Non-Linear Problems in Interacting Particle Systems

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INSTITUTO SUPERIOR TÉCNICO

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Supervisor : **Doctor** Ana Patrícia Carvalho Gonçalves
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Jury

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Funding Institutions - Fundação para a Ciência e a Tecnologia and Inria Lille

2024

Abstract

We present three works on Interacting Particle Systems where some form of non-linearity arises.

In the context of the description of Non-Equilibrium Steady States (NESS), we formulate a Matrix Product Ansatz in order to describe the NESS of a process defined on the lattice $\Lambda_N = \{1, \dots, N\}$, evolving as the SSEP in $\{2, \dots, N-1\}$ and coupled with two-site reaction-diffusion processes at the boundaries, acting on the pairs of sites $\{1, 2\}$ and $\{N-1, N\}$. These pairs of sites can take four different states each, leading to 12 possible transitions between them. We derive a set of constraints where the underlying quadratic algebra is consistent, which are related with the correlation of the reservoirs, present the representation of the objects in the formulation, and provide examples of rates satisfying these constraints.

In the second work, our focus is in generalizing the Porous Media Model (PMM), associated with the hydrodynamic equation $\partial_t \rho = \partial_u^2 \rho^m$ for $m \in \mathbb{N}_+$, into a universal exclusion family parametrized by $m \in \mathbb{R}$, and, in particular, representing the transition from the *slow* diffusion regime ($m > 1$) to the *fast* diffusion regime ($m < 1$). We successfully treat the case $m \in (0, 2)$, encompassing the phase transition $m = 1$, in this manner deriving the Porous Media Equation in the range $m \in (1, 2)$, and the Fast Diffusion Equation for $m > 0$.

Next, we generalize the PMM in another direction: constructing the diffusion coefficient $D(\rho) = \rho^n (1-\rho)^k$ with $n, k \in \mathbb{N}_+$. The generalized PMM that we introduce is a gradient model that inherits theoretical properties from PMM, as, for instance, the presence of mobile clusters and blocked configurations. The construction of the model is indirect and delicate, in order to maintain the gradient property. We then generalize this into a long-range dynamics inheriting the gradient property. The long-range generalization of the dynamics is simple and can be applied to any exclusion process.

Keywords

Matrix Ansatz, Hydrodynamic Limit, Non-linear Diffusion, Gradient Models, Porous Media Model

Resumo

Apresentamos três trabalhos em Sistemas de Particulas em Interação com natureza não-linear.

No contexto da descrição de *estados estáveis fora do equilíbrio*, reformulamos um Matrix Product Ansatz de modo a descrever um processo definido no reticulado $\Lambda_N = \{1, \dots, N\}$, evoluindo como o SSEP em $\{2, \dots, N\}$ e acoplado com processos de reação-difusão que atuam em dois pares de sítios na fronteira, $\{1, 2\}$ e $\{N-1, N\}$. Esses pares de sítios podem estar em quatro estados cada, num total de 12 possíveis transições entre eles. Derivamos um conjunto de restrições onde a algebra subjacente é consistente, que estão relacionadas com a correlação dos reservatórios, apresentamos a representação dos objetos na formulação, e damos exemplos de taxas que satisfazem essas restrições.

No segundo trabalho o nosso foco é em generalizar o Modelo em Meios Porosos (MMP), associado à equação hidrodinâmica $\partial_t \rho = \partial_u^2 \rho^m$ com $m \in \mathbb{N}_+$, para uma família de processos de exclusão universal, parametrizada por $m \in \mathbb{R}$, e, em particular, representar a transição entre o regime de difusão *lenta* ($m > 1$) e o de difusão *rápida* ($m < 1$). Fomos bem sucedidos para $m \in (0, 2)$, incluindo assim a mudança de fase $m = 1$, e desta forma derivando a Equação em Meios Porosos, para $m \in (1, 2)$, e a Equação de Difusão Rápida para $m > 0$.

Posteriormente, generalizamos o MMP noutra direção, construindo o coeficiente de difusão $D(\rho) = \rho^n (1-\rho)^k$ com $n, k \in \mathbb{N}_+$. O MMP generalizado que introduzimos é um modelo gradiente que herda propriedades teóricas do MMP, como a presença de "mobile clusters" e configurações bloqueadas. A construção do modelo é indireta e delicada, de modo a manter a propriedade gradiente. Generalizamos então esse modelo para uma dinâmica de saltos longos herdando a propriedade gradiente. A generalização para saltos longos é simples e pode ser aplicada a qualquer processo de exclusão.

Palavras Chave

Matrix Ansatz, Limite Hidrodinâmico, Difusão não-linear, Modelos Gradiente, Modelo em Meios Porosos

Para a minha Mãe, o meu Pai e a minha Namorida.

Acknowledgments

I would like to thank my supervisor, Patrícia Gonçalves, for this experience over the last six years, since my MSc. For the mentorship, trust, and interest in my work, but above all, for her dedication to me. I am grateful for having been her student and for learning so much about mathematics and research from her. Her insights and discussions were essential for the development of this manuscript.

I would like to acknowledge my sincere appreciation to my co-supervisor, Marielle Simon, for her support both personally and academically, for her many suggestions in all the works throughout my Ph.D., and specially for dedicating her time. In particular, I am grateful for all the in-person discussions, and I would like to express special thanks for the insights shared regarding the work in Chapter 4.

I want to express my thanks to Gunter Schütz for both meeting and working with him. His collaboration on the work in Chapter 2 was crucial for its development and personally very rewarding. I learned greatly, and his attitude towards research influenced all the subsequent works in this thesis.

A special thanks to the organization of SPSAS 2022, which not only provided an excellent academic experience but also a wonderful personal one, allowing me to reconnect with old friends and visit my home country. With no doubt, it is a memory I will keep.

To the Department of Mathematics, Instituto Superior Técnico, and all of its faculty and staff. It is interesting to look back and realize that I spent a decade here. I would like to tell the once struggling "freshman" that he will endure, teach and pursue a doctorate – he would probably laugh in my face! I am very grateful for having been formed by this department, for having had the opportunity to be a student of founding professors, and for the dedication of the faculty in shaping mathematicians. I extend my gratitude to the Statistics group, also for the opportunity to teach.

Additionally, I want to thank deeply my parents, Simone and Otaviano for their immense support, Sara and Bárbara to which I also extend this gratitude; and finally to Hugo and André for the friendship and many Pokémon games.

As a final acknowledgment, I want to thank the members of the jury for accepting to referee this dissertation, and for their dedicated commitment of time.

I have also been very fortunate regarding funding. I want to thank the *Lismath* program, and consequently, *FCT/Portugal*, for funding my Ph.D. and for the meetings with my co-supervisor. The latter was also made possible thanks to *Programa PESSOA cotutelas*. I also acknowledge funding from Lismath and CAMGSD for attending and presenting my work in multiple national and international conferences. Additionally, I thank *inria Lille* for the funding in the final stages of my Ph.D.

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Introduction

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1.1 Motivation

This thesis is a collection of three works in the context of Interacting Particle Systems, where technical difficulties arise due to various forms of non-linearity. In order to present our results, serving also as a brief introduction, it is convenient to introduce a seminal model, the Symmetric Simple Exclusion Process (often shortened as SSEP), that is related either directly, or indirectly, to all the models that will be introduced in the following chapters.

Let $\mathbb{N}_+$ be the set of positive natural numbers and fix a natural number $N \gg 1$. We denote by $\mathbb{T}_N$ the one dimensional discrete torus, that is, $\mathbb{T}_N = \{1, \dots, N\}$ with the identification $0 \equiv N$. Each element $x \in \mathbb{T}_N$ is called a *site*, and each unordered pair $\{x, y\} \subset \mathbb{T}_N$ is called a *bond*. For a configuration η and a site x, we denote by $\eta(x)$ the *occupation* at site x. For the SSEP and the models that will be studied throughout this thesis, $\eta(x) \in \{0, 1\}$ and we interpret $\eta(x) = 1$ as the site x being occupied by a particle, while $\eta(x) = 0$ as the site x being empty. The property that at each site the occupation is *at most* one is called in the literature as the *exclusion constraint*.

The SSEP, as well as all forthcoming models, are Markov Processes. In simple terms, they are stochastic processes that model a "goldfish memory" – where the probability of observing a future state depends solely on the present state. Formally, to each bond comprised by two neighboring sites, say $\{x, x+1\}$, a Poisson process is associated to it with parameter $\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))$, where for each bond the Poisson processes are independent. These independent Poisson processes are named *Poisson Clocks*, and the probability of two clocks ringing at the same time, with respect to the Lebesgue measure, is zero due to the continuity of the exponential random variable and independence of the clocks. These Poisson processes dictate the stochastic evolution in the following way. Writing η_t for the configuration at time $t > 0$, if a clock at $\{x, x+1\}$ rings at time t and $\eta_t(x) + \eta_t(x+1) = 1$, the occupations at the sites x and $x+1$ exchange ($\eta_{t+}(x) = \eta_t(x+1)$, $\eta_{t+}(x+1) = \eta_t(x)$); otherwise, which is the case when both sites x and $x+1$ are either occupied or empty, nothing happens. In this way, this simple process is symmetric (there is no bias in the direction of the jump – left or right) and satisfies the exclusion constraint. This dynamics is illustrated in Figure 1.1.

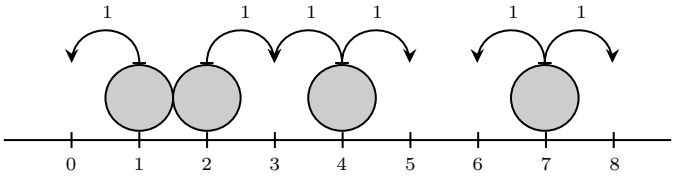


Figure 1.1: SSEP transition rates.

This process was first introduced in [35] as a mathematical model in the field of statistical mechanics, in the 70s, by Spitzer, as a particular case of a more general family of interacting particle systems. Concrete applications of the previously described (and other models) can be easily found in the literature. We refer the reader to [33] for the presentation of the SSEP as a model for the motion of a polymer chain, and applications of its asymmetric version and other lattice gas models.

Since the introduction of the SSEP there has been a large variety of works on this model, and many

other related processes, each exhibiting particular features of interest. The spectrum of models is remarkably diverse, being present in the literature dynamics that span from short to long-range interaction [7, 25], the occupation number may be discrete unbounded (as in the classical Zero Range model introduced in [35] and reviewed, for instance, in [26]), bounded (as in the SSEP(α), where the dynamics is similar to the SSEP but the occupation is constrained to be no larger than a fixed $\alpha \in \mathbb{N}_+$) continuous [15, 18, 37], and one can consider creation and annihilation of particles through Glauber processes or varied types of boundary dynamics [1, 3, 5, 17, 29]. The fundamental question of interest is how critical phenomena arise from the collective behaviour of a large number of interacting agents. A few classical books on the subject related to the work in this manuscript are [26, 36]. The former is focused on an analytical approach based on the differential entropy which describes, besides many other methodologies, the entropy method (introduced in [23]) that we are going to apply in Chapters 3 and 4 of this thesis. The book [36] focuses on the phenomenology of the models and is very physics oriented. We highlight that the work presented in Chapter 4 of this thesis is reminiscent of the exposition in the brief subsection on the gradient condition, [36, Subsection 2.4]. We shall now discuss briefly the scientific context of the results in this thesis. A more formal and detailed presentation will be given as an introduction to each of the subsequent chapters.

1.2 Scientific context and contributions

The work in Chapter 2 lives in the context of describing Non-Equilibrium Steady States (NESS), which are states of a system far from equilibrium but where a stable state is maintained through a constant flow of information. We investigate the applicability of the Matrix Product Ansatz (MPA) on the description of the open boundary SSEP coupled with generalized reservoirs. The MPA was first introduced in the context of Interacting Particle Systems in [11], where the authors provide exact results regarding the steady state of the open boundary, one dimensional Totally Asymmetric Exclusion Process, discuss generalizations for the ASEP (see [4, Section 3] for a complete study) and the case of second-class particles (see also [4, Section 7]). Beyond the previously referenced review article, we also direct the reader to [33, Chapter 1] for an introduction to exactly solvable models.

Avoiding a rigorous mathematical introduction on the subject, the formulation corresponds to a natural generalization of a product measure – the probability of observing any specific configuration in the stationary regime is formulated involving an ordered product of matrices, each corresponding to the occupation at a specific site. Precisely, writing μ_N^{ss} for the stationary measure, the starting point is that there exists a column and a line vector, $|V\rangle$ and $\langle W|$, respectively, and matrices D_i, E_i for $i \in \Lambda_N$, such that

$$\mu_N^{ss}(\eta) = Z_N^{-1} \langle W| \prod_{i=1}^{N} \left[D_i \eta(i) + E_i (1 - \eta(i)) \right] |V\rangle , \tag{1.1}$$

where Z_N is a normalization constant. These matrices, that can be of infinite dimension, belong to an algebra induced by the dynamics. In [28] the authors prove that in one dimension, under nearest-neighbour interactions in the bulk, and reservoirs acting only on the first and the last site this formulation is not an *ansatz* and the stationary state can always be written in that way. For different dynamics, one

must justify that the objects, satisfying the corresponding algebra, do exist. The goal is to study quantities of interest of the system (as the current, correlation functions, *etc* ...), however, a representation of the matrices may not be straightforwardly obtainable, or might be too complex to compute any quantity in a simple way. None of these issues hold for the description of the SSEP defined on the lattice $\Lambda_N \coloneqq \{1, \dots, N\}$ coupled with reservoirs at the boundary (sites 1 and N) that create and annihilate particles. A study is presented in [10, Section 6], and it turns out that the underlying algebra is simple enough to compute physical quantities without resorting to a specific representation (we will be more concrete about this in Chapter 2). These reservoirs are, in some sense, a natural continuation of the bulk dynamics. When the reservoirs act on more than one site at each boundary the picture changes non-trivially. We consider reservoirs acting on two sites at each boundary. At the those sites, reaction-diffusion processes dictate the dynamics, leading to 12 positive transition rates, presented in Figure 1.1.

Left-boundary Process		Rate
Diffusion $1 \to 2$	$\lvert 10 \to \lvert 01$	a_{23}
Diffusion $2 \to 1$	$\lvert 01 \to \lvert 10$	a_{32}
Pair annihilation	$\lvert 11 \to \lvert 00$	a_{14}
Pair creation	$\lvert 00 \to \lvert 11$	a_{41}
Fusion on 1	$\lvert 11 \to \lvert 10$	a_{34}
Fusion on 2	$\lvert 11 \to \lvert 01$	a_{24}
Branching to 1	$\lvert 01 \to \lvert 11$	a_{42}
Branching to 2	$\lvert 10 \to \lvert 11$	a_{43}
Death on 1	$\lvert 10 \to \lvert 00$	a_{13}
Death on 2	$\lvert 01 \to \lvert 00$	a_{12}
Birth on 1	$\lvert 00 \to \lvert 10$	a_{31}
Birth on 2	$\lvert 00 \to \lvert 01$	a_{21}

Right-boundary Process		Rate
Diffusion $N \to N-1$	$01\rvert \to 10\rvert$	b_{23}
Diffusion $N-1 \to N$	$10\rvert \to 01\rvert$	b_{32}
Pair annihilation	$11\rvert \to 00\rvert$	b_{14}
Pair creation	$00\rvert \to 11\rvert$	b_{41}
Fusion on N	$11\rvert \to 01\rvert$	b_{34}
Fusion on $N-1$	$11\rvert \to 10\rvert$	b_{24}
Branching to N	$10\rvert \to 11\rvert$	b_{42}
Branching to $N-1$	$01\rvert \to 11\rvert$	b_{43}
Death on N	$01\rvert \to 00\rvert$	b_{13}
Death on $N-1$	$10\rvert \to 00\rvert$	b_{12}
Birth on N	$00\rvert \to 01\rvert$	b_{31}
Birth on $N-1$	$00\rvert \to 10\rvert$	b_{21}

Figure 1.1: Two-site reaction-diffusion on the bods $\{1, 2\}$ and $\{N-1, N\}$ and their respective rates.

In the previous figure, the second column of each table represents the transition between two local configurations at a specific boundary: the "$\lvert \cdot\cdot$" (resp "$\cdot\cdot \rvert$") represents the left-boundary (resp. right-boundary), and the subsequent binary numbers represent the occupation value of the sites 1 and 2 (resp. $N-1$ and N). For example, $\lvert 11 \to \lvert 10$ corresponds to a fusion into the site 1, that occurs with expected rate $a_{34} \geq 0$.

The presence of these two-site reaction-diffusion processes only at the boundary is a novelty – in [34] the author introduced and studied in detail the dynamics comprised by these processes acting on the whole lattice. Motivated by the same aforementioned work, we classify the boundary dynamics as linear and non-linear, depending on whether the rates satisfy specific equations or not (see Definition 2.2.5 and the discussion just before it). To be concrete, we say that the left-boundary is *linear* if the rates satisfy the equations

$$a_{13} + a_{23} + a_{31} + a_{41} = a_{14} + a_{24} + a_{32} + a_{42},$$

$$a_{21} + a_{32} + a_{41} + a_{12} = a_{14} + a_{23} + a_{34} + a_{43},$$

and we refer to it as *non-linear* otherwise. These conditions arise in the equation for the stationary density, and lead to the cancellation of the non-linear terms, thus closing the equation. It turns out that this notion of linearity is related to the representation of the model, as we shall explain shortly. We also

note that a matrix approach for models where the reservoirs act on a pair of sites (as the bulk dynamics) is a novelty. In [27] the authors provided examples for the "opposite" case, when the local nearest neighbour dynamics depends on the occupation value of one site *more* than the boundary dynamics.

The formulation for the 1−site boundary case (corresponding to the homogeneous choice $D_i = D$ and $E_i = E$ for any i as in (1.1)) case does not encapsulate enough information of the dynamics in order to be applicable to our 2−site case. The reason for this is that it indirectly assumes that the expected current either in a node in the bulk or at the boundary is the same. We consider specific matrices associated with the sites 1 and N and obtain a system of non-linear equations characterizing the *consistency* of the underlying algebra. We analyse this set of constraints, and relate them with the correlation-function of the boundary reservoirs. Our emphasis lies in investigating if the underlying algebra is well-defined, and understanding the resulting constraints. No information about the representation, beyond the invertibility of a specific infinite matrix throughout some parts of our analysis, is assumed. The entries of the aforementioned inverse matrix are provided in terms of a system of recurrence relations, that we could not solve explicitly.

The works in Chapter 3 and 4 are intertwined, although not in an obvious way. We focus now on Chapter 3. A typical question in the field of statistical mechanics is related to the derivation of the macroscopic evolution equations from the stochastic dynamical interaction of microscopic particles. Coincidentally, this is at the heart of the aforementioned "description of the collective behaviour of a large number of agents". There has been a remarkable progress in the derivation of these equations, which are partial differential equations (PDEs), governing the space-time evolution of the conserved quantities of the microscopic system. These equations are referred to as *Hydrodynamic Equations*, obtained as *Hydrodynamic Limits*, *i.e.,* scaling limits in which the system of particles behave as a continuous "fluid" – thus justifying the terminology.

An equation which has received a lot of attention in the last years in the PDE's community is the following equation, that we pose for every $(t, u) \in \mathbb{R}_+ \times \mathbb{T}$ where $\mathbb{T}$ is the one-dimensional torus $[0, 1)$ with $0 \equiv 1$: it is given, for $m \in \mathbb{R}\backslash\{0\}$, by

$$\partial_t \rho = \partial_u^2(\rho^m), \qquad (t, u) \in \mathbb{R}_+ \times \mathbb{T}. \tag{1.2}$$

This is a parabolic equation, with diffusion coefficient given by $D(\rho) = m\rho^{m-1}$. For $m > 1$, (1.2) is the *porous medium equation*, referred to as PME; for $m = 1$ it is the heat equation (HE), while for $m < 1$ it belongs to the class of fast diffusion equations, and in this case we will refer to it as FDE. The rigorous analysis of (1.2) has attracted a lot of interest in the past decades, we refer the reader to [38] for a review on this subject.

From the particle systems' point of view, the rigorous derivation of (1.2) has been successfully achieved for particular values of m, in several different ways. The HE has been obtained as the hydrodynamic limit of the local empirical average of particles evolving according to the SSEP (see, for example, [26, Chapter 4]). In [21], the authors derived the PME, for any integer value of $m \geq 2$, by considering an exclusion process with *degenerate* rates. More precisely, particles evolve on the discrete torus $\mathbb{T}_N$ according to the exclusion rule, but the jump rate depends on the number of particles in the vicinity of the edge

where the jump occurs. To be concrete, if, for example, $m = 2$, then the jump rate from a site x to the site $x+1$ is given by $\eta(x)(1-\eta(x+1))(\eta(x-1)+\eta(x+2))$ and the rate from $x+1$ to x is given by $\eta(x+1)(1-\eta(x))(\eta(x-1)+\eta(x+2))$. This means that for a jump from x to $x+1$ to happen, one imposes to have *at least one* particle in the vicinity $\{x-1, x+2\}$, as in Figure 1.2. We note that this requirement does not characterize completely the rate – as the rate can be positive but taking different values depending on the local configuration. In fact, these particular rates were constructed in a way that makes the process being of *gradient* type, which is something we will explore in detail in Chapter 4.

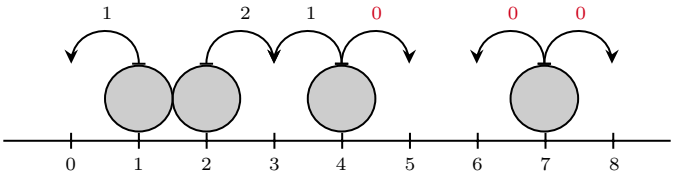


Figure 1.2: PMM transition rates for $m = 2$.

It is simple to compute the microscopic instantaneous current of the system, *i.e.* the difference between the jump rate from x to $x+1$ and the jump rate from $x+1$ to x, which is equal to $(\eta(x)-\eta(x+1))(\eta(x-1)+\eta(x+2))$. This microscopic current can be rewritten as a discrete gradient of some function $h(\eta)$, which in general is referred to in the literature [36, Subsection 2.4] as the *gradient property*. In fact, the choice for those specific rates was made in order to have this property, so that classical methods can be explored without too many complications, see [26, Chapters 5 and 6]. Since particles only swap positions on the torus, the number of particles is conserved by the dynamics. The PME (1.2) with $m = 2$ has then been obtained as the hydrodynamic limit of the empirical density of particles. This rationale was extended to any integer $m \geq 2$, and the resulting microscopic system is now called the *porous medium model*, that we denote by PMM$(m-1)$. Later in [5], the same PME for any integer $m \geq 2$ has been obtained on the interval $[0,1]$, with different types of boundary conditions (Dirichlet, Robin and Neumann), again as the hydrodynamic limit of the same constrained exclusion process, but in contact with stochastic reservoirs, which inject and destroy particles at the two extremities with some rate which is regulated by a parameter, giving rise to the aforementioned boundary conditions.

Another approach had previously been developed in [15, 18, 37]. The porous medium equation, when $m = 2$ was derived in [15, 37] from a model in which the occupation number is a *continuous* variable (therefore belonging to another class of models). More precisely, the model consists of configurations of "sticks" or "energies"; the configurations evolve randomly through exchanges of stick portions between nearest-neighbours through a *zero-range* pressure mechanism, and the conservation law is the total stick-length. Later in [18] the authors extended the derivation of the hydrodynamic limit from the previous model, and obtained the PME for all range $m > 1$.

Concerning the fast diffusion case, few results are available in the literature. In [24] the FDE, with $m = -1$, has been derived as the hydrodynamic limit of a *zero-range process* (the number of particles per site can be any non-negative integer) evolving on the discrete torus, with a jump rate function adjusted to observe frequently a large number of particles, with a specific "weight" associated with each particle. The formalization of the hydrodynamic limit was achieved by using Yau's relative entropy method (introduced

in [39] but also presented in [26]) with some adaptations including spectral gap estimates. The derivation of the FDE for general $m < 1$ was left there as an open problem, that we partially solve for $m \in (0,1)$ in the exclusion context.

Here, we address two questions: first, how can we generalize the family of PMMs, namely exclusion processes, to m not being an integer? Second, due to the different nature of the interacting particle systems constructed to derive (1.2) under the slow-diffusion regime and the fast-diffusion regime, is there a single family parameterized by m that interpolates between the slow and the fast diffusion? The first question regards the construction of a non-integer power starting from a finite, discrete scheme; while the second allow illustrating a phase transition between the fast and slow diffusion regimes. We give some answers in the direction of the first question, and a positive answer regarding the second. To be precise, the construction of our model is an application of the Generalized Binomial Theorem. As a consequence, the resulting family of models interpolates continuously in m between the SSEP and the PMM (for $m = 2$), while also going further into the fast-diffusion regime in the so-called *good exponent range* [38], that is $m \in (0,1)$, coinciding with the range where the potential ρ^m is finite at all times.

We now explain the contents of Chapter 4. In this chapter we focus only on the construction of models, and our goal is again twofold. First, we want to have a better understanding on how to construct *gradient* models systematically; next, we want to understand how to generalize a nearest-neighbour dynamics into a *long-jumps* dynamics, while maintaining the gradient property of the model. Regarding the first goal, we aimed at generalizing the PMM into a family of two-parameter, gradient models. Concretely, for each $n, k \in \mathbb{N}_+$ we will construct a model referred to as PMM(n,k), that generalizes the PMM(n) in the sense that PMM$(n,0)$=PMM(n), while also maintaining relevant properties of the one-parameter PMM family. In particular, the corresponding hydrodynamic equation for the density is

$$\partial_t \rho = \partial_u (D_{n,k}(\rho) \partial_u \rho) \quad \text{with} \quad D_{n,k}(\rho) = \rho^n (1-\rho)^k.$$

The choice of the PMM family as a starting point was not made without deliberate consideration: as already mentioned, in Chapter 3 we extend the PMM into a continuous family for the parameter in the range $(0,2)$, but an extension for $(2,+\infty)$ requires the definition of a model associated with the diffusion coefficient $D_{n,k}(\rho)$; moreover, we want to understand "how far one can go with gradient models", or to be precise, our main goal is to construct a family of constraints analogue to the Bernstein polynomial basis, which would allow deriving the differential equation $\partial_t \rho = \partial_u^2 H(\rho)$ for a large class of functions $H : \mathbb{T} \to \mathbb{R}$. With this in mind, we introduce in the literature a simple toy model for non-linear, not monotonic diffusion, and with relevant theoretical properties. We conclude the presentation of the model with a discussion of the proof of the hydrodynamic limit for the empirical measure.

Regarding the second goal, long-jumps dynamics have been used as a basis for deriving fractional operators [3, 7, 25]. However, it is not clear how one can start from a nearest-neighbour (n.n) dynamics and generalize it into a long-range (l.r.) in a "robust" way. To our knowledge the long-range porous media model introduced in [7] is one of the few examples of a l.r. exclusion dynamics in the IPS literature where the rate depends on a local configuration around the bond where an exchange is to be performed, and such that the l.r. dynamics is related to the n.n. one. There are two technical issues with the procedure

used in [7], when applied to other models: (1) it does not necessarily inherits the gradient property from its n.n. counterpart; (2) the l.r. rates, when restricted to jumps of length one, do not *always* coincide with the original n.n. rates. The first one evidences a gap in the literature regarding a "robust" extension of a n.n. dynamics; while the second is more of "philosophical" nature, as we want to *generalize* the dynamics. We note that (2) was already discussed in [7], being the case for the l.r. version of the PMM(1) but not the case for any other PMM(n) with $2 \leq n \in \mathbb{N}_+$. Regarding (1), it is indeed the case when extending the PMM(n, k).

In this manner, we present in Section 4.3 a *very simple* map to generalize any symmetric, exclusion n.n. dynamics where the l.r. version inherits the gradient property from the n.n. one, while also interpolating the n.n. dynamics in the sense previously explained. This map is a consequence of the simple realization that the gradient property should be lattice-invariant, and for that reason a l.r. dynamics can be seen as a n.n. dynamics in a stretched lattice. To conclude, we define the fractional Laplacian on the torus and analyze the correct scaling of its microscopic "version", which elucidates the discontinuity of the time-scale with respect to γ already known in the literature. We do not present the proof of the hydrodynamic limit, and focus only on the construction of models.

In Chapter 5, we gather the conclusions from the previously presented works, and discuss both future projects and those currently in development.

2

Matrix Product Ansatz for the SSEP with extended boundary.

Contents

2.1 Introduction

2.1.1 Statement of the results and general strategy

We formulate a matrix-product state representation of the stationary probability vector of the open-boundary SSEP, coupled with reservoirs that act on two sites at each boundary. Fixed $N \in \mathbb{N}_+$, we consider the discrete lattice $\Lambda_N = \{1, \dots, N\}$ and the state space $\Omega_N = \{0,1\}^{\Lambda_N}$. A configuration of particles in the system will be denoted by $\eta \in \Omega_N$, and $\eta(x) \in \{0,1\}$ will denote the occupation of the site $x \in \Lambda_N$. We consider a Markov process, that we write as $\{\eta_t\}_{t\geq 0}$, evolving according to the SSEP in the *bulk* $\{2, \dots, N-1\}$, and evolving at the boundary sites $\{1,2\}$ and $\{N-1, N\}$ with no constraints beyond the exclusion, in the sense that to each local configuration at $\{1,2\}$ (resp. $\{N-1,N\}$) there is a positive rate associated with the transition from one local configuration to any other local configuration at the same boundary. This dynamics was illustrated in Figure 1.1 and explained in the discussion just after it, in Chapter 1, yet we recall it for convenience in the next figure.

Left-boundary Process		Rate
Diffusion $1 \to 2$	$\|10 \to \|01$	a_{23}
Diffusion $2 \to 1$	$\|01 \to \|10$	a_{32}
Pair annihilation	$\|11 \to \|00$	a_{14}
Pair creation	$\|00 \to \|11$	a_{41}
Fusion on 1	$\|11 \to \|10$	a_{34}
Fusion on 2	$\|11 \to \|01$	a_{24}
Branching to 1	$\|01 \to \|11$	a_{42}
Branching to 2	$\|10 \to \|11$	a_{43}
Death on 1	$\|10 \to \|00$	a_{13}
Death on 2	$\|01 \to \|00$	a_{12}
Birth on 1	$\|00 \to \|10$	a_{31}
Birth on 2	$\|00 \to \|01$	a_{21}

Figure 2.1: Left-boundary dynamics.

Right-boundary Process		Rate
Diffusion $N \to N-1$	$01\| \to 10\|$	b_{23}
Diffusion $N-1 \to N$	$10\| \to 01\|$	b_{32}
Pair annihilation	$11\| \to 00\|$	b_{14}
Pair creation	$00\| \to 11\|$	b_{41}
Fusion on N	$11\| \to 01\|$	b_{34}
Fusion on $N-1$	$11\| \to 10\|$	b_{24}
Branching to N	$10\| \to 11\|$	b_{42}
Branching to $N-1$	$01\| \to 11\|$	b_{43}
Death on N	$01\| \to 00\|$	b_{13}
Death on $N-1$	$10\| \to 00\|$	b_{12}
Birth on N	$00\| \to 01\|$	b_{31}
Birth on $N-1$	$00\| \to 10\|$	b_{21}

Figure 2.2: Right-boundary dynamics

Figure 2.3: Two-site reaction-diffusion on the pairs of sites $\{1,2\}$ and $\{N-1,N\}$ and their rates.

Above, the notation "$|\cdot\cdot$" represents a particular configuration at the left-boundary, while "$\cdot\cdot|$" represents a particular configuration at the right-boundary. The third column on each of the two tables above corresponds to the transition rate between two specific local configurations.

We show that the standard approach imposes heavy constraints in the boundary parameters, that can be relaxed with the following formulation of the stationary measure, that we write as μ_N^{ss},

$$\mu_N^{ss}(\eta) = \frac{1}{Z_N} \langle W | \left(D^- \eta(1) + E^-(1-\eta(1))\right) \times \\ \times \prod_{x=2}^{N-1} \left[D\eta(x) + E(1-\eta(x))\right] \times \\ \times \left(D^+ \eta(N) + E^+(1-\eta(N))\right) |V\rangle, \quad (2.1)$$

and where $D, E, D^{\pm}, E^{\pm}$ are matrices and $\langle W|, |V\rangle$ vectors. The original approach [11] can be recovered by assuming that $D^{\pm} = D$ and $E^{\pm} = E$. These matrices belong to a particular algebra encoding the dynamics. The reason for considering specific boundary matrices is that the expected current flowing in $\{1,2\}$ or $\{N-1,N\}$ is different than the one flowing in a node $\{x, x+1\} \subset \{2, \dots, N-1\}$, in the bulk. This

is not encapsulated into the algebra induced by the process with Glauber dynamics acting only on the sites 1 and N, as in [11]. The relations involving $D, E, D^{\pm}$ and $E^{\pm}$ arise from the strategy to guarantee that the measure μ_N^{ss} is stationary. More concretely, under some assumptions on the rates, that we write for future reference as

$$H_0 \coloneqq (H_0^-) \vee (H_0^+), \tag{H_0}$$

and that can be found just after (2.17), where (H_0^-) (resp. (H_0^+)) corresponds to left-boundary (resp. right-boundary) constraints on the boundary parameters, we show that one obtains the *bulk relation* $[D, E] = D + E$, the *left-boundary algebra*

$$\langle W| D^- C = \langle W| d_1^- C^- C + d_2^- C^- + d_3^- D^-, \tag{R_1^-}$$

$$\langle W| C^- D = \langle W| q_1^- C^- C + q_2^- C^- + q_3^- D^-, \tag{R_2^-}$$

$$\langle W| D^- D = \langle W| f_1^- C^- C + f_2^- C^- + f_3^- D^-, \tag{R_{12}^-}$$

and the *right-boundary algebra*

$$\begin{aligned} CD^+ |V\rangle &= d_1^+ CC^+ - d_2^+ C^+ - d_3^+ D^+ |V\rangle, \\ DC^+ |V\rangle &= q_1^+ CC^+ - q_2^+ C^+ - q_3^+ D^+ |V\rangle, \\ DD^+ |V\rangle &= f_1^+ CC^+ - f_2^+ C^+ - f_3^+ D^+ |V\rangle, \end{aligned}$$

where $C = D + E$ and $C^{\pm} = D^{\pm} + E^{\pm}$. The coefficients above, $d_i^{\pm}, q_i^{\pm}, f_i^{\pm}$ with $i = 1, 2, 3$, depend in a non-linear way on the boundary rates and can be found in Appendix A.1. For the reader to have a more concrete picture in mind, the assumption (H_0) leads to the possibility of expressing the expectation value of products of occupation variables in terms of the bulk-current of the system defined on smaller latices, that is, $\{\langle j(\eta)\rangle_i\}_{i=4,\dots,N}$, with $j(\eta)$ being the current and $\langle\cdot\rangle_i$ the expectation with respect to μ_i^{ss}. For more details, see (2.17) and Appendix A.3, specifically (A.4).

It is not clear if this boundary algebra is well-defined. Precisely, let us focus on the left-boundary algebra, that we write as $\mathbb{A}^- = \{(R_1^-), (R_2^-), (R_{12}^-)\}$. Introducing the vectors $\langle W_0| = \langle W| E^-$ and $\langle W_1| = \langle W| D^-$ we see that, fixing a representation for the matrices D and E satisfying the bulk relation, one obtains three left-boundary relations to be solved for two vectors, and identical for the right-boundary. We show that there exists a manifold for the rates where the algebra is free from "contradictions", that is, when computing any probability weight by applying any sequence of identities always leads to the same result. We refer to this "contradiction free" property of the algebra as the algebra being *consistent*. Determining this set is the cornerstone of this work.

We are interested in the out-of-equilibrium behavior, and for that reason one needs to introduce an additional set of technical constraints,

$$\mathcal{T} = \{H_0,\ q_1^- \neq q_1^+\},$$

corresponding to the manifold where the parameters satisfy the aforementioned hypothesis (H_0) and also such that the coefficients q_1^- and q_1^+ are *not* equal. The latter corresponds to the density of the

reservoirs at the sites 2 and $N-1$ not being equal, and guarantees that the system is out-of-equilibrium (see Appendix A.2). In this thesis we investigate only the cases where $\mathcal{T}$ is satisfied.

Regarding the consistency of the algebra, we focus again on the left-boundary only. We show in Proposition 2.3.4 that it is possible to derive the identity

$$\begin{aligned}\langle W|\, D^- DC = \langle W|\, \big[C^- C^2 s_0 + C^- C s_1 + C^- s_2 + D^- s_3\big] \\ + \langle W|\, D^- D d_3^-\end{aligned} \tag{2.3}$$

from the identities (R_1^-),(R_2^-) and (R_0) – that is, without invoking (R_{12}^-) – and where the coefficients s_i, for $i = 0,1,2,3$, are quite long and are presented in (2.20). With this information, we show in Lemma 2.3.3 that if the set of constraints

$$\begin{aligned} f_1^- &= d_1^- q_1^-, \\ f_2^- + d_1^- f_3^- &= q_1^-(d_2^- + d_1^- d_3^-) + d_1^-(q_2^- + d_1^- q_3^-), \\ (q_1^- d_3^- + q_2^- + d_1^- q_3^- - f_3^-)(d_2^- + d_1^- d_3^-) &= -d_2^-, \\ q_3^-(d_2^- + d_1^- d_3^-) &= -d_3^- \end{aligned} \tag{C_1}$$

or

$$\begin{aligned} f_1^- &= d_1^- q_1^-, \\ f_2^- + d_1^- f_3^- &= d_1^-(q_2^- + d_1^- q_3^-), \\ 0 &= d_2^- + d_1^- d_3^- \end{aligned} \tag{C_2}$$

are satisfied, then (R_{12}^-) satisfies the identity (2.3). This leads to the content of Proposition 2.3.4, that can be summarized as follows: under (C_2), the subalgebra $\mathbb{A}^-\backslash\{(R_{12}^-)\}$ generates $\mathbb{A}^-$; and under (C_1), if $C - \mathbf{1}d_3^-$ is right-invertible, one can "drop" the identity (R_{12}^-), or, in other words, one can derive (R_{12}^-) from the (enlarged) algebra composed by (R_1^-),(R_2^-) and the bulk relation (R_0) with $C - \mathbf{1}d_3^-$ right-invertible. Analogous results hold for the right-boundary, but where one should now ask for $C + \mathbf{1}d_3^+$ to be *left*-invertible. This result allow one to "exchange" a boundary relation, per boundary, by the invertibility of the aforementioned objects, in this way leading to two boundary relations per boundary. While we do not present an explicit representation, this is discussed in Appendix A.5.

The analysis just presented does not directly imply that the formulation (2.1) is valid, as one should prove that, in general, $Z_N \neq 0$. In this vein, we show that the normalization constant is defined by the system of recurrence relations

$$\begin{pmatrix} (q_1^- - q_1^+) & 0 & 0 \\ -d_1^- & 1 & 0 \\ -d_1^+ & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_N \\ \langle \eta(1)\rangle_N^w \\ \langle \eta(N)\rangle_N^w \end{pmatrix} = \begin{pmatrix} N-3-(q_2^- + q_2^+) & -q_3^- & -q_3^+ \\ d_2^- & d_3^- & 0 \\ -d_2^+ & 0 & -d_3^+ \end{pmatrix} \begin{pmatrix} Z_{N-1} \\ \langle \eta(1)\rangle_{N-1}^w \\ \langle \eta(N-1)\rangle_{N-1}^w \end{pmatrix},$$

where $\langle \eta(i)\rangle_k^w := Z_k \langle \eta(i)\rangle_k$, for each $k \in \mathbb{N}_+$. This leads to the need of fixing the initial conditions of the recurrence relations in the previous display. These initial conditions involve the inner product of the boundary vectors, and these issues are solved in Subsection 2.3.5.

Next, we aim for a better understanding of the consistency conditions (C_1) and (C_2), as they are expressed in a very complex way involving the boundary rates. It turns out that the set of models satisfying them can be split into families with very particular properties (see Definition 2.4.2). The

models can be linear (family $\mathfrak{L}$); non-linear (family $\mathfrak{N}$); the stationary measure factorizes at the boundary (family $\mathfrak{F}$); or the choice $D^{\pm} = D$ and $E^{\pm} = E$, the original MPA formulation, is sufficient (family $\mathfrak{H}$). In Proposition 2.5.2 we also perform a link between these constraints and the behaviour of the correlation of the boundary reservoirs. Precisely, short-writing $\varphi_N(1,2) = \langle \eta(1)\eta(2)\rangle_N - \langle \eta(1)\rangle_N \langle \eta(2)\rangle_N$ and $j_2(\eta) = \eta(2) - \eta(3)$, we show that for *linear* models the constraints are equivalent to

$$\varphi_N(1,2) = \langle \eta(1) - d_1^- \rangle_N \langle j_2(\eta)\rangle_N + o(1/N^2) = O(1/N^2),$$

while for *non-linear* models they are equivalent to the previous two equalities plus

$$\lim_{N\to+\infty} \frac{\langle -j_2(\eta)\rangle_N}{\langle \eta(2) - q_1^-\rangle_N} = 1.$$

As a final note, we use our formulation to compute the macroscopic density, which turns out to be a discontinuous function if $d_1^{\pm} \neq q_1^{\pm}$:

$$\rho(u) = \begin{cases} d_1^-, & u = 0, \\ q_1^-(1-u) + q_1^+ u, & u \in (0,1), \\ d_1^+, & u = 1. \end{cases}$$

We present a non-exhaustive set of models satisfying (C_1) and (C_2) in Appendix A.4.

2.1.2 Outline of the chapter

The present work is organized as follows. Section 2.2 is devoted to introducing the SSEP with generalized boundary (that we refer to as SSEP$(2,2)$) and presenting the classical MPA approach to compute the stationary probability vector for the particular case of the classical 1−site boundary open SSEP [11] (that we refer to as SSEP$(1,1)$); particularly, in Subsection 2.2.1 we split the boundary rates of the SSEP$(2,2)$ into two classes – one where the corresponding models are termed *linear*, and another one where they are termed *non-linear*; in Section 2.3, we adapt the MPA approach to the SSEP$(2,2)$ and investigate under what conditions this yields a well-posed problem. The latter requires several steps: in Subsection 2.3.2 we manipulate conveniently the algebra induced by the MPA by performing a specific change of basis that allow deriving, in a simple manner, in Subsection 2.3.3, the maximal set of constraints acting on the boundary rates under which the representation problem has a solution. Some technical aspects arise, and in Subsection 2.3.4 we derive a system of recurrence relations involving the boundary density, correlation and normalizing factor; and in Subsection 2.3.5 we compute the initial conditions of the aforementioned system of recurrence relations for both linear or non-linear models.

Next, we completely characterize the set of constraints and discuss a matrix representation for the algebra: in Section 2.4 we split further the set of constraints into four classes, which we term: *non-linear ($\mathfrak{N}$), linear ($\mathfrak{L}$), factorizable ($\mathfrak{F}$) and homogenous ($\mathfrak{H}$).* Each of the last three classes are the object of study in Subsections 2.4.1, 2.4.2 and 2.4.3, respectively, where we see that for *non-linear* models a representation is not guaranteed from the existence of representations for the SSEP$(1,1)$. In Appendix A.5 we then provide a tridiagonal matrix representation of the objects D and E satisfying the relation $[D,E] = D + E$, present the entries of the resulting boundary vectors and discuss the invertibility of $C \pm d_3^{\pm}$.

We conclude the chapter with an alternative characterization of the set of constraints. Section 2.5 is devoted to computing the correlation between the two left-boundary reservoirs using the induced algebra and then using solely the stationary equations. Naturally, it is not possible to completely compute the correlation function directly from the stationary equations, but rewriting these equations conveniently and comparing with the expressions derived from the algebra allow us to characterize the set of constraints in terms of the behaviour of the correlation between the left-boundary reservoirs.

In the appendix is where we present the coefficients of the boundary algebra (Appendix A.1), concentrate long but straightforward computations (Appendix A.3), present the equilibrium case, where the stationary probability vector follows a (product) Bernoulli distribution (Appendix A.2) and, in Appendix A.4, a non-exhaustive set of models satisfying the aforementioned constraints.

2.2 Model

We consider the open SSEP defined on $\{2,\dots,N-1\}$ coupled with two-site reaction-diffusion processes acting on the pairs of sites $1,2$ and $N-1,N$. There is a total of 12 rates per boundary, each associated with a particular configuration at the corresponding boundary. In order to present a precise definition we need to introduce some notation. Let $\Lambda_N = \{1,\dots,N\}$ be the extended lattice, denote by $\Omega_N = \{0,1\}^{\Lambda_N}$ the state space of the aforementioned process. We shall recurrently denote a configuration by the greek letters $\xi,\eta \in \Omega_N$ and the sites by the latin letters $x,y,z \in \Lambda_N$. Furthermore, let $\eta(x) \in \{0,1\}$ be the occupation of a site $x \in \Lambda_N$.

Definition 2.2.1 (Exchange of occupation variables)**.** For any $x,y,z \in \Lambda_N$ let us consider the exchange of occupation variables $\eta \mapsto \eta^{x,y}$ given by

$$\eta^{x,y}(z) = \mathbf{1}_{z=y}\,\eta(x) + \mathbf{1}_{z=x}\,\eta(y) + \mathbf{1}_{z\neq x,y}\,\eta(z),$$

and for any set $A \subseteq \Lambda_N$ the flip $\eta \mapsto \eta^A$ given by

$$\eta^A(x) = \mathbf{1}_{x\in A}(1-\eta(x)) + \mathbf{1}_{x\notin A}\eta(x).$$

We can now introduce the model that we are going to study in this chapter.

Definition 2.2.2 (SSEP with extended boundary)**.** Introduce the infinitesimal generator

$$\mathcal{L} = \mathcal{L}_- + \mathcal{L}_0 + \mathcal{L}_+, \tag{2.4}$$

where $\mathcal{L}_0$ corresponds to the bulk dynamics and $\mathcal{L}_-$, $\mathcal{L}_+$ corresponds to the left and right boundary dynamics, respectively. Specifically,

$$\mathcal{L}_- = \mathcal{L}_1 + \mathcal{L}_{1,2} + \mathcal{L}_2, \quad \text{and} \quad \mathcal{L}_+ = \mathcal{L}_{N-1} + \mathcal{L}_{N-1,N} + \mathcal{L}_N,$$

with the subscripts in the right-hand side corresponding to the sites where the operator "acts". We can express each generator, acting on functions $f : \Omega_N \to \mathbb{R}$, as

$$(\mathcal{L}_0 f)(\eta) = \sum_{x=2}^{N-2} w_{x,x+1}(\eta)\left(f(\eta^{x,x+1}) - f(\eta)\right),$$

$$(\mathcal{L}_x f)(\eta) = w'_x(\eta)\left(f(\eta^{\{x\}}) - f(\eta)\right), \qquad \text{for} \quad x \in \{1,2,N-1,N\},$$
$$(\mathcal{L}_{x,y} f)(\eta) = w'_{x,y}(\eta)\left(f(\eta^{\{x,y\}}) - f(\eta)\right), \qquad \text{for} \quad (x,y) \in \{(1,2),(N-1,N)\},$$

where, given some configuration η, the occupation at site x flips with rate $w'_x(\eta)$; the occupation of both sites x, y flip with rate $w'_{x,y}(\eta)$; and the hopping in the bulk occurs with rate $w_{x,x+1}(\eta)$. These rates are further expressed as

$$\begin{pmatrix} w'_1(\eta) \\ w'_2(\eta) \\ w'_{12}(\eta) \end{pmatrix} = \begin{pmatrix} a_{31} & a_{42} & a_{13} & a_{24} \\ a_{21} & a_{12} & a_{43} & a_{34} \\ a_{41} & a_{32} & a_{23} & a_{14} \end{pmatrix} \begin{pmatrix} (1-\eta(1))(1-\eta(2)) \\ (1-\eta(1))\eta(2) \\ \eta(1)(1-\eta(2)) \\ \eta(1)\eta(2) \end{pmatrix}, \qquad a_{ij} \geq 0,$$

for the right boundary

$$\begin{pmatrix} w'_N(\eta) \\ w'_{N-1}(\eta) \\ w'_{N-2,N-1}(\eta) \end{pmatrix} = \begin{pmatrix} b_{31} & b_{42} & b_{13} & b_{24} \\ b_{21} & b_{12} & b_{43} & b_{34} \\ b_{41} & b_{32} & b_{23} & b_{14} \end{pmatrix} \begin{pmatrix} (1-\eta(N-1))(1-\eta(N)) \\ \eta(N-1)(1-\eta(N)) \\ (1-\eta(N-1))\eta(N) \\ \eta(N-1)\eta(N) \end{pmatrix}, \qquad b_{ij} \geq 0,$$

and for the bulk,

$$w_{x,x+1}(\eta) = (1-\eta(x))\eta(x+1) + \eta(x)(1-\eta(x+1)), \qquad x \in \{2, N-2\}.$$

Example 2.2.3 (SSEP$(1,1)$)**.** The 1–site reservoir case corresponds to the particular choice of rates

$$\begin{pmatrix} a_{31} & a_{42} & a_{13} & a_{24} \\ a_{21} & a_{12} & a_{43} & a_{34} \\ a_{41} & a_{32} & a_{23} & a_{14} \end{pmatrix} = \begin{pmatrix} a_{31} & a_{31} & a_{13} & a_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} b_{31} & b_{42} & b_{13} & b_{24} \\ b_{21} & b_{12} & b_{43} & b_{34} \\ b_{41} & b_{32} & b_{23} & b_{14} \end{pmatrix} = \begin{pmatrix} b_{31} & b_{31} & b_{13} & b_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

In other words, for the left-boundary, if the site 1 is empty, creation occurs at rate a_{31} independently of the occupation at site 2; if the site 1 is full, annihilation occurs at rate a_{13} also independently of the occupation of the site 2; the hopping dynamics takes place on any pair $\{x, x+1\}$ with $x \in \{1, N-1\}$. For the right-boundary the dynamics is analogous, with a_{ij} replaced by b_{ij} and the site 1 replaced by the site N. Regarding the description of the stationary states of the SSEP with 1–reservoir per boundary, it is straightforward to check that if both reservoirs are fixed at the same density, that is, if

$$\rho = \frac{a_{31}}{a_{13} + a_{31}} = \frac{b_{31}}{b_{13} + b_{31}}, \tag{2.5}$$

then the Bernoulli product measure parameterized by this density

$$\nu_\rho^N(\eta) = \prod_{x=1}^{N} \left[\rho\eta(x) + (1-\rho)(1-\eta(x))\right] \tag{2.6}$$

is invariant. In general, the stationary weights have the following factorization in one dimension

$$\mu_N^{ss}(\eta) = \frac{1}{\langle W| (D+E)^N |V\rangle} \langle W| \prod_{x=1}^{N} \left[D\eta(x) + E(1-\eta(x))\right] |V\rangle, \tag{2.7}$$

with the objects D, E being specific matrices satisfying some algebraic relations, not necessarily finite nor commuting, $\langle W|$ and $|V\rangle$ vectors and transposed vectors, respectively, and $\langle W| (D+E)^N |V\rangle =: Z_N$ a normalizing factor. The approach to see this can be succinctly summarized with the tensor product formalism, now very present in the literature. Under an appropriate basis, the vector of stationary probabilities equals

$$|P\rangle = \frac{1}{Z_N} \begin{pmatrix} \langle W| E \\ \langle W| D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes N-2} \otimes \begin{pmatrix} E|V\rangle \\ D|V\rangle \end{pmatrix}.$$

The stationary condition can be written in vector form, $H|P\rangle = 0$, where H corresponds to the generator written in matrix form. To be precise, $H = H_- + H_0 + H_+$ where

$$H_- = B_1 \otimes \mathbf{1}^{\otimes N-1}, \quad H_0 = \sum_{x=1}^{N} \mathbf{1}^{\otimes x-1} \otimes M \otimes \mathbf{1}^{\otimes L-(x+1)}, \quad H_+ = \mathbf{1}^{\otimes N-1} \otimes B_N,$$

and the intensity matrices are, with respect to the one and two sites ordered basis $\{E, D\}$ and $\{EE, ED, DE, DD\}$, equal to

$$B_1 = \begin{pmatrix} -a_{31} & a_{13} \\ a_{31} & -a_{13} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_N = \begin{pmatrix} -b_{31} & b_{13} \\ b_{31} & -b_{13} \end{pmatrix}.$$

Moreover, $\mathbf{1}$ is the identity matrix, with the convention $a^{\otimes 0} = \mathbf{1}$. The stationary condition can be recast as

$$\begin{aligned} 0 = &\left[B_1 \begin{pmatrix} \langle W|E \\ \langle W|D \end{pmatrix} \right] \otimes \begin{pmatrix} E|V\rangle \\ D|V\rangle \end{pmatrix}^{\otimes N-1} \\ &+ \sum_{x=1}^{N} \langle W| \begin{pmatrix} \langle W|E \\ \langle W|D \end{pmatrix}^{\otimes x-1} \otimes \left[M \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right] \otimes \begin{pmatrix} E|V\rangle \\ D|V\rangle \end{pmatrix}^{\otimes L-(x+1)} \\ &+ \begin{pmatrix} \langle W|E \\ \langle W|D \end{pmatrix}^{\otimes N-1} \otimes \left[B_N \begin{pmatrix} E|V\rangle \\ D|V\rangle \end{pmatrix} \right] \end{aligned}$$

and the algebraic relations for the matrices D, E arise by assuming the existence of auxiliary matrices X_1, X_2 leading to the telescopic relations

$$\begin{aligned} M \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} &= \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \\ B_1 \begin{pmatrix} \langle W|E \\ \langle W|D \end{pmatrix} &= \begin{pmatrix} \langle W|X_1 \\ \langle W|X_2 \end{pmatrix} \\ B_N \begin{pmatrix} E|V\rangle \\ D|V\rangle \end{pmatrix} &= -\begin{pmatrix} E|V\rangle \\ D|V\rangle \end{pmatrix} \end{aligned} \tag{2.8}$$

which guarantee stationarity. For $X_1 + X_2 = 0$ with $X_2 = \kappa\mathbf{1}$ and $\kappa \in \mathbb{R}$, this leads to the quadratic algebra

$$\begin{aligned} [D, E] &= \kappa(D + E), \\ \langle W| (a_{31}E - a_{13}D) &= \kappa \langle W| \mathbf{1}, \\ (b_{31}E - b_{13}D)|V\rangle &= -\kappa\mathbf{1}|V\rangle. \end{aligned}$$

It can be checked that the choice $\kappa = 0$ corresponds to the equilibrium constraint (2.5) in order for this algebra to be consistent. This allows for a constant choice $\rho^- = D = 1 - E$, which leads to (2.7) being identified with the Bernoulli product measure. If $\kappa \neq 0$ then it follows that one needs to prove that matrices and vectors satisfying these relations exist.

Remark 2.2.4 (about the values of κ)**.** Note that one can fix, without loss of generalization, $\kappa = 1$, by considering the parameterized matrices $D_\kappa = D/\kappa$ and $E_\kappa = E/\kappa$ which then satisfy $[D_\kappa, E_\kappa] = D_\kappa + E_\kappa$.

Using the quadratic algebra, it is possible to express any probability weight as a combination of the current over systems of equal of smaller size, $\{J_i\}_{i=0,\dots,N}$, where

$$J_N = \frac{Z_{N-1}}{Z_N}.$$

The current can be computed exactly. To that end, it is convenient to introduce the matrix $C = D + E$, allowing to rewrite the algebra as

$$\langle W| D = \langle W| \left(\frac{a_{31}}{a_{13}+a_{31}} C - \frac{\mathbf{1}}{a_{13}+a_{31}} \right), \quad [D,C] = C, \quad D|V\rangle = \left(\frac{b_{31}}{b_{13}+b_{31}} C + \frac{\mathbf{1}}{b_{13}+b_{31}} \right) |V\rangle . \tag{2.9}$$

The boundary densities can then be read from the algebra

$$\langle \eta(1) \rangle_N = \frac{a_{31}}{a_{13}+a_{31}} - \frac{1}{a_{13}+a_{31}} J_N, \qquad \langle \eta(N) \rangle_N = \frac{b_{31}}{b_{13}+b_{31}} - \frac{b_{31}}{b_{13}+b_{31}} J_N.$$

The bulk relation implies

$$DC^n = C^n D + nC,$$

which can be used to compute the normalization

$$Z_N \left(\frac{a_{31}}{a_{13}+a_{31}} - \frac{b_{31}}{b_{13}+b_{31}} \right) = Z_{N-1} \left(N - 1 + \frac{1}{a_{13}+a_{31}} + \frac{1}{b_{13}+b_{31}} \right).$$

Since the normalization follows a first order recurrence relation one can fix without loss of generalization, the inner product $\langle W|V\rangle = 1$.

The existence of bulk matrices D, E satisfying the bulk relation $[D,E] = D + E$ is well known [11]. Fixing a representation, the boundary relations allows one to solve system of linear recurrence relations for the entries of the boundary vectors $\langle W|$ and $|V\rangle$. It is worth mentioning that only infinite representations for these objects are known.

2.2.1 SSEP with generalized boundary

It turns out that the process given in (2.4) is in equilibrium when the densities of the reservoirs at site 2 and $N-1$ are equal plus some minor constraints on the rates (see Appendix A.2, specifically (A.2) for more details). In this case, the Bernoulli product measure parametrized by this density is invariant. Contrarily to the SSEP$(1,1)$, the stationary density cannot be computed directly from taking expectations with respect to the stationary measure. We focus on the left-boundary. To see this introduce

$$\begin{aligned}
A_1 &= a_{31} + a_{41} \\
B_1 &= a_{32} + a_{42} - (a_{31} + a_{41}) \\
C_1 &= a_{13} + a_{23} + a_{31} + a_{41} \\
D_1 &= C_1 - a_{32} - a_{42} - a_{14} - a_{24} \\
A_2 &= a_{21} + a_{41} \\
B_2 &= a_{23} + a_{43} - (a_{21} + a_{41}) \\
C_2 &= a_{12} + a_{32} + a_{21} + a_{41} \\
D_2 &= C_2 - a_{23} - a_{43} - a_{14} - a_{34}.
\end{aligned} \tag{2.10}$$

One finds that

$$\mathcal{L}(\eta(x)) = \begin{cases} A_1 + B_1\eta(2) - C_1\eta(1) + D_1\eta(1)\eta(2), & x = 1, \\ A_2 + B_2\eta(1) - C_2\eta(2) + D_2\eta(1)\eta(2) + (\eta(3) - \eta(2)), & x = 2. \end{cases} \tag{2.11}$$

Definition 2.2.5 (Linearity)**.** We call the models *left-linear* if $D_1 = D_2 = 0$, that is,

$$a_{13} + a_{23} + a_{31} + a_{41} = a_{14} + a_{24} + a_{32} + a_{42}$$
$$a_{21} + a_{32} + a_{41} + a_{12} = a_{14} + a_{23} + a_{34} + a_{43}$$

and *right-linear* if the equalities above hold with a_{ij} replaced by b_{ij}. If both the left and right boundaries are linear, we call the model *linear*.

Only for linear models the system $\{\langle \mathcal{L}\eta(x)\rangle_N = 0\}_{x=1,N}$ is sufficient to compute $\langle \eta(x)\rangle_N$. For non-linear models, computing the action of the generator on the correlation term $\eta(1)\eta(2)$ provides some insight on the macroscopic behaviour of the interacting reservoirs. Introduce

$$B_3 = a_{42} - a_{41},$$
$$C_3 = a_{41} - a_{43},$$
$$D_3 = a_{41} - (a_{14} + a_{24} + a_{34} + a_{42} + a_{43}).$$

It follows that

$$\mathcal{L}(\eta(1)\eta(2)) = \eta(1)\eta(2)D_3 - \eta(1)C_3 + \eta(2)B_3 + a_{41} - \eta(1)j_2(\eta)$$

where we defined $j_2(\eta) = \eta(2) - \eta(3)$. For $x, y = \{1, \dots, N\}$ with $x \neq y$ consider the 2–point correlation function

$$\varphi_N(x,y) := \langle \eta(x)\eta(y)\rangle_N - \langle \eta(x)\rangle_N \langle \eta(y)\rangle_N$$

and for $f : \Lambda_N \times \Lambda_N \to \mathbb{R}$ introduce the forward difference operator acting on the second variable,

$$(\nabla_2^+ f)(x,y) = f(x,y+1) - f(x,y).$$

We compute the correlation between the left-boundary reservoirs in Appendix A.3 (and perform an analysis in Section 2.5), where we see that this quantity can be rewritten as

$$\begin{aligned}
\varphi_N(1,2) = {} & (f_1^- - d_1^- q_1^-) \\
& + \langle j_2(\eta)\rangle_N \, (f_2^- + d_1^- f_3^- - q_1^-(d_2^- + d_1^- d_3^-) - d_1^-(q_2^- + d_1^- q_3^-)) \\
& + \langle j_2(\eta)\rangle_N^2 (d_2^- + d_1^- d_3^-) \left(\frac{f_3^- - q_1^- d_3^- - d_1^- q_3^-}{1 - d_3^- \langle j_2(\eta)\rangle_N} - q_2^- - d_1^- q_3^- \right) \\
& + \langle j_2(\eta)\rangle_N^3 \frac{d_2^- + d_1^- d_3^-}{1 - d_3^- \langle j_2(\eta)\rangle_N} \, (q_3^-(d_2^- + d_1^- d_3^-) + d_2^-(q_2^- + d_1^- q_3^-)) \\
& + \langle j_2(\eta)\rangle_N^4 d_3^- q_3^- (d_2^- + d_1^- d_3^-)^2 \frac{1}{(1 - d_3^- \langle j_2(\eta)\rangle_N)^2} \\
& - \frac{(\nabla_2^+ \varphi_N)(1,2)}{1 - d_3^- \langle j_2(\eta)\rangle_N} \, (f_3^- - q_1^- d_3^- - d_1^- q_3^-) \\
& - \frac{(\nabla_2^+ \varphi_N)(1,2)\langle j_2(\eta)\rangle_N}{1 - d_3^- \langle j_2(\eta)\rangle_N} \, (q_3^-(d_2^- + d_1^- d_3^-) + d_3^-(q_2^- + d_1^- q_3^-)) \\
& - \frac{(\nabla_2^+ \varphi_N)(1,2)\langle j_2(\eta)\rangle_N^2}{(1 - d_3^- \langle j_2(\eta)\rangle_N)^2} (d_2^- + d_1^- d_3^-) 2 \left(1 + \frac{d_3^- q_3^-}{1 - d_3^- \langle j_2(\eta)\rangle_N} \right) \\
& + \frac{((\nabla_2^+ \varphi_N)(1,2))^2}{(1 - d_3^- \langle j_2(\eta)\rangle_N)^2} d_3^- q_3^-
\end{aligned} \tag{2.12}$$

where the coefficients are defined through

$$\begin{pmatrix} d_1^- & d_2^- & d_3^- \\ q_1^- & q_2^- & q_3^- \\ f_1^- & f_2^- & f_3^- \end{pmatrix} := \begin{pmatrix} C_1 & -B_1 & -D_1 \\ -B_2 & C_2 & -D_2 \\ C_3 & -B_3 & -D_3 \end{pmatrix}^{-1} \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & -1 & 0 \\ a_{41} & 0 & 1 \end{pmatrix} \tag{2.13}$$

assuming invertibility of the first matrix on the right-hand side of the previous display. Analyzing each term above, we see that different models may be characterized by boundary correlations of different order. For example, if $f_1 = d_1 q_1$, then it can be shown that $\lim_{N\to+\infty} N\varphi(1,2) < \infty$. The equation $f_1 = d_1 q_1$ defines a particular class of models, as for arbitrary rates we can have $f_1 \neq d_1 q_1$. We elaborate on this in Section 2.5.

2.3 Well-posedness of the problem

2.3.1 Matrix Product formulation

Approaching the description of the stationary measure through the Matrix Product Ansatz introduces a series of technicalities that require an adjustment of the initial formulation (2.7). Following the same procedure as in Subsection 2.2.3, it can be checked that the equality

$$\langle W| [D, E] = \langle W| (D + E)$$

is valid only under heavy constraints on the boundary rates (this is further investigated in Subsection 2.4.3). The reason for this is that the current inside the reservoirs can be different from the bulk current. To circumvent this problem, opposed to (2.7), we consider matrices $D^{\pm}, E^{\pm}$ specific to the boundaries and make the ansatz

$$\mu_N^{ss}(\eta) = Z_N^{-1} \langle W| (D^-\eta(1) + E^-(1-\eta(1))) \prod_{i=2}^{N-1} [D\eta(i) + E(1-\eta(i))] (D^+\eta(N) + E^+(1-\eta(N))) |V\rangle,$$

now leading to the normalization $Z_N = \langle W| C^- C^{N-2} C^+ |V\rangle$, and where $C^{\pm} := D^{\pm} + E^{\pm} \neq C$. With this formulation, the probability vector is given by

$$|P\rangle = \frac{1}{Z_N} \begin{pmatrix} \langle W| E^- \\ \langle W| D^- \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes N-2} \otimes \begin{pmatrix} E^+ |V\rangle \\ D^+ |V\rangle \end{pmatrix}$$

and stationary condition can be cast in vector form as

$$\left(B^- \otimes \mathbf{1}^{\otimes N-2} + \sum_{x=2}^{N-2} \mathbf{1}^{\otimes x-1} \otimes M \otimes \mathbf{1}^{\otimes L-(x+1)} + \mathbf{1}^{\otimes N-2} \otimes B^+ \right) |P\rangle = 0,$$

where the bulk intensity matrix M is as in (2.8), while the boundary intensity matrices B_- and B_+ are expressed as

$$B^- = \begin{pmatrix} -(a_{21}+a_{31}+a_{41}) & a_{12} & a_{13} & a_{14} \\ a_{21} & -(a_{12}+a_{32}+a_{42}) & a_{23} & a_{24} \\ a_{31} & a_{32} & -(a_{13}+a_{23}+a_{43}) & a_{34} \\ a_{41} & a_{42} & a_{43} & -(a_{14}+a_{24}+a_{34}) \end{pmatrix},$$

$$B^+ = \begin{pmatrix} -(b_{31}+b_{21}+b_{41}) & b_{13} & b_{12} & b_{14} \\ b_{31} & -(b_{13}+b_{23}+b_{43}) & b_{32} & b_{34} \\ b_{21} & b_{23} & -(b_{12}+b_{32}+b_{42}) & b_{24} \\ b_{41} & b_{43} & b_{42} & -(b_{14}+b_{24}+b_{34}) \end{pmatrix},$$

with respect to the basis:

- left-boundary (B^-): $\{E^-E, E^-D, D^-E, D^-D\}$;
- right-boundary (B^+): $\{EE^+, ED^+, DE^+, DD^+\}$;
- bulk (M): $\{EE, ED, DE, DD\}$.

The telescopic relations, which guarantee stationarity, become

$$\begin{aligned} M\begin{pmatrix}E\\D\end{pmatrix}\otimes\begin{pmatrix}E\\D\end{pmatrix} &= \begin{pmatrix}E\\D\end{pmatrix}\otimes\begin{pmatrix}X_1\\X_2\end{pmatrix} - \begin{pmatrix}X_1\\X_2\end{pmatrix}\otimes\begin{pmatrix}E\\D\end{pmatrix}, \\ B^-\begin{pmatrix}\langle W|E^-\\\langle W|D^-\end{pmatrix}\otimes\begin{pmatrix}E\\D\end{pmatrix} &= \begin{pmatrix}\langle W|E^-\\\langle W|D^-\end{pmatrix}\otimes\begin{pmatrix}X_1\\X_2\end{pmatrix}, \\ B^+\begin{pmatrix}E\\D\end{pmatrix}\otimes\begin{pmatrix}E^+|V\rangle\\D^+|V\rangle\end{pmatrix} &= -\begin{pmatrix}X_1\\X_2\end{pmatrix}\otimes\begin{pmatrix}E^+|V\rangle\\D^+|V\rangle\end{pmatrix}. \end{aligned}$$

Because the bulk dynamics is still the SSEP, we fix $X_1 + X_2 = \kappa\mathbf{1}$ and let $\kappa = 1$, which guarantees that $\langle j_2(\eta)\rangle_N = O(1/N)$ and allow us to compute, in principle, any quantity using only the algebra. We end up with the following boundary relations.

$$\begin{aligned} &\begin{pmatrix} -(a_{21}+a_{31}+a_{41}) & a_{12} & a_{13} & a_{14} & 1 & 0 \\ a_{21} & -(a_{12}+a_{32}+a_{42}) & a_{23} & a_{24} & -1 & 0 \\ a_{31} & a_{32} & -(a_{13}+a_{23}+a_{43}) & a_{34} & 0 & 1 \\ a_{41} & a_{42} & a_{43} & -(a_{14}+a_{24}+a_{34}) & 0 & -1 \end{pmatrix}\begin{pmatrix}\langle W|E^-E\\\langle W|E^-D\\\langle W|D^-E\\\langle W|D^-D\\\langle W|E^-\\\langle W|D^-\end{pmatrix} = 0 \\ &\begin{pmatrix} -(b_{31}+b_{21}+b_{41}) & b_{13} & b_{12} & b_{14} & 1 & 0 \\ b_{31} & -(b_{13}+b_{23}+b_{43}) & b_{32} & b_{34} & 0 & 1 \\ b_{21} & b_{23} & -(b_{12}+b_{32}+b_{42}) & b_{24} & -1 & 0 \\ b_{41} & b_{43} & b_{42} & -(b_{14}+b_{24}+b_{34}) & 0 & -1 \end{pmatrix}\begin{pmatrix}EE^+|V\rangle\\ED^+|V\rangle\\DE^+|V\rangle\\DD^+|V\rangle\\-E^+|V\rangle\\-D^+|V\rangle\end{pmatrix} = 0. \end{aligned} \tag{2.14}$$

These boundary relations impose another difficulty. We first realize that since the sum of each column vanishes, one line must be linearly dependent of the remaining, thus we have three boundary rules per boundary. Introduce the boundary vectors

$$\begin{aligned} \langle W_0| &\coloneqq \langle W|E^-, \qquad |V_0\rangle \coloneqq E^+|V\rangle, \\ \langle W_1| &\coloneqq \langle W|D^-, \qquad |V_1\rangle \coloneqq D^+|V\rangle. \end{aligned} \tag{2.15}$$

Fixing a representation for D, E satisfying the bulk relation $[D, E] = D + E$, one still needs to fix the boundary vectors. Since there are two vectors and three equations per boundary, it is not clear if the representation problem is well-posed. On the following sections we show that there exists a set of constraints where the algebra "makes sense", and further characterize this set both microscopically and through the macroscopic behaviour of the reservoirs.

Remark 2.3.1. Note that we did not index the right-boundary parameters according to their matrix entries in B^+, as opposed to the left-boundary. Instead, we *associated* the left-boundary dynamics with the right-boundary dynamics. For example, a_{12} corresponds to the annihilation of a particle in the second site with respect to the left-boundary (site 2), when the first site (site 1) is empty ($|\, ED \mapsto |\, EE$); while b_{12} corresponds to the annihilation at the second site with respect to the right-boundary (site $N-1$), when the first site (site N) is empty ($DE\,|\mapsto EE\,|$).

This will be convenient in order to work with only the left-boundary, then deduce the results to the right-boundary by simply replacing a_{ij} by b_{ij} and κ by $-\kappa$.

2.3.2 Change of basis

We rewrite the boundary algebra analogously to (2.9). This allow us identify the density of each reservoir and their correlation in terms of the current. This facilitates substantially the study of the consistency of the boundary algebra. Recall (2.14), and denote the left-boundary matrix of coefficients there with its fourth line removed by $\mathcal{B}^-$, that is,

$$\mathcal{B}^- = \begin{pmatrix} -(a_{21}+a_{31}+a_{41}) & a_{12} & a_{13} & a_{14} & 1 & 0 \\ a_{21} & -(a_{12}+a_{32}+a_{42}) & a_{23} & a_{24} & -1 & 0 \\ a_{31} & a_{32} & -(a_{13}+a_{23}+a_{43}) & a_{34} & 0 & 1 \end{pmatrix}.$$

It holds that

$$\begin{pmatrix} E^-E \\ E^-D \\ D^-E \\ D^-D \\ E^- \\ D^- \end{pmatrix} = \mathcal{V}^- \begin{pmatrix} C^-C \\ D^-C \\ [D,E]^- \\ D^-D \\ C^- \\ D^- \end{pmatrix} \quad \text{with} \quad \mathcal{V}^- = \begin{pmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where we defined $[D,E]^- := D^-E - E^-D$; and rewriting

$$\mathcal{B}^-\mathcal{V}^- \begin{pmatrix} \langle W|C^-C \\ \langle W|D^-C \\ \langle W|[D,E]^- \\ \langle W|D^-D \\ \langle W|C^- \\ \langle W|D^- \end{pmatrix} = 0 \Leftrightarrow \mathcal{A}_1^- \begin{pmatrix} \langle W|D^-C \\ \langle W|[D,E]^-1 \\ \langle W|D^-D \end{pmatrix} = \mathcal{A}_2^- \begin{pmatrix} \langle W|C^-C \\ \langle W|C^- \\ \langle W|D^- \end{pmatrix}, \tag{2.16}$$

with

$$\mathcal{A}_1^- = \begin{pmatrix} a_{12}+a_{13}+2(a_{21}+a_{31}+a_{41}) & -a_{12}-a_{21}-a_{31}-a_{41} & -a_{12}-a_{13}+a_{14}-a_{21}-a_{31}-a_{41} \\ -a_{12}-2a_{21}+a_{23}-a_{32}-a_{42} & a_{12}+a_{21}+a_{32}+a_{42} & a_{12}+a_{21}-a_{23}+a_{24}+a_{32}+a_{42} \\ -a_{13}-a_{23}-2a_{31}+a_{32}-a_{43} & a_{31}-a_{32} & a_{13}+a_{23}+a_{31}-a_{32}+a_{34}+a_{43} \end{pmatrix} \tag{2.17}$$

$$\mathcal{A}_2^- = -\begin{pmatrix} -a_{21}-a_{31}-a_{41} & 1 & -1 \\ a_{21} & -1 & 1 \\ a_{31} & 0 & 1 \end{pmatrix},$$

Assuming that the rates are such that

$$\det(\mathcal{A}_1^-) \neq 0 \tag{H_0^-}$$

then $\mathcal{A}_1^-$ can be inverted. In particular, under (H_0) we introduce

$$\mathcal{A}^- := (\mathcal{A}_1^-)^{-1}\mathcal{A}_2^- = \begin{pmatrix} d_1^- & d_2^- & d_3^- \\ t_1^- & t_2^- & t_3^- \\ f_1^- & f_2^- & f_3^- \end{pmatrix}$$

where the entries of $\mathcal{A}^-$ are presented in Appendix A.1 and coincide with the coefficients in (2.13). One then arrives at the following three boundary relations

$$\begin{aligned} \langle W|D^-C &= \langle W|d_1^-C^-C + d_2^-C^- + d_3^-D^- \\ \langle W|[D,E]^- &= \langle W|t_1^-C^-C + t_2^-C^- + t_3^-D^- \end{aligned} \tag{2.18}$$

$$\langle W|\, D^- D = \langle W|\, f_1^- C^- C + f_2^- C^- + f_3^- D^-.$$

Note that we can extract the "density at the second site" from the equality $CD^- = D^- C - [D,E]^-$. Introducing for $i \in \{1,2,3\}$ the coefficient

$$q_i^- := d_i^- - t_i^- \tag{2.19}$$

we fix once and for all our left boundary algebra under the assumption (H_0^-) as

$$\langle W|\, D^- C = \langle W|\, d_1^- C^- C + d_2^- C^- + d_3^- D^- \tag{R_1^-}$$

$$\langle W|\, C^- D = \langle W|\, q_1^- C^- C + q_2^- C^- + q_3^- D^- \tag{R_2^-}$$

$$\langle W|\, D^- D = \langle W|\, f_1^- C^- C + f_2^- C^- + f_3^- D^-. \tag{R_{12}^-}$$

Moreover, we refer to the bulk relation by (R_0),

$$[D,C] = C, \tag{R_0}$$

and refer to the algebra composed by the left-boundary and bulk relations as

$$\mathbb{A}^- = \{(R_1^-),(R_2^-),(R_{12}^-)\} \cup \{(R_0)\}.$$

Analogous computations for the right-boundary yield

$$CD^+\,|V\rangle = d_1^+ CC^+ - d_2^+ C^+ - d_3^+ D^+\,|V\rangle \tag{R_1^+}$$

$$DC^+\,|V\rangle = q_1^+ CC^+ - q_2^+ C^+ - q_3^+ D^+\,|V\rangle \tag{R_2^+}$$

$$DD^+\,|V\rangle = f_1^+ CC^+ - f_2^+ C^+ - f_3^+ D^+\,|V\rangle \tag{R_{12}^+}$$

under the hypothesis

$$\det(\mathcal{A}_1^+) \neq 0, \tag{H_0^+}$$

where for $i = 1,2,3$ the coefficients d_i^+, q_i^+, f_i^+ are obtained by replacing $\{a_{ij}\}_{1\le i,j\le 4,\ i\neq j}$ by $\{b_{ij}\}_{1\le i,j\le 4,\ i\neq j}$ in d_i^-, q_i^-, f_i^-, respectively; and $\mathcal{A}_1^+$ obtained from $\mathcal{A}_1^-$ by the same substitution. We refer to the algebra composed by the right-boundary and bulk relations as $\mathbb{A}^+ = \{(R_1^+),(R_2^+),(R_{12}^+)\} \cup \{(R_0)\}$, and the complete algebra as

$$\mathbb{A} = \mathbb{A}^- \cup \mathbb{A}^+.$$

2.3.3 Consistency conditions

The idea underlying idea to obtain the set of constraints for the rates is that the expectation $\langle \eta(1)\eta(2)\rangle_N$ may take *formally* two values: by either computing $\langle \eta(1)\eta(2)\rangle_N$ using the rules (R_1^-),(R_2^-) and (R_0), or using (R_1^-),(R_2^-),(R_{12}^-) and (R_0). Since these values must match we extract, under (H_0^-), the two following set of constraints for $\kappa \neq 0$:

$$\begin{aligned} f_1^- &= d_1^- q_1^-, \\ f_2^- + d_1^- f_3^- &= q_1^-(d_2^- + d_1^- d_3^-) + d_1^-(q_2^- + d_1^- q_3^-), \\ (q_1^- d_3^- + q_2^- + d_1^- q_3^- - f_3^-)(d_2^- + d_1^- d_3^-) &= -d_2^-, \\ q_3^-(d_2^- + d_1^- d_3^-) &= -d_3^- \end{aligned} \tag{C_1}$$

or

$$\begin{aligned} f_1^- &= d_1^- q_1^-, \\ f_2^- + d_1^- f_3^- &= d_1^-(q_2^- + d_1^- q_3^-), \\ 0 &= d_2^- + d_1^- d_3^- \end{aligned} \tag{C_2}$$

where we recall again that the coefficients d^-, q^-, f^- can be found in Appendix A.1. If $\kappa = 0$ we obtain simply

$$f_1^{\pm} = d_1^{\pm} q_1^{\pm}$$

which, as it can be seen in Appendix A.2, corresponds to the equilibrium condition

$$q_1^- = q_1^+. \tag{H1}$$

Although these constraints are quite complex, they can be interpreted in a simpler manner from the macroscopic scale (Section 2.5). We remark that the non-linearity of the models can be identified from these constraints. Focus on the fourth condition in (C_1). Note that $q_3^- = 0 \implies d_3^- = 0$. From the expressions of the coefficients q_3^-, d_3^- it can be checked that

$$q_3^- = d_3^- = 0 \Leftrightarrow D_1 = D_2 = 0$$

in this way characterizing the left-linear class of models, as in Definition 2.2.5.

The goal now is to obtain a set of constraints where the algebra is well-defined. To do so, we compare $\mathbb{A}^-$ with $\mathbb{A}^- \backslash \{(R_{12}^-)\}$.

