The low energy effective action in superstring theory

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Preface

The present manuscript is prepared for my habilitation à diriger des recherches. Its purpose is to provide a summary of research, with the stated scope of proving the candidate's ability to master a research strategy in a sufficiently broad scientific field, and their ability to supervise PhD students.

I have chosen to write the manuscript on the subject of the low energy effective action in string theory, which concerns roughly about half of my research activity since 2013. Rather than summarising my list of publications for the last twelve years, I have chosen to expose my current understanding of the topic more broadly, with emphasis on my own work.

The manuscript does not discuss my work on supergravity black holes solutions and smooth geometries describing black hole microstates in a semi-classical approximation, nor does it review the construction of Kac–Moody exceptional field theories that formulate supergravity theories with infinite-dimensional duality group invariance.

During this period I have supervised three PhD students, Valentin Verschinin and Charles Cosnier-Horeau (co-direction with Boris Pioline) who have defended their PhD thesis in 2015 and 2018, and Adrien Loty who will defend this year.

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Introduction and summary

String theory is a consistent theory of quantum gravity in the sense that it defines unambiguously scattering amplitudes in asymptotically Minkowski spacetime and includes a massless spin two particle in its spectrum. The interpretation of the amplitudes at low energy in quantum field theory permits to derive the effective action that reproduces Einstein theory of general relativity coupled to matter and with infinitely many higher derivative corrections. However, string theory lacks a first principle non-perturbative definition, and one relies on perturbation theory to compute amplitudes. An n-point amplitude in D-dimensional Minkowski spacetime is written as an expansion over (super) Riemann surfaces of genus h with n punctures weighted by ${g_{\rm s}}^{2h-2}$. There is an effective dilaton e^{ϕ} that relates the string length square $\alpha' = \ell_{\rm s}^2$ to the Planck length ℓ in D dimensions through its expectation value $1/{g_{\rm s}}^2 = \langle e^{-2\phi} \rangle$

$$\ell^{D-2} = {\alpha'}^{\frac{D-2}{2}} {g_{\rm s}}^2 \ . \tag{0.1}$$

It appears therefore that at weak coupling $g_{\rm s} \ll 1$, one reaches the string scale much before the Planck scale and the physics becomes inherently stringy before one enters a quantum gravity regime. For many quantum gravity questions one wishes to understand string theory at strong coupling, meaning $g_{\rm s} \sim 1$ [1–4].

When the theory admits supersymmetry, some of the most relevant terms in the effective action are protected by supersymmetry and may be computed exactly in string theory. In particular, the scalar fields take value in a symmetric space when the supergravity theory admits more than twelve supercharges, and the two-derivative Lagrangian is determined by the number of vector multiplets and the gauging (i.e. the gauge group and its action on the symmetric space). Using moreover duality symmetries [5, 6], one can sometimes extract exact higher derivative corrections. Prototypical examples are the leading Green–Gutperle R^4 correction in type IIB string theory [7] and the Harvey–Moore R^2 correction in heterotic string theory on T^6 [8]. In the second example the correction is obtained by a one-loop computation in type IIB on K3$\times T^2$. In the first the correction is genuinely non-perturbative in string theory, but could be derived as a one-loop correction in eleven-dimensional supergravity on T^2 [9]. These coupling functions provide extremely useful informations about non-perturbative effects in string theory, as D-brane and Neveu–Schwarz five-brane instantons. The pioneering work [7] has been instrumental in computing instanton corrections from first principles [10–12].

In $D = 4$ space-time dimensions, the $\mathcal{N} = 1$ supergravity Lagrangian decomposes into the so-called D and F-terms, that are understood respectively to be the top components of an unconstrained scalar superfield and a chiral superfield. Chiral superfields that cannot be obtained by chiral projection of an unconstrained superfield determine protected couplings in supergravity. The scalar and Yukawa couplings are fixed in this way in terms of the Kähler potential and a holomorphic function, the superpotential $\mathcal{W}$. One determines similarly the protected couplings in $\mathcal{N}$-extended supergravity as preserving a fraction of the $\mathcal{N}$ left and $\mathcal{N}$ right-handed supersymmetries. Although one cannot write all the supersymmetry invariant couplings as superspace integrals for $\mathcal{N} \geq 4$,

BPS	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=4$	$\mathcal{N}=2$
$(4/\mathcal{N}, 1-4/\mathcal{N})$	$\mathcal{E}_{(0,0)}R^4$	$\mathcal{W}F^2R^2$	$\mathcal{W}(S)R^2$	
$(2/\mathcal{N}, 2/\mathcal{N})$	$\mathcal{E}_{(1,0)}\nabla^4R^4$	$\mathcal{E}_{(0,0,0)}R^4$	$\mathcal{E}_4(\phi)F^4$	
$(2/\mathcal{N}, 0)$	$\mathcal{W}_kF^{2k}\nabla^4R^4$	$\mathcal{W}_kF^{2k}R^4$	$\mathcal{W}_kF_{\rm gr}^{2k-2}F^2R^2$	$\mathcal{W}_0(t)F^2, \mathcal{W}_k(t)F_{\rm gr}^{2k-2}R^2$
$(1/\mathcal{N}, 1/\mathcal{N})$	$\mathcal{E}_{(0,1)}\nabla^6R^4$	$\mathcal{E}_{(1,0,0)}\nabla^2R^4$	$\mathcal{E}_{2,2}(\phi)\nabla^2F^4$	$\mathcal{G}(\phi)(\nabla\phi)^2$

Table 1: The supersymmetry invariants associated to $(k_L/\mathcal{N}, k_R/\mathcal{N})$-BPS operators in $\mathcal{N}$-extended supergravity [14–20]. We write $F_{\rm gr}$ the Maxwell field strengths in the gravity multiplet for $\mathcal{N}=2$ and 4 to distinguish them from the matter multiplet field strengths F. The notation is schematic, $\mathcal{G}(\phi)$ refers to the hyper-multiplet quaternionic Kähler metric and $\sum_k \mathcal{W}_k(t)T_{\rm gr}^{2k}$ to the special Kähler prepotential function of the Weyl multiplet superfield $T_{\rm gr}$ [21–23].

one can do it in the linearised approximation in terms of free superfields, see e.g. [13]. The notion of F-term is generalised to $(k_L/\mathcal{N}, k_R/\mathcal{N})$-BPS invariants that are written in the linearised approximation as superspace integrals

$$I_{(k_L/\mathcal{N},k_R/\mathcal{N})} \sim \int d^4x d^{2\mathcal{N}-2k_L}\theta d^{2\mathcal{N}-2k_L}\bar\theta\, \mathcal{O} \tag{0.2}$$

of some operator $\mathcal{O}$ annihilated by k_L left-handed and k_R right-handed supersymmetries that anticommute with each others. Most of the time one does not specify the chirality and simply refers to $\frac{k_L+k_R}{2\mathcal{N}}$-BPS invariants. We give the schematic form of such BPS invariants in Table 1.

The couplings in the same lines in Table 1 have in common the number of preserved supersymmetries indicated in the first column, as well as the degree of complexity of the coupling function. One can define an ordering in the set of differential equations a function on a symmetric space can satisfy. The coupling functions appearing in the first line satisfy the most constraining existing set of equations and are in this sense minimal. The coupling functions appearing in the second line are in the same sense next-to-minimal.

The couplings in the last line of table 1 are particularly interesting because they get corrections from generic $1/\mathcal{N}$-BPS D-brane instantons in type II string theory [24–28]. The partition function counting $1/\mathcal{N}$-BPS D-brane instantons grows exponentially with the instanton charge Q as $\sim e^{\pi Q^2}$. By duality, the partition function of such instantons is related to the helicity supertrace counting $1/\mathcal{N}$-BPS black hole microstates [29–32]. The exponential growth of the number of states reproduces the macroscopic Bekenstein–Hawking entropy $S^{\rm BH}=\pi Q^2$ [33, 34].

This implies that the sum over D-brane instantons does not converge since they behave as $e^{\pi Q^2-2\pi|Q|/g_{\rm s}}$. The corresponding asymptotic series can in principle be defined by Borel resummation and the ambiguities associated to poles in the Borel plane be fixed by Neveu–Schwarz five-brane instantons [35]. These coupling functions can alternatively be defined without ambiguities through their abelian Fourier expansion.

The general strategy to determine an exact coupling function $\mathcal{E}$ as the ones in table 1 is to use symmetries and perturbation theory. To simplify the discussion we assume that $\mathcal{N}\geq 4$ in four dimensions and the Minkowski vacuum is supersymmetric, such that the two-derivative Lagrangian

is uniquely determined (for a given number of vector multiplets) and the scalar fields parametrise a symmetric space $K\backslash G$, where K denotes the maximal compact subgroup.

- Supersymmetry Ward identities: $\mathcal{E}$ satisfies certain partial differential equations.
- U-duality symmetry: $\mathcal{E}$ is invariant under the action of an arithmetic subgroup $\Gamma \subset G$.
- Perturbative string theory: Determines the weak coupling expansion at $g_{\rm s} \ll 1$.
- Perturbative supergravity: Valid in the large volume limit of an internal subspace.

Note that there are often different non-equivalent perturbative definitions of the theory for $\mathcal{N} = 2$ and $\mathcal{N} = 4$, in which case all these couplings may be determined by perturbative computations. The partial differential equations satisfied by the coupling functions generally imply that they can only receive corrections up to a fixed order in the string theory genus expansion. The U-duality symmetry is generally conjectural, but the consistency between the various constraints usually permits to get compelling evidence of its validity. Combining supersymmetry and U-duality allow to pin down a finite set of solutions so that the coupling functions only depends on few parameters that can be determined by consistency with perturbative computations.

In this habilitation thesis we will concentrate on type II string theory on a torus T^{d-1}, for which the low-energy effective action describing massless particles is (ungauged) maximal supergravity in $D = 11 - d$ dimensions [36, 37]. The effective action then consists of the two-derivative Lagrangian determined by supersymmetry plus an infinity of higher derivative corrections themselves consistent with supersymmetry. One can understand the effective action as being obtained by integrating out all the massive fields in string theory, including non-perturbative states. A formal path integral description suggests that integrating out all string fields up to a scale Λ should give rise to a bare action $S^{\flat}_{\Lambda}$ that would allow to define the string amplitude through the supergravity path integral up the cut-off scale Λ. However, there is a priori no regularisation scheme that preserves maximal supersymmetry and the modern techniques to compute supergravity amplitudes do not rely on the Lagrangian, so the definition of this bare action $S^{\flat}_{\Lambda}$ is not very useful. We will rather introduce the Wilsonian effective action W_{μ} and determine the renormalisation scheme in supergravity by consistency with perturbative string theory.

The approach that has been followed in the literature is to decompose the four-graviton amplitude into analytic and non-analytic components depending of the Mandelstam variables [38]

$$\mathcal{M}_4(s,t,u) = \mathcal{M}_{4\,\mu}^{\text{analytic}}(s,t,u) + \mathcal{M}_{4\,\mu}^{\text{non-an}}(s,t,u)\ . \tag{0.3}$$

This split is of course not unique but there is a well justified choice that is defined up to a set of ambiguities in one-to-one correspondance with logarithmic divergences in supergravity. One may label them for short by a unique fiducial scale μ. It is useful to interpret this split as defining a Wilson scale above which all the states of the theory have been integrated out. One can apply a similar split to higher-point functions, but we shall restrict attention to the four-point super-amplitude that is determined by the four-graviton amplitude [39].

The Wilsonian effective action, as a local functional of the supergravity fields, must be invariant under supersymmetry transformations

$$\delta_\mu^{\rm susy} W_\mu = 0 \tag{0.4}$$

where both W_μ and the corresponding supersymmetry transformation $\delta_\mu^{\rm susy}$ admit expansions in the Planck length ℓ that involve higher order derivative terms. However, the local Wilsonian action W_μ is not invariant under the U-duality group of the theory.

Let us pause now and explain this subtlety in the example of the Harvey–Moore coupling in $\mathcal{N}=4$ supergravity [8]. The Riemann square type correction to the two-derivative action depends on an arbitrary holomorphic function of the axio-dilaton S in supergravity [19, 40]. The correction appearing in heterotic string theory is

$$-\frac{3}{2\pi^2}{\rm Im}\Big[\log\eta(S)\Big(R^{ab}\wedge R_{ab}+\frac{i}{2}\varepsilon_{abcd}R^{ab}\wedge R^{cd}\Big)\Big]+\dots \tag{0.5}$$

with $\eta(S)$ the Dedekind eta function, while the $SL(2,\mathbb{Z})$ invariant coupling function is

$$\mathcal{E}^{\rm HM}=-\frac{3}{8\pi^2}\log\big({\rm Im}S|\eta(S)|^4\big)\ , \tag{0.6}$$

and includes the linear dilaton term $\log({\rm Im}S)=-2\phi$. One explains such term from the change of frame, from string to Einstein frame, whenever there is a non-analytic dependence of the amplitude in the Mandelstam variables [41], here

$$\log(-\alpha' s)=\log(-\ell^2 s)-2\phi\ . \tag{0.7}$$

This interpretation only holds for the amplitude that depends on the boundary value $\phi(\infty)$, while the Lagrangian depends on the field $S(x)$. In supergravity the linear dilaton term only appears in the non-local effective action, i.e. the generating functional of one-particle-irreducible correlation functions. One cannot separate it from the one-loop correction with massless states in the loop and it is related to the $U(1)$ duality anomaly [42, 43, 19].

Non-analytic terms in the string coupling $g_{\rm s}$ appear in this way whenever there is a logarithmic divergence in supergravity. In practice it is easier to work with a decomposition of the amplitude in analytic and non-analytic components that are individually U-duality invariant [38]. One can interpret intuitively the associated 'Wilsonian' split in the analytic and the non-analytic components of the amplitude as being defined for a moduli dependent scale $\mu'(\varphi)$. Because W_μ is obtained by integrating out all the massive fields in the theory, it is natural to choose the Wilson scale μ as the lowest mass in the spectrum, that is moduli dependent in Einstein frame. In this heterotic example one may choose the mass of the first excited string states $\mu'(\varphi)=2e^{\phi}/\ell$. In type II string theory on T^{d-1} one can choose $\mu'(\varphi)=|Z|/\ell$ equal to the mass of any 1/2 BPS particle. The would-be 'effective action' $W_{\mu'(\varphi)}$ does not satisfy (0.4) as it involves the supersymmetry variation of the scale $\mu'(\varphi)$. There is in fact no local functional $W_{\mu'(\varphi)}$ of the fields, and the scale $\mu'(\varphi)$ only make sense for the asymptotic value of the scalar fields $\varphi(\infty)$. Nonetheless, one can define $\mathcal{M}_{4\,\mu}^{\rm analytic}$ with coupling functions that are U-duality invariant, but satisfy modified differential equations. These modifications of the supersymmetry differential equations are consistent with the action of the renormalisation group, and are associated to the logarithmic divergences in supergravity in a way that can intuitively be derived from the discussion above.

$D=11-d$	$K_d\backslash E_d$	$R(\Lambda_d)$	$R(\Lambda_1)$	$R(\Lambda_2)$
8	$SO(2)\backslash SL(2,\mathbb{R})\times SO(3)\backslash SL(3,\mathbb{R})$	$(\mathbf{2},\overline{\mathbf{3}})$	$(\mathbf{1},\mathbf{3})$	$(\mathbf{2},\mathbf{1})$
7	$SO(5)\backslash SL(5,\mathbb{R})$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$
6	$(Sp(2)\times_{\mathbb{Z}_2} Sp(2))\backslash Spin(5,5)$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$
5	$(Sp(4)/\mathbb{Z}_2)\backslash E_6$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$
4	$(SU(8)/\mathbb{Z}_2)\backslash E_7$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$
3	$(Spin(16)/\mathbb{Z}_2)\backslash E_8$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{147250}$

Table 2: Convention for representations of p-form fields with the Bourbaki labelling consistant with E_d by truncation [45]. These weights also label the parabolic gauges relevant to describe perturbative string theory with $v\in P_1\subset E_d$, and the large T^d volume limit in eleven dimensional supergravity with $v\in P_2\subset E_d$.

For example $\mathcal{E}^{\rm HM}$ is not the sum of a holomorphic and an anti-holomorphic function, but still satisfies

$$\Delta\mathcal{E}^{\rm HM}=-(S-\bar S)^2\partial_{\bar S}\partial_S\mathcal{E}^{\rm HM}=\frac{3}{8\pi^2}\ . \tag{0.8}$$

The right-hand-side is a correction proportional to 1, the coupling function appearing in the two-derivative Lagrangian. The fact that $\frac{3}{8\pi^2}$ corrects $\Delta\mathcal{E}^{\rm HM}$ is directly related to the fact that the one-loop supergravity amplitude diverges logarithmically in $\frac{3}{8\pi^2}R^2$.[2] Another canonical example is the holomorphic anomaly of the twisted topological string partition function [21] that can be understood as a consequence of logarithmic divergences in $\mathcal{N}=2$ supergravity [22].

The type II string theory effective action on T^{d-1} has been analysed in details over the years. In $D=11-d$ spacetime dimensions, maximal supergravity admits massless scalar fields φ parametrising the symmetric space $K_d\backslash E_d$, vector fields A_μ^M in the highest weight representation $R(\Lambda_d)$, two-form fields $B_{\mu\nu}^{MN}$ in $R(\Lambda_1)$, three-form fields $C_{\mu\nu\rho}^{MN,P}$ in $R(\Lambda_2)$ according to table 2 above. These fields couple respectively to the particles, the strings and the membranes of the theory. We will write $E_d\equiv E_{d(d)}$ for the exceptional groups, that are always assumed in their split real form.

The Wilsonian effective action takes the schematic form

$$W_\mu=\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\left(R+\cdots+\frac{\ell^6}{48}\Big(\sum_{p,q\geq0}\ell^{4p+6q}\mathcal{E}^{\rm W}_{(p,q)\,\mu}(\varphi)t_8t_8(\nabla^4)^p(\nabla^6)^qR^4+\dots\Big)\right)\ . \tag{0.9}$$

where $t_8t_8R^4$ generalises the square of the Bel–Robinson tensor in D dimensions [46, 47] and each ∇^4 and ∇^6 is Lorentz invariant and acts symmetrically on the four Riemann tensors. As explained above, one will rather use the amplitude coupling functions $\mathcal{E}_{(p,q)}(\varphi)\sim\mathcal{E}^{\rm W}_{(p,q)\,\mu'(\varphi)}(\varphi)$ that are not analytic in the moduli but are invariant under U-duality. One says that $\mathcal{E}_{(p,q)}(\varphi)$ are automorphic, i.e. functions on the moduli space $K_d\backslash E_d/E_d(\mathbb{Z})$. The U-duality arithmetic group $E_d(\mathbb{Z})$ can be defined as the set of matrices in the representation $R(\Lambda_d)$ that are integer valued in the Chevalley

[2] This divergence drops out in the amplitude because Gauss–Bonnet is topological in four dimensions, and one sometimes talks about an evanescent divergence [44].

basis (for $d \leq 7$). It is generated from the global diffeomorphisms $SL(d,\mathbb{Z})$ of the torus in eleven dimensions and the T-duality group $Spin(d-1,d-1,\mathbb{Z})$ of automorphisms of the Narain lattice [5].[3] The first three coupling functions $\mathcal{E}_{(0,0)}$, $\mathcal{E}_{(1,0)}$, $\mathcal{E}_{(0,1)}$ are protected by supersymmetry and satisfy very constraining differential equations [48–54].

For type IIB string theory, these three coupling functions can be computed from the one-loop and two-loop supergravity amplitude on T^2 in eleven dimensions [9, 55, 56]

$$\mathcal{E}_{(0,0)} = \sum_{m,n\in\mathbb{Z}}{}' \frac{(\mathrm{Im}S)^{\frac{3}{2}}}{|m+Sn|^3} = 2\zeta(3)E^{SL(2)}_{\frac{3}{2}\Lambda_1}\,, \quad \mathcal{E}_{(1,0)} = \frac{1}{2}\sum_{m,n\in\mathbb{Z}}{}' \frac{(\mathrm{Im}S)^{\frac{5}{2}}}{|m+Sn|^5} = \zeta(5)E^{SL(2)}_{\frac{5}{2}\Lambda_1}\,,$$
$$\mathcal{E}_{(0,1)} = \frac{2\pi^2}{9}\int_{\mathbb{R}^3_+}\frac{dL_1dL_2dL_3}{(\sum_{I<J}L_IL_J)^3}\Big(\sum_I L_I - \frac{5L_1L_2L_3}{\sum_{I<J}L_IL_J}\Big)\sum_{\substack{m_I,n_I\in\mathbb{Z}\\ \sum_I m_I=\sum_I n_I=0}}{}' e^{-\pi\sum_I L_I\frac{|m_I+Sn_I|^2}{\mathrm{Im}S}}\,, \quad (0.10)$$

where m,n are the Kaluza–Klein mode numbers and S the complex structure on T^2.[4] The first and the second function $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$ are known exactly in all dimensions $D \geq 3$, in terms of a special types of automorphic functions known as Eisenstein series (or Langlands–Eisenstein) [57–62, 38, 63].

We proposed in [64] to write an effective theory keeping all 1/2 BPS particules of the theory. In type II string theory on T^{d-1}, there is exactly one short spin 2 supermultiplet of particles for each charge Γ in the lattice $\mathbb{L}_d$ in the representation $R(\Lambda_d)$ satisfying the constraint that $\Gamma\times\Gamma = (\Gamma\otimes\Gamma)|_{R(\Lambda_1)} = 0$. This effective theory can formally be defined as an exceptional field theory on a generalised torus [65–73]. The truncation to 1/2 BPS states is not consistent, because two 1/2 BPS states can produce 1/4-BPS and even non-BPS states in the theory. One finds nonetheless that the resulting amplitude morally gives the correct answer for the protected coupling functions $\mathcal{E}_{(0,0)}$, $\mathcal{E}_{(1,0)}$ and $\mathcal{E}_{(0,1)}$. In particular, the one-loop and the two-loop exceptional field theory amplitude give for all $d \geq 3$ [64]

$$\mathcal{E}_{(0,0)} \underset{d\geq 2}{=} 4\pi\xi(d-3)\,E^{E_d}_{\frac{d-3}{2}\Lambda_d}\,, \quad \mathcal{E}_{(1,0)} \underset{d\geq 3}{=} 8\pi\xi(d-5)\Big(\xi(d-4)\,E^{E_d}_{\frac{d-4}{2}\Lambda_{d-1}} + \delta_{d,5}\xi(d-3)\,E^{E_d}_{\frac{d-3}{2}\Lambda_d}\Big)\,, \quad (0.11)$$

and reproduce the results of [63] using Langlands functional identities between different Eisenstein series. They will be defined in the main text, but let us simply say that $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function and $E^{E_d}_{s\Lambda_d}$ is the sum over $E_d(\mathbb{Z})$ orbits of a power $R^{2s\frac{10-d}{9-d}}$ of a circle radius $2\pi\ell R$ (or of a T^2 torus volume for $E^{E_d}_{s\Lambda_{d-1}}$). They are natural generalisations of the $SL(2)$ Eisenstein series in (0.10) and

$$2\zeta(2s)E^{E_d}_{s\Lambda_d} = \sum_{\substack{\Gamma\in\mathbb{L}_d\\ \Gamma\times\Gamma=0}}{}' \frac{1}{|Z(\Gamma)|^{2s}} \quad (0.12)$$

can for example be interpreted as a sum over 1/2 BPS supermultiplets weighted by the BPS mass $M = |Z(\Gamma)|/\ell$ of the particles [60].

[3] The automorphism group of the Narain lattice $I\!I_{d-1,d-1}$ is $O(d-1,d-1,\mathbb{Z})$, but the discrete $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup includes the exchange of left-handed and right-handed spinors that maps type IIA to type IIB and is not a symmetry of the effective action. It is nonetheless an accidental symmetry of the four-graviton scattering amplitude.

[4] The two-loop integral in the second line is formal and needs to be regularised [56].

Applying the same reasoning at the next order in derivatives and combining the two and three-loop contributions, we obtained a proposal for the next-to-next-to-leading coupling function for all $d \leq 3$ [64, 74]

$$\mathcal{E}_{(0,1)} = \frac{2\pi^2}{9} \int_{\mathbb{R}_+^3} \frac{dL_1 dL_2 dL_3}{(\sum_{I<J} L_I L_J)^{\frac{7-d}{2}}} \Big(\sum_I L_I - \frac{5 L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \Big) {\sum_{\substack{\Gamma_I \in \mathbb{L}_d \\ \sum_I \Gamma_I = 0 \\ \Gamma_I \times \Gamma_J = 0}}}' e^{-\pi \sum_I L_I |Z(\Gamma_I)|^2}$$
$$+ \frac{8\pi^4}{567} \xi(d+3) E^{E_d}_{\frac{d+3}{2}\Lambda_d} \,. \tag{0.13}$$

This formula is divergent and requires renormalisation. The infrared divergences can be interpreted in the effective theory and drop out in the full amplitude including massless states, up to the physical infrared divergences in $D \leq 4$ dimensions. The ultraviolet divergence must be regularised for both the two-loop integral above and the Eisenstein series. Because the Eisenstein series are meromorphic functions of their parameter, it is natural to regularise them by analytic continuation in the dimension $d \to d + 2\epsilon$. The naive analytic continuation of the two-loop formula above nevertheless turns out not to be correct, and one needs to introduce an appropriate deformation of the integrand at $\epsilon \neq 0$. We have given a precise definition in [27], that was checked to match string perturbation theory and is consistent with the successive decompactification limits of one circle becoming large in T^{d-1}.

This derivation does not follow from a first principle formulation of the theory and must be proved to give the correct coupling functions in string theory. The first element in the proof relies on the differential equations following from supersymmetry. There is a unique function $\mathcal{E}_{(0,0)}$ up to normalisation consistent with supersymmetry that is invariant under $E_d(\mathbb{Z})$ in dimensions $D \leq 7$ [62, 63, 51]. The function $\mathcal{E}_{(1,0)}$ is unique up to normalisation in dimension $D \leq 5$ [63, 52]. There are two $\nabla^6 R^4$ type supersymmetry invariants in dimensions $D \leq 6$. One inhomogeneous solution is uniquely determined from $\mathcal{E}_{(0,0)}$ and two functions invariant under $E_d(\mathbb{Z})$ solve the two corresponding homogeneous differential equations [53, 75]. We display the classification of supersymmetry invariant in Figure 1. One can fix the coefficients using string perturbation theory [38, 76, 27]. Because of the almost uniqueness of these coupling functions, they satisfy a large number of consistency conditions.

There are different ways to construct $E_d(\mathbb{Z})$ invariant solutions to the differential equations imposed by supersymmetry, but it is important to find a definition that can be manipulated to extract the perturbative limit. One can think of (0.13) as the symmetrisation of the eleven-dimensional supergravity amplitude on T^d. A similar formula was proposed in $D = 6$ dimensions by symmetrising the two-loop string amplitude through the extension of the Narain lattice $I\!I_{4,4}$ to a U-duality extended lattice $I\!I_{5,5}$ [77]. We checked in [27] that the two definitions give consistently the same coupling function.

We use this habilitation thesis as an occasion to sharpen the proof of this proposal. We check its consistency in the large M-theory torus volume limit with the supergravity amplitudes. We exhibit how the non-analytic terms in the moduli are connected through the renormalisation scale to the non-analytic components of the amplitude.

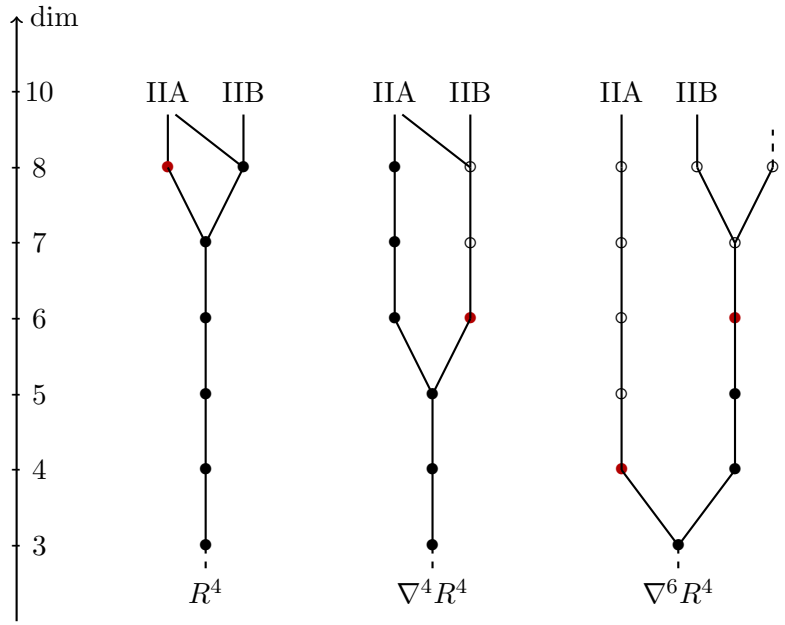

Figure 1: Each node corresponds to a supersymmetry invariant, white if it cannot be written in superspace in the linearised approximation, and red if the corresponding superspace integral is chiral (with $k_L \neq k_R$). For $\nabla^6 R^4$, the links to ten dimensions are valid for the homogeneous solution that do not take into account the modification of the supersymmetry transformations induced from the R^4 invariant.

The main interesting open question for future investigations is probably how to go beyond the BPS protected couplings and describe the low-energy effective action at higher orders in the Planck length. The four-graviton amplitude in type II string theory is known exactly up to two-loop in string theory [47, 24]. The expansion of the amplitude at low energy has been studied in details and is still under intense scrutiny, see [78–82] and [83–86] for important developments at one-loop and two-loop, respectively. However, the coupling functions starting from $\mathcal{E}_{(2,0)}$ onward receive corrections at all loop orders, as well as BPS and non-BPS instanton corrections. It is therefore a rather difficult problem that we leave for future investigations.

In this habilitation thesis we wish to review the construction explained above with particular emphasis on some subtleties that we believe have not yet been explained in details in the literature. We do not aim at an exhaustive review and will be rather brief on some aspects that can be read elsewhere.

In Section 1 we review several techniques to determine higher derivative corrections consistent with supersymmetry in the absence of off-shell formulation, i.e. when the supersymmetry algebra closes only modulo the equations of motion. We chose to concentrate on the case of half-maximal supersymmetry because it is better understood. We present several superspace methods through illustrative examples relevant to heterotic or type I string theory. We briefly review the maximal supergravity supersymmetry invariants.

Section 2 discusses the low energy expansion of perturbative string theory amplitudes at one and two-loop with particular emphasis on the non-analytic contributions involving intermediate massive

states. We compute the low energy expansion using successive approximations and in particular that a sphere amplitude emerges whenever several punctures are close enough on the worldsheet. We show how the non-analytic component of the amplitude can be written in terms of supergravity Feynman integrals involving the Wilsonian part of tree-level string amplitudes. In this way we derive one-loop and two-loop form-factors of higher derivative supersymmetric counter-terms in supergravity.

In Section 3 we review the definition of automorphic forms, automorphic representations and their relations to supersymmetry. We describe in details the relation between differential equations imposed by supersymmetry Ward identities and the set of allowed BPS instantons that can contribute in string theory. We explain how to get explicit BPS instanton solutions in general through the illustrative example of M2-brane instantons on T^7 and discuss their fermionic zero modes. We write down explicit formulas for Eisenstein series Fourier expansions relevant in the perturbative string limit and the M-theory large volume limit.

In Section 4 we revisit the eleven-dimensional supergravity amplitude on T^d to determine the contributions to the low energy effective action in $D = 11-d$ dimensions. We explain how to derive the form-factors for the insertion of the eleven-dimensional supergravity higher derivative corrections. We define the renormalised coupling functions in dimensional regularisation and exhibit the precise mixing between analytic and non-analytic components. These computations lead to further consistency checks for the coupling functions $\mathcal{E}_{(0,0)}$, $\mathcal{E}_{(1,0)}$ and $\mathcal{E}_{(0,1)}$.

Section 5 reviews the proposal for the coupling function (0.13) sketched above. We explain the constraints from supersymmetry and how we derived the precise renormalised expression. As a further consistency check, we compute its expansion in the large torus volume limit and check it reproduces the eleven-dimensional supergravity amplitude analysed in Section 4. We show that there is no purely non-perturbative $E_7(\mathbb{Z})$ invariant function consistent with supersymmetry, proving in this way that $\mathcal{E}_{(0,1)}$ is unique and not subject to ambiguities that could not be fixed using string perturbation theory in four dimension. Finally, we compute the instanton measure and compare it with the helicity supertrace counting 1/8 BPS black holes in four dimensions.

Contents

1 Supersymmetry of the low-energy effective action

Superstring theory in ten dimensions reduces at low energy to supergravity with $\mathcal{N}=1$ or $\mathcal{N}=2$ supersymmetry. The two-derivative Lagrangian is entirely determined by the spectrum and supersymmetry highly contrains the higher derivative corrections up to $8\mathcal{N}-2$ derivatives. To classify the possible supersymmetry invariants that can define higher derivative corrections, it is useful to combine several tools. It is extremely difficult to work out the Slavnov–Taylor identities of the full effective action generating the one-particle irreducible correlation functions, and one commonly restricts oneself to the Wilsonian effective action. The Wilsonian effective action is obtained after integrating out all the massive states in string theory, as a local effective action for the massless states in supergravity. This coarse graining operation is not directly doable in string theory, but we shall explain how this Wilsonian effective action can be extracted from superstring amplitudes by splitting them into analytic and non-analytic components. This split becomes ambiguous whenever there is a logarithmic divergence in supergravity, introducing a fiducial renormalisation scale that drops out in the amplitude. The Wilsonian effective action can be treated as a classical action with higher derivative corrections. Because the supersymmetry algebra only closes modulo the classical equations of motion, one must in principle consider the action with antifields $\varphi^*_{\rceil}$ associated to each field $\varphi^{\rceil}$ (including the ghosts), such that the supersymmetry Slavnov–Taylor identity can be written in terms of the master equation [87]

$$\int \sum_{\rceil} \frac{\delta^R \Sigma}{\delta \varphi^{\rceil}} \frac{\delta^L \Sigma}{\delta \varphi^*_{\rceil}} = 0 \tag{1.1}$$

with

$$\Sigma[\varphi,\varphi^*] = S[\varphi] + \int \sum_{\rceil} \varphi^*_{\rceil} s\varphi^{\rceil} + I[\varphi,\varphi^*] \,, \tag{1.2}$$

where $S[\varphi]$ is the Wilsonian action, s is the BRST operator defined as

$$s = \mathcal{L}_\xi \psi - \delta^{\text{susy}}(\epsilon) - \delta^{\text{Lorentz}}(\Omega) - \delta^{\text{gauge}}(c) \tag{1.3}$$

and $I[\varphi,\varphi^*]$ combines higher order terms in the antifields and circumvents the fact that s^2 only vanishes modulo the equations of motion. In the two-derivative approximation, $I[\varphi,\varphi^*]$ is quadratic in the antifields. We write the two-derivative action Σ for eleven-dimensional supergravity in Appendix A as an illustration. One can expand the Wilsonian effective action W_μ in the Planck length in D dimensions as

$$W_\mu = \frac{1}{\ell^{D-2}} \sum_{n\geq 0} \ell^{2n} S^{(n)} \,, \tag{1.4}$$

and similarly for the BRST operator $s = \sum_{n\geq 0} \ell^{2n} s^{(n)}$ and $I[\varphi,\varphi^*]$. The main part of the master equation is supersymmetry invariance

$$sW_\mu = \frac{1}{\ell^{D-2}} \sum_{n\geq 0} \ell^{2n} \sum_{k=0}^{n} s^{(k)} S^{(n-k)} = 0 \,. \tag{1.5}$$

Assuming the effective action and the BRST operator are determined up to order ℓ^{2n}, the order ℓ^{2n+2} can be decomposed into the particular solution $S^{(n+1)}_{\text{part}}$ and the homogeneous solution $S^{(n+1)}_{\text{hom}}$

satisfying respectively

$$s^{(0)}S^{(n+1)}_{\rm part}+s^{(2n+2)}_{\rm part}S^{(0)}=-\sum_{k=1}^{n}s^{(k)}S^{(n+1-k)} \tag{1.6}$$

and

$$s^{(0)}S^{(n+1)}_{\rm hom}+s^{(2n+2)}_{\rm hom}S^{(0)}=0\,. \tag{1.7}$$

The second equation only requires that

$$\delta^{\rm susy}_{(0)}(\epsilon)S^{(n+1)}_{\rm hom}\approx 0 \tag{1.8}$$

modulo the two derivative field equations of motion. Moreover, using field redefinitions, $S^{(n+1)}_{\rm hom}$ is itself only defined modulo the equations of motion and the problem can be considered for field satisfying the classical two-derivative equations on motion. This justifies the classification of higher derivative terms defined modulo the equations of motion and that are supersymmetric modulo the equations of motion, the so-called on-shell supersymmetry invariants. The particular solution $S^{(n+1)}_{\rm part}$ is generally extremely difficult to obtain, and the only non-trivial solution (without off-shell formulation) is the Bergshoeff and de Roo Lagrangian in $\mathcal{N}=1$ supergravity [88] and its direct generalisations.

We will first review quickly this emblematic case and describe the Green–Schwarz cancelation of the supersymmetric anomaly in ten dimensions. We will then explain the construction of some $\mathcal{N}=1$ supersymmetry invariants in eight dimensions, to pave the way to $\mathcal{N}=2$ supergravity.

1.1 Green–Schwarz counter-term and ectoplasm cohomology

We will start this section with a brief review of [88].

Bergshoeff and de Roo Lagrangian

It appears that the higher-derivative corrections in type I supergravity in ten dimensions are much more easily described in string frame. The string frame supersymmetry transformations (without vector multiplets) are

$$\begin{aligned}
\delta^{\rm susy}_{(0)}(\epsilon)e^a &= \bar\epsilon\gamma^a\psi\\
\delta^{\rm susy}_{(0)}(\epsilon)\psi &= d_{\hat\omega^+}\epsilon-\psi(\bar\epsilon\chi)+\epsilon(\bar\psi\chi)-\gamma^a\chi(\bar\epsilon\gamma_a\psi)\\
\delta^{\rm susy}_{(0)}(\epsilon)B &= -e^a_{\ \wedge}\,\bar\epsilon\gamma_a\psi\,,\qquad \delta^{\rm susy}_{(0)}(\epsilon)\phi=-\bar\epsilon\chi\\
\delta^{\rm susy}_{(0)}(\epsilon)\chi &= -\frac{1}{2}\gamma^a\epsilon\widehat{D_a\phi}-\frac{1}{24}\hat H_{abc}\gamma^{abc}\,\epsilon-\chi\bar\epsilon\chi
\end{aligned} \tag{1.9}$$

where the torsion-full spin connection is defined as

$$\hat\omega^{\pm}_{ab}=\hat\omega_{ab}\pm\frac{1}{2}\,\hat H_{abc}\,e^c\,,\qquad H=dB\,, \tag{1.10}$$

and the hats indicate the supercovariantisation. We recall that the supercovariantisation of a field is obtained by adding the ψ dependent components ensuring that its supersymmetry variation does

not involve the derivative of the spinor parameter. The trick of Bergshoeff and de Roo is to identify $\hat{\omega}^-_{ab}$ as a Spin(1,9) gauge field and the Rarita–Schwinger field strength ρ^+_{ab}

$$\frac{1}{2}\rho^+_{ab}\, e^a_\wedge e^b = \rho^+ = d_{\hat{\omega}^+}\psi + \frac{1}{2}\gamma_a\chi(\bar{\psi}\gamma^a\psi) - \psi(\bar{\psi}\chi) \tag{1.11}$$

as (minus) its Majorana–Weyl gaugino. Indeed, one finds modulo the fermions equations of motion the supersymmetry transformations

$$\begin{aligned}\delta^{\text{susy}}_{(0)}(\epsilon)\hat{\omega}^-_{ab} &= -e^c\bar{\epsilon}\gamma_c\rho^+_{ab}\,,\\ \delta^{\text{susy}}_{(0)}(\epsilon)\rho^+_{ab} &= \frac{1}{4}\hat{R}(\hat{\omega}^-)_{cdab}\gamma^{cd}\epsilon - \rho^+_{ab}\bar{\epsilon}\chi + \epsilon\bar{\chi}\rho^+_{ab} - \gamma^c\chi\bar{\epsilon}\gamma_c\rho^+_{ab}\,,\end{aligned} \tag{1.12}$$

which are identical to the ones of the supersymmetric Yang–Mills fields

$$\delta A = e^a\bar{\epsilon}\gamma_a\lambda\,, \qquad \delta\lambda = -\frac{1}{4}\gamma^{ab}\,\hat{F}_{ab}\epsilon - \lambda\bar{\epsilon}\chi + \epsilon\bar{\chi}\lambda - \gamma^a\chi\bar{\epsilon}\gamma_a\lambda\,. \tag{1.13}$$

One can therefore use the two-derivative Lagrangian for the vector multiplet and substitute the connection $\hat{\omega}^-_{ab}$ and $-\rho^+_{ab}$ to determine a correction at order α' to the effective action.

Including vectors multiplets the supersymmetry transformation of $\hat{\omega}^-_{ab}$ gets corrected to

$$\begin{aligned}\delta^{\text{susy}}_{(0)}(\epsilon)\hat{\omega}^-_{ab} &= -e^c\bar{\epsilon}\gamma_c\big(\rho^+_{ab} + \text{Tr}\big[\hat{F}_{ab}\lambda\big]\big) - 2e^c\bar{\epsilon}\gamma_{[a}\text{Tr}\big[\hat{F}_{b]c}\lambda\big]\\ \delta^{\text{susy}}_{(0)}(\epsilon)\rho^+_{ab} &= \frac{1}{4}\Big(\hat{R}(\hat{\omega}^-)_{cdab} + 3\text{Tr}\big[\hat{F}_{[ab}\hat{F}_{cd]}\big]\Big)\gamma^{cd}\epsilon + \ldots\end{aligned} \tag{1.14}$$

and the R^2 type correction comes together with an F^4 type correction as

$$\ell^2 S^{(2)} = \alpha'\int d^{10}x e\bigg(-\frac{1}{4}\hat{R}(\hat{\omega}^-)_{abcd}\hat{R}(\hat{\omega}^-)^{abcd} + \frac{3}{2}\text{Tr}\big[\hat{F}_{[ab}\hat{F}_{cd]}\big]\text{Tr}\big[\hat{F}^{ab}\hat{F}^{cd}\big] + \frac{1}{2}\text{Tr}\big[\hat{F}_a{}^c\hat{F}_{bc}\big]\text{Tr}\big[\hat{F}^a{}_d\hat{F}^{bd}\big]\bigg) \tag{1.15}$$

Bergshoeff and de Roo have computed the corresponding corrections to the effective action up to order α'^3. The calculation greatly simplifies through the introduction of the corrected three-form field strength H via the implicite equation [89]

$$H = dB + \text{Tr}\big[AdA + \tfrac{2}{3}A^3\big] - \alpha'\text{Tr}\big[\hat{\omega}^- d\hat{\omega}^- + \tfrac{2}{3}\hat{\omega}^{-3}\big] \tag{1.16}$$

that one solves perturbatively in α'. Up to order α'^3, the bosonic corrections to the supersymmetry transformation and the Lagrangian are simply obtained by substituting

$$\text{Tr}\big[\hat{F}_{ab}\hat{F}_{cd}\big] \to \text{Tr}\big[\hat{F}_{ab}\hat{F}_{cd}\big] + \alpha'\hat{R}(\hat{\omega}^-)_{abef}\hat{R}(\hat{\omega}^-)_{cd}{}^{ef} \tag{1.17}$$

in (1.14) and (1.15). To compare with the string theory amplitudes it is convenient to do field redefinitions to eliminate the higher derivative corrections to the propagators coming from $R(\omega^-)^2$ that would involve Ostrogradsky ghosts. The quartic term in the field strengths then recombine into [89, 90]

$$\frac{\alpha'}{32}t_8\Big(\text{Tr}[FF] - \alpha'\text{Tr}[R(\hat{\omega}^-)R(\hat{\omega}^-)]\Big)\Big(\text{Tr}[FF] - \alpha'\text{Tr}[R(\hat{\omega}^-)R(\hat{\omega}^-)]\Big)\,, \tag{1.18}$$

where the t_8 tensor is defined such that

$$\begin{aligned}t_8F_1F_2F_3F_4 = {}& 4F_{1\mu\nu}F_2^{\nu\sigma}F_{3\sigma\rho}F_4^{\rho\mu} + 4F_{3\mu\nu}F_2^{\nu\sigma}F_{1\sigma\rho}F_4^{\rho\mu} + 4F_{2\mu\nu}F_3^{\nu\sigma}F_{1\sigma\rho}F_4^{\rho\mu} + 4F_{1\mu\nu}F_3^{\nu\sigma}F_{2\sigma\rho}F_4^{\rho\mu}\\ &+4F_{3\mu\nu}F_1^{\nu\sigma}F_{2\sigma\rho}F_4^{\rho\mu} + 4F_{2\mu\nu}F_1^{\nu\sigma}F_{3\sigma\rho}F_4^{\rho\mu}\\ &-2F_1^{\mu\nu}F_{2\mu\nu}F_3^{\sigma\rho}F_{4\sigma\rho} - 2F_2^{\mu\nu}F_{3\mu\nu}F_1^{\sigma\rho}F_{4\sigma\rho} - 2F_3^{\mu\nu}F_{1\mu\nu}F_2^{\sigma\rho}F_{4\sigma\rho}\,.\end{aligned} \tag{1.19}$$

This determines the particular solution (1.6) for $S^{(4)}$ and $S^{(6)}$ induced by the $S^{(2)}$ correction (1.15).

Supergravity in superspace

We will now discuss the Green–Schwarz corrections to the effective action [91]. To analyse these corrections, we find convenient to work in superspace with coordinates $z^M = (x^\mu, \vartheta^\alpha)$. One defines the supervielbeins $E^A = (E^a, E^\alpha)$ and their connection $\Omega_B{}^A$ with the supertorsion and curvature

$$T^A = \mathrm{d}E^A + E^B \wedge \Omega_B{}^A \,, \qquad R_B{}^A = \mathrm{d}\Omega_B{}^A + \Omega_B{}^C \wedge \Omega_C{}^A \,. \tag{1.20}$$

$\mathcal{N} = 1$ supergravity in superspace was constructed in [92]. One decomposes the superspace exterior derivative as [93]

$$\mathrm{d} = \mathrm{d}z^M \partial_M = E^A D_A + T^A \iota_A \,, \tag{1.21}$$

where

$$D_A = E_A{}^M(\partial_M + \Omega_M) \,, \qquad \iota_A E^B = \delta_A^B \,. \tag{1.22}$$

The supermanifold $\mathcal{SM}$ is a fibre bundle over its bosonic base space $\mathcal{M}$ and we assume the existence of a global section $s : \mathcal{M} \to \mathcal{SM}$. In practice one uses the trivial injection $z^M = (x^\mu, 0)$. A homogeneous solution to (1.7) can be obtained from a Lagrangian density defined as the pull-back s^* to the bosonic space $\mathcal{M}$ of a closed D-superform L_D

$$\int s^* L_D = \frac{1}{D!} \int d^D x \, \varepsilon^{\mu_1 \dots \mu_D} \Big(E_{\mu_1}{}^{A_1} \dots E_{\mu_D}{}^{A_D} L_{A_1 \dots A_D} \Big) \Big|_{\vartheta = 0} \tag{1.23}$$

defined modulo an exact superform [94, 95]. The action of a superdiffeomorphism φ is defined by composition with the section s, and for a flow φ_t generated by a vector field Ξ^M one has

$$\frac{d}{dt} \int s^* \varphi_t^* L_D |_{t=0} = \int s^* \mathcal{L}_\Xi L_D = \int d\big(s^* \iota_\Xi L_D \big) \,. \tag{1.24}$$

Assuming the appropriate fall off of the fields at infinity one gets therefore that the integral is invariant under superdiffeomorphisms. If one writes explicitly the pull-back for $D = 10$ as

$$\begin{aligned} \int s^* L_{10} = \frac{1}{10!} \int \Big(& e^{a_1} \wedge e^{a_2} \wedge \dots e^{a_{10}} L_{a_1 \dots a_{10}} |_{\vartheta=0} + 10 \psi^\alpha \wedge e^{a_3} \dots e^{a_{10}} L_{\alpha a_2 \dots a_{10}} |_{\vartheta=0} \\ & + 45 \psi^\alpha \wedge \psi^\beta \wedge e^{a_3} \dots e^{a_{10}} L_{\alpha\beta a_3 \dots a_{10}} |_{\vartheta=0} + \dots \Big) \end{aligned} \tag{1.25}$$

the $L_{A_1 \dots A_{10}} |_{\vartheta=0}$ components only depend on the supercovariant fields ans this decomposition is very similar to the one appearing in the rheonomic approach [96].

The exterior differential can be decomposed in form-degree (b, f)

$$\mathrm{d} = \mathrm{d}_{(2,-1)} + \mathrm{d}_{(1,0)} + \mathrm{d}_{(0,1)} + \mathrm{d}_{(-1,2)} \,, \tag{1.26}$$

with b and f the bosonic and fermionic form-degrees, such that [5]

$$\begin{aligned} \mathrm{d}_{(2,-1)} &= \frac{1}{2} E^b \wedge E^a T_{ab}{}^\gamma \iota_\gamma \,, \quad \mathrm{d}_{(1,0)} = E^a D_a + \frac{1}{2} E^b \wedge E^a T_{ab}{}^c \iota_c + E^\beta \wedge E^a T_{a\beta}{}^\gamma \iota_\gamma \,, \\ \mathrm{d}_{(0,1)} &= E^\alpha D_\alpha + \frac{1}{2} E^\beta \wedge E^\alpha T_{\alpha\beta}{}^\gamma \iota_\gamma \,, \qquad \mathrm{d}_{(-1,2)} = \frac{1}{2} E^\beta \wedge E^\alpha T_{\alpha\beta}{}^c \iota_c \,. \end{aligned} \tag{1.27}$$

[5] We can generally assume $T_{\alpha b}{}^c = 0$ and it turns out that $T_{\alpha\beta}{}^\gamma = 0$ in ten dimensions [92].

The component of lowest bosonic form-degree $b = -1$

$$\mathrm{d}_{(-1,2)} \equiv t_0 = -\frac{i}{2} E^\beta \wedge E^\alpha \gamma^a{}_{\alpha\beta} \iota_a \tag{1.28}$$

is purely algebraic and nilpotent and must cancel the component $L_{(b,f)}$ of lowest bosonic form-degree b of L_D. This component must therefore be a cohomology class of t_0

$$t_0 L_{(b,f)} = 0 \ , \qquad L_{(b,f)} \approx L_{(b,f)} + t_0(\dots) \ . \tag{1.29}$$

The cohomology classes of t_0 have been classified in ten dimensions [97], and the lowest bosonic form-degree (b, f) component must have $b \leq 5$ to be non-trivial.

The simplest F^4 type closed-form can be obtained from the 11-superforms

$$W_{11} = H \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] \ , \qquad W'_{11} = H \wedge \mathrm{Tr}\big[F \wedge F\big] \wedge \mathrm{Tr}\big[F \wedge F\big] \ . \tag{1.30}$$

Note that we use the same letters to write the superforms H and F and their bosonic components $H|_{\vartheta=0}$ and $F|_{\vartheta=0}$, or more explicitly

$$H_{abc}(x, \vartheta = 0) = \hat{H}_{abc}(x) \ , \qquad F_{ab}(x, \vartheta = 0) = \hat{F}_{ab}(x) \ . \tag{1.31}$$

Hopefully the reader will distinguish them from the context. The superforms H and W_{11} are not closed because of the modified Bianchi identity

$$\mathrm{d}H = -\mathrm{Tr}\big[F \wedge F\big] + \alpha' \mathrm{Tr}\big[R(\omega^-) \wedge R(\omega^-)\big] \ . \tag{1.32}$$

We will see that this is related to the gauge anomaly in ten dimensions, but let us first neglect the right-hand-side and assume instead that $H = \mathrm{d}B$ and

$$H = \frac{1}{6} E^c \wedge E^b \wedge E^a H_{abc} - \frac{1}{2} E^c \wedge E^b \wedge E^\alpha \gamma_{bc\alpha}{}^\beta \chi_\beta - i E^c \wedge E^\beta \wedge E^\alpha \gamma_{c\,\alpha\beta} \ . \tag{1.33}$$

One can then write

$$H \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] = \mathrm{d}\Big(B \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big]\Big) \tag{1.34}$$

as a superform, whereas W_{11} is also the exterior derivative of a gauge invariant antecedent if one forgets again the Chern–Simons term. Using

$$F = \frac{1}{2} E^b \wedge E^a F_{ab} - i E^b \wedge E^\alpha \gamma_{b\,\alpha\beta} \lambda^\beta \ , \tag{1.35}$$

one computes that the lowest bosonic form-degree component

$$\mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big]_{abcd,\alpha\beta\gamma\delta} \sim \gamma_{abcde\,(\alpha\beta} \gamma^{fghij}{}_{\gamma\delta)} \mathrm{Tr}\big[\bar{\lambda} \gamma_{fgh} \lambda \bar{\lambda} \gamma_{ij}{}^e \lambda\big] \tag{1.36}$$

in t_0 cohomology [97], and therefore

$$5 \gamma_{[e\,(\varepsilon\zeta} \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big]_{abcd],|\alpha\beta\gamma\delta)} \sim \gamma^f{}_{(\varepsilon\zeta} \gamma_{abcde\,|\alpha\beta} \gamma^{ghijk}{}_{\gamma\delta)} \mathrm{Tr}\big[\bar{\lambda} \gamma_{fgh} \lambda \bar{\lambda} \gamma_{ijk} \lambda\big] \ . \tag{1.37}$$

Therefore the lowest bosonic form-degree component of (1.34) is t_0 trivial, and since there is no obstruction in t_0-cohomology with $b \geq 6$, one is ensured to find the gauge invariant antecedent recursively [97]

$$\begin{aligned}
W_{(5,6)} &= t_0 K_{(6,4)} \\
t_0\big(W_{(6,5)} - \mathrm{d}_{(0,1)} K_{(6,4)}\big) &= 0 \\
\Rightarrow \quad W_{(6,5)} - \mathrm{d}_{(0,1)} K_{(6,4)} &= t_0 K_{(7,3)} \\
t_0\big(W_{(7,4)} - \mathrm{d}_{(1,0)} K_{(6,4)} - \mathrm{d}_{(0,1)} K_{(7,3)}\big) &= 0 \\
\Rightarrow \quad W_{(7,4)} - \mathrm{d}_{(1,0)} K_{(6,4)} - \mathrm{d}_{(0,1)} K_{(7,3)} &= t_0 K_{(8,2)} \\
t_0\big(W_{(8,3)} - \mathrm{d}_{(2,-1)} K_{(6,4)} - \mathrm{d}_{(1,0)} K_{(7,3)} - \mathrm{d}_{(0,1)} K_{(8,2)}\big) &= 0 \\
\Rightarrow \quad W_{(8,3)} - \mathrm{d}_{(2,-1)} K_{(6,4)} - \mathrm{d}_{(1,0)} K_{(7,3)} - \mathrm{d}_{(0,1)} K_{(8,2)} &= t_0 K_{(9,1)} \\
t_0\big(W_{(9,2)} - \mathrm{d}_{(2,-1)} K_{(7,3)} - \mathrm{d}_{(1,0)} K_{(8,2)} - \mathrm{d}_{(0,1)} K_{(9,1)}\big) &= 0 \\
\Rightarrow \quad W_{(9,2)} - \mathrm{d}_{(2,-1)} K_{(7,3)} - \mathrm{d}_{(1,0)} K_{(8,2)} - \mathrm{d}_{(0,1)} K_{(9,1)} &= t_0 K_{(10,0)} \ .
\end{aligned} \tag{1.38}$$

Therefore

$$W_{11} \sim \mathrm{d} K_{10} \ , \tag{1.39}$$

and one has the closed superform

$$L_{10} = K_{10} - B \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] \ . \tag{1.40}$$

This closed superform would define a supersymmetry invariant if there was no Chern–Simons term in the three-form. This failure is related to the gauge anomaly in ten dimensions [98] and the fact that L_{10} should cancel both the gauge and the supersymmetry one-loop anomalies [91].

Anomaly in superspace

Let us now revisit the discussion above without neglecting the right-hand-side in the Bianchi identity (1.32). The closure of W_{11} gives

$$\mathrm{d} W_{11} = -\mathrm{Tr}\big[F \wedge F\big] \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] + \alpha' \mathrm{Tr}\big[R(\omega^-) \wedge R(\omega^-)\big] \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] \ . \tag{1.41}$$

One shows that

$$\mathrm{Tr}\big[F \wedge F\big] \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] = \mathrm{d}\big(\mathrm{Tr}\big[A\mathrm{d}A + \tfrac{2}{3} A^3\big] \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big]\big) = \mathrm{d} K_{11} \tag{1.42}$$

for a gauged invariant K_{11}. The existence of K_{11} follows from the fact that there is no (6,6)-form in t_0 cohomology [97]. The pull-back of the (10,1) component of

$$\mathrm{Tr}\big[A\mathrm{d}A + \tfrac{2}{3} A^3\big] \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] - K_{11} \tag{1.43}$$

to the bosonic subspace defines the supersymmetrisation of the gauge anomaly term

$$\mathrm{Tr}\big[c\, dA\big] \wedge \mathrm{Tr}\big[F \wedge F \wedge F \wedge F\big] \ . \tag{1.44}$$

This is somehow a natural extension of the descent equation relating a consistent anomaly to the invariant polynomial [99–101]. One can indeed relate the superform to the BRST extended form

appearing in the descent equation, by identifying the superspace exterior derivative d with the extended operator

$$\mathrm{d} \sim d + s - \mathcal{L}_\xi + \iota_{\frac{1}{2}\bar{\epsilon}\gamma\epsilon} \tag{1.45}$$

in components, where s is the BRST operator, and the fermionic components of the vielbeins and the gauge fields are identified with the ghosts for supersymmetry, gauge invariance and local Lorentz invariance, respectively

$$E^\alpha \sim -\epsilon^\alpha \qquad E^\alpha A_\alpha \sim -c\,, \qquad E^\alpha \Omega_{\alpha\, a}{}^b \sim -C_a{}^b\,. \tag{1.46}$$

Now, at zero order in α' we have that

$$\mathrm{d}(W_{11} + K_{11}) = 0\,, \tag{1.47}$$

and so we have a closed 11-form to obtain our 10-form Lagrangian density. The same argument as above shows that $W_{11} + K_{11}$ is $\mathrm{d}K_{10}$ for a gauge invariant K_{10}. This is because the proof is only based on the triviality of the t_0-cohomology for (b, f)-superforms with $b \geq 6$ and that the lowest bosonic degree component of K_{11} is $K_{(7,4)}$ while the one of W_{11} is $W_{(5,6)}$. Moreover

$$\mathrm{d}\Big(B \wedge \mathrm{Tr}\big[F\wedge F\wedge F\wedge F\big]\Big) = H \wedge \mathrm{Tr}\big[F\wedge F\wedge F\wedge F\big] + \mathrm{Tr}\big[A\mathrm{d}A + \tfrac{2}{3}A^3\big] \wedge \mathrm{Tr}\big[F\wedge F\wedge F\wedge F\big] \tag{1.48}$$

therefore the superform (1.40) satisfies

$$\mathrm{d}L_{10} = K_{11} - \mathrm{Tr}\big[A\mathrm{d}A + \tfrac{2}{3}A^3\big] \wedge \mathrm{Tr}\big[F\wedge F\wedge F\wedge F\big]\,. \tag{1.49}$$

The pull-back of L_{10} to the bosonic subspace is thus a counter-term that permits to restore gauge invariance and supersymmetry in the presence of a single trace anomaly (1.44).

Although the superform can be used in principle to extract the corresponding invariant, it is rather tedious to do it in practice and has not been done explicitly. For a superform with lowest bosonic degree 5, one finds in t_0-cohomology

$$L_{abcde,\alpha\beta\gamma\delta\varepsilon} \sim \gamma_{abcde\,(\alpha\beta}\Lambda_{\gamma\delta\varepsilon)} \tag{1.50}$$

and the linearised invariant can be extracted from the pure spinor measure [97]

$$L_{(10,0)} \sim (D^5)^{\alpha\beta\gamma}\Lambda_{\alpha\beta\gamma}\,. \tag{1.51}$$

Here D^5 is the quintic in the linearised superspace covariant derivative D^{lin}_α projected to the [0,0,0,3,0] irreducible representation of Spin(1,9). Using the background field $B = -iE^a\wedge E^\alpha\gamma_{a\,\alpha\beta}\vartheta^\beta$ in flat space one can write

$$\Lambda_{\alpha\beta\gamma} = \vartheta^\delta\gamma^a{}_{\delta(\alpha}\gamma^{bcdef}{}_{\beta\gamma)}\mathrm{Tr}\big[\bar{\lambda}\gamma_{abc}\lambda\bar{\lambda}\gamma_{def}\lambda\big] \tag{1.52}$$

and although $\Lambda_{\alpha\beta\gamma}$ depends explicitly on ϑ, the resulting integral is supersymmetric in the linearised approximation. However, this pure-spinor measure does not generalise at the non-linear level. The field strengths terms were originally computed by brute force in [102, 103] to be

$$L_{(10,0)} = \frac{1}{8}e\,t_8^{\mu\nu\rho\sigma\kappa\lambda\vartheta\tau}\mathrm{Tr}\big[F_{\mu\nu}F_{\rho\sigma}F_{\kappa\lambda}F_{\vartheta\tau}\big] - B\wedge\mathrm{Tr}\big[F\wedge F\wedge F\wedge F\big] + \ldots \tag{1.53}$$

The same construction goes through for the double trace, the mixed and the gravitational anomaly. Indeed, the Bergshoeff de Roo trick implies that the superforms $\mathrm{Tr}\big[R(\omega^-)^{2n}\big]$ can be treated exactly in the same way as $\mathrm{Tr}\big[F^{2n}\big]$.

To summarise, for a given 8-form invariant polynomial $P_8[F, R(\omega^-)]$, one can construct an anomaly canceling term. There exists a gauge invariant 11-superform K_{11} satisfying

$$\mathrm{d}K_{11} = \Big(\mathrm{Tr}\big[F\wedge F\big] - \alpha'\mathrm{Tr}\big[R(\omega^-)\wedge R(\omega^-)\big]\Big)\wedge P_8[F, R(\omega^-)]\ , \tag{1.54}$$

and a gauge invariant 10-superform K_{10} satisfying

$$\mathrm{d}K_{10} = H\wedge P_8[F, R(\omega^-)] + K_{11}\ . \tag{1.55}$$

The term that appears at one-loop in the Wilson effective action is the integral of the pull-back of the superform

$$L_{10} = K_{10} - B\wedge P_8[F, R(\omega^-)]\ , \tag{1.56}$$

which is not supersymmetric. But its supersymmetry variation is by construction the contraction with $\epsilon^\alpha\iota_\alpha$ of the superform

$$\mathrm{d}L_{10} = K_{11} - \Big(\mathrm{Tr}\big[A\mathrm{d}A + \tfrac{2}{3}A^3\big] - \alpha'\mathrm{Tr}\big[\omega^-\mathrm{d}\omega^- + \tfrac{2}{3}\omega^{-\,3}\big]\Big)\wedge P_8[F, R(\omega^-)] \tag{1.57}$$

which cancels the one-loop anomaly to the supersymmetry Ward identity in supergravity. Note however that K_{11} is only determined modulo an exact superform $\mathrm{d}\Delta_{10}$, which appears as an ambiguity in the definition of L_{10}. This ambiguity can only be fixed in the full effective action by ensuring that the supersymmetry Slavnov–Taylor identity is satisfied. In practice a reasonable regularisation involves a minimal solution K_{11} that starts with cubic terms in the fermions. In this way the bosonic part of the counter-term L_{10} and its quadratic terms in the fermions are uniquely fixed to the onces identified in [103].

There is a similar story for $\mathcal{N}=1$ supergravity in six dimensions, and in this case the counter-terms and the supersymmetrisation of the anomaly have been worked out explicitly in components for the gauge field part in [104].

Higher derivative corrections in the gauge fields can be defined [105, 106]. In particular the double-trace $\nabla\mathrm{Tr}F^2\nabla\mathrm{Tr}F^2$ type invariant is also protected and only recieves corrections up to two-loop in heterotic string theory.

Of particular interest are the R^4 type corrections. The only other invariant that can be constructed can be defined as the full-superspace integral of the supervielbein Berezinian times a function of the dilaton. Using consistency with dimensional reduction, one obtains schematically [19]

$$\begin{aligned}\int d^{10,16}z\mathrm{Ber}(E)K(\phi) \sim \big(\partial_\phi^4K - 12\partial_\phi^3K + 44\partial_\phi^2K - 48\partial_\phi K\big)\big(t_8t_8 + \tfrac{1}{8}\varepsilon\varepsilon\big)R^4 + \dots\\ + \big(\partial_\phi^2K - 6\partial_\phi K + 8K\big)\big(t_8\nabla R^2\nabla\mathrm{Tr}F^2 + \dots\big)\end{aligned} \tag{1.58}$$

The R^4 term therefore vanishes for $K = e^{(2\ell-2)\phi}$ for $\ell = 1, 2, 3, 4$. Because of the other non-vanishing terms, a linear dilaton pre-potential of the type $K = \phi\, e^{(2\ell-2)\phi}$ would lead to a linear dilaton term for $\ell = 1, 3, 4$. We conclude that there is a non-renormalisation theorem in heterotic string theory that forbids $t_8t_8R^4$ to appear in the four-graviton amplitude at 1, 2, 3 and 4-loop order. This is consistent with the two-loop non-renormalisation established in [24]. Because of the duality between type I and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string, one expects however that there should be a contribution to all loop orders $\ell \geq 5$ [107].

1.2 $\mathcal{N}=1$ chiral measure in eight dimensions

The analysis of supersymmetry invariants simplifies in eight dimensions because one can define chiral superspace integrals. The superspace fields on $\mathcal{N}=1$ supergravity in eight dimensions can be obtained by consistent truncation of $\mathcal{N}=2$ supergravity [51]. In this section we will be in Einstein frame.

The low dimension supertorsion components are

$$\begin{aligned} T_{\alpha\dot\beta}{}^{c} &= -i\gamma^{c}{}_{\alpha\dot\beta}\,, \qquad T_{\alpha\beta}{}^{c}=0\,, \qquad T_{\alpha\beta}{}^{\dot\gamma}=0\,, \qquad T_{\alpha b}{}^{c}=0\,, \\ T_{\alpha\beta}{}^{\gamma} &= \delta_{\alpha\beta}\chi^{\gamma}-\frac{1}{2}\delta^{\gamma}_{(\alpha}\chi_{\beta)}\,, \quad T_{\alpha\dot\beta}{}^{\dot\gamma}=-\frac{3}{4}\delta^{\dot\gamma}_{\dot\beta}\chi_{\alpha}+\frac{1}{2}\gamma^{a}{}_{\dot\beta\alpha}\gamma_{a}{}^{\delta\dot\gamma}\chi_{\delta}\,, \quad T_{ab}{}^{c}=0\,, \end{aligned} \tag{1.59}$$

where $a=0$ to 7 is the $SO(1,7)$ index and α and $\dot\alpha$ are the Weyl spinor indices. The gravity supermultiplets includes the field strengths

$$\begin{aligned} \bar F &= \frac{1}{2}E^{b}\wedge E^{a}\bar F_{ab}-iE^{b}\wedge E^{\dot\alpha}\gamma_{b\dot\alpha}{}^{\beta}\chi_{\beta}-E^{\dot\beta}\wedge E^{\dot\alpha}\delta_{\dot\alpha\dot\beta}\,, \\ H &= \frac{1}{6}E^{c}\wedge E^{b}\wedge E^{a}H_{abc}-\frac{1}{2}E^{c}\wedge E^{b}\wedge\big(E^{\alpha}\gamma_{ab\alpha}{}^{\beta}\chi_{\beta}+E^{\dot\alpha}\gamma_{ab\dot\alpha}{}^{\dot\beta}\bar\chi_{\dot\beta}\big)-iE^{c}\wedge E^{\dot\beta}\wedge E^{\alpha}\gamma_{c\dot\beta\alpha}\,. \end{aligned} \tag{1.60}$$

In this convention, the three-form H_{abc} and the graviphoton field strength $\bar F_{ab}$ are dressed with the scalar fields such that

$$H_{abc}(x,\vartheta=0)=e^{-\frac{2}{3}\phi}\hat H_{abc}(x)\,, \qquad \bar F_{ab}(x,\vartheta=0)=e^{-\frac{1}{3}\phi}\bar v_{I}\hat F^{I}_{ab}(x)\,, \tag{1.61}$$

where ϕ is the effective string coupling dilaton in eight dimensions and the complex $SO(2,n)$ vector v_I parametrises the symmetric space $SO(2,n)/\big(SO(2)\times SO(n)\big)$ for n abelian vector multiplets.[6] The left and right projectors on the Grassmanian are defined such that

$$-v_{AI}v^{A}{}_{J}+v_{I}\bar v_{J}+\bar v_{I}v_{J}=\eta_{IJ} \tag{1.62}$$

for the even bilinear form of signature $(n,2)$ (with two plus signs and n negative). In the quantum theory, this bilinear form defines an even lattice $\Lambda_{2,n}$ and the electric charges of the theory lie in the dual lattice $\Lambda^{*}_{2,n}$. It is useful to introduce complex coordinates $t=x+iy\in\mathbb{C}^{1,n-1}$ with y a positive norm vector in $\Lambda_{1,n-1}\otimes\mathbb{R}$ for a sublattice $\Lambda_{1,n-1}\subset\Lambda_{2,n}$ of signature $(n,1)$. One then defines the central charge for $Q\in\Lambda^{*}_{2,n}$

$$Z[Q]=2v_{I}Q^{I}=u\Big(m+(t,q)-\tfrac{1}{2}(t,t)n\Big)\,, \qquad u\bar u=\frac{2}{(y,y)}=e^{\mathcal{K}}\,, \tag{1.63}$$

with $Q^{I}=(m,q,n)$ and $q\in\Lambda^{*}_{1,n-1}$. The symmetric space admits the Kähler potential

$$\mathcal{K}(t,\bar t)=-\log\frac{-(t-\bar t,t-\bar t)}{8}\,, \tag{1.64}$$

and the phase α of $u=e^{\mathcal{K}/2+i\alpha}$ can be gauge-fixed in component. It is nevertheless useful to keep α free because the corresponding superfield is then chiral.

[6] If T_2 is the torus volume modulus then $e^{2\phi}=\frac{e^{2\phi_{\rm H}}}{T_2}$ where $\phi_{\rm H}$ is the dilaton in ten dimensions.

