xi^-} \tag{2.8}$$

$$R_{\xi'} a_{i,\xi} \geq -f_i [q_{\xi'} l_{i,\xi} + p_{\xi'} F_{i,\xi'}(l_{i,\xi})] \quad \forall \xi' \in \xi^+, \tag{2.9}$$

where $l_{i,\xi_0^-} > 0$ is given and $a_{i,\xi_0^-} = 0$. Here, the production function $F_{i,\xi}$ of agent i depend on both i and node ξ .

The deterministic model corresponds to the case where $\mathcal{D} = \{0, 1, 2, \dots\}$ and $u_{i,\xi}(c) = \beta_i^{t(\xi)} u_i(c)$. Another particular case of our model, where $F_{i,\xi} = 0$, $f_i = 0 \forall i, \forall \xi$, and there is no short-sale, corresponds to Pascoa et al. (2011). In this case, land can be interpreted as fiat money. However, Pascoa et al. (2011) assume that agents have money endowments at each node while we consider that agents have land endowments only at initial node.

The economy is denoted by \mathcal{E} characterized by a list of fundamentals

$$\mathcal{E} := \left((u_{i,\xi}, e_{i,\xi}, F_{i,\xi})_{\xi \in \mathcal{D}}, f_i, l_{i,\xi_0^-} \right)_{i \in I}.$$

Definition 4. Given the stochastic economy \mathcal{E} . A list $(\bar{p}_\xi, \bar{q}_\xi, \bar{R}_\xi, (\bar{c}_{i,\xi}, \bar{l}_{i,\xi}, \bar{a}_{i,\xi})_{i=1}^m)_{\xi \in \mathcal{D}}$ is an equilibrium if the following conditions are satisfied:

(i) Price positivity: $\bar{p}_\xi, \bar{q}_\xi, \bar{R}_\xi > 0$ for any ξ .

(ii) Market clearing: at each ξ ,

$$\text{good: } \sum_{i=1}^m \bar{c}_{i,\xi} = \sum_{i=1}^m (e_{i,\xi} + F_{i,\xi}(\bar{l}_{i,\xi^-})) \tag{2.10}$$

$$\text{land: } \sum_{i=1}^m \bar{l}_{i,\xi} = L \tag{2.11}$$

$$\text{financial asset: } \sum_{i=1}^m \bar{a}_{i,\xi} = 0. \tag{2.12}$$

(iii) *Agents' optimality: for each i , $(\bar{c}_{i,\xi}, \bar{l}_{i,\xi}, \bar{a}_{i,\xi})_{\xi \in \mathcal{D}}$ is a solution of the problem $P_i(\bar{p}, \bar{q}, \bar{R})$.*

Note that the financial asset in our framework is a short-lived asset with zero supply, which is different from the long-lived asset bringing exogenous positive dividends in [Lucas \(1978\)](#), [Santos and Woodford \(1997\)](#), [Le Van and Pham \(2016\)](#). Instead, when production functions are given by $F_{i,\xi}(x) = d_\xi x \forall i, \forall \xi$, land in our model corresponds to this asset with exogenous dividends; in particular, when $F_{i,\xi} = 0 \forall i, \forall \xi$, land becomes fiat money as in [Bewley \(1980\)](#) or pure bubble asset as in [Tirole \(1985\)](#).

Assumption 1 (production functions). *For each i , the function $F_{i,\xi}$ is concave, continuously differentiable, $F'_{i,\xi} > 0$, $F_{i,\xi}(0) = 0$.*

Assumption 2 (endowments). *$l_{i,\xi_0^-} > 0$ and $a_{i,\xi_0^-} = 0$ for any i . $e_{i,t} > 0$ for any i and for any t .*

Assumption 3 (borrowing limits). *$f_i \in (0, 1]$ for any i .*

Assumption 4 (utility functions). *For each i and for each $\xi \in \mathcal{D}$, the function $u_{i,\xi} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuously differentiable, concave, $u_{i,\xi}(0) = 0$, $u'_{i,\xi} > 0$.*

Assumption 5 (finite utility). *For each i ,*

$$\sum_{\xi \in \mathcal{D}} u_{i,\xi}(W_\xi) < \infty, \quad \text{where } W_\xi := \sum_{i=1}^m (e_{i,\xi} + F_{i,\xi}(L)). \quad (2.13)$$

Proposition 2 ([Bosi, Le Van and Pham \(2018\)](#)). *Under assumptions (1) to (5), there exists an equilibrium in the economy described in Definition 4.*

Note that a proof à la [Becker, Bosi, Le Van and Seegmuller \(2015\)](#) or [Le Van and Pham \(2016\)](#) no longer applies in our model because agents trade short-lived financial assets with zero supply instead of long-lived assets. The difficulty is to prove that individual asset volumes are bounded. To overcome the difficulty, we introduce an intermediate economy where the real asset is replaced by a nominal one.

Definition 5. *The intermediate economy $\tilde{\mathcal{E}}$ is defined as the original economy \mathcal{E} except the maximization problems of consumers are now*

$$P_i(p, q, r) : \quad \max_{(c_{i,\xi}, l_{i,\xi}, a_{i,\xi})} \sum_{\xi \in \mathcal{D}} u_{i,\xi}(c_{i,\xi})$$

subject to, for each $\xi \geq \xi_0$,

$$l_{i,\xi} \geq 0 \quad (2.14)$$

$$p_\xi c_{i,\xi} + q_\xi l_{i,\xi} + b_{i,\xi} \leq p_\xi e_{i,\xi} + q_\xi l_{i,\xi^-} + p_\xi F_{i,\xi}(l_{i,\xi^-}) + r_\xi b_{i,\xi^-} \quad (2.15)$$

$$r_{\xi'} b_{i,\xi} \geq -f_i [q_{\xi'} l_{i,\xi} + p_{\xi'} F_{i,\xi'}(l_{i,\xi})] \quad \forall \xi' \in \xi^+, \quad (2.16)$$

where $l_{i,\xi_0^-} > 0$ is given and $a_{i,\xi_0^-} = 0$.

In this intermediate economy, we can bound the volume of the financial asset, and so can prove the existence of equilibrium by adapting the method of [Becker, Bosi, Le Van and Seegmuller \(2015\)](#) and [Le Van and Pham \(2016\)](#): (1) we prove the existence of equilibrium for each T -truncated economy \mathcal{E}^T ; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy \mathcal{E} . Last, we construct an equilibrium for the original economy from an equilibrium of the intermediate economy.

Our proof can apply to a large class of general equilibrium models used in macroeconomics.

2.2 Existence of general equilibrium: role of Sperner's lemma

2.2.1 Sperner's lemma and the Gale-Nikaido-Debreu lemma

Let us start by recalling Sperner's lemma. Let Δ be the unit-simplex of \mathbb{R}^N

Definition 6. Consider a simplicial subdivision of Δ . Let V denote the set of vertices of all the subsimplices of Δ . A labeling R is a function from V into $\{1, 2, \dots, n\}$. A labeling R is said to be proper if it satisfies the **Sperner condition**:

$$\text{For any } m \leq n, \text{ if } x \in \text{ri}[[e^{i_1}, e^{i_2}, \dots, e^{i_m}]] \text{ then } R(x) \in \{i_1, i_2, \dots, i_m\}.$$
³

In particular, $R(e^i) = i, \forall i$.

Note that the Sperner condition implies that all vertices of the simplex are labeled distinctly. Moreover, the label of any vertex on the edge between the vertices of the original simplex matches with another label of these vertices. With these in mind, we can now state Sperner's lemma.

Lemma 2 (Sperner's lemma). Let $T = \{\Delta_1, \dots, \Delta_p\}$ be a simplicial subdivision of Δ . Let R be a labeling which satisfies the Sperner condition. Then there exists a subsimplex $\Delta_i \in T$ which is completely labeled, i.e., $\Delta_i = [[x^1(i), \dots, x^n(i)]]$ with $R(x^l(i)) = l, \forall l = 1, \dots, n$.

We now state the following version of the Gale-Nikaido-Debreu lemma (Theorem 1 in [Debreu \(1959\)](#), page 82).

Lemma 3 (Gale-Nikaido-Debreu lemma). Let Δ be the unit-simplex of \mathbb{R}^N . Let ζ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ into \mathbb{R}^N . Suppose ζ satisfies the following condition:

$$\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z \leq 0. \quad (2.17)$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}_-^N \neq \emptyset$.

We present a sketch of the proof in [Le, Le Van, Pham and Saglam \(2022\)](#).

Let $A = \max\{\|z\| \equiv (\sum_{i=1}^N z_i^2)^{1/2} : z \in \zeta(\Delta)\}$.

Step 0. Let $\epsilon \in (0, 1)$. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $(B(x^i(\epsilon), \epsilon))_{i=1, \dots, I(\epsilon)}$ and the non-negative continuous functions (α_i) such that $\text{Supp } \alpha_i \subset B(x^i(\epsilon), \epsilon)$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \in \Delta$.

Step 1. Take $y^i(\epsilon) \in \zeta(x^i(\epsilon)) \forall i$. We define the function $f^\epsilon : \Delta \rightarrow \mathbb{R}^N$ by $f^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon)$. This function is continuous.

Step 2. We claim that: $x \cdot f^\epsilon(x) \leq \epsilon A, \forall x \in \Delta$.

Step 3. We prove that:

$$\forall x \in \Delta, \text{ there exists } i \text{ satisfying: } f_i^\epsilon(x) \leq \epsilon A \text{ and } x_i \neq 0. \quad (2.18)$$

Step 4 (using the Sperner lemma). Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of \mathbb{R}^N such that $\text{Mesh}(T^K) < 1/K$ and define the labeling R as follows:

$$\forall x \in \Delta, R(x) = i, \text{ where } i \text{ is one of the indices satisfying } f_i^\epsilon(x) \leq \epsilon A \text{ and } x_i \neq 0.$$

³Recall that if $\Delta_i = [[x^{i_1}, x^{i_2}, \dots, x^{i_m}]]$, then $\text{ri}(\Delta_i) \equiv \{x \mid x = \sum_{k=1}^m \alpha_k x^k(i); \sum_k \alpha_k = 1; \text{ and } \forall k : \alpha(k) > 0\}$.

According to (2.18), this labeling is well-defined. It also satisfies the Sperner condition.

The Sperner lemma implies that there exists a completely labeled subsimplex $[[x^{K,1}, \dots, x^{K,N}]]$ with $R(x^{K,l}) = l, \forall l = 1, \dots, N$. Hence, we have $f_l^\epsilon(x^{K,l}) \leq \epsilon A, \forall l = 1, \dots, N$.

Let $K \rightarrow +\infty$, there is a subsequence (K_t) such that

$$\forall l, x^{K_t,l} \rightarrow x^\epsilon \in \Delta, \quad f^\epsilon(x^{K_t,l}) \rightarrow f^\epsilon(x^\epsilon)$$

and, therefore, $f_l^\epsilon(x^\epsilon) \leq \epsilon A, \forall l = 1, \dots, N$.

Step 5. Let $\epsilon \rightarrow 0$, without loss of generality, we can assume that $x^\epsilon \rightarrow \bar{x} \in \Delta$. Define $\bar{p} \equiv \bar{x}$, we can prove that $\zeta(\bar{p}) \cap \mathbb{R}_-^N \neq \emptyset$.

From Lemma 3, in Le, Le Van, Pham and Saglam (2022) we provide two stronger versions of the Gale-Nikaido-Debreu lemma. Each of them is stated and proved below.

Lemma 4. *Let Δ be the unit-simplex of \mathbb{R}^N . Let ζ be an upper semicontinuous correspondence with nonempty, compact, convex values from Δ into \mathbb{R}^N . Suppose ζ satisfies the condition*

$$\forall p \in \Delta, \exists z \in \zeta(p) \text{ which satisfies } p \cdot z \leq 0.$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}_-^N \neq \emptyset$.

Proof. For $p \in \Delta$, let $\tilde{\zeta}(p) = \{z \in \zeta(p) : z \cdot p \leq 0\}$. The correspondence $\tilde{\zeta}$ is upper semicontinuous, convex, and compact valued from Δ into \mathbb{R}^N . It satisfies the assumptions of Lemma 3. Hence there exist \bar{p} and $\bar{z} \in \tilde{\zeta}(\bar{p}) \subset \zeta(\bar{p})$, such that $\bar{z} \leq 0$. \square

Lemma 5. *Let Δ be the unit-simplex of \mathbb{R}^N . Let ζ be an upper semicontinuous correspondence with nonempty, compact, convex values from Δ into \mathbb{R}^N . Suppose ζ satisfies the condition*

$$\forall p \in \Delta, \forall z \in \zeta(p), \text{ we have } p \cdot z = 0.$$

Then there exist $\bar{p}, \bar{z} \in \zeta(\bar{p})$ such that (1) $\bar{z} \leq 0$, and (2) $\forall i = 1, \dots, N, \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = 0$.

Proof. Since " $\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z = 0$ " \Rightarrow " $\forall p \in \Delta, \forall z \in \zeta(p), p \cdot z \leq 0$ ", from Lemma 3, there exist \bar{p} and $\bar{z} \in \zeta(\bar{p})$ such that $\bar{z} \leq 0$. Since $\bar{p} \cdot \bar{z} = 0$, the conclusion is immediate. \square

2.2.2 Sperner's lemma and the existence of competitive equilibrium with incomplete markets

We consider a two-period stochastic model (see Magill and Quinzii (1996) and Florenzano (1999)). Our model economy has two periods ($t = 0$ and $t = 1$), L consumption goods, J financial assets, and I agents ($I \geq 2$). There is no uncertainty in period 0 while there are S possible states of nature in period 1. In period 0, each agent $i \leq I$ consumes and purchases assets. The consumption prices are denoted by $p_0 \in \mathbb{R}_+^L$ in the first period, $p_s \in \mathbb{R}_+^L$ in the state s of period 1.

Let $p \equiv (p_0, p_1, \dots, p_S) \in \mathbb{R}_+^{L(S+1)}$. Each consumer has endowments of consumption good $\omega_0^i \in \mathbb{R}_+^L$ in period 0 and $\omega_s^i \in \mathbb{R}_+^L$ in state s of period 1. Any agent i has a utility function $U^i(x_0^i, x_1^i, \dots, x_S^i)$ where x_s^i is her consumption at state s . There is a matrix of returns depending on p of financial assets which is the same for any agent. Typically, if agent $i \leq I$ purchases z^i quantity of assets in period 0, then in period 1, at state s , she/he will obtain an income (positive or negative) $\sum_{j=1}^J R_{s,j}(p)z^j$. where $R(p)$ is the payoff matrix.

We make use of the following set of assumptions.

Assumption 6. (i) For any $i = 1, \dots, I$, the consumption set is $X^i = \mathbb{R}_+^{L(S+1)}$, and the assets set is $Z^i = \mathbb{R}^J$.

(ii) For any $i = 1, \dots, I$, $\omega_0^i \in \mathbb{R}_{++}^L, \omega_s^i \in \mathbb{R}_{++}^L$ for any state s in period 1.

(iii) For any $i = 1, \dots, I$, U^i is strictly increasing, continuous, and strictly concave.

Assumption 7. The map $p \rightarrow R(p)$ is continuous. The matrix $R(p)$ is non-negative for any $p \geq 0$, $R(p)$ has rank J for any $p \gg 0$.

Definition 7. Consider the economy $\mathcal{E} = ((U^i, X^i, Z^i, \omega^i), R)$. An equilibrium of this economy is a list $((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*))$ where $(x^{i*}, z^{i*})_{i=1}^I \in (X^i)^I \times (Z^i)^I$, $(p^*, q^*) \in \mathbb{R}_{++}^{L(S+1)} \times \mathbb{R}_{++}^J$ such that

1. For any i , $(x^{i*}, z^{i*}) \in X^i \times Z^i$, $p_0^* \cdot (x_0^i - \omega_0^i) + q^* \cdot z^i = 0$, $p_s^* \cdot (x_s^i - \omega_s^i) = R_s(p^*) \cdot z^i$ $\forall s = 1, \dots, S$, and x^{i*} solves the problem

$$\max U^i(x_0^i, x_1^i, \dots, x_S^i) \text{ subject to: } x^i \in B^i(p^*, q^*) \quad (2.19a)$$

where we define

$$B^i(p, q) \equiv \{x^i \in X^i : \exists z^i \in Z^i, p_0 \cdot (x_0^i - \omega_0^i) + q \cdot z^i \leq 0 \\ p_s \cdot (x_s^i - \omega_s^i) \leq R_s(p) \cdot z^i, s = 1, \dots, S\}$$

2. $\sum_{i=1}^I (x_s^{i*} - \omega_s^i) = 0$ for any $s = 0, 1, \dots, S$ and $\sum_{i=1}^I z^{i*} = 0$.

We can now state our main result.

Theorem 8. Consider the economy \mathcal{E} . Let Assumptions 6 and 7 be satisfied. For any list $(\lambda_0, \lambda_1, \dots, \lambda_S)$ with $\lambda_0 = 1, \lambda_S > 0, s = 1, \dots, S$, there exists an equilibrium $((x^{i*}, z^{i*})_{i=1}^I, (p^*, q^*))$ with $p^* \in \Delta$ - the unit-simplex of $\mathbb{R}_+^{L(S+1)}$, and, more importantly,

$$q^* = \sum_{s=1}^S \lambda_s R_s(p^*), \text{ i.e., } q_j^* = \sum_{s=1}^S \lambda_s R_{s,j}(p^*), \forall j = 1, \dots, J. \quad (2.20)$$

Let us explain the intuition of our proof. The key point when applying the Sperner's lemma is to construct a labeling which is proper (i.e., it satisfies Sperner condition) and, more importantly, will generate a point corresponding to an equilibrium price.⁴ In an earlier attempt, Scarf (1982) (page 1024) also uses the Sperner's lemma to prove the existence of general equilibrium, but for a pure exchange economy. In an economy with production, thanks to the Weak Walras Law and by adapting the labeling in Scarf (1982), we can construct a proper labeling which generates an equilibrium price. While the labeling of Scarf (1982) can be adapted for an economy with production, it is not easy to construct a labeling In a two-period economy with incomplete financial markets, constructing a proper labeling is not easy because the budget sets may have empty interiors when some prices are null.

To overcome this difficulty, in Le, Le Van, Pham and Saglam (2024), we introduce an artificial economy where all agents except for one have an additional income ($\epsilon > 0$) in the first period so that their budget sets have a non-empty interior for any prices system in the simplex. The budget constraints of this economy are

$$p_0 \cdot (x_0^i - \omega_0^i) + \left(\sum_s \lambda_s R'_s(p, \epsilon) \right) \cdot z^i \leq \epsilon \quad (2.21)$$

$$p_s \cdot (x_s^i - \omega_s^i) \leq R'_s(p, \epsilon) \cdot z^i, \forall s \geq 1 \quad (2.22)$$

where the ϵ -payoff matrix $R'(p, \epsilon): R'((p_{sl})_{s,l}, \epsilon) = R((p_{sl} + \epsilon)_{s,l})$. Obviously, $R'(p, 0) = R(p)$ and $R'(p, \epsilon)$ is of rank J for any $\epsilon > 0$.

For this artificial economy, we use the excess-demand approach to construct a proper labeling and hence prove the existence of an equilibrium which depends on ϵ . Then, we let ϵ go to zero to get an equilibrium for the original economy.

⁴In applying the Kakutani fixed point theorem, a key issue is to construct the correspondence which generates the equilibrium prices. In general, this task is not trivial.

2.3 General equilibrium with non-normality and non-convexity

Demand and equilibrium with inferior and Giffen behaviors

In [Le Van and Pham \(2020\)](#), we present a class of differentiable, strictly increasing, concave utility functions exhibiting an explicit demand of a good which may have Giffen behavior. Our utility function is the following

$$U(c_1, c_2) = c_1 + bc_2 + A \frac{(ac_1 + c_2)^{1-\lambda}}{1-\lambda} \quad (2.23)$$

where $a > 0, b > 0, \lambda > 0, A > 0, \lambda \neq 1$. This function is strictly increasing, differentiable, concave. It is strictly quasi-concave if $ab \neq 1$.

Proposition 3 ([Le Van and Pham \(2020\)](#)). *Consider the utility function given by (2.23) with $ab \neq 1$. The demand function for good 1 is given by*

$$c_1 = \begin{cases} 0 & \text{if } \frac{a + \frac{1}{A}(\frac{w}{p_2})^\lambda}{1 + b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2 \leq p_1 \\ \frac{p_2 \left(A \frac{ap_2 - p_1}{bp_1 - p_2} \right)^{\frac{1}{\lambda}} - w}{ap_2 - p_1} & \text{if } \frac{a + \frac{1}{A}(\frac{w}{p_2})^\lambda}{1 + b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2 > p_1 > \frac{a + \frac{1}{A}(\frac{aw}{p_1})^\lambda}{1 + b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_2 \\ \frac{w}{p_1} & \text{if } p_1 \leq \frac{a + \frac{1}{A}(\frac{aw}{p_1})^\lambda}{1 + b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_2 \end{cases} \quad (2.24)$$

The demand function is continuous. Moreover, it is differentiable in $(w, p_1, p_2, a, b, \lambda)$ except points satisfying $p_1 = \frac{a + \frac{1}{A}(\frac{w}{p_2})^\lambda}{1 + b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2$ or $p_1 = \frac{a + \frac{1}{A}(\frac{aw}{p_1})^\lambda}{1 + b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_2$.

Notice that the demand function in (2.24) is computed for all possible parameters, including prices and income. The consumer does not buy good 1 (resp., good 2) if the price of good 1 (resp., good 2) is high in the sense that $p_1 \geq \frac{a + \frac{1}{A}(\frac{w}{p_2})^\lambda}{1 + b\frac{1}{A}(\frac{w}{p_2})^\lambda} p_2$ (resp., $p_2 > \frac{a + \frac{1}{A}(\frac{aw}{p_1})^\lambda}{1 + b\frac{1}{A}(\frac{aw}{p_1})^\lambda} p_1$). The solution is interior when prices and income have a middle level.

Proposition 3 allows us to identify conditions under which good 1 is normal, inferior or Giffen. The following result shows such conditions.