Lemma 2.3.2. *The identities* (R_1^-),(R_2^-) *and* (R_0) *imply the identity*

$$\begin{aligned} \langle W| D^- DC = \langle W| \left[C^- C^2 s_0 + C^- C s_1 + C^- s_2 + D^- s_3\right] \\ + \langle W| D^- D d_3^-, \end{aligned} \tag{$R_{12}^{-\prime}$}$$

with coefficients

$$\begin{aligned} s_0 &= d_1^- q_1^- \\ s_1 &= d_1^-(q_2^- + d_1^- q_3^-) + d_2^- q_1^- \\ s_2 &= p d_2^- + d_2^-(d_1^- q_3^- + q_2^-) \\ s_3 &= q_3^-(d_2^- + d_1^- d_3^-) + p d_3^-. \end{aligned} \tag{2.20}$$

Proof. Using (R_0) we compute

$$\begin{aligned} \langle W| D^- DC &= \langle W| D^-(CD + C) \\ &= \langle W| d_1^- C^- CD + d_2^- C^- D + d_3^- D^- D + d_1^- C^- C + d_2^- C^- + d_3^- D^- \\ &= \langle W| d_1^- C^- CD + C^- C + (d_2^- q_1^- + d_1^-) + C^-(d_2^- + d_2^- q_2^-) + D^-(d_2^- q_3^- + d_3^-) \\ &+ \langle W| d_3^- D^- D. \end{aligned} \tag{2.21}$$

Computing

$$\begin{aligned} \langle W| C^- CD &= \langle W| C^-(DC - C) \\ &= \langle W| C^- C^2 q_1^- + C^- C(q_2^- + d_1^- q_3^- - 1) + C^- d_2^- q_3^- + D^- d_3^- q_3^- \end{aligned}$$

and replacing this into (2.21) we obtain the identity appearing in the statement of the lemma. □

Lemma 2.3.3. *Identity* (R_{12}^{-}) *satisfies* $(R_{12}^{-'})$ *if, and only if,* (C_1) *or* (C_2) *are satisfied.*

Proof. Replace (R_{12}^{-}) on both sides of $(R_{12}^{-'})$ to obtain

$$\begin{aligned}\langle W|\, f_1^- C^- C^2 + f_2^- C^- C + f_3^- D^- C = \langle W|\, C^- C^2 s_0 + C^- C s_1 + C^- s_2 + D^- s_3 \\ + \langle W|\, d_3^- f_1^- C^- C + d_3^- f_2^- C^- + d_3^- f_3^- D^-.\end{aligned}$$

This can be rearranged into

$$\begin{aligned}f_3^- \langle W|\, D^- C = \langle W|\, C^- C^2 (s_0 - f_1^-) \\ + \langle W|\, C^- C (s_1 + d_3^- f_1^- - f_2^-) \\ + \langle W|\, C^- (s_2 + d_3^- f_2^-) \\ + \langle W|\, D^- (s_3 + d_3^- f_3^-).\end{aligned} \tag{2.22}$$

Recalling (R_1^-), if one imposes that

$$\begin{aligned}f_1^- &= s_0, \\ f_3^- d_1^- &= s_1 + d_3^- f_1^- - f_2^-, \\ f_3^- d_2^- &= s_2 + d_3^- f_2^-, \\ 0 &= s_3,\end{aligned}$$

then (2.22) is reduced to (R_1^-). These four conditions can be reduced to (C_1). Otherwise, replacing the right-hand side of (R_1^-) into the left-hand side of (2.22) we see that the equality

$$\begin{aligned}\langle W|\, f_3^- d_1^- C^- C + f_3^- d_2^- C^- + f_3^- d_3^- D^- = \langle W|\left[C^- C^2 (s_0 - f_1^-) + C^- C (s_1 + d_3^- f_1^- - f_2^-) + C^- (s_2 + d_3^- f_2^-)\right] \\ + \langle W|\, D^- (s_3 + d_3^- f_3^-)\end{aligned}$$

must hold. The previous equality can be rearranged into

$$\begin{aligned}0 = \langle W|\, C^- C^2 \left[f_1^- - s_0\right] \\ + \langle W|\, C^- C \left[f_3^- d_1^- - (s_1 + d_3^- f_1^- - f_2^-)\right] \\ + \langle W|\, C^- \left[f_3^- d_2^- - (s_2 + d_3^- f_2^-)\right] \\ + \langle W|\, D^- \left[-s_3\right].\end{aligned} \tag{2.23}$$

We show that this is true if, and only if, the requirements of (C_2) are met. Suppose that $s_3 = 0$. This implies that the normalization satisfies

$$0 = (f_1^- - s_0)\, Z_N + \left[f_3^- d_1^- - (s_1 + d_3^- f_1^- - f_2^-)\right] Z_{N-1} + \left[f_3^- d_2^- - (s_2 + d_3^- f_2^-)\right] Z_{N-2}.$$

The normalization satisfies the recurrence relation that will be presented in (2.27), therefore not satisfying the above unless the remaining coefficients all equal to zero, leading to (C_1).

Let then $s_3 \neq 0$ and introduce the coefficients

$$\begin{aligned}u_0 &= (f_1^- - s_0)/s_3, \\ u_1 &= (f_3^- d_1^- - (s_1 + d_3^- f_1^- - f_2^-))/s_3,\end{aligned}$$

$$u_2 = \left(f_3^- d_2^- - (s_2 + d_3^- f_2^-)\right)/s_3.$$

Equation (2.23) becomes

$$\langle W|\, D^- = \langle W|\, u_0 C^- C^2 + u_1 C^- C + u_2 C^-. \tag{2.24}$$

Replacing (2.24) on both sides of (R_1^-) leads to

$$0 = \langle W|\left[C^- C^3 u_0 + C^- C^2(u_1 - d_3 u_0) + C^- C(u_2 - d_1^- - d_3 u_1) - C^-(d_2^- + d_3^- u_2)\right],$$

inducing a normalization which must be compatible with (2.27). Since this is not the case, there is the need to impose

$$\begin{aligned} 0 &= u_0, \\ 0 &= u_1 - d_3 u_0, \\ 0 &= u_2 - d_1^- - d_3 u_1, \\ 0 &= d_2^- + d_3^- u_2, \end{aligned}$$

and this is the same as (C_2). □

As mentioned in the introduction, the main difficulty in finding a matrix representation for the algebra $\mathbb{A}$ is that it has three boundary relations for two boundary vectors, per boundary. With this in mind, let us focus on the identity $(R_{12}^{-\prime})$. Assuming that $C - \mathbf{1}d_3^-$ is right-invertible, then $(R_{12}^{-\prime})$ is equivalent to

$$\langle W|\, D^- D = \langle W|\, (C^- C^2 s_0 + C^- C s_1 + C^- s_2 + D^- s_3)(C - \mathbf{1}d_3^-)^{-1}.$$

We want to argue that the right-hand side above coincides with the right-hand side of (R_{12}^-), that is,

$$\langle W|\, (f_1^- C^- C + f_2^- C^- + f_3^- D^-) = \langle W|\, (C^- C^2 s_0 + C^- C s_1 + C^- s_2 + D^- s_3)(C - \mathbf{1}d_3^-)^{-1}.$$

We cannot do this by means of "derivations" since we do not have any identity involving $(C - \mathbf{1}d_3^-)^{-1}$, therefore we multiply by $(C - \mathbf{1}d_3^-)$ through the right on both sides of the equation in the previous display and reorganize the resulting equation, yielding $0 = 0$ by invoking the constraints on the parameters (we shall perform this computation carefully in the proof of the next proposition). For $(C - \mathbf{1}d_3^-)$ right-invertible, this means that one must have (under the aforementioned constraints) that

$$\begin{aligned} \langle W|\, D^- D &= \langle W|\, (C^- C^2 s_0 + C^- C s_1 + C^- s_2 + D^- s_3)(C - \mathbf{1}d_3^-)^{-1} \\ &= \langle W|\, (f_1^- C^- C + f_2^- C^- + f_3^- D^-). \end{aligned} \tag{2.25}$$

In this manner, we cannot *derive* the relation (R_{12}^-) from the subalgebra $\mathbb{A}^-\backslash\{(R_{12}^-)\}$, yet it is true that if we enlarge the algebra $\mathbb{A}^-\backslash\{(R_{12}^-)\}$ with the object A such that $(C - \mathbf{1}d_3^-)A = \mathbf{1}$ then (R_{12}^-) is satisfied (under the constraints on the rates). This can be thought of as "exchanging" the relation (R_{12}^-) by the invertibility of the aforementioned matrix. In this way, we are considering an alternative algebra with a larger set of objects, but one that still leads to the same stationary probability vector and, importantly, one that might be easier to argue that a matrix representation exists.

All of this could be avoided if one could compute any probability using two out of the three boundary relations, per boundary. As a consequence, no invertibility of any object would be required and it would

be enough to consider then a boundary subalgebra with those two identities, which has to generate the same probability vector. Moreover, finding a matrix representation for this subalgebra could also be, in principle, a simpler problem than for the original algebra $\mathbb{A}$, since one would have two identities per boundary. It turns out that to do so one needs to compute $\langle W|\, D^- D\cdots DD^+\,|V\rangle$ and for that having either (R_{12}^-) or its analogous right-boundary counterpart (R_{12}^+) is essential. In this vein, there are the following possible subalgebras that have an identity involving either $\langle W|\, D^- D$ or $DD^+\,|V\rangle$ but not both:

$$\mathbb{S}_0 \coloneqq \mathbb{A}\backslash\{(R_1^-)(R_{12}^+)\},\quad \mathbb{S}_1 \coloneqq \mathbb{A}\backslash\{(R_2^-)(R_{12}^+)\},\quad \mathbb{S}_2 \coloneqq \mathbb{A}\backslash\{(R_{12}^-)(R_1^+)\},\quad \mathbb{S}_3 \coloneqq \mathbb{A}\backslash\{(R_{12}^-)(R_2^+)\}.$$

It is important now to be specific regarding what "computing" a probability is meant to be understood. In the next section we are going to derive a recurrence relation characterizing the normalization Z_N, which can be solved in terms the initial conditions $\langle W|\, D^- CD^+\,|V\rangle$, $\langle W|\, D^- DD^+\,|V\rangle$, $\langle W|\, C^- DC^+\,|V\rangle$, $\langle W|\, C^- DD^+\,|V\rangle$, $\langle W|\, D^- DC^+\,|V\rangle$, and $\langle W|\, C^- C^+\,|V\rangle$. Since any sequence of D, E (resp. $D^\pm, E^\pm$) can be expressed as combination of C, D (resp. $C^\pm, D^\pm$) matrices, in order to obtain the exact value of some probability it is enough to express the probability weights in terms of $C, D, D^\pm, C^\pm$, then use the identities we can in order to have $C, C^\pm$ matrices only and the aforementioned initial conditions. One then should be able to fix the aforementioned "initial conditions", obtaining then the exact value of the probability, with no unknown quantities. The problem with this subalgebra approach is that for each $\mathbb{S}_i$, with $i = 0, 1, 2, 3$, there is a single weight that simply cannot be expressed in terms of the initial conditions or the normalization.

- $\mathbb{S}_0$: cannot compute $\langle W|\, D^- C\cdots CC^+\,|V\rangle$;
- $\mathbb{S}_1$: cannot express Z_N in terms of the initial conditions (see subsection 2.3.4);
- $\mathbb{S}_2$: cannot compute $\langle W|\, C^- C\cdots CD^+\,|V\rangle$;
- $\mathbb{S}_3$: cannot express Z_N in terms of the initial conditions.

This means that no relation can be removed from the original algebra $\mathbb{A}$ while being able to compute, without resorting to a specific representation, *any* probability weight. From this analysis we can extract the following possibilities for guaranteeing the representation of an algebra that generates the same probability vector:

1. Enlarge $\mathbb{A}$ with the right-inverse (resp. left-inverse) of $C - d_3^-$ (resp. $C - d_3^+$);
2. Fix a representation for $\mathbb{A}$ with $C - d_3^-$ (resp. $C - d_3^+$) *not* right-invertible (resp. left-invertible) and then check if the representation satisfies (R_{12}^-) and (R_{12}^+).

Note that the approach 1. guarantees that a representation must satisfy (R_{12}^-),(R_{12}^+) because the second equality in (2.25) holds, which is the case due to the left/right invertibility of the aforementioned objects. If the left-boundary is left-linear (resp. right-linear) then $d_3^- = 0$ (resp. $d_3^+ = 0$), reducing to the invertibility of C. We stress that, although no representation for $\mathbb{A}$ was found, Lemma 2.3.3 together with the computations in Subsection 2.3.5, show that $\mathbb{A}$ is free from contradictions, that is, computing any probability weight by applying any sequence of identities always leads to the same result.

Proposition 2.3.4. *Depending on whether the rates satisfy* (C_1) *or* (C_2) *the following holds:*

1. *Under* (C_2)*, the subalgebra* $\mathbb{A}^-\backslash\{(R_{12}^-)\}$ *generates* $\mathbb{A}^-$*.*

2. *Under* (C_1) *with* $C - \mathbf{1}d_3^-$ *right-invertible it holds that*

$$\begin{aligned}\langle W|\, D^-D &= \langle W|\,(C^-C^2s_0 + C^-Cs_1 + C^-s_2 + D^-s_3)(C - \mathbf{1}d_3^-)^{-1}\\ &= \langle W|\,(f_1^-C^-C + f_2^-C^- + f_3^-D^-);\end{aligned}$$

Proof. We start with the case 1. Under (C_2) the identity (2.24) is reduced to

$$\langle W|\, D^-C = u_2\,\langle W|\, C^-C$$

and the equalities in (C_2) imply that $u_2 = d_1^-$. One can check that it is sufficient to consider the identity

$$\langle W|\, D^- = d_1^-\,\langle W|\, C^- \tag{2.26}$$

instead of (R_1^-) by computing $\langle W|\, DD$ using (2.26), then comparing with (R_{12}^-) using the constraints.

We now focus on 2., that is $\mathbb{A}^-$ under (C_1). Recall that in Lemma 2.3.2 we derived

$$\langle W|\, D^-DC = \langle W|\, C^-C^2s_0 + C^-Cs_1 + C^-s_2 + D^-s_3 + D^-Dd_3^-$$

from $\mathbb{A}^-\backslash(R_{12}^-)$. Assuming the right-invertibility of $C - \mathbf{1}d_3^-$, to show that this identity is equivalent to (R_{12}^-), that is, that

$$\begin{aligned}\langle W|\, D^-D &= \langle W|\,(C^-C^2s_0 + C^-Cs_1 + C^-s_2 + D^-s_3)(C - \mathbf{1}d_3^-)^{-1}\\ &= \langle W|\,(f_1^-C^-C + f_2^-C^- + f_3^-D^-),\end{aligned}$$

we verify

$$\langle W|\,(C^-C^2s_0 + C^-Cs_1 + C^-s_2 + D^-s_3) = \langle W|\,(f_1^-C^-C + f_2^-C^- + f_3^-D^-)(C - \mathbf{1}d_3^-),$$

which can be rearranged into

$$\begin{aligned}0 = {}& \langle W|\, C^-C^2(s_0 - f_1^-)\\ &+ \langle W|\, C^-C(s_1 - (f_2^- + d_1^-f_3^- - d_3^-f_1^-))\\ &+ \langle W|\, C^-(s_2 - (d_2^-f_1^- - d_3^-f_2^-))\\ &+ \langle W|\, D^-s_3.\end{aligned}$$

Specifically under (C_1) this reads $0 = 0$. □

2.3.4 Normalization

At this point it is still not clear that $Z_N \neq 0$. Checking this from the representation becomes quickly intractable, while studying from the algebra point of view is much simpler. We first derive a system of recurrence relations for Z_N, which shows the need of investigating the specific value of the inner products of the boundary vectors. Consider the weights

$$\langle\eta\rangle_N^w := Z_N\langle\eta\rangle_N.$$

Analogously to the particular case of the SSEP$(1,1)$, (2.9), the bulk relation leads to

$$\langle \eta(2)\rangle_N^w = \langle \eta(N-1)\rangle_N^w - (N-3)Z_{N-1}.$$

From the boundary algebra it holds that

$$\begin{aligned}\langle \eta(2)\rangle_N^w &= q_1^- Z_N + q_2^- Z_{N-1} + q_3^- \langle \eta(1)\rangle_{N-1}^w \\ \langle \eta(N-1)\rangle_N^w &= q_1^+ Z_N - q_2^+ Z_{N-1} - q_3^+ \langle \eta(N-1)\rangle_{N-1}^w\end{aligned}$$

and we obtain the system

$$\begin{pmatrix} (q_1^- - q_1^+) & 0 & 0 \\ -d_1^- & 1 & 0 \\ -d_1^+ & 0 & 1 \end{pmatrix}\begin{pmatrix} Z_N \\ \langle \eta(1)\rangle_N^w \\ \langle \eta(N)\rangle_N^w \end{pmatrix} = \begin{pmatrix} N-3-(q_2^- + q_2^+) & -q_3^- & -q_3^+ \\ d_2^- & d_3^- & 0 \\ -d_2^+ & 0 & -d_3^+ \end{pmatrix}\begin{pmatrix} Z_{N-1} \\ \langle \eta(1)\rangle_{N-1}^w \\ \langle \eta(N-1)\rangle_{N-1}^w \end{pmatrix}. \tag{2.27}$$

This can be equivalently expressed in terms of the boundary densities and current, however, this leads to the same following problem. Define the vector $\mathcal{Z}_N = (Z_N,\ \langle \eta(1)\rangle_N^w,\ \langle \eta(N)\rangle_N^w)^\dagger$. Under (C_1) with $d_3^\pm = q_3^\pm = 0$ or under (C_2) we see that Z_N follows a first order recurrence relation and there are no initial conditions on (2.27) to be fixed. On the remaining cases, we need to fix the initial condition $\mathcal{Z}_2$. More precisely, if $\mathcal{Z}_2 = 0$ then $Z_N = 0$. We will show that $\mathcal{Z}_2 \neq 0$ and that the inner products can be fixed without invoking a particular representation. Our approach will be to exploit the freedom we have to compute quantities in different ways, using either the left or right-boundary relations.

We stress that $\langle \eta(1)\rangle_2^w / Z_2, \langle \eta(2)\rangle_2^w / Z_2$ should not be understood as the density at sites $1,2$ for a system with length $N = 2$, since our dynamics is only defined for $N \geq 4$.

2.3.5 Fixing the inner products

A straightforward way to fix $\mathcal{Z}_2$ is to try and solve the discrete equation $\langle \mathcal{L} f(\eta)\rangle_N = 0$ for specific functions f and N small. Unfortunately, this is not possible for non-linear models. To better present our argument we introduce some notation.

Notation 2.3.5. We denote by $\underline{\langle W| X}$ (resp. $\underline{X|V\rangle}$) if we use a left (resp. right) boundary relation to compute the local configuration X.

To have a well-defined algebra we must have

$$\begin{aligned}0 &= \underline{\langle W| D^- C} D^+ |V\rangle - \langle W| D^- \underline{C D^+ |V\rangle} \\ 0 &= \underline{\langle W| D^- D} D^+ |V\rangle - \langle W| D^- \underline{D D^+ |V\rangle} \\ 0 &= \underline{\langle W| C^- D} C^+ |V\rangle - \langle W| C^- \underline{D C^+ |V\rangle} \\ 0 &= \underline{\langle W| C^- D} D^+ |V\rangle - \langle W| C^- \underline{D D^+ |V\rangle} \\ 0 &= \underline{\langle W| D^- D} C^+ |V\rangle - \langle W| D^- \underline{D C^+ |V\rangle}\end{aligned}$$

Computing the right-hand sides above will lead to a linear system involving $\mathcal{Z}_2$.

1. $\underline{\langle W| D^- C} D^+ |V\rangle - \langle W| D^- \underline{C D^+ |V\rangle} = 0$:

$$\begin{aligned}\underline{\langle W| D^- C} D^+ |V\rangle &= d_1^- \langle W| C^- C D^+ |V\rangle + d_2^- \langle W| C^- D^+ |V\rangle + d_3^- \langle W| D^- D^+ |V\rangle \\ &\equiv d_1^- \langle W| C^- \underline{C D^+ |V\rangle} + d_2^- \langle W| C^- D^+ |V\rangle + d_3^- \langle W| D^- D^+ |V\rangle\end{aligned}$$

$$
\begin{aligned}
&= d_1^- \left(\langle W| C^- C C^+ |V\rangle \, d_1^+ - \langle W| C^- C^+ |V\rangle \, d_2^+ - \langle W| C^- D^+ |V\rangle \, d_3^+ \right) \\
&+ d_2^- \langle W| C^- D^+ |V\rangle + d_3^- \langle W| D^- D^+ |V\rangle \\
&= \langle W| C^- C C^+ |V\rangle \, d_1^- d_1^+ \\
&+ \langle W| C^- C^+ |V\rangle \, , (-d_1^- d_2^+) \\
&+ \langle W| D^- D^+ |V\rangle \, d_3^- \\
&+ \langle W| D^- C^+ |V\rangle \, 0 \\
&+ \langle W| C^- D^+ |V\rangle \, (-d_1^- d_3^+ + d_2^-)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle W| D^- \underline{C D^+ |V\rangle} &= \langle W| D^- C C^+ |V\rangle \, d_1^+ - \langle W| D^- C^+ |V\rangle \, d_2^+ - \langle W| D^- D^+ |V\rangle \, d_3^+ \\
&\equiv \underline{\langle W| D^- C} C^+ |V\rangle \, d_1^+ - \langle W| D^- C^+ |V\rangle \, d_2^+ - \langle W| D^- D^+ |V\rangle \, d_3^+ \\
&= d_1^+ \left(d_1^- \langle W| C^- C C^+ |V\rangle + d_2^- \langle W| C^- C^+ |V\rangle + d_3^- \langle W| D^- C^+ |V\rangle \right) \\
&- \langle W| D^- C^+ |V\rangle \, d_2^+ - \langle W| D^- D^+ |V\rangle \, d_3^+ \\
&= \langle W| C^- C C^+ |V\rangle \, d_1^- d_1^+ \\
&+ \langle W| C^- C^+ |V\rangle \, (d_1^+ d_2^-) \\
&+ \langle W| D^- D^+ |V\rangle \, (-d_3^+) \\
&+ \langle W| D^- C^+ |V\rangle \, (d_1^+ d_3^- - d_2^+) \\
&+ \langle W| C^- D^+ |V\rangle \, 0.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
0 &= \langle W| C^- C^+ |V\rangle \, (d_2^- d_1^+ + d_2^+ d_1^-) \\
&+ \langle W| D^- D^+ |V\rangle \, (-d_3^+ - d_3^-) \\
&+ \langle W| D^- C^+ |V\rangle \, (d_3^- d_1^+ - d_2^+) \\
&\langle W| C^- D^+ |V\rangle \, (d_3^+ d_1^- - d_2^-) \, .
\end{aligned}
$$

Remark 2.3.6. Note that we are assuming that computing $\langle W| C^- C D^+ |V\rangle$ and $\langle W| D^- C C^+ |V\rangle$ on different ways lead to the same value. This is not a problem and we will see why shortly.

2. $\underline{\langle W| D^- D} D^+ |V\rangle - \langle W| D^- \underline{D D^+ |V\rangle} = 0$:

$$
\begin{aligned}
\underline{\langle W| D^- D} D^+ |V\rangle &= f_1^- \langle W| C^- C D^+ |V\rangle + f_2^- \langle W| C^- D^+ |V\rangle + f_3^- \langle W| D^- D^+ |V\rangle \\
&= f_1^- \left(\langle W| C^- C C^+ |V\rangle \, d_1^+ - \langle W| C^- C^+ |V\rangle \, d_2^+ - \langle W| C^- D^+ |V\rangle \, d_3^+ \right) \\
&+ f_2^- \langle W| C^- D^+ |V\rangle + f_3^- \langle W| D^- D^+ |V\rangle \\
&= \langle W| C^- C C^+ |V\rangle \, f_1^- d_1^+ \\
&+ \langle W| C^- C^+ |V\rangle \, (-d_2^+ f_1^-) \\
&+ \langle W| D^- D^+ |V\rangle \, f_3^- \\
&+ \langle W| D^- C^+ |V\rangle \, 0 \\
&+ \langle W| C^- D^+ |V\rangle \, (-f_1^- d_3^+ + f_2^-),
\end{aligned}
$$

and

$$\begin{aligned}\langle W|\, D^- \underline{DD^+}\, |V\rangle &= \langle W|\, D^- CC^+\, |V\rangle\, f_1^+ - \langle W|\, D^- C^+\, |V\rangle\, f_2^+ - \langle W|\, D^- D^+\, |V\rangle\, f_3^+ \\ &= f_1^+ \left(\langle W|\, C^- CC^+\, |V\rangle\, d_1^- + \langle W|\, C^- C^+\, |V\rangle\, d_2^- + \langle W|\, D^- C^+\, |V\rangle\, d_3^-\right) \\ &\quad - \langle W|\, D^- C^+\, |V\rangle\, f_2^+ - \langle W|\, D^- D^+\, |V\rangle\, f_3^+ \\ &= \langle W|\, C^- CC^+\, |V\rangle\, f_1^+ d_1^- \\ &\quad + \langle W|\, C^- C^+\, |V\rangle\, (d_2^- f_1^+) \\ &\quad + \langle W|\, D^- D^+\, |V\rangle\, (-f_3^+) \\ &\quad + \langle W|\, D^- C^+\, |V\rangle\, (f_1^+ d_3^- - f_2^+) \\ &\quad + \langle W|\, C^- D^+\, |V\rangle\, 0,\end{aligned}$$

and we need

$$\begin{aligned}0 &= \langle W|\, C^- CC^+\, |V\rangle\, (d_1^- f_1^+ - d_1^+ f_1^-) \\ &\quad + \langle W|\, C^- C^+\, |V\rangle\, (d_2^- f_1^+ + d_2^+ f_1^-) \\ &\quad + \langle W|\, D^- D^+\, |V\rangle\, (-f_3^+ - f_3^-) \\ &\quad + \langle W|\, D^- C^+\, |V\rangle\, (-f_2^+ + d_3^- f_1^+) \\ &\quad + \langle W|\, C^- D^+\, |V\rangle\, (-f_2^- + d_3^+ f_1^-)\,.\end{aligned}$$

3. $\underline{\langle W|\, C^- D}C^+\, |V\rangle - \langle W|\, C^- \underline{DC^+\, |V\rangle} = 0$:

$$\underline{\langle W|\, C^- D}C^+\, |V\rangle = q_1^- \langle W|\, C^- CC_l + q_2^- \langle W|\, C^- C^+\, |V\rangle + q_3^- \langle W|\, D^- C^+\, |V\rangle\,, \tag{2.28}$$

while

$$\langle W|\, C^- \underline{DC^+\, |V\rangle} = \langle W|\, C^- CC^+\, |V\rangle\, q_1^+ - \langle W|\, C^- CC^+\, |V\rangle\, q_2^+ - \langle W|\, C^- D^+\, |V\rangle\, q_3^+,$$

which implies

$$\begin{aligned}0 &= \langle W|\, C^- CC^+\, |V\rangle\, (q_1^- - q_1^+) \\ &\quad + \langle W|\, C^- C^+\, |V\rangle\, (q_2^- + q_2^+) \\ &\quad + \langle W|\, D^- C^+\, |V\rangle\, (q_3^-) \\ &\quad + \langle W|\, C^- D^+\, |V\rangle\, (q_3^+)\,.\end{aligned} \tag{2.29}$$

4. $\underline{\langle W|\, C^- D}D^+\, |V\rangle - \langle W|\, C^- \underline{DD^+\, |V\rangle} = 0$:

$$\begin{aligned}\underline{\langle W|\, C^- D}D^+\, |V\rangle &= q_1^- \langle W|\, C^- \underline{CD^+\, |V\rangle} + q_2^- \langle W|\, C^- D^+\, |V\rangle + q_3^- \langle W|\, D^- D^+\, |V\rangle \\ &= q_1^- \langle W|\, C^- (d_1^+ CC^+\, |V\rangle - d_2^+ C^+\, |V\rangle - d_3^+ D^+\, |V\rangle) \\ &\quad + q_2^- \langle W|\, C^- D^+\, |V\rangle + q_3^- \langle W|\, D^- D^+\, |V\rangle \\ &= \langle W|\, C^- CC^+\, |V\rangle\, d_1^+ q_1^- \\ &\quad + \langle W|\, C^- C^+\, |V\rangle\, (-d_2^+ q_1^-) \\ &\quad + \langle W|\, D^- D^+\, |V\rangle\, q_3^- \\ &\quad + \langle W|\, C^- D^+\, |V\rangle\, (-d_3^+ q_1^- + q_2^-),\end{aligned}$$

while

$$\langle W|\, C^- \underline{DD^+}\, |V\rangle = f_1^+ \langle W|\, C^- CC^+ \,|V\rangle - f_2^+ \langle W|\, C^- C^+ \,|V\rangle - f_3^+ \langle W|\, C^- D^+ \,|V\rangle$$

leading to

$$\begin{aligned} 0 &= \langle W|\, C^- CC^+ \,|V\rangle \,(f_1^+ - d_1^+ q_1^-) \\ &+ \langle W|\, C^- C^+ \,|V\rangle \,(-f_2^+ + d_2^+ q_1^-) \\ &+ \langle W|\, D^- D^+ \,|V\rangle \,(-q_3^-) \\ &+ \langle W|\, C^- D^+ \,|V\rangle \,(-f_3^+ + d_3^+ q_1^- - q_2^-) \end{aligned}$$

5. $\underline{\langle W|\, D^-}\underline{D}C^+ \,|V\rangle - \langle W|\, D^- \underline{DC^+ \,|V\rangle} = 0$:

$$\underline{\langle W|\, D^- D}C^+ \,|V\rangle = f_1^- \langle W|\, C^- CC^+ \,|V\rangle + f_2^- \langle W|\, C^- C^+ \,|V\rangle + f_3^- \langle W|\, D^- C^+ \,|V\rangle$$

and

$$\begin{aligned} \langle W|\, D^- \underline{DC^+ \,|V\rangle} &= q_1^+ \langle W|\, D^- CC^+ \,|V\rangle - q_2^+ \langle W|\, D^- C^+ \,|V\rangle - q_3^+ \langle W|\, D^- D^+ \,|V\rangle \\ &= q_1^+ (d_1^- \langle W|\, C^- C + d_2^- \langle W|\, C^- + d_3^- \langle W|\, D^-) C^+ \,|V\rangle \\ &- q_2^+ \langle W|\, D^- C^+ \,|V\rangle - q_3^+ \langle W|\, D^- D^+ \,|V\rangle \\ &= \langle W|\, C^- CC^+ \,|V\rangle \,(d_1^- q_1^+) \\ &+ \langle W|\, C^- C^+ \,|V\rangle \,(d_2^- q_1^+) \\ &+ \langle W|\, D^- D^+ \,|V\rangle \,(-q_3^+) \\ &+ \langle W|\, D^- C^+ \,|V\rangle \,(d_3^- q_1^+ - q_2^+), \end{aligned}$$

and so

$$\begin{aligned} 0 &= \langle W|\, C^- CC^+ \,|V\rangle \,(f_1^- - d_1^- q_1^+) \\ &+ \langle W|\, C^- C^+ \,|V\rangle \,(f_2^- - d_2^- q_1^+) \\ &+ \langle W|\, D^- D^+ \,|V\rangle \, q_3^+ \\ &+ \langle W|\, D^- C^+ \,|V\rangle \,(f_3^- - d_3^- q_1^+ + q_2^+). \end{aligned}$$

Regarding Remark 2.3.6, the only other way to compute (for example) $\langle W|\, D^- CC^+ \,|V\rangle$ is to "send" the D^- matrix from one boundary to the other. For that, we note that $\langle W|\, D^- C = \langle W|\, C^- D + \langle W|\, [D, C]^-$, where $[D, C]^- = D^- C - C^- D$ (see (2.16)), and since $[D, C]^- = [D, E]^-$ we may look directly at the algebra where (R_2^-) is replaced by

$$\langle W|\, [D, C]^- = \langle W|\, t_1^- C^- C + t_2^- C^- + t_3^- D^-.$$

This leads to

$$\begin{aligned} \langle W|\, D^- CC^+ \,|V\rangle &= \langle W|\, C^- DC^+ \,|V\rangle + [D, C]_1 C^+ \,|V\rangle \\ &= \langle W|\, C^- DC^+ \,|V\rangle + t_1^- \langle W|\, C^- CC^+ \,|V\rangle + t_2^- \langle W|\, C^- C^+ \,|V\rangle + t_3^- \langle W|\, D^- D^+ \,|V\rangle, \end{aligned}$$

and so $\underline{\langle W|\, D^- C}C^+ \,|V\rangle = \langle W|\, D^- CC^+ \,|V\rangle$ is equivalent to

$$d_1^- \langle W|C^-CC^+|V\rangle + d_2^- \langle W|C^-C^+|V\rangle + d_3^- \langle W|D^-C^+|V\rangle$$
$$= \langle W|C^-DC^+|V\rangle + t_1^- \langle W|C^-CC^+|V\rangle + t_2^- \langle W|C^-C^+|V\rangle + t_3^- \langle W|D^-D^+|V\rangle$$

that is,

$$\langle W|C^-DC^+|V\rangle = (d_1^- - t_1^-)\langle W|C^-CC_l + (d_2^- - t_2^-)\langle W|C^-C^+|V\rangle + (d_3^- - t_3^-)\langle W|D^-C^+|V\rangle .$$

Recalling from (2.19) that $q_i^- \equiv d_i^- - t_i^-$ for $i \in \{1,2,3\}$, the equation above is the same as (2.28). The same procedure for $\langle W|C^-CD^+|V\rangle$ leads to

$$\langle W|C^-DC^+|V\rangle = (d_1^+ - t_1^+)\langle W|C^-CC^+|V\rangle - (d_2^+ - t_2^+)\langle W|C^-C^+|V\rangle - (d_3^+ - t_3^+)\langle W|D^-C^+|V\rangle .$$

Equating the last two expressions yields again (2.29).

We find that this is equivalent to the kernel problem

$$\begin{pmatrix} 0 & d_2^-d_1^+ + d_2^+d_1^- & -d_3^+ - d_3^- & d_3^-d_1^+ - d_2^+ & d_3^+d_1^- - d_2^- \\ d_1^-f_1^+ - d_1^+f_1^- & d_2^-f_1^+ + d_2^+f_1^- & -f_3^+ - f_3^- & -f_2^+ + d_3^-f_1^+ & -f_2^- + d_3^+f_1^- \\ q_1^+ - q_1^- & -q_2^- - q_2^+ & 0 & -q_3^- & -q_3^+ \\ f_1^+ - d_1^+q_1^- & -f_2^+ + d_2^+q_1^- & -q_3^- & 0 & -f_3^+ + d_3^+q_1^- - q_2^- \\ d_1^-q_1^+ - f_1^- & -f_2^- + d_2^-q_1^+ & -q_3^+ & -f_3^- + d_3^-q_1^+ - q_2^+ & 0 \end{pmatrix} \begin{pmatrix} Z_3 \\ Z_2 \\ \langle \eta(1)\eta(2)\rangle_2^w \\ \langle \eta(1)\rangle_2^w \\ \langle \eta(2)\rangle_2^w \end{pmatrix} = 0. \quad (2.30)$$

Denote the matrix on the left-hand side above by $\mathcal{I}$. For $\mathcal{I}$ to have a non-trivial kernel we need

$$\det(\mathcal{I}) = 0$$

Example 2.3.7 (Linear models)**.** If $d_3^\pm = q_3^\pm = 0$ and $d_2^\pm \neq 0$, then set of constraints (C_1) simplifies to

$$f_1^\pm = d_1^\pm q_1^\pm, \quad f_2^\pm = -d_1^\pm + d_2^\pm q_1^\pm, \quad f_3^\pm = 1 + q_2^\pm,$$

and the system (2.30) has the unique solution

$$\frac{Z_2}{Z_3} = \frac{q_1^+ - q_1^-}{q_2^- + q_2^+},$$
$$\langle \eta(1)\rangle_2 = d_1^- + d_2^- \frac{q_1^+ - q_1^-}{q_2^- + q_2^+ + 1},$$
$$\langle \eta(2)\rangle_2 = d_1^+ + d_2^+ \frac{q_1^- - q_1^+}{q_2^- + q_2^+ + 1},$$
$$\langle \eta(1)\eta(2)\rangle_2 = d_1^-d_1^+ - d_1^- \frac{d_2^+(q_1^+ - q_1^-)}{q_2^- + q_2^+ + 1} + \frac{d_2^-(q_1^+ - q_1^-)}{q_2^- + q_2^+ + 1}\left(d_1^+ - \frac{d_2^+(q_1^+ - q_1^-)}{q_2^- + q_2^+ + 2}\right).$$

We note that

$$\langle \eta(1)\eta(2)\rangle_2 = \langle \eta(1)\rangle_2\langle \eta(2)\rangle_2 + \frac{d_2^-d_2^+}{q_2^- + q_2^+ + 2}\left(\frac{q_1^- - q_1^+}{q_2^- + q_2^+ + 1}\right)^2 .$$

Example 2.3.8 (Non-linear models)**.** For (C_1) with $q_3^\pm \neq 0$ it is convenient to perform the substitutions

$$f_1^\pm = d_1^\pm q_1^\pm,$$
$$f_2^\pm = q_1^\pm d_2^\pm - d_1^\pm(1 + d_1^\pm q_3^\pm),$$
$$f_3^\pm = q_1^\pm d_3^\pm + 2d_1^\pm q_3^\pm + q_2^\pm + 1,$$
$$d_2^\pm = -\frac{d_3^\pm}{q_3^\pm} - d_1^\pm d_3^\pm.$$

It can be checked that this yields the unique solution

$$\begin{aligned}
\frac{Z_2}{Z_3} &= \frac{1}{z_0} d_3^- \left(q_1^- - q_1^+\right)^2 \left(-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 2\right) \\
&\quad + \frac{1}{z_0} \left(q_1^- - q_1^+\right) \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left(-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 1\right), \\
\langle \eta(1)\eta(2) \rangle_2 &= -\frac{1}{z_{12}} d_1^- q_3^- \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[d_1^+ q_3^+ (d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 1) - d_3^+ (d_1^+ q_3^+ + 1)(q_1^- - q_1^+)\right] \\
&\quad + \frac{1}{z_{12}} d_3^- \left(d_1^- q_3^- + 1\right) \left(q_1^- - q_1^+\right) \left[d_3^+ (d_1^+ q_3^+ + 1)(q_1^- - q_1^+) - d_1^+ q_3^+ (d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2)\right], \\
\langle \eta(1) \rangle_2 &= -\frac{1}{z_1} d_1^- q_3^- \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[d_1^- q_3^- + d_1^+ q_3^+ + d_3^+ (q_1^+ - q_1^-) + q_2^- + q_2^+ + 1\right] \\
&\quad + \frac{1}{z_1} d_3^- (d_1^- q_3^- + 1)(q_1^- - q_1^+) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 2\right], \\
\langle \eta(2) \rangle_2 &= -\frac{1}{z_2} d_1^+ q_3^+ \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[d_1^- q_3^- + d_1^+ q_3^+ + d_3^- (q_1^- - q_1^+) + q_2^- + q_2^+ + 1\right] \\
&\quad + \frac{1}{z_2} d_3^+ \left(d_1^+ q_3^+ + 1\right) \left(q_1^+ - q_1^-\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^- (q_1^+ - q_1^-) - q_2^- - q_2^+ - 2\right],
\end{aligned}$$

where

$$\begin{aligned}
z_0 &= -d_3^- \left(q_1^- - q_1^+\right) \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 1\right] \\
&\quad - \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 1\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+\right], \\
z_{12} &= d_3^- q_3^- q_3^+ \left(q_1^- - q_1^+\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 2\right] \\
&\quad + q_3^- q_3^+ \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 1\right], \\
z_1 &= d_3^- q_3^- \left(q_1^- - q_1^+\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 2\right] \\
&\quad + q_3^- \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 1\right], \\
z_2 &= d_3^- q_3^+ \left(q_1^- - q_1^+\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 2\right] \\
&\quad + q_3^+ \left(d_1^- q_3^- + d_1^+ q_3^+ + q_2^- + q_2^+ + 2\right) \left[-d_1^- q_3^- - d_1^+ q_3^+ + d_3^+ (q_1^- - q_1^+) - q_2^- - q_2^+ - 1\right].
\end{aligned}$$

We do not prove that the denominators $z_\cdot$, in the two previous examples, are not zero in general. However, it can be checked that for the examples that we provide in Appendix A.4 this is, indeed, the case.

2.4 Characterization of the consistency relations

The goal of this section is to have some better understanding of the constraints imposed by the algebra. To simplify the presentation we introduce some notation.

Definition 2.4.1. Denote by a and b the collection of the left and right boundary parameters, respectively:

$$a = \{a_{ij}\}_{\substack{i,j=1,\dots,4 \\ i \neq j}} \quad \text{and} \quad b = \{b_{ij}\}_{\substack{i,j=1,\dots,4 \\ i \neq j}}.$$

Denote also the set of parameters such that (C_1) is satisfied for the left-boundary (resp. right-boundary) by $\mathcal{C}_1^-$ (resp. $\mathcal{C}_1^+$) and the set of parameters satisfying (C_2) by $\mathcal{C}_2^-$ (resp. $\mathcal{C}_2^+$). With this notation, we say that a fixed choice of boundary rates a satisfies (C_1) by writing $a \in \mathcal{C}_1^-$. For the other set of constraints and boundary we write analogously.

For each boundary the full set of constraints $\mathcal{C}_1^{\pm} \cup \mathcal{C}_2^{\pm}$ can be split into three families. Again, we focus on the left-boundary. For the right-boundary one needs only to replace the superscript $-$ by $+$.

Recall the conditions (C_1) and (C_2) under (H_0^-). It is straightforward to check that $\{d_2^- , d_3^- = 0\} \cap \mathcal{C}_1 \subset \mathcal{C}_2$, while $\{d_2^- , d_3^- = 0\}^c \cap \mathcal{C}_1 \notin \mathcal{C}_2$ and so $\mathcal{C}_1^-$ and $\mathcal{C}_2^-$ have a non-empty intersection only if $d_2^- = d_3^- = 0$. This also trivially splits the set $\mathcal{C}_1$ into two non-intersecting sets

$$\mathcal{C}_1 = \Big(\mathcal{C}_1 \cap \{d_2^- , d_3^- = 0\}\Big) \cup \Big(\mathcal{C}_1 \cap \{d_2^- , d_3^- = 0\}^c\Big).$$

Let us now focus on the last equation in (C_1). Note that

$$d_3^- = 0 \implies (d_2^- = 0 \vee q_3^- = 0) \quad \text{and} \quad q_3^- = 0 \implies d_3^- = 0,$$

and so we can perform the decomposition

$$\begin{aligned}
\mathcal{C}_1 \cap \{d_2^- , d_3^- = 0\}^c &= \Big(\mathcal{C}_1 \cap \{d_2^- \neq 0\}\Big) \cup \Big(\mathcal{C}_1 \cap \{d_3^- \neq 0\}\Big) \\
&= \Big[\Big(\mathcal{C}_1 \cap \{d_2^- \neq 0,\ d_3^- = 0\}\Big) \cup \Big(\mathcal{C}_1 \cap \{d_2^- , d_3^- \neq 0\}\Big)\Big] \bigcup \Big(\mathcal{C}_1 \cap \{d_3^- , q_3^- \neq 0\}\Big) \\
&= \Big(\mathcal{C}_1 \cap \{d_2^- \neq 0,\ d_3^- , q_3^- = 0\}\Big) \bigcup \Big(\mathcal{C}_1 \cap \{d_2^- , d_3^- , q_3^- \neq 0\}\Big) \bigcup \Big(\mathcal{C}_1 \cap \{d_3^- , q_3^- \neq 0\}\Big) \\
&= \Big(\mathcal{C}_1 \cap \{d_2^- \neq 0,\ d_3^- , q_3^- = 0\}\Big) \bigcup \Big(\mathcal{C}_1 \cap \{d_3^- , q_3^- \neq 0\}\Big).
\end{aligned}$$

This motivates the introduction of the following families.

Definition 2.4.2. Define for the left-boundary

- Family $\mathfrak{F}^-$: models $a \in \mathcal{C}_2^-$;
- Family $\mathfrak{L}^-$: models $a \in \mathcal{C}_1^- \cap \{d_2^- \neq 0,\ d_3^- , q_3^- = 0\}$;
- Family $\mathfrak{N}^-$: models $a \in \mathcal{C}_1^- \cap \{d_3^- , q_3^- \neq 0\}$;
- Subfamily $\mathfrak{H}^-$: models $a \in (\mathcal{C}_1^- \cap \{t_1 = 0,\ t_2 = 1,\ t_3 = 0\}) \cup (\mathcal{C}_2^- \cap \{t_1 = 0,\ t_2 + d_1 t_3 = 1\})$

where we recall from (2.19) that $q_i^- := d_i^- - t_i^-$. For the right-boundary we introduce analogous families by simply replacing the superscript $-$ by $+$ and a by b. We introduce also the models with both boundaries in the same family:

$$\mathfrak{F} := \mathfrak{F}^- \times \mathfrak{F}^+, \quad \mathfrak{L} := \mathfrak{L}^- \times \mathfrak{L}^+, \quad \mathfrak{N} := \mathfrak{N}^- \times \mathfrak{N}^+, \quad \mathfrak{H} := \mathfrak{H}^- \times \mathfrak{H}^+.$$

We highlight that from the expression for the coefficients on Appendix A.1 we see that

$$d_3^- = 0,\ d_2^- = 0 \Leftrightarrow \begin{cases} a_{24} + a_{14} = a_{13} + a_{23}, \\ a_{42} + a_{32} = a_{31} + a_{41}. \end{cases}$$

$$d_3^- = 0, q_3^- = 0 \Leftrightarrow \begin{cases} a_{13} + a_{23} + a_{31} + a_{41} = a_{14} + a_{24} + a_{32} + a_{42}, \\ a_{21} + a_{32} + a_{41} + a_{12} = a_{14} + a_{23} + a_{34} + a_{43}. \end{cases}$$

It turns out that for each of the previous families the stationary measure has a different structure.

- For the family $\mathfrak{F}^-$ the stationary measure factorizes at site 1 and the reservoirs are uncorrelated;
- For the family $\mathfrak{L}^-$ the models are left-linear but the measure does not factorize at the left-boundary;

- The family $\mathfrak{N}^-$ corresponds to non-linear models where the measure also does not factorize at the left-boundary;
- The subfamily $\mathfrak{H}^-$ corresponds to models where the boundary matrices can be taken as being the same as the bulk ones (see Subsection 2.4.3). The corresponding models can be of linear or non-linear nature, and the stationary measure can be factorized at the boundary or not.

For both $\mathfrak{F}$ and $\mathfrak{L}$ a representation for the algebra can be guaranteed from the representation of the SSEP$(1,1)$. We note that there are models in $\mathfrak{F}^-$ whose left-boundary is also non-linear in the sense of the Definition 2.2.5 (see Appendix A.4). Studying these families in detail is the content of the next subsections. Some examples of models are given in Appendix A.4 although they are far from exhaustive.

2.4.1 Family $\mathfrak{L}$

Recall from (2.15) the boundary vectors

$$\langle W_0| := \langle W|E^-, \quad |V_0\rangle := E^+|V\rangle, \quad \langle W_1| := \langle W|D^-, \quad |V_1\rangle := D^+|V\rangle \tag{2.31}$$

and introduce

$$\langle A| := \langle W_0| + \langle W_1|, \quad |B\rangle = |V_0\rangle + |V_1\rangle.$$

With this notation the boundary algebra, assuming the right-invertibility (resp. left) of $C - \mathbf{1}d_3^-$ (resp. $C - \mathbf{1}d_3^+$) and therefore without (R_{12}^-), can be rewritten as

$$\begin{aligned} \langle W_1|C &= \langle A|(d_1^-C + d_2^-), & D|V_1\rangle &= d_1^-C - d_2^-|B\rangle, \\ \langle A|D &= \langle A|(q_1^-C - (-q_2^-)\mathbf{1}), & D|B\rangle &= q_1^-C + (-q_2^-)\mathbf{1}|B\rangle. \end{aligned} \tag{2.32}$$

Note that the last line is the boundary algebra for the open SSEP with boundary vectors $\langle A|$ and $|B\rangle$:

$$\langle A|D = \langle A|C\frac{\alpha}{\alpha+\gamma} - \frac{1}{\alpha+\gamma}\mathbf{1}, \qquad D|B\rangle = C\frac{\beta}{\beta+\delta} + \frac{1}{\beta+\delta}\mathbf{1}|B\rangle$$

under the identification $\alpha = \frac{q_1^-}{-q_2^-}$, $\gamma = \frac{1-q_1^-}{-q_2^-}$ and similar for the right-boundary. From the representation for the SSEP$(1,1)$ it is guaranteed the existence of vectors $\langle A|, |B\rangle$ and matrices D, E satisfying $[D,E] = D+E$ and the boundary relations in the second line of (2.32). These objects can then be plugged into the rest of the boundary algebra to construct the vectors $\langle W_1|$ and $|V_1\rangle$.

2.4.2 Family $\mathfrak{F}$

We already saw in Proposition 2.3.4 and in (2.26) that it is enough to consider the following subalgebra for the left-boundary

$$\begin{aligned} \langle W|D^- &= \langle W|d_1^-C^-, \\ \langle W|C^-D &= \langle W|q_1^-C^-C + q_2^-C^- + q_3^-D^-. \end{aligned}$$

It will be convenient to simplify this algebra. Recalling the boundary vectors (2.31), we rewrite

$$\langle W|(D^- - d_1^-C^-) = 0 \Leftrightarrow (1-d_1^-)\langle W_1| = d_1^-\langle W_0|.$$

For $d_1^- = 0$, we have $\langle W_1| = 0$, and the relation (R_2^-) becomes

$$\langle W_0| D = \langle W_0| (Cq_1^- + \mathbf{1}q_2^-),$$

while for $d_1^- \neq 0$ we have

$$\begin{aligned}
&(\langle W_0| + \langle W_1|)D = (\langle W_0| + \langle W_1|)(Cq_1^- + \mathbf{1}q_2^-) + q_3^- \langle W_1| \\
\Leftrightarrow\ & d_1^-(\langle W_0| + \langle W_1|)D = d_1^-(\langle W_0| + \langle W_1|)(Cq_1^- + \mathbf{1}q_2^-) + d_1^- q_3^- \langle W_1|, \\
\Leftrightarrow\ & \langle W_1| D = \langle W_1| Cq_1^- + (d_1^- q_3^- + q_2^-)\mathbf{1}.
\end{aligned}$$

This way, if both the left and right boundaries are in this family, $(a, b) \in \mathfrak{F}$, the complete boundary algebra reduces to

$$\begin{aligned}
&\langle W_0| d_1^- = \langle W_1| (1 - d_1^-) && \\
&\langle W_1| D = \langle W_1| q_1^- C + (d_1^- q_3^- + q_2^-)\mathbf{1}, && d_1^- \neq 0, \\
&\langle W_0| D = \langle W_0| Cq_1^- + q_2^- \mathbf{1}, && d_1^- = 0
\end{aligned}$$

$$\begin{aligned}
&d_1^+ |V_0\rangle = (1 - d_1^+) |V_1\rangle, && \\
&D |V_1\rangle = q_1^+ C - (d_1^+ q_3^+ + q_2^+)\mathbf{1} |V_1\rangle, && d_1^+ \neq 0, \\
&D |V_0\rangle = Cq_1^+ - q_2^+ \mathbf{1} |V_0\rangle, && d_1^+ = 0.
\end{aligned}$$

Proposition 2.4.3. *For models in $\mathfrak{F}$ the stationary measure μ_N^{ss} factorizes at the boundary in the following sense:*

$$\mu_N^{ss}(\eta) = \nu_{d_1^-}^1(\eta(1))\mu_b(\eta(2), \dots, \eta(N-1))\nu_{d_1^+}^1(\eta(N)), \tag{2.34}$$

where we recall from (2.6) *that for $x \in \Lambda$ and $\rho \in [0, 1]$ one has*

$$\nu_\rho(\eta(x)) = \rho^{\eta(x)}(1 - \rho)^{1-\eta(x)},$$

and μ_b is the stationary measure of the open SSEP defined in $\{2, \dots, N-1\}$ with injection (resp. removal) rate α (resp. γ) at the site 2, and injection (resp. removal) rate β (resp. δ) at the site $N-1$, with the identification

$$\begin{cases} \alpha = \frac{q_1^-}{\kappa^-} \\ \gamma = \frac{1-q_1^-}{\kappa^-} \end{cases} \quad \text{and} \quad \begin{cases} \beta = \frac{q_1^+}{\kappa^+} \\ \delta = \frac{1-q_1^+}{\kappa^+}, \end{cases} \quad \text{with} \quad \kappa^\pm = -(d_1^\pm q_3^\pm + q_2^\pm) \neq 0.$$

As a consequence, the boundary matrices can be taken as multiples of the identity satisfying the relations

$$0 = d_1^\pm E^\pm - (1 - d_1^\pm) D^\pm.$$

The reciprocal is also true: if $\exists \rho^\pm \in [0, 1]$ such that the stationary measure is factorized as

$$\mu_N^{ss}(\eta) = \nu_{\rho^-}(\eta(1))\mu_b(\eta(2), \dots, \eta(N-1))\nu_{\rho^+}(\eta(N))$$

where μ_b is the stationary measure of the open SSEP with with injection (resp. removal) rate α (resp. γ) at the first site, and injection (resp. removal) rate β (resp. δ) at the last site, then

$$\begin{cases} \alpha = \frac{q_1^-}{\kappa^-} \\ \gamma = \frac{1-q_1^-}{\kappa^-} \end{cases}, \quad \begin{cases} \beta = \frac{q_1^+}{\kappa^+} \\ \delta = \frac{1-q_1^+}{\kappa^+}, \end{cases} \quad \text{with} \quad \kappa^\pm = -(d_1^\pm q_3^\pm + q_2^\pm) \neq 0$$

and also $\rho^\pm = d_1^\pm$ with $(a, b) \in \mathfrak{F}$.

Proof. We show the result for $d_1^{\pm} \neq 0$ only since for the remaining cases the procedure is analogous. From the boundary algebra

$$\langle W| \eta(1) D^- + (1-\eta(1)) E^- = (d_1^- \eta(1) + (1-d_1^-)(1-\eta(1))) \frac{1}{d_1^-} \langle W_1| .$$

Similarly, since

$$\langle W| D^- + E^- \equiv \langle W_1 + W_0| = \left(\frac{1-d_1^-}{d_1^-} + 1 \right) \langle W_1| = \frac{1}{d_1^-} \langle W_1| ,$$

we can introduce

$$Z_N = \langle W| C^- C^{N-2} C^+ |V\rangle = \frac{1}{d_1^-} \langle W_1| C^{N-2} |V_1\rangle \frac{1}{d_1^+} =: \frac{1}{d_1^-} \tilde{Z}_{N-2} \frac{1}{d_1^+} .$$

Analogous computations for the right boundary then ends the proof for the factorization. Defining $\kappa^{\pm}$ as in the statement of the current proposition, the identification with the SSEP algebra comes from the identities

$$\langle W_1| D = \langle W_1| q_1^- C - \kappa^- \mathbf{1}, \qquad D|V_1\rangle = q_1^+ C + \kappa^+ \mathbf{1} |V_1\rangle , \qquad [D, C] = pC.$$

Since the stationary measure factorizes into a Bernoulli measure at the boundaries, it is clear that we can take the boundary matrices to be multiples of the identity. More precisely, letting them be as such, we have

$$d_1^- \langle W_0| = (1-d_1^-) \langle W_1| \Leftrightarrow [d_1^- E^- - (1-d_1^-) D^-] \langle W| \mathbf{1} = 0$$

and we need only to impose $d_1^- E^- - (1-d_1^-) D^- = 0$.

Now we prove the converse. The measure μ_b is completely described (see [11]) by the quadratic algebra

$$\langle W| \alpha E - \gamma D = \langle W| , \quad [D, E] = D + E, \quad \beta E - \delta D |V\rangle = -|V\rangle$$

and in particular

$$\mu_b(\eta) = \frac{1}{\langle W| C^{N-2} |V\rangle} \langle W| \prod_{x=2}^{N-1} [D\eta(x) + E(1-\eta(x))] |V\rangle .$$

Checking the stationary condition for μ_N^{ss} the operators acting on μ_b will lead to the telescopic relations and what remains can be identified with the algebra $\mathbb{A}$ where $D^{\pm}, E^{\pm}$ are multiples of the identity. The requirement for this algebra, with such choices of boundary matrices, to be well-defined is that $(a, b) \in \mathfrak{F}$. □

2.4.3 Family $\mathfrak{H}$

We characterize the models where the boundary matrices can be the same as in the bulk, $D^{\pm} = D$ and $E^{\pm} = E$. For these models the algebra for 1–site boundary is completely enough to characterize the stationary measure.

Recall from (2.18) that we can express the three left-boundary relation as

$$\begin{pmatrix} \langle W| D^- C \\ \langle W| [D, E]^- \\ \langle W| D^- D \end{pmatrix} = \begin{pmatrix} d_1^- & d_2^- & d_3^- \\ t_1^- & t_2^- & t_3^- \\ f_1^- & f_2^- & f_3^- \end{pmatrix} \begin{pmatrix} \langle W| C^- C \\ \langle W| C^- \\ \langle W| D^- \end{pmatrix}, \qquad [D, E]^- := D^- E - E^- D.$$

If $D^- = D$ and $E^- = E$ then

$$\langle W|\, C = \langle W|\, [D,E] = \langle W|\, [D,E]^- = \langle W|\, t_1^- C^- C + t_2^- C^- + t_3^- D^- = \langle W|\, t_1^- C^2 + t_2^- C + t_3^- D.$$

This motivates the following.

Proposition 2.4.4. *The boundary specific matrices D^-, E^- can be fixed as $D^-, E^- = D, E$ if and only if*

$$a \in (\mathcal{C}_1^- \cap \{t_1 = 0,\ t_2 = 1,\ t_3 = 0\}) \cup (\mathcal{C}_2^- \cap \{t_1 = 0,\ t_2 + d_1 t_3 = 1\})\,. \tag{2.35}$$

Proof. Suppose that $D^- = D$ and $E^- = E$. Then we must have

$$\langle W|\, C = \langle W|\, [D,E] = \langle W|\, t_1^- C^2 + t_2^- C + t_3^- D.$$

The constraints $t_3^- = 0, t_1^- = 0, t_2^- = 1$ are enough to guarantee that $[D,E] = C$ at the boundary, that is, $\langle W|\, C = \langle W|\, [D,E]$. In particular, we have $q_3^- = d_3^-$ because $t^- = d^- - q^-$ and we can have $d_3^-, q_3^- \neq 0$; moreover $d_2^- + d_1^- d_3^-$ also does not need to be zero. This way, the corresponding models can be on either (C_1) or (C_2).

If $t_3^- \neq 0$ we can write

$$\langle W|\, D = \langle W|\, -\frac{t_1^-}{t_3^-} C^2 + \frac{1 - t_2^-}{t_3^-} C$$

and compare with (R_1^-),

$$\langle W|\, -\frac{t_1^-}{t_3^-} C^3 + \frac{1 - t_2^-}{t_3^-} C^2 = \langle W|\, d_1^- C^2 + d_2^- C + d_3^- \left(-\frac{t_1^-}{t_3^-} C^2 + \frac{1 - t_2^-}{t_3^-} C \right). \tag{2.36}$$

That is,

$$0 = \langle W|\, C^3 \frac{t_1^-}{t_3^-} + C^2 \left(d_1^- - \frac{1 - t_2^-}{t_3^-} - d_3^- \frac{t_1^-}{t_3^-} \right) + C \left(d_3^- \frac{1 - t_2^-}{t_3^-} + d_2^- \right).$$

From the same argument for the normalization, on Lemma 2.3.3, we see that we need to set the coefficients to zero, which can be expressed as

$$0 = t_1^-, \quad 1 = t_2^- + d_1^- t_3^-, \quad 0 = d_2^- + d_1^- d_3^-, \quad 0 \neq t_3^-. \tag{2.37}$$

Since $d_2^- + d_1^- d_3^- = 0$, the full set of constraints is a subset of $\mathcal{C}_2^-$. Moreover, we end up with the single boundary relation

$$\langle W|\, D = \langle W|\, -\frac{t_1^-}{t_3^-} C^2 + \frac{1 - t_2^-}{t_3^-} C = \langle W|\, d_1^- C$$

because both (R_2^-) and (R_{12}^-) can be derived from the identity in the previous display plus (R_0).

For $t_3^- = 0$ it follows that $\mathcal{C}_2^- \cap \{t_1 = 0,\ t_2 = 1,\ t_3 = 0\} \subset \mathcal{C}_2^- \cap \{t_1 = 0,\ t_2 + d_1 t_3 = 1\}$, and we end up with the two constraints on (2.35).

Reciprocally, suppose the constraints (2.35) are satisfied. If $t_1^- = 0,\ t_2^- = 1,\ t_3^- = 0$ then

$$\langle W|\, D^- E - E^- D = \langle W|\, C^-.$$

Otherwise, we need

$$\langle W|\, C^- = \langle W|\, t_1^- C^- C + t_2^- C^- + t_3^- D^- \Leftrightarrow \langle W|\, D^- = \langle W|\, \frac{1 - t_2^-}{t_3^-} C^- - \frac{t_1^-}{t_3^-} C^- C. \tag{2.38}$$

We compare the previous relation with (R_1^-). Multiplying both sides of the previous display by C through the right and reorganizing the terms yields

$$\langle W|\, D^- C = \langle W|\, \frac{1-t_2^-}{t_3^-} C^- C - \frac{t_1^-}{t_3^-} C^- C^2, \tag{2.39}$$

which should be equivalent to (R_1^-). In particular, replacing (2.38) and (2.39) into (R_1^-), that we recall to be

$$\langle W|\, D^- C = \langle W|\, d_1^- C^- C + d_2^- C^- + d_3^- D,$$

and reorganizing the terms yields

$$\langle W|\, \frac{1-t_2^-}{t_3^-} C^- C - \frac{t_1^-}{t_3^-} C^- C^2 = \langle W| \left[d_1^- C^2 + d_2^- C^- + d_3^- \left(\frac{1-t_2^-}{t_3^-} C^- - \frac{t_1^-}{t_3^-} C^- C \right) \right],$$

analogously to (2.36). The previous normalization argument imposes (2.37). □

2.5 Density and reservoirs correlation

We focus on the left-boundary, specifically on models satisfying constraints (C_1), that is, in either of the families $\mathfrak{L}^-$ or $\mathfrak{N}^-$. Regarding family $\mathfrak{F}^-$ (see Subsection 2.4.2), the stationary measure has the factorization given in (2.34), therefore the boundary reservoirs are uncorrelated and the correlation function for particles in the bulk behaves as in the SSEP$(1,1)$.

We will compute the density and correlation between the left-boundary reservoirs using the algebra, then compare this with the expression for $\varphi_N(1,2)$ obtained in (2.12) from the stationary equations $\langle \mathcal{L} f(\eta) \rangle_N = 0$. From the boundary algebra,

$$\langle \eta(1) - d_1^- \rangle_N = (d_2^- + d_1^- d_3^-) J_N + d_3^- \langle \eta(1) - d_1^- \rangle_{N-1} J_N$$

and by iteration

$$\langle \eta(1) - d_1^- \rangle_N = J_N (d_2^- + d_1^- d_3^-) + (d_2^- + d_1^- d_3^-) \sum_{n=1}^{N-3} (d_3^-)^n \prod_{i=0}^{n} j_{L-i} + (d_3^-)^{N-2} \langle (\eta(1) - d_1^-) \rangle_3 \prod_{i=0}^{N-3} j_{L-i},$$

which can be replaced into the right-hand side of

$$\begin{aligned} \langle \eta(2) - q_1^- \rangle_N &= (q_2^- + d_1^- q_3^-) J_N + q_3^- J_N \langle \eta(1) - d_1^- \rangle_{N-1}, \\ \langle \eta(1)\eta(2) \rangle_N &= f_1^- + (f_1^- + d_1^- f_3^-) J_N + f_3^- J_N \langle \eta(1) - d_1^- \rangle_{N-1}. \end{aligned}$$

At this point we should compute the current. From (2.27) we have that

$$J_N \left(N - 3 - q_2^+ - q_2^- - q_3^+ \langle \eta(N-1) \rangle_{N-1} - q_3^- \langle \eta(1) \rangle_{N-1} \right) = q_1^- - q_1^+,$$

where $J_N = Z_{N-1}/Z_N$, and therefore

$$J_N = \frac{1}{N} \frac{q_1^- - q_1^+}{1 + O(1/N)}.$$

With this, we can show that

$$\begin{aligned} \langle \eta(1) \rangle_N &= d_1^- + J_N (d_2^- + d_1^- d_3^-) + J_N J_{N-1} d_3^- (d_2^- + d_1^- d_3^-) + O\left(1/N^3\right), \\ \langle \eta(2) \rangle_N &= q_1^- + J_N (q_2^- + d_1^- q_3^-) + J_N J_{N-1} q_3^- (d_2^- + d_1^- d_3^-) + O\left(1/N^3\right), \\ \langle \eta(1)\eta(2) \rangle_N &= f_1^- + J_N (f_2^- + d_1^- f_3^-) + J_N J_{N-1} f_3^- (d_2^- + d_1^- d_3^-) + O\left(1/N^3\right). \end{aligned} \tag{2.40}$$

The bulk relation implies that for any $x \in \{2, \dots, N-1\}$ it holds that

$$\langle \eta(x) \rangle_N = \langle \eta(2) \rangle_N - (x-2) J_N,$$

which leads to

$$\langle \eta(x) \rangle_N = \begin{cases} d_1^- + J_N(d_2^- + d_1^- d_3^-) + O(1/N^2), & x = 1, \\ q_1^- + J_N(q_2^- + d_1^- q_3^- - (x-2)) + O(1/N^2), & 2 \le x \le N-1, \\ d_1^+ - J_N(d_2^+ + d_1^+ d_3^+) + O(1/N^2), & x = N. \end{cases}$$

Performing the limit $N \to +\infty$ we write $u := \lim_{N\to+\infty} N^{-1}x$ and $\rho(u) := \lim_{N\to+\infty} \langle \eta(x) \rangle_N$, obtaining the linear profile

$$\rho(u) = \begin{cases} d_1^-, & u = 0, \\ q_1^-(1-u) + q_1^+ u, & u \in (0,1), \\ d_1^+, & u = 1. \end{cases}$$

We highlight that for $d_1^\pm \neq q_1^\pm$, that is, $t_1^\pm \neq 0$, the density is not continuous at the boundary. In the case $d_1^- = q_1^-$ the boundary derivatives are well defined. Concretely,

$$\begin{aligned} \partial_u \rho(0) &= \lim_{N\to+\infty} N(\langle \eta(1) \rangle_N - \langle \eta(2) \rangle_N) = (d_2^- + d_1^- d_3^- - (q_2^- + d_1^- q_3^-)) \lim_{N\to+\infty} N J_N \\ \partial_u \rho(1) &= \lim_{N\to+\infty} N(\langle \eta(N-1) \rangle_N - \langle \eta(N) \rangle_N) = (-d_2^+ - d_1^+ d_3^+ + q_2^+ + d_1^+ q_3^+) \lim_{N\to+\infty} N J_N, \end{aligned}$$

where $\lim_{N\to+\infty} N J_N = q_1^- - q_1^+ = d_1^- - d_1^+$.

For the correlation function, similarly,

$$\begin{aligned} \varphi_N(1,2) = {} & f_1^- - d_1^- q_1^- \\ & + J_N \left(f_2^- + d_1^- f_3^- - d_1^-(q_2^- + d_1^- q_3^-) - q_1^-(d_2^- + d_1^- d_3^-) \right) \\ & - (J_N)^2 (d_2^- + d_1^- d_3^-)(q_2^- + d_1^- q_3^-) \\ & + J_N J_{N-1} (d_2^- + d_1^- d_3^-)(f_3^- - d_1^- q_3^- - d_3^- q_1^-) + O(1/N^3). \end{aligned}$$

The algebra constraints (C_1) and (C_2) imply that the first two lines on the right-hand side equal zero, hence

$$\lim_{N\to+\infty} N^2 \varphi_N(1,2) = (q_1^- - q_1^+)^2 (d_2^- + d_1^- d_3^-) \left(f_3^- - d_1^- q_3^- - d_3^- q_1^- - (q_2^- + d_1^- q_3^-) \right).$$

For $d_2^- + d_1^- d_3^- \neq 0$ we can see that

$$f_3^- - d_1^- q_3^- - d_3^- q_1^- - (q_2^- + d_1^- q_3^-) = 1. \tag{2.41}$$

To prove this, note that

$$\begin{aligned} (d_2^- + d_1^- d_3^-)(f_3^- - d_1^- q_3^- - d_3^- q_1^- - (q_2^- + d_1^- q_3^-)) &= -(q_1^- d_3^- + q_2^- + d_1^- q_3^- - f_3^-)(d_2^- + d_1^- d_3^-) \\ &\quad - d_1^- q_3^- (d_2^- + d_1^- d_3^-) \\ &= d_2^- - d_1^- q_3^- (d_2^- + d_1^- d_3^-) \\ &= d_2^- + d_1^- d_3^- \end{aligned}$$

with the second equality in the previous display being a consequence of the fourth condition in (C_1). As such, we conclude that

$$\lim_{N\to+\infty} N^2\varphi_N(1,2) = (q_1^- - q_1^+)^2(d_2^- + d_1^- d_3^-). \tag{2.42}$$

This is in contrast with $\varphi_N(2,3)$, which should be scaled by a factor N: we can approximate this quantity with the same reasoning.

$$\begin{aligned}\langle\eta(2)\eta(3)\rangle_N &= q_1^-\langle\eta(3)\rangle_N + q_2^-\langle\eta(2)\rangle_{N-1}J_N + q_3^-\langle\eta(1)\eta(2)\rangle_{N-1}J_N,\\ \langle\eta(3)\rangle_N &= \langle\eta(2)\rangle_N - J_N\end{aligned}$$

and so

$$\begin{aligned}\varphi_N(2,3) &= q_1^-\langle\eta(3)\rangle_N + q_2^-\langle\eta(2)\rangle_{N-1}J_N + q_3^-\langle\eta(1)\eta(2)\rangle_{N-1}J_N - \langle\eta(2)\rangle_N(\langle\eta(2)\rangle_N - J_N)\\ &= -\langle\eta(2)\rangle_N(\langle\eta(2)\rangle_N - q_1^-) + J_N\left(\langle\eta(2)\rangle_N - q_1^- + q_2^-\langle\eta(2)\rangle_{N-1} + q_3^-\langle\eta(1)\eta(2)\rangle_{N-1}\right)\\ &= -(q_1^- + O(1/N))\left((q_2^- + d_1^- q_3^-)J_N + O(1/N(N-1))\right)\\ &+ J_N\left(q_1^- + O(1/N) - q_1^- + q_2^-(q_1^- + O(1/N-1)) + q_3^-(f_1^- + O(1/N-1))\right)\\ &= q_1^-(q_2^- + d_1^- q_3^-)J_N + J_N(q_1^- q_2^- + f_1^- q_3^-) + O(1/N(N-1))\end{aligned}$$

Since $f_1^- = d_1^- q_1^-$ this can be rearranged into

$$\varphi_N(2,3) = 2q_1^-(q_2^- + d_1 q_3)J_N + O(1/N(N-1)).$$

Notation 2.5.1. Hereafter we are going to use the following notation $g = o(f(N)) \Leftrightarrow g/f(N) \xrightarrow{N\to+\infty} 0$.

Proposition 2.5.2. *If the boundary parameters are such that* (H_0^-) *and* (H1) *are satisfied, then for left-linear models* ($a \in \mathfrak{L}^-$) *the constraints* (C_1) *are equivalent to*

$$\varphi_N(1,2) = \langle(\eta(1) - d_1^-)\rangle_N\langle j_2(\eta)\rangle_N + o(1/N^2) = O(1/N^2),$$

while for left-non-linear models ($a \in \mathfrak{N}^-$) *they are equivalent to the previous two equalities plus*

$$\lim_{N\to+\infty} \frac{\langle j_2(\eta)\rangle_N}{\langle\eta(2) - q_1^-\rangle_N} = -1. \tag{2.43}$$

Proof. We compute in Appendix A.3, solely from the equations $\langle\mathcal{L}\eta(1)\rangle_N = \langle\mathcal{L}\eta(2)\rangle_N = \langle\mathcal{L}\eta(1)\eta(2)\rangle_N = 0$, the correlation

$$\begin{aligned}\varphi_N(1,2) &= (f_1^- - d_1^- q_1^-) + \langle j_2(\eta)\rangle_N\left(f_2^- + d_1^- f_3^- - q_1^-(d_2^- + d_1^- d_3^-) - d_1^-(q_2^- + d_1^- q_3^-)\right)\\ &+ \langle j_2(\eta)\rangle_N^2(d_2^- + d_1^- d_3^-)\left(\frac{f_3^- - q_1^- d_3^- - d_1^- q_3^-}{1 - d_3^-\langle j_2(\eta)\rangle_N} - q_2^- - d_1^- q_3^-\right)\\ &+ \langle j_2(\eta)\rangle_N^3\frac{d_2^- + d_1^- d_3^-}{1 - d_3^-\langle j_2(\eta)\rangle_N}\left(q_3^-(d_2^- + d_1^- d_3^-) + d_2^-(q_2^- + d_1^- q_3^-)\right)\\ &+ \langle j_2(\eta)\rangle_N^4 d_3^- q_3^-(d_2^- + d_1^- d_3^-)^2\frac{1}{(1 - d_3^-\langle j_2(\eta)\rangle_N)^2}\\ &- \frac{(\nabla_2^+\varphi_N)(1,2)}{1 - d_3^-\langle j_2(\eta)\rangle_N}(f_3^- - q_1^- d_3^- - d_1^- q_3^-)\\ &- \frac{(\nabla_2^+\varphi_N)(1,2)\langle j_2(\eta)\rangle_N}{1 - d_3^-\langle j_2(\eta)\rangle_N}\left(q_3^-(d_2^- + d_1^- d_3^-) + d_3^-(q_2^- + d_1^- q_3^-)\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\nabla_2^+\varphi_N)(1,2)\langle j_2(\eta)\rangle_N^2}{(1-d_3^-\langle j_2(\eta)\rangle_N)^2}(d_2^- + d_1^- d_3^-)2\left(1 + \frac{d_3^- q_3^-}{1-d_3^-\langle j_2(\eta)\rangle_N}\right)\\
&+ \frac{((\nabla_2^+\varphi_N)(1,2))^2}{(1-d_3^-\langle j_2(\eta)\rangle_N)^2} d_3^- q_3^-,
\end{aligned}
$$

where we recall that $(\nabla_2^+\varphi_N)(1,2) = \varphi_N(1,3) - \varphi_N(1,2)$. The right-hand side of previous display is very general, with no constraints except (H_0^-) and with no algebra involved. The key quantity to focus on is the forward difference

$$
(\nabla_2^+\varphi_N)(1,2) = \frac{1}{N}(N\nabla_2^+\varphi)(1,2).
$$

We can either have $\lim_{N\to+\infty}(N\nabla_2^+\varphi)(1,2)$ finite or infinite. We suppose the former. This being finite means that there exists the partial derivative at the boundary, since $N\nabla_2^+$ should converge to the partial derivative operator acting on the second variable (which we write as ∂_2). In other words, $\lim_{N\to+\infty}\varphi_N(1,2)$ should be right-continuous at the macroscopic point $(u,v) = (0,0)$. In this case, we have

$$
N\varphi_N(1,2) = N(f_1^- - d_1^- q_1^-) + N\langle j_2(\eta)\rangle_N\,(f_2^- + d_1^- f_3^- - q_1^-(d_2^- + d_1^- d_3^-) - d_1^-(q_2^- + d_1^- q_3^-)) + O(1/N).
$$

This means that we can consider the cases

$$
\begin{aligned}
&0 \neq f_1^- - d_1^- q_1^- \implies \varphi_N(1,2) = O(1),\\
&0 = f_1^- - d_1^- q_1^- \neq f_2^- + d_1^- f_3^- - q_1^-(d_2^- + d_1^- d_3^-) - d_1^-(q_2^- + d_1^- q_3^-) \implies \varphi_N(1,2) = O(1/N),
\end{aligned}
$$

and

$$
0 = f_1^- - d_1^- q_1^- = f_2^- + d_1^- f_3^- - q_1^-(d_2^- + d_1^- d_3^-) - d_1^-(q_2^- + d_1^- q_3^-). \tag{2.44}
$$

We focus on the last case above, and highlight that the first two constraints in (C_1) correspond to the two constraints in (2.44).

Let us further suppose that $(\nabla_2^+\varphi_N)(1,2) \leq O(1/N^2)$. Then

$$
\lim_{N\to+\infty} N^2\varphi_N(1,2) = J^2(d_2^- + d_1^- d_3^-)\,(f_3^- - q_1^- d_3^- - d_1^- q_3^- - q_2^- - d_1^- q_3^-)\,, \qquad J = \lim_{N\to+\infty} N\langle j_2(\eta)\rangle_N.
$$

Imposing (2.41) also, one obtains the same as in (2.42), from the MPA. Since

$$
\langle(\eta(1) - d_1^-)j_2(\eta)\rangle_N = \langle(\eta(1) - d_1^-)\rangle_N\langle j_2(\eta)\rangle_N - (\nabla_2^+\varphi_N)(1,2),
$$

from (2.40) the constraint (2.41) can be encapsulated into

$$
\lim_{N\to+\infty} N^2\varphi_N(1,2) = J^2(d_2^- + d_1^- d_3^-) = \lim_{N\to+\infty} N^2\langle(\eta(1) - d_1^-)\rangle_N\langle j_2(\eta)\rangle_N = \lim_{N\to+\infty} N^2\langle(\eta(1) - d_1^-)j_2(\eta)\rangle_N.
$$

We highlight that from the MPA we can compute

$$
\varphi_N(1,3) = \varphi_N(1,2) + (d_2^- + d_1^- d_3^-)J_N(J_N - J_{N-1}) + O(1/N^4),
$$

and so $(\nabla_2^+\varphi_N)(1,2) = O(1/N^3)$ assuming (C_1), since we already saw that the consistency of the algebra implies that $\varphi_N(1,2) = O(1/N^2)$.

To see that the last constraint, $q_3(d_2 + d_1 d_3) = -d_3$, can be encapsulated into (2.43), from (A.5) and (A.7) it holds that

$$\langle \eta(1) - d_1^- \rangle_N = (d_2^- + d_1^- d_3^-)\langle j_2(\eta)\rangle_N + d_3^-(d_2^- + d_1^- d_3^-)\langle j_2(\eta)\rangle_N^2 \frac{1}{1 - d_3^- \langle j_2(\eta)\rangle_N} - (\nabla_2^+ \varphi_N)(1,2)\frac{1}{1 - d_3^- \langle j_2(\eta)\rangle_N} \tag{2.45}$$

hence

$$d_2 + d_1 d_3 = \lim_{N\to+\infty} \frac{\langle \eta(1) - d_1^- \rangle_N}{\langle j_2(\eta)\rangle_N} \quad \text{and} \quad \frac{d_3}{q_3} = \frac{\langle \eta(1) - d_1^- \rangle_N - (d_2 + d_1 d_3)\langle j_2(\eta)\rangle_N}{\langle \eta(2) - q_1^- \rangle_N - (q_2 + d_1 q_3)\langle j_2(\eta)\rangle_N} = \lim_{N\to+\infty} \frac{\langle (\eta(1) - d_1^-)\rangle_N}{\langle \eta(2) - q_1^- \rangle_N}.$$

Since $d_2^- + d_1^- d_3^- \neq 0$ by hypothesis, from (2.45) and again (A.5) we obtain (2.43). □

3

From exclusion to slow and fast diffusion

Contents

3.1 Introduction

3.1.1 Main result and strategy

We construct a family of exclusion processes parametrized by $m \in (0,2]$ and evolving on the one-dimensional (discrete) torus $\mathbb{T}_N$ and we prove that their hydrodynamic limit is given by

$$\partial_t \rho = \partial_u^2(\rho^m), \qquad (t,u) \in \mathbb{R}_+ \times \mathbb{T}.$$

The motivation for the definition of our models comes from the analysis of the diffusion coefficient $D(\rho) = m\rho^{m-1}$ and the generalized binomial theorem (Proposition 3.2.7 below). As a consequence, the resulting family of models interpolates continuously in m between the SSEP and the so-called PMM(1). For the reader to have a better picture of the processes that will interplay in this chapter present the constraints for the occupation exchange in the node $\{0,1\}$ of the relevant processes. The SSEP and the PMM(k), for any $k \in \mathbb{N}_+$, are defined, for each $\eta \in \Omega_N$, through the rates

$$\mathbf{a}(\eta) := \eta(0)(1-\eta(1)) + (1-\eta(0))\eta(1) \quad \text{and} \quad \mathbf{r}^{(k)}(\eta) := \mathbf{a}(\eta) \sum_{j=1}^{k+1} \prod_{\substack{i=-(k+1)+j \\ i\neq 0,1}}^{j} \eta(i),$$

respectively. In particular, the rate of the PMM(1) is given by $\mathbf{r}^{(1)} = (\eta(-1)+\eta(2))\mathbf{a}(\eta)$. The rates acting in a generic node $\{x, x+1\}$ are obtained by a translation of the maps just introduced. For the moment, let us refer to the rate of the interpolating model simply by $r_N^{(m-1)}$, for $m \in (0,2)$ and $N \gg 1$ a fixed natural number. The interpolating family is presented in (3.2) below and, more detailed, in Definition 3.2.8; the PMM is presented in Definition 3.2.3 and the interpolating property presented in Proposition 3.2.12.

The starting point is to represent the diffusion coefficient $D(\rho) = m(1-(1-\rho))^{m-1}$ in terms of the series

$$D(\rho) = \sum_{k\geq 1} \binom{m}{k} (-1)^{k-1} k (1-\rho)^{k-1}, \tag{3.1}$$

where $\binom{m}{k}$ is the generalized binomial coefficient (see (3.7) for the definition). The diffusion coefficient $k\rho^{k-1}$ is associated with the PMM(k), and for this reason the family $\{\mathrm{PMM}(k)\}_{k\geq 0}$ can be seen as a "polynomial basis". To be precise, to generate a discrete version of (3.1) we consider as basis the family of processes $\overline{\mathrm{PMM}}$, corresponding to the PMM constraints with holes and particles exchanged; and since the lattice is finite, the series is truncated at a step $\ell_N \leq N$ with $\ell_N \xrightarrow{N\to+\infty} +\infty$.