We define the scalar momentum superform

$$\bar{P}^A = E^a \bar{P}_a^A - 2E^{\dot\alpha}\bar\lambda_{\dot\alpha}^A \tag{1.65}$$

where

$$P_a^A(x,\vartheta=0) = \widehat{v^A{}_I D_a v^I}(x)\ . \tag{1.66}$$

The superfields u and t are chiral. The spinor χ_α is identified with the dilatino and

$$D_\alpha\phi = -\frac{3}{2}\chi_\alpha\ ,\quad D_\alpha\chi_\beta = \frac{1}{12}\gamma^{ab}{}_{\alpha\beta}\big(\bar F_{ab} - \tfrac{1}{16}\chi\gamma_{ab}\chi - \tfrac{1}{2}\bar\lambda^A\gamma_{ab}\bar\lambda_A\big)\ . \tag{1.67}$$

The supermultiplet structure can be represented schematically as

$$\begin{array}{ccccc}
\phi & \underset{D_\alpha}{\to} & \chi_\alpha & \underset{D_\alpha}{\to} & \bar F_{ab} \\
\downarrow{\scriptstyle \bar D_{\dot\alpha}} & & \downarrow{\scriptstyle \bar D_{\dot\alpha}} & & \downarrow{\scriptstyle \bar D_{\dot\alpha}} \\
\bar\chi_{\dot\alpha} & \underset{D_\alpha}{\to} & H_{abc} & \underset{D_\alpha}{\to} & \bar\rho_{ab\,\dot\gamma} \\
\downarrow{\scriptstyle \bar D_{\dot\alpha}} & & \downarrow{\scriptstyle \bar D_{\dot\alpha}} & & \downarrow{\scriptstyle \bar D_{\dot\alpha}} \\
F_{ab} & \underset{D_\alpha}{\to} & \rho_{ab\,\gamma} & \underset{D_\alpha}{\to} & R_{abcd}
\end{array}
\qquad,\qquad
\begin{array}{ccccc}
\bar t \\
\downarrow{\scriptstyle \bar D_{\dot\alpha}} \\
\bar\lambda_{\dot\alpha}^A \\
\downarrow{\scriptstyle \bar D_{\dot\alpha}} \\
F_{ab}^A
\end{array}
\qquad,\qquad
\begin{array}{ccccc}
t & \underset{D_\alpha}{\to} & \lambda_\alpha^A & \underset{D_\alpha}{\to} & F_{ab}^A
\end{array}
\tag{1.68}$$

where $\rho = d_{\hat\omega}\psi$ is the Rarita–Schwinger field strength.

We will be mostly interested in the chiral derivatives D_α that satisfy

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^\gamma D_\gamma + R_{\alpha\beta\,cd}J^{cd}\ , \tag{1.69}$$

with J_{ab} the Lorentz generators. The Riemann tensor component is

$$R_{\alpha\beta\,cd} = C_{\alpha\beta}\big(\bar F_{cd} - \tfrac{1}{2}\chi\gamma_{cd}\chi + \tfrac{1}{2}\bar\lambda^A\gamma_{cd}\bar\lambda_A\big) + \frac{1}{12}(\gamma_{cd}\gamma^{ab})_{(\alpha\beta)}\big(\bar F_{ab} - \chi\gamma_{ab}\chi + \bar\lambda^A\gamma_{ab}\bar\lambda_A\big)\ . \tag{1.70}$$

Note that because $\bar P_\alpha^A = 0$, the axial $U(1)$ and the $SO(n)$ component of the Riemann tensor $R_{\alpha\beta\,B}{}^A$ both vanish. One has for the gluinos

$$D_\alpha\bar\lambda_{\dot\beta A} = \frac{1}{2}\gamma^a{}_{\alpha\dot\beta}\big(-i\bar P_{aA} + \bar\lambda_A\gamma_a\chi\big) + \frac{3}{4}\chi_\alpha\bar\lambda_{\dot\beta A}\ . \tag{1.71}$$

It is convenient to introduce the combinations

$$\bar M_{ab} = \bar F_{ab} + \tfrac{1}{2}\chi\gamma_{ab}\chi\ ,\qquad \bar N_a^A = \bar P_a^A - i\bar\lambda^A\gamma_a\chi\ , \tag{1.72}$$

of which the chiral derivative simplifies to

$$\begin{aligned}
D_\alpha\bar M_{ab} &= -(\gamma_{[a}{}^c\chi)_\alpha\bar M_{b]c} - \tfrac{1}{2}\chi_\alpha\bar M_{ab} - 2i(\gamma_{[a}\bar\lambda_A)_\alpha\bar N_{b]}^A + \dots \\
D_\alpha\bar N_{aA} &= \frac{1}{2}(\gamma_a{}^b\chi)_\alpha\bar N_b^A - \tfrac{1}{2}\chi_\alpha\bar N_a^A + i(\gamma^b\bar\lambda^A)_\alpha\bar M_{ab} + \dots\ ,
\end{aligned} \tag{1.73}$$

up to cubic terms in the fermions.

Because $T_{\alpha\beta}{}^{\dot\gamma} = 0$ and the axial $U(1)$ Riemann tensor $R_{\alpha\beta} = 0$, the Weyl spinor vector fields E_α close under their Lie bracket as

$$\{E_\alpha, E_\beta\} = \big(\Omega_{\alpha\beta}{}^\gamma + \Omega_{\beta\alpha}{}^\gamma - T_{\alpha\beta}{}^\gamma\big)E_\gamma \tag{1.74}$$

and one can expand any scalar superfield in the associated normal coordinates ζ^α. Following the construction established in [108], one can then expand the supervielbein Beresinian [7]

$$\zeta^{\mathfrak{a}}\partial_{\mathfrak{a}}\log \mathrm{Ber}(E) = \zeta^\alpha D_\alpha \log \mathrm{Ber}(E) = (-1)^A T_{\alpha A}{}^A - \Omega_{\beta\alpha}{}^\beta \zeta^\alpha - \delta^\alpha_{\mathfrak{a}}(E_\alpha{}^{\mathfrak{a}} - \delta^{\mathfrak{a}}_\alpha)\ . \tag{1.75}$$

It follows that there exists a chiral measure $\mathcal{E}$, such that the full-superspace integral of a scalar superfield L can be written as a chiral superspace integral as follows

$$\int d^{8,8,8}z\,\mathrm{Ber}(E)\ L = \int d^{8,8}z \int d^8\zeta\,\mathrm{Ber}(E)\ L = \int d^{8,8}z\mathcal{E}\ [D^8]L \tag{1.76}$$

where the chiral projector

$$[D^8]L = \frac{1}{8!}\varepsilon^{\alpha\beta\gamma\ldots\zeta}\Big(D_\alpha D_\beta D_\gamma\cdots D_\zeta + 21\chi_\alpha D_\beta D_\gamma\cdots D_\zeta + 28 e_{\alpha\beta}D_\gamma\cdots D_\zeta + \ldots\Big)L \tag{1.77}$$

is alternatively determined such that [19]

$$[D^8]\Big(D_\alpha\Xi^\alpha + \frac{13}{4}\chi_\alpha\Xi^\alpha\Big) = 0 \tag{1.78}$$

for an arbitrary spinor superfield Ξ^α. The term in χ_α comes from the supertrace

$$(-1)^A T_{\alpha A}{}^A = \frac{13}{4}\chi_\alpha\ . \tag{1.79}$$

The computation of the chiral projector is rather involved and we shall not attempt to do it here. One can check nonetheless that it satisfies

$$(D_\alpha - T_{\alpha\dot\beta}{}^{\dot\beta})[D^8] = (D_\alpha + 2\chi_\alpha)[D^8] = 0\ , \tag{1.80}$$

and one can therefore define the superspace integral

$$\int d^{8,8}z\,\mathcal{E}\,W \tag{1.81}$$

for any chiral Lorentz scalar superfield W of $U(1)$ weight 4 satisfying

$$(D_\alpha + 2\chi_\alpha)W = 0\ . \tag{1.82}$$

Note that this condition is integrable because

$$D_\alpha\chi_\beta + D_\beta\chi_\alpha = 0\ , \qquad T_{\alpha\beta}{}^\gamma\chi_\gamma = 0\ . \tag{1.83}$$

For example one can take any anti-holomorphic function of the vector multiplets scalar

$$W = e^{\frac{4}{3}\phi}\bar u^4 W_4(\bar t)\ . \tag{1.84}$$

Note that the power of the dilaton is precisely such that the corresponding supersymmetry invariant includes a quartic term in the vector multiplet field strengths that does not depend on the

[7] Here $\mathfrak{a}$ denotes the local fermionic index and $E_\alpha{}^{\mathfrak{a}}$ is the component of the inverse supervielbein.

dilaton, and therefore corresponds to a one-loop correction in heterotic string theory. The corresponding t_8F^4 type supersymmetry invariant is the eight-dimensional parent of the ten-dimensional Green–Schwarz anomaly canceling term in ten dimension. In eight dimensions it is of course fully supersymmetric.

To understand the Riemann tensor terms one needs to construct a chiral superfield quartic in the modified graviphoton field strength $\bar{M}_{ab}$. Let us first discuss the case with no vector multiplet. One can then compute that (1.73) reduces to

$$\big(D_\alpha + \tfrac{1}{2}\chi_\alpha\big)\bar{M}_{ab} = -(\gamma_{[a}{}^c\chi)_\alpha \bar{M}_{b]c}\ , \tag{1.85}$$

where the right-hand-side is a Lorentz transformation acting on $\bar{M}_{ab}$. Note that this equation is exact in the absence of vector multiplets, and not only valid up to cubic terms in the fermions. It follows that any Lorentz invariant polynomial of order k in $\bar{M}_{ab}$ satisfies

$$\big(D_\alpha + \tfrac{k}{2}\chi_\alpha\big)X_k(\bar{M}) = 0 \tag{1.86}$$

and can be used to define an integrand for the chiral measure. The supermultiplet structure is not as simple in the presence of vector multiplets. One can nonetheless find that $D_\alpha + \frac{1}{2}\chi_\alpha$ acting on the triplets $(\bar{M}_{ab}, \bar{N}^A_a, \bar{\lambda}^A\bar{\lambda}^B)$ gives a right-hand-side that simply rotates them under an ancillary $\mathfrak{so}(1,7+n)$ fermionic rotation

$$\begin{aligned}
(D_\alpha + \tfrac{1}{2}\chi_\alpha)\bar{M}_{ab} &= -(\gamma_{[a}{}^c\chi)_\alpha\bar{M}_{b]c} - 2i(\gamma_{[a}\bar{\lambda}_A)_\alpha\bar{N}^A_{b]} + \dots \\
(D_\alpha + \tfrac{1}{2}\chi_\alpha)\bar{N}_{aA} &= \frac{1}{2}(\gamma_a{}^b\chi)_\alpha\bar{N}^A_b + i(\gamma^b\bar{\lambda}^A)_\alpha\bar{M}_{ab} - 2i(\gamma_a\bar{\lambda}_B)_\alpha(\bar{\lambda}^A\bar{\lambda}^B) + \dots \\
(D_\alpha + \tfrac{1}{2}\chi_\alpha)\bar{\lambda}^A\bar{\lambda}^B &= i(\gamma^a\bar{\lambda}^{[A})_\alpha N^{B]}_a\ ,
\end{aligned} \tag{1.87}$$

up to cubic terms in the fermions. This structure breaks down at higher order in the fermions, but is good enough to determine the chiral superfield up to quartic terms in the fermions. This therefore strongly suggests that there exists X_2 satisfying the chirality constraint

$$(D_\alpha + \chi_\alpha)X_2[\bar{M},\bar{P}] = 0\ . \tag{1.88}$$

with up to quartic terms in the fermions.

$$X_2[\bar{M},\bar{P}] = \bar{M}_{ab}\bar{M}^{ab} - 2\bar{N}^A_a\bar{N}^a_A + \dots \tag{1.89}$$

Similarly, we assume that one can define X_4 satisfying

$$(D_\alpha + 2\chi_\alpha)X_4[\bar{M},\bar{P}] = 0\ , \tag{1.90}$$

which, up to quartic terms in the fermions, reads

$$X_4[\bar{M},\bar{P}] = \bar{M}_{ab}\bar{M}^{bc}\bar{M}_{cd}\bar{M}^{da} - 4\bar{N}^A_a\bar{N}^b_A\bar{M}^{ac}\bar{M}_{bc} + 2\bar{N}^A_a\bar{N}^a_B\bar{N}^B_b\bar{N}^b_A - 8\bar{\lambda}^A\bar{\lambda}^B\bar{N}^a_A\bar{N}^b_B\bar{M}_{ab} + \dots \tag{1.91}$$

One can therefore write the chiral superspace integral

$$\int d^{8,8}z\,\mathcal{E}\left(W_0(\bar{t})X_4[\bar{M},\bar{P}] + W_0'(\bar{t})X_2[\bar{M},\bar{P}]^2 + e^{\frac{2}{3}\phi}\bar{u}^2W_2(\bar{t})X_2[\bar{M},\bar{P}] + e^{\frac{4}{3}\phi}\bar{u}^4W_4(\bar{t})\right) \tag{1.92}$$

in eight dimensions. This integral is easy to analyse in the linearised approximation since the integrand is already quartic. One directly obtains from the linearised derivative that it gives rise to a Riemann to the four term

$$\int d^{8,8}z\,\mathcal{E}\left(W_0(\bar{t})X_4[\bar{M},\bar{P}]+W_0'(\bar{t})X_2[\bar{M},\bar{P}]^2\right)\sim(t_8-\tfrac{i}{2}\varepsilon)\Big(W_0(\bar{t})\mathrm{Tr}\big[\bar{R}^4\big]+W_0'(\bar{t})\mathrm{Tr}\big[\bar{R}^2\big]^2\Big)\ . \quad (1.93)$$

These terms also involve $\nabla^2F^2R^2$ and ∇^4F^4 type terms with the structure coming from the approximate $SO(1,7+n)$ symmetry of the integrand.

In eight dimension this defines a fully supersymmetric correction to the effective action that is associated to a modular anomaly. The pentagone diagram contribution to the axial $U(1)$ current conservation [98, 109, 42] gives the $SO(2,n)$ anomaly of the 1PI effective action [110, 111]

$$\Gamma^{1\mathrm{PI}}|_\gamma=\Gamma^{1\mathrm{PI}}+\frac{1}{128\pi^4}\int\log\big[d+(\gamma,t)+\tfrac{c}{2}(t,t)\big]\Big[(246+n)\Big(\frac{1}{360}\mathrm{Tr}[R^4]+\frac{1}{288}\mathrm{Tr}[R^2]^2\Big)-\mathrm{Tr}[R^2]^2\Big] \quad (1.94)$$

for the $SO(2,n)$ transformation g

$$t|_g=\frac{-\frac{\alpha}{2}(t,t)+At+\beta}{d+(\gamma,t)-\frac{c}{2}(t,t)}\ ,\quad g^{-1}Q=\begin{pmatrix}a&\delta&b\\ \alpha&A&\beta\\ c&\gamma&d\end{pmatrix}\begin{pmatrix}m\\ q\\ n\end{pmatrix}\ , \quad (1.95)$$

and in particular

$$(\gamma,\gamma)=-2cd\ . \quad (1.96)$$

It can be compensated for a discrete subgroup $\Gamma\subset SO(2,n)$ if $W_0(\bar{t})$ and $W_0'(\bar{t})$ are defined as logarithms of modular forms of the appropriate weight.

This is the case for $W_0(t)$ in all string theories with $\mathcal{N}=1$ supersymmetry in eight dimensions, which are believed to only exist for $n=2,10$ and 18 according to [112, 113]. The elliptic genus allows to compute this coupling at one-loop, and gives for the maximal rank case [114]

$$\begin{aligned}&t_8\Big(\mathcal{E}(t,\bar{t})\mathrm{Tr}[R^4]+\mathcal{E}'(t,\bar{t})\mathrm{Tr}[R^2]^2+\mathcal{G}_{AB}(t,\bar{t})\mathrm{Tr}[R^2]F^AF^B+\mathcal{F}_{ABCD}(t,\bar{t})F^AF^BF^BF^C\Big)\\ &=\frac{\pi}{3}t_8\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{1}{\Delta(\tau)}\sum_{Q\in I\!I_{2,2}\oplus D_{16}^+}e^{-\pi i\tau(Q,Q)-4\pi\tau_2|Z(Q)|^2}\Big(\frac{\alpha'^3}{60}E_4(\tau)\mathrm{Tr}[R^4]+\frac{\alpha'^3}{48}\widehat{E}_2(\tau)^2\mathrm{Tr}[R^2]^2\\ &\qquad+\alpha'^2\widehat{E}_2(\tau)\Big(v_{AI}Q^Iv_{BJ}Q^J-\tfrac{1}{4\pi\tau_2}\delta_{AB}\Big)F^AF^B\mathrm{Tr}[R^2]\\ &\quad+\alpha'\Big(8v_{AI}Q^Iv_{BJ}Q^Jv_{CK}Q^Kv_{DL}Q^L-\tfrac{12}{\pi\tau_2}\delta_{AB}v_{CI}Q^Iv_{DJ}Q^J+\tfrac{3}{2\pi^2\tau_2^2}\delta_{AB}\delta_{CD}\Big)F^AF^BF^CF^D\Big)\ .\end{aligned} \quad (1.97)$$

From this formula one computes using the unfolding method [115–118]

$$\frac{1}{\alpha'^3}\mathcal{E}(t,\bar{t})=\frac{22\pi}{15}\mathcal{K}-\frac{\pi}{90}\sum_{q\in I\!I_{1,1}\oplus D_{16}^+}\log\big|c((q,q)/2)e^{2\pi i(q,t)}\big|^2\ ,\qquad\frac{E_4(\tau)}{\Delta(\tau)}=\sum_{n\geq-1}c(n)e^{2\pi in\tau}\ , \quad (1.98)$$

so that $W_0(t)$ is the logarithm of the Borcherds product associated to the modular form E_4/Δ [119]

$$\frac{1}{\alpha'^3}W_0(t)=-\frac{\pi}{90}\sum_{q\in I\!I_{1,1}\oplus D_{16}^+}\log\big[c((q,q)/2)e^{2\pi i(q,t)}\big]\ . \quad (1.99)$$

Note however that the function multiplying $\mathrm{Tr}[R^2]^2$ does not give the real part of a locally holomorphic modular form because $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$ is not holomorphic [118]. This failure is due to the α' correction at tree-level that we have neglected in this analysis. The corrected Bianchi identity for the three-form is

$$\mathrm{d}H = -\frac{2}{3}\mathrm{d}\phi \wedge H - F \wedge \bar{F} + \frac{1}{2}F_A \wedge F^A - \alpha' e^{-\frac{2}{3}\phi} R_{ab} \wedge R^{ab} \,. \tag{1.100}$$

Defining the covariant derivatives

$$\mathcal{D}_A v_I = \frac{1}{2} v_{AI} \,, \qquad \mathcal{D}_A v_{BI} = \frac{1}{2}\delta_{AB}\bar{v}_I \,, \tag{1.101}$$

and its complex conjugate, one works out the differential equations [120]

$$\begin{aligned}
\mathcal{D}_A\bar{\mathcal{D}}_B\mathcal{F}_{CDEF} &= 5\delta_{(BC}\mathcal{F}_{DEF)A} \\
\mathcal{D}_A\bar{\mathcal{D}}_B\mathcal{G}_{CD} &= \frac{3}{2}\delta_{(AB}\mathcal{G}_{CD)} + \frac{3}{4}\alpha'\mathcal{F}_{ABCD} \\
\mathcal{D}_A\bar{\mathcal{D}}_B\mathcal{E}' &= -\frac{\pi}{24}\alpha'^3\delta_{AB} + \frac{1}{4}\alpha'\mathcal{G}_{AB} \,, \qquad \mathcal{D}_A\bar{\mathcal{D}}_B\mathcal{E} = \frac{11\pi}{30}\alpha'^3\delta_{AB} \,.
\end{aligned} \tag{1.102}$$

This gives an illustrative example of a general property. The superspace integral defined in this section does not take into account the $\alpha' R^2$ correction and therefore does not predict the terms in α' in the differential equations. The terms linear in α' can still be interpreted within the Wilsonian effective action, and correspond to the modification of the supersymmetry transformation linear in α'. The terms in α'^3 come instead from the terms linear in the Kähler potential $\mathcal{K}$ similarly as in the Harvey–Moore coupling discussed in the introduction. They appear in the amplitude but not in the Wilsonian effective action and are directly associated to the one-loop supergravity divergence in eight dimensions and the presence of a $U(1)$-anomaly.

1.3 Higher derivative corrections in maximal supergravity

The methods described in the previous sections to determine higher derivative corrections preserving supersymmetry fail to provide a complete proof of the existence of higher derivative corrections with maximal supersymmetry in general. We refer to [121] for a review. These higher derivative corrections must nonetheless exist by consequence of the existence of supersymmetric scattering amplitudes in type II string theory and supergravity. The tree-level type II string theory amplitudes in ten dimensions define an effective action that preserves supersymmetry and therefore implies the existence of R^4, $\nabla^4 R^4$ type corrections to the effective action [122, 123]. In supergravity, supersymmetry Slavnov–Taylor identities imply that there must be a supersymmetric counter-term whenever there is a logarithmic divergence in perturbation theory. The supergravity logarithmic divergences in eight, seven and six dimensions at one, two and three-loop order imply respectively the existence of R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ type corrections to the effective action [124–126]. Consistency of M-theory also requires the existence of a R^4 type supersymmetric invariant in eleven dimensions [6, 127].

In principle the superspace construction of the Green–Schwarz counter-terms explained in section 1.1 generalises to eleven dimensional supergravity [97, 128]. It is believed that $F_4 \wedge (4\mathrm{Tr}R^4 - \mathrm{Tr}R^2 \wedge \mathrm{Tr}R^2)$ is t_0-trivial, which would imply the existence of the R^4 type supersymmetry invariant

with the expected Chern–Simons coupling $A_3 \wedge (4\mathrm{Tr}R^4 - \mathrm{Tr}R^2 \wedge \mathrm{Tr}R^2)$. It is in principle possible to prove t_0-triviality, however, this computation is extremely difficult and has not been achieved.

Assuming these higher derivative terms exist, one can determine very strong constraints on them. They can be defined in linearised superspace [129, 130, 48, 131, 132]. From the linearised superspace structure, one can derive the generic form of the non-linear invariant and checks some of the supersymmetry variations.

This approach was taken in [49] to prove that the leading Wilson coefficient in type IIB string theory satisfies to a Poisson equation. Type IIB supergravity in superspace was formulated in [133]. The linearised theory is determined by a chiral superfield that satisfies $\bar{D}_\alpha W = 0$ and the reality constraint that $D^4 W = \bar{D}^4 \bar{W}$ in both irreducible representations associated respectively to the Weyl tensor $R_{ab,cd}$ and the self-dual field strength gradient $\nabla_a F^+_{bcdef}$. The field content of the theory includes a complex axio-dilaton $S = C_0 + ie^{-\phi}$, a left-handed Weyl fermion λ_α, two two-form potentials transforming as a doublet of $SL(2,\mathbb{R})$ with complex field strength $G = e^{\phi/2}(dC_2 - SdB_2)$, the right-handed Weyl gravitino field ψ^α_μ, the metric and the four-form with self-dual field strength. The R^4 type invariant can be defined in the linearised approximation from the superspace integral

$$D^{16} W^4 \sim t_8 t_8 R^4 + \dots \tag{1.103}$$

and more generally one can define the $U(1)$ violating linearised invariant at $4+n$ points

$$D^{16} W^{4+n} \sim W^n t_8 t_8 R^4 + \dots \tag{1.104}$$

The set of linearised invariants determines the constraints on the function $f(S)$ defining the invariant as

$$\mathcal{L} = \mathrm{det}e\, f(S) t_8 t_8 R^4 + \dots \tag{1.105}$$

in the linearised approximation. The property that there is no linearised invariant with $\bar{W}W$ does not imply that $f(S)$ is holomorphic, but rather that $\bar{\mathcal{D}}\mathcal{D} f(S) \approx 0$ in the linearised approximation, which in this case implies that $\bar{\mathcal{D}}\mathcal{D} f(S) = s(s-1) f(S)$ for some s. We define the covariant derivative $\mathcal{D} = 2i\mathrm{Im}S\partial_S + w$ on a weight w form $f_w(S)$, such that $\mathcal{D} f_w$ has weight $w+1$.[8] Taking a generic ansatz of the form

$$\mathcal{L} = \mathrm{det}e\Big(f(S) t_8 t_8 R^4 + \dots + c_{11} \mathcal{D}^{11} f(S) \hat{G} \lambda^{14} + c'_{11} \mathcal{D}^{11} f(S) \bar{\psi}_a \gamma^a \lambda^{15} + c_{12} \mathcal{D}^{12} f(S) \lambda^{16} \Big) \tag{1.106}$$

supersymmetry implies $c'_{11} = 104 c_{12}$ and $\bar{\mathcal{D}}\mathcal{D}^{12} f(S) = -\frac{525}{4} \mathcal{D}^{11} f(S)$ and therefore [49]

$$\bar{\mathcal{D}}\mathcal{D} f(S) = \frac{3}{4} f(S)\,. \tag{1.107}$$

The interaction terms have been computed from string amplitudes up to six-point at tree-level [134–138] and up to five-point at one-loop [139, 140].

We did a similar computation in eight dimensions in [51], for which there are two types of supersymmetry invariants. The field content of the theory is summarised in figure 2. The chiral

[8] We call weight w what is refered to as weight $(w,-w)$ in [49] for short. With this second definition one recovers the standard weight $(w,0)$ of modular forms commonly used in the literature with the derivative $\partial_S + \frac{w}{2iS_2}$ shifting weight by $(2,0)$.

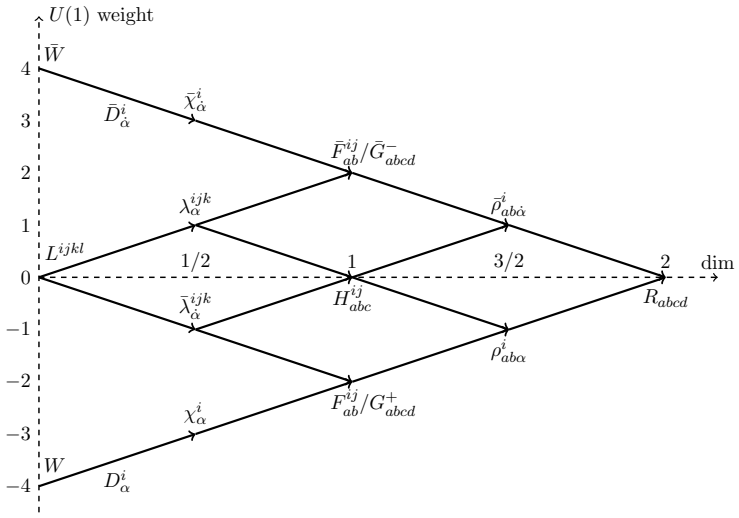

Figure 2: Structure of the supergravity supermultiplet in the linearised approximation. It includes a chiral superfield W and a tensor superfield L^{ijkl} related through their second derivative. The symmetry with respect to the horizontal axe defines complex conjugation.

invariant is complex and was found to expand as

$$(t_8+\tfrac{i}{2})^2 f(U)R^4+\cdots+\frac{1}{128}\mathcal{D}^{11}f(U)\Big(\frac{1}{6}\hat G^+_{abcd}(\chi^{14})^{abcd}+4\hat F^{ij}_{ab}(\chi^{14})^{ab}_{ij}-(\bar\lambda\bar\lambda)^{ij}_{ab}(\chi^{14})^{ab}_{ij}-3(\bar\lambda\bar\lambda)(\chi^{14})\Big)$$
$$+\frac{i}{2}\mathcal{D}^{11}f(U)\bar\psi_i\gamma^a\chi^{15\,i}+\mathcal{D}^{12}f(U)\chi^{16}\,, \quad (1.108)$$

where $f(U)$ is a holomorphic function of the complex structure modulus in type IIB. Note that this supersymmetry invariant gives under truncation to $\mathcal{N}=1$ supergravity the chiral superspace integral written in the previous section. There is also a parity symmetric invariant for which a similar analysis determines a tensorial quadratic equation satisfied by the function of the $SO(3)\backslash SL(3,\mathbb{R})$ moduli [51].

In lower dimensions it appears that the symbol of the tensorial differential equations satisfied by the protected couplings determines by integrability the full differential equations. This method is particularly powerful in four dimensions because the linearised invariants can be determined from superconformal primary operators according to [16, 141]. In dimensions $D\geq 5$ not all linearised supersymmetry invariants can be defined as harmonic superspace integrals as one can read in Figure 1.

In $\mathcal{N}=8$ supergravity the linearised theory is described by the superfield W^{ijkl} satisfying the complex-selfduality relation

$$\bar W_{ijkl}=\frac{1}{24}\varepsilon_{ijklpqrs}W^{pqrs}\,, \quad (1.109)$$

and the linear constraints that

$$D^i_\alpha W^{jklp}=\frac{1}{12}\varepsilon^{ijklpqrs}\chi_{\alpha qrs}\,, \qquad \bar D_{\dot\alpha i}W^{jklp}=2\delta_i^{[j}\bar\chi^{klp]}_{\dot\alpha}\,. \quad (1.110)$$

We shall write the fundamental weights of $SU(8)$ as Υ_i, such that the representation of highest weight $\lambda = n^i\Upsilon_i$ admits Dynkin labels $[n^1, n^2, n^3, n^4, n^5, n^6, n^7]$ and in particular the fundamental representation $R(\Upsilon_1)$ has highest weight Υ_1 (as D^i_α).[9] In this convention the linearised superfield is in the representation $R(\Upsilon_4)$, and one can construct 1/2 BPS superfields as the monomials W^n in the irreducible representation $R(n\Upsilon_4)$ [130]. This is most easily understood in harmonic superspace [142], in which case one can write $SU(8)/S(U(4)\times U(4))$ harmonic variables $(u^r{}_i, u^{\hat{r}}{}_i)$ with $r = 1,2,3,4$ and $\hat{r} = 5,6,7,8$ of the respective $U(4)$ subgroups. Defining

$$D^r_\alpha = u^r{}_i D^i_\alpha \ , \qquad \bar{D}_{\dot{\alpha}\hat{r}} = u^i{}_{\hat{r}} \bar{D}_{\dot{\alpha}i} \ , \quad W = u^1{}_i u^2{}_j u^3{}_k u^4{}_l W^{ijkl} \ , \tag{1.111}$$

the superfield W is $(4,4)$-Grassmann analytic

$$D^r_\alpha W = 0 \ , \qquad \bar{D}_{\dot{\alpha}\, r} W = 0 \ . \tag{1.112}$$

So W is annihilated by half of the superspace derivatives and so is any monomial in W. The other derivatives define an order sixteen measure $D^8\bar{D}^8$ that gives a non-zero result (up to total derivative) for W^{4+n} and any $n \geq 0$. This gives the analogue of (1.104) in type IIB supergravity. This implies that the non-linear ansatz for the supersymmetry invariant can be defined in terms of a single function of the scalar fields in $(SU(8)/\mathbb{Z}_2)\backslash E_7$ and the ansatz only involves covariant derivatives of order n in the irreducible representation $R(n\Upsilon)$. This constraint turns out to determine uniquely the differential equations satisfied by the function $\mathcal{E}_{(0,0)}$ and one finds in particular

$$\Delta\mathcal{E}_{(0,0)} = -42\mathcal{E}_{(0,0)} \ . \tag{1.113}$$

This discussion generalises to the next to leading correction of type $\nabla^4 R^4$. One can construct 1/4 BPS superfields as the monomials W^{2m+n} in the irreducible representation $R(m\Upsilon_2{+}n\Upsilon_4{+}m\Upsilon_6)$ [16,141]. This is also described in harmonic superspace [142] using harmonic variables parametrising $SU(8)/S(U(2)\times U(4)\times U(2))$ with $r = 3,4,5,6$ of $U(4)$, $\hat{r} = 1,2$ and $\check{r} = 7,8$ of the two $U(2)$'s. One defines then

$$D^{\hat{r}}_\alpha = u^{\hat{r}}{}_i D^i_\alpha \ , \qquad \bar{D}_{\dot{\alpha}\check{r}} = u^i{}_{\check{r}} \bar{D}_{\dot{\alpha}i} \ , \quad W^{rs} = u^1{}_i u^2{}_j u^r{}_k u^s{}_l W^{ijkl} \ , \tag{1.114}$$

and the superfield W^{rs} is $(2,2)$-Grassmann analytic

$$D^{\hat{r}}_\alpha W^{rs} = 0 \ , \qquad \bar{D}_{\dot{\alpha}\,\check{r}} W^{rs} = 0 \ . \tag{1.115}$$

One can write the $(2,2)$-Grassmann analytic superfields W^{4+2m+n} in the representation $[0,n,0]$ of $U(4)$ that give rise to invariants in $R(m\Upsilon_2{+}n\Upsilon_4{+}m\Upsilon_6)$ of $SU(8)$. One concludes that the ansatz in the coupling function only involves covariant derivatives in the irreducible representations $R(m\Upsilon_2{+}n\Upsilon_4{+}m\Upsilon_6)$ [52]. Once again this determines entirely the set of differential equations satisfied by the function $\mathcal{E}_{(1,0)}$ and in particular

$$\Delta\mathcal{E}_{(1,0)} = -60\mathcal{E}_{(1,0)} \ . \tag{1.116}$$

For $\nabla^6 R^4$ one finds two kinds of linearised supersymmetry invariants [53]. The first one is defined in harmonic superspace using $U(8)/S(U(1)\times U(6)\times U(1))$ with $r = 2,3,4,5,6,7$ [16,141]. One introduces

$$D^1_\alpha = u^1{}_i D^i_\alpha \ , \qquad \bar{D}_{\dot{\alpha}8} = u^i{}_8 \bar{D}_{\dot{\alpha}i} \ , \quad W^{rst} = u^1{}_i u^r{}_j u^s{}_k u^t{}_l W^{ijkl} \ , \tag{1.117}$$

[9] The corresponding Young diagram admits $\sum_i i n_i$ boxes with n_7 columns of hight 7, n_6 columns of hight 6, etc.

and the superfield W^{rst} is $(1,1)$-Grassmann analytic

$$D^1_\alpha W^{rst} = 0 \ , \qquad \bar{D}_{\dot\alpha\, 8} W^{rst} = 0 \ . \tag{1.118}$$

This harmonic superspace can be defined in the non-linear theory [143], which proves in this case that there exists a supersymmetric correction to the two-derivative Lagrangian without relying on indirect results from amplitudes. One can integrate the monomials $W^{4+n_1+2n_2+3n_3+4n_4+4n'_4}$ in the irreducible representation $[n_2, n_4, n_1{+}n_3, n_4, n_2]$ of $U(6)$ to get linearised invariants in the representation $[n_3{+}n_4{+}2n'_4, n_2, n_4, n_1{+}n_3, n_4, n_2, n_3{+}n_4{+}2n'_4]$ of $SU(8)$. These are the only allowed representations in the covariant Taylor expansion of the function $\mathcal{E}^{(1)}_{(0,1)}$ defining the coupling at the non-linear level, and they determine in this way the symbol of the differential equations satisfied by $\mathcal{E}^{(1)}_{(0,1)}$. This in turn determines the differential equation at the non-linear level up to the eigen-value of the Laplacian.

There is a second class of supersymmetry invariant that does not include an $SU(8)$ invariant representative and was therefore disregarded in [141]. It is complex and defined in harmonic superspace using $U(8)/S(U(2)\times U(6))$ with $r = 3,4,5,6,7,8$ and $\hat r = 1,2$. One introduces

$$D^{\hat r}_\alpha = u^{\hat r}{}_i D^i_\alpha \ , \quad W^{rs} = u^1{}_i u^2{}_j u^r{}_k u^s{}_l W^{ijkl} \ , \tag{1.119}$$

and the superfield W^{rs} is $(2,0)$-Grassmann analytic

$$D^{\hat r}_\alpha W^{rs} = 0 \ . \tag{1.120}$$

One can integrate monomials $W^{6+n_1+2n_2+3n_3}$ in the representation $[0, n_1, 0, n_2, 0]$ of $U(6)$ to get a invariants with a scalar monomials in the representation $[0, n_2+2n_3, 0, n_1, 0, n_2, 0]$ of $SU(8)$ multiplying $\bar F^2 \nabla^4 R^4$. This set of complex invariants determine another class of non-linear invariant of the form $\mathcal{E}^{(7)}_{(0,1)} \nabla^6 R^4$ such that the covariant Taylor expansion of $\mathcal{E}^{(7)}_{(0,1)}$ is restricted to the irreducible representations $[0, n_2+2n_3, 0, n_1, 0, n_2, 0]$ and their complex conjugates. This set of representations determines the entire set of differential equation and in particular the eigen-value of the Laplace operator to be -60 as for (1.116) [143].

To discuss these differential equations and their solutions in more details we find convenient to first introduce some other concepts that appear in the description of automorphic forms and in particular the notion of nilpotent orbit. At this stage we can simply identify that the harmonic superspace can be labelled by an integral weight $\upsilon = n^i \Upsilon_i$ of $SU(8)$. The weight determines a parabolic subgroup $P_\upsilon \subset SL(8,\mathbb{C})$ as the group generated by the Cartan subalgebra and the roots satisfying $(\upsilon, \alpha) \geq 0$. The harmonic variables are valued in $SL(8,\mathbb{C})/P_\upsilon = SU(8)/(P_\upsilon \cap SU(8))$. The Grassmann analytic structure is then associated to the highest weight vector spaces in $R(\Upsilon_1) = \mathbf{8}$ and $R(\Upsilon_7) = \overline{\mathbf{8}}$ with respect to υ, and the Grassmann analytic superfield is valued in the highest weight vector space in $R(\Upsilon_4)$. We will see in Section 3 that this gives a one-to-one correspondence between harmonic superspaces and nilpotent orbits and in fact automorphic representations.

But before to do so we want to discuss the effective action in string theory.

2 String perturbation theory

In this section we shall describe the perturbative four-graviton scattering amplitude in type II string theory on a torus. We shall concentrate on the low energy limit, with particular emphasise on the Wilsonian effective action obtained by integrating out massive string states. We shall not review the computation of the amplitude from first principles.

The perturbative states of type IIB string theory on $\mathbb{R}^{1,9-d} \times T^d$ are counted by the one-loop partition function

$$\mathrm{Tr}'[(-1)^F e^{-2\pi\tau_2 H + 2\pi i z\cdot J}] = \int_{-\frac12}^{\frac12} d\tau_1 \left| \frac{\sum\limits_{p\in D_4+\mathrm{v}} q^{\frac{p^2}{2}} e^{2\pi i z\cdot p} - \sum\limits_{p\in D_4+\mathrm{s}} q^{\frac{p^2}{2}} e^{2\pi i z\cdot p}}{\eta^4 q^{\frac13} \prod\limits_{a=1}^{4}\prod\limits_{n=1}^{\infty}(1-e^{2\pi i z_a} q^n)(1-e^{-2\pi i z_a} q^n)} \right|^2 \sum_{m,n\in\mathbb{Z}^d} q^{\frac{p_L^2}{2}} \bar q^{\frac{p_R^2}{2}} \tag{2.1}$$

where J_a for $a = 1$ to $\lfloor \frac{8-d}{2} \rfloor$ are the Lorentz currents in a Cartan subalgebra of $\mathfrak{so}(8-d)$ and $z_a = 0$ for $a > \lfloor \frac{8-d}{2} \rfloor$ corresponding to the internal torus. The prime on the trace excludes the zero modes on $\mathbb{R}^{1,9-d}$, that would involve a divergent integral over the external momenta. Here D_4 is the root lattice and v and s (or c) are the vector and Weyl spinor representation highest weight vectors. We choose conventions such that

$$\begin{aligned} p_L^2 &= \frac12 G^{IJ}\big(m_I + (B_{IK}+G_{IK})n^K\big)\big(m_J + (B_{JL}+G_{JL})n^L\big)\ , \\ p_R^2 &= \frac12 G^{IJ}\big(m_I + (B_{IK}-G_{IK})n^K\big)\big(m_J + (B_{JL}-G_{JL})n^L\big)\ , \end{aligned} \tag{2.2}$$

with the dimensionless torus metric G_{IJ} and two-form B_{IJ} in sting frame. m_I are the momentum mode numbers of the string along the torus, and n^I the winding numbers. In light-cone gauge the factor $\frac{1}{\eta^8}$ comes from the eight transverse world-sheet bosons while the world-sheet fermions contribute to $V_8 = \sum\limits_{D_4+\mathrm{v}} q^{\frac{p^2}{2}}/\eta^4$ in the (antiperiodic) NS sector and to $S_8 = \sum\limits_{D_4+\mathrm{s}} q^{\frac{p^2}{2}}/\eta^4$ in the (periodic) R sector. The type IIA partition function is obtained by changing the chirality of the Weyl spinor in the right-moving sector.

In ten dimensions, the little group $Spin(8)$ representations of the states are determined by the expansions

$$\begin{aligned} \frac{\sum\limits_{p\in D_4+\mathrm{v}} q^{\frac{p^2}{2}} e^{2\pi i z\cdot p}}{\eta^4 q^{\frac13}} &= (1+q+2q^2+\dots)\mathrm{tr}_{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]}\zeta + (q+2q^2+\dots)\mathrm{tr}_{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]}\zeta + q^2\mathrm{tr}_{\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]}\zeta + \dots \\ \frac{\sum\limits_{p\in D_4+\mathrm{s}} q^{\frac{p^2}{2}} e^{2\pi i z\cdot p}}{\eta^4 q^{\frac13}} &= (1+q+2q^2+\dots)\mathrm{tr}_{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]}\zeta + (q+2q^2+\dots)\mathrm{tr}_{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}\zeta + q^2\mathrm{tr}_{\left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right]}\zeta + \dots \\ \frac{\sum\limits_{p\in D_4+\mathrm{c}} q^{\frac{p^2}{2}} e^{2\pi i z\cdot p}}{\eta^4 q^{\frac13}} &= (1+q+2q^2+\dots)\mathrm{tr}_{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]}\zeta + (q+2q^2+\dots)\mathrm{tr}_{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}\zeta + \dots \end{aligned} \tag{2.3}$$

where $\zeta = e^{2\pi i z\cdot J}$ is defined in the corresponding Spin(8) representations. In particular the massless spectrum at q^0 is consistent with maximal supergravity in $10-d$ dimensions. In this section we want to describe the process of integrating out the massive excitations.

2.1 Type II superstring four-graviton amplitude at tree-level

We will study the low-energy effective action through the four-graviton scattering amplitude. The four-graviton scattering amplitude obtained from the Einstein–Hilbert Lagrangian

$$\frac{1}{2\kappa_D^2}\int d^D x\sqrt{-g}R \tag{2.4}$$

in D spacetimes dimensions can be written as

$$\mathcal{M}_{4\,\text{tree}}^{\text{sugra}} = -i\frac{\kappa_D^2}{2^{10}}t_8 t_8 \prod_{a=1}^{4} R(k_a,\epsilon_a)\frac{64}{stu} \tag{2.5}$$

where t_8 is the rank eight tensor determined by its contraction with four antisymmetric tensors F_a as

$$\begin{aligned} t_8 F_1 F_2 F_3 F_4 \,=\, & 4F_{1\mu\nu}F_2^{\nu\sigma}F_{3\sigma\rho}F_4^{\rho\mu} + 4F_{3\mu\nu}F_2^{\nu\sigma}F_{1\sigma\rho}F_4^{\rho\mu} + 4F_{2\mu\nu}F_3^{\nu\sigma}F_{1\sigma\rho}F_4^{\rho\mu} + 4F_{1\mu\nu}F_3^{\nu\sigma}F_{2\sigma\rho}F_4^{\rho\mu} \\ & +4F_{3\mu\nu}F_1^{\nu\sigma}F_{2\sigma\rho}F_4^{\rho\mu} + 4F_{2\mu\nu}F_1^{\nu\sigma}F_{3\sigma\rho}F_4^{\rho\mu} \\ & -2F_1^{\mu\nu}F_{2\mu\nu}F_3^{\sigma\rho}F_{4\sigma\rho} - 2F_2^{\mu\nu}F_{3\mu\nu}F_1^{\sigma\rho}F_{4\sigma\rho} - 2F_3^{\mu\nu}F_{1\mu\nu}F_2^{\sigma\rho}F_{4\sigma\rho} \end{aligned} \tag{2.6}$$

and $R(k_a,\epsilon_a)$ is the linearised Riemann tensor

$$R(k_a,\epsilon_a)_{\mu\nu}{}^{\sigma\rho} = -4k_{a[\mu}k_a^{[\sigma}\epsilon_{a\nu]}{}^{\rho]} \tag{2.7}$$

for the external momenta k_a and the polarisation tensors ϵ_a. We use the Mandelstam variables s, t, u with all momenta incoming

$$s = -(k_1+k_2)^2\,, \quad t = -(k_2+k_3)^2\,, \quad u = -(k_1+k_3)^2\,. \tag{2.8}$$

On the other-hand, the sphere string theory four-graviton amplitude on $\mathbb{R}^{1,10-d}\times T^d$ gives

$$\mathcal{M}_{4\,\text{tree}}^{\text{string}} = i\frac{(2\pi)^7\alpha'^7}{2^{11}}\frac{e^{2\phi}}{(2\pi\sqrt{\alpha'})^d\,\upsilon_{\text{s}}}t_8 t_8 \prod_{a=1}^{4} R(k_a,\epsilon_a)\frac{\Gamma(-\frac{\alpha'}{4}s)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(1+\frac{\alpha'}{4}s)\Gamma(1+\frac{\alpha'}{4}t)\Gamma(1+\frac{\alpha'}{4}u)} \tag{2.9}$$

where e^{ϕ} is the ten-dimensional dilaton and $(2\pi\sqrt{\alpha'})^d\,\upsilon_{\text{s}}$ is the volume of T^d parametrised by the dimensionless modulus υ_{s}. This gives the identification

$$\kappa_D^2 = \frac{1}{2}(2\pi)^{7-d}\alpha'^{\frac{8-d}{2}}\frac{e^{2\phi}}{\upsilon_{\text{s}}} \tag{2.10}$$

and one defines accordingly the effective dilaton in dimension $D = 10-d$

$$e^{2\phi_d} = \frac{e^{2\phi}}{\upsilon_{\text{s}}} \tag{2.11}$$

and the Planck length ℓ is related to the string length as

$$\ell = e^{\frac{2}{8-d}\phi_d}\sqrt{\alpha'}\,. \tag{2.12}$$

We use the same symbol ℓ for the Planck length in all dimensions and define the torus coordinates with period $2\pi\ell$. The dependence of the Planck length in the torus dimension is absorbed in the Weyl rescaling of the metric in supergravity.

Supersymmetry implies that the total amplitude can be written in the same way as

$$\mathcal{M}_4^{\text{string}} = -i\frac{\kappa_D^2}{2^{10}} t_8 t_8 \prod_{a=1}^{4} R(k_a, \epsilon_a) A(s,t,u,\varphi) \tag{2.13}$$

for a function $A(s,t,u,\varphi)$ that is invariant under permutations of the Mandelstam variables and depends on the moduli φ of the theory. We define accordingly the perturbative series

$$A(s,t,u,\varphi) = \alpha'^3 \sum_{n=0}^{\infty} e^{2n\phi_d} A^{n\text{-loop}}(s,t,u,G,B) + \mathcal{O}(e^{-2\pi e^{-\phi_d}}) \,. \tag{2.14}$$

At tree-level one obtains [78]

$$\begin{aligned} A^{\text{tree}}(s,t,u) &= -\frac{\Gamma(-\frac{\alpha'}{4}s)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(1+\frac{\alpha'}{4}s)\Gamma(1+\frac{\alpha'}{4}t)\Gamma(1+\frac{\alpha'}{4}u)} \\ &= \frac{64}{\alpha'^3\, stu} \exp\Big(\sum_{n=1}^{\infty} \frac{2\zeta(2n+1)}{2n+1} \big(\tfrac{\alpha'}{4}\big)^{2n+1} (s^{2n+1}+t^{2n+1}+u^{2n+1})\Big) \\ &= \frac{64}{\alpha'^3\, stu} + \sum_{p,q\geq 0} \big(\tfrac{\alpha'}{4}\big)^{2p+3q} c_{(p,q)} (s^2+t^2+u^2)^p (s^3+t^3+u^3)^q \,, \end{aligned} \tag{2.15}$$

where $c_{(p,q)}$ are constants that are polynomial in the odd zeta values $\zeta(2n+1)$.

The corresponding effective action in string frame takes the schematic form

$$\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\bigg(R + \frac{\ell^6}{48} e^{-\frac{12}{8-d}\phi_d}\Big(\sum_{p,q\geq 0} \big(\ell e^{-\frac{2}{8-d}\phi_d}\big)^{4p+6q} c_{(p,q)} t_8 t_8 (\nabla^4)^p (\nabla^6)^q R^4\Big)\bigg) \,. \tag{2.16}$$

In this action, the higher derivative terms take into account the effect of the massive string states at tree-level. Of course they must be completed by the corresponding supersymmetry completion and higher order terms in the Riemann tensor. The complete low energy effective action taking into account both perturbative and non-perturbative quantum corrections take a similar form where the coefficients $c_{(p,q)} e^{-\frac{12+8p+12q}{8-d}\phi_d}$ are replaced by non-perturbative coupling functions $\mathcal{E}^{\text{W}}_{(p,q)\,\mu}$ that are functions of all scalar fields

$$W_\mu \sim \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\bigg(R + \frac{\ell^6}{48}\Big(\sum_{p,q\geq 0} \ell^{4p+6q} \mathcal{E}^{\text{W}}_{(p,q)\,\mu}\, t_8 t_8 \nabla^{4p+6q} R^4\Big)\bigg) \,. \tag{2.17}$$

In this section we shall concentrate on their perturbative component. As explained in the introduction, it is convenient to do not work with the Wilsonian effective action W_μ, but with the coupling functions appearing in the amplitude such that each $\mathcal{E}_{(p,q)}(\varphi)$ is U-duality invariant.

2.2 Low-energy expansion at one-loop

The one-loop contribution to the four-graviton scattering amplitude is written as the integral [47]

$$A^{\text{1-loop}} = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Gamma_{I\!I_{d,d}} \prod_{a=1}^{4} \int_{\Sigma} \frac{d^2 z_a}{\tau_2} \tau_2 \delta^{(2)}(z_4) e^{-\frac{\alpha'}{2}\sum_{a>b} G(z_a - z_b) k_a \cdot k_b} \tag{2.18}$$

where

$$\Gamma_{I\!I_{d,d}} = \tau_2^{\frac{d}{2}} \sum_{m,n \in \mathbb{Z}^d} e^{\pi i \tau p_L^2 - \pi i \bar{\tau} p_R^2} \,, \tag{2.19}$$

and the torus Green function is defined as

$$G(\tau, z) = -\log \left| \frac{\vartheta_1(\tau, z)}{\vartheta_1'(\tau)} \right|^2 + \frac{2\pi}{\tau_2} (\text{Im} z)^2 \,. \tag{2.20}$$

In this section we will review the low energy limit $\alpha' s \ll 1$ of this amplitude that was analysed in detail in [78, 81, 144] in the case $D = 10$. This expression is well defined for $0 \leq d \leq 5$, but suffers from infrared divergences in $D = 4$. Because the infrared divergences are well understood in supergravity [145, 146, 126], we will discuss the infrared singularities after the low energy expansion will have been derived. For $d = 6$ it must therefore be understood that an infrared regularisation has been introduced in (2.18).

Note that because of conservation of momentum, the Green function in the Koba–Nielsen exponent can be shifted by an arbitrary constant function of τ. It is convenient to replace the Green functions in the exponent by the Arakelov Green function [81, 144]

$$\mathcal{G}(\tau, z) = -\log \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^2 + \frac{2\pi}{\tau_2} (\text{Im} z)^2 \tag{2.21}$$

that satisfies $\int d^2 z \, \mathcal{G}(\tau, z) = 0$ and $\mathcal{G}(\tau, z) = G(\tau, z) - 2\log\big(2\pi |\eta(\tau)|^2\big)$.

To carry out the low energy expansion of the amplitude, it is convenient to introduce a fiducial parameter Λ to split the moduli space integral into two pieces [78]

$$A^{\text{1-loop}} = A^{\text{1-loop}}_{<\Lambda} + A^{\text{1-loop}}_{>\Lambda} \,. \tag{2.22}$$

One defines accordingly the truncated fundamental domain

$$\mathcal{F}_\Lambda = \big\{ \tau_2 < \Lambda, \, -\tfrac{1}{2} \leq \tau_1 \leq \tfrac{1}{2}, \, |\tau| > 1 \big\} \tag{2.23}$$

as the intersection of the fundamental domain with the region $\tau_2 < \Lambda$ for $1 \ll \Lambda \ll \frac{1}{\alpha' s}$ and

$$\mathcal{F} = SO(2) \backslash SL(2, \mathbb{R}) / PSL(2, \mathbb{Z}) = \mathcal{F}_\Lambda \cup \big\{ \tau_2 \geq \Lambda, -\tfrac{1}{2} \leq \tau_1 \leq \tfrac{1}{2} \big\} \,. \tag{2.24}$$

This split defines accordingly the Wilsonian component

$$A^{\text{1-loop}}_{<\Lambda} = 2\pi \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \Gamma_{I\!I_{d,d}} \prod_{a=1}^{4} \int_{\Sigma} \frac{d^2 z_a}{\tau_2} \tau_2 \delta^{(2)}(z_4) \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\alpha'}{2} \sum_{a>b} \mathcal{G}(z_a - z_b) k_a \cdot k_b \right)^n \tag{2.25}$$

for which one can expand the Koba–Nielsen factors such that

$$A^{\text{1-loop}}_{<\Lambda} = \sum_{m,n\geq 0} \mathcal{E}^{\text{1-loop}}_{(m,n)\,\Lambda} \sigma_2^m \sigma_3^n \,, \tag{2.26}$$

with

$$\sigma_k = (\tfrac{\alpha'}{4})^k (s^k + t^k + u^k) \,, \tag{2.27}$$

and

$$\mathcal{E}^{\text{1-loop}}_{(m,n)\,\Lambda} = 2\pi \int_{\mathcal{F}_\Lambda} \frac{d^2\tau}{\tau_2^2} \Gamma_{II_{d,d}} \mathcal{B}_{(m,n)}(\tau) \,, \tag{2.28}$$

where $\mathcal{B}_{(m,n)}(\tau)$ are the graph functions introduced in [80]. The first graph functions are Eisenstein series [78]

$$\mathcal{B}_{(0,0)}(\tau) = 1 \,, \quad \mathcal{B}_{(1,0)}(\tau) = \frac{\pi^2}{45} E_2(\tau) \,, \quad \mathcal{B}_{(0,1)}(\tau) = \frac{2\pi^3}{567} E_3(\tau) + \frac{\zeta(3)}{3} \,, \tag{2.29}$$

where the normalisation is such that $E_s(\tau) \sim \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)} \tau_2^{1-s}$. The graph functions have been studied extensively, see for example [147–152] for an non-exhaustive list. They behave at large τ_2 as [80]

$$\mathcal{B}_{(m,n)}(\tau) = \sum_{w=0}^{2m+3n-1} b_{(m,n)}(w) (\pi\tau_2)^{2m+3n-w} + \mathcal{O}(e^{-2\pi\tau_2}) \tag{2.30}$$

such that the cutoff dependent part of these couplings takes the form

$$\mathcal{E}^{\text{1-loop}}_{(m,n)\,\Lambda} = \mathcal{E}^{\text{1-loop}}_{(m,n)\,\epsilon} + 2\pi \sum_{w=0}^{2m+3n-1} \pi^{2m+3n-w} b_{(m,n)}(w) \frac{\Lambda^{\frac{d-2}{2}+2m+3n-w}}{\frac{d-2}{2}+2m+3n-w} \tag{2.31}$$

for constant coefficients $b_{(m,n)}(w)$ that are linear combinations over $\mathbb{Q}$ of single-valued multi-zeta values of weight w. To avoid to replace the power of Λ by a $\log\Lambda$ whenever $\frac{d-2}{2}+2m+3n-w=0$, we have introduced an analytic continuation in d (with d replaced by $d+2\epsilon$ corresponding to the spacetime dimension $D-2\epsilon$) and the coupling function $\mathcal{E}^{\text{1-loop}}_{(m,n)\,\epsilon}$ is defined as

$$\mathcal{E}^{\text{1-loop}}_{(m,n)\,\epsilon} = 2\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^\epsilon \, \Gamma_{II_{d,d}} \mathcal{B}_{(m,n)}(\tau) \,. \tag{2.32}$$

This integral converges for $\epsilon < 1 - \frac{d}{2} - 2m - 3n$ and is defined by analytic continuation near $\epsilon \approx 0$.

Let us now discuss the non-analytic component of the amplitude defined by the complementary set integral

$$A^{\text{1-loop}}_{>\Lambda} = 2\pi \int_\Lambda^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 d\tau_1 \Gamma_{II_{d,d}} \prod_{a=1}^4 \int_\Sigma \frac{d^2 z_a}{\tau_2} \tau_2 \delta^{(2)}(z_4) e^{-\frac{\alpha'}{2}\sum_{a>b} \mathcal{G}(z_a - z_b) k_a \cdot k_b} \,. \tag{2.33}$$

It can be interpreted in supergravity as the sum of five terms

$$A^{\text{1-loop}}_{>\Lambda} = A^{\text{1-loop}}_{>\Lambda\,\square} + A^{\text{1-loop}}_{>\Lambda\,\blacksquare} + A^{\text{1-loop}}_{>\Lambda\,\bullet\!\!\circ\!\!\bullet} + A^{\text{1-loop}}_{>\Lambda\,\blacktriangle} + A^{\text{1-loop}}_{>\Lambda\,\bigstar} \,, \tag{2.34}$$

corresponding respectively to the supergravity amplitude, the four-point insertion form-factor, the two four-point insertions form-factor, the five-point insertion form-factor and the six-point insertion form-factor.

In supergravity, the only form-factors that make sense as gauge invariant observables are the insertions of the integrated non-linear invariants, so the four-point, five-point and six-point insertions must be combined together to define observables. We will nevertheless call them form-factors for short.

To compute $A^{\text{1-loop}}_{>\Lambda}$ one expands the Green function $\mathcal{G}(z)$ in the tropical limit [153] as

$$\mathcal{G}(x+\tau y) = 2\pi\tau_2\big(\tfrac{1}{6} - |y| + y^2\big) - \log\big|1 - e^{-2\pi\tau_2|y|+2\pi i(x+\tau_1 y)}\big|^2 + \mathcal{O}(e^{-\pi\tau_2}) \tag{2.35}$$

for $-\frac{1}{2} < y < \frac{1}{2}$, and where all the terms in $\mathcal{O}(e^{-\pi\tau_2})$ can be neglected because they only contribute to terms exponentially suppressed in the limit $\Lambda \to \infty$. The logarithmic term can also be neglected as long as y is away from zero. There are accordingly five different terms one must take into account depending of the number of distances $|y_{ij}|$ going to zero, corresponding to the split in five terms in (2.34).

At a generic point, one just gets the supergravity one-loop amplitude with the ultra-violet cutoff $\frac{1}{\sqrt{\pi\alpha'}\Lambda}$ in Schwinger parameter space [10]

$$\begin{aligned} A^{\text{1-loop}}_{>\Lambda\,\square} &= 2\pi \int_\Lambda^\infty \frac{d\tau_2}{\tau_2^2} \int_0^1 d\tau_1 \tau_2^{\frac{d}{2}} \prod_{a=1}^4 \int_0^1 dy_a \delta(y_4) e^{-\alpha'\pi\tau_2 \sum_{a>b}\left(-|y_a-y_b|+(y_a-y_b)^2\right) k_a\cdot k_b} \\ &= 4\pi \int_\Lambda^\infty d\tau_2 \tau_2^{\frac{d}{2}-2} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \Big(e^{\pi\tau_2\alpha'[(x_2-x_1)(1-x_3)t + x_1(x_3-x_2)u]} \\ &\qquad + e^{\pi\tau_2\alpha'[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} + e^{\pi\tau_2\alpha'[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)u]} \Big) . \end{aligned} \tag{2.36}$$

In the computation we have decomposed the integral over the three y_a variables into the 6 ordered integrals, that give two times the three integrals with the orderings $0 \le y_1 \le y_2 \le y_3$, $0 \le y_2 \le y_3 \le y_1$ and $0 \le y_3 \le y_1 \le y_2$. This gives in total two times the three corresponding integrals over $0 \le x_1 \le x_2 \le x_3 \le 1$ with the three respective changes of variables

$$y_1 = 1 - x_3 , \qquad y_2 = 1 - x_2 , \qquad y_3 = 1 - x_1 , \tag{2.37}$$

$$y_1 = 1 - x_1 , \qquad y_2 = 1 - x_3 , \qquad y_3 = 1 - x_2 , \tag{2.38}$$

$$y_1 = x_2 , \qquad y_2 = x_3 , \qquad y_3 = x_1 . \tag{2.39}$$

This computation can be understood more generally from the point of view of tropical geometry [153] by recognising (2.101) as the worldline Green function [154].

[10] The parameter $\Lambda \gg 1$ but the energy cutoff $\frac{1}{\sqrt{\pi\alpha'}\Lambda}$ is large compare to the energy scale of the amplitude since $\alpha' s \Lambda \ll 1$.

The one-loop integral can be computed as follows [155]

$$
\begin{aligned}
& 4\pi \int_\Lambda^\infty d\tau_2 \tau_2^{\frac{d}{2}-2} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 e^{\pi\tau_2\alpha'[(x_2-x_1)(1-x_3)s+x_1(x_3-x_2)t]} \\
= \; & 4\pi^{2-\frac{d}{2}} \alpha'^{1-\frac{d}{2}} \Gamma(\tfrac{d-2}{2}) \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \big(-(x_2-x_1)(1-x_3)s - x_1(x_3-x_2)t\big)^{1-\frac{d}{2}} \\
& - \sum_{k=0}^\infty \frac{(4\pi)^{k+1}}{k!} \frac{\Lambda^{\frac{d}{2}-1+k}}{\frac{d}{2}-1+k} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \big((x_2-x_1)(1-x_3)\tfrac{\alpha' s}{4} + x_1(x_3-x_2)\tfrac{\alpha' t}{4}\big)^k \\
= \; & -8\pi^{2-\frac{d}{2}} \frac{\Gamma(\frac{d-6}{2})\Gamma(\frac{8-d}{2})^2}{\Gamma(7-d)} \int_0^1 dx \, \Big((-\alpha' s)^{\frac{2-d}{2}} \frac{(1-x)^{\frac{4-d}{2}}}{(1+\frac{t}{s})x-1} + (-\alpha' t)^{\frac{2-d}{2}} \frac{(1-x)^{\frac{4-d}{2}}}{(1+\frac{s}{t})x-1} \Big) \\
& - \sum_{k=0}^\infty \frac{(4\pi)^{k+1}(k+1)!}{(2k+3)!} \frac{\Lambda^{\frac{d}{2}-1+k}}{\frac{d}{2}-1+k} \int_0^1 dx \, \frac{((1-x)\frac{\alpha' s}{4})^{k+1} - (x\frac{\alpha' t}{4})^{k+1}}{(1-x)\frac{\alpha' s}{4} - x\frac{\alpha' t}{4}} \, ,
\end{aligned}
\tag{2.40}
$$

where one must take the analytic continuation of $d \leq 5$ to complex values $d+2\epsilon$ in order to make sense of the two terms in the last line separately. By construction the sum of the two terms is analytic at $\epsilon = 0$ and the regular parts of the two terms are uniquely defined from the prescription that d is replaced by $d+2\epsilon$. Note that although $\Lambda \gg 1$, the parameter of the incomplete Gamma function

$$
\Gamma(\tfrac{d-2}{2}, -\pi\Lambda\alpha'[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]) \tag{2.41}
$$

is very small because $\alpha' s \Lambda \ll 1$ and the last line is an expansion in $\alpha' s \Lambda$. One defines accordingly the supergravity four-point amplitude

$$
\begin{aligned}
A^{\text{1-loop}}_{\Box\,\epsilon} &= -8\pi^{2-\frac{d}{2}} \frac{\Gamma(\frac{d-6}{2})\Gamma(\frac{8-d}{2})^2}{\Gamma(7-d)} \int_0^1 dx \, \Big((-\alpha' s)^{\frac{2-d}{2}} \frac{(1-x)^{\frac{4-d}{2}}}{(1+\frac{t}{s})x-1} + (-\alpha' t)^{\frac{2-d}{2}} \frac{(1-x)^{\frac{4-d}{2}}}{(1+\frac{s}{t})x-1} \Big) + \circlearrowleft \\
&= 32(2\pi)^{7-d} \alpha'^{\frac{2-d}{2}} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p+k_4)^2} + \circlearrowleft
\end{aligned}
\tag{2.42}
$$

where $\circlearrowleft$ is the sum over the two other cyclic permutations of the Mandelstam variables and the Λ dependent terms expand as

$$
A^{\text{1-loop}}_{>\Lambda\,\Box} = A^{\text{1-loop}}_{\Box\,\epsilon} - \Big(2\pi \frac{\Lambda^{\frac{d-2}{2}}}{\frac{d-2}{2}} + \frac{2\pi^3}{45} \frac{\Lambda^{\frac{d+2}{2}}}{\frac{d+2}{2}} \sigma_2 + \frac{4\pi^4}{567} \frac{\Lambda^{\frac{d+4}{2}}}{\frac{d+4}{2}} \sigma_3 + \frac{16\pi^5}{14175} \frac{\Lambda^{\frac{d+6}{2}}}{\frac{d+6}{2}} \frac{\sigma_2^2}{2} + \dots \Big) \tag{2.43}
$$

For $d = 6$ the integral suffers from infrared divergences and $A^{\text{1-loop}}_{\Box\,\epsilon}$ must be regularised. To avoid the confusion between ultraviolet and infrared divergences we consider that the infrared regulator is defined through the introduction of a small mass scale μ, for example by giving a mass μ to the external momenta in (2.42). We write this integral $A^{\text{1-loop}}_{\Box\,\epsilon,\mu}$.

The second contribution $A^{\text{1-loop}}_{>\Lambda\,\blacksquare}$ corresponds to two coincident points. It splits into three contributions corresponding to the s, t, and u-channels. For the s-channel one can consider equivalently $z_1 \approx z_2$ or $z_3 \approx 0$. They can all be obtained by permutation from the case $z_2 - z_1 = x + y\tau_2$ with y small. They get contributions from the different domains with y negative or positive, and, $y_1 < y_3$

for which one sets $y_1 = x_1$ and $y_3 = x_2$, or, $y_3 < y_1$ for which one sets $y_1 = 1 - x_1$ and $y_3 = 1 - x_2$. With this change of variables one computes

$$\begin{aligned} & e^{-\frac{\alpha'}{2}\sum_{a>b}\mathcal{G}(z_a-z_b)k_a\cdot k_b} \\ \sim\ & e^{\alpha' s\pi\tau_2 x_1(x_2-x_1)}\Big|1-e^{-2\pi\tau_2|y|+2\pi i x}\Big|^{-\frac{\alpha' s}{2}} e^{-\alpha' s\pi\tau_2|y|(1-x_1)}\big(e^{\alpha' t\pi\tau_2|y|(1-x_2)}+e^{\alpha' u\pi\tau_2|y|(1-x_2)}\big) \end{aligned} \tag{2.44}$$

where the integration domain is now from $0 < x_1 < x_2 < 1$, $0 < x < 1$ and $-\delta < y < \delta$ with $0 < \delta \ll 1$. Up to terms that are exponentially suppressed in $e^{-2\pi\delta\Lambda}$ one can extend the integration domain to $y \in \mathbb{R}$, that we write as twice the integral over $|y| \in \mathbb{R}_+$. Using the change of variable

$$z = e^{-2\pi\tau_2|y|+2\pi i x} \tag{2.45}$$

one obtains eventually the contribution

$$\begin{aligned} & 2\pi\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-2}\int_0^1 dx_2\int_0^{x_2} dx_1 e^{\alpha' s\pi\tau_2 x_1(x_2-x_1)} \\ & \qquad\qquad \times\frac{2}{4\pi^2\tau_2}\int_{|z|<1}\frac{d^2z}{|z|^2}\big|1-z\big|^{-\frac{\alpha' s}{2}}\Big(|z|^{\frac{\alpha'}{2}(x_1 s-(1-x_2)t)}+|z|^{\frac{\alpha'}{2}(x_1 s-(1-x_2)u)}\Big) \\ = \ & \frac{1}{\pi}\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-3}\int_0^1 dx_2\int_0^{x_2} dx_1 e^{\alpha' s\pi\tau_2 x_1(x_2-x_1)}\int_{\mathbb{C}} d^2z\big|1-z\big|^{-\frac{\alpha' s}{2}}|z|^{\frac{\alpha'}{2}(x_1 s-(1-x_2)t)-2} \\ = \ & \big(\tfrac{\alpha' s}{4}\big)^2\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-3}\int_0^1 dx_2\int_0^{x_2} dx_1 e^{\alpha' s\pi\tau_2 x_1(x_2-x_1)}\frac{-\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t_1}{4})\Gamma(-\frac{\alpha' u_1}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t_1}{4})\Gamma(1+\frac{\alpha' u_1}{4})} \end{aligned} \tag{2.46}$$

where

$$t_1 = (1-x_2)t - x_1 s\ , \quad u_1 = (1-x_2)u - (x_2-x_1)s\ . \tag{2.47}$$

We recognise the integral over $\mathbb{C}$ as the sphere four-point amplitude. This limit $z_1 \approx z_2$ corresponds indeed physically to the exchange of massive string states in the s-channel. To obtain the contribution to $A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\!\triangleleft}$ one must subtract the contribution to the supergravity amplitude at coincident points, which here comes from the supergravity tree-level component of the string theory sphere four-point amplitude. Taking into account the six possible coincident point singularities and subtracting the three-level supergravity contribution one obtains

$$A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\!\triangleleft} = 2\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-3}\int_0^1 dx_2\int_0^{x_2} dx_1\Big(e^{\alpha' s\pi\tau_2 x_1(x_2-x_1)}\big(\tfrac{\alpha' s}{4}\big)^2 W\big(\tfrac{\alpha' s}{4},\tfrac{\alpha'}{4}((1-x_2)t-x_1 s)\big)+\circlearrowleft\Big) \tag{2.48}$$

where we have introduced the Wilsonian part of the Virasoro–Shapiro amplitude

$$W(s,t) = -\frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} - \frac{1}{stu}\ , \tag{2.49}$$

as in [144]. It is convenient to integrate τ_2 and to split this integral as

$$\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-3}\int_0^1 dx_2\int_0^{x_2} dx_1 e^{\alpha' s\pi\tau_2 x_1(x_2-x_1)}\big(\tfrac{\alpha' s}{4}\big)^2 W\big(\tfrac{\alpha' s}{4},\tfrac{\alpha'}{4}((1-x_2)t-x_1 s)\big) \tag{2.50}$$

$$= (-\tfrac{\alpha' s}{4})^{\frac{8-d}{2}}(4\pi)^{\frac{4-d}{2}}\Gamma(\tfrac{d-4}{2})\int_0^1 dx_2\int_0^{x_2} dx_1\big(x_1(x_2-x_1)\big)^{2-\frac{d}{2}}W\big(\tfrac{\alpha' s}{4},\tfrac{\alpha'}{4}((1-x_2)t-x_1 s)\big)$$
$$-\sum_{k=0}^{\infty}(\tfrac{\alpha' s}{4})^{2+k}\frac{(4\pi)^k}{k!}\frac{\Lambda^{\frac{d}{2}-2+k}}{\frac{d}{2}-2+k}\int_0^1 dx_2\int_0^{x_2} dx_1\big(x_1(x_2-x_1)\big)^{k}W\big(\tfrac{\alpha' s}{4},\tfrac{\alpha'}{4}((1-x_2)t-x_1 s)\big)$$

where once again this split is only defined for d away from the integral dimension. Taking $d=2\epsilon$ one reproduces the contribution to $A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\!\triangleleft}$ derived in [144] in $D=10$ dimensions.