Proposition 4 ([Le Van and Pham \(2020\)](#)). *Let assumptions in Proposition 3 be satisfied. Consider the case of interior solution.*

1. Good 1 is normal (i.e., $\partial c_1 / \partial w > 0$) if and only if $ap_2 < p_1$.
2. Good 1 is inferior (i.e., $\partial c_1 / \partial w < 0$) if and only if $ap_2 > p_1$.
3. Good 1 has Giffen behavior (i.e., $\partial c_1 / \partial p_1 > 0$) if and only if

$$(bp_1 - p_2) \left(\frac{w}{p_2} \right)^\lambda < A(ap_2 - p_1) < (bp_1 - p_2) \left(\frac{aw}{p_1} \right)^\lambda \quad (2.25a)$$

$$p_2 \left(A \frac{ap_2 - p_1}{bp_1 - p_2} \right)^{\frac{1}{\lambda}} \left(1 - \frac{p_2(ab - 1)}{\lambda(bp_1 - p_2)} \right) - w > 0. \quad (2.25b)$$

Moreover, there exists a positive list $(p_1, p_2, a, b, \lambda, A, w)$ such that (2.25a) and (2.25b) hold.

We then consider an exchange economy whose agent's utility function is given by (2.23). Our utility function leads to an interesting point in general equilibrium context: the price of a good may be an increasing function of the aggregate supply of this good. Moreover, we show that the Giffen behavior may arise in equilibrium when preferences or/and endowments of agents change.

Equilibrium with non-convex preferences: some insights

In [Le Van and Pham \(2023\)](#), we consider a two-agent, two-good exchange economy with a risk averse agent and an agent who is neither risk loving nor risk averse.

Assume that the consumption set is \mathbb{R}_+^2 .

Assume that agent A is risk averse and the utility of this agent is $U^A(c_1, c_2) = \ln(c_1) + \ln(c_2)$. This function is strictly concave.

Assume that agent B has utility function $U^B(c_1, c_2) = \frac{c_1^2}{2} + \mathcal{D} \ln(c_2)$, where $\mathcal{D} > 0$. This agent is risk loving with good 1 but risk averse with good 2. Note that this function is neither quasiconcave nor quasiconvex on the consumption set \mathbb{R}_+^2 . With this specification, we can explicitly compute the demand for each good. Assume that agent A has endowments $(e_1^A, e_2^A) \gg 0$ and agent B has endowments $(e_1^B, e_2^B) \gg 0$.

Let us denote $\pi^{cor} \equiv \frac{2e_1^B + e_1^A}{e_2^A}$ and x^* the unique solution to the equation $g(x) = 0$ where

$$g(x) \equiv \frac{1}{8}(x + \sqrt{x^2 - 4\mathcal{D}})^2 + \mathcal{D}(\ln(1 - \sqrt{1 - 4\mathcal{D}x^{-2}}) - \ln(2)). \quad (2.26)$$

The function g is increasing in x and $g(w/p_1) = V(w, p_1) - \mathcal{D} \ln(w)$.

Notice that $x^* > 2\sqrt{\mathcal{D}}$ and x^* only depends on \mathcal{D} (so we write $x^* = x^*(\mathcal{D})$). Observe that $x^*(\mathcal{D})$ is an increasing function of \mathcal{D} and it converges to 0 when \mathcal{D} converges to 0.

Denote π^{int} the smallest root (if there exists a root) of the function

$$F(X) \equiv [(e_2^B + e_2^A)^2 - (e_2^B)^2]X^2 - 2[(e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^B e_2^B]X + (e_1^B + e_1^A)^2 + 4\mathcal{D} - (e_1^B)^2.$$

Observe that

$$0 < \pi^{int} < \frac{e_1^B + e_1^A}{e_2^B + e_2^A} < X^* < \frac{2e_1^B + e_1^A}{e_2^A} = \pi^{cor} \quad (2.27)$$

where $X^* \equiv \frac{(e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^B e_2^B}{(e_2^B + e_2^A)^2 - (e_2^B)^2}$ which satisfies $F'(X^*) = 0$.

By Inada condition, c_1^A, c_2^A and c_2^B are strictly positive while c_1^B may be zero or strictly positive at equilibrium. Our main result is to prove a full characterization of general equilibrium.

Proposition 5 (existence and computation of equilibrium). *Let us consider the exchange economy with two agents described above.*

1. There exists an equilibrium $(p_1, p_2, c_1^A, c_2^A, c_1^B, c_2^B)$ with $c_1^B = 0$ if and only if condition

$$e_1^B + e_2^B \frac{2e_1^B + e_1^A}{e_2^A} \leq x^*(\mathcal{D}). \quad (2.28)$$

Such an equilibrium is unique, up to a positive scalar for the prices. The equilibrium relative price $p_2/p_1 = \pi^{cor}$, where $\pi^{cor} \equiv \frac{2e_1^B + e_1^A}{e_2^A}$.

2. There exists an equilibrium $(p_1, p_2, c_1^A, c_2^A, c_1^B, c_2^B)$ with $c_1^B > 0$ if and only if condition

$$e_1^B + e_2^B \pi^{int} \geq x^*(\mathcal{D}). \quad (2.29)$$

Such an equilibrium is unique, up to a positive scalar for the prices. The equilibrium relative price $p_2/p_1 = \pi^{int}$ and $c_1^B = \frac{1}{2}(e_1^B + \pi^{int} e_2^B + \sqrt{(e_1^B + \pi^{int} e_2^B)^2 - 4\mathcal{D}}) > 0$.

3. There is no equilibrium if and only if $e_1^B + e_2^B \pi^{cor} > x^*(\mathcal{D}) > e_1^B + e_2^B \pi^{int}$.

Proposition 9 allows us to understand the roles of agents' endowments and preferences on the existence of equilibrium.

Corollary 1 (role of the risk averse agent's endowments). 1. When e_1^A is high enough, there exist a unique equilibrium and $c_1^B > 0$ at equilibrium.

2. When e_1^B is high enough, condition (2.29) holds. Thus, there exists a unique equilibrium and $c_1^B > 0$ at equilibrium.

3. There exists an equilibrium for any e_2^A high enough (because point 3 in Proposition 9 cannot happen when e_2^A is high enough). More precisely, we have that:

(a) If $x^*(\mathcal{D}) > e_1^B$, then when e_2^A is very large, there exist a unique and $c_1^B = 0$ at equilibrium.

(b) If $x^*(\mathcal{D}) \leq e_1^B$, then when e_2^A is very large, there exist a unique and $c_1^B > 0$ at equilibrium.

4. When e_2^B is high enough, we have that:

(a) If $\frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B} < x^*(\mathcal{D})$,⁵ then there is no equilibrium.

(b) If $\frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B} > x^*(\mathcal{D})$, then there exists a unique equilibrium and $c_1^B > 0$ at equilibrium.

According to Corollary 1, there exists an equilibrium if the endowment of good 1 or good 2 of the risk-averse agent is high enough. This point is consistent with the main finding in Araujo, Chateauneuf, Gama and Novinski (2018). Notice that Araujo, Chateauneuf, Gama and Novinski (2018) consider general utility functions but that of type A agent is concave and that of type B agent is convex. Their main result is to prove that there exists an equilibrium when the endowment of risk averse agent e_2^A or e_1^A is high enough.

Corollary 1 is distinct from Araujo, Chateauneuf, Gama and Novinski (2018) in two ways. First, although we work with specific preferences, the utility function of agent B is neither concave nor convex. By consequence, the method of Araujo, Chateauneuf, Gama and Novinski (2018) cannot be applied to our model. Second, in Araujo, Chateauneuf, Gama and Novinski (2018), the optimal allocation of the type B agent in equilibrium is always in the corner (which corresponds to the case $c_1^B = 0$ in our model) because this agent's utility is strictly convex. By contrast, in our model, when the risk-aversion agent's endowment is high enough, the equilibrium may be interior. Indeed, this happens if (i) e_2^A is high enough and $x^*(\mathcal{D}) \leq e_1^B$ (see point 1.b of Corollary 1) or (ii) e_1^A is high enough (see point 2 of Corollary 1).

Corollary 2 (role of agent B's endowments).

1. When e_1^B is high enough, condition (2.29) holds. Thus, there exists a unique equilibrium and $c_1^B > 0$ at equilibrium.

2. Observe that $\lim_{e_2^B \rightarrow \infty} (e_1^B + e_2^B \pi^{int}) = \frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B}$. So, when e_2^B is high enough, we have that:

(a) If $\frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B} < x^*(\mathcal{D})$,⁶ then there is no equilibrium.

⁵This is satisfied if $e_1^A + 2e_1^B = 2\sqrt{\mathcal{D}}$ because $2\sqrt{\mathcal{D}} < x^*(\mathcal{D})$.

⁶This is satisfied if $e_1^A + 2e_1^B = 2\sqrt{\mathcal{D}}$ because $2\sqrt{\mathcal{D}} < x^*(\mathcal{D})$.

(b) If $\frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B} > x^*(\mathcal{D})$, then there exists a unique equilibrium and $c_1^B > 0$ at equilibrium.

Proposition 9 also provides a necessary and sufficient condition for the non-existence of equilibrium: $e_1^B + e_2^B \pi^{cor} > x^*(\mathcal{D}) > e_1^B + e_2^B \pi^{int}$ (notice that this happens if \mathcal{D} has a middle level).

Chapter 3

Financial frictions and macroeconomics: A general equilibrium approach

3.1 Asset valuation and asset price bubbles in general equilibrium

3.1.1 Asset price bubbles in general equilibrium models with infinitely-lived heterogeneous agents

In [Le Van and Pham \(2016\)](#), [Bosi, Le Van and Pham \(2022\)](#), we address the issues of asset price bubbles by using different general equilibrium models with infinitely-lived heterogeneous agents .

Firstly, we summarize main results concerning the issue of bubbles in [Le Van and Pham \(2016\)](#) whose model is described in Section 2.1.1. The following result provides some conditions ruling out bubbles in equilibrium.

Proposition 6 (Propositions 8 and 9 in [Le Van and Pham \(2016\)](#)). *Consider the model in Section 2.1.1. Consider an equilibrium. We have that*

$$\frac{q_t}{p_t} = \gamma_{t+1} \left(\frac{q_{t+1}}{p_{t+1}} + d_{t+1} \right) \quad (3.1)$$

where $\gamma_{t+1} := \max_{i \in \{1, \dots, m\}} \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}$. Denote Q_t the discount factor of the economy from

initial date to date t defined by $Q_0 := 1, Q_t := \prod_{s=1}^t \gamma_s, t \geq 1$. We have

$$\underbrace{\frac{q_0}{p_0}}_{\text{Asset equilibrium price}} = \underbrace{\sum_{t=1}^{+\infty} Q_t \xi_t}_{\text{Fundamental value}} + \underbrace{\lim_{t \rightarrow +\infty} Q_t \frac{q_t}{p_t}}_{\text{Asset price bubble}} \quad (3.2)$$

1. Assume that the production functions are stationary, $F'(\infty) < \delta$ and $0 < \liminf_{t \rightarrow \infty} \xi_t \leq \limsup_{t \rightarrow \infty} \xi_t < \infty$. Then there is no bubble on the financial asset market: $\frac{q_0}{p_0} = FV_0 := \sum_{t=1}^{+\infty} Q_t \xi_t$, or equivalently, $\lim_{t \rightarrow +\infty} Q_t \frac{q_t}{p_t} = 0$.
2. Assume that $f^i = 1$ for every i and the production functions are stationary and linear, i.e., $F_t(K) = AK$ for every t . Then there is no financial bubble and every equilibrium is efficient.

Conditions $F'(\infty) < \delta$ and $\limsup_{t \rightarrow \infty} \xi_t < \infty$ are natural and to ensure that the aggregate capital stocks are uniformly bounded from above. We also require $\liminf_{t \rightarrow \infty} \xi_t > 0$ under which we get that $\sum_{t=1}^{+\infty} Q_t < \infty$.¹ Since the output is uniformly bounded from above, the present value of the aggregate consumption good is finite, i.e.,

$$\sum_{t=0}^{\infty} Q_t Y_t < \infty,$$

where $Y_t := F(K_t) + (1 - \delta)K_t$. Our result is consistent with the following well-known result (see Kocherlakota (1992), Santos and Woodford (1997), Huang and Werner (2000)): there is no financial bubble if the present value of the aggregate endowment is finite.² However, this is only a sufficient condition ruling out financial bubble.

A key point in establishing the above result concerns the role of borrowing constraints and transversality conditions:

Lemma 6. (1) For each agent i , we define $S_{i,0} = 1$, $S_{i,t} := \frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}$ is the agent i 's discount factor from initial period to period t . Then $\lim_{t \rightarrow \infty} S_{i,t} (\frac{q_t}{p_t} a_{i,t} + f^i k_{i,t+1}) = 0$.

(2) If the borrowing constraints of agent i are not binding from t_0 to t , then $\frac{Q_t}{Q_{t_0}} = \frac{S_{i,t}}{S_{i,t_0}}$.

(3) If there exists a time t_0 such that $\frac{Q_t}{Q_{t_0}} = \frac{S_{i,t}}{S_{i,t_0}}, \forall i, \forall t \geq t_0$, then there is no bubble.

Le Van and Pham (2016) (Section 6.1) and Bosi, Le Van and Pham (2017) provide examples of bubbles of the Lucas' tree, where the asset price may be multiple (due to the portfolio effect) but the consumption is not affected by the existence of bubbles. However, these examples do not provide a complete picture showing the dynamics of asset price bubbles and how it depend on fundamentals. This is the main motivation in Bosi, Le Van and Pham (2022).

Let us describe quickly the model in Bosi, Le Van and Pham (2022). This is an infinite-horizon discrete-time model with short-sale as in Kocherlakota (1992). There are a finite number m of agents, a single consumption good and an asset. The asset structure is similar to Lucas' tree (Lucas, 1978) with exogenous dividend stream but we introduce a short-sale constraint. Denote $c_{i,t}, b_{i,t}$ the consumption and asset holding of agent i at date t while q_t is the asset price at date t . Agent i maximizes her intertemporal utility $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t})$ subject to the following constraints:

(1) Physical constraints: $c_{i,t} \geq 0 \forall t, \forall i$.

(2) Budget constraint: $c_{i,t} + q_t b_{i,t} \leq e_{i,t} + (q_t + d_t) b_{i,t-1} \forall t, \forall i$, where $e_{i,t} > 0$ is the exogenous endowment of agent i at date t and $b_{i,-1}$ is exogenously given.

(3) Borrowing constraint (or short-sale constraint): $b_{i,t} \geq -b_i^* \forall t, \forall i$, where $b_i^* \geq 0$ is an exogenous borrowing limit.

Definition 9. An equilibrium is a list of prices and allocations $(q_t, (c_{i,t}, b_{i,t})_i)_{t \geq 0}$ satisfying three conditions: (1) given price, for any i , the allocation $(c_{i,t}, b_{i,t})_i$ is a solution of the optimization problem of agent i (i.e., $\sum_{t=0}^{+\infty} \beta_{i,t} u_i(c_{i,t}) \geq \limsup_{T \rightarrow \infty} \sum_{t=0}^T \beta_{i,t} u_i(c'_{i,t})$ for any sequence (c'_i, b'_i) satisfying physical, budget and borrowing constraints), and (2) market clearing conditions: $\sum_i b_{i,t} = L$ and $\sum_i c_{i,t} = \sum_i e_{i,t} + L d_t \forall t \geq 0$, where L is the net asset supply, and (3) $q_t > 0 \forall t \geq 0$.

Denote $W_t \equiv \sum_i e_{i,t} + L d_t$ the aggregate resource at date t . We require standard assumptions in the rest of the paper.

¹Because we always have $\sum_{t=1}^{+\infty} Q_t \xi_t \leq \frac{q_0}{p_0} < \infty$.

²In their frameworks, the present value of the aggregate endowment is finite if and only if the the present value of the aggregate consumption good is finite

Assumption 8. Assume that u_i is concave, strictly increasing, and continuously differentiable for any i . We also assume that $\beta_{i,t} > 0$, $e_{i,t} > 0$, $b_{i,-1} \geq -b_i^*$, $d_t \geq 0$, $\sum_t \beta_{i,t} u_i(W_t) < \infty$, $\lim_{t \rightarrow \infty} \beta_{i,t} = 0$, $\forall i, t$, and the net asset supply is positive ($L > 0$).

Assumption 9. There exists an increasing function $v(c)$ such that $u'_i(c)c \leq v(c) \forall c$ and $\sum_t \beta_{i,t} v(W_t) < \infty \forall i$.

Definition 10. Consider an equilibrium in Definition 9. We define discount factors $(R_t)_t$ by $R_{t+1}q_t = q_{t+1} + d_{t+1}$. The fundamental value of the asset is $FV_0 \equiv \sum_{t=1}^{\infty} Q_t d_t$ where $Q_t \equiv \frac{1}{R_1 \dots R_t}$. We say that there is a bubble in this equilibrium if $q_0 > FV_0$. In this case, this equilibrium is called bubbly. Otherwise, it is called bubbleless.

Remark 1. One can prove that $1 = R_{t+1} \max_i \frac{\beta_{i,t+1} u'_i(c_{i,t+1})}{\beta_{i,t} u'_i(c_{i,t})} \forall t \geq 0$.³

Proposition 7 (Bosi, Le Van and Pham (2022)). Let Assumptions 8, 9 be satisfied. If there is a bubble in equilibrium, then we have that:

1. For each i , at least one of the two following statements is true: (i) there exists an infinite increasing sequence i_n of time such that $b_{i,i_n} + b_i^* = 0$, $\forall n$; (ii) $\lim_{t \rightarrow \infty} (b_{i,t} + b_i^*) = 0$.
2. There exist at least 2 agents i and j such that their asset holding sequences $(b_{i,t})_t$ and $(b_{j,t})_t$ do not converge. Moreover, their borrowing constraints bind infinitely often: there exist 2 infinite increasing sequences $(i_n)_n, (j_n)_n$ such that $b_{i,i_n} + b_i^* = 0$ and $b_{j,j_n} + b_j^* = 0$ for all n .

Corollary 3. Let Assumptions 8, 9 be satisfied. There is no equilibrium with bubble if one of the following conditions hold:

1. There is a date T such that $b_i^* d_t > e_{i,t} \forall i, \forall t \geq T$.
2. $\sum_{t \geq 0} Q_t (\sum_i e_{i,t}) < \infty$.
3. $\liminf_{t \rightarrow \infty} \frac{d_t}{\sum_i e_{i,t}} > 0$.

General equilibrium models with bubbles and indeterminacy

We are now interested in constructing model economies in which bubbles exist. Proposition 7 shows that such models must contain at least 2 heterogeneous agents. So, we should focus on a model with two types of agents, say 1 and 2. Suggesting by Proposition 7, we should look at equilibria in which borrowing constraints of agent 1 (agent 2) bind at any even (odd) date because this is the simplest model under which bubbles may exist. Formally, we aim to find economies where there is an equilibrium such that

$$b_{1,2t} = -b_1^*, \quad b_{2,2t} = L + b_1^*, \quad b_{1,2t+1} = L + b_2^*, \quad b_{2,2t+1} = -b_2^*. \quad (3.3)$$

With these asset holdings, the consumptions are given by

$$c_{1,0} = e_{1,0} + (q_0 + d_0)b_{1,-1} + q_0 b_1^*, \quad c_{2,0} = e_{2,0} + (q_0 + d_0)b_{2,-1} - q_0(L + b_1^*) \quad (3.4a)$$

$$c_{1,2t-1} = e_{1,2t-1} - b_1^* d_{2t-1} - q_{2t-1} H, \quad c_{2,2t-1} = e_{2,2t-1} + d_{2t-1}(L + b_1^*) + q_{2t-1} H \quad (3.4b)$$

$$c_{1,2t} = e_{1,2t} + d_{2t}(L + b_2^*) + q_{2t} H, \quad c_{2,2t} = e_{2,2t} - d_{2t} b_2^* - q_{2t} H \quad (3.4c)$$

³Indeed, let $t \geq 0$ arbitrary, then FOCs imply that $q_t \geq (q_{t+1} + d_{t+1}) \max_i \frac{\beta_{i,t+1} u'_i(c_{i,t+1})}{\beta_{i,t} u'_i(c_{i,t})}$. Since $\sum_i b_{i,t} = L > 0$, there is an agent i_t such that $b_{i_t,t} > 0$. Hence, $\eta_{i_t,t} = 0$. By consequence, $q_t = (q_{t+1} + d_{t+1}) \frac{\beta_{i_t,t+1} u'_{i_t}(c_{i_t,t+1})}{\beta_{i_t,t} u'_{i_t}(c_{i_t,t})}$. Therefore, we obtain our result.

for any t , where $b_{1,-1}, b_{2,-1}$ are given and $H \equiv L + b_1^* + b_2^*$.

To understand conditions (based on fundamentals) under which such equilibrium may have a bubble, our idea is to look at the benchmark economy (i.e., the economy without assets). Let us define the exogenous sequences $(R_{1,t}^*), (R_{2,t}^*), (R_t^*)$ by

$$1 = \frac{\beta_{1,t} u_1'(e_{1,t})}{\beta_{1,t-1} u_1'(e_{1,t-1})} R_{1,t}^*, \quad 1 = \frac{\beta_{2,t} u_2'(e_{2,t})}{\beta_{2,t-1} u_2'(e_{2,t-1})} R_{2,t}^*, \quad \text{and } R_t^* \equiv \min(R_{1,t}^*, R_{2,t}^*). \quad (3.5)$$

$R_{1,t}^*$ (resp., $R_{2,t}^*$) can be interpreted as the *subjective real interest rate* of agent 1 (resp., 2) and R_t^* as *the real interest rate* between dates $(t-1)$ and t in the benchmark economy. Notice that $R_t \geq R_t^* \forall t \geq 2$ which means that the interest rate of the benchmark economy is lower than that of our economy with asset.

We have the following result providing necessary conditions for the existence of (bubbly) equilibrium in this two-agent economy.

Proposition 8 (Bosi, Le Van and Pham (2022)). *(The role of interest rates of the benchmark economy.) Consider a two-agent model described above. Focus on an equilibrium whose asset holdings are given by (3.3).*

1. We have

$$R_{2,2t}^* \geq R_{1,2t}^*, \quad R_{1,2t+1}^* \geq R_{2,2t+1}^* \quad \forall t \geq 1 \quad (\text{seesaw property}). \quad (3.6)$$

2. Moreover, there is no bubble if

$$\lim_{t \rightarrow \infty} \frac{e_t}{R_1^* \cdots R_t^*} = 0. \quad (3.7)$$

In a particular case, where $e_t = e$ and $R_t^* = R^* \forall t$, there is no bubble if $R^* > 1$.

The seesaw property is necessary to ensure that (3.3) can be part of equilibrium. This suggests that an equilibrium with bubble should have a fluctuation.

The novelty of condition (3.7) is to show the importance of interest rates of the economy without asset (these interest rates are exogenous) on the existence of bubbles in the economy with assets.