The porous medium models $\{\mathrm{PMM}(k)\}_{k\in\mathbb{N}}$ (first considered in [21]) are of gradient type, and the Bernoulli product measures with constant parameter are invariant for each PMM(k). In this way, the interpolating model keeps both properties, and moreover it becomes irreducible, in the sense that every particle configuration can be changed into any other configuration with the same number of particles through successive jumps that happen with positive probability. We note that this *irreducibility property* was not verified for the original PMM(k), and in our case is consequence of the interpolating process being "closer" to the SSEP than the PMM(1) is.

We denote by $\{\eta_t\}_{t\geq 0}$ the Markov process on the lattice $\Omega_N = \{0,1\}^{\mathbb{T}_N}$, which is univocally defined through its *infinitesimal generator*, denoted below by $\mathcal{L}_N^{m-1}$, which is an operator acting on functions

defined on Ω_N. In order to give a precise definition, we first need to introduce the infinitesimal generators related to the basis mentioned above: let $\mathcal{L}_N^{\overline{\mathrm{PMM}}(k)}$ be the generator of a process defined as the PMM(k), but with the constraints acting on *empty* sites, instead of particles, as previously mentioned. The infinitesimal generator of the interpolating model, which is a linear combination of the latter, is defined for any $m \in (0,2]$ by

$$\mathcal{L}_N^{(m-1)} = \sum_{k=1}^{\ell_N} \binom{m}{k} (-1)^{k-1} \mathcal{L}_N^{\overline{\mathrm{PMM}}(k-1)}, \quad \text{where} \quad 2 \le \ell_N \xrightarrow[N\to+\infty]{} +\infty. \tag{3.2}$$

The treatment of a linear combination of nearest neighbour models with $\ell_N \to +\infty$ as $N \to +\infty$ is one of the novelties of this work. It is also worth pointing out that although (1.2) only has local interactions, it is *not* required that $\ell_N = o(N)$, and it can be of any order, as long as $N \geqslant \ell_N \to +\infty$. In fact, several difficulties arise from maintaining ℓ_N with no order restrictions. To achieve this, some new ideas and properties of the family $\{\mathrm{PMM}(k)\}_{k\geq 0}$ are explored. The interpolating property invoked above is a consequence of the definition of the generalized binomial coefficients. Concretely, for any fixed configuration η and fixed N it holds that

$$\lim_{m\nearrow 1} r_N^{(m-1)}(\eta) = \mathbf{a}(\eta) = \lim_{m\searrow 1} r_N^{(m-1)}(\eta) \qquad \text{and} \qquad \lim_{m\nearrow 2} r_N^{(m-1)}(\eta) = \mathbf{r}^{(1)}(\eta).$$

To better visualize how these rates can deform the SSEP into a slow or fast diffusion model we refer the reader to Figure 3.5 and to the discussion just before it.

We remark that the sign of the generalized binomial coefficients $\binom{m}{k}$ changes according to the values of m and k. This oscillating nature is the reason why one may find rates for which (3.2) is not well-defined for $m > 2$. For $m \in (0,2)$, the sign of these coefficients lead to an interpretation of the resulting models as the SSEP with either a *penalization* or *reinforcement* given by porous medium models (with constraints on the empty sites), as explained in (3.10), and this also explains why the interpolating model becomes irreducible. This is presented in more details in Proposition 3.2.10.

Proving a *hydrodynamic limit* is, in plain terms, a law of large numbers for a specific quantity of the system – in our case the density of particles, which is also a conserved quantity since we will work in the torus. Concretely, the (time accelerated) empirical measure associated with the particle density at time $t > 0$ is defined for any $\eta \in \Omega_N$, as follows

$$\pi_t^N(\eta, \mathrm{d}u) = \frac{1}{N} \sum_{x\in\mathbb{T}_N} \eta_{N^2 t}(x) \delta_{x/N}(\mathrm{d}u).$$

In other words $\pi_t^N(\eta, \mathrm{d}u)$ is a random measure on the continuous torus $\mathbb{T}$ and performs the link between the microscopic and macroscopic space scales, *via* the space scaling $x \mapsto N^{-1}x$ and the so-called *diffusive time scaling* $t \mapsto N^2 t$. Our main result states that starting from a *local equilibrium* distribution (see Definition 3.2.23), this random empirical measure converges in probability as $N \to +\infty$, to a deterministic measure $\rho_t(u)du$, where $\rho_t(u)$ is the *unique* weak solution of the *hydrodynamic equation* (1.2) for $m \in (0,2)$.

Our proof follows the entropy method introduced by [23], which highly relies on the fact that the microscopic model of particles is gradient and has the irreducibility property. The overall strategy can be split into three steps: (i) prove tightness of the sequence of measures induced by the density empirical

measure; (ii) obtain an *energy estimate* which gives information on the regularity of the density profile, which is crucial for the proof of uniqueness of weak solutions; (iii) characterize uniquely the limiting points. Different technical problems arise for both slow ($m > 1$) and fast ($m < 1$) regimes. In plain terms, a particularly delicate and crucial step is showing that the occupation variables can be approximated by local (empirical) averages, which is done in several steps. The main difficulty, in our case, lies in that we have a large product of occupation variables, which is where the "nonlinearity" of the dynamics arises. This approximation scheme is known in the literature as the *replacement lemmas*. In particular, the replacement lemmas are specific to each regime (see Lemmas B.1.3, B.1.5 for the slow regime and Lemmas B.1.6, B.1.7 for the fast regime). Fundamental to the proof of those lemmas is the energy lower bound (Proposition 3.4.2) which compares the Dirichlet form of our process with the "Carré-du-Champ" operator, and the results of Subsection 3.2.1, where we derive some new properties of the family $\{\mathrm{PMM}(k)\}_{k\geq 0}$. In the fast regime, it is surprising that the tightness step requires the replacement Lemma B.1.6, due to the supremum of the rates being unbounded as $N \to +\infty$. The characterization of the limit points is the most technical part, and also uses several replacement lemmas.

The application of those replacement lemmas involves some novelties due to the summation with binomial coefficients in the definition of $\mathcal{L}^{(m-1)}$. The replacement lemmas link the microscopic and macroscopic scales by approximating the product of k occupation variables by k empirical averages over independent boxes – first by *microscopic* boxes ("one-block estimate"), then by approximating the microscopic boxes by *mesoscopic* boxes ("two-blocks estimate"). Here, it is important to adjust the size of these boxes dynamically with k for the series of errors to vanish in the limit $N \to +\infty$. However, this dynamical argument alone would require to impose assumptions on the explosion of ℓ_N. To avoid this, it is fundamental to first slow down the explosion by replacing ℓ_N by $(\ell_N)^n$ with $0 < n < 1$. This argument depends on the order of the tail of the series $\sum_{k\geq 1} |\binom{m}{k}|$. The treatment of this series also requires a sharp *non-asymptotic* estimate on the binomial coefficients which, surprisingly, was absent in the literature (see Lemma B.3.1) and can be of independent interest.

There were also some technical issues regarding the *energy estimate*, precisely when showing that the (weak) solution of (1.2) (Definition 3.2.22) belongs to the target Sobolev space. The weak differentiability of specific functions of ρ is needed to prove uniqueness, giving us that the whole sequence of measures converges thanks to tightness. Specifically, if ρ^m belongs to the target Sobolev space (which is the case for $m \in (1, 2)$), uniqueness follows by simple energy arguments (see Lemma B.2.3), while if ρ only belongs to the target Sobolev space (which is the case when $m \in (0, 1)$), then the proof is more involved (see Lemma B.2.5), and it is an adaptation of the argument for *very weak* solutions in [38].

3.1.2 Outline of the chapter

The present chapter is organized as follows: Section 3.2 is devoted to introducing the family of porous medium models which will be the building blocks to construct our new models and used to prove some of the important properties of the latter; particularly, in Subsection 3.2.1 we construct the interpolating models, prove that they are well-defined, and in Subsection 3.2.2 we study some of their monotonicity properties and present our main result. Then we prove the convergence towards the hydrodynamic limit

in Section 3.3. In Section 3.4 we obtain the *energy estimates* for the different regimes.

Appendix B.1 is devoted to the statement and proof of the so-called *replacement Lemmas*, which are in the heart of the proof of the hydrodynamic limit. We concentrate the PDE's results in Appendix B.2: uniqueness and regularity of the weak solution of the hydrodynamic equations. In Appendix B.3 we prove the auxiliary results regarding the generalized binomial coefficients and a quantity arising from the gradient condition.

3.2 Microscopic models and Main Result

The microscopic dynamics at the core of this subsection is a system of particles which evolves according to a Markov process, satisfying the exclusion rule and situated on the discrete torus $\mathbb{T}_N$. A particle configuration η is an element of $\Omega_N = \{0,1\}^{\mathbb{T}_N}$, namely $\eta(x) \in \{0,1\}$ for any $x \in \mathbb{T}_N$. Particles can jump to nearest-neighbour sites only, providing the latter are not already occupied. Before defining the generator of the dynamics, let us introduce some operators and recall others from the beginning of the previous chapter.

Definition 3.2.1 (Exchange of occupation variables)**.** For any $x, y, z \in \mathbb{T}_N$ let us consider the exchange of occupation variables $\eta \mapsto \eta^{x,y}$ given by

$$\eta^{x,y}(z) = \mathbf{1}_{z\neq x,y}\,\eta(z) + \mathbf{1}_{z=x}\,\eta(y) + \mathbf{1}_{z=y}\,\eta(x).$$

We define the operator $\nabla_{x,y}$ associated with the occupation exchange, given on any $f : \Omega_N \to \mathbb{R}$ by

$$\nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta).$$

Finally, for any $x \in \mathbb{T}_N$, define the translation $\tau_x\eta(y) = \eta(x+y)$ for $y \in \mathbb{T}_N$, and extend it to functions $f : \Omega_N \to \mathbb{R}$ by $\tau_x f(\eta) = f(\tau_x\eta)$.

Let us now introduce rigorously the known models which will come into play.

Definition 3.2.2 (Symmetric Simple Exclusion Process)**.** We denote by SSEP on $\mathbb{T}_N$ the Markov process with state space Ω_N generated by the following operator $\mathcal{L}_N^{\mathrm{SSEP}}$, which acts on $f : \Omega_N \to \mathbb{R}$, for any $\eta \in \Omega_N$, as:

$$(\mathcal{L}_N^{\mathrm{SSEP}} f)(\eta) = \sum_{x\in\mathbb{T}_N} \mathbf{a}_{x,x+1}(\eta)(\nabla_{x,x+1} f)(\eta),$$

where $\mathbf{a}_{0,1}(\eta) = \eta(0)(1-\eta(1)) + \eta(1)(1-\eta(0))$ and $\mathbf{a}_{x,x+1}(\eta) = \mathbf{a}_{x+1,x}(\eta) = \tau_x\mathbf{a}_{0,1}(\eta)$. Due to the symmetry of the rates we will short-write $\mathbf{a} := \mathbf{a}_{0,1} = \mathbf{a}_{1,0}$.

Definition 3.2.3 (Porous Medium Model for any integer $k \geq 1$, [21])**.** For any $k \in \mathbb{N}_+$ let us denote by $\mathrm{PMM}(k)$ the *porous medium model* on $\mathbb{T}_N$ with parameter k, as the Markov process with state space Ω_N generated by the following operator $\mathcal{L}_N^{\mathrm{PMM}(k)}$, which acts on $f : \Omega_N \to \mathbb{R}$, for any $\eta \in \Omega_N$, as:

$$(\mathcal{L}_N^{\mathrm{PMM}(k)} f)(\eta) = \sum_{x\in\mathbb{T}_N} \mathbf{r}_{x,x+1}^{(k)}(\eta)\mathbf{a}_{x,x+1}(\eta)(\nabla_{x,x+1} f)(\eta) \quad \text{with} \quad \mathbf{r}_{x,x+1}^{(k)}(\eta) = \mathbf{c}_{x,x+1}^{(k)}(\eta)\mathbf{a}_{x,x+1}(\eta) \tag{3.3}$$

and where $\mathbf{c}^{(k)}_{x,x+1}(\eta) = \tau_x \mathbf{c}^{(k)}_{0,1}(\eta)$ with

$$\mathbf{c}^{(k)}_{0,1}(\eta) = \sum_{j=1}^{k+1} \mathbf{s}^{(k)}_j(\eta) \quad \text{and} \quad \mathbf{s}^{(k)}_j(\eta) = \prod_{\substack{i=-(k+1)+j \\ i\neq 0,1}}^{j} \eta(i). \tag{3.4}$$

The quantity $\mathbf{c}^{(k)}_{x,x+1}(\eta)$ is the constraint to be satisfied for the jump to happen. Again due to the symmetry of the rate and constraint, we short-write $\mathbf{c}^{(k)}(\eta) \equiv \mathbf{c}^{(k)}_{0,1}(\eta)$ and $\mathbf{r}^{(k)}(\eta) \equiv \mathbf{r}^{(k)}_{0,1}(\eta)$. As it can be seen from (3.4) and Figure 3.1, a jump crossing the bond $\{x, x+1\}$ is allowed only if at least k consecutive particles out of the edge $\{x, x+1\}$ are situated in the box $[\![x-k, x+(k+1)]\!]\backslash\{x, x+1\}$, where for any $a, b \in \mathbb{N}$ such that $a < b$ we short-write $\{a, \dots, b\} = [\![a, b]\!]$.

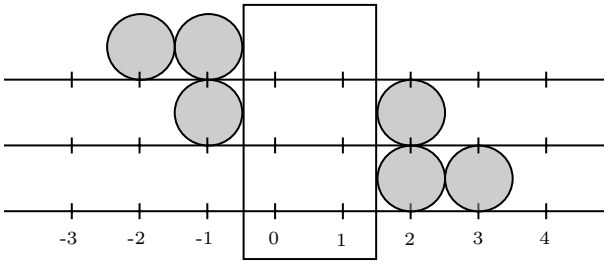


Figure 3.1: PMM(2) valid local configurations for which a particle swaps positions in the edge $\{0, 1\}$.

An illustration of the dynamics for $k = 1$ is also provided in Figure 3.2. We remark that the observable value of the constraint just described equals the number of fully occupied "windows" of the collection $\{[\![x-(k+1)+j,\ x+j]\!]\backslash\{x, x+1\}\}_{j=1,\dots,k+1}$, with $x \in \mathbb{T}_N$ fixed. This constraint was delicately constructed so that the dynamics satisfies the *gradient* property (that we will explore and further reference throughout this chapter and its subsequent). Limiting the consideration to the indicator function of having at least one fully occupied window, without reinforcement dependent on the specific number of occupied windows, results in a non-gradient model with very few results in the existing literature [30] .

Remark 3.2.4 ($k = 0$)**.** Note that for $k = 0$, $\mathbf{c}^{(0)}(\eta) \equiv 1$ and therefore $\mathbf{r}^{(0)}(\eta) = \mathbf{a}(\eta)$, which corresponds to the exchange rate in SSEP. It will be useful to interpret PMM(0) = SSEP.

Definition 3.2.5 (Flipped configuration)**.** For any $\eta \in \Omega_N$, let $\eta \mapsto \overline{\eta}$ be the map that flips holes with particles, namely: for any $x \in \mathbb{T}_N$, $\overline{\eta}(x) = 1 - \eta(x)$.

We are now ready to introduce the flipped porous medium model.

Definition 3.2.6. For any $k \in \mathbb{N}_+$, let us denote by $\overline{\text{PMM}}(k)$ the *flipped porous medium model* with parameter k with dynamical constraints on the vacant sites, as the Markov process on Ω_N generated by the following operator $\mathcal{L}_N^{\overline{\text{PMM}}(k)}$, which acts on functions $f : \Omega_N \to \mathbb{R}$, for any $\eta \in \Omega_N$, as

$$\big(\mathcal{L}_N^{\overline{\text{PMM}}(k)} f\big)(\eta) = \sum_{x\in\mathbb{T}_N} \mathbf{c}^{(k)}_{x,x+1}(\overline{\eta})\mathbf{a}_{x,x+1}(\eta)(\nabla_{x,x+1} f)(\eta). \tag{3.5}$$

Note that the process above can be interpreted as the *empty sites* following the same constraint as in PMM(k): a jump crossing the bond $\{x, x+1\}$ is allowed only if at least k "consecutive" *empty sites* out of the edge $\{x, x+1\}$ are situated in the box $[\![x-k, x+(k+1)]\!]$. An illustration of the dynamics is provided

in Figure 3.3. We also highlight that the parameter k in the PMM(k) corresponds to the exponent of the diffusion coefficient, $D(\rho) = (k+1)\rho^k$, hence to the equation (1.2) with $m = k+1$.

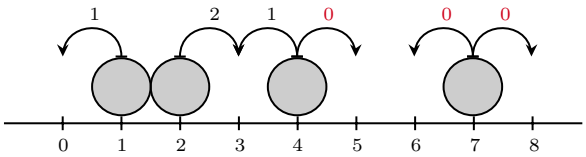


Figure 3.2: PMM(1) transition rates.

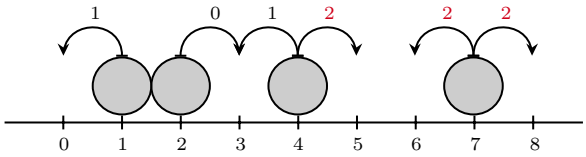


Figure 3.3: $\overline{\text{PMM}}(1)$ transition rates.

3.2.1 The interpolating model

Recall Remark 3.2.4, where we made the observation that SSEP=PMM(0). The construction of the interpolating model will be based on two main ingredients: the generalized binomial theorem and the fact that the family $\{\text{PMM}(k)\}_{k\geq 0}$ can be seen as a "polynomial basis" for the diffusion coefficient $D(\rho) = m\rho^{m-1}$. We base our analysis in the next identity: for any $\rho \in (0,1)$

$$m\rho^{m-1} = m(1-(1-\rho))^{m-1} = m\sum_{k\geq 0}\binom{m-1}{k}(-1)^k(1-\rho)^k = \sum_{k\geq 1}\binom{m}{k}(-1)^{k-1}k(1-\rho)^{k-1} \tag{3.6}$$

where the generalized binomial coefficient is given by the formula

$$\binom{c}{k} = \frac{(c)_k}{k!} = \frac{c(c-1)\cdots(c-(k-1))}{k!}, \qquad c \in \mathbb{R} \tag{3.7}$$

and therefore we have the identity $m\binom{m-1}{k} = (k+1)\binom{m}{k+1}$. This is a particular case of the generalized binomial expansion for real coefficients:

Proposition 3.2.7 (Generalized Binomial Theorem)**.** *For any $x, y, c \in \mathbb{R}$ such that $|x| > |y|$ we have that*

$$(x+y)^c = \sum_{k=0}^{\infty}\binom{c}{k}x^{c-k}y^k,$$

where $\binom{c}{k}$ has been defined in (3.7)*.*

Proof. The proof is standard and as such we only outline the main steps. Without loss of generalization let $x \neq 0$. Writing $z = y/x$ we have $(x+y)^c = x^c(1+z)^c$. Let $f(z) = (1+z)^c$ be defined for $|z| < 1$. Then, by induction we see that $\frac{d^k f}{dz^k}(z) = (c)_k(1+z)^{c-k}$ for any $k \geq 1$ integer. To conclude we recall the Taylor expansion of f and apply Lemma B.3.1 stated below, which guarantees the convergence. □

Proposition 3.2.7 implies the convergence of the series appearing in (3.6) for any $\rho \in (0,1)$. For $\rho \in \{0,1\}$ and $m \in (1,2)$ or $\rho = 1$ and $m \in (0,1)$ one can also easily guarantee the convergence by replacing ρ by 1 or 0 in each term of the series as written in (3.6). For $m \in (0,1)$ and $\rho = 0$ the series is divergent. This will not be a problem, since due to the gradient property of the model we shall see that the object of study will be ρ^m and not ρ^{m-1}.

Definition 3.2.8 (Interpolating model)**.** Let $m \in [0,2]$, $N \in \mathbb{N}_+$ and $\ell_N \in \mathbb{N}$, with $\ell_N \geq 2$. We define the generator

$$\mathcal{L}_N^{(m-1)} := \sum_{k=1}^{\ell_N}\binom{m}{k}(-1)^{k-1}\mathcal{L}_N^{\overline{\text{PMM}}(k-1)} \tag{3.8}$$

where $\mathcal{L}_N^{\overline{\mathrm{PMM}}(k)}$ has been defined in (3.5). More precisely, this generator acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$(\mathcal{L}_N^{(m-1)} f)(\eta) = \sum_{x \in \mathbb{T}_N} c_N^{(m-1)}(\tau_x \eta) \mathbf{a}_{x,x+1}(\eta)(\nabla_{x,x+1} f)(\eta),$$

where

$$c_N^{(m-1)}(\eta) = \sum_{k=1}^{\ell_N} \binom{m}{k} (-1)^{k-1} \mathbf{c}^{(k-1)}(\overline{\eta}) \tag{3.9}$$

and we shorten the rate $r_N^{(m-1)}(\eta) = c_N^{(m-1)}(\eta)\, \mathbf{a}(\eta)$. We call *non integer porous medium model* (resp. *fast diffusion model*), and we denote it by PMM$(m-1)$ (resp. by FDM$(m-1)$), the Markov process whose infinitesimal generator is given by (3.8) with $m \in (1,2)$ (resp. $m \in (0,1)$).

Remark 3.2.9 (About the restrictions on ℓ_N)**.** Although there is no particular assumption on the order at which $\ell_N \to +\infty$, note that if $\ell_N > N$ then for $N \le k \le \ell_N$ we have that $\mathbf{r}^{(k)}(\eta) \neq 0$ if, and only if, every site is occupied except one at the node $\{0,1\}$. Due to the mass conservation, this would be achievable only by starting from a configuration with one empty site only, and we would see no macroscopic evolution of the local density. This is a particular technical consequence of working on the torus, therefore we assume throughout this work that $\ell_N \le N$.

The goal now is to show that the model is well-defined. In other words, we are going to prove that the map $\eta \mapsto c_N^{(m-1)}(\eta)$ is non-negative. The key argument is the following remark about the sign of $(-1)^{k-1}\binom{m}{k}$. By definition,

- if $m \in (0,1)$, then $(-1)^{k-1}\binom{m}{k} > 0$ for any $k \ge 1$,
- if $m \in (1,2)$, then

$$(-1)^{k-1}\binom{m}{k} > 0 \quad \text{if } k = 1, \quad \text{and} \quad (-1)^{k-1}\binom{m}{k} < 0 \quad \text{if } k \ge 2.$$

Therefore we can rewrite

$$\mathcal{L}_N^{(m-1)} = m\mathcal{L}_N^{\mathrm{SSEP}} - \mathrm{sign}(m-1) \sum_{k=2}^{\ell_N} \left|\binom{m}{k}\right| \mathcal{L}_N^{\overline{\mathrm{PMM}}(k-1)}, \qquad m \in (0,2)\backslash\{1\}. \tag{3.10}$$

We also need non-asymptotic bounds for the generalized binomial coefficients: from Lemma B.3.1 one can extract that for $m \in \mathbb{R}$ and $k \ge 2$

$$\frac{1}{(k+1)^m} \lesssim \left|\binom{m-1}{k}\right| \lesssim \frac{1}{k^m}. \tag{3.11}$$

The notation $f(k) \lesssim g(k)$ shortens that there exists $C > 0$, such that for all $k \in \mathbb{N}$, $|f(k)| \le C|g(k)|$.

Now we state and prove the main technical result of this section, which contains two estimates: the lower bounds show that the generators are well-defined and permit to prove an energy bound (given in Proposition 3.4.2), which is essential to the proof of the forthcoming *replacement lemmas*; the upper bounds reflect the boundedness of the rates as $N \to +\infty$.

Proposition 3.2.10. *If $\ell_N \gg 1$, then for any $\eta \in \Omega_N$,*

$$r_N^{(m-1)}(\eta) \ge \begin{cases} m\, \mathbf{r}^{(0)}(\eta), & m \in (0,1), \\ m\delta_N\, \mathbf{r}^{(0)}(\eta) + \binom{m}{2} \mathbf{r}^{(1)}(\eta), & m \in (1,2), \end{cases} \quad \textit{and} \quad r_N^{(m-1)}(\eta) \le \begin{cases} \sum_{k=1}^{\ell_N} \left|\binom{m}{k}\right| k, & m \in (0,1), \\ m\mathbf{r}^{(0)}(\eta), & m \in (1,2), \end{cases}$$

where $(\ell_N+1)^{-(m-1)} \lesssim \delta_N = \sum_{k\geq \ell_N} |\binom{m-1}{k}| \lesssim (\ell_N)^{-(m-1)}$. *Moreover, when* $m \in (0,1)$,

$$\sum_{k=1}^{\ell_N} \left|\binom{m}{k}\right| k = \max_{\eta\in\Omega_N} r_N^{(m-1)}(\eta) \xrightarrow[N\to+\infty]{} +\infty.$$

Proof. We start with the case $m \in (1,2)$. From (3.10), we rewrite

$$r_N^{(m-1)}(\eta) = m - \sum_{k=2}^{\ell_N} \left|\binom{m}{k}\right| k + \sum_{k=2}^{\ell_N} \left|\binom{m}{k}\right| \big(k - \mathbf{r}^{(k-1)}(\overline{\eta})\big) \geq m - \sum_{k=2}^{\ell_N} \left|\binom{m}{k}\right| k + \binom{m}{2}\big(2 - \mathbf{r}^{(1)}(\overline{\eta})\big),$$

where for the last inequality we used the fact that, by definition, $\mathbf{c}^{(k-1)}(\overline{\eta}) \leq k$, and we bounded from below all but the first term of the second summation in k by zero. Then, since the alternating sum of the binomial coefficients vanishes, we obtain, for any $\ell_N \in \mathbb{N}_+$, that

$$m - \sum_{k=2}^{\ell_N} \left|\binom{m}{k}\right| k = m\left(1 - \sum_{k=1}^{\ell_N-1} \left|\binom{m-1}{k}\right|\right) > m\left(1 - \sum_{k=1}^{+\infty} \left|\binom{m-1}{k}\right|\right) = 0 \tag{3.12}$$

and therefore we get that $r_N^{(m-1)} > 0$. To conclude, we note that $2 - \mathbf{c}^{(1)}(\overline{\eta}) = \mathbf{c}^{(1)}(\eta)$ and we set

$$\delta_N := 1 - \sum_{k=1}^{\ell_N-1} \left|\binom{m-1}{k}\right| = \sum_{k\geq \ell_N} \left|\binom{m-1}{k}\right| > 0.$$

Recalling (3.11), we are reduced to estimate the tail of the $m-$series:

$$\frac{c}{(m-1)(\ell_N+1)^{m-1}} \leq \sum_{k\geq \ell_N+1} \frac{1}{k^m} \leq \frac{C}{(m-1)(\ell_N)^{m-1}}$$

with $c, C > 0$ being constants independent of N. Putting the inequalities together, the proof of the lower bound follows. To prove the upper bound, we only keep the first term in the definition (3.9) of $r_N^{(m-1)}$, since the other ones are negative.

The case $m \in (0,1)$ is straightforward from (3.10). To conclude, we see that the maximum is obtained when $\mathbf{r}^{(k-1)}(\overline{\eta}) = k$, that is, when the window $[\![-\ell_N+1, \ell_N]\!]\backslash\{0,1\}$ is completely empty and $\eta(0)+\eta(1) = 1$. The lower bound for the binomial coefficients in (3.11) then shows that this maximum tends to infinity as $N \to +\infty$. □

Remark 3.2.11 (On the sharpness of the bounds in Proposition 3.2.10)**.** The estimates of Proposition 3.2.10 are not sharp. Instead, the goal of the lower bound for $m \in (1,2)$ is to relate our process with the simpler process induced by the generator

$$m\delta_N \mathcal{L}_N^{\mathrm{PMM}(0)} + \tfrac{m(m-1)}{2} \mathcal{L}_N^{\mathrm{PMM}(1)},$$

which is very close to the one studied in [21], where the porous medium model is perturbed by a "small" SSEP dynamics.

The lower bound for $m \in (0,1)$ is here to emphasize that the transition rates will always be *greater* than those of the SSEP (modulo a multiplicative constant), as expected, since under this regime the macroscopic diffusion is faster than the one of the heat equation ($m = 1$). This will be useful, in particular, for the proof of the replacement Lemma B.1.7.

Finally, let us highlight that the divergence $\max_{\eta\in\Omega_N} r_N^{(m-1)}(\eta) \to +\infty$ as $N \to +\infty$ gives us an extra difficulty in the proof of tightness (see in particular (3.27)) and makes it impossible to argue, as for $m \in (1,2)$, that ρ^m is weak differentiable (see the last step in the proof of Proposition 3.4.9).

3.2.2 Characterization of the interpolating family

In this subsection we present further properties of the interpolating model. We start by explaining how this model interpolates between the SSEP and the PMM(1).

Proposition 3.2.12 (Interpolation property)**.** *For $m \in (1,2)$, $N \in \mathbb{N}$ and $\ell_N \geq 2$ fixed, the process $\mathcal{L}_N^{(m-1)}$ interpolates between $\mathcal{L}_N^{\mathrm{PMM}(0)}$ and $\mathcal{L}_N^{\mathrm{PMM}(1)}$ in the following sense: for all $\eta \in \Omega_N$,*

$$\lim_{m \nearrow 1} r_N^{(m-1)}(\eta) = \mathbf{r}^{\mathrm{PMM}(0)}(\eta) = \lim_{m \searrow 1} r_N^{(m-1)}(\eta) \qquad \text{and} \qquad \lim_{m \nearrow 2} r_N^{(m-1)}(\eta) = \mathbf{r}^{\mathrm{PMM}(1)}(\eta).$$

Proof. The limit to SSEP as $m \to 1$ from either above or below is a direct consequence of the interpolation property of the binomial coefficients, while the limit to PMM(1) is both consequence of this, but also of some rearrangement in the summation which defines the rates, and which implies $2 - \mathbf{c}^{(1)}(\overline{\eta}) = \mathbf{c}^{(1)}(\eta)$, see also (3.18) below. □

From [21], or a simple computation, the *grand-canonical invariant measures* for the PMM(k) (and therefore for the $\overline{\mathrm{PMM}}(k)$) are the Bernoulli product measures ν_ρ^N of parameter $\rho \in [0,1]$, namely, their marginal is given on $x \in \mathbb{T}_N$ by

$$\nu_\rho^N\big(\eta \in \Omega_N : \ \eta(x) = 1\big) = \rho. \tag{3.13}$$

The next lemma gives information on the invariant measures of our models.

Lemma 3.2.13 (Invariant measures and irreducibility)**.** *Let $m \in (0,2)$. For any $\rho \in [0,1]$, the Bernoulli product measure ν_ρ^N defined in* (3.13) *is invariant for the Markov process generated by $\mathcal{L}_N^{(m-1)}$. Moreover, for any $k \in \{0,\dots,N\}$, the hyperplane*

$$\mathcal{H}_{N,k} = \Big\{\eta \in \Omega_N \ : \ \sum_{x \in \mathbb{T}_N} \eta(x) = k\Big\}$$

is irreducible under the Markov process generated by $\mathcal{L}_N^{(m-1)}$.

Proof. The irreducibility of the process on the above hyperplanes is consequence of the fact that $c_N^{(m-1)}(\eta) > 0$ for any $\eta \in \Omega_N$, as shown in Proposition 3.2.10, and so the exclusion rule is the only constraint. We already know from [21] that the product measure ν_ρ^N is invariant for PMM(k), for any $k \in \mathbb{N}_+$, hence also for $\overline{\mathrm{PMM}}(k)$. In particular, it is also invariant for linear combinations of such models. □

For a good understanding of the interpolating model it is important to describe some properties of the integer family $\{\mathrm{PMM}(k)\}_{k \in \mathbb{N}}$, which is what we do in the following Lemma 3.2.16 and Proposition 3.2.18. Moreover we explain how to identify the macroscopic diffusion coefficient (due to ν_ρ^N being invariant for $0 \leq \rho \leq 1$). More precisely, let us introduce the following operator:

Definition 3.2.14 (Translation operators)**.** Let $\mathbf{1}$ be the identity function on Ω_N, and consider the operators $\nabla^\pm$ associated with the translation operator given by $\nabla^+ = \tau_1 - \mathbf{1}$ and $\nabla^- = \mathbf{1} - \tau_{-1}$, that is, for any function $f : \Omega_N \to \mathbb{R}$, we define $(\nabla^+ f)(\eta) = f(\tau_1\eta) - f(\eta), (\nabla^- f)(\eta) = f(\eta) - f(\tau_{-1}\eta)$, and for any $x \in \mathbb{T}_N$ consider $(\nabla_x^\pm f)(\eta) = (\nabla^\pm f)(\tau_x \eta)$.

It is straightforward to check that, for any $x \in \mathbb{T}_N$,

$$\mathcal{L}_N^{\mathrm{PMM}(k)}\big(\eta(x)\big) = \nabla^-\left(\mathbf{c}^{(k)}(\tau_x\eta)\nabla^+\eta(x)\right). \tag{3.14}$$

Therefore, the *microscopic density current* for PMM(k) between sites x and $x+1$, is equal to

$$-\mathbf{c}^{(k)}(\tau_x\eta)\nabla^+\eta(x) =: \mathbf{j}^{(k)}_{\{x,x+1\}}(\eta).$$

It turns out that this quantity is itself a discrete gradient, namely

$$\mathbf{j}^{(k)}_{\{x,x+1\}}(\eta) = -\nabla^+\mathbf{h}^{(k)}(\eta),$$

where $\mathbf{h}^{(k)}$ is given in Lemma 3.2.15. We highlight that although this gradient property was already known (see [9] for instance), the expression (3.15) for $\mathbf{h}^{(k)}$ is new (we give the original expression of $\mathbf{h}^{(k)}$ in the appendix, see (B.11)). Then, note that the expectation of $\mathbf{c}^{(k)}(\tau_x\eta)$ under the invariant measure ν_ρ^N is

$$\int \mathbf{c}^{(k)}(\tau_x\eta)\mathrm{d}\nu_\rho^N(\eta) = (k+1)\rho^k = D(\rho)$$

which is the diffusion coefficient of the PME(k) (1.2), *i.e.*, for $m = k+1 \in \mathbb{N}_+$. Similarly, since $\eta(1) - \eta(0) = -(\overline{\eta}(1) - \overline{\eta}(0))$, the gradient property is also true for $\overline{\mathrm{PMM}}(k)$. One can readily check that the expected diffusion equation associated with the microscopic dynamics of $\overline{\mathrm{PMM}}(k)$ has diffusion coefficient $\overline{D}(\rho) = (k+1)(1-\rho)^k$.

Let us now state more precisely the aforementioned gradient property, which will be proved in Appendix B.3. We recall the definition of $\mathbf{s}_j^{(k)}$ in (3.4).

Lemma 3.2.15 (Gradient property)**.** *For any $k \in \mathbb{N}$,* PMM*(k) is a gradient model. Precisely, for any $\eta \in \Omega_N$ we have that $\mathbf{c}^{(k)}(\eta)\nabla^+\eta(0) = \nabla^+\mathbf{h}^{(k)}(\eta)$, where*

$$\mathbf{h}^{(k)}(\eta) = \prod_{i=0}^{k}\eta(i) - \sum_{j=1}^{k}\sum_{n=0}^{k-j}(\nabla^+\eta)(n)\mathbf{s}_j^{(k)}(\tau_n\eta). \tag{3.15}$$

Now, for the interpolating model generated by $\mathcal{L}_N^{(m-1)}$, similarly to (3.14), a straightforward computation gives for all $x \in \mathbb{T}_N$

$$\mathcal{L}^{(m-1)}\big(\eta(x)\big) = \nabla^-\left(c_N^{(m-1)}(\tau_x\eta)\nabla^+\eta(x)\right),$$

and we can easily deduce from the previous lemma that

$$c_N^{(m-1)}(\eta)\nabla^+\eta(0) = \nabla^+h_N^{(m-1)}(\eta), \quad \text{where} \quad h_N^{(m-1)}(\eta) = \sum_{k=1}^{\ell_N}\binom{m}{k}(-1)^k\mathbf{h}^{(k-1)}(\overline{\eta}). \tag{3.16}$$

3.2.3 Properties on the rates

We start by stating and proving two important properties of the basis family $\{\mathrm{PMM}(k)\}_{k\in\mathbb{N}}$. The first one (Lemma 3.2.16) will be used later in Propositions 3.3.1, 3.4.2 and Lemma B.1.6, while the second one (Proposition 3.2.17) will provide some interesting monotonicity property of the rates for both the integer and non-integer families, see Propositions 3.2.18 and 3.2.19 at the end of this section. Recall the definition of $\mathbf{r}^{(k)}$ from (3.3).

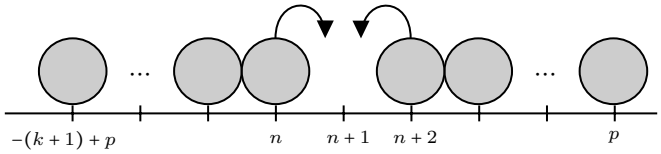


Figure 3.4: Configuration with $\mathbf{a}(\tau_n\eta)\mathbf{s}_j^{(k)}(\tau_n\eta) = \mathbf{a}(\tau_{n+1}\eta)\mathbf{s}_{j-1}^{(k)}(\tau_{n+1}\eta) = 1$ and p fixed.

Lemma 3.2.16 (Bound on the rates)**.** *For all $\ell, k \in \mathbb{N}_+$ such that $\ell \geq k$ and any $\eta \in \Omega_N$ we have that*

$$\sum_{n=1}^{\ell} \mathbf{r}^{(k)}(\tau_n\eta) \leq 2(\ell + k).$$

Proof. Note that

$$\sum_{n=1}^{\ell} \mathbf{r}^{(k)}(\tau_n\eta) = \sum_{n=1}^{\ell} \mathbf{a}(\tau_n\eta) \sum_{j=1}^{k+1} \mathbf{s}_j^{(k)}(\tau_n\eta) = \sum_{p=2}^{\ell+k+1} \sum_{n=1}^{\ell} \sum_{j=1}^{k+1} \mathbf{a}(\tau_n\eta)\mathbf{s}_j^{(k)}(\tau_n\eta)\mathbf{1}_{\{j+n=p\}} \leq \sum_{p=2}^{\ell+k+1} 2 = 2(\ell + k).$$

The inequality can be justified as follows. Fixed p, the quantity $\mathbf{s}_j^{(k)}(\tau_n\eta)$ depends on the occupation of the sites

$$[\![-(k+1)+j+n, j+n]\!] \backslash \{n, n+1\} = [\![-(k+1)+p, p]\!] \backslash \{n, n+1\}.$$

Because $1 \leq j \leq k+1$, then $\{n, n+1\} \in [\![-(k+1)+p, p]\!]$ for sure. There are a number of pairs (j, n) such that $j + n = p$, but for all of those pairs the box $[\![-(k+1)+p, p]\!]$ is the same. Thus, for each p fixed, there are at most two pairs $(n, j), (n', j')$ such that $p = n + j = n' + j'$ and $\mathbf{a}(\tau_n\eta)\mathbf{s}_j^{(k)}(\tau_n\eta) = \mathbf{a}(\tau_{n'}\eta)\mathbf{s}_{j'}^{(k)}(\tau_{n'}\eta) = 1$. Specifically, if (n, j) is as previously, then $(n', j') = (n+1, j-1)$ or $(n', j') = (n-1, j+1)$. □

Now we state a monotonicity property. The following proposition is used right after in Proposition 3.2.19 to prove an analogous property for the interpolating model.

Proposition 3.2.17. *For any $\eta \in \Omega_N$, the sequence $\left\{\frac{1}{k}\mathbf{c}^{(k-1)}(\eta)\right\}_{k\geq 1}$ is non-increasing.*

Proof. In order to prove the result, it is enough to show that

$$u_k(\eta) := \frac{k+1}{k}\mathbf{c}^{(k-1)}(\eta) - \mathbf{c}^{(k)}(\eta) \geqslant 0, \tag{3.17}$$

for any $\eta \in \Omega_N$. It turns out that this expression can we rewritten in terms of the products $\mathbf{s}_j^{(k)}$ defined in (3.4), after flipping some of the configuration values $\eta(x)$. Let us be more precise.

To simplify the presentation let us introduce some notation: for any $A \subseteq \mathbb{T}_N$ define the flip $\eta \mapsto \overline{\eta}^A$ as $\overline{\eta}^A(x) = \overline{\eta}(x)\mathbf{1}_{x\in A} + \eta(x)\mathbf{1}_{x\notin A}$. Straightforward computations show that

$$u_k(\eta) = \sum_{j=1}^{k+1} \left\{ \frac{k-(j-1)}{k}\mathbf{s}_j^{(k)}\left(\overline{\eta}^{\{-(k+1)+j\}}\right) + \frac{j-1}{k}\mathbf{s}_j^{(k)}\left(\overline{\eta}^{\{j\}}\right) \right\}. \tag{3.18}$$

Indeed, this is a consequence of the fact that

- for any $j \in [\![1, k]\!]$ it holds

$$\mathbf{s}_j^{(k)}\left(\eta^{\{-(k+1)+j\}}\right) = \overline{\eta}(-(k+1)+j)\mathbf{s}_j^{(k-1)}(\eta) = \mathbf{s}_j^{(k-1)}(\eta) - \mathbf{s}_j^{(k)}(\eta)$$

- and for any $j \in [\![2, k+1]\!]$ we have

$$\mathbf{s}_j^{(k)}\left(\eta^{\{j\}}\right) = \mathbf{s}_{j-1}^{(k-1)}(\eta)\overline{\eta}(j) = \mathbf{s}_{j-1}^{(k-1)}(\eta) - \mathbf{s}_j^{(k)}(\eta).$$

Two changes of variables in the two terms of the summation in (3.18) then lead to the desired result:

$$\begin{aligned}\mathbf{c}^{(k-1,1)}(\eta) &= \sum_{j=1}^{k} \frac{k-(j-1)}{k}\left(\mathbf{s}_j^{(k-1)}(\eta) - \mathbf{s}_j^{(k)}(\eta)\right) + \sum_{j=1}^{k} \frac{j}{k}\left(\mathbf{s}_j^{(k-1)}(\eta) - \mathbf{s}_{j+1}^{(k)}(\eta)\right) \\ &= \frac{k+1}{k}\mathbf{c}^{(k-1)}(\eta) - \sum_{j=1}^{k}\left(\frac{k-(j-1)}{k}\mathbf{s}_j^{(k)}(\eta) + \frac{j}{k}\mathbf{s}_{j+1}^{(k)}(\eta)\right).\end{aligned}$$

From another change of variables, the summation term above equals

$$\sum_{j=0}^{k-1} \frac{k-j}{k}\mathbf{s}_{j+1}^{(k)}(\eta) + \sum_{j=1}^{k} \frac{j}{k}\mathbf{s}_{j+1}^{(k)}(\eta) = \mathbf{s}_1^{(k)}(\eta) + \mathbf{s}_{k+1}^{(k)}(\eta) + \sum_{j=1}^{k-1} \mathbf{s}_{j+1}^{(k)}(\eta)\left(\frac{k-j}{k} + \frac{j}{k}\right) = \mathbf{c}^{(k)}(\eta).$$

□

Due to the analytical nature of the generalized binomial coefficients, a combinatorial interpretation of the whole model is not appropriate, as opposed to the integer case. Additionally, the problem of quantifying how, fixed some configuration, the rates change by varying m is not easy since the rates depend in a complex manner on m and the behaviour of the rate (with respect to m) is different for distinct configurations. Instead of doing an extensive study of the form of the rates, we gather information about some simple monotonicity aspects of the model. We show that the reinforcement/penalization of the SSEP given in (3.10) is non-increasing in k; then we derive a property of the interpolating family analogous to Proposition 3.2.17; and finally we plot in Figure 3.5 the rates in some equivalence classes of configurations which cover the values of $c_N^{(m-1)}(\eta)$. This is, to our mind, a satisfying solution to observe the continuous deformation of the SSEP into a slow or fast diffusion model.

Proposition 3.2.18. *Fixed any $\eta \in \Omega_N$ and $m \in [0,2]$ the sequence $\{|\binom{m}{k}|\mathbf{c}^{(k-1)}(\overline{\eta})\}_{k\geq 2}$ is decreasing up to the smallest k such that $\mathbf{c}^{(k-1)}(\overline{\eta}) = 0$.*

Proof. Recall that we proved in Proposition 3.2.17 that for any $\eta \in \Omega_N$ the sequence $\left\{\frac{1}{k}\mathbf{c}^{(k-1)}(\eta)\right\}_{k\geq 1}$ is non-increasing. From the definition of the binomial coefficients, for $m \in (0,2)$ the sequence $\left\{k|\binom{m}{k}|\right\}_{k\geq 2}$ is decreasing, since

$$(k+1)\left|\binom{m}{k+1}\right| = k\left|\binom{m}{k}\right| \frac{|m-k|}{k},$$

and whenever $k \geq 2$ and $m \in (0,2)$ we have $|m-k| = k - m < k$. □

Before stating the monotonicity property, note that we have the following limit

$$\lim_{m\searrow 0} \frac{1}{m} c_N^{(m-1)}(\eta) = \sum_{k=0}^{\ell_N - 1} \frac{\mathbf{c}^{(k)}(\overline{\eta})}{k+1}.$$

Proposition 3.2.19. *For any $\eta \in \Omega_N$ the sequence $\{\frac{1}{m}c_N^{(m-1)}(\eta)\}_{m\in[0,2]}$ is non-increasing.*

Proof. From Proposition 3.2.10 we can extract that $\frac{1}{m}c_N^{(m-1)} \geq \mathbf{c}^{(0)}$ for $m \in (0,1)$, and $\mathbf{c}^{(0)} \geq \frac{1}{m}c_N^{(m-1)}$ for $m \in (1,2)$. It remains to see the monotonicity of the sequence in the statement according to the values of $m \in [0,2]\backslash\{1\}$. Assuming that the aforementioned sequence is non-increasing, since the binomial coefficients are continuous functions of m the interpolation property allows us to take the limit $m \to 2$ and as such we only need to focus on $m \in [0,2)\backslash\{1\}$. Rewrite

$$\frac{1}{m}c_N^{(m-1)}(\eta) = \mathbf{1}_{\{m\in(0,1)\}} \sum_{k=0}^{\ell_N-1} \left|\binom{m-1}{k}\right| \frac{\mathbf{c}^{(k)}(\overline{\eta})}{k+1} + \mathbf{1}_{\{m\in(1,2)\}}\left(1 - \sum_{k=1}^{\ell_N-1}\left|\binom{m-1}{k}\right| \frac{\mathbf{c}^{(k)}(\overline{\eta})}{k+1}\right).$$

For any $k \geq 2$ we compute

$$\frac{\mathrm{d}}{\mathrm{d}m}|(m-1)_k| = -|(m-1)_k| f_k(m) \quad \text{where} \quad f_k(m) := \sum_{j=1}^{k} \frac{1}{j-m}.$$

This means that

$$\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{1}{m} c_N^{(m-1)}(\eta)\right) = -\frac{1}{2}\mathbf{c}^{(1)}(\overline{\eta}) + \operatorname{sign}(m-1) \sum_{k=2}^{\ell_N - 1} \left|\binom{m-1}{k}\right| f_k(m) \frac{\mathbf{c}^{(k)}(\overline{\eta})}{k+1}. \tag{3.19}$$

If $m \in [0,1)$ then $f_k(m) > 0$ which concludes the proof. For $m \in (1,2)$ we need some extra work. We claim that differentiating with respect to m both sides of

$$0 = 1 - \sum_{k=1}^{+\infty} \left|\binom{m-1}{k}\right| \quad \text{one obtains that} \quad 1 = \sum_{k=2}^{+\infty} \left|\binom{m-1}{k}\right| f_k(m).$$

For $m \in [\frac{3}{2}, 2)$ we have $f_k(m) > 0$ for all $k \geq 2$ since $f_2(m) > 0$ and $f_k(m)$ is increasing in k. If $m \in (1, \frac{3}{2})$ then for each m there must be some $k_0 > 2$ such that $f_k > 0$ for all $k \geq k_0$ so that the second summation on the previous display is equal to one. Let ℓ_N be large enough so that $k_0 < \ell_N$ (otherwise the result is obvious). Then we can bound (3.19) from above by taking the limit $\ell_N \to +\infty$. Since the sequence of maps $\{\frac{1}{k+1}\mathbf{c}^{(k)}\}_{k\geq 0}$ is non-increasing, for any $j \leq k_0 \leq i$ we have

$$\frac{1}{i+1}\mathbf{c}^{(i)} \leq \frac{1}{k_0+1}\mathbf{c}^{(k_0)} \leq \frac{1}{j+1}\mathbf{c}^{(j)}.$$

Then we can bound

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{1}{m} c_N^{(m-1)}(\eta)\right) &\leq -\frac{1}{2}\mathbf{c}^{(1)}(\overline{\eta}) + \sum_{k=2}^{k_0-1} \left|\binom{m-1}{k}\right| f_k(m) \frac{\mathbf{c}^{(k_0)}(\overline{\eta})}{k_0+1} + \sum_{k\geq k_0} \left|\binom{m-1}{k}\right| f_k(m) \frac{\mathbf{c}^{(k_0)}(\overline{\eta})}{k_0+1} \\ &= -\frac{\mathbf{c}^{(1)}(\overline{\eta})}{2} + \frac{\mathbf{c}^{(k_0)}(\overline{\eta})}{k_0+1} \leq 0.\end{aligned}$$

To conclude the proof, it is enough to show that the sequence $(a_n)_{n\geq 2}$ given by $0 < a_n := \sum_{k=2}^{n} |\binom{m-1}{k}| f_k(m)$ is uniformly bounded. Since $f_1(m) < 0$, we first bound $f_k(m)$ by the corresponding integral for $k \geq 2$:

$$f_k(m) \lesssim \log(k-m) - \log(2-m).$$

Recall the inequality $\log x \leq \frac{1}{s} x^s$ for any $x, s \in \mathbb{R}_+$. From this and (3.11) it holds

$$\sum_{k=2}^{n} \left|\binom{m-1}{k}\right| f_k(m) \lesssim \sum_{k=2}^{n} \frac{1}{(k-m)^{m-s}} - \log(2-m) \sum_{k=2}^{n} \left|\binom{m-1}{k}\right|.$$

Setting $0 < s$ such that $m - s > 1$, observing that the quantity on the right-hand side of the previous display is increasing in n and taking $n \to +\infty$ we end the proof. □

We now plot the evolution of $c_N^{(m-1)}(\eta)$ with respect to m (for a fixed configuration η). To that aim, let us start with the following remark: for any $k \geq 1$ the value of $\mathbf{c}^{(k)}(\overline{\eta})$ is uniquely determined by the positions of the first particle to the left of 0 and the first particle to the right of 1. More precisely, for any $x_0, x_1 \in \mathbb{T}_N$ consider the set

$$\Omega_N^{x_0,x_1} = \big\{\eta \in \Omega_N \,:\, \eta(-x_0) = \eta(x_1) = 1,\ \eta(x) = 0, \ \text{ for all } \ x \in [\![-x_0+1, x_1-1]\!] \backslash \{0,1\}\big\}.$$

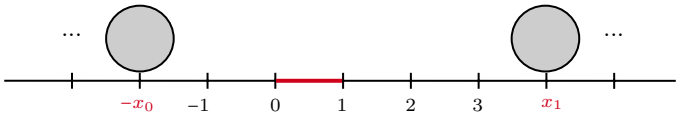


Figure 3.5: Configuration belonging to $\Omega_N^{2,4}$.

It is simple to see that if $\eta_0, \eta_1 \in \Omega_N^{x_0,x_1}$ then $\mathbf{r}^{(k)}(\overline{\eta}_0) = \mathbf{r}^{(k)}(\overline{\eta}_1)$ for all $k \geq 1$. Therefore we obtain $c_N^{(m-1)}(\eta_0) = c_N^{(m-1)}(\eta_1)$, and for every $\eta \in \Omega_N^{x_0,x_1}$ one can plot $c_N^{(m)}(\eta)$ as a function of m, as in Figure 3.5. To that end, for each m, ℓ_N, x_0 and x_1 fixed and $\xi \in \Omega_N^{x_0,x_1}$ we introduce $\tilde{c}_N(x_0, x_1, m) \equiv c_N^{(m-1)}(\xi)$.

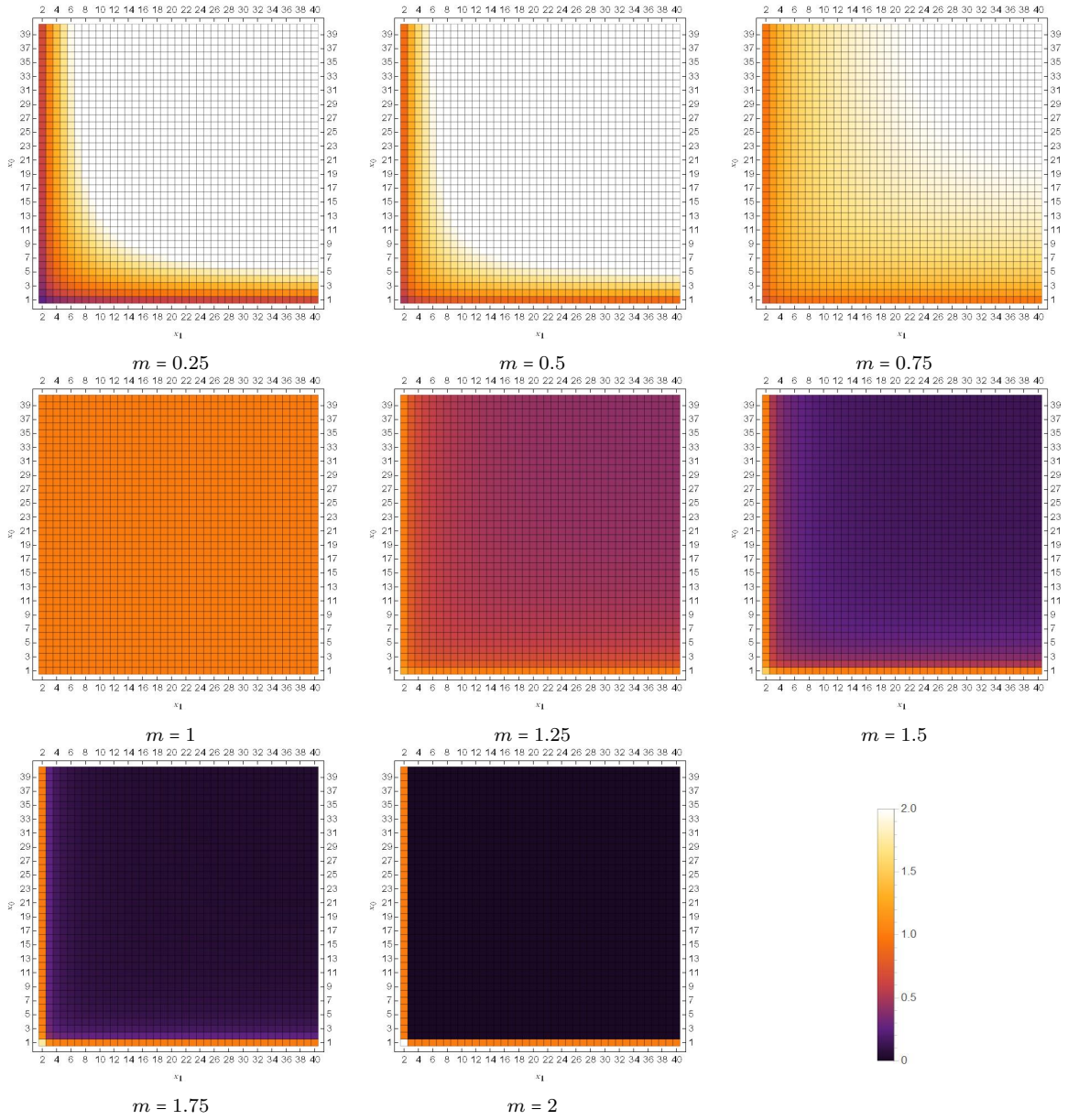


Figure 3.5: Evolution of $\tilde{c}_N(x_0, x_1, m)$ for $\ell_N = 40$

We stress that the previous figure presents the value of the constraint $c_N^{(m-1)}(\eta)$ (equivalently, the rate $r_N^{(m-1)}(\eta)$ when $\eta(0) + \eta(1) = 1$) fixed x_0, x_1 and a representative $\eta \in \Omega_N^{x_0,x_1}$, that is, a configuration

with the first particle to the *left* of the site 0 located at the site $-x_0$, and the first particle to the *right* of the site 1 located at the site x_1. Note the symmetry of the plots with respect to $x_0 = x_1$, which is consequence of the symmetry of the jumps. Fixed m and ℓ_N, varying x_0 and x_1 allow us to see all the possible values of the constraints. For example, for $m = 1$ the rate is equal to 1 independently of x_0, x_1, hence the sub-figure, in this case, has the same colour for all $x_0, x_1 \leq 40$. For $m = 2$ the rate is non-zero if and only if there is at least one particle located at the site -1 or at the site 2. In other words, $\mathbf{c}^{(1)}(\eta) = \eta(-1)+\eta(2)$. Therefore, we obtain in the respective sub-figure the horizontal and vertical orange lines, where $\mathbf{c}^{(1)}(\eta) = 1$ for any $\eta \in \Omega_N^{1,x_1} \cup \Omega_N^{x_0,2}$ with $x_0 \geq 2$ and $x_1 \geq 3$ and $\mathbf{c}^{(1)} = 0$ otherwise; while at $x_0 = 1$ and $x_1 = 2$ the constraint attains its largest value, *i.e.,* $\mathbf{c}^{(1)}(\eta) = 2$ for all $\eta \in \Omega_N^{1,2}$. In the *fast-diffusion* regime, we see a "continuous" increase of the rates as x_0, x_1 increase, while the opposite in the *slow-diffusion* regime. This is a clear consequence of the penalization/reinforcement terms, as seen in (3.10).

3.2.4 Main result

To expose our main result about the hydrodynamic limit of the interpolating model we first introduce some definitions. Let us fix a finite time horizon $[0, T]$, let μ_N be a probability measure on Ω_N, and let $\{\eta_{N^2 t}\}_{t\geq 0}$ be the Markov process generated by $N^2\mathcal{L}_N^{(m-1)}$ for $m \in (0, 2)\backslash\{1\}$, given in (3.8).

Definition 3.2.20 (Empirical measure)**.** For any $\eta \in \Omega_N$ define the empirical measure $\pi^N(\eta, \mathrm{d}u)$ on the continuous torus $\mathbb{T}$ by

$$\pi^N(\eta, \mathrm{d}u) = \frac{1}{N} \sum_{x\in\mathbb{T}_N} \eta(x)\delta_{x/N}(\mathrm{d}u)$$

where $\delta_{x/N}$ is the Dirac measure at the macroscopic point x/N. Moreover, we define its time evolution in the diffusive time scale by $\pi_t^N(\eta, \mathrm{d}u) = \pi^N(\eta_{N^2 t}, \mathrm{d}u)$. For any function $G : \mathbb{T} \to \mathbb{R}$, we define the integral of G with respect to the empirical measure as

$$\langle \pi_t^N, G\rangle = \int_{\mathbb{T}} G(u)\pi_t^N(\eta, \mathrm{d}u) = \frac{1}{N} \sum_{x\in\mathbb{T}_N} G(\tfrac{x}{N})\eta_{N^2 t}(x). \tag{3.20}$$

Let $\mathcal{M}_+$ be the space of positive measures on $[0, 1]$ with total mass no larger than 1 and endowed with the weak topology. Let $\mathcal{D}([0, T], \Omega_N)$ be the Skorokhod space of trajectories induced by $\{\eta_{N^2 t}\}_{t\in[0,T]}$ with initial measure μ_N. Denote by $\mathbb{P}_{\mu_N}$ the induced probability measure on the space of trajectories $\mathcal{D}([0, T], \Omega_N)$ and by $\mathbb{Q}_N = \mathbb{P}_{\mu_N} \circ (\pi^N)^{-1}$ the probability measure on $\mathcal{D}([0, T], \mathcal{M}_+)$ induced by $\{\pi_t^N\}_{t\in[0,T]}$ and μ_N.

Now we introduce the notion of weak solutions to equation (1.2) for $m \in (0, 2)$. For that purpose, for $n \in \mathbb{N}_+ \cup \{\infty\}$ let $C^n(\mathbb{T})$ be the set of n times continuously differentiable, real-valued functions defined on $\mathbb{T}$; and let $C^{n,p}([0, T] \times \mathbb{T})$ be the set of all real-valued functions defined on $[0, T] \times \mathbb{T}$ that are n times differentiable on the first variable and p times differentiable on the second variable and with continuous derivatives. Finally, for two functions $f, g \in L^2(\mathbb{T})$, $\langle f, g\rangle$ denotes their standard euclidean product in $L^2(\mathbb{T})$ and $\|\cdot\|_{L^2(\mathbb{T})}$ is the associated norm. We remark that we use the notation $\langle\cdot,\cdot\rangle$ twice, for the inner-product just introduced, and also in (3.20), although their difference will be clear from the context.

Definition 3.2.21 (Sobolev space)**.** The semi inner-product $\langle\cdot,\cdot\rangle_1$ on the set $C^\infty(\mathbb{T})$ is given on $G, H \in C^\infty(\mathbb{T})$ by $\langle G, H\rangle_1 = \langle \partial_u G, \partial_u H\rangle = \int_{\mathbb{T}} \partial_u G(u)\partial_u H(u)\mathrm{d}u$, and the associated semi-norm is denoted by $\|\cdot\|_1$. Let $\mathcal{H}^1(\mathbb{T})$ be the Sobolev space on $\mathbb{T}$, defined as the completion of $C^\infty(\mathbb{T})$ for the norm $\|\cdot\|^2_{\mathcal{H}^1(\mathbb{T})} = \|\cdot\|^2_{L^2} + \|\cdot\|^2_1$, and let $L^2([0,T];\mathcal{H}^1(\mathbb{T}))$ be the set of measurable functions $f : [0,T] \to \mathcal{H}^1(\mathbb{T})$ such that $\int_0^T \|f_s\|^2_{\mathcal{H}^1(\mathbb{T})}\mathrm{d}s < \infty$.

Definition 3.2.22 (Weak solutions to (1.2))**.** Let $\rho^{\text{ini}} : \mathbb{T} \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times \mathbb{T} \mapsto [0,1]$ is a weak solution of the FDE (resp. PME) with $m \in (0,1)$ (resp. $m \in (1,2)$) if

1. (a) For $m \in (0,1)$ it holds $\rho \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$,
   (b) For $m \in (1,2)$ it holds $\rho^m \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$.

2. For any $t \in [0,T]$ and $G \in C^{1,2}([0,T]\times\mathbb{T})$ it holds that

$$F(\rho^{\text{ini}}, \rho, G, t) := \langle \rho_t, G_t\rangle - \langle \rho^{\text{ini}}, G_0\rangle - \int_0^t \left\{\langle \rho_s, \partial_s G_s\rangle + \langle (\rho_s)^m, \partial_u^2 G_s\rangle\right\}\mathrm{d}s \equiv 0. \tag{3.21}$$

In the appendix, Lemmas B.2.5 and B.2.3, we will show that the weak solution given by last definition is unique, for $m \in (0,1)$ and $m \in (1,2)$, respectively.

One can also extract from the weak differentiability of ρ and ρ^m that the solution ρ is $\frac{1}{2}$–Hölder continuous, for $m \in (0,1)$, and $\frac{1}{4}$–Hölder continuous, for $m \in (1,2)$. This is proved in Proposition B.2.4 and Corollary B.2.2, respectively. Although we were able to avoid needing these regularity results, we found them interesting by themselves as an additional characterization of the solution; and as seen in [5] there is the need to show some regularity of the solution in order to study the system with open boundaries, in this way also preparing future work.

Definition 3.2.23 (Local equilibrium distribution)**.** Let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures on Ω_N, and let $f : \mathbb{T} \to [0,1]$ be a measurable function. If for any continuous function $G : \mathbb{T} \to \mathbb{R}$ and every $\delta > 0$ it holds

$$\lim_{N\to+\infty} \mu_N\left(\eta \in \Omega_N : \left|\langle \pi^N, G\rangle - \langle f, G\rangle\right| > \delta\right) = 0,$$

we say that the sequence $\{\mu_N\}_{N\geq 1}$ is a *local equilibrium measure* associated with the profile f.

Example 3.2.24. An example of a measure satisfying Definition 3.2.23 is the product Bernoulli measure, given on $x \in \mathbb{T}_N$ by

$$\nu^N_{\rho^{\text{ini}}(\cdot)}(\eta \in \Omega_N :\ \eta(x) = 1) = \rho^{\text{ini}}\left(\tfrac{x}{N}\right),$$

where $\rho^{\text{ini}} : \mathbb{T} \to [0,1]$ is a measurable Lipschitz profile. Then $\nu^N_{\rho^{\text{ini}}(\cdot)}$ is a local equilibrium measure associated with ρ^{ini}.

We are now ready to state the main result of this paper:

Theorem 3.2.25 (Hydrodynamic limit)**.** *Let $\rho^{\text{ini}} : \mathbb{T} \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a local equilibrium measure associated with it. Then, for any $t \in [0,T]$, $\delta > 0$ and $\mathbb{N} \ni \ell_N \to \infty$ such that $2 \leq \ell_N \leq N$, it holds*

$$\lim_{N\to+\infty} \mathbb{P}_{\mu_N}\left(\left|\langle \pi^N_t, G\rangle - \langle \rho_t, G\rangle\right| > \delta\right) = 0,$$

where ρ is the unique weak solution of (1.2) *in the sense of Definition 3.2.22, with initial condition ρ^{ini}.*

3.3 Proof of Theorem 3.2.25

We first outline the proof. As previously mentioned, to prove the hydrodynamic limit we use the classical entropy method introduced in [23]. The general scheme is the following: we prove that the sequence of empirical measures is tight (as proved in Subsection 3.3.1), which implies the existence of weakly convergent subsequences; and then we prove that the limiting measure is concentrated on paths of absolutely continuous measures with respect to the Lebesgue measure, whose density is a weak solution to the hydrodynamic equation (1.2) (proved in Section 3.3.2). To do so we shall need an energy estimate (Appendix 3.4), which gives us some regularity of the solution to the PDE, and replacement lemmas (Section B.1) whose role is to close the equations for the limiting profile at the microscopic level. Proving uniqueness of weak solutions (see Appendix B.2), we see that the limit of the sequence of measures is then unique and we can conclude that the whole sequence converges to that limit.

We introduce some discrete operators that will be important in what follows. Let us extend Definition 3.2.14 to functions defined on $\mathbb{T}_N$, instead of Ω_N. Without loss of generality, we adopt the same notation. Namely, if $f : \mathbb{T}_N \to \mathbb{R}$ then its gradients are $\nabla^+ f = (\tau_1 - \mathbf{1})f$ and $\nabla^- f = (\mathbf{1} - \tau_{-1})f$, where $\mathbf{1}$ is now the identity function defined on $\mathbb{T}_N$. Finally, for any $N \in \mathbb{N}_+$, we also define the *rescaled gradients* on $\mathbb{T}_N$ as $\nabla^{\pm,N} = N\nabla^{\pm}$, and the *rescaled Laplacian* as $\Delta^N = \nabla^{+,N} \circ \nabla^{-,N} = \nabla^{-,N} \circ \nabla^{+,N}$.

3.3.1 Tightness

Let us start by exploiting the gradient property of our model. Recall that we consider the evolution in the diffusive time scale tN^2, that is, given by the generator $\mathcal{L} := N^2\mathcal{L}_N^{(m-1)}$. From Dynkin's formula [26, Appendix 1, Lemma 5.1], we know that for any $G \in C^{1,2}([0,T] \times \mathbb{T})$

$$M_t^N(G) := \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t (\partial_s + \mathcal{L}) \langle \pi_s^N, G_s \rangle \mathrm{d}s \tag{3.22}$$

is a martingale with respect to the natural filtration of the process. From Lemma 3.2.15 and two summations by parts we see that

$$\begin{aligned} M_t^N(G) = \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, \partial_s G_s \rangle \mathrm{d}s \\ - \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \Delta^N G_s(\tfrac{x}{N}) \sum_{k=1}^{\ell_N} \binom{m}{k} (-1)^k \mathbf{h}_s^{(k-1)}(\tau_x \overline{\eta}) \mathrm{d}s, \end{aligned} \tag{3.23}$$

where we defined for any $k \in \mathbb{N}$ and any $s \in [0,t]$ the time evolution $\mathbf{h}_s^{(k-1)}(\eta) = \mathbf{h}^{(k-1)}(\eta_{N^2 s})$.

Proposition 3.3.1 (Tightness)**.** *The sequence of probability measures $(\mathbb{Q}_N)_{N \in \mathbb{N}}$ is tight with respect to the Skorokhod topology of $\mathcal{D}([0,T], \mathcal{M}_+)$.*

Proof. To prove tightness we resort to Aldous' conditions (see, for instance, [22, proof of Proposition 4.1] or, equivalently, [5, Proposition 3.3] for more details). Since the occupation variable is bounded by 1, it is enough to show that for all $\epsilon > 0$

$$\limsup_{\alpha \to 0} \limsup_{N \to +\infty} \mathbb{P}_{\mu_N} \left(\sup_{|t-s| \le \alpha} \left| \langle \pi_t^N, G \rangle - \langle \pi_s^N, G \rangle \right| > \epsilon \right),$$

where G is a time-independent function belonging to a dense subset of $C([0,1])$ with respect to the uniform topology (for the details regarding the reduction of the space of test functions see, for instance, [6]). From the fact that $M_t^N(G)$ is a martingale (with respect to the natural filtration of the process), the previous condition can be reduced to the study of the quadratic variation of (3.22) and the boundedness of the generator, *i.e.,* it is enough to prove that

$$\lim_{\alpha\to 0}\limsup_{N\to+\infty}\left\{\mathbb{P}_{\mu_N}\left(\sup_{|t-s|\le\alpha}\left|M_t^N(G)-M_s^N(G)\right|>\frac{\epsilon}{2}\right)+\mathbb{P}_{\mu_N}\left(\sup_{|t-s|\le\alpha}\left|\int_s^t\mathcal{L}\langle\pi_s^N,G\rangle\mathrm{d}s\right|>\frac{\epsilon}{2}\right)\right\}=0. \tag{3.24}$$

We apply the triangular, Jensen and Doob's inequalities in the first term above, and Proposition B.1.2 in the second term, reducing to the treatment of

$$\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left(M_T^N(G)\right)^2\right]^{\frac{1}{2}}=0\quad\text{and}\quad\lim_{\alpha\to 0}\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left|\int_s^t\mathcal{L}\langle\pi_s^N,G\rangle\mathrm{d}s\right|\right]=0, \tag{3.25}$$

where $|t-s|\le\alpha$ in the second expectation, that we omit further references in the next computations. Recalling from [26, Appendix A, Lemma 5.1] the expression for the quadratic variation of the martingale, we have that the first expectation in (3.25) equals

$$\mathbb{E}_{\mu_N}\left[\int_0^T F_s^N(G)\mathrm{d}s\right],\quad\text{where}\quad F_s^N(G)=N^2\left(\mathcal{L}_N^{(m-1)}\langle\pi_s^N,G\rangle^2-2\langle\pi_s^N,G\rangle\mathcal{L}_N^{(m-1)}\langle\pi_s^N,G\rangle\right).$$

Since our transition rates are symmetric, we get

$$\begin{aligned}F_s^N(G)&=\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}c_N^{(m-1)}(\tau_x\eta_{N^2s})\left(\eta_{N^2s}(x+1)-\eta_{N^2s}(x)\right)^2\left(\nabla^{+,N}G(\tfrac{x}{N})\right)^2\\&\lesssim\frac{1}{N^2}\|\partial_uG\|^2_{L^\infty(\mathbb{T})}\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{x\in\mathbb{T}_N}\mathbf{r}^{(k-1)}(\tau_x\eta_{N^2s})\lesssim\frac{1}{N}\|\partial_uG\|^2_{L^\infty(\mathbb{T})},\end{aligned}$$

where we used Lemma 3.2.16 for the last inequality. This concludes the proof of the first condition in (3.25). For the second, we split the proof in two cases $m\in(0,1)$ and $m\in(1,2)$.

• Assume first that $m\in(1,2)$. From (3.15) (or more obviously (B.11)) we have that $|\mathbf{h}^{(k-1)}(\eta)|\le k$. Therefore, using the inequality (3.12), the integrand in the second limit in (3.25) can be bounded from above by

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\Delta^NG_s(\tfrac{x}{N})\right|\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|k\lesssim\left\|\partial_u^2G\right\|_{L^1(\mathbb{T})}+\frac{1}{N},$$

which implies the second requirement in (3.25). This finishes the proof in the case $m\in(1,2)$.

• For $m\in(0,1)$ we need some extra work. Recalling that in the fast diffusion case the generator can be rewritten as in (3.10), we see that the second expectation in (3.25) equals

$$\mathbb{E}_{\mu_N}\left[\left|\int_s^t\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Delta_NG(\tfrac{x}{N})\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\tau_x\mathbf{h}^{(k-1)}(\overline{\eta}_{N^2s})\mathrm{d}s\right|\right]. \tag{3.26}$$

It will be fundamental to identify $\mathbf{h}^{(k-1)}$ as a function of the constraints $\mathbf{c}^{(k-1)}$, as in (3.15). From the triangular inequality we bound the expectation (3.26) from above by

$$\begin{aligned}&\mathbb{E}_{\mu_N}\left[\left|\int_s^t\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Delta_NG(\tfrac{x}{N})\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\prod_{j=0}^{k-1}\overline{\eta}_{N^2s}(x+j)\mathrm{d}s\right|\right]\\&\quad+\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\Delta_NG(\tfrac{x}{N})\right|\mathbb{E}_{\mu_N}\left[\left|\int_s^t\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\tau_x\left\{\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^+\overline{\eta}_{N^2s})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta}_{N^2s})\right\}\mathrm{d}s\right|\right]\end{aligned} \tag{3.27}$$

where, by convention, $\sum_{\varnothing} \equiv 0$. Since $m \in (0,1)$ and the process is of exclusion type, we have

$$\sum_{k=1}^{\ell_N} \left|\binom{m}{k}\right| \prod_{j=0}^{k-1} \overline{\eta}_{N^2 s}(x+j) \leq \sum_{k=1}^{\ell_N} \left|\binom{m}{k}\right| < 1$$

and due to the regularity of the test function the first expectation in (3.27) can be bounded as:

$$\mathbb{E}_{\mu_N}\left[\left| \int_s^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \Delta_N G(\tfrac{x}{N}) \sum_{k=1}^{\ell_N} \left|\binom{m}{k}\right| \prod_{j=0}^{k-1} \overline{\eta}_{N^2 s}(x+j) \mathrm{d}s \right|\right] \lesssim |t-s| \left(\left\| \partial_u^2 G \right\|_{L^1(\mathbb{T})} + \frac{1}{N} \right).$$

The treatment of the second expectation in (3.27) is more demanding. Concretely, since $m \in (0,1)$ the tail of the series $\sum_{k \geq 1} |\binom{m}{k}|$ is too heavy to either argue directly via Lemma B.1.7 or slow down the speed of explosion of ℓ_N (as we shall do in a different context shortly), while maintaining ℓ_N with no particular order of explosion. One then needs to invoke the forthcoming replacement Lemma B.1.6 instead, which takes advantage of the particular expression of $\mathbf{h}^{(k-1)}$ in (3.15). In this way, applying Lemma B.1.6 for each term of the summation over $x \in \mathbb{T}_N$ we obtain the final upper bound

$$\frac{1}{B} + \sigma B \frac{(\ell_N)^{1-m}}{N}.$$

Recalling that $\ell_N \leq N$ and $1-m \in (0,1)$, taking the limit $N \to +\infty$ and then $B \to +\infty$ we finish the proof.

□

3.3.2 Characterization of limit points

The goal of this subsection is to show that the limiting points of $(\mathbb{Q}_N)_{N \in \mathbb{N}}$, which we know to exist as a consequence of the results of the previous section, are concentrated on trajectories of absolutely continuous measures with respect to the Lebesgue measure, whose density is a weak solution to either the FDE or the PME, depending on the value of m. Showing the aforementioned absolute continuity is simple since we deal with an exclusion process, and its proof can be found (modulo small adaptations) for instance in [26, page 57]. From this and the previous proposition, we know (without loss of generality) that for any $t \in [0,T]$, the sequence $(\pi_t^N(\eta, \mathrm{d}u))_{N \in \mathbb{N}}$ converges weakly with respect to $\mathbb{Q}_N$ to an absolutely continuous measure $\pi_\cdot(\mathrm{d}u) = \rho_\cdot(u)\mathrm{d}u$. In the next result we obtain information about the profile ρ.

Proposition 3.3.2. *Let $a(m) = \mathbf{1}_{m \in (0,1)} + m\mathbf{1}_{m \in (1,2)}$. For any limit point $\mathbb{Q}$ of $(\mathbb{Q}_N)_{N \in \mathbb{N}}$ it holds*

$$\mathbb{Q}\Big(\pi \in \mathcal{D}([0,T], \mathcal{M}_+) \,:\, \forall t \in [0,T],\ \pi_t(du) = \rho_t(u)du,\ \textit{where } \rho^{a(m)} \in L^2([0,T]; \mathcal{H}^1(\mathbb{T})), \\ \textit{and for all } G \in C^{1,2}([0,T] \times \mathbb{T}),\ F(\rho^{\mathrm{ini}}, \rho, G, t) = 0 \Big) = 1,$$

where $F(\rho^{\mathrm{ini}}, \rho, G, t)$ is given in (3.21).

Before showing Proposition 3.3.2, we introduce some definitions and technical results.

Definition 3.3.3. For any $x \in \mathbb{T}_N$ and $\ell \in \mathbb{N}$ consider the following microscopic box of size ℓ, and the empirical average over it, given by

$$\Lambda_x^\ell = [\![x, x+\ell-1]\!], \quad \text{and} \quad \eta^\ell(x) = \frac{1}{\ell} \sum_{y \in \Lambda_x^\ell} \eta(y).$$

Moreover, for $\epsilon > 0$ and $u, v \in \mathbb{T}$, let $\iota_\epsilon^u(v) = \frac{1}{\epsilon}\mathbf{1}_{v \in [u, u+\epsilon)}$.

The starting rationale for the law of large numbers for the empirical measure is that one expects the solution ρ to be "close" to a local average a.e.. Formally, one expects that $\rho_s(u) \sim \frac{1}{\epsilon}\int_u^{u+\epsilon} \rho_s(u)\mathrm{d}v$ for a.e. $(s,u) \in [0,T]\times\mathbb{T}$ with $\epsilon > 0$ arbitrarily small. Since $\rho_s(\cdot)$ is integrable with respect to the Lebesgue measure (recall that ρ is a density defined on $\mathbb{T}$), the previous approximation is justified from Lebesgue's differentiation theorem. In our case, we have two additional technical problems: • for $k \geq 2$ the PME(k) describes non-linear diffusion, as opposed to $k = 1$; • for $m \in (0,1)$ the series $\sum_{k\geq 1}\left|\binom{m}{k}\right|k$ is *divergent*. Those two technical "features" ramify into the methodology, and a more detailed study than what is present in the literature must be performed in order to guarantee the aforementioned local approximation, and also so that the errors due to the forthcoming replacement lemmas vanish in the limit. With that in mind, next lemma solves the first issue.