One defines accordingly the form-factor in dimensional regularisation

$$A^{\text{1-loop}}_{\blacksquare\!\!\triangleleft\,\epsilon} = 2(-\tfrac{\alpha' s}{4})^{\frac{8-d}{2}}(4\pi)^{\frac{4-d}{2}}\Gamma(\tfrac{d-4}{2})\int_0^1 dx_2\int_0^{x_2} dx_1\big(x_1(x_2-x_1)\big)^{2-\frac{d}{2}}W\big(\tfrac{\alpha' s}{4},\tfrac{\alpha'}{4}((1-x_2)t-x_1 s)\big)+\circlearrowleft$$
$$= (2\pi)^{7-d}\alpha'^{\frac{8-d}{2}}\int\frac{d^Dp}{(2\pi)^D}\frac{s^2W\big(\frac{\alpha' s}{4},-\frac{\alpha'}{2}k_4\cdot p\big)}{p^2(p-k_1)^2(p-k_1-k_2)^2}+\circlearrowleft\ . \tag{2.51}$$

This is indeed the expected structure for the supergravity form-factor, with the insertion of the four-point interaction with momenta $k_3,k_4,p,-p-k_3-k_4$. Expanding $W\big(\frac{\alpha' s}{4},\frac{\alpha'}{4}((1-x_2)t-x_1 s)$, one can extract the Λ dependent terms

$$A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\!\triangleleft} = A^{\text{1-loop}}_{\blacksquare\!\!\triangleleft\,\epsilon} - 2\zeta(3)\left(\sigma_2\frac{\Lambda^{\frac{d-4}{2}}}{\frac{d-4}{2}}+\frac{\pi}{3}\sigma_3\frac{\Lambda^{\frac{d-2}{2}}}{\frac{d-4}{2}}+\frac{4\pi^2}{45}\frac{\sigma_2^2}{2}\frac{\Lambda^{\frac{d}{2}}}{\frac{d}{2}}+\dots\right)-\zeta(5)\left(\frac{5}{3}\frac{\sigma_2^2}{2}\frac{\Lambda^{\frac{d-4}{2}}}{\frac{d-4}{2}}+\dots\right)+\dots \tag{2.52}$$

For $d=6$ this form-factor suffers from infrared divergences and we write $A^{\text{1-loop}}_{\blacksquare\!\!\triangleleft\,\epsilon,\mu}$ for the same Feynman integral (2.51) with the small mass scale μ. Recall that we call this contribution the supergravity form-factor, while the only consistent form-factor in supergravity is the insertion of a supersymmetric Lagrangian correction that combines the four-point, the five-point and the six-point insertions.

The third contribution $A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\!\bigcirc\!\!\blacksquare}$ corresponds to two pairs of coincident points. They give three different terms. For the s-channel one has $z_1\approx z_2$ and $z_3\approx 0$. To avoid to have to expand the propagator at both $y_3\approx 0$ and $y_3\approx 1$ we shift the integration domain such that $-\delta<y_3<\delta$ with $0<\delta\ll 1$, as for $y=y_2-y_1$. There are four different contributions depending of the signs of y and y_3. We have $0<|y_3|<y_1$ and $0<|y_3|<y_2$ and one has the two possible signs for $y=y_2-y_1$. For $y>0$ one sets $y_1=x$ and for $y<0$, $y_1=1-x$. One gets two different terms depending on the sign of yy_3

$$e^{-\frac{\alpha'}{2}\sum_{a>b}\mathcal{G}(z_a-z_b)k_a\cdot k_b} \tag{2.53}$$
$$\sim\ e^{\alpha' s\pi\tau_2 x(1-x)}\Big|1-e^{-2\pi\tau_2|y|+2\pi i x_{12}}\Big|^{-\frac{\alpha' s}{2}}e^{-\alpha' s\pi\tau_2|y|x}\Big|1-e^{-2\pi\tau_2|y_3|+2\pi i x_3}\Big|^{-\frac{\alpha' s}{2}}$$
$$\times\Big(\theta(-yy_3)e^{-\alpha' s\pi\tau_2|y_3|x}e^{\alpha' t\pi\tau_2|yy_3|}+\theta(yy_3)e^{-\alpha' s\pi\tau_2|y_3|(1-x)}e^{-\alpha' t\pi\tau_2|yy_3|}\Big)$$

Once again, up to terms that are exponentially suppressed in $e^{-2\pi\delta\Lambda}$ one can extend the integration domain to $y\in\mathbb{R}$ and $y_3\in\mathbb{R}$, that we write as four times the integral over $|y|\in\mathbb{R}_+$ and $|y_3|\in\mathbb{R}_+$. Using the change of variables

$$z=e^{-2\pi\tau_2|y|+2\pi i x_{12}}\ ,\qquad w=e^{-2\pi\tau_2|y_3|+2\pi i x_3}\ , \tag{2.54}$$

one obtains eventually the contribution

$$
\begin{aligned}
&2\pi\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-2}\int_0^1 dx e^{\alpha' s\pi\tau_2 x(1-x)}\frac{2}{4\pi^2\tau_2}\int_{|z|<1}\frac{d^2z}{|z|^2}\frac{1}{4\pi^2\tau_2}\int_{|w|<1}\frac{d^2w}{|w|^2}\big|1-z\big|^{-\frac{\alpha' s}{2}}|z|^{\frac{\alpha' s}{2}x}\\
&\qquad\qquad\times\big|1-w\big|^{-\frac{\alpha' s}{2}}\Big(|w|^{\frac{\alpha'}{2}sx}e^{\frac{\alpha' t}{4\pi\tau_2}\log|z|\log|w|}+|w|^{\frac{\alpha'}{2}s(1-x)}e^{-\frac{\alpha' t}{4\pi\tau_2}\log|z|\log|w|}\Big)\\
=\;&\frac{1}{8\pi^3}\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-4}\int_0^1 dx e^{\alpha' s\pi\tau_2 x(1-x)}\int_{\mathbb{C}}d^2z\int_{\mathbb{C}}d^2w\,\big|1-z\big|^{-\frac{\alpha' s}{2}}|z|^{\frac{\alpha' s}{2}x-2}\\
&\qquad\qquad\times\big|1-w\big|^{-\frac{\alpha' s}{2}}|w|^{\frac{\alpha'}{2}sx-2}e^{\frac{\alpha' t}{4\pi\tau_2}\log|z|\log|w|}\\
=\;&\frac{1}{8\pi^3}\sum_{k=0}^\infty\frac{(\frac{\alpha' t}{4})^k}{\pi^k k!}\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-4-k}\int_0^1 dx\, e^{\alpha' s\pi\tau_2 x(1-x)}\left(\int_{\mathbb{C}}d^2z\big|1-z\big|^{-\frac{\alpha' s}{2}}|z|^{\frac{\alpha' s}{2}x-2}\log^k|z|\right)^2
\end{aligned}
\tag{2.55}
$$

where we recognise the square of the sphere amplitude for $k=0$. Once again one must subtract the contribution from the domain already integrated in $A^{\text{1-loop}}_{>\Lambda\,\square}$ and $A^{\text{1-loop}}_{>\Lambda\,\blacksquare}$. The contribution from $A^{\text{1-loop}}_{>\Lambda\,\square}$ corresponds to the case where the sphere amplitudes are both replaced by the supergravity tree-level amplitude while the contribution from $A^{\text{1-loop}}_{>\Lambda\,\blacksquare}$ to the case where only one is. One checks that this subtraction amounts to replace the sphere amplitude by its Wilsonian part and the logarithm terms by derivatives as in [144] to obtain

$$
A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\bigcirc\!\blacksquare}=\frac{(\frac{\alpha' s}{4})^4}{8\pi}\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-4}\int_0^1 dx\; e^{\alpha' s\pi\tau_2 x(1-x)}\sum_{k=0}^\infty\frac{(\frac{\alpha' t}{4})^k}{(4\pi\tau_2)^k k!}\Big(\frac{\partial^k W(\frac{\alpha' s}{4},-\frac{\alpha' s}{4}x+\zeta)}{\partial\zeta^k}\Big)^2\Big|_{\zeta=0}+\circlearrowleft
\tag{2.56}
$$

One computes then

$$
\begin{aligned}
A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\bigcirc\!\blacksquare}=A^{\text{1-loop}}_{\blacksquare\!\bigcirc\!\blacksquare\,\epsilon}-\frac{1}{8\pi}\sum_{m,n\geq 0}&\frac{(4\pi)^{m-n}(\frac{\alpha' s}{4})^{m+4}(\frac{\alpha' t}{4})^n}{m!n!}\frac{\Lambda^{\frac{d}{2}-3+m-n}}{\frac{d}{2}-3+m-n}\\
&\times\int_0^1 dx\big(x(1-x)\big)^m\Big(\frac{\partial^n W(\frac{\alpha' s}{4},-\frac{\alpha' s}{4}x+\zeta)}{\partial\zeta^n}\Big)^2\Big|_{\zeta=0}+\circlearrowleft
\end{aligned}
\tag{2.57}
$$

with

$$
\begin{aligned}
A^{\text{1-loop}}_{\blacksquare\!\bigcirc\!\blacksquare\,\epsilon}\;&=\;\frac{1}{2}\sum_{k=0}^\infty\frac{1}{k!}(-\tfrac{\alpha' s}{4})^{\frac{14-d}{2}+k}(\tfrac{\alpha' t}{4})^k(4\pi)^{\frac{4-d}{2}}\Gamma(\tfrac{d-6}{2}+k)\\
&\qquad\times\int_0^1 dx\big(x(1-x)\big)^{3+k-\frac{d}{2}}\Big(\frac{\partial^k W(\frac{\alpha' s}{4},-\frac{\alpha' s}{4}x+\zeta)}{\partial\zeta^k}\Big)^2\Big|_{\zeta=0}+\circlearrowleft\\
&=\;(2\pi)^{7-d}\alpha'^{\frac{6-d}{2}}\int\frac{d^Dp}{(2\pi)^D}\frac{(\frac{\alpha' s}{4})^4 W\big(\frac{\alpha' s}{4},\frac{\alpha'}{2}k_2\cdot p\big)W\big(\frac{\alpha' s}{4},-\frac{\alpha'}{2}k_3\cdot p\big)}{p^2(p-k_1-k_2)^2}+\circlearrowleft
\end{aligned}
\tag{2.58}
$$

In this last formula we see that the expansion in $(\frac{\alpha' t}{4})^k$ simply follows from the Feynman integral of the two insertions form-factor. Taking $d=2\epsilon$ reproduces the contribution to $A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\bigcirc\!\blacksquare}$ derived in [144]. Expanding $W(\frac{\alpha' s}{4},-\frac{\alpha' s}{4}x+\zeta)$ one extracts the first Λ dependent term

$$
A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\bigcirc\!\blacksquare}=A^{\text{1-loop}}_{\blacksquare\!\bigcirc\!\blacksquare\,\epsilon}-\frac{1}{2\pi}\zeta(3)^2\frac{\sigma_2^2}{2}\frac{\Lambda^{\frac{d-6}{2}}}{\frac{d-6}{2}}+\dots
\tag{2.59}
$$

In this case $A^{\text{1-loop}}_{\blacksquare\!\bigcirc\!\blacksquare\,\epsilon}$ is safe from infrared singularities in $d = 6$ and we do not need to introduce a mass scale.

The fourth term $A^{\text{1-loop}}_{>\Lambda\,\not\bigstar}$ corresponds to three coincident points. There are four choices of triplets, and for each choice there is one Green function argument that goes to zero for each Mandelstam variable. Each contribution is therefore invariant under permutation of the Mandelstam variables, although it will not be manifest in the computation. We consider $z_2 \approx z_1 \approx 0$ while keeping $z_3 \neq 0$. To avoid to have to expand the propagator at both $y_1 \approx 0$ and $y_1 \approx 1$ we shift the integration domain such that $-\delta < y_1 < \delta$, as for $y = y_2 - y_1$. There are six different contributions depending on the sign of y and y_1 and $y + y_1$, but using the appropriate redefinition of $y_3 = x$ or $1 - x$ one obtains that the only sign that matters is the one of yy_1 and

$$\begin{aligned} & e^{-\frac{\alpha'}{2}\sum_{a>b}\mathcal{G}(z_a-z_b)k_a\cdot k_b} \qquad (2.60)\\ \sim\ & \left|1-e^{-2\pi\tau_2|y|+2\pi i x_{12}}\right|^{-\frac{\alpha' s}{2}} e^{\alpha' s\pi\tau_2|y_1|(x-|y_1|)}\left|1-e^{-2\pi\tau_2|y_1|+2\pi i x_1}\right|^{-\frac{\alpha' t}{2}} \\ & \times\Bigg(\theta(-yy_1)e^{\alpha' t\pi\tau_2|y|x}e^{\alpha' s\pi\tau_2|yy_1|}\left|e^{-2\pi\tau_2|y|+2\pi i x_{12}}-e^{-2\pi\tau_2|y_1|+2\pi i x_1}\right|^{-\frac{\alpha' u}{2}} \\ & \qquad +\theta(yy_3)e^{\alpha' t\pi\tau_2|y|(1-x)}e^{-\alpha' s\pi\tau_2|yy_1|}\left|1-e^{-2\pi\tau_2|y|+2\pi i x_{12}}e^{-2\pi\tau_2|y_1|+2\pi i x_1}\right|^{-\frac{\alpha' u}{2}}\Bigg)\,. \end{aligned}$$

Using the change of variables

$$z = e^{-2\pi\tau_2|y|+2\pi i x_{12}}\,, \qquad w = e^{-2\pi\tau_2|y_1|+2\pi i x_1}\,, \qquad (2.61)$$

one obtains eventually

$$\begin{aligned} & 2\pi\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-2}\int_0^1 dx\frac{2}{4\pi^2\tau_2}\int_{|z|<1}\frac{d^2z}{|z|^2}\frac{1}{4\pi^2\tau_2}\int_{|w|<1}\frac{d^2w}{|w|^2}\left|1-z\right|^{-\frac{\alpha' s}{2}}\left|1-w\right|^{-\frac{\alpha' t}{2}}|w|^{-\frac{\alpha' s}{2}x} \\ & \qquad\times\left(\left|z-w\right|^{-\frac{\alpha' u}{2}}|z|^{-\frac{\alpha'}{2}tx}e^{\frac{\alpha' s}{4\pi\tau_2}\log|w|\log\frac{|z|}{|w|}}+\left|1-zw\right|^{-\frac{\alpha' u}{2}}|z|^{-\frac{\alpha'}{2}t(1-x)}e^{-\frac{\alpha' s}{4\pi\tau_2}\log|w|\log|z||w|}\right) \\ = \ & \frac{1}{8\pi^3}\int_\Lambda^\infty d\tau_2\tau_2^{\frac{d}{2}-4}\int_0^1 dx\int_{\mathbb{C}}d^2z\int_{\mathbb{C}}d^2w\left|1-z\right|^{-\frac{\alpha' s}{2}}|z|^{-\frac{\alpha' t}{2}x-2}\left|1-w\right|^{-\frac{\alpha' t}{2}}|w|^{-\frac{\alpha' s}{2}x-2}\left|z-w\right|^{-\frac{\alpha' u}{2}} \\ & \qquad\times e^{\frac{\alpha' s}{4\pi\tau_2}\log|w|\log\frac{|z|}{|w|}} \qquad (2.62) \end{aligned}$$

where one can identify the sphere five-point amplitude. As for (2.49) we need to extract the analytic part of the five-point amplitude to do not overcount the contribution common to $A^{\text{1-loop}}_{>\Lambda\,\square}$ and $A^{\text{1-loop}}_{>\Lambda\,\blacksquare\!\!\blacktriangleleft}$.

We therefore subtract the poles as follows

$$
\begin{aligned}
W_5(s_1,s_2,s_3,s_4,s_5) \;=\; & \frac{1}{\pi^2}\int_{\mathbb{C}} d^2z \int_{\mathbb{C}} d^2w\; |z|^{-2s_1-2}\big|1-z\big|^{-2s_2}|w|^{-2s_3-2}\big|1-w\big|^{-2s_4}\big|z-w\big|^{-2s_5} \\
& + \frac{1}{s_3(s_1{+}s_2{+}s_5)} + \frac{1}{s_1(s_3{+}s_4{+}s_5)} - \frac{1}{(s_1{+}s_2{+}s_5)(s_1{+}s_2{+}s_3{+}s_4{+}s_5)} \\
& - \frac{1}{(s_3{+}s_4{+}s_5)(s_1{+}s_2{+}s_3{+}s_4{+}s_5)} - \frac{1}{s_1(s_1{+}s_3{+}s_5)} - \frac{1}{s_3(s_1{+}s_3{+}s_5)} \\
& + \frac{1}{2}\frac{s_2^2}{s_3}\big(W(s_1{+}s_3{+}s_5,s_2)+W(-s_1{-}s_2{-}s_5,s_2)\big) \\
& - \frac{1}{2}\frac{s_2^2}{s_3{+}s_4{+}s_5}\big(W(s_1,s_2)+W(-s_1{-}s_2{-}s_3{-}s_4{-}s_5,s_2)\big) \\
& + \frac{1}{2}\frac{s_4^2}{s_1}\big(W(s_1{+}s_3{+}s_5,s_4)+W(-s_3{-}s_4{-}s_5,s_4)\big) \\
& - \frac{1}{2}\frac{s_4^2}{s_1{+}s_2{+}s_5}\big(W(s_3,s_4)+W(-s_1{-}s_2{-}s_3{-}s_4{-}s_5,s_4)\big) \\
& + \frac{1}{2}\frac{s_5^2}{s_1{+}s_3{+}s_5}\big(W(s_1,s_5)+W(s_3,s_5)\big) \\
& - \frac{1}{2}\frac{s_5^2}{s_1{+}s_2{+}s_3{+}s_4{+}s_5}\big(W(-s_1{-}s_2{-}s_5,s_5)+W(-s_3{-}s_4{-}s_5,s_5)\big)\,.
\end{aligned}
\tag{2.63}
$$

We check that $W_5(s_i)$ is indeed analytic in Appendix B and obtain the α' expansion (B.5)

$$
W_5(s_1,s_2,s_3,s_4,s_5) = \big(15s_2s_4s_5 - (s_2s_4{+}s_4s_5{+}s_5s_2)(s_2{+}s_4{+}s_5)\big)\zeta(5) + \mathcal{O}(s_i^4)\,. \tag{2.64}
$$

Note that the this term does not depend on s_1 and s_3 and is manifestly symmetric under permutations of s_2, s_4, s_5. As a five-point amplitude with punctures at $z_1 = 0, z_2 = z, z_3 = w, z_4 = 1, z_5 = \infty$, (2.63) is invariant under permutations of (z_2, z_3, z_4). It follows that $W_5(s_i)$ is invariant under the combined permutations of (s_2, s_4, s_5) and $(s_3, s_1, -\sum_i s_i)$, consistently with (2.62) being invariant under permutations of the Mandelstam variables.

Summing (2.62) over the four choices of distinguished external momentum and expanding the integral

$$
\int_\Lambda^\infty d\tau_2 \tau_2^{\frac{d}{2}-4} e^{\frac{\alpha' s}{4\pi\tau_2}\partial_\zeta\partial_\xi} = \Big(-\tfrac{\alpha' s}{4\pi}\partial_\zeta\partial_\xi\Big)^{\frac{d-6}{2}}\big(\Gamma(\tfrac{6-d}{2}) - \Gamma(\tfrac{6-d}{2}, -\tfrac{\alpha' s}{4\pi\Lambda}\partial_\zeta\partial_\xi)\big) \tag{2.65}
$$

at small $\frac{s}{\Lambda}$, one obtains finally

$$
\begin{aligned}
A^{\text{1-loop}}_{>\Lambda\,\star} \;&=\; -\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(\frac{\alpha' s}{4})^k}{(4\pi)^k k!}\frac{\Lambda^{\frac{d}{2}-3-k}}{\frac{d}{2}-3-k}\int_0^1 dx\Big(\frac{\partial^{2k} W_5(\frac{\alpha'}{4}xt+\zeta, \frac{\alpha'}{4}s, \frac{\alpha'}{4}xs+\xi-\zeta, \frac{\alpha'}{4}t, \frac{\alpha'}{4}u)}{\partial\zeta^k\partial\xi^k}\Big)\Big|_{\zeta=0,\xi=0} \\
&=\; -\frac{5}{2\pi}\zeta(5)\sigma_3\frac{\Lambda^{\frac{d-6}{2}}}{\frac{d-6}{2}} - \frac{1}{2\pi}\zeta(3)^2\frac{\sigma_2^2}{2}\frac{\Lambda^{\frac{d-6}{2}}}{\frac{d-6}{2}} + \dots
\end{aligned}
\tag{2.66}
$$

This contribution is therefore irrelevant for $d \leq 5$. This is why it was disregarded in [144] that focuses on ten dimensions, i.e. $d = 0$. However it contributes to the logarithmic ultraviolet divergence in Schwinger parameter space for $d = 6$, and the integral (2.62) then diverges in the limit $\tau_2 \to \infty$ exhibiting that this form-factor suffers from infrared divergences. One may introduce

an infrared regulator through a cut-off at large τ_2, but it is more convenient to first rewrite this contribution as a momentum space integral

$$A^{\text{1-loop}}_{\text{⧋}\,\epsilon,\mu} = 4(2\pi)^{7-d}\alpha'^{\frac{6-d}{2}} \int \frac{d^Dp}{(2\pi)^D} \frac{W_5(-\frac{\alpha'}{2}p\cdot k_4, \frac{\alpha'}{4}s, -\frac{\alpha'}{2}p\cdot(k_1+k_2), \frac{\alpha'}{4}t, \frac{\alpha'}{4}u)}{p^2(p-k_1)^2} \,, \tag{2.67}$$

where the mass scale μ can for example be introduced by giving a mass to the external momenta.

The last term $A^{\text{1-loop}}_{>\Lambda\,\text{⧋}}$ corresponds to four coincident points $z_1 \approx z_2 \approx z_3 \approx 0$. In this case one gets the six-point sphere amplitude with an internal supergravity bubble loop

$$(2\pi)^{7-d}\alpha'^{\frac{4-d}{2}} \int \frac{d^Dp}{(2\pi)^D} \frac{W_6(k_1,k_2,k_3,k_4,p,-p)}{p^2} \tag{2.68}$$

that only contributes to power-low divergences in the Schwinger parameter space ultraviolet cutoff for $d \leq 7$. It is therefore irrelevant for $d \leq 6$ and we shall not compute it.

Eventually one gets that all the Λ dependent parts between the analytic and the non-analytic components of the amplitude cancel each other. We have checked this explicitly up to σ_2^2 using [156]. The complete amplitude can be rewritten as the limit $\epsilon \to 0$ of

$$A^{\text{1-loop}} = A^{\text{1-loop}}_{\square\,\epsilon} + A^{\text{1-loop}}_{\text{◧}\,\epsilon} + A^{\text{1-loop}}_{\text{◨}\,\epsilon} + 2\pi \sum_{m,n\geq 0} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^\epsilon\, \Gamma_{II_{d,d}} \mathcal{B}_{(m,n)}(\tau) \sigma_2^m \sigma_3^n \,, \tag{2.69}$$

for $0 \leq d \leq 5$, with the supergravity amplitudes and the form-factors defined in dimensional regularisation with the coupling constant

$$\kappa_D^2 = \frac{1}{2}(2\pi)^{7-d}\alpha'^{\frac{8-d}{2}} e^{2\phi_d} \tag{2.70}$$

function of ϵ through d as well. For $d = 6$ one must also regularise the infrared divergences, which requires to consider the full form-factor including the five-point insertion

$$A^{\text{1-loop}}_{\mu} = A^{\text{1-loop}}_{\square\,\epsilon,\mu} + A^{\text{1-loop}}_{\text{◧}\,\epsilon,\mu} + A^{\text{1-loop}}_{\text{⧋}\,\epsilon,\mu} + A^{\text{1-loop}}_{\text{◨}\,\epsilon} + 2\pi \sum_{m,n\geq 0} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^\epsilon\, \Gamma_{II_{6,6}} \mathcal{B}_{(m,n)}(\tau) \sigma_2^m \sigma_3^n \,. \tag{2.71}$$

It was observed in [144] that the non-analytic component of the four-point amplitude does not involve irreducible multiple zeta-values in ten dimensions. It is manifestly the case in $d \leq 5$ since these form-factors only depend on the Virasoro–Shapiro amplitude. In $D = 4$ the infrared regulator dependent terms in $A^{\text{1-loop}}_{\text{⧋}\,\epsilon,\mu}$ could in principle involve irreducible multiple zeta-values, but one checks that the first three have vanishing coefficients in Appendix B.

Let us finally comment on the supergravity amplitude and form-factors in dimensional regularisation. It is known that the dimensional reduction prescription [157] that preserves supersymmetry is not a consistent regularisation scheme [158–161]. The total amplitude (2.71) is supersymmetric, and one expects to be able to split it into the sum of the supersymmetric supergravity amplitude, the form-factor and the analytic terms contributing to the Wilsonian effective action. As we discussed in the introduction, this split is only supersymmetric if we include some logarithmic term

in the moduli in the non-analytic amplitude. The supersymmetric regularisation in supergravity can be expressed as a dimensional regularisation with coupling constant

$$\kappa^2_{D-2\epsilon} = \kappa^2_D \Big(\frac{e^{\frac{2}{8-d}\phi_d} \sqrt{G^{IJ} m_I m_J}}{2\pi\ell} \Big)^\epsilon \tag{2.72}$$

for any $m_I \in \mathbb{Z}$ with $\gcd(m) = 1$. This factor is the mass of Kaluza–Klein states on the torus, and more generally can be replaced by the mass of any 1/2 BPS state in the theory with $\kappa^2_{D-2\epsilon} = \kappa^2_D \big(\frac{M}{2\pi}\big)^\epsilon$.

2.3 The two-loop amplitude

The two-loop amplitude was studied in a series of papers by d'Hoker and Phong [162–165, 24, 24, 166, 167]. We will not attempt to give a review of this gigantic work, but will try to give enough definitions to make sense of the four-graviton amplitude and its low energy limit.

We write $\omega^i(z) = \omega^i_z(z)dz$ the two holomorphic abelian differentials on the genus two surface Σ, They define the symmetric period matrix $\Omega^{ij} = \Omega^{ij}_1 + i\Omega^{ij}_2$ in the Siegel upper half-plane

$$\oint_{A_i} \omega^j(z) = \delta^j_i \ , \qquad \Omega^{ij} = \oint_{B^i} \omega^j(z) \ , \tag{2.73}$$

with positive definite imaginary part Ω_2. The period matrix parameterizes the arithmetic quotient $\mathcal{F}_2 = U(2)\backslash Sp(4,\mathbb{R})/PSp(4,\mathbb{Z})$ of the Siegel symmetric space. We write its components as

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} = \begin{pmatrix} \rho & u_1 + \rho u_2 \\ u_1 + \rho u_2 & \varsigma + iL + \rho u_2^2 \end{pmatrix} \ . \tag{2.74}$$

The separating degeneration locus $v = 0$ is a regular point of $\mathcal{F}_2$ that corresponds to a singular genus two Riemann surfaces made of two regular genus one surfaces connected by a very thin cylinder. The non-separating degeneration locus $L \to \infty$ is an asymptotic boundary of $\mathcal{F}_2$, and corresponds to the singular genus two Riemann surface when the cycle B_2 has an infinite length. A useful basis of modular forms is defined by the 16 theta series $\vartheta[\delta](\Omega, Z)$

$$\vartheta[{}^{a_1\,a_2}_{b^1\,b^2}](\Omega, Z) = \sum_{n_i \in \mathbb{Z}} e^{i\pi\Omega^{ij}(n_i + a_i/2)(n_j + a_j/2) + 2\pi i (n_i + a_i/2)(b^i/2 + Z^i)} \ , \tag{2.75}$$

for $a_i, b^i = 0, 1$, and $\vartheta[\delta](\Omega) \equiv \vartheta[\delta](\Omega, 0)$. One defines the derivatives

$$\partial_{ij} = \frac{1}{2}\frac{\partial}{\partial\Omega^{ij}} + \frac{1}{2}\delta_{ij}\frac{\partial}{\partial\Omega^{ii}} \ , \qquad \partial_i = \frac{\partial}{\partial Z^i} \ . \tag{2.76}$$

The 16 matrices $\delta \in \mathbb{Z}_2^{2\times 2}$ are the spin structures on Σ and parametrise the periodicity conditions for the spinors around A_i and B^i cycles. If $a_i b^i = 0 \bmod 2$ one says that the spin structure is even, and odd otherwise. The theta functions $\vartheta[\delta](\Omega)$ vanish for the six odd spin structures ν and their derivatives $(h_\nu)^2 = \partial_i\vartheta[\nu](\Omega, 0)\omega^i(z)$ carry only double poles and admit a unique spinor square root h_ν up to sign. One defines the prime form as the square root [168]

$$E(z, w) = \frac{\vartheta[\nu](\Omega, \int_w^z \omega)}{\sqrt{\partial_i\vartheta[\nu](\Omega, 0)\omega^i_z(z)\partial_j\vartheta[\nu](\Omega, 0)\omega^j_w(w)}} \tag{2.77}$$

for any odd spin structure, which generalises the genus one prime form $E(z,w) = \frac{\vartheta_1(z-w)}{\vartheta_1'(0)}$. The prime form is independent of the choice of odd spin structure ν and has a single pole at $z=w$. It is single valued around A_I cycles and has monodromy around a B_I cycle such that the Green function

$$G(z,w) = -\log|E(z,w)|^2 + 2\pi \mathrm{Im}\int_w^z \omega^i \; \Omega_{2\,ij}^{-1} \; \mathrm{Im}\int_w^z \omega^j \;, \tag{2.78}$$

is single valued. The Weyl fermion two-point function is defined as the square root $S_\delta(z,w)$ of

$$S_\delta(z,w)^2 = \partial_z\partial_w \log E(z,w) + 4\pi i \omega_z^i(z)\omega_w^j(w)\frac{\partial_{ij}\vartheta[\delta]}{\vartheta[\delta]} \;. \tag{2.79}$$

The genus two partition function is [165]

$$Z = \frac{1}{3(4\pi^3)^4}\int_{\mathcal{F}_2}\frac{d^6\Omega}{\det\Omega_2^5}\left|\frac{\sum_\delta \Xi_6[\delta]\vartheta[\delta]^4}{\Psi_{10}}\right|^2 \Gamma^{\text{2-loop}}_{I\!I_{d,d}} \;, \tag{2.80}$$

where the inverse of the Igusa cusp form

$$\Psi_{10} = \frac{1}{4^6}\prod_\delta \vartheta[\delta]^2 \;, \tag{2.81}$$

and the genus two Narain partition function are the contributions from the worldsheet bosons

$$\Gamma^{\text{2-loop}}_{I\!I_{d,d}} = \det\Omega_2^{\frac{d}{2}} \sum_{Q_i\in I\!I_{d,d}} e^{i\pi\Omega^{ij}p_L(Q_i)p_L(Q_j) - i\pi\bar\Omega^{ij}p_R(Q_i)p_R(Q_j)} \;, \tag{2.82}$$

while $\Xi_6[\delta]\vartheta[\delta]^4$ comes from the worldsheet fermions with spin structure δ [165]. One defines for each even spin structure two equivalent triplets of odd spin structures $\nu_1+\nu_2+\nu_3 = \nu_4+\nu_5+\nu_6 = \delta$ mod 2 and introducing the scalar product

$$\langle\nu|\delta\rangle = e^{i\pi(\delta_{11}\nu_{21}+\delta_{12}\nu_{22}-\delta_{21}\nu_{11}-\delta_{22}\nu_{12})} \;, \tag{2.83}$$

d'Hoker and Phong define

$$\Xi_6[\nu_1+\nu_2+\nu_3] = \sum_{1\le i<j\le 3}\langle\nu_i|\nu_j\rangle\prod_{k=4}^6 \vartheta[\nu_i+\nu_j+\nu_k]^4 \;. \tag{2.84}$$

The modular form $\Xi_6[\delta]$ does not depend on the choice of odd spin structure, as can be checked using the Riemann identities for any odd spin structure ν

$$\sum_\delta \langle\nu|\delta\rangle\vartheta[\delta]^4 = 0 \;. \tag{2.85}$$

The theta functions satisfy moreover the following identity for each non-zero even spin structure ε [169]

$$\sum_{I=1}^3 (-1)^I \vartheta[\delta_I^+]^2\vartheta[\delta_I^-]^2 = 0 \;, \tag{2.86}$$

where $\delta_I^\pm$ are the six even spin structures associated to ε such that $\delta_I^- = \delta_I^+ + \varepsilon$, and the $(-1)^I$ sign convention is defined such that $\delta_2^\pm$ is chosen as follows: δ_2^+ is $\delta_2^+ = \binom{10}{00}$ for $\varepsilon = \binom{10}{01}$, $\delta_2^+ = \binom{00}{10}$ for $\varepsilon = \binom{01}{10}$, $\delta_2^+ = \binom{11}{00}$ for $\varepsilon = \binom{11}{11}$, and $\delta_2^+ = \binom{00}{00}$ for the six other cases. Among these fifteen relations only six are independent, and they allow to determine all theta series in function of the four first $\vartheta[\binom{00}{rs}]$ and the choice of nine sign ambiguities for the roots appearing in their expressions. These signs depend on the sheet on the double cover of $\mathcal{F}_2$. Using both the Riemann identities (2.85) and (2.86) one checks that

$$\sum_\delta \Xi_6[\delta]\vartheta[\delta]^4 = 0 \tag{2.87}$$

which ensures that the two-loop contribution to the cosmological constant vanishes [165].

The four-graviton amplitude was derived in [24] and reads using (2.13)

$$A^{\text{2-loop}} = \frac{\pi}{128}\int_{\mathcal{F}_2}\frac{d^6\Omega}{(\det\Omega_2)^5}\Gamma^{\text{2-loop}}_{II_{d,d}}\int_{\Sigma^4}\mathcal{Y}_S\wedge\overline{\mathcal{Y}_S}\exp\left(-\frac{\alpha'}{2}\sum_{a>b}k_a\cdot k_b\,G(z_a,z_b)\right) \tag{2.88}$$

where $\mathcal{Y}_S$ is the holomorphic 4-form on four copies of the Riemann genus two surface Σ defined as [24]

$$\mathcal{Y}_S = \frac{\alpha'}{3}\big((t-u)\,\varepsilon_{ij}\varepsilon_{kl}+(s-t)\varepsilon_{ik}\varepsilon_{lj}+(u-s)\varepsilon_{il}\varepsilon_{jk}\big)\omega^i(z_1)\omega^j(z_2)\omega^k(z_3)\omega^l(z_4)\ . \tag{2.89}$$

The fundamental domain $\mathcal{F}_2$ can be chosen as [170]

$$-1/2\le\rho_1,v_1,\sigma_1\le 1/2\ ,\qquad 0\le 2v_2\le\rho_2\le\sigma_2\ ,\qquad |\det(C\Omega+D)|\ge 1\ , \tag{2.90}$$

for all C,D such that $\gamma\in\binom{AB}{CD}\in Sp(4,\mathbb{Z})$. This latter condition needs only to be checked for a finite set of matrices C,D.

In order to describe the low energy effective action we need now to consider three regions, the truncated fundamental domain $\mathcal{F}_{2,\Lambda}$ on which $L\le\Lambda$, the non-separating degeneration intermediate region $L\ge\Lambda\ge\Lambda_1\ge\rho_2$, and the tropical region $L\ge\Lambda,\rho_2\ge\Lambda_1$ [171]. We split accordingly

$$A^{\text{2-loop}} = A^{\text{2-loop}}_{<\Lambda} + A^{\text{2-loop}}_{<\Lambda_1>\Lambda} + A^{\text{2-loop}}_{>\Lambda_1}\ . \tag{2.91}$$

In the domain $\mathcal{F}_{2,\Lambda}$ the amplitude $A^{\text{2-loop}}_{<\Lambda}$ is analytic in the Mandelstam variables and one can expend the Koba–Nielsen factor as in the preceding section to obtain

$$A^{\text{2-loop}}_{<\Lambda} = \sum_{m,n\ge 0}\mathcal{E}^{\text{2-loop}}_{(m,n)\,\Lambda}\sigma_2^m\sigma_3^n\ , \tag{2.92}$$

with

$$\mathcal{E}^{\text{2-loop}}_{(m,n)\,\Lambda} = 2\pi\int_{\mathcal{F}_{2\Lambda}}\frac{d^6\Omega}{\det\Omega_2^3}\Gamma_{II_{d,d}}\mathcal{B}^{(2)}_{(m,n)}(\Omega)\ . \tag{2.93}$$

One finds directly that $\mathcal{B}^{(2)}_{(0,0)}(\Omega)=0$ [24], and using

$$\int_\Sigma\omega^i\wedge\overline{\omega^j} = -2i\Omega_2^{ij} \tag{2.94}$$

one computes that $\mathcal{B}^{(2)}_{(1,0)}(\Omega) = 2$ [166], consistently with the two-loop computation in supergravity [55]. The first non-trivial genus two graph function is the Kawazumi–Zhang invariant [83]

$$\mathcal{B}^{(2)}_{(0,1)}(\Omega) = 4\varphi_{\rm KZ}(\Omega) = -\int_{\Sigma^2} \big(\Omega^{-1}_{2ik}\Omega^{-1}_{2jl} - \tfrac{1}{2}\Omega^{-1}_{2ij}\Omega^{-1}_{2kl}\big)\omega^i(z_1)\wedge\overline{\omega^j(z_1)}\wedge\omega^k(z_2)\wedge\overline{\omega^l(z_2)}G(z_1,z_2,\Omega)\ . \tag{2.95}$$

It is a distribution on $\mathcal{F}_2$ satisfying the Poisson equation [84]

$$\Delta\varphi_{\rm KZ}(\Omega) = 5\varphi_{\rm KZ}(\Omega) - \pi\det\Omega_2\,\delta(v_1)\delta(v_2)\ , \tag{2.96}$$

and is a well defined function on the moduli space of genus two Riemann surfaces. This singularity is a consequence of unitarity and the factorisation of the two-loop amplitude at the massive poles $\frac{\alpha'}{4}s = \mathbb{Z}_{>0}$ [166]. The next genus two graph function $\mathcal{B}^{(2)}_{(2,0)}(\Omega)$ was analysed in [55], but the general graph functions $\mathcal{B}^{(2)}_{(p,q)}(\Omega)$ remain to be understood [86].

Let us now discuss the integration region corresponding to the tropical limit, when both ρ_2 and σ_2 are large. In this limit the A_i cycles remain unmodified, while the B^i cycles have infinite length. In this integration domain $\rho_2 \geq \Lambda_1$ we have $\det\Omega_2 \geq \Lambda\Lambda_1 \gg 1$ and this region of moduli space is

$$\mathcal{F}_2 \cap \{\Omega\,|\,\det\Omega_2 \gg 1\} = SO(2)\backslash GL(2,\mathbb{R})/PGL(2,\mathbb{Z}) \times \mathbb{R}^3/\mathbb{Z}^3 \cap \{\Omega\,|\,\det\Omega_2 \gg 1\} \tag{2.97}$$

We then write

$$\Omega_2 = \begin{pmatrix} L_1{+}L_3 & L_3 \\ L_3 & L_2{+}L_3 \end{pmatrix}\ , \tag{2.98}$$

where L_I for $I = 1,2,3$ are the lengths of the three lines in the degenerate surface

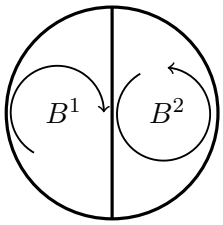

Figure 3: Tropical genus-two vacuum diagram

with the ordering $0 \leq L_3 \leq L_1 \leq L_2$ and the abelian one-forms $\omega^i(z)$ are locally constant and defined such that they have support on the cycle B^i [153]. Note that this is only a valid approximation for z away from the branching points. For a path γ from z to w in the surface Σ, one defines the geometric length $d(\gamma)$. For example taking $z = w$ one has by definition $d(B^1) = L_1{+}L_3$ and $d(B^2) = L_2{+}L_3$. For a given path that does not include closed loops, there always exists $\alpha_i \in \{-1,0,1\}$ such that [153]

$$\alpha_i\,{\rm Im}\int_\gamma \omega^i = d(\gamma)\ . \tag{2.99}$$

If γ is a shortest path from z to w, it is convenient to use an odd theta series of parameter $a_i = |\alpha_i|$ as above and b^i chosen such that $a_ib^i = 1$ to compute the prime form. Up to exponentially suppressed contributions, the prime form then simplifies to [153]

$$E(z,w) \sim \frac{2i\sin\big(\pi\alpha_i\int_\gamma\omega^i\big)}{2\pi\sqrt{\alpha_i\omega^i_z(z)\alpha_j\omega^j_z(w)}} \tag{2.100}$$

and one obtains the tropical limit of the Green function

$$G(z,w) \sim 2\pi\Big(-d(\gamma) + \ \mathrm{Im}\int_\gamma \omega^i \ \Omega_{2\,ij}^{-1} \ \mathrm{Im}\int_\gamma \omega^j\Big) + 2\log 2\pi - \log|1-e^{-2\pi d(\gamma)+2\pi i x_\gamma}|^2 \ . \qquad (2.101)$$

The first term reproduces the worldline Green function [154] that does not depend on the path γ, while the logarithmic term must be replaced by the sum over all paths in the graph to exhibit that the Green function is single valued. In the tropical limit, only the shortest path can be non-negligible at coincident points, whereas all the other contributions are always exponentially suppressed. This expression is only valid when the punctures are away from the branching points.

By the property of $\mathcal{Y}_S$ in (2.89), the only contributions in the tropical limit are when at least two marked points are on the cycle B^1, at least two on B^2 and at most two on $B^1 \cap B^2$. This implies that there must be two punctures on one of the three lines and the two others can be either on one other line or on two, i.e. a permutation of

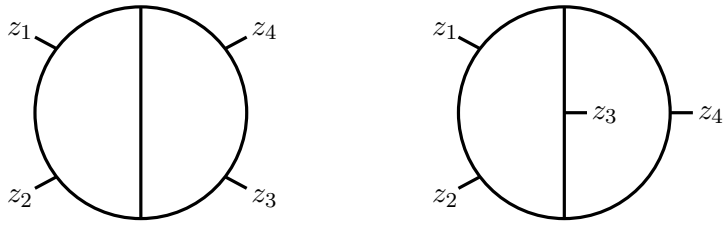

Figure 4: Planar and non-planar diagrams

corresponding to the planar and the non-planar diagrams in supergravity. There are 18 choices of planar diagrams and for each we may choose four different orderings of punctures on two lines, so 72 ordered choices in total. There are 36 choices of non-planar diagrams and for each one may choose two different orderings of punctures on one line, so 72 ordered choices in total. These cases involve the three choices of channel and the six permutations of the lines, while the extra four choices give the same result up to crossing and a change of variables for the punctures positions. It is convenient to absorb the six permutations of the lines in an unfolding of the $SO(2)\backslash GL(2,\mathbb{R})/PGL(2,\mathbb{Z})$ fundamental domain to the Schwinger parameter space of the vacuum diagram.

The Schwinger parameters $L_I \in \mathbb{R}_+$ are then not ordered and the cut-off is determined by $\sum_{I<J} L_I L_J \geq \Lambda\Lambda_1$ and $(L_I+L_J) \geq \Lambda_1$ for all pairs I,J. Doing so, there is a single integral with multiplicity two for the planar and the non-planar diagrams that we must sum over the six permutations of the Mandelstam variables. One can therefore write the contribution to the

amplitude when the punctures are separated from each others and from the branching points as

$$A^{\text{2-loop}}_{>\Lambda_1\,\square\square} = \frac{\pi}{4}\int\limits_{\substack{L_I\geq 0\\ \sum_{I<J}L_IL_J\geq\Lambda\Lambda_1\\ L_I+L_J\geq\Lambda_1}}\frac{dL_1dL_2dL_3}{\left(\sum_{I<J}L_IL_J\right)^{5-\frac{d}{2}}}(\alpha' s)^2\int_{0\leq y_1\leq y_2\leq 1}dy_1dy_2\;L_1^{\,2}\;e^{\pi\alpha' sL_1y_1(1-y_2)}\Bigg(\tag{2.102}$$

$$\int_{0\leq y_3\leq y_4\leq 1}dy_3dy_4\;L_2^{\,2}\;e^{\pi\alpha'\left(sL_2y_3(1-y_4)+\frac{L_1L_2L_3}{\sum_{I<J}L_IL_J}\left(t(y_2-y_1)(y_4-y_3)+s(1-y_1-y_4)(1-y_2-y_3)\right)\right)}$$

$$+\int_0^1 dy_3\int_0^1 dy_4\;L_2L_3\;e^{\pi\alpha'\frac{L_1L_2L_3}{\sum_{I<J}L_IL_J}\left(t(y_2-y_1)(y_4-y_3)+s(1-y_1-y_4)(1-y_2-y_3)\right)}\Bigg)+\text{perm.}$$

which gives the four-graviton supergravity amplitude [153]. Introducing the deformation $d\to d+2\epsilon$ one can write the divergent part of the integral as a sum of powers of $\alpha' s\Lambda_1\ll 1$ and $\alpha' s\Lambda\ll 1$, and the cut-off independent part gives the supergravity four-graviton amplitude in dimensional regularisation [39]

$$A^{\text{2-loop}}_{\square\square\,\epsilon}\tag{2.103}$$

$$= 16(2\pi)^{14-2d}\alpha'^{5-d}\int\frac{d^Dpd^Dq}{(2\pi)^{2D}}\Bigg(\frac{s^2}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2q^2(q-k_4)^2(q-k_3-k_4)^2}$$

$$+\frac{s^2}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2(p+q+k_3)^2q^2(q-k_4)^2}+\text{perm.}\Bigg)\,.$$

When two points are coincident

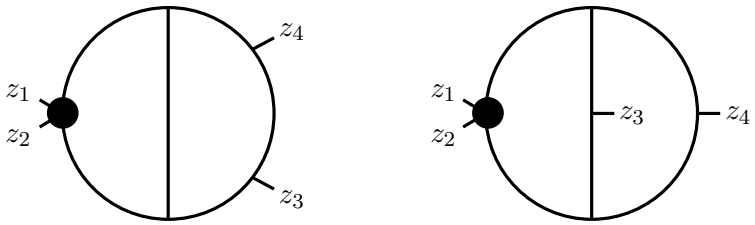

Figure 5: Planar and non-planar diagrams with coincident points

we cannot neglect the logarithmic term in (2.101) and we introduce the complex variable

$$z = e^{2\pi i\int_{z_2}^{z_1}\omega^1}\tag{2.104}$$

that is interpreted as the coordinate of the fourth puncture on the sphere. Similarly as in the preceding section one has then

$$G(z_1,z_2)\sim\log|z|-\log|1-z|^2+\frac{(\log|z|)^2}{2\pi L_1}\,,\tag{2.105}$$

with $|z|<1$ and combining the permutation $t\leftrightarrow u$ with the change of variable $y_1\to 1-y_1$, $y_3\to 1-y_4$, $y_4\to 1-y_3$ we obtain the integral

$$\begin{aligned}&\int_{|z|<1} d^2z\big|1-z\big|^{-\frac{\alpha' s}{2}}\left(|z|^{\frac{\alpha'}{2}\left(y_1 s-\frac{L_2L_3}{\sum_{I<J}L_IL_J}X\right)-2}+|z|^{\frac{\alpha'}{2}\left((1-y_1)s+\frac{L_2L_3}{\sum_{I<J}L_IL_J}X\right)-2}\right)\\ =\;&\pi\big(\tfrac{\alpha' s}{4}\big)^2\left(W\Big(\tfrac{\alpha'}{4}s,\tfrac{\alpha'}{4}\big(\tfrac{L_2L_3}{\sum_{I<J}L_IL_J}X-y_1s\big)\Big)+\frac{64}{\alpha'^3stu}\right)\end{aligned}\tag{2.106}$$

with

$$X=\big(t(y_4-y_3)-s(1-y_1-y_4)\big)\,.\tag{2.107}$$

The non-analytic term in $\frac{1}{stu}$ was already taken into account in the supergravity two-loop integral, and we subtract it to obtain the contribution

$$\begin{aligned}A^{\text{2-loop}}_{>\Lambda_1\,\blacksquare\square}\;=\;&\frac{1}{256}\int_{\substack{L_I\geq 0\\ \sum_{I<J}L_IL_J\geq\Lambda\Lambda_1\\ L_I+L_J\geq\Lambda_1}}\frac{dL_1dL_2dL_3}{\big(\sum_{I<J}L_IL_J\big)^{5-\frac{d}{2}}}(\alpha' s)^4\int_0^1 dy_1\;L_1\,e^{\pi\alpha' sL_1y_1(1-y_1)}\\ &\left(\int_{0\leq y_3\leq y_4\leq 1}dy_3dy_4\;L_2^2\;e^{\pi\alpha' sL_2y_3(1-y_4)}+\frac{1}{2}\int_0^1 dy_3\int_0^1 dy_4\;L_2L_3\right)\\ &e^{\pi\alpha'\frac{L_1L_2L_3}{\sum_{I<J}L_IL_J}s(1-y_1-y_4)(1-y_1-y_3)}W\Big(\tfrac{\alpha'}{4}s,\tfrac{\alpha'}{4}\big(\tfrac{L_2L_3}{\sum_{I<J}L_IL_J}\big(t(y_4-y_3)-s(1-y_1-y_4)\big)-y_1s\big)\Big)+\text{perm.}\end{aligned}\tag{2.108}$$

We can finally write this expression as a sum of powers in the cut-off plus the dimensionally regularised supergravity form-factor diagram

$$\begin{aligned}A^{\text{2-loop}}_{\blacksquare\square\;\epsilon}\;=\;&\frac{1}{2}(2\pi)^{14-2d}\alpha'^{8-d}\int\frac{d^Dpd^Dq}{(2\pi)^{2D}}\left(\frac{2s^4W(\frac{\alpha'}{4}s,\frac{\alpha'}{2}p\cdot k_1)}{p^2(p-k_1-k_2)^2(p+q)^2q^2(q-k_3)^2(q-k_3-k_4)^2}\right.\\ &\left.+\frac{s^4W(\frac{\alpha'}{4}s,-\frac{\alpha'}{2}p\cdot k_2-\frac{\alpha'}{4}s)}{p^2(p-k_1-k_2)^2(p+q)^2(p+q+k_3)^2q^2(q-k_4)^2}+\text{perm.}\right)\,.\end{aligned}\tag{2.109}$$

We will now consider the limit in which one puncture approaches a branching point, i.e. permutations of

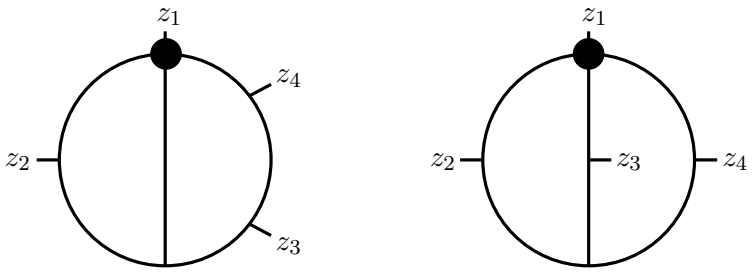

Figure 6: Diagrams with a puncture at a branching point

The abelian one-forms near the branching point can be written as

$$\omega^1(z)=\frac{i}{2\pi}\frac{dz}{z}\,,\qquad \omega^2(z)=\frac{i}{2\pi}\frac{dz}{z-1}\,,\tag{2.110}$$

such that one recovers the tropical limits as z approches the first line (of length L_1) at $z=0$, the second (of length L_2) at $z=1$ and the third (of length L_3) at $z=\infty$. For z_2 on the first line of length L_1, one computes the prime form using (2.100) and either $\vec{\alpha}=(1,0)$ or $\vec{\alpha}=(1,-1)$. One consistently obtains in both cases [11]

$$
\begin{aligned}
|E(z_1,z_2)|^2 &\sim \frac{|z|^2 e^{2\pi L_1 y_2}}{(2\pi)^2}\,,\\
G(z_1,z_2) &\sim -2\pi L_1 y_2 - \log|z|^2 + \frac{1}{2\pi}\Omega_2^{-1}[(2\pi L_1 y_2 + \log|z|, \log|1-z|)]\,.
\end{aligned}
\tag{2.111}
$$

Similarly for z_4 on the second line of length L_2 one finds

$$
\begin{aligned}
|E(z_1,z_4)|^2 &\sim \frac{|1-z|^2 e^{2\pi L_2(1-y_4)}}{(2\pi)^2}\,,\\
G(z_1,z_4) &\sim -2\pi L_2(1-y_4) - \log|1-z|^2 + \frac{1}{2\pi}\Omega_2^{-1}[(\log|z|, 2\pi L_2(1-y_4) + \log|1-z|)]\,,
\end{aligned}
\tag{2.112}
$$

and for z_3 on the third of length L_3

$$
\begin{aligned}
|E(z_1,z_3)|^2 &\sim \frac{e^{2\pi L_3(1-y_3)}}{(2\pi)^2}\,,\\
G(z_1,z_3) &\sim -2\pi L_3(1-y_3) + \frac{1}{2\pi}\Omega_2^{-1}[(2\pi L_3(1-y_3) - \log|z|, 2\pi L_3(1-y_3) - \log|1-z|)]\,.
\end{aligned}
\tag{2.113}
$$

For the first diagram the measure can be computed as for the previous case, but for the second, one must take into account that $\omega^i(z_1)$ is non-zero on both cycles and one gets

$$
\begin{aligned}
\mathcal{Y}_S \wedge \overline{\mathcal{Y}_S} &\sim \frac{4\alpha'}{\pi^2}\Big|\frac{s}{z} + \frac{t}{z-1}\Big|^2 L_1L_2L_3 d^2z dy_2 dy_3 dy_4\\
&\sim -\frac{4\alpha'}{\pi^2}\Big(\frac{su}{|z|^2} + \frac{tu}{|1-z|^2} + \frac{st}{|z|^2|1-z|^2}\Big) L_1L_2L_3 d^2z dy_2 dy_3 dy_4\,.
\end{aligned}
\tag{2.114}
$$

One obtains the amplitude

$$
\begin{aligned}
A^{\text{2-loop}}_{>\Lambda_1} = \frac{\alpha'^4}{64} \int_{\substack{L_I\geq 0\\ \sum_{I<J} L_I L_J \geq \Lambda\Lambda_1\\ L_I+L_J\geq\Lambda_1}} \frac{dL_1 dL_2 dL_3}{\big(\sum_{I<J} L_I L_J\big)^{5-\frac{d}{2}}} \int_0^1 dy_2\; L_1 \Bigg(\\
\int_{0\leq y_3\leq y_4\leq 1} dy_3 dy_4\; L_2^2\; e^{\pi\alpha' s L_2 y_3(1-y_4)} s^2 (t_1+uy_3)^2 W(\tfrac{\alpha'}{4}s_1, \tfrac{\alpha'}{4}(t_1+uy_3))\\
-\frac{1}{6}\int_0^1 dy_3 \int_0^1 dy_4\; L_2L_3\big(sut_1^2 + tus_1^2 + stu_1^2\big) W(\tfrac{\alpha'}{4}s_1, \tfrac{\alpha'}{4}t_1)\Bigg)\\
\times e^{\pi\alpha' \frac{L_1L_2L_3}{\sum_{I<J} L_I L_J}\left(s(1-y_4)(1-y_2-y_3) + ty_2(y_4-y_3)\right)} + \text{perm.}\,,
\end{aligned}
\tag{2.115}
$$

[11] We do not keep track of the additive constant $2\log(2\pi)$ in the Green function because it drops out in the amplitude. We write $\Omega_2^{-1}{}_{ij} Z^i Z^j = \Omega_2^{-1}[(Z^1, Z^2)]$.

with

$$\begin{aligned} s_1 &= s(1-y_2) - \tfrac{L_2L_3}{\sum_{I<J} L_IL_J}\big(t(y_4-y_3)+s(1-y_2-y_3)\big)\,, \\ t_1 &= ty_4 - \tfrac{L_1L_3}{\sum_{I<J} L_IL_J}\big(t(y_4-y_3)+s(1-y_2-y_3)\big)\,, \\ u_1 &= uy_3 - \tfrac{L_1L_2}{\sum_{I<J} L_IL_J}\big(t(y_4-y_3)+s(1-y_2-y_3)\big)\,. \end{aligned} \tag{2.116}$$

We can finally write this expression as the sum of powers in the cut-off and a dimensionally regularised supergravity form-factor diagram

$$\begin{aligned} A^{\text{2-loop}}_{\blacksquare\!\!\triangleleft\,\epsilon} = (2\pi)^{14-2d}\alpha'^{8-d}\int\frac{d^Dpd^Dq}{(2\pi)^{2D}}\Bigg(&\frac{s^2(2q\cdot k_1-s)^2W(\frac{\alpha'}{4}s+\frac{\alpha'}{2}p\cdot k_1,\frac{\alpha'}{2}q\cdot k_1-\frac{\alpha'}{4}s)}{p^2(p-k_2)^2(p+q)^2q^2(q-k_3)^2(q-k_3-k_4)^2} \\ &+\frac{2(s\,q\cdot k_1-t\,p\cdot k_1)^2W(\frac{\alpha'}{4}s+\frac{\alpha'}{2}p\cdot k_1,\frac{\alpha'}{2}q\cdot k_1+\frac{\alpha'}{4}t)}{p^2(p-k_2)^2(p+q)^2(p+q+k_3)^2q^2(q-k_4)^2}+\text{perm.}\Bigg)\,. \end{aligned} \tag{2.117}$$

At the next step we consider the case in which z_2 also approaches the same branching point

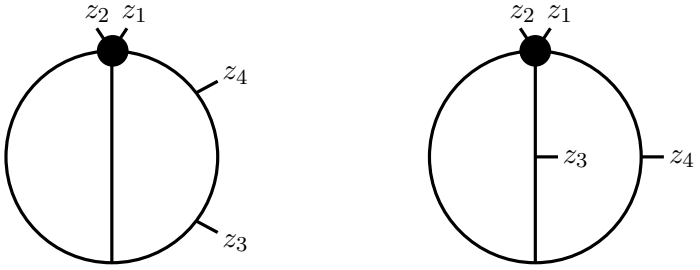

Figure 7: Diagrams with coincident punctures at a branching point

in which case we define

$$\frac{z}{w} = e^{2\pi i\int_{z_2}^{z_1}\omega^1}\,, \qquad \frac{1-z}{1-w} = e^{2\pi i\int_{z_2}^{z_1}\omega^2}\,, \tag{2.118}$$

and

$$\begin{aligned} |E(z_1,z_2)|^2 &\sim \frac{|z-w|^2}{(2\pi)^2}\,, \\ G(z_1,z_2) &\sim -\log|z-w|^2 + \frac{1}{2\pi}\Omega_2^{-1}\Big[\Big(\log\Big|\frac{z}{w}\Big|,\log\frac{|1-z|}{|1-w|}\Big)\Big]\,. \end{aligned} \tag{2.119}$$

For the first diagram, the measure is unchanged because both z_3 and z_4 are on the second line, and one computes the contribution

$$\begin{aligned}
&\frac{\alpha'^2 s^2}{64\pi^3} \sum_{\substack{L_I\geq 0\\ \sum_{I<J} L_I L_J \geq \Lambda\Lambda_1\\ L_I+L_J\geq\Lambda_1}} \frac{dL_1 dL_2 dL_3}{\left(\sum_{I<J} L_I L_J\right)^{5-\frac{d}{2}}} \int_{0\leq y_3\leq y_4\leq 1} dy_3 dy_4\; L_2^2\; e^{\pi\alpha' s L_2(1-y_4)\left(y_3+\frac{L_1L_3}{\sum_{I<J}L_IL_J}(1-y_3)\right)} \\
&\int_{\mathbb{C}} d^2z \int_{\mathbb{C}} d^2w\; |z|^{\frac{\alpha'}{2}\frac{L_2L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+t(y_4-y_3)\right)-2} |w|^{\frac{\alpha'}{2}\frac{L_2L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+u(y_4-y_3)\right)-2} |w-z|^{-\frac{\alpha'}{2}s} \\
&\times|1-z|^{\frac{\alpha'}{2}\left(-uy_3-ty_4+\frac{L_1L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+t(y_4-y_3)\right)\right)} |1-w|^{\frac{\alpha'}{2}\left(-ty_3-uy_4+\frac{L_1L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+u(y_4-y_3)\right)\right)} \\
&\times e^{\frac{\alpha' s}{4\pi}\frac{1}{\sum_{I<J}L_IL_J}\left(L_2\log|z|\log|w|+L_3(\log|z|-\log|1-z|)(\log|w|-\log|1-w|)+L_1\log|1-z|\log|1-w|\right)}
\end{aligned} \tag{2.120}$$

plus permutations of the Mandelstam variables. Using (2.63) one finds that the non-analytic pieces are already included in the previous integral and we conclude that the corresponding contribution only involves the five-point function $W_5(s_1, s_2, s_3, s_4, s_5)$.

For the second diagram for which z_3 and z_4 are on different lines, we must consider that $\omega^i(z_1)$ and $\omega^i(z_2)$ are non-zero on both cycles. One computes

$$\mathcal{Y}_S \wedge \overline{\mathcal{Y}_S} \sim \frac{\alpha'}{\pi^4}\left|\frac{s}{zw}+\frac{t}{(z-1)w}+\frac{u}{z(w-1)}\right|^2 L_2L_3 d^2z d^2w dy_3 dy_4\,. \tag{2.121}$$

and one gets

$$\begin{aligned}
&\frac{\alpha'^2}{64\pi^3} \sum_{\substack{L_I\geq 0\\ \sum_{I<J} L_I L_J \geq \Lambda\Lambda_1\\ L_I+L_J\geq\Lambda_1}} \frac{dL_1 dL_2 dL_3}{\left(\sum_{I<J} L_I L_J\right)^{5-\frac{d}{2}}} \int_0^1 dy_3 \int_0^1 dy_4\; L_2L_3\; e^{\pi\alpha' s\frac{L_1L_2L_3}{\sum_{I<J}L_IL_J}(1-y_3)(1-y_4)} \\
&\int_{\mathbb{C}} d^2z \int_{\mathbb{C}} d^2w\; \left|\tfrac{s}{zw}+\tfrac{t}{(z-1)w}+\tfrac{u}{z(w-1)}\right|^2 |z|^{\frac{\alpha'}{2}\frac{L_2L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+t(y_4-y_3)\right)} |1-z|^{\frac{\alpha'}{2}\left(-ty_3+\frac{L_1L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+t(y_4-y_3)\right)\right)} \\
&\times|w|^{\frac{\alpha'}{2}\frac{L_2L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+u(y_4-y_3)\right)} |1-w|^{\frac{\alpha'}{2}\left(-uy_3+\frac{L_1L_3}{\sum_{I<J}L_IL_J}\left(s(1-y_3)+u(y_4-y_3)\right)\right)} |w-z|^{-\frac{\alpha'}{2}s} \\
&\times e^{\frac{\alpha' s}{4\pi}\frac{1}{\sum_{I<J}L_IL_J}\left(L_2\log|z|\log|w|+L_3(\log|z|-\log|1-z|)(\log|w|-\log|1-w|)+L_1\log|1-z|\log|1-w|\right)}\,.
\end{aligned} \tag{2.122}$$

To take into account the subtraction of the poles that were already taken into account in the previous limits one defines

$$\begin{aligned}
&\frac{1}{\pi^2}\int_{\mathbb{C}}d^2z\int_{\mathbb{C}}d^2w\,\left|\frac{s}{zw}+\frac{t}{(z-1)w}+\frac{u}{z(w-1)}\right|^2|z|^{-2s_1}|1-z|^{-2s_2}|w|^{-2s_3}|1-w|^{-2s_4}|w-z|^{-2s_5}\\
&=F_5(s_1,s_2,s_3,s_4,s_5)+\frac{s^2}{s_1(s_1+s_3+s_5)}+\frac{s^2}{s_3(s_1+s_3+s_5)}-\frac{t^2}{s_1(s_3+s_4+s_5)}-\frac{u^2}{s_3(s_1+s_2+s_5)}\\
&\quad-\frac{t^2}{s_2(s_3+s_4+s_5)}-\frac{u^2}{s_4(s_1+s_2+s_5)}+\frac{t^2}{s_2s_3}+\frac{u^2}{s_1s_4}-\frac{1}{2}\frac{s^2s_5^2}{s_1+s_3+s_5}\big(W(s_1,s_5)+W(s_3,s_5)\big)\\
&\quad-\frac{s^2s_4(s_3+s_4+s_5)-t^2s_4(s_3+s_5)+u^2(s_3+s_5)(s_3+s_4+s_5)}{s_1}W(s_3+s_5,s_4)\\
&\quad-\frac{s^2s_2(s_1+s_2+s_5)+t^2(s_1+s_5)(s_1+s_2+s_5)-u^2s_2(s_1+s_5)}{s_3}W(s_1+s_5,s_2)\\
&\quad+\frac{t^2(s_1+s_2)^2}{s_3+s_4+s_5}W(s_1,s_2)+\frac{u^2(s_3+s_4)^2}{s_1+s_2+s_5}W(s_3,s_4)\\
&\quad-\frac{t^2(s_4+s_5)^2}{s_2}W(s_4+s_5,s_3)-\frac{u^2(s_2+s_5)^2}{s_4}W(s_2+s_5,s_1)\,.
\end{aligned}\tag{2.123}$$

Here the double pole in $-\frac{t^2}{s_1(s_3+s_4+s_5)}-\frac{u^2}{s_3(s_1+s_2+s_5)}$ comes from the planar diagram 4, at $y_1\sim 1-y_4\sim 0$ and $1-y_2\sim y_3\sim 0$, the double pole in $\frac{s^2}{s_1(s_1+s_3+s_5)}+\frac{s^2}{s_3(s_1+s_3+s_5)}$ comes from the non-planar diagram at $y_1\sim y_2\sim 0$ or 1, while all the other double poles come from the non-planar diagram with $y_1\sim 1-y_{3,4}\sim 0$ and $1-y_2\sim y_{3,4}\sim 0$. The simple pole in $\frac{s^2s_5^2}{s_1+s_3+s_5}$ comes from the non-planar diagram 5, the four last poles from the first diagram 5 and the two terms in $W(s_3+s_5,s_4)$ and $W(s_1+s_5,s_2)$ from the second diagram 5.

To compute the α' expansion of the function $F(s_i)$, it is convenient to use the single-valued map [172] as proposed in [135].[12] We compute in Appendix B the leading contribution

$$F_5(s_1,s_2,s_3,s_4,s_5)=2\zeta(3)s_5(2s^2-3t^2-3u^2)+s^2\mathcal{O}(s_i^3)+t^2\mathcal{O}(s_i^3)+u^2\mathcal{O}(s_i^3)\,.\tag{2.124}$$

Combing these results one obtains the corresponding supergravity form-factor in dimensional regularisation

$$\begin{aligned}
A^{\text{2-loop}}_{\text{\Large$\Theta$}_\epsilon}=(2\pi)^{14-2d}\alpha'^{7-d}\int\frac{d^Dpd^Dq}{(2\pi)^{2D}}\Bigg(&\frac{s^2W_5(\frac{\alpha'}{2}p\cdot k_1,\frac{\alpha'}{2}q\cdot k_1,\frac{\alpha'}{2}p\cdot k_2,\frac{\alpha'}{2}q\cdot k_2,\frac{\alpha'}{4}s)}{p^2(p+q-k_1-k_2)^2q^2(q+k_4)^2(q-k_1-k_2)^2}\\
&+\frac{F_5(\frac{\alpha'}{2}p\cdot k_1,\frac{\alpha'}{2}q\cdot k_1,\frac{\alpha'}{2}p\cdot k_2,\frac{\alpha'}{2}q\cdot k_2,\frac{\alpha'}{4}s)}{p^2(p+q-k_1-k_2)^2(p+q+k_4)^2q^2(q+k_4)^2}+\text{perm.}\Bigg)\,.
\end{aligned}\tag{2.125}$$

One may then consider the case in which z_1,z_2,z_3 all coincide at a branching point

[12] We thank Oliver Schlotterer for guiding us in the literature.

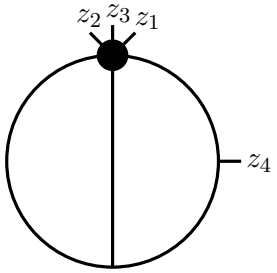

Figure 8: Diagram with three coincident punctures at a branching point

but this limit only contributes a cut-off dependent constant and we shall not compute it. It involves a tree-level six-point function and a priori contribute to the bubble diagram divergence of the form-factor in four dimensions. The last contribution to the one insertion form-factor comes from four coincident punctures at a branching point and does not contribute to the amplitude for $d \leq 6$.

At this order one obtains the contribution to the power-low terms in the cut-off from the α' expansion

$$A^{\text{2-loop}}_{>\Lambda_1} \sim \frac{1}{6} \int\limits_{\substack{L_I \geq 0 \\ \sum_{I<J} L_I L_J \geq \Lambda\Lambda_1 \\ L_I + L_J \geq \Lambda_1}} \frac{dL_1 dL_2 dL_3}{\left(\sum_{I<J} L_I L_J\right)^{3-\frac{d}{2}}} \left(4\pi \left(\tfrac{\alpha'}{4}\right)^2 (s^2+t^2+u^2) + \tfrac{4\pi^2}{3} \left(\tfrac{\alpha'}{4}\right)^3 (s^3+t^3+u^3) \left(\sum_I L_I - \frac{5 L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \right) \right.$$
$$\left. + \frac{2\zeta(3)}{\sum_{I<J} L_I L_J} \left(\frac{5}{\pi} \left(\tfrac{\alpha'}{4}\right)^3 (s^3+t^3+u^3) + \frac{1}{6} \left(\tfrac{\alpha'}{4}\right)^4 (s^2+t^2+u^2)^2 \left(7 \sum_I L_I - \frac{8 L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \right) \right) + \dots \right) \tag{2.126}$$

which matches the behaviour of the analytic component at large L_I according to [173, 174]

$$\mathcal{B}^{(2)}_{(1,0)}(\Omega) = 2 \tag{2.127}$$
$$\mathcal{B}^{(2)}_{(0,1)}(\Omega) \sim \frac{2\pi}{3} \left(\sum_I L_I - \frac{5 L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \right) + \frac{5\zeta(3)}{\pi^2 \sum_{I<J} L_I L_J}$$
$$\mathcal{B}^{(2)}_{(2,0)}(\Omega) \sim \frac{\pi^2}{90} \left(4 \left(\sum_I L_I \right)^2 - 22 \sum_I L_I \frac{L_1 L_2 L_3}{\sum_{I<J} L_I L_J} - 3 \left(\sum_{I<J} L_I L_J \right)^2 + 32 \left(\frac{L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \right)^2 \right)$$
$$+ \frac{\zeta(3)}{6\pi} \left(7 \sum_I L_I - \frac{8 L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \right) + \frac{3\zeta(5)}{4\pi^3} \frac{\sum_I L_I}{\left(\sum_{I<J} L_I L_J \right)^2} + \beta \frac{\zeta(3)^2}{2\pi^4 \left(\sum_{I<J} L_I L_J \right)^2} \,.$$

For consistency the term in $\zeta(5)$ above must come from diagram 8 that gives a tree-level six-point insertion. The last term with the undetermined coefficient β must come from the diagrams with all coincident points at the branching points.