Condition (3.7) allows us to establish the connection between the literature of bubbles in OLG models and that in infinite-horizon models. Indeed, let us compare it with the main result in the influential paper of Tirole (1985) who studies a pure bubble asset (i.e., asset pays no dividend) in an OLG model. He provides a no-bubble condition based on fundamentals: there is no bubble if the steady state interest rate of the economy without bubble asset is higher than the population growth rate. Condition (3.7), also based on exogenous variables, can be interpreted as a high interest rates condition (indeed, it becomes $R^* > 1$ if $e_t = e$, $R_t^* = R^* \forall t$). So, our result is consistent with that in Tirole (1985). The difference is that we do not require the convergence of interest rates R_t^* as in Tirole (1985) or in Farhi and Tirole (2012).

Condition (3.7) helps us to understand better a number of examples of bubbles in the literature. Indeed, in Example 1 in Kocherlakota (1992) and Example 4.2 in Santos and Woodford (1997) of fiat money, we can verify that $\lim_{t \rightarrow \infty} \frac{e_t}{R_1^* \cdots R_t^*} = \infty$, i.e., condition (3.7) is violated. Moreover, in examples of bubbles in Bosi, Le Van and Pham (2018), we have $R_t^* = 0$, and hence condition (3.7) is also violated.

Asset without dividends (or fiat money)

We begin our exposition by studying a specific case.

Example 1 (Bosi, Le Van and Pham (2022)). Assume that $u_i(c) = \ln(c)$, $\beta_{i,t} = \beta^t$ where $\beta \in (0, 1)$ and there is no dividend ($d_t = 0 \forall t$). Assume also that $b_{1,-1} = L + b_2^*$, $b_{2,-1} = -b_2^*$, and endowments are periodic:

$$(e_{1,t})_{t \geq 0} = (w, e, w, e, \dots), \quad (e_{2,t})_{t \geq 0} = (e, w, e, w, \dots), \quad (3.8a)$$

where $e, w > 0$ (so $e_t = e > 0, w_t = w > 0 \forall t$).

Let us focus on equilibrium satisfying (3.3) and (3.4a-3.4c).

1. If $\frac{\beta e}{w} \leq 1$ (i.e., $R^* \geq 1$), there is no bubble.
2. If $\frac{\beta e}{w} > 1$ (i.e., $R^* < 1$: low interest rate condition), then the initial price of any equilibrium with bubble must satisfy condition $q_0 \leq \frac{1}{H} \frac{\beta e - w}{1 + \beta}$. Conversely, we have:

(a) There is a unique equilibrium with initial price $q_0 = \frac{1}{H} \frac{\beta e - w}{1 + \beta}$. Moreover, we have $q_t = \frac{1}{H} \frac{\beta e - w}{1 + \beta} > 0 \forall t$.

(b) (Continuum of equilibria with bubble) For any value x in the interval $[0, \frac{1}{H} \frac{\beta e - w}{1 + \beta})$, the sequence (q_t) determined by $q_0 = x$ and $\frac{1}{H q_{t+1}} = \frac{\beta e}{w} \frac{1}{H q_t} - \frac{1 + \beta}{w} \forall t \geq 0$,⁴ is a system of price of an equilibrium with bubble. Moreover, (1) q_t is decreasing in t and converges to zero, (2) the interest rate $R_t \equiv q_t / q_{t-1}$ is decreasing in t and converges to $R^* = \frac{w}{\beta e} < 1$.

Notice that in the case of bubbles in Example 1, the seesaw property (3.6) holds and high interest rate condition (3.7) is violated (because $e_t = e$ and $R^* < 1$)

Example 1 is related to several models of bubbles in general equilibrium, for instance, Example 4.2 in Santos and Woodford (1997), Townsend (1980), Chapter 27 in Ljungqvist and Sargent (2012) (their model corresponds to the case $e = 1, w = 0$ in our model), Section 2 in Bloise and Citanna (2019). An added value of Example 1 is to show that multiple equilibria may exist and we completely characterize the set of multiple equilibria. By the way, it complements Example 4.2 in Santos and Woodford (1997), which only examines the steady state $q_t = q > 0, \forall t$ (but with a general utility function).

However, Example 1 and those in the existing literature do not clearly show us how the existence of bubbles depends on the dynamics of economic fundamentals and on the asset structure (dividends and borrowing limits). Our models contribute to address this issue.

We firstly focus on the case of fiat money or pure bubble asset (i.e., $d_t = 0 \forall t$). To simplify our exposition, we introduce some notations.

$$\gamma_{i,t} = \frac{\beta_{i,t+1}}{\beta_{i,t}}, \quad \gamma_{2t} \equiv \gamma_{2,2t}, \quad \gamma_{2t+1} \equiv \gamma_{1,2t+1}, \quad \mu_{2t} \equiv \gamma_{1,2t}, \quad \mu_{2t+1} \equiv \gamma_{2,2t+1}, \quad \forall t \geq 0 \quad (3.9a)$$

$$e_{2t} \equiv e_{2,2t}, \quad e_{2t+1} \equiv e_{1,2t+1}, \quad w_{2t} \equiv e_{1,2t}, \quad w_{2t+1} \equiv e_{2,2t+1}, \quad \forall t \geq 0 \quad (3.9b)$$

$$\Gamma_t \equiv \frac{\gamma_{t-1} e_{t-1} \dots \gamma_0 e_0}{w_t} = \frac{1}{R_1^* \dots R_t^*}, \quad \forall t \geq 1 \quad (3.9c)$$

$$D_t \equiv \frac{1 + \gamma_{t-1}}{w_t} + \frac{1}{R_t^*} \frac{1 + \gamma_{t-2}}{w_{t-1}} + \dots + \frac{1}{R_t^* \dots R_2^*} \frac{1 + \gamma_0 \frac{L + b_1^* - b_{2,-1}}{L + b_1^* + b_2^*}}{w_1}, \quad \forall t \geq 2 \quad (3.9d)$$

$$D_1 \equiv \frac{1 + \gamma_0 \frac{L + b_1^* - b_{2,-1}}{L + b_1^* + b_2^*}}{w_1}, \quad \forall t \geq 1. \quad (3.9e)$$

⁴By convention, if $q_0 = 0$, we determine $q_t = 0 \forall t \geq 1$.

The following result provides a necessary and sufficient condition under which bubbles exist in equilibrium.

Proposition 9 ((Bosi, Le Van and Pham, 2022)). (*continuum equilibria with bubble*)

Assume that $d_t = 0 \forall t$ and $u_i(c) = \ln(c) \forall i = 1, 2$. The sequences $(b_{i,t})$ given by (3.3), $(c_{i,t})$ given by (3.4a-3.4c), and $(q_t)_{t \geq 0}$ constitute an equilibrium with bubble if and only if

$$X_t \equiv \frac{\gamma_t e_t}{w_{t+1}} - \frac{\mu_t w_t}{e_{t+1}} > 0, \forall t \geq 0 \quad (3.10)$$

$$\sup_{t \geq 1} \left\{ \frac{\frac{1+\gamma_t}{w_{t+1}} + \frac{1+\mu_t}{e_{t+1}}}{\frac{\gamma_t e_t}{w_{t+1}} - \frac{\mu_t w_t}{e_{t+1}}} R_1^* \cdots R_t^* \right\} < \infty \quad (3.11)$$

$$\sup_{t \geq 2} \left\{ \frac{R_1^* \cdots R_{t-1}^*}{e_{t-1}} \left(\frac{1}{\gamma_{t-1}} + 1 \right) + \cdots + \frac{1}{e_0} \left(\frac{1}{\gamma_0} + \frac{L + b_1^* - b_{2,-1}}{L + b_1^* + b_2^*} \right) \right\} < \infty, \quad (3.12)$$

and the sequence of asset prices $(q_t)_{t \geq 0}$ is determined by

$$q_0 \in (0, \bar{q}], \quad \frac{1}{Hq_t} = \frac{1}{Hq_0} \Gamma_t - D_t \quad \forall t \geq 1, \quad (3.13)$$

where the upper bound \bar{q} is defined as follows:

$$\bar{q} \equiv \frac{1}{L + b_1^* + b_2^*} \min \left\{ \frac{\frac{\gamma_{2,0} e_{2,0}}{e_{2,1}} - \frac{\gamma_{1,0} e_{1,0}}{e_{1,1}}}{\frac{1+\gamma_{2,0}}{1+\gamma_{2,0}} \frac{L+b_1^*-b_{2,-1}}{L+b_1^*+b_2^*} + \frac{1+\gamma_{1,0}}{1+\gamma_{1,0}} \frac{b_1^*+b_{1,-1}}{L+b_1^*+b_2^*}}, \inf_{t \geq 1} \frac{X_t \Gamma_t}{X_t D_t + Y_t} \right\}, \quad (3.14)$$

where $Y_t \equiv \frac{1+\gamma_t}{w_{t+1}} + \frac{1+\mu_t}{e_{t+1}}$.

By consequence, there is a continuum of equilibria and all such equilibria are bubbly if (3.10-3.12) are satisfied.

Proposition 9 is a generalized version of Example 1 for the case where endowments and time preference rates $\beta_{i,t}$ are time dependent.⁵ It provides a complete characterization of all equilibria satisfying (3.3) and (3.4a-3.4c). Importantly, we can explicitly describe all such equilibria by using fundamentals. Conditions (3.10-3.12) are necessary and sufficient for the existence of bubble, and they are satisfied for a large class of parameters. Condition (3.10) implies the seesaw property (3.6), i.e., $R_{2,t}^* \geq R_{1,t}^*$ and $R_{1,2t+1}^* \geq R_{2,2t+1}^*$, $\forall t$, for the logarithmic utility $u(c) = \ln(c)$. Conditions (3.11-3.12) can be interpreted as interest rates of the economy without asset (R_t^*) are low enough. It implies that, when $u_i(c) = \ln(c)$, there is no bubble if $\sum_t \frac{R_1^* \cdots R_t^*}{e_t} = \infty$. This is consistent but much stronger than condition (3.7) in Proposition 8 with a general utility function.

Under conditions (3.10-3.12), the sequence (q_t) is part of equilibrium with bubble if and only if $0 < q_0 \leq \bar{q}$. So, the value \bar{q} can be interpreted as the *maximum value of bubble*. The higher the value of \bar{q} , the more chance to have a bubble in equilibrium. So, it is important to understand how the upper bound \bar{q} depends on fundamentals. Observe that the maximum value \bar{q} of bubble defined by (3.14) is decreasing in the asset supply L , borrowing limits b_1^*, b_2^* , the endowment ratio $\frac{w_t}{e_t}$, the initial asset holding $b_{1,-1}$ of agent 1. Moreover, \bar{q} is increasing in the rate of time preference γ_t , the initial asset holding $b_{2,-1}$ of agent 2. Therefore, the following conditions contribute to promote the existence of bubble:

1. Asset supply L is low. This means that asset shortage matters.

⁵Indeed, when $\beta_{i,t} = \beta^t$, $e_t = e$, $w_t = w$, $\forall t, \forall i = 1, 2$, we recover Example 1; in this case, we can compute that $\bar{q} = \frac{\beta e}{w}$.

2. Borrowing limits b_1^* and b_2^* are low. This shows that financial frictions matter.
3. The initial asset $b_{2,-1}$ is high and/or the initial asset $b_{1,-1}$ is low, the endowment ratios $\frac{e_{2,2t}}{e_{1,2t}}, \frac{e_{1,2t+1}}{e_{2,2t+1}}$ are high, and the rates of time preference $\frac{\beta_{2,2t+1}}{\beta_{2,2t}}, \frac{\beta_{1,2t}}{\beta_{1,2t-1}}$ are high.⁶ These conditions indicate that heterogeneity matters.

Making clear the role of these factors on the existence and value of bubbles is a contribution of Proposition 9 with respect to Example 1 and the examples in the literature. For instance, Bewley (1980) (Section 13), Townsend (1980), Kocherlakota (1992) (Example 1), Scheinkman and Weiss (1986), Santos and Woodford (1997) (Example 4.2), borrowing is not allowed, which corresponds to the case $b_i^* = 0, \forall i$, in our model.

Proposition 9 can be viewed as a version of the classical result in Tirole (1985) (Proposition 1) for an exchange general equilibrium model with infinitely lived agents and short-sale constraints. With our specification, we explicitly compute the maximum level \bar{q} of initial price bubble while it is implicit in more general models. Moreover, we do not require the convergence of interest rate as in Tirole (1985) and Farhi and Tirole (2012).

Dynamics of bubbles in the case of positive dividends

The FOCs give the following system:

$$\begin{cases} \frac{e_{2,1} - b_2^* d_1}{q_1 + d_1} = \frac{\beta_{2,1}(e_{2,0} + d_0 b_{2,-1})}{\beta_{2,0} q_0} - \frac{\beta_{2,1}}{\beta_{2,0}}(L + b_1^* - b_{2,-1}) - H \\ \frac{e_{1,2t} - d_{2t} b_1^*}{q_{2t} + d_{2t}} = \frac{\beta_{1,2t}(e_{1,2t-1} - b_1^* d_{2t-1})}{\beta_{1,2t-1} q_{2t-1}} - H \left(\frac{\beta_{1,2t}}{\beta_{1,2t-1}} + 1 \right) \\ \frac{e_{2,2t+1} - d_{2t+1} b_2^*}{q_{2t+1} + d_{2t+1}} = \frac{\beta_{2,2t+1}(e_{2,2t} - b_2^* d_{2t})}{\beta_{2,2t} q_{2t}} - H \left(\frac{\beta_{2,2t+1}}{\beta_{2,2t}} + 1 \right). \end{cases} \quad (3.15)$$

where recall that $H \equiv L + b_1^* + b_2^*$.

Denote

$$\begin{cases} a_1 \equiv \frac{\gamma_{2,0}(e_{2,0} + d_0 b_{2,-1})}{e_{2,1} - b_2^* d_1} \\ a_{2t} \equiv \frac{\gamma_{1,2t-1}(e_{1,2t-1} - b_1^* d_{2t-1})}{e_{1,2t} - b_1^* d_{2t}} \\ a_{2t+1} \equiv \frac{\gamma_{2,2t}(e_{2,2t} - b_2^* d_{2t})}{e_{2,2t+1} - b_2^* d_{2t+1}} \end{cases} \quad \begin{cases} H_1 \equiv \frac{\gamma_{2,0}(L + b_1^* - b_{2,-1}) + H}{e_{2,1} - b_2^* d_1} \\ H_{2t} \equiv \frac{H(1 + \gamma_{1,2t-1})}{e_{1,2t} - b_1^* d_{2t}} \\ H_{2t+1} \equiv \frac{H(1 + \gamma_{2,2t})}{e_{2,2t+1} - b_2^* d_{2t+1}} \end{cases} \quad \begin{cases} \bar{q}_0 \equiv \frac{e_{2,0} - e_{1,0} - d_0(b_{1,-1} - b_{2,-1})}{L + 2b_1^* + b_{1,-1} - b_{2,-1}} \\ \bar{q}_{2t-1} \equiv \frac{e_{1,2t-1} - e_{2,2t-1} - (L + 2b_1^*)d_{2t-1}}{2H} \\ \bar{q}_{2t} \equiv \frac{e_{2,2t} - e_{1,2t} - (L + 2b_2^*)d_{2t}}{2H}. \end{cases}$$

Our system can be rewritten as

$$\frac{1}{q_t + d_t} = \frac{a_t}{q_{t-1}} - H_t \quad \forall t \geq 1, \text{ or, equivalently, } q_t = \frac{q_{t-1}}{a_t - H_t q_{t-1}} - d_t \quad \forall t \geq 1 \quad (3.16)$$

We obtain the following result.

Proposition 10 (Bosi, Le Van and Pham (2022)). *Let $u_i(c) = \ln(c) \forall i = 1, 2$ and $d_t > 0, \forall t$. Assume that $H_t > 0, a_{t+1}/H_{t+1} < \bar{q}_t \forall t$ and there are sequences $(\alpha_t)_{t \geq 1}, (\sigma_t)_{t \geq 1}$ satisfying $0 < \alpha_t < 1 < \sigma_t$ and*

$$\text{Strong heterogeneity: } \frac{a_{t+1} H_t}{H_{t+1}} > \frac{\alpha_t}{\alpha_{t+1}(1 - \alpha_t)} \quad (3.17a)$$

$$\text{Low dividend condition: } \begin{cases} \frac{d_t}{d_{t+1}} > \frac{\sigma_{t+1}}{\sigma_t - 1} a_{t+1} \\ 1 - (\sigma_t - 1)d_t H_t > 0 \\ \text{and } \frac{\sigma_1 a_1 d_1}{1 + d_1 H_1} < \frac{\alpha_1 a_1}{H_1} \end{cases} \quad (3.17b)$$

⁶When $\beta_{i,t} = \beta^t \forall i, t$, the existence of bubbles requires that β must be high enough (this is consistent with the finding in Proposition 3 in Bosi, Le Van and Pham (2022)).

Then, there is a continuum of bubbly equilibria. More precisely, any sequence $(q_t)_{t \geq 0}$ determined by

$$q_0 \in \left(\frac{\sigma_1 a_1 d_1}{1 + d_1 H_1}, \frac{\alpha_1 a_1}{H_1} \right) \text{ and } (q_t)_{t \geq 1} \text{ is computed by the system (3.15)} \quad (3.18)$$

is a system of prices of an equilibrium in which asset holdings are given by (3.3) and agents' consumptions are given by (3.4a-3.4c). Moreover for such equilibrium, we have

$$\frac{\sigma_t a_t d_t}{1 + d_t H_t} < q_{t-1} < \frac{\alpha_t a_t}{H_t} \quad \forall t \geq 1. \quad (3.19)$$

Proposition 10 partially extends Proposition 9 to the case where dividends are time dependent. To the best of our knowledge, Proposition 10 is the first result showing the existence of multiple equilibria with bubbles of assets with positive dividends in deterministic general equilibrium models. Observe that the equilibrium indeterminacy in our model is real (in the sense that different equilibria have different consumption allocations) and the asset price affects agents' consumptions. Note that dividends and endowments are time dependent.

Look at condition (3.17a), we observe that

$$\begin{aligned} \frac{a_{2t+1} H_{2t}}{H_{2t+1}} &= \frac{\gamma_{2,2t}(1 + \gamma_{1,2t-1}) e_{2,2t} - b_2^* d_{2t}}{1 + \gamma_{2,2t} e_{1,2t} - b_1^* d_{2t}} \\ \frac{a_{2t} H_{2t-1}}{H_{2t}} &= \frac{\gamma_{1,2t-1}(1 + \gamma_{2,2t-2}) e_{1,2t-1} - b_1^* d_{2t-1}}{1 + \gamma_{1,2t-1} e_{2,2t-1} - b_2^* d_{2t-1}}. \end{aligned}$$

Therefore, condition (3.17a) ensures that there are a strong heterogeneity and a seesaw property in our model. It can also be viewed as a "low interest rate condition".

3.1.2 Asset price bubbles in OLG models

In Bosi and Pham (2016), we study the interplay between taxation, bubble formation and economic growth. We introduce a pure bubble asset and a tax on this asset in the standard OLG model. In our model, each agent faces two budget constraints (one per period):

$$\begin{aligned} c_t + s_t + q_t a_t &\leq (1 - \tau) w_t \\ d_{t+1} &\leq (1 - \tau_k) R_{t+1} s_t + (1 - \tau_b) q_{t+1} a_t \end{aligned}$$

where a_t is the pure bubble asset and q_t is its price at date t . The tax rate of labor income, capital income and financial asset incomes are represented by τ , τ_k and τ_b respectively.

The public investment G_t equals

$$G_t = \tau w_t + \tau_k \frac{R_t s_{t-1}}{n} + \tau_b \frac{q_t a_{t-1}}{n} \quad (3.20)$$

Then, this investment affects the TFP of the firm in the spirit of Barro (1990):

$$A_t = \theta G_t^{1-\alpha}. \quad (3.21)$$

In this framework, we prove that a rational bubble may be beneficial when growth is fuelled by R&D externalities and the government levies taxes on bubble returns to finance this R&D. We provide a bigger picture concerning the effect of bubbles in endogenous growth which complements the conventional view about the negative effect of bubbles in endogenous growth (Grossman and Yanagawa, 1993).

In [Bosi, Ha-Huy, Le Van, Pham and Pham \(2018a\)](#), we introduce descendant altruism and an asset with positive dividend in an OLG model by modifying the budget constraints of household born at date t as follows

$$c_t + s_t + q_t a_t \leq w_t + g_t \quad (3.22)$$

$$d_{t+1} + n g_{t+1} \leq R_{t+1} s_t + (q_{t+1} + \delta_{t+1}) a_t \quad (3.23)$$

$$x d_{t+1} \leq n g_{t+1} \quad (3.24)$$

where g_{t+1} represents the bequests from parents to offspring and x is the degree of forward (or descending) altruism. Our modeling is of altruistic preferences. Instead of considering as in [Barro \(1974\)](#) the utility of children in the utility of parents, we introduce a "moral" constraint (which can be interpreted either as naive behavior or the result of social pressures (either moral or religious)): parents leave a share of their wealth when old to offspring.

We then prove the existence of equilibrium and provide global dynamics of capital stocks and asset values as well as the interplay between them. Asset price bubbles are also investigated. Our result suggests that the forward altruism promotes pure bubble à la [Tirole \(1985\)](#) and has a positive impact on asset values but a negative impact on the capital stocks along the transition sequence of an asymptotically bubbly equilibrium. Some examples of multiple bubbly equilibria have been provided. Note that when outputs are not bounded, bubbles may appear even if the interest rates are greater than the population growth rates or even if dividends do not converge to zero (or even if they tend to infinity).

In [Bosi, Ha-Huy, Le Van, Pham and Pham \(2018a\)](#), we consider ascendant altruism that we model as follows: The household born at date t chooses positive vector $(c_t, d_{t+1}, s_t, a_t, h_t)$ to maximize their utility subject to the following constraints:

$$c_t + s_t + q_t a_t + h_t \leq w_t \quad (3.25a)$$

$$d_{t+1} \leq R_{t+1} s_t + q_{t+1} a_t + n h_{t+1} \quad (3.25b)$$

$$h_t = y c_t. \quad (3.25c)$$

where h_t represents the gifts from children to parents and y is the degree of ascendant altruism.

Several empirical studies show the existence of this transfer in Burkina Faso ([Kazianga, 2006](#)), China ([Cai, Giles and Meng, 2006](#)), South Korea ([Park, 2014](#)), and Vietnam ([Nguyen, Liu and Booth, 2012](#)). For example, [Nguyen, Liu and Booth \(2012\)](#) focus on the case of Vietnam and find that transfers from children to parents are motivated by the desire to provide old age support and are closely related to the ability of children to give and the needs of parents.⁷ These transfers are present in many countries where public old-age insurance is not well-developed.