Lemma 3.3.4. *Let $m \in (0,2)\backslash\{1\}$ and $\epsilon > 0$ be fixed, and consider some sequence $0 < (\epsilon_k)_{k\geq 1}$ where for any k it holds that $\epsilon_k \to 0$ as $\epsilon \to 0$. Then, for any $k \in \mathbb{N}_+$, a.e. $u \in \mathbb{T}$ and $s \in [0,T]$ it holds that*

$$\limsup_{\epsilon\to 0}\left|\sum_{k\geq 2}\binom{m}{k}(-1)^k(1-\rho_s(u))^k - \sum_{k\geq 2}\binom{m}{k}(-1)^k\prod_{j=0}^{k-1}\left(1-\langle \pi_s, \iota_{\epsilon_k}^{u+j\epsilon_k}\rangle\right)\right| = 0.$$

Proof. Fixed $A \in \mathbb{N}$, because for any $u \in \mathbb{T}$, $s \in [0,T]$ it holds $\rho_s(u) \leq 1$. Then from Lemma B.3.1

$$\sum_{k\geq 2}\binom{m}{k}(-1)^k(1-\rho_s(u))^k = \sum_{k=2}^{A}\binom{m}{k}(-1)^k(1-\rho_s(u))^k + O(1/A^m).$$

For any a_0, b_0, a_1, b_1 one can rewrite $a_0a_1 = a_0(a_1-b_1) + (a_0-b_0)b_1 + b_0b_1$. With this rationale,

$$(1-\rho_s(u))^k = \prod_{j=0}^{k-1}\left(1-\langle \pi_s, \iota_{\epsilon_k}^{u+j\epsilon_k}\rangle\right) + \delta_{k,s}(u), \quad \text{with} \quad \delta_{k,s}(u) \leq \sum_{i=0}^{k-1}\left|\rho_s(u) - \langle \pi_s, \iota_{\epsilon_k}^{u+i\epsilon_k}\rangle\right|$$

since also $\langle \pi_s, \iota_{\epsilon_k}^{u+i\epsilon_k}\rangle \leq 1$. Without invoking any type of regularity of the density ρ it is not obvious that Lebesgue's differentiation theorem implies that $|\rho_s(u) - \langle \pi_s, \iota_{\epsilon_k}^{u+i\epsilon_k}\rangle| \to 0$ as $\epsilon \to 0$. To see this one can estimate as follows.

$$\begin{aligned}\left|\rho_s(u) - \langle \pi_s, \iota_\epsilon^{u+i\epsilon}\rangle\right| &= \left|\rho(u) - \frac{1}{\epsilon}\int_{u+i\epsilon}^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v\right| \\ &\leq \left|\rho(u) - \frac{1}{(i+1)\epsilon}\int_u^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v\right| + \left|\frac{1}{(i+1)\epsilon}\int_u^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v - \frac{1}{\epsilon}\int_{u+i\epsilon}^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v\right|.\end{aligned}$$

Now focus on the second term on the right-hand side, which equals

$$\begin{aligned}&\left|\frac{1}{(i+1)\epsilon}\int_u^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v - \left(\frac{i+1}{(i+1)\epsilon}\int_u^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v - \frac{i}{i\epsilon}\int_u^{u+i\epsilon}\rho_s(v)\mathrm{d}v\right)\right| \\ &\qquad = i\left|\frac{1}{(i+1)\epsilon}\int_u^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v - \frac{1}{i\epsilon}\int_u^{u+i\epsilon}\rho_s(v)\mathrm{d}v\right| \\ &\qquad\qquad \leq i\left|\rho_s(u) - \frac{1}{(i+1)\epsilon}\int_u^{u+(i+1)\epsilon}\rho_s(v)\mathrm{d}v\right| + i\left|\rho_s(u) - \frac{1}{i\epsilon}\int_u^{u+i\epsilon}\rho_s(v)\mathrm{d}v\right|.\end{aligned}$$

For i fixed, since $i\epsilon \to 0$ as $\epsilon \to 0$, Lebesgue's differentiation theorem concludes that $F_i(\epsilon,s,u) \to 0$ for any s,u,i, fixed as $\epsilon \to 0$. However, in our case i is not fixed and so we need some extra care. Introducing

$$F_i(\epsilon,s,u) = \left|\rho_s(u) - \frac{1}{i\epsilon}\int_u^{u+i\epsilon}\rho_s(v)\mathrm{d}v\right|,$$

from the previous computations we can compare

$$\begin{aligned}\left|\sum_{k=2}^{A}\binom{m}{k}(-1)^k\left\{(1-\rho_s(u))^k-\prod_{j=0}^{k-1}\left(1-\langle\pi_s,\iota_{\epsilon_k}^{u+j\epsilon_k}\rangle\right)\right\}\right| &\leq \sum_{k=2}^{A}\left|\binom{m}{k}\right|\sum_{i=0}^{k-1}\{(i+1)F_{i+1}(\epsilon_k,s,u)+iF_i(\epsilon_k,s,u)\}\\ &\lesssim \sum_{k=2}^{A}\left|\binom{m}{k}\right|k^2\sup_{0\leq i\leq A}F_i(\epsilon_k,s,u)\\ &\lesssim \frac{A^2}{A^m}\sup_{0\leq i\leq A}F_i(\epsilon_k,s,u).\end{aligned}$$

Although $A^2/A^m\to+\infty$ as $A\to+\infty$, we first take the limit $\epsilon\to 0$ and then $A\to+\infty$ to conclude the proof. □

The largest issue now is how to handle the products of occupation variables in the martingale decomposition (3.23). The final goal is to close the equation, relating the correlation terms with the power terms in the weak formulation (3.21). The idea behind the forthcoming approach is to replace a product of $\rho's$ by a product of empirical averages with respect to different, non-intersecting boxes. This last requirement avoids the correlations between the occupation variables on these microscopic boxes.

In order to prove the Proposition 3.3.2 we will make use of several replacement lemmas, whose statements and proofs will be given in Section B.1. The fact that the limiting measure $\mathbb{Q}$ concentrates on absolute continuous trajectories of measures that have a density in the right Sobolev space is also provided by Proposition 3.4.8, proved in Section 3.4.

In what follows, consider the sequence $(\epsilon_k)_{k\geq 1}$ introduced in Lemma 3.3.4 to be defined, for each k, as

$$\epsilon_k=\left(\frac{1}{k^3}\mathbf{1}_{m\in(0,1)}+\frac{1}{k}\mathbf{1}_{m\in(1,2)}\right)\epsilon\quad\text{and}\quad 0<n<\frac{1}{4-m}\mathbf{1}_{m\in(0,1)}+\frac{2-m}{3-m}\mathbf{1}_{m\in(1,2)}<1. \tag{3.28}$$

The role of the two elements just introduced is intertwined. The sequence $(\epsilon_k)_{k\geq 1}$ is chosen such that the series of errors from the discrete approximations vanish. This introduces constraints on the rate of explosion of ℓ_N, which are then corrected through n as above. To be concrete, because the series $\sum_{k\geq 1}\left|\binom{m}{k}\right|$ has a very light tail, we can reduce the study of the next summations over $0\leq k\leq \ell_N$ to $0\leq k\leq(\ell_N)^n$. In this way, modulo some technicalities, one can impose explosion constraints on $(\ell_N)^n$ which can then be encapsulated into n.

Proof of Proposition 3.3.2. From Proposition 3.4.8 we know that if $m\in(0,1)$ then $\rho\in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$, while if $m\in(1,2)$ then $\rho^m\in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$. Recalling that $a(m)=\mathbf{1}_{m\in(0,1)}+m\mathbf{1}_{m\in(1,2)}$, if $\mathbb{Q}$ is a limit point of $(\mathbb{Q}_N)_{N\in\mathbb{N}}$ then

$$\mathbb{Q}\big(\text{for any } t\in[0,T],\ \pi_t(du)=\rho_t(u)du,\ \text{where } \rho^{a(m)}\in L^2([0,T];\mathcal{H}^1(\mathbb{T}))\big)=1.$$

In the weak formulation (3.21), let us replace ρ^m by its binomial expansion as in (3.6). Since we are on the torus we have $\langle\partial_u^2 G,1\rangle=0$, and therefore the binomial series starts from the second term. Otherwise, this would lead to boundary conditions. In this way, it is enough to show that for any $\delta>0$ it holds

$$\mathbb{Q}\left(\sup_{t\in[0,T]}\left|\langle G_t,\rho_t\rangle-\langle G_0,\rho^{\text{ini}}\rangle-\int_0^t\langle\rho_s,\partial_s G_s\rangle\mathrm{d}s-\int_0^t\left\langle\partial_u^2 G_s,\sum_{k\geq 1}\binom{m}{k}(-1)^k(1-\rho_s)^k\mathrm{d}s\right\rangle\right|>\delta\right)=0.$$

Last probability is bounded from above by

$$\mathbb{Q}\Bigg(\sup_{t\in[0,T]} \Bigg| \langle G_t, \rho_t\rangle - \langle G_0, \rho_0\rangle - \int_0^t \langle \rho_s, \partial_s G_s\rangle \mathrm{d}s + m\int_0^t \langle \partial_u^2 G_s, 1-\rho_s\rangle \mathrm{d}s$$

$$- \int_0^t \sum_{k\geq 2} \binom{m}{k} (-1)^k \Big\langle \partial_u^2 G_s, \prod_{j=0}^{k-1} \big\langle 1-\pi_s, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\big\rangle \Big\rangle \mathrm{d}s \Bigg| > \frac{\delta}{2^2} \Bigg) \tag{3.29}$$

$$+ \mathbb{Q}\left(\sup_{t\in[0,T]} \left| \int_0^t \sum_{k\geq 2} \binom{m}{k} (-1)^k \Big\langle \partial_u^2 G_s, (1-\rho_s)^k - \prod_{j=0}^{k-1} \big\langle 1-\pi_s, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\big\rangle \Big\rangle \mathrm{d}s \right| > \frac{\delta}{2} \right) \tag{3.30}$$

$$+ \mathbb{Q}\left(\left| \langle G_0, \rho_0 - \rho^{\mathrm{ini}}\rangle \right| > \frac{\delta}{2^2} \right), \tag{3.31}$$

with $(\epsilon_k)_{k\geq 0}$ as in 3.28. Observe that the third probability (3.31) is equal to zero since the initial probability measure μ_N is a local equilibrium measure associated with the profile ρ^{ini}. From Markov's inequality and Lemma 3.3.4, the second probability (3.30) vanishes as $\epsilon \to 0$, reducing us to treat the first probability (3.29).

We now want to apply Portmanteau's Theorem, and relate the micro and macro scales. For that purpose we need to argue that the whole function of our trajectories is continuous with respect to the Skorokhod topology, thus preserving the open sets. Although this is not the case due to the cutoff functions ι_ε, it is standard in the literature to approximate these functions by continuous functions. Additionally, one has to relate the weak formulation with the martingale (3.23), which involves a *finite* sum. We first treat the truncation problem, then the continuity. Let us fix $1 < \ell_{1/\epsilon} \xrightarrow[\epsilon\to 0]{} +\infty$. Since for any bounded sequence $(a_k)_{k\geq 1}$ we have

$$\left| \sum_{k\geq 2} \binom{m}{k} (-1)^k a_k - \sum_{k=2}^{\ell_{1/\epsilon}} \binom{m}{k} (-1)^k a_k \right| \lesssim \frac{1}{(\ell_{1/\epsilon})^m},$$

we truncate the sum in some $\ell_{1/\epsilon}$ step. To approximate the required functions by continuous functions it is important to analyze the case $m \in (0,1)$ because in this case the series $\sum_{k\geq 11} \left|\binom{m}{k}\right| k$ diverges. We consider the following approximation of the indicator function, from [6, Proposition 4.3.3]. For each $\epsilon > 0$ and $u \in \mathbb{T}$ let $\zeta_\epsilon^u \in C(\mathbb{T})$ be defined as

$$\zeta_\epsilon^u(v) = \begin{cases} v/\epsilon^2, & v \in u + [0,\epsilon^2], \\ 1, & v \in u + (\epsilon^2, \epsilon - \epsilon^2], \\ 1/\epsilon - v/\epsilon^2, & v \in u + (\epsilon - \epsilon^2, \epsilon], \\ 0, & \text{otherwise.} \end{cases}$$

It is simple to see that $\left\| \mathbf{1}_{[0,\epsilon)} - \zeta_\epsilon^0 \right\|_1 = \epsilon^2$, therefore by translation $\left\| \iota_\epsilon^u - \frac{1}{\epsilon}\zeta_\epsilon^u \right\|_1 = \epsilon$ for any $u \in \mathbb{T}$. In particular,

$$\left| \big\langle \rho_s, \iota_{\epsilon_k}^{\cdot+j\epsilon_k} - \frac{1}{\epsilon}\zeta_{\epsilon_k}^{\cdot+j\epsilon_k} \big\rangle \right| \leq \left\| \iota_{\epsilon_k}^{\cdot+j\epsilon_k} - \frac{1}{\epsilon_k}\zeta_{\epsilon_k}^{\cdot+j\epsilon_k} \right\|_1 = \epsilon_k \tag{3.32}$$

and one can replace $\iota_{\epsilon_k}^{\cdot+j\epsilon_k}$ by its continuous approximation with total error of the order of

$$\sum_{k\geq 1} \left| \binom{m}{k} \right| k\epsilon_k \lesssim \epsilon$$

thanks to (3.28). From [19, Proposition A.3] it is enough to show the continuity, with respect to the Skorokhod weak topology, of the map

$$\pi \mapsto \sup_{t\in[0,T]} \left| \int_0^t \sum_{k=2}^{\ell_{1/\epsilon}} \binom{m}{k} (-1)^k \Big\langle \partial_u^2 G_s, \prod_{j=0}^{k-1} \big\langle 1-\pi_s, \frac{1}{\epsilon_k}\zeta_{\epsilon_k}^{\cdot+j\epsilon_k}\big\rangle \Big\rangle \mathrm{d}s \right|,$$

which can be done using the definition of the Skorokhod metric and is also consequence of our definition of the sequence $(\epsilon_k)_{k\geq 1}$. It is now possible to apply Portmanteau's Theorem and then replace back the approximated functions with a vanishing error, as $\varepsilon \to 0$. We are then reduced to treat

$$\begin{aligned}\liminf_{N\to+\infty} \mathbb{Q}_N\Bigg(\sup_{t\in[0,T]} \Bigg| \langle \pi_t^N, G_t\rangle - \langle G_0, \rho_0\rangle - \int_0^t \langle \pi_s^N, \partial_s G_s\rangle \mathrm{d}s + m\int_0^t \langle \partial_u^2 G_s, 1-\pi_s^N\rangle \mathrm{d}s \\ - \int_0^t \sum_{k=2}^{\ell_{1/\epsilon}} \binom{m}{k} (-1)^k \Big\langle \partial_u^2 G_s, \prod_{j=0}^{k-1} \langle 1-\pi_s^N, \frac{1}{\epsilon_k} \zeta_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle \Big\rangle \mathrm{d}s \Bigg| > \frac{\delta}{2^4}\Bigg).\end{aligned} \tag{3.33}$$

We stress that, although for small $\epsilon > 0$ we can have $\ell_{1/\epsilon} > N$, for N fixed, the sum

$$\sum_{k=2}^{\ell_{1/\epsilon}} \binom{m}{k} (-1)^k \prod_{j=0}^{k} \langle 1-\pi_s^N, \frac{1}{\epsilon_k} \zeta_{\epsilon_k}^{\cdot\, u+j\epsilon_k}\rangle$$

is indeed well-defined for any $u \in \mathbb{T}$ and one obtains, for k large enough, repeated terms in the product above. Now we can replace back $\langle \pi_s^N, \frac{1}{\epsilon_k}\zeta_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle$ by $\langle \pi_s^N, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle$ with the previous rationale. Fixed N, since the martingale (3.23) involves a sum up to ℓ_N, we compare again

$$\Bigg| \sum_{k=2}^{\ell_N} \binom{m}{k} (-1)^k a_k - \sum_{k=2}^{\ell_{1/\epsilon}} \binom{m}{k} (-1)^k a_k \Bigg| \lesssim \big| (\ell_{1/\epsilon})^{-m} - (\ell_N)^{-m} \big|. \tag{3.34}$$

Summing and subtracting the appropriate terms, and recalling (3.23), the first probability (3.29), after the aforementioned replacements, is no larger than the sum of terms of order $\epsilon, (\ell_{1/\epsilon})^{-m}, (\ell_N)^{-m}$, plus

$$\begin{aligned}&\liminf_{N\to+\infty} \mathbb{Q}_N \Bigg(\sup_{t\in[0,T]} \Bigg| M_t^N(G) + \sum_{k=2}^{\ell_N} \binom{m}{k} (-1)^k \int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} \Delta^N G_s(\tfrac{x}{N}) \mathbf{h}_s^{(k-1)}(\tau_x\overline{\eta}) \mathrm{d}s \\ &\qquad + \sum_{k=2}^{\ell_N} \binom{m}{k} (-1)^k \int_0^t \Big\langle \partial_u^2 G_s, \prod_{j=0}^{k-1} \langle 1-\pi_s^N, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle \Big\rangle \mathrm{d}s \Bigg| > \frac{\delta}{2^6} \Bigg) \\ &\leq \mathbb{P}_{\mu_N} \Bigg(\sup_{t\in[0,T]} \Bigg| \sum_{k=2}^{\ell_N} \binom{m}{k} (-1)^k \int_0^t \Big\langle \partial_u^2 G_s - \Delta^N G_s, \prod_{j=0}^{k-1} \langle 1-\pi_s^N, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle \Big\rangle \mathrm{d}s \Bigg| > \frac{\delta}{3\times 2^6} \Bigg) \\ &\quad + \mathbb{P}_{\mu_N} \Bigg(\sup_{t\in[0,T]} \Bigg| \sum_{k=2}^{\ell_N} \binom{m}{k} (-1)^k \int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} \Delta^N G_s(\tfrac{x}{N}) \Big[\prod_{j=0}^{k-1} \langle 1-\pi_s^N, \iota_{\frac{x}{N}+j\epsilon_k}\rangle - \mathbf{h}_s^{(k-1)}(\tau_x\overline{\eta}) \Big] \mathrm{d}s \Bigg| > \frac{\delta}{3\times 2^6} \Bigg) \\ &\quad + \mathbb{P}_{\mu_N} \Bigg(\sup_{t\in[0,T]} \big| M_t^N(G) \big| > \frac{\delta}{3\times 2^6} \Bigg).\end{aligned}$$

Note that the linear term $\langle \partial_u^2 G_s, 1-\pi_s^N\rangle$ in (3.33) was absorbed into the martingale $M_t^N(G)$, and so the challenge is to treat the non-linear terms. The first probability on the right-hand side above vanishes as $N \to +\infty$ since $G_s \in C^2(\mathbb{T})$ for all $s \in [0,t]$; the second probability is treated using the replacement lemmas with a scheme that we present shortly; the third with Doob's inequality and the proof of the first condition in (3.25). Let us give more details for the second one. Recall the expression of $\mathbf{h}^{(k)}$ as in Lemma 3.2.15. We split the second probability on the right-hand side of last display into

$$\begin{aligned}&\mathbb{P}_{\mu_N} \Bigg(\sup_{t\in[0,T]} \Bigg| \sum_{k=2}^{\ell_N} \binom{m}{k} (-1)^k \int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} \Delta^N G_s(\tfrac{x}{N}) \times \\ &\qquad \times \tau_x \Bigg\{ \sum_{j=1}^{k-1} \sum_{n=0}^{k-1-j} (\nabla^+ \overline{\eta}_{N^2 s})(n) \mathbf{s}_j^{(k-1)}(\tau_n \overline{\eta}_{N^2 s}) \Bigg\} \mathrm{d}s \Bigg| > \frac{\delta}{3\times 2^7} \Bigg)\end{aligned} \tag{3.35}$$

$$+\mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\sum_{k=2}^{\ell_N}\binom{m}{k}(-1)^k\int_0^t\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Delta^N G_s(\tfrac{x}{N})\left[\prod_{j=0}^{k-1}\langle 1-\pi_s^N,\iota_{\epsilon_k}^{\frac{x}{N}+j\epsilon_k}\rangle-\prod_{i=0}^{k-1}\overline{\eta}_{N^2s}(x+i)\right]\mathrm{d}s\right|>\frac{\delta}{3\times 2^7}\right).$$

Focus on the first probability in the previous display. We apply Proposition B.1.2 and triangle's inequality and then pass the summation over x to outside the expectation. For $m\in(0,1)$, since the summation starts at $k=2$, the resulting quantity is treated using both Lemma B.1.6 and Lemma B.1.7 (to introduce the term corresponding to $k=1$) for each term of the summation over x, estimating it by

$$\frac{1}{B}+TB\frac{(\ell_N)^{1-m}}{N},$$

for any $B>0$, which will be taken to infinity after $N\to+\infty$.

For $m\in(1,2)$ we could either prove an analogue of Lemma B.1.6 for the slow regime, or take advantage of the tail of the sum of the binomial coefficients being just light enough, in this regime, to slow down the explosion of ℓ_N, avoiding further restrictions. We present the second alternative, and in this way let n be as in (3.28).

Since $\left|\mathbf{h}^{(k)}(\eta)\right|\le k$ for any configuration η, from Lemma (3.2.15) and the fact that $\eta(0)\cdots\eta(k)\le 1$ we see that

$$\left|\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^+\overline{\eta})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta})\right|\le k$$

and we can estimate

$$\begin{aligned}
&\sum_{k=(\ell_N)^n+1}^{\ell_N}\binom{m}{k}(-1)^k\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Delta^N G_s(\tfrac{x}{N})\tau_x\left\{\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^+\overline{\eta}_{N^2s})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta}_{N^2s})\right\}\\
&\qquad\le\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\Delta^N G_s(\tfrac{x}{N})\right|\sum_{k=(\ell_N)^n}^{\ell_N}\left|\binom{m}{k}\right|k\\
&\qquad\lesssim\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\Delta^N G_s(\tfrac{x}{N})\right|\left(\frac{1}{(\ell_N)^{n(m-1)}}-\frac{1}{(\ell_N)^{m-1}}\right),
\end{aligned}$$

which vanishes by taking the limit $N\to+\infty$. This means that we can replace the summation up to ℓ_N by a summation up to $(\ell_N)^n$. In this way, from Proposition B.1.2, the previous truncation at $(\ell_N)^n$ and triangle's inequalities, we are then reduced to treating

$$\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right|\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}\frac{1}{N}\sum_{x\in\mathbb{T}_N}\mathbb{E}_{\mu_N}\left[\left|\int_0^t\Delta^N G_s(\tfrac{x}{N})\tau_x\left\{(\nabla^+\overline{\eta}_{N^2s})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta}_{N^2s})\right\}\mathrm{d}s\right|\right].$$

Applying the replacement Lemma B.1.3 to each term of the sum over j with $\varphi(s,\eta)=\Delta^N G_s(\frac{x}{N})\mathbf{s}_j^{(k-1)}(\tau_{n+x}\overline{\eta})$ we obtain an upper bound of the order of

$$\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right|k^2\left(\frac{1}{B_k}+TB_k\frac{(\ell_N)^{m-1}}{N}\right).$$

Let $B_k=kB>0$. Then last display is bounded from above by some constant times

$$\frac{1}{B}\sum_{k=2}^{(\ell_N)^n}\frac{1}{k^m}+TB\frac{(\ell_N)^{m-1}}{N}\sum_{k=2}^{(\ell_N)^n}\frac{1}{k^{m-2}}\lesssim\frac{1}{B}+TB\frac{(\ell_N)^{m-1+n(3-m)}}{N},$$

and the right-hand side converges to zero as $N\to+\infty$ and $B\to+\infty$ since by the definition of n in (3.28) we have $m-1+n(3-m)<1$.

Now the main goal is to estimate for $m \in (0,2)\backslash\{1\}$ the quantity

$$\sum_{k=2}^{\ell_N} \left|\binom{m}{k}\right| \mathbb{E}_{\mu_N}\left[\left|\int_0^t \frac{1}{N}\sum_{x\in\mathbb{T}_N} \Delta^N G_s(\tfrac{x}{N})\left(\prod_{j=0}^{k-1}\left(1-\langle \pi_s^N, \iota_{\epsilon_k}^{\frac{x}{N}+j\epsilon_k}\rangle\right) - \prod_{i=0}^{k-1}\overline{\eta}_{N^2 s}(x+i)\right)\mathrm{d}s\right|\right] \tag{3.36}$$

where, again, we applied Proposition B.1.2. It will be important to slow down the explosion $\ell_N \to +\infty$ for $m \in (0,1)$ too before applying repeatedly the replacement lemmas. Consider the sequence $(a_k)_{k\geq 1}$ with $a_k \equiv a_k(t,G,\eta)$ defined by

$$a_k = \left|\int_0^t \frac{1}{N}\sum_{x\in\mathbb{T}_N} \Delta^N G_s(\tfrac{x}{N})\left(\prod_{j=0}^{k-1}\left(1-\langle \pi_s^N, \iota_{\epsilon_k}^{\frac{x}{N}+j\epsilon_k}\rangle\right) - \prod_{i=0}^{k-1}\overline{\eta}_{N^2 s}(x+i)\right)\mathrm{d}s\right|.$$

From the triangle inequality and the fact that $G_s \in C^2(\mathbb{T})$ it is simple to see that the sequence $(a_k)_k$ is uniformly bounded by $\int_0^t N^{-1}\sum_{x\in\mathbb{T}_N}|\Delta^N G_s(\frac{x}{N})|\mathrm{d}s$ which tends to $\left\|\partial_u^2 G\right\|_{L^1([0,T]\times\mathbb{T})} < \infty$ as $N\to\infty$. In particular,

$$\left|\sum_{k=2}^{\ell_N}\left|\binom{m}{k}\right| a_k - \sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right| a_k\right| = \sum_{k=(\ell_N)^n+1}^{\ell_N}\left|\binom{m}{k}\right| a_k \lesssim (\ell_N)^{-nm} - (\ell_N)^{-m} \xrightarrow[N\to+\infty]{} 0.$$

In this way, the treatment of (3.36) gives place to the treatment of

$$\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right| \frac{1}{N}\sum_{x\in\mathbb{T}_N} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \Delta^N G_s(\tfrac{x}{N})\left(\prod_{j=0}^{k-1}\left(1-\langle \pi_s^N, \iota_{\epsilon_k}^{\frac{x}{N}+j\epsilon_k}\rangle\right) - \prod_{i=0}^{k-1}\overline{\eta}_{N^2 s}(x+i)\right)\mathrm{d}s\right|\right]. \tag{3.37}$$

To treat (3.37) we now split into the slow and fast diffusion cases. In what follows, for each $k \geq 1$ let

$$L_k = \frac{1}{k}(\ell_N)^p \quad \text{with} \quad p = \frac{n+1}{2}(2-m), \quad m \in (1,2). \tag{3.38}$$

As we argued that $\rho_s(u)$ is approximated locally by an average (Lemma 3.3.4), we continued further into the discrete scale by approximating the local average by its correspondent in the mesoscopic scale, with the average over "small macroscopic" boxes. Now we proceed similarly into the microscopic scale, with the average over "large microscopic" boxes, which is then shown to be close to the occupation variable, akin to a discrete version of Lebesgue's differentiation theorem. The sequence $(L_k)_{k\geq 1}$ is precisely the length of the microscopic boxes for the replacements associated with a power k of the density.

• Slow-diffusion, $m \in (1,2)$: We follow a simplified version of the scheme in [5, Subsection 5.2]. The forthcoming lemmas will be applied to each term of the summation over $x \in \mathbb{T}_N$.

1. *Rearrangements*: rewrite

$$\prod_{j=0}^{k-1}\overline{\eta}(jL_k) - \prod_{j=0}^{k-1}\overline{\eta}(j) = \sum_{i=1}^{k-1}\left(\eta(iL_k) - \eta(i)\right)\tilde{\varphi}_i^{(1)}(\eta) \tag{3.39}$$

where for each $i \in \{1,\dots,k-1\}$ we defined $\tilde{\varphi}_i^{(1)}(\eta) = \prod_{j=0}^{i-1}\overline{\eta}(j)\prod_{j=i+1}^{k-1}\overline{\eta}(jL_k)$. The random variable $\varphi_i^{(1)}(s,\eta) \equiv \Delta^N G_s(\frac{x}{N})\tilde{\varphi}_i^{(1)}(\tau_x\eta)$ is independent of the occupation variables at sites $[\![i, iL_k]\!]$ and, fixed x and applying the triangle inequality we treat each term of the summation over i in (3.39) with Lemma B.1.3. For any $B>0$ we obtain an upper bound of the order of

$$\begin{aligned}\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right|\sum_{i=1}^{k-1}\left\{\frac{1}{B} + TBiL_k\frac{(\ell_N)^{m-1}}{N}\right\} &\lesssim \sum_{k=2}^{(\ell_N)^n}\frac{1}{k^m}\left\{\frac{1}{B} + \frac{(\ell_N)^{m-1}}{N}TBL_k k\right\} \\ &\lesssim \frac{1}{B} + TB\frac{(\ell_N)^{m-1+p}}{N}.\end{aligned} \tag{3.40}$$

Since $m-1+p<1 \Leftrightarrow n<1$ which is indeed satisfied, one can take the limit $N\to+\infty$ and then $B\to+\infty$ in the previous display;

2. *One-block estimates*: rewrite

$$\prod_{j=0}^{k-1}\overline{\eta}^{L_k}(jL_k)-\prod_{j=0}^{k-1}\overline{\eta}(jL_k)=\sum_{i=0}^{k-1}\left(\eta^{L_k}(iL_k)-\eta(iL_k)\right)\tilde{\varphi}_i^{(2)}(\eta) \tag{3.41}$$

where for every $i\in\{1,\dots,k-1\}$ we defined $\tilde{\varphi}_i^{(2)}(\eta)=\prod_{j=0}^{i-1}\overline{\eta}_k^L(jL_k)\prod_{j=i+1}^{k-1}\overline{\eta}(jL_k)$. The random variable $\varphi_i^{(2)}(s,\eta)\equiv\Delta^N G_s(\frac{x}{N})\tilde{\varphi}_i^{(2)}(\tau_x\eta)$ is independent of the occupation variables at sites $[\![iL_k,(i+1)L_k-1]\!]$ and, fixed x and applying the triangle inequality we treat each term of the summation over i in (3.41) with Corollary B.1.4. We obtain an upper bound of the order of

$$\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right|\sum_{i=0}^{k-1}\left\{\frac{1}{B}+TBL_k\frac{(\ell_N)^{m-1}}{N}\right\}.$$

This quantity is no larger than the quantity on the left-hand side of (3.40), therefore the same rationale used there guarantees that these errors vanish by taking the limits;

3. *Two-block estimates*: rewrite

$$\prod_{j=0}^{k-1}\overline{\eta}^{\lfloor N\epsilon_k\rfloor}(j\lfloor N\epsilon_k\rfloor)-\prod_{j=0}^{k-1}\overline{\eta}^{L_k}(jL_k)=\sum_{i=0}^{k-1}\left(\eta^{\lfloor N\epsilon_k\rfloor}(i\lfloor N\epsilon_k\rfloor)-\eta^{L_k}(iL_k)\right)\tilde{\varphi}_i^{(3)}(\eta) \tag{3.42}$$

where for every $i\in\{1,k-1\}$ we defined $\tilde{\varphi}_i^{(3)}(\eta)=\prod_{j=0}^{i-1}\overline{\eta}^{L_k}(jL_k)\prod_{j=i+1}^{k-1}\overline{\eta}^{\lfloor N\epsilon_k\rfloor}(j\lfloor N\epsilon_k\rfloor)$. The random variable $\varphi_i^{(3)}(s,\eta)\equiv\Delta^N G_s(\frac{x}{N})\tilde{\varphi}_i^{(3)}(\tau_x\eta)$ is independent of the occupation variables at sites contained in

$$[\![iL_k,(i+1)\lfloor N\epsilon_k\rfloor-1]\!]\cup[\![iL_k,iL_k+\lfloor N\epsilon_k\rfloor-1]\!]$$

provided $\lfloor N\epsilon_k\rfloor\geq L_k$, that is, $\lfloor N\epsilon\rfloor\geq L$, which is true for N large enough from (3.38). Fixed x and applying the triangle inequality we treat each term of the summation over i in (3.42) with Lemma B.1.5, leading to an upper bound of the order of

$$\begin{aligned}
&\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right|\sum_{i=0}^{k-1}\left\{\frac{1}{B}+T\left[\frac{1}{L_k}+B\left(\frac{L_k(\ell_N)^{m-1}}{N}+\frac{iL_k}{N}+\epsilon_k(i+1)\right)\right]\right\}\\
&\qquad\lesssim\sum_{k=2}^{(\ell_N)^n}\frac{1}{Bk^m}+T\sum_{k=2}^{(\ell_N)^n}\frac{1}{L_kk^m}+\frac{T(\ell_N)^{m-1}}{N}\sum_{k=2}^{(\ell_N)^n}\frac{BL_k}{k^m}+T\sum_{k=2}^{(\ell_N)^n}\frac{B}{k^{m-1}}\left(\frac{L_k}{N}+\epsilon_k\right)\\
&\qquad\lesssim\frac{1}{B}+T\frac{(\ell_N)^{n(2-m)}}{L}+TB\frac{(\ell_N)^{m-1}L}{N}+TB\frac{L}{N}+TB\epsilon.
\end{aligned}$$

Since $L=(\ell_N)^p$ with $n(2-m)-p<0$, $m-1+p<1$ and $p<1$ the right-hand side above vanishes taking the appropriate limits;

4. *Conclusion*: rewrite

$$\begin{aligned}
&\prod_{j=0}^{k-1}\left(1-\langle\pi^N,\iota_{\epsilon_k}^{j\epsilon_k}\rangle\right)-\prod_{j=0}^{k-1}\left(1-\eta^{\lfloor N\epsilon_k\rfloor}(j\lfloor N\epsilon_k\rfloor)\right)\\
&\qquad=\sum_{i=0}^{k-1}\left\{\left[\prod_{j=1}^{i-1}1-\eta^{\lfloor N\epsilon_k\rfloor}(j\lfloor N\epsilon_k\rfloor)\right]\left(\eta^{\lfloor N\epsilon_k\rfloor}(i\lfloor N\epsilon_k\rfloor)-\langle\pi^N,\iota_{\epsilon_k}^{i\epsilon_k}\rangle\right)\left[\prod_{j=i+1}^{k-1}1-\langle\pi^N,\iota_{\epsilon_k}^{j\epsilon_k}\rangle\right]\right\},
\end{aligned} \tag{3.43}$$

and since $|\langle\pi_s^N,\iota_\epsilon^{\frac{x}{N}}\rangle-\eta_{N^2s}^{\lfloor N\epsilon\rfloor}(x)|\leq\lfloor N\epsilon\rfloor^{-1}$, previous display is no larger than $k\lfloor N\epsilon\rfloor^{-1}$. This way, we need to bound from above

$$\sum_{k=2}^{(\ell_N)^n}\left|\binom{m}{k}\right|\frac{k}{\lfloor N\epsilon_k\rfloor}\lesssim\frac{1}{\lfloor N\epsilon\rfloor}\sum_{k=2}^{(\ell_N)^n}\frac{k}{k^m}\lesssim\frac{(\ell_N)^{n(2-m)}}{\lfloor N\epsilon\rfloor}.$$

Since by the definition of n in (3.28) it holds that $n < 1$, to conclude the proof it is enough to recall that $\ell_N \le N$, and then take the limit $N \to +\infty$ and $\epsilon \to 0$ and then $B \to +\infty$.

• FAST-DIFFUSION, $m \in (0,1)$: Recall that the goal is to treat (3.37). The strategy now is similar but simpler than for the slow diffusion case. The specific maps $\varphi : \Omega_N \to \mathbb{R}$ in the statement of the replacement lemmas in Subsection B.1.6 can be introduced analogously to the slow-diffusion case, therefore we omit their definition.

1. *Rearrangements*: rewrite

$$\prod_{j=0}^{k-1} \overline{\eta}(j\lfloor N\epsilon_k \rfloor) - \prod_{j=0}^{k-1} \overline{\eta}(j) = \sum_{i=1}^{k-1} \left[\prod_{j=0}^{i-1} \overline{\eta}(j) \right] (\eta(i\lfloor N\epsilon_k \rfloor) - \eta(i)) \left[\prod_{j=i+1}^{k-1} \overline{\eta}(j\lfloor N\epsilon_k \rfloor) \right]$$

and apply Lemma B.1.7 to each term of the summation in i. Recalling that in this regime $\epsilon_k = \epsilon/k^3$, as in (3.28), one obtains an error of the order of

$$\sum_{k=2}^{(\ell_N)^n} \left| \binom{m}{k} \right| \sum_{i=1}^{k-1} \left\{ \frac{1}{B_k} + TB_k \frac{i(\lfloor N\epsilon_k \rfloor - 1)}{N} \right\} \lesssim \sum_{k=2}^{(\ell_N)^n} \left\{ \frac{1}{B_k k^m} + \epsilon T B_k k^2 \frac{1/k^3}{k^{1+m}} \right\} \lesssim \frac{1}{B} + TB\epsilon,$$

where we fixed $B_k = Bk > 0$ with $B > 0$, guaranteeing the convergence of the series as $N \to +\infty$.

2. *One-block and two-blocks estimates*: rewrite

$$\prod_{j=0}^{k-1} \overline{\eta}^{\lfloor N\epsilon_k \rfloor}(j\lfloor N\epsilon_k \rfloor) - \prod_{j=0}^{k-1} \overline{\eta}(j\lfloor N\epsilon_k \rfloor) = \sum_{i=0}^{k-1} \left[\prod_{j=0}^{i-1} \overline{\eta}^{\lfloor N\epsilon_k \rfloor}(j\lfloor N\epsilon_k \rfloor) \right] \left(\eta^{\lfloor N\epsilon_k \rfloor}(i\lfloor N\epsilon_k \rfloor) - \eta(i\lfloor N\epsilon_k \rfloor) \right) \left[\prod_{j=i+1}^{k-1} \overline{\eta}(j\lfloor N\epsilon_k \rfloor) \right]$$

and apply Lemma B.1.7 to each term of the summation in i, leading to an upper bound of the order of

$$\sum_{k=2}^{(\ell_N)^n} \left| \binom{m}{k} \right| \sum_{i=0}^{k-1} \left\{ \frac{1}{B_k} + TB_k \frac{1}{\lfloor N\epsilon_k \rfloor} \sum_{y \in \Lambda_{i\lfloor N\epsilon_k \rfloor}^{\lfloor N\epsilon_k \rfloor}} \frac{|i\lfloor N\epsilon_k \rfloor - y|}{N} \right\} \lesssim \sum_{k=2}^{(\ell_N)^n} \frac{1}{k^{1+m}} \left(\frac{k}{Bk} + TB\epsilon k^2 \frac{1}{k^3} \right) \lesssim \frac{1}{B} + TB\epsilon$$

where we chose $B_k = Bk > 0$ with $B > 0$ in order to guarantee the convergence as $N \to +\infty$.

3. *Conclusion*: rewrite

$$\prod_{j=0}^{k-1} \left(1 - \langle \pi^N, \iota_{\epsilon_k}^{j\epsilon_k} \rangle\right) - \prod_{j=0}^{k-1} \left(1 - \eta^{\lfloor N\epsilon_k \rfloor}(j\lfloor N\epsilon_k \rfloor)\right) = \sum_{i=0}^{k-1} \left[\prod_{j=1}^{i-1} 1 - \eta^{\lfloor N\epsilon_k \rfloor}(j\lfloor N\epsilon_k \rfloor) \right] \left(\eta^{\lfloor N\epsilon_k \rfloor}(i\lfloor N\epsilon_k \rfloor) - \langle \pi^N, \iota_{\epsilon_k}^{i\epsilon_k} \rangle \right) \left[\prod_{j=i+1}^{k-1} 1 - \langle \pi^N, \iota_{\epsilon_k}^{j\epsilon_k} \rangle \right],$$

and proceed as in (3.43), leading to an upper bound of the order of

$$\sum_{k=2}^{(\ell_N)^n} \left| \binom{m}{k} \right| \frac{k}{\lfloor N\epsilon_k \rfloor} \lesssim \frac{1}{\lfloor N\epsilon \rfloor} \sum_{k=2}^{(\ell_N)^n} \frac{1}{k^{m-3}} \lesssim \frac{1}{\lfloor N\epsilon \rfloor} (\ell_N)^{n(4-m)}.$$

It is then enough to fix $0 < n < 1$ such that $n(4-m) < 1$. To conclude one takes the corresponding limits as previously. □

3.4 Energy Estimate

Before studying the differentiability of quantities related to the solution we need to introduce some objects that will play an important role both in the present section and in the proof of the replacement lemmas in Appendix B.1.

Definition 3.4.1 (Dirichlet Form and Carré du Champ)**.** For a probability measure μ on Ω_N and $f:\Omega_N\to\mathbb{R}$ density with respect to μ, we define the Dirichlet form for any $m\in(0,2]$ as

$$\mathcal{E}_N^{(m-1)}(f,\mu)=\langle f,\big(-\mathcal{L}_N^{(m-1)}\big)f\rangle_\mu=\int_{\Omega_N}f(\eta)\sum_{x\in\mathbb{T}_N}\big(-r_N^{(m-1)}(\tau_x\eta)\big)(\nabla_{x,x+1}f)(\eta)\mu(\mathrm{d}\eta)$$

and the non-negative quadratic form

$$\Gamma_N^{(m-1)}(f,\mu)=\int_{\Omega_N}\sum_{x\in\mathbb{T}_N}r_N^{(m-1)}(\tau_x\eta)\left[(\nabla_{x,x+1}f)(\eta)\right]^2\mu(\mathrm{d}\eta),$$

where we recall the introduction of the rate $r_N^{(m-1)}$ in Definition 3.2.8.

We remark that rewriting $-a(b-a)=(a-b)^2/2+(a^2-b^2)/2$, one obtains the identity

$$\frac{1}{2}\Gamma_N^{(m-1)}(\sqrt{f},\mu)=\mathcal{E}_N^{(m-1)}(\sqrt{f},\mu)+\frac{1}{2}\left\langle\mathcal{L}_N^{(m-1)}f\right\rangle_\mu. \tag{3.44}$$

The key observation in order to proceed similarly to [5] is the following proposition.

Proposition 3.4.2 (Energy lower bound)**.** *Let ν_α^N be the Bernoulli product measure on Ω_N where $\alpha:[0,1]\to(0,1)$ is either a Lipschitz function or constant, and let f be a density with respect to ν_α^N. For any $m\in(0,2)\backslash\{1\}$ and any $N\in\mathbb{N}_+$ such that $\ell_N\geq 2$ it holds*

$$\begin{aligned}\mathcal{E}_N^{(m-1)}(\sqrt{f},\nu_\alpha^N)\geq\ &\mathbf{1}_{\{m\in(1,2)\}}\frac{m}{4}\left(\delta_N\Gamma_N^{(0)}(\sqrt{f},\nu_\alpha^N)+\frac{m-1}{2}\Gamma_N^{(1)}(\sqrt{f},\nu_\alpha^N)\right)\\&+\mathbf{1}_{\{m\in(0,1)\}}\frac{1}{4}\Gamma_N^{(0)}(\sqrt{f},\mu)-\frac{\mathbf{c}_\alpha}{4N}\end{aligned} \tag{3.45}$$

where $\mathbf{c}_\alpha>0$ if α is Lipschitz non-constant, or $\mathbf{c}_\alpha=0$ if α is constant.

Remark 3.4.3. We highlight that throughout the rest of this chapter we will fix ν_α^N, with $\alpha(\cdot)\equiv\alpha\in(0,1)$ a constant function as reference measure. We chose to present the previous result with $\alpha(\cdot)$ also not being constant since it is necessary in order to extend the present model to the open boundary setting.

We recall some classical results that we will invoke throughout this section. Let $\mathcal{H}$ be a Hilbert space with corresponding norm $\|\cdot\|_{\mathcal{H}}$ and $f:\mathcal{H}\to\mathbb{R}$ a linear functional. The (dual) norm of the linear functional f is defined as

$$|\!|\!|f|\!|\!|=\sup_{\|x\|_{\mathcal{H}}\leq 1,x\in\mathcal{H}}|f(x)|.$$

We know that (see for instance [31, Proposition A.1.1.]) if there exists $K_0>0$ and a positive real number κ such that $\sup_{x\in\mathcal{H}}\left\{f(x)-\kappa\|x\|_{\mathcal{H}}^2\right\}\leq K_0$, then f is bounded. Let us now introduce:

Definition 3.4.4. Let $L^2([0,T]\times\mathbb{T})$ be the (Hilbert) space of measurable functions $G:[0,T]\times\mathbb{T}\to\mathbb{R}$ such that

$$\int_0^T\|G_s\|_{L^2(\mathbb{T})}^2\mathrm{d}s<\infty,$$

endowed with the scalar product $\langle\!\langle G, H \rangle\!\rangle$ defined by

$$\langle\!\langle G, H \rangle\!\rangle = \int_0^T \langle G_s, H_s \rangle \mathrm{d}s.$$

For any $r \in \mathbb{R}_+$ fixed, define the linear functional $\ell^{(r)}$ on $C^{0,1}([0,T] \times \mathbb{T})$ by $\ell_\rho^{(r)}(G) = \langle\!\langle \partial_u G, \rho^r \rangle\!\rangle$.

An important result is the following:

Lemma 3.4.5. *[31, Lemma A.1.9]. If $\xi \in L^2([0,T] \times \mathbb{T})$ is such that there exists a function $\partial\xi \in L^2([0,T] \times \mathbb{T})$ satisfying for all $G \in C^{0,1}([0,T] \times \mathbb{T})$ the identity*

$$\langle\!\langle \partial_u G, \xi \rangle\!\rangle = -\langle\!\langle G, \partial\xi \rangle\!\rangle,$$

then $\xi \in L^2([0,T]; \mathcal{H}^1(\mathbb{T}))$.

Definition 3.4.6. For $G \in C^{0,1}([0,T] \times \mathbb{T})$, $r, \kappa \in \mathbb{R}_+$ define $\mathcal{E}_{G,\kappa}^{(m)} : \mathcal{D}([0,T], \mathcal{M}_+) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_{G,\kappa}^{(r)}(\pi) = \begin{cases} \ell^{(r)}(G) - \kappa \|G\|_2^2, & \text{if } \pi \in \mathcal{D}([0,T], \mathcal{M}_+), \\ +\infty, & \text{otherwise,} \end{cases}$$

and the energy functional $\mathcal{E}_\kappa^{(r)} : \mathcal{D}([0,T], \mathcal{M}_+) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_\kappa^{(r)}(\pi) = \sup_{G \in C^{0,1}([0,T] \times \mathbb{T})} \mathcal{E}_{G,\kappa}^{(r)}(\pi).$$

Remark 3.4.7. Note that $\mathcal{E}_\kappa^{(r)}(\pi) \geq 0$. To see this it is enough to take $G = 0$.

Recall that the measure $\mathbb{Q}$ is the weak limit of a subsequence of $\mathbb{Q}_N$ as $N \to +\infty$, where $\mathbb{Q}_N$ is the measure induced by the empirical measure in the Skorokhod space of trajectories $\mathcal{D}([0,T], \Omega_N)$. Recall also the definition of the target Sobolev space (Definition 3.2.21). The main goal of this section is to prove the next proposition.

Proposition 3.4.8. *The measure $\mathbb{Q}$ is concentrated on trajectories of absolutely continuous measures with respect to the Lebesgue measure, $\pi_\cdot(\mathrm{d}u) = \rho_\cdot(u)\mathrm{d}u$, such that $\rho^m \in L^2([0,T]; \mathcal{H}^1(\mathbb{T}))$, for $m \in (1,2)$, and $\rho \in L^2([0,T], \mathcal{H}^1(\mathbb{T}))$, for $m \in (0,1)$.*

This will be shown to be a consequence of existing positive real numbers κ_0, κ_1, K_0 and K_1 such that: for $m \in (0,1)$ holds $E_\mathbb{Q}\left[\mathcal{E}_{\kappa_0}^{(1)}(\pi)\right] \leq K_0$, and for $m \in (1,2)$ holds $E_\mathbb{Q}\left[\mathcal{E}_{\kappa_1}^{(m)}(\pi)\right] \leq K_1$, where $E_\mathbb{Q}$ denotes the expectation with respect to $\mathbb{Q}$. This will be proved in Proposition 3.4.10 and 3.4.9, respectively. Recall (3.16). For the slow diffusion case, the argument is analogous to [5, Section 6] but we make evident that this argument works due to the fact that the rates are uniformly bounded by a constant independent of N and the fact that the model is gradient. In particular, the argument is suited to show that the "macroscopic" quantity

$$\rho^m = \lim_{N \to +\infty} \int h_N^{(m-1)}(\eta) \nu_\rho^N(\mathrm{d}\eta)$$

lives in the target Sobolev space, where $\rho(\cdot) \in (0,1)$ is a constant function. As in [5], the argument does not allow us to show that ρ has a weak derivative, the reason being that $\rho^m \leq \rho$.

For $m \in (0,1)$ we have the opposite problem. Without imposing any restriction on the initial profile, we cannot show that $\rho^m \in L^2([0,T],\mathcal{H}^1(\mathbb{T}))$, the reason being that (see Remark 3.2.11)

$$\lim_{N\to+\infty} \sup_{\eta\in\Omega_N} r_N^{(m-1)}(\eta) = +\infty.$$

This is the discrete analogous to $\rho^{m-1} \to +\infty$ as $\rho \to 0$. Yet, we can show that $\rho \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$ because the transition rates, in this case, are larger than the ones for the SSEP (analogous to $\rho \le \rho^m$ in this case), which is again a gradient model.

Proof of Proposition 3.4.8. Recall that up to this point we have proved that the measure $\mathbb{Q}$ is a *Dirac measure*, namely $\mathbb{Q} = \delta_\pi$ with $\pi_\cdot$ the trajectory of absolutely continuous measures $\pi_\cdot(\mathrm{d}u) = \rho_\cdot(u)\mathrm{d}u$, where ρ is integrable and its time evolution is described by the equation (3.21), that is,

$$\langle \rho_t, G_t\rangle = \langle \rho^{\mathrm{ini}}, G_0\rangle + \int_0^t \left\{\langle \rho_s, \partial_s G_s\rangle + \langle (\rho_s)^m, \partial_u^2 G_s\rangle\right\}\mathrm{d}s$$

with $G \in C^{1,2}([0,T]\times\mathbb{T})$ a test function. For $m \in (1,2)$, from Proposition 3.4.9 the functional $\ell^{(m)}$ is bounded $\mathbb{Q}$–a.s. Since $C^{0,1}([0,T]\times\mathbb{T})$ is dense in $L^2([0,T]\times\mathbb{T})$, we can extend $\ell^{(m)}$ to a $\mathbb{Q}$–a.s. bounded functional in $L^2([0,T]\times\mathbb{T})$. One can thus invoke Riesz's representation Theorem and conclude that for any $m \in (1,2)$ there exists a function $\partial\rho^m \in L^2([0,T]\times\mathbb{T})$ such that

$$\ell_\rho^{(m)}(G) = -\langle\!\langle G, \partial\rho^m\rangle\!\rangle.$$

To finish the proof, since $\rho^m \in L^2([0,T]\times\mathbb{T})$, one invokes Lemma 3.4.5.

For $m \in (0,1)$ the same argument leads to $\rho \in L^2([0,T],\mathcal{H}^1(\mathbb{T}))$ but now one should invoke instead Proposition 3.4.10 which states that the functional $\ell^{(1)}$ is bounded. □

Proposition 3.4.9. *For any $m \in (1,2)$ there are finite constants $\kappa, K > 0$ independent of N such that*

$$E_{\mathbb{Q}}\left[\mathcal{E}_\kappa^{(m)}(\pi)\right] \le K.$$

Proof. Recall that from the binomial theorem we can expand

$$\rho^m = \sum_{k\ge 0}\binom{m}{k}(-1)^k(1-\rho)^k,$$

and since we are on the torus we can treat

$$E_{\mathbb{Q}}\left[\sup_{G\in C^{0,1}([0,T]\times\mathbb{T}_N)}\left\{\sum_{k\ge1}\binom{m}{k}(-1)^k\langle\!\langle(1-\rho)^k, \partial_u G\rangle\!\rangle - \kappa_1\|G\|_2^2\right\}\right].$$

For the open boundary case it is enough to focus on showing the (weak) differentiability of the quantity $\rho^m - 1$, that implies the differentiability of ρ^m. Recalling that $C^{0,1}([0,T]\times\mathbb{T})$ is separable with respect to the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{T})}$, consider a countable dense subset, $\{G_p\}_{p\in\mathbb{N}}$, in $C^{0,1}([0,T]\times\mathbb{T})$. An application of the monotone convergence theorem then reduces the problem to that of treating

$$\lim_{\ell\to+\infty} E_{\mathbb{Q}}\left[\max_{\substack{G_p\\ p\le\ell}}\left\{\mathcal{E}_{G_p}(\pi)\right\}\right].$$

Fixed G_p, Lemma 3.3.4 allow us to replace $(1-\rho)^k$ by $\prod_{j=0}^{k-1}\left(1-\langle \pi, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle\right)$, with the sequence $(\epsilon_k)_{k\geq 0}$ depending on the regime of m, leaving us with

$$E_{\mathbb{Q}}\left[\max_{\substack{G_p\\ p\leq \ell}}\left\{\mathcal{E}_{G_p}(\pi)\right\}\right] \leq E_{\mathbb{Q}}\left[\max_{\substack{G_p\\ p\leq \ell}}\left\{\sum_{k\geq 1}\binom{m}{k}(-1)^k\left\langle\!\!\left\langle \prod_{j=0}^{k-1}\left(1-\langle \pi, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle\right), \partial_u G_p\right\rangle\!\!\right\rangle - \kappa_1\|G_p\|_2^2\right\}\right].$$

Note that we need to take the $\limsup_{\epsilon\to 0}$ outside of the expectation, since otherwise we get from the reverse of Fatou's lemma that $E_{\mathbb{Q}}\limsup \geq \limsup E_{\mathbb{Q}}$. And so, we further reduce the problem to the study of

$$\lim_{\ell\to+\infty}\limsup_{\epsilon\to 0} E_{\mathbb{Q}}\left[\max_{\substack{G_p\\ p\leq \ell}}\left\{\sum_{k\geq 1}\binom{m}{k}(-1)^k\left\langle\!\!\left\langle \prod_{j=0}^{k-1}\left(1-\langle \pi, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle\right), \partial_u G_p\right\rangle\!\!\right\rangle - \kappa_1\|G_p\|_2^2\right\}\right].$$

To make the link between the microscopic system and the macroscopic PDE we want to express $\mathbb{Q}$ as the limit of a subsequence of $(\mathbb{Q}_N)_{N\geq 0}$, thus replacing π by π^N and then recovering the occupation variables from the application of replacement lemmas. To do this, as previously, one wants to argue that the map

$$\pi \mapsto \Psi(\pi) = \max_{\substack{G_p\\ p\leq \ell}}\left\{\sum_{k\geq 1}\binom{m}{k}(-1)^k\left\langle\!\!\left\langle \prod_{j=0}^{k-1}\left(1-\langle \pi, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle\right), \partial_u G_p\right\rangle\!\!\right\rangle - \kappa_1\|G\|_2^2\right\}$$

is continuous with respect to the Skorokhod topology, hence lower semicontinuous and therefore $\Psi(\pi) \leq \liminf_{N\to+\infty}\Psi(\pi^N)$. Although this is not the case, one can first truncate the series at an $\ell_{1/\epsilon}$ step, then replace $\iota_{\epsilon_k}^{\cdot+j\epsilon_k}$ by a continuous approximation as in (3.32), and then argue by lower semicontinuity. Next, we replace the sum up to $\ell_{1/\epsilon}$ by a sum up to ℓ_N, as in (3.34), and finally we replace back $\langle \pi^N, \frac{1}{\epsilon_k}\zeta_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle$ by $\langle \pi^N, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle$. In this way, we have to treat

$$\lim_{\ell\to+\infty}\limsup_{\epsilon\to 0} E_{\mathbb{Q}_N}\left[\liminf_{N\to+\infty}\max_{\substack{G_p\\ p\leq \ell}}\left\{\sum_{k=1}^{\ell_N}\binom{m}{k}(-1)^k\left\langle\!\!\left\langle \prod_{j=0}^{k-1}\left(1-\langle \pi^N, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle\right), \partial_u G_p\right\rangle\!\!\right\rangle - \kappa_1\|G_p\|_2^2\right\}\right],$$

where we recall again that $\mathbb{Q}$ is a Dirac measure. Now we apply Fatou's lemma to exchange the expectation and the $\liminf$. Hence, it is enough to show that there exists some constant K_1 independent of $\{G_p\}_{p\leq\ell}$ for any $\ell\in\mathbb{N}$ such that

$$\mathbb{E}_{\mu_N}\left[\max_{\substack{G_p\\ p\leq \ell}}\left\{\sum_{k=1}^{\ell_N}\binom{m}{k}(-1)^k\left\langle\!\!\left\langle \prod_{j=0}^{k-1}\left(1-\langle \pi^N, \iota_{\epsilon_k}^{\cdot+j\epsilon_k}\rangle\right), \partial_u G_p\right\rangle\!\!\right\rangle - \kappa_1\|G_p\|_2^2\right\}\right] \leq K_1.$$

Because $\partial_u G_p$ is bounded in L^1 and the products involving the empirical measure are bounded by 1, we can replace ℓ_N by $(\ell_N)^n$ with n as in (3.28). Now we are able to proceed backwards in the replacement lemmas' scheme (from (3.39) to (3.43)), approximating the space integral by the Riemann sum along the way. At this point we have to estimate

$$\mathbb{E}_{\mu_N}\left[\max_{\substack{G_p\\ p\leq \ell}}\left\{\int_0^T\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N}\sum_{k=1}^{(\ell_N)^n}\binom{m}{k}(-1)^k\prod_{j=0}^{k-1}\overline{\eta}_{N^2 s}(x+j)\partial_u G_p(s,\tfrac{x}{N}) - \kappa_1\|G_p(s,\cdot)\|_2^2\right)\mathrm{d}s\right\}\right],$$

where we recall that $(\ell_N)^n$ can be replaced back by ℓ_N since the terms involving η are bounded and $\partial_u G_p$ is bounded in L^1. We are able to introduce

$$\tau_x\left\{\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^+\overline{\eta})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta})\right\}$$

inside the summations over x and k (see the treatment of the first probability in (3.35) and Definition 3.2.3 for the expression for $\mathbf{s}_j^{(k-1)}$). This is important because now we have

$$\mathbb{E}_{\mu_N}\left[\max_{\substack{G_p\\ p\le\ell}}\left\{\int_0^T\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N}h_N^{(m-1)}(\tau_x\eta_{N^2s})\partial_u G_p(s,\tfrac{x}{N})-\kappa_1\|G_p(s,\cdot)\|_2^2\right)\mathrm{d}s\right\}\right]$$

which will be used to exploit the gradient property of the model. Analogously to the replacement lemmas, and with the same choice of a constant profile $\alpha(\cdot)\in(0,1)$ we obtain the upper bound

$$c_\alpha+\int_0^T\sup_f\left\{\left\langle\frac{1}{N}\sum_{x\in\mathbb{T}_N}h_N^{(m-1)}(\tau_x\overline{\eta})\partial_u G_p(s,\tfrac{x}{N}),\,f\right\rangle_{\nu_\alpha^N}-\kappa_1\|G_p(s,\cdot)\|_2^2-N\mathcal{E}_N^{(m-1)}(\sqrt{f},\nu_\alpha^N)\right\}\mathrm{d}s \quad (3.46)$$

where $c_\alpha>0$ is a constant. Let us now focus on the inner product above, specifically on

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N}\partial_u G_p(s,\tfrac{x}{N})\sum_{\eta\in\Omega_N}h_N^{(m-1)}(\tau_x\overline{\eta})f(\eta)\nu_\alpha^N(\eta).$$

One can replace the space derivative by its discrete version with a cost

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N}\left|\left(\partial_u-N\nabla^+\right)G_p(s,\tfrac{x}{N})\right|\sum_{\eta\in\Omega_N}h_N^{(m-1)}(\tau_x\overline{\eta})f(\eta)\nu_\alpha^N(\eta)\le\left\|\left(\partial_u-N\nabla^+\right)G_p(s,\cdot)\right\|_\infty\sup_{\eta\in\Omega_N}h_N^{(m-1)}(\eta).$$

This vanishes on the limit $N\to+\infty$ since $\left\|\left(\partial_u-N\nabla^+\right)G_p(s,\cdot)\right\|_\infty\lesssim\frac{1}{N}$ and, since $\mathbf{h}^{(k)}\le k$, we have that $h_N^{(m-1)}(\eta)\lesssim 1$ (recall that $m\in(1,2)$).

At this point the discrete derivative can be passed to $h_N^{(m-1)}$ by performing a summation by parts, which puts us in place to use the gradient property of the model:

$$\sum_{x\in\mathbb{T}_N}\nabla^+G_p(s,\tfrac{x}{N})\sum_{\eta\in\Omega_N}h_N^{(m-1)}(\tau_x\overline{\eta})f(\eta)\nu_\alpha^N(\eta)=-\sum_{x\in\mathbb{T}_N}G_p(s,\tfrac{x+1}{N})\sum_{\eta\in\Omega_N}c_N^{(m-1)}(\overline{\eta})\nabla^+\eta(x)f(\eta)\nu_\alpha^N(\eta).$$

From Lemma B.1.1,

$$\sum_{\eta\in\Omega_N}c_N^{(m-1)}(\tau_x\overline{\eta})\nabla^+\eta(x)f(\eta)\nu_\alpha^N(\eta)=-\frac{1}{2}\sum_{\eta\in\Omega_N}c_N^{(m-1)}(\tau_x\overline{\eta})\nabla^+\eta(x)\nabla_{x,x+1}f(\eta)\nu_\alpha^N(\eta).$$

and we are left with

$$\begin{aligned}
&\frac{1}{2}\sum_{x\in\mathbb{T}_N}G_p(s,\tfrac{x+1}{N})\sum_{\eta\in\Omega_N}c_N^{(m-1)}(\tau_x\overline{\eta})\nabla^+\eta(x)\nabla_{x,x+1}f(\eta)\nu_\alpha^N(\eta)\\
&\quad\le\frac{1}{4A}\sum_{x\in\mathbb{T}_N}\sum_{\eta\in\Omega_N}c_N^{(m-1)}(\tau_x\overline{\eta})\left(G_p(s,\tfrac{x+1}{N})\right)^2\left(\sqrt{f}(\eta)+\sqrt{f}(\eta^{x,x+1})\right)^2\nu_\alpha^N(\eta)\\
&\qquad+\frac{A}{4}\sum_{x\in\mathbb{T}_N}\sum_{\eta\in\Omega_N}c_N^{(m-1)}(\tau_x\overline{\eta})\left(\nabla^+\eta(x)\right)^2\left((\nabla_{x,x+1}\sqrt{f})(\eta)\right)^2\nu_\alpha^N(\eta)\\
&\quad\le\frac{1}{A}\sup_{\eta\in\Omega_N}\{c_N^{(m-1)}(\eta)\}\sum_{x\in\mathbb{T}_N}\left(G_p(s,\tfrac{x+1}{N})\right)^2+\frac{A}{4}\Gamma_N^{(m-1)}(\sqrt{f},\nu_\alpha^N).
\end{aligned}$$

Recalling that $\sup_{\eta\in\Omega_N}\{c_N^{(m-1)}(\eta)\}\le m$, fixing $A=N$ and replacing all this into (3.46), then taking the corresponding limits we finish the proof. □

Proposition 3.4.10. *For any $m\in(0,1)$ there are finite constants $\kappa,K>0$ such that*

$$E_{\mathbb{Q}}\left[\mathcal{E}_\kappa^{(1)}(\pi)\right]\le K.$$

Proof. The proof is almost identical to the one for the SSEP (see for example [17, Proposition B.1]). Besides the fact that there the authors have a boundary term, the differences lie in that we apply the replacement lemmas in the present text, and that in the final step we need to invoke Proposition 3.4.2. We outline the main steps. The treatment of the expectation in the statement can be reduced to the treatment of

$$\sup_f \left\{ \left\langle \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta^{\epsilon N}(x) \partial_u G_p(s, \tfrac{x}{N}) - \kappa_0 \|G_p(s,\cdot)\|_2^2, f \right\rangle_{\nu_\alpha^N} - N \mathcal{E}_N^{(m-1)}(\sqrt{f}, \nu_\alpha^N) \right\}. \tag{3.47}$$

Above, the sup is taken over the set of densities with respect to ν_α^N, and $\{G_p\}_{p \in \mathbb{N}}$ is a countable dense subset in $C^{0,1}([0,T] \times \mathbb{T})$. Exchanging the continuous derivative by a discrete one, then performing a summation by parts we end up having to treat

$$\sup_f \left\{ \left\langle - \sum_{x \in \mathbb{T}_N} \nabla^+ \eta^{\epsilon N}(x) G_p(s, \tfrac{x}{N}) - \kappa_0 \|G_p(s,\cdot)\|_2^2, f \right\rangle_{\nu_\alpha^N} - N \mathcal{E}_N^{(m-1)}(\sqrt{f}, \nu_\alpha^N) \right\}.$$

Then again, from Lemma B.1.1 we have that

$$\int_{\Omega_N} \nabla^+ \eta^{\epsilon N}(x) f(\eta) \nu_\alpha^N(\mathrm{d}\eta) = \frac{1}{2\epsilon N} \sum_{i \in \Lambda_N^{\epsilon N}} \int_{\Omega_N} (\eta(x+i+1) - \eta(x+i)) \left(f(\eta) - f(\eta^{x+i, x+i+1}) \right) \nu_\alpha^N(\mathrm{d}\eta).$$

Taking our function G_p back into consideration and recalling that the process is of exclusion type we have that

$$\begin{aligned} &\sum_{x \in \mathbb{T}_N} G_p(s, \tfrac{x}{N}) \frac{1}{2\epsilon N} \sum_{i \in \Lambda_N^{\epsilon N}} \int_{\Omega_N} (\eta(x+i+1) - \eta(x+i)) \left(f(\eta) - f(\eta^{x+i, x+i+1}) \right) \nu_\alpha^N(\mathrm{d}\eta) \\ &\qquad \le \frac{1}{4A} \sum_{x \in \mathbb{T}_N} \left(G_p(s, \tfrac{x}{N}) \right)^2 \sum_{i \in \Lambda_N^{\epsilon N}} \int_{\Omega_N} \left(\sqrt{f}(\eta) + \sqrt{f}(\eta^{x+i, x+i+1}) \right)^2 \nu_\alpha^N(\mathrm{d}\eta) + \frac{A}{4} \Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N). \end{aligned}$$

Since f is a density and α is constant, last display is no larger than

$$\frac{1}{2A} \sum_{x \in \mathbb{T}_N} \left(G_p(s, \tfrac{x}{N}) \right)^2 + \frac{A}{4} \Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N).$$

Plugging this into the first term inside the sup in (3.47), we obtain

$$\sup_f \left\{ \frac{1}{2A} \sum_{x \in \mathbb{T}_N} \left(G_p(s, \tfrac{x}{N}) \right)^2 - \kappa \|G_p(s,\cdot)\|_2^2 + \frac{A}{4} \Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N) - N \mathcal{E}_N^{(m-1)}(\sqrt{f}, \nu_\alpha^N) \right\}.$$

Applying the lower bound for $\mathcal{E}_N^{(m-1)}(\sqrt{f}, \nu_\alpha^N)$ from Proposition 3.4.2 and hence setting $A = 4N$ and $\kappa = 1/8$, to conclude it is enough to perform the limit $N \to +\infty$. □

4

Generalized Porous Media Model

Contents

4.1 Introduction

4.1.1 Main result and strategy

This chapter is split into three parts. In the first part, we construct a family of exclusion processes parameterized by $(n,k) \in \mathbb{N}\times\mathbb{N}$, evolving in the same setting as in the previous chapter, in order to model the equation

$$\partial_t \rho = \partial_u (D_{n,k}(\rho)\partial_u \rho) \quad \text{with} \quad D_{n,k}(\rho) = \rho^n (1-\rho)^k. \tag{4.1}$$

Before advancing with specific details, we make some observations regarding the diffusion coefficient $D_{n,k}(\rho)$. For $(n,k) = (1,0)$ or $(n,k) = (0,1)$ the function $D_{n,k}$ is *linear*; if $n \geq 2, k = 0$ (resp. $n = 0, k \geq 2$) then it is *non-linear* but monotonic increasing (resp. decreasing), and its maximum is attained at the boundary, at $\rho = 1$ (resp. $\rho = 0$); while for $n \geq 2, k \geq 1$ (resp. $n \geq 1, k \geq 2$), the function $D_{n,k}$ is non-linear and *not* monotonic. Concretely, it attains its maximum in the interval $(0,1)$, and it can be checked that

$$\max_{\rho\in[0,1]} D_{n,k}(\rho) = \left(\frac{n}{n+k}\right)^n \left(1 - \frac{n}{n+k}\right)^k.$$

Note that for any $n,k \geq 1$ there is no diffusion when $\rho = 0$ or $\rho = 1$, and as n,k increases so does the multiplicity of the roots of the polynomial $D_{n,k}(\rho)$. All of this is represented in Figure 4.1.

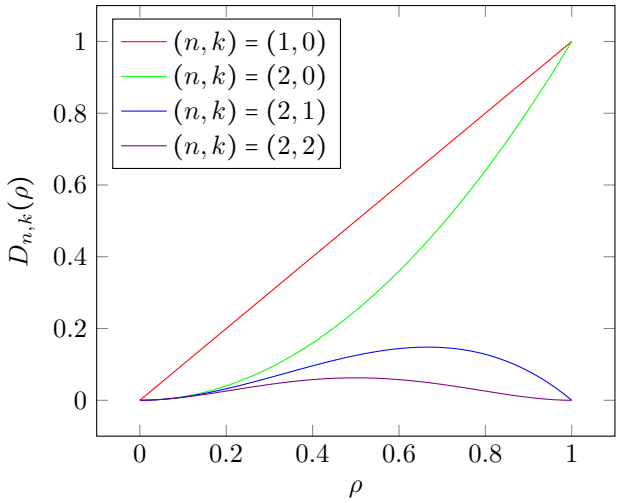


Figure 4.1: Plot of $D_{n,k}(\rho)$ for different values of $n,k \in \mathbb{N}_+$.

The differential equation (4.1) can be alternatively recast as

$$\partial_t \rho = (\partial_u^2 H_{n,k})(\rho), \quad \text{with} \quad (\partial_u^2 H_{n,k})(\rho) = \partial_u (D_{n,k}(\rho)\partial_u \rho),$$

where we note that $H_{n,k}$ can be identified as a primitive of $D_{n,k}$. We will often refer to both the map $H_{n,k}$ and its discrete version (to be introduced shortly) as a *potential*. Precisely, from (4.1) one derives the continuity equation $J(\rho) + D_{n,k}(\rho)\partial_u \rho = 0$, with

$$J = -\partial_u H_{n,k} \tag{4.2}$$

the current, and as such the current is a *potential gradient*, associated with the potential $H_{n,k}$. The quantity $H_{n,k}(\rho)$ has no "simple" expression in terms of elementary functions, and can be computed

invoking the Binomial Theorem,

$$H_{n,k}(\rho) = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \frac{1}{n+\ell+1} \rho^{n+\ell+1}.$$

As explained previously, the gradient property can be seen as (4.2) being satisfied at a discrete level.

Our discrete set-up will also be the same as in the previous chapter: consider the one-dimensional torus, $\mathbb{T}_N := \mathbb{R}/N\mathbb{Z}$, where $1 \ll N \in \mathbb{N}$ is a fixed number; the state-space is the set $\Omega_N = \{0,1\}^{\mathbb{T}_N}$ and we recurrently denote the configurations by the Greek letters η and ξ.

The problem of defining a microscopic model for $\rho^n(1-\rho)^k$ turns out to be a very delicate task if one aims to maintain the gradient property. Our approach is to characterize the gradient condition in terms of a linear system, which is performed in Subsection 4.2.1. The underlying idea is that products of occupation variables over specific sites can be seen as functions on the state space applied to a set. This translates the verification of the gradient property to a series of simple combinatorial problems: given some kinetic constraint $\mathbf{c}$, we identify sets that are going to be associated with $\mathbf{c}(\eta)\eta(1)$ as translations of sets associated with $\mathbf{c}(\eta)\eta(0)$. Because translations are "gradient invariant", in the sense that $\mathbf{1} = -\nabla^+ + \tau_1$ (see more details throughout this chapter), the identification of the aforementioned sets will induce a linear system. In this way, each (exclusion) gradient model is related to a particular solution of an underlying linear system that characterizes the gradient condition.

Our starting point is to consider the PMM mechanism, already discussed in the previous chapter, where for an exchange of occupations between the sites 0 and 1 to happen at least one of the "windows" $\{[\![-k-1+j, j]\!]\backslash\{0,1\}\}_{j=1,\dots,k+1}$, for $k \in \mathbb{N}_+$ fixed, must be completely occupied (recall the discussion just before Figure 3.1). The extension of this model is realized by associating, to each local configuration with exactly n particles in the window

$$W_j := [\![-j, -j+n+k+1]\!]\backslash\{0,1\},$$

for each $0 \le j \le n+k$, a non-negative weight, as in Definition 4.2.4. The normalized weights can be seen as the probability distribution of the position of the n particles with respect to each window W_j. One can then show that the corresponding model will satisfy the gradient condition if these weights are solution to a linear system. This is the content of Proposition 4.2.11, with the linear system as in (4.13). As a consequence (see Corollary 4.2.12), we discover, for each $0 \le \ell \le k$, a family of gradient models associated with the diffusion coefficient $(n+\ell+1)D_{n+\ell,0}(\rho)$. This family is associated with the aforementioned weights being a solution of a reduced linear system (presented in (4.21)), which the PMM is a particular solution to.

The linear system (4.13) is expressed in terms of different maps, and its solution seems to be not unique for general $n, k \in \mathbb{N}_+$. In Appendix C.1 we present the linear system for the particular case of $n, k = 2$. In the general case, we study throughout this chapter a particular (uniform) solution, presented in Proposition 4.2.13, that permits to solve some problems which arise in the following section through combinatoric arguments. This solution corresponds to the following very simple kinetic constraint for the exchange of the occupations between the sites 0 and 1: the number of windows $\{W_j\}_{0 \le j \le k}$ such that the density of particles in it is $n/(n+k)$, then normalized by a factor $\binom{n+k}{k}^{-1}$. Formally, writing

$\langle\eta\rangle_A = \frac{1}{|A|}\sum_{z\in A}\eta(z)$ for any $A \subseteq \mathbb{T}_N$, we define the constraint

$$\mathbf{c}^{(n,k)}(\eta) = \frac{1}{\binom{n+k}{k}}\sum_{j=0}^{n+k}\mathbf{1}\left\{\langle\eta\rangle_{W_j} = \frac{n}{n+k}\right\},$$

which is related to the diffusion coefficient $(n+k+1)\rho^n(1-\rho)^k$. The constraint in the previous display is valid for any $n, k \in \mathbb{N}_+$, and the generality of n and k will indeed be our focus. The model induced by the constraint $\mathbf{c}^{(n,k)}$ will be named PMM(n,k) (see Definition 4.2.14).

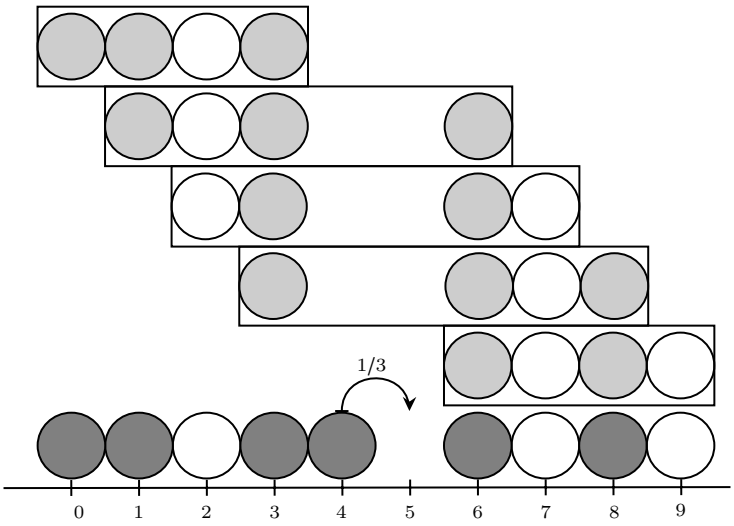


Figure 4.2: PMM$(2,2)$ transition rates.

In the previous figure, the windows W_j, for $0 \le j \le 4$, are represented as the rectangles containing segments of the configuration of particles, in light gray. The family $\{\text{PMM}(n,k)\}_{n,k\in\mathbb{N}_+}$ enjoys of several properties:

- If $\eta \in \Omega_N$ is such that there are less than n particles on each of the windows $\{W_j\}_{0\le j\le k}$, then $\mathbf{c}^{(n,k)}(\eta) = 0$, in this way PMM$(n,k)$ is a kinetically constrained model;
- It can be checked that $\max_\eta \mathbf{c}^{(n,k)}(\eta) = (n+k+1)/\binom{n+k}{k}$. In other words, there is a local configuration such that $\langle\eta\rangle_{W_j} = \frac{n}{n+k}$ for each $0 \le j \le n+k$. An example of such configuration is given in the next figure:

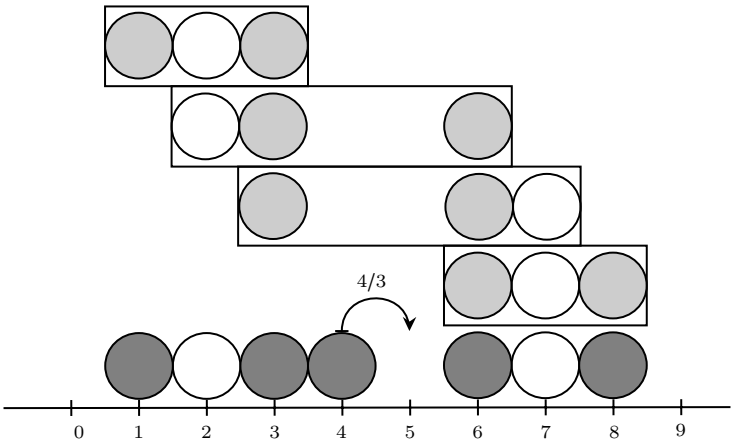


Figure 4.3: Configuration corresponding to the maximum rate, for the PMM$(2,1)$.

- The minimum number of particles required on the whole window $W = \cup_{j=0}^{k} W_j$ that guarantees a positive rate is $2(n-1)+1$;

- Fixed $L \in \mathbb{N}_+$ such that $L < N$, there is a sequence of normalizing factors $\mathcal{Z}_{L,n} > 0$ such that

$$\mathcal{L}_N^{\mathrm{SSEP}} = \sum_{n=0}^{L} \frac{1}{\mathcal{Z}_{L,n}} \mathcal{L}_N^{(n,L-n)}$$

  where $\mathcal{L}_N^{(n,L-n)}$ is the infinitesimal generator of the process we are just describing, the PMM$(n, L-n)$, as is introduced in Definition 4.2.14. In other words, the rescaled constraints $\{\frac{1}{\mathcal{Z}_{L,n}} \mathbf{c}^{(n,L-n)}\}_{0 \le n \le L}$ forms a partition of the unity in Ω_N (Proposition 4.2.15);

- There are *blocked configurations* depending on the number of particles in the system. A simple example is the configuration where each particle is at a distance larger than $n+k$ from each other;

- There exists *mobile clusters*. Specifically, if $\langle \eta \rangle_{W_j} = \frac{n}{n+k}$ for some $0 \le j \le k$, then the local configuration $\eta|_{W_j}$ (that is, the configuration η restricted to the window W_j) is a mobile cluster.

In the second part of this chapter, Section 4.3, we extend the previous model to a long-range dynamics. This extension is performed, again, on the discrete torus, although it is straightforward to adapt it to some other lattices. The main result of this section is the introduction of an operator acting on any nearest neighbour (that we short-write n.n.) constraints that defines a jump of arbitrary length $r \in \mathbb{N}_+$ and such that the corresponding long-range (that we short-write l.r.) constraint satisfies the gradient condition. The operator is introduced in Definition 4.3.3, and the proof of the gradient property is given in Proposition 4.3.4. The idea behind the definition is quite simple and can be explained with the diagram in Figure 4.4

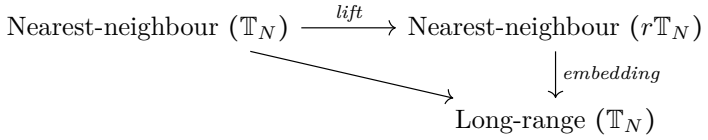


Figure 4.4: Long-range extension.

In other words, given some n.n. constraint $\mathbf{c}$ defined on $\mathbb{T}_N$, we can define a n.n. constraint on the lattice $r\mathbb{T}_N$, for any $r \in \mathbb{N}_+$ fixed, which we can write as $\mathbf{c}_r$. This, in turn, defines a jump of length r on the original lattice $\mathbb{T}_N$. Because $\mathbf{c}_r$ satisfies the gradient condition on $r\mathbb{T}_N$, it will also satisfy this condition in $\mathbb{T}_N$. We then introduce the long-range version of the PMM(n,k) for each jump of range r in Definition 4.3.5, which we refer to as PMM$_r(n,k)$. From its definition, it holds naturally that

$$\mathrm{PMM}_1(n,k) = \mathrm{PMM}(n,k).$$

We then use the family $\{\mathrm{PMM}_r(n,k)\}_{1 \le r \le \lfloor N/2 \rfloor}$ as a basis to introduce what we call *Fractional Porous Media Model*, in Definition 4.3.14, and refer to as PMM$^\gamma(n,k)$, where $\gamma > 0$ is a parameter that will be associated with the fractional Laplacian $\Delta^{\gamma/2}$, if $\gamma \in (0,2)$, and the classical Laplacian if $\gamma \ge 2$.

4.1.2 Outline of the chapter

Now we explain the content of the next sections in a more technical level. We found appropriate to present an example in order to motivate our subsequent proofs, and in this manner the present subsection deviates, in length and amount of detail, from the corresponding subsection of the previous chapters.

Section 4.2 is devoted to deriving the aforementioned linear system (Subsection 4.2.1), and expressing the potential associated with the microscopic current in a convenient manner (Subsection 4.2.2). We focus now on the content of Subsection 4.2.1. The main result is Proposition 4.2.11, with the linear system as in (4.13). The approach to obtain the linear system can be succinctly presented in the following example for the case $n, k = 1$, although it does not reveal all the technical difficulties.