We will not analyse the limits with more than one insertion and the single degeneration limits. The diagrams with two or three insertions are

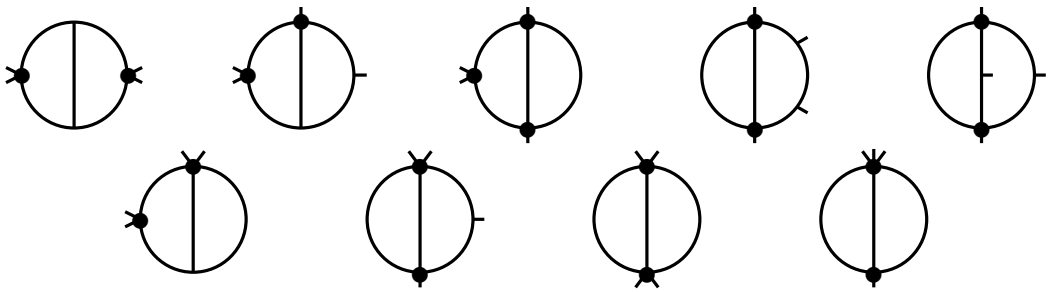

Figure 9: Multiple insertions diagrams

and could be computed in the same way as for a single insertion.

There are other contributions from the non-separating degeneration limit in which only the B^2 cycle becomes very large. The corresponding diagrams correspond to form-factors with at least one insertion of a one-loop correction (2.26). They can be drown as a genus one surface with two punctures connected by a tropical line of length L, and the four punctures can be either on the genus one surface or the tropical line. The tropical line corresponds to the loop in supergravity while the genus one surface gives rise to the insertion of a term in the string theory one-loop effective action. To compute such corrections one needs to analyse the Green function $G(z,w)$ in the non-separating degeneration limit when one or the two points are on the tropical line. The non-separating degeneration limit was analysed in [85,174], based on the general description [168]. In this limit the genus two surface with four punctures z_1, z_2, z_3, z_4 is described as a genus one surface with six punctures z_1, z_2, z_3, z_4, 0 and v and the identification of the local coordinates z near 0 and w near v as

$$z(w-v) = e^{-2\pi L + 2\pi i \varsigma} \,, \tag{2.128}$$

with v, L and ς defining the period matrix (2.74). However, the authors only consider the case in which z and w remain at finite distance of 0 and v in the limit $L \gg 1$. One needs to relax this assumption to compute the diagrams for which one or two punctures are on the tropical line of length L. There is also the further degeneration limit in which the B^1 cycle of the genus one surface also becomes infinite while the points v remain close to 0. The corresponding diagrams can be drown as a sphere with four punctures that are connected in pairs with two tropical lines of length L_1 and L_2. The four punctures can then be on the tropical lines or on the sphere. The corresponding corrections also contribute to the two-loop form factors in supergravity for which the sphere gives rise to the insertion of a term in the string theory tree-level effective action. They correspond for example to diagrams of the form

Figure 10: Eight shape diagrams

The maximal degeneration limit that we have analysed in this section is obtained as the further degeneration in which L_3 becomes also infinite, as one can see in figure one of [175]. We have not computed these contributions but we shall see in Section 4 that consistency with eleven-dimensional supergravity requires that the first diagram in (10) does not vanish.

3 Automorphic forms

The couplings in type II string theory on T^{d-1} are tensors on the symmetric space $K_d\backslash E_d$ that are invariant under the U-duality group $E_d(\mathbb{Z})$, defined as the arithmetic subgroup preserving the highest weight lattice $\mathbb{L}_d \subset R(\Lambda_d)$. As such, the couplings are functions f_w on E_d with the transformation rule

$$f_w(kv\gamma) = \rho_w(k) f_w(v) \tag{3.1}$$

for all $k \in K_d$ and $\gamma \in E_d(\mathbb{Z})$. Here ρ_w is a finite-dimensional representation of K_d indexed by the weight $w = n^i \Upsilon_i$.

One defines a parabolic subgroup P_λ from a weight λ such that it is generated by the Cartan subalgebra and the root generators E_α satisfying $(\lambda, \alpha) \geq 0$. The Borel subgroup is defined for $\lambda = \varrho$ the Weyl vector. $P_\lambda = LU$ where L is the Levi subgroup generated by the Cartan subgroup and the root generators E_α satisfying $(\lambda, \alpha) = 0$, while U is the unipotent radical generated by the root generators E_α satisfying $(\lambda, \alpha) > 0$. We define the fundamental weights Λ_i dual to the simple roots $(\Lambda_i, \alpha^j) = \delta_i^j$, such that any dominant weight can be written as $\lambda = n^i \Lambda_i$ for $n_i \in \mathbb{Z}_{\geq 0}$ and (Λ_i, Λ_j) is the inverse of the Cartan matrix. In particular $\varrho = \sum_i \Lambda_i$. For short we write the maximal parabolic subgroup $P_i \equiv P_{\Lambda_i}$. We use the Bourbaki labelling such that the exceptional node of E_d groups is Λ_2 and the Weyl spinor nodes of $SO(d,d)$ are Λ_{d-1} and Λ_d.

It will be natural to consider a coset representative of $v \sim kv \in E_d$ in a maximal parabolic subgroup relevant to a perturbative limit in string theory. In the perturbative string theory parabolic $P_1 = GL(1) \times Spin(d-1, d-1) \ltimes U_1$, the cusp is defined at small effective dilaton $e^{\phi_{d-1}} \ll 1$, where ϕ_{d-1} is defined in (2.11). Consistently with string perturbation theory, the leading term in the effective string coupling constant is proportional to $e^{-2\phi_{d-1}}$ in string frame, and generally power-low in Einstein frame. Therefore each coupling is bounded by a power-low function at the cusp $e^{\phi_{d-1}} \to 0$.

Similarly, in the M-theory parabolic $P_2 = GL(d) \ltimes U_2$, the cusp is defined at large torus volume and the corresponding supergravity amplitude on T^d gives terms that are power-low in the volume.

More generally one concludes that the coupling is of moderate growth, i.e. is bounded by a power-low behaviour at all cusps. For each parabolic subgroup $P_\lambda \subset E_d$ one defines a coset representative of $v \in P_\lambda$ such that $v = alu$ with $a \in GL(1)^r$, l in the semi-simple component $[L, L]$ of the Levi subgroup L and $u \in U$ the unipotent radical. Moderate growth implies that there exists coefficients C and $s = \{s_1, s_2, \dots s_r\}$ such that $|f_w(v)| < Ca^s$ at large a.

In string theory it is moreover natural to expect that the coupling functions are smooth almost everywhere in moduli space. Generally one may expect to have poles at locus in moduli space where additional fields become massless, but this does not happen in type II string theory on T^{d-1}. Assuming that the coupling function is smooth, one can define differential operators through the left-action of the universal enveloping algebra $\mathcal{U}(\mathfrak{e}_d)$ generated by $\mathfrak{e}_d$ through

$$(X \cdot f_w)(v) = \Big(\frac{d}{dt} f_w(e^{tX} v)\Big)\Big|_{t=0} . \tag{3.2}$$

For $X \in \mathfrak{e}_d \ominus \mathfrak{k}_d$ in the coset component, this definition coincides with the covariant derivative on the symmetric space

$$(X \cdot f_w)(v) = \kappa^{\alpha\beta} X_\alpha \mathcal{D}_\beta f_w(v) \tag{3.3}$$

with $\kappa^{\alpha\beta}$ the Cartan–Killing form and $X = X_\alpha T^\alpha$ in a basis of generators T^α. The action of $X \in \mathfrak{k}_d$ gives instead the linear transformation under $\mathfrak{k}_d$ induced from the K representation ρ_w

$$(X \cdot f_w)(v) = \mathrm{d}\rho_w(X) f_w(v) \ . \tag{3.4}$$

One decomposes the Maurer–Cartan form

$$\partial_\mu v \cdot v^{-1} = P_\mu - \omega_\mu \tag{3.5}$$

with $\omega \in \mathfrak{k}_d$ and P in the complement $\mathfrak{e}_d \ominus \mathfrak{k}_d$, and write the symmetric space metric

$$G_{\mu\nu} = \kappa^{\alpha\beta} P_{\mu\alpha} P_{\nu\beta} \tag{3.6}$$

such that the covariant derivative is defined as

$$\mathcal{D}_\alpha f_w(v) = P_{\mu\alpha} G^{\mu\nu} (\partial_\nu + \rho_w(\omega_\nu)) f_w(v) \ . \tag{3.7}$$

The covariant derivative is usually defined in representation theory as in (3.3) with (3.2), while it is more commonly defined in the physics literature and in Riemannian geometry as (3.7).

For any element in the centre $\mathcal{Z}(\mathfrak{e}_d)$ of $\mathcal{U}(\mathfrak{e}_d)$ one defines a left-right invariant differential operator. The canonical example is defined as the quadratic Casimir, which gives the Laplace–Beltrami operator for the trivial $K(E_d)$ representation

$$\Delta f_0(v) = \kappa^{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta f_0(v) = \kappa_{\alpha\beta} (T^\alpha \cdot (T^\beta \cdot f_0))(v) \ . \tag{3.8}$$

More generally the centre $\mathcal{Z}(\mathfrak{e}_d)$ can be defined as the set of K_d-invariant polynomials in the differential operator $\mathcal{D}_\alpha$ generated by the d Casimirs.

A modular form is defined in mathematics as a vector of smooth functions on $E_d/E_d(\mathbb{Z})$ with uniform moderate growth, the left-action (3.1), and which defines a finite dimensional representation of the centre $\mathcal{Z}(\mathfrak{e}_d)$. In the simplest case it is a dimension one representation of $\mathcal{Z}(\mathfrak{e}_d)$, which means that the function is an eigen-function of all invariant operators. The string theory coupling functions are generically not in finite-dimensional representations of $\mathcal{Z}(\mathfrak{e}_d)$. In particular the function (0.13) is not and belongs to a class of functions on $E_d/E_d(\mathbb{Z})$ that generalises the notion of automorphic forms. Examples of modular forms are defined by Eisenstein series. We shall first define Eisenstein series and describe their automorphic representations. We will only introduce the function (0.13) in Section 5.

3.1 Eisenstein series and automorphic representations

The simplest example of an Eisenstein series are the real analytic Eisenstein series for the group $SL(2)$ that can be defined in two ways

$$E_s^{SL(2)}(\tau) = \frac{1}{2\zeta(2s)} {\sum_{m,n\in\mathbb{Z}}}' \frac{\tau_2^s}{|m+\tau n|^{2s}} = \sum_{\gamma \in SL(2,\mathbb{Z})/P_1(\mathbb{Z})} \tau_2^s\big|_\gamma \ , \tag{3.9}$$

either as the sum of vectors $(m,n) \in \mathbb{Z}^2$ in the lattice $\mathbb{L}_1$ or as the Poincaré sum over coset representatives $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acting on $\tau = \tau_1 + i\tau_2$ with a Möbius tranformation

$$\tau|_\gamma = \frac{a\tau + b}{c\tau + d} \,. \tag{3.10}$$

This definition has a natural generalisation to maximal parabolic Eisenstein series for arbitrary simple groups G. The maximal parabolic P_i is defined such that $g \in P_i$ acts on the highest weight Λ_i as a rescaling [13]

$$g\Lambda_i = \frac{1}{y_i}\Lambda_i \tag{3.11}$$

and one can define the maximal parabolic Eisenstein series in two ways [176]

$$E^G_{P_i,s}(v) = \frac{1}{2\zeta(2s)} \sum_{\substack{Q\in\mathbb{L}_i \\ Q\times Q=0}}{}' \frac{1}{|vQ|^{2s}} = \sum_{\gamma\in G(\mathbb{Z})/P_i(\mathbb{Z})} y_i^s\big|_\gamma \,. \tag{3.12}$$

In the first sum we write the weight lattice $\mathbb{L}_i \subset R(\Lambda_i)$, and $Q\times Q \in \mathbb{L}_i \otimes \mathbb{L}_i$ is defined as

$$Q\times Q = \kappa_{\alpha\beta} T^\alpha Q \otimes T^\beta Q - (\Lambda_i,\Lambda_i) Q\otimes Q \,, \tag{3.13}$$

and $|v(Q)|^2$ is the K-invariant norm square in the representation normalised such that for $g\in P_i$ we have $|g\Lambda_i|^2 = 1/y_i^2$. The second is the Poincaré sum that generalises the sum over pairs of relative primes in (3.9). Both infinite sums are absolutely convergent for $\mathrm{Re}[s] > \frac{(\Lambda_i,\varrho)}{(\Lambda_i,\Lambda_i)}$ and admit an analytic continuation to a meromorphic function of s over $\mathbb{C}$ [176].

Coming back to the original example of $SL(2,\mathbb{Z})$, one computes in the domain of absolute convergence $\mathrm{Re}[s]>1$ that

$$E^{SL(2)}_s(\tau) = \tau_2^s + \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} + \frac{2\sqrt{\tau_2}}{\xi(2s)} \sum_{n\in\mathbb{Z}}{}' \frac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac12}} K_{s-\frac12}(2\pi|n|\tau_2) e^{2\pi i n\tau_1} \,, \tag{3.14}$$

where $\sigma_s(n) = \sum_{d|n} d^s$ is the divisor sum and K_s is the modified Bessel function of the second kind that behaves asymptotically as $2\sqrt{\tau_2}K_s(2\pi\tau_2) \sim e^{-2\pi\tau_2}$. This Fourier expansion of the Eisenstein series is absolutely convergent and is manifestly analytic in $s\neq 1$, while it admits a simple pole $\frac{3}{\pi(s-1)}$ at $s=1$.

The maximal parabolic Eisenstein series can naturally be interpreted as sum over the 1/2 BPS sates with

$$2\zeta(2s) E^{E_d}_{P_d,s}(v) = \sum_{\substack{\Gamma\in\mathbb{L}_d \\ \Gamma\times\Gamma=0}}{}' \frac{1}{|Z(\Gamma)|^{2s}} \tag{3.15}$$

and $\ell M = |Z(\Gamma)|$ the mass of the particles in the short multiplet [60].

The definition generalises to arbitrary parabolic subgroup $P_{i_1,i_2,\dots i_r}$ associated to the weight $\Lambda_{i_1}+\Lambda_{i_2}+\ldots+\Lambda_{i_r}$ such that

$$E^G_\lambda(v) = \sum_{\gamma\in G(\mathbb{Z})/P_{i_1,i_2,\dots i_r}(\mathbb{Z})} y_{i_1}^{s_1} y_{i_2}^{s_2} \dots y_{i_r}^{s_r}\big|_\gamma \tag{3.16}$$

[13] For short we write the basis vector e_{Λ_i} in the representation space as the highest weight Λ_i itself. This abuse of language would not accommodate easily weights with multiplicity, but we will not need to write them.

where the weight λ is defined with r complex parameters s_i as

$$\lambda = s_{i_1}\Lambda_{i_1} + s_{i_2}\Lambda_{i_2} + \cdots + s_{i_r}\Lambda_{i_r} \ . \tag{3.17}$$

The sum is absolutely convergent if $\mathrm{Re}[(\Lambda_i, \lambda)] > (\Lambda_i, \varrho)$ for all $i = i_1, i_2, \dots i_r$ and admits a meromorphic continuation to the complex value $s \in \mathbb{C}^r$ [176]. Moreover, the Eisenstein series is meromorphic in the weight λ independently of the choice of parabolic, and the limit $s_{i_k} = 0$ for $1 \leq k \leq r$ gives the Eisenstein series for a bigger parabolic. In this way they are all defined from the Borel Eisenstein series

$$E^G_\lambda(v) = \sum_{\gamma \in G(\mathbb{Z})/B(\mathbb{Z})} \prod_{i=1}^{\mathrm{rk}(G)} y_i^{s_i}\big|_\gamma \ . \tag{3.18}$$

In the domain of absolute convergence away from the poles in $\mathbb{C}^{\mathrm{rk}(G)}$ they are eigen-functions of all invariant operators and in particular of the Laplace operator

$$\Delta E^G_\lambda(v) = 2(\lambda, \lambda - \varrho) E^G_\lambda(v) \ . \tag{3.19}$$

They satisfy the Langlands functional relation for any Weyl reflection w [14]

$$E^G_\lambda(v) = \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{\xi((2\lambda - \varrho, \alpha))}{\xi((2\lambda - \varrho, \alpha) + 1)} E^G_{w\lambda + \frac{1}{2}(1-w)\varrho}(v) \tag{3.20}$$

where the product is over all positive roots α reflected to negative ones by w and $\xi(s)$ is the completed Riemann zeta function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{3.21}$$

that satisfies $\xi(1-s) = \xi(s)$. See [176, 177] for an exhaustive exposition involving their definition over the ring of adèles of $\mathbb{Q}$. We shall use extensively the Langlands functional relations above in the next section to relate $SL(d)$ Eisenstein series for different weight. We are very grateful to Axel Kleinschmidt who shared with us a nice program that computes these relations very efficiently.

It may happen that we need to evaluate a Langlands Eisenstein series at a pole. For example for $SL(2)$ one wants to define

$$\widehat{E}^{SL(2)}_1(\tau) = \lim_{\epsilon \to 0}\left(E^{SL(2)}_{1+\epsilon}(\tau) - \frac{\xi(1+2\epsilon)}{\xi(2+2\epsilon)} \right) = -\frac{3}{\pi}\log\big(\tau_2 |\eta(\tau)|^4\big) \ . \tag{3.22}$$

More generally we will need to consider maximal parabolic Eisenstein series that diverge at $s = j$ for some half-integer $j \in \mathbb{Z}/2$. In practice the pole always comes from a Langlands functional relation involving $\xi(1+2\epsilon)$ or $\xi(2\epsilon)$ in the numerator, so for the appropriate Weyl reflection w such that the limit below is finite, we define the renormalised Eisenstein series

$$\widehat{E}^G_{j\Lambda_i} = \lim_{\epsilon \to 0}\left(E^G_{(j+\epsilon)\Lambda_i} - \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{\xi((2(j+\epsilon)\Lambda_i - \varrho, \alpha))}{\xi((2(j+\epsilon)\Lambda_i - \varrho, \alpha) + 1)} E^G_{jw\Lambda_i + \frac{1}{2}(1-w)\varrho} \right) . \tag{3.23}$$

[14] In the literature one defines normally the Eisenstein series in function of the weight $2\lambda - \varrho$ that transforms under the Weyl group by the associated reflection [176]. Because we consider mainly maximal parabolic Eisenstein series we find convenient to label them by $s\Lambda_i$ rather than $2s\Lambda_i - \varrho$.

This Weyl group element w is generally not unique, and the definition of $\widehat{E}^G_{j\Lambda_i}$ depends typically on the choice of w through terms proportional to $E^G_{jw\Lambda_i+\frac{1}{2}(1-w)\varrho}$.

These definitions can be generalised to non-trivial representation ρ_υ of $K \subset G$ according to (3.1) and one can also average a cuspidal form on the Levi subgroup semi-simple component $[L_P, L_P] \subset L_P$

$$E^G_{\lambda,\upsilon,f}(v) = \sum_{\gamma \in G(\mathbb{Z})/P_{i_1,i_2,\dots i_r}(\mathbb{Z})} y_{i_1}^{s_1} y_{i_2}^{s_2} \dots y_{i_r}^{s_r} f(l)\big|_{\gamma,\upsilon} \,. \tag{3.24}$$

We shall not need these generalisations and will only consider so-called spherical parabolic Eisenstein series (with $\upsilon = 0$ and $\rho_\upsilon = 1$) and their derivatives.

An Eisenstein series defines a representation of the group G that is characterised by the action of the universal enveloping algebra acting as in (3.2). One can define accordingly the right ideal $\mathcal{I}_\lambda$ of $\mathcal{U}(\mathfrak{g})$ as the annihilator of E^G_λ. Because we define the Eisenstein series to be spherical, the ideal always includes $\mathfrak{k}$. At generic weight (where all $s_i \neq 0$ and no Weyl image does) the ideal is generated by $\mathfrak{k}$ and the eigen-value equations of the invariant operators in $\mathcal{Z}(\mathfrak{g})$.[15] For larger parabolic, one gets generally a larger ideal. This ideal is described in part by the associated variety $\mathcal{A}_\mathcal{I}$ [178]. We define the standard filtration $\mathcal{U}_n$ of $\mathcal{U}(\mathfrak{g})$ defined by polynomials of order n in $\mathfrak{g}$ and

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathcal{U}_n/\mathcal{U}_{n-1} \,. \tag{3.25}$$

By the Poincaré–Birkhoff–Witt theorem, there is a natural G_{ad}-equivariant isomorphism of graded algebras $\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$, where $S(\mathfrak{g})$ is the algebra of polynomial functions on $\mathfrak{g}^*_\mathbb{C}$. One defines accordingly $\mathrm{Gr}\,\mathcal{I}_\lambda$ as the symbol of the ideal $\mathcal{I}$ in $\mathrm{Gr}\,\mathcal{U}(\mathfrak{g})$ and the associated variety is the set of zeros of $\mathrm{Gr}\,\mathcal{I}_\lambda \subset S(\mathfrak{g})$ in $\mathfrak{g}^*_\mathbb{C}$. The associated variety is the closure of the union of complex nilpotent co-adjoint orbits $\mathcal{O} \in \mathfrak{g}^*_\mathbb{C}$ [179].

The set of nilpotent elements in $\mathfrak{g}^*_\mathbb{C}$ admits a stratified structure into open nilpotent orbits $\mathcal{O} \in \mathfrak{g}^*_\mathbb{C}$ and a partial ordering defined by the inclusion of their closure such that $\mathcal{O}_A \preccurlyeq \mathcal{O}_B$ if $\overline{\mathcal{O}_A} \subset \overline{\mathcal{O}_B}$. One says that an automorphic representation is small if the associated variety of the corresponding annihilator ideal is a small nilpotent orbit. In particular there is a single minimal non-trivial complex orbit $\mathcal{O}_{\mathrm{min}}$ and the associated automorphic representations are called minimal.

For a spherical function $f(g) = f(kg)$ for all $k \in K$, it is natural to define the graded module defined by the action of $\mathcal{U}(\mathfrak{g})$ on the spherical function

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{g})f = \bigoplus_{n=0}^{\infty} \mathcal{U}_n/\mathcal{U}_{n-1}f \,. \tag{3.26}$$

There is then a natural K-equivariant isomorphism of graded modules $\mathrm{Gr}\,\mathcal{U}(\mathfrak{g})f = S(\mathfrak{p})/\mathcal{I}_f$ where $S(\mathfrak{p})$ is the algebra of polynomial in $\mathfrak{p}^*_\mathbb{C} = (\mathfrak{g}_\mathbb{C} \ominus \mathfrak{k}_\mathbb{C})^*$ and $\mathcal{I}_f$ the ideal associated to f. One can define similarly the real associated variety of the function f as the set of zeros of $\mathcal{I}_f \subset S(\mathfrak{p})$ in $\mathfrak{p}^*_\mathbb{C}$. The real associated variety is the closure of the union of complex K orbits in $\mathcal{O} \cap \mathfrak{p}^*_\mathbb{C}$.[16]

[15] Only regularised Eisenstein series at poles define higher-dimensional representations of $\mathcal{Z}(\mathfrak{g})$. For example a regularised Eisenstein series (3.23) would define a two-dimensional representation mixing $\widehat{E}_{j\Lambda_i}$ and $E_{jw\Lambda_i+1/2(1-w)\varrho}$.

[16] We are grateful to Dmitry Gourevitch for explaining this to us.

For a function f defining a coupling protected by supersymmetry, the corresponding ideal $\mathcal{I}_\lambda$ is the set of differential equations that follow from supersymmetry and the ideal $\mathcal{I}_f \in S(\mathfrak{p})$ defines the symbol of these differential operators. The quotient $S(\mathfrak{p})/\mathcal{I}_f$ is then identified with the set of Grassmann analytic polynomials in the scalar superfield in $\mathfrak{p}$. For small representation it appears that $\mathcal{I}_f$ may determine $\mathcal{I}_\lambda$ so that the linearised analysis in supergravity is sufficient to determine the non-linear equations [51, 52].

In the simplest example of an $SL(2,\mathbb{Z})$ Eisenstein series we have $\mathcal{D}\bar{\mathcal{D}}E_s^{SL(2)} = s(s-1)E_s^{SL(2)}$ and

$$\mathcal{U}(\mathfrak{sl}_2)E_s^{SL(2)} \cong E_s^{SL(2)} \oplus \bigoplus_{n=1}^{\infty} \mathcal{D}^n E_s^{SL(2)} \oplus \bigoplus_{n=1}^{\infty} \bar{\mathcal{D}}^n E_s^{SL(2)} \tag{3.27}$$

as a vector space, with the relations

$$\bar{\mathcal{D}}\mathcal{D}^n E_s^{SL(2)} = \big(s(s-1) - n(n-1)\big)\mathcal{D}^{n-1}E_s^{SL(2)} \ , \quad [\mathcal{D},\bar{\mathcal{D}}]\mathcal{D}^n E_s^{SL(2)} = 2n\mathcal{D}^n E_s^{SL(2)} \ . \tag{3.28}$$

The $U(1)$ module $\mathrm{Gr}\,\mathcal{U}(\mathfrak{sl}_2)E_s^{SL(2)}$ forgets the right-hand-sides that are set to zero, so that it is identified with the set of holomorphic plus anti-holomorphic functions of $W \in \mathbb{C}$.

Another way to associate a nilpotent orbit to an automorphic form is through its Fourier coefficients. For a parabolic subgroup $P \subset G$ that decomposes into a Levi subgroup L and a unipotent radical U, the intersection $U(\mathbb{Z}) = U \cap G(\mathbb{Z})$ defines periodicity conditions on the unipotent component of $v \in P$. One decomposes $v = lu$ and $u \in U$ is parametrised by axion fields in physics. The action of $U(\mathbb{Z})$ then corresponds to discrete gauge transformations of the axions. One can define accordingly a Fourier expansion of an automorphic form, generalising (3.14). The abelian Fourier coefficients are defined for characters of U that have support on $U/[U,U]$. One writes $\psi_q(u) = e^{2\pi i q\cdot a}$ the corresponding character, where a is the axion parametrising $U/[U,U]$ and q is a vector in the lattice $\mathfrak{u}_1^*(\mathbb{Z})$ such that $\psi_q(u)$ is invariant under $U(\mathbb{Z})$. One defines the Fourier coefficient

$$\mathcal{F}_q[E_\lambda^G](v) = \int_{U/U(\mathbb{Z})} du\ \overline{\psi_q(u)} E_\lambda^G(vu) \ . \tag{3.29}$$

If the unipotent radical is abelian, one can reconstruct the function from its convergent Fourier expansion

$$E_\lambda^G(v) = \sum_{q\in\mathfrak{u}_1^*(\mathbb{Z})} \mathcal{F}_q[E_\lambda^G](v) = \sum_{q\in\mathfrak{u}_1^*(\mathbb{Z})} E_{\lambda,q}^G(l) e^{2\pi i q\cdot a} \ . \tag{3.30}$$

This generalises to non-abelian unipotent subgroups U as we shall discuss in sections 3.4 and 3.5. As an element in $\mathfrak{u}_1^*(\mathbb{Z}) \subset \mathfrak{g}^*$, q is nilpotent and belongs to a nilpotent co-adjoint orbit. One defines the wave-front set as the union of nilpotent orbits $\mathcal{O}$ for which there exists $q \in \mathcal{O}$ and a parabolic $P = LU$ such that $\mathcal{F}_q[E_\lambda^G](v) \neq 0$. The wave-front set is the closure of the union of nilpotent orbits. For protected couplings in string theory one can interpret the Fourier coefficients as being associated to non-perturbative effects, for example D-brane instantons. The more BPS is the coupling, the smallest is the orbit and more restricted are the D-brane instanton charges [62,63].

In all cases that we shall consider in this text, the wave-front set and the associated variety define the same complex co-adjoint orbits.

It will be convenient to introduce some notations for the nilpotent orbits [178]. Complex nilpotent orbits of $G_{\mathbb{C}}$ are classified by conjugacy classes of homomorphisms $\mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}_{\mathbb{C}}$, i.e. by standard triples (e, f, h) of elements in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ satisfying the commutation relations

$$[e, f] = h \ , \quad [h, e] = 2e \ , \quad [h, f] = -2f \ . \tag{3.31}$$

The $G_{\mathbb{C}}$-orbit of h is determined by a dominant weight $\lambda = n^i \Lambda_i$ with $n^i \in \{0, 1, 2\}$ such that for each root generator $[h, E_\alpha] = (\lambda, \alpha) E_\alpha$. A representative of the nilpotent orbit is a generic element $e = \sum_{(\lambda,\alpha)=2} x_\alpha E_\alpha$ in the vector space of eigen values 2 that admits a unique complex open orbit. We write $\mathcal{O}_\lambda$ the corresponding niplotent $G_{\mathbb{C}}$-orbit. When λ is even, i.e. all $n^i \in \{0, 2\}$ in λ, the intersection of the nilpotent orbit with the nilpotent algebra $\mathfrak{u}$ of $P_\lambda = LU$ is dense in $\mathfrak{u}$. The Eisenstein series of parabolic P_λ at generic values of s admits then as wave-front set the real elements in the closure of the nilpotent orbit $\overline{\mathcal{O}_\lambda} \cap \mathfrak{g}$.

Kostant–Sekiguchi correspondence relates real G-orbits in $\mathfrak{g}$ to $K_{\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathbb{C}} \ominus \mathfrak{k}_{\mathbb{C}}$. The $K_{\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathbb{C}} \ominus \mathfrak{k}_{\mathbb{C}}$ are themselves determined by normal triples (e, f, h) where e and f are in $\mathfrak{g}_{\mathbb{C}} \ominus \mathfrak{k}_{\mathbb{C}}$ and $h \in \mathfrak{k}_{\mathbb{C}}$. They are therefore classified by dominant weights $\upsilon = n^i \Upsilon_i$ of K and we write the real orbit $\mathcal{O}_\upsilon$. We will see that for couplings protected by supersymmetry, the associated variety of the corresponding function is the closure of the real orbit $\mathcal{O}_\upsilon$ for υ the weight defining the harmonic coset space as $K_{\mathbb{C}}/P_\upsilon = K/(K \cap P_\upsilon)$.

We shall now describe the relation between the wave-front set, the associated variety and $SU(8)$ harmonic homogeneous spaces for small orbits of E_7.

3.2 Example of E_7

It will be convenient to consider coordinates ϕ^{μ} for the symmetric space $(SU(8)/\mathbb{Z}_2)\backslash E_7$. For this we decompose $\mathbf{56} = \mathbf{28} \oplus \overline{\mathbf{28}}$ to write the 56 by 56 E_7 matrices in the fundamental representation. The indices $i = 1$ to 8 correspond to the fundamental of $SU(8)$. One defines the Maurer–Cartan form

$$d\mathcal{V}\,\mathcal{V}^{-1} = \begin{pmatrix} 2\delta^{[k}_{[i}\omega^{l]}{}_{j]} & P_{ijkl} \\ P^{ijkl} & -2\delta^{[i}_{[k}\omega^{j]}{}_{l]} \end{pmatrix} \ , \tag{3.32}$$

with $\omega^i{}_j$ the $\mathfrak{su}(8)$ component written as an 8 by 8 matrix and

$$P^{ijkl} = \frac{1}{24}\varepsilon^{ijklpqrs} P_{pqrs} \ , \tag{3.33}$$

the component in $\mathfrak{e}_7 \ominus \mathfrak{su}(8)$. The metric on $(SU(8)/\mathbb{Z}_2)\backslash E_7$ is defined as

$$G_{\mu\nu}(\phi) d\phi^{\mu} d\phi^{\nu} = \frac{1}{3} P_{ijkl} P^{ijkl} \ , \tag{3.34}$$

and the derivative in tangent frame is defined such that for any function

$$d\mathcal{E} = 3P^{ijkl}\mathcal{D}_{ijkl}\mathcal{E} \ . \tag{3.35}$$

For a spherical function $f(g) = f(kg)$ on E_7, one has

$$[\mathcal{D}^{ijkl}, \mathcal{D}_{pqrs}]\mathcal{D}_{tuvw} f(\phi) = -24\delta^{ijkl}_{qrs][t}\mathcal{D}_{uvw][p} f(\phi) + 3\delta^{ijkl}_{pqrs}\mathcal{D}_{tuvw} f(\phi) \ , \tag{3.36}$$

and the Laplace operator is

$$\Delta f(\phi) = \frac{1}{3}\mathcal{D}^{ijkl}\mathcal{D}_{ijkl}f(\phi) \ . \tag{3.37}$$

The higher order Casimir operators can be defined similarly as

$$\Delta_n f(\phi) = \frac{1}{3}\mathrm{tr}_{\mathbf{28}}[(\bar{\mathcal{D}}\mathcal{D})^n]f(\phi) = \frac{1}{3}\mathcal{D}^{ijkl}\mathcal{D}_{klpq}\ldots\mathcal{D}^{mnrs}\mathcal{D}_{rsij}f(\phi) \tag{3.38}$$

for $n = 3, 4, 5, 6, 7, 9$. They correspond to $\frac{1}{6}$ of the trace of Q^{2n} in the fundamental representation, which can be checked to define a basis for the linearly independent Casimirs.

Fourier support

Consider a spherical function $f(g)$ expanded in Fourier coefficients in the abelian parabolic P_7. We write $g = ve^{a\cdot E}$ the representive of E_7 in the abelian parabolic with $v \in GL(1)\times E_6$ and $a \in \mathbb{R}^{27}$ the axions, such that

$$f(g) = \sum_{q\in\mathbb{Z}^{27}} f_q(v)e^{2\pi i q\cdot a} \ . \tag{3.39}$$

The wave-front set of $f(g)$ determines for which E_6 orbit of q this function is non-vanishing. The associated variety determines the set of differential operators in the enveloping algebra $\mathcal{U}(\mathfrak{e}_7)$ that annihilate it. In this subsection we will show that they are identical for small spherical representations.

According to [180, 181] one has the E_6 orbits of charge $q \in \mathbb{Z}^{27}$

$$\begin{aligned} q &= 0\,, && \dim = 0\,,\\ q\times q &= 0\,, && \dim = 17\,,\\ \det q &= 0\,, && \dim = 26\,,\\ \det q &\neq 0\,, && \dim = 27\,. \end{aligned} \tag{3.40}$$

If $f_q(v) = 0$ for all q the function is in the trivial representation, i.e. it is constant. If $f_q(v) = 0$ for all q such that the Jordan product $q\times q \neq 0$ the function is in the minimal representation. If $f_q(v) = 0$ for all q with $\det q \neq 0$ then these Fourier coefficients are in the next to minimal orbit and the wave-front set must not include the nilpotent orbit $\mathcal{O}_{2\Lambda_7}$. It follows that the wave-front set is then included in the closure of the nilpotent orbit $\mathcal{O}_{2\Lambda_1}$ according to the Hasse diagram 12.

In supergravity, the parabolic P_7 corresponds to the symmetry that is manifest in the large circle radius limit in which one recovers five-dimensional supergravity. The radius gives the $GL(1)$ modulus, the scalar fields $(Sp(4)/\mathbb{Z}_2)\backslash E_6$, and the vector fields along the circle the axions a. These Fourier coefficients can be interpreted physically as instantons associated to BPS Euclidean black hole of electric charge $q \in \mathbb{Z}^{27}$ wrapping the large circle [182]. According to [180], the constraint $q\times q = 0$ is satisfied for 1/2 BPS instantons and $\det q \neq 0$ for 1/4 BPS instantons.

Using

$$\begin{aligned} \frac{\partial}{\partial x}\times\frac{\partial}{\partial x}f(e^{x\cdot E}g)\Big|_{x=0} &= \frac{1}{2}t^{abc}\frac{\partial}{\partial x^b}\frac{\partial}{\partial x^c}f(e^{x\cdot E}g)\Big|_{x=0}\\ &= \frac{1}{2}t^{abc}\frac{\partial}{\partial x^b}\frac{\partial}{\partial x^c}f(ve^{x\cdot v^{-1}Ev+a\cdot E})\Big|_{x=0}\\ &= (2\pi i)^2\sum_{q\in\mathbb{Z}^{27}} v^{-1\intercal}(q\times q)f_q(v)e^{2\pi i q\cdot a}\ , \end{aligned} \tag{3.41}$$

one obtains that for a function $f^{\min}$ in the minimal representation, such that $f^{\min}_q(v) = 0$ for all q with non-vanishing Jordan product $q \times q \neq 0$, one has the differential equation

$$\frac{\partial}{\partial a} \times \frac{\partial}{\partial a} f^{\min}(g) = 0 \,. \tag{3.42}$$

Here we define $a = 1$ to 27 and switch from a notation with explicit indices to a notation without. Because $f^{\min}(g) = f^{\min}(kg\gamma)$ for all $k \in SU(8)$ and $\gamma \in E_7(\mathbb{Z})$, one obtains that all the differential operators in the $SU(8)$ orbit of $\frac{\partial}{\partial a} \times \frac{\partial}{\partial a}$ must annihilate the function $f^{\min}(g)$. This may be easier to understand using a partially fixed gauge for the scalar fields, enforcing $g = ve^{a\cdot E} \in P_7$. Acting on the right with $\gamma \in E_7(\mathbb{Z})$ and on the left with $k(g,\gamma)$ such that $k(g,\gamma)g\gamma \in P_7$, one obtains that $\frac{\partial}{\partial a} \times \frac{\partial}{\partial a} f(g) = k(g,\gamma) \cdot \frac{\partial}{\partial a} \times \frac{\partial}{\partial a} f(g)$ for any such $k(g,\gamma)$. It is easy to convince oneself that one can find enough $\gamma \in E_7(\mathbb{Z})$ to span all the irreducible $Sp(4)$ representations that appear in branching the irreducible representation $R(\Upsilon_2{+}\Upsilon_6)$ of $SU(8)$, see [51, Eq. 4.31-4.35].[17] We conclude that $f^{\min}_q = 0$ for all q satisfying $q \times q = 0$ implies

$$\Big(28\mathcal{D}_{ijpq}\mathcal{D}^{klpq} - 3\delta^{kl}_{ij}\Delta\Big) f^{\min} = 0 \,. \tag{3.43}$$

In this case the differential equation determines the functional form of the Fourier coefficients with $q \times q = 0$ as [62, 51]

$$f^{\min}_q = 2\mu(q)\frac{(\det v)^2}{|v(q)|^3}(1 + 2\pi|v(q)|)e^{-2\pi|v(q)|} \,. \tag{3.44}$$

The cubic determinant gives in the same way

$$\begin{aligned} \det \frac{\partial}{\partial x} f(e^{x\cdot E}g)\Big|_{x=0} &= \frac{1}{6}t^{abc}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\frac{\partial}{\partial x^c} f(e^{x\cdot E}g)\Big|_{x=0} \\ &= \frac{1}{6}t^{abc}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\frac{\partial}{\partial x^c} f(ve^{x\cdot v^{-1}Ev + a\cdot E})\Big|_{x=0} \\ &= (2\pi i)^3 \det v \sum_{q\in\mathbb{Z}^{27}} \det(q) f_q(v) e^{2\pi i q\cdot a} \,. \end{aligned} \tag{3.45}$$

For a function associated to an orbit in the closure of $\mathcal{O}_{2\Lambda_1}$ we have therefore

$$\det \frac{\partial}{\partial x} f^{\text{n-min}}(e^{x\cdot E}g)\Big|_{x=0} = 0 \,. \tag{3.46}$$

We want to identify the $SU(8)$ module in which this differential operator belongs, i.e. the set of differential equations generated by $E_7(\mathbb{Z})$ on $\det \frac{\partial}{\partial x} f^{\text{n-min}}(e^{x\cdot E}k(g,\gamma)g\gamma) = 0$. Let us write a basis of all third order differential operators acting on a spherical function. The third order differential operators can be decomposed in the irreducible representation in the symmetric tensor product of three $\mathfrak{p} = R(\Upsilon_4)$. One finds three rank four tensors

$$[\mathcal{D}^3_{2\Upsilon_2}]_{ij,kl} \,, \qquad [\mathcal{D}^3_{2\Upsilon_6}]_{ij,kl} \,, \qquad [\mathcal{D}^3_{\Upsilon_4}]_{ijkl} \tag{3.47}$$

with

$$[\mathcal{D}^3_{2\Upsilon_2}]_{ij,kl} = \mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} - \mathcal{D}_{ijkl}\big(\tfrac{1}{4}\Delta + 6\big) \,, \tag{3.48}$$

[17] This is the analogue of the the result that a modular form of $SL(2,\mathbb{Z})$ with all Fourier coefficients zero must be constant. Assuming $\partial_{\tau_1} f(\tau) = 0$ on a function invariant under $\tau \to -\frac{1}{\tau}$ implies $\partial_{\tau_2} f(\tau) = 0$.

and

$$[\mathcal{D}^3_{\Upsilon_4}]_{ijkl} = \mathcal{D}_{ijkl}\Delta \,, \tag{3.49}$$

one rank six tensor

$$[\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^{i} = \mathcal{D}_{jr[kl}\mathcal{D}^{irmn}\mathcal{D}_{pq]mn} - \tfrac{1}{36}\delta^i_j\mathcal{D}_{klpq}(\Delta+42) + \tfrac{1}{36}\delta^i_{[k}\mathcal{D}_{lpq]j}(\Delta-120) \,, \tag{3.50}$$

one rank eight and one rank twelve tensor

$$[\mathcal{D}^3_{\Upsilon_2+\Upsilon_4+\Upsilon_6}]_{ij,klpq}{}^{rs} \,, \qquad [\mathcal{D}^3_{3\Upsilon_4}]_{ijkl,pqrs,tuvw} \,. \tag{3.51}$$

These differential operators are defined on a spherical function, and the terms linear in $\mathcal{D}_{ijkl}$ are determined such that they are in the corresponding irreducible representations of $SU(8)$. These representations do not appear in the symmetric tensor product of two $\mathfrak{p} = R(\Upsilon_4)$ by property of E_7 because $\mathfrak{p}$ is odd under the central element $e^{\frac{i\pi}{4}}$ of $SU(8)$. At first order one gets back the first derivative $\mathcal{D}_{ijkl}$.

To identify the differential operator, one can check the common E_7 module in the symmetric tensor product of three adjoints. The only E_7 representation including the $3\Upsilon_4$ is $3\Lambda_1$. Branching $3\Lambda_1$ under E_6 one concludes that it contains the $3\Lambda_1$ of E_6, i.e.

$$[\mathcal{D}^3_{3\Upsilon_4}]_{ijkl,pqrs,tuvw} f(g) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\frac{\partial}{\partial x^c} f(e^{x\cdot E}g)\Big|_{x=0} = 0 \tag{3.52}$$

and so the function satisfying this equation must be constant. Branching $\Lambda_1+\Lambda_3$, one gets $\Upsilon_2+\Upsilon_4+\Upsilon_6$ of $SU(8)$ and $\Lambda_1+\Lambda_6$ of E_6, i.e. that

$$[\mathcal{D}^3_{\Upsilon_2+\Upsilon_4+\Upsilon_6}]_{ij,klpq}{}^{rs} f(g) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^a} t^{bcd}\frac{\partial}{\partial x^c}\frac{\partial}{\partial x^d} f(e^{x\cdot E}g)\Big|_{x=0} = 0 \tag{3.53}$$

so that a function satisfying this equation must be in the minimal representation. The only remaining E_7 representation that includes a cubic term in the **27** of degree 2 is the $2\Lambda_7$, and this is precisely the determinant. It is easy to check that the matrix $\mathcal{Q} = \mathfrak{p}$ in the $\mathbf{56} = R(\Lambda_7)$ representation satisfies

$$\begin{pmatrix} 0 & \mathcal{Q}_{ijkl} \\ \mathcal{Q}^{ijkl} & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & \mathcal{Q}_{ijpq}\mathcal{Q}^{pqrs}\mathcal{Q}_{rskl} \\ \mathcal{Q}^{ijpq}\mathcal{Q}_{pqrs}\mathcal{Q}^{rskl} & 0 \end{pmatrix} \tag{3.54}$$

while $(q\cdot E)^3$ in the **56** gives a term proportional to $\det q$.

One concludes that the $SU(8)$ modules of differential operators that include $\det\frac{\partial}{\partial x}$ are $2\Upsilon_2$ and $2\Upsilon_6$. $2\Upsilon_2$ and $2\Upsilon_6$ are complex conjugate to each others, so for a spherical function the two corresponding equations must be satisfied. We conclude therefore that the following conditions are equivalent

$$f(g) = \sum_{\substack{q\in\mathbb{Z}^{27}\\ \det q=0}} f_q(v)e^{2\pi i q\cdot a} \qquad \Leftrightarrow \qquad [\mathcal{D}^3_{2\Upsilon_2}]_{ij,kl} f(g) = [\mathcal{D}^3_{2\Upsilon_6}]^{ij,kl} f(g) = 0 \,. \tag{3.55}$$

Let us now consider the Heisenberg parabolic $P_1 = GL(1)\times Spin(6,6)\ltimes\mathbb{R}^{32+1}$ relevant in string perturbation theory. In this case the effective string coupling constant e^{ϕ_6} parametrises

$GL(1)$, the Narain moduli $SO(6,6)$, the Ramond-Ramond fields along T^6 parametrise $\mathbb{R}^{32}$ and the Kalb-Ramond axion the central part of the Heisenberg algebra. According to [181], the abelian Fourier coefficients in P_1 are associated to unipotent characters of charge $Q \in S_- = \mathbb{Z}^{32}$ of vanishing cubic product in $\mathbb{Z}^{32}$ for a function in the next to minimal representation. This is more generally the case for any function in the representation associated to the closure of $\mathcal{O}_{2\Lambda_7}$. Indeed, there are five $Spin(6,6)$ orbits of Majorana-Weyl spinors [183]

$$\begin{aligned} Q &= 0\,, && \dim = 0\,,\\ Q\gamma^{ab}Q &= 0\,, && \dim = 16\,,\\ (Q\gamma^{ab}Q)\gamma_{ab}Q &= 0\,, && \dim = 25\,,\\ 2(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) &= 0\,, && \dim = 31\,,\\ 2(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) &\neq 0\,, && \dim = 32\,. \end{aligned} \tag{3.56}$$

Physically, these Fourier coefficients are associated to Euclidean D-brane instantons. The 1/2 BPS instantons satisfy $(Q\gamma_2 Q) = 0$ and the 1/4 BPS brane instantons $(Q\gamma_2 Q)\cdot\gamma_2 Q = 0$. We write $g = ve^{a\cdot E}e^{bE}$ the representative of E_7 in the Heisenberg parabolic, with b the Kalb-Ramond axion and a^α the Ramond-Ramond spinor. The Abelian Fourier expansion reads

$$\int_{[0,1]} db\; f(g) = \sum_{Q\in\mathbb{Z}^{32}} f_Q(v) e^{2\pi i Q\cdot a}\,. \tag{3.57}$$

We write similarly

$$t^{\alpha\beta\gamma\delta}\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial x^\delta}\int_{[0,1]} db\; f(e^{x\cdot E}g)\Big|_{x=0} = (2\pi i)^3 \sum_{Q\in\mathbb{Z}^{32}} v[(Q\gamma_2 Q)\cdot\gamma_2 Q]^\alpha\, f_Q(v) e^{2\pi i Q\cdot a} \tag{3.58}$$

Therefore this differential operator annihilates a function in the next to minimal representation.

If we write $\mathcal{Q} = Q\cdot E$, the condition $(Q\gamma_2 Q)\cdot\gamma_2 Q = 0$ is equivalent to having $\mathcal{Q}^3_{\mathbf{133}} = 0$ in the adjoint representation. The components of $\mathcal{Q}^3_{\mathbf{133}} = 0$ are in the symmetric tensor product $\mathrm{Sym}^3 R(\Lambda_1)$ as a cubic monomial and in $R(\Lambda_1)\otimes R(\Lambda_1)$ as a adjoint representation matrix. The intersection gives the two representations $R(\Lambda_1)$ and $R(\Lambda_3)$. The component in $R(\Lambda_1)$ vanishes for any nilpotent element so $(Q\gamma_2 Q)\cdot\gamma_2 Q \in R(\Lambda_3)$ of E_7. Branching $R(\Lambda_3)$ under $SU(8)$ one obtains that

$$R(\Lambda_3)\cap \mathrm{Sym}^3 R(\Upsilon_4) = R(\Upsilon_1{+}\Upsilon_4{+}\Upsilon_7)\,, \tag{3.59}$$

corresponding to the differential operator [52]

$$[\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^i = \mathcal{D}_{jr[kl}\mathcal{D}^{irmn}\mathcal{D}_{pq]mn} - \tfrac{1}{36}\delta^i_j\mathcal{D}_{klpq}(\Delta+42) + \tfrac{1}{36}\delta^i_{[k}\mathcal{D}_{lpq]j}(\Delta-120)\,, \tag{3.60}$$

satisfying

$$[\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{[j,klpq]}{}^i = 0\,,\quad [\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^j = 0\,,\quad [\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^q = 0\,. \tag{3.61}$$

We conclude that

$$\int_{[0,1]} db\; f(g) = \sum_{\substack{q\in\mathbb{Z}^{32}\\ (Q\gamma_2 Q)\cdot\gamma_2 Q=0}} f_Q(v) e^{2\pi i Q\cdot a} \qquad\Leftrightarrow\qquad [\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^i \int_{[0,1]} db\; f(g) = 0\,. \tag{3.62}$$

For an automorphic form this should imply that the complete function $f(g)$ satisfies

$$[\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^i f(g) = 0 \,. \tag{3.63}$$

One may then consider the parabolic $P_2 = GL(7) \ltimes \mathbb{R}^{35+7}$ relevant in the large T^7 volume limit in eleven-dimensional supergravity. The group $GL(7)$ is parametrised by the metric on T^7, $\mathbb{R}^{35}$ by the three-form gauge field along T^7 and $\mathbb{R}^7$ by the six-form potential along T^7. The generic abelian Fourier coefficient in P_2 corresponds to the nilpotent orbit $\mathcal{O}_{2\Lambda_2}$ [181], and there are ten distinct $SL(7)$ orbits corresponding to the ten nilpotent orbits in the closure of $\mathcal{O}_{2\Lambda_2}$

$$\begin{aligned}
N^{IJK} &= 0\,, && \dim = 0\,, \\
N^{[I_1I_2I_3}N^{I_4I_5]J} &= 0\,, && \dim = 13\,, \\
N^{I[K_1K_2}N^{K_3K_4K_5}N^{K_6K_7]J} &= 0\,, && N^{I_1I_2][K_1}N^{K_2K_3K_4}N^{K_5K_6][I_3} = 0\,, \quad \dim = 20\,, \\
N^{I_1I_2][K_1}N^{K_2K_3K_4}N^{K_5K_6][I_3} &= 0\,, && \dim = 21\,, \\
N^{I[K_1K_2}N^{K_3K_4K_5}N^{K_6K_7]J} &= 0\,, && N^{I_4I_5I_6}N^{I_1I_2][K_1}N^{K_2K_3K_4}N^{K_5K_6][I_3} = 0\,, \quad \dim = 25\,, \\
N^{I[K_1K_2}N^{K_3K_4K_5}N^{K_6K_7]J} &= 0\,, && \dim = 26\,, \\
N^{I_4I_5I_6}N^{I_1I_2][K_1}N^{K_2K_3K_4}N^{K_5K_6][I_3} &= 0\,, && \dim = 28\,, \\
N^{J_1J_2J_3}N^{J_4][I_1I_2}N^{I_3I_4][K_1}N^{|J_5J_6|K_2}N^{K_3K_4K_5}N^{K_6K_7][J_7} &= 0\,, && \dim = 31\,, \\
N^{J_1J_2J_3}N^{J_4][I_1I_2}N^{I_3I_4I_5}N^{I_6I_7][K_1}N^{|J_5J_6|K_2}N^{K_3K_4K_5}N^{K_6K_7][J_7} &= 0\,, && \dim = 34\,, \\
N^{J_1J_2J_3}N^{J_4][I_1I_2}N^{I_3I_4I_5}N^{I_6I_7][K_1}N^{|J_5J_6|K_2}N^{K_3K_4K_5}N^{K_6K_7][J_7} &\neq 0\,, && \dim = 35\,.
\end{aligned} \tag{3.64}$$

One finds in this case that the constraint $(N^3)^{IJ} = 0$ above with solution of dimension 26 corresponds to equation (3.55) and the constraint $(N^3)^{I_1I_2I_3}{}_K = 0$ with solution of dimension 21 corresponds to equation (3.63). More generally one finds

$$\begin{aligned}
\int_{[0,1]} db\, f(g) &= \sum_{\substack{N\in\wedge^3\mathbb{Z}^7 \\ N^{2I_1I_2I_3I_4I_5,J}=0}} f_N(v) e^{2\pi i N\cdot a} && \Leftrightarrow && [\mathcal{D}^2_{\Upsilon_2+\Upsilon_6}]_{ij}{}^{kl} \int_{[0,1]} db\, f(g) = 0\,, \\
\int_{[0,1]} db\, f(g) &= \sum_{\substack{N\in\wedge^3\mathbb{Z}^7 \\ N^{3\,IJ}=0}} f_N(v) e^{2\pi i N\cdot a} && \Leftrightarrow && [\mathcal{D}^3_{2\Upsilon_2}]_{ij,kl} \int_{[0,1]} db\, f(g) = 0\,, \\
\int_{[0,1]} db\, f(g) &= \sum_{\substack{N\in\wedge^3\mathbb{Z}^7 \\ N^{3\,I_1I_2I_3}{}_K=0}} f_N(v) e^{2\pi i N\cdot a} && \Leftrightarrow && [\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^i \int_{[0,1]} db\, f(g) = 0\,, \\
\int_{[0,1]} db\, f(g) &= \sum_{\substack{N\in\wedge^3\mathbb{Z}^7 \\ N^4_{IJ}=0}} f_N(v) e^{2\pi i N\cdot a} && \Leftrightarrow && [\mathcal{D}^4_{2\Upsilon_1+2\Upsilon_7}]_{i,j}{}^{k,l} \int_{[0,1]} db\, f(g) = 0\,, \\
\int_{[0,1]} db\, f(g) &= \sum_{\substack{N\in\wedge^3\mathbb{Z}^7 \\ N^6_{I_1I_2I_3}=0}} f_N(v) e^{2\pi i N\cdot a} && \Leftrightarrow && [\mathcal{D}^6_{3\Upsilon_1+\Upsilon_5}]_{i,j,k}{}^{lpq} \int_{[0,1]} db\, f(g) = 0\,, \\
\int_{[0,1]} db\, f(g) &= \sum_{\substack{N\in\wedge^3\mathbb{Z}^7 \\ N^7=0}} f_N(v) e^{2\pi i N\cdot a} && \Leftrightarrow && [\mathcal{D}^7_{4\Upsilon_1}]_{i,j,k,l} \int_{[0,1]} db\, f(g) = 0\,.
\end{aligned} \tag{3.65}$$

We will now describe the integrability conditions for these differential equations.

Integrability conditions

• Let us first review the case of the minimal representation in these conventions. The absence of irreducible representation in the Joseph ideal $\mathcal{I}^2_{\Lambda_1} = R(\Upsilon_2+\Upsilon_6)$ at the second oder, implies for a spherical function

$$\Big(28\mathcal{D}_{ijpq}\mathcal{D}^{klpq} - 3\delta^{kl}_{ij}\Delta\Big)f^{\min} = 0 \ . \tag{3.66}$$

We write the projector

$$[\mathcal{D}^2_{\Upsilon_2+\Upsilon_6}]_{ij}{}^{kl} = \mathcal{D}_{ijpq}\mathcal{D}^{klpq} - \tfrac{3}{28}\delta^{kl}_{ij}\Delta \ , \tag{3.67}$$

such that $[\mathcal{D}^2_{\Upsilon_2+\Upsilon_6}]_{ik}{}^{jk} = 0$. Clearly all the higher order Casimir operators eigenvalues are determined from the quadratic one since they can be defined as traces of the operator $\mathcal{D}_{ijpq}\mathcal{D}^{klpq}$ to higher powers. We have in this way

$$\Delta_n f^{\min} = (\tfrac{3}{28})^{n-1}\Delta^n f^{\min} \ . \tag{3.68}$$

The third order ideal component in $R(2\Upsilon_2)$ implies moreover

$$\Big(4\mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} - \mathcal{D}_{ijkl}\big(\Delta+24\big)\Big)f^{\min} = 0 \ , \tag{3.69}$$

where

$$[\mathcal{D}^3_{2\Upsilon_2}]_{ij,kl} = \mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} - \mathcal{D}_{ijkl}\big(\tfrac{1}{4}\Delta+6\big) \ . \tag{3.70}$$

One computes indeed using (3.36) that

$$18\mathcal{D}_{pq[ij}\mathcal{D}^{pqrs}\mathcal{D}_{kl]rs} = \frac{9}{2}\mathcal{D}_{ijkl}\Delta + 108\mathcal{D}_{ijkl} \ . \tag{3.71}$$

Now using moreover (3.66) one obtains that

$$\Big(4\mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} - \mathcal{D}_{ijkl}\big(\Delta+24\big)\Big)f^{\min} = -16\,\mathcal{D}_{ijkl}\big(\Delta+42\big)f^{\min} = 0 \ , \tag{3.72}$$

so either $f^{\min}$ is constant and in the trivial representation, or $f^{\min}$ is an eigen-function of the Laplacian with eigenvalue -42. The spherical representation is therefore unique and is determined by the Joseph ideal equation

$$\mathcal{D}_{ijpq}\mathcal{D}^{klpq}f^{\min} = -\frac{9}{2}\delta^{kl}_{ij}f^{\min} \ . \tag{3.73}$$

• For the next to minimal we write the two cubic ideals, the one in $R(2\Upsilon_2)$

$$\Big(4\mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} - \mathcal{D}_{ijkl}\big(\Delta+24\big)\Big)f^{\text{n-min}} = 0 \ , \tag{3.74}$$

and the ideal (3.60) in the $R(\Upsilon_1+\Upsilon_4+\Upsilon_7)$

$$\Big(36\mathcal{D}_{jr[kl}\mathcal{D}^{irmn}\mathcal{D}_{pq]mn} - \delta^i_j\mathcal{D}_{klpq}(\Delta+42) + \delta^i_{[k}\mathcal{D}_{lpq]j}(\Delta-120)\Big)f^{\text{n-min}} = 0 \ . \tag{3.75}$$

Using again (3.36) one can act with an additional derivative on (3.74) and use (3.75) to obtain

$$\mathcal{D}^{ijpq}\Big(4\mathcal{D}_{pqrs}\mathcal{D}^{rsmn}\mathcal{D}_{mnkl}-\mathcal{D}_{pqkl}\big(\Delta+24\big)\Big)f^{\text{n-min}}=\frac{1}{12}\Big(28\mathcal{D}^{ijpq}\mathcal{D}_{klpq}-3\delta^{ij}_{kl}\Delta\Big)\big(\Delta+60\big)f^{\text{n-min}} \tag{3.76}$$

We assume that $\Big(28\mathcal{D}^{ijpq}\mathcal{D}_{klpq}-3\delta^{ij}_{kl}\Delta\Big)f^{\text{n-min}}\neq 0$ because otherwise we would be back to the minimal or the trivial representation, so we obtain that $f^{\text{n-min}}$ is an eigen-function of the Laplacian with eigenvalue -60. The higher order Casimir are determined using (3.74) as

$$\Delta_n f^{\text{n-min}}=-60(-9)^{n-1}f^{\text{n-min}}\ . \tag{3.77}$$

Associated variety and harmonic superspace

The one-half BPS Grassmann analytic structure defined in (1.111) is associated to the weight Υ_4 of $SU(8)$ that labels the minimal $SU(8)$ nilpotent orbit $\mathcal{O}_{\Upsilon_4}$ in $\mathfrak{p}_{\mathbb{C}}=\mathfrak{e}_7(\mathbb{C})\ominus\mathfrak{sl}(8,\mathbb{C})$ [178]. Defining the normal triple $(h_{\Upsilon_4},e_{\Upsilon_4},f_{\Upsilon_4})$, the Grassmann analytic superfield W is precisely defined in (1.111) to be the component along $e_{\Upsilon_4}\in\mathfrak{p}_{\mathbb{C}}$ of W^{ijkl} and the harmonic variables parametrise the $SU(8)$ orbit of $h_{\Upsilon_4}\in\mathfrak{su}(8)$. As such, W is in the minimal nilpotent orbit $\mathcal{O}_{\Upsilon_4}$ and the set of polynomials in W is isomorphic to $\text{Gr}\,\mathcal{U}(\mathfrak{e}_7)f^{\text{min}}$ as an $SU(8)$ submodule $S(\mathfrak{p}^*_{\mathbb{C}})$ of the polynomial functions in W^{ijkl}, i.e.

$$\text{Gr}\,\mathcal{U}(\mathfrak{e}_7)f^{\text{min}}\cong\bigoplus_{n=0}^{\infty}R(n\Upsilon_4)\ . \tag{3.78}$$

We find therefore that supersymmetry implies equations (3.66) and (3.69) and so by integrability equation (3.73).

The one-quarter BPS Grassmann analytic structure defined in (1.114) is associated to the weight $\Upsilon_2+\Upsilon_6$ of $SU(8)$ that labels the next-to-minimal $SU(8)$ nilpotent orbit $\mathcal{O}_{\Upsilon_2+\Upsilon_6}$ in $\mathfrak{p}_{\mathbb{C}}$ [178]. Defining the normal triple $(h_{\Upsilon_2+\Upsilon_6},e_{\Upsilon_2+\Upsilon_6},f_{\Upsilon_2+\Upsilon_6})$, the Grassmann analytic superfield W^{rs} defined in (1.114) is in the weight 2 vector space and therefore includes the component along $e_{\Upsilon_2+\Upsilon_6}\in\mathfrak{p}_{\mathbb{C}}$ of W^{ijkl} and the harmonic variables parametrise the $SU(8)$ orbit of $h_{\Upsilon_2+\Upsilon_6}\in\mathfrak{su}(8)$. As such, W^{rs} is in the next-to-minimal nilpotent orbit $\mathcal{O}_{\Upsilon_2+\Upsilon_6}$ and the set of polynomials in W^{rs} is isomorphic to $\text{Gr}\,\mathcal{U}(\mathfrak{e}_7)f^{\text{n-min}}$ as an $SU(8)$ submodule of $S(\mathfrak{p}^*_{\mathbb{C}})$

$$\text{Gr}\,\mathcal{U}(\mathfrak{e}_7)f^{\text{n-min}}\cong\bigoplus_{n,m\geq 0}R(m\Upsilon_2+n\Upsilon_4+m\Upsilon_6)\ . \tag{3.79}$$

We find therefore that supersymmetry implies the equations (3.74) and (3.75) and so by integrability equation (3.77).

The one-eight BPS Grassmann analytic structure of type $(1/8,1/8)$ defined in (1.117) is associated to the weight $2\Upsilon_1+2\Upsilon_7$ of $SU(8)$ that labels the $SU(8)$ nilpotent orbit $\mathcal{O}_{2\Upsilon_1+2\Upsilon_7}$ in $\mathfrak{p}_{\mathbb{C}}$ [178]. This real orbit is inside the even complex orbit $\mathcal{O}_{2\Lambda_1}$ associated to the Heisenberg parabolic P_1. The complex orbit $\mathcal{O}_{2\Lambda_1}$ includes two real orbits, $\mathcal{O}_{2\Upsilon_1+2\Upsilon_7}$ and $\mathcal{O}_{2\Upsilon_4}$. The closure of their union $\overline{\mathcal{O}_{2\Lambda_1}}\cup\overline{\mathcal{O}_{2\Upsilon_4}}$ is expected to define the associated variety of the Heisenberg parabolic Eisenstein series $E^{E_7}_{s\Lambda_1}$ at generic values of $s\in\mathbb{C}$. Defining the normal triple $(h_{2\Upsilon_1+2\Upsilon_7},e_{2\Upsilon_1+2\Upsilon_7},f_{2\Upsilon_1+2\Upsilon_7})$, the

Grassmann analytic superfield W^{rst} defined in (1.117) is in the weight 2 vector space and therefore includes the component along $e_{2\Upsilon_1+2\Upsilon_7} \in \mathfrak{p}_{\mathbb{C}}$ of W^{ijkl} and the harmonic variables parametrise the $SU(8)$ orbit of $h_{2\Upsilon_1+2\Upsilon_7} \in \mathfrak{su}(8)$. As such, W^{rst} is in the nilpotent orbit $\mathcal{O}_{2\Upsilon_1+2\Upsilon_7}$ and the set of polynomials in W^{rst} is isomorphic to the $SU(8)$ submodule of $S(\mathfrak{p}^*_{\mathbb{C}})$

$$\begin{aligned} & S(\mathfrak{p}^*_{\mathbb{C}})|_{\mathcal{O}_{2\Upsilon_1+2\Upsilon_7}} \\ \cong & \bigoplus_{n_1,n_2,n_3,n_4,n_4'\geq 0} R((n_3{+}n_4{+}2n_4')\Upsilon_1{+}n_2\Upsilon_2{+}n_4\Upsilon_3{+}(n_1{+}n_3)\Upsilon_4{+}n_4\Upsilon_5{+}n_2\Upsilon_6{+}(n_3{+}n_4{+}2n_4')\Upsilon_7) \ . \end{aligned} \tag{3.80}$$

The module $\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{s\Lambda_1}$ also includes polynomial with support on $\mathcal{O}_{2\Upsilon_4}$ (see Appendix C.2)

$$S(\mathfrak{p}^*_{\mathbb{C}})|_{\mathcal{O}_{2\Upsilon_4}} = \bigoplus_{n_1,n_2,n_3,n_4\geq 0} R(n_3\Upsilon_1{+}n_2\Upsilon_2{+}(n_1{+}n_3{+}2n_4)\Upsilon_4{+}n_2\Upsilon_6{+}n_3\Upsilon_7) \tag{3.81}$$

but this does not give any new $SU(8)$ representation. The complete module $\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{s\Lambda_1}$ only admits $SU(8)$ irreducible representations that appear in $S(\mathfrak{p}^*_{\mathbb{C}})|_{\mathcal{O}_{2\Upsilon_1+2\Upsilon_7}}$ as suggested by linearised supersymmetry. The multiplicities of the representations are different, and in particular the irreducible representation $R(2\Upsilon_4)$ appears with two distinct derivative of the function. In general the associate variety associated to a spherical automorphic representation typically includes the union of all real orbits in the same complex orbit. We can still conclude that supersymmetry implies equation (3.74). This analysis does not allow to determine the eigen-value of the Laplacian, although we know that supersymmetry fixes it to be -60. We obtain that the abelian Fourier coefficients in the decompactification limit are not generic according to (3.55).

The other one-eight BPS Grassmann analytic structure of type $(1/4, 0)$ defined in (1.119) is associated to the weight $2\Upsilon_2$ of $SU(8)$ that labels the $SU(8)$ nilpotent orbit $\mathcal{O}_{2\Upsilon_2}$ in $\mathfrak{p}_{\mathbb{C}}$ [178]. It is equivalent to its complex conjugate $\mathcal{O}_{2\Upsilon_6}$ that must be considered together for a non-linear supersymmetry invariant. This real orbit is inside the even complex orbit $\mathcal{O}_{2\Lambda_7}$ associated to the abelian parabolic P_7. The closure $\overline{\mathcal{O}_{2\Lambda_2} \cup \mathcal{O}_{2\Lambda_6}}$ defines therefore the associated variety of the abelian parabolic Eisenstein series $E^{E_7}_{s\Lambda_7}$ at generic values of $s \in \mathbb{C}$. Defining the normal triple $(h_{2\Upsilon_2}, e_{2\Upsilon_2}, f_{2\Upsilon_2})$, the Grassmann analytic superfield W^{rs} defined in (1.119) is in the weight 2 vector space and therefore includes the component along $e_{2\Upsilon_2} \in \mathfrak{p}_{\mathbb{C}}$ of W^{ijkl} and the harmonic variables parametrise the $SU(8)$ orbit of $h_{2\Upsilon_2} \in \mathfrak{su}(8)$. As such, W^{rs} is in the nilpotent orbit $\mathcal{O}_{2\Upsilon_2}$ and the set of polynomials in W^{rs} is isomorphic to the $SU(8)$ submodule of $S(\mathfrak{p}^*_{\mathbb{C}})$

$$S(\mathfrak{p}^*_{\mathbb{C}})|_{\mathcal{O}_{2\Upsilon_2}} \cong \bigoplus_{n_1,n_2,n_3\geq 0} R((n_2{+}2n_3)\Upsilon_2 + n_1\Upsilon_4 + n_2\Upsilon_6) \ . \tag{3.82}$$

For generic values of s one has

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{s\Lambda_7} \cong \bigoplus_{\substack{n_1,n_2,n_3,n_3'\geq 0 \\ n_3 n_3'=0}} R((n_2{+}2n_3)\Upsilon_2 + n_1\Upsilon_4 + (n_2{+}2n_3')\Upsilon_6) \ , \tag{3.83}$$

which is the union of $S(\mathfrak{p}^*_{\mathbb{C}})|_{\mathcal{O}_{2\Upsilon_2}}$ and $S(\mathfrak{p}^*_{\mathbb{C}})|_{\mathcal{O}_{2\Upsilon_6}}$. The chiral harmonic superspace suggests that one can define a submodule for $n_3 \geq k \geq 1$. One shows in Appendix C.1 that such module can be

defined starting from a non-trivial $SU(8)$ representation, and $k \geq 1$ integer

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)\mathcal{D}^{3k}_{2k\Upsilon_2}E^{E_7}_{(4+k)\Lambda_7} \cong \bigoplus_{\substack{n_1,n_2\geq 0\\ n_3\geq k}} R((n_2{+}2n_3)\Upsilon_2 + n_1\Upsilon_4 + n_2\Upsilon_6) \,\oplus\, \delta_{k,1}\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{4\Lambda_7}\,. \quad (3.84)$$

For the $\nabla^6 R^4$ type invariant this selects the regularised Eisenstein series at $s = 5$, for which

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)\widehat{E}^{E_7}_{5\Lambda_7} \cong \bigoplus_{\substack{n_1,n_2,n_3,n_3'\geq 0\\ n_3 n_3'=0}} R((n_2{+}2n_3)\Upsilon_2 + n_1\Upsilon_4 + (n_2{+}2n_3')\Upsilon_6) \,\oplus\, \mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{4\Lambda_7}\,, \quad (3.85)$$

is consistent with the linearised supersymmetry analysis. The mixing with $\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{4\Lambda_7}$ which expands as (3.79) is due to the one-loop divergence of the $\nabla^4 R^4$ form-factor that can be read from (2.66). We find therefore that supersymmetry implies equation (3.74) and that the Laplacian eigen-value is -60. The abelian Fourier coefficients in the string perturbative limit are not generic in this automorphic representation and satisfy (3.62).