Using this model, we explore the role of ascendant altruism on the dynamic properties of equilibrium and rational bubbles in the cases of exogenous and endogenous growth. We obtain a generalized version of [Tirole \(1985\)](#): bubbles exist only if the so-called *modified interest rate* (i.e., the interest rate in the economy with ascendant altruism and golden rule capital stock but without bubble) is lower than the population growth rate.

The modified interest rate may not be monotonic in the ascendant altruism degree because the altruism generates two effects: the income when young is decreasing in y but the income when old is increasing in y . So, our low interest rate condition leads to important implications. Let us mention two of them.

⁷More precisely, using the Vietnam Living Standard Survey conducted in 1997/1998, [Nguyen, Liu and Booth \(2012\)](#) show that monetary transfers from children depend on the needs of parents and the ability of children. On average, 31% of elderly parents receive transfers from their non-coresiding children. For these receiving parents, transfers from children represent 58% of per capita household expenditure.

- First, if the elasticity of intertemporal substitution (EIS, for short) is less than 1, there exists a steady state with bubble if and only if the benchmark interest rate is lower than the population growth rate and the ascendant altruism is moderate (in the sense that it is lower than an exogenous threshold); in this case, the value of bubble is decreasing in the altruism degree. The intuition is that when the EIS is less than 1, the total savings is decreasing in the ascendant altruism degree. Since agents use part of the total savings to finance the purchase of bubble asset, the ascendant altruism must not be too high.
- Second, if the EIS is higher than 1, a bubble may exist even the benchmark interest rate is higher than the population growth rate. This scenario represents an important difference between our framework and that of [Tirole \(1985\)](#). The basic reason is that there exists a steady state with bubble if and only if that the modified interest rate (but not the benchmark interest rate) is low. In this scenario, the modified interest rate can be lower than the population growth rate (that is lower than the benchmark interest rate) if the ascendant altruism is strong enough. This in turn ensures the existence of bubble. Moreover, the value of bubble is increasing in the ascendant altruism degree.

[Bosi, Ha-Huy, Pham and Pham \(2019\)](#) investigate bubbles in an OLG model where altruism à la [Barro \(1974\)](#) is introduced through a recursive utility. Under specific utility functions, we compute the global dynamics and we show that, in the case of low altruism, bequests are zero and our model works exactly as the that in [Tirole \(1985\)](#) where rational bubbles can arise, while, in the case of high altruism, bequests are positive and bubbles are ruled out.

3.1.3 Extension 1: Valuation and bubbles of productive assets

[Becker, Bosi, Le Van and Seegmuller \(2015\)](#) is the first paper introducing the notion of "bubble of physical capital". In [Bosi, Le Van and Pham \(2017\)](#), we extend this notion to the case of physical capital good.

Consider an infinite-horizon general equilibrium model with a finite number of households and two firms. A representative firm (firm c) produces an aggregate good by using the same good and a capital good. This firm rents K_t units of aggregate good and H_t^c units of capital good to produce $F_t(K_t, H_t^c)$ units of aggregate good. F_t represents a (possibly) non-stationary technology. Its profit maximization problem is written

$$P(p_t, r_t, r_{h,t}) : \quad \pi_{c,t}(p_t, r_t, r_{h,t}) \equiv \max_{K_t, H_t^c \geq 0} [p_t F_t(K_t, H_t^c) - r_t K_t - r_{h,t} H_t^c]$$

where r_t (resp., $r_{h,t}$) denotes the return of aggregate (resp., capital) good at date t .

In the capital good sector, a representative firm (firm k) produces the capital good by using the same good. Formally, the firm decides the demand of capital good H_t^k to maximize its profit.

$$P(q_t, r_{h,t}) : \quad \pi_{k,t}(q_t, r_{h,t}) \equiv \max_{H_t^k \geq 0} [q_t G_t(H_t^k) - r_{h,t} H_t^k]$$

where G_t is a non-stationary production function.

For notational parsimony, we will write $\pi_{c,t}$ and $\pi_{k,t}$ instead of $\pi_{c,t}(p_t, r_t, r_{h,t})$ and $\pi_{k,t}(q_t, r_{h,t})$. $\pi_t \equiv \pi_{c,t} + \pi_{k,t}$ will denote the aggregate profit.

Definition 11. *A sequence of prices and quantities*

$(p_t, q_t, r_t, r_{h,t}, (c_{i,t}, k_{i,t+1}, h_{i,t+1})_{i \in I}, K_t, H_t, H_t^c, H_t^k)_{t \geq 0}$ *is an equilibrium of the economy* $\mathcal{E} = ((u_i, \beta_i, (e_{i,t})_t, k_{i,0}, h_{i,0}, \theta_i)_{i \in I}, \delta, \delta_h, (F_t, G_t)_{t \geq 0})$ *if the following conditions hold.*

(i) *Price positivity:* $p_t, r_t, r_{h,t} > 0, q_t \geq 0$ for any $t \geq 0$.

(ii) *Market clearing:* for each $t \geq 0$,

$$\text{aggregate good: } \sum_{i \in I} (c_{i,t} + k_{i,t+1}) = \sum_{i \in I} (1 - \delta)k_{i,t} + F_t(K_t, H_t^c) + e_t \quad (3.26)$$

$$\text{capital good: } \sum_{i \in I} h_{i,t+1} \leq (1 - \delta_h) \sum_{i \in I} h_{i,t} + G_t(H_t^k) \quad (3.27)$$

$$\text{and } q_t \left(\sum_{i \in I} h_{i,t+1} - (1 - \delta_h) \sum_{i \in I} h_{i,t} - G_t(H_t^k) \right) = 0 \quad (3.28)$$

$$\text{rental markets: } K_t = \sum_{i \in I} k_{i,t} \text{ and } H_t^c + H_t^k = H_t \equiv \sum_{i \in I} h_{i,t}. \quad (3.29)$$

where $e_t \equiv \sum_{i \in I} e_{i,t}$.

(iii) *Optimal consumption plans:* $(c_{i,t}, k_{i,t+1}, h_{i,t+1})_{t \geq 0}^\infty$ is a solution to the problem $(P_i(p, r))$ for any i .

$$P_i(p, q, r) : \max_{(c_{i,t}, k_{i,t+1}, h_{i,t+1})_{t=0}^\infty} \sum_{t=0}^\infty \beta_i^t u_i(c_{i,t})$$

facing sequences of borrowing and budget constraints:

$$\begin{aligned} k_{i,t+1} &\geq 0, \quad h_{i,t+1} \geq 0, \\ p_t(c_{i,t} + k_{i,t+1}) + q_t h_{i,t+1} &\leq p_t(1 - \delta)k_{i,t} + r_t k_{i,t} + q_t(1 - \delta_h)h_{i,t} + r_{h,t} h_{i,t} + E_{i,t}. \end{aligned}$$

Here, $\beta_i \in (0, 1)$ captures the time preference of consumer i , u_i denotes her utility function. And $E_{i,t} \equiv p_t e_{i,t} + \theta_{c,t}^i \pi_{c,t} + \theta_{k,t}^i \pi_{k,t}$ where $e_{i,t}$ denotes her endowment in terms of consumption good at date t .

(iv) *Optimal production plans:* (K_t, H_t^c) is a solution to the problem $P(p_t, r_{c,t}, r_{k,t})$ and (H_t^k) is a solution to the problem $P(q_t, r_{k,t})$ for any $t \geq 0$.

Before presenting equilibrium analysis, we state basic assumptions in the paper.

Assumption 10. (H1): For each i , the utility function u_i is strictly increasing, strictly concave, continuously differentiable, and $u_i'(0) = \infty$.

(H2): For any t , $F_t(\cdot, \cdot)$ is continuously differentiable, increasing, concave, and $F_t(0, 0) = 0$. For any t , $G_t(\cdot)$ is continuously differentiable, increasing, concave, and $G_t(0) = 0$.

(H3): $\delta, \delta_h \in [0, 1)$; $k_{i,0}, h_{i,0} \geq 0$ for every i ; $e_{i,t} \geq 0$ for any i and t . We also assume that $(k_{i,0}, h_{i,0}, e_{i,0}) \neq (0, 0, 0)$ for each i , and $H_0 \equiv \sum_i h_{i,0} > 0$.

(H4): Let (X_t) and (D_t) be defined by

$$\begin{aligned} D_t &= (1 - \delta_h)D_{t-1} + F_t(D_{t-1}, X_{t-1}) + e_t \quad \forall t \geq 1, \quad D_0 \equiv (1 - \delta_h)K_0 + F_0(K_0, H_0) + e_0 \\ X_t &= (1 - \delta_h)X_{t-1} + G_t(X_{t-1}) \quad \forall t \geq 0, \quad X_{-1} \equiv H_0. \end{aligned}$$

We assume that, for each i , the maximal utility of agent i is finite:

$$\sum_{t=0}^\infty \beta_i^t u_i(D_t) < \infty.$$

Under these assumptions, we have that:

Lemma 7 (asset-pricing for aggregate and capital goods). *We have, for any $t \geq 0$,*

$$(\text{Capital good}): \frac{q_t}{p_t} = \gamma_{t+1} \left(\frac{r_{h,t+1}}{p_{t+1}} + (1 - \delta_h) \frac{q_{t+1}}{p_{t+1}} \right) \quad (3.30)$$

$$(\text{Aggregate good}): 1 = \gamma_{t+1} \left(\frac{r_{t+1}}{p_{t+1}} + 1 - \delta \right). \quad (3.31)$$

Definition of bubble in capital good

The capital good is a long-lived asset whose price (in terms of aggregate good) at the initial date equals q_0 . Agents buy the capital good at date 0 taking in account what they will receive in the future.

1. At date 1, one unit purchased at date 0 will bring $r_{h,1}$ units of aggregate good and $1 - \delta_h$ unit of capital good. This is formally represented by $q_0 = r_{h,1}Q_1 + (1 - \delta_h)q_1Q_1$.
2. At date 2, $1 - \delta_h$ unit of the capital good will bring $(1 - \delta_h)r_{h,2}$ units of aggregate good and $(1 - \delta_h)^2$ unit of capital good. Formally, we have $(1 - \delta_h)q_1Q_1 = (1 - \delta_h)r_{h,2}Q_2 + (1 - \delta_h)^2q_2Q_2$.

Iterating the argument, we find that, for any T ,

$$\begin{aligned} q_0 &= (r_{h,1} + (1 - \delta_h)q_1)Q_1 = r_{h,1}Q_1 + (1 - \delta_h)q_1Q_1 \\ &= r_{h,1}Q_1 + (1 - \delta_h)(r_{h,2} + (1 - \delta_h)q_2)Q_2 = r_{h,1}Q_1 + (1 - \delta_h)r_{h,2}Q_2 + (1 - \delta_h)^2q_2Q_2 \\ &= \dots \\ &= \sum_{t=1}^T [(1 - \delta_h)^{t-1}r_{h,t}Q_t] + (1 - \delta_h)^T q_T Q_T. \end{aligned} \quad (3.32)$$

According to (3.32), q_0 – what we pay (in terms of aggregate good) at the initial date to hold 1 unit of capital good – equals what we expect to receive in the future which consists of two terms: (i) the first term in (3.32) is what the production process brings, and (ii) the second term is what any agent receives by reselling the capital good at date T .

Definition 12 (bubble in capital good). *The fundamental value of capital good at date 0 is defined as the sum of discounted values of its returns (net of depreciation):*

$$FV_k \equiv \sum_{t \geq 1} (1 - \delta_h)^{t-1} r_{h,t} Q_t.$$

We say that there is a bubble in capital good if the equilibrium price of capital good exceeds its fundamental value: $q_0 > FV_k$, or, equivalently, $\lim_{t \rightarrow \infty} (1 - \delta_h)^T q_T Q_T > 0$.

Full depreciation. Notice that if the depreciation of capital good is full ($\delta_h = 1$), there is no bubble. The equilibrium price of capital becomes $q_0 = r_{h,1}Q_1$. For this reason, we will consider only the case of partial depreciation ($\delta_h < 1$) in the sequel.

Comments. One unit of capital good at the initial date will be depreciated to $(1 - \delta_h)^t$ unit of the same good at date t . The discounted value of this quantity is $Q_t q_t (1 - \delta_h)^t$. Therefore, the bubble value in capital good is interpreted as the discounted market value (at infinity) of one unit of capital good at the initial date.

The capital good that we consider is more general than the financial asset with exogenous dividends of standard literature (see Kocherlakota (1992) among others). Indeed, the capital good is also a long-lived asset (it is resold and gives dividends at each date). That being said,

their models differ from ours in three main respects: (1) the capital good depreciates while financial assets do not, (2) the sequence of returns $(r_{h,t})$ is endogenous while the sequence of financial dividends (ξ_t) is exogenous, (3) the supply of capital good is also endogenous while the asset supply is not.

Definition of bubble in aggregate good

According to (3.31) and using the same argument in (3.32), we get that, for any $T \geq 1$,

$$1 = \sum_{t=1}^T \left[(1-\delta)^{t-1} r_t Q_t \right] + (1-\delta)^T Q_T. \quad (3.33)$$

This leads to the following concepts:

Definition 13. *The fundamental value of aggregate good at date 0 is defined as the sum of discounted values of its returns (net of depreciation):*

$$FV_f \equiv \sum_{t \geq 1} (1-\delta)^{t-1} r_t Q_t.$$

We say that there is a bubble in aggregate good if the price of aggregate good exceeds its fundamental value: $1 > FV_f$, or, equivalently, $\lim_{T \rightarrow \infty} (1-\delta)^T Q_T > 0$.⁸

Full depreciation. When $\delta = 1$, there is no bubble in aggregate good. In this case, $1 = r_1 Q_1$, i.e., the price of this good at the initial date equals the discounted value of 1 unit of this good at date 1.

Proposition 11 (Bosi, Le Van and Pham (2017)). *(1) There exists a bubble in capital good (or equivalently $\lim_{t \rightarrow \infty} (1-\delta_h)^t Q_t q_t > 0$) if and only if $\sum_{t \geq 1} r_{h,t}/q_t < \infty$.*

(2) There exists a bubble in aggregate good (or equivalently $\lim_{t \rightarrow \infty} (1-\delta)^t Q_t > 0$) if and only if $\sum_{t \geq 1} r_t < \infty$.

Proposition 12. *(i) There is no bubble in capital good if one of the following conditions is satisfied.*

1. *There exists t_0 such that $\beta_i u'_i(c_{i,t})/u'_i(c_{i,t-1}) = \gamma_t$ for any i and $t \geq t_0$.*
2. *There exists t_0 such that $(k_{i,t}, h_{i,t}) \neq (0, 0)$ for any i and $t \geq t_0$.*
3. $\sum_{t \geq 0} Q_t E_t < \infty$.

⁸Our definition of bubble in aggregate good can be presented in nominal terms as follows. We may write (3.31) as $p_t = \gamma'_{t+1} (r_{t+1} + (1-\delta)p_{t+1})$, where $\gamma'_{t+1} \equiv \max_i (\lambda_{i,t+1}/\lambda_{i,t})$. Then we have the following decomposition

$$p_0 = \sum_{t=1}^T \left[(1-\delta)^{t-1} r_t Q'_t \right] + (1-\delta)^T Q'_T p_T. \quad (3.34)$$

Note that $Q'_t \equiv \gamma'_1 \cdots \gamma'_t = Q_t p_0 / p_t$ for any t . So, we see that $\lim_{T \rightarrow \infty} (1-\delta)^T Q'_T p_T > 0$ is equivalent to $\lim_{T \rightarrow \infty} (1-\delta)^T Q_T > 0$, which implies that $p_0 > \sum_{t=1}^{\infty} \left[(1-\delta)^{t-1} r_t Q'_t \right]$ is equivalent to $1 > \sum_{t=1}^{\infty} \left[(1-\delta)^{t-1} \frac{r_t}{p_t} Q_t \right]$. By normalizing $p_t = 1$ for any t , we recover Definition 13.

(ii) Let F, G be increasing and concave production functions with $F(0, 0) = G(0) = 0$. Assume that (1) $F_t = a_t F$ for every t where $a_t \in [\underline{a}, \bar{a}]$ with $\underline{a}, \bar{a} \in (0, \infty)$, and (2) $G_t = b_t G$ for every t where $b_t \in [\underline{b}, \bar{b}]$ with $\underline{b}, \bar{b} \in (0, \infty)$. Assume also that $0 < \bar{b}G'(\infty) < \delta_h$ and $\sup_t e_t < \infty$. Then, there is no bubble in aggregate good.

Bosi, Le Van and Pham (2017) provide some models where bubbles in capital good (and aggregate good) can exist in equilibrium.

Differences between two kinds of bubbles

We have seen that a bubble in aggregate good may exist even if (1) the present value of outputs is finite, (2) all consumers are identical, (3) borrowing constraints of consumers are never binding. However, Proposition 12 indicates that under one of these three conditions, there is no bubble in capital good.

| Statement | Capital good (Lucas' tree) | Aggregate good |
|---|-------------------------------|-------------------|
| No bubble if the present value of profits is finite | holds | may fail |
| No bubble if there is one consumer | holds | may fail |
| No bubble if borrowing constraints are not binding | holds | may fail |

Interestingly, bubbles in capital and aggregate goods are not incompatible. In some cases, only one of these bubbles or no bubble at all exists.

These interesting differences come from the structural difference between aggregate and capital goods: the aggregate good is not only consumed but also used to produce while the capital good is only processed in production.

It should be noticed that this structural difference leads to a difference in the decomposition of asset prices in a finite T -horizon model in which we have

$$q_0 = \sum_{t=1}^T \left[(1 - \delta_h)^{t-1} r_{h,t} Q_t \right] + (1 - \delta_h)^T Q_T q_T = \sum_{t=1}^T \left[(1 - \delta_h)^{t-1} r_{h,t} Q_t \right] \quad (3.35)$$

$$1 = \sum_{t=1}^T \left[(1 - \delta)^{t-1} r_t Q_t \right] + (1 - \delta)^T Q_T. \quad (3.36)$$

At the last date (date T), the capital good price equals zero ($q_T = 0$) while the aggregate good price is strictly positive (households buy aggregate good at the last date because this good can be consumed). So, the price of capital good is equal to the present value of its returns while the price of aggregate good is the sum of the present value of its returns and the discounted value of $(1 - \delta)^T$ units of this good which is strictly positive.

3.1.4 Extension 2: Valuation and bubbles of land and house

In Bosi, Le Van and Pham (2017c), Bosi, Le Van and Pham (2018), we propose a theory of valuation of the assets which can be used by different economic agents or give

Let us start with the land. Assume that the land has an exogenous total supply L . Denote q_t the of land and its price at date t . At this date, agent i buys $l_{i,t}$ units of land at price q_t . This is used to produce $F_i(l_{i,t})$ units of consumption good and can be sold at a price q_{t+1} at date $t + 1$, where F_i is the production function of agent i . The budget constraint of the agent i at date t is

$$c_{i,t} + q_t l_{i,t} + r_t a_{i,t} \leq e_{i,t} + q_t l_{i,t-1} + F_i(l_{i,t-1}) + a_{i,t-1} \quad (3.37)$$

while the borrowing constraint is

$$a_{i,t} \geq -f_i[q_{t+1}l_{i,t} + F_i(l_{i,t})] \quad (3.38)$$

Under mild conditions, in equilibrium, we can write the FOCs of each agent and obtain that

$$\begin{aligned} q_t &= \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + F'_i(l_{i,t})) + \frac{\eta_{i,t}}{\lambda_{i,t}} \\ &= \frac{\lambda_{i,t+1}}{\lambda_{i,t}} \underbrace{\left(q_{t+1} + F'_i(l_{i,t}) \right)}_{\text{Production return}} + \frac{\eta_{i,t}}{\lambda_{i,t+1}} + \underbrace{\frac{f_i\mu_{i,t+1}}{\lambda_{i,t+1}}(q_{t+1} + F'_i(l_{i,t}))}_{\text{Collateral return}} \end{aligned}$$

where $\lambda_{i,t} = \beta_i^t u'_i(c_{i,t})$ while $\mu_{i,t}$ and $\eta_{i,t}$ are the Lagrange multipliers associated the borrowing constraint (3.38) and the physical constraint $l_{i,t} \geq 0$. We rewrite

$$q_t = \gamma_{i,t+1}(q_{t+1} + d_{i,t+1}) \quad (3.39)$$

and call $d_{i,t+1}$ the individual dividend of agent i at date $t+1$. Here $d_{i,t+1}$ includes two terms. The first one is $X_{i,t+1} := F'_i(l_{i,t}) + \frac{\eta_{i,t}}{\lambda_{i,t+1}}$ which represents the return from the production process.⁹ The second term $\frac{f_i\mu_{i,t+1}}{\lambda_{i,t+1}}(q_{t+1} + F'_i(l_{i,t}))$ can be interpreted as a collateral return. Note that the collateral return is equal to zero if $f_i = 0$ or $\mu_{i,t+1} = 0$ (happen if borrowing constraint is not binding).

We define the discount factor γ_{t+1} ($\gamma_{i,t+1}$) of the economy (agent i) from date t to date $t+1$, and the discount factor Q_t ($Q_{i,t}$) of the economy (agent i) from the initial date to date t as follows

$$\begin{aligned} \gamma_{t+1} &:= \max_{i \in \{1, \dots, m\}} \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}, \quad Q_0 := 1, \quad Q_t := \gamma_1 \dots \gamma_t = r_0 \dots r_{t-1} \\ \gamma_{i,t+1} &:= \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}, \quad Q_{i,0} := 1, \quad Q_{i,t} := \gamma_{i,1} \dots \gamma_{i,t} = \frac{\beta_i^t u'_i(c_{i,t})}{u'_i(c_{i,0})} \end{aligned}$$

Definition 14 (dividends of land). The dividends of land $(d_t)_t$ is defined by the following no-arbitrage condition

$$q_t = \gamma_{t+1}(q_{t+1} + d_{t+1}) \quad (3.40)$$

We can prove that

$$r_t = \max_{i \in \{1, \dots, m\}} \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})} \quad (3.41)$$

The asset-pricing equations (3.39) and (3.40) give us different ways to evaluate the price/the value of land.

$$\begin{aligned} q_0 &= \sum_{t=1}^T Q_{i,t} d_{i,t} + Q_{i,T} q_T \\ q_0 &= \sum_{t=1}^T Q_t d_t + Q_T q_T. \end{aligned}$$

This leads us to introduce the so-called notions of individual and strong bubbles.