Example 4.1.1. For each $(j,i) \in \{0,1,2\} \times \{1,2\}$ let $\mathfrak{a} = \{a_{ij}\}_{(j,i)\in\{0,1,2\}\times\{1,2\}}$ be a family of non-negative constants and introduce the following (symmetric) constraint for the occupation exchange of the sites 0 and 1

$$\begin{aligned}\mathbf{c}(\eta) = {} & a_{10}\overline{\eta}(2)\eta(3) + a_{20}\eta(2)\overline{\eta}(3) \\ & + a_{11}\overline{\eta}(-1)\eta(2) + a_{21}\eta(-1)\overline{\eta}(2) \\ & + a_{12}\overline{\eta}(-2)\eta(-1) + a_{22}\eta(-2)\overline{\eta}(-1).\end{aligned} \tag{4.3}$$

Distributing the products and multiplying by $(\nabla^+\eta)(0)$ yields

$$\begin{aligned}\mathbf{c}(\eta)(\nabla^+\eta)(0) = {} & a_{10}\eta(1)\eta(3) + (a_{20}+a_{11})\eta(1)\eta(2) + (a_{21}+a_{12})\eta(-1)\eta(1) + a_{22}\eta(-2)\eta(1) \\ & - a_{10}\eta(0)\eta(3) - (a_{20}+a_{11})\eta(0)\eta(2) - (a_{21}+a_{12})\eta(-1)\eta(0) - a_{22}\eta(-2)\eta(0) \\ & - \left(a^0\eta(2)\eta(3) + a^1\eta(-1)\eta(2) + a^2\eta(-2)\eta(-1)\right)(\nabla^+\eta)(0)\end{aligned} \tag{4.4}$$

where we shortened $a^j = a_{1j} + a_{2j}$ for $j = 0,1,2$. It is simple to show that the third line in the previous display is the gradient of some function if $a^{j_1} = a^{j_2}$ for any $j_1, j_2 \in \{0,1,2\}$. Considering the normalized weights $a^j = 1$ for every j as previously, the aforementioned quantity equals $(\nabla^+\mathbf{h}^{(2)})(\eta)$, with $\mathbf{h}^{(2)}$ as in (3.15). We now focus on the first two lines of the previous display. The first one can be expressed as

$$\begin{aligned}& (a_{10}+a_{21}+a_{12})\eta(0)\eta(2) + (a_{20}+a_{11})\eta(0)\eta(1) + a_{22}\eta(0)\eta(3) \\ & + a_{10}(\tau_1 - \mathbf{1})\eta(0)\eta(2) + (a_{20}+a_{11})(\tau_1-\mathbf{1})\eta(0)\eta(1) \\ & + (a_{21}+a_{12})(\tau_{-1}-\mathbf{1})\eta(0)\eta(2) + a_{22}(\tau_{-2}-\mathbf{1})\eta(0)\eta(3).\end{aligned} \tag{4.5}$$

It is clear that the quantities in the last two lines of the previous display are (discrete) gradients of some quantity. Repeating the same procedure for the quantity in the second line of (4.4) leads to

$$-a_{10}\eta(0)\eta(3) - (a_{20}+a_{11}+a_{22})\eta(0)\eta(2) - (a_{21}+a_{12})\eta(0)\eta(1)$$

plus the discrete gradient of some function of the configurations. Comparing the terms in the first line of (4.5) with the ones in the previous display, we see that $\mathbf{c}(\eta)(\nabla^+\eta)(0)$ satisfies the gradient condition if

$$\begin{aligned}0 = {} & \eta(0)\eta(1)\left[a_{20}+a_{11}-a_{21}-a_{12}\right] + \eta(0)\eta(3)\left[a_{22}-a_{10}\right] \\ & + \eta(0)\eta(2)\left[(a_{21}+a_{12}-a_{20}-a_{11}) + (a_{10}-a_{22})\right].\end{aligned} \tag{4.6}$$

One must then solve the system $a_{20}+a_{11}-a_{21}-a_{12}=a_{22}-a_{10}=0$, and $a^j=1$, with $j=0,1,2$ (recall (4.4) and the discussion just after it). This system is equivalent to

$$\begin{cases} a_{10}=a_{22}, \\ a_{20}=a_{12}, \\ a_{11}=a_{21}=1/2, \\ a_{10}+a_{20}=1. \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} 1&0&0&0&-1&0 \\ 0&1&0&0&0&-1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&1&0&0&0&0 \end{pmatrix} \begin{pmatrix} a_{10} \\ a_{20} \\ a_{11} \\ a_{21} \\ a_{22} \\ a_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1 \end{pmatrix} \tag{4.7}$$

and it does not have a unique solution.

Note that, in the previous example, the particular choice $(a_{10},a_{20})=(0,1)$, $(a_{11},a_{21})=(1/2,1/2)$ and $(a_{12},a_{22})=(1,0)$ corresponds to the quantity $u_k(\eta)$ in (3.17) with $k=2$. Indeed, our motivation for considering as starting point (4.3) was the next rearrangement. Fixed $k\in\mathbb{N}_+$ and defining, for each $\eta\in\Omega_N$, the constraint

$$\mathbf{c}^{(k-1,1)}(\eta):=\frac{k+1}{k}\mathbf{c}^{(k-1)}(\overline{\eta})-\mathbf{c}^{(k)}(\overline{\eta})=\sum_{j=0}^{k}\left\{\frac{k-j}{k}\mathbf{s}_{j+1}^{(k)}\left(\eta^{\{-k+j\}}\right)+\frac{j}{k}\mathbf{s}_{j+1}^{(k)}\left(\eta^{\{j+1\}}\right)\right\}, \tag{4.8}$$

we expect that to the model induced by $\mathbf{c}^{(1,k-1)}$ corresponds the diffusion coefficient $(k+1)D_{k-1,1}(\rho)$. For $k=2$ we observe an empirical average

$$\mathbf{c}^{(1,1)}(\eta)=\eta(-2)\overline{\eta}(-1)+\frac{\eta(-1)\overline{\eta}(2)+\overline{\eta}(-1)\eta(2)}{2}+\overline{\eta}(2)\eta(3),$$

while for $k=3$ a weighted sum:

$$\begin{aligned}\mathbf{c}^{(2,1)}(\overline{\eta})=\eta(-3)\overline{\eta}(-2)\overline{\eta}(-1)+\frac{\overline{\eta}(-2)\overline{\eta}(-1)\eta(2)}{3}+2\frac{\eta(-2)\overline{\eta}(-1)\overline{\eta}(2)}{3}\\ +2\frac{\overline{\eta}(-1)\overline{\eta}(2)\eta(3)}{3}+\frac{\eta(-1)\overline{\eta}(2)\overline{\eta}(3)}{3}+\overline{\eta}(2)\overline{\eta}(3)\eta(4),\end{aligned}$$

in this way successfully generalizing the integer porous medium model in a specific direction. While it is not clear for us the reason why the quantity in the right-hand side of the previous display follows a "combinatoric" reasoning, this is encapsulated in the forthcoming linear system (as it is encapsulated into (4.7) for the case $n=1$ and $k=1$). The reason why (4.8) can be taken as constraints ($\mathbf{c}^{(k-1,1)}\geq 0$) can be shown to be "independent" of the rearrangement (4.8) and consequence of the sequence $\{\frac{1}{k+1}\mathbf{c}^{(k)}(\eta)\}_{k\geq 0}$ being non-increasing, for every $\eta\in\Omega_N$, to which the aforementioned rearrangement is a consequence thereof. It turns out that the rationale to define $\mathbf{c}^{(k-1,1)}$ in (4.8) does not generalize in the most natural way: defining $\mathbf{c}^{(k-1,2)}:=\frac{1}{k+1}\mathbf{c}^{(k-1,1)}-\frac{1}{k+2}\mathbf{c}^{(k,1)}$, in analogy with the binomial expansion of $\rho^{k-1}(1-\rho)^2$ as a polynomial in ρ, leads to possible negative rates because the sequence $\{\frac{1}{k+i}\mathbf{c}^{(k+i-1,1)}\}_{i\in\mathbb{N}}$ is not monotone.

One can identify in Example 4.1.1 three main steps:

1. Distribution of the products of flipped occupation variables, as in (4.4);
2. Translation of each product to start at zero, as in (4.5);
3. The terms translated to zero associated with a multiplication by $\eta(1)$ are compared with the terms translated to zero associated with a multiplication by $\eta(0)$, as in (4.6).

Those are also the main steps in the proof of Proposition 4.2.11. However, performing these computations for general n and k and proving that the resulting linear system admits a solution is much more demanding, and needs to be done in an organized way. With this in mind, one needs to keep track of the sets of sites that need to be flipped and the indexes of the weights associated with them, when the translations of these sets are going to be performed. In order to do so, we work with equivalence classes over sets and their indexes (see Definitions 4.2.3 and 4.2.6). The aforementioned step (1) is encapsulated into Lemma 4.2.5; step (2) is performed in two sub-steps: first we shift each *window* W_j, with $0 \le j \le k$, to the origin, identifying a first gradient quantity, and then we shift each product of occupation variables to the origin. This allows us to identify a quantity which is also present in the potential associated with the PMM (namely, $\mathbf{h}_0^{(n,k;\mathfrak{a})}$, as in (4.14)), and which vanishes in the limit $N \to +\infty$. Step (3) is encapsulated into Lemma 4.2.7 and its Corollary 4.2.8.

After the proof of Proposition 4.2.11, we focus on a particular solution of the system (4.13), the *uniform* solution presented in Proposition 4.2.13, where all the weights equal $\binom{n+k}{k}^{-1}$, and prove some properties of the resulting model, namely, the partition of the unity, in Proposition 4.2.15. We also provide a very precise upper-bound for the sum of the rates over a discrete interval, in Lemma 4.2.16. The latter, as seen in the previous chapter (Lemma 3.2.16), is an important ingredient for considering linear combinations of models in the same family. Only when referring to the case where the uniform solution is chosen, the superscript $(n,k;\mathfrak{a})$ will be replaced by (n,k). The potential arising from the gradient property, $\mathbf{h}^{(n,k;\mathfrak{a})}$, is in general the sum of two quantities ($\mathbf{h}^{(n,k;\mathfrak{a})} = \mathbf{h}_0^{(n,k;\mathfrak{a})} + \mathbf{h}_1^{(n,k;\mathfrak{a})}$), as in (4.14). The quantity $\mathbf{h}_1^{(n,k;\mathfrak{a})}$ is the one that "survives" in the limit $N \to +\infty$, however, it is not presented in the most convenient way, as it depends notably on the particular choice of labels of the sets that dictate the flips, the equivalence classes of those sets and of their indexes. Subsection 4.2.2 is devoted to its simplification for the case of the uniform solution and for the case of a particular subclass of solutions of (4.13). For that purpose, one needs a complete characterization of the above mentioned equivalence classes, which is the content of the series of Lemmas 4.2.17, 4.2.19, 4.2.20 and 4.2.21. We then perform several simplifications of $\mathbf{h}^{(n,k;\mathfrak{a})}$ for the uniform choice $\mathfrak{a} = \binom{n+k}{k}^{-1}$, in which case we drop the dependence on $\mathfrak{a}$ from the notation. Concretely, it holds that

$$\mathbf{h}^{(n,k)} = \mathbf{H}^{(n,k)}(\eta) + \nabla^+ \mathbf{g}^{(n,k)} + \mathbf{f}^{(n,k)}(\eta),$$

with $\mathbf{g}^{(n,k)}$ as in (4.26), $\mathbf{f}^{(n,k)} = \mathbf{h}_0^{(n,k)} + \mathbf{H}_0^{(n,k)}$ with the terms as in (4.30) and (4.14).

In Proposition 4.2.23 we also show that if an extended linear system is satisfied, then it holds the "invariance principle" $\mathbf{h}_1^{(n,k;\mathfrak{a})} = \mathbf{h}_1^{(n,k)}$. This concludes the content of Subsection 4.2.1.

As explained in the introduction, the main result of Section 4.3 is Proposition (4.3.4), where we show that the long-range extension is a gradient model. In Definition 4.3.14 we introduce a (time re-scaled) linear combination of long-range versions of the PMM(n,k) that can be used in order to obtain the fractional Laplacian $(-\Delta)^{\gamma/2}$, for $\gamma \in (0,2)$. We direct the reader to the two detailed PhD thesis [6, 32] on long-range dynamics and the papers [2, 7] for a proof of the hydrodynamic limit for the empirical measure, as we derive solely the correct time-scale and prove the convergence and interpolation of the relevant operators. We also prove in appendix C.2 a regularity result of independent importance, which

is also crucial for studying the open-boundary case.

4.2 Microscopic Models and Linear System

4.2.1 Generalized Porous Media Model

Our first goal is to define a specific constraint such that the diffusion coefficient of the corresponding hydrodynamic equation of the process induced by it is given by $D_{n,k}(\rho) = \rho^n(1-\rho)^k$, for $n, k \in \mathbb{N}_+$; and, importantly, such that the induced model has the gradient property. The models that we are going to introduce are derived from two main mechanisms. To present them we need to introduce some notation.

Notation 4.2.1.

- For $A = \mathbb{T}_N$ we write $\eta(A) \equiv \prod_{i\in A}\eta(i)$. If $A = \varnothing$ it is defined by convention that $\prod_{\varnothing} \equiv 1$;
- For $A \subseteq \mathbb{T}_N$ we write η^A as the configuration where for any $x \in \mathbb{T}_N$ it holds $\eta^A(x) = (1-\eta(x))\mathbf{1}\{x \in A\} + \eta(x)\mathbf{1}\{x \notin A\}$;
- Given two sets $A, B \subset \mathbb{T}_N$ such that $A \cap B = \varnothing$ we denote by $A \sqcup B$ their disjoint union;
- For any $x \in \mathbb{T}_N$ we write $x + A = \{x + a : a \in A\}$;
- For any $r \in \mathbb{N}_+$ and $A \subseteq \mathbb{T}_N$ we write $rA = \{ra : a \in A\}$.

The first mechanism is the symmetry that makes the PMM a family of gradient models. We recall its introduction in Definition 3.2.3. Fixed $k \in \mathbb{N}_+$ one can think of the PMM(k) as a model generated by translations of sets around the node $\{0, 1\}$. With the previous notation, one can express the constraint $\mathbf{c}^{(k)}$ as

$$\mathbf{c}^{(k)}(\eta) = \sum_{j=0}^{k}(\tau_{-j}\eta)(\llbracket 0, k+1\rrbracket\backslash\{j, j+1\}).$$

The second mechanism regards on how to obtain "mixed powers", of the form $\rho^n(1-\rho)^k$ for any $n, k \in \mathbb{N}_+$. The constraint of Example (4.8), for $n, k = 1$, can be recast as

$$\sum_{j=0}^{k}\sum_{i=1}^{2} a_{ij}\eta^{-j+P_i}(-j + \llbracket 0, k+1\rrbracket\backslash\{j, j+1\}),$$

where for each $j \in \llbracket 0, k\rrbracket$ we have

$$a_{ij} = \begin{cases} \frac{j}{k}, & i = 1, \\ \frac{k-j}{k}, & i = 2 \end{cases} \qquad \text{and} \qquad P_i = \begin{cases} \{0\}, & i = 1, \\ \{k+1\}, & i = 2. \end{cases}$$

This suggests that there might be different "P–sets", hence weights $\mathfrak{a} = \{a_{ij}\}_{0\le j\le k\,, i=1,2}$, such that the resulting constraint still satisfies the gradient condition. In this way, we start with the "prototype" model in Definition 4.2.4 and we introduce some more definitions and notation.

Definition 4.2.2. Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{T}_N$ be a set of size k. We denote the family of subsets of A with size $\ell \le k$ by

$$\mathcal{P}_\ell(A) = \{A' : A' \subseteq A\,,\ |A'| = \ell\}.$$

We now introduce some notation that will simplify the presentation of the next results.

Definition 4.2.3. Hereafter we fix $1 \ll N \in \mathbb{N}$ and $n, k \in \mathbb{N}_+$ such that $n + k \leq N$. For each $0 \leq j \leq n + k$ and $0 \leq \ell \leq k$ fixed we consider the set of *indexes* $I_\ell \times J$ where

$$I_\ell = \left\{1, \dots, \binom{n+k}{k-\ell}\right\} \quad \text{and} \quad J = \{0, \dots, n+k\}.$$

If $\ell = 0$ we write $I \equiv I_0$.

Let us introduce the following auxiliary sets, that will provide the basis for the construction of the family of constraints and their respective study:

- $M = [\![0, n + k + 1]\!]$;
- $M_j = M \backslash \{j, j + 1\}$, for each $j \in J$;
- $P_{ij}^{n+\ell}$, for $0 \leq \ell \leq k$ and $(i, j) \in I_\ell \times J$: which are such that $\mathcal{P}_{n+\ell}(M_j) = \{P_{ij}^{n+\ell}\}_{i \in I_\ell}$;
- P_{ij}, for each $(i, j) \in I \times J$: fixed some label for the elements of $\mathcal{P}_k(M_j)$ we write $\mathcal{P}_k(M_j) = \{P_{ij}\}_{i \in I}$;
- Q_{ijq}^ℓ, with $(i, j) \in I \times J$, $0 \leq \ell \leq k$, $1 \leq q \leq \binom{k}{\ell}$: fixed $(i, j) \in I_\ell \times J$, we write $\mathcal{P}_\ell(P_{ij}) = \{Q_{ijq}^\ell\}_{1 \leq q \leq \binom{k}{\ell}}$.

Note that if $n+\ell = k$ we *can* have $P_{ij}^{n+\ell} = P_{ij}$ with an appropriate choice of indexes. Finally, fixed $0 \leq \ell \leq k$, we shorten for each $(i, j) \in I_\ell \times J$ and $x = 0, 1$,

$$p_{ij}^{\ell,x} = \min\{P_{ij}^{n+\ell} \sqcup \{j + x\}\} \geq 0 \quad \text{and} \quad A_{ij}^{n+\ell,x} = -p_{ij}^{\ell,x} + (P_{ij}^{n+\ell} \sqcup \{j + x\}).$$

Although all the previously defined sets depend on n and k, we do not make that dependence always explicit in order to simplify the presentation. The reader should have in mind, however, that $n, k \in \mathbb{N}_+$ are fixed. We remark that, for each ℓ and j as previously, the set I_ℓ corresponds to the indexes of the elements of $\mathcal{P}_{n+\ell}(M_j)$, and the set J corresponds to the indexes of the window M_j.

We are going to associate, for each $j \in J$ and every element of $\mathcal{P}_k(M_j) = \{P_{ij}\}_{i \in I}$ a non-negative weight.

Definition 4.2.4. Fix a family of non-negative constants $\mathfrak{a} = \{a_{ij}\}_{(i,j) \in I \times J}$ where for each $j \in J$ there exists at least one $i \in I$ such that $a_{ij} > 0$, and define the maps $\mathbf{s}_j^{(n,k;\mathfrak{a})}, \mathbf{c}^{(n,k;\mathfrak{a})} : \Omega_N \to \mathbb{R}_+ \cup \{0\}$ through

$$\mathbf{c}^{(n,k;\mathfrak{a})} = \sum_{j=0}^{n+k} \mathbf{s}_j^{(n,k;\mathfrak{a})} \quad \text{where} \quad \mathbf{s}_j^{(n,k;\mathfrak{a})}(\eta) = \sum_{i \in I} a_{ij} \eta^{-j+P_{ij}}(-j + M_j).$$

We define the Markov process induced by the infinitesimal generator $\mathcal{L}_N^{(n,k;\mathfrak{a})}$ acting on functions $f : \Omega_N \to \mathbb{R}$ as

$$(\mathcal{L}_N^{(n,k;\mathfrak{a})} f)(\eta) = \sum_{x \in \mathbb{T}_N} \mathbf{c}^{(n,k;\mathfrak{a})}(\tau_x \eta)(\nabla_{x,x+1} f)(\eta).$$

In the next two figures we present the sets of interest in the previous definition, for the particular case of $n, k = 2$.

j	$-j+M_j$
0	$\{2,3,4,5\}$
1	$\{-1,2,3,4\}$
2	$\{-2,-1,2,3\}$
3	$\{-3,-2,-1,2\}$
4	$\{-4,-3,-2,-1\}$

Figure 4.1: Windows where the constraints are imposed.

	$-j+P_{ij}$				
i/j	0	1	2	3	4
1	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1\}$	$\{-3,-2\}$	$\{-4,-3\}$
2	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3,-1\}$	$\{-4,-2\}$
3	$\{2,5\}$	$\{-1,4\}$	$\{-2,3\}$	$\{-3,2\}$	$\{-4,-1\}$
4	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1\}$	$\{-3,-2\}$
5	$\{3,5\}$	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3,-1\}$
6	$\{4,5\}$	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1\}$

Figure 4.2: Sets corresponding to the sites with flipped occupation ($\eta^{-j+P_{ij}}$).

Before characterizing the weights $\mathfrak{a}$ such that $\mathbf{c}^{(n,k,\mathfrak{a})}$ satisfies the gradient property we need to present several technical results regarding the sets introduced in Definition 4.2.3. The next lemma will be used at the start of the proof of Proposition 4.2.11, in order to group the common terms as in (4.4), arising from an application of the distributive rule to the products of flipped occupation variables (related to $\eta^{-j+P_{ij}}$ in the previous definition); and in Proposition 4.2.13, providing the existence of a particular solution to the aforementioned linear system characterizing the gradient condition.

Lemma 4.2.5. *For each $j \in J$ and $0 \le \ell \le k$ there is a non-injective but surjective map $\psi_{j,\ell} : I \times [\![1, \binom{k}{\ell}]\!] \to I_\ell$ such that for each $(i,q) \in I \times \{1,\dots,\binom{k}{\ell}\}$ there exists $i' \in I_\ell$ such that*

$$[M_j \backslash P_{ij}] \sqcup Q^\ell_{ijq} = P^{n+\ell}_{i'j} \quad \textit{where} \quad \psi_{j,\ell}(i,q) = i'.$$

Moreover, for any $(i',j) \in I_\ell \times J$ it holds that $\left|(\psi_{j,\ell})^{-1}(i')\right| = \binom{n+\ell}{\ell}$.

Proof. Let $0 \le \ell \le k$ be fixed and introduce the set $M' = [\![0, n+k-1]\!]$. We write $\mathcal{P}_k(M') = \{P_i\}_{i\in I}$ and $\mathcal{P}_\ell(P_i) = \{Q^\ell_{iq}\}_{1\le q\le \binom{k}{\ell}}$. For each $j \in J$, introduce also the map Φ_j through

$$\Phi_j(A) = \{a \in A : a < j\} \sqcup \{a+2 : \ a \in A,\ a \ge j\}, \qquad \text{where} \qquad A \subseteq \mathbb{T}_N. \tag{4.9}$$

Note that $|\Phi_j(A)| = |A|$ and that $M_j = \Phi_j(M')$, $P_{ij} = \Phi_j(P_i)$, $Q^\ell_{ijq} = \Phi_j(Q^\ell_{iq})$ and $P^{n+\ell}_{ij} = \Phi_j(P^{n+\ell}_i)$, for any $0 \le \ell \le k$, $(i,j) \in I_\ell \times J$.

It is clear that

$$\mathcal{P}_{n+\ell}(M') \subseteq \{[M'\backslash P_i] \sqcup Q^\ell_{iq}\}_{(i,q)\in I\times\{1,\dots,\binom{k}{\ell}\}}$$

where, for $\ell \ne k$, since $|\mathcal{P}_{n+\ell}(M_j)| < \binom{k}{\ell}|I|$ the right-hand-side above must have "repeated terms". Let then ψ_ℓ be the map such that $[M'\backslash P_i] \sqcup Q^\ell_{iq} = P^{n+\ell}_{i'} \Leftrightarrow \psi_\ell(i,q) = i'$ for some $i' \in I_\ell$. Then to compute $\left|(\psi_\ell)^{-1}(i')\right|$ for any particular $i' \in I_\ell$ one needs to count the number of pairs (P,Q) such that $P \in \mathcal{P}_k(M'), Q \in \mathcal{P}_\ell(P)$ and

$$(M'\backslash P) \sqcup Q = P^{n+\ell}_{i'}$$

In order to do so, we can pick any ℓ elements of $P^{n+\ell}_{i'}$ and construct Q, existing some set $A \subseteq M'$ such that $P'\backslash Q = A$. In particular, there exists a unique $P \in \mathcal{P}_k(M')$ such that $A = M'\backslash P$, that is, $P = M'\backslash A$. To see that $Q \subseteq P$ it is enough to note that $M'\backslash P = P'\backslash Q$ and so $Q \notin M'\backslash P$, hence $Q \subseteq P$. Since there are $\binom{n+\ell}{\ell}$ ways to choose ℓ elements of $P^{n+\ell}_{i'}$, we conclude that $\left|(\psi_\ell)^{-1}(i')\right| = \binom{n+\ell}{\ell}$.

In particular, it also holds that $(\Phi_j(M')/\Phi_j(P)) \sqcup \Phi_j(Q) = P^{n+\ell}_{i'j}$ and one can define $\psi_{j,\ell} := \psi_\ell$, where now $\psi_{j,\ell}(i,q) = i' \Leftrightarrow (\Phi_j(M')/\Phi_j(P_i)) \sqcup \Phi_j(Q^\ell_{iq}) = P^{n+\ell}_{i'j}$. Since $\Phi_j(M) = M_j$ and there is exactly one $i \in I$ such that $\Phi_j(P) = P_{ij}$ and $\Phi_j(Q^\ell_{iq}) = Q^\ell_{ijq}$, and $\Phi_j(M'\backslash P_i) = \Phi_j(M')\backslash\Phi_j(P_i)$, it is simple to see that

$$[M_j\backslash P_{ij}] \sqcup Q^\ell_{ijq} = P^{n+\ell}_{i'j} \Leftrightarrow [M'\backslash P_i] \sqcup Q^\ell_{iq} = P^{n+\ell}_{i'}$$

which concludes the proof. □

The arguments for the main result of this section revolve around translating appropriate sets in Definition 4.2.3, organizing them in a specific manner. But to do so, we need to keep track of their respective indexes or, to be precise, of particular equivalence classes of indexes.

Definition 4.2.6. Define over the set composed by all the non-empty subsets of $\mathbb{T}_N$ the equivalence relation $\equiv$ through

$$A \equiv B \Leftrightarrow \exists a \in \mathbb{Z} : \ A = a + B, \quad A, B \subset \mathbb{T}_N.$$

For each $0 \le \ell \le k$ and $x = 0, 1$, define the equivalence relation $\overset{\ell,x}{\sim}$ over $I_\ell \times J$ as

$$(i,j) \overset{\ell,x}{\sim} (i',j') \Leftrightarrow P^{n+\ell}_{ij} \sqcup \{j + x\} \equiv P^{n+\ell}_{i'j'} \sqcup \{j' + x\}.$$

We shorten $\mathcal{C}_{\ell,x} = I_\ell \times J \,/\, \overset{\ell,x}{\sim}$.

The next lemma provides the essential ingredient for constructing a gradient model, starting from the PMM mechanism. It will be invoked in the proof of the main result of this section, Proposition 4.2.11.

Lemma 4.2.7. *For each $0 \le \ell \le k$ and $(i,j) \in I_\ell \times J$ there exists $(i',j') \in I_\ell \times J$ such that*

$$P^{n+\ell}_{ij} \sqcup \{j + 1\} \equiv P^{n+\ell}_{i'j'} \sqcup \{j'\}.$$

Similarly, the converse also holds: for each $0 \le \ell \le k$ and $(i,j) \in I_\ell \times J$ there exists $(i',j') \in I_\ell \times J$ such that

$$P^{n+\ell}_{ij} \sqcup \{j\} \equiv P^{n+\ell}_{i'j'} \sqcup \{j' + 1\}.$$

Proof. It is convenient to see the "P–sets" as binary strings. For each $A \subseteq \mathbb{T}_N$, let us consider the configuration $\xi_A \in \Omega_N$ where $\xi_A(y) = \mathbf{1}\{y \in A\}$.

Let us start by fixing $\ell = k$ and $j \in J$. Note that in this case $|I_\ell| = 1$ and $P^{n+\ell}_{1j} = M\backslash\{j, j+1\}$, where we recall from Definition 4.2.3 that $M = [\![0,\ n + k + 1]\!]$. Then

$$\xi_{P^{n+\ell}_{1,j} \sqcup \{j+1\}} = \begin{cases} \tau_{-1}\xi_{P^{n+\ell}_{1,n+\ell} \sqcup \{n+\ell\}}, & j = 0, \\ \xi_{P^{n+\ell}_{1,j-1} \sqcup \{j-1\}}, & j \neq 0. \end{cases}$$

For $\ell \le k - 1$ the rationale is analogous. Fixed $(i,j) \in I_\ell \times J$, if $\xi_{P^{n+\ell}_{ij}}$ consists of $n + \ell$ *consecutive* particles *inside* the box M with $\xi_{P^{n+\ell}_{ij}}(n+k+1) = 1$, that is, $\xi_{P^{n+\ell}_{ij}}([\![k+1-\ell, n+k+1]\!]) = 1$, and $j+1 = k-\ell$, then

$$\xi_{P^{n+\ell}_{i,k-1-\ell} \sqcup \{k-\ell\}} = \tau_{-1}\xi_{P^{n+\ell}_{i',n+k+1} \sqcup \{n+k+1\}}$$

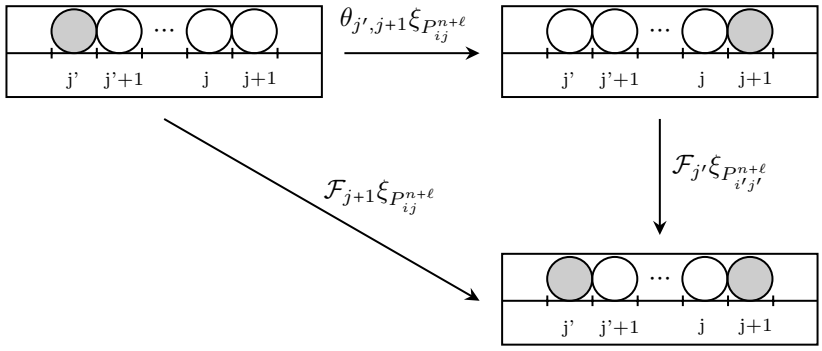
where $P^{n+\ell}_{i',n+k+1}$ corresponds to $\xi_{P^{n+\ell}_{i',n+k+1}}(\llbracket k-\ell, n+k-1\rrbracket) = 1$. If $j+1 \neq k-\ell$ or if we do not have $n+\ell$ consecutive particles, then there exists a local configuration of the form $(1,0)$ in the window M. Let us then say that we have $(1,0)$ at $\{j', j'+1\}$ for some $j' \in \llbracket 0,\ n+k\rrbracket$. Since $\{P^{n+\ell}_{\tilde{i}\tilde{j}}\}_{\tilde{i}\in I_\ell} = \mathcal{P}_{n+\ell}(M_{\tilde{j}})$ for all $(\tilde{i},\tilde{j}) \in I_\ell \times J$, there is also some $i' \in I_\ell$ such that

$$\theta_{j',j+1}\xi_{P^{n+\ell}_{ij}} = \xi_{P^{n+\ell}_{i'j'}}.$$

For any $x, y \in \mathbb{T}_N$, let $\theta_{x,y}$ be the operator acting on Ω_N that exchanges the occupations at x, y and let $\mathcal{F}_x$ be the operator that flips the occupation at x. In this case, specifically, holds that $\mathcal{F}_{j+1} = \mathcal{F}_{j'}\theta_{j',j+1}$, that is,

$$\xi_{P^{n+\ell}_{ij}\sqcup\{j+1\}} = \mathcal{F}_{j+1}\xi_{P^{n+\ell}_{ij}} = \mathcal{F}_{j'}\theta_{j',j+1}\xi_{P^{n+\ell}_{ij}} = \xi_{P^{n+\ell}_{i'j'}\sqcup\{j'\}}.$$

This rationale is represented in the following commutative diagram.



The converse is analogous. □

The previous lemma has the following simple, but important corollary, which can be seen as the analogue of the previous lemma in terms of the indexes.

Corollary 4.2.8. *For each $0 \le \ell \le k$ there exists a permutation ϕ_ℓ on the set of indexes $(i,j) \in I_\ell \times J$ such that*

$$-p^{\ell,1}_{ij} + P^{n+\ell}_{ij} \sqcup \{j+1\} = -p^{\ell,0}_{j'i'} + P^{n+\ell}_{i'j'} \sqcup \{j'\} \quad \textit{where} \quad \phi_\ell(i,j) = (i',j'). \tag{4.10}$$

In particular, it holds that for each $\underline{c} \in \mathcal{C}_{\ell,1}$

$$|\underline{c}| = |\phi_\ell(\underline{c})|. \tag{4.11}$$

Proof. The equality (4.10) is a consequence of shifting the equivalent sets (with respect to the equivalence relation $\equiv$) to the origin; while the equality (4.11) holds because from Lemma 4.2.7 each set of the form $P^{n+\ell}_{ij}\sqcup\{j+1\}$, with $(i,j) \in I_\ell \times J$, can be seen as a translation of some set $P^{n+\ell}_{i'j'}\sqcup\{j'\}$, for some $(i',j') \in I_\ell \times J$ (and, naturally, vice-versa). As such, the respective classes of those sets (therefore the classes of their indexes) must be of equal size. □

Notation 4.2.9. Fixed $x \in \{0,1\}$ and $0 \le \ell \le k$, given any $\underline{c} \in \mathcal{C}_{\ell,x}$ we shall write as $A_{\underline{c}}$ one set representative of the class $\underline{c}$, that is, such that $A_{\underline{c}} = A^{n+\ell,x}_{ij}$ for every $(i,j) \in \underline{c}$.

Remark 4.2.10. We remark that, from Corollary 4.2.8, for any $\underline{c_1} \in \mathcal{C}_{\ell,1}$ there is a corresponding $\underline{c_0} \in \mathcal{C}_{\ell,0}$ such that $\underline{c_0} = \phi_\ell(\underline{c_1})$, and the converse is also true since ϕ_ℓ is a bijection. In this way, $\mathcal{C}_{\ell,0} = \phi_\ell(\mathcal{C}_{\ell,1})$, and in particular $A_{\underline{c_1}} = A_{\phi_\ell(\underline{c_1})}$.

We are now ready to present the main result of this section. Recall Definition 4.2.4. The next proposition gives us necessary conditions over $\mathfrak{a}$ such that the Markov process induced by the constraint $\mathbf{c}^{(n,k;\mathfrak{a})}$ satisfies the gradient property. As we shall see in the proof below (and as we saw in Example 4.1.1), the idea is that the distributive property allows us to develop the products in $s_1 := \{\mathbf{s}_j^{(n,k;\mathfrak{a})}(\eta)\eta(1)\}_{j\in J}$, generating a linear combination of products of occupation variables, where each product can be associated with a set that is equivalent, with respect to $\equiv$, to the sets "generated" by $s_0 := \{\mathbf{s}_j^{(n,k;\mathfrak{a})}\eta(0)\}_{j\in J}$. Since translations can be seen as "gradient preserving operations", in the sense that $\mathbf{1} \pm \tau_1 = \tau_1 - \nabla^+$, identifying correctly the terms generated by s_1 with the terms generated by s_0 induces a linear system for the weights $\mathfrak{a}$ characterizing a dynamics of gradient type. We provide in Appendix C.1 a detailed example for the case $n, k = 2$, that we encourage the reader to follow through the next proof. Although this particular case is still quite involved in terms of computations, it is the simplest case that exhibits all the novelties, with respect to the PMM, and difficulties that we are going to face while studying the general case (as in the next proposition), and also gives some intuition regarding the results for the present and next sections. In what follows recall Definition 4.2.3.

Proposition 4.2.11. *For each $(i,j) \in I_\ell \times J$ with $0 \le \ell \le k$ fixed, consider the quantity*

$$b_{ij}^{\ell} = \sum_{(i',q)\in(\psi_{j,\ell})^{-1}(i)} a_{i'j}. \tag{4.12}$$

If the family $\mathfrak{a}$ is such that the linear system

$$\left\{\sum_{(i,j)\in\underline{c}} b_{ij}^{\ell} = \sum_{(i,j)\in\phi_\ell(\underline{c})} b_{ij}^{\ell}\right\}_{0\le\ell\le k-1,\ \underline{c}\in\mathcal{C}_{\ell,1}} \quad \wedge \quad \left\{\sum_{i\in I} a_{ij} = 1\right\}_{j\in J} \tag{4.13}$$

has a solution, then the generator $\mathcal{L}_N^{(n,k;\mathfrak{a})}$ induces a gradient model. Specifically, in this case, it holds that $\mathbf{c}^{(n,k;\mathfrak{a})}(\eta)(\nabla^+\eta)(0) = (\nabla^+\mathbf{h}^{(n,k;\mathfrak{a})})(\eta)$ with the potential $\mathbf{h}^{(n,k,a)} = \mathbf{h}_0^{(n,k,a)} + \mathbf{h}_1^{(n,k,a)}$ where

$$\begin{aligned}
\mathbf{h}_0^{(n,k;\mathfrak{a})} &= \sum_{j=0}^{n+k}\sum_{y=1}^{j}(\nabla^+\tau_{-y}\eta)(j)\mathbf{s}_j^{(n,k;\mathfrak{a})}(\tau_{-y}\eta)\\
\mathbf{h}_1^{(n,k;\mathfrak{a})} &= \sum_{\ell=0}^{k}(-1)^\ell\sum_{\underline{c}\in\mathcal{C}_{\ell,1}}\sum_{(i,j)\in\underline{c}}\left\{b_{ij}^{\ell}\sum_{y=0}^{p_{ij}^{\ell,1}-1}(\tau_y\eta)(A_{\underline{c}}) - b_{\phi_\ell(i,j)}^{\ell}\sum_{y=0}^{p_{\phi_\ell(i,j)}^{\ell,0}-1}(\tau_y\eta)(A_{\underline{c}})\right\}.
\end{aligned} \tag{4.14}$$

Proof. As a first step, we use the distributive property to expand the constraints into a linear combination of "pure occupation variables", with no flips. Recall from Definition 4.2.3 that for each $(i,j) \in I \times J$ we write $\{Q_{ijq}^{\ell}\}_{1\le q\le\binom{k}{\ell}} = \mathcal{P}_\ell(P_{ij})$. We develop

$$\begin{aligned}
\mathbf{c}^{(n,k;\mathfrak{a})}(\eta) &= \sum_{(i,j)\in I\times J} a_{ij}(\tau_{-j}\eta)(M_j\backslash P_{ij})\tau_{-j}\left\{(-1)^k\eta(P_{ij}) + \sum_{\ell=0}^{k-1}\sum_{q=1}^{\binom{k}{\ell}}(-1)^\ell\eta(Q_{ijq}^{\ell})\right\}\\
&= (-1)^k\sum_{j\in J}(\tau_{-j}\eta)(M_j)\sum_{i\in I}a_{ij} + \sum_{\ell=0}^{k-1}\sum_{(i,j)\in I\times J}a_{ij}\sum_{q=1}^{\binom{k}{\ell}}(-1)^\ell(\tau_{-j}\eta)([M_j\backslash P_{ij}]\sqcup Q_{ijq}^{\ell}).
\end{aligned} \tag{4.15}$$

We are going to multiply the quantity above by $(\nabla^+\eta)(0)$, but before that note the following. If $\sum_{i\in I} a_{ij} = \sum_{i\in I} a_{ij'}$ for every $j, j' \in J$, then the term associated with the first summation in the second line of the previous display is identified with a multiple of the PMM$(n+k)$ constraint, $\mathbf{c}^{(n+k)}(\eta)$, and in this case $(-1)^k \sum_{j\in J}(\tau_{-j}\eta)(M_j)\sum_{i\in I} a_{ij} = (-1)^k \sum_{i\in I} a_{ij}\nabla^+\mathbf{h}^{(n+k)}(\eta)$. In this way, we consider without loss of generalization the normalized weights

$$\sum_{i\in I} a_{ij} = 1, \quad \forall j \in J. \tag{4.16}$$

Fixed ℓ in (4.15), there are repeated elements in $\{[M_j\backslash P_{ij}] \sqcup Q^\ell_{ijq}\}_{(i,j)\in I\times J, 1\le q\le \binom{k}{\ell}}$. We want to group the coefficients associated with these repeated sets. Recalling Lemma 4.2.5 and Definition 4.2.6, introducing, for each $(i,j) \in I \times J$ and $0 \le \ell \le k$, the new weights

$$b^\ell_{ij} = \sum_{(i',q)\in(\psi_{j,\ell})^{-1}(i)} a_{i'j}$$

we can rewrite (4.15) as

$$\sum_{\ell=0}^{k}(-1)^\ell \sum_{(i,j)\in I_\ell\times J} b^\ell_{ij}(\tau_{-j}\eta)(P^{n+\ell}_{ij}).$$

We will now focus on the quantity $\mathbf{c}^{(n,k;\mathfrak{a})}(\eta)(\nabla^+\eta)(0)$, that is,

$$\sum_{x=0,1}(-1)^{x-1}\sum_{\ell=0}^{k}(-1)^\ell \sum_{(i,j)\in I_\ell\times J} b^\ell_{ij}(\tau_{-j}\eta)(P^{n+\ell}_{ij}\sqcup\{j+x\}).$$

We want to remove the translations τ_{-j}, having then to work only with subsets of $M_j = M\backslash\{j, j+1\}$, for each $j \in J$. Since $\mathbf{1} - \tau_{-j} = \nabla^+ \circ \sum_{y=1}^{j}\tau_{-y}$, summing and subtracting the appropriate terms in the quantity in the previous display we obtain

$$(\nabla^+\mathbf{h}_0^{(n,k;\mathfrak{a})})(\eta) + \sum_{x=0,1}(-1)^{x-1}\sum_{\ell=0}^{k}(-1)^\ell \sum_{(i,j)\in I_\ell\times J} b^\ell_{ij}\eta(P^{n+\ell}_{ij}\sqcup\{j+x\}) \tag{4.17}$$

with

$$\begin{aligned}
\mathbf{h}_0^{(n,k;\mathfrak{a})}(\eta) &= \sum_{x=0,1}(-1)^{x-1}\sum_{\ell=0}^{k}(-1)^\ell \sum_{(i,j)\in I\times J} a_{ij}\sum_{q=1}^{\binom{k}{\ell}}\sum_{y=1}^{j}(\tau_{-y}\eta)([M_j\backslash P_{ij}]\sqcup Q^\ell_{ijq}\sqcup\{j+x\}).\\
&= \sum_{(i,j)\in I\times J} a_{ij}\sum_{q=1}^{\binom{k}{\ell}}\sum_{y=1}^{j}\nabla^+(\tau_{-y}\eta)(j)(\tau_{-y}\eta)(M_j\backslash P_{ij})\sum_{\ell=0}^{k}(-1)^\ell(\tau_{-y}\eta)(Q^\ell_{ijq})\\
&= \sum_{(i,j)\in I\times J} a_{ij}\sum_{y=1}^{j}\nabla^+(\tau_{-y}\eta)(j)(\tau_{-y}\eta^{P_{ij}})(M_j),
\end{aligned}$$

where in the last equality we applied the distributive rule.

At this point, for each x fixed in the second term in (4.17) we want to translate every set to the origin, in this way facilitating the comparison between the terms associated with $x = 0$ with the terms associated with $x = 1$. Recall then from Definition 4.2.3 that for every $(i,j) \in I_\ell \times J$, $0 \le \ell \le k$ and $x = 0, 1$ we short-write

$$p^{\ell,x}_{ij} = \min\{P^{n+\ell}_{ij}\sqcup\{j+x\}\} \ge 0 \quad \text{and} \quad A^{n+\ell,x}_{ij} = -p^{\ell,x}_{ij} + (P^{n+\ell}_{ij}\sqcup\{j+x\}).$$

Naturally, for all the indexes i, j, ℓ, x it holds that $p_{ij}^{\ell,x} \geq 0$ and $0 \in A_{ij}^{n+\ell,x}$. With this, we can rewrite the second term in (4.17) as

$$(\nabla^+ \mathbf{h}_1^{(n,k,\mathfrak{a})})(\eta) + \sum_{x=0,1} (-1)^{x-1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{(i,j)\in I_\ell \times J} b_{ij}^{\ell} \eta(A_{ij}^{n+\ell,x}) \tag{4.18}$$

where, recalling Notation 4.2.9,

$$\mathbf{h}_1^{(n,k,\mathfrak{a})}(\eta) = \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c_1}\in\mathcal{C}_{\ell,1}} \sum_{(i,j)\in\underline{c_1}} b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,1}-1} (\tau_y \eta)(A_{\underline{c_1}}) - \sum_{\underline{c_0}\in\mathcal{C}_{\ell,0}} \sum_{(i,j)\in\underline{c_0}} b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,0}-1} (\tau_y \eta)(A_{\underline{c_0}}) \tag{4.19}$$

and where we used that $\tau_p - \mathbf{1} = \nabla^+ \circ \sum_{y=0}^{p-1} \tau_y$ for any $p \in \mathbb{N}$. We stress that $\sum_{\varnothing} := 0$ and in the previous display one has $p_{ij}^{\ell,x} > 0$ for all i, j, ℓ, x, as we only translate the sets that do not start at the origin. Due to the shifting of the sets $P_{ij}^{n+\ell} \sqcup \{j + x\}$ for all the indexes i, j, for each x and ℓ fixed the collection $\{A_{ij}^{n+\ell,x}\}_{(i,j)\in I_\ell \times J}$ may have repeated elements. Grouping the repeated sets we can rewrite the second term in (4.18) as

$$\sum_{x=0,1} (-1)^{x-1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c_x}\in\mathcal{C}_{\ell,x}} \eta(A_{\underline{c_x}}) \sum_{(i,j)\in\underline{c_x}} b_{ij}^{\ell} \tag{4.20}$$

where $A_{\underline{c_x}}$ is the set representative of the class $\underline{c_x}$, that is, such that $A_{\underline{c_x}} = A_{ij}^{n+\ell,x}$ for all $(i,j) \in \underline{c_x}$ with $x = 0$ and $x = 1$.

The final step consists in invoking Lemma 4.2.7, which provides, through its Corollary 4.2.8, the existence, for each ℓ, of a permutation, ϕ_ℓ, over the set of indexes, $(i,j) \in I_\ell \times J$, such that

$$\phi_\ell(j,i) = (j', i') \quad \text{where} \quad -p_{ij}^{\ell,1} + (P_{ij}^{n+\ell} \sqcup \{j+1\}) = -p_{j',i'}^{\ell,0} + (P_{i'}^{j',n+\ell} \sqcup \{j'\}).$$

With this, it is clear that a sufficient condition for the model to have the gradient property is

$$\left\{ \sum_{(i,j)\in\underline{c_1}} b_{ij}^{\ell} = \sum_{(i,j)\in\phi_\ell(\underline{c_1})} b_{ij}^{\ell} \right\}_{0\leq\ell\leq k,\ \underline{c_1}\in\mathcal{C}_{\ell,1}},$$

since it leads to the quantity (4.20) being equal to zero. Note that for $\ell = k$ we simply have (4.16).

The map ϕ_ℓ allows us to rewrite (4.19) as

$$\mathbf{h}_1^{(n,k;\mathfrak{a})} = \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c}\in\mathcal{C}_{\ell,1}} \sum_{(i,j)\in\underline{c}} \left\{ b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,1}-1} (\tau_y \eta)(A_{\underline{c}}) - b_{\phi_\ell(i,j)}^{\ell} \sum_{y=0}^{p_{\phi_\ell(i,j)}^{\ell,0}-1} (\tau_y \eta)(A_{\underline{c}}) \right\},$$

as a consequence of the following. For $(i,j) \in I_\ell \times J$ it holds that $A_{ij}^{n+\ell,1} = A_{\phi_\ell(i,j)}^{n+\ell,1} = A_{\underline{c}}$ where $(i,j) \in \underline{c} \in \mathcal{C}_{\ell,1}$; and since ϕ_ℓ is a permutation over the indexes induced by a bijection between the classes, for each $\underline{c_0} \in \mathcal{C}_{\ell,0}$ there exists one $\underline{c_1} \in \mathcal{C}_{\ell,1}$ such that $\underline{c_0} = \phi_\ell(\underline{c_1})$. Therefore, one can replace the summation over $\mathcal{C}_{\ell,0}$ by a summation over $\mathcal{C}_{\ell,1}$ and apply the map ϕ_ℓ to the indexes:

$$\sum_{\underline{c_0}\in\mathcal{C}_{\ell,0}} \sum_{(i,j)\in\underline{c_0}} b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,0}-1} (\tau_y \eta)(A_{\underline{c_0}}) = \sum_{\underline{c_1}\in\mathcal{C}_{\ell,1}} \sum_{(i,j)\in\underline{c_1}} b_{\phi_\ell(i,j)}^{\ell} \sum_{y=0}^{p_{\phi_\ell(i,j)}^{\ell,0}-1} (\tau_y \eta)(A_{\underline{c_1}}).$$

□

As a corollary we see that for each fixed $0 \le \ell \le k$ the linear system

$$\left\{ \sum_{(i,j)\in \underline{c}} b_{ij}^{\ell} = \sum_{(i,j)\in \phi_\ell(\underline{c})} b_{ij}^{\ell} \right\}_{\underline{c}\in \mathcal{C}_{\ell,1}} \tag{4.21}$$

is associated with a gradient model, in the following sense.

Corollary 4.2.12. *For each $0 \le \ell \le k$ introduce the constraint*

$$\mathbf{c}^{(n+\ell;\mathfrak{a})}(\eta) = \sum_{j=0}^{n+k} \sum_{i=1}^{\binom{n+k}{k-\ell}} b_{ij}^{\ell}(\tau_{-j}\eta)(P_{ij}^{n+\ell})$$

with $b_{ij}^{\ell} \equiv b_{ij}^{\ell}(\mathfrak{a})$ as in Proposition 4.2.11. Then, if the linear system (4.21) *is satisfied, $\mathbf{c}^{(n+\ell)}$ satisfies the gradient condition.*

We recall from the proof of the previous proposition that the linear system above reduces, for $\ell = k$, to $\sum_{i\in I} a_{ij} = 1$ and $\mathbf{c}^{(n+k;\mathfrak{a})}$ is identified with the PMM$(n+k)$ constraint. It is also simple to see that for each $0 \le \ell \le k$ there is a solution to the system (4.21) such that $\mathbf{c}^{(n+\ell;\mathfrak{a})} = \mathbf{c}^{(n+\ell)}$, given by $b_{ij}^{\ell} = \mathbf{1}\{P_{ij}^{n+\ell} = [\![0, n+\ell+1]\!]\backslash\{j, j+1\}\}$. Moreover, for each $0 \le \ell \le k$, following the techniques in the previous chapter, or arguing with the relative entropy method as in [21] we expect that the renormalized constraint

$$\frac{1}{z_{n,k}(\ell)}\mathbf{c}^{(n+\ell;\mathfrak{a})}, \quad \text{with} \quad z_{n,k}(\ell) = \binom{k}{\ell}\frac{n+k+1}{n+\ell+1}$$

leads to the diffusion coefficient $(n+\ell+1)D_{n+\ell,0}(\rho) = (n+\ell+1)\rho^{n+\ell}$. In this manner, we can see the integer Porous Media Model as a particular example of the large class of models corresponding to each solution of (4.21), with adjusted constraints as in the previous display. We recall that the linear systems when $n, k = 2$ and $0 \le \ell \le 2$ can be found in Appendix C.1, specifically, in (C.3).

For general $n, k \in \mathbb{N}_+$ the uniform choice of the weights corresponds to a solution of (4.13). This choice leads to a series of technical challenges in the study of the potential $\mathbf{h}^{(n,k;\mathfrak{a})}$, but the resulting model is simple enough to argue via combinatoric arguments. In the present subsection we are going to prove some properties of the dynamics induced by the uniform choice; simplify the expression for $\mathbf{h}^{(n,k;\mathfrak{a})}$ in (4.14) and characterize each of its constituents in order to provide a solid basis for future works.

Proposition 4.2.13. *For any fixed $n, k \in \mathbb{N}_+$, the uniform weights $a_{ij} = \frac{1}{|I|} = \binom{n+k}{k}^{-1}$ for $(i,j) \in I \times J$ is a solution of the system* (4.13).

Proof. From Lemma 4.2.5 we know that for every $(i,j) \in I_\ell \times J$ the quantity $\left|(\psi_{j,\ell})^{-1}(i)\right|$ depends only on k and ℓ. This, coupled with the existence of a particular permutation on $(i,j) \in I_\ell \times J$, for each $0 \le \ell \le k$, as in Corollary 4.2.8, allow us to readily extract as a solution the uniform choice:

$$a_{ij} = \frac{1}{|I|} = \binom{n+k}{k}^{-1}, \qquad \forall (i,j) \in I \times J.$$

□

The uniform choice for the weights has the particularity that for each $0 \le j \le n+k$ the process only looks at the *density* of particles in the window $W_j = -j + M_j$ (recall Definition 4.2.3); while for a non-uniform solution of (4.13) the constraints depend also on the *distribution* of vacant sites in the

aforementioned window. For the latter case, this leads further to, for each j as previously, the weights $\{a_{ij}\}_{i\in I}$ being interpreted as a probability distribution of the vacant sites inside the window W_j, since the weights can be normalized ($\sum_{i\in I} a_{ij} = 1$ for any $j \in J$). To be precise, let us introduce for each $A \subseteq \mathbb{T}_N$ the map $\eta \mapsto \langle\eta\rangle_A = \frac{1}{|A|}\sum_{z\in A}\eta(z)$. Then, for the uniform choice, we can express

$$\mathbf{c}^{(n,k\,\mathfrak{a})}(\eta) = \frac{1}{\binom{n+k}{k}}\sum_{j=0}^{n+k}\mathbf{1}\left\{\langle\eta\rangle_{W_j} = \frac{n}{n+k}\right\},$$

while in general

$$\mathbf{c}^{(n,k\,\mathfrak{a})}(\eta) = \sum_{j=0}^{n+k}\mathbf{1}\left\{\langle\eta\rangle_{W_j} = \frac{n}{n+k}\right\}\sum_{i=1}^{\binom{n+k}{k}}\mathbf{1}\{\overline{\eta}(P_{ij}) = 1\}a_{ij}.$$

Note that for $k = 0$ one obtains the PMM(n) constraint, which can be recast as $\sum_{j=0}^{n}\mathbf{1}\{\langle\eta\rangle_{W_j} = 1\}$, where we remind the reader that W_j also depends on n and k.

We fix once and for all the weights, extending the $\{\text{PMM }(n)\}_{n\in\mathbb{N}_+}$ family to a 2–parameter family.

Definition 4.2.14 (Generalized PMM)**.** For each $n, k \in \mathbb{N}$ fixed, let $\mathfrak{a} = \{a_{ij}\}_{(i,j)\in I\times J}$ with $a_{ij} = \binom{n+k}{k}^{-1}$. We short-write $\mathbf{c}^{(n,k)} \equiv \mathbf{c}^{(n,k;a)}$, $\mathbf{s}_j^{(n,k)} \equiv \mathbf{s}_j^{(n,k;a)}$ and $\mathcal{L}_N^{(n,k)} \equiv \mathcal{L}_N^{(n,k;a)}$, with the previous quantities as in Definition 4.2.4. We refer to the Markov process defined through the infinitesimal generator $\mathcal{L}_N^{(n,k)}$ as PMM(n,k).

We stress that if any other solution of (4.13) exists, it should lead to the same continuous diffusion coefficient – yet with a different discrete dynamics.

The first property of the previously defined family that we are going to show is a link with the SSEP, which is a microscopic version of the identity $1 = \sum_{n=0}^{L}\binom{L}{n}\rho^n(1-\rho)^{L-n}$ for any $L \in \mathbb{N}_+$.

Proposition 4.2.15. *Fixed $L \in \mathbb{N}_+$ such that $L < N$, the family of rescaled constraints $\{\frac{1}{\mathcal{Z}_{L,n}}\mathbf{c}^{(n,L-n)}\}_{0\le n\le L}$, with $\mathcal{Z}_{L,n} = (L+1)/\binom{L}{n}$, is a partition of the unity in Ω_N. In other words,*

$$\mathcal{L}_N^{SSEP} = \sum_{n=0}^{L}\frac{1}{\mathcal{Z}_{L,n}}\mathcal{L}_N^{(n,L-n)}$$

and in this way the SSEP can be seen as a superposition of porous media models.

Proof. Simply note that for any $\eta \in \Omega_N$ fixed it holds that

$$\sum_{n=0}^{L}\frac{1}{\mathcal{Z}_{L,n}}\mathbf{c}^{(n,L-n)}(\eta) = \frac{1}{L+1}\sum_{j=0}^{L}\left(\sum_{n=0}^{L}\mathbf{1}\left\{\langle\eta\rangle_{W_j} = \frac{n}{L}\right\}\right) = 1.$$

The second equality is justified with the simple observation that for any fixed η and $0 \le j \le L$ (hence, fixed a window W_j), there is exactly one density $\rho_j \in \{0, 1/L, 2/L, \dots, 1\}$ such that $\langle W_j\rangle = \rho_j$. In this way, for each j, the summation over n in the previous display equals one. □

As an observation, note that the extended PMM family has the following discrete interpolation property, relating the PMM and the $\overline{\text{PMM}}$ families, with the latter defined through the constraints $\{\mathbf{c}^{(n)}\circ\mathcal{F}_{\mathbb{T}_N}\}_{n,k\in\mathbb{N}_+}$, as in Definition 3.2.6:

$$\text{PMM}(0,0)=\text{SSEP}, \quad \text{PMM}(n,0)=\text{PMM}(n) \quad \text{and} \quad \text{PMM}(0,k)=\overline{\text{PMM}}(k).$$

Lemma 4.2.16. *Let $\mathbf{c}^{(n,k)}$ be the constraint produced by the uniform choice of weights $\mathfrak{a}$. For any $n,k,\ell \in \mathbb{N}$ such that $\ell, n+k+1 \le N$, it holds that*

$$\sum_{x=0}^{\ell} \mathbf{c}^{(n,k)}(\tau_x\eta)\mathbf{a}(\tau_x\eta) \le \frac{1+2k}{\binom{n+k}{k}}(\ell+n+k+1).$$

Proof. Fix $\eta \in \Omega_N$ and consider the set $B_\eta = \{x \in [\![0,\ \ell]\!] : \eta(x)+\eta(x+1) = 1\}$. One can then express

$$\sum_{x=0}^{\ell} \mathbf{c}^{(n,k)}(\tau_x\eta)\mathbf{a}(\tau_x\eta) = \frac{1}{\binom{n+k}{k}} \sum_{p=-(n+k)}^{\ell} \sum_{j=0}^{n+k} \sum_{x\in B_\eta} \mathbf{1}\{\langle\eta\rangle_{M_p'\setminus\{x,x+1\}} = \frac{n}{n+k}\}\mathbf{1}\{x = j+p\}, \tag{4.22}$$

where $M_p' = [\![p, n+k+1+p]\!]$. Fixed $-n-k \le p \le \ell$, there are a number of pairs $(x,j) \in [\![0,\ \ell]\!] \times [\![0,\ n+k+1]\!]$ such that $p = x-j$, but we are interested only on the pairs such that $x \in B_\eta$. In this manner, we can bound

$$\sum_{x\in B_\eta} \mathbf{1}\left\{\langle\eta\rangle_{M_p'\setminus\{x,x+1\}} = \frac{n}{n+k}\right\} \sum_{j=0}^{n+k} \mathbf{1}\{x = j+p\} \le 2(1+k)$$

with the following reasoning. Suppose $\langle\eta\rangle_{M_p'\setminus\{x,x+1\}} = \frac{n}{n+k}$, for some p and x. Thus, there exists exactly k empty sites in the window M_p'. Note also that, trivially, fixed x and p, there exists a unique j such that $x = j+p$. The upper bound in the previous display is then obtained by maximizing the number of local configurations $(1,0)$ and $(0,1)$ in the window M_p' taking into account that one has exactly k holes, that is, where each empty site is followed by a particle and vice-versa. Plugging the previous upper bound in the right-hand side of (4.22) finishes the proof. □

Note that for $k = 0$ the upper bound in the previous lemma corresponds to the upper bound in Lemma 3.2.16.

4.2.2 Simplification of $\mathbf{h}^{(n,k)}$.

The present subsection is devoted to the simplification of the expression for the potential introduced in (4.14) for the particular case of the uniform solution (the map $\mathbf{h}^{(n,k)}$), as in Proposition 4.2.13. The aforementioned map is composed by two parts, $\mathbf{h}_0^{(n,k)}$ and $\mathbf{h}_1^{(n,k)}$. The term $\mathbf{h}_0^{(n,k)}$ can be shown to vanish in the limit $N \to +\infty$ in probability, while the second is expected to converge to a primitive of the diffusion coefficient $(n+k+1)\rho^n(1-\rho)^k$. For the *uniform* solution one can "remove", in $\mathbf{h}_1^{(n,k)}$, the dependence over the particular choice of the labels of the elements of $\{\mathcal{P}_k(-j+M)\}_{j\in J}$. Note that this dependence is made implicit through the objects $\phi_\ell, p_{ij}^\ell, \underline{c} \in \mathcal{C}_{\ell,1}$. Instead, we have the expression in (4.26), which is much simpler than the one in (4.14). This expression is itself the sum of two terms, $\nabla^+\mathbf{g}^{(n,k)}$ and $\mathbf{H}^{(n,k)}$. Both of them do not depend on the labelling of the aforementioned sets, and are linear combinations of $\{\eta(\{0\} \sqcup A)\}_{A\in\mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\})}$, with coefficients involving a quantity $s_M(A)$ that expresses how much "space" the set A occupies in M (see Definition 4.2.22). This is all proved in Proposition 4.2.23, which invokes all the lemmas of the present section. We also show that if the weights $\mathfrak{a}$ are *not* uniform but satisfy both the linear system (4.13) and some extra equations, then $\mathbf{h}_0^{(n,k;\mathfrak{a})} = \mathbf{h}_0^{(n,k)}$ – this is part of the content of Proposition 4.2.23. In Appendix C.1.2 we present a particular, non-uniform, solution of the linear system extended with these extra equations, for the particular case of $n,k = 2$. We chose to present the simplest solution we could find, as the system has many solutions. We remark again that we

did not prove that for general n and k a non-uniform solution to the system (4.13) exists, but only for the particular case $n = k = 2$, and in this manner the content of Proposition 4.2.23 regarding the extended system is in fact currently verified only for $n = k = 2$.

The forthcoming Lemma 4.2.17 will be invoked in Lemmas 4.2.19 and 4.2.20, as it allows us to identify all the sets equal to either $P_{ij}^{n+\ell} \sqcup \{j+1\}$ or $P_{ij}^{n+\ell} \sqcup \{j\}$, for any fixed $0 \leq \ell \leq k$ and $(i,j) \in I_\ell \times J$, by analysing their particular structure. This identification, in turn, facilitates the study of the numbers $\{p_{ij}^{\ell,x}\}_{(i,j)\in I_\ell\times J,\, x=0,1}$ (see Lemma 4.2.20) and a better understanding of the classes in $\mathcal{C}_{\ell,x}$, for $x = 0,1$, which will be important to express the potential as in Proposition 4.2.23.

Lemma 4.2.17. *Fixed $0 \leq \ell \leq k$ and $(i,j) \in I_\ell \times J$, for each integer $0 \leq w \leq n+k$ such that $w \notin P_{ij}^{n+\ell} \sqcup \{j+1\} \ni w+1$ there exists $i' \in I_\ell$ such that $P_{i'w}^{n+\ell} \sqcup \{w+1\} = P_{ij}^{n+\ell} \sqcup \{j+1\}$.*

Analogously, for each integer $0 \leq w \leq n+k$ such that $w+1 \notin P_{ij}^{n+\ell} \sqcup \{j+1\} \ni w$ there exists $i' \in I_\ell$ such that it holds $P_{i'w}^{n+\ell} \sqcup \{w\} = P_{ij}^{n+\ell} \sqcup \{j\}$.

Proof. Let $0 \leq w \leq n+k$ be such that $w \notin P_{ij}^{n+\ell} \sqcup \{j+1\} \ni w+1$. It is enough to note that because

$$P_{ij}^{n+\ell} \sqcup \{j+1\} = \left[(P_{ij}^{n+\ell} \sqcup \{j+1\}) \backslash \{w+1\}\right] \sqcup \{w+1\}$$

and since $(P_{ij}^{n+\ell} \sqcup \{j+1\}) \backslash \{w+1\} \in \mathcal{P}_{n+\ell}(M_w)$, there is a unique $i' \in I_\ell$ such that

$$(P_{ij}^{n+\ell} \sqcup \{j+1\}) \backslash \{w+1\} = P_{i'w}^{n+\ell}.$$

To see the second assertion in the statement the proof is analogous. □

The goal now is to characterize the sets associated with a fixed equivalence class of the indexes.

Definition 4.2.18. For each $j \in J$, $0 \leq \ell \leq k$ fixed and $x = 0,1$ introduce the families of sets $\mathcal{P}^{\ell,x} = \{\mathcal{P}_j^{\ell,x}\}_{j\in J}$, where $\mathcal{P}_j^{\ell,x} := \{\{j+x\} \sqcup A \,:\, A \in \mathcal{P}_{n+\ell}(M \backslash \{j, j+1\})\}$.

Note that if $P \in \mathcal{P}_j^{\ell,x}$ for some j, ℓ, x as in the previous definition, then there exists $i \in I_\ell$ such that $P = P_{ij}^{n+\ell} \sqcup \{j+x\}$. Moreover, note that $\{0, \ldots, n+\ell\} \notin \mathcal{P}^{\ell,1}$ and $\{k+1-\ell, \ldots, n+k+1\} \notin \mathcal{P}^{\ell,0}$.

We are going to show that fixed some ℓ and given some set $P_{ij}^{n+\ell}$ with index $(i,j) \in \underline{c_x} \in \mathcal{C}_{\ell,x}$ with $x = 0$ or $x = 1$, if $P_{ij}^{n+\ell} \sqcup \{j+x\} \not\equiv \{0, \ldots, n+\ell\}$ then *all* the sets corresponding to translations of $P_{ij}^{n+\ell} \sqcup \{j+1\}$ inside M are of the form $P_{i'j'}^{n+\ell} \sqcup \{j'+x\}$ where the index (i',j') is in the same class $\underline{c_x}$; while if $P_{ij}^{n+\ell} \sqcup \{j+x\} \equiv \{0, \ldots, n+\ell\}$ then one has all the translations except a particular one for $x = 0$ and another one for $x = 1$. This will be important for the proof of Lemmas 4.2.20 and 4.2.21.

Lemma 4.2.19. *Let $0 \leq \ell \leq k$ and $x \in \{0,1\}$ be fixed, and consider some class $\underline{c_x} \in \mathcal{C}_{\ell,x}$.*

1. *If $P_{ij}^{n+\ell} \sqcup \{j+x\} \equiv \{0, \ldots, n+\ell\}$ then for each $z \in \mathbb{Z}\backslash\{0\}$ such that*

$$z + (P_{ij}^{n+\ell} \sqcup \{j+x\}) \subset M \quad \text{and} \quad z \neq \begin{cases} -\min(P), & x = 1, \\ n+k+1-\max(P), & x = 0, \end{cases}$$

   there exists $(i',j') \in \underline{c_x}$ such that $z + P_{ij}^{n+\ell} = P_{i'j'}^{n+\ell}$.

2. *If $P_{ij}^{n+\ell} \sqcup \{j+x\} \not\equiv \{0, \ldots, n+\ell\}$ then for each $z \in \mathbb{Z}\backslash\{0\}$ such that $z + (P_{ij}^{n+\ell} \sqcup \{j+x\}) \subset M$, there exists $(i',j') \in \underline{c_x}$ such that $z + P_{ij}^{n+\ell} = P_{i'j'}^{n+\ell}$.*

Proof. The proof is split in several cases. Let $x = 1$ and take some set $P \in \mathcal{P}^{\ell,1}$. Assume first that P is *not* a translation of $\{0,\dots,n+\ell\}$. There is some $j \in J$ such that $P \in \mathcal{P}^1_j = \{j+1\} \sqcup \mathcal{P}_{n+\ell}(M\backslash\{j\})$. Let z be as in the statement and note that $z + P \in \{z+j+1\} \sqcup \mathcal{P}_{n+\ell}(z + M\backslash\{j\})$. Then there is some set $Q^1 \in \mathcal{P}_{n+\ell}(z+M\backslash\{j\})$ with the property that $\{z+j+1\} \sqcup Q^1 \subseteq M$. Therefore, we want to show that there exists some $j' \in J$ such that $Q^1_z := \{z+j+1\} \sqcup Q^1 \in \mathcal{P}^{\ell,1}_{j'}$. From Lemma 4.2.17 such j' exists as long as there is some $q \geq 0$ such that $q \notin Q^1_z \ni q+1 \leq n+k+1$, and in that case it is then enough to fix $j' = q$. Since $P \in \mathcal{P}^1$ and $P \not\equiv \{0,\dots,n+\ell\}$, there exists $0 \leq p \notin P \ni p+1 \neq \min(P)$ and as such we can fix $q = z + p = j'$.

If P is a translation of $\{0,\dots,n+\ell\}$, then the previous argument holds for all z such that $z + P \subset M$ *except* for $z = -\min(P)$, since $-\min(P) + P = \{0,\dots,n+\ell\} \notin \mathcal{P}^{\ell,1}$.

For $x = 0$ the argument is identical and as such we provide only the main steps. If P is not a translation of $\{k+1-\ell,\dots,n+k+1\}$, letting z be as in the statement, one needs to argue that there is some $j' \in J$ such that $Q^0_z := \{z+j\} \sqcup Q^0 \in \mathcal{P}^{\ell,0}_{j'}$, where $Q^0 \in \mathcal{P}_{n+\ell}(z + M\backslash\{j+1\})$. It is then enough to show that there is some $q+1 \notin Q^0_z \ni q$ with $q+1 \leq n+k+1 \neq \max(P)$, which is done as previously, and invoking Lemma 4.2.17. If P is a translation of $\{k+1-\ell,\dots,n+k\}$, then the previous argument holds for all z in the statement *except* for $z = n+k+1-\max(P)$, since $(n+k+1-\max(P)) + P = \{k+1-\ell,\dots,n+k\} \notin \mathcal{P}^{\ell,0}$. □

In order to simplify the quantities arising from the gradient property it will be important to characterize the set $\{p^{\ell,x}_{ij}\}_{(i,j)\in I_\ell \times J,\ x=0,1}$ which will, in turn, allows us to fix the map ϕ_ℓ conveniently, with respect to each class. We note that for any i,j,ℓ,x as previously, $p^{\ell,x}_{ij} \in \{0,\dots,n+k+1-\max(A^{\ell,x}_{ij})\}$, with $A^{\ell,x}_{ij}$ as in Definition 4.2.3.

Lemma 4.2.20. *Let $0 \leq \ell \leq k$ and $\underline{c} \in \mathcal{C}_{\ell,1}$ be fixed. If $\max(A_{\underline{c}}) \neq n+k+1$ then:*

(1) For any $p \neq 0$, $n+k+1-\max(A_{\underline{c}})$ it holds that $\left|\{(i,j) \in \underline{c} : p^{\ell,1}_{ij} = p\}\right| = \left|\{(i,j) \in \phi_\ell(\underline{c}) : p^{\ell,0}_{ij} = p\}\right|$;

(2) $\left|\{(i,j) \in \underline{c} : p^{\ell,1}_{ij} = 0\}\right| = \left|\{(i,j) \in \phi_\ell(\underline{c}) : p^{\ell,0}_{ij} = 0\}\right| - 1$;

(3) $\left|\{(i,j) \in \underline{c} : p^{\ell,1}_{ij} = n+k+1-\max(A_{\underline{c}})\}\right| = \left|\{(i,j) \in \phi_\ell(\underline{c}) : p^{\ell,0}_{ij} = n+k+1-\max(A_{\underline{c}})\}\right| + 1$.

Proof. In order to show (1) we will follow a "diagonal" argument. Let $\underline{c}$ be a fixed class as in the statement of the current lemma. We start by showing that for each $(i,j) \in \underline{c}$ such that $p^{\ell,1}_{ij} \geq 1$ there is some $(i',j') \in \phi_\ell(\underline{c})$ such that $p^{\ell,1}_{ij} - 1 = p^{\ell,0}_{i'j'}$; and its converse. Next, we show that for any $1 \leq p \leq n+k+1-\max(A_{\underline{c}})-1$ it holds that

$$\{(i,j) \in \underline{c} : p^{\ell,1}_{ij} = p\} = \{(i,j) \in \underline{c} : p^{\ell,1}_{ij} = p+1\}. \tag{4.23}$$

This directly implies the property (1). In particular, it also implies that

$$\left|\{(i,j) \in \underline{c} : p^{\ell,1}_{ij} = 0\}\right| = \left|\{(i,j) \in \phi_\ell(\underline{c}) : p^{\ell,0}_{ij} = n+k+1-\max(A_{\underline{c}})\}\right|,$$

which will then be used to show (2) and (3).

In this way, recall Definition 4.2.18. If $p^{\ell,1}_{ij} + A^{n+\ell,1}_{ij} = P^{n+\ell}_{ij} \sqcup \{j+1\} \in \mathcal{P}^1_j$, then $P' := (p^{\ell,1}_{ij} - 1) + A^{n+\ell,1}_{ij} \in \mathcal{P}^0$ since one can take $j' = \max(P')$ and in this way $n+k+1 \geq j'+1 \notin P' \ni j' \geq 0$ and from Lemma 4.2.17 there is some $i' \in I_\ell$ such that $P' = p^{\ell,0}_{i'j'} + A^{n+\ell,0}_{i'j'}$ (note that $(i',j') \in \phi_\ell(\underline{c})$). The argument to show the

converse: that for each $(i,j) \in \phi_\ell(\underline{c})$ there is one $(i',j') \in \underline{c}$ such that $p_{ij}^{\ell,0} + 1 = p_{i'j'}^{\ell,1}$ is identical and so we omit it.

We now aim to show (4.23), which is consequence of Lemma 4.2.19. Fix $1 \le p \le n+k+1-\max(A_{\underline{c}}^{n+\ell,1})-1$. If $p_{ij}^{\ell,1} = p$ then $p + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$ and from the aforementioned lemma, $(p+1) + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$, because $p+1 \le n+k+1-\max(A_{\underline{c}}^{n+\ell,1})$. Likewise, if $p_{ij}^{\ell,1} = p+1$ then $(p+1) + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$ and $p + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$ because $p \ge 1$. This concludes the proof of (4.23).

As previously explained, this reduces the proof of (2) to that of

$$\left|\{(i,j) \in \phi_\ell(\underline{c}) : p_{ij}^{\ell,0} = n+k+1-\max(A_{\underline{c}})\}\right| = \left|\{(i,j) \in \phi_\ell(\underline{c}) : p_{ij}^{\ell,0} = 0\}\right| - 1.$$

This is also consequence of Lemma 4.2.19. Suppose that $p_{ij}^{\ell,0} = n+k+1-\max(A_{\underline{c}})$ where $(i,j) \in \phi_\ell(\underline{c})$ and $\underline{c}$ is such that $A_{ij}^{n+\ell,1} \equiv \{0,\dots,n+\ell\}$. Since $p_{ij}^{\ell,0} + A_{ij}^{n+\ell,0} \in \mathcal{P}_j^0$ and all the translations of $p_{ij}^{\ell,0} + A_{ij}^{n+\ell,0}$ inside M correspond to sets whose index is in the same class $\phi_\ell(\underline{c})$, for each $(i,j) \in \phi_\ell(\underline{c})$ such that $p_{ij}^{\ell,0} = n+k+1-\max(A_{\underline{c}})$ there is one $(i',j') \in \phi_\ell(\underline{c})$ such that $p_{i'j'}^{\ell,0} = 0$. The only set without this correspondence is the set $(n+k+1-(n+\ell)) + \{0,\dots,n+\ell\}$. To see (3) the argument is also analogous. □

Now we are going to characterize the sets associated with each element of $\mathcal{C}_{\ell,1}$.

Lemma 4.2.21. *It holds that*

$$\{A_{\underline{c}}\}_{\underline{c} \in \mathcal{C}_{\ell,1}} = \{\{0\} \sqcup P \,:\, P \in \mathcal{P}_{n+\ell}(M\backslash\{0\})\}\,. \tag{4.24}$$

Proof. Clearly the collection on the left-hand side is contained on the collection in the right-hand side. To see the converse, it is enough to note that any particular set of the collection on the right-hand side of (4.24) that can be translated (non-trivially) inside M corresponds to a set in $\mathcal{P}^{\ell,1}$ (introduced in Definition 4.2.18), since for any particular $P \in \mathcal{P}_{n+\ell}(M\backslash\{0\})$ with $\max(P) \neq n+k+1$ it holds that $z-1 \notin z + (\{0\} \sqcup P) \ni z$ for $z \in \mathbb{N}_+$ and such that $z + (\{0\} \sqcup P) \subseteq M$. In this way, each $\{0\} \sqcup P$ that can be translated inside M can be seen as a shift to the origin of some set in $\mathcal{P}^{\ell,1}$, and as such for each $P \in \mathcal{P}_{n+\ell}(M\backslash\{0\})$ with $\max(P) \neq n+k+1$ there is some class $\underline{c} \in \mathcal{C}_{\ell,1}$ such that $\{0\} \sqcup P = A_{\underline{c}}^{n+\ell+1}$.

To conclude the proof, we see that if $P' \in \{\{0\} \sqcup P \,:\, P \in \mathcal{P}_{n+\ell}(M\backslash\{0\}),\ \max(P) = n+k+1\}$ then $P' \in \{A_{\underline{c}}\}_{\underline{c} \in \mathcal{C}_{\ell,1}}$ also, since $\exists z \in M$ such that $z \notin \{0\} \sqcup P \ni z+1$ (recall that $0 \le \ell \le k$), and as such the proof is done by invoking Lemma 4.2.17. □

We now introduce a key map that quantifies how "spread" are the elements of any given subset of M.

Definition 4.2.22. For each $A \subseteq M = [\![0, n+k+1]\!]$ introduce $s(A) = \max(A) - \min(A)$ and the quantity

$$s_M(A) = 1 - \frac{s(A)}{n+k+1}.$$

With the previous results and definitions we are finally able to remove the dependence of any particular choice of labels for the elements of $\{\mathcal{P}_k(M_j)\}_{j \in J}$, in the expression for $\mathbf{h}_1^{(n,k)}(\eta)$ (for the uniform case), and in the expression for $\mathbf{h}_1^{(n,k;\mathfrak{a})}(\eta)$, if $\mathfrak{a}$ is solution to an *additional* linear system. The latter is proved in Proposition C.1.1. Note that for the uniform case the labels of the aforementioned sets do *not* appear in the expression for the quantity $\mathbf{h}_0^{(n,k)}(\eta)$ (in (4.14)), since for any $j \in J$ the quantity $\mathbf{s}_j^{(n,k)}(\eta)$ depends only on the density at the window $-j + M_j$.

Proposition 4.2.23. *It holds that*

$$\mathbf{h}_1^{(n,k)} = \nabla^+ \mathbf{g}^{(n,k)} + \mathbf{V}^{(n,k)} \tag{4.25}$$

with

$$\begin{aligned}
\mathbf{g}^{(n,k)}(\eta) &= (n+k+1)\sum_{\ell=0}^{k}(-1)^\ell \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \times \\
&\quad \times \sum_{A\in\mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\})} \sum_{y=1}^{n+k-\max(A)} \left(s_M(\{0\}\sqcup A) - \frac{y}{n+k+1}\right)(\tau_{y-1}\eta)(\{0\}\sqcup A), \qquad (4.26)\\
\mathbf{V}^{(n,k)}(\eta) &= (n+k+1)\sum_{\ell=0}^{k}(-1)^\ell \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \sum_{A\in\mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\})} s_M(\{0\}\sqcup A)\eta(\{0\}\sqcup A).
\end{aligned}$$

Proof. For the uniform choice the expression for $\mathbf{h}_1^{(n,k;\mathfrak{a})}(\eta)$ in (4.14) simplifies to

$$\mathbf{h}_1^{(n,k)} = \frac{1}{\binom{n+k}{k}}\sum_{\ell=0}^{k}(-1)^\ell \binom{n+\ell}{\ell} \sum_{\underline{c}\in\mathcal{C}_{\ell,1}} \sum_{(i,j)\in\underline{c}} \left\{ \sum_{y=0}^{p_{ij}^{\ell,1}-1} (\tau_y\eta)(A_{\underline{c}}) - \sum_{y=0}^{p_{\phi_\ell(i,j)}^{\ell,0}-1} (\tau_y\eta)(A_{\underline{c}}) \right\}.$$

Fixed ℓ and $\underline{c}\in\mathcal{C}_{\ell,1}$, from the property (1) in Lemma 4.2.20 the summation over $(i,j)\in\underline{c}$ in the previous display can be expressed as

$$\begin{aligned}
&\sum_{y=0}^{n+k+1-\max(A_{\underline{c}})-1} (\tau_y\eta)(A_{\underline{c}})\times \\
&\qquad \times \left\{ \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = n+k+1-\max(A_{\underline{c}}) \neq 0\} - \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_\ell(i,j)}^{\ell,0} = n+k+1-\max(A_{\underline{c}}) \neq 0\} \right\}.
\end{aligned}$$

We stress that we can apply Lemma 4.2.20 since for any $\underline{c}\in\mathcal{C}_{\ell,1}$ in the summations above it holds that $\max(A_{\underline{c}}) \neq n+k+1$, since otherwise $p_{ij}^{\ell,1} = 0$ for any $(i,j)\in\underline{c}$, and these elements are not present in those summations, as explained just after (4.19). Now we apply Lemma 4.2.20 again, concretely, property (3), obtaining that

$$\mathbf{h}_1^{(n,k)}(\eta) = \sum_{\ell=0}^{k}(-1)^\ell \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \sum_{\underline{c}\in\mathcal{C}_{\ell,1}} \sum_{y=0}^{n+k-\max(A_{\underline{c}})} (\tau_y\eta)(A_{\underline{c}}), \tag{4.27}$$

and from Lemma 4.2.21,

$$\mathbf{h}_1^{(n,k)}(\eta) = \sum_{\ell=0}^{k}(-1)^\ell \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \sum_{A\in\mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\})} \sum_{y=0}^{n+k-\max(A)} (\tau_y\eta)(\{0\}\sqcup A). \tag{4.28}$$

Above, we have the set $M\setminus\{0,n+k+1\}$ instead of $M\setminus\{0\}$ as in Lemma 4.2.21 because in (4.27) are only present the indexes such that $p_{ij}^{\ell,1}, p_{ij}^{\ell,0} \neq 0$; or, equivalently, if $\max(A_{\underline{c}}) = n+k+1$ in (4.27) then the summation over y in the previous display is empty.