We can summarise the results in this section by associating a nilpotent orbit to each supersymmetry invariant, as displayed in Figure 11. We can similarly associate Eisenstein series to nilpotent

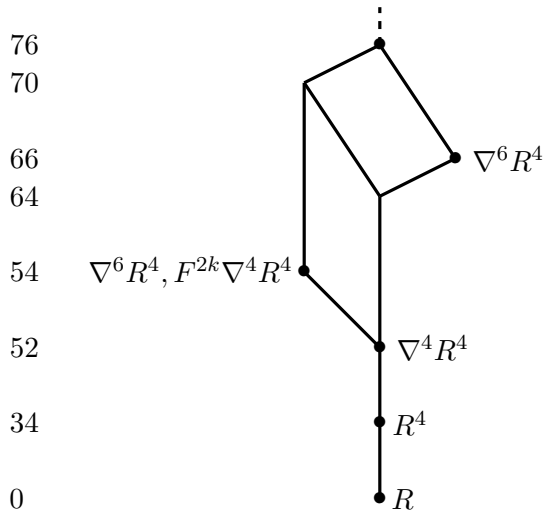

Figure 11: Nilpotent orbits associated to supersymmetry invariants

orbits as displayed in Figure 12. For completeness we have included nilpotent orbits beyond the ones known to appear in string theory.

3.3 BPS instantons as supergravity solutions

The D(-1) brane instanton was described as a supergravity Euclidean solution in [7]. The metric is flat $g_{\mu\nu} = \delta_{\mu\nu}$ and all the fields but the dilaton and the Ramond-Ramond axion vanish. The

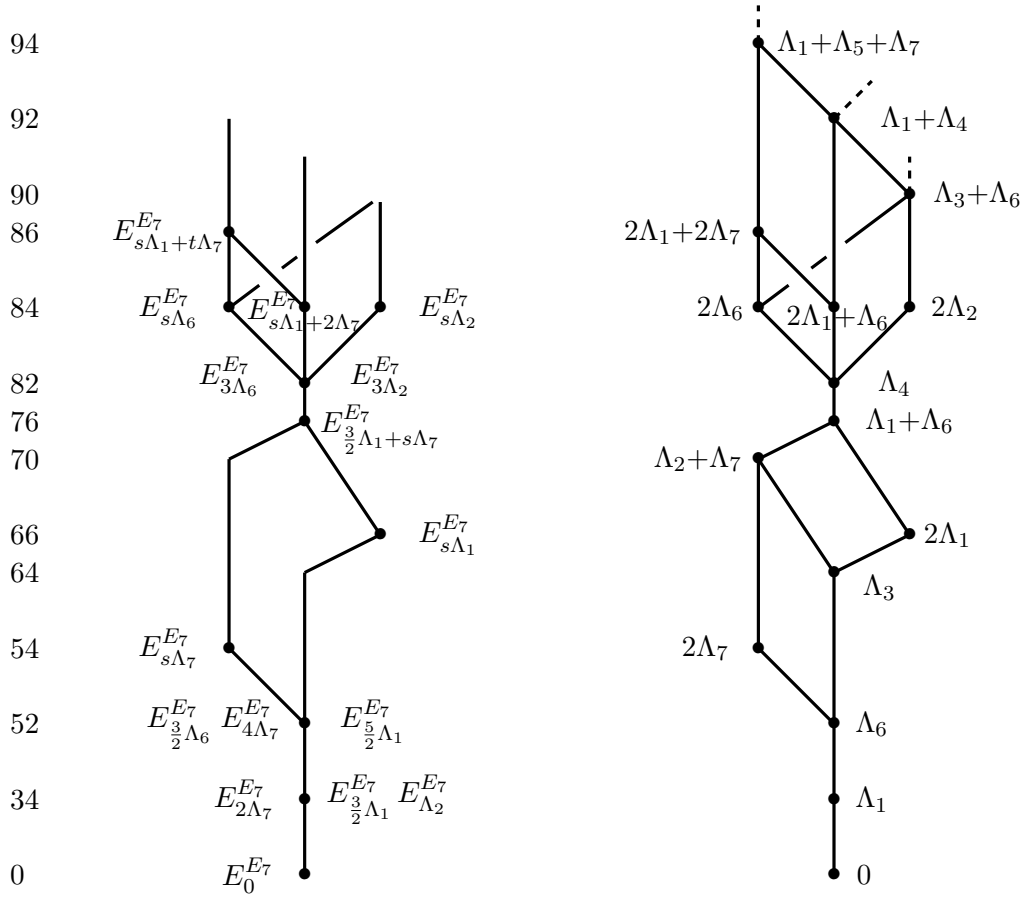

Figure 12: Nilpotent orbits associated to Eisenstein series in the E_7 closure diagram, where we removed the non-special orbits on the left for which there is no automorphic representation. If generic parameters s (or t) are chosen one obtains the orbits shown; for specific values the wavefront set of the Eisenstein series (or its leading residue) can be smaller as indicated for smaller orbits. Where we write different Eisenstein series, they are all related by functional relations. The reduction of the wavefront set have be studied by analysing the degenerate Whittaker vectors using Casselman–Shalika formula for Fourier coefficients [184, 185, 75].

Lagrangian then reduces to

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sqrt{-g}\big(\partial_\mu\phi\partial^\mu\phi - e^{2\phi}\partial_\mu C\partial^\mu C\big)\\ &\cong \frac{1}{2}d\phi\star d\phi + \frac{1}{2}e^{-2\phi}F\star F - id\big(CF\big)\end{aligned} \tag{3.86}$$

where we used that the kinetic term for the dual nine-form field strength F changes sign by duality in Euclidean signature and the pure imaginary total derivative remains after duality using C as a Lagrange multiplier for $dF = 0$. The BPS instantons can be obtained by writing the action as a square plus a topological term

$$\mathcal{L} = \frac{1}{2}e^{-2\phi}(de^\phi \mp \star F)\star(de^\phi \mp \star F) + d\big((\pm e^{-\phi} - iC)F\big)\ . \tag{3.87}$$

One finds that e^ϕ is harmonic and the single instanton solution reads

$$F = \frac{12}{\pi^5}nd\Omega_9\ , \qquad e^\phi = e^{\phi_\infty} + \frac{3|n|}{2\pi^5 r^8} \tag{3.88}$$

with Euclidean action $S = |n|e^{-\phi_\infty} + inC_\infty$. Without introducing the eight-form potential one can interpret the solution in the symmetric space $SO(1,1)\backslash SL(2,\mathbb{R})$ as

$$v = e^{\phi/2}\begin{pmatrix} 1 & C\\ 0 & e^{-\phi}\end{pmatrix}\ , \qquad C = C_\infty + \frac{3ne^{-\phi_\infty}}{2\pi^5 e^{\phi_\infty}r^8 + 3|n|}\ , \tag{3.89}$$

and the conserved current

$$J = v^{-1}\big(\star dvv^{-1} + \eta^{-1}(\star dvv^{-1})^{\intercal}\eta\big)v = \frac{12n}{\pi^5}d\Omega_9\begin{pmatrix} C_\infty + e^{-\phi_\infty} & (C_\infty + e^{-\phi_\infty})^2\\ -1 & -C_\infty - e^{-\phi_\infty}\end{pmatrix}\ , \tag{3.90}$$

with $\eta = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ (for the pseudo-Riemannian quotient by $SO(1,1)$) is nilpotent, i.e. $J^2 = 0$ as a matrix.

This generalises to all the symmetric spaces and the BPS instantons contributing in string theory. We will only discuss the case of E_7, but the results of this section can easily be generalised. For instantons contributing to abelian Fourier coefficients in a parabolic $P_i \subset E_7$, one needs to consider a Euclidean solution in which all the axions parametrising $U_i/[U_i, U_i]$ have been dualised. In this way the dual field strength charge defines the instanton number, as the nine-form F for the D(-1)-instanton secribed above. Because of the change of sign of the kinetic term in Euclidean signature, this implies that for P_7 one must look at solutions in $(SU(8)^*/\mathbb{Z}_2)\backslash E_7 \cong \mathbb{R}_+ \times (Sp(4)/\mathbb{Z}_2)\backslash E_6 \ltimes \mathbb{R}^{27}$. This is the real form of the coset space that arises when considering the consistent truncation to stationary solutions in five dimensions. The corresponding instantons are then BPS black holes in five dimensions with electric charge $q \in \mathbb{Z}^{27}$ [186], reduced on the time-like isometry interpreted as a thermal circle [182].

For P_1 the relevant real form is $(SU(4,4)/\mathbb{Z}_2)\backslash E_7 \cong \mathbb{R}_+ \times (SO(6)\times SO(6))\backslash SO(6,6) \ltimes \mathbb{R}^{32+1}$. In this case the vector fields do not admit a real duality equation compatible with E_7 symmetry and the spinors are complex. One finds nonetheless that it seems to be the correct real form to describe Euclidean D-brane solutions for which the truncation to the scalar sector is well defined [51].

For P_2 the relevant real form is $(SL(8)/\mathbb{Z}_2)\backslash E_7 \cong SO(7)\backslash GL(7) \ltimes \mathbb{R}^{35+7}$. In this case the vector fields do admit a real duality equation compatible with E_7 symmetry but the spinors are complex. Having complex spinors is not a problem in Euclidean signature [187]. We will discuss this example is some details using the approach proposed in [188]. The Euclidean Lagrangian for the scalar fields is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}dG_{IJ}\star dG^{IJ} + \frac{1}{8}G^{IJ}dG_{IJ}\star G^{KL}dG_{KL} - \frac{1}{12}G^{I_1J_1}G^{I_2J_2}G^{I_3J_3}da_{I_1I_2I_3}\star da_{J_1J_2J_3}\\ &+\frac{1}{2}\det G^{-1}G_{IJ}\Big(db^I - \frac{1}{72}\varepsilon^{IK_1\dots K_6}a_{K_1K_2K_3}da_{K_4K_5K_6}\Big)\star\Big(db^J - \frac{1}{72}\varepsilon^{JL_1\dots L_6}a_{L_1L_2L_3}da_{L_4L_5L_6}\Big)\end{aligned} \tag{3.91}$$

where only the axions a^{IJK} have a kinetic term with negative sign. Here $G_{IJ} = E_I{}^a E_{Ja}$ is the metric on T^7 with vielbeins $E_I{}^a$, a_{IJK} the three-form and b^I the six-form. One defines the dual three-form field strengths on $\mathbb{R}^4$

$$\begin{aligned}F^{I_1I_2I_3} &= G^{I_1J_1}G^{I_2J_2}G^{I_3J_3}\star da_{J_1J_2J_3} - \frac{1}{6}\varepsilon^{I_1I_2I_3J_1J_2J_3K}a_{J_1J_2J_3}H_K\ ,\\ H_I &= \det G^{-1}G_{IJ}\star\Big(db^J - \frac{1}{72}\varepsilon^{JL_1\dots L_6}a_{L_1L_2L_3}da_{L_4L_5L_6}\Big) = 0\ .\end{aligned} \tag{3.92}$$

The superymmetry variation of the Weyl fermions

$$\delta\chi_{\alpha ijk} = (\not{P}_{ijkl}\bar{\epsilon}^l)_\alpha\ ,\quad \delta\bar{\chi}^{ijk}_{\dot\alpha} = (\not{P}^{ijkl}\epsilon_l)_{\dot\alpha} \tag{3.93}$$

implies that one gets a BPS solution if $P^{abcd} = 0$ for $a,b,c,d \le 7$, so that $P_{abc8} = 0$. The generic BPS solution only preserves 1/16 of the supersymmetry through a single right-handed Weyl Killing spinor $\bar{\epsilon}^8_{\dot\alpha}$ with $SL(7) \subset SL(8)$ stabiliser. Written in terms of F^{IJK} and $E_I{}^a$, only $SO(7)$ is manifest with stabiliser $G_2 \subset SO(7)$. One can parametrise $SO(7)/G_2$ by the orbit of the antisymmetric tensor C_{abc} that defines the structure constants of the octonions algebra. It is of course expected that G_2 appears in breaking supersymmetry to $\mathcal{N} = 1$ [189], but here we only keep one chirality. The tensor C_{abc} is determined for a given solution by maximisation of the asymptotic boundary integral

$$M(N) = \frac{1}{6}\int_{\partial M} C_{abc}E_I{}^aE_J{}^bE_K{}^cF^{IJK}\ , \tag{3.94}$$

which gives the real part of the instanton action. The constraints $P_{abc8} = 0$ can be solved by using the equation $[\mathrm{h}, P] = 2P$ for the $\mathfrak{sl}_8$ generator $\mathrm{h} = \frac{7}{2}\mathrm{T}^8{}_8$, imposing that P belongs to the nilpotent orbit $\mathcal{O}_{4\Upsilon_7} \subset \mathcal{O}_{2\Lambda_2}$. Then the $\mathfrak{e}_7$ conserved current is automatically in the nilpotent orbit $\mathcal{O}_{2\Lambda_2}$.

In the P_2 basis with manifest $SL(7)$ symmetry, the diagonalising element h can be written in terms of the generators $\mathrm{E}^{abc} \in \mathbf{35}^{(2)}$ and $\mathrm{F}_{abc} \in \mathbf{35}^{(-2)}$ as

$$\mathrm{h} = \frac{1}{6}C_{abc}\mathrm{E}^{abc} + \frac{1}{6}C^{abc}\mathrm{F}_{abc} \in \mathfrak{sl}_8\ . \tag{3.95}$$

The equation $[\mathrm{h}, P] = 2P$ ensuring that $P \in \mathcal{O}_{2\Lambda_2}$ gives

$$\star da_{IJK} = -C_{abc}\star D(E_I{}^aE_J{}^bE_K{}^c) - \frac{1}{12}E_I{}^aE_J{}^bE_K{}^c\varepsilon_{abcdefg}C^{def}\det EE^{gI}H_I\ , \tag{3.96}$$

while $[\mathrm{h}, d_\omega \mathrm{h}] = 4 d_\omega \mathrm{h}$ ensuring that this condition is integrable gives

$$\star D C^{abc} = \frac{1}{12} \varepsilon^{abcdefg} C_{def} \det E E_g{}^I H_I \,, \tag{3.97}$$

with D the $SO(7)$-covariant derivative. It follows that

$$d a_{IJK} = -d(E_I{}^a E_J{}^b E_K{}^c C_{abc}) \;\Rightarrow\; a_{IJK} = -E_I{}^a E_J{}^b E_K{}^c C_{abc} + \alpha_{IJK} \,, \tag{3.98}$$

for constant coefficients α_{IJK}. Using the expression for F^{IJK} in (3.92) and (3.96) again we obtain

$$F^{IJK} - \frac{1}{6} \varepsilon^{IJKPQRS} \alpha_{PQR} H_S = \star D(E_a{}^I E_b{}^J E_c{}^K C^{abc}) \,. \tag{3.99}$$

Because the left-hand-side is a closed three-form on $\mathbb{R}^4$, $E_a{}^I E_b{}^J E_c{}^K C^{abc}$ are harmonic functions $\mathcal{K}^{IJK}$

$$E_a{}^I E_b{}^J E_c{}^K C^{abc} = \mathcal{K}^{IJK} = \mathcal{K}^{IJK}_\infty + \sum_p \frac{N^{IJK}}{4\pi^2 |x - x_p|^2} \,. \tag{3.100}$$

These 35 harmonic functions determine completely the vielbeins $E_a{}^I$ up the $G_2 \subset SO(7)$ stabiliser that is pure gauge. The three-forms H_I satisfy

$$H_I = -\frac{1}{72} \varepsilon_{IJKLPQR} \mathcal{K}^{JKL} \star d\mathcal{K}^{PQR} \,, \tag{3.101}$$

which is an exact three-form everywhere on $\mathbb{R}^4$ provided the harmonic functions satisfy a bubble equation of the same type as for four-dimensional multi-black hole solutions to do not admit closed time-like curves [190]. In the present case this condition imposes the absence of M5-brane instanton charge, which is required for instanton corrections associated to abelian Fourier coefficients. The non-abelian Fourier coefficients with non-vanishing M5-brane instanton charge must be defined in a different signature, see e.g. [120] for a discussion of Taub-NUT instantons.

The single instanton solution takes the form

$$F^{IJK} = \frac{N^{IJK}}{2\pi^2} d\Omega_3 \,, \qquad E_a{}^I E_b{}^J E_c{}^K C^{abc} = \mathcal{K}^{IJK}_\infty + \frac{N^{IJK}}{4\pi^2 r^2} \,, \tag{3.102}$$

with the constraint $dH_I = 0$ at $r = 0$ that gives

$$\varepsilon_{IJKLPQR} \mathcal{K}^{JKL}_\infty N^{PQR} = 0 \,. \tag{3.103}$$

For $H_I = 0$ one can rewrite the Lagrangian as the sum of a square plus a total derivative

$$\begin{aligned} \mathcal{L} = \frac{1}{12} G_{IL} G_{JP} G_{KQ} \Big(F^{IJK} - \star d\big(E_a{}^I E_b{}^J E_c{}^K C^{abc}\big) \Big) \star \Big(F^{LPQ} - \star d\big(E_d{}^L E_e{}^P E_f{}^Q C^{def}\big) \Big) \\ + \frac{1}{6} d\Big((C_{abc} E_I{}^a E_J{}^b E_K{}^c - i a_{IJK}) F^{IJK} \Big) \,, \end{aligned} \tag{3.104}$$

which gives the action $\int \mathcal{L} = M(N) - \frac{i}{6} a_{IJK} N^{IJK}$. One can always choose the asymptotic vielbeins representative $E_I{}^a$ such that C_{abc} takes a canonical form at infinity. In this case defining $Z^{abc} = E_I{}^a E_J{}^b E_K{}^c N^{IJK}$ at infinity, one must maximise

$$M(N) = \frac{1}{6} \int_{\partial M} C_{abc} E_I{}^a E_J{}^b E_K{}^c F^{IJK} = Z_{123} - Z_{156} + Z_{246} - Z_{345} + Z_{147} + Z_{257} + Z_{367} \tag{3.105}$$

with respect to $SO(7)$ acting on ${E_I}^a$ on the right to determine the Euclidean action. The function $M(N)$ is maximised if these seven Z's are the only non-vanishing components of Z_{abc} and they all contribute positively. One finds representatives of the ten $SL(7)$ orbits described in (3.64) depending of the set of non-vanishing components Z_{abc} among the seven above. One gets a 1/2 BPS charge with only $Z_{123} \neq 0$ and all the other components vanishing, a 1/4 BPS charge with only Z_{123}, Z_{156} non-zero. A 1/8 BPS charge in $\mathcal{O}_{2\Lambda_7}$ can be obtained for only $Z_{123}, Z_{156}, Z_{147}$ non-zero, while a 1/8 BPS charge is in $\mathcal{O}_{2\Lambda_1}$ for only $Z_{123}, Z_{156}, Z_{246}, Z_{345}$ non-zero. This permits to write the covariant form of the instanton action as the largest root of a polynomial in Z_{abc}.

For the appropriate choice of asymptotic values of the scalar fields one has then

$$P^{abc8} = Z^{abc} d\frac{1}{4\pi^2 r^2} , \qquad P_{abcd} = \frac{1}{6}\varepsilon_{abcdefg} Z^{efg} d\frac{1}{4\pi^2 r^2} , \tag{3.106}$$

while the other components vanish. This implies for the solution with only $Z_{123}, Z_{156}, Z_{147}$ non-zero that the only non-vanishing components of P^{ijkl} and P_{ijkl} are up to permutations

$$P^{1238} , \ P^{1568} , \ P^{1478} , \ P_{4567} , \ P_{2347} , \ P_{2356} \tag{3.107}$$

which gives the four right-handed Killing spinors $\bar\epsilon^1_{\dot\alpha}$, $\bar\epsilon^8_{\dot\alpha}$ using (3.93) and the fact that $P_{ijkl} = 0$ if any of the indices is 1 or 8. On the contrary $P^{ijkl} \neq 0$ for any value of i. Following the same argument as in [7], the fermionic zero modes of the solutions are determined by the unbroken supersymmetries [191] that give rise to a superspace integral of the type discussed in Section 1.3 for the G-analytic superfield (1.119). One expects therefore such instanton to contribute to the corresponding $(0, 1/4)$-BPS supersymmetry invariant consistently with the analysis of the preceding section.

Similarly for the solution with only $Z_{123}, Z_{156}, Z_{246}, Z_{345}$ non-zero one obtains that the only non-vanishing components of P^{ijkl} and P_{ijkl} are up to permutations

$$P^{1238} , \ P^{1568} , \ P^{2468} , \ P^{3468} , \ P_{4567} , \ P_{2347} , \ P_{1357} , \ P_{1257} \tag{3.108}$$

which gives the two right-handed Killing spinors $\bar\epsilon^8_{\dot\alpha}$ and the two left-handed Killing spinors $\epsilon_{\alpha 7}$ using (3.93). The integration over the fermionic zero modes in this background give rise to a superspace integral of the type discussed in Section 1.3 for the G-analytic superfield (1.117). One expects therefore such instanton to contribute to the corresponding $(1/8, 1/8)$-BPS supersymmetry invariant consistently with the analysis of the preceding section.

3.4 Eisenstein series in the string perturbative limit

In this section we shall give the explicit form of the $E^{E_d}_{s\Lambda_d}$ Eisenstein series in the parabolic P_1 relevant to describe the string theory perturbative limit $g_s = e^{\phi_{d-1}} \ll 1$. We use the notation of table 2. The computation is based on the formula (3.12), which reads for E_7

$$2\zeta(2s) E^{E_7}_{s\Lambda_7} = \sum_{\substack{\Gamma\in\mathbb{Z}^{56} \\ (\Gamma\times\Gamma)|_{\mathbf{133}}=0}}{}' \frac{1}{|Z(\Gamma)|^{2s}} , \tag{3.109}$$

and that has a natural interpretation as a sum over massive 1/2 BPS supermultiplets with BPS mass $\ell M = |Z(\Gamma)|$ [64]. The computation is explained in Appendix D.1. We will write generally S_-

for the lattice of D-brane charges in the left-handed spinor representation of the T-duality group $Spin(d-1,d-1,\mathbb{Z})$. In our notation, the action of the 1/4-BPS Euclidean D-brane instanton of charge $Q \in S_-$ is [192] [18]

$$S^{\text{D-brane}}(Q) = 2\pi \frac{\sqrt{|v(Q)|^2 + 2|v(Q\times Q)|}}{g_{\rm s}} - 2\pi i (Q,a) \tag{3.110}$$

with $v \in Spin(d-1,d-1)$ parametrising the Narrain moduli and $a \in \overline{S_-(\mathbb{R})}$ the Ramond-Ramond axions. For short we write $Q\times Q = (Q\Gamma_{d-5}Q)$ the $(d-5)$-form quadratic in Q defined by the gamma matrix. For $d=5$, $(QQ)\in\mathbb{Z}$ is related by triality to half the even scalar product on $I\!I_{4,4}$. For $d=6$, $Q\gamma^a Q$ is normalised such that it is in $I\!I_{5,5}$. For $d=7$, $Q\gamma^{ab}Q$ is normalised such that it is a vector in $\mathbb{L}_2$, so all its components are integer except one that is possibly half-integer, when $2(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) = 1 \bmod 4$. We normalise $|v(Q\times Q)|^2$ with a factor of $\frac{1}{(d-5)!}$.

For $d\leq 6$, the parabolic P_1 has an abelian unipotent radical, so that one can write a standard Fourier expansion

$$\begin{aligned}
g_{\rm s}^{\frac{4}{9-d}s} E^{E_d}_{s\Lambda_d} = {} & E^{D_{d-1}}_{s\Lambda_1} + \frac{\xi(2s-d+1)}{\xi(2s)} g_{\rm s}^{2s-d+1} E^{D_{d-1}}_{(s-\frac{d-3}{2})\Lambda_{d-2}} + \delta_{d,6}\frac{\xi(2s-8)\xi(2s-11)}{\xi(2s)\xi(2s-3)} g_{\rm s}^{4s-16} \\
& + 2\frac{g_{\rm s}^{s-\frac{d-1}{2}}}{\xi(2s)} {\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \frac{\sigma_{d-1-2s}(Q)}{\gcd Q^{\frac{d-3}{d-1}(d-3-2s)}} E^{SL(d-1)}_{(s-\frac{d-3}{2})\Lambda_{d-2}}(v_Q) \frac{K_{s-\frac{d-1}{2}}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{(1-\frac{4}{d-1})s+\frac{9-d}{2}-\frac{4}{d-1}}} e^{2\pi i(Q,a)} \\
& + 2\delta_{d,6}\frac{\xi(2s-8) g_{\rm s}^{3s-\frac{21}{2}}}{\xi(2s)\xi(2s-3)} {\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \sigma_{11-2s}(Q) \frac{K_{s-\frac{11}{2}}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{11}{2}-s}} e^{2\pi i(Q,a)} \\
& + 2\delta_{d,6}\frac{g_{\rm s}^{2s-8}}{\xi(2s)\xi(2s-3)} \sum_{\substack{Q\in S_-\\ Q\times Q\neq 0}} \sum_{n|Q} n^{11-2s}\sigma_{8-2s}(\tfrac{Q\times Q}{n^2}) \frac{B_{\frac{3}{2},s-4}(\frac{|v(Q)|^2}{g_{\rm s}^2},\frac{|v(Q\times Q)|}{g_{\rm s}^2})}{|v(Q\times Q)|^{4-s}} e^{2\pi i(Q,a)}
\end{aligned} \tag{3.111}$$

where D_{n-1} is the split real form $Spin(d-1,d-1)$, the $SL(d-1)$ Eisenstein series is evaluated on the Levi stabiliser subgroup $SL(d-1)\subset P_{d-2}\subset Spin(d-1,d-1)$ of the one-half BPS pure spinor Q, and $B_{j,s}$ is the integral

$$\begin{aligned}
B_{j,s}(x,y) &= \int_0^\infty \frac{{\rm d}t}{t^{1+j}} e^{-\pi t - \pi x/t}\, K_s(2\pi y/t)\ , \qquad x,y>0 \\
&= \sum_{k=0}^\infty \frac{\Gamma(s+k+\frac12)\, K_{j-k-\frac12}\left(2\pi\sqrt{x+2y}\right)}{k!\,\Gamma(s-k+\frac12)\,(4\pi)^k\, y^{k+\frac12}\,(x+2y)^{\frac{2j-2k-1}{4}}}\ ,
\end{aligned} \tag{3.112}$$

that behaves as

$$B_{j,s}(\tfrac{|v(Q)|^2}{g_{\rm s}^2},\tfrac{|v(Q\times Q)|}{g_{\rm s}^2}) \sim g_{\rm s}^{j+1} \frac{e^{-2\pi\frac{\sqrt{|v(Q)|^2+2|v(Q\times Q)|}}{g_{\rm s}}}}{2\sqrt{|v(Q\times Q)|}(\sqrt{|v(Q)|^2+2|v(Q\times Q)|})^j} \tag{3.113}$$

[18] For $d=7$ this formula is only valid for 1/4 BPS brane instantons, but 1/8 BPS instanton do not contribute to $E^{E_7}_{s\Lambda_7}$.

at small string coupling $g_s \ll 1$. We recall that $\sigma_s(Q) = \sum_{n|Q} d^s$ is the sigma divisor sum of the greatest divisor n of Q such that $Q/n \in S_-$. Note that this expression (3.111) is an absolutely convergent sum that is real analytic in the moduli and meromorphic in s. The 1/4 BPS instanton corrections are exponentially suppressed with the expected weight defined by the classical world-volume action on the Euclidean brane. The measure factor $\sum_{n|Q} n^4 \sigma_1(\frac{Q\times Q}{n^2})$ is generic for $d \geq 5$ and will be discussed below. The fact that the generic abelian Fourier coefficient is factorised has been proved in general for the next-to-minimal representation [193–195].

Let us mention that the method developed in [194] provides an alternative way to derive minimal and next-to-minimal Fourier coefficients from Borel Fourier coefficients that can be computed efficiently using Casselman–Shalika formula [184].

In four dimensions there are also Neveu–Schwarz Euclidean five-brane corrections. We write the NS5-brane charge k and the corresponding axion b (dual to the Kalb-Ramond two-form). Using the antisymmetric $Spin(6,6)$ scalar product $\langle\,,\rangle$ on S_- defined by the charge conjugation matrix C, one obtains that the unipotent generators define the derivative

$$\frac{\partial}{\partial a^\alpha} + \frac{1}{2} C_{\alpha\beta} a^\beta \frac{\partial}{\partial b}\ , \quad \frac{\partial}{\partial b} \quad \Rightarrow \quad Q\cdot\frac{\partial}{\partial a} + \frac{1}{2}\langle Q, a\rangle \frac{\partial}{\partial b}\ , \quad k\frac{\partial}{\partial b}\ , \tag{3.114}$$

that satisfy the Heisenberg algebra with central charge $\frac{\partial}{\partial b}$. To define the non-abelian Fourier coefficients one must therefore choose a polariation [196]. For this we need to break the symmetry $SO(6,6)$ by choosing a Lagrangian subspace in S_-. A convenient way to do this is to take the further decomposition that appears in choosing a specific polarisation circle in T^6

$$\begin{aligned} \mathfrak{so}(6,6) &\cong \mathbf{10}^{(-2)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{so}(5,5))^{(0)} \oplus \mathbf{10}^{(2)}\ , \\ \mathbf{32}_- &\cong \mathbf{16}^{(-1)} \oplus \overline{\mathbf{16}}^{(1)}\ . \end{aligned} \tag{3.115}$$

We write accordingly $Q = (q, \bar q)$, $a = (a, \bar a)$, and

$$k\frac{\partial}{\partial b}\ , \quad q\cdot\frac{\partial}{\partial a} - \bar a q \frac{\partial}{\partial b}\ , \quad \bar q\cdot\frac{\partial}{\partial \bar a} + \bar q a \frac{\partial}{\partial b}\ , \quad p\cdot\frac{\partial}{\partial c} - \bar a \not{p}\frac{\partial}{\partial a}\ , \tag{3.116}$$

where we added the derivative with respect to the $SO(6,6)$ axions c in the $\mathbf{10}^{(2)}$ nilpotent component. We can therefore consider the Fourier decomposition with respect to the unipotent character

$$\psi_{k,q,p} = e^{2\pi i\left(k(b+\bar a a + \bar a \not{q}\bar a) + q(a + 2\not{q}\bar a) + p\cdot c\right)}\ , \tag{3.117}$$

where $q\times q$ is the product defined by the $Spin(5,5)$ gamma matrices, and we use the notation that $p\cdot(q\times q) = q\not{p}q$. In this form $q \in \mathbb{Z}^{16}$ is the charge of the Euclidean D-branes wrapping the polarisation circle, whereas the charge $\bar q$ of the D-branes not wrapping the polarisation circle is not defined.

After several manipulations explained in Appendix D.1, one obtains the R^4 coupling function in four dimensions

$$\begin{aligned} g_{\rm s}^4\mathcal{E}_{(0,0)} &= 4\pi\xi(4)g_{\rm s}^4E^{E_7}_{2\Lambda_7} \\ &= 2\zeta(3)g_{\rm s}^{-2}+4\pi\xi(4)E^{D_6}_{2\Lambda_1}+\frac{8\pi}{g_{\rm s}}{\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \sigma_2(Q)\frac{K_1(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|}e^{2\pi i(Q,a)} \\ &\quad +\frac{8\pi}{g_{\rm s}}{\sum_{k\in\mathbb{Z}}}'\sum_{\substack{q\in\mathbb{Z}^{16}\\ k|q\times q}} R_{\rm s}^3\sigma_3(k,q,\tfrac{q\times q}{k})\frac{K_{\frac{3}{2}}(2\pi\frac{\sqrt{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar{a}k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar{a}k)\times(q+\bar{a}k)}{k})|^2}}{g_{\rm s}^2})}{\left(k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar{a}k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar{a}k)\times(q+\bar{a}k)}{k})|^2\right)^{\frac{3}{4}}}\psi_{k,q,\frac{q\times q}{k}} \ , \end{aligned} \tag{3.118}$$

where $R_{\rm s}$ is the radius in string lengths of the polarisation circle. The one-loop correction reproduces (2.29) as was identified in [76]

$$4\pi\xi(4)E^{D_6}_{2\Lambda_1}=2\pi\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\tau_2^{\epsilon}\,\Gamma_{I\!I_{6,6}} \ . \tag{3.119}$$

The Euclidean D-brane instanton is the same as in ten-dimensions

$$\frac{8\pi}{g_{\rm s}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\sigma_2(|n|)\frac{K_1(2\pi\frac{|n|}{g_{\rm s}})}{|n|}e^{2\pi inC} \ . \tag{3.120}$$

It was recently computed in [10–12]. The instanton measure $\sigma_{-2}(Q)=\frac{\sigma_2(Q)}{\gcd Q^2}$ is the appropriately normalised partition function of maximal super Yang–Mills on the torus with gauge group $SU(\gcd Q)/\mathbb{Z}_{\gcd Q}$, which was computed in [197]. The NS5-brane instanton corrections have not been computed and are only known through duality relations in general. One may observe that one finds the same measure and functional dependence in the string coupling constant as in the F^4 threshold function computed in type IIB on $K3\times T^2$ at the $T^4/\mathbb{Z}_2$ orbifold point in [198].[19]

One can check by consistency that the function is invariant under the Ramond-Ramond axion shift $\bar{a}\to\bar{a}+x$ for $x\in\mathbb{Z}^{16}$. Indeed

$$\psi_{k,q,p}\big|_{\bar{a}\to\bar{+}x}=\psi_{k,q+xk,p+2q\times u+u\times uk} \ , \tag{3.121}$$

and the Bessel function argument satisfies the same property such that the total function is invariant under the shift when including the sum over all $k\in\mathbb{Z}$ and $q\in\mathbb{Z}^{16}$ satisfying $k|q\times q$. It is the case that if k divides $q\times q$ then it divides $(q+xk)\times(q+xk)=q\times q+2q\times xk+x\times xk^2$.

[19] Note however that the measure factor at a generic point of K3 is very different, and gives instead a result similar to the heterotic NS5 brane instanton corrections [120].

The $\nabla^4 R^4$ coupling function is also calculated in Appendix D.1 as

$$
\begin{aligned}
g_{\rm s}^8 \mathcal{E}_{(1,0)} &= 8\pi\xi(4)\xi(8) g_{\rm s}^8 E^{E_7}_{4\Lambda_7} \qquad (3.122)\\
&= \zeta(5) g_{\rm s}^{-2} + \frac{4\pi^3}{45}\xi(8) E^{D_6}_{4\Lambda_1} + \frac{4}{3} g_{\rm s}^2 \zeta(4) E^{D_6}_{2\Lambda_5}\\
&\quad +16 \sum_{\substack{Q\in S_-\\ Q\times Q=0}}{}' \left(\frac{g_{\rm s}}{\pi} \frac{\sigma_2(Q)}{\gcd Q^{-\frac{2}{3}}} \zeta(4) E^{SL(6)}_{2\Lambda_5}(v_Q) \frac{K_1(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{5}{3}}} + \frac{\pi^2}{6}\sigma_4(Q)\frac{K_2(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^2}\right) e^{2\pi i (Q,a)}\\
&\quad + \frac{16\pi}{g_{\rm s}} \sum_{\substack{Q\in S_-\\ Q\times Q\neq 0\\ Q\cdot(Q\times Q)=0}} \sum_{d|Q} d^4 \sigma_1(\tfrac{Q\times Q}{d^2}) \frac{K_1(2\pi\frac{\sqrt{|v(Q)|^2+2|v(Q\times Q)|}}{g_{\rm s}})}{|v(Q\times Q)|\sqrt{|v(Q)|^2+2|v(Q\times Q)|}} e^{2\pi i(Q,a)}\\
&\quad +8 g_{\rm s}^7 \sum_{k\in\mathbb{Z}}{}' \sum_{\substack{q\in\mathbb{Z}^{16}\\ k|q\times q}} \psi_{k,q,\frac{q\times q}{k}} \left(\frac{\pi^3 R_{\rm s}^8}{45}\sigma_7(k,q,\tfrac{q\times q}{k}) \frac{K_{\frac{7}{2}}(2\pi\frac{\sqrt{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2}}{g_{\rm s}^2})}{\left(k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2\right)^{\frac{7}{4}}}\right.\\
&\qquad \left. + R_{\rm s}^5 \frac{\sigma_1(k,q,\frac{q\times q}{k})}{\gcd(k,q,\frac{q\times q}{k})^{-\frac{3}{2}}}\zeta(3) E^{D_5}_{\frac{3}{2}\Lambda_1}(v_{k,q}) \frac{K_{\frac{1}{2}}(2\pi\frac{\sqrt{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2}}{g_{\rm s}^2})}{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2}\right)\\
&\quad +16\pi g_{\rm s}^3 R_{\rm s}^8 \sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{\substack{q\in\mathbb{Z}^{16}\\ \frac{k|p|^2}{2}=q\not p q}} \sum_{p\in\mathbb{Z}^{10}} \psi_{k,q,p} \sum_{n|(k,q,p)} n^7 \sigma_3(\tfrac{(\frac{|p|^2}{2},\not p q, kp-q\times q)}{n^2})\\
&\qquad \frac{B_{2,\frac{3}{2}}(\frac{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(p+2\bar a\times q+\bar a\times\bar a k)|^2}{g_{\rm s}^4}, \frac{\sqrt{R_{\rm s}^2|v(kp-q\times q)|^2+g_{\rm s}^2R_{\rm s}^3|v(\not p q+(k\not p-q\times\not q)\bar a)|^2+g_{\rm s}^4R_{\rm s}^4|v(\frac{|p|^2}{2}+2\bar a\not p\bar a+\bar a(k\not p-q\times\not q)\bar a)|^2}}{g_{\rm s}^4})}{\left(R_{\rm s}^2|v(kp-q\times q)|^2+g_{\rm s}^2R_{\rm s}^3|v(\not p q+(k\not p-q\times\not q)\bar a)|^2+g_{\rm s}^4R_{\rm s}^4|v(\frac{|p|^2}{2}+2\bar a\not p\bar a+\bar a(k\not p-q\times\not q)\bar a)|^2\right)^{\frac{3}{4}}}\,.
\end{aligned}
$$

The one-loop and two loop corrections agree with the perturbative computation [76] [20]

$$
\frac{4\pi^3}{45}\xi(8)E^{D_6}_{4\Lambda_1} = \frac{2\pi^3}{45}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}E_2(\tau)\tau_2^{\epsilon}\Gamma_{I\!I_{6,6}}\,,\quad \frac{4}{3}\zeta(4)E^{D_6}_{2\Lambda_5} = 4\pi\int_{\mathcal{F}_2}\frac{d^6\Omega}{\det\Omega_2^3}\det\Omega_2^{\epsilon}\Gamma_{I\!I_{6,6}}\,. \qquad (3.123)
$$

It is interesting to look at the generic abelian Fourier coefficient

$$
\frac{16\pi}{g_{\rm s}} \sum_{\substack{Q\in S_-\\ Q\times Q\neq 0\\ Q\cdot(Q\times Q)=0}} \sum_{d|Q} d^4 \sigma_1(\tfrac{Q\times Q}{d^2}) \frac{K_1(2\pi\frac{\sqrt{|v(Q)|^2+2|v(Q\times Q)|}}{g_{\rm s}})}{|v(Q\times Q)|\sqrt{|v(Q)|^2+2|v(Q\times Q)|}} e^{2\pi i(Q,a)} \qquad (3.124)
$$

corresponding to 1/4 BPS D-brane instanton corrections. As the 1/2 BPS instanton corrections in (3.119), one expects them to be universal in all dimensions $D\leq 6$ by T-duality. We need at least T^4 to have such a configuration of brane breaking 3/4 of the supersymmetries. One easily reads from (3.111) that the correction is indeed the same in $D=5$. In $D=6$ 1/4 BPS instantons only contribute to $\zeta(5)\widehat{E}^{D_5}_{5/2\Lambda_1}$ and give again the same formula (3.124) [63].[21]

[20] Where one uses the functional relation $\xi(2)E^{D_6}_{\frac{3}{2}\Lambda_2} = \xi(4)E^{D_6}_{2\Lambda_6} = \xi(4)E^{D_6}_{2\Lambda_5}$ to simplify the genus one integral.

[21] In $D=6$, $Q\in I\!I_{4,4}$ by triality and one can use $|v(Q)|^2 = p_L(Q)^2+p_R(Q)^2$ and $2|v(Q\times Q)| = |p_L(Q)^2 - p_R(Q)^2|$. See for example [27] for the complete Fourier expansion of $\zeta(5)\widehat{E}^{D_5}_{5/2\Lambda_1}$.

The 1/2 BPS index in [197] suggests that the supersymmetric index of D0 particles on T^9 is equal to the partition function of the Euclidean D3 brane partition function on T^4. This generalised T-duality suggests then that the helicity supertrace of D1-D5 states on T^5 should be identical to the partition function of Euclidean D1-D5 branes on T^6 [199], or similar T-dual configurations. Using [30], one can derive the following helicity supertrace of the D1-D5 CFT on T^4, with Q_1 and Q_5 relative primes, as

$$\underset{L_0=\bar{L}_0=J_0^3=0}{\mathrm{Tr}}\Big[(-1)^{2\tilde{J}_0^3}(2\tilde{J}_0^3)^2\Big] = \sum_{j_L,j_R}(-1)^{j_R}(2j_R+1)D_{\frac{1}{4}\text{-BPS}}(Q_1,Q_5,j_L,j_R) = \sigma_1(Q_1Q_5)\ , \quad (3.125)$$

which is consistent with the instanton measure $\sum_{d|Q} d^4\sigma_1(\frac{Q\times Q}{d^2})$. However, this consistency check should not be overestimated since there are other choices of helicity supertraces and the projection to states with $J_0^3=0$ is not justified. This result could be derived in principle in string field theory and one expects the measure factor to be the partition function of the $(U(Q_1)\times U(Q_5))/U(1)$ half-maximal super Yang–Mills theory coupled to $(Q_1,\bar{Q}_5)$ hyper-multiplets on a torus [200]. However, the equivariant localisation formula used in [197] does not directly apply because the Ω-deformed partition function is a rational function of the ε_I deformation parameters and the limit $\varepsilon_I\to 0$ is not unique.

The ∇^6R^4 coupling admits a contribution from two different supersymmetry invariants, and the one associated to the chiral harmonic superspace (and nilpotent orbit) of weight Υ_2 is given by the regularised Eisenstein series $\frac{64\zeta(10)}{189}\widehat{E}^{E_7}_{5\Lambda_7}$ [64]. The abelian Fourier expansion can more generally be computed as

$$\begin{aligned}
g_{\rm s}^{2k+8}\int_{[0,1]}db\ E^{E_7}_{(k+4)\Lambda_7} &= E^{D_6}_{(k+4)\Lambda_1} + \frac{\xi(2k+2)}{\xi(2k+8)}g_{\rm s}^{2+2k}E^{D_6}_{(k+2)\Lambda_5} + \frac{\xi(2k-1)\xi(2k-4)}{\xi(2k+8)\xi(2k+8)}g_{\rm s}^{4k-2}E^{D_6}_{k\Lambda_1}\\
&+2\frac{g_{\rm s}^{k+1}}{\xi(2k+8)}\sum_{\substack{Q\in S_-\\ Q\times Q=0}}{}'\frac{\sigma_{2k+2}(Q)}{\mathrm{gcd}Q^{\frac{2}{3}(k+1)}}E^{SL(6)}_{(k+2)\Lambda_5}(v_Q)\frac{K_{k+1}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{k+5}{3}}}e^{2\pi i(Q,a)}\\
&+2\frac{\xi(2k-1)g_{\rm s}^{3k}}{\xi(2k+8)\xi(2k+4)}\sum_{\substack{Q\in S_-\\ Q\times Q=0}}{}'\frac{\sigma_{4-2k}(Q)}{\mathrm{gcd}Q^{\frac{-k}{3}}}E^{SL(6)}_{k\Lambda_1}(v_Q)\frac{K_{k-2}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{6-k}{3}}}e^{2\pi i(Q,a)}\\
+2\frac{g_{\rm s}^{2k-1}}{\xi(2k+8)\xi(2k+4)}&\sum_{\substack{Q\in S_-\\ Q\times Q\neq 0\\ Q\cdot(Q\times Q)=0}}\ \sum_{d|Q}\frac{d^{2+2k}\sigma_{2k-1}(\frac{Q\times Q}{d^2})}{(\mathrm{gcd}Q\times Q)^{k-1}}E^{SL(2)}_k(U_Q)\frac{B_{\frac{3}{2},k-\frac{1}{2}}(\frac{|v(Q)|^2}{g_{\rm s}^2},\frac{|v(Q\times Q)|}{g_{\rm s}^2})}{\sqrt{|v(Q\times Q)|}}e^{2\pi i(Q,a)}
\end{aligned} \quad (3.126)$$

where U_Q in the last line parametrises the stabiliser $SL(2)\subset SL(2)\times Spin(3,4)\ltimes\mathbb{R}^{2\times 8+1}\subset Spin(6,6)$ of the instanton charge Q.[22] One can compare these results to perturbative string

[22] More precisely $Q\times Q\in\mathbb{L}_2\subset\mathfrak{so}(6,6)$ and belongs to the $SO(6,6,\mathbb{Z})$ orbit of the highest weight vector $\mathrm{gcd}(Q\times Q)\Lambda_2$. One defines $kv\gamma$ such that its $SL(2)$ Levi factor in $P_2\subset SO(6,6)$ is the $SL(2)$ matrix parametrised by U_Q and $\mathrm{gcd}(Q\times Q)\gamma\Lambda_2 = Q\times Q$.

theory

$$\frac{64\zeta(10)}{189}E^{D_6}_{5\Lambda_1}=\frac{4\pi^4}{567}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}E_3(\tau)\tau_2^6\Gamma_{I\!I_{6,6}}\ ,\quad 20\int_{\mathcal{F}_3}\frac{d^{12}\Omega}{\det\Omega_2^4}\det\Omega_2^6\Gamma_{I\!I_{6,6}}=\frac{4\zeta(6)}{27}\big(\widehat{E}^{D_6}_{3\Lambda_5}+\widehat{E}^{D_6}_{3\Lambda_6}\big) \tag{3.127}$$

so that this function reproduces the above one-loop component and the $\widehat{E}^{D_6}_{3\Lambda_5}$ Eisenstein series part in the three-loop amplitude [201, 84].

Let us end this section by a short analysis of the $\mathcal{U}(\mathfrak{e}_7)$ module structure (3.84). Using (3.116) one finds that the differential operator $D^3_{2\Upsilon_2}$ defined in (3.45) can be written as

$$\Big(\frac{\partial}{\partial c^I}-\bar a\gamma_I\frac{\partial}{\partial a}\Big)\Big(\frac{\partial}{\partial a}-\bar a\frac{\partial}{\partial b}\Big)\gamma^I\Big(\frac{\partial}{\partial a}-\bar a\frac{\partial}{\partial b}\Big)-\frac{1}{2}\frac{\partial}{\partial b}\Big(\frac{\partial}{\partial c}-\bar a\gamma\frac{\partial}{\partial a},\frac{\partial}{\partial c}-\bar a\gamma\frac{\partial}{\partial a}\Big)\ . \tag{3.128}$$

By the cubic constraint $Q\cdot(Q\times Q)=0$, this derivative $\frac{\partial}{\partial c^I}$ is projected to the Levi stabiliser $SL(2)$. Writing the projection of $D^3_{2\Upsilon_2}$ to the $(\mathbf{20},\mathbf{1})$ irreducible representation of $SO(6)\times SO(6)$ one obtains therefore

$$\begin{aligned}\mathcal{D}_{(a}{}^{\hat c}v\Big(\frac{\partial}{\partial a}\times\frac{\partial}{\partial a}\Big)_{b)'\hat c}\int_{[0,1]}db\ E^{E_7}_{(k+4)\Lambda_7}=-\frac{8\pi^2g_{\rm s}^{-9}}{\xi(2k+8)\xi(2k+4)}\sum_{\substack{Q\in S_-\\ Q\times Q\neq0\\ Q\cdot(Q\times Q)=0}}\sum_{d|Q}\frac{d^{2+2k}\sigma_{2k-1}(\frac{Q\times Q}{d^2})}{(\gcd Q\times Q)^{k-1}}\\ \times\mathcal{D}_{(a}{}^{\hat c}E^{SL(2)}_k(U_Q)v(Q\times Q)_{b)'\hat c}\frac{B_{\frac32,k-\frac12}(\frac{|v(Q)|^2}{g_{\rm s}^2},\frac{|v(Q\times Q)|}{g_{\rm s}^2})}{\sqrt{|v(Q\times Q)|}}e^{2\pi i(Q,a)}\end{aligned}$$

where $a,b,\ \hat c,\hat d$ are the vector indices of $SO(6)\times SO(6)$ and $(ab)'$ is projected to the traceless symmetric component $\mathbf{20}$. Using the property of the $SL(2)$ Eisenstein series we conclude that applying the component of $D^{3k}_{2k\Upsilon_2}$ in the $2k\Upsilon_2$ irreducible representation of $SO(6)$ gives accordingly the holomorphic Eisenstein series

$$\mathcal{D}^kE^{SL(2)}_k(U)=\frac{\Gamma(2k)}{\Gamma(k)}U_2^kG_{2k}(U)=\frac{\Gamma(2k)}{\Gamma(k)}U_2^k\Big(1+\frac{(2\pi i)^k}{\Gamma(2k)\zeta(2k)}\sum_{n=1}^\infty\sigma_{2k-1}(n)e^{2\pi inU}\Big)-\delta_{k,1}\frac{3}{\pi}\ . \tag{3.129}$$

For the $k=1$ case one gets the quasi-holomorphic Eisenstein series $\hat G_2=G_2-\frac{3}{\pi U_2}$, because of the pole at $s=5$. Using the definition

$$\widehat{E}^{E_7}_{5\Lambda_7}=\lim_{\epsilon\to0}\Big(E^{E_7}_{(5+\epsilon)\Lambda_7}-\frac{\xi(5)\xi(1+2\epsilon)}{\xi(6)\xi(10)}E^{E_7}_{\frac52\Lambda_1}\Big)\ , \tag{3.130}$$

it follows accordingly from

$$\mathcal{D}^{78ij}[\mathcal{D}^3_{2\Upsilon_2}]_{ij,12}E^{E_7}_{s\Lambda_7}=\frac{7}{12}\mathcal{D}^{78ij}\mathcal{D}_{ij12}(\Delta+60)E^{E_7}_{s\Lambda_7} \tag{3.131}$$

and

$$(\Delta+60)\frac{64\zeta(10)}{189\pi}\widehat{E}^{E_7}_{5\Lambda_7}=\frac{45}{2\pi}\zeta(5)E^{E_7}_{\frac52\Lambda_1}\ , \tag{3.132}$$

that

$$\mathcal{D}^{78ij}[\mathcal{D}^3_{2\Upsilon_2}]_{ij,12}\Big(\frac{64\zeta(10)}{189\pi}\widehat{E}^{E_7}_{5\Lambda_7}\Big) = \frac{35}{8\pi}\mathcal{D}^{78ij}\mathcal{D}_{ij12}\big(\zeta(5)E^{E_7}_{\frac{5}{2}\Lambda_1}\big)\ . \tag{3.133}$$

Here we have used the normalisation justified by the fact that $\mathcal{E}_{(0,1)} = \frac{64\zeta(10)}{189\pi}\widehat{E}^{E_7}_{5\Lambda_7} + \mathcal{E}^{\rm ExFT}_{(0,1)}$. The relation above does apply to the exact coupling functions

$$\mathcal{D}^{78ij}[\mathcal{D}^3_{2\Upsilon_2}]_{ij,12}\mathcal{E}_{(0,1)} = \frac{35}{8}\mathcal{D}^{78ij}\mathcal{D}_{ij12}\mathcal{E}_{(1,0)}\ . \tag{3.134}$$

In analogy with the discrete series, $\mathcal{D}^{3k}_{2k\Upsilon_2}E^{E_7}_{(4+k)\Lambda_7}$ defines a submodule of $\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{(4+k)\Lambda_7}$ exactly as G_{2k} defines a submodule of $\mathcal{U}(\mathfrak{sl}_2)E^{SL(2)}_k$ for $k \geq 2$. This tensor $\mathcal{D}^{3k}_{2k\Upsilon_2}E^{E_7}_{(4+k)\Lambda_7}$ is a natural generalisation of a holomorphic function for the real symmetric space $(SU(8)/\mathbb{Z}_2)\backslash E_7$. In this interpretation, (3.134) is the equivalent of a holomorphic anomaly that is due to the supergravity logarithmic divergence.[23]

3.5 Eisenstein series in the M-theory large volume limit

Let us consider the Eisenstein series $E^{E_d}_{s\Lambda_d}$ in the parabolic P_2 corresponding to the large torus volume limit in eleven-dimensional supergravity. We define the volume of T^d as $(2\pi\ell V^{1/3})^d$, such that the volume spanned by a Euclidean M2-brane scales linearly in V.

For $d \geq 5$ we define $N \times N \in \wedge^{5,1}\mathbb{Z} = \wedge^5\mathbb{Z}\otimes\mathbb{Z}\ominus\wedge^6\mathbb{Z}$. The 1/2 BPS M2-brane winding matrix N satisfies $N \times N = 0$ and the Euclidean action reads

$$\begin{aligned} S^{\rm M2}(N) &= 2\pi\sqrt{\tfrac{1}{6}G_{IL}G_{JP}G_{KQ}N^{IJK}N^{LPQ}} - \frac{\pi i}{3}a_{IJK}N^{IJK} \\ &= 2\pi V|Z(N)| - 2\pi i(N,a)\ , \end{aligned} \tag{3.135}$$

with G_{IJ} the torus metric. The stabiliser of N is then $SL(3)\times SL(d-3)\ltimes\mathbb{R}^{3\times(d-3)} \subset SL(d)$. We write $v_N \in SL(3)$ and $v'_N \in SL(d-3)$ for the Levi stabiliser components of $v \in GL(d)$.

For $d \geq 6$ we define $(N\times N)\cdot N \in \wedge^{6,3}\mathbb{Z}^d$ the projection of $\wedge^6\mathbb{Z}\otimes\wedge^3\mathbb{Z}$ to the $R(\Lambda_3{+}\Lambda_6)$ irreducible representation. The 1/4 BPS M2-brane winding matrix N satisfies $N\times N \neq 0$ but $(N\times N)\cdot N = 0$. The stabiliser of N for $d=5$ is then $Sp(4,\mathbb{R})\times\mathbb{R}^4$, and for $d\geq 6$ one gets $GL(d-5)\times Sp(4,\mathbb{R})\ltimes\mathbb{R}^{4+4\times(d-5)+d-5} \subset P_{1,5} \subset SL(d)$. We will write ν_N the corresponding $GL(1)$ stabiliser for $d\geq 6$ (See Appendix D.2 for the precise normalisation). The Euclidean action for 1/4 BPS Euclidean M2-brane instantons of winding matrix $N \in \wedge^3\mathbb{Z}^d$ can be worked out from (3.105) as [24]

$$\begin{aligned} S^{\rm M2}(N) &= 2\pi\sqrt{\tfrac{1}{6}G_{IL}G_{JP}G_{KQ}N^{IJK}N^{LPQ} + \tfrac{1}{12}\sqrt{\tfrac{1}{5!}G_{I_1J_2}\cdots G_{I_5J_5}G_{KL}N^{[I_1I_2I_3}N^{I_4I_5]K}N^{J_1J_2J_3}N^{J_4J_5L}}} \\ &\quad -\frac{\pi i}{3}a_{IJK}N^{IJK} \\ &= 2\pi V\sqrt{|Z(N)|^2 + 2|z(N\times N)|} - 2\pi i(N,a) \end{aligned} \tag{3.136}$$

[23] There is a similar construction for $D^{7k}_{4k\Upsilon_1}E^{E_7}_{(2+k)\Lambda_2}$ that defines a submodule of $\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{(2+k)\Lambda_2}$ for $k\geq 2$. These representations are natural candidates to define cuspidal automorphic representations with abelian Fourier coefficients supported on the set of generic M2-brane instanton charges with compact stabiliser $G_2 \subset SL(7)$. They may contribute to 1/16 BPS couplings in string theory, and the coupling function could receive corrections not accessible by any perturbative methods.

[24] One must obtain $|Z_{123}| + |Z_{156}|$ for Z_{abc} with only non-zero components Z_{123} and Z_{156}. This is the case if one takes the formula below.

We will use the short notation in the second line, where the normalisations of $|Z(N)|$ and $|z(N\times N)|$ are multiplicative characters of the parabolic P_3 and $P_{5,1}$ for the highest weight representatives of N and $N\times N$, respectively.

We compute in Appendix D.2 for $d\leq 5$

$$
\begin{aligned}
V^{-\frac{6}{9-d}s}E^{E_d}_{s\Lambda_d} &= E^{SL(d)}_{s\Lambda_{d-1}} + \frac{\xi(2s-d+2)}{\xi(2s)}V^{d-2-2s}E^{SL(d)}_{(s-\frac{d-3}{2})\Lambda_2} + \delta_{d,5}\frac{\xi(2s-5)\xi(2s-7)}{\xi(2s)\xi(2s-2)}V^{10-4s} \\
&+ 2\frac{V^{\frac{d-2}{2}-s}}{\xi(2s)}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}{}' \frac{\sigma_{d-2-2s}(N)}{\mathrm{gcd}N^{\frac{2}{3}(d-3-2s)}}E^{SL(3)}_{(s-\frac{d-3}{2})\Lambda_2}(v_N)\frac{K_{s-\frac{d-2}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{2s-d+6}{6}}}e^{2\pi i(N,a)} \\
&+ 2\delta_{d,5}\frac{\xi(2s-5)V^{\frac{13}{2}-3s}}{\xi(2s)\xi(2s-2)}\sum_{\substack{N\in\wedge^3\mathbb{Z}^5\\ N\times N=0}}{}' \sigma_{7-2s}(N)E^{SL(2)}_{(s-\frac{5}{2})\Lambda_2}(v_N)\frac{K_{s-\frac{7}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{7}{2}-s}}e^{2\pi i(N,a)} \\
&+ 2\delta_{d,5}\tfrac{V^{5-2s}}{\xi(2s)\xi(2s-2)}\sum_{\substack{N\in\wedge^3\mathbb{Z}^5\\ N\times N\neq 0}}{}' \sum_{n|N} n^{7-2s}\sigma_{5-2s}(\tfrac{N\times N}{n^2})\frac{B_{\frac{3}{2},s-\frac{5}{2}}(V^2|Z(N)|^2, V^2|z(N\times N)|)}{|z(N\times N)|^{\frac{5}{2}-s}}e^{2\pi i(N,a)} \ . \qquad (3.137)
\end{aligned}
$$

For $d=6$ the parabolic P_2 is not abelian and in particular there are Euclidean M5-brane corrections to be taken into account. We write the M5-brane axion b. Let us now consider the non-abelian Fourier coefficient for E_6. We introduce the antisymmetric $SL(6)$ scalar product over $\wedge^3\mathbb{Z}^6$. Similarly as in the preceding section, one can define the generator of the Heisenberg algebra

$$
\frac{\partial}{\partial b}\ , \quad \frac{\partial}{\partial a_{IJK}} + \frac{1}{12}\varepsilon^{IJKLPQ}a_{LPQ}\frac{\partial}{\partial b}\ . \qquad (3.138)
$$

To define the non-abelian Fourier coefficients one must therefore choose a polariation. Once again we choose a polarisation circle in T^6, that involves the decomposition of $\mathfrak{sl}_6$ into P_5

$$
\begin{aligned}
\mathfrak{sl}_6 &\cong \overline{\mathbf{5}}^{(-2)}\oplus(\mathfrak{gl}_1\oplus\mathfrak{sl}_5)^{(0)}\oplus\mathbf{5}^{(2)}\ , \\
\mathbf{20} &\cong \mathbf{10}^{(-1)}\oplus\overline{\mathbf{10}}^{(1)}\ . \qquad (3.139)
\end{aligned}
$$

In this way one obtains the decomposition of $N=(q,\bar{q})$, $a=(a,\bar{a})$, and

$$
k\frac{\partial}{\partial b}\ , \quad q\cdot\frac{\partial}{\partial a} - \bar{a}q\frac{\partial}{\partial b}\ , \quad \bar{q}\cdot\frac{\partial}{\partial\bar{a}} + \bar{q}a\frac{\partial}{\partial b}\ , \quad p\cdot\frac{\partial}{\partial c} - \bar{a}\not{p}\frac{\partial}{\partial a}\ , \qquad (3.140)
$$

where we added the derivative with respect to the $SL(6)$ axions in $\mathbf{5}^{(2)}$. We can therefore consider the Fourier decomposition with respect to the unipotent character

$$
\psi_{k,q,p} = e^{2\pi i\left(k(b+\bar{a}\wedge a+\bar{a}\wedge\bar{a}\wedge c)+q(a+2c\wedge\bar{a})+pc\right)}\ , \qquad (3.141)
$$

where we define the wedge products such that c is a 1-form, q and $\bar{a}$ are 2-form, a and $\bar{q}$ are 3-forms, and b is a 5-form. One finds that this is a unipotent character of the parabolic P_6, for which we know the Fourier decomposition for any s [202]. Physically, we first expand in the T^6 M-theory volume V^2 and then on a particular circle radius r_5 to choose a polarisation. The M2-brane wrapping the polarisation circle have a well defined charge q, whereas the charge q of the M2-brane not wrapping that circle is not determined.

One obtains in this way the complete expansion of the E_6 series

$$
\begin{aligned}
E^{E_6}_{s\Lambda_6} &= V^{2s}E^{SL(6)}_{s\Lambda_5} + \frac{\xi(2s-4)}{\xi(2s)}V^4 E^{SL(6)}_{(s-\frac{3}{2})\Lambda_2} + \frac{\xi(2s-6)\xi(2s-8)}{\xi(2s)\xi(2s-3)}V^{12-2s}E^{SL(6)}_{(s-3)\Lambda_5} \\
&\quad + 2\frac{V^{s+2}}{\xi(2s)}\sum_{\substack{N\in\wedge^3\mathbb{Z}^6\\ N\times N=0}}{}' \frac{\sigma_{4-2s}(N)}{\mathrm{gcd}N^{2-\frac{4s}{3}}}E^{SL(3)}_{(s-\frac{3}{2})\Lambda_2}(v_N)\frac{K_{s-2}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{s}{3}}}e^{2\pi i(N,a)} \\
&\quad + 2\frac{\xi(2s-6)V^{8-s}}{\xi(2s)\xi(2s-3)}\sum_{\substack{N\in\wedge^3\mathbb{Z}^6\\ N\times N=0}}{}' \frac{\sigma_{8-2s}(N)}{\mathrm{gcd}N^{\frac{6-2s}{3}}}E^{SL(3)}_{(s-3)\Lambda_2}(v'_N)\frac{K_{s-4}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{6-s}{3}}}e^{2\pi i(N,a)} \\
&\quad + 2\frac{V^6}{\xi(2s)\xi(2s-3)}\sum_{\substack{N\in\wedge^3\mathbb{Z}^6\\ N\times N\neq 0\\ (N\times N)\cdot N=0}}{}' \sum_{n|N}\frac{n^{8-2s}\sigma_{6-2s}(\frac{N\times N}{n^2})}{\mathrm{gcd}(N\times N)^{\frac{s-3}{2}}}\frac{B_{\frac{3}{2},s-3}(V^2|Z(N)|^2, V^2|z(N\times N)|)}{\nu_N^{2s-6}|z(N\times N)|^{\frac{3(3-s)}{2}}}e^{2\pi i(N,a)} \\
&\quad + 2\frac{V}{\xi(2s)}\sum_{k\in\mathbb{Z}}{}'\sum_{\substack{q\in\mathbb{Z}^{10}\\ k|q\wedge q}}\psi_{k,q,\frac{q\wedge q}{k}}\Bigg(r_5^{\frac{10s}{3}}\sigma_{2s-1}(k,q,\tfrac{q\wedge q}{k})\frac{K_{s-\frac{1}{2}}\big(2\pi\sqrt{V^4k^2+V^2r_5^2|v(q+\bar{a}k)|^2+r_5^4|v(\frac{(q+\bar{a}k)\wedge(q+\bar{a}k)}{k})|^2}\big)}{\big(k^2+\frac{r_5^2}{V^2}|v(q+\bar{a}k)|^2+\frac{r_5^4}{V^4}|v(\frac{(q+\bar{a}k)\wedge(q+\bar{a}k)}{k})|^2\big)^{\frac{2s-1}{4}}} \\
&\quad + \frac{\xi(2s-4)r_5^{8-\frac{2s}{3}}}{\xi(2s-3)}\frac{\sigma_{7-2s}(k,q,\frac{q\times q}{k})}{\mathrm{gcd}(k,q,\frac{q\times q}{k})^{\frac{4}{5}(2-s)}}E^{SL(5)}_{(s-2)\Lambda_1}(\tilde{v}_{k,q})\frac{K_{s-\frac{7}{2}}\big(2\pi\sqrt{V^4k^2+V^2r_5^2|v(q+\bar{a}k)|^2+r_5^4|v(\frac{(q+\bar{a}k)\wedge(q+\bar{a}k)}{k})|^2}\big)}{\big(k^2+\frac{r_5^2}{V^2}|v(q+\bar{a}k)|^2+\frac{r_5^4}{V^4}|v(\frac{(q+\bar{a}k)\wedge(q+\bar{a}k)}{k})|^2\big)^{\frac{19-2s}{20}}}\Bigg) \\
&\quad + 2\frac{V^4 r_5^{\frac{10s}{3}}}{\xi(2s)\xi(2s-3)}\sum_{\substack{k\in\mathbb{Z}\\ q\in\mathbb{Z}^{10}}}{}'\sum_{p\in\mathbb{Z}^5}\psi_{k,q,p}\sum_{n|(k,q,p)}n^{2s-1}\sigma_{2s-4}\big(\tfrac{(kp-q\wedge q,p\wedge q)}{n^2}\big) \\
&\quad \times \frac{B_{\frac{3}{2},s-2}\big(V^4k^2+V^2r_5^2|v(q+\bar{a}k)|^2+r_5^4|v(p+2\bar{a}\wedge q+\bar{a}\wedge\bar{a}k)|^2,\ \sqrt{V^4r_5^4|v(kp-q\wedge q)|^2+V^2r_5^6|v(p\wedge q+\bar{a}\wedge(kp-q\wedge q))|^2}\big)}{\big(r_5^4|v(kp-q\wedge q)|^2+\frac{r_5^6}{V^2}|v(p\wedge q+\bar{a}\wedge(kp-q\wedge q))|^2\big)^{\frac{s-2}{2}}}\,. \qquad (3.142)
\end{aligned}
$$

For example for the 1/2 BPS coupling, one gets

$$
\begin{aligned}
\mathcal{E}_{(0,0)} &= 2\zeta(3)E^{E_6}_{\frac{3}{2}\Lambda_d} = 2V^3\zeta(3)E^{SL(6)}_{\frac{3}{2}\Lambda_5} + \frac{2\pi^2}{3}V^4 + 4\pi V^3\sum_{\substack{N\in\wedge^3\mathbb{Z}^6\\ N\times N=0}}{}'\sigma_1(N)\frac{e^{-2\pi V|Z(N)|}}{|Z(N)|}e^{2\pi i(N,a)} \\
&\quad + \frac{4\pi^3}{45}V\sum_{k\in\mathbb{Z}}{}'\sum_{\substack{q\in\mathbb{Z}^{10}\\ k|q\wedge q}}\psi_{k,q,\frac{q\wedge q}{k}}r_5{}^5\sigma_2(k,q,\tfrac{q\wedge q}{k})\frac{K_1\big(2\pi\sqrt{V^4k^2+V^2r_5^2|v(q+\bar{a}k)|^2+r_5^4|v(\frac{(q+\bar{a}k)\wedge(q+\bar{a}k)}{k})|^2}\big)}{\big(k^2+\frac{r_5^2}{V^2}|v(q+\bar{a}k)|^2+\frac{r_5^4}{V^4}|v(\frac{(q+\bar{a}k)\wedge(q+\bar{a}k)}{k})|^2\big)^{\frac{1}{2}}} \qquad (3.143)
\end{aligned}
$$

The M2-brane instanton corrections are of the same form in all dimensions, and the measure factor $\sigma_1(N)$ has been computed in a matrix model [203]. The partition function of k parallel M5-branes wrapping T^6 with the centre of mass motion factorized out gives the partition function of $SU(k)/\mathbb{Z}_k$ $\mathcal{N}=4$ Yang–Mills on T^4 [204], that is $\sigma_{-2}(k)=\frac{\sigma_2(k)}{k^2}$ [197].

We have $\mathcal{E}_{(1,0)} = \frac{\zeta(7)}{6}E^{E_6}_{\frac{7}{2}\Lambda_6} = \zeta(5)E^{E_6}_{\frac{5}{2}\Lambda_1}$ in five dimensions, and the coupling function $\mathcal{E}_{(1,0)}$ can therefore be read from (3.142) at $s=\frac{7}{2}$. We do not write the expression since there is no particular

simplification and the reader can set $s = \frac{7}{2}$ in (3.142). Let use note nonetheless that the generic abelian Fourier coefficient

$$8\pi V^6 \sum_{\substack{N\in\wedge^3\mathbb{Z}^6 \\ N\times N\neq 0 \\ (N\times N)\cdot N=0}}{}' \sum_{n|N} n\sigma_{-1}(\tfrac{N\times N}{n^2}) \frac{|z(N\times N)|^{\frac{1}{4}} K_1(2\pi V\sqrt{|Z(N)|^2+2|z(N\times N)|})}{\nu_N \mathrm{gcd}(N\times N)^{\frac{1}{4}}\sqrt{|Z(N)|^2+2|z(N\times N)|}} e^{2\pi i(N,a)} \tag{3.144}$$

is factorised and $\sum_{n|N} n\sigma_{-1}(\frac{N\times N}{n^2})$ should therefore be the partition function for 1/4 BPS M2-brane instantons. As expected, this Fourier coefficient is universal in all dimensions $D \leq 6$, see (D.36) for the expression in $D = 5$ and $D = 4$.

4 Eleven-dimensional supergravity on T^d

Eleven-dimensional supergravity on a circle is conjectured to describe type IIA string theory in the strong coupling limit $e^{\phi_{\rm A}} \gg 1$ at low energy $s\ell^2 \ll 1$ [6]. Compactification on an additional circle in string frame allows to identify both type IIA and type IIB moduli in nine dimensions, with the metric

$$\begin{aligned} ds^2_{\rm 11D} &= e^{\frac{4}{3}\phi_{\rm A}}\left(dy^{10} + C_9 dy^9\right)^2 + e^{-\frac{2}{3}\phi_{\rm A}}\left(R_{\rm A}^2 (dy^9)^2 + ds^2_{\rm 9D}\right) \\ &= \left(\frac{e^{\phi_{\rm B}}}{R_{\rm B}^4}\right)^{\frac{1}{3}}\left(e^{\phi_{\rm B}}\left(dy^{10} + C dy^9\right)^2 + e^{-\phi_{\rm B}}(dy^9)^2\right) + \left(\frac{R_{\rm B}}{e^{\phi_{\rm B}}}\right)^{\frac{2}{3}} ds^2_{\rm 9D} \,, \end{aligned} \tag{4.1}$$

where we write $\phi_{\rm A}$ and $R_{\rm A}$ the type IIA dilaton and circle radius in string length, and respespectively $\phi_{\rm B}$ and $R_{\rm B}$ in type IIB, while $C_9 = C$ for the Ramond-Ramond one-form and axion. The small torus volume limit corresponds in type IIB to the large radius limit such that one retrieves type IIB string theory in ten dimensions. The $SL(2,\mathbb{Z})$ S-duality symmetry of type IIB string theory is then realised geometrically as the group of global diffeomorphisms of the M-theory torus.