⁹Note that $X_{i,t+1}l_{i,t} = F'_i(l_{i,t})l_{i,t}$.

Definition 15 (individual bubble). 1. $FV_0 := \sum_{t=1}^{\infty} Q_t d_t$ is the fundamental value of land. We say that a land bubble exists if the market price of land (in term of consumption good) exceeds its fundamental value: $q_0 > FV_0$.

2. $FV_i := \sum_{t=1}^{\infty} Q_{i,t} d_{i,t}$ is the i -fundamental value of land. We say that a i -land bubble exists if $q_0 > \sum_{t=1}^{\infty} Q_{i,t} d_{i,t}$.

3. A strong bubble exists if the asset price exceeds any individual value of land, that is $q_0 > \max_i FV_i, \forall i$.

The following result shows the connection between the concepts of bubble and i -bubble.

Proposition 13. 1. $FV_0 \leq FV_i \leq q_0 \forall i$. By consequence, if an i -land bubble exists for some agent i , then a land bubble exists.

2. There is an agent i such that her i -bubble is ruled out. Consequently, there is no strong land bubble, that is $q_0 = \max_i FV_i$.

3. If $FV_0 = FV_i \forall i$, then $FV_0 = FV_i = q_0 \forall i$, that is, there is no room for bubble nor i -bubble.

Note that [Bosi, Le Van and Pham \(2017c\)](#) also consider the evaluation of house. We assume that houses are traded every period as follows. The agent i buys $h_{i,t}$ units of house at date $t-1$. At date t , she enjoys the house services, that is a utility $v_{i,t}(h_{i,t})$, and resells her house at price q_t . Taking the sequence of prices $(p, q) = (p_t, q_t)_{t=0}^{\infty}$ as given, each household i chooses the sequence of goods $(c_i, h_i) := (c_{i,t}, h_{i,t})_{t=0}^{\infty}$ and solves a program to maximize her intertemporal utility function:

$$R_i(p, q) : \max \sum_{t=0}^{\infty} \beta_i^t [u_i(c_{i,t}) + v_{i,t}(h_{i,t})] \text{ subject to } : h_{i,t+1} \geq 0$$

$$p_t c_{i,t} + q_t h_{i,t+1} \leq p_t e_{i,t} + q_t h_{i,t}.$$

In [Bosi, Le Van and Pham \(2017c\)](#), [Bosi, Le Van and Pham \(2018\)](#), we provide some models where (individual) bubbles of land and house may exist in equilibrium. Then, in [Bosi, Le Van and Pham \(2018\)](#), we extend this idea and the idea of state-price in [Santos and Woodford \(1997\)](#) to study the asset price bubbles on land in the stochastic model described in Section 2.1.2. In this framework, constraints of agent i at node ξ is

$$l_{i,\xi} \geq 0 \tag{3.42}$$

$$c_{i,\xi} + q_{\xi} l_{i,\xi} + r_{\xi} a_{i,\xi} \leq e_{i,\xi} + q_{\xi} l_{i,\xi^-} + F_{i,\xi}(l_{i,\xi^-}) + a_{i,\xi^-} \tag{3.43}$$

$$a_{i,\xi} \geq -f_i[q_{\xi'} l_{i,\xi} + F_{i,\xi'}(l_{i,\xi})] \quad \forall \xi' \in \xi^+, \tag{3.44}$$

Definition 16. Consider an equilibrium and fix a node ξ . $\Gamma_{\xi} := (\gamma_{\xi'}, d_{\xi'})_{\xi' \in \xi^+}$ is called a state-price (or discount factor) and land dividend process if

$$d_{\xi'} \geq \min_i F'_{i,\xi'}(l_{i,\xi}) \quad \forall \xi' \in \xi^+ \tag{3.45}$$

$$q_{\xi} = \sum_{\xi' \in \xi^+} \gamma_{\xi'} (q_{\xi'} + d_{\xi'}) \tag{3.46}$$

Given a process of state-price and land dividend $\Gamma := (\gamma_{\xi}, d_{\xi})_{\xi \in \mathcal{D}}$, let us denote $Q_{\xi} := \prod_{\xi' \leq \xi} \gamma_{\xi'}$. Hence, we can provide

$$q_{\xi_0} = \sum_{t=1}^T \sum_{\xi \in D_t} Q_{\xi} d_{\xi} + \sum_{\xi \in D_T} Q_{\xi} q_{\xi} \tag{3.47}$$

Definition 17. Given a process of state-price and land dividend $\Gamma := (\gamma_\xi, d_\xi)_{\xi \in \mathcal{D}}$, the fundamental value of land associated to this process is defined by $FV_\Gamma := \sum_{t=1}^{\infty} \sum_{\xi \in D_t} Q_\xi d_\xi$. We say that a Γ -land bubble exists if $q_{\xi_0} > FV_\Gamma$.

Definitions 16 and 17 cover the traditional intertemporal pricing and price bubbles of assets with exogenous dividends (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997; Montrucchio, 2004). They also cover the concepts of (individual) dividends of land in our deterministic case. Moreover, we propose an approach for valuation of productive assets which are more general than the fiat money in Pascoa et al. (2011).

In Bosi, Le Van and Pham (2018), we prove that the set of all state-price and land dividend process is not empty. We also provide conditions ruling bubbles in this stochastic framework.

3.2 Equilibrium efficiency under borrowing constraints

3.2.1 Efficiency of intertemporal equilibrium in a one-sector model

Definition 18 (Malinvaud (1953)). Let F_t be the production function at date t , δ be capital depreciation rate. A feasible path of capital is a positive sequence $(K_t)_{t=0}^{\infty}$ such that $0 \leq K_{t+1} \leq F_t(K_t) + (1 - \delta)K_t$ for every $t \geq 0$ and K_0 is given.

A feasible path is efficient if there is no other feasible path (K'_t) such that

$$F_t(K'_t) + (1 - \delta)K'_t - K'_{t+1} \geq F_t(K_t) + (1 - \delta)K_t - K_{t+1}$$

for every t with strict inequality for some t .

Here, aggregate feasible consumption at date t is defined by $C_t := F_t(K_t) + (1 - \delta)K_t - K_{t+1}$.

We recall some classical results:

Theorem 19 (Malinvaud (1953)). Assume that $F_t = F$ for every t , where F is strictly increasing, strictly concave, twice continuously differentiable, and $F(0) = 0$, $F'(\infty) = 0$, $F'(0) = \infty$.

A feasible path (K_t) is efficient if

$$\lim_{t \rightarrow \infty} \frac{K_t}{\prod_{s=0}^{t-1} (1 - \delta + F'(K_s))} = 0. \quad (3.48)$$

Theorem 20 (Cass (1972)). Consider capital paths with $K_t \geq \underline{k} > 0$ for every t . Assume that $F_t = F$ for every t , where F is strictly increasing, strictly concave, twice continuously differentiable, and $F(0) = 0$, $0 \leq F'(\infty) < \delta < F'(\underline{k})$. A feasible path (K_t) is inefficient if and only if

$$\sum_{t=1}^{\infty} \prod_{s=0}^{t-1} (1 - \delta + F'(K_s)) < \infty. \quad (3.49)$$

Theorem 21 (Cass and Yaari (1971)). Assume that for each t , F_t is strictly increasing, strictly concave, continuously differentiable, and $F(0) = 0$. The feasible path (K_t) is efficient if and only if

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T \frac{c'_t - c_t}{\Pi_t} \leq 0 \quad (3.50)$$

for every feasible capital path (K'_t) , where $\Pi_t := \prod_{s=0}^{t-1} (1 - \delta + F'_s(K_s))$.

We now define the efficiency of intertemporal equilibrium.

Definition 22. We say that an intertemporal equilibrium is efficient if its aggregate feasible capital path (K_t) is efficient in sense of [Malinvaud \(1953\)](#).¹⁰

Our first finding can be stated as follows:

Proposition 14. Consider the general equilibrium model à la Ramsey described in Section 2.1.1. Let assumptions in Proposition 1 be satisfied.

$$\gamma_{t+1} := \max_{i \in \{1, \dots, m\}} \frac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})} \quad (3.51)$$

$$Q_0 := 1, Q_t := \prod_{s=1}^t \gamma_s, t \geq 1. \quad (3.52)$$

Q_t the discount factor of the economy from initial date to date t .

1. An equilibrium is efficient if $\liminf_{t \rightarrow \infty} Q_t K_{t+1} = 0$.
2. Assume that the production functions are linear. Then every equilibrium path is efficient.

Note that this result does not require any conditions about the convergence or boundedness of the capital path as in the existing literature.

The following result shows the role of financial dividend on the equilibrium efficiency.

Proposition 15. Consider the general equilibrium model à la Ramsey described in Section 2.1.1. Let assumptions in Proposition 1 be satisfied. We also assume that the production function F is strictly concave, $F'(\infty) < \delta$, and $\limsup_{t \rightarrow \infty} \xi_t < \infty$. If the asset dividend is quite good, i.e., $\limsup_{t \rightarrow \infty} \xi_t > 0$, every equilibrium is efficient.

Our results are related to [Becker and Mitra \(2012\)](#) where they proved that a Ramsey equilibrium is efficient if the most patient household is not credit constrained from some date. However, their result is based on the fact that consumption of each household is uniformly bounded away from zero. In Proposition 4, we do not need this condition. Instead, the efficient capital path in our model may converge to zero. [Mitra and Ray \(2012\)](#) studied the efficiency of a capital path with nonconvex production technologies and examined whether the Phelps-Koopmans theorem is valid. However, their results are no longer valid without the convergence or the boundedness of capital paths.

[Becker, Dubey and Mitra \(2014\)](#) give an example of inefficient Ramsey equilibrium in a model with only physical capital. The production function in their model satisfies $F'(\infty) = 0$ and they consider full depreciation of the capital. The following result shows that financial dividends, for such models, may make production paths efficient. Actually, our result is more general.

¹⁰Another concept of efficiency is *constrained efficiency*. About the constrained efficiency in general equilibrium models with financial asset, see [Kehoe and Levine \(1993\)](#), [Alvarez and Jermann \(2000\)](#), [Bloise and Reichlin \(2011\)](#). About the constrained efficiency in the neoclassical growth model, see [Davila et al. \(2012\)](#).

3.2.2 Efficiency of intertemporal equilibrium in a two-sector model

We adapt to our model the concept of efficiency introduced by [Malinvaud \(1953\)](#).

Definition 23. Consider a two-sector model described in [Section 3.1.3](#). Let F_t, G_t be production functions. Let δ and δ_h be the depreciation rate of the aggregate and capital goods respectively.

A feasible path of production plan is a positive sequence (K_t, H_t, H_t^c, K_t^k) such that

$$\begin{aligned} C_t &\equiv (1 - \delta)K_t + F_t(K_t, H_t^c) - K_{t+1} \geq 0 \\ H_{t+1} &= (1 - \delta_h)H_t + G_t(H_t^k) \\ H_t^c + H_t^k &= H_t \end{aligned}$$

for every t , where (K_0, H_0, H_0^c, H_0^k) is given with $H_0 = H_0^c + H_0^k$.

For each feasible production plan, a feasible path is efficient if there is no other feasible path $(K'_t, H'_t, H'^c_t, H'^k_t)$ such that $C'_t \geq C_t$ for every t with strict inequality for some t .

Definition 24. An intertemporal equilibrium is efficient if its production plan (K_t, H_t, H_t^c, H_t^k) is efficient.

We have the following sufficient condition ensuring the efficiency of equilibrium.

Lemma 8. An equilibrium is efficient if $\lim_{t \rightarrow \infty} Q_t(K_{t+1} + q_t H_{t+1}) = 0$.

We have the following result.

Proposition 16 (Proposition 4 in [Bosi, Le Van and Pham \(2017\)](#)). Consider a two-sector model in [Section 3.1.3](#). Let [Assumption 10](#) be satisfied. An equilibrium is efficient if one of the following conditions is satisfied.

1. There exists t_0 such that $\gamma_t = \beta_i u'_i(c_{i,t})/u'_i(c_{i,t-1})$ for any i and $t \geq t_0$.
2. There exists t_0 such that $(h_{i,t}, k_{i,t}) \neq (0, 0)$ for any i and $t \geq t_0$.
3. $\sum_{t=0}^{\infty} Q_t E_t < \infty$.

We should mention that [Becker, Dubey and Mitra \(2014\)](#) provide an example of inefficient Ramsey equilibrium with a cycle of period three, where borrowing constraints are binding at infinitely many dates.

According to [Proposition 16](#), when there is no endowment and profits equal zero, every equilibrium is efficient. The particular (but prominent) case of zero profit CRS technologies is considered in the following corollary.

Corollary 4. Assume that $e_t = 0$ for any t . Assume also that F_t and G_t are constant returns to scale for any t . Then, any equilibrium path is efficient.

Notice that [Assumption 10](#) $e_t = 0 \forall t$ holds in models of [Cass \(1972\)](#), [Becker and Mitra \(2012\)](#), [Mitra and Ray \(2012\)](#). [Corollary 4](#) differs from those obtained by these authors in two respects: (1) we allow for linear technologies (while they consider strictly concave production functions), (2) we do not need bounded capital stocks (as it is the case in their papers).

3.2.3 Discussion: efficiency and existence of bubbles

In this section, we discuss the connection between efficiency and existence of bubbles in aggregate and capital goods.

First, focus on the side of aggregate good.

1. According to Corollary 4, the equilibrium is efficient for any sequence (a_t) . Moreover, this efficient equilibrium exhibits a bubble in aggregate good if and only if $\sum_{t=1}^{\infty} a_t < \infty$. So, an efficient equilibrium may (or may not) exhibit a bubble in aggregate good.
2. According to Proposition 12, there is no bubble in aggregate good in a one-sector model with stationary technologies. In a similar framework, Becker, Dubey and Mitra (2014) provide an example of inefficient Ramsey equilibrium with a cycle of period three. In this framework, one can prove that there is no bubble in aggregate good. This shows that an inefficient equilibrium may have no bubble in aggregate good.

This discussion suggests that there is no robust causal relationship between the existence of bubble in aggregate good and the efficiency of equilibrium. We should not be surprised because the existence of bubble rests on the low total returns while equilibrium efficiency on capital distribution does.

Second, we look at the connection between the efficiency and the existence of bubble in capital good.

1. As mentioned, a pure bubble asset à la Tirole (1985) is a particular case of our capital good. There is a large literature on this kind of bubble and most of papers focus on OLG frameworks. In a standard OLG model of bubble, Tirole (1985) shows that a pure bubble may occur only if the economy is dynamically inefficient.¹¹ However, Farhi and Tirole (2012) point out that with imperfect capital markets, a bubble may exist even when the economy is efficient. So, there is no robust causal relationship between the existence of bubble in capital good and the efficiency (in the sense of Tirole (1985)).
2. We now consider general case and point out that an efficient equilibrium may or may not have bubble in capital good.

First, according to Propositions 12 and 16, an efficient equilibrium may have no bubble in capital good.

Second, consider a particular case of Example in Section 1, where $k_{i,0} = 0$ for any i , $F_t(K, H) = b_t H$, $G_t(H) = 0$ for any t , $\delta = 1$, $\delta_h = 0$. Budget constraints write

$$p_t c_{i,t} + q_t h_{i,t+1} \leq (q_t + r_{h,t}) h_{i,t} + p_t e_{i,t}.$$

We have equilibrium efficiency in this case. However, as shown in Section 1, a bubble in capital good may arise.

It remains a question: is there an inefficient equilibrium which experiences a bubble in capital good? This question is hard to tackle because there is no general sufficient condition (based on exogenous parameters) for equilibrium efficiency or for existence of bubble in capital good.¹²

¹¹According to Tirole (1985), an allocation is dynamically efficient if it is not possible to improve the welfare of all generations (and this strictly for at least one of them).

¹²In Example in Section 1, when a bubble in capital good arises, we have $\lim_{t \rightarrow \infty} a_t = 0$ which implies that $\lim_{t \rightarrow \infty} Q_t(K_{t+1} + q_t H_{t+1}) = \lim_{t \rightarrow \infty} \frac{\beta w}{1+\beta} [(1 - \delta + a_1) \cdots (1 - \delta + a_t)]^{-1} = \infty$. So, we cannot apply Lemma 8.

3.3 (Non-Monotonic) impacts of productivity and credit changes on equilibrium aggregate production

Let us start with a two-period general equilibrium with borrowing constraint.

Definition 25. Assume that there are m agents. A list of allocations and gross interest rate $((k_i, a_i)_{i=1}^m, r)$ is a general equilibrium if it satisfies two conditions:

1. For each i , given r , the allocation (k_i, a_i) is a solution to the maximization problem of the agent i is

$$(P_i) \quad \pi_i = \max_{k_i, a_i} [F_i(k_i) - ra_i] \quad (3.53)$$

$$\text{subject to} \quad 0 \leq k_i \leq S_i + a_i \quad (3.54)$$

$$ra_i \leq \gamma_i F_i(k_i). \quad (3.55)$$

where γ_i is the credit limit, F_i the production function and S_i is the initial capital of agent i .

2. The financial market clears: $\sum_{i=1}^m a_i = 0$.

The following table from the [Enterprise Surveys \(2018\)](#)'s panel datasets suggests that borrowing and collateral constraints matter for the development of firms. In [Pham and](#)

| Economy | Proportion of loans requiring collateral (%) | Value of collateral needed for a loan (% of the loan amount) | Percent of firms not needing a loan | Percent of firms whose recent loan application was rejected | Proportion of investments financed internally (%) |
|----------------------------|--|--|-------------------------------------|---|---|
| All Countries | 79.1 | 205.8 | 46.4 | 11.0 | 71.0 |
| East Asia & Pacific | 82.6 | 238.4 | 50.7 | 6.4 | 77.8 |
| Europe & Central Asia | 78.7 | 191.9 | 54.3 | 10.9 | 72.4 |
| Latin America & Caribbean | 71.3 | 198.5 | 45.0 | 3.1 | 62.7 |
| Middle East & North Africa | 77.4 | 183.0 | 51.8 | 10.2 | 71.1 |
| South Asia | 81.1 | 236.0 | 44.7 | 14.4 | 73.9 |
| Sub-Saharan Africa | 85.3 | 214.8 | 37.4 | 15.3 | 73.9 |

[Pham \(2021\)](#), we consider some specific cases (linear or Cobb-Douglas, we can explicitly compute the equilibrium outcomes. For example, if $F_i(k) = A_i k$, $\forall i, \forall k$ with $A_1 < A_2$, then we prove that the consumption allocation of equilibrium is efficient if and only if

$$\gamma_2 \geq \gamma_2^* \equiv \frac{A_1}{A_2} \frac{S_1}{S_1 + S_2}.$$

This happens if the credit limit γ_2 is high (in the sense that $\gamma_2 \geq \gamma_2^* \equiv \frac{A_2}{A_1} \frac{S_2}{S_1 + S_2}$). The lower productivity dispersion (A_1/A_2) and initial wealth dispersion (S_1/S_2), the lower the threshold level $\gamma_2^* \equiv \frac{A_2}{A_1} \frac{S_2}{S_1 + S_2}$, and the easier we can obtain equilibrium efficiency.

We now look at the equilibrium effects of productivity changes. Let both productivities A_1 and A_2 vary.

Proposition 17 ([Pham \(2022\)](#)). Consider a two-agent economy having linear technologies $F_i(k) = A_i k$ $\forall i = 1, 2$ with $\gamma_2 < A_1 < A_2$, and borrowing constraints: $Rb_i \leq \gamma_i A_i k_i$.

Assume that there is a productivity shock that changes the productivity of agents from (A_1, A_2) to (A'_1, A'_2) . Assume that $A'_2 > A'_1$. Assume that the credit constraint of agent 2 is

low so that $\gamma_2 < \frac{A_1}{A_2} \frac{S_1}{S_1+S_2}$ and $\gamma_2 < \frac{A'_1}{A'_2} \frac{S_1}{S_1+S_2}$. Then, the output change is

$$Y(A'_1, A'_2) - Y(A_1, A_2) = (A'_1 - A_1)S_1 + A_2S_2(1 - \gamma_2) \frac{A_1A'_2 - A'_1A_2}{(A_1 - \gamma_2A_2)(A'_1 - \gamma_2A'_2)} \quad (3.56)$$

(1) We have that:

$$\text{If } \frac{A'_2}{A_2} \geq \frac{A'_1}{A_1} \geq 1, \text{ then } Y(A'_1, A'_2) \geq Y(A_1, A_2) \quad (3.57)$$

(2) Assume that

$$S_2A_2(1 - \gamma_2) \frac{\gamma_2A_2}{(A_1 - \gamma_2A_2)^2} - S_1 > 0, \text{ i.e., } \frac{S_1}{S_2} \left(\frac{A_1}{A_2} - \gamma_2 \right)^2 < (1 - \gamma_2)\gamma_2 \quad (3.58)$$

Then, there is a neighborhood \mathcal{B} of (A_1, A_2) such that

$$\frac{Y(A'_1, A'_2) - Y(A_1, A_2)}{A'_1 - A_1} < 0 \quad (3.59a)$$

$$\forall (A'_1, A'_2) \in \mathcal{B} \text{ satisfying } \frac{\frac{A'_2}{A_2} - 1}{\frac{A'_1}{A_1} - 1} < \frac{\gamma_2A_2}{A_1} - \frac{S_1(A_1 - \gamma_2A_2)^2}{S_2A_1A_2(1 - \gamma_2)} \text{ and } A'_1 \neq A_1. \quad (3.59b)$$

Condition (3.57) indicates that the aggregate output increases if the productivity of both firms increases and the productivity of the most productive agent increases faster than that of the less productive one.

Let us now focus on point 2 of Proposition 17. It shows that when productivity changes are not proportional, positive productivity changes may reduce the aggregate output. Here, condition (3.58) plays a very important role. It is satisfied if the ratio $\frac{A_1}{A_2}$ is low in the sense that $\frac{A_1}{A_2} < \gamma_2 + \left(\frac{\gamma_2(1-\gamma_2)S_2}{S_1} \right)^{0.5}$. This can be interpreted as a *high productivity dispersion*. Under this condition, we see that $\frac{\gamma_2A_2}{A_1} - \frac{S_1(A_1 - \gamma_2A_2)^2}{S_2A_1A_2(1 - \gamma_2)} \in (0, 1)$. According to conditions (3.58) and (3.59a), under a positive shock that improves the TFP of all agents, the aggregate output may decrease:

$$Y(A'_1, A'_2) < Y(A_1, A_2), \forall A'_1 > A_1, A'_2 > A_2, (A'_1, A'_2) \in \mathcal{B} \text{ satisfying (3.59b)}.$$

Note that Pham (2022) generalizes this result in a more general setups (a finite number of agents, a general production function). We also provide conditions under which the aggregate output is increasing or decreasing in the credit limits (γ_i).