Using the fact that for any $w\in\mathbb{N}_+$ it holds that

$$\sum_{y=0}^{w}\tau_y = (w+1)\mathbf{1} + \nabla^+ \circ \sum_{y=1}^{w}(w+1-y)\tau_{y-1}$$

and $n+k-\max(A) = (n+k+1)s_M(\{0\}\sqcup A)-1$, with A as in (4.28), we can recast $\mathbf{h}_1^{(n,k)}(\eta) = (\nabla^+\mathbf{g}^{(n,k)})(\eta) + \mathbf{V}^{(n,k)}(\eta)$ with $\mathbf{g}$ and $\mathbf{V}^{(n,k)}$ as in the statement of the current lemma. □

In the previous chapter (specifically, in the proof of tightness, Proposition 3.3.1) it was shown that a requirement for studying the model defined by the superposition of Porous Media Model is a precise upper bound on the potential arising from the gradient condition. The next proposition has the goal of both paving the way for such study, while also elucidating the difficulties one might encounter. In particular, it relates $\mathbf{V}^{(n,k)}$ with a surprisingly difficult combinatorial problem, with the Taylor expansion of the exponential function, and also the Harmonic series.

Proposition 4.2.24. *Fixed $n, k \in \mathbb{N}_+$ and $\eta \in \Omega_N$, if $\langle \eta \rangle_M < \frac{n}{n+k+1}$ or $\eta(0) = 0$ then $\mathbf{V}^{(n,k)} = 0$. Otherwise, let $0 \leq \ell^\star \leq k$ be the largest integer such that there is some $A^\star \in \mathcal{P}_{n+\ell^\star}(M \backslash \{0, n+k+1\})$ where $\eta(A^\star) = 1$. Then*

$$\frac{\binom{n+k}{k}}{n+k+1}\mathbf{V}^{(n,k)}(\eta) = \sum_{A \in \mathcal{P}_n(A^\star)} s_M(\{0\} \sqcup A) \sum_{\ell=0}^{\ell^\star} \frac{(-1)^\ell}{\ell!} - \frac{1}{n+k+1} \sum_{\ell=0}^{\ell^\star} (-1)^\ell \binom{n+\ell}{\ell} \sum_{i=n+1}^{n+\ell} \frac{1}{i}.$$

Before proceeding with the proof we make some comments. It is simple to see that the summation $\sum_{\ell=0}^{\ell^\star}(-1)^\ell \binom{n+\ell}{\ell} \sum_{i=n+1}^{n+\ell} \frac{1}{i}$ takes positive values for $\ell^\star$ even, negative values for $\ell^\star$ odd, and is divergent in the limit $\ell^\star \to +\infty$ when restricted to either $\ell^\star$ even or odd. In this way, for $0 \leq \ell^\star \leq k$, it attains its minimum when $\ell^\star = k$ with k even. Moreover, the alternating sum $\sum_{\ell=0}^{\ell^\star}(-1)^\ell \binom{n+\ell}{\ell}$ has no simple closed-form expression, and is the "alternating version" of the well-known *Hockey-Stick Identity*, that can be expressed as

$$\sum_{\ell=0}^{\ell^\star} \binom{n+\ell}{\ell} = \binom{n+\ell^\star+1}{\ell^\star}.$$

Proof of Proposition 4.2.24. The cases where $\mathbf{V}^{(n,k)}(\eta) = 0$ as mentioned in the statement are very simple to see and so we omit the proof. For the remaining, we start by deriving a recurrence relation for the sequence $(c_\ell(\eta))_{0 \leq \ell \leq \ell^\star}$, where

$$c_\ell(\eta) = \sum_{A \in \mathcal{P}_{n+\ell}(M \backslash \{0\})} \eta(\{0\} \sqcup A) s_M(\{0, n+k+1\} \sqcup A).$$

Fixed some $\eta \in \Omega_N$, which leads to some $\ell^\star$ and $A^\star$, the former set generates a simple hierarchy that will be exploited bellow, by noticing that $\forall S \subset A^\star$ it holds that $\eta(S) = 1$.

We then have that $c_{\ell^\star}(\eta) = s_M(\{0\} \sqcup A^\star)$, while for $\ell^\star - 1 \geq 0$ we have

$$c_{\ell^\star-1}(\eta) = \sum_{A \in \mathcal{P}_{n+\ell^\star-1}(A^\star)} s_M(\{0\} \sqcup A)\eta(\{0\} \sqcup A) = (n + \ell^\star - 1) s_M(\{0\} \sqcup A^\star) + (1 - \frac{\max\{A^\star\} - 1}{n+k+1}).$$

The number $n + \ell^\star - 1$ is associated with each set resulting of removing one element of $A^\star$ that is not $\max\{A^\star\}$, while $1 - \frac{\max\{A^\star\}-1}{n+k+1}$ is associated with removing $\max\{A^\star\}$. Therefore,

$$c_{\ell^\star-1} = (n + \ell^\star) c_{\ell^\star} + \frac{1}{n+k+1}.$$

In particular, solving the recurrence relation above yields, for any $0 \leq \ell \leq \ell^\star$, the formula

$$c_\ell = c_0 \frac{n!}{(\ell+n)!} - \frac{1}{n+k+1} \sum_{i=n+1}^{n+\ell} \frac{1}{i}.$$

To conclude, we replace this into (4.26) and simplify the resulting expression.

□

The expression for $\mathbf{V}^{(n,k)}$ in (4.26) is to us very satisfying: independent of the labels of the sets in the definition of the model; involving a quantification of the "space" a subset of M occupies (through the map s_M); making evident the constant $(n+k+1)$ that will appear multiplied in the macroscopic diffusion coefficient $D(\rho) = (n+k+1)\rho^n(1-\rho)^k$; suggesting the multiplication of the polynomial $\rho^n(1-\rho)^k$ by the binomial coefficient $\binom{n+k}{k}$ as in the Bernstein Polynomial Basis; and its value involves an approximation of an irrational number ($\sum_{\ell=0}^{\ell^\star} \frac{(-1)^\ell}{\ell!} \approx e^{-1}$), the truncated Harmonic series ($\sum_{i=n+1}^{n+\ell} \frac{1}{i}$); and an alternating version of a known identity related to Pascal's triangle (Hockey-Stick Identity). While this is enough to prove the hydrodynamic limit for the PMM(n,k), it will be very useful to make evident the identification of $\mathbf{V}^{(n,k)}$ with a primitive of $D(\rho)$, which is the content of the next and last proposition.

Lemma 4.2.25. *For any $n,k \in \mathbb{N}_+$ and $0 \le \ell \le k$ fixed it holds that*

$$\sum_{A \in \mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\})} s_M(\{0\} \sqcup A) = \frac{1}{n+\ell+1}\binom{n+k}{n+\ell} \tag{4.29}$$

and, in particular, $\mathbf{V}^{(n,k)} = \mathbf{H}_0^{(n,k)} + \mathbf{H}^{(n,k)}$, where

$$\mathbf{H}^{(n,k)}(\eta) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{n+k+1}{n+\ell+1} \eta(\llbracket 0, n+\ell \rrbracket) \tag{4.30}$$

$$\mathbf{H}_0^{(n,k)}(\eta) = \frac{n+k+1}{\binom{n+k}{k}} \sum_{\ell=0}^{k} (-1)^\ell \binom{n+\ell}{\ell} \eta(0) \sum_{\substack{A \in \mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\}) \\ A=\{a_1,\dots,a_{n+\ell}\} \\ a_1<\cdots<a_{n+\ell}}} s_M(\{0\} \sqcup A) \sum_{y=1}^{n+\ell} (\eta(a_y) - \eta(y))\, \varphi_{y,A}^{\ell}(\eta).$$

and with

$$\varphi_{y,A}^{\ell}(\eta) = \left[\prod_{i=1}^{y-1} \eta(i)\right]\left[\prod_{j=y+1}^{n+\ell} \eta(a_j)\right].$$

Proof. From the definition of the rates it holds that $E_{\nu_\rho}[\mathbf{h}^{(n,k)}(\eta)] = (n+k+1)\rho^n(1-\rho)^k$, for ρ a constant profile, while for $\rho \in C^1(\mathbb{T})$,

$$\rho^n(1-\rho)^k\rho' = \left(\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{1}{n+\ell+1} \rho^{n+\ell+1}\right)',$$

leading directly to

$$\frac{1}{\binom{n+k}{k}}\binom{n+\ell}{\ell} \sum_{A \in \mathcal{P}_{n+\ell}(M\setminus\{0\})} s_M(\{0,n+k+1\} \sqcup A) = \binom{k}{\ell}\frac{1}{n+\ell+1}$$

which can be reorganized into (4.29).

Now let $A \in \mathcal{P}_{n+\ell}(M\setminus\{0,n+k+1\})$ and write its elements as $\{a_1,\dots,a_{n+\ell}\}$, with $a_i < a_{i+1}$ for any $1 \le i \le n+\ell-1$. We can rewrite

$$\eta(A) = \eta(\llbracket 1, n+\ell \rrbracket) + \sum_{y=1}^{n+\ell} \left[\prod_{i=1}^{y-1} \eta(i)\right](\eta(a_y) - \eta(y))\left[\prod_{j=y+1}^{n+\ell} \eta(a_j)\right].$$

Replacing the above and (4.29) into $\mathbf{V}^{(n,k)}$ yields (4.30).

□

With all of this, introducing $\mathbf{f}^{(n,k)} := \mathbf{h}_0^{(n,k)} + \mathbf{H}_0^{(n,k)}$, we conclude that

$$\mathbf{h}^{(n,k)} = \mathbf{H}^{(n,k)}(\eta) + \nabla^+ \mathbf{g}^{(n,k)} + \mathbf{f}^{(n,k)}(\eta), \tag{4.31}$$

where we recall the definition of $\mathbf{g}^{(n,k)}$ in (4.26). The reason for the distinction between all of the terms of $\mathbf{h}^{(n,k)}$, is that one can show that both $\mathbf{f}^{(n,k)}$ and $\nabla^+\mathbf{g}^{(n,k)}$ vanish in the limit $N \to +\infty$ – the former by arguing via replacement lemmas, the latter through an appropriate summation by parts and continuity argument – while $\mathbf{H}^{(n,k)}$ approximates a primitive of the diffusion coefficient.

4.2.3 Comments on the Hydrodynamic Limit

Regarding a proof of the hydrodynamic limit for the empirical measure (recall its introduction in Definition 3.2.20), one can study the macroscopic behaviour of the empirical measure induced by the *perturbed* PMM(n,k), defined through the generator $\mathcal{L} = N^2\mathcal{L}_N$, where

$$\mathcal{L}_N = N^{-a}\mathcal{L}_N^{\mathrm{SSEP}} + \mathcal{L}_N^{(n,k)} \quad \text{and} \quad a \in (0,1), \tag{4.32}$$

by following the very same scheme presented in detail in Chapter 3. We now explain the technical differences arising in the entropy method. The proof in Chapter 3, involving a superposition of Porous Media Models is, in fact, more evolved than the proof for a single, perturbed, PMM, which is technically very similar to the perturbed PMM(n,k). We recall that the difficulties in Chapter 3 were, in a large part, a consequence of considering a linear combination of models up to ℓ_N, with $\ell_N \xrightarrow{N\to+\infty} +\infty$, that led to difficulties in the proof of the replacement lemmas and in the convergence of the errors introduced by these lemmas. Since for $\mathcal{L}$ there is no linear combination of models (beyond the perturbation in (4.32)), all these issues are absent. For the PMM(n,k), one should follow the proof in the previous chapter, regarding the *slow-diffusion* regime. In this regime, the rates are bounded above by a constant independent of N, and for this reason no replacement lemmas are required to prove the tightness step (Subsection 3.3.1). Regarding the characterization of the limit points, Lemma 3.3.4 simplifies substantially: there is no need of fixing a sequence $(\epsilon_k)_{k\geq 1}$ since the "series" presented there now only has one term. In fact, technical differences should only arise in the application and proof of the replacement lemmas. In the previous chapter, we invoked recurrently Proposition 3.4.2 in the proof of the replacement lemmas. We recall that this proposition compares the Dirichlet form of the PMM$(m-1)$, for $m \in (1,2)$, with that of the process $\delta_N\mathcal{L}_N^{\mathrm{SSEP}} + \frac{m-1}{2}\mathcal{L}_N^{(1)}$, which was required for treating the linear combination of models. This is not needed for the perturbed PMM(n,k) and, except for the two-blocks estimate, one should be able to prove the replacements in the same fashion. Regarding the two-blocks estimate, the difference lies in the "path argument" in Lemma B.1.5. Precisely, for the PMM(n,k), the mobile clusters have a different structure than the mobile clusters of the PMM(1) – yet the main argument is analogous: instead of conditioning that there are at least 3 particles in the specific boxes of length L presented there (see (B.3)), in order to construct a mobile cluster and perform the transportation of mass in the system, for the PMM(n,k) one needs n particles and k holes to construct a mobile cluster, and in this manner one should condition to the presence of $n+1$ particles and k holes, and then repeat the argument presented there. The role of the SSEP perturbation is to mix the configuration in order to construct a mobile cluster. Naturally, the length L of the boxes needs to be adjusted accordingly.

Regarding the uniqueness of weak solutions (see Lemma B.2.3), the attentive reader can see that the expression for the potential in the previous chapter (that we recall to be ρ^m) plays no particular

role in the proof, and one only needs for it to be regular enough, which in that case meant that $\rho^m \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$. Indeed, it can be shown through an adaptation of the proof of Proposition 3.4.9, in Section 3.4, that for the (perturbed) PMM(n,k) it holds that $H_{n,k}(\rho) \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$, which is enough to prove the uniqueness of solutions. We recall that the properties of the model used in the proof of Proposition 3.4.9 were the gradient property and the fact that the rates are bounded above independently of N. The weak formulation of the hydrodynamic equation is as follows.

Definition 4.2.26. Let $\rho^{\text{ini}} : \mathbb{T} \to [0,1]$ be a measurable function and $\gamma > 0$ be fixed. We say that $\rho : [0,T] \times \mathbb{T} \to [0,1]$ is a weak solution of the differential equation

$$\begin{cases} \partial_t \rho_t(u) = \partial_u^2 H_{n,k}(\rho_t(u)), & (t,u) \in [0,T] \times \mathbb{T}, \\ \rho_0(u) = \rho^{\text{ini}}(u), & u \in \mathbb{T}, \end{cases}$$

if

1. $H_{n,k}(\rho) \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$;

2. For any $t \in [0,T]$ and $G \in C^{1,2}([0,T] \times \mathbb{T})$ it holds that

$$F_\gamma(\rho^{\text{ini}}, \rho, G, t) := \langle \rho_t, G_t \rangle - \langle \rho^{\text{ini}}, G_0 \rangle - \int_0^t \left\{ \langle \rho_s, \partial_s G_s \rangle + \langle H(\rho_s), \partial_u^2 G_s \rangle \mathrm{d}s \right\} \equiv 0.$$

The main novelty, and largest difficulty, is in fact the further characterization of the regularity of the solution ρ, that is required for both an open-boundary study [8] and a slow-barrier [6]. We showed in Corollary B.2.2 of Proposition B.2.1 that the macroscopic density associated with the PMM$(m-1)$ is $\frac{1}{4}$–Hölder continuous, for $m \in (1,2)$, through an adaptation of the proof in [5, Lemma 6.2]. For the PMM(n,k), the argument presented in Proposition B.2.1 seems too challenging to apply, since it was based on the explicit expression of the potential ρ^m, which for the PMM(n,k) is $H_{n,k}(\rho)$ and much more difficult to treat algebrically:

$$H_{n,k}(\rho) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{1}{n+\ell+1} \rho^{n+\ell+1}.$$

For this reason, we present in Appendix C.2 a generalization of the argument that can be adapted to other functions (see Lemma C.2.1 and the discussion just before it), which can be of independent interest. In particular, we prove in Corollary C.2.3 that for the PMM(n,k) it holds that ρ, as in Definition 4.2.26, is $\frac{1}{2(n+k+1)}$–Hölder continuous.

4.3 Long-range dynamics

4.3.1 Long-range basis

The goal of this section is to extend the PMM(n,k) into a long-jumps dynamics. Our approach will be different than the one in [7], used for generalizing the PMM(n) for any $n \in \mathbb{N}_+$, and for that reason we review briefly how it was there constructed. One starts, for example, with the PMM(1), where $\mathbf{c}^{(1)}(\tau_x \eta) = \eta(x-1) + \eta(x+2)$. This defines the rate for the exchange of occupations between the sites $x, x+1$. Noticing that $x+2 = (x+1)+1$, one can define an exchange between x and y as $\mathbf{c}^{(1)}_{x,y}(\eta) = \eta(x-1) + \eta(y+1)$. Note, however, that $\mathbf{r}^{(1)}_{x,y} \neq \mathbf{r}^{(1)}_{y,x}$. Therefore, in order to maintain the

symmetry of the jumps, one can define the constraint $\mathbf{r}^{(1)}_{x,y} = \mathbf{c}^{(1)}_{x,y} + \mathbf{c}^{(1)}_{y,x}$. This reasoning is then extended to any PMM(n), yielding for each $n \in \mathbb{N}_+$ a well-defined long-jumps version of the PMM(n). As it can be seen there, for each n as previously the model does satisfy the gradient condition. There is a technical subtlety regarding its relationship with its nearest-neighbour version, as explained in [7], which is that $\mathbf{r}^{(1)}_{x,x+1} \neq \mathbf{c}^{(1)} \circ \tau_x$, and it turns out that the model induced by $\mathbf{r}^{(1)}_{x,x+1}$ is irreducible. Precisely,

$$\mathbf{r}^{(1)}(\eta) = \eta(x-1) + (\eta(x) + \eta(x+1)) + \eta(x+2).$$

The previous approach for extending the PMM to a long-range dynamics, in our case, does not lead necessarily to a gradient model. In fact, repeating the arguments of the previous section, one obtains a different linear system characterizing the gradient condition and all the combinatoric arguments need to be adapted, since the sets arising after the application of the distributive rule, in the first step of the proof of Proposition 4.2.11, will have a different structure than the ones studied here.

In order to extend this nearest neighbour dynamics to a long-jumps dynamics we shall focus instead on the "distance of the jump", and not on the particular positions of the sites where the exchange of occupations will be performed. For the occupations of the sites $x, y \in \mathbb{T}_N$ to exchange, we are going to define a map $\mathbf{c}_r$, where r is the length of the shortest path between x, y. This map defines an occupation exchange of distance r, between the sites 0 and r, which is then translated into a reference point, either x or y, and the exchange is performed.

Definition 4.3.1. Define the distance $\tilde{d}$ on the torus as, for each $x, y \in \mathbb{T}_N$

$$\tilde{d}(x,y) = \min\{|y-x|, N - x \vee y + x \wedge y\}.$$

Moreover, define the map $(x,y) \mapsto x\tilde{\wedge}y$ as

$$x\tilde{\wedge}y = \begin{cases} x \wedge y, & |y-x| \leq \lfloor N/2 \rfloor, \\ x \vee y, & \text{otherwise.} \end{cases}$$

In this way, for each $x, y \in \mathbb{T}_N$ we want to construct a non-negative map $\mathbf{c}_{x,y} : \Omega_N \to \mathbb{R}_+ \cup \{0\}$ that must be symmetric, in the sense that $\mathbf{c}_{x,y} = \mathbf{c}_{y,x}$, and such that $\mathbf{c}_{x,x+1} = \mathbf{c} \circ \tau_x$. Parametrizing the constraint by $\tilde{d}(\cdot,\cdot)$, one has no option but to consider

$$\mathbf{c}_{x,y} := \mathbf{c}_{\tilde{d}(x,y)} \circ \tau_{x\tilde{\wedge}y},$$

where for each $r \in \mathbb{N}_+$ the map $\mathbf{c}_r : \Omega \to \mathbb{R}_+ \cup \{0\}$ is to be defined. The necessity of introducing $\tilde{\wedge}$ is motivated by reducing the problem to that of defining an exchange of "range" $\tilde{d}$. Concretely, it holds that $\tilde{d}(N-1, 1) = 2$, therefore, we want to apply an exchange with range 2: we fix $N-1$, and perform an exchange with two sites "next" to it: $N-1 \longleftrightarrow N-1+2 \equiv 1$. In this way, there is no fixed orientation on the torus, and a "jump" from $N-1$ to 1 does not have length $N-2$. Exchanges with a range larger then half the length of the torus do not formally occur, since there is a shorter path, with length no larger than $\lfloor N/2 \rfloor$, between the two sites. As such, there is no ambiguity between the *direction* of the jump.

We now explain our construction of $\mathbf{c}_r$. Consider the pair $(\mathbf{c}_1, \mathbb{T}_N)$, corresponding to the nearest neighbour constraint, and the corresponding lattice where it is defined. Then one can define a nearest

neighbour constraint in $r\mathbb{T}_N$, considering the pair $(\mathbf{c}_r, r\mathbb{T}_N)$, where $\mathbf{c}_r$ is simply $\mathbf{c}_1$ but defined in the lattice $r\mathbb{T}_N$, that we define as $r\mathbb{T}_N = \{rx : x \in \mathbb{T}_N\}$. Algebrically, the constraint in the new lattice corresponds to a constraint in the original lattice $\mathbb{T}_N$ with the transformation $\eta(x) \to \eta(rx)$ applied to each occupation variable. This rationale is represented in following diagram.

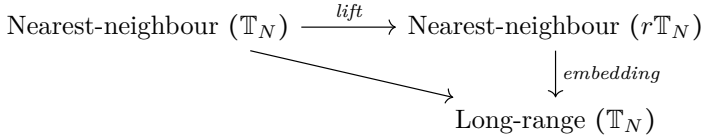


Figure 4.5: From nearest-neighbour to long-range.

Because each occupation variable is applied to a multiple r of the original site, each set involved in the constraint has the same structure – with its elements a multiple r of the original. In this way the gradient condition is naturally preserved (see Proposition 4.3.4).

This approach has the subtlety that if $x \in \mathbb{T}_N$ then $rx \in \mathbb{T}_N$ also. Note that this is also preserved in the lattice $\mathbb{Z}$. It is not the case, for example, for the lattice $\Sigma = [\![1, N]\!]$. In this case, one can define jumps "out of Σ" (see [32]), necessarily introducing artificial reservoirs.

One can formalize all this in the following way.

For any $X \in \mathcal{P}(\mathbb{T}_N)$ let $\mathbf{e}_X : \Omega_N \to \{0, 1\}$ be the dual map defined through

$$\mathbf{e}_X(\eta) = \eta(X).$$

It is simple to see that the collection $\{\mathbf{e}_X\}_{X \in \mathcal{P}(\mathbb{T}_N)}$ forms a basis for the space of functions from Ω_N to $\mathbb{R}$, and in this manner there are real constants $\{a_X\}_{X \in \mathcal{P}(\mathbb{T}_N)}$ such that

$$\mathbf{c} = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} a_X \mathbf{e}_X.$$

Notation 4.3.2. For each $r \in \mathbb{N}_+$ we write $\nabla_r^{\pm} = \pm(\tau_{\pm r} - \mathbf{1})$ and $\nabla_r^- \circ \nabla_r^+ = \Delta_r$. If $r = 1$ we omit the subscript. We also perform an abuse of notation and write, for any $u, v \in \mathbb{T}$ and $G : \mathbb{T} \to \mathbb{T}$, $\Delta_v G(u) = G(u + v) - 2G(u) + G(u - v)$, analogously to the discrete case.

Note that the operators $\nabla_r^{\pm}$ can be seen as the operators $\nabla^{\pm}$ defined on $r\mathbb{T}_N$, then embedded into $\mathbb{T}_N$.

Definition 4.3.3. For each $r \in \mathbb{N}_+$, define the linear operator $\upsilon^r \in \mathcal{L}(\Omega_N; \mathbb{R})$ through its action on the basis $\{\mathbf{e}_X\}_{X \in \mathcal{P}(\mathbb{T}_N)}$ as

$$\upsilon^r \mathbf{e}_X = \mathbf{e}_{rX},$$

and define the map $\mathbf{c}_r := \upsilon^r \mathbf{c}$, where

$$\upsilon^r \mathbf{c} = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} a_X \mathbf{e}_{rX}.$$

Proposition 4.3.4. *If there exists* $\mathbf{h} : \Omega_N \to \mathbb{R}$ *such that* $\mathbf{c}(\eta)(\nabla^+\eta)(0) = (\nabla^+\mathbf{h})(\eta)$ *then for every* $r \in \mathbb{N}_+$ *it holds that*

$$\mathbf{c}_r(\eta)(\nabla_r^+\eta)(0) = (\nabla_r^+ \upsilon^r \mathbf{h})(\eta).$$

Proof. To make the presentation more succinct, let us fix some label for the elements of $\mathcal{P}(\mathbb{T}_N)$ and define the vector $|\mathbf{e}\rangle$, whose i–th entry is given by $\mathbf{e}_X$ for some specific $X \in \mathcal{P}(\mathbb{T}_N)$ which can be identified from the choice of the labels. Define also the vector $|\upsilon^r(\mathbf{e})\rangle := \upsilon^r |\mathbf{e}\rangle$, which corresponds to the application of υ^r to each element of $|\mathbf{e}\rangle$.

It holds that

$$\mathbf{c}_r(\eta)(\eta(r) - \eta(0)) = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} a_X(\mathbf{e}_{rX \cup \{r\}} - \mathbf{e}_{rX \cup \{0\}}). \tag{4.33}$$

Likewise, because $1 + X \in \mathcal{P}(\mathbb{T}_N)$, there are constants $\{h_X\}_{X \in \mathcal{P}(\mathbb{T}_N)}$ and $\{g^h_X\}_{X \in \mathcal{P}(\mathbb{T}_N)}$ such that

$$\nabla^+ \mathbf{h} = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} h_X(\mathbf{e}_{1+X} - \mathbf{e}_X) = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} g^h_X \mathbf{e}_X =: G^h |\mathbf{e}\rangle$$

and because $X \cup \{x\} \in \mathcal{P}(\mathbb{T}_N)$, for any $x \in \mathbb{N}$, one can also construct a matrix B such that

$$B |\mathbf{e}\rangle = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} a_X(\mathbf{e}_{X \cup \{1\}} - \mathbf{e}_{X \cup \{0\}}) = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} b_X \mathbf{e}_X.$$

Note that there is a correspondence with b_X and a linear combination of a_X coefficients induced by the identification of $X \cup \{0\}, X \cup \{1\}$ with elements of $\mathcal{P}(\mathbb{T}_N)$. In other words, there exists a non-necessarily invertible matrix L such that $LA = B$. This is akin to the "b–coefficients" being related to the "a–coefficients", in Proposition 4.2.11. In particular,

$$LA = G^h.$$

Having in mind that there is a one-to-one correspondence between X and rX given by the scalar multiplication r, one can express

$$(\nabla^+_r \upsilon^r \mathbf{h})(\eta) = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} h_X(\mathbf{e}_{r(1+X)} - \mathbf{e}_{rX}) = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} g^h_X \mathbf{e}_{rX} = G^h |\upsilon^r(\mathbf{e})\rangle.$$

As such,

$$G^h |\upsilon^r(\mathbf{e})\rangle = LA |\upsilon^r(\mathbf{e})\rangle = \sum_{X \in \mathcal{P}(\mathbb{T}_N)} a_X(\mathbf{e}_{r(X \cup \{1\})} - \mathbf{e}_{r(X \cup \{0\})})$$

which equals (4.33).

□

We have now all the ingredients to define the long-jump version of a nearest neighbour dynamics, in such a way that it inherits the gradient property.

Definition 4.3.5 (Long-Range PMM(n,k))**.** For $n, k \in \mathbb{N}_+$ fixed, define, for all $x, y \in \mathbb{N}$, the constraint

$$\mathbf{c}^{(n,k)}_{x,y} = \upsilon^{\tilde{d}(x,y)} \mathbf{c}^{(n,k)} \circ \tau_{x \tilde{\wedge} y}.$$

For each $1 \le r \le \lfloor N/2 \rfloor$ define the Markov generator $\mathcal{L}^{(n,k):r}_N$, as acting on functions $f : \Omega_N \to \mathbb{R}$ through

$$\mathcal{L}^{(n,k):r}_N f(\eta) = \sum_{x,y \in \mathbb{T}_N} \mathbf{1}_{\{\tilde{d}(x,y)=r\}} \frac{1}{2} \mathbf{c}^{(n,k)}_{x,y}(\eta)(\nabla_{x,y} f)(\eta). \tag{4.34}$$

For each r as previously we refer to this process as *PMM of range* r, or PMM$_r(n,k)$.

Note, in the previous definition, the factor 1/2. This is due to each exchange being counted twice: for example, a jump from $0 \to r$ and from $r \to 0$. It is also very convenient to recast (4.34) in a different form. Introduce the jump "backwards" of length r, that is,

$$\upsilon^{-r}\mathbf{c}^{(n,k)} := \upsilon^{r}\mathbf{c}^{(n,k)} \circ \tau_{-2r}$$

and from now on short-write $\mathbf{c}_r^{(n,k)} \equiv \upsilon^r \mathbf{c}^{(n,k)}$. Then one can write

$$(\mathcal{L}_N^{(n,k):r} f)(\eta) = \frac{1}{2}\sum_{x\in\mathbb{T}_N} \mathbf{c}_{-r}^{(n,k)}(\tau_x\eta)(\nabla_{x,x-r}f)(\eta) + \frac{1}{2}\sum_{x\in\mathbb{T}_N} \mathbf{c}_{r}^{(n,k)}(\tau_x\eta)(\nabla_{x,x+r}f)(\eta).$$

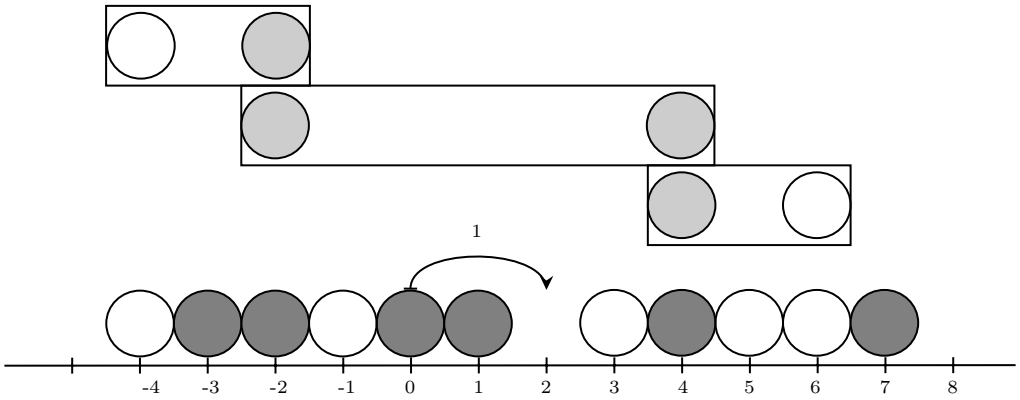


Figure 4.6: $\mathrm{PMM}_2(2,1)$ rate.

In the previous figure, the rectangles correspond to the windows $2 \times W_j$, for $j = 0, 1, 2$.

4.3.2 The fractional process

The "long-jumps version" of a nearest-neighbour exclusion process can be seen as a basis for fractional operators, in the sense that we explain shortly. Given that we presented in the previous section an general extension of a symmetric, exclusion nearest-neighbour dynamics into a long range dynamics, we take the opportunity to explain how one can consider linear combinations or long-range symmetric exclusion processes *in the torus* in order to derive a fractional partial differential equation. We first introduce a fractional Sobolev space and Laplacian on $\mathbb{T}$, in the same spirit of [14] (where it was defined in $\mathbb{R}$), and understand the scaling limit of the family of Markov chains associated to it. We start with some motivation. If $\mathcal{L}^r$ is the Markov generator induced by a symmetric long-range constraint $\mathbf{c}_r$ that defines a jump of length r and satisfies the gradient condition, as in Definition 4.3.3, then one finds that

$$(\mathcal{L}^r\eta)(0) = (\Delta_r \mathbf{h})(\eta) + (\Delta_{-r}\mathbf{h})(\eta)$$

with $\Delta_{\pm r}$ as in Notation 4.3.2. We now introduce a key operator.

Definition 4.3.6. Let $\gamma \geq 0$ and introduce the operator $\mathbb{L}_{\gamma,N}$ acting on functions $G : \mathbb{T} \to \mathbb{R}$ defined as

$$\mathbb{L}_{\gamma,N} = \sum_{|r|=1}^{\lfloor N/2 \rfloor} \frac{1}{|r|^{1+\gamma}} \Delta_{\frac{r}{N}}.$$

Considering the process given by the (rescaled) superposition of long-range dynamics

$$\sum_{r=1}^{\lfloor N/2 \rfloor} \frac{1}{r^{1+\gamma}} \mathcal{L}_N^r,$$

from the gradient property one finds that

$$\mathcal{L}_N^r \langle \pi^N(\eta), G \rangle = \langle \mathbb{L}_{\gamma,N} G, \mathbf{h}(\eta) \rangle,$$

with $\mathbf{h}$ the potential associated with the nearest-neighbour dynamics, arising as consequence of the gradient property. In this way, identifying the correct time-scale $\Theta_\gamma(N)$ and the correct space of test functions, for $\|\mathbf{h}\|_\infty < c$ with $c > 0$ a constant independent of N one should be able to argue that $\Theta_\gamma(N)\mathbb{L}_{\gamma,N} \xrightarrow{N\to+\infty} \mathbb{L}_\gamma$ in a sense that we will explain shortly, and where $\mathbb{L}_\gamma$ is an operator acting on functions in $\mathbb{T}$.

Definition 4.3.7. We extend the distance $\tilde{d}$ on $\mathbb{T}_N$, in Definition 4.3.1, to the continuous torus $\mathbb{T}$. We will use the same notation and differentiate between them whenever it is not clear from the context. Concretely, for any $u, v \in \mathbb{T}$ let

$$\tilde{d}(u,v) = \min\{|u-v|\,,\ 1 - u \vee v + u \wedge v\}.$$

If $v = 0$ we write simply $|u|_{\mathbb{T}} = \tilde{d}(u,0)$.

Definition 4.3.8 (Fractional Sobolev space $\mathcal{H}^{\gamma/2}(\mathbb{T})$)**.** Fixed $0 < \gamma < 2$ define the Hilbert space

$$\mathcal{H}^{\gamma/2}(\mathbb{T}) = \left\{ G \in L^2(\mathbb{T}) : \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|G(u) - G(v)|^2}{\tilde{d}(u,v)^{1+\gamma}} \mathrm{d}u\mathrm{d}v < \infty \right\},$$

endowed with the norm $\|\cdot\|_{\gamma/2}$ defined as

$$\|G\|^2_{\mathcal{H}^{\gamma/2}} := \|G\|_2^2 + [G]^2_{\gamma/2} \quad \text{where} \quad [G]^2_{\gamma/2} = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|G(u) - G(v)|^2}{\tilde{d}(u,v)^{1+\gamma}} \mathrm{d}u\mathrm{d}v.$$

The space $\mathcal{H}^{\gamma/2}$ can be extended to $\gamma \geq 2$, in that case coinciding with functions in $\mathcal{H}^{\lfloor \gamma/2 \rfloor}$ whose $\lfloor \gamma/2 \rfloor$–th order distributional derivative belongs to $\mathcal{H}^{\gamma/2 - \lfloor \gamma/2 \rfloor}$. Other properties of this space, as Sobolev embeddings and dense subspaces can be found in [14, Section 2]. Its definition is closely related to a fractional Laplacian operator, as in the next definition.

Definition 4.3.9. For each $\gamma \in (0,2)$ define the fractional Laplacian operator $\mathbb{L}_\gamma := -(-\Delta)^{\gamma/2}$ acting on functions $G : \mathbb{T} \to \mathbb{R}$ as

$$\mathbb{L}_\gamma G(u) := \frac{1}{c_\gamma} \int_{\mathbb{T}} \frac{G(u+v) - 2G(u) + G(u-v)}{|v|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v \quad \text{where} \quad c_\gamma = 2 \lim_{N\to+\infty} \frac{N^\gamma}{N^2} \sum_{r=1}^{N} \frac{r}{r^\gamma} > 0$$

whenever $\|\mathbb{L}_\gamma G\|_\infty < \infty$.

Modulo the space where it is defined and the multiplicative constant c_γ, the fractional Laplacian in the previous definition can be shown to be equivalent to the one in [14, Section 3] through the equality

$$\lim_{\epsilon \to 0} \int_{(B_\epsilon(u))^c} \frac{G(v) - G(u)}{\tilde{d}(v,u)^{1+\gamma}} \mathrm{d}v = \frac{1}{2} \int_{\mathbb{T}} \frac{1}{|v|_{\mathbb{T}}} \Delta_v G(u) \mathrm{d}v$$

which can be proved through an adaptation of the proof of [14, Lemma 3.2]. The constant c_γ was chosen so that the interpolation property in Proposition 4.3.11 is satisfied. This has consequences in the particular scaling limit of the Markov chains that we are going to define next.

Definition 4.3.10. For each $\gamma > 0$ introduce the scaling $\Theta_\gamma \equiv \Theta_\gamma(N)$ given by

$$\Theta_\gamma(N) := \frac{1}{2}\frac{1}{\sum_{r=1}^{N} r^{-(\gamma-1)}} N^2.$$

Proposition 4.3.11. *Let $G : \mathbb{T} \to \mathbb{R}$ and $x \in \mathbb{T}_N$ be arbitrary. It holds that*

(1) $\lim_{N\to+\infty} \left|\Theta_\gamma(N)\mathbb{L}_{\gamma,N}G(\frac{x}{N}) - \mathbb{L}_\gamma G(\frac{x}{N})\right| = 0$ *for* $0 < \gamma < 2$ *and* $G \in C^1(\mathbb{T})$ *such that* $\|\mathbb{L}_\gamma G\|_\infty < \infty$*;*

(2) $\lim_{N\to+\infty} \left|\Theta_\gamma(N)\mathbb{L}_{\gamma,N}G(\frac{x}{N}) - G''(\frac{x}{N})\right|_\infty = 0$*, for* $\gamma \geq 2$ *and* $G \in C^2(\mathbb{T})$*.*

The proof of the previous proposition can be found in Appendix C.3. As a corollary, one can show the next continuity result.

Corollary 4.3.12. *For any $G : \mathbb{T} \to \mathbb{R}$ such that $G \in C^2(\mathbb{T})$ and $\|\mathbb{L}_\gamma G\|_\infty < \infty$, for any $u \in \mathbb{T}$ it holds that*

$$\lim_{\gamma\to 2^-} |\mathbb{L}_\gamma G(u) - G''(u)| = 0.$$

Proof. It is enough to estimate

$$|\mathbb{L}_\gamma G(u) - G''(u)| \leq |\mathbb{L}_\gamma G(u) - \Theta_\gamma \mathbb{L}_{\gamma,N} G(u)| + |\Theta_\gamma \mathbb{L}_{\gamma,N} G(u) - \Theta_2 \mathbb{L}_{2,N} G(u)| + |\Theta_2 \mathbb{L}_{2,N} G(u) - G''(u)|.$$

The first term in the right-hand side is estimated as in the case $0 < \gamma < 2$ in the previous proposition, then one takes the limit $\gamma \to 2^-$ and $N \to +\infty$. For the second one, since N is fixed it is enough to take the limit $\gamma \to 2^-$. The third one is consequence of the previous proposition. □

Remark 4.3.13 (On the scaling)**.** The results available in the literature (see for example [2, 7, 25] and references therein) present the scaling $N^\gamma \mathbf{1}_{0<\gamma<2} + \frac{N^2}{\log(N)}\mathbf{1}_{\gamma=2} + N^2\mathbf{1}_{\gamma>2}$, which shows a discontinuity in $\gamma = 2$. As seen in the proof of the previous proposition, our scaling Θ_γ is *equivalent* to the one just presented, in the sense that $\Theta_\gamma(N) = O(N^\gamma)\mathbf{1}_{0<\gamma<2} + O(N^2/\log(N))\mathbf{1}_{\gamma=2} + O(N^2)\mathbf{1}_{\gamma>2}$. The reason for this subtle difference originates from the definition of $\mathbb{L}_\gamma$ – specifically, with the multiplicative constant $1/c_\gamma$. We believe that the scaling Θ_γ is the most natural one that guarantees the interpolation property relating $\mathbb{L}_\gamma$ and ∂_u^2, while providing a clear explanation of the time-scale commonly provided in the literature. We find also worth noting that the multiplicative factor 2 in the constant c_γ (see Definition 4.3.9) is consequence of our imposition that the jumps can have (formally) length up to $\lfloor N/2 \rfloor$, microscopically, or $1/2$, macroscopically. Indeed, it is simple to see that

$$\int_{\mathbb{T}} \frac{\Delta_v G(u)}{|v|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v = 2\int_{\mathbb{T}\cap[0,\frac{1}{2}]} \frac{\Delta_v G(u)}{|v|^{1+\gamma}} \mathrm{d}v.$$

It is also interesting to note that if one considers the summation over r up to some ℓ_N such that $\ell_N = o(N)$, then

$$\sum_{|r|=1}^{\ell_N} \frac{1}{|r|^{1+\gamma}} \Delta_{\frac{r}{N}} G(\tfrac{x}{N}) = G''(\tfrac{x}{N}) \frac{1}{N^2} \sum_{|r|=1}^{\ell_N} \frac{1}{|r|^{\gamma-1}} + O\left(\frac{1}{\ell_N^\gamma}\left(\frac{\ell_N}{N}\right)^4\right),$$

and for this reason the time-scale $N^2/\sum_{|r|=1}^{\ell_N} |r|^{-(\gamma-1)}$ should be the correct scale in order to derive the classical Laplacian, for any $\gamma > 0$.

We conclude this chapter with a last definition. As explained previously, in order to "construct" the operator $\mathbb{L}_{\gamma,N}$ starting from the collection $\{\mathrm{PMM}_r(n,k)\}_r$ one should consider the process defined through the next generator.

Definition 4.3.14. We call *fractional* porous media model the process defined through the generator

$$\sum_{r=1}^{\lfloor N/2 \rfloor} \frac{1}{r^{1+\gamma}} \mathcal{L}_N^{(n,k):r}.$$

We expect this process to lead, under the $\Theta_\gamma(N)$–time scale, to the hydrodynamic equation

$$\begin{cases} \partial_t \rho = \mathbb{L}_\gamma H_{n,k}(\rho), & 0 < \gamma < 2, \\ \partial_t \rho = \partial_u^2 H_{n,k}(\rho), & \gamma \geq 2. \end{cases}$$

5

Conclusions and Future Work

The works presented in this thesis open a few questions, some of which are currently in development. Regarding Chapter 2, it is not clear if when the assumption (H_0) is not satisfied it is still possible to construct a matrix representation, or even if in this case the resulting models are interesting in some sense. We recall that (H_0) arises in two different settings, yet related to the same goal: (1) In the matrix formulation, so that we can express the relations in terms of $\langle W|\, D^- C$, $\langle W|\, C^- D$ and $\langle W|\, D^- D$, in this way obtaining indirectly equations involving the density in the first and second sites and the correlation between the reservoirs. As we saw, this facilitates the study of the consistency of the algebra in an intuitive way; (2) and in the stationary equations, so that we can express those very same quantities in terms of the current, obtaining, in particular, an equation for the correlation that allows us to better understand the consistency constraints. Very recently, we were able to consider a general change of basis that allows for introducing more parameters into the formulation, with a particular choice corresponding to the one presented in this thesis. We are currently trying to understand if this approach relaxes the constraints. With this same goal of reducing the constraints, it is natural to ask if considering different matrices at more sites would lead to better results. We expect that not to be true since the expected (stationary) current in the bulk is node-independent, and for that reason any relation involving bulk matrices must also hold at all the bulk sites. This suggests that one should look for a different *bulk* algebra. For the moment, it is not clear if the identity $[D, E] = D + E$ is the unique "solution" to the telescopic relations, or if a different cancellation mechanism (in order to guarantee the stationarity condition) should be formulated.

Regarding direct applications of our formulation, one can follow a similar path as that for the open SSEP with the classical reservoirs [12, 13], and study the effect of the reaction-diffusion processes on the stationary density large deviations' when coupled with slow/fast reservoirs. By slow/fast reservoirs we mean that the boundary rates are scaled by a factor $1/N^\theta$ with $\theta \in \mathbb{R}$ – in this way being *slow* when $\theta > 0$ and *fast* when $\theta < 0$. In our case, however, this scaling must be made with some care due to the non-linearity of the constraints (C_1) and (C_2). To be precise, multiplying all the rates by the aforementioned factor leads to the constraints being dependent of N, and for that reason one should consider instead a particular factor $1/N^{\theta_{ij}}$ associated with each rate a_{ij}, and then find the values for θ_{ij} so that the constraints becomes independent of N. We note that no aspect of our formulation needs to be adapted in this case.

We focus now on Chapter 3. As already mentioned throughout this manuscript, the extension of the model to $m > 2$ requires the modelling of the diffusion coefficient in Chapter 4. To finally be concrete with respect to this extension, the starting point is the binomial expansion

$$\rho^{m-1} = \sum_{k\geq 0} \binom{m - \lfloor m \rfloor}{k} (-1)^k \rho^{\lfloor m-1 \rfloor} (1-\rho)^k,$$

and for this reason there is the need to model the diffusion coefficient $D_{\lfloor m \rfloor, k}(\rho) = \rho^{\lfloor m \rfloor}(1-\rho)^k$. Indeed, the nearest-neighbour model introduced in Chapter 4 will allow us to model $D_{\lfloor m \rfloor, k}(\rho)$ and to derive the PME for any $m > 2$, with the proof being very similar to the case $m \in (1, 2)$. It is important to note that the resulting model will interpolate between the PMM($\lfloor m \rfloor$) and PMM($\lceil m \rceil$), consequently being *not* irreducible and a perturbation must be added in order to follow the entropy method – or, alternatively,

one can impose that the initial data ρ^{ini} does not equal 0 or 1 anywhere, and since a L^1 contraction principle holds for the PME with $m > 2$, one can follow the relative entropy method as in [21]. Regarding future works related Chapter 3, in a collaboration with Pedro Cardoso we are currently in the final stages of adapting the ideas of that chapter to the long-range setting in the lattice $\mathbb{Z}$, with a slow barrier at the origin (in the same spirit as in [6]), but considering instead a linear combination of Porous Media Models with general coefficients that can differ from the generalized binomials presented here.

An open problem is that of deriving the complete *fast* diffusion regime $m < 0$. A well-defined model can be constructed using both the geometric series and the generalized binomial expansion, but because in this case the diffusion coefficient can be infinite and the solution of the hydrodynamic equation blows-up in finite time, several problems arise in the entropy method, which cannot be resolved by specifying the order of the explosion ℓ_N. We believe that in this case the relative entropy method might be more appropriate, specifically adapting the approach in [24] to the exclusion setting, but for the moment we are not sure and more work needs to be done.

Regarding the work in Chapter 4, there is a technical subtlety in the definition of the model that suggests that the uniform solution does not lead to the "true" generalization of the PMM to $m > 2$, which is that the resulting model will lose the monotonicity property stated in Proposition 3.2.19 due to the sequence $\{\frac{1}{n+k+1}c^{(n,k)}\}_{k\geq 1}$ not being monotone decreasing for $n \geq 1$. We find this property to be important to have, since it corresponds to the microscopic version of $\{\rho^n(1-\rho)^k\}_{k\geq 1}$ being decreasing (for $\rho \neq 0, 1$). In order to maintain this monotonicity, one should look for a specific non-uniform solution of the linear system (4.13). Ideally, this solution should be general in some form, with the weights following some specific recurrence relation involving n and k, but the model might not be algebrically as simple as the one presented in this thesis. Our starting point is to look for non-uniform solutions in the class presented in Proposition 4.2.23, where even though the solution is non-uniform, the potential is expressed as in the uniform case. This work is currently in development.

Another, already mentioned, application of the work in Chapter 4 is the construction of a (microscopic) analogue of the Bernstein polynomial basis:

$$\mathcal{L}_N := \frac{1}{\ell_N + 1} \sum_{n=0}^{\ell_N} \beta_n \binom{\ell_N}{n} \mathcal{L}_N^{(n,\ell_N - n)} \tag{5.1}$$

where $\{\beta_n\}_{n\geq 0}$ are the Bézier coefficients. The constraints resulting from the generator $\binom{\ell_N}{n}\mathcal{L}_N^{(n,\ell_N-n)}$ satisfy some interesting properties of the Bernstein basis, such as symmetry and partition of the unit (which was already proved in Proposition 4.2.15), but it does *not* satisfy its characteristic recurrence relation. The reason for this is that the normalized constraints $\mathbf{p}^{(n,k)} := \frac{1}{n+k+1}\mathbf{c}^{(n,k)}$ do not satisfy

$$\mathbf{p}^{(n,k+1)} + \mathbf{p}^{(n+1,k)} = \mathbf{p}^{(n,k)}.$$

We are confident that the proper generalization of the PMM that maintains the aforementioned monotonicity property should satisfy the recurrence above, which serves as a motivation for our guess that the desired particular non-uniform solution of (4.13) should follow some general recurrence relation. Nevertheless, even without the satisfaction of this recurrence relation, a proof of the hydrodynamic limit for the empirical measure induced by (5.1) is very challenging, with most of the difficulties arising due to the

complexity of the potential $\mathbf{h}^{(n,k)}$, as expressed in (4.31):

$$\mathbf{h}^{(n,k)} = \mathbf{H}^{(n,k)}(\eta) + \nabla^{+}\mathbf{g}^{(n,k)} + \mathbf{f}^{(n,k)}(\eta).$$

Above, the dominating term is $\mathbf{H}^{(n,k)}$, and the remaining terms converge to zero in probability, in the limit $N \to +\infty$. Considering linear combinations of the potential, as in (5.1), adds another layer of difficulty compared to the work in Chapter 3 because both $\mathbf{g}^{(n,k)}$ and $\mathbf{f}^{(n,k)}$ are expressed as an alternating sum that is not absolutely convergent, hence the approximation errors (as in the replacement lemmas) do not converge either. Because the expressions of those quantities are too complex to be studied through combinatorial means, this requires not only very sharp estimates but possibly new tools.

The works in Chapter 4 and 3 were developed on the one dimensional torus, hence the total number of particles is a conserved quantity of the system. It is possible to extend our results to the open boundary setting, with appropriate boundary dynamics. In fact, the work in Chapter 3 was developed with this in mind, as we provided all the necessary ingredients with respect to the bulk dynamics for such an extension to be performed. Following [5, 17], it is possible to derive Robin, Dirichlet and Neumann boundary conditions in the slow-diffusion range $m \in (1, 2)$, while Robin and Neumann in the fast-diffusion range $m \in (0, 1)$. In the fast diffusion regime, since the rates can blow-up as $N \to +\infty$, the standard procedure in the literature does not allow us to derive Dirichlet boundary conditions. This turns out to be, in nature, a problem very similar to the derivation of Dirichlet boundary conditions for the case of $K \geq 2$ in [17]. There, the authors couple the SSEP with boundary dynamics inspired by [29], and where a particle is created in the first empty site and removed from the first occupied site, in a window of length K at the boundary. For $K \geq 2$ the choice of a Lipschitz reference profile is not enough to argue that the occupation at the boundary is approximately constant, and the analysis of some correlation terms are required. This led to the work developed in [16], where the authors use very different tools, involving the study of the 2−point correlation field and duality arguments to treat solely the non-linear case $K = 2$. However, our model is too different in nature to argue as in [16] and a simple dual process is not available, and for that reason a different approach must be taken.

Unfortunately, we had to leave this and many other interesting problems for future works, as there is a lot to do but time, existing, seems to run quite fast.

A

Matrix Product Ansatz for the SSEP with extended boundary.

Contents

A.1 Coefficients in the boundary algebra

We consider the left-boundary coefficients only. For the right-boundary one needs to replace the superscript $^-$ by $^+$ and a_{ij} by b_{ij}. Let $i = 1, 2, 3$. Recall that $q_i^- = d_i^- - t_i^-$. We have

$$d_i^- = \frac{\tilde{d}_i^-}{\det(\mathcal{A}_1^-)}, \quad t_i^- = \frac{\tilde{t}_i^-}{\det(\mathcal{A}_1^-)}, \quad f_i^- = \frac{\tilde{f}_i^-}{\det(\mathcal{A}_1^-)},$$

with $\mathcal{A}_1^-$ is as in (2.16):

$$\mathcal{A}_1^- = \begin{pmatrix} a_{12}+a_{13}+2(a_{21}+a_{31}+a_{41}) & -a_{12}-a_{21}-a_{31}-a_{41} & -a_{12}-a_{13}+a_{14}-a_{21}-a_{31}-a_{41} \\ -a_{12}-2a_{21}+a_{23}-a_{32}-a_{42} & a_{12}+a_{21}+a_{32}+a_{42} & a_{12}+a_{21}-a_{23}+a_{24}+a_{32}+a_{42} \\ -a_{13}-a_{23}-2a_{31}+a_{32}-a_{43} & a_{31}-a_{32} & a_{13}+a_{23}+a_{31}-a_{32}+a_{34}+a_{43} \end{pmatrix}.$$

The numerators are given by

$$\begin{aligned}
\tilde{d}_1^- &= (a_{13}+a_{24}+a_{34}+a_{43})a_{32}a_{41} + (a_{14}+a_{23}+a_{34}+a_{43})a_{31}a_{42} \\
&\quad + (a_{14}+a_{24}+a_{34}+a_{43})(a_{31}(a_{12}+a_{32})+a_{21}a_{32}) + (a_{13}+a_{23}+a_{34}+a_{43})(a_{41}(a_{12}+a_{42})+a_{21}a_{42}), \\
\tilde{d}_2^- &= (-a_{13}+a_{14}-a_{23}+a_{24})(a_{42}-a_{41}) + (a_{31}-a_{32}+a_{41}-a_{42})(a_{14}+a_{24}+a_{34}+a_{43}), \\
\tilde{d}_3^- &= (-a_{13}+a_{14}-a_{23}+a_{24})(a_{12}+a_{21}+a_{32}+a_{41}) - (a_{31}-a_{32}+a_{41}-a_{42})(a_{14}+a_{23}+a_{34}+a_{43}),
\end{aligned}$$

$$\begin{aligned}
\tilde{t}_1^- &= a_{14}\{a_{31}(a_{12}-a_{23}+a_{32}+a_{42}) - a_{21}(a_{13}+a_{23}-a_{32}+a_{43})\} \\
&\quad + a_{24}\{a_{31}(a_{12}+a_{13}) - (a_{21}+a_{31}+a_{41})(a_{13}+a_{23}-a_{32}+a_{43})\} \\
&\quad + a_{34}\{(a_{21}+a_{31}+a_{41})(a_{12}-a_{23}+a_{32}+a_{42}) - a_{21}(a_{12}+a_{13})\}, \\
\tilde{t}_2^- &= (a_{14}+a_{24}+a_{34}+a_{41})(a_{13}+a_{23}+2a_{31}-a_{32}+a_{43}) - (a_{31}-a_{34})(2a_{41}-a_{42}-a_{43}), \\
\tilde{t}_3^- &= (a_{14}+a_{24}+a_{31}+a_{41})(a_{12}-2a_{14}-a_{23}+a_{32}-2a_{34}-a_{43}) \\
&\quad - (a_{14}+a_{21}+a_{34}+a_{41})(a_{13}-2a_{14}+a_{23}-2a_{24}-a_{32}-a_{42}),
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_1^- &= a_{12}(a_{41}(a_{13}+a_{23}) + a_{43}(a_{31}+a_{41})) + a_{13}(a_{42}(a_{21}+a_{41}) + a_{32}a_{41}) \\
&\quad + (a_{21}+a_{31}+a_{41})(a_{23}a_{42} + a_{43}(a_{32}+a_{42})), \\
\tilde{f}_2^- &= (a_{41}-a_{42})(a_{13}+a_{23}+a_{32}+a_{43}) - (a_{31}-a_{32})(a_{42}-a_{43}), \\
\tilde{f}_3^- &= (a_{12}+a_{23}+a_{32}+a_{43})(-a_{31}+a_{32}-a_{41}+a_{42}) - (a_{12}+a_{21}+a_{32}+a_{41})(a_{13}+a_{23}+a_{32}+a_{42}).
\end{aligned}$$

Regarding the denominator, one finds that

$$\begin{aligned}
\det(\mathcal{A}_1) &= \tilde{d}_1^- + (a_{12}+a_{21}+a_{32}+a_{41})a_{13}a_{24} + (a_{12}+a_{21}+a_{31}+a_{42})a_{14}a_{23} \\
&\quad + (a_{12}+a_{21}+a_{31}+a_{41})(a_{23}(a_{24}+a_{34}) + a_{24}a_{43}) \\
&\quad + (a_{12}+a_{21}+a_{32}+a_{42})(a_{13}(a_{14}+a_{34}) + a_{14}a_{43}).
\end{aligned}$$

It can also be checked by substitution that the coefficients associated with the reservoirs' correlation, f_1^-, f_2^- and f_3^-, can be recast as linear combinations of the remaining coefficients of the algebra

$$\begin{aligned}
f_1^-\alpha_0 &= d_1^-(a_{43}-a_{41}) + q_1^-(a_{42}-a_{41}) + a_{41}, \\
f_2^-\alpha_0 &= d_2^-(a_{43}-a_{41}) + q_2^-(a_{42}-a_{41}), \\
f_3^-\alpha_0 &= d_3^-(a_{43}-a_{41}) + q_3^-(a_{42}-a_{41}) - 1,
\end{aligned} \tag{A.1}$$

where

$$\alpha_0 = a_{14} + a_{24} + a_{34} + a_{42} + a_{43} - a_{41}.$$

Additionally, the remaining coefficients can be expressed in terms of the coefficients A_i, B_i, C_i, D_i with $i = 1, 2$ as in (2.10). Specifically, for linear models (as introduced in Definition 2.4.2), one sees that

$$d_1^- = \frac{A_1}{C_1} + \frac{B_1}{C_1} q_1^- \quad \text{and} \quad d_2^- = \frac{B_1}{C_1} q_2^-,$$

where we recall that in this case $q_3^-, d_3^- = 0$; while for non-linear models (also as in Definition 2.4.2),

$$d_1^- \alpha_1 = \left(\frac{A_1}{D_1} - \frac{A_2}{D_2}\right) + q_1^- \alpha_2, \quad d_2^- \alpha_1 = \left(\frac{1}{D_2}\right) + q_2^- \alpha_2 \quad \text{and} \quad d_3^- \alpha_1 = q_3^- \alpha_2$$

where

$$\alpha_1 = \frac{C_1}{D_1} + \frac{B_2}{D_2} \quad \text{and} \quad \alpha_2 = \frac{C_2}{D_2} + \frac{B_1}{D_1}.$$

A.2 Product Measures

It is expected that letting $x = 0$, the MPA formulation expresses the equilibrium state. This is indeed the case: for $x = 0$ the bulk relation is $[D, E] = 0$, and it can be checked that the constraints (C_1) and (C_2) both reduce to

$$f_1^\pm = d_1^\pm q_1^\pm. \tag{A.2}$$

Moreover, $[D, E] = 0$ leads to $\langle \eta(2) \rangle_N^w = \langle \eta(N-1) \rangle_N^w$, that is,

$$q_1^- = q_1^+.$$

The probability of observing a fixed configuration can be computed. Fix the length of the system, $N \geq 4$, and let η be any configuration with $n \geq 0$ particles in the bulk. Since D and E commute,

$$\begin{aligned}
\mu_N^{ss}(\eta) &= \frac{1}{Z_N} \langle W | \left(D^- \eta(1) + E^-(1 - \eta(1))\right) D^n E^{N-2-n} \left(D^+ \eta(N) + E^+(1 - \eta(N))\right) |V\rangle \\
&= \frac{1}{Z_N} \sum_{j=0}^{N-2-n} \binom{L-2-n}{j} (-1)^j \times \\
&\quad \times \langle W | \left(D^- \eta(1) + E^-(1 - \eta(1))\right) D^{n+j} C^{N-2-n-j} \left(D^+ \eta(N) + E^+(1 - \eta(N))\right) |V\rangle .
\end{aligned}$$

From the boundary relations, that for $x = 0$ reduce to

$$\langle W | D^- C = \langle W | d_1^- C^- C, \quad \langle W | C^- D = \langle W | q_1^- C^- C, \quad \langle W | D^- D = \langle W | f_1^- C^- C$$

and

$$C D^+ |V\rangle = d_1^+ C C^+ |V\rangle , \quad D C^+ |V\rangle = q_1^+ C C^+ |V\rangle , \quad D D^+ |V\rangle = f_1^+ C C^+ |V\rangle ,$$

we see that

$$\begin{aligned}
\mu_N^{ss}(\eta) = (q_1^-)^n \sum_{j=0}^{N-2-n} \binom{N-2-n}{j} (-1)^j (q_1^-)^j \left(\eta(1)\eta(N) d_1^- d_1^+ + \eta(1)(1 - \eta(N)) d_1^- (1 - d_1^+)\right. \\
\left. + (1 - \eta(1))\eta(N)(1 - d_1^-) d_1^+ + (1 - \eta(1))(1 - \eta(N))(1 - d_1^-)(1 - d_1^+)\right)
\end{aligned}$$

$$= (d_1^-)^{\eta(1)}(1-d_1^-)^{1-\eta(1)}(q_1^-)^n(1-q_1^-)^{N-2-n}(d_1^+)^{\eta(N)}(1-d_1^+)^{1-\eta(N)}.$$

In other words,

$$\mu_N^{ss} = \nu_{d_1^-}^1 \otimes \nu_{q_1^-}^{N-2} \otimes \nu_{d_1^+}^1.$$

A.3 Computing $\varphi_N(1,2)$ from the stationary equations

Recall that $j_2(\eta) \equiv \eta(2) - \eta(3)$. We compute

$$\begin{aligned}\mathcal{L}(\eta(1)\eta(2)) &= \eta(1)\eta(2)(a_{41} - (a_{14}+a_{24}+a_{34}+a_{42}+a_{43})) \\ &\quad + \eta(1)(a_{43}-a_{41}) + \eta(2)(a_{42}-a_{41}) + a_{41} - \eta(1)j_2(\eta).\end{aligned} \tag{A.3}$$

Introduce the coefficients

$$\begin{aligned}B_3 &= a_{42} - a_{41},\\ C_3 &= -(a_{43}-a_{41}),\\ D_3 &= a_{41} - (a_{14}+a_{24}+a_{34}+a_{42}+a_{43}).\end{aligned}$$

Taking the expectation with respect to the stationary measure on (A.3), together with (2.11) we obtain the system

$$\begin{aligned}D_1\langle\eta(1)\eta(2)\rangle_N &= -A_1 - B_1\langle\eta(2)\rangle_N + C_1\langle\eta(1)\rangle_N,\\ D_2\langle\eta(1)\eta(2)\rangle_N &= -A_2 - B_2\langle\eta(1)\rangle_N + C_2\langle\eta(2)\rangle_N + \langle j_2\rangle_N,\\ D_3\langle\eta(1)\eta(2)\rangle_N &= -a_{41} - B_3\langle\eta(2)\rangle_N + C_3\langle\eta(1)\rangle_N + \langle\eta(1)j_2\rangle_N,\end{aligned}$$

with A_i, B_i and D_i, for $i = 1,2$, as in (2.10). Equivalently,

$$\begin{pmatrix} C_1 & -B_1 & -D_1\\ -B_2 & C_2 & -D_2\\ C_3 & -B_3 & -D_3\end{pmatrix}\begin{pmatrix}\langle\eta(1)\rangle_N\\ \langle\eta(2)\rangle_N\\ \langle\eta(1)\eta(2)\rangle_N\end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0\\ A_2 & -1 & 0\\ a_{41} & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ \langle j_2\rangle_N\\ \langle\eta(1)j_2\rangle_N\end{pmatrix}.$$

Introducing

$$\tilde{\mathcal{A}}_1 = \begin{pmatrix} C_1 & -B_1 & -D_1\\ -B_2 & C_2 & -D_2\\ C_3 & -B_3 & -D_3\end{pmatrix}, \quad \tilde{\mathcal{A}}_2 = \begin{pmatrix} A_1 & 0 & 0\\ A_2 & -1 & 0\\ a_{41} & 0 & 1\end{pmatrix}$$

one can check that $\det\big(\tilde{\mathcal{A}}_1\big) = \det(\mathcal{A}_1)$ with $\mathcal{A}_1$ given in (2.17). Moreover,

$$(\tilde{\mathcal{A}}_1)^{-1}\tilde{\mathcal{A}}_2 = \begin{pmatrix} d_1^- & d_2^- & d_3^-\\ q_1^- & q_2^- & q_3^-\\ f_1^- & f_2^- & f_3^-\end{pmatrix}$$

with the coefficients d^-, q^-, f^- as previously. In this way, under (H_0^-) we have that

$$\begin{pmatrix}\langle\eta(1)\rangle_N\\ \langle\eta(2)\rangle_N\\ \langle\eta(1)\eta(2)\rangle_N\end{pmatrix} = \begin{pmatrix} d_1^- & d_2^- & d_3^-\\ q_1^- & q_2^- & q_3^-\\ f_1^- & f_2^- & f_3^-\end{pmatrix}\begin{pmatrix}1\\ \langle j_2\rangle_N\\ \langle\eta(1)j_2\rangle_N\end{pmatrix} \tag{A.4}$$

in complete analogy to the boundary algebra $\{(R_1^-),(R_2^-),(R_{12}^-)\}$. We short-write, for any $x \in \Lambda_N$,

$$\varphi_N(1,x) = \langle\eta(1)\eta(x)\rangle_N - \langle\eta(1)\rangle_N\langle\eta(x)\rangle_N$$

and rewrite the first two equations in (A.4) as

$$\begin{aligned}\langle \eta(1) - d_1^- \rangle_N &= (d_2^- + d_1^- d_3^-)\langle j_2 \rangle_N + d_3^- \langle (\eta(1) - d_1^-) j_2 \rangle_N, \\ \langle \eta(2) - q_1^- \rangle_N &= (q_2^- + d_1^- q_3^-)\langle j_2 \rangle_N + q_3^- \langle (\eta(1) - d_1^-) j_2 \rangle_N.\end{aligned} \tag{A.5}$$

Consequently,

$$\begin{aligned}\langle \eta(1) - d_1^- \rangle_N \langle \eta(2) - q_1^- \rangle_N = {} & \langle j_2 \rangle_N^2 (d_2^- + d_1^- d_3^-) \\ & + \langle j_2 \rangle_N \langle (\eta(1) - d_1^-) j_2 \rangle_N \left(q_3^- (d_2^- + d_1^- d_3^-) + d_3^- (q_2^- + d_1^- q_3^-) \right) \\ & + \langle (\eta(1) - d_1^-) j_2 \rangle_N^2 d_3^- q_3^-,\end{aligned}$$

and since

$$\begin{aligned}\langle \eta(1) \rangle_N \langle \eta(2) \rangle_N = {} & d_1^- q_1^- + \langle \eta(1) - d_1^- \rangle_N \langle \eta(2) - q_1^- \rangle_N \\ & + \langle j_2 \rangle_N \left(q_1^- (d_2^- + d_1^- d_3^-) + d_1^- (q_2^- + d_1^- q_3^-) \right) \\ & + \langle (\eta(1) - d_1^-) j_2 \rangle_N (q_1^- d_3^- + d_1^- q_3^-),\end{aligned}$$

we find that

$$\begin{aligned}\varphi_N(1,2) = {} & f_1^- - d_1^- q_1^- \\ & + \langle j_2 \rangle_N \left(f_2^- + d_1^- f_3^- - q_1^- (d_2^- + d_1^- d_3^-) - d_1^- (q_2^- + d_1^- q_3^-) \right) \\ & - \langle j_2 \rangle_N^2 (d_2^- + d_1^- d_3^-)(q_2^- + d_1^- q_3^-) \\ & + \langle (\eta(1) - d_1^-) j_2 \rangle_N (f_3^- - q_1^- d_3^- - d_1^- q_3^-) \\ & + \langle (\eta(1) - d_1^-) j_2 \rangle_N \langle j_2 \rangle_N \left(q_3^- (d_2^- + d_1^- d_3^-) + d_3^- (q_2^- + d_1^- q_3^-) \right) \\ & + \langle (\eta(1) - d_1^-) j_2 \rangle_N^2 d_3^- q_3^-.\end{aligned} \tag{A.6}$$

Some computations yield

$$\begin{aligned}\langle (\eta(1) - d_1^-) j_2 \rangle_N &= \langle \eta(1) - d_1^- \rangle_N \langle j_2 \rangle_N - (\nabla_2^+ \varphi_N)(1,2) \\ &= (d_2^- + d_1^- d_3^-)\langle j_2 \rangle_N^2 + d_3^- \langle (\eta(1) - d_1^-) j_2 \rangle_N \langle j_2 \rangle_N - (\nabla_2^+ \varphi_N)(1,2).\end{aligned}$$

By successive iterations it follows that

$$\begin{aligned}\langle (\eta(1) - d_1^-) j_2 \rangle_N = {} & (d_2^- + d_1^- d_3^-) \sum_{i=2}^{n} (d_3^-)^{i-2} \langle j_2 \rangle_N^i \\ & + (d_3^-)^{n-1} \langle \eta(1) - d_1^- \rangle_N \langle j_2 \rangle_N^n - (\nabla_2^+ \varphi_N)(1,2) \sum_{i=0}^{n-1} (d_3^- \langle j_2 \rangle_N)^i.\end{aligned}$$

Since $\langle j_2 \rangle_N = O(1/L)$, taking the limit $n \to +\infty$ and using the geometric series one obtains

$$\langle (\eta(1) - d_1^-) j_2 \rangle_N = (d_2^- + d_1^- d_3^-)\langle j_2 \rangle_N^2 \frac{1}{1 - d_3^- \langle j_2 \rangle_N} - (\nabla_2^+ \varphi_N)(1,2) \frac{1}{1 - d_3^- \langle j_2 \rangle_N}. \tag{A.7}$$

Note that this is valid also for $d_3^- = 0$. Replacing this into (A.6) and rearranging the terms leads to (2.12).

A.4 Models

In order to facilitate solving the constraints in terms of the rates with the assistance of a computer, it turns out to be convenient to consider the following factorization of the boundary rates

$$a_{ij} \equiv \frac{1}{p^-} a_{ij} \quad \text{and} \quad b_{ij} \equiv \frac{1}{p^+} b_{ij}, \qquad p^\pm > 0. \tag{A.8}$$

Doing so, it is straightforward to see that the last two constraints in (C_1) take the form of

$$(q_1^- d_3^- + q_2^- + d_1^- q_3^- - f_3^-)(d_2^- + d_1^- d_3^-) = -d_2^-/p^-,$$
$$q_3^-(d_2^- + d_1^- d_3^-) = -d_3^-/p^-,$$

while the factor p^- does not appear in (C_2) nor in the *remaining* constraints of (C_1). We note that for $p^- = p^+$ equal to some $p > 0$, one can interpret p as the hopping rate in the bulk, instead of a factor of the boundary rates as in (A.8). In what follows, we present admissible left-boundary rates only. Specifically, we present particular solutions of the constraints in terms of the rates, and present the algebra coefficients d_i^-, q_i^- for $i = 1, 2, 3$. The coefficients f_i^- with $i = 1, 2, 3$ can be deduced from (A.1), and the right-boundary rates with the same dynamics as the left-boundary ones can be obtained by replacing directly a_{ij} by b_{ij} and p^- by p^+.

A.4.1 Family $\mathfrak{N}$

As seen in Section 2.4, this family is defined as $\mathfrak{N} := \mathcal{C}_1^{\pm} \cap \{d_3^{\pm}, q_3^{\pm} \neq 0\}$ and corresponds to non-linear models (in the sense of Definition 2.2.5) such that the stationary measure does not factorize at the boundary. The subfamily $\mathfrak{N} \cap \{t_1^{\pm} = 0,\ t_2^{\pm} = 1/p^{\pm},\ t_3^{\pm} = 0\} \subset \mathfrak{H}$ corresponds to non-linear models where the stationary measure can be described by the homogenous choice of matrices $D^{\pm} = D$ and $E^{\pm} = E$.

In what follows recall (A.8).

A.4.1.A Full dynamics

The following model is the simplest one we could find where no rate is zero.

$$\begin{aligned}
a_{31} &= a_{32},\\
\frac{a_{32}+a_{21}}{2} &= a_{43} = a_{42} = a_{41} = a_{14} = a_{13},\\
a_{34} &= a_{12} + a_{13} - a_{14},\\
a_{23} &= \frac{a_{32}(a_{12}+2a_{13}) + a_{13}(a_{32}-2a_{24})}{a_{12}+a_{13}},\\
p^- &= \frac{a_{12}+3a_{13}}{a_{13}(a_{24}-a_{32})}(a_{12}(a_{13}+a_{32}) + a_{13}(a_{13}+2a_{32}-a_{24})),
\end{aligned}$$

and the algebra coefficients equal to

$$\begin{aligned}
d_1^- &= \frac{1}{2},\\
d_2^- &= 0,\\
d_3^- &= \frac{(a_{24}-a_{32})/2}{(a_{12}+a_{13})(a_{13}+a_{32}) + a_{13}(a_{32}-a_{24})},\\
q_1^- &= \frac{2a_{13}}{a_{12}+3a_{13}},\\
q_2^- &= \frac{-1}{a_{12}+3a_{13}},\\
q_3^- &= \frac{-2a_{13}(a_{24}-a_{32})}{(a_{12}+3a_{13})(a_{12}+a_{13})(a_{13}+a_{32}) + a_{13}(a_{32}-a_{24})}.
\end{aligned}$$

The model is well-defined when all the rates are non-negative. It is simple to obtain a sufficient condition for this. From $(a_{32}+a_{21})/2 = a_{13}$ we need $2a_{13} - a_{32} \geq 0$ and from a_{34} we need $a_{12} + a_{13} - a_{14} \geq 0$.

The expression for p^- can be rewritten as

$$p^- = (3 + a_{12}/a_{13})\left(\frac{(1 + a_{12}/a_{13})(1 + a_{32}/a_{13})}{a_{24} - a_{32}} - 1\right)$$

and to obtain $p > 0$ it is enough to impose $0 < a_{24} - a_{31} \leq 1$. Similarly, from a_{23} it is enough to let $a_{32} - 2a_{24} \geq 0$. We conclude that a sufficient condition for the model to be well-defined is

$$2a_{31} < 2a_{24} \leq a_{32} \leq 2a_{13} \ \wedge \ a_{24} \leq 1 + a_{31} \ \wedge \ a_{14} < a_{12} + a_{13}.$$

If we consider the right-boundary with the same *dynamics*, then the *nonequilibrium* condition $q_1^- \neq q_1^+$ translates to

$$\frac{2a_{13}}{a_{12} + 3a_{13}} \neq \frac{2b_{13}}{b_{12} + 3b_{13}}.$$

Moreover, it can be checked that

$$\begin{aligned}
t_1^- &= \frac{a_{12} - a_{13}}{2a_{12} + 6a_{13}}, \\
t_2^- - 1/p^- &= \frac{a_{12}^2(a_{13} + a_{32}) + a_{12}a_{13}(a_{32} - a_{24}) - a_{13}\left(a_{13}^2 - a_{13}a_{24} + 2a_{13}a_{32} + 2a_{24} - 2a_{32}\right)}{2(a_{12} + 3a_{13})(a_{12}(a_{13} + a_{32}) + a_{13}(a_{13} - a_{24} + 2a_{32}))}, \\
t_3^- &= \frac{(a_{12} + 7a_{13})(a_{24} - a_{32})}{2(a_{12} + 3a_{13})(a_{12}(a_{13} + a_{32}) + a_{13}(a_{13} - a_{24} + 2a_{32}))}.
\end{aligned}$$

It is straightforward to see that imposing $t_1^- = 0$ (that is, $a_{12} = a_{13}$), for the remaining homogeneity conditions to be satisfied one needs also to impose $a_{24} = a_{32}$, which is not possible due to the expression for p. This means that the homogeneity conditions cannot be imposed on the dynamics

A.4.1.B No annihilation

We provide the two models with the simplest expressions for the rates that we could find.

- No annihilation I:

$$\begin{aligned}
&a_{34} = a_{24} = a_{14} = a_{13} = a_{12} = 0, \\
&a_{21} = \frac{a_{31}(a_{43} - a_{41} - (a_{31} - a_{32})) + (a_{41} - a_{42})(a_{23} - (a_{31} - a_{32}) - (a_{41} - a_{42}))}{a_{31} - a_{32} + a_{41} - a_{42}}, \\
&p^- = (a_{23}a_{42} + a_{43}(a_{32} + a_{42}))\frac{a_{43}(a_{31} - a_{32}) + (a_{23} + a_{43})(a_{41} - a_{42})}{a_{23}(a_{32}a_{41} - a_{31}a_{42})},
\end{aligned}$$

and the algebra coefficients reduce to

$$\begin{aligned}
d_1^- &= 1, \\
d_2^- &= \frac{a_{31} - a_{32} + a_{41} - a_{42}}{a_{23}a_{42} + a_{43}(a_{32} + a_{42})}, \\
d_3^- &= -\frac{a_{23} + a_{31} - a_{32} + a_{41} - a_{42}}{a_{23}a_{42} + a_{43}(a_{32} + a_{42})}, \\
q_1^- &= 1, \\
q_2^- &= -\frac{(a_{31} - a_{32} + a_{41} - a_{42})(a_{42}(a_{23} + a_{31} + a_{43}) + a_{32}(a_{43} - a_{41}))}{(a_{23}a_{42} + a_{43}(a_{32} + a_{42}))(a_{23}(a_{41} - a_{42}) + a_{43}(a_{31} - a_{32} + a_{41} - a_{42}))}, \\
q_3^- &= \frac{(a_{31}a_{42} - a_{32}a_{41})(a_{23} + a_{31} - a_{32} + a_{41} - a_{42})}{(a_{23}a_{42} + a_{43}(a_{32} + a_{42}))(a_{23}(a_{41} - a_{42}) + a_{43}(a_{31} - a_{32} + a_{41} - a_{42}))}..
\end{aligned}$$

It is straightforward to see that a sufficient condition for the model to be well-defined is

$$0 \leq a_{31} - a_{32} \leq a_{43} - a_{41} \ \wedge \ 0 \leq a_{41} - a_{42} \ \wedge \ (a_{41} - a_{42}) + (a_{31} - a_{32}) \leq a_{23} \ \wedge \ 0 < a_{32}a_{41} - a_{31}a_{42}.$$

There are two particular cases of interest. We have that

$$\begin{aligned}
t_1^- &= 0, \\
t_2^- - 1/p^- &= \frac{a_{23}a_{31}(a_{41}+a_{42}) + a_{23}a_{41}(-2a_{32}+a_{41}-a_{42}) + (a_{31}-a_{32}+a_{41}-a_{42})(a_{31}(a_{42}+a_{43}) - a_{32}a_{41} + a_{41}a_{43})}{(a_{23}a_{42}+a_{43}(a_{32}+a_{42}))(a_{23}(a_{41}-a_{42})+a_{43}(a_{31}-a_{32}+a_{41}-a_{42}))}, \\
t_3^- &= -\frac{(a_{23}+a_{31}-a_{32}+a_{41}-a_{42})(a_{23}(a_{41}-a_{42}) + a_{31}a_{42} + a_{31}a_{43} - a_{32}(a_{41}+a_{43}) + a_{41}a_{43} - a_{42}a_{43})}{(a_{23}a_{42}+a_{43}(a_{32}+a_{42}))(a_{23}(a_{41}-a_{42})+a_{43}(a_{31}-a_{32}+a_{41}-a_{42}))},
\end{aligned}$$

and further imposing the constraints

$$a_{23} = \frac{a_{32}+a_{42}}{2} \quad \text{and} \quad a_{43} = \frac{a_{42}(-2a_{31}-a_{41}+a_{42}) + a_{32}(a_{41}+a_{42})}{2(a_{31}-a_{32}+a_{41}-a_{42})},$$

the resulting model belongs to $\mathfrak{H}$ and the SSEP$(1,1)$ algebra can describe the stationary measure.