Eleven-dimensional supergravity is instead a good approximation at large torus volume, corresponding to small type IIB radius. In this limit one expects the supergravity four-graviton amplitude to reproduce accurately the string theory amplitude at low energy. The supergravity four-graviton amplitude in eleven-dimensional supergravity has been computed up to five-loop order [205–208]. The low-momenta expansion of the amplitude exhibits that the lowest order in momenta only get contributions from the first loop orders. In particular, $\mathcal{E}_{(0,0)}$ only gets contributions at one-loop, $\mathcal{E}_{(1,0)}$ up to two-loop and $\mathcal{E}_{(0,1)}$ up to three-loop. Moreover, the non-perturbative corrections in M-theory due to BPS M2-brane instantons do not exist on T^2. Thanks to these properties one can then safely take the small torus volume limit. The four-graviton amplitude in eleven dimensions on $\mathbb{R}^{1,8} \times T^2$ was analysed in [9, 55, 56, 209, 210] up to three-loop order and the one-loop and the two-loop supergravity amplitudes determine the exact coupling functions $\mathcal{E}_{(0,0)}$, $\mathcal{E}_{(1,0)}$ and $\mathcal{E}_{(0,1)}$ in type IIB string theory [9, 55, 56]. The validity of this construction may rely on some conjectures, but have been checked to match string theory computations [166, 201, 84, 10–12].

In this section we repeat this computation on a torus T^d for $3 \leq d \leq 7$. We will use this result to check consistency with the non-perturbative coupling functions in the large volume limit.

4.1 Coupling functions from the supergravity amplitude

The supergravity limit does not give the exact coupling functions $\mathcal{E}_{(0,0)}$, $\mathcal{E}_{(1,0)}$ and $\mathcal{E}_{(0,1)}$ in general. Including the eleven-dimensional effective action, one expects nevertheless to get all the perturbative power-low terms in the torus volume. The only BPS non-perturbative corrections are due to M2 and M5-brane instantons, as exhibited explicitly in (3.142) and (3.143) in the preceding section.

We define the eleven-dimensional metric in Einstein frame as

$$ds^2_{\rm 11D} = r^{\frac{9-d}{3}} U_{IJ} dy^I dy^J + r^{-\frac{d}{3}} g_{\mu\nu} dx^\mu dx^\nu \,, \tag{4.2}$$

with U unimodular and r defined such that the torus volume is $\mathrm{Vol}(T^d) = (2\pi\ell r^{\frac{9-d}{6}})^d$. The relevant physical modulus used in section 3.5 is $V = r^{\frac{9-d}{2}}$, which scales with the volume spanned by the

Euclidean M2-brane inside the torus. We write the four-graviton amplitude in terms of the scalar function $A(s,t,u,\phi)$ (2.13) with $2\kappa_D^2 = (2\pi)^{8-d}\ell^{9-d}$ and the torus T^d has coordinates y^I with periodicity $y^I \cong y^I + 2\pi\ell$. It admits the loop expansion

$$A = \sum_{n=1}^{\infty} \kappa_D^{2n} \Big(A^{\text{n-loop}} + c\ell^6 A^{\text{n-loop}} \big[\int t_8 t_8 R^4 + \dots \big] + \frac{1}{2} c^2 \ell^{12} A^{\text{n-loop}} \big[\tfrac{1}{5} \int t_8 t_8 \nabla^6 R^4 + \dots + \big(\int t_8 t_8 R^4 + \dots \big)^2 \big] + \dots \tag{4.3}$$

where the first term is the supergravity amplitude, the second represents the form-factor with the insertion of the leading higher derivative correction in the Wilsonian action in eleven dimensions. The last line includes the form-factor with the insertion of the next-to-leading correction and the double insertion of the leading correction. In eleven dimensions there is a unique BPS counter-term that starts as $\mathcal{R}^4$, and that we must include with a fixed coefficient $c = \frac{2\pi^2}{3}$ to match the type II string theory four-graviton amplitude [9]. This coefficient can be fixed in many ways, for example through the cancelation of the M5-brane world-volume anomaly [127], or the cancelation of the U-duality $SL(2,\mathbb{Z})$ anomaly in $D \leq 8$ [42, 51].

The one-loop amplitude in eleven dimensions on T^d gives rise to the sum over the Kaluza–Klein states

$$\begin{aligned} A^{\text{1-loop}}_{\square^\epsilon} &= 64\kappa_D^2 \sum_{n\in\mathbb{Z}^d} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2+M^2)((p-k_1)^2+M^2)((p-k_1-k_2)^2+M^2)((p+k_4)^2+M^2)} + \circlearrowleft \\ &= -8\pi^{\frac{5-d}{2}} \ell^6 \frac{\Gamma(\frac{d-7}{2})\Gamma(\frac{9-d}{2})^2}{\Gamma(8-d)} \int_0^1 dx \left((-\ell^2 s)^{\frac{3-d}{2}} \frac{(1-x)^{\frac{5-d}{2}}}{(1+\frac{t}{s})x-1} + (-\ell^2 t)^{\frac{3-d}{2}} \frac{(1-x)^{\frac{5-d}{2}}}{(1+\frac{s}{t})x-1} \right) \\ &\quad + 4\pi\ell^6 \int_0^\infty \frac{dL}{L^{\frac{5-d}{2}}} {\sum_{n\in\mathbb{Z}^d}}' \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \; e^{\pi L\ell^2((1-x_1)(x_2-x_3)s + x_3(x_1-x_2)t) - \pi L r^{-3} U^{-1}[n])} + \circlearrowleft \end{aligned} \tag{4.4}$$

where the Kaluza–Klein mass is $\ell^2 M^2 = r^{-3}U^{-1}[n]$ and the second term including only the sum over non-zero modes $n \in \mathbb{Z}^d$ is analytic. The dimensional regularisation $d \to d + 2\epsilon$ is chosen to agree with the one derived in string theory in Section 2.2. The two-loop amplitude is written in Schwinger parameter space as in (2.102)

$$\begin{aligned} A^{\text{2-loop}}_{\square\square^\epsilon} &= \frac{\pi}{4}\ell^{10} \sum_{n_i\in\mathbb{Z}^d} \int_{\mathbb{R}_+^3} \frac{dL_1 dL_2 dL_3}{\big(\sum_{I<J} L_I L_J\big)^{5-\frac{d}{2}}} s^2 e^{-\pi r^{-3}(L_1 U^{-1}[n_1] + L_2 U^{-1}[n_2] + L_3 U^{-1}[n_1+n_2])} \\ &\quad \int_{0\le y_1\le y_2\le 1} dy_1 dy_2 \; L_1^2 \; e^{\pi\ell^2 s L_1 y_1 (1-y_2)} \Bigg(\int_{0\le y_3\le y_4\le 1} dy_3 dy_4 \; L_2^2 \\ &\quad e^{\pi\ell^2 \left(s L_2 y_3 (1-y_4) + \frac{L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \left(t(y_2-y_1)(y_4-y_3) + s(1-y_1-y_4)(1-y_2-y_3) \right) \right)} \\ &\quad + \int_0^1 dy_3 \int_0^1 dy_4 \; L_2 L_3 \; e^{\pi\ell^2 \frac{L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \left(t(y_2-y_1)(y_4-y_3) + s(1-y_1-y_4)(1-y_2-y_3) \right)} \Bigg) + \text{perm.} \end{aligned} \tag{4.5}$$

It must be decomposed into the component with $n_i = 0$ corresponding to the two-loop amplitude in D dimensions, the component with $n_1 = 0$, $n_2 = 0$ or $n_1 + n_2 = 0$ corresponding to a one-loop

form-factor (when one of the particles in the loops is massless) and the component with $n_i \neq 0$ and $n_1 + n_2 \neq 0$ that contributes to the Wilsonian effective action

$$A_{\mathrm{W}\,\epsilon}^{\text{2-loop}} = \frac{\ell^6}{6}\int_{\mathbb{R}_+^3} \frac{dL_1 dL_2 dL_3}{\left(\sum_{I<J} L_I L_J\right)^{3-\frac{d}{2}}} \left(4\pi\sigma_2 + \tfrac{4\pi^2}{3}\sigma_3\Big(\sum_I L_I - \frac{5L_1L_2L_3}{\sum_{I<J}L_IL_J}\Big) + \dots\right) \sum_{\substack{n_i\in\mathbb{Z}^d \\ n_i\neq 0 \\ n_1+n_2\neq 0}} e^{-\pi\Omega_2^{ij}\frac{r^{-3}}{\ell^2} n_i^\intercal U^{-1} n_j} \tag{4.6}$$

where we introduced

$$\sigma_k = (\tfrac{\ell}{2})^{2k}(s^k + t^k + u^k)\ . \tag{4.7}$$

In critical dimensions in which the form-factor diverges logarithmically the sum over degenerate Kaluza–Klein modes also contribute to the Wilsonian component of the amplitude, as we shall describe in the next section.

The supergravity three-loop four-graviton amplitude was derived in [125], and further simplified in [126, 205]. To compute the three-loop amplitude in Schwinger parameter space, it is convenient to use the so-called BCJ integrand [205]. There are twelve diagrams in total, nine are associated to the vacuum tetrahedron and can be written in the schematic form

$$\begin{aligned} A_{\boxplus\,\epsilon}^{\text{3-loop}} &= \frac{\pi}{16}\sum_{S_3}\sum_{n_i\in\mathbb{Z}^d}\int d^{3D}\ell \int_{\mathcal{S}_+} d^6\Omega P_E^{(4)}(\Omega)\int_{\mathcal{F}_E} d^4x\,(N^{ij}\ell_i\cdot k_j + f(s,t))^2 e^{-\pi\left[\Omega^{ij}(\ell_i\cdot\ell_j + \frac{r^{-3}}{\ell^2}n_i^\intercal U^{-1}n_j) + 2M^{ij}\ell_i\cdot k_j + g(s,t)\right]} \\ &= \frac{\pi}{16}\sum_{S_3}\sum_{n_i\in\mathbb{Z}^d}\int_{\mathcal{S}_+}\frac{d^6\Omega}{\det\Omega^{\frac{D}{2}}} P_E^{(4)}(\Omega)\int_{\mathcal{F}_E} d^4x\left(\frac{1}{2\pi}\Omega_{kl}^{-1}N^{ki}N^{lj}k_i\cdot k_j + (-\Omega_{kl}^{-1}M^{ki}N^{lj}k_i\cdot k_j + f(s,t))^2\right) \\ &\qquad\qquad \times e^{-\pi\Omega^{ij}\frac{r^{-3}}{\ell^2}n_i^\intercal U^{-1}n_j + \pi\Omega_{kl}^{-1}M^{ki}M^{lj}k_i\cdot k_j - \pi g(s,t)} \end{aligned} \tag{4.8}$$

where S_+ is the subset of the three by three positive matrices

$$\Omega = \begin{pmatrix} L_{23}+L_{03}+L_{02} & -L_{03} & -L_{02} \\ -L_{03} & L_{31}+L_{03}+L_{01} & -L_{01} \\ -L_{02} & -L_{01} & L_{12}+L_{01}+L_{02} \end{pmatrix} \tag{4.9}$$

where $L_{\mu\nu} = L_{\nu\mu} \geq 0$. The matrices M^{ij} are linear in the $L_{\mu\nu}$ with coefficients that are affine in the Feynman parameters x and depend on each diagrams. The Feynman parameters come with their integration domain $\mathcal{F}_A$ and a polynomial $P_E^{(4)}(\Omega)$ in the Schwinger parameters coming from the change of variables. The matrices N^{ij} have constant coefficients and determine the kinetic BCJ numerators. f and g are linear in the Mandelstam variables, linear in the Schwinger parameters and polynomial in the Feynman parameters. We used the explicit expressions in a Mathematica file, but they are rather lengthly and not so illuminating so we chose to do not display them.

The ladder skeleton is obtained by setting $L_{02} = 0$. To combine the two types of diagrams it is convenient to use the same Schwinger parameter space S_+ with a Dirac distribution $\delta(L_{02})$. We write schematically the three diagrams associated to the vacuum ladder diagram as

$$\begin{aligned} A_{\square\square\,\epsilon}^{\text{3-loop}} &= \frac{\pi}{16}\sum_{S_3}\sum_{n_i\in\mathbb{Z}^d}\int d^{3D}\ell\int_{\mathcal{S}_+} d^6\Omega\,\delta(\Omega^{13})P_A^{(5)}(\Omega)\int_{\mathcal{F}_A} d^5x\, s^4 e^{-\pi\left[\Omega^{ij}(\ell_i\cdot\ell_j + \frac{r^{-3}}{\ell^2}n_i^\intercal U^{-1}n_j) + 2M^{ij}\ell_i\cdot k_j + g(s,t)\right]} \\ &= \frac{\pi}{16}\sum_{S_3}\sum_{n_i\in\mathbb{Z}^d}\int_{\mathcal{S}_+}\frac{d^6\Omega}{\det\Omega^{\frac{D}{2}}}\delta(\Omega^{13})P_A^{(5)}(\Omega)\int_{\mathcal{F}_A} d^5x\, s^4 e^{-\pi\Omega^{ij}\frac{r^{-3}}{\ell^2}n_i^\intercal U^{-1}n_j + \pi\Omega_{kl}^{-1}M^{ki}M^{lj}k_i\cdot k_j - \pi g(s,t)}. \end{aligned} \tag{4.10}$$

Writing the ladder diagrams contribution this way is justified if we want to interpret the three-loop supergravity amplitude as a tropical limit of the three-loop string amplitude. The moduli space of genus three Riemann surfaces can be identified with the Siegel symmetric space $U(3)\backslash Sp(6,\mathbb{R})/Sp(6,\mathbb{Z})$ with the separating degeneration locus removed. The tetrahedron and ladder diagrams then arise from the two maximal non-degeneration limits [175,211]. Note nonetheless that one does not expect to be able to write the four-point amplitude as an integral over the moduli space of genus three Riemann surfaces with four punctures [212]. The fact that it is possible for the leading Wilson coefficient might be related to the fact that it takes the form of a vacuum diagram [201].

We shall only describe in detail the leading contribution to the Wilsonian effective action that arises in combining the contributions at low energy

$$A_{\mathrm{W}\,\epsilon}^{\text{3-loop}} = \int_{\mathcal{S}_+} \frac{d^6\Omega}{\det\Omega^{\frac{9-d}{2}}} \Big(\frac{5}{6}\sigma_3 + \frac{\frac{1}{6}P^{(4)}(\Omega) + \delta(\Omega^{13})P^{(5)}(\Omega)}{\det\Omega}\sigma_2^2 + \dots\Big) \sum_{\substack{n_i\in\mathbb{Z}^d \\ n_i\neq 0 \\ n_i+n_j\neq 0}} e^{-\pi\Omega^{ij}\frac{r^{-3}}{\ell^2}n_i^{\intercal}U^{-1}n_j} \quad (4.11)$$

with the definitions

$$\begin{aligned} P^{(4)}(\Omega) &= \frac{\pi}{6}\sum_{S_4}\Big(\det\Omega\, L_{12} - \frac{17}{2}L_{23}L_{31}L_{12}L_{03} - \frac{19}{8}L_{23}L_{31}L_{01}L_{02}\Big) + \frac{\pi}{64}\Big(\sum_{S_4}L_{23}L_{01}\Big)^2 \\ P^{(5)}(\Omega) &= \frac{\pi}{4}\sum_{\mathbb{Z}_2\times\mathbb{Z}_2}\Big(L_{23}L_{31}L_{12}(L_{01}L_{03}+2L_{23}L_{01}+L_{23}L_{12})\Big) \end{aligned} \quad (4.12)$$

where $\mathbb{Z}_2\times\mathbb{Z}_2$ is the stabilizer of L_{02} in S_4 permuting the vertices 0 and 2 and the vertices 1 and 3.

The Schwinger parameter space is a 24^{th} order unfolding of the $SL(3,\mathbb{Z})$ fundamental domain $\mathcal{G}_3 = \mathcal{S}_+/S_4$, where S_4 acts as the permutations of the indices μ of the tetrahedron vertices [210, 74]. In the same way at two-loop, the Schwinger parameter space is a 6^{th} order unfolding of the $PGL(2,\mathbb{Z})$ fundamental domain $\mathcal{G}_2 = \mathcal{S}_+/S_3$ [55]. We shall write the analytic component of the amplitude in Schwinger parameter space using the tropical moduli space $\mathcal{G}_h \cong SO(h)\backslash GL(h,\mathbb{R})/PGL(h,\mathbb{Z})$ for $h = 1,2,3$ of the vacuum diagrams

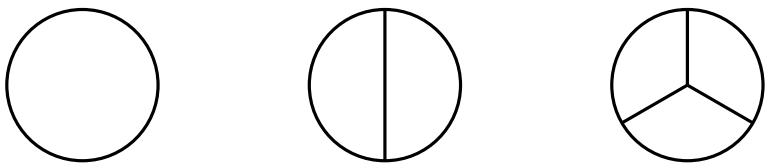

Figure 13: The skeletons graphs with respective symmetry $\{1\}$, S_3 and S_4.

We must also take into account the insertions of the higher-derivative terms in the eleven-dimensional supergravity effective action. The form-factors have not been derived directly in supergravity, but one can deduce them from the perturbative string theory amplitude or from the sub-divergences of the supergravity amplitude. The second method was used in [209] to derive the contribution of the R^4 form-factor. Because the power-low divergence does not preserve supersymmetry a priori, one must be careful and it is more safe to only consider logarithmic divergences in

the critical dimensions. The pole in $\frac{1}{\epsilon}$ in dimensional regularisation must define supersymmetry invariants as the first logarithmic divergence in the theory in a given dimension. For R^4 it turns out to give the same result as in string theory, so we can simply deduce from (2.51) that

$$\begin{aligned} A^{\text{1-loop}}_{\blacksquare\!\!\triangleleft\,\epsilon} &= 2cr^d\,\ell^6\kappa_D^2\sum_{n\in\mathbb{Z}^d}\int\frac{d^Dp}{(2\pi)^D}\frac{s^2}{(p^2+M^2)((p-k_1)^2+M^2)((p-k_1-k_2)^2+M^2)}+\circlearrowleft \\ &= \frac{cr^d}{8}\,\ell^{10}\sum_{n\in\mathbb{Z}^d}\int_0^\infty\frac{dL}{L^{3-\frac{d}{2}}}\int_0^1dx_2\int_0^{x_2}dx_1\Big(s^2e^{\pi\ell^2sLx_1(x_2-x_1)-\pi Lr^{-3}U^{-1}[n]}+\circlearrowleft\Big)\,. \end{aligned} \tag{4.13}$$

One can obtain similarly the two-loop form-factor starting from the three-loop amplitude and taking a subdivergence in $D=8-2\epsilon$ dimensions

$$\begin{aligned} A^{\text{2-loop}}_{\blacksquare\!\!\triangleleft\,\epsilon} = cr^d\ell^6\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{9-d}{2}}}\sum_{n_i\in\mathbb{Z}^d}e^{-\pi r^{-3}\Omega_2^{ij}n_i^{\intercal}U^{-1}n_j}\Big(\frac{5}{\pi}\sigma_3 \\ +\Big(\frac{1}{6}\Big(7\sum_I L_I-\frac{8L_1L_2L_3}{\sum_{I<J}L_IL_J}\Big)+\frac{1}{4}\sum_I L_{I+1}L_{I+2}\delta(L_I)\Big)\sigma_2^2+\dots\Big)\,. \end{aligned} \tag{4.14}$$

As expected from supersymmetry, this result is consistant with the form-factor contribution at two-loop in string theory (2.126), upon replacing $2\zeta(3)$ by cr^d. The term in $\delta(L_I)$ corresponds to a eight-shape Feynman diagram [25]

$$2cr^d\,\ell^6\kappa_D^4\Big(\sum_{n\in\mathbb{Z}^d}\int\frac{d^Dp}{(2\pi)^D}\frac{s^2}{(p^2+M^2)((p-k_1)^2+M^2)((p-k_1-k_2)^2+M^2)}\Big)^2+\circlearrowleft \tag{4.15}$$

that should appear in the two-loop string amplitude. One expects such term to appear in the further degeneration of the non-separating degeneration limit through the first diagram in figure 10, but we have not carried out this computation.

Here we shall consider the low energy effective action up to fourteen derivatives, for which only the first three loop orders described above contribute in supergravity, including the $\mathcal{R}^4$ counter-term with coefficient $c=\frac{2\pi^2}{3}$. Combining all the contributions described above, we get

$$\begin{aligned} A_{W\epsilon} &= \ell^6\sum_{p,q}\sigma_2^p\sigma_3^q\,\mathcal{E}^{\text{sugra}}_{(p,q)} \\ &= 4\pi\ell^6\int_0^\infty\frac{dL}{L^{\frac{5-d}{2}}}{\sum_{n\in\mathbb{Z}^d}}'\int_0^1dx_1\int_0^{x_1}dx_2\int_0^{x_2}dx_3\;e^{\pi L\ell^2((1-x_1)(x_2-x_3)s+x_3(x_1-x_2)t)-\pi Lr^{-3}U^{-1}[n])}+\circlearrowleft \\ &\quad+4\pi\ell^6\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\Big(\sigma_2+2\sigma_3\varphi^{\text{tr}}_{\text{KZ}}(\Omega_2)+\dots\Big){\sum_{n_i\in\mathbb{Z}^{2\times d}}}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^{\intercal}U^{-1}n_j} \\ &\quad+20\ell^6\sigma_3\int_{\mathcal{G}_3}\frac{d^6\Omega_3}{|\Omega_3|^{\frac{9-d}{2}}}{\sum_{n_i\in\mathbb{Z}^{3\times d}}}'e^{-\pi\Omega_3^{ij}r^{-3}n_i^{\intercal}U^{-1}n_j} \end{aligned}$$

[25] Off-shell one should write one integrand in function of k_1,k_2 and the other in function of k_3,k_4, but because the integral only depends on $s=-(k_1+k_2)^2=-(k_3+k_4)^2$ we can write it as a square.

$$
\begin{aligned}
&+\ell^6\Big(cr^d+\frac{(cr^d)^2}{10}\sigma_3+\dots\Big)\\
&+\frac{\ell^{10}}{8}cr^d\Big(s^2\int_0^\infty\frac{dL}{L^{\frac{7-d}{2}}}{\sum_{n\in\mathbb{Z}^d}}'\int_0^1dx_1\int_0^{x_1}dx_2\;e^{\pi L\ell^2(1-x_1)x_2s-\pi Lr^{-3}U^{-1}[n]}+\circlearrowleft\Big)\\
&+\frac{5}{\pi}\ell^6cr^d\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{9-d}{2}}}(\sigma_3+\dots){\sum_{n_i\in\mathbb{Z}^{2\times d}}}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j}+\dots
\end{aligned}
\tag{4.16}
$$

The three first lines follow from the loop expansion in two-derivative supergravity. The fourth line is the invariant leading Wilsonian correction while the two last lines are the one-loop and two-loop contributions with one insertion of the leading Wilsonian correction. The final ellipses state for higher loop contributions that only contribute to higher derivative terms. We have introduced the tropical Kawazumi–Zhang invariant according to [173]

$$
\varphi^{\rm tr}_{\rm KZ}(\Omega_2)=\frac{\pi}{6}\Big(\sum_I L_I-\frac{5L_1L_2L_3}{\sum_{I<J}L_IL_J}\Big)\,. \tag{4.17}
$$

One obtains formally for all d

$$
\mathcal{E}^{\rm sugra}_{(0,0)}\;\sim\;V^{3\frac{d-3}{9-d}}\Big(\frac{2\pi^2}{3}V+2\zeta(3)E^{SL(d)}_{\frac{3}{2}\Lambda_1}\Big) \tag{4.18}
$$

$$
\mathcal{E}^{\rm sugra}_{(1,0)}\;\sim\;8\pi V^{3\frac{d+1}{9-d}}\Big(\xi(4)\xi(d+1)E^{SL(d)}_{\frac{d+1}{2}\Lambda_{d-1}}+\frac{\xi(2)\xi(5)}{V^2}E^{SL(d)}_{\frac{5}{2}\Lambda_1}+\frac{\xi(3)\xi(4)}{V^3}E^{SL(d)}_{2\Lambda_2}\Big) \tag{4.19}
$$

$$
\begin{aligned}
\mathcal{E}^{\rm sugra}_{(0,1)}\;\sim\;&V^{3\frac{d+3}{9-d}}\Big(\frac{8\pi^4}{567}\xi(d+3)E^{SL(d)}_{\frac{d+3}{2}\Lambda_{d-1}}+\frac{\pi^4}{27V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(d)}_{\frac{3}{2}\Lambda_1}\\
&+\frac{8\pi}{V^3}\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\varphi^{\rm tr}_{\rm KZ}(\Omega_2){\sum_{n_i\in\mathbb{Z}^d}}'e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}+\frac{40\xi(2)}{V^5}\xi(6)\xi(5)E^{SL(d)}_{3\Lambda_2}+\frac{40}{V^6}\xi(3)\xi(4)\xi(5)E^{SL(d)}_{\frac{5}{2}\Lambda_3}\Big)
\end{aligned}
\tag{4.20}
$$

where we use the functional relations following from (3.20)

$$
\prod_{i=0}^{k-1}\big[\xi(2s-i)\big]E^{SL(d)}_{s\Lambda_k}=\prod_{i=0}^{k-1}\big[\xi(d-2s-i)\big]E^{SL(d)}_{(\frac{d}{2}-s)\Lambda_{d-k}}\,. \tag{4.21}
$$

These expressions are formally correct, but to give a meaningful answer we need to be more careful whenever there is a pole in the Eisenstein series associated to an ultra-violet divergence in supergravity. Then the duality invariant coupling functions get a contribution from the non-analytic component of the amplitude. This is the subject of the next section, which is admittedly aimed at the most committed readers.

4.2 Cancelation of divergences and logarithmic terms

One needs to combine the analytic and non-analytic components of the amplitude in the Mandelstam variables to define the finite amplitude, as we did in Section 2 in string theory. Accordingly we can use the dimensional regularisation $d\to d+2\epsilon$ with the definition of $\kappa_D^2=\frac{1}{2}(2\pi)^{8-d}\ell^{9-d}$ to extract the finite amplitude, up to orders in which the perturbative supergravity amplitude is finite

in dimensional regularisation and the low energy effective action is under control. The first pole in the dimensional regularisation parameter ϵ appears at two-loop order and involves a $\nabla^{12}R^4$ type counter-term in eleven dimensions. Fortunately this is far beyond the terms we shall consider and this will not affect our computations. Therefore we will not need to renormalise eleven-dimensional supergravity in dimensional regularisation.

We could in principle extract the complete amplitude with a well defined split in the analytic and the non-analytic components of the amplitude as we did in section 2.2. In this section we will not define carefully the non-analytic component and we shall instead cancel the divergences between analytic and non-analytic components by using an ad hoc infrared regularisation of the amplitudes with a massless particle in the loop. In practice we shall give a mass μ to the massless propagators and only keep the logarithmic terms in $\log\mu$. Fixing these constant is exactly equivalent as determining the renormalisation scheme determined by string theory. To determine the scheme, we would not only need to compute the contributions from the non-analytic components carefully, but also to keep track of all the precise pole subtractions used to define renormalised $SL(d)$ Langlands Eisenstein series. This is a doable but rather tedious computation. Testing the coupling functions up to the match of the renormalisation scheme is already a very strong consistency check.

$\mathcal{R}^4$ at 1-loop

At one-loop we therefore include the contribution from the massless mode with an infrared mass μ as follows

$$\begin{aligned} & 2\pi\int_0^\infty \frac{dL}{L^{\frac{5-d-2\epsilon}{2}}}\Big(\sum_{n\in\mathbb{Z}^d}{}' e^{-\pi L r^{-3}U^{-1}[n]}+e^{-\pi L\mu^2}\Big) \\ = \;& 4\pi\xi(d+2\epsilon-3)r^{\frac{3}{2}(d+2\epsilon-3)}E^{SL(d)}_{\frac{d+2\epsilon-3}{2}\Lambda_{d-1}}+2\pi\Gamma(\tfrac{d+2\epsilon-3}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-3}{2}} \\ \underset{\epsilon\to0}{=}\;& 4\pi\xi(3)\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}-2\pi\log V-4\pi\log(2\pi\mu)\ . \end{aligned} \tag{4.22}$$

One concludes that formula (4.18) for $\mathcal{E}^{\text{sugra}}_{(0,0)}$ is accurate, except for $d=3$ in which case we have

$$\mathcal{E}^{\text{sugra}}_{(0,0)}\underset{d=3}{=}\frac{2\pi^2}{3}V+2\zeta(3)\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}-2\pi\log V\ . \tag{4.23}$$

This is indeed the result obtained for the non-perturbative coupling function [38]. The renomalisation prescription compatible with the string theory amplitude has been written explicitly in [213], which gives an additional $22\pi/3$.

Here and below we define the regularised Eisenstein series $\widehat{E}^{SL(d)}_{\frac{d}{2}\Lambda_k}$ such that

$$\xi(2\epsilon)\prod_{i=1}^{k-1}\big[\xi(2\epsilon-i)\big]E^{SL(d)}_{\epsilon\Lambda_k}=\xi(2\epsilon)\prod_{i=1}^{k-1}\big[\xi(1+i)\big]+\prod_{i=0}^{k-1}\big[\xi(d-i)\big]\widehat{E}^{SL(d)}_{\frac{d}{2}\Lambda_{d-k}}+\mathcal{O}(\epsilon)\ . \tag{4.24}$$

More generally we write $\widehat{E}^{G}_{j\Lambda_k}$ for the renormalised value of a maximal parabolic Eisenstein series as defined from (3.23). The precise pole subtraction prescription will be mostly irrelevant since we do not compute carefully the constants fixing the renormalisation scheme determined by string theory.

$\sigma_2\mathcal{R}^4$ up to 2-loop

The 1-loop contribution to $\mathcal{E}^{\text{sugra}}_{(1,0)}$ is regular in all dimensions

$$4\pi\xi(4)\int_0^\infty \frac{dL}{L^{\frac{1-d-2\epsilon}{2}}}\sum_{n\in\mathbb{Z}^d} e^{-\pi L r^{-3}U^{-1}[n]} = 8\pi V^{3\frac{d+1}{9-d}}\xi(4)\xi(d+1)E^{SL(d)}_{\frac{d+1}{2}\Lambda_{d-1}}\,. \tag{4.25}$$

For the 2-loop contribution to $\mathcal{E}^{\text{sugra}}_{(1,0)}$ we decompose the sum over pairs of Kaluza–Klein momenta n_i into the sum over linearly independent n_i, the linearly dependent n_i with an infrared regulating mass on the orthogonal component, and the massless contribution with infrared regulator $R(\mu) = \det\Omega_2\mu^2$. The latter is not very natural, as one may rather define $\mathrm{Tr}\Omega_2\mu^2$ to give a mass μ to each internal propagator, but the leading logarithm is universal and we can use this definition to simplify the computation.

For linearly independent momenta n_i we can always map them using $SL(d,\mathbb{Z})$ to the set of momenta on a torus $T^2\subset T^d$, so we can write the sum over two-by-two matrices combined with a Poincaré sum over $P_{d-2}(\mathbb{Z})\backslash SL(d,\mathbb{Z})$. Recall that $P_{d-2} = GL(2)\times SL(d-2)\ltimes\mathbb{R}^{2\times(d-2)}$. Similarly for a single charge we can write the sum over momenta n_i for a preferred circle with a Poincaré sum over $P_{d-1}(\mathbb{Z})\backslash SL(d,\mathbb{Z})$. This gives

$$\begin{aligned}
&4\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d-2\epsilon}{2}}}\sum_{n_i\in\mathbb{Z}^d}{}' e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j-\pi R(\mu)} \qquad (4.26)\\
=\;& 8\pi\sum_{\gamma\in P_{d-2}\backslash SL(d)}\sum_{\substack{N\in\mathbb{Z}^{2\times2}/GL(2,\mathbb{Z})\\ \det N\neq0}}\int_{\mathcal{H}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{3}{2}-\frac{d+2\epsilon-4}{2}}}e^{\pi r^{-3}y_{d-2}\mathrm{Tr}\,\Omega_2 N^\intercal U^{-1}N}\Big|_\gamma\\
&+4\pi\sum_{\gamma\in P_{d-1}\backslash SL(d)}\sum_{n=1}^\infty\int_0^\infty\frac{dL}{L^{1-\frac{d+2\epsilon-3}{2}}}e^{\pi L r^{-3}y_{d-1}^2n^2}\Big|_\gamma\int_0^\infty\frac{dt}{t^{1-\frac{d-5}{2}}}e^{-\pi t\mu^2}\\
&\qquad+4\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{3}{2}-\frac{d+2\epsilon-4}{2}}}e^{-\pi\det\Omega_2\mu^2}\\
=\;& 8\pi\xi(d+2\epsilon-4)\xi(d+2\epsilon-5)r^{3(d+2\epsilon-4)}E^{SL(d)}_{\frac{d+2\epsilon-4}{2}\Lambda_{d-2}}\\
&+4\pi\xi(d+2\epsilon-3)r^{\frac{3}{2}(d+2\epsilon-3)}E^{SL(d)}_{\frac{d+2\epsilon-3}{2}\Lambda_{d-1}}\frac{\Gamma(\frac{d+2\epsilon-5}{2})}{(\pi\mu^2)^{\frac{d+2\epsilon-5}{2}}}+4\pi\xi(2)\frac{\Gamma(\frac{d+2\epsilon-4}{2})}{(\pi\mu^2)^{\frac{d+2\epsilon-4}{2}}}\\
\underset{\epsilon\to0}{=}\;& 8\pi\Big(\xi(4)\xi(3)V^{3\frac{d+1}{9-d}-3}\widehat{E}^{SL(d)}_{2\Lambda_2}-\delta_{d,4}\xi(2)\big(\tfrac{6}{5}\log V+\log(2\pi\mu)\big)\\
&\qquad-\delta_{d,5}\xi(3)V^{\frac{3}{2}}\Big(\big(\log V+\log(2\pi\mu)\big)E^{SL(5)}_{\frac{3}{2}\Lambda_1}++\tfrac{1}{2}\partial_\epsilon E^{SL(5)}_{(\frac{3}{2}+\epsilon)\Lambda_1}\big|_{\epsilon=0}\Big)\Big)\,,
\end{aligned}$$

where for $d=4$ we define $\widehat{E}^{SL(4)}_{2\Lambda_2}$ from (4.24), for $d=5$ we define $\widehat{E}^{SL(5)}_{2\Lambda_2}$ such that

$$\begin{aligned}
\xi(1+2\epsilon)\xi(2\epsilon)E^{SL(5)}_{(\frac{1}{2}+\epsilon)\Lambda_3} &= \xi(2\epsilon)\xi(3-2\epsilon)E^{SL(5)}_{(\frac{3}{2}-\epsilon)\Lambda_1+\epsilon\Lambda_4} = \xi(4-2\epsilon)\xi(3-2\epsilon)E^{SL(5)}_{(2-\epsilon)\Lambda_2}\\
&= \xi(2\epsilon)\xi(3-2\epsilon)E^{SL(5)}_{\frac{3}{2}\Lambda_1}+\xi(4)\xi(3)\widehat{E}^{SL(5)}_{2\Lambda_2}+\mathcal{O}(\epsilon) \qquad (4.27)
\end{aligned}$$

and $E^{SL(d)}_{2\Lambda_2}$ is finite for $d \neq 4,5$. To write this we need to use the Langlands functional relations on non-maximal parabolic Eisenstein series (3.20). The explicit computations are cumbersome and are done with a computer so we do not display the details.

We must also consider the 1-loop form-factor of the leading correction to the effective action. Its contribution to $\mathcal{E}^{\rm sugra}_{(1,0)}$ gives

$$\begin{aligned} & 4\pi\xi(2)r^d\int_0^\infty \frac{dL}{L^{\frac{7-d-2\epsilon}{2}}}\Big(\sum_{n\in\mathbb{Z}^d}{}' e^{-\pi L r^{-3}U^{-1}[n]} + e^{-\pi L\mu^2}\Big) \\ = \;& 8\pi\xi(2)r^d\xi(d+2\epsilon-5)r^{\frac{3}{2}(d+2\epsilon-5)}E^{SL(d)}_{\frac{d+2\epsilon-5}{2}\Lambda_{d-1}} + 4\pi\xi(2)r^d\Gamma(\tfrac{d+2\epsilon-5}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-5}{2}} \\ \underset{\epsilon\to 0}{=}\;& 8\pi\xi(2)V^{\frac{5}{2}}\Big(\xi(5)\widehat{E}^{SL(5)}_{\frac{5}{2}\Lambda_1} - \frac{3}{4}\log V - \log(2\pi\mu)\Big)\,. \end{aligned} \tag{4.28}$$

We conclude that (4.18) for $\mathcal{E}^{\rm sugra}_{(1,0)}$ is accurate for $d=4,5$, while for $d=4,5$ we have

$$\begin{aligned} \mathcal{E}^{\rm sugra}_{(1,0)} &\underset{d=4}{=} 8\pi\Big(\xi(4)\xi(5)V^3E^{SL(4)}_{\frac{5}{2}\Lambda_3} + \xi(2)\xi(5)VE^{SL(4)}_{\frac{5}{2}\Lambda_1} + \xi(4)\xi(3)\widehat{E}^{SL(4)}_{2\Lambda_2} - \frac{\pi}{5}\log(V) - \xi(2)\log(2\pi\mu)\Big) \\ \mathcal{E}^{\rm sugra}_{(1,0)} &\underset{d=5}{=} 8\pi V^{\frac{9}{2}}\Big(\xi(4)\xi(5)E^{SL(5)}_{3\Lambda_3} + \frac{\xi(2)}{V^2}\big(\xi(5)\widehat{E}^{SL(5)}_{\frac{5}{2}\Lambda_1} - \tfrac{3}{4}\log V\big) \\ &\quad + \frac{\xi(3)}{V^3}\big(\xi(4)\widehat{E}^{SL(5)}_{2\Lambda_2} - \log V\, E^{SL(5)}_{\frac{3}{2}\Lambda_1} - \tfrac{1}{2}\partial_\epsilon E^{SL(5)}_{(\frac{3}{2}+\epsilon)\Lambda_1}\big|_{\epsilon=0}\big)\Big) - 2\log(2\pi\mu)\mathcal{E}^{\rm sugra}_{(0,0)}\,. \end{aligned} \tag{4.29}$$

This result is indeed consistent with the large radius limit of the exact coupling function $\mathcal{E}_{(1,0)}$ [38], as one can see explicitly by comparing with (D.39) and (D.40). Note that the terms proportional to $\log V$ do not need to recombine into the coupling $\mathcal{E}^{\rm sugra}_{(0,0)}$ and they do not. However, the $\log(2\pi\mu)$ terms must recombine into $\log(2\pi\mu)\mathcal{E}^{\rm sugra}_{(0,0)}$ in $D=6$ dimensions because the ambiguity in the definition of the infrared cutoff is fixed by adding the one-loop form factor of the $\mathcal{E}^{\rm sugra}_{(0,0)}R^4$ type invariant insertion

$$\begin{aligned} A[\mathcal{E}^{\rm sugra}_{(0,0)}\mathcal{R}^4] &\underset{d=5}{=} \frac{\ell^{10}}{8}\mathcal{E}^{\rm sugra}_{(0,0)}\left(s^2\int_0^\infty\frac{dL}{L^{1-\epsilon}}\int_0^1 dx_1\int_0^{x_1}dx_2\; e^{\pi L\ell^2(1-x_1)x_2 s} + \circlearrowleft\right) \\ &= \frac{\ell^{10}}{8}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{(1-\epsilon)\Gamma(3-2\epsilon)}\mathcal{E}^{\rm sugra}_{(0,0)}\Big(s^2(-\pi\ell^2 s)^{-\epsilon} + t^2(-\pi\ell^2 t)^{-\epsilon} + u^2(-\pi\ell^2 u)^{-\epsilon}\Big) \\ &= \ell^6\sigma_2\mathcal{E}^{\rm sugra}_{(0,0)}\Gamma(\epsilon)(\pi\mu^2)^{-\epsilon} \\ &\quad + \frac{\ell^{10}}{16}\mathcal{E}^{\rm sugra}_{(0,0)}\Big(s^2\big(3-\log(-\tfrac{\ell^2 s}{\mu^2})\big) + t^2\big(3-\log(-\tfrac{\ell^2 t}{\mu^2})\big) + u^2\big(3-\log(-\tfrac{\ell^2 u}{\mu^2})\big)\Big) + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{4.30}$$

so that the total amplitude does not depend on μ.

$\sigma_3\mathcal{R}^4$ up to 3-loop

The 1-loop contribution to $\mathcal{E}^{\rm sugra}_{(0,1)}$ is finite in all dimensions

$$\frac{4\pi^4}{567}\int_0^\infty\frac{dL}{L^{-\frac{1+d+2\epsilon}{2}}}\sum_{n\in\mathbb{Z}^d}e^{-\pi L r^{-3}U^{-1}[n]} = 40\xi(2)\xi(6)\xi(d+3)V^{3\frac{d+3}{9-d}}E^{SL(d)}_{\frac{d+3}{2}\Lambda_{d-1}}\,. \tag{4.31}$$

The 2-loop contribution to $\mathcal{E}^{\text{sugra}}_{(0,1)}$ has a divergence for $d=3$ and $d=7$. To compute them it is convenient to unfold the integral to Schwinger parameter space and use some simplifications of the integral explained in Appendix F.1. Using those one gets

$$
\begin{aligned}
&8\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d-2\epsilon}{2}}}\varphi^{\text{tr}}_{\text{KZ}}(\Omega_2)\sum_{n_i\in\mathbb{Z}^d}{}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j-\pi R(\mu)}\\
=\;&\frac{2\pi^2}{9}\int_0^\infty\frac{d\rho_2}{\rho_2^{\frac{5-d-2\epsilon}{2}}}\int_0^1du\int_{\rho_2u(1-u)}^\infty\frac{dt}{t^{\frac{5-d-2\epsilon}{2}}}\Big(1+\frac{\rho_2}{t}\big((1-6u(1-u)\big)+\frac{5\rho_2^2}{t^2}u^2(1-u)^2\Big)\\
&\times\Bigg(\sum_{\substack{n_i\in\mathbb{Z}^d\\ n_i\neq0,\,n_1+n_2\neq0}}{}'e^{-\pi r^{-3}\left(\rho_2(n_1+un_2)^\intercal U^{-1}(n_1+un_2)+tU^{-1}[n_2]\right)}+3\sum_{n\in\mathbb{Z}^d}{}'e^{-\pi\rho_2r^{-3}U^{-1}[n]-\pi t\mu^2}+e^{-\pi(\rho_2+t)\mu^2}\Bigg)\\
=\;&8\pi r^{3(d+2\epsilon-3)}\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d-2\epsilon}{2}}}\varphi^{\text{tr}}_{\text{KZ}}(\Omega_2)\sum_{\substack{n_i\in\mathbb{Z}^d\\ n_1\wedge n_2\neq0}}{}'e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}\\
&+\frac{4\pi^2}{3}\frac{\Gamma(\frac{d+2\epsilon-3}{2})}{(\pi\mu^2)^{\frac{d+2\epsilon-3}{2}}}\xi(d+2\epsilon-3)r^{\frac{3}{2}(d+2\epsilon-3)}E^{SL(d)}_{\frac{d+2\epsilon-3}{2}\Lambda_{d-1}}+\frac{2\pi^2}{9}\frac{9-d-2\epsilon}{7-d-2\epsilon}\frac{\Gamma(\frac{d+2\epsilon-3}{2})^2}{(\pi\mu^2)^{d+2\epsilon-3}}\\
&+\frac{2\pi^2}{9}\frac{\Gamma(\frac{d+2\epsilon-7}{2})}{(\pi\mu^2)^{\frac{d+2\epsilon-7}{2}}}\xi(d+2\epsilon+1)r^{\frac{3}{2}(d+2\epsilon+1)}E^{SL(d)}_{\frac{d+2\epsilon+1}{2}\Lambda_{d-1}}\,.
\end{aligned}
\tag{4.32}
$$

For $d=3$ one gets

$$
\begin{aligned}
&8\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{2-\epsilon}}\varphi^{\text{tr}}_{\text{KZ}}(\Omega_2)\sum_{n_i\in\mathbb{Z}^3}{}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j-\pi R(\mu)}\\
=\;&8\pi V^{2\epsilon}\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{2-\epsilon}}\varphi^{\text{tr}}_{\text{KZ}}(\Omega_2)\sum_{\substack{n_i\in\mathbb{Z}^3\\ n_1\wedge n_2\neq0}}e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}+\frac{4\pi^2}{3}\frac{\Gamma(\epsilon)}{(\pi\mu^2)^\epsilon}\xi(2\epsilon)V^\epsilon E^{SL(3)}_{\epsilon\Lambda_2}+\frac{2\pi^2}{9}\frac{3-\epsilon}{2-\epsilon}\frac{\Gamma(\epsilon)^2}{(\pi\mu^2)^{2\epsilon}}\\
\underset{\epsilon\to0}{=}\;&\mathcal{E}^{SL(3)}_{(0,1)}+\frac{\pi^2}{3}\log(V)^2-\frac{\pi}{3}\log V\Big(2\zeta(3)\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}+\frac{\pi}{3}\Big)\\
&+\frac{4\pi^2}{3}\log(2\pi\mu)^2-\frac{2\pi}{3}\log(2\pi\mu)\Big(2\zeta(3)\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}-2\pi\log V+\frac{\pi}{3}\Big)\,,
\end{aligned}
\tag{4.33}
$$

where $\mathcal{E}^{SL(3)}_{(0,1)}$ is a function of the $SL(3)$ moduli only, that is defined as

$$
\mathcal{E}^{SL(3)}_{(0,1)}=\lim_{\epsilon\to0}\Bigg(8\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{2-\epsilon}}\varphi^{\text{tr}}_{\text{KZ}}(\Omega_2)\sum_{\substack{n_i\in\mathbb{Z}^3\\ n_1\wedge n_2\neq0}}e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}-\frac{16\pi^2\xi(2\epsilon)^2}{(4-2\epsilon)(3+2\epsilon)}E^{SL(3)}_{2\epsilon\Lambda_2}\Bigg)\,.
\tag{4.34}
$$

We prove that this limit is finite in Appendix F.2. For $d=7$ one gets

$$\begin{aligned}
&8\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{-\epsilon}}\varphi^{\rm tr}_{\rm KZ}(\Omega_2)\sum_{n_i\in\mathbb{Z}^7}{}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j-\pi R(\mu)}\\
&= 8\pi V^{12+6\epsilon}\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{-\epsilon}}\varphi^{\rm tr}_{\rm KZ}(\Omega_2)\sum_{\substack{n_i\in\mathbb{Z}^7\\ n_1\wedge n_2\neq 0}}{}'e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}+\frac{2\pi^2}{9}\frac{\Gamma(\epsilon)}{(\pi\mu^2)^\epsilon}\xi(8+2\epsilon)V^{12+3\epsilon}E^{SL(7)}_{(4+\epsilon)\Lambda_6}\\
&\underset{\epsilon\to 0}{=} V^{12}\mathcal{E}^{SL(7)}_{(0,1)}-60\xi(4)\xi(8)V^{12}\log V\,E^{SL(7)}_{4\Lambda_6}-40\xi(4)\xi(8)\log(2\pi\mu)V^{12}E^{SL(7)}_{4\Lambda_6}\,. \qquad (4.35)
\end{aligned}$$

where $\mathcal{E}^{SL(7)}_{(0,1)}$ is defined from the finite limit (see Appendix F.2)

$$\mathcal{E}^{SL(7)}_{(0,1)}=\lim_{\epsilon\to 0}\left(8\pi\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{-\epsilon}}\varphi^{\rm tr}_{\rm KZ}(\Omega_2)\sum_{n_i\in\mathbb{Z}^7}{}'e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}-\frac{4\pi^2}{9}\xi(2\epsilon)\xi(8+2\epsilon)E^{SL(7)}_{(4+\epsilon)\Lambda_6}\right). \qquad (4.36)$$

The 3-loop contribution to $\mathcal{E}^{\rm sugra}_{(0,1)}$ can be computed as in [74] by splitting the sum over Kaluza–Klein momenta depending on the rank of n_{iI}. We will not explain the details in this case and just give

$$\begin{aligned}
&20\int_{\mathcal{G}_3}\frac{d^6\Omega_2}{\det\Omega_2^{\frac{9-d-2\epsilon}{2}}}\sum_{n_i\in\mathbb{Z}^d}{}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j-\pi R(\mu)}\\
&= 40\sum_{\gamma\in P_{d-3}\backslash SL(d)}\sum_{\substack{N\in\mathbb{Z}^{3\times 3}/GL(3,\mathbb{Z})\\ \det N\neq 0}}\int_{\mathcal{H}_3}\frac{d^6\Omega_3}{|\Omega_3|^{2-\frac{d+2\epsilon-5}{2}}}e^{-\pi r^{-3}y_{d-3}^{\frac{2}{3}}\mathrm{Tr}\,\Omega_3 N^\intercal U^{-1}N}\Big|_\gamma\\
&\quad+20\sum_{\gamma\in P_{d-2}\backslash SL(d)}\sum_{\substack{N\in\mathbb{Z}^{2\times 2}/GL(2,\mathbb{Z})\\ \det N\neq 0}}\int_{\mathcal{H}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{3}{2}-\frac{d+2\epsilon-4}{2}}}e^{\pi r^{-3}y_{d-2}\mathrm{Tr}\,\Omega_2 N^\intercal U^{-1}N}\Big|_\gamma\int_0^\infty\frac{dt}{t^{1-\frac{d+2\epsilon-7}{2}}}e^{-\pi t\mu^2}\\
&\quad+20\sum_{\gamma\in P_{d-1}\backslash SL(d)}\sum_{n=1}^\infty\int_0^\infty\frac{dL}{L^{1-\frac{d-3+2\epsilon}{2}}}e^{\pi Lr^{-3}y_{d-1}^2n^2}\Big|_\gamma\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{3}{2}-\frac{d+2\epsilon-6}{2}}}e^{-\pi\det\Omega_2\mu^2}\\
&\qquad+20\int_{\mathcal{G}_3}\frac{d^6\Omega_2}{\det\Omega_2^{2-\frac{d+2\epsilon-5}{2}}}e^{-\pi\det\Omega_2\mu^2}\\
&= 40\xi(d+2\epsilon-5)\xi(d+2\epsilon-6)\xi(d+2\epsilon-7)r^{\frac{9}{2}(d+2\epsilon-5)}E^{SL(d)}_{\frac{d+2\epsilon-5}{2}\Lambda_{d-3}}\\
&\quad+20\Gamma(\tfrac{d+2\epsilon-7}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-7}{2}}\xi(d+2\epsilon-4)\xi(d+2\epsilon-5)r^{3(d+2\epsilon-4)}E^{SL(d)}_{\frac{d+2\epsilon-4}{2}\Lambda_{d-2}}\\
&\quad+20\xi(2)\Gamma(\tfrac{d+2\epsilon-6}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-6}{2}}\xi(d+2\epsilon-3)r^{\frac{3}{2}(d+2\epsilon-3)}E^{SL(d)}_{\frac{d+2\epsilon-3}{2}\Lambda_{d-1}}\\
&\quad+20\xi(2)\xi(3)\Gamma(\tfrac{d+2\epsilon-5}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-5}{2}} \qquad (4.37)
\end{aligned}$$

We need to check the cases for which there is a logarithmic correction in μ for each d separately. For this we use the Langlands functional relations to simplify the result. For $d=5$ we get

$$\begin{aligned}
&40\xi(2\epsilon)\xi(2\epsilon-1)\xi(2\epsilon-3)V^{\frac{9}{2}\epsilon}E^{SL(5)}_{\epsilon\Lambda_2}+20\xi(2)\xi(3)\Gamma(\epsilon)(\pi\mu^2)^{-\epsilon}\\
&= 40\xi(5)\xi(4)\xi(3)\widehat{E}^{SL(5)}_{\frac{5}{2}\Lambda_3}-90\xi(2)\xi(3)\log V-40\xi(2)\xi(3)\log(2\pi\mu)\,, \qquad (4.38)
\end{aligned}$$

for $d=6$

$$40\xi(1+2\epsilon)\xi(2\epsilon)\xi(2\epsilon-1)V^{3+6\epsilon}E^{SL(6)}_{(\frac{1}{2}+\epsilon)\Lambda_3}+20\xi(2)\Gamma(\epsilon)(\pi\mu^2)^{-\epsilon}\xi(3+2\epsilon)V^{3+2\epsilon}E^{SL(6)}_{(\frac{3}{2}+\epsilon)\Lambda_5} \tag{4.39}$$
$$=40\xi(3)V^3\Big(\xi(4)\xi(5)\widehat{E}^{SL(6)}_{\frac{5}{2}\Lambda_3}+\tfrac{1}{2}\xi(2)\partial_\epsilon E^{SL(6)}_{(\frac{3}{2}+\epsilon)\Lambda_5}\big|_{\epsilon=0}-2\xi(2)\log V E^{SL(6)}_{\frac{3}{2}\Lambda_5}-\xi(2)\log(2\pi\mu)E^{SL(6)}_{\frac{3}{2}\Lambda_5}\Big)\,,$$

where we define $\widehat{E}^{SL(d)}_{\frac{d-1}{2}\Lambda_{d-3}}$ for $d\geq 6$ such that

$$\begin{aligned}&\xi(1+2\epsilon)\xi(2\epsilon)\xi(2\epsilon-1)E^{SL(d)}_{(\frac{1}{2}+\epsilon)\Lambda_3}=\xi(2\epsilon)\xi(3-2\epsilon)\xi(2-2\epsilon)E^{SL(d)}_{(\frac{3}{2}-\epsilon)\Lambda_1+\epsilon\Lambda_4}\\ =\;&\xi(d-1-2\epsilon)\xi(d-2-2\epsilon)\xi(d-3-2\epsilon)E^{SL(d)}_{(\frac{d-1}{2}-\epsilon)\Lambda_{d-3}}\\ =\;&\xi(2)\xi(2\epsilon)\xi(3)E^{SL(d)}_{\frac{3}{2}\Lambda_1}+\xi(d-1)\xi(d-2)\xi(d-3)\widehat{E}^{SL(d)}_{\frac{d-1}{2}\Lambda_{d-3}}+\mathcal{O}(\epsilon)\,.\end{aligned} \tag{4.40}$$

For $d=7$ this gives

$$\begin{aligned}&40\xi(2+2\epsilon)\xi(1+2\epsilon)\xi(2\epsilon)V^{9+9\epsilon}E^{SL(7)}_{(1+\epsilon)\Lambda_4}+20\Gamma(\epsilon)(\pi\mu^2)^{-\epsilon}\xi(3+2\epsilon)\xi(2+2\epsilon)V^{9+6\epsilon}E^{SL(7)}_{(\frac{3}{2}+\epsilon)\Lambda_5}\\ =\;&40\xi(3)V^9\Big(\xi(5)\xi(4)\widehat{E}^{SL(7)}_{\frac{5}{2}\Lambda_3}+\tfrac{1}{2}\xi(2)\partial_\epsilon E^{SL(7)}_{(\frac{3}{2}+\epsilon)\Lambda_5}\big|_{\epsilon=0}\Big)-60\xi(4)\xi(3)V^9\log V E^{SL(6)}_{2\Lambda_2}\\ &-40\xi(4)\xi(3)\log(2\pi\mu)V^9E^{SL(6)}_{2\Lambda_2}\,,\end{aligned} \tag{4.41}$$

where we used

$$\begin{aligned}&\xi(2+2\epsilon)\xi(1+2\epsilon)\xi(2\epsilon)E^{SL(7)}_{(1+\epsilon)\Lambda_4}=\xi(2\epsilon)\xi(4-2\epsilon)\xi(3-2\epsilon)E^{SL(7)}_{(2-\epsilon)\Lambda_2+\epsilon\Lambda_6}\\ =\;&\xi(5-2\epsilon)\xi(4-2\epsilon)\xi(3-2\epsilon)E^{SL(7)}_{(\frac{5}{2}-\epsilon)\Lambda_3}\,.\end{aligned} \tag{4.42}$$

We must now consider the form-factor for the leading correction to the Wilson effective action in eleven dimensions. The 2-loop R^4 type form-factor contributes through the integral

$$20\xi(2)r^d\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{9-d}{2}}}\sum_{n_i\in\mathbb{Z}^d}{}'\,e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j-\pi R(\mu)} \tag{4.43}$$
$$\begin{aligned}=\;&40\xi(2)r^d\sum_{\gamma\in P_{d-2}\backslash SL(d)}\sum_{\substack{N\in\mathbb{Z}^{2\times 2}/GL(2,\mathbb{Z})\\ \det N\neq 0}}\int_{\mathcal{H}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{3}{2}-\frac{d+2\epsilon-6}{2}}}e^{\pi r^{-3}y_{d-2}\mathrm{Tr}\,\Omega_2N^\intercal U^{-1}N}\Big|_\gamma\\ &+20\xi(2)r^d\sum_{\gamma\in P_{d-1}\backslash SL(d)}\sum_{n=1}^\infty\int_0^\infty\frac{dL}{L^{1-\frac{d+2\epsilon-5}{2}}}e^{\pi Lr^{-3}y_{d-1}^2n^2}\Big|_\gamma\int_0^\infty\frac{dt}{t^{1-\frac{d-7}{2}}}e^{-\pi t\mu^2}\\ &+20\xi(2)r^d\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{3}{2}-\frac{d+2\epsilon-6}{2}}}e^{-\pi\,\det\Omega_2\mu^2}\\ =\;&40\xi(2)\xi(d-6)\xi(d-7)r^{d+3(d+2\epsilon-6)}E^{SL(d)}_{\frac{d+2\epsilon-6}{2}\Lambda_{d-2}}\\ &+20\xi(2)\Gamma(\tfrac{d-7}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-7}{2}}\xi(d-7)r^{d+\frac{3}{2}(d+2\epsilon-5)}E^{SL(d)}_{\frac{d-5}{2}\Lambda_{d-1}}+20\xi(2)^2\Gamma(\tfrac{d-6}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-6}{2}}r^d\,,\end{aligned}$$

For $d=6$ one obtains

$$\begin{aligned} & 40\xi(2)\xi(2\epsilon)\xi(2\epsilon-1)V^{4+4\epsilon}E^{SL(6)}_{\epsilon\Lambda_4}+20\xi(2)^2\Gamma(\epsilon)(\pi\mu^2)^{-\epsilon}V^4 \\ = \;& 40\xi(2)\xi(3)\xi(2)V^4\widehat{E}^{SL(6)}_{3\Lambda_2}-80\xi(2)^2V^4\log V-40\xi(2)^2\log(2\pi\mu)V^4\;, \end{aligned} \tag{4.44}$$

and for $d=7$

$$\begin{aligned} & 40\xi(2)\xi(1+2\epsilon)\xi(2\epsilon)V^{10+6\epsilon}E^{SL(7)}_{(\frac{1}{2}+\epsilon)\Lambda_5}+20\xi(2)\Gamma(\epsilon)(\pi\mu^2)^{-\epsilon}\xi(2+2\epsilon)V^{10+3\epsilon}E^{SL(7)}_{(1+\epsilon)\Lambda_6} \\ = \;& 40\xi(2)\xi(5)V^{10}\big(\xi(6)\widehat{E}^{SL(7)}_{3\Lambda_2}-\tfrac{1}{2}\partial_\epsilon E^{SL(7)}_{(\frac{5}{2}+\epsilon)\Lambda_1}|_{\epsilon=0}\big)-60\xi(2)\xi(5)V^{10}\log V E^{SL(7)}_{\frac{5}{2}\Lambda_1} \\ & -40\xi(2)\xi(5)\log(2\pi\mu)V^{10}E^{SL(7)}_{\frac{5}{2}\Lambda_1}\;. \end{aligned} \tag{4.45}$$

The 1-loop R^4 type form-factor contributes through the integral

$$\begin{aligned} & \frac{4\pi^2}{3}\xi(2)r^d\int_0^\infty\frac{dL}{L^{\frac{5-d-2\epsilon}{2}}}\Big(\sum_{n\in\mathbb{Z}^d}{}'\,e^{-\pi Lr^{-3}U^{-1}[n]}+e^{-\pi L\mu^2}\Big) \\ = \;& \frac{8\pi^2}{3}\xi(2)\xi(d+2\epsilon-3)r^{d+\frac{3}{2}(d+2\epsilon-3)}E^{SL(d)}_{\frac{d+2\epsilon-3}{2}\Lambda_{d-1}}+\frac{4\pi^2}{3}\Gamma(\tfrac{d+2\epsilon-3}{2})(\pi\mu^2)^{-\frac{d+2\epsilon-3}{2}}r^d \\ \underset{\epsilon\to0}{\overset{d=3}{=}} \;& \frac{8\pi^2}{3}\xi(2)\xi(3)V\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}-\frac{4\pi^2}{3}\xi(2)V\log V-\frac{4\pi^2}{3}\xi(2)\log(2\pi\mu)V\;. \end{aligned} \tag{4.46}$$

In total for $d=7$ on gets

$$\begin{aligned} \mathcal{E}^{\rm sugra}_{(0,1)} \underset{d=7}{=} V^{15}\Big(&\frac{8\pi^4}{567}\xi(10)E^{SL(7)}_{5\Lambda_6}+\frac{4\pi^2\xi(2)^2}{3V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(7)}_{\frac{3}{2}\Lambda_1} \\ &+\frac{1}{V^3}\mathcal{E}^{SL(7)}_{(0,1)}+\frac{40\xi(2)}{V^5}\xi(5)\Big(\xi(6)\widehat{E}^{SL(7)}_{3\Lambda_2}-\tfrac{1}{2}\partial_\epsilon E^{SL(7)}_{(\frac{5}{2}+\epsilon)\Lambda_1}\big|_{\epsilon=0}\Big) \\ &+\frac{40}{V^6}\xi(3)\Big(\xi(5)\xi(4)\widehat{E}^{SL(7)}_{\frac{5}{2}\Lambda_3}+\tfrac{1}{2}\xi(2)\partial_\epsilon E^{SL(7)}_{(\frac{3}{2}+\epsilon)\Lambda_5}\big|_{\epsilon=0}\Big)\Big) \\ &-60V^{12}\log V\Big(\xi(4)\xi(8)E^{SL(7)}_{4\Lambda_6}+\frac{\xi(2)\xi(5)}{V^2}E^{SL(7)}_{\frac{5}{2}\Lambda_1}+\frac{\xi(3)\xi(4)}{V^3}E^{SL(7)}_{2\Lambda_2}\Big)-\frac{5}{\pi}\log(2\pi\mu)\mathcal{E}^{\rm sugra}_{(1,0)}\;. \end{aligned} \tag{4.47}$$

Note that the $\log(2\pi\mu)$ term is proportional to the coupling function $\mathcal{E}^{\rm sugra}_{(1,0)}$, as it must be since this terms comes from the non-analytic $\mathcal{E}^{\rm sugra}_{(1,0)}\nabla^4\mathcal{R}^4$ one-loop form-factor, that includes a logarithmic term in $\log(-s\ell^2)\sigma_3\mathcal{E}^{\rm sugra}_{(1,0)}$.

For $d=6$

$$\begin{aligned} \mathcal{E}^{\rm sugra}_{(0,1)} \underset{d=6}{=} V^9\Big(&\frac{8\pi^4}{567}\xi(9)E^{SL(6)}_{\frac{9}{2}\Lambda_5}+\frac{4\pi^2\xi(2)^2}{3V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(6)}_{\frac{3}{2}\Lambda_1}+\frac{40\xi(2)}{V^5}\xi(6)\xi(5)\widehat{E}^{SL(6)}_{3\Lambda_2} \\ &+\frac{8\pi}{V^3}\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2^{\frac{1}{2}}}\varphi^{\rm tr}_{\rm KZ}(\Omega_2)\sum_{n_i\in\mathbb{Z}^6}{}'e^{-\pi\Omega_2^{ij}n_i^{\intercal}U^{-1}n_j}+\frac{40}{V^6}\xi(3)\Big(\xi(4)\xi(5)\widehat{E}^{SL(6)}_{\frac{5}{2}\Lambda_3}+\tfrac{1}{2}\xi(2)\partial_\epsilon E^{SL(6)}_{(\frac{3}{2}+\epsilon)\Lambda_5}\big|_{\epsilon=0}\Big)\Big) \\ &-\frac{40\pi}{3}V^3\log V\Big(\xi(2)V+\xi(3)E^{SL(6)}_{\frac{3}{2}\Lambda_1}\Big)-\frac{5}{3}\log(2\pi\mu)\mathcal{E}^{\rm sugra}_{(0,0)}\;. \end{aligned} \tag{4.48}$$

In this case the $\log(2\pi\mu)$ term comes from the 2-loop $\mathcal{R}^4\mathcal{E}^{\rm sugra}_{(0,0)}$ form-factor that diverges logarithmically as we exhibited in the 2-loop string theory amplitude (2.125),(2.126).

For $d=5$

$$\begin{aligned}\mathcal{E}^{\rm sugra}_{(0,1)} \underset{d=5}{=} V^6\Big(&\frac{8\pi^4}{567}\xi(d+3)E^{SL(5)}_{4\Lambda_4}+\frac{4\pi^2\xi(2)^2}{3V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(5)}_{\frac{3}{2}\Lambda_1}\\&+\frac{8\pi}{V^3}\int_{\mathcal{G}_2}\frac{d^3\Omega_2}{\det\Omega_2}\varphi^{\rm tr}_{\rm KZ}(\Omega_2)\sum_{n_i\in\mathbb{Z}^5}{}'e^{-\pi\Omega_2^{ij}n_i^{\intercal}U^{-1}n_j}+\frac{40\xi(2)}{V^5}\xi(6)\xi(5)E^{SL(5)}_{3\Lambda_2}+\frac{40}{V^6}\xi(5)\xi(4)\xi(3)\widehat{E}^{SL(5)}_{\frac{5}{2}\Lambda_3}\Big)\\&-\frac{15\zeta(3)}{2}\log V-\frac{10\zeta(3)}{3}\log(2\pi\mu)\ ,\end{aligned}\tag{4.49}$$

where the last term is consistent with the 3-loop divergence in six dimensions.

For $d=3$

$$\begin{aligned}\mathcal{E}^{\rm sugra}_{(0,1)} \underset{d=3}{=} V^3\Big(&\frac{8\pi^4}{567}\xi(6)E^{SL(3)}_{3\Lambda_2}+\frac{4\pi^2\xi(2)^2}{3V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(3)}_{\frac{3}{2}\Lambda_1}\\&+\frac{1}{V^3}\mathcal{E}^{SL(3)}_{(0,1)}+\frac{40\xi(2)}{V^5}\xi(6)\xi(5)E^{SL(3)}_{3\Lambda_2}+\frac{40}{V^6}\xi(5)\xi(4)\xi(3)\Big)\\&+\frac{\pi^2}{3}\log(V)^2-\frac{\pi}{3}\log V\Big(2\zeta(3)\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}+\frac{\pi}{3}\Big)+\frac{4\pi^2}{3}\log(2\pi\mu)^2-\frac{2\pi^2}{3}\log(2\pi\mu)\Big(\mathcal{E}^{\rm sugra}_{(0,0)}+\frac{\pi}{3}\Big)\ .\end{aligned}\tag{4.50}$$

We have now derived the precise contribution from eleven-dimensional supergravity to the low energy effective action on T^d for $3\leq d\leq 7$. We find perfect agreement with the results of [76] derived from the Langlands constant term formula applied to the non-perturbative coupling functions $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$. We will see in the next section that the non-perturbative function $\mathcal{E}_{(0,1)}$ proposed in [64, 27] does give the same function $\mathcal{E}^{\rm sugra}_{(0,1)}$ in the large volume limit.

5 Beyond automorphic forms

As we said previously, the study of coupling functions in string theory requires to generalise the notion of automorphic forms. In general the coupling functions do satisfy the condition of automorphicity $f(kv\gamma) = f(v)$ for $k \in K$ and $\gamma \in G(\mathbb{Z})$, the condition of uniform moderate growth such that $f(v)$ and all its covariant derivatives are polynomially bounded, but they are not eigen functions of the invariant differential operators on G.

Let us first recall the prototypal example of the coupling function $\mathcal{E}_{(0,1)}$ in type IIB string theory. This function was computed from the two-loop amplitude in eleven-dimensional supergravity on T^2 as (0.10). This integral can be regularised using a formal Poisson summation formula, and defining the regularised integral as [56]

$$\mathcal{E}_{(0,1)} = \frac{2\pi^2}{9} \int_{\mathbb{R}^3_+} \frac{dL_1 dL_2 dL_3}{\sqrt{\sum_{I<J} L_I L_J}} \Big(\sum_I L_I - \frac{5 L_1 L_2 L_3}{\sum_{I<J} L_I L_J} \Big) {\sum_{m_I, n_I \in \mathbb{Z}}}' e^{-\pi \sum_I L_I \frac{|m_I + S n_I|^2}{\mathrm{Im}S}} \,. \qquad (5.1)$$

$$\sum_I m_I = \sum_I n_I = 0$$

The same authors showed that this function satisfies the Poisson equation

$$(\Delta - 12)\mathcal{E}_{(0,1)} = -(\mathcal{E}_{(0,0)})^2 \,, \qquad (5.2)$$

where $\Delta = -(S-\bar{S})^2 \partial_{\bar{S}} \partial_S$ and $\mathcal{E}_{(0,0)} = 2\zeta(3) E^{SL(2)}_{3/2}$. This equation together with the appropriate boundary condition at $\mathrm{Im}S \to \infty$ was proved in [214] to determine uniquely the function, consistently with expectations from non-perturbative effects in string theory. It was proved to be fully consistent with string perturbation theory [84], as well as the leading instanton corrections including the first instanton anti-instanton correction [12].

In this section we discuss the generalisation of this function for $D \leq 8$ proposed in [64]. The function in $D = 8$ dimensions was already determined in [215] and in seven and six dimensions in [77]. The precise definition of this coupling function was given in [27], and shown to agree with all previous proposals.

5.1 Supersymmetry and automorphic representations

The supersymmetry analysis explained in sections 1.3 and 3.2 did not take into account the presence of the R^4 type correction in analysing the $\nabla^6 R^4$ type supersymmetry invariant. The presence of $S^{(3)} \sim \int \mathcal{E}_{(0,0)} t_8 t_8 R^4$ at order ℓ^6 requires to modify supersymmetry by a variation $\delta^{(3)}$ linear in the function $\mathcal{E}_{(0,0)}$ and its derivatives, such that $S^{(6)} \sim \int \mathcal{E}_{(0,1)} t_8 t_8 \nabla^6 R^4$ must be a particular solution to the supersymmetry invariance identity

$$\delta^{(0)} S^{(6)} + \delta^{(3)} S^{(3)} + \delta^{(6)} S^{(0)} = 0 \,. \qquad (5.3)$$

The explicit computation is very difficult, but one can nonetheless conclude that the differential equation satisfied by $\mathcal{E}_{(0,1)}$ must be modified by terms quadratic in $\mathcal{E}_{(0,0)}$ and its derivative.

There are two distinct $\nabla^6 R^4$ type invariants in $4 \leq D \leq 7$ dimensions, and only one of them is modified by the R^4 type supersymmetric correction [53]. In this discussion we will concentrate on $D = 4$, as in Section 3.2.