In Pham (2022), we also investigate the effects of credit limits on the equilibrium outcomes in an infinite-horizon model. In such a model, there are a finite number of agents. Each agent i maximizes her intertemporal utility subject to budget and borrowing constraints

$$\max_{(c_i, k_i, b_i)} \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \quad (3.60a)$$

$$\text{subject to: } c_{i,t} + k_{i,t} + R_t b_{i,t-1} \leq F_{i,t}(k_{i,t-1}) + b_{i,t} \quad (3.60b)$$

$$R_{t+1} b_{i,t} \leq \gamma_i F_{i,t}(k_{i,t}) \leq 0 \quad (3.60c)$$

where we assume that $b_{i,-1} = 0 \forall i$ and denote $w_{i,0} = F_{i,0}(k_{i,-1})$.

Definition 26. An equilibrium is a list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_{t \geq 0}$ such that (1) given (R_t) , the allocation $(c_{i,t}, k_{i,t}, b_{i,t})$ is a solution of the above maximization problem, and (2) markets clear: $\sum_i b_{i,t} = 0$, $\sum_i (c_{i,t} + k_{i,t}) = \sum_i F_{i,t}(k_{t-1})$, $\forall t$.¹³

In Pham (2022), we provide a generalized version of Proposition 17 concerning the effects of productivity changes. We point out that, due to financial frictions (credit constraints), the equilibrium aggregate production may be non-monotonic in both individual productivity and credit limit. We provide conditions under which this phenomenon happens. By consequence, the emergence of some firms (for example, improving productivity or relaxing credit limit) may not necessarily be beneficial to economic development.

Let us formally present a result showing the effects of credit limits on the aggregate output along intertemporal equilibrium.

Proposition 18 (Pham (2022)). Consider a two-agent economy with linear production functions $F_i(k) = A_i k$, $\forall i = 1, 2$, where $\gamma_2 A_2 < A_1$. Assume that

$$\beta_2 > \beta_1, \quad A_1 < \gamma_2 A_2 \left(1 + \frac{s_{2,0}}{s_{1,0}}\right) < A_2 \quad (3.61)$$

where $s_{i,0} \equiv \beta_i w_{i,0}$ and $w_{i,0}$ is the endowment of agent i at the initial date.

Then there exists an equilibrium with the interest rates¹⁴

$$R_1 = \gamma_2 A_2 \left(1 + \frac{s_{2,0}}{s_{1,0}}\right) \in (A_1, A_2) \quad (3.62a)$$

$$R_t = A_2, \forall t \geq 2. \quad (3.62b)$$

The aggregate output is given by

$$Y_1 = A_2 k_{2,0} = \frac{A_2 R_1}{R_1 - \gamma_2 A_2} s_{2,0} = A_2 (s_{1,0} + s_{2,0})$$

$$\forall t \geq 2, \quad Y_t = A_2 k_{2,t-1} = A_2^t \left(\beta_1^{t-1} \gamma_2 + \beta_2^{t-1} (1 - \gamma_2) \right) (s_{1,0} + s_{2,0})$$

The aggregate output at date 1 does not depend on the credit limit γ_2 of the most productive agent and

$$\frac{\partial Y_t}{\partial \gamma_2} < 0, \forall t \geq 2. \quad (3.63)$$

It means that the aggregate output from date 2 on decreases when the most productive agent's credit limit increases. This is different from the standard view on the effects of financial constraints as shown in Buera and Shin (2013), Khan and Thomas (2013), Midrigan and Xu (2014), Moll (2014), Catherine, Chaney, Huang, Sraer and Thesmar (2017, 2022).

¹³Here, we do not focus the equilibrium existence in the general case (see, for instance, Bosi, Le Van and Pham (2018)).

¹⁴Notice that Kiyotaki (1998)'s Section 2 only focuses on the case where the equilibrium interest rate equals the rate of return on investment of unproductive agents, i.e., $R_t = A_1$, $\forall t$.

Chapter 4

Economic growth: roles of financial markets and globalization

4.1 Financial market, recession and economic growth

4.1.1 Financial market versus productive sector

To present our ideas in a simple way, consider an agent whose initial endowment is S . Agent has two choices to invest: to produce or to invest in financial asset. She may produce $AF(K)$ units of consumption good by using K units of physical capital. If she buys a units of financial asset with price q , she will receive ξa units of consumption good, where ξ is the dividend of the financial asset.

$$\max_{K, a \geq 0} AF(K) + \xi a \quad (4.1)$$

$$K + qa \leq S \quad (4.2)$$

Proposition 19. (i) If $AF'(0) \leq \frac{\xi}{q}$, agent does not produce, i.e., $K = 0$ and $a = S$.

(ii) If $AF'(S) \geq \frac{\xi}{q}$, agent does not invest in financial asset, i.e., $a = 0$ and $K = S$.

(iii) If $AF'(S) \leq \frac{\xi}{q} \leq AF'(0)$, agent produces and invests in financial asset. K is determined by $AF'(K) = \frac{\xi}{q}$ and $a = S - K$.

The intuition is very clear: We invest in the highest return asset. Point (i) says that we do not produce if the maximum return of the productive sector is less than the return of the financial sector. The main implication of Proposition 19 is that the productive sector will disappear if its productivity is low.

We embed this idea in an infinite horizon dynamic general equilibrium model presented in Section 2.1.1. We explore conditions under which a recession in the production sector appear and point out the role of financial dividend. Our finding is summarized as follows.

Proposition 20 (Le Van and Pham (2016)). Consider the general equilibrium model in Section 2.1.1. Assume that there exists $\xi > 0$ such that $\xi_t \geq \xi$ for every $t \geq 0$ and $F'(0) \leq \delta$. Then there is an infinite sequence $(t_n)_{n=0}^{\infty}$ such that $K_{t_n} = 0$ for every $n \geq 0$.

However, even when the productivity is low, a recession may be avoided thanks to financial asset. This ideas is formalized by the following result.

Proposition 21 (Le Van and Pham (2016)). *Consider the general equilibrium model in Section 2.1.1. Assume that for every i ,*

$$\beta_i(F'(0) + 1 - \delta)u'_i(\xi_{t+1}) > u'_i\left(\frac{\xi_t}{m}\right).$$

We have $K_{t+1} > 0$.

When ξ_t is high enough, this condition is satisfied, and so $K_{t+1} > 0$. We also see that $K_{t+1} > 0$ if $F'(0) = \infty$.

In our framework, a fluctuation of dividends (ξ_t) may create a fluctuation of capital stock (K_t).

Proposition 22 (Le Van and Pham (2016)). *(fluctuation of the capital stocks) Consider the general equilibrium model in Section 2.1.1. Assume that*

- (i) $\beta_i = \beta$, $u_i(c) = \frac{c^{1-\sigma}}{1-\sigma}$, and $F'(0) \leq \delta$.
- (ii) $d_{2t} \rightarrow \xi^e$, $d_{2t+1} \rightarrow \xi^o$ when $t \rightarrow \infty$.
- (iii) $\xi^e > \frac{m\xi^o}{(\beta(F'(0) + 1 - \delta))^{\frac{1}{\sigma}}}$.

We have that:

- (i) There is an infinite sequence $(t_n)_{n=0}^{\infty}$ s.t. $K_{t_n} = 0$ for every $n \geq 0$.
- (ii) $\limsup_{t \rightarrow \infty} K_t > 0$.

4.1.2 Dividend taxation, public spending and economic growth

In Pham (2023), we extend Le Van and Pham (2016) by introducing a government who imposes a dividend tax and uses it to finance productive government spending which in turn improves the productivity of firms. The budget constraint of household i is

$$p_t(c_{i,t} + k_{i,t+1} - (1 - \delta)k_{i,t}) + q_t a_{i,t} \leq r_t k_{i,t} + q_t a_{i,t-1} + p_t \xi_t (1 - \tau) a_{i,t-1} + \theta_t^i \pi_t. \quad (4.3)$$

The aggregate tax revenue T_t (in terms of consumption good) is

$$T_t = \sum_{i=1}^m \tau \xi_t a_{i,t-1}.$$

Let us denote by G_t the productive government spending at date t . In the spirit of Barro (1990) (see all Barro and Sala-i-Martin (1995), Section 4.4), we assume that the productive government spending will improve the productivity of all firms at the next date. More precisely, the production function at date t is given by

$$F_g(G_{t-1}, K) = f(G_{t-1})F(K)$$

where f is an increasing function and $f(0) = 1$. When there is no productive government spending, we have $F_g(G_{t-1}, K) = f(0)F(K) = F(K)$.

The value $f(G)$ represents the effect of the productive government spending on the productivity of firms. This effect depends not only on the spending G but also on the governance quality.

We prove that, along intertemporal equilibrium, we may avoid recessions and potentially get economic growth. This happens if: (1) the productive government spending is productive enough and (2) dividends are high. We have also investigated the optimal dividend taxation. When the government's objective function is the total consumption of households at the steady state, the optimal level of dividend taxation increases in the firm's productivity and the governance quality but decreases in the size of dividends.

4.1.3 Bubble asset taxation, public spending and economic growth

In [Bosi and Pham \(2016\)](#), we investigate the effects of a tax on the pure bubble asset. The model is briefly described in Section 3.1.2.

Denote ρ_0 and ρ_1 the growth rate of the aggregate capital when the value of the pure bubble asset is asymptotically negligible and non-negligible respectively. We can prove that these growth rates ρ_0 and ρ_1 are increasing in the government's efficiency. We compare these rates and explore conditions under which $\rho_0 < \rho_1$ (or $\rho_0 > \rho_1$).

Since the R&D process is financed by taxes, the growth rate depends on the fiscal policy. When the labor and capital income taxes are low, the key instrument becomes the tax rate τ_b on the asset bubble. In this case, when this tax rate is sufficiently high ($\tau_b > \underline{\tau}_b$), the growth factor ρ_1 (with asymptotically non-negligible bubble) turns out to be higher than the growth factor ρ_0 (with asymptotically negligible bubble).

Our findings suggest that the existence of a bubble asset (such as a housing bubble) and a good tax policy may be beneficial to economic growth. This point of view challenges the one in [Grossman and Yanagawa \(1993\)](#) where it is shown that an asset bubble absorbs the savings of a market economy experiencing under-accumulation (because of positive productive externalities), and, in the end, makes the situation worse.

However, when the government applies a higher tax rate on capital income, the growth rate with asymptotically negligible bubble is higher than the one with asymptotically non-negligible bubble. In this case, we recover the conventional result by [Grossman and Yanagawa \(1993\)](#).

4.2 Foreign aid and economic growth

4.2.1 Foreign grants and economic growth

[Pham and Pham \(2020\)](#) consider a discrete-time infinite-horizon frameworks where public investment, partially financed by aid, may improve the recipient economy's TFP. We model aid per capita at period t as

$$a_t = \max(\bar{a} - \phi k_t, 0)$$

where the couple (\bar{a}, ϕ) is interpreted as the aid rule imposed by the donor and represents aid conditionalities while the physical capital per capita k_t represents the recipient country's need.

We then investigate the effects of foreign aid in two models: infinite-horizon general equilibrium model and an optimal growth model with externality. We present here the main economic insights in [Pham and Pham \(2020\)](#). With AK technology and logarithmic utility function, we obtain the dynamics of capital stock

$$k_{t+1} = G(k_t) \equiv f(k_t)k_t \tag{4.4a}$$

$$\text{where } f(k_t) \equiv \beta \frac{1 - \delta + A \left[1 + (\sigma(\tau k_t + \alpha_i(\bar{a} - \phi k_t)^+) - b)^+ \right]}{1 + \tau}. \tag{4.4b}$$

where β represents the rate of preference time of households, δ is the depreciation rate, b is the fixed cost, A is the autonomous TFP, τ may be interpreted as the government effort in financing public investment, α_i represents the efficiency in the use of aid, parameter $\sigma \in (0, \infty)$ measures the extent to which the public investment translates into technology and it reflects the efficiency of public investment.

The endogenous TFP is

$$A_t \equiv A \left[1 + (\sigma B_t - b)^+ \right], \quad B_t = \tau k_t + \alpha_i a_t = \tau k_t + \alpha_i (\bar{a} - \phi k_t)^+. \quad (4.5)$$

which depend on the domestic public investment and foreign aid.

This dynamic system is non-linear and non-monotonic. We provide a full analysis of transitional dynamics of capital in order to explain how foreign aid can help a small recipient country to escape the poverty trap and potentially get economic growth in the long run. The global dynamics of equilibrium is quite complex: the capital path may diverge or converge to a steady-state, or fluctuate around it. There may also be a two-period cycle. These different outcomes are analyzed in four distinguished levels of circumstances depending on the degree of corruption in the use of aid, the autonomous technology, the fixed cost and efficiency of public investment, as well as the donor's rules.

It should be noticed that the effectiveness of aid depends on how aid is used in recipient countries and on the absorptive capacity of these countries. Two questions arise: Does aid reduce recipient governments' effort in financing public expenditure? If so, what is the impact of the crowding-out effect on economic outputs in a small recipient country? [Pham and Pham \(2019\)](#) address these questions in a two-period model by endogenizing the use of foreign aid and fiscal policy in a recipient country. They assume that overall public expenditures, comprising public services and public investment, are financed by two sources: foreign aid and tax revenue. The recipient government makes its decisions regarding public services, public investment, and the manner to use foreign aid when caring about the population's welfare and its own interest. Their framework indicates a crowding-out effect on the tax effort, which is reinforced by public investment's low efficiency. However, due to a positive effect of aid on public investment, the final effect of aid on economic growth is positive. The aid effect depends on the recipient circumstances, reflected by the efficiency of public investment, as well as the donors' sensitivity with respect to this efficiency.

4.2.2 Development loans and economic dynamics

These above papers focus on the case of grants. In [Le Van, Pham and Pham \(2023\)](#), we study the effects of development loans in a Solow model with a fixed costs. At each date t , the wealth W_t , defined as the real income net after the repayment, is divided between consumption and saving.

$$c_t + S_t = W_t \quad (4.6a)$$

$$S_t = sW_t \quad (4.6b)$$

$$k_{t+1} = (1 - \delta)k_t + I_t \quad (4.6c)$$

where c_t, S_t, I_t are consumption, saving, and investment at the period t ($t = 0, 1, \dots, +\infty$), $s \in (0, 1)$ is the exogenous saving rate. k_t represents the physical capital stock at date t ($k_0 > 0$ is given) while $\delta \in [0, 1]$ is the capital depreciation rate.

Following [Le Van, Saglam and Turan \(2016\)](#), we introduce a fixed cost in the production process by assuming that the production function is given by

$$F(k_t) = \begin{cases} 0 & \text{if } k_t < b_0 \\ Af(k_t - b_0) & \text{if } k_t \geq b_0 \end{cases} \quad (4.7)$$

where A represents the exogenous total factor productivity, and $b_0 \geq 0$ the fixed cost.

The total investment of the recipient country at t equals

$$I_t = S_t + \lambda a_t \quad (4.8)$$

where a_t represents the foreign capital flow while $\lambda \in (0, 1]$ is an exogenous parameter which can be interpreted as the efficient amount of aid. The amount a_t is defined by the following rule:

$$a_t = \begin{cases} x_t & \text{if } k_t < b_1 \\ 0 & \text{if } k_t \geq b_1 \end{cases} \quad (4.9)$$

where x_t is the solution of the following problem (P_t).

$$(P_t): \quad W_{t+1} \equiv \max_{0 \leq x \leq \bar{a}} \{F(k_{t+1}) - Rx\} = \max_{0 \leq x \leq \bar{a}} \{F[(1 - \delta)k_t + S_t + \lambda x] - Rx\}$$

When $k_t < b_1$, the recipient country can borrow from foreign organizations, but it cannot borrow more than a borrowing limit \bar{a} , which is exogenous. The amount a_t is chosen by the recipient country so that it maximizes the country's wealth in the next period.

The aid rule is represented by three exogenous parameters (b_1, \bar{a}, R) .

Summing up, the dynamics of capital accumulation becomes:

$$k_{t+1} = (1 - \delta)k_t + sW_t + \lambda a_t = \begin{cases} (1 - \delta)k_t + sW_t + \lambda x_t & \text{if } k_t < b_1 \\ (1 - \delta)k_t + sW_t & \text{if } k_t \geq b_1 \end{cases} \quad (4.10)$$

while the recipient's wealth at date $t + 1$ is determined by:

$$W_{t+1} = \begin{cases} \max_{0 \leq x \leq \bar{a}} \{F[(1 - \delta)k_t + S_t + \lambda x] - Rx\} & \text{if } k_t < b_1 \\ F[(1 - \delta)k_t + sW_t] & \text{if } k_t \geq b_1 \end{cases} \quad (4.11)$$

Here, we assume, by convention, that W_0 is exogenously given.

Our model brings many insights. First, the presence of this fixed cost generates a poverty trap. Second, we show that foreign aid may help the country to escape from the poverty trap and converge to a stable steady state in the long run, but only if (i) the country's characteristics, such as saving rate, initial capital, governance quality, and productivity are good enough, (ii) the fixed cost is relatively low, and (iii) the loan rule is generous enough. We also show that our model with foreign aid has room for endogenous cycles, unlike the standard Solow model.

Our results in [Pham and Pham \(2019\)](#), [Pham and Pham \(2019\)](#), [Le Van, Pham and Pham \(2023\)](#) show that whether or not the country can overcome the poverty or middle-income trap depends not only on the foreign aid but also, and mainly, on its capacity (such as the TFP, the depreciation rate, the saving rate, and the governance quality). Moreover, under the presence of development loans, an endogenous cycle may arise.

4.3 FDI and economic growth

In [Nguyen-Huu and Pham \(2018\)](#), we consider a two-period model. This is a small open economy with two productive sectors. The first (called old sector) produces the consumption good by using physical capital good. Assume that there is a unique representative domestic firm (called consumption good firm) in this sector and its production function is given by

$$F^c(K_c) = A_c K_c^{\alpha_c}, \text{ where } A_c > 0 \text{ and } \alpha_c \in (0, 1). \quad (4.12)$$

The second sector (called new sector or new industry) produces a new good by using physical capital and specific labor. There are two kinds of firm: a MNF (or foreign firm) and a potential domestic one (domestic firm for short) in this sector. The foreign firm is well-planted in the country and has the production function

$$F^e(K_e, L_e) = A_e K_e^{\alpha_e} L_e^{\beta_e} \quad (4.13)$$

where $A_e > 0$ and $\alpha_e \in (0, 1), \beta_e \in (0, 1), \alpha_e + \beta_e \leq 1$. The foreign firm (without market power) maximizes its profit

$$(F) : \quad \max_{K_{e,1}, L_{e,1}^D \geq 0} \left[p_n F^e(K_{e,1}, L_{e,1}^D) - p K_{e,1} - w_1 L_{e,1}^D \right].$$

The potential domestic firm in this sector has the production function

$$F^d(K_d, L_d) = A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{\beta_d} \quad (4.14)$$

where $A_d > 0$ and $\alpha_d \in (0, 1), \beta_d \in (0, 1), \alpha_d + \beta_d \leq 1$. To enter the new industry, the domestic firm must make an initial investment. We model this investment by the fixed cost, \bar{L} , representing the level of specific labor needed to ensure the setup of production process. Since the foreign firm can receive specific labor from its parent company, it does not suffer such investment.

There is a representative consumer in the host country, who takes prices as given and chooses $(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1})$ to maximize the economy's GNP at the second period

$$(P) : \quad \max_{(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1})} \left[U := F^c(K_{c,1}) + w_1 L_{e,1} + p_n F^d(K_{d,1}, L_{d,1}) \right]$$

subject to

$$H_1 + p(K_{c,1} + K_{d,1}) \leq S \quad (4.15)$$

$$L_{d,1} + L_{e,1} \leq L_0 + T_0 + \epsilon H_1, \quad (4.16)$$

$$K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1} \geq 0. \quad (4.17)$$

In the first period (date 0), the representative consumer invests H_1 units of consumption good to train specific labor. She buys $K_{c,1}$ and $K_{d,1}$ units of physical capital for the consumption sector and the new sector respectively. In the second period (date 1), an amount of specific labor $L_{e,1}$ is used by the MNF and another amount of specific labor $L_{d,1}$ is used by the domestic firm. Here, the representative agent only consumes consumption good. She can trade the new good in an international market in order to get consumption good.

Definition 27. Consider the economy $\mathcal{E} := (F^c, F^d, F^e, S, p, p_n, \epsilon, L_0, T_0, \bar{L})$. An equilibrium is a list $(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1}, K_{e,1}, w_1)$ such that

(i) Given labor price w_1 , $(K_{c,1}, K_{d,1}, H_1, L_{d,1}, L_{e,1})$ is a solution for the problem (P).

(ii) Given labor price w_1 , $(L_{e,1}^D, K_{e,1})$ is a solution for the problem (F).

(iii) Labor market clears: $L_{e,1}^D = L_{e,1}$.

Using this model, we investigate the role of different factors (productivity of domestic and multinational firms, development level, return of training, exogenous prices, elasticity,

credit constraint) on the outcomes in order to understand the optimal strategy of the host country.

First, poor countries with low FDI spillovers cannot invest in a new industry requiring a high entry cost. In this case, all specific labor will be used by MNFs. Second, FDI spillovers can help poor or developing countries to create a new firm. If FDI spillovers are high, these country may set up a new firm without training specific labor. By contrast, if FDI spillovers are not high, these countries must train specific workers in order to create a new industry. Third, once the host country has a sufficient level of specific labor to cover the fixed cost in the new industry, the domestic firm's efficiency is necessary and sufficient to ensure its entry.

The competition between multinational and domestic firms depends on several factors. The key factor is their productivity. However, credit constraints also play an important role. Due to credit constraints and fixed costs, the domestic firm may be eliminated even if its productivity is higher than that of its foreign counterpart.

In [Nguyen-Huu and Pham \(2024\)](#), we study the role of FDI and the host country's optimal strategy in a dynamic setting. Hence, the MNE's maximization problem is:

$$(F_t) : \quad \max_{K_{e,t}, L_{e,t}^D \geq 0} \left[p_n F^e(K_{e,t}, L_{e,t}^D) + p(1 - \delta_e)K_{e,t} - pK_{e,t} - w_t L_{e,t}^D \right] \quad (4.18)$$

where w_t is the wage (in terms of consumption good). We assume that $\delta_e \in (0, 1]$ and $F^e(K, L) = A_e K^{\alpha_e} L^{1-\alpha_e}$, $\forall K, L \geq 0$, where $\alpha_e \in (0, 1)$ and $A_e > 0$.