Another particular case is to impose $a_{42} = a_{23} + a_{31} - a_{32} + a_{41}$, that results in a *linear* model, where a_{21} and p^- take the form of

$$\begin{aligned}
a_{21} &= a_{23} - a_{32} - a_{41} + a_{43}, \\
p^- &= \frac{(a_{23}+a_{31}-a_{32}+a_{43})\left(a_{23}^2 + a_{23}(a_{31}-a_{32}+a_{41}+a_{43}) + a_{43}(a_{31}+a_{41})\right)}{a_{23}a_{31} + (a_{31}-a_{32})(a_{31}+a_{41})}.
\end{aligned}$$

- No annihilation II:

The choice of rates

$$\begin{aligned}
a_{34} &= a_{24} = a_{14} = a_{13} = a_{12} = 0, \\
a_{41} &= a_{42}, \\
a_{31} &= a_{32}, \\
p^- &= \frac{(a_{21}+a_{32}+a_{42})(a_{23}a_{42}+a_{43}(a_{32}+a_{42}))}{a_{21}(a_{32}+a_{42}) + a_{42}(-a_{23}+a_{42}-a_{43}) + a_{32}^2 + a_{32}(2a_{42}-a_{43})},
\end{aligned}$$

leads to the algebra coefficients

$$\begin{aligned}
d_1^- &= 1, \\
d_2^- &= 0, \\
d_3^- &= -\frac{a_{23}}{a_{23}a_{42}+a_{43}(a_{32}+a_{42})}, \\
q_1^- &= 1, \\
q_2^- &= -\frac{1}{a_{21}+a_{32}+a_{42}}, \\
q_3^- &= \frac{-a_{21}(a_{32}+a_{42}) + a_{42}(a_{23}-a_{42}+a_{43}) - a_{32}^2 + a_{32}(a_{43}-2a_{42})}{(a_{21}+a_{32}+a_{42})(a_{23}a_{42}+a_{43}(a_{32}+a_{42}))}..
\end{aligned}$$

The model is well-defined, for example, for

$$a_{42} \geq a_{23} + a_{43}.$$

It can be checked that there is no particular choice of rates that leads to a linear model. Regarding the homogeneous choice of matrices,

$$\begin{aligned} t_1^- &= 0, \\ t_2^- - 1/p^- &= -\frac{a_{21}(a_{32}+a_{42}) + a_{42}(-2a_{23}+a_{42}-2a_{43}) + a_{32}^2 + 2a_{32}(a_{42}-a_{43})}{(a_{21}+a_{32}+a_{42})(a_{23}a_{42}+a_{43}(a_{32}+a_{42}))}, \\ t_3^- &= \frac{a_{21}(-a_{23}+a_{32}+a_{42}) - a_{23}(a_{32}+2a_{42}) + (a_{32}+a_{42})(a_{32}+a_{42}-a_{43})}{(a_{21}+a_{32}+a_{42})(a_{23}a_{42}+a_{43}(a_{32}+a_{42}))}, \end{aligned}$$

and fixing also

$$a_{23} = \frac{a_{32}+a_{42}}{2} \quad \text{and} \quad a_{43} = \frac{a_{21}+a_{32}}{2}$$

leads to a model in $\mathfrak{H}$.

A.4.1.C No Creation

The *complementary dynamics* to the previous two models also satisfies the consistency conditions. Precisely, exchanging

$$(a_{21}, a_{31}, a_{41}, a_{42}, a_{43}, a_{23}) \quad \text{by} \quad (a_{34}, a_{24}, a_{14}, a_{13}, a_{12}, a_{32})$$

in the models "No annihilation I" and "No annihilation II" gives, respectively, the complementary "No creation I" and "No creation II" below, suggesting a particle-hole symmetry.

- No creation I

The choice

$$\begin{aligned} a_{21} &= a_{31} = a_{41} = a_{42} = a_{43} = 0, \\ a_{34} &= \frac{a_{24}(a_{12}-a_{14}-(a_{24}-a_{23})) + (a_{14}-a_{13})(a_{32}-(a_{24}-a_{23})-(a_{14}-a_{13}))}{a_{24}-a_{23}+a_{14}-a_{13}}, \\ p^- &= (a_{32}a_{13}+a_{12}(a_{23}+a_{13}))\frac{a_{12}(a_{24}-a_{23}) + (a_{32}+a_{12})(a_{14}-a_{13})}{a_{32}(a_{23}a_{14}-a_{24}a_{13})}, \end{aligned}$$

leads to well-defined rates when, for example,

$$0 \le a_{24}-a_{23} \le a_{12}-a_{14} \ \wedge\ 0 \le a_{14}-a_{13} \ \wedge\ (a_{14}-a_{13})+(a_{24}-a_{23}) \le a_{32} \ \wedge\ 0 < a_{23}a_{14}-a_{24}a_{13}.$$

The algebra coefficients reduce to

$$\begin{aligned} d_1^- &= 0, \\ d_2^- &= -\frac{a_{32}}{a_{12}(a_{13}+a_{23})+a_{13}a_{32}}, \\ d_3^- &= \frac{-a_{13}+a_{14}-a_{23}+a_{24}+a_{32}}{a_{12}(a_{13}+a_{23})+a_{13}a_{32}}, \\ q_1^- &= 0 \\ q_2^- &= -\frac{a_{13}+a_{23}}{a_{12}(a_{13}+a_{23})+a_{13}a_{32}}, \\ q_3^- &= -\frac{(a_{14}a_{23}-a_{13}a_{24})(-a_{13}+a_{14}-a_{23}+a_{24}+a_{32})}{(a_{12}(a_{13}+a_{23})+a_{13}a_{32})(a_{12}(a_{13}-a_{14}+a_{23}-a_{24})+a_{32}(a_{13}-a_{14}))}. \end{aligned}$$

In particular,

$$
\begin{aligned}
t_1^- &= 0,\\
t_2^- - 1/p^- &= \frac{a_{12}(a_{13}+a_{23}-a_{32})(a_{13}-a_{14}+a_{23}-a_{24}) + a_{32}\left(a_{13}^2 - a_{13}(a_{14}-a_{23}+a_{24}+a_{32}) + a_{14}a_{32}\right)}{(a_{12}(a_{13}+a_{23})+a_{13}a_{32})(a_{12}(a_{13}-a_{14}+a_{23}-a_{24})+a_{32}(a_{13}-a_{14}))},\\
t_3^- &= \frac{(a_{13}-a_{14}+a_{23}-a_{24}-a_{32})(a_{12}(-a_{13}+a_{14}-a_{23}+a_{24})+a_{13}(a_{24}-a_{32})+a_{14}(a_{32}-a_{23}))}{(a_{12}(a_{13}+a_{23})+a_{13}a_{32})(a_{12}(a_{13}-a_{14}+a_{23}-a_{24})+a_{32}(a_{13}-a_{14}))},
\end{aligned}
$$

and imposing additionally

$$
a_{32} = \frac{a_{23}+a_{13}}{2} \quad \text{and} \quad a_{12} = \frac{a_{32}(a_{13}+a_{14}) - a_{13}(a_{14}+a_{24})}{a_{14}+a_{24}-2a_{32}}
$$

leads to a model in $\mathfrak{H}$; while the choice

$$
a_{13} = a_{32} + a_{14} + a_{24} - a_{23}
$$

leads to a model in $\mathfrak{L}$.

- No creation II

$$
\begin{aligned}
&a_{21} = a_{31} = a_{41} = a_{42} = a_{43} = 0,\\
&a_{14} = a_{13},\\
&a_{24} = a_{23},\\
&p^- = \frac{(a_{34}+a_{23}+a_{13})(a_{32}a_{13}+a_{12}(a_{23}+a_{13}))}{a_{34}(a_{23}+a_{13}) + a_{13}(-a_{32}+a_{13}-a_{12}) + a_{23}^2 + a_{23}(2a_{13}-a_{12})}.
\end{aligned}
$$

The model is well-defined when, for example,

$$
a_{13} \geq a_{32} + a_{12}.
$$

The algebra coefficients are equal to

$$
\begin{aligned}
d_1^- &= 0,\\
d_2^- &= -\frac{a_{32}}{a_{12}(a_{13}+a_{23})+a_{13}a_{32}},\\
d_3^- &= \frac{a_{32}}{a_{12}(a_{13}+a_{23})+a_{13}a_{32}},\\
q_1^- &= 0,\\
q_2^- &= -\frac{a_{13}+a_{23}}{a_{12}(a_{13}+a_{23})+a_{13}a_{32}},\\
q_3^- &= \frac{-a_{12}(a_{13}+a_{23}) + a_{13}^2 + a_{13}(2a_{23}-a_{32}+a_{34}) + a_{23}(a_{23}+a_{34})}{(a_{13}+a_{23}+a_{34})(a_{12}(a_{13}+a_{23})+a_{13}a_{32})},
\end{aligned}
$$

and the particular choice

$$
a_{32} = \frac{a_{13}+a_{23}}{2} \quad \text{and} \quad a_{12} = \frac{a_{34}+a_{23}}{2}
$$

leads to a model in $\mathfrak{H}$, while there is no particular choice in $\mathfrak{L}$.

A.4.2 Family $\mathfrak{L}$

The linear family is defined as $\mathfrak{L} := \mathcal{C}_1^{\pm} \cap \{d_2^{\pm} \neq 0,\ d_3^{\pm}, q_3^{\pm} = 0\}$ and every model satisfies the linear condition (2.2.5), while the stationary measure still does not factorize at the boundary (condition $d_2^{\pm} \neq 0$). Models of this family can also be described by the SSEP$(1,1)$ algebra whenever it is also satisfied $t_1^{\pm} = 0$, $t_2^{\pm} = 1/p^{\pm}$ and $t_3^{\pm} = 0$.

A.4.2.A Model I

The choice

$$
\begin{aligned}
a_{32} &= a_{31} = a_{34},\\
a_{14} &= a_{21} = a_{41} = 0,\\
a_{13} &= \frac{(2a_{34} - a_{12})(a_{34} + a_{12})}{a_{12} + 2a_{34}},\\
a_{43} &= \frac{a_{12}}{4a_{34}} \frac{a_{12}^2 + 4a_{34}^2}{a_{12} + 2a_{34}},\\
a_{42} &= \frac{(2a_{34} - a_{12})(a_{34} + a_{12})}{4a_{34}},\\
a_{23} &= \frac{a_{12}}{4a_{34}} \frac{(2a_{34} + a_{12})(2a_{34} - a_{12}) + 4a_{12}a_{34}}{a_{12} + 2a_{34}},\\
a_{24} &= \frac{1}{4}(a_{12} + 2a_{34}),\\
p^- &= \frac{8a_{34}^2(a_{12} + a_{34})}{(2a_{34} - a_{12})(2a_{34} + a_{12})},
\end{aligned}
$$

leads to the algebra coefficients

$$
\begin{aligned}
d_1^- &= \frac{1}{2}, ,\\
d_2^- &= \frac{a_{12} - 2a_{34}}{8a_{34}^2}, ,\\
d_3^- &= 0,\\
q_1^- &= \frac{a_{12}}{2(a_{12} + a_{34})}, ,\\
q_2^- &= \frac{(a_{12} - 4a_{34})(a_{12} + 2a_{34})}{8a_{34}^2(a_{12} + a_{34})}, ,\\
q_3^- &= 0.
\end{aligned}
$$

Non-negativity of the rates is achieved if and only if

$$2a_{34} - a_{12} > 0.$$

Note that $d_1^- \neq q_1^-$ and we cannot choose $a_{34} = 0$ (see, for example, the expression for a_{42}), meaning that the model above cannot be represented by the homogeneous choice of matrices $D^{\pm}, E^{\pm} = D, E$. Moreover, one cannot have $d_2^- = 0$ due to the expression for p^-, and as such there is no particular choice of rates where the model belongs to $\mathfrak{F}$ either.

The conditions $a_{41} = 0$ and $a_{32} = a_{34} = a_{31}$ can be relaxed. However, the resulting expressions for the rates are much longer and the study of their non-negativity becomes much more involved.

A.4.2.B Model II

Fixing

$$
\begin{aligned}
a_{13} &= a_{42},\\
a_{43} &= a_{24} + a_{34} - a_{42},\\
a_{21} &= a_{34},
\end{aligned}
$$

$$
\begin{aligned}
a_{41} &= a_{14},\\
a_{23} &= a_{32},\\
a_{31} &= a_{24},\\
a_{12} &= a_{24} + a_{34} - a_{42},\\
p^- &= \frac{(a_{24}+a_{34})\,(a_{14}(a_{24}+2a_{32}+a_{34}) + a_{24}(a_{24}+a_{32}+a_{34}-a_{42}) + a_{34}(a_{32}+a_{42}))}{a_{24}a_{42} - a_{14}a_{32}},
\end{aligned}
$$

the rates are well-defined if, for example,

$$
a_{14}a_{32} < a_{24}a_{42} \quad \text{and} \quad a_{42} \le a_{24} + a_{32} + a_{34}.
$$

The algebra coefficients are equal to

$$
\begin{aligned}
d_1^- &= \frac{1}{2},\\
d_2^- &= \frac{a_{14} + a_{24} - a_{32} - a_{42}}{2\,(a_{14}(a_{24}+2a_{32}+a_{34}) + a_{24}^2 + a_{24}(a_{32}+a_{34}-a_{42}) + a_{34}(a_{32}+a_{42}))},\\
d_3^- &= 0,\\
q_1^- &= \frac{1}{2},\\
q_2^- &= -\frac{a_{14} + a_{24} + a_{32} + a_{42}}{2\,(a_{14}(a_{24}+2a_{32}+a_{34}) + a_{24}^2 + a_{24}(a_{32}+a_{34}-a_{42}) + a_{34}(a_{32}+a_{42}))},\\
q_3^- &= 0,
\end{aligned}
$$

and we have that

$$
\begin{aligned}
t_1^- &= 0,\\
t_2^- - 1/p^- &= \frac{a_{14}(a_{24}+a_{32}+a_{34}) + a_{24}(a_{24}+a_{34}-a_{42})}{(a_{24}+a_{34})\,(a_{14}(a_{24}+2a_{32}+a_{34}) + a_{24}^2 + a_{24}(a_{32}+a_{34}-a_{42}) + a_{34}(a_{32}+a_{42}))},\\
t_3^- &= 0.
\end{aligned}
$$

The particular choice

$$
a_{42} = a_{34} + a_{24} \quad \text{and} \quad a_{14} = 0
$$

leads to a model in $\mathfrak{H}$, while

$$
a_{32} + a_{42} = a_{14} + a_{24}
$$

leads to a model in $\mathfrak{F}$.

A.5 Representation

We present in Lemma A.5.1 a system of recurrence relations characterizing the entries of the left-boundary row vectors $\langle W_0| = \langle W|\,E$ and $\langle W_1| = \langle W|\,D$, provided that the matrices D and E satisfying the relation $[D, E] = D+E$ are tridiagonal and infinite. In Proposition A.5.2 we consider a particular family of representations where the right-inverse of $C - \mathbf{1}d_3^-$ is lower triangular. For the right-boundary the results can be deduced by analogous arguments. This proves the expression of the stationary probabilities as in (2.1).

In what follows, for each $k \in \mathbb{N}_+$ we refer to the entries of the infinite row vectors $\langle W_0|$ and $\langle W_1|$ as $w_k^0 = \langle W_0|k\rangle$ and $w_k^1 = \langle W_1|k\rangle$, respectively, where $(|k\rangle)_i = \delta_{k,i}$, for $i \geq 1$, and $\delta_{k,i}$ equals one if $k = i$, and zero otherwise. Consider, additionally, the sequence of column vectors $(|U_k\rangle)_{k\geq 1}$, where $|U_k\rangle := \left|w_k^0, w_k^1\right\rangle$.

Lemma A.5.1. *Let D and E be infinite tridiagonal matrices with entries $(D)_{ij} = d_{ij}(\delta_{i-1,j} + \delta_{i,j} + \delta_{i+1,j})$ and $(E)_{ij} = e_{ij}(\delta_{i-1,j} + \delta_{i,j} + \delta_{i+1,j})$, for $i, j \geq 1$, and such that $[D, E] = D + E$. In matrix form,*

$$D = \begin{pmatrix} d_{11} & d_{12} & 0 & 0 & \cdots \\ d_{21} & d_{22} & d_{23} & 0 & \cdots \\ 0 & d_{32} & d_{33} & d_{34} & \cdots \\ 0 & 0 & d_{43} & d_{4,4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} e_{11} & e_{12} & 0 & 0 & \cdots \\ e_{21} & e_{22} & e_{23} & 0 & \cdots \\ 0 & e_{32} & e_{33} & e_{34} & \cdots \\ 0 & 0 & e_{43} & e_{4,4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row vectors $\langle W_0|$ and $\langle W_1|$ satisfy the boundary equations (R_1^-) and (R_2^-) if the following recurrence relation is satisfied:

$$\begin{cases} 0 = A_k\,|U_{k+1}\rangle + B_k\,|U_k\rangle + C_k\,|U_{k-1}\rangle\,, & k \geq 2, \\ 0 = A_1\,|U_2\rangle + B_1\,|U_1\rangle\,, & k = 1, \end{cases} \tag{A.9}$$

with coefficient matrices

$$\begin{aligned} A_k &= \begin{pmatrix} d_1^- c_{k+1k} & (d_1^- - 1)c_{k+1k} \\ q_1^- c_{k+1k} - d_{k+1k} & q_1^- c_{k+1k} - d_{k+1k} \end{pmatrix}, \\ B_k &= \begin{pmatrix} d_1^- c_{kk} + d_2^- & (d_1^- - 1)c_{kk} + d_3^- \\ q_1^- c_{kk} - d_{kk} + q_2^- & q_1^- c_{kk} - d_{kk} + q_2^- + q_3^- \end{pmatrix}, \\ C_k &= \begin{pmatrix} d_1^- c_{k-1k} & (d_1^- - 1)c_{k-1k} \\ q_1^- c_{k-1k} - d_{k-1k} & q_1^- c_{k-1k} - d_{k-1k} \end{pmatrix}, \end{aligned} \tag{A.10}$$

where $c_{ij} = d_{ij} + e_{ij}$ for $i, j \geq 1$.

Proof. The recurrence relation (A.9) is obtained by replacing the specific tridiagonal representations of D and $C = D + E$ in the boundary relations, then reorganizing the terms. □

Note that if D' and E' are matrices such that $[D', E'] = D' + E'$, then the matrices parameterized by $\lambda \in \mathbb{R}$, that we write as D_λ and E_λ and which are given by

$$D_\lambda = (1 + \lambda)D' + \lambda E' \quad \text{and} \quad E_\lambda = (1 - \lambda)E' - \lambda D'$$

also satisfy $[D_\lambda, E_\lambda] = D_\lambda + E_\lambda$.

With this observation in mind, let $(u_k)_{k\geq 0}$ by any real sequence with $u_k \neq 0$ for any $k \geq 0$. The bidiagonal matrices D' and E' given by

$$(D')_{ij} = \delta_{i,j}(i - 1 + d) + \delta_{i+1,j} i u_{i-1} \quad \text{and} \quad (E')_{ij} = \delta_{i-1,j}\frac{i - 2 + d + e}{u_{i-2}} + \delta_{i,j}(i - 1 + e),$$

or, in matrix form

$$D' = \begin{pmatrix} d & u_0 & 0 & 0 & \dots \\ 0 & 1 + d & 2u_1 & 0 & \dots \\ 0 & 0 & 2 + d & 3u_2 & \dots \\ 0 & 0 & 0 & 3 + d & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad E' = \begin{pmatrix} e & 0 & 0 & 0 & \dots \\ \frac{d+e}{u_0} & 1 + e & 0 & 0 & \dots \\ 0 & \frac{1+d+e}{u_1} & 2 + e & 0 & \dots \\ 0 & 0 & \frac{2+d+e}{u_2} & 3 + e & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

satisfy the commuting relation $[D', E'] = D' + E'$. In particular, for any $\lambda \in \mathbb{R}$ the parameterized matrices D_λ and E_λ are tridiagonal and satisfy the same relation.

Proposition A.5.2. *Let $\lambda, d, e \in \mathbb{R}$ be fixed such that $\lambda \neq q_1^-$, and such that for each $k \in \mathbb{N}$ it holds $d+e+k-1 \neq 0$. A particular representation of the algebra $\{(R_1^-), (R_2^-), (R_0)\}$ is given by $D = D_\lambda$, $E = E_\lambda$ and row vectors $\langle W_0|$ and $\langle W_1|$, defined through a sequence $(|U_k\rangle)_{k\geq 1}$, where $|U_k\rangle = \left|w_k^0, w_k^1\right\rangle$, satisfying the following second-order recurrence relation:*

$$|U_{k+1}\rangle = -A_k^{-1} B_k |U_k\rangle - A_k^{-1} C_k |U_{k-1}\rangle, \quad \text{for } k \geq 2,$$

with initial conditions

$$\begin{cases} |U_2\rangle = -A_1^{-1} B_1 |U_1\rangle, \\ |U_1\rangle \neq 0, \end{cases}$$

and with coefficients, for each $k \in \mathbb{N}_+$, given by

$$A_k = \frac{d+e+k-1}{u_{k-1}} \begin{pmatrix} d_1^- & d_1^- - 1 \\ q_1^- - \lambda & q_1^- - \lambda \end{pmatrix},$$

$$B_k = 2(k-1) \begin{pmatrix} d_1^- & d_1^- - 1 \\ (q_1^- - \lambda) - 1/2 & (q_1^- - \lambda) - 1/2 \end{pmatrix} + \begin{pmatrix} d_1^-(d+e) + d_2^- & (d_1^- - 1)(d+e) + d_3^- \\ (q_1^- - \lambda)(d+e) - d + q_2^- & (q_1^- - \lambda)(d+e) - d + q_2^- + q_3^- \end{pmatrix},$$

$$C_k = (k-1) u_{k-2} \begin{pmatrix} d_1^- & d_1^- - 1 \\ q_1^- - (1+\lambda) & q_1^- - (1+\lambda) \end{pmatrix}.$$

In particular, there exists a right-inverse of $C - \mathbf{1} d_3^-$, that we write as A, which is an infinite lower-triangular matrix with entries equal to zero at the diagonal. Concretely, the entries of A, that we write as $a_{ij} = (A)_{ij}$ for $i, j \geq 1$, satisfy the second-order, non-homogeneous recurrence relation

$$a_{i+1,j} i u_{i-1} + a_{i,j}(2(i-1) + d + e - d_3^-) + a_{i-1,j} \frac{i - 2 + d + e}{u_{i-2}} = \delta_{i,j}, \quad \text{for } i, j \geq 1, \tag{A.11}$$

with boundary conditions

$$\begin{cases} a_{0,j} := 0, & j \geq 1, \\ a_{1,j} = 0, & j \geq 1. \end{cases}$$

Proof. The recurrence relation defining the sequence $(|U_k\rangle)_{k\geq 1}$, in the statement of the current proposition, is obtained through (A.9) by imposing that A_k is invertible for every k. From the particular representation of the matrices D and E, that we recall to be both parameterized by λ, this condition is verified for $(d+e+k-1)(q_1^- - \lambda) \neq 0$. The particular coefficients A_k, B_k, C_k are obtained by direct substitution of the entries of the matrices D and E in the expressions for the coefficients in (A.10).

Regarding the existence of a matrix A such that $(C - \mathbf{1} d_3^-) A = \mathbf{1}$, let $C' := C - \mathbf{1} d_3^-$ and denote its entries by $c'_{i,j}$, for $i, j \geq 1$. Because C' is a tridiagonal matrix, the equality $C'A = \mathbf{1}$ is translated, entry-wise, to

$$\begin{cases} c'_{1,1} a_{1,1} + c'_{1,2} a_{2,1} = 1, \\ c'_{1,1} a_{1,j} + c'_{1,2} a_{2,j} = 0, \quad j \neq 1, \end{cases} \quad \text{and} \quad \begin{cases} c'_{i,i-1} a_{i-1,i} + c'_{i,i} a_{i,i} + c'_{i,i+1} a_{i+1,i} = 1, & i \neq 1, \\ c'_{i,i-1} a_{i-1,j} + c'_{i,i} a_{i,j} + c'_{i,i+1} a_{i+1,j} = 0, & i \neq 1,\ j \neq i, \end{cases}$$

which can be succinctly presented as in (A.11). We now iterate the recurrence for the initial values of $i \in \mathbb{N}_+$. Precisely, fixed $j \geq 1$, for $i = 1, 2, 3$, respectively,

$$c'_{1,1} a_{1,j} + c'_{1,2} a_{2,j} = \delta_{j,1},$$

$$c'_{2,1} a_{1,j} + c'_{2,2} a_{2,j} + c'_{2,3} a_{3,j} = \delta_{j,2},$$

$$c'_{3,2} a_{2,j} + c'_{3,3} a_{3,j} + c'_{3,4} a_{4,j} = \delta_{j,3}.$$

Therefore,

$$
\begin{aligned}
a_{2,j} &= -\frac{c'_{1,1}}{c'_{1,2}} a_{1,j} + \frac{1}{c'_{1,2}} \delta_{j,1}, \\
a_{3,j} &= a_{1,j} \left(\frac{c'_{2,2}}{c'_{2,3}} \frac{c'_{1,1}}{c'_{1,2}} - \frac{c'_{2,1}}{c'_{2,3}} \right) - \frac{1}{c'_{1,2}} \frac{c'_{2,2}}{c'_{2,3}} \delta_{j,1} + \frac{1}{c'_{2,3}} \delta_{j,2}, \\
a_{4,j} &= a_{1,j} \left(\frac{c'_{2,1}}{c'_{2,3}} \frac{c'_{3,3}}{c'_{3,4}} - \frac{c'_{2,2}}{c'_{2,3}} \frac{c'_{1,1}}{c'_{1,2}} \frac{c'_{3,3}}{c'_{3,4}} + \frac{c'_{1,1}}{c'_{1,2}} \frac{c'_{3,2}}{c'_{3,4}} \right) + \delta_{j,1} \frac{1}{c'_{3,4}} \left(\frac{c'_{3,3}}{c'_{1,2}} \frac{c'_{2,2}}{c'_{2,3}} - \frac{c'_{3,2}}{c'_{1,2}} \right) - \delta_{j,2} \frac{1}{c'_{3,1}} \frac{c'_{3,3}}{c'_{2,3}} + \delta_{j,3} \frac{1}{c'_{3,4}},
\end{aligned}
$$

and so on. Fixing each entry of the first line of A as equal to zero ($a_{1,j} = 0$ for all $j \geq 1$) makes A a lower-triangular matrix with entries equal to zero at the diagonal. □

B

From exclusion to slow and fast diffusion

Contents

B.1 Replacement Lemmas

Proof of Proposition 3.4.2. Recalling the identity (3.44), let us focus on the rightmost term there. Note that $b-a=\sqrt{a}(\sqrt{b}-\sqrt{a})+\sqrt{b}(\sqrt{b}-\sqrt{a})$, thus

$$\begin{aligned}\left\langle \mathcal{L}_N^{(m-1)} f\right\rangle_{\nu_\alpha^N} &= \sum_{x\in\mathbb{T}_N}\int_{\eta\in\Omega_N} r_N^{(m-1)}(\tau_x\eta)\sqrt{f}(\eta)\left(\nabla_{x,x+1}\sqrt{f}\right)(\eta)\nu_\alpha^N(\mathrm{d}\eta)\\ &\quad+\sum_{x\in\mathbb{T}_N}\int_{\eta\in\Omega_N} r_N^{(m-1)}(\tau_x\eta)\sqrt{f}(\eta^{x,x+1})\left(\nabla_{x,x+1}\sqrt{f}\right)(\eta)\nu_\alpha^N(\mathrm{d}\eta).\end{aligned}$$

Performing the change of variables $\eta\mapsto\eta^{x,x+1}$ on the second term above and using the symmetry of the rates, $r_N^{(m-1)}(\tau_x\eta^{x,x+1})=r_N^{(m-1)}(\tau_x\eta)$, we obtain

$$\left\langle \mathcal{L}_N^{(m-1)} f\right\rangle_{\nu_\alpha^N} = \sum_{x\in\mathbb{T}_N}\sum_{\eta\in\Omega_N} r_N^{(m-1)}(\tau_x\eta)\sqrt{f}(\eta)\left(\nabla_{x,x+1}\sqrt{f}\right)(\eta)\left(1-\frac{\nu_\alpha^N(\eta^{x,x+1})}{\nu_\alpha^N(\eta)}\right)\nu_\alpha^N(\eta).$$

Note that the previous quantity equals zero if $\alpha(\cdot)$ is constant. Otherwise, applying Young's inequality with $A>0$,

$$\sqrt{f}(\eta)\left(\nabla_{x,x+1}\sqrt{f}\right)(\eta)\left(1-\frac{\nu_\alpha^N(\eta^{x,x+1})}{\nu_\alpha^N(\eta)}\right)\le\frac{1}{2A}\left|\left(\nabla_{x,x+1}\sqrt{f}\right)(\eta)\right|^2+\frac{A}{2}f(\eta)\left|1-\frac{\nu_\alpha^N(\eta^{x,x+1})}{\nu_\alpha^N(\eta)}\right|^2$$

and therefore

$$\left\langle \mathcal{L}_N^{(m-1)} f\right\rangle_{\nu_\alpha^N}\le\frac{1}{2A}\Gamma_N^{(m-1)}(\sqrt{f},\nu_\alpha^N)+\frac{A}{2}\sum_{x\in\mathbb{T}_N}\sum_{\eta\in\Omega_N} r_N^{(m-1)}(\tau_x\eta)\left|1-\frac{\nu_\alpha^N(\eta^{x,x+1})}{\nu_\alpha^N(\eta)}\right|^2 f(\eta)\nu_\alpha^N(\eta),$$

where $\left|1-\nu_\alpha^N(\eta^{x,x+1})/\nu_\alpha^N(\eta)\right|^2\le\mathbf{c}_\alpha N^{-2}$ with $\mathbf{c}_\alpha>0$ for $\alpha(\cdot)$ a Lipschitz function. From Lemma 3.2.16 we can bound

$$\sum_{x\in\mathbb{T}_N} r_N^{(m-1)}(\tau_x\eta)\le\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{x\in\mathbb{T}_N}\mathbf{r}^{(k-1)}(\tau_x\eta)\le 2\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|(N+k-1)\lesssim N.$$

In this way, recalling that f is a density with respect to ν_α^N, we obtain the upper bound

$$\left\langle \mathcal{L}_N^{(m-1)} f\right\rangle_{\nu_\alpha^N}\le\frac{1}{2A}\Gamma_N^{(m-1)}(\sqrt{f},\nu_\alpha^N)+\frac{A}{2}\frac{\mathbf{c}_\alpha}{N}.$$

Plugging this upper bound into identity (3.44) with the choice $A=1$ we obtain

$$\mathcal{E}_N^{(m-1)}\ge\frac{1}{4}\Gamma_N^{(m-1)}(\sqrt{f},\mu)-\frac{\mathbf{c}_\alpha}{4N}.$$

To finish the proof, we see that the lower bounds in Proposition 3.2.10 imply that

$$\Gamma_N^{(m-1)}(\sqrt{f},\mu)\ge\mathbf{1}_{m\in(1,2)}m\left[\delta_N\Gamma_N^{(0)}(\sqrt{f},\mu)+\frac{m-1}{2}\Gamma_N^{(1)}(\sqrt{f},\mu)\right]+\mathbf{1}_{m\in(0,1)}\Gamma_N^{(0)}(\sqrt{f},\mu).$$

□

The next two technical results are standard but will be invoked in the proof of the replacement lemmas and in their applications.

Lemma B.1.1. *Consider $x,y\in\mathbb{T}$ and let $\varphi:[0,T]\times\Omega_N\to\mathbb{R}$ be invariant for the map $\eta\mapsto\eta^{x,y}$. Moreover, consider the measure ν_α^N with $\alpha(\cdot)\in(0,1)$ a constant function and let $f:\Omega_N\to\mathbb{R}$. For all $s\in[0,T]$ it holds that*

$$\int_{\Omega_N}\varphi(s,\eta)(\eta(x)-\eta(y))f(\eta)\nu_\alpha^N(\mathrm{d}\eta)=\frac{1}{2}\int_{\Omega_N}\varphi(s,\eta)(\eta(y)-\eta(x))(f(\eta^{x,y})-f(\eta))\nu_\alpha^N(\mathrm{d}\eta)$$

Proof. Summing and subtracting the appropriate term we have

$$\int_{\Omega_N} \varphi(s,\eta)(\eta(x)-\eta(y))f(\eta)\nu_\alpha^N(\mathrm{d}\eta) = \frac{1}{2}\int_{\Omega_N} \varphi(s,\eta)(\eta(x)-\eta(y))(f(\eta)-f(\eta^{x,y}))\nu_\alpha^N(\mathrm{d}\eta) + \frac{1}{2}\int_{\Omega_N} \varphi(s,\eta)(\eta(x)-\eta(y))(f(\eta)+f(\eta^{x,y}))\nu_\alpha^N(\mathrm{d}\eta).$$

To see that the second term in the right-hand side equals zero, simply note that performing the change of variables $\eta \mapsto \eta^{x,y}$ and using that $\varphi(s,\eta^{x,y}) = \varphi(s,\eta)$ and $\nu_\alpha^N(\eta^{x,y}) = \nu_\alpha^N(\eta)$ we obtain

$$\int_{\Omega_N} \varphi(s,\eta)(\eta(x)-\eta(y))f(\eta^{x,y})\nu_\alpha^N(\mathrm{d}\eta) = -\int_{\Omega_N} \varphi(s,\eta)(\eta(x)-\eta(y))f(\eta)\nu_\alpha^N(\mathrm{d}\eta).$$

□

The next proposition is applied in the second term in (3.24) and in (3.35)

Proposition B.1.2. *[6, Lemma 4.3.2]. Assume there exists a family $\mathcal{F}$ of functions $F_{N,\epsilon} : [0,T] \times \mathcal{D}([0,T],\Omega) \to \mathbb{R}$ satisfying*

$$\sup_{\substack{\epsilon\in(0,1),N\geq 1\\ s\in[0,T],\eta_\cdot\in\mathcal{D}([0,T],\Omega)}} \left|F_{N,\epsilon}(s,\eta_\cdot)\right| \leq M < \infty.$$

Above, the interval for (0,1) for ϵ is arbitrary. We also assume that for all $t \in [0,T]$,

$$\limsup_{\epsilon\to 0^+}\limsup_{N\to+\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t F_{N,\epsilon}(s,\eta_s)\mathrm{d}s\right|\right] = 0.$$

Then we have for all $\delta > 0$,

$$\limsup_{\epsilon\to 0^+}\limsup_{N\to+\infty} \mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\int_0^t F_{N,\epsilon}(s,\eta_s)\mathrm{d}s\right| > \delta\right) = 0.$$

B.1.1 Replacement Lemmas for $m \in (1,2)$

Lemma B.1.3. *Consider $x, y \in \mathbb{T}_N$. Let $\varphi : [0,T]\times\Omega_N \to \mathbb{R}$ be such that $\|\varphi\|_{L^\infty([0,T]\times\Omega_N)} \leq c_\varphi < \infty$ and invariant for the map $\eta \mapsto \eta^{z,z+1}$ with $z \in [\![x, y-1]\!]$. Then for all $B > 0$ and for all $t \in [0,T]$*

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \varphi(s,\eta_{N^2 s})(\eta_{N^2 s}(x)-\eta_{N^2 s}(y))\mathrm{d}s\right|\right] \lesssim \frac{1}{B} + TB|y-x|\frac{(\ell_N)^{m-1}}{N}.$$

Proof. From the entropy inequality (see [26, Appendix 1, Chapter 8]) with ν_α^N as reference measure and Feynman Kac's formula (see [1, page 14] for instance), we bound the previous expectation from above by

$$\frac{c_\alpha}{B} + \int_0^T \sup_f \left\{\left|\langle\varphi(s,\eta)(\eta(x)-\eta(y)), f\rangle_{\nu_\alpha^N}\right| - \frac{N}{B}\mathcal{E}_N^{(m-1)}(\sqrt{f},\nu_\alpha^N)\right\}\mathrm{d}s, \tag{B.1}$$

where the supremum is over densities with respect to ν_α^N. Rewriting $\eta(x)-\eta(y) = \sum_{z=x}^{y-1}\eta(z)-\eta(z+1)$, from Lemma B.1.1 the first term inside the supremum in (B.1) can be rewritten as

$$\frac{1}{2}\int_{\Omega_N}\sum_{z=x}^{y-1}\varphi(s,\eta)(\eta(z)-\eta(z+1))(f(\eta)-f(\eta^{z,z+1}))\nu_\alpha^N(\mathrm{d}\eta).$$

From Young's inequality we bound this from above by c_φ times

$$\frac{1}{4A}\int_{\Omega_N}\sum_{z=x}^{y-1}\left(\sqrt{f}(\eta^{z,z+1})+\sqrt{f}(\eta)\right)^2\nu_\alpha^N(\mathrm{d}\eta) + \frac{A}{4}\int_{\Omega_N}\sum_{z=x}^{y-1}\left(\nabla_{z,z+1}\sqrt{f}(\eta)\right)^2\nu_\alpha^N(\mathrm{d}\eta)$$

$$\leq \frac{|y-x|}{2A} + \frac{A}{2}\Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N),$$

where we performed a change of variables on the first term. Summarizing, applying Proposition 3.4.2 on (B.1) we bound (B.1) from above by

$$\frac{c_\alpha}{B} + T\left(\frac{1}{2}\Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N)\left(\frac{c_\varphi}{4}A - \frac{N}{B}\delta_N\right) + c_\varphi\frac{|y-x|}{2A}\right).$$

Fixing $A = 4N\delta_N/c_\varphi B$ and recalling from Proposition 3.2.10 that $0 < \delta_N = \sum_{k\geq \ell_N} |\binom{m-1}{k}| \lesssim (\ell_N)^{-(m-1)}$, the proof is concluded. □

Corollary B.1.4. *Fixed N, for any $i \in \mathbb{N}_+$ and $L \in \mathbb{N}_+$ such that $L < N$, let $\varphi : [0,T] \times \Omega_N \to \mathbb{R}$ be such that $\|\varphi\|_{L^\infty([0,T]\times\Omega_N)} \leq c_\varphi < \infty$ and invariant for the map $\eta \mapsto \eta^{z,z+1}$ with $z \in [\![iL, (i+1)L-2]\!]$. Then, for all $B > 0$ and for all $t \in [0,T]$ it holds*

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \varphi(s, \eta_{N^2 s})\left(\eta_{N^2 s}(iL) - \eta^L_{N^2 s}(iL)\right)\mathrm{d}s\right|\right] \lesssim \frac{1}{B} + TB\frac{(L+1)(\ell_N)^{m-1}}{N}.$$

Proof. Observing that $\eta(0) - \eta^L(0) = \frac{1}{L}\sum_{y\in\Lambda_0^L}(\eta(0) - \eta(y))$, from Lemma B.1.3 we can bound from above the expectation in the statement of the corollary by a constant times

$$\frac{1}{L}\sum_{y\in\Lambda_0^L}\frac{1}{B} + TBy\frac{(\ell_N)^{m-1}}{N} \lesssim \frac{1}{B} + TB\frac{(L+1)(\ell_N)^{m-1}}{N}.$$

□

Let us now state the two-blocks estimate:

Lemma B.1.5. *Fix $\epsilon > 0$ and $N \in \mathbb{N}$. For $i \in \mathbb{N}$ and $L < \epsilon N$ fixed, let $\varphi : [0,T] \times \Omega_N \to \mathbb{R}_+$ be such that $\|\varphi\|_{L^\infty([0,T]\times\Omega_N)} \leq c_\varphi < \infty$ and invariant for the map $\eta \mapsto \eta^{z,z+1}$ with $z \in [\![iL, iL + \lfloor N\epsilon\rfloor - 1]\!]$. Then for all $B > 0$ and for all $t \in [0,T]$ it holds that*

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \varphi(s, \eta_{N^2 s})\left(\eta^{\lfloor N\epsilon\rfloor}_{N^2 s}(i\lfloor N\epsilon\rfloor) - \eta^L_{N^2 s}(iL)\right)\mathrm{d}s\right|\right] \lesssim \frac{1}{B} + T\left[\frac{1}{L} + B\left(\frac{L(\ell_N)^{m-1}}{N} + i\frac{L}{N} + \epsilon(i+1)\right)\right].$$

Before proving this lemma, let us comment on the proof: we will follow closely the path argument in [5], although with some warm up before its application and some minor adjustments. Although for $m \in (1,2)$ the state-space is irreducible, the exclusion rates are not fast enough to travel along $\lfloor N\epsilon\rfloor$–distances for every configuration, which would avoid the use of the path argument below (as it is the case for $m \in (0,1)$). A simple way to see this quantitatively is to take $|y-x| = \epsilon N$ in Lemma B.1.3. The main reason for the resulting blow up is that the rate decreases as $\inf\{k \in [\![1, \ell_N]\!] : \mathbf{r}^{(k)}(\overline{\eta}) = 0\}$ increases, and so for certain configurations the jumping rate can be as small as $\delta_N \lesssim (\ell_N)^{-(m-1)}$ (see Proposition 3.2.10).

In order not to use the path argument we would need to replace the lower bound (3.45) by $\kappa\Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N)$, for some constant $\kappa > 0$ independent of N, which cannot be done because there is no such constant such that $\inf_{\eta\in\Omega_N} c_N^{(m-1)}(\eta) \geq \kappa$. Then again, we cannot relate the function inside the expectation in the statement of the two-blocks estimate solely with $\Gamma_N^{(1)}$, since this would require an initial shuffling of the configuration in order to move the particles with the PMM(1), hence there is the need to compare it with a SSEP term as well. In this way, we are restricted to finding some useful lower bound, such as (3.45). This introduces a second issue: in [5], the replacement scheme relies on the treatment of

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \eta^L_{N^2 s}(-L)\left(\eta^{\lfloor N\epsilon\rfloor}_{N^2 s}(0) - \eta^L_{N^2 s}(0)\right)\mathrm{d}s\right|\right],$$

analogously to (3.42). There, the authors start by conditioning on the number of particles in $\eta^L_{N^2 s}(-L)$, which allows them to introduce the PMM(1) rates via Young's inequality. In our case however, we have $\overline{\eta}^L_{N^2 s}(0)$, meaning that we must condition on the number of *holes* instead. Controlling the holes does not allow us to introduce the PMM(1) rates, but the $\overline{\text{PMM}}(1)$ rates instead, which are incompatible with the lower bound (3.45). To avoid this, one could distribute the products of empirical averages in (3.42), but doing so would necessarily lead to restrictions on the explosion rate of ℓ_N. The simple workaround is to replace *directly* $\eta^{\lfloor N\epsilon\rfloor}_{N^2 s}(0)$ by $\eta^L_{N^2 s}(0)$ with the conditioning happening inside either the $\lfloor N\epsilon\rfloor$ or L–boxes and not outside, and at the final step of the proof invoke Proposition 3.4.2.

Let us now go into the proof.

Proof of Lemma B.1.5. Analogously to the previous replacement lemmas, the expectation in the statement of the lemma can be estimated by some constant times

$$\frac{c_\alpha}{B} + \int_0^t \sup_f \left\{ \left| \left\langle \varphi(s,\eta) \left(\eta^{\lfloor N\epsilon\rfloor}(i\lfloor N\epsilon\rfloor) - \eta^L(iL) \right), f \right\rangle_{\nu^N_\alpha} \right| - \frac{N}{B} \mathcal{E}^{(m-1)}_N(\sqrt{f}, \nu^N_\alpha) \right\} \mathrm{d}s, \tag{B.2}$$

where the supremum is over densities with respect to ν^N_α and $\alpha(\cdot) \in (0,1)$ is a constant function. Now we break the box $\Lambda_0^{\lfloor N\epsilon\rfloor}$ into K smaller L–sized boxes:

$$[\![0, \lfloor N\epsilon\rfloor - 1]\!] = [\![0, KL-1]\!] = \bigcup_{j=1}^K [\![(j-1)L, jL-1]\!], \qquad K = \frac{\lfloor N\epsilon\rfloor}{L},$$

leading to

$$\eta^{\lfloor N\epsilon\rfloor}(i\lfloor N\epsilon\rfloor) - \eta^L(iL) = \frac{1}{K} \sum_{j=1}^K \left(\eta^L(i\lfloor N\epsilon\rfloor + (j-1)L) - \eta^L(iL) \right).$$

Note that we can do this only if $\lfloor N\epsilon\rfloor > L$, which is the case given that $L/N < \epsilon$. Moreover, K might not be an integer. Nevertheless, since for any bounded function $\psi : \Omega_N \to \mathbb{R}$ we have

$$\frac{1}{\lceil K\rceil} \sum_{j=1}^{\lceil K\rceil} \psi(\tau_j\eta) - \frac{1}{\lfloor K\rfloor} \sum_{j=1}^{\lfloor K\rfloor} \psi(\tau_j\eta) \lesssim \frac{1}{\lceil K\rceil} + \left(1 - \frac{\lfloor K\rfloor}{\lceil K\rceil}\right) \xrightarrow{K\to+\infty} 0$$

we proceed as if $K \in \mathbb{N}_+$.

For each $j \in \{1, \dots, K\}$ consider the event

$$X_j = \left\{ \eta \in \Omega_N : \eta^L(iL) \geq \frac{3}{L} \right\} \bigcup \left\{ \eta \in \Omega_N : \eta^L(i\lfloor N\epsilon\rfloor + (j-1)L) \geq \frac{3}{L} \right\}, \tag{B.3}$$

meaning that there are at least 3 particles in at least one of the boxes

$$\Lambda^L_{iL} = [\![iL, (i+1)L-1]\!] \quad \text{or} \quad \Lambda^L_{i\lfloor N\epsilon\rfloor + (j-1)L} = [\![i\lfloor N\epsilon\rfloor + (j-1)L, i\lfloor N\epsilon\rfloor + jL - 1]\!].$$

The integral, over $(X_j)^c$, of the first term in the variational formula (B.2) is of order L^{-1}, therefore we can bound from above the first term in the aforementioned variational formula by a term of order L^{-1} plus

$$\frac{1}{2KL} \sum_{j=1}^K \sum_{z\in\Lambda^L} \left| \int_{X_j} \varphi(s,\eta) \left(\eta(z + i\lfloor N\epsilon\rfloor + (j-1)L) - \eta(z+iL) \right) \left(f(\eta) - f(\eta^{z+i\lfloor N\epsilon\rfloor + (j-1)L, z+iL}) \right) \nu^N_\alpha(\mathrm{d}\eta) \right|, \tag{B.4}$$

where we used Lemma B.1.1. To estimate the quantity in the previous display, we use a path argument in the same spirit as in [5, Lemma 5.8], we claim that we can decompose

$$f(\eta) - f(\eta^{z+i\lfloor N\epsilon\rfloor+(j-1)L, z+iL}) = \sum_{n\in J^{\mathrm{PMM}(0)}} \big(f(\eta^{(n-1)}) - f(\eta^{(n)})\big) + \sum_{n\in J^{\mathrm{PMM}(1)}} \big(f(\eta^{(n-1)}) - f(\eta^{(n)})\big) \tag{B.5}$$

where

- $\eta^{(0)} = \eta$, $\eta^{(n+1)} = (\eta^{(n)})^{x(n),x(n)+1}$;
- $(x(n))_{n=0,\dots,N(x_1)}$ is a sequence of moves (following the procedure to be described shortly) taking values in the set $\{x_1, \dots, z+i\lfloor N\epsilon\rfloor+(j-1)L\}$, with $N(x_1)$ the number of nodes we have to exchange;
- $J^{\mathrm{PMM}(0)}, J^{\mathrm{PMM}(1)}$ are the sets of indexes that count the nodes used with the PMM(0) and PMM(1) dynamics, respectively, and are such that

  $$\big|J^{\mathrm{PMM}(0)}\big| \le J_0 L \quad \text{and} \quad \big|J^{\mathrm{PMM}(1)}\big| \le J_1(iL + jL + i\lfloor N\epsilon\rfloor)$$

  for some finite constants $J_0, J_1 > 0$;
- for each $n \in J^{\mathrm{PMM}(1)}$ we have $\mathbf{c}^{(1)}(\tau_{x(n-1)}\eta^{(n-1)}) > 0$.

Assuming all this, for $i \in \{0,1\}$ and $j \in \{1,\dots,K\}$ we have that

$$\begin{aligned}
&\sum_{n\in J^{\mathrm{PMM}(i)}} \int_{X_j} \big|f(\eta^{(n-1)}) - f(\eta^{(n)})\big| \nu_\alpha^N(\mathrm{d}\eta) \\
&\quad = \sum_{n\in J^{\mathrm{PMM}(i)}} \int_{X_j} \Big|\sqrt{f}(\eta^{(n-1)}) - \sqrt{f}(\eta^{(n)})\Big|\Big|\sqrt{f}(\eta^{(n-1)}) + \sqrt{f}(\eta^{(n)})\Big| \nu_\alpha^N(\mathrm{d}\eta) \\
&\quad \le \frac{A_i}{2} \sum_{n\in J^{\mathrm{PMM}(i)}} \int_{X_j} \mathbf{r}^{(i)}(\tau_{x(n-1)}\eta^{(n-1)}) \Big|\sqrt{f}(\eta^{(n-1)}) - \sqrt{f}(\eta^{(n)})\Big|^2 \nu_\alpha^N(\mathrm{d}\eta) \\
&\qquad + \frac{1}{2A_i} \sum_{n\in J^{\mathrm{PMM}(i)}} \int_{X_j} \frac{1}{\mathbf{r}^{(i)}(\tau_{x(n-1)}\eta^{(n-1)})} \Big|\sqrt{f}(\eta^{(n-1)}) + \sqrt{f}(\eta^{(n)})\Big|^2 \nu_\alpha^N(\mathrm{d}\eta)
\end{aligned} \tag{B.6}$$

for any $A_i > 0$. The inequality requires some justification. Fix some $n \in J^{\mathrm{PMM}(0)} \cup J^{\mathrm{PMM}(1)}$ and let us write $\xi = \eta^{(n-1)}$. Then

$$\begin{aligned}
f(\eta^{(n-1)}) - f(\eta^{(n)}) &= \mathbf{1}_{\{\eta\in\Omega_N:\, \eta_{x(n-1)}+\eta_{x(n-1)+1}=1\}}(\xi)\big(f(\xi) - f(\xi^{x(n-1),x(n-1)+1})\big) \\
&= \mathbf{r}^{(0)}(\tau_{x(n-1)}\xi)\big(f(\xi) - f(\xi^{x(n-1),x(n-1)+1})\big).
\end{aligned}$$

If $n \in J^{\mathrm{PMM}(0)}$ we are done. Otherwise, since $n \in J^{\mathrm{PMM}(1)}$, we have $\xi_{x(n-1)-1} + \xi_{x(n-1)+2} > 0$. Consider the set $\Omega_x^{(1)} = \{\eta \in \Omega_N :\ \mathbf{c}^{(1)}(\tau_x\eta) > 0\}$. Then $f(\xi) = f(\xi)\mathbf{1}_{\{\Omega^{(2)}_{x(n-1)}\}}(\xi)$, and since the constraints are independent of the occupation at the sites $x(n-1), x(n-1)+1$ we also have that $\xi^{x(n-1),x(n-1)+1} \in \Omega^{(1)}_{x(n-1)}$. As such,

$$f(\eta^{(n-1)}) - f(\eta^{(n)}) = \mathbf{r}^{(0)}(\tau_{x(n-1)}\xi)\left(f(\xi)\mathbf{1}_{\Omega^{(1)}_{x(n-1)}}(\xi) - f(\xi^{x(n-1),x(n-1)+1})\mathbf{1}_{\Omega^{(1)}_{x(n-1)}}(\xi^{x(n-1),x(n-1)+1})\right).$$

And since the change of variables $\xi \mapsto \xi^{x(n-1),x(n-1)+1} \in \Omega^{(1)}_{x(n-1)}$ is a bijection of $\Omega^{(1)}_{x(n-1)}$, we conclude that

$$f(\eta^{(n-1)}) - f(\eta^{(n)}) = \mathbf{1}_{\Omega^{(1)}_{x(n-1)}}(\xi)\mathbf{r}^{(0)}(\tau_{x(n-1)}\xi)\big(f(\xi) - f(\xi^{x(n-1),x(n-1)+1})\big)$$

and the rates for the PMM(1) can be introduced by using Young's inequality.

We treat the integral on the first term on the right-hand side of (B.6). Recall that $\xi = \eta^{(n-1)}$. For $i \in \{0,1\}$ and $j \in \{1,\dots,K\}$,

$$\sum_{\eta \in X_j} \mathbf{r}^{(i)}(\tau_{x(n-1)}\xi)\Big|\sqrt{f}(\xi) - \sqrt{f}(\xi^{x(n-1),x(n-1)+1})\Big|^2 \nu_\alpha^N(\eta) \\ \leq \sum_{\eta \in \Omega_N} \mathbf{r}^{(i)}(\tau_{x(n-1)}\xi)\Big|\sqrt{f}(\xi) - \sqrt{f}(\xi^{x(n-1),x(n-1)+1})\Big|^2 \nu_\alpha^N(\xi).$$

Since $\eta \in \Omega_N \Leftrightarrow \xi = \eta^{(n-1)} \in \Omega_N$, rearranging the first summation in the previous display and relabelling the terms yields

$$\sum_{\eta \in \Omega_N} \mathbf{r}^{(i)}(\tau_{x(n-1)}\eta)\Big|\sqrt{f}(\eta) - \sqrt{f}(\eta^{x(n-1),x(n-1)+1})\Big|^2 \nu_\alpha^N(\eta).$$

Consequently, the first term on the right-hand side of (B.6) can be bounded from above by $A_i \Gamma_N^{(i)}(\sqrt{f}, \nu_\alpha^N)$, while the second can be bounded from above by

$$\frac{1}{A_i} \sum_{n \in J^{\mathrm{PMM}(i)}} \int_{\Omega_N} \left(f(\eta^{(n-1)}) + f(\eta^{(n)})\right) \nu_\alpha^N(\mathrm{d}\eta) = \frac{2}{A_i}\left|J^{\mathrm{PMM}(i)}\right|.$$

In this way, (B.4) is no larger than

$$\frac{1}{2}A_0\Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N) + \frac{1}{2}A_1\Gamma_N^{(1)}(\sqrt{f}, \nu_\alpha^N) + J_0\frac{L}{A_0} + J_1\left(\frac{iL + i\lfloor N\epsilon\rfloor}{A_1} + \frac{KL}{A_1}\right).$$

Recalling Proposition 3.4.2, the quantity (B.2) is overestimated by

$$\frac{c_\alpha}{B} + T\sup_f \Bigg\{ 3\frac{c_\varphi}{L} + J_0\frac{L}{A_0} + J_1\frac{iL + (i+1)\lfloor N\epsilon\rfloor}{A_1} \\ + \frac{1}{2}\Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N)\left(A_0 - \frac{N}{B}\frac{m}{2}\delta_N\right) + \frac{1}{2}\Gamma_N^{(1)}(\sqrt{f}, \nu_\alpha^N)\left(A_1 - \frac{N}{B}\frac{m-1}{4}\right)\Bigg\},$$

where we recall that $KL = \lfloor N\epsilon\rfloor$. Setting

$$A_0 = \delta_N \frac{N}{B}\frac{m}{2} \quad \text{and} \quad A_1 = \frac{N}{B}\frac{m-1}{4}$$

we obtain an upper bound of the order of

$$\frac{1}{B} + T\left[\frac{1}{L} + B\left(\frac{L(\ell_N)^{m-1}}{N} + i\frac{L}{N} + \epsilon(i+1)\right)\right].$$

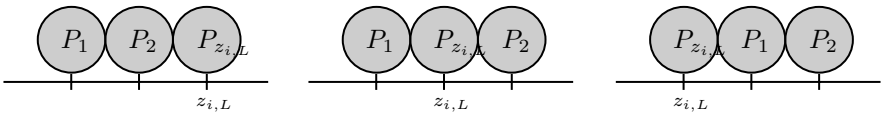
Now we prove our claim with the path argument. The goal is to exchange the occupation variables of the sites

$$z_{i,\epsilon,L} := z + i\lfloor N\epsilon\rfloor + (j-1)L \quad \text{and} \quad z_{i,L} := z + iL, \qquad z \in [\![0\,, L-1]\!].$$

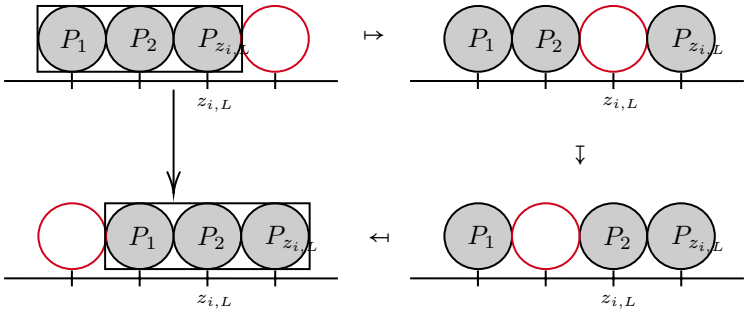
Recall that there are at least three particles either in Λ_{iL}^L or in $\Lambda_{i\lfloor N\epsilon\rfloor+(j-1)L}^L$. We outline the argument only for the case of at least three particles in Λ_{iL}^L since the other one is analogous and leads to an equivalent estimate. It is sufficient to consider configurations in (B.4) such that $\eta(z_{i,\epsilon,L}) + \eta(z_{i,L}) = 1$. The decomposition (B.5) illustrates a path on the state-space starting from the configuration η and ending at $\eta^{z_{i,\epsilon,L},z_{i,L}}$. Note that we can consider without loss of generalization that $\eta(z_{i,L}) = 1$, since if $\eta(z_{i,L}) = 0$ then we construct an analogous path starting from $\eta^{z_{i,\epsilon,L},z_{i,L}}$ and ending at η.

Recall that a *mobile cluster* with respect to the PMM(1) is a local configuration which can be translated on the lattice by a sequence of jumps dictated by the PMM(1). For example, the smallest mobile cluster for the PMM(1) corresponds to a local configuration where $\eta(x)+\eta(x+1)+\eta(x+2)=2$, for some $x \in \mathbb{T}_N$.

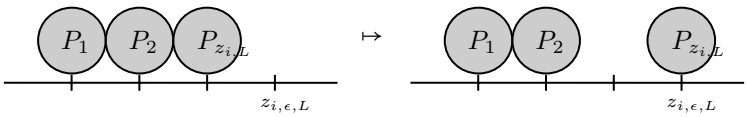
Since $\eta(z+iL)=1$, there are at least two other particles in Λ_{iL}^{L}. Pick the two closest to the site $z+iL$ and label them as P_1 and P_2. Let us also denote the particle at site $z_{i,L}$ by $P_{z_{i,L}}$. We use the SSEP dynamics to move P_1 and P_2 to the vicinity of $P_{z_{i,L}}$, forming a "mobile cluster". This can be done with a number of steps of order L. We arrive at one of the following three local configurations.



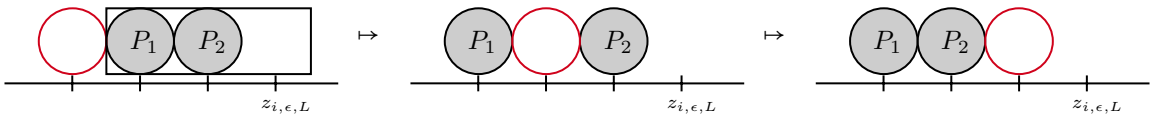
Note that we still need an empty site in the vicinity of these three particles to construct a mobile cluster. Nevertheless, if this is not the case we can assume that they are part of a larger mobile cluster. Moreover, we can relabel the particles and use the SSEP dynamics to have the local configuration (for example) as in the first case of the previous figure. Now we move this mobile cluster to the left of the (empty) site $z_{i,\epsilon,L}$ with the PMM(1) dynamics.



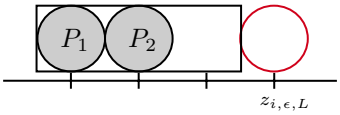
The number of steps can be crudely bounded above by a term of order $L+(i\lfloor N\epsilon\rfloor+(j-1)L)$. By hypothesis, $\eta(z_{i,\epsilon,L})=0$ and so we leave $P_{z_{i,L}}$ at site $z_{i,\epsilon,L}$ using either the SSEP or the PMM(1) dynamics, and transport the hole to the site $z_{i,L}$ with the PMM(1) dynamics.



If the site to the left of P_1 is either empty or occupied, we can perform the following transport with either the PMM(1) or a relabelling in the last step.



If the aforementioned site was occupied, we can exchange the hole and the particle at site $z+i\lfloor N\epsilon\rfloor+(j-1)L-4$ with the PMM(1) dynamics, otherwise there is nothing to do and we relabel the hole, obtaining



This procedure is repeated at most an order of $L+(i\lfloor N\epsilon\rfloor+(j-1)L)$ steps, moving the mobile cluster to the vicinity of the site $z_{i,L}$. The SSEP dynamics is then used to shuffle the configuration restricted to the box Λ_{iL}^{L}, moving P_1 and P_2 to their original sites with a cost of at most an order of L steps.

□

B.1.2 Replacement Lemmas for $m \in (0,1)$

Lemma B.1.6. *For all $B>0$ and $\mathcal{T}\subseteq[0,T]$ it holds that*

$$\mathbb{E}_{\mu_N}\left[\left|\int_{\mathcal{T}}\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^{+}\overline{\eta}_{N^2s})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta}_{N^2s})\mathrm{d}s\right|\right]\lesssim\frac{1}{B}+|\mathcal{T}|B\frac{(\ell_N)^{1-m}}{N}.$$

Proof. Proceeding as previously, we have to estimate

$$\frac{c_\alpha}{B}+\int_{\mathcal{T}}\sup_f\left\{\left|\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\langle\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^{+}\overline{\eta})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta}),f\rangle_{\nu_\alpha^N}\right|-\frac{N}{B}\mathcal{E}_N^{(m-1)}(\sqrt{f},\nu_\alpha^N)\right\}\mathrm{d}s. \tag{B.7}$$

Note that we have the following inequality

$$\sum_{j=1}^{k-1}\sum_{n=0}^{k-1-j}(\nabla^{+}\overline{\eta})(n)\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta})\le\sum_{j=1}^{k-1}\sum_{n=0}^{k-1}\left|(\nabla^{+}\overline{\eta})(n)\right|\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta})\le\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta}).$$

Since $\mathbf{s}_j^{(k-1)}(\tau_n\overline{\eta})$ is invariant for the map $\eta\mapsto\eta^{n,n+1}$, from Lemma B.1.1, Young's inequality and the inequality in the previous display we can bound from above the summation over j in (B.7) by:

$$\frac{1}{4A}\int_{\Omega_N}\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta})\left(\sqrt{f}(\eta^{n,n+1})+\sqrt{f}(\eta)\right)^2\nu_\alpha^N(\mathrm{d}\eta)$$
$$+\frac{A}{4}\int_{\Omega_N}\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta})\left(\nabla_{n,n+1}\sqrt{f}(\eta)\right)^2\nu_\alpha^N(\mathrm{d}\eta).$$

Taking the binomial coefficients into consideration, we bound from above

$$\int_{\Omega_N}\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta})\left(\nabla_{n,n+1}\sqrt{f}(\eta)\right)^2\nu_\alpha^N(\mathrm{d}\eta)\le\Gamma_N^{(m-1)}(\sqrt{f},\nu_\alpha^N).$$

We have the following upper bounds

$$\begin{aligned}
&\int_{\Omega_N}\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta})\left(\sqrt{f}(\eta^{n,n+1})+\sqrt{f}(\eta)\right)^2\nu_\alpha^N(\mathrm{d}\eta)\\
&\qquad\le 2\int_{\Omega_N}\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta})\left(f(\eta^{n,n+1})+f(\eta)\right)\nu_\alpha^N(\mathrm{d}\eta)\\
&\qquad=4\int_{\Omega_N}\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|\sum_{n=1}^{k-1}\mathbf{r}^{(k-1)}(\tau_n\overline{\eta})f(\eta)\nu_\alpha^N(\mathrm{d}\eta)\\
&\qquad\le 16\sum_{k=1}^{\ell_N}\left|\binom{m}{k}\right|k\int_{\Omega_N}f(\eta)\nu_\alpha^N(\mathrm{d}\eta)\le c(\ell_N)^{1-m}
\end{aligned}$$

where $c>0$ is some constant independent of N. The previous inequalities follow, respectively, from Young's inequality, the fact that the profile α is constant, Lemma 3.2.16, the fact of f being a density

and then Lemma B.3.1 and an integral comparison. With all this, we obtain the following estimate for (B.7)

$$\frac{c_\alpha}{B} + |\mathcal{T}|\left(\frac{c}{4}\frac{(\ell_N)^{1-m}}{A} + \Gamma_N^{(m-1)}(\sqrt{f}, \nu_\alpha^N)\left(\frac{1}{4}A - \frac{N}{B}\right)\right).$$

Fixing $A = 4N/B$ concludes the proof. □

Lemma B.1.7. *Consider $x, y \in \mathbb{T}_N$. Let $\varphi : [0,T]\times\Omega_N \to \mathbb{R}$ such that $\|\varphi\|_{L^\infty([0,T]\times\Omega_N)} < \infty$ and invariant for the map $\eta \mapsto \eta^{z,z+1}$ with $z \in [\![x, y-1]\!]$. Then, for all $B > 0$ and for all $t \in [0,T]$ it holds*

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \varphi(s, \eta_{N^2 s})(\eta_{N^2 s}(x) - \eta_{N^2 s}(y))\mathrm{d}s\right|\right] \lesssim \frac{1}{B} + TB\frac{|y-x|}{N}.$$

Proof. Repeating the computations in the proof of Lemma B.1.3, there exist constants $c_0, c_1, c_2 > 0$ such that we can overestimate the expectation by

$$\frac{c_0}{B} + T\sup_f\left\{c_1 A\Gamma_N^{(0)}(\sqrt{f}, \nu_\alpha^N) + c_2\frac{|y-x|}{A} - \frac{N}{B}\mathcal{E}_N^{(m-1)}(\sqrt{f}, \nu_\alpha^N)\right\}.$$

Recalling the lower bound for the Dirichlet form in Proposition 3.4.2 we can choose $A = mN/c_1 B$. □

B.2 PDE results

B.2.1 Slow diffusion

The following result extends [5, Lemma 6.2] to the case $m \in (1,2)$.

Proposition B.2.1. *Let $f, g \in [0,1]$ with $f \neq g$. If $m \in (1,2)$ then, for all $A > 0$ we have*

$$|f-g| \le \frac{|(f)^m - (g)^m|}{V^{(m)}(f,g) + A} + A\frac{2}{m(m-1)}.$$

where

$$0 < V^{(m)}(f,g) = \sum_{k\ge 1}\binom{m}{k}(-1)^{k+1}v_k(1-f, 1-g) < \infty$$

and

$$v_k(f,g) = \mathbf{1}_{k=1} + \mathbf{1}_{k=2}(f+g) + \mathbf{1}_{k\ge 3}\left(f^{k-1} + g^{k-1} + \sum_{i=1}^{k-2} g^i f^{k-1-i}\right).$$

Proof. We start with $f, g \in (0,1)$.

$$(f)^m - (g)^m = \sum_{k\ge 1}\binom{m}{k}(-1)^k\left((1-f)^k - (1-g)^k\right).$$

One can rewrite, for any $k \in \mathbb{N}_+$,

$$a^k - b^k = (a-b)\left[\mathbf{1}_{k=1} + \mathbf{1}_{k=2}(a+b) + \mathbf{1}_{k\ge 3}\left(a^{k-1} + b^{k-1} + \sum_{i=1}^{k-2} b^i a^{k-1-i}\right)\right] = (a-b)v_k(a,b). \tag{B.8}$$

In this way,

$$(f)^m - (g)^m = (f-g)\sum_{k\ge 1}\binom{m}{k}(-1)^{k+1}v_k(1-f, 1-g) = (f-g)V^{(m)}(f,g).$$

We show that $V^{(m)}(f,g) > 0$. Assume $f, g \in (0,1)$ with $f > g$. Then, $f^m - g^m > 0$ implies $V^{(m)}(f,g) > 0$. Similarly, if $f < g$ then $f^m - g^m < 0 \implies V^{(m)}(f,g) > 0$. With this in mind, we can rewrite

$$(f)^m - (g)^m = (f-g)\left(V^{(m)}(f,g) \pm A\right) \Leftrightarrow f - g = \frac{(f)^m - (g)^m}{V^{(m)}(f,g) + A} + A\frac{f-g}{V^{(m)}(f,g)+A}, \quad \text{for any } A > 0.$$

Now we will treat the second term on the right-hand side of last display. Note that

$$V^{(m)}(f,g) = m\sum_{k\geq 0}\binom{m-1}{k}(-1)^k \frac{v_{k+1}(1-f,1-g)}{k+1}.$$

Since $m \in (1,2)$ and $v_1(1-f,1-g) = 1$, then

$$V^{(m)}(f,g) = m\left(1 - \sum_{k\geq 1}\left|\binom{m-1}{k}\right|\frac{v_{k+1}(1-f,1-g)}{k+1}\right) = m\sum_{k\geq 1}\left|\binom{m-1}{k}\right|\left(1 - \frac{v_{k+1}(1-f,1-g)}{k+1}\right),$$

where we note that

$$1 - \sum_{k\geq 1}\left|\binom{m-1}{k}\right| = 0.$$

Since $f, g \in (0,1)$ we also have $0 < \frac{v_{k+1}(1-f,1-g)}{k+1} < 1$, and so let us introduce

$$W^{(m)}(f,g) = m\sum_{k\geq 2}\left|\binom{m-1}{k}\right|\left(1 - \frac{v_{k+1}(1-f,1-g)}{k+1}\right) > 0.$$

In this way, we can write

$$V^{(m)}(f,g) = m(m-1)\left(1 - \frac{v_2(1-f,1-g)}{2}\right) + W^{(m)}(f,g) = m\frac{m-1}{2}(f+g) + W^{(m)}(f,g).$$

Now back to our main problem,

$$A\frac{|f-g|}{V^{(m)}(f,g)+A} = A\frac{|f-g|}{m\frac{m-1}{2}(f+g) + W^{(m)}(f,g) + A} \leq A\frac{2}{m(m-1)}\frac{|f-g|}{f+g} \leq A\frac{2}{m(m-1)}.$$

If $f = 1$ we can write $1 - (g)^m = (1-g)V(1,g)$, while if $f = 0$, we use instead that $0 = \sum_{k\geq 0}\binom{m}{k}(-1)^k$. For either $f \in \{0,1\}$, the rest of the proof is analogous.

To check that $V^{(m)}$ is bounded is enough to bound from above $v_k \leq k$ and use the estimate for the binomial coefficients from Lemma B.3.1. □

Corollary B.2.2 ($\frac{1}{4}$–Hölder continuity)**.** *If $\rho^m \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$, with $m \in (1,2)$, then for any $t \in [0,T]$*

$$|\rho_t(u) - \rho_t(v)| \leq |v-u|^{\frac{1}{4}}\left(\frac{2}{m(m-1)} + \|\partial_u(\rho_t^m)\|_{L^2(\mathbb{T})}\right) \quad \text{a.e. } u, v \in \mathbb{T}.$$

Proof. Since ρ^m is in the target Sobolev space, we have a weak derivative of ρ and can write a.e., from the previous proposition

$$|\rho_t(u) - \rho_t(v)| \leq \frac{\int_u^v \partial_w(\rho_t^m)\mathrm{d}w}{V^{(m)}(\rho_t(u),\rho_t(v)) + A} + \frac{2A}{m(m-1)} \leq \frac{1}{A}\int_u^v \partial_w(\rho_t^m)\mathrm{d}w + \frac{2A}{m(m-1)}.$$

We now apply Cauchy-Schwarz's inequality and set $A = |v-u|^{\frac{1}{4}}$. □

Lemma B.2.3 (Uniqueness of weak solutions)**.** *For $\rho^{\mathrm{ini}} : \mathbb{T} \to [0,1]$ a measurable initial profile the weak solution of* (1.2)*, in the sense of Definition 3.2.22, is unique.*

Proof. The proof relies on the same choice of test function as in [24, Lemma 6.3], there for solutions of the FDE with $m = -1$. Note that for $m \in (1,2)$ holds $\rho^m \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$. A solution ρ of (1.2) satisfies then the formulation (3.21) or, equivalently,

$$0 = \langle \rho_t, G_t\rangle - \langle \rho^{\mathrm{ini}}, G_0\rangle - \int_0^t \langle \rho_s, \partial_s G_s\rangle \mathrm{d}s + \int_0^t \langle \partial_u(\rho_s)^m, \partial_u G_s\rangle \mathrm{d}s$$

for any $G \in C^{1,2}([0,T]\times\mathbb{T})$. In particular, one can consider the alternative formulation where the regularity of G above is reduced to $G \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$ and $\partial_t G \in L^2([0,T];L^2[0,1])$ (satisfying the equality on the previous display), and then show the equivalence of formulations by approximating G by a sequence of functions in $C^{1,2}([0,T]\times\mathbb{T})$. Assume that $\rho^{(1)}, \rho^{(2)}$ are two solutions starting from the same profile ρ^{ini} and write $w = \rho^{(1)} - \rho^{(2)}$. Then w satisfies the equality

$$\langle w_t, G_t\rangle = \int_0^t \langle w_s(u), \partial_s G_s\rangle \mathrm{d}s - \int_0^t \langle \partial_u \left((\rho_s^{(1)})^m - (\rho_s^{(2)})^m \right), \partial_u G_s\rangle \mathrm{d}s.$$

With the choice of test function

$$G_s(u) = \int_s^t (\rho_r^{(1)}(u))^m - (\rho_r^{(2)}(u))^m \mathrm{d}r,$$

we obtain

$$\langle w_t, G_t\rangle = 0 = -\int_0^t \left\langle w_s, \left((\rho_s^{(1)})^m - (\rho_s^{(2)})^m \right)\right\rangle \mathrm{d}s - \frac{1}{2}\left\| \int_0^t \partial_u \left((\rho_r^{(1)})^m - (\rho_r^{(2)})^m \right) \mathrm{d}r \right\|_2^2.$$

It is simple to see that $w_s(u)\left(\rho_1^m(s,u) - \rho_2^m(s,u)\right) \geq 0$ for a.e. $u \in \mathbb{T}$, implying $w = 0$ almost everywhere. □

B.2.2 Fast diffusion

Proposition B.2.4 ($\frac{1}{2}$–Hölder continuity)**.** *If $\rho \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$ then for any $t \in [0,T]$ it holds that*

$$|\rho_t(u) - \rho_t(v)| \leq |u-v|^{\frac{1}{2}} \|\partial \rho_t\|_{L^2(\mathbb{T})} \quad a.e.\ u, v \in \mathbb{T}$$

Proof. This is a simple consequence of Cauchy-Schwarz's inequality. □

Lemma B.2.5 (Uniqueness of weak solutions)**.** *For $\rho^{\mathrm{ini}} : \mathbb{T} \to [0,1]$ a measurable initial profile the weak solution of* (1.2) *in the sense of Definition 3.2.22 is unique.*

Proof. For $m \in (0,1)$ our weak formulation can be shown to be equivalent to

$$\langle \rho_t, G_t\rangle - \langle \rho^{\mathrm{ini}}, G_0\rangle = \int_0^t \left\{ \langle \rho_s, \partial_s G_s\rangle + \langle (\rho_s)^m, \partial_{uu} G_s\rangle \right\} \mathrm{d}s, \qquad \forall t \in (0,T],$$

where $G \in C^{1,2}([0,T]\times\mathbb{T})$. Recall also that we already showed, in Proposition 3.4.8, that there exists a solution $\rho \in L^2([0,T];\mathcal{H}^1(\mathbb{T}))$. Let $\rho^{(1)}, \rho^{(2)}$ be two solutions starting from the same initial data and write $w = \rho^{(1)} - \rho^{(2)}$. Then we have the following equation

$$\langle w_t, G_t\rangle = \int_0^t \left\{ \langle w_s, \partial_s G_s\rangle + \langle (\rho_s^{(1)})^m - (\rho_s^{(2)})^m, \partial_{uu} G_s\rangle \right\} \mathrm{d}s = 0.$$

We will write $(\rho^{(1)})^m - (\rho^{(2)})^m$ as a function of w. To do so, we consider the binomial expansion of these powers. Since $m \in (0,1)$ we have

$$(\rho^{(1)})^m - (\rho^{(2)})^m = \sum_{k\geq 1} \left| \binom{m}{k} \right| \left((1-\rho^{(2)})^k - (1-\rho^{(1)})^k \right).$$

It is important to truncate now the series at some step ℓ which will be taken to infinity later on. Let $\ell \in \mathbb{N}_+$. Then

$$\sum_{k\geq \ell+1} \left|\binom{m}{k}\right| \left((1-\rho^{(2)})^k - (1-\rho^{(1)})^k\right) \leq \sum_{k\geq \ell+1} \left|\binom{m}{k}\right| = \mathcal{O}\left(\ell^{-m}\right).$$

As such, from (B.8)

$$\sum_{k=1}^{\ell} \left|\binom{m}{k}\right| \left((1-\rho^{(2)})^k - (1-\rho^{(1)})^k\right) = w \sum_{k=1}^{\ell} \left|\binom{m}{k}\right| v_k(1-\rho^{(2)}, 1-\rho^{(1)}) =: wV^{\ell}$$

where we shorten $V_s^{\ell}(u) \equiv V^{\ell}(\rho_s^{(1)}(u), \rho_s^{(2)}(u))$ and $v_k(s,u) \equiv v_k(1-\rho_s^{(1)}(u), 1-\rho_s^{(2)}(u))$. Note that for each ℓ fixed we have the crude upper bound

$$V_s^{\ell}(u) \leq \sum_{k=1}^{\ell} \left|\binom{m}{k}\right| k = \mathcal{O}\left(\ell^{1-m}\right).$$

This truncation allows us to obtain

$$\int_0^t \langle (\rho_s^1)^m - (\rho_s^2)^m, \partial_{uu} G_s\rangle \mathrm{d}s \lesssim \int_0^t \langle w_s V_s^{\ell}, \partial_{uu} G_s\rangle \mathrm{d}s + \frac{1}{\ell^m} \int_0^t \int_{\mathbb{T}} |\partial_{uu} G_s(u)| \mathrm{d}u\mathrm{d}s.$$

Because for each fixed ℓ we have $V^{\ell} \in L^p([0,t]\times\mathbb{T})$, for any $1 \leq p \leq \infty$, one can approximate V^{ℓ} by a sequence of functions in $C^{\infty}([0,t]; L^{\infty}(\mathbb{T}))$, with $t \in [0,T]$, and with respect to the $L^p([0,t]\times\mathbb{T})$ norm. Let φ be some positive mollifier and define $\varphi_\epsilon = \epsilon^{-1}\varphi(\epsilon^{-1}\,\cdot)$ for $\epsilon > 0$. Define

$$V_{\cdot}^{\ell,\epsilon}(u) = V_{\cdot}^{\ell}(u) \star \varphi_\epsilon.$$

Note that $V^{\ell,\epsilon} \in L^p([0,T]\times\mathbb{T})$ for any $1 \leq p \leq \infty$ because V^{ℓ} is uniformly bounded in both time and space. Denote by $\hat{f}$ the Fourier transformation of a function f defined on $[0,t]$. From Parseval-Plancherel's identity we have the isometry

$$\left\|V_{\cdot}^{\ell,\epsilon}(u) - V_{\cdot}^{\ell}(u)\right\|_{L^2([0,t])} = \left\|\widehat{V_{\cdot}^{\ell,\epsilon}(u)} - \widehat{V_{\cdot}^{\ell}(u)}\right\|_{L^2([0,t])} = \left[\int_0^t \left|\widehat{V_{\cdot}^{\ell}(u)}(\xi)\right|^2 |1-\widehat{\varphi_\epsilon}(\xi)|^2 \mathrm{d}\xi\right]^{\frac{1}{2}}.$$

Because the mollifier is normalized and positive,

$$|1-\widehat{\varphi_\epsilon}(\xi)| \leq \int_{B_\epsilon(0)} \varphi_\epsilon(v) \left|(1-e^{-iv\xi})\right| \mathrm{d}v,$$

where $B_\epsilon(0)$ is the open ball in $\mathbb{T}$ centred in zero and with radius $\epsilon > 0$. Since $e^{-x} \geq 1-x$ we can see that

$$\sup_{v\in B_\epsilon(0)} \left|(1-e^{-iv\xi})\right| \leq \sup_{v\in B_\epsilon(0)} |iv\xi| \leq \epsilon|\xi|.$$

With this we obtain the estimate

$$\left\|V_{\cdot}^{\ell,\epsilon}(u) - V_{\cdot}^{\ell}(u)\right\|_{L^2([0,t])} \leq \epsilon \left[\int_0^t \left|\widehat{V_{\cdot}^{\ell}(u)}(\xi)\right|^2 |\xi|^2 \mathrm{d}\xi\right]^{\frac{1}{2}} \leq t\epsilon \left[\int_0^t \left|\widehat{V_{\cdot}^{\ell}(u)}(\xi)\right|^2 \mathrm{d}\xi\right]^{\frac{1}{2}} = t\epsilon \left\|V_{\cdot}^{\ell}(u)\right\|_{L^2([0,t])}$$

and the right-hand side of the previous display is no larger than a constant times $t^{\frac{3}{2}}\epsilon\ell^{1-m}$. In particular, from Cauchy-Schwarz's inequality

$$\begin{aligned}\int_0^t \langle w_s V_s^{\ell}, \partial_{uu} G_s\rangle \mathrm{d}s \leq{}& \int_0^t \langle w_s V_s^{\ell,\epsilon}, \partial_{uu} G_s\rangle \mathrm{d}s \\ &+ \int_{\mathbb{T}} \left[\int_0^t \left|V_s^{\ell}(u) - V_s^{\ell,\epsilon}(u)\right|^2 \mathrm{d}s\right]^{\frac{1}{2}} \left[\int_0^t |\partial_{uu} G_s(u)|^2 \mathrm{d}s\right]^{\frac{1}{2}} \mathrm{d}u.\end{aligned}$$

From the previous computations and again from the Cauchy-Schwarz's inequality, the second line in last display is bounded above by $t^{\frac{3}{2}}\epsilon\ell^{1-m}\|\partial_{uu}G\|_{L^2([0,t]\times\mathbb{T})}$. We just showed that

$$\langle w_t, G_t\rangle \lesssim \int_0^t \int_{\mathbb{T}} w_s(u)\big\{\partial_s G_s(u) + V_s^{\ell,\epsilon}(u)\partial_{uu}G_s(u)\big\}\mathrm{d}u\mathrm{d}s + t^{\frac{1}{2}}\ell^{-m}\left(1+\epsilon t\ell\right)\|\partial_{uu}G\|_{L^2([0,t]\times\mathbb{T})}.$$

We want to fix G as a solution to the backwards problem

$$\begin{cases}\partial_s f + \lambda\partial_{uu}f = 0, & (s,u)\in[0,t)\times\mathbb{T},\\ f(t,u) = \phi(u), & u\in\mathbb{T},\end{cases} \tag{B.9}$$

with ϕ to be chosen suitably later on. This is a well-posed problem and has a solution $f\in C^{1,2}([0,t]\times\mathbb{T})$ given some conditions on ϕ and λ: under the new time $\tau = t-s$ a solution to this problem is equivalently a solution to

$$\begin{cases}\partial_\tau g = \lambda\partial_{uu}g, & (\tau,u)\in(0,t]\times\mathbb{T},\\ g(0,u) = \phi(u), & u\in\mathbb{T}.\end{cases}$$

According to [20, Thm. 4.5, Ch. 6, Sec. 4], for λ positive and bounded uniformly in $[0,t]\times\mathbb{T}$, continuous with respect to time (uniformly in $\mathbb{T}$) and $\alpha-$Hölder continuous with respect to the space variable; and ϕ a continuous function, there exists a solution to this Cauchy problem in $C^{1,2}([0,t]\times\mathbb{T})$. Note that we have already checked that $V^{\ell,\epsilon}$ satisfies all the requirements for λ above (for ℓ fixed) except the Hölder continuity condition. Noting that $\rho^{(1)},\rho^{(2)}$ is $\frac{1}{2}-$Hölder so is V_ϵ. To see this we sum and subtract appropriate terms and use the triangle inequality to estimate

$$\begin{aligned}|v_k(s,x)-v_k(s,y)| \le{}& \big|\rho_s^{(1)}(y)-\rho_s^{(1)}(x)\big|\sum_{i=0}^{k-1} v_i(1-\rho_s^{(1)}(x),1-\rho_s^{(1)}(y))(1-\rho_s^{(2)}(x))^{k-1-i}\\ &+\big|\rho_s^{(2)}(y)-\rho_s^{(2)}(x)\big|\sum_{i=0}^{k-1} v_{k-1-i}(1-\rho_s^{(2)}(x),1-\rho_s^{(2)}(y))(1-\rho_s^{(1)}(x))^{i} \lesssim k^2|x-y|^{\frac{1}{2}}.\end{aligned}$$

In this way,

$$|(v_k(\cdot,x)-v_k(\cdot,y))\star\varphi_\epsilon(s)| = \int_0^t \varphi_\epsilon(s-r)(v_k(r,x)-v_k(r,y))\mathrm{d}r \le k^2|x-y|^{\frac{1}{2}}\int_0^t\varphi_\epsilon(s-r)\mathrm{d}r.$$

Recalling that the integral on the right-hand side equals one, we see that

$$\big|V_s^{\ell,\epsilon}(x)-V_s^{\ell,\epsilon}(y)\big| \le \sum_{k=1}^{\ell}\left|\binom{m}{k}\right||v_k(\cdot,x)-v_k(\cdot,y)\star\varphi_\epsilon(s)| \lesssim |x-y|^{\frac{1}{2}}\ell^{2-m}.$$

In this way, fixing our test function as $G=f$ with $\lambda = V^{\ell,\epsilon}$ we see that

$$\langle w_t,\phi\rangle \lesssim t^{\frac{1}{2}}\ell^{-m}(1+\epsilon t\ell)\|\partial_{uu}G\|_{L^2([0,t]\times\mathbb{T})}$$

and we need to estimate the integral on the right-hand side above.