Let us first analyse the set of differential equations satisfied by $(\mathcal{E}_{(0,0)})^2$. The function $\mathcal{E}_{(0,0)}$ satisfies (3.73), which implies equivalently

$$\mathcal{D}_{ij[pq}\mathcal{D}_{rs]kl}\mathcal{E}_{(0,0)} = \frac{1}{6}\mathcal{D}_{ijkl}\mathcal{D}_{pqrs}\mathcal{E}_{(0,0)} - \frac{1}{16}\varepsilon_{ijklpqrs}\mathcal{E}_{(0,0)} \tag{5.4}$$

and therefore

$$\Big(\mathcal{D}_{ijpq}\mathcal{D}^{pqrs}\mathcal{D}_{rskl} - \mathcal{D}_{ijkl}\big(6+\tfrac{1}{4}\Delta\big)\Big)(\mathcal{E}_{(0,0)})^2 = 0\ . \tag{5.5}$$

Note that this is precisely the equation that follows from the condition that the Fourier coefficients are non-generic in the decompactification limit in which one circle becomes large (3.55). In this decomposition we have the explicit formula

$$\mathcal{E}^{D=4}_{(0,0)} = R^3\mathcal{E}^{D=5}_{(0,0)} + \frac{2\pi^3}{45}R^6 + 4\pi R^{\frac{9}{2}}\sum_{\substack{q\in\mathbb{Z}^{27}\\ q\times q=0}}{}' \sigma_3(q)\frac{K_{\frac{3}{2}}(2\pi R|Z(q)|)}{|Z(q)|^{\frac{3}{2}}}e^{2\pi i q\cdot a}\ , \tag{5.6}$$

where R is the circle radius in Planck length and $|Z(q)|^2 = |v(q)|^2$ for $v\in E_6$. The Fourier expansion of $(\mathcal{E}_{(0,0)})^2$ is therefore easily found to be

$$\begin{aligned}(\mathcal{E}^{D=4}_{(0,0)})^2 = &\left(R^3\mathcal{E}^{D=5}_{(0,0)} + \frac{2\pi^3}{45}R^6\right)^2 + 16\pi^2R^9\sum_{\substack{q\in\mathbb{Z}^{27}\\ q\times q=0}}{}'\left(\sigma_3(q)\frac{K_{\frac{3}{2}}(2\pi R|Z(q)|)}{|Z(q)|^{\frac{3}{2}}}\right)^2\\ &+ 8\pi R^{\frac{15}{2}}\left(\mathcal{E}^{D=5}_{(0,0)} + \frac{2\pi^3}{45}R^3\right)\sum_{\substack{q\in\mathbb{Z}^{27}\\ q\times q=0}}{}'\sigma_3(q)\frac{K_{\frac{3}{2}}(2\pi R|Z(q)|)}{|Z(q)|^{\frac{3}{2}}}e^{2\pi i q\cdot a}\\ &+ 16\pi^2R^9\sum_{\substack{q_1,q_2\in\mathbb{Z}^{27}\\ q_1\times q_1=0\\ q_2\times q_2=0\\ q_1\neq q_2\neq 0}}{}'\sigma_3(q_1)\sigma_3(q_2)\frac{K_{\frac{3}{2}}(2\pi R|Z(q_1)|)}{|Z(q_1)|^{\frac{3}{2}}}\frac{K_{\frac{3}{2}}(2\pi R|Z(q_2)|)}{|Z(q_2)|^{\frac{3}{2}}}e^{2\pi i(q_1-q_2)\cdot a}\ .\end{aligned} \tag{5.7}$$

One can see that the first line includes exponentially suppressed terms at large radius that do not depend on the axion a. They correspond to instanton anti-instanton effects found previously in [56, 214] in type IIB. By property of the E_6 cubic invariant

$$\det(q_1-q_2) = \det q_1 - \mathrm{tr}[q_2\cdot(q_1\times q_1)] + \mathrm{tr}[q_1\cdot(q_2\times q_2)] - \det q_2 = 0 \tag{5.8}$$

and $(\mathcal{E}_{(0,0)})^2$ indeed satisfies (3.55), and therefore (5.5).

A similar computation in string perturbation theory using (3.119) shows that $(\mathcal{E}_{(0,0)})^2$ admits generic Fourier coefficients in P_1 and therefore does not satisfy (3.62). Physically $\mathcal{E}_{(0,0)}$ admits corrections from D(-1) and D5 instantons in type IIB. Therefore $(\mathcal{E}_{(0,0)})^2$ includes D(-1)-D5 instanton corrections, that correspond to a generic charge $Q\in S_-$ with $(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) < 0$. Therefore the abelian Fourier coefficient of $(\mathcal{E}_{(0,0)})^2$ with Q generic is generally non-zero provided $(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) < 0$. It turns out that the Fourier coefficients vanish for $(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) > 0$.

According to the analysis of section (3.2) we find that $(\mathcal{E}_{(0,0)})^2$ is compatible with the $(1/8,1/8)$-BPS supersymmetry invariant associated to the harmonic variables in $SU(8)/S(U(1)\times U(6)\times U(1))$, but not the $(1/4,0)$-BPS supersymmetry invariant associated to the harmonic variables in $SU(8)/S(U(2)\times U(6))$ [53].

We have accordingly

$$\mathcal{E}_{(0,1)} = \frac{8\pi^4}{567}\xi(10)\widehat{E}^{E_7}_{5\Lambda_7} + \mathcal{E}^{\text{ExFT}}_{(0,1)} \tag{5.9}$$

where the Eisenstein series $\widehat{E}^{E_7}_{5\Lambda_7}$ defined in (3.130) satisfies

$$\Big(36\mathcal{D}_{jr[kl}\mathcal{D}^{irmn}\mathcal{D}_{pq]mn} - \delta^i_j\mathcal{D}_{klpq}(\Delta+42) + \delta^i_{[k}\mathcal{D}_{lpq]j}(\Delta-120)\Big)\widehat{E}^{E_7}_{5\Lambda_7} = 0\ , \tag{5.10}$$

as well as (3.132) and (3.134). One cannot add a source $(\mathcal{E}_{(0,0)})^2$ in the right-hand-side of (3.132) without violating equation (5.10).

The function $\hat{\mathcal{E}}^{\text{ExFT}}_{(0,1)}$ instead satisfies

$$\big(\mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} - \mathcal{D}_{ijkl}\big(\tfrac{1}{4}\Delta+6\big)\big)\hat{\mathcal{E}}^{\text{ExFT}}_{(0,1)} = 0\ , \tag{5.11}$$

and the Poisson equation

$$\big(\Delta+60\big)\hat{\mathcal{E}}^{\text{ExFT}}_{(0,1)} = \frac{20}{\pi}\mathcal{E}_{(1,0)} - (\mathcal{E}_{(0,0)})^2\ . \tag{5.12}$$

Here we must distinguish the two terms on the right-hand-side of this equation. As discussed above, the source term in $(\mathcal{E}_{(0,0)})^2$ comes from the modification of the supersymmetry transformations at order ℓ^6 (5.3), while the term linear in $\mathcal{E}_{(1,0)}$ comes from the logarithmic divergence of the ∇^4R^4 type form-factor in four dimensions [53]. The logarithmic divergence in the one-loop form factor comes from the five-point insertion (2.66)

$$A^{\text{1-loop}}_{\epsilon\,\measuredangle\bigstar} \sim -\frac{5}{2\pi}\frac{1}{\epsilon}\sigma_3\,\mathcal{E}_{(1,0)}\ . \tag{5.13}$$

Following the reasoning of [216] this implies that the full coupling function depends logarithmically on the string coupling constant

$$\mathcal{E}_{(0,1)} \underset{\log g_{\text{s}}}{\sim} \frac{5}{\pi}\log g_{\text{s}}\,\mathcal{E}_{(1,0)}\ . \tag{5.14}$$

On a function of the dilaton $g_{\text{s}} = e^{\phi_6}$ only, the E_7 Laplace operator gives

$$\Delta f(\phi_6) = \frac{1}{4}\frac{\partial}{\partial\phi_6}\Big(\frac{\partial}{\partial\phi_6}+34\Big)f(\phi_6) \tag{5.15}$$

so that

$$\Delta\big(\frac{5}{\pi}\log g_{\text{s}}\big) = \frac{85}{2\pi}\ . \tag{5.16}$$

Therefore consistency with the logarithmic divergence requires that the full function (5.9) satisfies [77]

$$\big(\Delta+60\big)\mathcal{E}_{(0,1)} = \frac{85}{2\pi}\mathcal{E}_{(1,0)} - (\mathcal{E}_{(0,0)})^2\ . \tag{5.17}$$

Because of the quadratic source term in (5.12), the function $\mathcal{E}^{\text{ExFT}}_{(0,1)}$ does not define an automorphic representation [217]. In particular the decomposition of $\mathcal{U}(\mathfrak{e}_7)\mathcal{E}_{(0,1)}$ admits infinite multiplicities for

the irreducible representations of $SU(8)$. Nonetheless, the representations with non-zero multiplicity (and then infinite) are the same as for the Heisenberg parabolic Eisenstein series $\mathrm{Gr}\,\mathcal{U}(\mathfrak{e}_7)E^{E_7}_{s\Lambda_1}$, consistently with the linearised supersymmetry analysis. The associated variety of the ideal is therefore still formally the closure of the nilpotent orbit $\mathcal{O}_{2\Lambda_1}$.

The Fourier coefficients of $\hat{\mathcal{E}}^{\mathrm{ExFT}}_{(0,1)}$ in any parabolic subgroup, for any unipotent character that does not define a charge in the closure of the complex nilpotent orbit $\mathcal{O}_{2\Lambda_1}$, vanish, as in (3.55) and the equivalent in (3.65). This means physically that $\hat{\mathcal{E}}^{\mathrm{ExFT}}_{(0,1)}$ only receives non-perturbative correction from $(1/8,1/8)$-BPS instantons, but no corrections from $(1/4,0)$-BPS instantons, or even less BPS instantons. The notion of wavefront set attached to the orbit $\mathcal{O}_{2\Lambda_1}$ is therefore well defined and very constraining.

We will now explain the formula for $\mathcal{E}^{\mathrm{ExFT}}_{(0,1)}$ anticipated in [64] and determined in [27].

5.2 Exceptional field theory amplitude

It was proposed in [64] to consider an effective theory of one-half BPS states in string theory. From the point of view of eleven-dimensional supergravity, one can define the conserved charge

$$\Gamma = \left(m_I, n^{IJ}, n^{I_1\dots I_5}, k^{I_1\dots I_7,J}\right) \tag{5.18}$$

where m_I is the momentum mode on T^7, n^{IJ} the winding numbers of the M2-brane, $n^{I_1\dots I_5}$ the winding numbers of the M5-brane and $n^{I_1\dots I_7,J}$ the Kaluza-Klein monopole charge. There is exactly one massive 1/2 BPS state spin 2 supermultiplet for each $\Gamma\neq 0$ in $\mathbb{L}_7=\mathbb{Z}^{56}$ satisfying the constraint $\Gamma\times\Gamma=0$, i.e. (see e.g. [218, 192, 60])

$$\begin{aligned} n^{IJ}m_J &= 0\,, \qquad 3n^{[IJ}n^{KL]} = n^{IJKLP}m_P\,,\\ 6n^{I[J}n^{KLPQR]} &= -k^{I,JKLPQRS}m_S + k^{S,IJKLPQR}m_S\,,\\ 7n^{IJK[PQ}n^{RSTUV]} &= 2n^{[IJ}k^{K],PQRSTUV}\,, \qquad n^{[IJKLP}k^{Q],RSTUVWX}=0\,. \end{aligned} \tag{5.19}$$

Because of $E_7(\mathbb{Z})$ U-duality invariance, these multiplets interact mutually when $\Gamma_1\times\Gamma_2=0$, with the same interactions as Kaluza–Klein states in eleven dimensions.

Whenever $\Gamma_1\times\Gamma_2\neq 0$, the two 1/2 BPS particles can produce a 1/4 BPS state. This can easily be understood in type IIA, calling y^7 the M-theory circle coordinate and y^i with $i=1,6$ the type IIA T^6 coordinates we have the identification

$$q=(m_i,n^{i7})\,, \qquad \chi=(m_7,n^{ij},n^{ijkl7},k^{ijklpq7,7})\,, \qquad p=(n^{ijklp},k^{1234567,i})\,, \tag{5.20}$$

with $q\in I\!I_{6,6}$ the world-sheet zero modes, $\chi\in S_-$ the D-brane charges (in this case D0-D2-D4-D6) and $p\in I\!I_{6,6}$ the NS5-brane and Kaluza–Klein monopole charges. A perturbative string state carries a charge q while $\chi=p=0$, and the BPS string states are 1/4 BPS whenever $m_in^{i7}\neq 0$. In string theory two 1/2 BPS states can interact to give a generic 1/4 BPS state. According to (2.1) the perturbative 1/4 BPS states are counted by the partition function [219][26]

$$\int_{-\frac12}^{\frac12}d\tau_1\left(\prod_{n=1}^{\infty}\frac{(1-e^{\pi iz}q^n)^4(1-e^{-\pi iz}q^n)^4}{(1-q^n)^6(1-e^{2\pi iz}q^n)(1-e^{-2\pi iz}q^n)}-1\right)\left(2\sin(\pi z/2)\right)^8\sum_{m,n\in\mathbb{Z}^d}q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}\,, \tag{5.21}$$

[26] The reader should not confuse the charge $q\in I\!I_{6,6}$ with the $q=e^{2\pi i\tau}$ parameter in the partition function.

and its complex conjugate.

In fact all 1/4 BPS states of the theory can be mapped to perturbative 1/4 BPS states in string theory by U-duality. This is because $\Gamma \times \Gamma$ is always in the minimal nilpotent orbit of E_7, therefore there exists $\gamma \in E_7(\mathbb{Z})$ such that $\gamma(\Gamma \times \Gamma) = \big((q,q),0,0,0,0\big)$ and $\gamma\Gamma = (q,0,0)$ in the P_1 decomposition.

Up to and including $\nabla^6 R^4$, the only diagram that contribute are displayed in figure 13 and such that only three-point interactions with charges satisfying $\Gamma_1 \times \Gamma_2 \neq 0$ are involved in the loops. One can therefore extract the contribution from the supergravity amplitude [64]. Because this truncation to one-half BPS states is naturally interpreted as an exceptional field theory amplitude on a generalised torus [65–73], we shall refer to them as exceptional field theory amplitudes.

The direct computation in $d = 7{+}2\epsilon$ dimensional regularisation gives [64, 74]

$$\begin{aligned}
\mathcal{E}^{\text{ExFT 1-loop}}_{(0,1)\,\epsilon} &= \frac{8\pi^4}{567}\xi(10{+}2\epsilon)E^{E_7}_{(5+\epsilon)\Lambda_7}\,,\\
\mathcal{E}^{\text{ExFT 2-loop}}_{(0,1)\,\epsilon} &= \frac{2\pi^2}{9}\int_{\mathbb{R}^3_+}\frac{dL_1dL_2dL_3}{(\sum_{I<J}L_IL_J)^{-\epsilon}}\left(\sum_I L_I - \frac{5L_1L_2L_3}{\sum_{I<J}L_IL_J}\right)\sum_{\substack{\Gamma_I\in\mathbb{Z}^{56}\\ \sum_I\Gamma_I=0\\ \Gamma_I\times\Gamma_J=0}}{}' e^{-\pi\sum_I L_I|Z(\Gamma_I)|^2}\,,\\
\mathcal{E}^{\text{ExFT 3-loop}}_{(0,1)\,\epsilon} &= 40\xi(2{+}2\epsilon)\xi(1{+}2\epsilon)\xi(2\epsilon)E^{E_7}_{(1+\epsilon)\Lambda_5} + 20\frac{\Gamma(\epsilon)}{(\pi\mu^2)^\epsilon}\xi(2{+}2\epsilon)\xi(3{+}2\epsilon)E^{E_7}_{(\frac{3}{2}+\epsilon)\Lambda_6}\,,
\end{aligned} \tag{5.22}$$

where an infrared regularisation was introduced at three-loop because of the one-loop subdivergence when one internal propagator is massless. The three-loop result turns out to be finite, and up to a part linear in $\mathcal{E}_{(1,0)}$ one gets

$$\mathcal{E}^{\text{ExFT 3-loop}}_{(0,1)} = 40\xi(8)\xi(9)\xi(12)\widehat{E}^{E_7}_{6\Lambda_1} + \frac{64}{189\pi}\zeta(10)\widehat{E}^{E_7}_{5\Lambda_7}\,. \tag{5.23}$$

The second term is identical to the one-loop contribution whereas they each have the coefficient expected in string theory. The first term is not present is string theory, and satisfies the same differential equations as $\mathcal{E}^{\text{ExFT}}_{(0,1)}$ without the quadratic source term in (5.12).

The contributions from 1/2 BPS states seem to give all the ingredients to define the non-perturbative result, but there is some mismatch. This should not be unexpected since we neglected 1/4 BPS states in the loops that have no reason to do not contribute to ∇^4R^4 and ∇^6R^4 couplings. In [74] and [27] we argued that one can obtain such contribution using the one-loop and the two-loop string theory integrands. We extracted the contribution from BPS states in the perturbative string amplitude and summed over all U-duality copies to obtain the non-perturbative contribution. This sum does not converge, and the computation is only defined through an analytic continuation that we could not justify. Doing so we obtain nonetheless the following result including these expected

contributions from 1/4 BPS states

$$\mathcal{E}^{\text{BPS 1-loop}}_{(0,1)\,\epsilon} = 0\,,$$

$$\mathcal{E}^{\text{BPS 2-loop}}_{(0,1)\,\epsilon} = \frac{2\pi^2}{9}\int_{\mathbb{R}^3_+}\frac{d^3\Omega_2}{(\det\Omega_2)^{-\epsilon}}\Big(\sum_I L_I - \frac{5L_1L_2L_3}{\sum_{I<J}L_IL_J} - \frac{\sqrt{\det\Omega_2}}{6}E^{SL(2)}_{-3}(\tau)\Big)\sum_{\substack{\Gamma_I\in\mathbb{Z}^{56}\\ \sum_I\Gamma_I=0\\ \Gamma_I\times\Gamma_J=0}}{}' e^{-\pi\sum_I L_I|Z(\Gamma_I)|^2}\,,$$

$$\mathcal{E}^{\text{BPS 3-loop}}_{(0,1)\,\epsilon} = \frac{2\pi^2}{9}\int_{\mathbb{R}^3_+}d^3\Omega_2\frac{\sqrt{\det\Omega_2}}{6}E^{SL(2)}_{-3+2\epsilon}(\tau)\sum_{\substack{\Gamma_I\in\mathbb{Z}^{56}\\ \sum_I\Gamma_I=0\\ \Gamma_I\times\Gamma_J=0}}{}' e^{-\pi\sum_I L_I|Z(\Gamma_I)|^2} + \frac{8\pi^4}{567}\xi(10{+}2\epsilon)E^{E_7}_{(5+\epsilon)\Lambda_7}\,, \tag{5.24}$$

where we used

$$\Omega_2 = \begin{pmatrix} L_1{+}L_3 & L_3\\ L_3 & L_2{+}L_3\end{pmatrix} = \frac{\sqrt{\det\Omega_2}}{\tau_2}\begin{pmatrix} 1 & \tau_1\\ \tau_1 & |\tau|^2\end{pmatrix}\,. \tag{5.25}$$

The total one-loop contribution is found to vanish and the contribution from 1/4 BPS states at two-loop makes the total result finite in the limit $\epsilon\to 0$. There is no contribution from 1/4 BPS states at three-loop because the three-loop string theory amplitude is just the volume of the moduli space of genus-three surfaces and the sum over string zero modes are level-matched. The function $\mathcal{E}^{\text{BPS 3-loop}}_{(0,1)\,\epsilon}$ is simply a rewriting of the exceptional field theory amplitude

$$\mathcal{E}^{\text{ExFT 3-loop}}_{(0,1)\,\epsilon} = \mathcal{E}^{\text{BPS 3-loop}}_{(0,1)\,\epsilon} - \frac{5}{\pi}\big(\log(2\pi\mu)+\dots\big)\mathcal{E}_{(1,0)} + \mathcal{O}(\epsilon)\,. \tag{5.26}$$

To obtain this form it is convenient to introduce the variable $t=\sqrt{\det\Omega_2}$ to derive [74]

$$\begin{aligned}
&\frac{\pi^2}{27}\int_{\mathbb{R}^3_+}d^3\Omega_2\frac{(\det\Omega_2)^{\frac12+\epsilon}}{6}E^{SL(2)}_{-3+2\delta}(\tau)\sum_{\substack{\Gamma_I\in\mathbb{Z}^{56}\\ \sum_I\Gamma_I=0\\ \Gamma_I\times\Gamma_J=0}}{}' e^{-\pi\sum_I L_I|Z(\Gamma_I)|^2}\\
&= \frac{4\pi^2}{9}\int_0^\infty dt\,t^{3+2\epsilon}\int_{\mathcal{F}/\mathbb{Z}_2}\frac{d^2\tau}{\tau_2^2}E^{SL(2)}_{-3+2\delta}(\tau)\sum_{\substack{\Gamma_i\in\mathbb{Z}^{56}\\ \Gamma_i\times\Gamma_j=0}}{}' e^{-\pi t\frac{1}{\tau_2}(\Gamma_1+\bar\tau\Gamma_2)^{\intercal}v^{\intercal}v(\Gamma_1+\tau\Gamma_2)}\\
&= \frac{4\pi^2}{9}\xi(2\epsilon+2\delta)\xi(7-2\delta+2\epsilon)E^{E_7}_{(\frac72-\delta+\epsilon)\Lambda_6+(-3+2\delta)\Lambda_7}\\
&= 40\frac{\xi(4)\xi(7{+}2\epsilon{-}2\delta)}{\xi(4{-}2\epsilon{-}2\delta)\xi(7{-}4\delta)}\xi(8{-}4\delta)\xi(9{-}2\epsilon{-}2\delta)\xi(12{-}2\epsilon{-}2\delta)E^{E_7}_{(6-\epsilon-\delta)\Lambda_1+2\epsilon\Lambda_7}
\end{aligned} \tag{5.27}$$

where $\mathcal{F}/\mathbb{Z}_2 = SO(2)\backslash SL(2)/PGL(2,\mathbb{Z})$.

This justifies the definition

$$\mathcal{E}_{(0,1)} = \lim_{\epsilon\to0}\Big(\mathcal{E}^{\text{ExFT}}_{(0,1),\epsilon} + \frac{8\pi^4}{567}\xi(10{+}2\epsilon)E^{E_7}_{(5+\epsilon)\Lambda_7}\Big)\,, \tag{5.28}$$

with

$$\mathcal{E}^{\rm ExFT}_{(0,1),\epsilon}=8\pi\int_{\mathcal{G}_2}\mathrm{d}^3\Omega_2\left((\det\Omega_2)^{\epsilon}\varphi^{\rm tr}_{\rm KZ}-\frac{\pi}{36}(\det\Omega_2)^{\frac{1}{2}+\epsilon}E^{SL(2)}_{-3}+\frac{\pi}{36}\sqrt{\det\Omega_2}E^{SL(2)}_{-3+2\epsilon}\right)\sum_{\substack{\Gamma_i\in\mathbb{Z}^{56}\\ \Gamma_i\times\Gamma_j=0}}{}'\,e^{-\pi\Omega^{ij}\Gamma_i^{\intercal}v^{\intercal}v\Gamma_j} \tag{5.29}$$

where we folded the integration domain to $\mathcal{G}_2=\mathbb{R}^3_+/S_3$, use $\varphi^{\rm tr}_{\rm KZ}$ in (4.17), and wrote the sum over Γ_i's in a manifestly $GL(2,\mathbb{Z})$ invariant form.

The justifications to arrive to this formula are not fully satisfying. However, taking this as a working hypothesis turns out to give the right answer, as we can check using all possible limits.

We analysed this function (5.28) in [27] in details and showed that it is consistent with string perturbation theory and the large radius limit. We found that the additional Eisenstein series $E^{SL(2)}_{-3}$ and $E^{SL(2)}_{-3+2\epsilon}$ in the integrand are necessary for

$$\hat{\mathcal{E}}^{\rm ExFT}_{(0,1)}=\lim_{\epsilon\to0}\left(\mathcal{E}^{\rm ExFT}_{(0,1),\epsilon}+40\xi(4)\xi(8)\xi(1{+}2\epsilon)E^{E_7}_{4\Lambda_7}\right) \tag{5.30}$$

to satisfy the differential equations (5.11) and (5.12). Independently of the argument we gave to justify the definition of $\hat{\mathcal{E}}^{\rm ExFT}_{(0,1)}$, one can consider the consistency checks as a proof that it defines the exact coupling function in string theory.

The same construction applies to all dimensions $D\geq4$ and one gets the direct generalisation

$$\begin{aligned}&\mathcal{E}^{\rm ExFT}_{(0,1),\epsilon}\\&=8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\left((\det\Omega_2)^{\epsilon}\varphi^{\rm tr}_{\rm KZ}-\frac{\pi}{36}(\det\Omega_2)^{\frac{1}{2}+\epsilon}E^{SL(2)}_{-3}+\frac{\pi}{36}\sqrt{\det\Omega_2}E^{SL(2)}_{-3+2\epsilon}\right)\sum_{\substack{\Gamma_i\in\mathbb{L}_d\\ \Gamma_i\times\Gamma_j=0}}{}'\,e^{-\pi\Omega^{ij}\Gamma_i^{\intercal}v^{\intercal}v\Gamma_j}\end{aligned} \tag{5.31}$$

and

$$\mathcal{E}_{(0,1)}=\lim_{\epsilon\to0}\left[\mathcal{E}^{\rm ExFT}_{(0,1),\epsilon}+\frac{8\pi^4}{567}\xi(d{+}2\epsilon{+}3)E^{E_d}_{\frac{d+2\epsilon+3}{2}\Lambda_d}+\frac{5\zeta(3)}{3\epsilon}\delta_{d,5}+\frac{\pi^2}{3\epsilon}\delta_{d,3}\left(\frac{1}{\epsilon}+\frac{1}{6}+\frac{1}{\pi}\mathcal{E}_{(0,0)\,\epsilon}\right)\right] \tag{5.32}$$

which is consistent with string perturbation theory and the large radius limit [27]. The poles in ϵ come from the three-loop divergence in $D=6$ and the two-loop divergence in $D=8$ [125,126]. For $d=2,3,4$ one does not need to introduce the two terms with the Eisenstein series $E^{SL(2)}_{-3}$ because there is no pole.

5.3 Matching the eleven-dimensional supergravity limit

It may seem natural that these formulas give back the correct eleven-dimensional supergravity contribution but it turns out to be a rather non-trivial computation. We find after some efforts that formulas (5.28) and (5.29) gives back the formula obtained from supergravity in (4.47), with for $4\leq d\leq7$

$$\mathcal{E}^{\text{ExFT}}_{(0,1)\,\epsilon} \sim V^{3\frac{d+3}{9-d}}\Big(\frac{4\pi^2\xi(2)^2}{3V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(d)}_{\frac{3}{2}\Lambda_1}+\frac{40}{V^{6-2\epsilon}}\xi(5{-}2\epsilon)\xi(4{-}2\epsilon)\xi(3{-}2\epsilon)E^{SL(d)}_{(\frac{5}{2}-\epsilon)\Lambda_3}$$
$$+\frac{8\pi}{V^3}\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d-2\epsilon}{2}}}\varphi^{\text{tr}}_{\text{KZ}}{\sum_{n_i\in\mathbb{Z}^d}}'e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}+\frac{40}{V^3}\delta_{d,7}\xi(4)\xi(8)\xi(2\epsilon)\Big(E^{SL(7)}_{(4-2\epsilon)\Lambda_6}-E^{SL(7)}_{4\Lambda_6}\Big)$$
$$+\frac{40}{V^{5-4\epsilon}}\xi(2{+}2\epsilon)\xi(2\epsilon)\Big(\delta_{d,6}\xi(2-2\epsilon)+\delta_{d,7}\xi(5{+}2\epsilon)E^{SL(7)}_{(\frac{5}{2}+\epsilon)\Lambda_1}\Big)\Big) \tag{5.33}$$

and

$$40\xi(2{+}2\epsilon)\xi(6{+}2\epsilon)\xi(d{+}2\epsilon{+}3)E^{E_d}_{\frac{d+2\epsilon+3}{2}\Lambda_d}$$
$$\sim 40V^{3\frac{d+3+2\epsilon}{9-d}}\Big(\xi(2{+}2\epsilon)\xi(6{+}2\epsilon)\xi(d{+}2\epsilon{+}3)E^{SL(d)}_{\frac{d+2\epsilon+3}{2}\Lambda_{d-1}}+\delta_{d,7}\frac{\xi(4-2\epsilon)\xi(8-2\epsilon)\xi(1{+}2\epsilon)}{V^{3+6\epsilon}}E^{SL(d)}_{(4-\epsilon)\Lambda_6}$$
$$+\frac{\xi(2{+}2\epsilon)\xi(5{+}2\epsilon)\xi(6{+}2\epsilon)}{V^{5+2\epsilon}}E^{SL(d)}_{(3+\epsilon)\Lambda_2}+\delta_{d\geq 5}\frac{\xi(2{+}2\epsilon)\xi(3{+}2\epsilon)\xi(1{+}2\epsilon)}{V^{6+4\epsilon}}E^{SL(d)}_{(\frac{3}{2}+\epsilon)\Lambda_5}\Big)\,. \tag{5.34}$$

This is indeed in perfect agreement with (4.18) including the logarithmic terms computed in section 4.2. To see this one uses

$$\begin{aligned}
\xi(5{-}2\epsilon)\xi(4{-}2\epsilon)\xi(3{-}2\epsilon)E^{SL(d)}_{(\frac{5}{2}-\epsilon)\Lambda_3} &\underset{d\geq 5}{=} \xi(2{-}2\epsilon)\xi(3{-}2\epsilon)\xi(1{-}2\epsilon)E^{SL(d)}_{\epsilon\Lambda_2+(\frac{3}{2}-\epsilon)\Lambda_5}\\
\xi(2{+}2\epsilon)\xi(5{+}2\epsilon)\xi(6{+}2\epsilon)E^{SL(7)}_{(3+\epsilon)\Lambda_2} &= \xi(2{+}2\epsilon)\xi(1{+}2\epsilon)\xi(5{+}2\epsilon)E^{SL(7)}_{(\frac{5}{2}+\epsilon)\Lambda_1-\epsilon_7}\\
\xi(2{+}2\epsilon)\xi(5{+}2\epsilon)\xi(6{+}2\epsilon)E^{SL(6)}_{(3+\epsilon)\Lambda_2} &= \xi(2{+}2\epsilon)^2\xi(1{+}2\epsilon)E^{SL(6)}_{-\epsilon\Lambda_4}\,.
\end{aligned} \tag{5.35}$$

We will now explain how we obtained formula (5.33). For this purpose we decompose the pairs of charges according to (5.18), and carry out the sums over layers, including an increasing number of non-vanishing components. We start by the layer for which all the brane charges vanish

$$8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\Big((\det\Omega_2)^\epsilon\varphi^{\text{tr}}_{\text{KZ}}-\frac{\pi}{36}(\det\Omega_2)^{\frac{1}{2}+\epsilon}E^{SL(2)}_{-3}+\frac{\pi}{36}\sqrt{\det\Omega_2}E^{SL(2)}_{-3+2\epsilon}\Big){\sum_{m_i\in\mathbb{Z}^d}}'e^{-\frac{\pi}{r^3}\Omega_2^{ij}m_i^\intercal U^{-1}m_j}$$
$$=8\pi V^{3\frac{d+3+4\epsilon}{9-d}-3}\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}(\det\Omega_2)^\epsilon\varphi^{\text{tr}}_{\text{KZ}}{\sum_{m_i\in\mathbb{Z}^d}}'e^{-\pi\Omega_2^{ij}m_i^\intercal U^{-1}m_j} \tag{5.36}$$
$$+40\delta_{d,7}\xi(4)V^{12}\Big(-V^{6\epsilon}\frac{\xi(7{+}2\epsilon)}{\xi(7)}\xi(8)\xi(2\epsilon)E^{SL(7)}_{\epsilon\Lambda_5+4\Lambda_6}+\frac{\xi(7{-}2\epsilon)}{\xi(7{-}4\epsilon)}\xi(8{-}4\epsilon)\xi(2\epsilon)E^{SL(7)}_{\epsilon\Lambda_5+(4-2\epsilon)\Lambda_6}\Big)\,.$$

Using simplifications at first order in ϵ one obtains for $d=7$

$$8\pi\int_{\mathcal{G}_2}\mathrm{d}^3\Omega_2\Big((\det\Omega_2)^\epsilon\varphi^{\text{tr}}_{\text{KZ}}-\frac{\pi}{36}(\det\Omega_2)^{\frac{1}{2}+\epsilon}E^{SL(2)}_{-3}+\frac{\pi}{36}\sqrt{\det\Omega_2}E^{SL(2)}_{-3+2\epsilon}\Big){\sum_{n_i\in\mathbb{Z}^7}}'e^{-\pi\Omega_2^{ij}r^{-3}n_i^\intercal U^{-1}n_j}$$
$$=8\pi V^{12}\int_{\mathcal{G}_2}\mathrm{d}^3\Omega_2(\det\Omega_2)^\epsilon\varphi^{\text{tr}}_{\text{KZ}}{\sum_{n_i\in\mathbb{Z}^7}}'e^{-\pi\Omega_2^{ij}n_i^\intercal U^{-1}n_j}$$
$$+40\xi(4)\xi(8)V^{12}\Big(-\xi(2\epsilon)E^{SL(7)}_{4\Lambda_6}+\xi(2\epsilon)E^{SL(7)}_{(4-2\epsilon)\Lambda_6}+c\,E^{SL(7)}_{4\Lambda_6}\Big)\,, \tag{5.37}$$

for a constant c. The determination of the constant c is irrelevant at this level since we did not determine the finite amplitude from string theory with the explicit split in analytic and non-analytic components.

At the next step we consider that the M2-brane windings n_1^{IJ} and n_2^{IJ} are non-zero but linearly dependent. Because of the constraint (5.19) they are in the $SL(d,\mathbb{Z})$ orbit of the highest weight representative, and the sum can be written as a Poincaré sum over $P_2(\mathbb{Z})\backslash SL(d,\mathbb{Z})$ and pairs of integers $n_i = \gcd(n_i^{IJ})$. We have therefore

$$\begin{aligned}
\theta_d^{(2)} &= \sum_{\substack{n_i^{IJ}\in\wedge^2\mathbb{Z}^d \\ n_i^{I[J}n_j^{KL]}=0}}{}' \sum_{\substack{m_{iI}\in\mathbb{Z}^d \\ n_i^{IJ}m_{jJ}=0}} e^{-\pi\Omega_2^{ij}\left(r^{-3}U^{-1IJ}(m_{iI}+1/2a_{IKL}n_i^{KL})(m_{jJ}+1/2a_{JPQ}n_j^{PQ})+1/2r^{6-d}U_{IK}U_{JL}n_i^{IJ}n_j^{KL}\right)} \\
&= \sum_{\gamma\in P_2\backslash SL(d)} \sum_{n_i\in\mathbb{Z}^2}{}' \sum_{m_i\in\mathbb{Z}^{d-2}} e^{-\pi\Omega_2^{ij}\left(r^{-3}y^{\frac{2}{d-2}}u^{-1}(m_i+an_i,m_j+an_j)+r^{6-d}y^2n_in_j\right)}\Big|_\gamma \qquad (5.38) \\
&= \sum_{\gamma\in P_2\backslash SL(d)} \frac{r^{3(d-2)}y^{-2}}{\det\Omega_2^{\frac{d-2}{2}}} \sum_{n_i\in\mathbb{Z}^2}{}' \sum_{m^i\in\mathbb{Z}^{d-2}} e^{-\pi\Omega_2^{ij}r^{6-d}y^2n_in_j-\pi\Omega_{ij}^{-1}r^3y^{-\frac{2}{d-2}}u(m^i,m^j)+2\pi in_i(m^i,a)}\Big|_\gamma
\end{aligned}$$

and one computes the integral using Appendix F.3

$$\begin{aligned}
&8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d-2\epsilon}{2}}}\varphi_{\mathrm{KZ}}^{\mathrm{tr}}\theta_d^{(2)} \\
\sim\ & \frac{16\pi^2\xi(2-2\epsilon)^2}{(3+2\epsilon)(4-2\epsilon)}r^{2d-2\epsilon(6-d)}E^{SL(d)}_{2\epsilon\Lambda_2}+\frac{8\pi^2}{3}r^{\frac{5d-9}{2}+(d-3)\epsilon}\xi(1-2\epsilon)\xi(2-2\epsilon)E^{SL(d)}_{2\epsilon\Lambda_2+(\frac{1}{2}-\epsilon)\Lambda_3} \\
&+\frac{4\pi^2}{9}r^{\frac{9}{2}(d-5)+(d-3)\epsilon}\xi(5-2\epsilon)\xi(3+2\epsilon)E^{SL(d)}_{2\epsilon\Lambda_2+(\frac{5}{2}-\epsilon)\Lambda_3}+\dots \\
\sim\ & \frac{4\pi^2\xi(2)^2}{3}r^{2d}+\frac{8\pi^2}{3}r^{\frac{5d-9}{2}}\xi(2)\xi(3)E^{SL(d)}_{\frac{3}{2}\Lambda_1} \\
&+\frac{4\pi^2}{9}r^{\frac{9}{2}(d-5)+(d-3)\epsilon}\xi(5-2\epsilon)\xi(3+2\epsilon)E^{SL(d)}_{2\epsilon\Lambda_2+(\frac{5}{2}-\epsilon)\Lambda_3}+\dots \qquad (5.39)
\end{aligned}$$

up to terms that are exponentially suppressed at large V. The first line is finite, except for $d=3$ in which case one has instead $\widehat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}$. For the complete function one computes similarly

$$\begin{aligned}
&8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\left((\det\Omega_2)^\epsilon\varphi_{\mathrm{KZ}}^{\mathrm{tr}}-\frac{\pi}{36}(\det\Omega_2)^{\frac{1}{2}+\epsilon}E^{SL(2)}_{-3}+\frac{\pi}{36}\sqrt{\det\Omega_2}E^{SL(2)}_{-3+2\epsilon}\right)\theta_d^{(2)} \\
&= V^{3\frac{d+3}{9-d}}\left(\frac{4\pi^2\xi(2)^2}{3V}+\frac{8\pi^2}{3V^2}\xi(2)\xi(3)E^{SL(d)}_{\frac{3}{2}\Lambda_1}+\frac{40}{V^{6-2\epsilon}}\xi(5-2\epsilon)\xi(4-2\epsilon)\xi(3-2\epsilon)E^{SL(d)}_{(\frac{5}{2}-\epsilon)\Lambda_3}\right). \qquad (5.40)
\end{aligned}$$

There are many more pairs of charges Γ_i to be considered, but similarly as for the Eisenstein series $E^{E_d}_{\frac{d-3}{2}\Lambda_d}$ and $E^{E_d}_{\frac{d+1}{2}\Lambda_d}$ most of the contributions turn out to vanish at $\epsilon\to 0$. For this to be true it is crucial to take the definition (5.31), and not the naive expression (5.22). We relegate this computation to Appendix E. One obtains that there are no other contributions for $d=3,4,5$. For $d=6,7$ one must include the contribution for which the M5-brane charges n^{IJKLP} are non-zero and linearly independent. They give the last line in (5.33).

5.4 1/8-BPS brane instanton measure and uniqueness

In this last section we will show that there is a unique coupling function $\mathcal{E}_{(0,1)}$ compatible with supersymmetry and string perturbation theory in four dimensions. For this purpose we consider a generic D-brane instanton correction, i.e. an abelian Fourier coefficient in the parabolic $P_1 \subset E_7$

$$F_Q(g_s, v) = \int_{[0,1]^{32+1}} d^{32}a\, db\, \mathcal{E}_{(0,1)} e^{-2\pi i(Q,a)} \,, \tag{5.41}$$

for a generic D-brane charge $Q \in S_-$. There are two real orbits of charges $Q \in S_-$. Either the quartic invariant

$$I_4(Q) = 2(Q\gamma^{ab}Q)(Q\gamma_{ab}Q) \tag{5.42}$$

is positive and the Levi stabiliser is $SU(3,3) \subset Spin(6,6)$, or it is negative and the Levi stabiliser is $SL(6) \subset Spin(6,6)$. For $I_4 < 0$ one can find a representative of the character $e^{-2\pi i(Q,a)}$ with support on two real roots, as for example a D(-1)-D5 brane instanton. For $I_4 > 0$ one needs at least four roots, corresponding for example to a D1-D1-D1-D5 instanton, with three orthogonal Euclidean D1-branes inside the Euclidean D5 [220]. One works out in general that two 1/2-BPS charges Q_1 and Q_2 satisfy that $I_4(Q_1{+}Q_2) \leq 0$. This implies that the source in equation (5.12) drops out for the Fourier coefficient $F_Q(g_s, v)$ when $I_4(Q) > 0$,

$$\big(\Delta + 60\big)\big(F_Q(g_s, v) e^{2\pi i(Q,a)}\big) \underset{I_4(Q)>0}{=} 0 \,, \tag{5.43}$$

and $F_Q(g_s, v)$ satisfies the same differential equations as the Eisenstein series $\widehat{E}^{E_7}_{6\Lambda_1}$ Fourier coefficient

$$\big(\mathcal{D}_{ijpq}\mathcal{D}^{pqmn}\mathcal{D}_{mnkl} + 9\mathcal{D}_{ijkl}\big)\big(F_Q(g_s, v) e^{2\pi i(Q,a)}\big) \underset{I_4(Q)>0}{=} 0 \,. \tag{5.44}$$

One proves that there is a unique solution to these differential equations with moderate growth such that [27]

$$F_Q(g_s, v) = \mu(Q) g_s^{-\beta} B(vQ/g_s) \,, \tag{5.45}$$

where $\mu(Q)$ is a function of Q that only depends on its arithmetic $Spin(6,6,\mathbb{Z})$ invariants. This instanton measure $\mu(Q)$ is expected to be the partition function for the corresponding Euclidean brane effective theory. The function $B(vQ/g_s)$ only depends of the $SU(4) \times SU(4)$ invariant polynomials in the complex central charge $v_{i\hat{j}}{}^{\alpha}Q_{\alpha} = Z_{i\hat{j}}(Q)$ in the bi-fundamental representation of $SU(4) \times SU(4)$. As for BPS black holes in four dimensions, these invariants are polynomials in the four modules $|z_i|$ of the eigenvalues of $Z_{i\hat{j}}$ and the phase of its determinant [221]. The Euclidean action of the instanton is the largest eigenvalue, and the function $B(Z/g_s)$ is exponential suppressed in $e^{-2\pi \max_i |z_i|/g_s}$. We computed these Fourier coefficients in [27] up to a final Poincaré sum that we could not simplify, but which exhibits the structure explained above. We will see below that a more precise result can be obtained directly from the two-loop string amplitude.

The Fourier coefficient $F_Q(g_s, v)$ does not depend on the Levi stabiliser $SU(3,3)$ of the D-brane charge Q. We can therefore consider the limit in which one dilaton in $SU(3,3)$ goes to infinity while keeping $F_Q(g_s, v)$ finite. To see this, consider the parabolic decomposition

$$\mathfrak{su}(3,3) \supset \big(\mathfrak{gl}_1 \oplus \mathfrak{u}(2,2)\big)^{(0)} \oplus (\mathbb{C}^4)^{(1)} \oplus \mathbb{R}^{(2)} \,. \tag{5.46}$$

[27] The proof will be published in a collaboration with Friedberg, Gourevitch, Kleinschmidt and Persson.

brane	T^2	T^2	T^4	T^4	T^4	T^4
D(-1)	○	○	○	○	○	○
D1	●	●	○	○	○	○
D1	○	○	●	●	○	○
D1	○	○	○	○	●	●
D5	●	●	●	●	●	●

$\Rightarrow$

brane	T^2	T^2	T^4	T^4	T^4	T^4
D1	○	●	●	○	○	○
D1	●	○	●	○	○	○
D1	○	●	○	●	○	○
D3	○	●	●	○	●	●
D3	●	○	○	●	●	●

Table 3: 1/8-BPS configurations of D-branes related by T-duality on one circle in T^2 and one in T^4

It embeds in the parabolic P_2 of $SO(6,6)$

$$\mathfrak{so}(6,6) \supset \big(\mathfrak{gl}_2 \oplus \mathfrak{so}(4,4)\big)^{(0)} \oplus (\mathbf{2},\mathbf{8}_{\rm v})^{(1)} \oplus \mathbb{R}^{(2)} \,, \tag{5.47}$$

such that the spinor representation decomposes as

$$\mathbf{32} \cong \mathbf{8}_{\rm s}^{(-1)} \oplus (\mathbf{2},\mathbf{8}_{\rm c})^{(0)} \oplus \mathbf{8}_{\rm s}^{(1)} \,. \tag{5.48}$$

We recognise up to triality the decomposition of the parabolic $P_1 \subset E_7$. We find therefore that the $GL(1) \subset SU(3,3)$ in the stabiliser of the charge Q is conjugate under E_7 to the $GL(1) \subset E_7$ defining P_1. The corresponding dilatons are not conjugate under $E_7(\mathbb{Z})$, however, we can interpret this parabolic P_2 in the limit in which one $T^2 \subset T^6$ admits a large volume $\mathrm{Vol}(T^2) = (2\pi)^2\alpha'/y^2$, and there is an $E_7(\mathbb{Z})$ transformation that exchanges y and $g_{\rm s}$ and acts by triality on $Spin(4,4)$.

Let us choose $Q = (0, q_i, 0)$ in this decomposition with $I_4(Q) = q_1^2 q_2^2 - (q_1 \cdot q_2)^2 > 0$, which corresponds for example to a 1/8 BPS Euclidean D1-D1-D1-D3-D3 instanton as displayed in Table 3. We write $\frac{1}{y^2 U_2}\left(\begin{smallmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{smallmatrix}\right)$ the metric on T^2, $\upsilon_{\rm v} \in SO(4,4)$ the Narain moduli on T^4 and c^1, c^2 the Narain axions parametrising the off-diagonal components between T^2 and T^4. We write $\upsilon_{\rm c}$ and $\upsilon_{\rm s}$ for the spinor representations. We have then for $Q = (p, q_i, 0)$

$$vQ = \Big(y\upsilon_{\rm c}(p + \not{q}_i c^i), \frac{1}{\sqrt{U_2}}\upsilon_{\rm s}(q_1 + U q_2), 0\Big) \,. \tag{5.49}$$

We would like to interpret the Fourier coefficient $F_Q(g_{\rm s}, v)$ in (5.45) as a Fourier coefficient of a perturbative term in the large T^2 volume limit after duality $g_{\rm s} \leftrightarrow y$. The axions c^i become then Ramond-Ramond fields, so we need to integrate them out. But to do so we need to include all Fourier coefficients that will contribute, so we must include the D-branes wrapping T^4 only with $p \neq 0$. We compute using Poisson summation over $p \in \mathbb{Z}^8$ that

$$\begin{aligned} &\int_{[0,1]^{16}} d^{16}c \sum_{p\in\mathbb{Z}^8} \mu(p,q,0) g_{\rm s}^{-\beta} B\Big(\frac{y}{g_{\rm s}}\upsilon_{\rm c}(p+\not{q}_i c^i), \frac{1}{g_{\rm s}\sqrt{U_2}}\upsilon_{\rm s}(q_1+Uq_2), 0\Big) \\ = \;& \frac{g_{\rm s}^{8-\beta}}{y^8} \sum_{p \bmod \not{q}} \frac{\mu(p,q,0)}{\det{}'\not{q}} \widetilde{B}\Big(0, \frac{1}{g_{\rm s}\sqrt{U_2}}\upsilon_{\rm s}(q_1+Uq_2), 0\Big) \end{aligned} \tag{5.50}$$

where

$$\widetilde{B}(a,z,b) = \int_{\mathbb{R}^8} d^8x \; B(x,z,b) e^{2\pi i (x,a)} \tag{5.51}$$

and the sum is over the equivalence classes of $p \in \mathbb{Z}^8 \subset \mathbf{8}_c$ modulo $p \sim p + \not{q}_i x^i$ for x^i in $\mathbb{Z}^{2\times 8} \subset (\mathbf{2}, \mathbf{8}_v)$, while $\det'\not{q}$ is the number of such classes. Here we have used that the measure is invariant under the T-duality unipotent subgroup $U(\mathbb{Z}) \subset SO(6,6,\mathbb{Z})$ of parameter $x^i \in \mathbb{Z}^{2\times 8} \subset (\mathbf{2}, \mathbf{8}_v)$

$$\mu(p + \not{q}_i x^i, q_i, 0) = \mu(p, q_i, 0) \ . \tag{5.52}$$

Using the duality that exchanges y and g_s, one concludes that the same Fourier coefficient appears as a Fourier coefficient of the two-loop amplitude in the large T^2 torus volume limit. If we assume moreover $\det'\not{q} = 1$ this gives

$$\int_{[0,1]^{16}} d^{16}c \ \Big(\int_{[0,1]^{32+1}} d^{32}a\, db\, \mathcal{E}_{(0,1)}\Big) e^{-2\pi i(q,c)} = \mu(0,q,0)\frac{y^{8-\beta}}{g_s^8}\widetilde{B}\Big(0, \frac{1}{y\sqrt{U_2}}v_v(q_1+Uq_2), 0\Big) \ . \tag{5.53}$$

We find therefore that the existence of a non-zero Fourier coefficient $F_Q(g_s, v)$ implies by $E_7(\mathbb{Z})$ invariance that there are some terms that are not exponentially suppressed in the weak coupling limit. A similar argument works for smaller orbits, so we conclude that a function with wavefront set $\overline{O}_{2\Lambda_1}$ cannot be exponentially suppressed at small $g_s \ll 1$. This implies that finding a U-duality invariant function $\mathcal{E}_{(0,1)}$ compatible with the differential equations imposed by supersymmetry and string perturbation theory must give the exact string theory coupling function.

This formula is also useful because it determines the D-brane instanton from a world-sheet instanton. Using the two-loop amplitude [83] we obtain

$$8\pi \int_{[0,1]^{16}} d^{16}c \int_{\mathcal{F}_2} \frac{d^6\Omega}{\det\Omega_2^3} \Gamma_{II_{6,6}} \varphi_{KZ}(\Omega) e^{-2\pi i(q,c)} = \sum_{p \bmod \not{q}} \frac{\mu(p,q,0)}{\det'\not{q}} y^{8-\beta} \widetilde{B}\Big(0, \frac{1}{y\sqrt{U_2}}v_v(q_1+Uq_2), 0\Big) \ . \tag{5.54}$$

Using the Fourier expansion of the Kawazumi-Zhang invariant φ_{KZ} derived in [173] together with the unfolding method [117],[28] one computes straightforwardly that $\beta = 10$ and

$$\begin{aligned} &\widetilde{B}\Big(0, \frac{1}{g_s\sqrt{U_2}}v_v(q_1+Uq_2), 0\Big) \qquad (5.55)\\ = \ & 4\pi \int_{SO(2)\backslash GL(2,\mathbb{R})} \frac{d^3\Omega_2}{\det\Omega_2} \left(\frac{I_4(q)}{g_s^4} + \frac{5}{4\pi^2 \det\Omega_2}\Big(1 + \frac{\pi \mathrm{tr}\Omega_2 Z(q) Z(q)^\intercal}{g_s^2}\Big)\right) e^{-\frac{2\pi \mathrm{tr}\Omega_2 Z_L(q) Z_L(q)^\intercal}{g_s^2} - \pi \mathrm{tr}\Omega_2^{-1}} \ , \end{aligned}$$

with

$$\begin{aligned} Z(q)Z(q)^\intercal &= Z_L(q)Z_L(q)^\intercal + Z_R(q)Z_R(q)^\intercal \ , \\ Z_L(q)Z_L(q)^\intercal &= \frac{1}{U_2}\begin{pmatrix} 1 & U_1 \\ 0 & U_2 \end{pmatrix}\begin{pmatrix} p_L(q_1)^2 & p_L(q_1)p_L(q_2) \\ p_L(q_1)p_L(q_2) & p_L(q_2)^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ U_1 & U_2 \end{pmatrix} \ . \end{aligned} \tag{5.56}$$

[28] In fact the unfolding method does not obviously make sense because the integral diverges at the cusp and φ_{KZ} is a distribution. One convenient way to regularise the integral is to introduce a differential operator acting with Maass raising and lowering operators that satisfy $\overline{\Box}_2\Box\varphi_{KZ} = \frac{15}{32}\varphi_{KZ}$ away from the separating degeneration locus $\Omega^{12} = 0$ [173]. Up to terms that involve a distribution on this singular locus, one can define $\int d\mu\, \varphi_{KZ}\Gamma \sim \frac{32}{15}\int d\mu\, \varphi_{KZ}\overline{\Box}_2\Box\Gamma$, which is regular at the cusp. We thank Solomon Friedberg for this explanation. One must treat separately the regularisation of the integral at the singular locus, but this does not affect the generic Fourier coefficients with $I_4 > 0$.

The measure satisfies

$$\sum_{p \bmod \not{q}} \frac{\mu(p,q,0)}{\det' \not{q}} = \sum_{\substack{A \in \mathbb{Z}^{2\times 2}/GL(2,\mathbb{Z}) \\ A^{-1}q \in \mathbb{Z}^2 \otimes I\!I_{4,4}}} \sum_{d | A^{-1} q^{\intercal} q A^{-1 \intercal}} \frac{1}{d^3 \det A} \tilde{c}\Big(\frac{I_4(q)}{d^2 \det A^2}\Big) \,, \tag{5.57}$$

with $\tilde{c}$ the Fourier coefficients of

$$-\frac{\vartheta_4(2\tau)}{\eta(4\tau)^6} = \sum_{n \geq -1} \tilde{c}(n) e^{2\pi i n \tau} \,, \tag{5.58}$$

and $d | q^{\intercal} q$ means that d divides $\gcd(q_1^2/2, q_2^2/2, q_1 \cdot q_2)$.

The function $\widetilde{B}$ above and therefore B itself exhibits the exponential suppression in the Euclidean action for a 1/8 BPS Euclidean brane instanton. The measure factor $\mu(Q)$ involves the Fourier coefficients $\tilde{c}(I_4(q))$ that counts the number of 1/8 BPS D-brane bound-states through the helicity supertrace $\Omega_{14}(Q)$ [30–32].

We will now precise the relation between the instanton measure and the 1/8-BPS helicity supertrace. One checks that $\det' \not{q} = 1$ if we take q with $\gcd(q) = 1$ and such that

$$\gcd q q^{\intercal} \equiv \gcd(q_1^2/2, q_2^2/2, q_1 \cdot q_2) = 1 \,. \tag{5.59}$$

If the D-brane charge Q is primitive in S_-, i.e. with $\gcd(Q) = 1$, one can always find $\gamma \in Spin(6,6,\mathbb{Z})$ to bring it into a diagonal reduced form $Q = (0, q, 0)$ with [183]

$$q_1 = e_{1+} + n_1 e_{1-} \,, \qquad q_2 = n_2 e_{2+} + n_3 e_{2-} + k e_{1-} \,, \tag{5.60}$$

where the $e_{i\pm}$ define a basis of null vectors

$$(e_{i\pm}, e_{j\pm}) = 0 \,, \qquad (e_{i+}, e_{j-}) = \delta_{ij} \,. \tag{5.61}$$

These charges correspond in Table 3 to the brane configuration T-dual to a single D5-brane, n_I D1-brane in each orthogonal $T^2 \subset T^6$ and k D(-1)-branes. The T-duality subgroup $SL(3,\mathbb{Z}) \times SL(3,\mathbb{Z}) \subset SL(6,\mathbb{Z}) \subset Spin(6,6,\mathbb{Z})$ acts on the three by three matrix of D1-brane charges that wrap one even and one odd coordinate in the bi-fundamental. One can find such a transformation to bring (n_1, n_2, n_3) in a form in which n_1 divides n_2 and n_2 divides n_3. We will write for short $n_1 | n_2 | n_3$. One computes then for a given basis of gamma matrices that

$$\det' \not{q} = \gcd(k, n_1, n_2)^2 \gcd(k, n_1, n_3)^2 \underset{n_1|n_2|n_3}{=} \gcd(k, n_1)^4 \,, \tag{5.62}$$

so $\det' \not{q} = 1$ if $\gcd q q^{\intercal} = \gcd(k, n_1, n_2 n_3) = 1$. We will therefore assume $\gcd q q^{\intercal} = 1$ to avoid the sum over p classes in (5.57). For such q one obtains moreover that the only matrices A modulo $GL(2,\mathbb{Z})$ dividing q are $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right)$ with $r | \gcd(k, n_2, n_3)$ and no $d > 1$ divides $q q^{\intercal}$. Using

$$\gcd(q_1 \wedge q_2) = \gcd(k, n_2, n_3, n_1 n_2, n_1 n_3) = \gcd(k, n_2, n_3) \,, \tag{5.63}$$

we obtain that the instanton measure takes the form

$$\mu(0,q,0) \underset{\substack{\gcd(q)=1 \\ \gcd(q q^{\intercal})=1}}{=} \sum_{r | q_1 \wedge q_2} r^{-1} \tilde{c}\Big(\frac{I_4(q)}{r^2}\Big) \,. \tag{5.64}$$

We can write this expression in a manifestly $Spin(6,6,\mathbb{Z})$ invariant form following [222]

$$\mu(Q) \underset{\substack{\gcd(Q)=1\\ \gcd(Q\times Q)'=1}}{=} \sum_{d|\frac{1}{4}Q\wedge I_4'(Q)} d^{-1}\tilde{c}\Big(\frac{I_4(Q)}{d^2}\Big)\ , \tag{5.65}$$

where $(Q\times Q)'$ is $Q\gamma_{ab}Q$ with the component in $\mathbb{Z}/2$ excluded.[29] $I_4'(Q)$ is the derivative of $I_4(Q)$ and $\frac{1}{4}Q\wedge I_4'(Q)$ includes the projection to $\wedge^4 I\!I_{6,6}$ and $I_4(Q)$. They are defined for a general Q in diagonal reduced form as

$$\gcd(Q\times Q)' = \gcd(k, n_I, n_{I+1}n_{I+2})\ , \quad \gcd(\tfrac{1}{4}Q\wedge I_4'(Q)) = \gcd(k^2, kn_I, n_{I+1}n_{I+2})\ . \tag{5.66}$$

Choosing Q in a diagonal reduced form for which n_1 divides n_2 divides n_3, we obtain

$$\gcd(Q\times Q)' = \gcd(k,n_1) = 1\ , \quad \gcd(\tfrac{1}{4}Q\wedge I_4'(Q)) \underset{\gcd(k,n_1)}{=} \gcd(k,n_2)\ , \tag{5.67}$$

consistently with (5.64).

The instanton measure differs slightly from the helicity supertrace counting 1/8 BPS black holes [30–32, 223, 222]

$$\Omega_{14}(Q) \underset{\substack{\gcd(Q)=1\\ \gcd(Q\times Q)'=1}}{=} (-1)^{I_4(Q)} \sum_{d|\frac{1}{4}Q\wedge I_4'(Q)} d\ \tilde{c}\Big(\frac{I_4(Q)}{d^2}\Big)\ . \tag{5.68}$$

One finds a similar result in $\mathcal{N}=4$ theories defined by orbifolds of type II on K3$\times T^2$ [20]. It would be interesting to compute the instanton measure in three dimensions in the $P_8\subset E_8$ parabolic, where one expects the instanton measure to match precisely the helicity supertrace $\Omega_{14}(\Gamma)$ counting 1/8 BPS black holes of charge $\Gamma\in\mathbb{Z}^{56}$.

[29] For the set of charges (5.60) the components of $Q\gamma_{ab}Q$ include $(k, n_I, n_{I+1}n_{J+1}, 3k/2)$ while $(Q\times Q)'$ only includes $(k, n_I, n_{I+1}n_{J+1})$, which greatest common divisor is $Spin(6,6,\mathbb{Z})$ invariant [183].

6 Outlook

In this concluding section we will discuss some ongoing and future projects.

• D-brane instanton corrections in string theory:

One important problem is to derive the non-perturbative D-brane instanton corrections from first principle. This has been achieved for 1/2 BPS multi-instantons in ten dimensions [11] and for a single instanton anti-instanton in [12]. It would be interesting to generalise this computation to 1/4 BPS instantons in $D \leq 6$ dimensions and even 1/8 BPS instantons in $D = 4$ dimensions. We are currently working on the calculation of the partition function in the corresponding quiver gauge theory on a torus. In a first stage we are revisiting the computation of [197] using the more precise contour prescription of Jeffrey–Kirwan [224].

On the other hand the coupling function Fourier coefficients provide a prediction for the instanton measure of 1/8-BPS D-branes instantons in string theory. It would be interesting to repeat the same computation to see if the measure of black hole instantons in three dimensions reproduces the helicity supertrace counting 1/8 BPS black holes. A similar analyses can also be applied to less supersymmetric string theories, we had analysed the case of black holes in $\mathcal{N} = 4$ supergravity from heterotic CHL orbifolds in [20] and we are currently working on a similar study in $\mathcal{N} = 6$ supergravity.

It would be interesting to understand D-brane instanton corrections for the higher derivative couplings, not protected by supersymmetry. Because $\mathcal{E}_{(2,0)}$ does a priori receives corrections to all orders in string perturbation theory, it does not make sense to try to make the perturbative function U-duality invariant starting from tree, one-loop and two-loop corrections. We have checked in [64] in particular that the two-loop exceptional field theory amplitude is inconsistent with string perturbation theory. However, the method introduced in Section 5.4 can in principle be used to compute the D-brane instantons dual to the two-loop wordsheet instantons. To do so, we would need first to derive the Siegel parabolic Fourier expansion of the function $\mathcal{B}_{(2,0)}$ studied in [174].

• Effective theory of BPS particles:

The effective theory of 1/2 BPS states that we proposed in [64] provides an efficient tool to compute BPS protected couplings in string theory with maximal supersymmetry. It is interesting to investigate if this same tool can be applied with less supersymmetry. We have already analysed in some details the case of $\mathcal{N} = 6$ supergravity in four dimensions, for which the 1/2 BPS supermultiplets are either spin 2 or spin 3/2, and can therefore be analysed in supergravity. We found already some compelling evidence that this procedure works, generalising the preliminary results obtained in [28].

For $\mathcal{N} \leq 4$ there are 1/2 BPS supermultiplets of arbitrary high spin in string theory. Nonetheless, it may be that one only needs to consider supermultiplets up to spin 2 in the Horava–Witten formulation of the theory in eleven dimensions [225, 226]. In the future we would like to come with a formulation of the theory involving all the massive spin 1 supermultiplets with their non-abelian interactions that are standard non-abelian interactions at the enhanced gauge symmetry points. This effective theory should be formalised in the framework of double field theory [227].

• Uniqueness of string theory:

One related open question is about the possible existence of consistent quantum theories of gravity that could not be realised as string theories. There are compelling evidences that all consistent supergravity theories with extended supersymmetry are string theories. However, there is no proof of this conjecture, even for maximal supergravity.

This is the subject of the PhD thesis of my student Adrien Loty, with who we have analysed the minima of the coupling function $\mathcal{E}_{(0,0)}$ [213] and consistency with unitarity of the S-matrix [228,229]. The coupling function $\mathcal{E}_{(0,0)}$ can be negative in dimensions $D \leq 7$, and its minimum is at a symmetric point, which represents the Gram matrix of the densest lattice sphere packing for $SL(5)$ in $D = 7$ dimensions.

Considering the perturbative limit in extended supergravity where some of the radii are large, one generally gets logarithmic singularities in the amplitude that can be interpreted as Kaluza–Klein particles becoming massless in this limit. In string theory we must always get the entire Kaluza–Klein tower and their interactions can be inferred from supergravity. We are investigating how unique these interactions are and if one can prove directly in field theory that recovering the Kaluza–Klein tower is the only consistent resolution of the singularity.

If we make the hypothesis that the effective theory admits more than sixteen supersymmetries and a U-duality group congruent subgroup $\Gamma \subset G(\mathbb{Z})$ of the maximal possible symmetry group, one can derive the possible corrections to the low energy effective action that satisfy all the consistency conditions with the perturbative and decompactification limits. One may hope to come up with a finite list of theories compatible with these conditions and it would be interesting to check if they all admit a realisation in string theory.

• Generalised automorphic representations:

The notion of automorphic representations defined in mathematics does not apply directly to the coupling function defined in Section 5. We are currently working on a mathematical definition of representations that are not $\mathcal{Z}(\mathfrak{g})$ finite and including this set of functions. We are particularly interested in functions for which there exist specific Fourier coefficients that define an arithmetic number as the instanton measure in (5.45). We have already evidence that there are natural candidates when the source term generalising $(\mathcal{E}_{(0,0)})^2$ is associated to rigid nilpotent orbits. It would be interesting to analyse the kind of combinatoric objects these Fourier coefficients count and if they have a meaning in string theory for $E_7(\mathbb{Z})$ or other U-duality groups.

• Other topics:

We have other ongoing and future projects in supergravity applications that are not obviously related to the subject of this habilitation thesis. They include the study of Kac–Moody exceptional field theories and their applications to the quantum mechanics formulations of M-theory through the D0 matrix model [230] and the cosmological billiards [231].

The analysis of supersymmetry invariants in presence of a cosmological constant is almost inexistent beyond $\mathcal{N} = 2$, despite its relevance to the study of string theory on AdS backgrounds. The presence of gauge couplings gives a relevant deformation of the supersymmetry invariants that potentially involves lower derivative couplings in the low energy effective action.