The maximization problem of the representative agent now becomes:

$$(P) : \quad \max_{(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \quad (4.19)$$

subject to

$$0 \leq K_{c,t}, K_{d,t}, L_{d,t}, L_{e,t}, H_t, \forall t \geq 1 \quad (4.20a)$$

$$c_t + S_{t+1} \leq A_c K_{c,t}^\alpha + p(1 - \delta)K_{c,t} + w_t L_{e,t} + p_n F^d(K_{d,t}, L_{d,t}) + p(1 - \delta_d)K_{d,t}, \forall t \geq 0 \quad (4.20b)$$

$$S_{t+1} = p(K_{c,t+1} + K_{d,t+1}) + H_{t+1}, \forall t \geq 0 \quad (4.20c)$$

$$L_{e,t} \leq A_h H_t^{\alpha_h}, \forall t \geq 1 \quad (4.20d)$$

$$L_{d,t} \leq (A_h H_t^{\alpha_h} - L_{e,t}) + \frac{BA_e}{1 + S_t} L_{e,t}, \forall t \geq 1, \quad (4.20e)$$

where $K_{d,0}, L_{d,0}, K_{c,0}, L_{e,0} \geq 0$ are given, and δ_d represents the capital depreciation rate in the new sector.

We assume that $\delta_d \in (0, 1]$ and the function $F^d : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined by

$$F^d(K, L) = A_d K^{\alpha_d} ((L - \bar{L})^+)^{1-\alpha_d}, \forall K, L \geq 0, \quad (4.21)$$

where $\alpha_d \in (0, 1)$ and $\bar{L} \geq 0$ is a fixed entry-cost.

Then, we study the global properties of the equilibrium. Our theoretical results lead to an interesting implication for a low-income country having high productivity in both old sectors (A_c) and new sectors (A_d). The new sectors are underdeveloped owing to high fixed costs. Hence, this country could apply the following optimal development strategy:

- First, the country attracts FDI and trains specific workers for the new sectors.

- Then, those workers work for MNEs (located in the new sectors) to get high salaries and high-skill knowledge (through learning by doing effects or specific training), consequently improving the country's income.
- Once its income reaches a critical threshold that allows the country to cover the fixed costs, new domestic firms can enter the new industries and eventually dominate the MNE.

Figure 4.1 below displays the domestic and foreign productions in two sectors "Electrical equipment manufacturing" and "Radio, Television, and Communication equipment manufacturing" during the period 2000-2016.

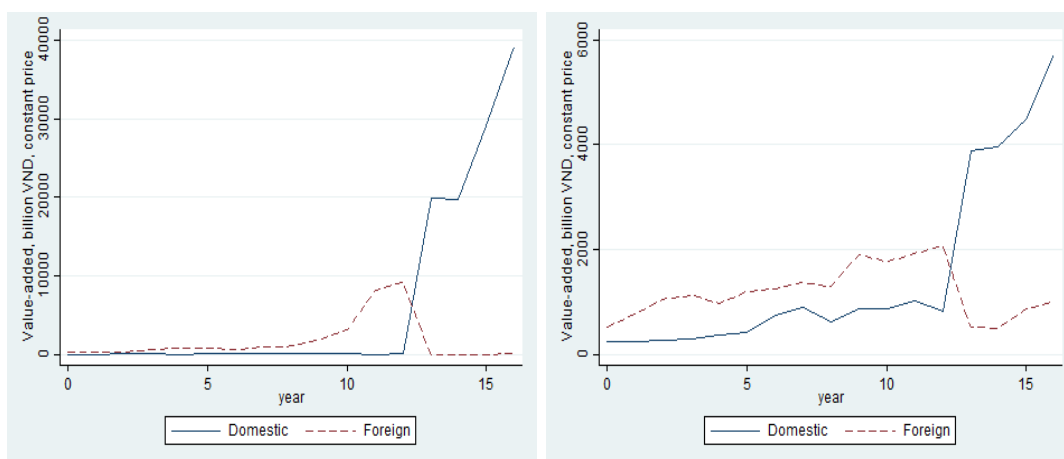


Figure 4.1: Domestic and foreign productions of two sectors during the period 2000-2016. The LHS: Sector "Electrical equipment manufacturing". The RHS: Sector "Radio, Television, and Communication equipment manufacturing".

Our results suggest that whether the host country should (or should not) develop a new industry where MNEs have been well installed depends on different factors, including the setup cost, the productivity of domestic and foreign firms, FDI spillovers, and timing.

Chapter 5

Research program

5.1 Dynamics of economic distributions

In a Solow model (Solow, 1956) with standard assumptions, we can prove the convergence of the capital stock and study its dynamics. This project aims to study a similar question but for the convergence and evolution of the distributions of capital/wealth/income of households and of firms.

Our project has a strong empirical motivation. For instance, a very active debate concerns the slowdown of productivity in many countries (see Goldin, Koutroumpis, Lafond, and Winkler (2021) for an excellent review on the slowdown in productivity growth). By using data in 23 OECD countries over the 2000s, Andrews, Criscuolo and Gal (2015) document a slowdown in aggregate productivity growth, a rising productivity gap between the global frontier and other firms, and that productivity growth at the global frontier remained robust. Bouche, Cette and Lecat (2021) present empirical evidence showing an increase in productivity dispersion between French firms during the period 1991-2016, with a growing productivity gap between frontier and laggard firms.

Summing up, the productivity distribution matters for the aggregate productivity growth.

To address our above question, we need to work in a model with heterogeneous agents (households and firms). A model with a continuum of producers is a promising candidate.

Let us attempt to build a discrete-time model by extending that in Pham (2022).¹ There are a continuum agents with the mass 1. Each agent $i \in [0, 1]$ maximizes her intertemporal utility subject to budget and borrowing constraints

$$\max_{(c_i, k_i, b_i)} \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \quad (5.1a)$$

$$\text{subject to: } c_{i,t} + k_{i,t} + R_t b_{i,t-1} \leq F_{i,t}(k_{i,t-1}) + b_{i,t} \quad (5.1b)$$

$$R_{t+1} b_{i,t} \leq \gamma_i F_{i,t}(k_{i,t}) \quad (5.1c)$$

where we assume that $b_{i,-1} = 0 \forall i$ and denote $w_{i,0} = F_{i,0}(k_{i,-1})$.

Definition 28. *An equilibrium is a list $((c_{i,t}, k_{i,t}, b_{i,t})_i, R_t)_{t \geq 0}$ such that (1) given (R_t) , the allocation $(c_{i,t}, k_{i,t}, b_{i,t})$ is a solution of the above maximization problem, and (2) markets clear: $\int_{i=0}^1 b_{i,t} di = 0$, $\int_{i=0}^1 (c_{i,t} + k_{i,t}) di = \int_{i=0}^1 F_{i,t}(k_{i,t-1}) di$, $\forall t$.*

We mention some avenues of research:

¹There is a growing literature studying the economic distribution by using the continuous time approach and the mean-field game (see (Achdou, Han, Lasry, Lions and Moll, 2022) and references therein).

1. Study the existence of equilibrium. Here, we have to take care of the topologies and the integrals we use in the model.
2. Then, we theoretically study how the distributions of economic variables like income $\mathcal{C}_t \equiv ((c_{i,t})_{i \in [0,1]})_t$ and capital $\mathcal{K}_t \equiv ((k_{i,t})_{i \in [0,1]})_t$ change over time. This is far from trivial. So, it would be valuable to start with a specific utility function (logarithmic, for example) and linear technology.
3. The next step is to construct a model from a quantitative point of view which helps us to run simulations which can fit empirical data.

5.2 Corruption, inequality, and economic development

Corruption is one of the classical topics in economics. The objective of this project is to investigate the interplay between corruption, redistribution and economic development. I plan to work with Le Van Cuong, Pham Thi Kim Cuong and Tran Nam Binh.

Let us start with a simple model. Consider a competitive economy having four agents: two groups of consumers (A, B), a representative firm, and a government. We are interested in the following questions:

1. What happens if the government takes a fraction of the output and uses this amount to do two things: (1) buys the good and services from abroad (in an unofficial way) and (2) give an amount to an agent, say agent B , of the economy?
2. What are the effects of this action on the output and the economic inequality?

We consider a two-period general equilibrium model. Assume that there is a single good at each date. At each date t , the representative firm has the production function $F_t(\cdot)$ and maximizes its profit $(1 - \tau)F_t(K_t) - r_t K_t$ by choosing the capital level K_t . Here r_t is the capital return at date t . The firm takes r_t, τ as given.

Denote π_t the maximum profit of the firm at date t .

The rate $\tau \in [0, 1]$ is set by the government. The government uses $(\tau - \tau_B)F(K_t)$ to buy goods and services from abroad (in an unofficial way) for its consumption and gives $\tau_B F(K_t)$ to the agent B , where $\tau \geq \tau_B \geq 0$. The agent B takes the amount $\tau_B F(K_t)$ as given.

The maximization problem of agent B is

$$\max_{(c_{B0}, c_{B1}, k_{B1})} u_B(c_{B0}) + \beta_B u(c_{B1}) \quad (5.2)$$

$$\text{subject to constraints: } c_{B0} + k_{B1} \leq r_0 k_{B0} + \theta_B \pi_0 + \tau_B F(K_0) \quad (5.3)$$

$$c_{B1} \leq r_1 k_{B1} + \theta_B \pi_1 \quad (5.4)$$

$$c_{B0} \geq 0, c_{B1} \geq 0, k_{B1} \geq 0 \quad (5.5)$$

Here, k_{B0} is the capital of agent A , which is exogenous. $\beta_B \in (0, 1)$ is the rate of time preference of the agent B . $\theta_b \in [0, 1]$ is the profit share of the agent B , π_t is the profit of the firm at date $t = 0, 1$.

The maximization problem of agent A is

$$\max_{(c_{A0}, c_{A1}, k_{A1})} u_A(c_{A0}) + \beta_A u(c_{A1}) \quad (5.6)$$

$$\text{subject to constraints: } c_{A0} + k_{A1} \leq r_0 k_{A0} + \theta_A \pi_0 \quad (5.7)$$

$$c_{A1} \leq r_1 k_{A1} + \theta_A \pi_1 \quad (5.8)$$

$$c_{A0} \geq 0, c_{A1} \geq 0, k_{A1} \geq 0 \quad (5.9)$$

Here, k_{A0} is the capital of agent A , which is exogenous. $\theta_A \in [0, 1]$ is the profit share of the agent A .

Definition 29. Let τ, τ_B be given. A general equilibrium is a list $(c_{A0}, c_{A1}, k_{A1}, c_{B0}, c_{B1}, k_{B1}, K_1, r_1)$ satisfying the following conditions:

1. Given r_1 , the allocations c_{i0}, c_{i1}, k_{i1} is a solution to the maximization problem of agent i , for $i = A, B$.
2. Given r_1 , the allocation K_1 is a solution to the profit maximization problem of the firm at date 1.
3. The markets clear

$$K_t = k_{At} + k_{Bt} \quad (5.10)$$

$$\sum_i c_{i0} + K_1 = (1 - \tau)F_0(K_0) + \tau_B F_0(K_0) \quad (5.11)$$

$$\sum_i c_{i1} = (1 - \tau)F_1(K_1) + \tau_B F_1(K_1). \quad (5.12)$$

Remark 2. The market clearing condition (5.11) suggests that the amount $\tau_B F_0(K_0)$ may improve K_1 and hence the output at the date 1.

For the sake of tractability, we assume that

Assumption 11. $F_t(K) = A_t K, \forall K \geq 0, \forall t = 0, 1$. $u_A(c) = u_B(c) = \ln(c), \forall c$. Assume also that $k_{A0} > 0, k_{B0} > 0$.

In this case, we obtain the following result.

Proposition 23. Denote $K_1^* \equiv K_1(\tau = 0, \tau_B = 0), Y_1^* \equiv Y_1(\tau = 0, \tau_B = 0)$ the aggregate capital and the GDP when $\tau = 0$ and $\tau_B = 0$, i.e., in the standard economy.

1. The capital gap $K_1 - K_1^*$ equals

$$K_1 - K_1^* = A_0 k_{A0} \left(\tau_B \frac{\beta_B}{1 + \beta_B} - \tau \frac{\beta_A}{1 + \beta_A} \right) - \frac{\beta_B}{1 + \beta_B} A_0 k_{B0} (\tau - \tau_B) \quad (5.13)$$

$$K_1 - K_1^* \geq 0 \iff \tau_B k_{A0} \frac{\beta_B}{1 + \beta_B} \geq \tau k_{A0} \frac{\beta_A}{1 + \beta_A} + (\tau - \tau_B) \frac{\beta_B}{1 + \beta_B} A_0 k_{B0}. \quad (5.14)$$

The capital gap $K_1 - K_1^*$ can be negative or positive depending strongly on the saving rate $\frac{\beta_B}{1 + \beta_B}$ of agent B . Moreover, it is increasing in τ_B and decreasing in τ .

2. The GDP gap $Y_1 - Y_1^*$ equals

$$\begin{aligned} Y_1 - Y_1^* = & A_0 k_{A0} \left(\tau_B (1 - \tau + \tau_B) \frac{\beta_B}{1 + \beta_B} - (1 - (1 - \tau)(1 - \tau + \tau_B)) \frac{\beta_A}{1 + \beta_A} \right) \\ & - \frac{\beta_B}{1 + \beta_B} A_0 k_{B0} (1 - (1 - \tau + \tau_B)^2). \end{aligned} \quad (5.15)$$

The GDP gap $Y_1 - Y_1^*$ can be negative or positive.

3. The consumption inequality at date 0 is

$$\frac{c_{A0}}{c_{B0}} = \frac{\frac{(1-\tau)A_0k_{A0}}{1+\beta_A}}{\frac{(1-\tau)A_0k_{B0}+\tau_B A_0(k_{A0}+k_{B0})}{1+\beta_B}} \quad (5.16)$$

The consumption inequality $\frac{c_{A0}}{c_{B0}}$ is decreasing in $\frac{\tau_B}{1-\tau}$. So, it is increasing in τ and τ_B .

Interpretations. Observe that both capital and GDP gap is increasing in τ_B . Moreover, we always have

$$\text{Min}(K_1, K^*) > K_1(\tau_B = 0, \tau > 0) \quad (5.17)$$

$$\text{Min}(Y_1, Y_1^*) > Y_1(\tau_B = 0, \tau > 0). \quad (5.18)$$

It means that when the whole amount $\tau F_0(K_0)$ is taken out of the economy, both the aggregate capital and output decrease.

However, when $\tau_B > 0$, it means that a fraction τ_B of the amount $\tau F_0(K_0)$ is put in the economy via the agent B , then the aggregate capital can be higher or lower than that in the standard economy. Precisely, we get a higher aggregate capital if condition (5.14) is satisfied. This is more likely to hold if the rate τ_B , the agent B 's rate of time preference are high, and the rate τ is low.

Our result leads to two implications:

1. **A pure corruption ($\tau > 0 = \tau_B$) is always worse than the standard situation.**
2. **A corruption with good redistribution and investment ($\tau > \tau_B > 0$) may improve the aggregate output but increase the consumption inequality.**

We now consider an infinite-horizon general equilibrium with corruption. As above, the representative firm maximizes its profit.

$$\pi_t \equiv \text{Max}_{K_t \geq 0} [(1-\tau)F_t(K_t) - r_t K_t]. \quad (5.19)$$

The maximization problem of agent B :

$$\max_{(c_{Bt}, k_{Bt+1})} \sum_t \beta_B^t u_B(c_{Bt}) \quad (5.20)$$

$$\text{subject to constraints: } c_{Bt} + k_{Bt+1} \leq r_t k_{Bt} + \theta_B \pi_t + \tau_B F(K_t) \quad (5.21)$$

$$c_{Bt} \geq 0, k_{Bt+1} \geq 0. \quad (5.22)$$

The maximization problem of agent A :

$$\max_{(c_{At}, k_{At+1})} \sum_t \beta_A^t u_A(c_{At}) \quad (5.23)$$

$$\text{subject to constraints: } c_{At} + k_{At+1} \leq r_t k_{At} + \theta_A \pi_t \quad (5.24)$$

$$c_{At} \geq 0, k_{At+1} \geq 0. \quad (5.25)$$

Definition 30. Let τ, τ_B be given. A general equilibrium is a list of $(c_{At}, k_{At+1}, c_{Bt}, k_{Bt+1}, K_t, r_t)$ satisfying the following conditions:

1. Given the sequence (r_t) , the allocation (c_{it}, k_{it+1}) is a solution to the maximization problem of agent i , for $i = A, B$.
2. Given the sequence (r_t) , the allocation K_t is a solution to the profit maximization problem of the firm at date t .
3. The markets clear

$$K_t = k_{At} + k_{Bt} \quad (5.26)$$

$$\sum_i c_{it} + K_{t+1} = (1 - \tau)F_t(K_t) + \tau_B F_t(K_t). \quad (5.27)$$

In the future, we plan to continue our research in two directions:

1. Extend the above result to an infinite-horizon general equilibrium with corruption and general functions. It would be interesting to study the dynamics of the economy under different regimes.
2. Then, we may endogenize the corruption level τ by introducing an agent who takes care of his interest and the welfare of the population. This agent can impose the rate τ . By studying the notion of political-economic equilibrium as in [Pham and Pham \(2019\)](#), we may understand the interplay between the dynamics of the economy and of the corruption level.
3. We collect data from different countries and do an empirical study.

5.3 FDI and endogenous growth

In [Nguyen-Huu and Pham \(2018\)](#), [Nguyen-Huu and Pham \(2024\)](#), the productivity of firms remain exogenous. With this assumption, the host country can suffer a middle income trap when domestic firms have low productivity. This point leads to a natural question: How a host country could avoid such a middle income trap, and get sustainable growth in the long-run? We address this question by endogenizing the TFP of domestic firms and studying the host country's optimal strategy as well as the role of FDI in an endogenous growth model.

In [Nguyen-Huu and Pham \(2024\)](#), we extend the model in [Nguyen-Huu and Pham \(2024\)](#) by allowing an investment in R&D which in turns creates new technology: If the host country invests N_{t+1} units of the consumption good in R&D at period t , it will obtain bN_{t+1}^σ units of new technology in period $t+1$, where b represents the efficiency of the research process). We assume that $\sigma \in (0, 1)$. The new technologies can improve the old sector's productivity but only if the amount of investment in R&D exceeds a critical threshold such that $bN_{t+1}^\sigma > \bar{x}$, where $\bar{x} > 0$ represents a fixed cost. In this case, the productivity goes up to $A_c + a(bN_{t+1}^\sigma - \bar{x})$ where the parameter a indicates the efficiency or the leverage of the new technology.²

To sum up, the representative agent solves the dynamic growth problem below:

$$(P) : \quad \max_{(c_t, K_{c,t}, H_t, N_t, L_{e,t})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \quad (5.28)$$

²To introduce R&D, we can also write, for example, $A_c + \gamma((N_{t+1} - N^*)^+)^{\sigma}$ instead of $A_c + a(bN_{t+1}^\sigma - \bar{x})^+$. However, the main results have similar insights.

subject to

$$0 \leq c_t, K_{c,t}, H_t, L_{e,t}, N_t \quad (5.29a)$$

$$c_t + pK_{c,t+1} + N_{t+1} + H_{t+1} \leq \left(A_c + a(bN_t^\sigma - \bar{x})^+ \right) K_{c,t}^\alpha + w_t L_{e,t} \quad (5.29b)$$

$$L_{e,t} \leq A_h H_t^{\alpha_h}. \quad (5.29c)$$

for every $t \geq 1$. Here, $\beta \in (0, 1)$ is a rate of time preference while u is the instantaneous utility function.

We require the following assumption.

Assumption 12. *The utility function u is in C^1 , strictly increasing, concave, and $u'(0) = \infty$. Assume that $A_c > 0, A_h > 0, \alpha \in (0, 1), \alpha_h \in (0, 1)$.*

We assume that $a\bar{x} > A_c$, i.e., the fixed cost \bar{x} is not too low.

At initial date, assume that $N_0 = 0$ while $K_{c,0}, L_{e,0} > 0$ are given.

We provide a formal definition of equilibrium.

Definition 31. *An intertemporal equilibrium is a list $(c_t, K_t, H_t, N_t, L_{e,t}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^\infty$ satisfying 3 conditions: (1) Given $(w_t)_{t=0}^\infty$, $(c_t, K_t, H_t, N_t, L_{e,t})_{t=0}^\infty$ is a solution of the problem (P_1) , (2) Given w_t , $(L_{e,t}^D, K_{e,t}^D)$ is a solution of the multinational firm's maximization problem, (3) Labor market clears: $L_{e,t}^D = L_{e,t}$.*

Under this setup, we have provided conditions under which the equilibrium capital path converges to a finite value or to infinity. We have also analyzed the role of different factors (FDI, firms' productivity, fixed cost, elasticity, ...).

This model with specific production functions is quite tractable because it allows us to do comparative statics in an explicit way.

We mention two on-going topics:

1. We are trying to extend our results to the case with general production functions.
2. We are collecting data in order to test our model's insights.

5.4 Wariness

We are living in a world where many things can make people worry. An example is the emergence of the COVID-19 pandemic.³ In this world, some people may worry about the future, prefer to live better now, and enjoy the present. However, others may save more in order to face uncertainties in the future. This behavior concerns the notion of wariness. Motivated by these observations, we aim to investigate the role of wariness in economic growth.

To model the wariness in a simple way, we consider that individuals take care not only of the discounted sum of utilities but also the minimum utility across time (i.e., the worse situation). Formally, when a consumer lives for two periods, we assume that her intertemporal utility is given by

$$\text{Intertemporal utility} = (u(c) + \beta u(d)) + \gamma \min \{u(c), u(d)\}, \quad (5.30)$$

where c, d represent the consumer's consumption in the present and the future respectively, β is the rate of time preference. The parameter $\gamma \in [0, \infty)$ represents the wariness of the household. When $\lambda = 0$, we recover the standard case.

³See Baker, Bloom and Davis (2016) and the website <http://www.policyuncertainty.com> for the economic Policy Uncertainty Index, Jurado, Ludvigson and Ng. (2015) for macroeconomic uncertainty index for 12 month horizon, and Jurado, Ludvigson and Ng. (2015), Ludvigson, Ma and Ng (2021) for financial uncertainty index for 12 month horizon.