Let us multiply both sides of (B.9) by $\partial_{uu}G$ and integrate once in space and time, obtaining

$$0 = \int_0^t\int_{\mathbb{T}}\partial_s G\partial_{uu}G\mathrm{d}u\mathrm{d}s + \int_0^t\int_{\mathbb{T}}V^{\ell,\epsilon}|\partial_{uu}G|^2\mathrm{d}u\mathrm{d}s.$$

An integration by parts on the first integral on the right-hand side above yields

$$-\int_0^t\int_{\mathbb{T}}\partial_u(\partial_s G)\partial_u G\mathrm{d}u\mathrm{d}s = -\frac{1}{2}\int_0^t\int_{\mathbb{T}}\partial_s\left(\partial_u G\right)^2\mathrm{d}u\mathrm{d}s$$

$$= -\frac{1}{2}\int_{\mathbb{T}} \left\{ (\partial_u G_t(u))^2 - (\partial_u G_0(u))^2 \right\} \mathrm{d}u.$$

Using the terminal condition and bounding from below $(\partial_u G_0(u))^2 \geq 0$ and $V_\epsilon^\ell > m$ we conclude that

$$\int_0^t \int_{\mathbb{T}} |\partial_{uu} G|^2 \mathrm{d}u\mathrm{d}s \leq \frac{1}{2m} \|\phi'\|^2_{L^2(\mathbb{T})}.$$

With this, and fixing $\epsilon = 1/\ell$ we obtain

$$\langle w_t, \phi \rangle \lesssim t^{\frac{1}{2}} \ell^{-m} (1+t) \|\phi'\|_{L^2(\mathbb{T})}. \tag{B.10}$$

Denoting by $w^\pm$ the positive/negative part of w, we want to fix $\phi(\cdot) = \mathbf{1}_{\{u\in\mathbb{T}:\, w_t(u)\geq 0\}}(t,\cdot)$, obtaining that $\rho^{(1)} \leq \rho^{(2)}$ a.e., and analogously take $\phi(\cdot) = \mathbf{1}_{\{u\in\mathbb{T}:\, w_t(u)\leq 0\}}(t,\cdot)$, obtaining instead $\rho^{(1)} \geq \rho^{(2)}$ and leading to $\rho^{(1)} = \rho^{(2)}$ a.e. To do so we need to consider in (B.10) a sequence $(\phi_n)_n \subset C(\mathbb{T})$ converging to ϕ at least in L^2 and such that $\|\phi_n'\|_{L^2(\mathbb{T})} < \infty$ for all $n > 0$. Regarding the convergence, since $\phi \in L^2(\mathbb{T})$ and $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ for all $1 \leq p < \infty$, there is a sequence of continuous functions $(\phi_n)_n$ approximating ϕ in $L^2(\mathbb{T})$. This sequence of continuous functions can be approximated (in L^2) by a sequence of smooth functions $(\phi_{n,k})_k$ *via* mollification. We fix one of these smooth representatives as the terminal condition on the problem (B.9). Taking the limit $\ell \to +\infty$ in (B.10) and then the limits on n and k, and recalling that $t \in [0,T]$ is arbitrary concludes the proof.

□

B.3 Auxiliary results

Lemma B.3.1. *For any $m \in \mathbb{R}_+$ and any $k \in \mathbb{N}_+$ such that $k \geq 2$, it holds*

$$\frac{\Gamma(m)|\sin(\pi(k-m))|}{\pi(k+1)^m} < \left| \binom{m-1}{k} \right| < \frac{\Gamma(m)|\sin(\pi(k-m))|}{\pi(k-m)^m} \lesssim \frac{1}{k^m},$$

where the Γ−function is defined, for any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, as

$$\Gamma(z) = \int_0^{+\infty} u^{z-1} e^{-u} \mathrm{d}u.$$

Proof. The binomial coefficients have the following classical representation in terms of the Γ−function

$$\binom{m-1}{k} = \frac{\Gamma(m)}{\Gamma(k+1)\Gamma(m-k)}.$$

From the reflection formula

$$\Gamma(m-k)\Gamma(k+1-m) = \frac{\pi}{\sin(\pi(m-k))},$$

we can rewrite

$$\binom{m-1}{k} = \frac{\sin(\pi(m-k))}{\pi} \frac{\Gamma(m)\Gamma(k+1-m)}{\Gamma(k+1)}.$$

Recall now the B−function, defined on $z, w \in \mathbb{C} :\ \mathrm{Re}(z), \mathrm{Re}(w) > 0$, as

$$\mathrm{B}(z,w) = \int_0^1 v^{z-1}(1-v)^{w-1} \mathrm{d}v = \int_0^{+\infty} \frac{s^{w-1}}{(s+1)^{w+z}} \mathrm{d}s,$$

where the equality of the representations above can be checked by performing the change of variables $v = s/(s+1)$ on the first integral. From the definition of Γ, one can show the following classical relationship between the Γ and B functions, for all $z, w \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Re}(w) > 0$:

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

In this way, we can rewrite

$$\binom{m-1}{k} = \frac{\sin(\pi(m-k))}{\pi}\mathrm{B}(m, k+1-m).$$

Recall that for $k \geq 2$ holds $(m-1)_k = (-1)^{k-\lfloor m \rfloor}|(m-1)_k|$. Noticing that $\mathrm{B}(m, k+1-m) > 0$, we then have that $\sin(\pi(m-k)) = (-1)^{k-\lfloor m \rfloor}|\sin(\pi(m-k))|$ and we need only to find an upper and lower bound for the B−function. From the inequality $e^x \geq 1 + x$, valid for $x \in \mathbb{R}$, the rescaling $v = u/(w-1)$ with $w > 1$ on

$$\Gamma(z) = \int_0^{+\infty} u^{z-1}e^{-u}du = (w-1)^z \int_0^{+\infty} v^{z-1}e^{-(w-1)v}dv > (w-1)^z\mathrm{B}(z,w),$$

and from the rescaling $v = u/(z+w)^z$,

$$\Gamma(z) = \int_0^{+\infty} u^{z-1}e^{-u}du = (z+w)^z \int_0^{+\infty} v^{z-1}e^{-(w-1)v}dv < (z+w)^z\mathrm{B}(z,w).$$

We conclude that

$$\frac{\Gamma(m)}{(k+1)^m} < \mathrm{B}(m, k+1-m) < \frac{\Gamma(m)}{(k-m)^m}.$$

□

We now prove Lemma 3.2.15.

Proof. From [9] we have the following expression

$$\mathbf{h}^{(k)}(\eta) = \sum_{j=1}^{k+1} \prod_{i=j-(k+1)}^{j-1} \eta(i) - \sum_{j=1}^{k} \prod_{\substack{i=-(k+1)+j \\ i\neq 0}}^{j} \eta(i). \tag{B.11}$$

Expression (3.15) is a consequence of a rearrangement which turns out to be fundamental for maintaining ℓ_N with no restrictions. Indeed, we can rewrite

$$\sum_{j=1}^{k+1} \prod_{i=j-(k+1)}^{j-1} \eta(i) - \sum_{j=1}^{k} \prod_{\substack{i=-(k+1)+j \\ i\neq 0}}^{j} \eta(i) = \prod_{i=0}^{k} \eta(i) + \sum_{j=1}^{k} (\eta(0) - \eta(j)) \prod_{\substack{i=-(k+1)+j \\ i\neq 0}}^{j-1} \eta(i).$$

Note that

$$(\eta(0) - \eta(j)) \prod_{\substack{i=-(k+1)+j \\ i\neq 0}}^{j-1} \eta(i) = \prod_{i=-(k+1)+j}^{j-1} \eta(i) - \prod_{\substack{i=-(k+1)+j \\ i\neq 0}}^{j} \eta(i).$$

Now we reorganize the products on the second term above. For $n \in [\![-(k+1)+j, j-1]\!]$ we have

$$\prod_{\substack{i=-(k+1)+j \\ i\neq n+1}}^{j} \eta(i) = (\eta(n) - \eta(n+1)) \prod_{\substack{i=-(k+1)+j \\ i\neq n, n+1}}^{j} \eta(i) + \prod_{\substack{i=-(k+1)+j \\ i\neq n}}^{j} \eta(i).$$

Observing that a change of variables yields

$$\prod_{\substack{i=-(k+1)+j \\ i\neq n,n+1}}^{j} \eta(i) = \prod_{\substack{i=-(k+1)+j-n \\ i\neq 0,1}}^{j-n} \eta(i+n) = \mathbf{s}_{j-n}^{(k)}(\tau_n\eta),$$

by iteration we see that

$$\prod_{\substack{i=-(k+1)+j \\ i\neq 0}}^{j} \eta(i) = \prod_{\substack{i=-(k+1)+j \\ i\neq j}}^{j} \eta(i) - \sum_{i=0}^{j-1}(\eta(i)-\eta(i+1))\mathbf{s}_{j-i}^{(k)}(\tau_i\eta).$$

Exchanging the summations and performing a change of variables,

$$\sum_{j=1}^{k}\sum_{i=0}^{j-1}(\eta(i)-\eta(i+1))\mathbf{s}_{j-i}^{(k)}(\tau_i\eta) = \sum_{i=0}^{k-1}(\eta(i)-\eta(i+1))\sum_{j=1}^{k-i}\mathbf{s}_{j}^{(k)}(\tau_i\eta),$$

which ends the proof. □

C

Generalized Porous Media Model

Contents

C.1 Case $n, k = 2$

Here we exemplify, for the case $n = 2$ and $k = 2$, the approach to derive the linear system characterizing the gradient condition, (4.13). For the convenience of the reader, we recall from 4.2.4 that our kinetic constraints take, in general, the form of

$$\mathbf{c}^{(n,k;\mathfrak{a})}(\eta) = \sum_{j=0}^{n+k} \sum_{i \in I} a_{ij} \eta^{-j+P_{ij}}(-j + M_j), \tag{C.1}$$

and that for the particular case of $n, k = 2$ the sets in the previous display are given by

j	$-j + M_j$
0	$\{2,3,4,5\}$
1	$\{-1,2,3,4\}$
2	$\{-2,-1,2,3\}$
3	$\{-3,-2,-1,2\}$
4	$\{-4,-3,-2,-1\}$

Figure C.1: Windows where the constraints are imposed.

	$-j + P_{ij}$				
i/j	0	1	2	3	4
1	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1\}$	$\{-3,-2\}$	$\{-4,-3\}$
2	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3,-1\}$	$\{-4,-2\}$
3	$\{2,5\}$	$\{-1,4\}$	$\{-2,3\}$	$\{-3,2\}$	$\{-4,-1\}$
4	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1\}$	$\{-3,-2\}$
5	$\{3,5\}$	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3,-1\}$
6	$\{4,5\}$	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1\}$

Figure C.2: Sets corresponding to the sites with flipped occupation ($\eta^{-j+P_{ij}}$).

We also find worth recalling Definition 4.2.3 and Lemma 4.2.5. With the aid of the two previous tables we can see, for example, that for $(i,j) = (1,0)$ we have $\eta^{P_{10}}(-0 + M_0) = (1 - \eta(2))(1 - \eta(3))\eta(4)\eta(5)$.

The starting point is to distribute the products of functions of the occupation variables in (C.1). This leads, in general, to

$$\mathbf{c}^{(n,k)}(\eta) = \frac{1}{\binom{n+k}{k}} \sum_{\ell=0}^{k} \sum_{(i,j) \in I \times J} \sum_{q=1}^{\binom{k}{\ell}} (-1)^{\ell} (\tau_{-j}\eta)([M_j \backslash P_{ij}] \sqcup Q_{ijq}^{\ell}),$$

with the sets of the form Q_{ijq}^{ℓ} as in Definition 4.2.3. We will focus on each term of the summation over ℓ separately. The first goal is to identify each set of the form $[M_j \backslash P_{ij}] \sqcup Q_{ijq}^{\ell}$ as an element of $\mathcal{P}_{n+\ell}(M_j)$. In order to do so, recalling the introduction of the map Φ in (4.9), it is enough to consider the sets $M = \Phi_j^{-1}(M_j)$, $P_i = \Phi_j^{-1}(P_{ij})$ and $Q_{iq} = \Phi_j^{-1}(Q_{ijq})$ and to identify $[M \backslash P_i] \sqcup Q_{iq}^{\ell}$ as an element of $\mathcal{P}_{n+\ell}(M)$. This will induce a map ψ_ℓ defined through $\psi_\ell(i,q) = i' \Leftrightarrow [M \backslash P_i] \sqcup Q_{iq}^{\ell} = P_{i'}^{n+\ell} \in \mathcal{P}_{n+\ell}(M)$. The rationale is the following

$$P_i \to Q_{iq}^{\ell} \in \mathcal{P}_{\ell}(P_i) \to P_{i'}^{n+\ell} = [M \backslash P_i] \sqcup Q_{iq}^{\ell} \in \mathcal{P}_{n+\ell}(M) \to \psi_\ell(i,q) = i'. \tag{C.2}$$

One can then "introduce" the $j-$th window by an application of the map Φ_j. Concretely, by observing that

$$[M \backslash P_i] \sqcup Q_{iq}^{\ell} = P_{i'}^{n+\ell} \in \mathcal{P}_{n+\ell}(M) \Leftrightarrow [M_j \backslash P_{ij}] \sqcup Q_{ijq}^{\ell} = P_{i'j}^{n+\ell} \in \mathcal{P}_{n+\ell}(M_j),$$

where $P_{i'j}^{n+\ell} = \Phi_j(P_i^{n+\ell})$, and as such one can extend the map ψ_ℓ by defining $\psi_{j,\ell}(i,q) = \psi_\ell(i,q) = i'$.

- Term $\ell = 0$:

Because $\ell = 0$ we have that $\mathcal{P}_\ell(P_i) = \{\varnothing\}$ and as consequence every "Q–set" is identified with $\varnothing$, hence $M \backslash P_i$ is simply the complement of P_i in M. One then needs only to fix some index for the elements of $\mathcal{P}_{n+\ell}(M)$. The rationale in (C.2) is presented in the next figure.

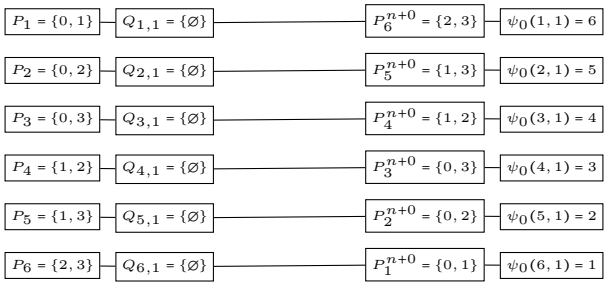


Figure C.1: $\ell = 0$: Construction of the map ψ_0.

Identifying the map $\psi_{j,\ell}$ leads to the sets in the next figure.

	$-j + P_{ij}^{n+0}$				
$i \backslash j$	0	1	2	3	4
1	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1,\}$	$\{-3,-2,\}$	$\{-4,-3,\}$
2	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3,-1,\}$	$\{-4,-2,\}$
3	$\{2,5\}$	$\{-1,4\}$	$\{-2,3\}$	$\{-3,2\}$	$\{-4,-1,\}$
4	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1,\}$	$\{-3,-2,\}$
5	$\{3,5\}$	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3,-1,\}$
6	$\{4,5\}$	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2,-1,\}$

Figure C.3: $\ell = 0$: Sets generated by $\{-j + [M_j \backslash P_{ij}] \sqcup Q_{ijq}^\ell\}_{(i,j)\in I_\ell \times J, 1 \le q \le \binom{k}{\ell}}$.

A multiplication of the rates by $\eta(1) - \eta(0)$ leads then to the sets in the next two figures.

	$[-j + P_{ij}^{n+0}] \sqcup \{0\} = -j + (P_{ij}^{n+0} \sqcup \{j\})$				
$i \backslash j$	0	1	2	3	4
1	$\{0,2,3\}$	$\{-1,0,2\}$	$\{-2,-1,0\}$	$\{-3,-2,0\}$	$\{-4,-3,0\}$
2	$\{0,2,4\}$	$\{-1,0,3\}$	$\{-2,0,2\}$	$\{-3,-1,0\}$	$\{-4,-2,0\}$
3	$\{0,2,5\}$	$\{-1,0,4\}$	$\{-2,0,3\}$	$\{-3,0,2\}$	$\{-4,-1,0\}$
4	$\{0,3,4\}$	$\{0,2,3\}$	$\{-1,0,2\}$	$\{-2,-1,0\}$	$\{-3,-2,0\}$
5	$\{0,3,5\}$	$\{0,2,4\}$	$\{-1,0,3\}$	$\{-2,0,2\}$	$\{-3,-1,0\}$
6	$\{0,4,5\}$	$\{0,3,4\}$	$\{0,2,3\}$	$\{-1,0,2\}$	$\{-2,-1,0\}$

Figure C.4: $\ell = 0$: Sets resulting from the multiplication with $\eta(0)$.

	$[-j + P_{ij}^{n+0}] \sqcup \{1\}$				
$i \backslash j$	0	1	2	3	4
1	$\{1,2,3\}$	$\{-1,1,2\}$	$\{-2,-1,1\}$	$\{-3,-2,1\}$	$\{-4,-3,1\}$
2	$\{1,2,4\}$	$\{-1,1,3\}$	$\{-2,1,2\}$	$\{-3,-1,1\}$	$\{-4,-2,1\}$
3	$\{1,2,5\}$	$\{-1,1,4\}$	$\{-2,1,3\}$	$\{-3,1,2\}$	$\{-4,-1,1\}$
4	$\{1,3,4\}$	$\{1,2,3\}$	$\{-1,1,2\}$	$\{-2,-1,1\}$	$\{-3,-2,1\}$
5	$\{1,3,5\}$	$\{1,2,4\}$	$\{-1,1,3\}$	$\{-2,1,2\}$	$\{-3,-1,1\}$
6	$\{1,4,5\}$	$\{1,3,4\}$	$\{1,2,3\}$	$\{-1,1,2\}$	$\{-2,-1,1\}$

Figure C.5: $\ell = 0$: Sets resulting from the multiplication with $\eta(1)$.

The first layer of translations corresponds to translating each window (with its respective constraints)

to the origin. With this, one obtains the sets in the next two figures.

	$P_{ij}^{n+0} \sqcup \{j+1\}$				
$i\backslash j$	0	1	2	3	4
1	$\{1,2,3\}$	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,4\}$	$\{0,1,5\}$
2	$\{1,2,4\}$	$\{0,2,4\}$	$\{0,3,4\}$	$\{0,2,4\}$	$\{0,2,5\}$
3	$\{1,2,5\}$	$\{0,2,5\}$	$\{0,3,5\}$	$\{0,4,5\}$	$\{0,3,5\}$
4	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,3,4\}$	$\{1,2,4\}$	$\{1,2,5\}$
5	$\{1,3,5\}$	$\{2,3,5\}$	$\{1,3,5\}$	$\{1,4,5\}$	$\{1,3,5\}$
6	$\{1,4,5\}$	$\{2,4,5\}$	$\{3,4,5\}$	$\{2,4,5\}$	$\{2,3,5\}$

Figure C.6: $\ell = 0$: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(1)$.

	$P_{ij}^{n+0} \sqcup \{j\}$				
$i\backslash j$	0	1	2	3	4
1	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,2\}$	$\{0,1,3\}$	$\{0,1,4\}$
2	$\{0,2,4\}$	$\{0,1,4\}$	$\{0,2,4\}$	$\{0,2,3\}$	$\{0,2,4\}$
3	$\{0,2,5\}$	$\{0,1,5\}$	$\{0,2,5\}$	$\{0,3,5\}$	$\{0,3,4\}$
4	$\{0,3,4\}$	$\{1,3,4\}$	$\{1,2,4\}$	$\{1,2,3\}$	$\{1,2,4\}$
5	$\{0,3,5\}$	$\{1,3,5\}$	$\{1,2,5\}$	$\{1,3,5\}$	$\{1,3,4\}$
6	$\{0,4,5\}$	$\{1,4,5\}$	$\{2,4,5\}$	$\{2,3,5\}$	$\{2,3,4\}$

Figure C.7: $\ell = 0$: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(0)$.

The second layer of translations corresponds to translating to the origin each set in the previous two figures. This leads to the sets in the next two figures.

	$A_{ij}^{n+0,0}$				
$i\backslash j$	0	1	2	3	4
1	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,2\}$	$\{0,1,3\}$	$\{0,1,4\}$
2	$\{0,2,4\}$	$\{0,1,4\}$	$\{0,2,4\}$	$\{0,2,3\}$	$\{0,2,4\}$
3	$\{0,2,5\}$	$\{0,1,5\}$	$\{0,2,5\}$	$\{0,3,5\}$	$\{0,3,4\}$
4	$\{0,3,4\}$	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,2\}$	$\{0,1,3\}$
5	$\{0,3,5\}$	$\{0,2,4\}$	$\{0,1,4\}$	$\{0,2,4\}$	$\{0,2,3\}$
6	$\{0,4,5\}$	$\{0,3,4\}$	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,2\}$

Figure C.8: $\ell = 0$: Sets associated with the multiplication by $\eta(0)$ translated to the origin.

	$A_{ij}^{n+0,1}$				
$i\backslash j$	0	1	2	3	4
1	$\{0,1,2\}$	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,4\}$	$\{0,1,5\}$
2	$\{0,1,3\}$	$\{0,2,4\}$	$\{0,3,4\}$	$\{0,2,4\}$	$\{0,2,5\}$
3	$\{0,1,4\}$	$\{0,2,5\}$	$\{0,3,5\}$	$\{0,4,5\}$	$\{0,3,5\}$
4	$\{0,2,3\}$	$\{0,1,2\}$	$\{0,2,3\}$	$\{0,1,3\}$	$\{0,1,4\}$
5	$\{0,2,4\}$	$\{0,1,3\}$	$\{0,2,4\}$	$\{0,3,4\}$	$\{0,2,4\}$
6	$\{0,3,4\}$	$\{0,2,3\}$	$\{0,1,2\}$	$\{0,2,3\}$	$\{0,1,3\}$

Figure C.9: $\ell = 0$: Sets associated with the multiplication by $\eta(1)$ translated to the origin.

In the next table, the first column corresponds to all the *unique* sets in Figures C.9 and C.8; the second (resp. third) column corresponds to the indexes $(i,j) \in I_\ell \times J$ of the sets in Figure C.8 (resp. Figure C.9) associated with the set in the first column. A concrete example is the following. Consider

$x = 0$, $(i,j) = (3,0)$ and the set $A_{ij}^{n+0,x} = A_{3,0}^{n+0,0} = \{0,2,5\}$, presented in Figure C.8. For $x = 0$ still, we also have that $(i,j) = (3,2)$ corresponds to $A_{3,2}^{n+0,0} = \{0,2,5\}$ and so the indexes $(i,j) \in \{(3,0),(3,2)\}$ all correspond to the set $\{0,2,5\}$, for $x = 0$, and in this way they belong to the same class in the quotient space $\mathcal{C}_{\ell,0}$. For $x = 1$ we have that $(i,j) \in \{(3,1),(2,4)\}$ is also associated with the set $\{0,2,5\}$ (see Figure C.9).

A^{ℓ}	$\mathcal{C}_{\ell,0}$	$\mathcal{C}_{\ell,1}$
$\{0,2,3\}$	$\{(1,0),(4,1),(6,2),(2,3),(5,4)\}$	$\{(4,0),(1,1),(6,1),(4,2),(6,3)\}$
$\{0,2,4\}$	$\{(2,0),(5,1),(2,2),(5,3),(2,4)\}$	$\{(5,0),(2,1),(5,2),(2,3),(5,4)\}$
$\{0,2,5\}$	$\{(3,0),(3,2)\}$	$\{(3,1),(2,4)\}$
$\{0,3,4\}$	$\{(4,0),(6,1),(3,4)\}$	$\{(6,0),(2,2),(5,3)\}$
$\{0,3,5\}$	$\{(5,0),(3,3)\}$	$\{(3,2),(3,4)\}$
$\{0,4,5\}$	$\{(6,0)\}$	$\{(3,3)\}$
$\{0,1,3\}$	$\{(1,1),(4,2),(1,3),(6,3),(4,4)\}$	$\{(2,0),(5,1),(1,2),(4,3),(6,4)\}$
$\{0,1,4\}$	$\{(2,1),(5,2),(1,4)\}$	$\{(3,0),(1,3),(4,4)\}$
$\{0,1,5\}$	$\{(3,1)\}$	$\{(1,4)\}$
$\{0,1,2\}$	$\{(1,2),(4,3),(6,4)\}$	$\{(1,0),(4,1),(6,2)\}$

Figure C.10: $\ell = 0$: Equivalence classes of indexes and the corresponding "A−set".

In conclusion, in the second (resp. third) column above we have the equivalence classes of the indexes that originate from multiplying the constraints by $\eta(0)$ (resp. $\eta(1)$); in the first column the unique sets that are obtained by translating every element of $\mathcal{P}_{n+0}(M_j)$, for each $0 \leq j \leq n+k = 4$, to the origin; and the correspondence between the second and third columns provides a bijection between $\mathcal{C}_{\ell,0}$ and $\mathcal{C}_{\ell,1}$, which can be extended into a permutation ϕ_{ℓ} over $I_{\ell} \times J$.

- Term $\ell = 1$:

Following the same procedure, the rationale in (C.2) provides the map ψ_{ℓ} which, in turn, provides $\psi_{j,\ell}$.

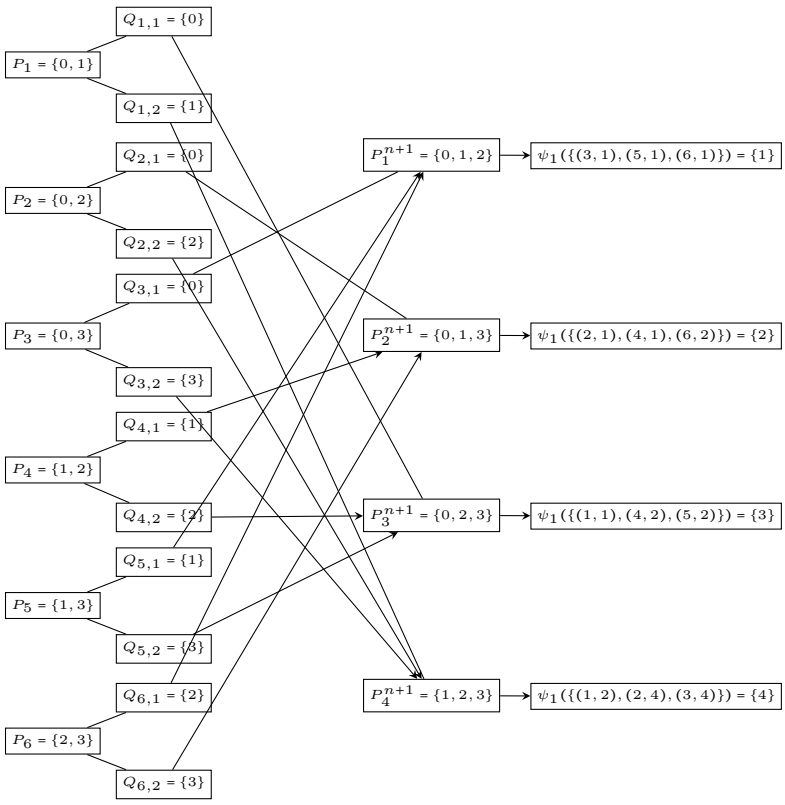


Figure C.2: $\ell = 1$: Construction of the map ψ_1.

The sets appearing in the product $\mathbf{c}^{(n,k;\mathfrak{a})}(\eta)(\eta(1) - \eta(0))$ are presented in the next two figures.

	$[-j + P_{ij}^{n+1}] \sqcup \{0\}$				
$i \backslash j$	0	1	2	3	4
1	$\{0, 2, 3, 4\}$	$\{-1, 0, 2, 3\}$	$\{-2, -1, 0, 2\}$	$\{-3, -2, -1, 0\}$	$\{-4, -3, -2, 0\}$
2	$\{0, 2, 3, 5\}$	$\{-1, 0, 2, 4\}$	$\{-2, -1, 0, 3\}$	$\{-3, -2, 0, 2\}$	$\{-4, -3, -1, 0\}$
3	$\{0, 2, 4, 5\}$	$\{-1, 0, 3, 4\}$	$\{-2, 0, 2, 3\}$	$\{-3, -1, 0, 2\}$	$\{-4, -2, -1, 0\}$
4	$\{0, 3, 4, 5\}$	$\{0, 2, 3, 4\}$	$\{-1, 0, 2, 3\}$	$\{-2, -1, 0, 2\}$	$\{-3, -2, -1, 0\}$

Figure C.11: $\ell = 1$: Sets resulting from the multiplication with $\eta(0)$.

	$[-j + P_{ij}^{n+1}] \sqcup \{1\}$				
$i \backslash j$	0	1	2	3	4
1	$\{1, 2, 3, 4\}$	$\{-1, 1, 2, 3\}$	$\{-2, -1, 1, 2\}$	$\{-3, -2, -1, 1\}$	$\{-4, -3, -2, 1\}$
2	$\{1, 2, 3, 5\}$	$\{-1, 1, 2, 4\}$	$\{-2, -1, 1, 3\}$	$\{-3, -2, 1, 2\}$	$\{-4, -3, -1, 1\}$
3	$\{1, 2, 4, 5\}$	$\{-1, 1, 3, 4\}$	$\{-2, 1, 2, 3\}$	$\{-3, -1, 1, 2\}$	$\{-4, -2, -1, 1\}$
4	$\{1, 3, 4, 5\}$	$\{1, 2, 3, 4\}$	$\{-1, 1, 2, 3\}$	$\{-2, -1, 1, 2\}$	$\{-3, -2, -1, 1\}$

Figure C.12: $\ell = 1$: Sets resulting from the multiplication with $\eta(1)$.

The first layer of translations leads to the following.

	$P_{ij}^{n+1} \sqcup \{j\}$				
$i \backslash j$	0	1	2	3	4
1	$\{0,2,3,4\}$	$\{0,1,3,4\}$	$\{0,1,2,4\}$	$\{0,1,2,3\}$	$\{0,1,2,4\}$
2	$\{0,2,3,5\}$	$\{0,1,3,5\}$	$\{0,1,2,5\}$	$\{0,1,3,5\}$	$\{0,1,3,4\}$
3	$\{0,2,4,5\}$	$\{0,1,4,5\}$	$\{0,2,4,5\}$	$\{0,2,3,5\}$	$\{0,2,3,4\}$
4	$\{0,3,4,5\}$	$\{1,3,4,5\}$	$\{1,2,4,5\}$	$\{1,2,3,5\}$	$\{1,2,3,4\}$

Figure C.13: ℓ = 1: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(0)$.

	$P_{ij}^{n+1} \sqcup \{j+1\}$				
$i \backslash j$	0	1	2	3	4
1	$\{1,2,3,4\}$	$\{0,2,3,4\}$	$\{0,1,3,4\}$	$\{0,1,2,4\}$	$\{0,1,2,5\}$
2	$\{1,2,3,5\}$	$\{0,2,3,5\}$	$\{0,1,3,5\}$	$\{0,1,4,5\}$	$\{0,1,3,5\}$
3	$\{1,2,4,5\}$	$\{0,2,4,5\}$	$\{0,3,4,5\}$	$\{0,2,4,5\}$	$\{0,2,3,5\}$
4	$\{1,3,4,5\}$	$\{2,3,4,5\}$	$\{1,3,4,5\}$	$\{1,2,4,5\}$	$\{1,2,3,5\}$

Figure C.14: ℓ = 1: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(1)$.

The "A–sets" corresponding to the second layer of translations are presented in the next two figures.

	$A_{ij}^{n+1,0}$				
$i \backslash j$	0	1	2	3	4
1	$\{0,2,3,4\}$	$\{0,1,3,4\}$	$\{0,1,2,4\}$	$\{0,1,2,3\}$	$\{0,1,2,4\}$
2	$\{0,2,3,5\}$	$\{0,1,3,5\}$	$\{0,1,2,5\}$	$\{0,1,3,5\}$	$\{0,1,3,4\}$
3	$\{0,2,4,5\}$	$\{0,1,4,5\}$	$\{0,2,4,5\}$	$\{0,2,3,5\}$	$\{0,2,3,4\}$
4	$\{0,3,4,5\}$	$\{0,2,3,4\}$	$\{0,1,3,4\}$	$\{0,1,2,4\}$	$\{0,1,2,3\}$

Figure C.15: ℓ = 1: Sets associated with the multiplication by $\eta(0)$ translated to the origin.

	$A_{ij}^{n+1,1}$				
$i \backslash j$	0	1	2	3	4
1	$\{0,1,2,3\}$	$\{0,2,3,4\}$	$\{0,1,3,4\}$	$\{0,1,2,4\}$	$\{0,1,2,5\}$
2	$\{0,1,2,4\}$	$\{0,2,3,5\}$	$\{0,1,3,5\}$	$\{0,1,4,5\}$	$\{0,1,3,5\}$
3	$\{0,1,3,4\}$	$\{0,2,4,5\}$	$\{0,3,4,5\}$	$\{0,2,4,5\}$	$\{0,2,3,5\}$
4	$\{0,2,3,4\}$	$\{0,1,2,3\}$	$\{0,2,3,4\}$	$\{0,1,3,4\}$	$\{0,1,2,4\}$

Figure C.16: ℓ = 1: Sets associated with the multiplication by $\eta(1)$ translated to the origin.

Comparing the previous two figures leads to the identification of the equivalence classes of the indexes.

$A_{\underline{c}}$	$\mathcal{C}_{\ell,0}$	$\mathcal{C}_{\ell,1}$
$\{0,1,2,3\}$	$\{(1,3),(4,4)\}$	$\{(1,0),(4,1)\}$
$\{0,1,2,4\}$	$\{(1,2),(1,4),(4,3)\}$	$\{(2,0),(1,3),(4,4)\}$
$\{0,1,2,5\}$	$\{(2,2)\}$	$\{(1,4)\}$
$\{0,1,3,4\}$	$\{(1,1),(2,4),(4,2)\}$	$\{(1,2),(3,0),(4,3)\}$
$\{0,1,3,5\}$	$\{(2,1),(2,3)\}$	$\{(2,2),(2,4)\}$
$\{0,1,4,5\}$	$\{(3,1)\}$	$\{(2,3)\}$
$\{0,2,3,4\}$	$\{(1,0),(4,1),(3,4)\}$	$\{(1,1),(4,0),(4,2)\}$
$\{0,2,3,5\}$	$\{(2,0),(3,3)\}$	$\{(2,1),(3,4)\}$
$\{0,2,4,5\}$	$\{(3,0),(3,2)\}$	$\{(3,1),(3,3)\}$
$\{0,3,4,5\}$	$\{(4,0)\}$	$\{(3,2)\}$

Figure C.17: ℓ = 1: Equivalence classes of indexes and the corresponding "A–set".

C.1.1 Linear system characterizing the gradient property.

The correspondence between the equivalence classes in Figure C.10 and C.17 provide a linear system for the "b–coefficients" (as in (4.12)). From the map $\psi_{j,\ell}$, for each $j \in J$, $0 \leq \ell \leq n+k$, the original "a–coefficients" (as in Definition 4.2.4) can be recovered, yielding the following linear system, where we remark that the last line corresponds to the equations for the case $\ell = 2$, which in turn corresponds simply to the PMM(4), as explained just before (4.16).

$$a_{1,2} + a_{2,4} + a_{3,1} + a_{5,3} + a_{6,0} = a_{1,1} + a_{1,3} + a_{3,0} + a_{3,2} + a_{6,1}$$

$$a_{2,1} + a_{2,3} + a_{5,0} + a_{5,2} + a_{5,4} = a_{2,0} + a_{2,2} + a_{2,4} + a_{5,1} + a_{5,3}$$

$$a_{4,0} + a_{4,2} = a_{4,1} + a_{5,4}$$

$$a_{1,1} + a_{3,0} + a_{4,4} = a_{1,0} + a_{2,3} + a_{5,2}$$

$$a_{2,0} + a_{4,3} = a_{4,2} + a_{4,4}$$

$$a_{1,0} = a_{4,3}$$

$$a_{1,3} + a_{3,2} + a_{3,4} + a_{6,1} + a_{6,3} = a_{1,4} + a_{2,1} + a_{3,3} + a_{5,0} + a_{6,2}$$

$$a_{2,2} + a_{5,1} + a_{6,4} = a_{3,4} + a_{4,0} + a_{6,3}$$

$$a_{4,1} = a_{6,4}$$

$$a_{1,4} + a_{3,3} + a_{6,2} = a_{1,2} + a_{3,1} + a_{6,0}$$

$$a_{1,1} + a_{1,4} + a_{2,1} + a_{3,0} + a_{3,1} + a_{4,4} + a_{5,0} + a_{5,4} + a_{6,0} = a_{1,0} + a_{1,2} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,1} + a_{3,2} + a_{5,1} + a_{6,1}$$

$$a_{1,3} + a_{2,0} + a_{4,0} + a_{4,3} + a_{5,3} + a_{6,0} = a_{1,4} + a_{2,1} + a_{4,1} + a_{4,4} + a_{5,4} + a_{6,1}$$

$$a_{1,0} + a_{1,2} + a_{4,0} + a_{4,2} + a_{5,0} + a_{5,2} = a_{1,1} + a_{1,3} + a_{4,1} + a_{4,3} + a_{5,1} + a_{5,3}$$

$$a_{1,0} + a_{2,0} + a_{3,0} = a_{1,2} + a_{4,2} + a_{5,2}$$

$$a_{1,2} + a_{2,2} + a_{2,4} + a_{3,1} + a_{3,2} + a_{4,4} + a_{5,1} + a_{6,1} + a_{6,4} = a_{1,0} + a_{1,3} + a_{2,3} + a_{3,2} + a_{3,3} + a_{4,0} + a_{5,0} + a_{5,2} + a_{6,2}$$

$$a_{2,1} + a_{2,3} + a_{4,1} + a_{4,3} + a_{6,1} + a_{6,3} = a_{2,2} + a_{2,4} + a_{4,2} + a_{4,4} + a_{6,2} + a_{6,4}$$

$$a_{1,1} + a_{4,1} + a_{5,1} = a_{2,3} + a_{4,3} + a_{6,3}$$

$$a_{1,3} + a_{2,3} + a_{3,2} + a_{3,3} + a_{3,4} + a_{5,2} + a_{5,4} + a_{6,2} + a_{6,4} = a_{1,4} + a_{2,0} + a_{2,4} + a_{3,3} + a_{3,4} + a_{4,0} + a_{5,3} + a_{6,0} + a_{6,3}$$

$$a_{2,2} + a_{4,2} + a_{6,2} = a_{3,4} + a_{5,4} + a_{6,4}$$

$$a_{1,4} + a_{2,4} + a_{3,3} + a_{3,4} + a_{5,3} + a_{6,3} = a_{1,1} + a_{2,1} + a_{3,0} + a_{3,1} + a_{5,0} + a_{6,0}$$

$$\sum_{i \in I} a_{ij} = 1, \ \forall j \in J$$

Above, the first 10 equations correspond to $\ell = 0$; the equations $\sum_{i\in I} a_{ij} = 1$ for all $j \in J$ correspond to $\ell = 2$; while the rest to $\ell = 1$. The previous system can then be reduced to

$$\begin{aligned}
a_{1,0} + a_{2,3} &= a_{1,1} + a_{1,2} \\
a_{1,0} + a_{2,0} + a_{2,2} + a_{2,4} &= 2a_{1,1} + a_{1,3} + a_{2,1} \\
2a_{1,2} + a_{3,1} &= a_{1,1} + a_{1,3} + a_{1,4} \\
a_{1,0} + a_{1,2} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,2} &= 2a_{1,1} + 2a_{1,4} + 2a_{2,1} \\
a_{3,4} + a_{4,0} &= a_{1,2} + a_{2,2} \\
a_{4,3} &= a_{1,0} \\
a_{4,2} + a_{4,4} &= a_{1,0} + a_{2,0} \\
a_{1,0} + 2a_{1,2} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,3} + a_{5,0} &= 3a_{1,1} + 2a_{1,3} + a_{1,4} + a_{2,1} + a_{3,4} \\
a_{1,2} + a_{4,2} + a_{5,2} &= a_{1,0} + a_{2,0} + a_{3,0} \\
a_{1,2} + a_{3,3} + a_{4,1} + a_{5,1} + a_{5,3} &= 2a_{1,1} + a_{1,3} + a_{1,4} + a_{2,1} \\
a_{3,4} + a_{4,1} + a_{5,4} &= a_{1,2} + a_{2,2} + a_{4,2} \\
a_{1,1} + a_{1,3} + a_{6,0} &= 2a_{1,2} + a_{3,3} \\
a_{4,1} + a_{5,1} + a_{6,1} &= a_{1,1} + a_{1,2} + a_{1,3} \\
a_{6,2} &= a_{1,2} \\
a_{1,2} + a_{6,3} &= a_{4,1} + a_{5,1} \\
a_{6,4} &= a_{4,1}
\end{aligned} \tag{C.3}$$

C.1.2 Linear system characterizing the potential's invariance.

In this subsection we: prove Proposition C.1.1, where we derive an additional set of conditions on the weights $\mathfrak{a}$ so that the potential $\mathbf{h}^{(n,k;\mathfrak{a})}$ is related with the potential corresponding to the uniform choice, $\mathbf{h}^{(n,k)}$; and show that the linear system characterizing the gradient condition (see 4.13), when extended with the aforementioned conditions, has a non-uniform solution for the particular case $n, k = 2$. We developed a *Mathematica* routine in order to obtain and help us solve these linear systems for any value of n and k, and we were able to obtain multiple non-uniform solutions for varied values of n and k. However, we were not able to prove this rigorously. For that reason, Proposition C.1.1 is not empty for $n, k = 2$ only, as we provide an example, yet we are confident that non-uniform solutions exist for every value of n and k.

Proposition C.1.1. *If, for each* $0 \le \ell \le k$ *and* $\underline{c} \in \mathcal{C}_{\ell,1}$ *such that* $\max(A_{\underline{c}}) \neq n + k + 1$*, the following equations are satisfied, for* $1 \le p \le n + k - \max(A_{\underline{c}})$*,*

$$\begin{aligned}
\sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = p\} b_{ij}^{\ell} &= \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_\ell(i,j)}^{\ell,0} = p\} b_{\phi_\ell(i,j)}^{\ell} \\
\sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = n + k + 1 - \max(A_{\underline{c}})\} b_{ij}^{\ell} &= \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} + \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_\ell(i,j)}^{\ell,0} = n + k + 1 - \max(A_{\underline{c}})\} b_{ij}^{\ell},
\end{aligned} \tag{C.4}$$

then $\mathbf{h}_1^{(n,k;\mathfrak{a})} = \mathbf{h}_1^{(n,k)}$*, with* $\mathbf{h}_1^{(n,k)}$ *as in* (4.25)*.*

Proof. Recalling the expression for $\mathbf{h}_1^{(n,k;\mathfrak{a})}(\eta)$ from (4.14),

$$\mathbf{h}_1^{(n,k;\mathfrak{a})}(\eta) = \sum_{\ell=0}^{k}(-1)^{\ell}\sum_{\underline{c}\in\mathcal{C}_{\ell,1}}\sum_{(i,j)\in\underline{c}}\left\{b_{ij}^{\ell}\sum_{y=0}^{p_{ij}^{\ell,1}-1}(\tau_y\eta)(A_{\underline{c}}) - b_{\phi_\ell(i,j)}^{\ell}\sum_{y=0}^{p_{\phi_\ell(i,j)}^{\ell,0}-1}(\tau_y\eta)(A_{\underline{c}})\right\}, \tag{C.5}$$

fixed ℓ and $\underline{c}\in\mathcal{C}_{\ell,1}$, from the property (1) in Lemma 4.2.20 the summation over $(i,j)\in\underline{c}$ in $\mathbf{h}_1^{(n,k;\mathfrak{a})}(\eta)$ can be expressed as

$$\begin{aligned}
&\sum_{p=1}^{n+k-\max(A_{\underline{c}})}\sum_{y=0}^{p-1}(\tau_y\eta)(A_{\underline{c}})\left\{\sum_{(i,j)\in\underline{c}}\mathbf{1}\{p_{ij}^{\ell,1}=p\}b_{ij}^{\ell} - \sum_{(i,j)\in\underline{c}}\mathbf{1}\{p_{\phi_\ell(i,j)}^{\ell,0}=p\}b_{\phi_\ell(i,j)}^{\ell}\right\}\\
&+\sum_{y=0}^{n+k+1-\max(A_{\underline{c}})-1}(\tau_y\eta)(A_{\underline{c}})\times\\
&\times\left\{\sum_{(i,j)\in\underline{c}}\mathbf{1}\{p_{ij}^{\ell,1}=n+k+1-\max(A_{\underline{c}})\neq 0\}b_{ij}^{\ell} - \sum_{(i,j)\in\underline{c}}\mathbf{1}\{p_{\phi_\ell(i,j)}^{\ell,0}=n+k+1-\max(A_{\underline{c}})\neq 0\}b_{\phi_\ell(i,j)}^{\ell}\right\}.
\end{aligned}$$

If the weights $\mathfrak{a}$ are such that (C.4) is satisfied, one obtains that (C.5) equals (4.27), which concludes the proof. □

We now present the additional linear system in Proposition C.1.1 for the case $n,k=2$. We omit the case $\ell=2$ as it is associated with the PMM(4). The equations in the first line of (C.4) are, in this case,

$$\begin{aligned}
a_{2,4}+a_{3,1} &= a_{3,0}+a_{3,2}\\
a_{3,2}+a_{3,4} &= a_{3,3}+a_{5,0}\\
a_{1,4}+a_{3,3} &= a_{3,1}+a_{6,0}\\
a_{1,4}+a_{2,4}+a_{3,4} &= a_{3,0}+a_{5,0}+a_{6,0},
\end{aligned} \tag{C.6}$$

with the first 3 equations corresponding to $\ell=0$ and the last to $\ell=1$. The equations in the second line in (C.4) are

$$\begin{aligned}
1/6+a_{1,2} &= a_{1,1}+a_{1,3}\\
1/6+a_{2,1}+a_{2,3} &= a_{2,0}+a_{2,2}+a_{2,4}\\
1/6+a_{1,1} &= a_{1,0}+a_{2,3}\\
1/6+a_{1,3} &= a_{1,4}+a_{2,1}\\
1/6+a_{2,2} &= a_{3,4}+a_{4,0}\\
1/6 &= a_{1,2}\\
1/2+a_{1,1}+a_{2,1}+a_{3,1} &= a_{1,0}+a_{1,2}+a_{2,0}+a_{2,2}+a_{3,0}+a_{3,2}\\
1/2+a_{1,2}+a_{2,2}+a_{3,2} &= a_{1,0}+a_{1,3}+a_{2,3}+a_{3,3}+a_{4,0}+a_{5,0}\\
1/2+a_{1,3}+a_{2,3}+a_{3,3} &= a_{1,4}+a_{2,0}+a_{2,4}+a_{3,4}+a_{4,0}+a_{6,0}\\
1/2 &= a_{1,1}+a_{2,1}+a_{3,1}
\end{aligned} \tag{C.7}$$

and the first 6 equations correspond to $\ell=0$. The *extended* system composed by the equations in (C.3),(C.6) and (C.7) can be reduced to

$$\begin{aligned}
a_{3,1} &= a_{1,4}\\
a_{4,3} &= a_{1,0}
\end{aligned}$$

$$
\begin{aligned}
a_{6,0} &= a_{3,3}\\
a_{6,4} &= a_{4,1}\\
a_{4,2} + a_{4,4} &= a_{1,0} + a_{2,0}\\
a_{1,2} &= 1/6\\
a_{6,2} &= 1/6\\
a_{1,0} + a_{2,3} &= 1/6 + a_{1,1}\\
a_{3,4} + a_{4,0} &= 1/6 + a_{2,2}\\
a_{4,1} + a_{5,1} &= 1/6 + a_{6,3}\\
a_{1,0} + a_{2,0} + a_{3,0} &= 1/6 + a_{4,2} + a_{5,2}\\
a_{3,4} + a_{4,1} + a_{5,4} &= 1/6 + a_{2,2} + a_{4,2}\\
a_{1,1} + a_{1,3} &= 1/3\\
a_{1,1} + a_{1,4} + a_{2,1} &= 1/2\\
a_{4,1} + a_{5,1} + a_{6,1} &= 1/2\\
a_{3,3} + a_{4,1} + a_{5,1} + a_{5,3} &= 2/3\\
a_{1,0} + a_{1,4} + a_{2,0} + a_{2,2} + a_{2,4} &= 5/6\\
a_{1,0} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,2} &= 5/6\\
a_{1,0} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,3} + a_{5,0} &= 5/6 + a_{3,4}.
\end{aligned}
$$

A particular solution is

	a_{ij}				
i/j	0	1	2	3	4
1	0	0	1/6	1/3	0
2	0	1/2	0	1/6	5/6
3	1/6	0	2/3	0	0
4	1/6	0	0	0	0
5	2/3	1/2	0	1/6	1/6
6	0	0	1/6	1/3	0

Figure C.18: $n, k = 2$: Particular solution of the extended system.

C.2 Regularity of the solution

Let $H \in C^2([0,1])$ be an increasing non-negative function (in particular, H' is non-negative). Our goal is to deduce the regularity of ρ from the regularity of $H(\rho)$. Consider the map V defined for any $x, y \in [0,1]$ through

$$V(x,y) = \frac{H(x) - H(y)}{x - y}. \tag{C.8}$$

Note that $V(x,x)$ is well-defined because H is differentiable. Later on, the points x, y will represent $\rho(u), \rho(v)$, respectively, with $u, v \in \mathbb{T}$. In this way, in order to obtain an upper bound for $|x - y|$ it is

enough to obtain a lower bound for $V(x,y)$. The main argument seems to be in the same spirit as in [8, Lemma 7.0.2.], where $H(x) = x^m$ and $m \in \mathbb{N}_+$ with $m \geq 3$. For $m \leq 2$ the proof is much simpler than for $m \geq 3$, and we recall that our proof of Proposition B.2.1 was based on it. In both of the previously mentioned manuscripts, the expression for V is simple enough for the authors to be able to bound it from below algebricaly. In our case, as presented in (C.10), the expression is quite complex and obtaining a lower bound algebricaly seems intractable. We will first study some general properties of (C.8), then particularize to our case and proceed with monotony arguments.

Because H is differentiable, V coincides with H' at the diagonal. Precisely, for any $x \in [0,1]$ it holds that

0. $V(x,x) = H(x)$.

One finds that the map V enjoys the following properties, for any $x, y \in [0,1]$ arbitrary:

1. Symmetry: $V(x,y) = V(y,x)$;
2. Non-negativity: $V(x,y) > 0$ for any $x \neq y$;
3. Zeroes: $V(x,y) = 0 \Leftrightarrow (x,y) \in \{(r,r) \,:\, H'(r) = 0\}$.

Regarding the monotonicity of V in any direction, the partial derivatives provide important information. Since V is symmetric and equals H' at the diagonal, let us consider only the region below the diagonal,

$$\{(x,y) \in [0,1]^2 \,:\, x > y\}.$$

Fixed $y \in [0,1]$ it holds that

$$\partial_x V(x,y) = \frac{H'(x) - V(x,y)}{x-y} \quad \text{and} \quad \partial_x^2 V(x,y) = \frac{H''(x) - 2\partial_x V(x,y)}{x-y}. \tag{C.9}$$

In this way, we see that

4. The critical points of $V(\cdot, y)$ are the points where $V(\cdot, y)$ intersects the graph of H';
5. $V(\cdot, y)$ is *increasing* if and only if $V(\cdot, y)$ is *below* the graph of H';
6. Let $x = p$ be a critical point.
   (a) If $H''(p) > 0$ then $V(p,y)$ is a minimum;
   (b) If $H''(p) < 0$ then $V(p,y)$ is a maximum;
   (c) If $H''(p) = 0$ then $H'(p) = H'(y)$. This is consequence of $\partial_y V(p,y) = 0$ and property (7) that we will state shortly.

Regarding the y–axis direction, fixed $x \in [0,1]$ it holds that

$$\partial_y V(x,y) = \frac{V(x,y) - H'(y)}{x-y} \quad \text{and} \quad \partial_y^2 V(x,y) = \frac{2\partial_y V(x,y) - H''(y)}{x-y},$$

which provide us directly with the following information.

7. The critical points of $V(x,\cdot)$ are the points where $V(x,\cdot)$ intersects the graph of H';

8. $V(x,\cdot)$ is *increasing* if and only if $V(x,\cdot)$ is *above* the graph of H';

9. Let $y = p$ be a critical point.

   (a) If $H''(p) > 0$ then $V(p,y)$ is a maximum;

   (b) If $H''(p) < 0$ then $V(p,y)$ is a minimum;

   (c) If $H''(p) = 0$ then $H'(p) = H'(x)$. This is consequence of $\partial_x V(x,p) = 0$ and property (4).

The previous properties gives us a full picture of the surface V on any fixed direction. We will now particularize to our specific case of $H = H_{n,k}$ where

$$H_{n,k}(x) = \int_0^x u^n(1-u)^k \mathrm{d}u \quad \text{and} \quad V(x,y) = \sum_{\ell=0}^{k} (-1)^k \binom{k}{\ell} \frac{1}{n+\ell+1} \sum_{i=0}^{n+\ell} x^i y^{(n+\ell-i)}. \tag{C.10}$$

The expression for the map V is consequence of the equality $x^a - y^a = (x-y)\sum_{i=0}^{a-1} x^a y^{a-1-i}$ for any $a \in \mathbb{N}_+$.

Let us fix V as in the previous display and write $\rho \equiv n/(n+k)$. We represent the behaviour of $V(x,y)$, that we are now going to prove, in Figure C.3 in the x–axis, and in Figure C.4 in the y–axis direction.

Let us fix $y \in [0,1)$ and focus on the region $\{0 \le y < x \le 1\}$. As $x \searrow y$ then $V(x,y) \to V(y,y)$ because $V(\cdot,y)$ is continuous. Note that $y = x$ is a critical point and from (C.9) we should consider two cases: $y < \rho$ or $y \ge \rho$.

Lemma C.2.1. *Consider the region $\{(x,y) \in [0,1]^2 : x > y\}$ and fix $y \in [0,1)$.*

- *If $y < \rho$ then there exists a unique critical point $x = x_0 > \rho$, $V(\cdot,y)$ is increasing in (y,x_0) and $V(\cdot,y)$ is decreasing in $(x_0,1]$;*
- *If $y \ge \rho$ then $V(\cdot,y)$ is decreasing in $(y,1]$.*

Fixed $x \in (0,1]$, it holds the following.

- *If $x < \rho$ then $V(x,\cdot)$ is increasing in $(0,x)$;*
- *If $x \ge \rho$ then there exists a unique critical point $y = y_0 < \rho$, $V(x,\cdot)$ is increasing in $[0,y_0)$ and decreasing in (y_0,x).*

Proof. We start with the following case.

- $y < \rho$, region $\{0 \le y < x \le \rho \le 1\} \cup \{0 \le y < \rho < x \le 1\}$.

  In this case, $V(\cdot,y)$ is concave up at $x = y$, hence $V(y,y)$ is a local minimum. In particular $V(x,y)$ is increasing for values of x close to y, and $V(x,y)$ is below the graph of H', that is, $V(x,y) < H'(x)$. The graph of H' is an "arc", with $H'(0) = H'(1) = 0$ and $H'(w)$ increasing for $w \in [0,\rho)$, where $H'(\rho) = \max_{x\in[0,1]} H'(x)$, and decreasing for $w \in (\rho,1]$. For this reason, since $V(\cdot,y)$ is increasing there must be some x_0 such that $V(x_0,y) = H'(x_0)$. Note that $x_0 \ge \rho$, because if $y < x_0 < \rho$ then $V(\cdot,y)$ is also concave up at $x = x_0$, just as in $x = y$, which is not possible. Since $x \ne y$, it must be the case that $x_0 > \rho$: if $x_0 = \rho$ then from property 6(c) it holds that $H'(y) = H'(\rho)$ which is equivalent to $y = \rho$. Thus, $V(\cdot,y)$ must be decreasing in $(x_0,1]$. There are no more critical points because $x > \rho$.

- $y > \rho$, region $\{\rho < y < x \leq 1\}$.

  As $x \searrow y$ then $V(x,y) \to V(y,y)$ and $x = y$ is a critical point larger than ρ, so $V(\cdot, y)$ is concave down at $x = y$, hence $V(\cdot, y)$ is decreasing. Arguing as previously, since $y > \rho$, there is no x_0 such that $V(x_0, y) = H'(x_0)$.

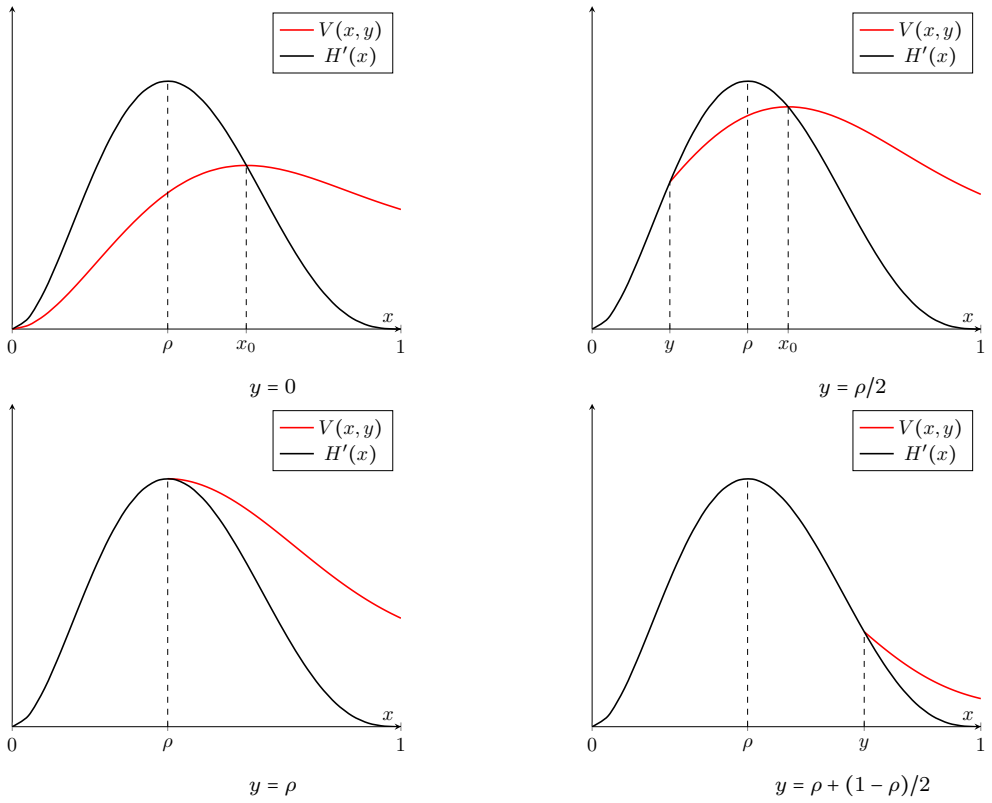


Figure C.3: Plot of $V(\cdot, y)$ with y fixed and $n = 2, k = 3$.

Now we fix x and consider $\{0 \leq y < x \leq 1\}$. As $y \nearrow x$ then $V(x,y) \to V(y,y)$ and $y = x$ is a critical point. Again, we split into two cases. The arguments will be identical to the case of x varying, and for that reason we explain the main steps only.

- $x < \rho$, region $\{0 \leq y < x < \rho\}$.

  Since $x < \rho$ then $V(x, \cdot)$ concave down at $V(x,x)$, hence $V(x, \cdot)$ is increasing in $0 \leq y < x < \rho$. Note that there can be no $y_0 < x$ such that $V(x, y_0) = H'(y_0)$.

- $x \geq \rho$, region $\{0 \leq y < \rho \leq x \leq 1\} \cup \{0 < \rho \leq y < x \leq 1\}$.

  We focus on $\{0 < \rho \leq y < x \leq 1\}$, with y decreasing up to entering the region $\{0 \leq y < \rho \leq x \leq 1\}$. In this way, we see that $V(x, \cdot)$ is concave up at $y = x$ since $x > \rho$. Then as y decreases, $V(x,y)$ increases. It must also be the case that there is some y_0 such that $V(x, y_0) = H'(y_0)$, and we can show that $0 \leq y_0 < \rho$ for $x \neq 0$, and $y_0 = \rho$ for $x = \rho$. For $y \in [0, y_0)$, because $V(x, \cdot)$ is decreasing if and only if is below the graph of H', we see that $V(x, \cdot)$ must be increasing.

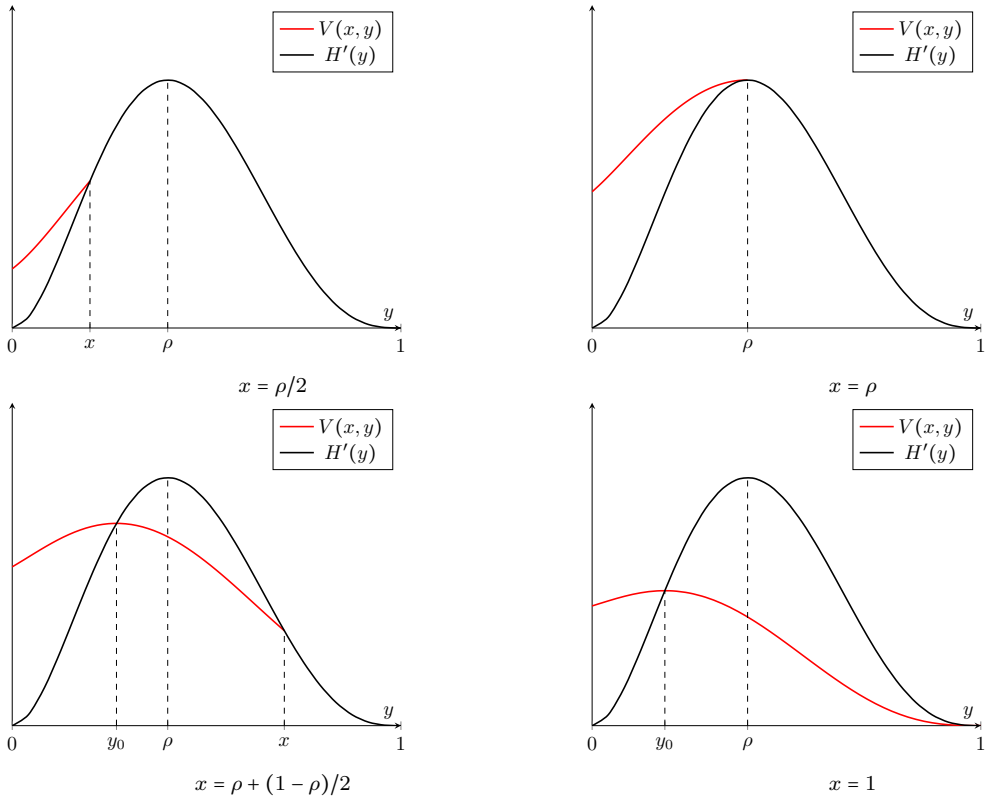


Figure C.4: Plot of $V(x,\cdot)$ with x fixed and $n = 2, k = 3$.

□

With the previous analysis we can show the following key result.

Lemma C.2.2. *For any $x, y \in [0,1]$ and any $\epsilon > 0$ it holds that*

$$|x - y| \leq 4\epsilon + \frac{1}{n+k+1}\epsilon^{-(n+k)}|H_{n,k}(x) - H_{n,k}(y)|.$$

Proof. Note that if $x = y$ the inequality is trivially true. Due to the symmetry of V, it is enough to consider again only the region below the diagonal. Let then $x > y$ and split

$$x - y = (x - y)\left(\mathbf{1}\{y < x < \rho\} + \mathbf{1}\{y \leq \rho < x\} + \mathbf{1}\{\rho \leq y < x\}\right).$$

Consider the following regions

- $A_0 = \{y < x < \rho\} \cap \{x, y < \epsilon\} = \{y < x < \epsilon < \rho\}$;
- $A_1 = \{y < x < \rho\} \cap \{x, y < \epsilon\}^c = \{y < \epsilon \leq x < \rho\}$;
- $B_0 = \{\rho \leq y < x\} \cap \{1 - x, 1 - y < \epsilon\} = \{\rho < 1 - \epsilon < y < x \leq 1\}$;
- $B_1 = \{\rho \leq y < x\} \cap \{1 - x, 1 - y < \epsilon\}^c = \{\rho < y \leq 1 - \epsilon < x \leq 1\}$.

It is clear that $(x - y)(\mathbf{1}_{A_0} + \mathbf{1}_{B_0}) < 4\epsilon$. Now we study V when restricted to A_1. Fixed $y < \rho$, the map $V(\cdot, y)$ is increasing in $[\epsilon, \rho)$ and $V(\epsilon, \cdot)$ is increasing in $[0, \epsilon)$ since $\epsilon < \rho$. In this way,

$$V(x,y)\mathbf{1}_{A_1} > V(\epsilon, y) > V(\epsilon, 0) = \frac{1}{\epsilon}\int_0^\epsilon u^n(1-u)^k \mathrm{d}u > \frac{1}{\epsilon}\int_0^\epsilon u^{n+k}\mathrm{d}u = \frac{1}{n+k+1}\epsilon^{n+k}$$

where we used that $u < \epsilon$ implies $1 - u > \epsilon > u$.

Restricted to B_1, the map $V(\cdot, y)$ is decreasing in $(1-\epsilon, 1]$ and $V(1, \cdot)$ is decreasing in $(\rho, 1-\epsilon]$. In this way,

$$V(x,y)\mathbf{1}_{B_1} > V(1,y) > V(1,1-\epsilon) = \frac{1}{\epsilon}\int_1^{1-\epsilon} u^n(1-u)^k \mathrm{d}u > \frac{1}{\epsilon}\int_{1-\epsilon}^{1}(1-u)^{n+k}\mathrm{d}u = \frac{1}{n+k+1}\epsilon^{n+k}.$$

Now it remains to study the region $\{y \le \rho < x\}$. In this case, $V(\cdot, y)$ is increasing in $(\rho, x_0]$ but decreasing in $(x_0, 1]$ and so

$$V(x,y)\mathbf{1}\{y \le \rho < x\} > \min\{V(1,y), V(\rho,y)\}\mathbf{1}\{y \le \rho < x\}.$$

The map $V(1,\cdot)$ is increasing in $[0, y_0)$ and decreasing in $(y_0, \rho]$, thus in this case $V(1,y) > \min\{V(1,0), V(1,\rho)\}$. Similarly, $V(\rho,\cdot)$ is increasing in $[0,\rho)$ and $V(\rho,y) > V(\rho,0)$. We conclude that

$$V(x,y)\mathbf{1}\{y \le \rho < x\} > \min\{V(1,\rho), V(1,0), V(\rho,0)\} =: \underline{V} > 0.$$

Note that since $\underline{V}$ is a fixed positive constant independent of ϵ, then $\underline{V} > \epsilon^{n+k}/(n+k+1)$ for $\epsilon > 0$ small enough. To conclude, one gathers the lower bounds in the previous analysis. □

As consequence of the previous lemma we obtain the regularity of ρ from that of $H_{n,k}$.

Corollary C.2.3. *If $H_{n+k}(\rho) \in \mathcal{H}^1(\mathbb{T})$, then ρ is $\frac{1}{2(n+k+1)}$−Hölder continuous. Precisely,*

$$|\rho(u) - \rho(v)| \le \left(4 + \frac{1}{n+k+1}\|H_{n+k}(\rho)\|_{L^2(\mathbb{T})}\right)|u-v|^{\frac{1}{2(n+k+1)}}.$$

Proof. From Lemma C.2.2, the regularity of H and Cauchy-Schwarz' inequality, for any $u, v \in \mathbb{T}$ it holds that

$$|\rho(u) - \rho(v)| \le 4\epsilon + \frac{1}{n+k+1}\frac{|u-v|^{\frac{1}{2}}}{\epsilon^{n+k}}\|H_{n+k}(\rho)\|_{L^2(\mathbb{T})}.$$

Fixing $\epsilon = |u-v|^{\frac{1}{2(n+k+1)}}$ leads to the upper bound in the statement. □

As a final remark, our case is in fact the basis for proving the previous result for any $H \in C^2([0,1])$ monotonic with $H' \ge 0$: the essential ingredient is the description of V, in some fixed direction, restricted to where the graph of H' is an "arc". In our case, $H'(0) = H'(1)$ and there is a global maximum at ρ, but for general H' one can split the domain $[0,1] = \cup_i [x_i, x_{i+1})$ where x_i, x_{i+1} are two consecutive local minimizers. Then, the description of $V(x,\cdot)$ or $V(\cdot, y)$ restricted to $[x_i, x_{i+1})$ will be identical to the case in Lemma C.2.1. Regarding Lemma C.2.2, one should focus on the sets where x_i and/or x_{i+1} are zeroes of H'. The argument is analogous: for each zero of H', one fixes two points very close to it and analyse V when at least one of those points are farther away from the zero. As in our case, when at least one point is far from the zero, $V(x,y)$ must be larger than when the two are close to the zero. In our case, this argument lead to some technical issues – one must split the domain into intervals containing only one zero of H' so that when we focus on one zero of H' the minimum value of V does not "escape" to another one which provides the trivial lower bound $V \ge 0$. This is the same difficulty for general H and is solved with the same approach, where instead of considering values of x or y larger or smaller than

ρ, one replaces ρ by the local maximizer, between the two local minimizers x_i, x_{i+1}. This provides the following lower bound

$$V(x,y) > \min_{r:\, H'(r)=0} \left\{ \frac{1}{\epsilon} \int_{r-\epsilon}^{r} H'(u)\mathrm{d}u \,,\, \frac{1}{\epsilon} \int_{r}^{r+\epsilon} H'(u)\mathrm{d}u \right\},$$

which can be studied, for example, with residue techniques. In order to present the arguments in a clear way and directly related to the work in this chapter we chose to consider $H = H_{n,k}$, but the general case will be presented by necessity in a forthcoming work.

C.3 Convergence of the fractional operators

Proof of Proposition 4.3.11.

(1) $0 < \gamma < 2$:

Because $N^\gamma \frac{1}{N^2} \sum_{|r|=1}^{\lfloor N/2 \rfloor} \frac{|r|}{|r|^\gamma} \xrightarrow{N\to+\infty} c_\gamma$ it is enough to study

$$\left| \int_{\mathbb{T}} \frac{\Delta_v G(\frac{x}{N})}{|v|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v - N^\gamma \sum_{r=1}^{N-1} \frac{\Delta_{\frac{r}{N}} G(\frac{x}{N})}{|r|_{\mathbb{T}_N}^{1+\gamma}} \right|. \tag{C.11}$$

One can write

$$\int_{\mathbb{T}} \frac{\Delta_v G(\frac{x}{N})}{|v|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v = \sum_{r=1}^{N-1} \int_{\frac{r}{N}}^{\frac{r+1}{N}} \frac{\Delta_v G(\frac{x}{N})}{|v|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v + \int_0^{\frac{1}{N}} \frac{\Delta_v G(\frac{x}{N})}{v^{1+\gamma}} \mathrm{d}v$$

with the second integral on the right-hand side vanishing in the limit $N \to +\infty$ uniformly:

$$\left| \int_0^{\frac{1}{N}} \frac{\Delta_v G(\frac{x}{N})}{v^{1+\gamma}} \mathrm{d}v \right| \lesssim \int_0^{\frac{1}{N}} \frac{v^2}{v^{1+\gamma}} \mathrm{d}v = \int_0^{\frac{1}{N}} v^{1-\gamma} \mathrm{d}v = \frac{1}{2-\gamma} \left[v^{2-\gamma} \right]_{v=0}^{v=\frac{1}{N}} \lesssim \frac{1}{N^{2-\gamma}}.$$

In this way, the treatment of (C.11) is reduced to the treatment of

$$\sum_{r=1}^{N-1} \left| \int_{\frac{r}{N}}^{\frac{r+1}{N}} \left\{ \frac{\Delta_v G(\frac{x}{N})}{|v|_{\mathbb{T}}^{1+\gamma}} - \frac{\Delta_{\frac{r}{N}} G(\frac{x}{N})}{\left|\frac{r}{N}\right|_{\mathbb{T}}^{1+\gamma}} \right\} \mathrm{d}v \right|.$$

Introducing $\varphi(v) = |v|_{\mathbb{T}}^{-(1+\gamma)} \Delta_v G(\frac{x}{N})$, we see that

$$|\varphi'(v)| = \left| \frac{\Delta_v G(\frac{x}{N})}{|v|_{\mathbb{T}}^{1+\gamma}} - \frac{\Delta_v G(\frac{x}{N})}{(1+\gamma)|v|_{\mathbb{T}}^{2+\gamma}} \right| \lesssim \frac{1}{|v|_{\mathbb{T}}^{1+\gamma}}$$

and from the Mean-Value Theorem,

$$\left| \frac{\Delta_v G(\frac{x}{N})}{|v|_{\mathbb{T}}^{1+\gamma}} - \frac{\Delta_{\frac{r}{N}} G(\frac{x}{N})}{\left|\frac{r}{N}\right|_{\mathbb{T}}^{1+\gamma}} \right| \lesssim \frac{\left|v - \frac{r}{N}\right|}{|v|_{\mathbb{T}}^{1+\gamma}}.$$

In this way, (C.11) is no larger than a positive constant times

$$\sum_{r=1}^{N-1} \int_{\frac{r}{N}}^{\frac{r+1}{N}} \frac{\left|v - \frac{r}{N}\right|}{|v|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v \leq \sum_{r=1}^{N-1} \int_{\frac{r}{N}}^{\frac{r+1}{N}} \frac{\frac{1}{N}}{\left|\frac{r}{N}\right|_{\mathbb{T}}^{1+\gamma}} \mathrm{d}v = \frac{N^\gamma}{N} \sum_{r=1}^{N-1} \frac{1}{|r|_{\mathbb{T}_N}^{1+\gamma}} \lesssim \frac{1}{N}.$$

Considering the scaling Θ_γ and taking the limit $N \to +\infty$ concludes the proof.

(2) $\gamma = 2$:

We want to analyse the cases when $r = o(N)$ or not, and for that we split the summation over r into $|r| \le \epsilon N$ and $|r| > \epsilon N$ for $\epsilon > 0$ and $N \gg 1$. For $r < \epsilon N$, from the fact that $G \in C^2(\mathbb{T})$ it holds that

$$\sum_{|r|=1}^{\epsilon N} \frac{\left|\frac{r}{N}\right|^2}{|r|^3} \frac{1}{\left|\frac{r}{N}\right|^2} \Delta_{\frac{r}{N}} G(\tfrac{x}{N}) = G''(\tfrac{x}{N}) \frac{1}{N^2} \sum_{|r|=1}^{\epsilon N} \frac{|r|}{|r|^2} + O\left(\frac{1}{N^4} \sum_{|r|=1}^{\epsilon N} \frac{|r|^3}{|r|^2} \right)$$

where

$$\frac{1}{N^2} \sum_{|r|=1}^{\epsilon N} \frac{|r|}{|r|^2} = O\left(\frac{\log(\epsilon N)}{N^2} \right) \quad \text{and} \quad \frac{1}{N^4} \sum_{|r|=1}^{\epsilon N} \frac{|r|^3}{|r|^2} = O\left(\frac{\epsilon^2}{N^2} \right),$$

and in particular

$$\frac{\frac{1}{N^4} \sum_{|r|=1}^{\epsilon N} \frac{|r|^3}{|r|^2}}{\frac{1}{N^2} \sum_{|r|=1}^{\epsilon N} \frac{|r|}{|r|^2}} = O\left(\frac{\epsilon^2}{\log(\epsilon N)} \right).$$

For $|r| > \epsilon N$, one obtains

$$\sum_{|r|=\epsilon N+1}^{\lfloor N/2 \rfloor} \frac{1}{|r|^3} \Delta_{\frac{r}{N}} G(\tfrac{x}{N}) \le 4 \|G\|_\infty \sum_{|r|=\epsilon N+1}^{\lfloor N/2 \rfloor} \frac{1}{|r|^3} \lesssim \frac{1}{(\epsilon N)^2}.$$

Considering the scaling Θ_γ then taking the limit $N \to +\infty$ and then $\epsilon \to 0$ concludes the proof.

(3) $\gamma > 2$:

In the same vein as in the previous case, for $|r| \le \epsilon N$

$$\sum_{|r|=1}^{\epsilon N} \frac{\left|\frac{r}{N}\right|^2}{|r|^{1+\gamma}} \frac{1}{\left|\frac{r}{N}\right|^2} \Delta_{\frac{r}{N}} G(\tfrac{x}{N}) = G''(\tfrac{x}{N}) \frac{1}{N^2} \sum_{|r|=1}^{\epsilon N} \frac{|r|}{|r|^\gamma} + O\left(\frac{\epsilon^{4-\gamma}}{N^\gamma} \right)$$

where now the series $\sum_{r \ge 1} r^{-(\gamma-1)}$ is convergent since $\gamma > 2$.

For $|r| > \epsilon N$ one obtains

$$\sum_{|r|=\epsilon N+1}^{\lfloor N/2 \rfloor} \frac{1}{|r|^{1+\gamma}} \Delta_{\frac{r}{N}} G(\tfrac{x}{N}) \le 4 \|G\|_\infty \sum_{|r|=\epsilon N+1}^{\lfloor N/2 \rfloor} \frac{1}{|r|^{1+\gamma}} \lesssim \frac{1}{(\epsilon N)^\gamma}$$

and as such one multiplies by Θ_γ and take the appropriate limits.

□

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