5.4.1 Wariness and economic growth

In [Pham and Pham \(2024\)](#), we embed the modeling (5.30) of wariness in a standard overlapping generations model. We then study the role of wariness on the dynamics of capital stocks and the interplay between wariness and economic development.

Definition 32. *Given $\beta > 0, w > 0, R > 0$, we define $s_\beta(w, R)$ the unique solution of the following equation $u'(w - s) = \beta R u'(Rs)$.*

Lemma 9. *Let $w > 0, R > 0$ be given. If $\beta > \beta' > 0$, then $s_\beta(w, R) > s_{\beta'}(w, R)$.*

Denote $\beta_1 \equiv \beta + \gamma, \beta_2 \equiv \frac{\beta}{1+\gamma}$, we obtain the following dynamics

$$nk_{t+1} = s(w_t, R_{t+1}) = \begin{cases} s_{\beta_1}(w_t, R_{t+1}) & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ s_{\beta_2}(w_t, R_{t+1}) & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases}. \quad (5.31)$$

where w_t, R_{t+1} are the wage and capital return.

Under standard assumptions of utility and production functions, we can prove that the capital stock of intertemporal equilibrium, in the exogenous growth economy, converges to a steady state. We have also studied how this steady state depends on the wariness. We argue that whether the effects are positive or negative depend on the structure of the economy, specially the interest rate of the economy without wariness. This insight holds not only for the exogenous growth model but also for the endogenous growth framework à la [Romer \(1986\)](#).

5.4.2 Wariness, altruism, and their macroeconomic impacts

[Bosi, Ha-Huy, Le Van, Pham and Pham \(2018a\)](#), [Bosi, Ha-Huy, Pham and Pham \(2022\)](#) consider two ah doc kinds of altruism. These modelings are tractable. However, a huge literature uses the approach of [Barro \(1974\)](#). An interesting question is to study the interaction between wariness (of parents) and altruism and the dynamics of bequests.

We follow the modeling the bequest motive as in [Barro \(1974\)](#) (see in Section 2.2 in [Michel, Thibault and Vidal \(2006\)](#)). The households at date t take care of their intertemporal utility and the welfare of their children.

$$(P_{c,t}) : \quad V_t = \max_{c_t, d_{t+1}, s_t, x_{t+1} \geq 0} [U(c_t, d_{t+1}) + \delta V_{t+1}]$$

$$c_t + s_t \leq w_t + x_t \quad (5.32)$$

$$d_{t+1} + (1+n)x_{t+1} \leq R_{t+1}s_t \quad (5.33)$$

$$c_t, d_{t+1}, s_t, x_{t+1} \geq 0, \quad (5.34)$$

Assume that $\delta^T V_T \rightarrow 0$. The above problem becomes

$$(P_{c,t}) : \quad V_t = \max_{(c_s, d_{s+1}, s_s, x_{s+1})_{s \geq t}} \sum_s \delta^{s-t} U(c_s, d_{s+1})$$

$$c_s, d_{s+1}, x_{s+1}, s_s \geq 0, \forall s \geq t \quad (5.35a)$$

$$c_s + s_s \leq w_s + x_s \quad (5.35b)$$

$$d_{s+1} + (1+n)x_{s+1} \leq R_{s+1}s_s \quad (5.35c)$$

where $x_t \geq 0$ is given.

The wariness degree γ is described in the utility function.

Assumption 13. Assume that $U(c_t, d_{t+1}) \equiv u(c_t) + \beta u(d_{t+1}) + \gamma \min(u(c_t), u(d_{t+1}))$, where u is strictly concave, strictly increasing.

Given the initial value x_0 and the sequence of prices $(w_t, R_{t+1})_{t \geq 0}$, we consider the problem.

$$(P_{2,t}) : \quad V_0 \equiv V_0(x_0; (w_t, R_{t+1})_{t \geq 0}) \equiv \max_{(c_t, d_{t+1}, x_{t+1})_{t \geq 0}} \sum_t \delta^t U(c_t, d_{t+1})$$

$$c_t, d_{t+1}, x_{t+1} \geq 0, \quad (5.36a)$$

$$R_{t+1}c_t + d_{t+1} + (1+n)x_{t+1} \leq R_{t+1}(w_t + x_t). \quad (5.36b)$$

According to Proposition A.17 in [de la Croix and Michel \(2002\)](#), we have the following result.

Proposition 24. Assume that $w_t > 0, R_{t+1} > 0, \forall t \geq 0$.

Assume that $\sum_t \delta^t U(c_t, d_{t+1})$ is bounded for all feasible paths of consumptions and bequests.

Assume also that there exists a path of consumptions and zero bequests such that $\sum_t \delta^t U(c_t, d_{t+1})$ is finite.

Define

$$L_t = U(c_t, d_{t+1}) + p_{t+1} \frac{\delta}{1+n} \left(R_{t+1}(w_t + x_t - c_t) - d_{t+1} \right) - p_t x_t. \quad (5.37)$$

Under our assumptions above, $(c_t^*, d_{t+1}^*, x_{t+1}^*)_{t \geq 0}$ is a solution to the problem $(P_{2,t})$ if and only if there exists a sequence of so-called shadow prices (p_t) such that

1. For each t , the triple $(c_t^*, d_{t+1}^*, x_t^*)$ is a solution to the problem

$$\max_{(c_t, d_{t+1}, x_t)} U(c_t, d_{t+1}) + p_{t+1} \frac{\delta}{1+n} \left(R_{t+1}(w_t + x_t - c_t) - d_{t+1} \right) - p_t x_t \quad (5.38a)$$

$$c_t, d_{t+1}, x_t \geq 0, \quad (5.38b)$$

$$R_{t+1}c_t + d_{t+1} \leq R_{t+1}(w_t + x_t) \quad (5.38c)$$

2. the transversality condition: $\lim_{t \rightarrow \infty} \delta^t p_t x_t^* = 0$.

Proof. Observe that the function $U(c_t, d_{t+1}) \equiv u(c_t) + \beta u(d_{t+1}) + \gamma \min(u(c_t), u(d_{t+1}))$ is strictly concave if u is strictly concave. We can verify that Assumptions B0, B1, B2, and B3 in Proposition A.17 in [de la Croix and Michel \(2002\)](#) are satisfied. So, we obtain our result. \square

In the future research we expect to understand the behavior of households under altruism and wariness. Several questions will be investigated:

1. Comparative statics at the steady state. How do the capital stock and the output depends on the wariness and altruism levels? Under which conditions the bequests are strictly positive? What is the role of wariness?
2. The convergence of equilibrium. It is interesting but seems to be difficult. We may make use of the approach of supermodular ([Amir, 1996](#)).
3. We may introduce and study education/ human capital in this model to better understand the effects of wariness.

I am working on this project with Pham Hai Ha, Nguyen Thi Do Hanh and Nguyen Thi Tuyet Mai.

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Appendix A

Curriculum Vitae

Full name: Pham Ngoc Sang

Born: December 1985 in Hung Yen (Viet Nam)

Nationalities: French and Vietnamese.

Email addresses: npham@em-normandie.fr (professional) and pns.pham@gmail.com (personal)

Phone: + 33250320408

Address: EM Normandie Business School, 9 Rue Claude Bloch, 14000 Caen, France

Website: <https://phamns.wordpress.com/>

Research interests

- Economic theory (optimal growth, dynamic general equilibrium with heterogeneous agents and incomplete/imperfect markets)
- Financial economics (money theory, asset pricing, bubbles)
- Macroeconomics (DSGE models, growth models, FDI, foreign aid)

A.1 Positions and educations

Academic positions

- 2020–: Professor of Economics, EM Normandie Business School
- 2016–2020: Assistant Professor of Economics, Montpellier Business School
- 2015–2016: Lecturer (ATER), LEM, University of Lille 3
- 2014–2015: Postdoctoral fellow, Labex MME-DII, EPEE, University of Evry
- 2011–2014: Doctoral fellow, Centre d’Economie de la Sorbonne, University of Paris 1

Educations

- 2011–2014: Ph.D. in Economics, University of Paris 1, Paris School of Economics.
Dissertation: "Essays in General Equilibrium with Borrowing Constraints, Optimal Growth, and FDI". The dissertation is proposed by the Committee for an award of the "Chancellerie des Universités de Paris".

Committee: Jean-Marc Bonnisseau, Stefano Bosi, Gael Giraud (Advisor), Cuong Le Van, Filipe Martins-Da-Rocha (Referee), Yiannis Vailakis (Referee).

- 2010–2011: M.S. in Quantitative Finance, University of Paris 1, France
- 2008–2009: M.S. in Mathematics, University of Strasbourg, France
- 2007–2008: Master 1 in Mathematics, HNUE and VAST, Viet Nam
- 2003–2007: B.A. in Mathematics, HNUE, Viet Nam

Grant, awards, and fellowships

- 2018: Research Excellence Award, Montpellier Business School
- 12/2014: Institut Europlace de Finance’s research fund (10000 euros), with Stefano Bosi and Cuong Le Van
- 2014–2015: Postdoctoral Fellowship, Labex MME-DII, EPEE, University of Evry
- 2011–2014: Doctoral Fellowship (Allocation de Recherche), CES, University of Paris 1
- 2008–2009: Master Scholarship, Vietnamese Ministry of Education and Training
- 2003: Third Prize, Vietnamese Mathematical Olympiad

A.2 Teachings and supervisions

2020-present

- Statistics (English and French), EM Normandie Business School

2023-present

- Introduction to Economics (English), EM Normandie Business School

2020-2021

- Alternative and Applied Economics (French), EM Normandie Business School

2016-2020

- Macroeconomics (English and French), Montpellier Business School
- Introduction to economics (English and French), Montpellier Business School

Spring 2016

- Economic policy in the long run (French), 24h, under-graduate, Uni. of Lille 3
- International macroeconomics (French), 24h, under-graduate, Uni. of Lille 3
- The economics of european integration (French), 24h, under-graduate, Uni. of Lille 3
- Macroeconomic Dynamics, 12h (T.A., French), under-graduate, Uni. of Lille 3
- Macroeconomics: intergenerational transfers, 8h (T.A., French), under-graduate, Uni. of Lille 3
- Industrial economics, 12h (T.A., French), under-graduate, Uni. of Lille 3

- Microeconomics (risk and uncertainty), 2×15 h (T.A., French), under-graduate, Uni. of Lille 1

Fall 2015

- Microeconomics (consumer theory), 16h (T.A., French), under-graduate, Uni. of Lille 3
- Microeconomics, 15h (T.A., French), under-graduate, Uni. of Lille 1

Other teaching experiences

- 09/2023: Macroeconomics (growth models and general equilibrium) (English), 24h, pre-master, CASED
- 03/2021: Mathematics for Economists: finite dimensional optimization and discrete time dynamical system (English), 21h, pre-master, CASED
- 08/2016: Microeconomics (English), 21h, pre-master, VCREME
- 06/2015: Microeconomics and general equilibrium theory (English), 31h, pre-master, VCREME
- Fall 2013: Mathematics for economists (Vietnamese), 6h, under-graduate, Café-Séminaire
- Fall 2007: Classical analysis (Vietnamese), 48h, under-graduate, Hanoi University of Industry
- Spring 2007: Teacher in mathematics (Vietnamese), Kien Xuong High School
- Spring 2006: Teacher in mathematics (Vietnamese), Kien Xuong High School

Supervisions

- 2023-present: Nguyen Trong Tin's Ph.D. thesis, "Corruption, efficiency, productivity, and economic growth: the case of developing countries" (Co-Advisor with Pham Thi Kim Cuong), Uni. of Paris Nanterre.
- 2017-present: 5-10 master theses/year (in finance, economics and management), EM Normandie Business School and Montpellier Business School.
- 2016-2017: Pham Thi Hien, "Collateral requirements, capital flows, and welfare: a general equilibrium approach", Paris School of Economics, Master thesis 2017 (Co-Advisor).

A.3 Visits and professional activities

Academic visits

- 06/2017: BETA, University of Strasbourg (one week, invited by P. Nguyen-Van and T.K.C. Pham)
- 10/2015: Department of Economics, Bilkent University (one week, invited by Cagri Saglam)

Professional activities

- **Referee:** Theoretical Economics, Economic Theory, Journal of Mathematical Economics, Journal of Macroeconomics, Journal of Public Economic Theory, Energy Economics, Mathematical Social Sciences, Australian Economic Papers, Economics Bulletin.
- **Membership:** Econometric Society, American Economic Association, ISVE

A.4 Talks

Conference and workshop talks

- 2023: PET 2024 (Lyon), EWET 2024 (Manchester)
- 2023: SAET 2023 (Paris)
- 2022: PET 2022 (Marseille), Workshop Sustainability, Public Action, and Well-Being (Hanoi), VEAM 2012 (Hanoi).
- 2019: The XXVIII European Workshop on Economic Theory EWET 2019 (Berlin), VEAM 2019 (Dalat)
- 2018: 17e Journées Louis-André Gérard-Varet in Public Economics (Aix-en-Provence), the XXVII European Workshop on General Equilibrium Theory EWGET 2018 (Paris), the 2018 Public Economic Theory conference PET 2018 (Hue), the 11th Vietnam Economist Annual Meeting (Hanoi), the 71st European Summer Meeting of the Econometric Society ESEM (Cologne)
- 2017: Workshop "Finance and growth" (Florence), PET 2017 (U. Paris 2), SAET 2017 (Faro), 66th Annual Meeting of the French Economic Association (Nice), Workshop "Economic Growth, Macroeconomic Dynamics and Agents' Heterogeneity" (St. Petersburg)
- 2016: SAET 2016 (IMPA, Rio de Janeiro), PET 2016 (FGV, Rio de Janeiro), VEAM 2016 (Da Nang), European Winter Meeting of the Econometric Society (Edinburgh)
- 2015: UECE Lisbon Meetings in Game Theory and Applications 2015 (Lisbon), Sustainability workshop (Strasbourg), Econometric Society World Congress (Montreal), Association of Public Economic Theory Conference - PET (Luxembourg), Vietnam Economist Annual Meeting - VEAM (Thai Nguyen), ADRES (Paris), Royal Economic Society Conference (Manchester)
- 2014: General Equilibrium Days (York), XXIII EWGET (Paris)
- 2013: XXII EWGET (Vienna), General Equilibrium Days (York), Financial Regulation and Systemic Risk (Paris), Vietnam Economist Annual Meeting (Hue), 13th SAET Society for the Advancement of Economic Theory (Paris)

Seminar talks

- 2022: ISER Macro/Trade Workshop (Osaka University), GATE Lyon Saint-Étienne
- 2018: Seminar EFEC (Uni. of Montpellier)
- 2017: CES-PSE (Uni. of Paris I), BETA (Uni. of Strasbourg)

- 2016: CES-PSE (Uni. of Paris I), Department of Economics (Uni. of Namur), GATE (Uni. of Lyon)
- 2015: CES-PSE (Uni. of Paris I), BETA (Uni. of Strasbourg), Department of Economics (Bilkent University), THEMA (Uni. of Cergy-Pontoise), LEM (Uni. of Lille), GREQAM (Aix-Marseille School of Economics), Department of Banking and Finance (Uni. of Zurich), LEDa (Uni. of Paris Dauphine), GREDEG (Uni. of Nice), Department of Economics (Uni. of Evry)
- 2014: Methods of economic dynamics (Uni. of Paris I-PSE), Department of Economics (Uni. of Evry), PhD student seminar (Uni. of Paris I)
- 2013: Methods of economic dynamics (Uni. of Paris I-PSE)

A.5 Others

- Co-founder of Café-Séminaire

Languages and computer skills

| | |
|-------------------------|--------|
| English | Fluent |
| French | Fluent |
| Vietnamese | Native |
| Matlab, R, Python, SPSS | |
| Latex, Office | |

A.6 Publications

A.6.1 Journal articles

1. Le, T., Le Van, C., Pham, N.-S., Saglam, S. (forthcoming) Sperner's lemma and competitive equilibrium with incomplete financial markets. *Economics Letters*, Volume 238, May 2024, 111720.
2. Nguyen-Huu, T. T., Pham, N.-S. (2024) FDI spillovers, new industry development, and economic growth. *Journal of Public Economic Theory*, Volume 26, Issue 1, February 2024, e12670.
3. Le Van, C., Pham, N.-S., Pham, T.K.C. (2023) Effects of development aid (grants and loans) on the economic dynamics of the recipient country. *Mathematical Social Sciences*, vol. 125, p. 101-112.
4. Pham, N.-S. (2023) Intertemporal equilibrium with physical capital and financial asset: role of dividend taxation, *Mathematical Social Sciences*, vol. 123, p. 95-104.
5. Bosi, S., Ha-Huy, T., Pham, C.-T., Pham, N.-S. (2022) Ascendant altruism and asset price bubbles, *International Journal of Economic Theory* vol. 18 (4), p. 532-551.
6. Le, T., Le Van, C., Pham, N.-S., Saglam, S. (2022) A Direct Proof of the Gale–Nikaido–Debreu Lemma Using Sperner's Lemma. *Journal of Optimization Theory and Applications*, vol. 194, pp. 1072–1080.

7. Bosi, S., Le Van, C, Pham, N.-S. (2022) Real indeterminacy and dynamics of asset price bubbles in general equilibrium. *Journal of Mathematical Economics*, vol. 100, 102651.
8. Pham, N.-S., Pham, H. (2021) Effects of credit limit on efficiency and welfare in a simple general equilibrium model. *International Journal of Economic Theory*, vol. 17, p. 446-470.
9. Le Van, C, Pham, N.-S. (2020) Demand and equilibrium with inferior and Giffen behaviors. *Mathematical Social Sciences*, vol. 103, p. 45-50.
10. Pham, N.-S., Pham, T.K.C. (2020) Effects of foreign aid on the recipient country's economic growth". *Journal of Mathematical Economics*, vol. 86, p. 52-68.
11. Pham, N.-S., Pham, T.K.C. (2019). Foreign aid, recipient government's fiscal behavior, and economic growth. *Economics Bulletin*, vol. 39, issue 4, p. 2457-2466.
12. Bosi, S., Ha-Huy, T., Pham, C.-T., Pham, N.-S. (2019) Rational bubbles in altruistic economies: when Tirole meets Ramsey. *Economics Bulletin*, vol 39, issue 2, p. 816-829.
13. Bosi, S., Le Van, C, Pham, N.-S. (2018) Intertemporal equilibrium with heterogeneous agents, endogenous dividends and borrowing constraints. *Journal of Mathematical Economics*, vol. 76, p. 1-20.
14. Bosi, S., Ha-Huy, T., Le Van, C., Pham, C.-T., Pham, N.-S. (2018) Financial bubbles and capital accumulation in altruistic economies. *Journal of Mathematical Economics*, vol. 75, p. 125-139.
15. Nguyen-Huu, T.T., Pham, N.-S. (2018) Should the host economy invest in a new industry?. *Revue Economique* vol. 69, p. 29-65.
16. Bosi, S., Le Van, C., Pham, N.-S. (2017) Asset bubbles and efficiency in a generalized two-sector model. *Mathematical Social Sciences*, vol. 88, p. 37-48.
17. Bosi, S., Pham, N.-S. (2016) Taxation, bubbles and endogenous growth. *Economics Letters*, vol. 143, p. 73-76.
18. Le Van, C., Pham, N.-S. (2016) "Intertemporal equilibrium with financial asset and physical capital" (with C. Le Van), *Economic Theory*, vol. 62, p. 155-199.
19. Acurio-Vasconez, V., Giraud, G., Mc Isaac, F., Pham, N.-S. (2015) The effects of oil price shocks in a new-Keynesian framework with capital accumulation. *Energy Policy*, vol. 86, p. 844-854.
20. Bonnel, H., Pham, N.-S. (2011) Nonsmooth optimization over the (weakly or properly) Pareto set of a linear-quadratic multi-objective control problem: Explicit optimality conditions". *Journal of Industrial and Management Optimization*, vol. 7, No. 4, p. 789-809.

A.6.2 Book chapters

21. Le Van, C, Pham, N.-S. (2022) "Why Does Productivity Matter?. In Le Van et al. (Eds.), *International Trade, Economic Development, and the Vietnamese Economy, New Frontiers in Regional Science: Asian Perspectives* 61.

22. Pham, N.-S., Pham, T.K.C. (2022) Allocation and Effectiveness of Foreign Aid: an Overview. In Le Van et al. (Eds.), *International Trade, Economic Development, and the Vietnamese Economy, New Frontiers in Regional Science: Asian Perspectives* 61.
23. Le Van, C., Nguyen, T.D.H., Pham, N.-S., Tran-Nam, B. (2018) A General Equilibrium Model in Which Consumption Takes Time. In Tran-Nam et al. (Eds.), *Recent Developments in Normative Trade Theory and Welfare Economics*, Springer-Verlag.
24. Bosi, S., Le Van, C., Pham, N.-S. (2017) Rational land and housing bubbles in infinite-horizon economies. In Nishimura, K., Venditti, A., Yannelis, N. (Eds.), *Sunspots and Non-Linear Dynamics – Essays in honor of Jean-Michel Grandmont, Series Studies in Economic Theory*, Springer-Verlag.

A.7 Lectures

1. Pham, N.-S. (2023) Some Lectures on Macroeconomics.
2. Le Van, C., Pham, N.-S. (2021) Lecture Notes: Mathematics for Economics.

A.8 Working papers

1. Pham, H.-H., Pham, N.-S. Effects of Wariness on Economic Growth in Overlapping Generations Models
2. Le Van, C., Pham, N.-S. Equilibrium with non-convex preferences: some insights.
3. Nguyen-Huu, T. T., Pham, N.-S. The role of FDI along transitional dynamics of the host country in an endogenous growth model.
4. Gourdel, P., Le Van, C., Pham, N.-S., Tran Viet, C. Hartman-Stampacchia theorem, Gale-Nikaidô-Debreu lemma, and Brouwer and Kakutani fixed-point theorem.
5. Pham, N.-S. Impacts of (individual and aggregate) productivity and credit shocks on equilibrium aggregate production.
6. Pham, N.-S. Collateral monetary equilibrium with liquidity constraints in an infinite-horizon economy.
7. Pham, N.-S. Assets with possibly negative dividends.

A.9 Work in progress

1. Pham, N.-S., "The dynamics of economic distributions".
2. Le Van, C., Pham, N.-S., Pham, T.K.C., Binh Tran-Nam, "Corruption, inequality, and economic development".
3. Nguyen, T.D.H., Nguyen, T.T.M., Pham, H.-H., Pham, N.-S., "Wariness, altruism, and their macroeconomic impacts".
4. Nguyen, T.T., Pham, N.-S., Pham, T.K.C., "Environmental quality and foreign aid".