A Eleven-dimensional supergravity with antifields

We write the Cremmer–Julia–Scherk supersymmetry transformations [36]

$$\begin{aligned}
\delta^{\rm susy}(\epsilon)e^a &= \big(\bar\epsilon\gamma^a\psi\big)\\
\delta^{\rm susy}(\epsilon)A &= \tfrac{1}{4}e^a_{\wedge}e^b\,\big(\bar\epsilon\gamma_{ab}\psi\big)\\
\delta^{\rm susy}(\epsilon)\psi &= d_{\hat\omega}\epsilon - \frac{1}{6}e^a\Big(\frac{1}{24}\gamma_a{}^{bcde} - \frac{1}{3}\delta_a^{[b}\gamma^{cde]}\Big)\hat F_{bcde}\,\epsilon
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
\delta^{\rm susy}(\epsilon)\hat\omega_{ab} &= e^c\big(\bar\epsilon[\gamma_{[a}\hat\rho_{b]c} - \tfrac{1}{2}\gamma_c\hat\rho_{ab}]\big) + \frac{1}{3}\Big(\bar\epsilon\Big[\frac{1}{24}\gamma_{abcdef}\hat F^{cdef} + \hat F_{abcd}\gamma^{cd}\Big]\psi\Big)\\
\delta^{\rm susy}(\epsilon)\hat F_{abcd} &= -3\big(\bar\epsilon\gamma_{[ab}\hat\rho_{cd]}\big)
\end{aligned}\tag{A.2}$$

where

$$\hat\rho = d_{\hat\omega}\psi - e^a_{\wedge}\mathbb{T}_a\psi\,,\qquad \hat F = F - \frac{1}{8}e^a_{\wedge}e^b\bar\psi\gamma_{ab}\psi\,,\qquad d_{\hat\omega}e^a + \frac{1}{2}\big(\overline{\psi}\gamma^a\psi\big) = 0\,,\tag{A.3}$$

are the supercovariant field strengths and the supertorsion tensor $\mathbb{T}_a$ is

$$\mathbb{T}_a \equiv \frac{1}{6}\Big(\frac{1}{24}\gamma_a{}^{bcde} - \frac{1}{3}\delta_a^{[b}\gamma^{cde]}\Big)\hat F_{bcde}\,.\tag{A.4}$$

In this convention the Lagrangian is [36]

$$\begin{aligned}
\mathcal{L} = e\Big(\frac{1}{4}R(\omega) - \frac{1}{48}\hat F_{\mu\nu\rho\sigma}\hat F^{\mu\nu\rho\sigma} - \frac{1}{2}\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu(\tfrac{\omega+\hat\omega}{2})\psi_\rho + \frac{1}{192}\bar\psi_\mu\gamma^{\mu\nu\rho\sigma\kappa\lambda}\psi_\nu\big(F_{\rho\sigma\kappa\lambda} + \hat F_{\rho\sigma\kappa\lambda}\big)\Big)\\
- \frac{1}{10368}\varepsilon^{\mu\nu\rho\sigma\kappa\lambda\vartheta\upsilon\epsilon\iota\tau}A_{\mu\nu\rho}F_{\sigma\kappa\lambda\vartheta}F_{\upsilon\epsilon\iota\tau}\,,
\end{aligned}\tag{A.5}$$

where one introduces the non-supercovariant spin-connection ω that solves the first order Euler–Lagrange equation

$$d_\omega e^a + \frac{1}{2}\big(\overline{\psi}\gamma^a\psi\big) + \frac{1}{4}e^b_{\wedge}e^c\big(\overline{\psi}_d\gamma^{ade}{}_{bc}\psi_e\big) = 0\,.\tag{A.6}$$

The supersymmetry algebra (with commuting spinor ϵ) closes on the elfbein e^a and the 3-form A as

$$\delta^{\rm susy}(\epsilon)^2 = \hat{\mathcal{L}}_{\frac{1}{2}(\bar\epsilon\gamma\epsilon)} - \delta^{\rm Lorentz}(\bar\epsilon\mathbb{R}\epsilon) - \delta^{\rm gauge}(\tfrac{1}{4}\bar\epsilon\gamma_2\epsilon)\tag{A.7}$$

with

$$\mathbb{R}_{ab} \equiv \frac{1}{144}\gamma_{abcdef}\hat F^{cdef} + \frac{1}{6}\hat F_{abcd}\gamma^{cd}\,,\qquad \big(\bar\epsilon\gamma_2\epsilon\big) = \frac{1}{2}e^a_{\wedge}e^b\big(\bar\epsilon\gamma_{ab}\epsilon\big)\,.\tag{A.8}$$

However, for the gravitino field, this relation is only satisfied modulo its own equation of motion as

$$\delta^{\rm susy}(\epsilon)^2\psi = \hat{\mathcal{L}}_{\frac{1}{2}(\bar\epsilon\gamma\epsilon)}\psi - \delta^{\rm Lorentz}(\bar\epsilon\mathbb{R}\epsilon)\psi + e^a\mathbb{K}_{ab}\gamma^{bcd}\hat\rho_{cd}\tag{A.9}$$

where

$$\mathbb{K}_{ab} \equiv \frac{1}{12}\Big(\gamma^c[\epsilon\bar\epsilon]\gamma_c + \frac{1}{2}\gamma^{cd}[\epsilon\bar\epsilon]\gamma_{cd}\Big)\eta_{ab} - \frac{1}{36}\gamma_a[\epsilon\bar\epsilon]\gamma_b + \frac{1}{12}\gamma_b[\epsilon\bar\epsilon]\gamma_a + \frac{1}{24}\gamma_b{}^c[\epsilon\bar\epsilon]\gamma_{ac} - \frac{1}{12}\big(\bar\epsilon\gamma_{ab}\epsilon\big)\,.\tag{A.10}$$

We therefore define the BRST transformation of the gravitino field as

$$s\,\psi = \mathcal{L}_\xi\psi - \tfrac{1}{4}\Omega_{ab}\gamma^{ab}\psi - d_{\hat\omega}\epsilon + e^a\mathbb{T}^*_a(\epsilon) \tag{A.11}$$

with the anti-field dependent supertorsion term

$$\begin{aligned}\mathbb{T}^*_a(\epsilon) \equiv \frac{1}{6}\Big(\frac{1}{24}\gamma_a{}^{bcde} - \frac{1}{3}\delta_a^{[b}\gamma^{cde]}\Big)\hat F_{bcde}\epsilon + \frac{1}{6}\Big(\gamma^c[\epsilon\overline{\epsilon}]\gamma_c + \frac{1}{2}\gamma^{cd}[\epsilon\overline{\epsilon}]\gamma_{cd}\Big)\Psi^*_a \\ - \frac{1}{18}\gamma_a[\epsilon\overline{\epsilon}]\gamma^b\Psi^*_b + \frac{1}{6}\gamma^b[\epsilon\overline{\epsilon}]\gamma_a\Psi^*_b + \frac{1}{12}\gamma^{bc}[\epsilon\overline{\epsilon}]\gamma_{ac}\Psi^*_b - \frac{1}{6}\big(\overline{\epsilon}\gamma_{ab}\epsilon\big)\Psi^{*\,b}\end{aligned} \tag{A.12}$$

where the antifield combination Ψ^*_a is defined as

$$\Psi^*_a \equiv \psi^*_a - \Omega^*_{ab}\gamma^b\epsilon\,. \tag{A.13}$$

Here Ω_{ab} is the Lorentz ghost and Ω^*_{ab} its antifield. One checks that this does not affect the nilpotency of the BRST operator on the bosonic fields using

$$\overline{\epsilon}\gamma_{[ab}\mathbb{K}_{c]d} = 0\,, \qquad \overline{\epsilon}\gamma_{\{a}\mathbb{K}_{b\}c} = 0\,. \tag{A.14}$$

The closure of the supersymmetry transformation then involves the antifield dependent Lorentz transformation with

$$\begin{aligned}\mathbb{R}^*_{ab}(\epsilon) \equiv \big(\overline{\epsilon}\,\gamma_a\mathbb{T}^*_b(\epsilon)\big) = \frac{1}{6}\big(\overline{\epsilon}\big[\frac{1}{24}\gamma_{abcdef}\hat F^{cdef} + \hat F_{abcd}\gamma^{cd}\big]\epsilon\big) \\ + \frac{1}{4}\big(\overline{\epsilon}\gamma^c\epsilon\big)\big(\overline{\epsilon}\gamma_{[ab}\Psi^*_{c]}\big) + \frac{1}{3}\big(\overline{\epsilon}\gamma_{[a}{}^c\epsilon\big)\big(\overline{\epsilon}\gamma_{b]}\Psi^*_c\big) - \frac{1}{18}\big(\overline{\epsilon}\gamma_{ab}\epsilon\big)\big(\overline{\epsilon}\gamma^c\Psi^*_c\big)\,.\end{aligned} \tag{A.15}$$

This way one determines the solution of the master equation [87]

$$\int\sum_{\lrcorner}\frac{\delta^R\Sigma}{\delta\varphi^{\lrcorner}}\frac{\delta^L\Sigma}{\delta\varphi^*_{\lrcorner}} = 0 \tag{A.16}$$

as

$$\begin{aligned}\Sigma = \int\Big(-\frac{1}{4{\cdot}9!}\varepsilon_{ab[9]}e^{[9]}_\wedge R^{ab}(\omega) + \frac{1}{2}\hat F_\wedge\star\hat F + \frac{1}{12{\cdot}8!}\varepsilon_{abc[8]}e^{[8]}_\wedge\big(\overline{\psi}\gamma^{abc}d_{\frac{1}{2}(\omega+\hat\omega)}\psi\big) \\ - \frac{1}{8}e^{[5]}_\wedge\big(\overline{\psi}\gamma_{[5]}\psi\big)\big(F+\hat F\big) - \frac{1}{3}A_\wedge F_\wedge F \\ + e^*_{a\,\wedge}\Big(\mathcal{L}_\xi e^a - \Omega^a{}_b e^b - \big(\overline{\epsilon}\gamma^a\psi\big)\Big) + \overline{\psi}^*_\wedge\Big(\mathcal{L}_\xi - \frac{1}{4}\not{\Omega}\psi - d_{\hat\omega}\epsilon + e^a\mathbb{T}_a\epsilon\Big) \\ + A^*_{(8,-1)\,\wedge}\Big(\mathcal{L}_\xi A_{(3,0)} - \tfrac{1}{4}\big(\overline{\epsilon}\,e^a_\wedge e^b_\wedge\gamma_{ab}\psi\big) - dA_{(2,1)}\Big) \\ + A^*_{(9,-2)\,\wedge}\Big(\mathcal{L}_\xi A_{(2,1)} - \tfrac{1}{8}\big(\overline{\epsilon}\,e^a_\wedge e^b\,\gamma_{ab}\epsilon\big) - dA_{(1,2)} + \iota_{\frac{1}{2}(\overline{\epsilon}\gamma\epsilon)}A_{(3,0)}\Big) \\ + A^*_{(10,-3)\,\wedge}\Big(\mathcal{L}_\xi A_{(1,2)} - dA_{(0,3)} + \iota_{\frac{1}{2}(\overline{\epsilon}\gamma\epsilon)}A_{(2,1)}\Big) + A^*_{(11,-4)\,\wedge}\Big(\mathcal{L}_\xi A_{(0,3)} + \iota_{\frac{1}{2}(\overline{\epsilon}\gamma\epsilon)}A_{(1,2)}\Big) \\ + \xi^*\cdot\Big(\tfrac{1}{2}\{\xi,\xi\} - \tfrac{1}{2}\big(\overline{\epsilon}\gamma\epsilon\big)\Big) + \Omega^{*\,ab}\Big(\mathcal{L}_\xi\Omega_{ab} + \Omega_a{}^c\Omega_{bc} - \big(\overline{\epsilon}\,\mathbb{R}_{ab}\epsilon\big) + \iota_{\frac{1}{2}(\overline{\epsilon}\gamma\epsilon)}\hat\omega\Big) \\ + \overline{\epsilon}^*\Big(\mathcal{L}_\xi\epsilon - \tfrac{1}{4}\not{\Omega}\epsilon + \iota_{\frac{1}{2}(\overline{\epsilon}\gamma\epsilon)}\psi\Big) - [\overline{\psi}^*_a + \overline{\epsilon}\gamma^c\Omega^*_{ac}]\mathbb{K}^{ab}[\psi^*_b - \Omega^*_{bd}\gamma^d\epsilon]\Big)\end{aligned} \tag{A.17}$$

where we used the form notation and $[n]$ is a short for

$$\varepsilon_{ab[9]}e^{[9]} = \varepsilon_{abcdefghijk}e^c_\wedge e^d_\wedge e^e_\wedge e^f_\wedge e^g_\wedge e^h_\wedge e^i_\wedge e^j_\wedge e^k\,. \tag{A.18}$$

B Closed string sphere integrals

In this appendix we discuss some details about the five-point sphere integrals. To check that $W_5(s_i)$ in (2.63) is indeed analytic in its arguments, one decomposes the integral with respect to the six possible orderings of $|z|$, $|w|$ and 1. The arguments s_i are chosen such that the boundary contributions where z and w are either infinity or zero vanish. The integral in the first line gives six permutations of the integral

$$
\begin{aligned}
&\frac{1}{\pi^2}\int_0^{2\pi}d\theta\int_0^{2\pi}d\varphi\int_0^{\infty}d\alpha\int_0^{\infty}d\beta\; e^{2\alpha t+2\beta v}|1-e^{-\alpha+i\theta}|^{-2s}|1-e^{-\beta+i\varphi}|^{-2u}|1-e^{-\alpha-\beta+i\theta+i\varphi}|^{-2w}\\
=\;&\frac{1}{tv}-\frac{s^2}{v}\sum_{n\geq1}t^{n-1}\Big(\zeta(2{+}n)+\sum_{m\geq3}s^{m-2}\tfrac{2^n}{m!}S(m,n)\Big)-\frac{u^2}{t}\sum_{n\geq1}v^{n-1}\Big(\zeta(2{+}n)+\sum_{m\geq3}u^{m-2}\tfrac{2^n}{m!}S(m,n)\Big)\\
&+s^2u^2\bigg(\sum_{n\geq1}t^{n-1}\Big(\zeta(2{+}n)+\sum_{m\geq3}s^{m-2}\tfrac{2^n}{m!}S(m,n)\Big)\bigg)\bigg(\sum_{n\geq1}v^{n-1}\Big(\zeta(2{+}n)+\sum_{m\geq3}u^{m-2}\tfrac{2^n}{m!}S(m,n)\Big)\bigg)\\
&+w^2\sum_{m,n\geq1}t^{m-1}v^{n-1}\Big(\zeta(2{+}m{+}n)+\sum_{k\geq3}w^{k-2}\tfrac{2^{m+n}}{(m+n)!}S(k,m{+}n)\Big)+2suw\sum_{m,n\geq1}t^{m-1}v^{n-1}\zeta(3{+}m{+}n)\\
&+suw\sum_{m,n\geq1}(ut^{m-1}v^{n-1}+st^{n-1}v^{m-1})\big(2S(2{+}m,1;1{+}n)+S(1,1;2{+}m{+}n)\big)\\
&+suw^2\sum_{m,n\geq1}t^{m-1}v^{n-1}\big(2S(2,1;1{+}m{+}n)+S(1,1;2{+}m{+}n)\big)\\
&+s^2w^2\sum_{m,n\geq1}t^{m-1}v^{n-1}S(2{+}n,2;m)+u^2w^2\sum_{m,n\geq1}t^{n-1}v^{m-1}S(2{+}n,2;m)+\dots
\end{aligned}
\tag{B.1}
$$

where one uses

$$
-\log|1-e^{-\alpha+i\theta}|^2=\sum_{n\neq0}\frac{1}{|n|}e^{-|n|\alpha+in\theta}\,,
\tag{B.2}
$$

and the infinite sums $S(m,n;k)$ and $S(m,n)$ are defined as

$$
S(m,n)=\sum_{k_i|_{i=1}^m\neq0}\frac{\delta_{\sum_{i=1}^m k_i}}{\prod_{i=1}^m|k_i|(\sum_{i=1}^m|k_i|)^n}\,,\qquad S(m,n;k)=\sum_{i,j\geq1}\frac{1}{i^m j^n(i+j)^k}\,.
\tag{B.3}
$$

They can be computed using [79, Appendix A.3] as

$$
\begin{aligned}
&\frac{2}{6}S(3,1)=\frac{\zeta(4)}{2}\,,\quad \frac{2^2}{6}S(3,2)=4\zeta(5)-2\zeta(2)\zeta(3)\,,\quad \frac{2^3}{6}S(3,3)=\frac{3}{2}\zeta(6)-\zeta(3)^2\,,\\
&\frac{2}{24}S(4,1)=\frac{5}{2}\zeta(5)-\zeta(2)\zeta(3)\,,\quad \frac{2^2}{24}S(4,2)=\frac{53}{12}\zeta(6)-3\zeta(3)^2\,,\\
&\frac{2}{5!}S(5,1)=\frac{47}{24}\zeta(6)-\zeta(3)^2\,,\quad S(3,2;1)=\frac{1}{2}\zeta(3)^2\,,\\
&S(1,1;4)+2S(2,1;3)=\frac{11}{6}\zeta(6)-\zeta(3)^2\,,\quad 2S(3,1;2)+S(1,1;4)=\frac{5}{6}\zeta(6)\,.
\end{aligned}
\tag{B.4}
$$

This gives up to quartic order

$$
\begin{aligned}
&W_5(s_1,s_2,s_3,s_4,s_5)=\big(15s_2s_4s_5-(s_2s_4{+}s_4s_5{+}s_5s_2)(s_2{+}s_4{+}s_5)\big)\zeta(5)\\
&+\Big(\tfrac{1}{2}(s_2^2{+}s_4^2{+}s_5^2)^2+\tfrac{3}{2}(s_2^2{+}s_4^2{+}s_5^2)(s_2{+}s_4{+}s_5)^2-\big(5s_2s_4s_5+2(s_2^3{+}s_4^3{+}s_5^3)\big)(s_2{+}s_4{+}s_5)\Big)\zeta(3)^2\\
&\qquad\qquad+\mathcal{O}(s_i^5)
\end{aligned}
\tag{B.5}
$$

To compute more systematically the α' expansion of closed string sphere integrals, it is convenient to use the single-valued map [172]. It was conjectured in [135, 136] and proved in [138] that one can obtain the closed string amplitude sphere integrals from the open string amplitude disk integrals using the single valued map [172] that satisfies

$$\mathrm{sv}\zeta(2n) = 0\,, \qquad \mathrm{sv}\zeta(2n+1) = 2\zeta(2n+1)\,. \tag{B.6}$$

The single-valued map efficiently packages the KLT relations [232] in a way that avoids to compute the KLT matrices. Following [138], one introduces the sphere integrals with Park–Taylor denominator associated to two permutations τ and ρ

$$J(\tau|\rho) = \frac{1}{\pi^2}\int_{\mathbb{C}^5}\frac{d^2z_1d^2z_2d^2z_3d^2z_4d^2z_5}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})}\frac{\prod_{1\leq i<j\leq 5}|z_i-z_j|^{-2s_{ij}}}{\prod_{i=1}^4(z_{\rho(i+1)}-z_{\rho(i)})\prod_{i=1}^4(\bar{z}_{\tau(i+1)}-\bar{z}_{\tau(i)})}\,, \tag{B.7}$$

that can be obtained from the single-valued map applied to a disk integral

$$J(\tau|\rho) = \mathrm{sv}\int_{z_{\tau(i)}<z_{\tau(i+1)}}\frac{dz_1dz_2dz_3dz_4dz_5}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{R})}\frac{\prod_{1\leq i<j\leq 5}|z_i-z_j|^{-s_{ij}}}{\prod_{i=1}^4(z_{\rho(i+1)}-z_{\rho(i)})}\,. \tag{B.8}$$

This generalises to arbitrary numbers of points, but we shall only use it for five-point integrals.

Using the identity $\frac{1}{zw} = \frac{1}{z(w-z)}+\frac{1}{w(z-w)}$, and a change of variables to get the right-moving Park–Taylor factor in the trivial permutation one computes that

$$\begin{aligned}
&= \frac{1}{\pi^2}\int_{\mathbb{C}}d^2z\int_{\mathbb{C}}d^2w\,\frac{1}{|z|^2|w|^2}|z|^{-2s_{12}}|1-z|^{-2s_{24}}|w|^{-2s_{23}}|1-w|^{-2s_{34}}|w-z|^{-2s_{23}}\\
&= J(\sigma_{45}|\sigma_{45}) + J(\sigma_{45}\sigma_{23}|\sigma_{45}) + J(\sigma_{45}|\sigma_{45}\sigma_{23}) + J(\sigma_{45}\sigma_{23}|\sigma_{45}\sigma_{23})\\
&= \mathrm{sv}\int_{0<x<y<1}dxdy\,\Bigg[\frac{1}{x(1-x)y(1-y)}|x|^{-s_{12}}|1-x|^{-s_{25}}|y|^{-s_{13}}|1-y|^{-s_{35}}|y-z|^{-s_{23}}\\
&\qquad\qquad + \frac{1}{x(1-x)y(1-y)}|x|^{-s_{13}}|1-x|^{-s_{35}}|y|^{-s_{12}}|1-y|^{-s_{25}}|y-z|^{-s_{23}}\Bigg]\,.
\end{aligned} \tag{B.9}$$

One can use this form to extract the analytic function $W_5(s_i)$ in (2.63) by subtracting the poles as described in [134]. The basis proposed in [134] does not give directly (2.63), but the α' expansion of the basis functions given in [137] allows to obtain the expansion (2.64) of $W_5(s_i)$ straightforwardly using [135] and the datas provided by Broedel–Schlotterer–Stieberger. We will not display the expressions that are not very illuminating, but we observe that the first irreducible multiple zeta-values $\zeta(3,3,5)$, $\zeta(3,5,5)$ and $\zeta(3,3,7)$ appear with a polynomial in the Mandelstam variables proportional to $s_2s_4s_5(s_2+s_4+s_5)$ and therefore drop out in (2.66).

One uses the same method to compute the integral (2.123) as

$$
\begin{aligned}
&= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2z \int_{\mathbb{C}} d^2w \left|\frac{s}{zw}+\frac{t}{(z-1)w}+\frac{u}{z(w-1)}\right|^2 |z|^{-2s_{12}}|1-z|^{-2s_{24}}|w|^{-2s_{23}}|1-w|^{-2s_{34}}|w-z|^{-2s_{23}} \\
&= s^2 J(\sigma_{45}|\sigma_{45}) + s^2 J(\sigma_{45}\sigma_{23}|\sigma_{45}) + s^2 J(\sigma_{45}|\sigma_{45}\sigma_{23}) + s^2 J(\sigma_{45}\sigma_{23}|\sigma_{45}\sigma_{23}) \\
&\quad -stJ(\sigma_{45}|\sigma_{45}\sigma_{23}\sigma_{34}) - suJ(\sigma_{45}|\sigma_{45}\sigma_{34}) - stJ(\sigma_{45}\sigma_{23}|\sigma_{45}\sigma_{23}\sigma_{34}) - suJ(\sigma_{45}\sigma_{23}|\sigma_{45}\sigma_{34}) \\
&\quad -tsJ(\sigma_{45}\sigma_{23}\sigma_{34}|\sigma_{45}) - usJ(\sigma_{45}\sigma_{34}|\sigma_{45}) - tsJ(\sigma_{45}\sigma_{23}\sigma_{34}|\sigma_{45}\sigma_{23}) - usJ(\sigma_{45}\sigma_{34}|\sigma_{45}\sigma_{23}) \\
&+t^2 J(\sigma_{45}\sigma_{23}\sigma_{34}|\sigma_{45}\sigma_{23}\sigma_{34}) + utJ(\sigma_{45}\sigma_{34}|\sigma_{45}\sigma_{23}\sigma_{34}) + tuJ(\sigma_{45}\sigma_{23}\sigma_{34}|\sigma_{45}\sigma_{34}) + u^2 J(\sigma_{45}\sigma_{34}|\sigma_{45}\sigma_{34}) \\
&= \mathrm{sv}\int_{0<x<y<1} dxdy \left[s\Big(\frac{s}{xy}+\frac{t}{x(y-1)}+\frac{u}{(x-1)y}\Big)|x|^{-s_{12}}|1-x|^{-s_{25}}|y|^{-s_{13}}|1-y|^{-s_{35}}|y-z|^{-s_{23}} \right. \\
&\qquad +s\Big(\frac{s}{xy}+\frac{t}{(x-1)y}+\frac{u}{x(y-1)}\Big)|x|^{-s_{13}}|1-x|^{-s_{35}}|y|^{-s_{12}}|1-y|^{-s_{25}}|y-z|^{-s_{23}} \\
&\qquad +t\Big(\frac{t}{x(y-x)}+\frac{u}{x(y-1)}\Big)|x|^{-s_{13}}|1-x|^{-s_{23}}|y|^{-s_{15}}|1-y|^{-s_{25}}|y-z|^{-s_{35}} \\
&\qquad \left. +u\Big(\frac{t}{x(y-x)}+\frac{u}{x(y-1)}\Big)|x|^{-s_{12}}|1-x|^{-s_{23}}|y|^{-s_{15}}|1-y|^{-s_{35}}|y-z|^{-s_{25}} \right] . \qquad \text{(B.10)}
\end{aligned}
$$

One can use this form to extract the analytic function $F_5(s_i)$ in (2.123). The α' expansion of the basis functions given in [137] allows to obtain the expansion of $F_5(s_i)$ as

$$
\begin{aligned}
&F_5(s_1,s_2,s_3,s_4,s_5) \\
&= 2\zeta(3)s_5(2s^2-3t^2-3u^2) \\
&\quad +\zeta(5)s_5\big(s^2\big(6(s_1+s_3)^2+2s_1(s_1+s_4)+2s_3(s_2+s_3)+4s_2s_4+7(s_1+s_2+s_3+s_4+s_5)(s_2+s_4+s_5)+s_5(7s_1+2s_2+7s_3+2s_4+6s_5)\big) \\
&\qquad -t^2\big(5\big((s_1+s_3)(2s_1+3s_2+2s_3+s_4)+(s_1+s_2)^2+2(s_2+s_4)^2+s_1^2+s_4^2\big)+s_5(29s_1+28s_2+21s_3+22s_4+19s_5)\big)\big) \\
&\qquad -u^2\big(5\big((s_1+s_3)(2s_1+s_2+2s_3+3s_4)+(s_3+s_4)^2+2(s_2+s_4)^2+s_3^2+s_2^2\big)+s_5(21s_1+22s_2+29s_3+28s_4+19s_5)\big)\big) \\
&\quad +(s^2,t^2,u^2)\mathcal{O}(s_i^4) . \qquad \text{(B.11)}
\end{aligned}
$$

The first irreducible multiple zeta-values $\zeta(3,3,5)$, $\zeta(3,5,5)$ and $\zeta(3,3,7)$ appear with a polynomial in the Mandelstam variables proportional to $s_5(s_2+s_4+s_5)(s_1+s_2+s_3+s_4+s_5)$ and therefore vanish in the two-loop four-point amplitude in which F_5 appears evaluated at $s_1+s_2+s_3+s_4+s_5=0$.

C E_7 parabolic multiplicative characters

In this section we derive some of the differential equations satisfied by the Eisenstein series by using their formula as Poincaré sums of a multiplicative character (3.12). In the domain of absolute convergence for Re[s] sufficiently large this implies the same differential equation for the Eisenstein series. When the equation is satisfied as an analytic equation in s it extends by analytic continuation to all values of s except at the poles.

C.1 Abelian parabolic character

To compute the differential operators acting on a abelian parabolic character of E_7 we use the property that it is obtained from the E_7 fundamental representation on a vector Γ in the minimal

orbit. One can choose such a Γ, and define $Z(\Gamma)_{ij} = \mathcal{V}_{ij}{}^{I}\Gamma_I$ and its complex conjugate from the action of $g \in E_7$ on the vector Γ. For simplicity we shall write Z_{ij} and its complex conjugate Z^{ij}. The minimal orbit condition for Γ implies the equation

$$Z_{[ij}Z_{kl]} = \frac{1}{24}\varepsilon_{ijklpqrs}Z^{pq}Z^{rs} \ , \qquad Z_{ik}Z^{jk} = \frac{1}{8}\delta_i^j Z_{kl}Z^{kl} \ , \tag{C.1}$$

and the differential operator acts on Z_{ij} as an element of $\mathfrak{e}_{7(7)}$

$$\mathcal{D}_{ijkl}Z^{pq} = 3\delta^{pq}_{[ij}Z_{kl]} \ , \qquad \mathcal{D}_{ijkl}Z_{pq} = \frac{1}{8}\varepsilon_{ijklpqrs}Z^{rs} \ . \tag{C.2}$$

Using the definition $|Z|^2 = Z_{ij}Z^{ij}$, the spherical abelian parabolic character can be defined as the function $y_7^s = |Z|^{-2s}$ in (3.12). One computes that it satisfies

$$\begin{aligned}
\mathcal{D}_{ijpq}\mathcal{D}^{klpq}|Z|^{-2s} &= 2s(s-2)Z_{ij}Z^{kl}|Z|^{-2s-2} + \frac{s(s-11)}{4}\delta_{ij}^{kl}|Z|^{-2s} \ , \\
\mathcal{D}_{ijpq}\mathcal{D}^{pqrs}\mathcal{D}_{rskl}|Z|^{-2s} &= -3s(s-2)(s-4)Z_{ij}Z_{kl}|Z|^{-2s-2} + \frac{s^2-15s+8}{4}\mathcal{D}_{ijkl}|Z|^{-2s} \ , \\
\mathcal{D}_{jr[kl}\mathcal{D}^{irmn}\mathcal{D}_{pq]mn}|Z|^{-2s} &= \frac{(s-2)(s-7)}{12}\delta^i_j\mathcal{D}_{klpq}|Z|^{-2s} - \frac{s^2-9s-40}{12}\delta^i_{[k}\mathcal{D}_{pql]j}|Z|^{-2s} \ ,
\end{aligned} \tag{C.3}$$

and the Laplace equation

$$\Delta|Z|^{-2s} = 3s(s-9)|Z|^{-2s} \ . \tag{C.4}$$

To exhibit the module structure (3.83), it is convenient to consider a restricted set of indices as follows

$$\begin{aligned}
&\left(\mathcal{D}_{12ij}\mathcal{D}^{ijkl}\mathcal{D}_{kl12}\right)^{n_3}\left(\mathcal{D}_{12pq}\mathcal{D}^{78pq}\right)^{n_2}\left(\mathcal{D}_{1234}\right)^{n_1}|Z|^{-2s} \qquad (C.5)\\
&= \tfrac{(s+n_1+n_2+n_3-1)!(s+n_2+n_3-3)!(s+n_3-5)!}{(s-1)!(s-3)!(s-5)!}\left(-3Z_{12}^2\right)^{n_3}\left(2Z_{12}Z^{78}\right)^{n_2}\left(-6Z_{[12}Z_{34]}\right)^{n_1}|Z|^{-2(s+n_1+n_2+n_3)} \ .
\end{aligned}$$

One computes moreover that for $m \leq n$

$$\begin{aligned}
&\left(\mathcal{D}^{78ij}\mathcal{D}_{ijkl}\mathcal{D}^{kl78}\right)^{m}\left(\mathcal{D}_{12pq}\mathcal{D}^{pqrs}\mathcal{D}_{rs12}\right)^{n}|Z|^{-2s} \qquad (C.6)\\
&= \tfrac{(s+n-1)!(s+n-3)!(s+n-5)!(s+n+m-1)!(s+n+m-3)!(s-n+m-5)!}{(s-1)!(s-3)!(s-5)!(s+n-1)!(s+n-3)!(s-n-5)!}\left(-3Z^{78\,2}\right)^{m}\left(-3Z_{12}^2\right)^{n}|Z|^{-2(s+n+m)} \\
&= \left(-\tfrac{3}{2}\right)^{n+m}\tfrac{(s+n-5)!(s+n+m-1)!(s+n+m-3)!(s-n+m-5)!}{(s+n-m-5)!(s+2n-1)!(s+2n-3)!(s-n-5)!}\left(\mathcal{D}_{12ij}\mathcal{D}^{ijkl}\mathcal{D}_{kl12}\right)^{n-m}\left(\mathcal{D}_{12pq}\mathcal{D}^{78pq}\right)^{n+m}|Z|^{-2s}
\end{aligned}$$

such that acting with a derivative operator in the conjugate representation $2m\Upsilon_6$ does not produce an independent tensor. One has in particular for $s = k+4$ an integer greater than 5

$$\left(\mathcal{D}^{78ij}\mathcal{D}_{ijkl}\mathcal{D}^{kl78}\right)\left(\mathcal{D}_{12pq}\mathcal{D}^{pqrs}\mathcal{D}_{rs12}\right)^{k}|Z|^{-2k-8} = 0 \ . \tag{C.7}$$

The restriction of the derivative $\mathcal{D}^{3n}|Z|^{-2s}$ to the $R(2n\Upsilon_2)$ with two free indices reads

$$\begin{aligned}
&[\mathcal{D}^{3n}_{2n\Upsilon_2}]_{ij1^{2n-1}2^{2n-1}}|Z|^{-2s} \\
&= \tfrac{(s+n-1)!(s+n-3)!(s+n-5)!}{(s-1)!(s-3)!(s-5)!}\frac{(-3)^n}{n+\frac{1}{2}}\left(Z_{ij}Z_{12}^{2n-1} - (2n-1)Z_{1[i}Z_{j]2}Z_{12}^{2n-2}\right)|Z|^{-2(s+n)} \ ,
\end{aligned} \tag{C.8}$$

and one computes that

$$\begin{aligned}
&\mathcal{D}^{78ij}\frac{1}{n+\frac{1}{2}}\Big(Z_{ij}Z_{12}^{2n-1}-(2n-1)Z_{1[i}Z_{j]2}Z_{12}^{2n-2}\Big)|Z|^{-2(s+n)}\\
&= \frac{(2n+5)(n-s+4)}{4n+2}Z^{78}Z_{12}^{2n-1}|Z|^{-2s}\ ,
\end{aligned}\tag{C.9}$$

such that

$$\begin{aligned}
&\mathcal{D}^{78ij}[\mathcal{D}_{2n}^{3n}\Upsilon_2]_{ij1^{2n-1}2^{2n-1}}|Z|^{-2s}\\
&= \frac{3(s+n-5)(2n+5)(s-n-4)}{8n+4}\big(\mathcal{D}_{12ij}\mathcal{D}^{ijkl}\mathcal{D}_{kl12}\big)^{n-1}\big(\mathcal{D}_{12pq}\mathcal{D}^{78pq}\big)|Z|^{-2s}\ .
\end{aligned}\tag{C.10}$$

In particular for $s=k+4$ integer we have that

$$\mathcal{D}^{78ij}[\mathcal{D}_{2k}^{3k}\Upsilon_2]_{ij1^{2k-1}2^{2k-1}}|Z|^{-2k-8}=0\ ,\tag{C.11}$$

consistently with the assumption that no lower order tensor is produced. This identity can be trusted for the Eisenstein series in the domain of absolute convergence $k\geq 6$. There are poles at $k=1,3,5$, but since the residue is in a representation in which the operator vanish one expects to have no correction for $k\geq 2$.

C.2 Heisenberg parabolic character

One can also consider the Heisenberg parabolic character using the adjoint action on an element Q in the minimal nilpotent orbit. The element $\mathcal{V}Q\mathcal{V}^{-1}$ decomposes into the anti-Hermitian traceless matrix $\Lambda^i{}_j$ and the complex-selfdual antisymmetric tensor X_{ijkl} satisfying to the constraints

$$\begin{aligned}
\Lambda^i{}_k\Lambda^k{}_j &= -\frac{1}{48}\delta^i_j X^{klpq}X_{klpq}\ ,\\
\Lambda^{[i}{}_{[k}\Lambda^{j]}{}_{l]} &= -\frac{1}{2}X^{ijpq}X_{klpq}+\frac{1}{48}\delta^{ij}_{kl}X^{pqrs}X_{pqrs}\ ,\\
\Lambda^{[i}{}_pX^{j]pkl} &= \Lambda^{[k}{}_pX^{l]pij}\ .
\end{aligned}\tag{C.12}$$

The action of the derivative on these tensors is defined as the $\mathfrak{e}_{7(7)}$ action

$$\mathcal{D}_{ijkl}X^{pqrs}=12\delta^{[pqr}_{[ijk}\Lambda^{s]}{}_{l]}\ ,\qquad \mathcal{D}_{ijkl}\Lambda^p{}_q=2\delta^p_{[i}X_{jkl]q}+\frac{1}{4}\delta^p_qX_{ijkl}\ .\tag{C.13}$$

One computes for $|X|^2=X_{ijkl}X^{ijkl}$ that

$$\begin{aligned}
\mathcal{D}_{ijkl}|X|^2 &= -24X_{p[ijk}\Lambda^p{}_{l]}\ , & \mathcal{D}_{ijpq}X^{klpq} &= 10\delta^{[k}_{[i}\Lambda^{j]}{}_{l]}\ ,\\
\mathcal{D}_{ijpq}\mathcal{D}^{klpq}|X|^2 &= 30X_{ijpq}X^{klpq}+3\delta^{kl}_{ij}|X|^2\ , & \mathcal{D}_{ijpq}|X|^2\mathcal{D}^{klpq}|X|^2 &= 12X_{ijpq}X^{klpq}|X|^2\ ,
\end{aligned}\tag{C.14}$$

which permits to derive that

$$\mathcal{D}_{ijpq}\mathcal{D}^{klpq}|X|^{-2s}=6s(2s-3)X_{ijpq}X^{klpq}|X|^{-2s-2}-3s\delta^{kl}_{ij}|X|^{-2s}\ ,\tag{C.15}$$

and

$$\Delta|X|^{-2s}=2s(2s-17)|X|^{-2s}\ .\tag{C.16}$$

One gets therefore a solution to the equation

$$\mathcal{D}_{ijpq}\mathcal{D}^{klpq}f^{\min} = -\frac{9}{2}\delta^{kl}_{ij}f^{\min}\ , \tag{C.17}$$

for $s=\frac{3}{2}$. One computes in general that

$$\mathcal{D}_{ijpq}\mathcal{D}^{pqrs}\mathcal{D}_{rskl}|X|^{-2s} = \big(s^2-\tfrac{17}{2}s+6\big)\mathcal{D}_{ijkl}|X|^{-2s}\ , \tag{C.18}$$

and the function satisfies to (3.69) and its complex conjugate for all s. The restriction of the third order derivative to the $\Upsilon_1+\Upsilon_4+\Upsilon_7$ gives

$$\mathcal{D}_{1k[12}\mathcal{D}_{34]ij}\mathcal{D}^{8ijk}|X|^{-2s} = -\frac{3}{4}s(2s-3)(2s-5)\Lambda^8{}_1X_{1234}|X|^{-2s-2}\ , \tag{C.19}$$

which gives for the irreducible representation

$$\begin{aligned}&[\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{j,klpq}{}^i|X|^{-2s} \qquad\qquad (C.20)\\ &= -\frac{3}{4}\frac{s(2s-3)(2s-5)}{|X|^{2s+2}}\Big(\Lambda^i{}_jX_{klpq}+\frac{4}{5}\Lambda^r{}_j\delta^i_{[k}X_{lpq]r}-\frac{4}{45}\delta^i_j\Lambda^r{}_{[k}X_{lpq]r}-\Lambda^i{}_{[j}X_{klpq]}-\frac{8}{9}\Lambda^r{}_{[j}\delta^i_kX_{lpq]r}\Big)\end{aligned}$$

showing that the function solves the cubic equation (3.75) for $s=\frac{5}{2}$.

One then computes that

$$\begin{aligned}&\mathcal{D}_{i[123}[\mathcal{D}^3_{\Upsilon_1+\Upsilon_4+\Upsilon_7}]_{4],1234}{}^i|X|^{-2s} \qquad\qquad (C.21)\\ =\ &\frac{3}{4}s(2s-3)(2s-5)\Big(\frac{8}{3}(s+1)(\Lambda^i{}_{[1}X_{234]i})^2|X|^{-2s-4}+\Big(\frac{s+1}{2}-\frac{20}{9}\Big)(X_{1234})^2|X|^{-2s-2}\Big)\end{aligned}$$

whereas

$$\mathcal{D}_{1234}\mathcal{D}_{1234}|X|^{-2s} = 6s\Big(96(s+1)(\Lambda^i{}_{[1}X_{234]i})^2|X|^{-2s-4}+(X_{1234})^2|X|^{-2s-2}\Big)\ , \tag{C.22}$$

exhibiting the multiplicity 2 of the representation $2\Upsilon_4$ of the Eisenstein series.[30]

One can also consider the restriction of the fourth order derivative to the $2\Upsilon_1+2\Upsilon_7$

$$\mathcal{D}^{8kij}\mathcal{D}_{1lij}\mathcal{D}_{1kpq}\mathcal{D}^{8lpq}|X|^{-2s} = -\frac{9}{2}s(2s-3)(2s-5)(s-4)\Lambda^8{}_1\Lambda^8{}_1|X|^{-2s-2}\ . \tag{C.23}$$

This does not prove that the Eisenstein series $E^{E_7}_{4\Lambda_1}$ satisfies the same differential equation. Nevertheless, the function $E^{E_7}_{4\Lambda_1}$ has a vanishing constant term and turns out to be square integrable as $E^{E_7}_{\frac{3}{2}\Lambda_1}$ and $E^{E_7}_{\frac{5}{2}\Lambda_1}$.

D Computation of $E^{E_d}_{s\Lambda_d}$ Fourier expansions

We compute the Fourier expansion of the Eisenstein series $E^{E_d}_{s\Lambda_d}$ in the parabolic subgroups P_1 and P_2. The case of P_d was analysed in [202]. The method can be generalised to P_3 and $P_{1,2}$.

[30] For the multiplicative character one finds that they are proportional for $s=\frac{7}{2}$, but $s=\frac{7}{2}$ is a regular point for the Eisenstein series in the critical strip and the functional relation to $s=5$ shows that the Eisenstein series does not satisfy any particular equation at this point.

D.1 E_d Eisenstein series in P_1

Let us consider the Eisenstein series $E^{E_d}_{s\Lambda_d}$ of E_d for $6 \le d \le 7$ in the parabolic P_1 corresponding to the perturbative string limit. For this purpose we write the decomposition P_1

$$\begin{aligned}
\mathfrak{e}_d &\cong (\wedge^{d-7}V)^{(-2)} \oplus S_-^{(-1)} \oplus (\mathfrak{gl}_1 \oplus \mathfrak{so}(d-1,d-1))^{(0)} \oplus \overline{S}_-^{(1)} \oplus (\wedge^{d-7}V)^{(2)} \\
R(\Lambda_d) &\cong (\wedge^{d-6}V)^{(2\frac{d-8}{9-d})} \oplus S_+^{(\frac{d-7}{9-d})} \oplus V^{(\frac{2}{9-d})} \\
R(\Lambda_1) &\cong \delta_{d,7}\mathbf{1}^{(-2)} \oplus (\wedge^{d-7}V \otimes S_-)^{(\frac{3d-23}{9-d})} \oplus (\wedge^{d-7}V \oplus \wedge^{d-5}V)^{(\frac{2d-14}{9-d})} \oplus S_-^{(\frac{d-5}{9-d})} \oplus \mathbf{1}^{(\frac{4}{9-d})} \,. \qquad \text{(D.1)}
\end{aligned}$$

We decompose the sum over all $\Gamma \in \mathbb{L}_d$ into layers according to the number of non-zero components

$$\Gamma = (q, \chi, p) \,. \qquad \text{(D.2)}$$

The first includes only non-zero string zero modes $q \in I\!I_{d-1,d-1} = V(\mathbb{Z})$. The second includes the sum over D-brane winding numbers $\chi \in S_+(\mathbb{Z})$. The third includes the sum over NS5-branes $p \in \wedge^{d-6}V(\mathbb{Z})$ for $d \ge 6$.

The string zero-modes must be level matched so the norm $(q,q) = 0$. For the second layer the D-brane winding must be half-BPS, which is the case if $\chi \times \chi \equiv \bar{\chi}\gamma_{d-5}\chi = 0$ where γ_{d-5} is the antisymmetric product of $d-5$ gamma matrices for $d \ge 5$. The constraint $\not{q}\chi = 0$ implies then that $(q,q) = 0$. The third layer appears for $d \ge 6$. For $d = 6$ the NS5-brane in an integer $p \in \mathbb{Z}$ and the constraint is

$$pq^a = \bar{\chi}\gamma^a\chi \,, \qquad \not{q}\chi = 0 \,, \qquad (q,q) = 0 \qquad \text{(D.3)}$$

that can be solved for

$$q^a = \frac{1}{p}\bar{\chi}\gamma^a\chi \qquad \text{(D.4)}$$

provided p divides $\chi \times \chi$ in $I\!I_{5,5}$. For $d = 7$ we have a vector $p \in I\!I_{6,6}$ of NS5-branes and KK-monopoles. The constraints are

$$(p,p) = 0 \,, \quad \not{p}\chi = 0 \,, \quad p_a q_b - p_b q_a = \bar{\chi}\gamma_{ab}\chi \,, \quad \not{q}\chi = 0 \,, \quad (q,q) = 0 \,. \qquad \text{(D.5)}$$

We write these constraints schematically for all d and the Eisenstein series sum can be written as

$$\begin{aligned}
E^{E_d}_{s\Lambda_d} &= \frac{1}{2\xi(2s)}\int_0^\infty \frac{dt}{t^{1+s}}\Bigg({\sum_{\substack{q\in V\\(q,q)=0}}}' e^{-\frac{\pi}{t}g_s^{\frac{4}{9-d}}|v(q)|^2} + {\sum_{\substack{\chi\in S_+\\ \chi\times\chi=0}}}' \sum_{\substack{q\in V\\ q\chi=0}} e^{-\frac{\pi}{t}(g_s^{\frac{4}{9-d}}|v(q+2a\times\chi)|^2 + g_s^{2\frac{d-7}{9-d}}|v(\chi)|^2)} \\
&\quad + {\sum_{\substack{p\in\wedge^{d-6}V\\ p\times p=0}}}' \sum_{\substack{\chi\in S_+\\ p\times\chi=0}} \sum_{\substack{q\in V\\ \chi\times\chi=pq}} e^{-\frac{\pi}{t}(g_s^{\frac{4}{9-d}}|v(q+2a\times\chi+(a\times a+b)p)|^2 + g_s^{2\frac{d-7}{9-d}}|v(\chi+ap)|^2 + g_s^{4\frac{d-8}{9-d}}|v(p)|^2)} \Bigg) \\
&= \frac{1}{\xi(2s)}\int_0^\infty \frac{dt}{t^{1+s}}\Bigg(\sum_{\gamma\in P_1\setminus D_{d-1}} {\sum_{n\in\mathbb{N}}}' e^{-\frac{\pi}{t}g_s^{\frac{4}{9-d}}y_\gamma^2 n^2} \\
&\qquad + \sum_{\gamma\in P_{d-1}\setminus D_{d-1}} {\sum_{n\in\mathbb{N}}}' \sum_{m\in\mathbb{Z}^{d-1}} e^{-\frac{\pi}{t}(g_s^{\frac{4}{9-d}}y_\gamma^{\frac{4}{d-1}}|v_\gamma(m+a_\gamma n)|^2 + g_s^{2\frac{d-7}{9-d}}y_\gamma^2 n^2)} \\
&\qquad + \sum_{\gamma\in P_{d-6}\setminus D_{d-1}} {\sum_{n\in\mathbb{N}}}' \sum_{\substack{\chi\in\mathbb{Z}^{16}\\ q\in\mathbb{Z}^{10}\\ \chi\times\chi=nq}} \sum_{m\in\mathbb{Z}^{d-6}} e^{-\frac{\pi}{t}|Z_\gamma(n,\chi,q,m)|^2} \Bigg) \qquad \text{(D.6)}
\end{aligned}$$

where for $d = 7$

$$|Z(n,\chi,q,m)|^2 = g_s^2\Big(y_\gamma^2|v(m+\bar{a}\chi+(\bar{a}a+b)n+c(q+2a\times\chi+a\times an))|^2 + |v(q+2a\times\chi+a\times an))|^2\Big) + y_\gamma|v(\chi+an)|^2 + g_s^{-2}y_\gamma^2 n^2 \ , \quad \text{(D.7)}$$

where all the moduli but g_s should have a label γ, since they are different for each representative γ. For $d = 6$ there is no Poincaré sum and

$$|Z(n,\chi,q,m)|^2 = g_s^{\frac{4}{3}}|v(q+2a\times\chi+a\times an))|^2 + g_s^{-\frac{2}{3}}|v(\chi+an)|^2 + g_s^{-\frac{8}{3}}n^2 \ . \quad \text{(D.8)}$$

We write y_γ the P_k multiplicative character in each Poincaré sum over $P_k(\mathbb{Z})\backslash Spin(d-1,d-1)$ that we write $P_k\backslash D_{d-1}$ for brevity.

Let us explain how to get these Poincaré sums. The first is the definition of the orbit of light-like vectors. The second comes by using that $\bar{\chi}\gamma_{d-5}\chi = 0$ implies that there exist $\gamma \in P_{d-1}(\mathbb{Z})\backslash Spin(d-1,d-1,\mathbb{Z})$ such that $\chi = \gamma n\Lambda_d$, the highest weight representative. P_{d-1} is the decomposition relevant in the large torus volume $\mathrm{Vol}(T^{d-1}) = (2\pi)^{d-1}\alpha'^{\frac{d-1}{2}}y_\gamma^{-2}$ limit, so that n is the number of D0-branes and the one-half BPS constraint $\not{q}\chi = 0$ implies that $q = (m,0) \in \mathbb{Z}^{d-1} \subset I\!I_{d-1,d-1}$ does not carry winding number.

The last Poincaré sum is only non-trivial for $d = 7$.[31] There exists $\gamma \in P_1(\mathbb{Z})\backslash Spin(6,6,\mathbb{Z})$ to bring p into a highest weight vector $\gamma n\Lambda_1$. In this parabolic $\not{p}\chi = 0$ implies that $\chi \in \mathbb{Z}^{16}$, a spinor of $Spin(5,5)$. The vector q satisfies $(p,q) = 0$, so it splits into a vector in $I\!I_{5,5}$ and the highest weight vector

$$q = \gamma\Big(\frac{1}{n}\bar{\chi}\gamma^1\chi + m\Lambda_1\Big) \ . \quad \text{(D.9)}$$

The remaining constraints are then automatically satisfied using the $Spin(5,5)$ identity

$$\gamma^a\chi\bar{\chi}\gamma_a\chi = 0 \quad \text{(D.10)}$$

familiar from super Yang-Mills theory in ten dimensions. Thinking of this parabolic P_1 has a decompactification limit, we can split the R-R axions in $\overline{S}_-^{(1)}$ into two spinors of $Spin(5,5)$ that we write a and $\bar{a}$. We write the Kalb-Ramond two-form axion b.

The first Poincaré sum gives by definition $g_{\rm s}^{-\frac{4}{9-d}s}E^{D_{d-1}}_{s\Lambda_1}$. The second can be computed by Poisson resummation over $q \in \mathbb{Z}^{d-1}$. One can then exchange the Poincaré sum over $P_{d-1}\backslash D_{d-1}$ for the spinor in S_+ and the sum over $q \in \overline{\mathbb{Z}}^{d-1}$ by a Poincaré sum over $P_{d-2}\backslash D_{d-1}$ for the Fourier coefficient $Q = \not{q}\chi$ in S_-, and the sum over $\chi \in \overline{\mathbb{Z}}^{d-1}$ of gcd dividing Q. The result is

$$\begin{aligned}&\frac{\xi(2s-d+1)}{\xi(2s)}g_{\rm s}^{2\frac{7-d}{9-d}s-d+1}E^{D_{d-1}}_{(s-\frac{d-3}{2})\Lambda_{d-2}} \\ &+2\frac{g_{\rm s}^{\frac{5-d}{9-d}s-\frac{d-1}{2}}}{\xi(2s)}\sum_{\substack{Q\in S_-\\ Q\times Q=0}}{}'\frac{\sigma_{d-1-2s}(Q)}{\mathrm{gcd}Q^{\frac{d-3}{d-1}(d-3-2s)}}E^{SL(d-1)}_{(s-\frac{d-3}{2})\Lambda_{d-2}}(v_Q)\frac{K_{s-\frac{d-1}{2}}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{(1-\frac{4}{d-1})s+\frac{9-d}{2}-\frac{4}{d-1}}}e^{2\pi i(Q,a)} \ . \quad \text{(D.11)}\end{aligned}$$

[31] Although there are additional layers of charges for $d > 7$, all the expressions we use extend to $d > 7$.

For the third and last Poincaré sum in (D.6) one carries out the Poisson resummation over $m \in \mathbb{Z}^{d-6}$ and the change of variable $t \to y_\gamma^{\frac{2}{3}} g_{\rm s}^{\frac{4}{3}\frac{d-6}{9-d}} t$ that gives the Poincaré sum over $P_{d-6}\backslash D_{d-1}$ of

$$\frac{g_{\rm s}^{-\frac{4}{3}(\frac{d-6}{9-d}s+\frac{d-6}{2})} y_\gamma^{\frac{d-9-2s}{3}}}{\xi(2s)} {\sum_{n\in\mathbb{N}}}' \sum_{\substack{\chi\in\mathbb{Z}^{16}\\ q\in\mathbb{Z}^{10}\\ \chi\times\chi=nq}} \int_0^\infty \frac{dt}{t^{1+s-\frac{d-6}{2}}} \Psi(n,\chi,q) \times e^{-\frac{\pi}{t}\left((\frac{g_{\rm S}}{\sqrt{y_\gamma}})^{\frac{4}{3}}|v(q+2a\times\chi+a\times an))|^2+(\frac{g_{\rm S}}{\sqrt{y_\gamma}})^{-\frac{2}{3}}|v(\chi+an)|^2+(\frac{g_{\rm S}}{\sqrt{y_\gamma}})^{-\frac{8}{3}}n^2\right)} \tag{D.12}$$

with

$$\Psi(n,\chi,q) = \sum_{m\in\mathbb{Z}^{d-6}} e^{-\pi t(\frac{g_{\rm s}}{\sqrt{y_\gamma}})^{-\frac{4}{3}} y_\gamma^{-\frac{2}{d-6}}|v^{-T}(m)|^2+2\pi i m(\bar{a}\chi+(b+\bar{a}a)n+c(q+2a\times\chi+a\times an))} . \tag{D.13}$$

In this form mn is manifestly the Fourier coefficient of $e^{2\pi i m n b}$, *i.e.* the NS5-brane instanton charge. To compute the abelian Fourier coefficient we use $\int db\ \Psi(n,\chi,q) = 1$, which sets $m = 0$. The constrained lattice sum is then the same for all $6 \le d \le 8$ up to the shift in s and the rescaling of $g_{\rm s}$ that are absent in $d = 6$.

We don't know how to compute this sum directly, but using the exact expansion of $E^{E_6}_{s\Lambda_6}$ in P_6 computed in [202] and the Langlands functional relation

$$E^{E_6}_{s\Lambda_6} = \frac{\xi(2s-8)\xi(2s-11)}{\xi(2s)\xi(2s-3)} E^{E_6}_{(6-s)\Lambda_1} \tag{D.14}$$

altogether with the partial expansion already carried out in this section for $d = 6$, one obtains that

$$\begin{aligned}
&{\sum_{n\in\mathbb{N}}}' \sum_{\substack{\chi\in\mathbb{Z}^{16}\\ q\in\mathbb{Z}^{10}\\ \chi\times\chi=nq}} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}\left(g_{\rm s}^{\frac{4}{3}}|v(q+2a\times\chi+a\times an))|^2+g_{\rm s}^{-\frac{2}{3}}|v(\chi+an)|^2+g_{\rm s}^{-\frac{8}{3}}n^2\right)} \\
= \ & \frac{g_{\rm s}^{-\frac{4}{3}s}}{\xi(2s-3)} \Bigg(\xi(2s-8)\xi(2s-11) g_{\rm s}^{4s-16} \\
&+2\xi(2s-8) g_{\rm s}^{3s-\frac{21}{2}} {\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \sigma_{11-2s}(Q) \frac{K_{s-\frac{11}{2}}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{11-2s}{2}}} e^{2\pi i(Q,a)} \\
&+2 g_{\rm s}^{2s-8} \sum_{\substack{Q\in S_{-*}\\ Q\times Q\neq 0}} \sum_{d|Q} d^{11-2s} \sigma_{8-2s}(\tfrac{Q\times Q}{d^2}) \frac{B_{\frac{3}{2},s-4}(\frac{|v(Q)|^2}{g_{\rm s}^2}, \frac{|v(Q\times Q)|}{g_{\rm s}^2})}{|v(Q\times Q)|^{4-s}} e^{2\pi i(Q,a)} \Bigg) .
\end{aligned} \tag{D.15}$$

Note that this gives the expected overal factor of $\frac{1}{\xi(2s-3)}$ that exhibits that this contribution vanishes for the minimal series at $s = \frac{3}{2}$. Including this result in (D.12) one finds more generally that the abelian component of this sum vanishes for $s = \frac{d-3}{2}$. This should not be the case for the full

series (D.12) including the non-abelian Fourier coefficients since the minimal series also carries non-abelian Fourier coefficients. One obtains then

$$
\begin{aligned}
& g_{\rm s}^{-\frac{4}{3}(\frac{d-6}{9-d}s+\frac{d-6}{2})} y_\gamma^{\frac{d-9-2s}{3}} {\sum_{n\in\mathbb{N}}}' \sum_{\substack{\chi\in\mathbb{Z}^{16}\\ q\in\mathbb{Z}^{10}\\ \chi\times\chi=nq}} \int_0^\infty \frac{dt}{t^{1+s-\frac{d-6}{2}}} e^{-\frac{\pi}{t}\left((\frac{g_{\rm s}}{\sqrt{y_\gamma}})^{\frac{4}{3}}|v(q+2a\times\chi+a\times an))|^2+(\frac{g_{\rm s}}{\sqrt{y_\gamma}})^{-\frac{2}{3}}|v(\chi+an)|^2+(\frac{g_{\rm s}}{\sqrt{y_\gamma}})^{-\frac{8}{3}}n^2\right)} \\
= \; & \frac{g_{\rm s}^{-\frac{4}{9-d}s}}{\xi(2s-d+3)} \Bigg(\xi(2s-d-2)\xi(2s-d-5) g_{\rm s}^{4s-2d-4} y_\gamma^{-2s+d+1} \\
& + 2\xi(2s-d-2) g_{\rm s}^{3(s-\frac{d+1}{2})} y_\gamma^{\frac{3d-1-6s}{4}} {\sum_{\substack{Q\in\overline{\mathbf{16}}\\ Q\times Q=0}}}' \sigma_{d+5-2s}(Q) \frac{K_{s-\frac{d+5}{2}}(2\pi\frac{\sqrt{y_\gamma}|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{d+5}{2}-s}} e^{2\pi i(Q,a)} \\
& + 2 g_{\rm s}^{2s-d-2} y_\gamma^{d+1-2s} {\sum_{\substack{Q\in\overline{\mathbf{16}}\\ Q\times Q\neq 0}}}' \sum_{n|Q} n^{d+5-2s}\sigma_{d+2-2s}(\tfrac{Q\times Q}{n^2}) \frac{B_{\frac{3}{2},s-\frac{d+2}{2}}(\frac{y_\gamma|v(Q)|^2}{g_{\rm s}^2},\frac{y_\gamma|v(Q\times Q)|}{g_{\rm s}^2})}{|y_\gamma v(Q\times Q)|^{\frac{d+2}{2}-s}} e^{2\pi i(Q,a)} \Bigg)
\end{aligned}
\tag{D.16}
$$

which we must summed over $\gamma \in Spin(6,6,\mathbb{Z})/P_1$ for $d=7$. The first Poincaré sum gives by definition the Eisenstein series $E^{D_6}_{(s-\frac{d+1}{2})\Lambda_1}$. For the other terms one identifies the character y_γ as the norm for a primitive vector $\omega \in V$ and Q as a spinor in S_- satisfying $\not{\omega} \times Q = 0$. The sum over such ω and Q indeed reproduces the Poincaré sum with $\omega \in \gamma\Lambda_1$ and $Q \in \overline{\mathbf{16}}^{(1)}$. One can rewrite the second sum, for which $Q\times Q = 0$, as a Poincaré sum over $P_5\backslash D_6$ with $Q \in \gamma \mathrm{gcd}(Q)\Lambda_5$ and $\omega \in (\mathbb{Z}^6)^{(1)}$. One gets therefore the change of variables

$$
\sqrt{y_\gamma}|v_\gamma(Q)| = |v(Q)| \ , \qquad y_\gamma = \left|\frac{v(Q)}{\mathrm{gcd}Q}\right|^{\frac{1}{6}} y_{\Lambda_1} \ , \tag{D.17}
$$

with the right-hand-sides defined in the parabolic P_5, and y_{Λ_1} the character of the parabolic subgroup $P_1^{SL(6)}$ of the Levi stabilizer $SL(6)$ of $Q \in S_-$.

For the last sum over rank 2 Q one considers instead the Heisenberg parabolic P_2 decomposition. In this case $Q\gamma_{ab}Q \neq 0$, and defines an element in the minimal nilpotent orbit of $\mathfrak{so}(6,6)$, and therefore $Q\gamma_{ab}Q = \gamma(\mathrm{gcd}(Q\gamma_{ab}Q)\Lambda_2)$. In this decomposition Q can be written as an $SO(4,4)$ vector by triality. We obtain therefore a Poincaré sum over $Spin(6,6)/P_2$ of spinors Q realised as vectors of $SO(4,4)$ with non-zero norm square equal $\mathrm{gcd}(Q\gamma_{ab}Q)$. This parabolic corresponds to the large T^2 torus limit. In this basis the constraint $\not{\omega}Q = 0$ implies that $\omega \in \mathbb{Z}^2$, i.e. a doublet of momenta along T^2. The Poincaré sum over $P_1(\mathbb{Z})\backslash Spin(6,6,\mathbb{Z})$ and the sum over generic $Q \in \overline{\mathbf{16}}^{(1)}$ can therefore be rewritten as a sum over rank 2 $Q \in S_-$ and a Poincaré sum over $P_1\backslash SL(2)$ of the Levi stabilizer $SL(2)\times Spin(3,4)$ of Q. The relevant change of variables is then

$$
\sqrt{y_\gamma}|v_\gamma(Q)| = |v(Q)| \ , \quad y_\gamma|v_\gamma(Q\times Q)| = |v(Q\times Q)| \ , \quad y_\gamma = \left|\frac{v(Q\times Q)}{\mathrm{gcd}(Q\times Q)}\right|^{\frac{1}{2}} U_2 \ , \tag{D.18}
$$

where U_2 is now the imaginary part of the large volume T^2 complex structure.

Using these formula and manipulating the Poincaré sums as explained above, one obtains eventually

$$\int_{[0,1]} db\; E^{E_d}_{s\Lambda_d} = g_{\rm s}^{-\frac{4}{9-d}s}\Bigg(E^{D_{d-1}}_{s\Lambda_1} + \frac{\xi(2s-d+1)}{\xi(2s)} g_{\rm s}^{2s-d+1} E^{D_{d-1}}_{(s-\frac{d-3}{2})\Lambda_{d-2}}$$
$$+ \frac{\xi(2s-d-2)\xi(2s-d-5)}{\xi(2s)\xi(2s-d+3)} g_{\rm s}^{4s-2d-4} E^{D_{d-1}}_{(s-\frac{d+1}{2})\Lambda_{d-6}}$$
$$+ 2\frac{g_{\rm s}^{s-\frac{d-1}{2}}}{\xi(2s)} {\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \frac{\sigma_{d-1-2s}(Q)}{\gcd Q^{\frac{d-3}{d-1}(d-3-2s)}} E^{SL(d-1)}_{(s-\frac{d-3}{2})\Lambda_{d-2}}(v_Q) \frac{K_{s-\frac{d-1}{2}}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{(1-\frac{4}{d-1})s+\frac{9-d}{2}-\frac{4}{d-1}}} e^{2\pi i(Q,a)}$$
$$+ 2\frac{\xi(2s-d-2) g_{\rm s}^{3s-\frac{3(d+1)}{2}}}{\xi(2s)\xi(2s-d+3)} {\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \frac{\sigma_{d+5-2s}(Q)}{\gcd Q^{2\frac{d-6}{d-1}(d+1-2s)}} E^{SL(d-1)}_{(s-\frac{d+1}{2})\Lambda_{d-6}}(v_Q) \frac{K_{s-\frac{d+5}{2}}(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{(6d-46)s+3d(8-d)+19}{2(d-1)}}} e^{2\pi i(Q,a)}$$
$$+ 2\frac{g_{\rm s}^{2s-d-2}}{\xi(2s)\xi(2s-d+3)} \sum_{\substack{Q\in S_{-*}\\ Q\times Q\neq 0}} \sum_{n|Q} \frac{n^{d+5-2s}\sigma_{d+2-2s}(\frac{Q\times Q}{n^2})}{\gcd_{Q\times Q} Q^{\frac{d-6}{d-5}(d+1-2s)}} E^{SL(d-5)}_{(s-\frac{d+1}{2})\Lambda_{d-6}}(v_Q) \frac{B_{\frac{3}{2},s-\frac{d+2}{2}}(\frac{|v(Q)|^2}{g_{\rm s}^2},\frac{|v(Q\times Q)|}{g_{\rm s}^2})}{|v(Q\times Q)|^{\frac{(d-7)s+1-\frac{d(d-7)}{2}}{d-5}}} e^{2\pi i(Q,a)}\Bigg) \tag{D.19}$$

Here we define $v(Q)$ as the Narain moduli matrix acting on the spinor $Q \in S_-$, and $Q\times Q = (Q\Gamma_{d-5}Q)$ is the $(d-5)$-form quadratic in Q, such that $v(Q\times Q) = v(Q)\times v(Q)$, and its norm is normalised with the apporiate normalisation for a $(d-5)$-form with the $\frac{1}{(d-5)!}$ factor, such that $|v(Q)|$ and $|v(Q\times Q)|$ are $Spin(d-1,d-1,\mathbb{Z})$ conjugate to multiplicative Borel characters for primitive charges Q and $Q\times Q$, respectively. For E_7 the Gelfand–Kirillov dimension 27 is saturated by the rank 2 spinor charge lattice of dimension 25 plus 1 for the $SL(2)$ Eisenstein series of its Levi stabilizer, if one includes the non-abelian Fourier support, *i.e.* $27 = 25+1+1$.

This computation is in fact the same for E_8, although this formula only gives a small part of the Fourier coefficients at generic values of s in this case. However, one expects that it gives the complete abelian Fourier coefficients for $s=\frac{5}{2}$ and $s=\frac{9}{2}$ and in particular

$$8\pi\xi(4)\xi(9) g_{\rm s}^{18} \int_{[0,1]} db\; E^{E_8}_{\frac{9}{2}\Lambda_8} = \zeta(5) g_{\rm s}^{-2} + \frac{4\pi^3}{45}\xi(9) E^{D_7}_{\frac{9}{2}\Lambda_1} + \frac{4}{3}\zeta(4) g_{\rm s}^2 E^{D_7}_{2\Lambda_6}$$
$$+ 16 {\sum_{\substack{Q\in S_-\\ Q\times Q=0}}}' \left(\frac{g_{\rm s}}{\pi} \frac{\sigma_2(Q)}{\gcd Q^{-\frac{6}{7}}} \zeta(4) E^{SL(7)}_{2\Lambda_6}(v_Q) \frac{K_1(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^{\frac{13}{7}}} + \frac{\pi^2}{6}\sigma_4(Q)\frac{K_2(2\pi\frac{|v(Q)|}{g_{\rm s}})}{|v(Q)|^2}\right) e^{2\pi i(Q,a)}$$
$$+ 16\pi g_{\rm s}^{-1} \sum_{\substack{Q\in S_{-*}\\ Q\times Q\neq 0}} \sum_{d|Q} d^4\sigma_1(\tfrac{Q\times Q}{d^2}) \frac{K_1(2\pi\frac{\sqrt{|v(Q)|^2+2|v(Q\times Q)|}}{g_{\rm s}})}{|v(Q\times Q)|\sqrt{|v(Q)|^2+2|v(Q\times Q)|}} e^{2\pi i(Q,a)}\,. \tag{D.20}$$

Let us now consider the non-abelian Fourier coefficient for $E_{7(7)}$. We use the polarisation (3.116) and the unipotent character (3.117). One finds that this is a unipotent character of the parabolic P_7, for which we know the Fourier decomposition at the special values of $s=2$ and 4 [202]. One can therefore consider this expansion of the Eisenstein series, and further decompose it with respect to the P_1 parabolic of E_6. Physically the non-abelian Fourier coefficients in the parabolic P_1 correspond to first expand in the 4D string coupling $g_{\rm s}$ and then in the string frame

radus $R_{\rm s}$ to choose a polarisation, whereas the further decomposition of the Fourier expansion in P_7 corresponds to expand in the Einstein frame radius R first and then in the 5D string coupling g_5. They are related to each others through

$$R = \left(\frac{R_{\rm s}}{g_{\rm s}}\right)^{\frac{2}{3}} , \qquad g_5 = g_{\rm s} R_{\rm s}^{\frac{1}{2}} . \tag{D.21}$$

One obtains in this way using the expression of [202], up to contributions of rank three charge $q \in \mathbb{Z}^{27}$ that vanish for $s = 2,\, 4$,

$$\begin{aligned}
&2\frac{g_{\rm s}^{-1}}{\xi(2s)} {\sum_{k\in\mathbb{Z}}}' \sum_{\substack{q\in\mathbb{Z}^{16}\\ k|q\times q}} \psi_{k,q,\frac{q\times q}{k}} \Bigg(R_{\rm s}^{2s}\sigma_{2s-1}(k,q,\tfrac{q\times q}{k}) \frac{K_{s-\frac12}(2\pi\frac{\sqrt{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2}}{g_{\rm s}^2})}{\left(k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2\right)^{\frac{s-\frac12}{2}}} \\
&\quad + \frac{\xi(2s-5)R_{\rm s}^5}{\xi(2s-4)} \frac{\sigma_{9-2s}(k,q,\frac{q\times q}{k})}{\gcd(k,q,\frac{q\times q}{k})^{\frac52-s}} E^{D_5}_{(s-\frac52)\Lambda_1}(g_{k,q}) \frac{K_{s-\frac92}(2\pi\frac{\sqrt{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2}}{g_{\rm s}^2})}{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(\frac{(q+\bar a k)\times(q+\bar a k)}{k})|^2} \Bigg) \\
&\qquad + 2\frac{g_{\rm s}^{-5}R_{\rm s}^{2s}}{\xi(2s)\xi(2s-4)} {\sum_{\substack{k\in\mathbb{Z}\\ q\in\mathbb{Z}^{16}}}}' \sum_{\substack{p\in\mathbb{Z}^{10}\\ \frac{k|p|^2}{2}=q\not{p}q}} \psi_{k,q,p} \sum_{n|(k,q,p)} n^{2s-1}\sigma_{2s-5}(\tfrac{(\frac{|p|^2}{2},\not{p}q,kp-q\times q)}{n^2}) \\
&\qquad\qquad \frac{B_{2,s-\frac52}(\frac{k^2+g_{\rm s}^2R_{\rm s}|v(q+\bar a k)|^2+g_{\rm s}^4R_{\rm s}^2|v(p+2\bar a\times q+\bar a\times\bar a k)|^2}{g_{\rm s}^4}, \frac{\sqrt{R_{\rm s}^2|v(kp-q\times q)|^2+g_{\rm s}^2R_{\rm s}^3|v(\not{p}q+(k\not{p}-q\not{\times}q)\bar a)|^2+g_{\rm s}^4R_{\rm s}^4|v(\frac{|p|^2}{2}+2\bar a\not{p}\bar a+\bar a(k\not{p}-q\not{\times}q)\bar a)|^2}}{g_{\rm s}^4})}{\left(R_{\rm s}^2|v(kp-q\times q)|^2+g_{\rm s}^2R_{\rm s}^3|v(\not{p}q+(k\not{p}-q\not{\times}q)\bar a)|^2+g_{\rm s}^4R_{\rm s}^4|v(\frac{|p|^2}{2}+2\bar a\not{p}\bar a+\bar a(k\not{p}-q\not{\times}q)\bar a)|^2\right)^{\frac{2s-5}{4}}}
\end{aligned} \tag{D.22}$$

where $g_{k,q}$ is the $Spin(5,5)$ element function of $g_{\rm s}R_s^{\frac12}$, v and $\bar a$ that stabilize the vector $(k,q,\frac{q\times q}{k}) \in \mathbf{27}$. For $q=0$ it is obtained by changing basis from the positive parabolic P_6 to its transpose inside E_6 (Going from KLU to $KL\bar U$). More generally it is then obtained from the former by the discrete $E_6(\mathbb{Z})$ transformation that relates $(k,q,\frac{q\times q}{k})$ to $(\gcd(k,q),0,0)$.

D.2 E_d Eisenstein series in P_2

Let us consider the Eisenstein series $E^{E_d}_{s\Lambda_d}$ of E_d for $3 \le d \le 7$ in the parabolic P_2 corresponding to the large volume limite in M-theory. For this purpose we write the decomposition P_2 [32]

$$\begin{aligned}
\mathfrak{e}_d &\cong \cdots \oplus (\mathfrak{gl}_d)^{(0)} \oplus (\wedge^3\overline{\mathbb{Z}}^d)^{(9-d)} \oplus (\wedge^6\overline{\mathbb{Z}}^d)^{(18-2d)} \oplus (\wedge^{8,1}\overline{\mathbb{Z}}^d)^{(27-3d)} \\
R(\Lambda_d) &\cong (\wedge^7\mathbb{Z}^d\otimes\mathbb{Z}^d)^{(3d-24)} \oplus (\wedge^5\mathbb{Z}^d)^{(2d-15)} \oplus (\wedge^2\mathbb{Z}^d)^{(d-6)} \oplus (\overline{\mathbb{Z}}^d)^{(3)} \\
R(\Lambda_1) &\cong \cdots \oplus (\wedge^7\mathbb{Z}^d\otimes\wedge^3\mathbb{Z}^d)^{(4d-30)} \oplus (\wedge^6\mathbb{Z}^d\otimes\mathbb{Z}^d)^{(3d-21)} \oplus (\wedge^4\mathbb{Z}^d)^{(2d-12)} \oplus (\mathbb{Z}^d)^{(d-3)} ,
\end{aligned} \tag{D.23}$$

where we normalised the grading such that it corresponds to the powers of r with $(2\pi\ell r^{\frac{9-d}{6}})^d = (2\pi\ell V^{\frac13})^d$ the volume of the torus T^d.

We will cary out the sum over $\Gamma \in \mathbb{L}_d$ with the first layer with only the Kaluza-Klein momentum $p \in \mathbb{Z}^d$ non-zero, the second layer with non-zero M2-winding $Q \in \wedge^2\mathbb{Z}^d$ and finally the third layer with non-zero M5-brane winding number $N \in \wedge^5\mathbb{Z}^d$. For $d=7$ one should also consider a fourth layer with Kaluza-Klein monopole charge $k \in \mathbb{Z}^7$, but we will not do this computation.

[32] We use the notation $\wedge^{p,q}\mathbb{Z}^d$ for the irreducible representation $R(\Lambda_p+\Lambda_q)$.

Layer decomposition

The three first grades of the Epstein series, *i.e.* the complete Epstein series for $3 \leq d \leq 6$, reads

$$\begin{aligned} E^{E_d}_{s\Lambda_d} &= \frac{1}{2\xi(2s)}\int_0^\infty \frac{dt}{t^{1+s}}\Bigg(\sum_{p\in\mathbb{Z}^d}{}' e^{-\frac{\pi}{t}r^{-3}|U^{-1}(p)|^2} + \sum_{\substack{Q\in\wedge^2\mathbb{Z}^d_* \\ Q\wedge Q=0}}\sum_{\substack{p\in\mathbb{Z}^d \\ Q\cdot p=0}} e^{-\frac{\pi}{t}(r^{-3}|U^{-1}(p+a\cdot Q)|^2+r^{6-d}|U(Q)|^2)} \\ &\quad + \sum_{\substack{N\in\wedge^5\mathbb{Z}^d_* \\ N\times N=0}}\sum_{\substack{Q\in\wedge^2\mathbb{Z}^d \\ Q\times N=0}}\sum_{\substack{p\in\mathbb{Z}^d \\ Q\wedge Q=N\cdot p}} e^{-\frac{\pi}{t}(r^{-3}|U^{-1}(p+a\cdot Q+(a\times a+b)\cdot N)|^2+r^{6-d}|U(Q+a\cdot N)|^2+r^{15-2d}|U(N)|^2)}\Bigg) \\ &= \frac{1}{\xi(2s)}\int_0^\infty \frac{dt}{t^{1+s}}\Bigg(\sum_{\gamma\in P_d\backslash SL(d)}\sum_{n\in\mathbb{N}}{}' e^{-\frac{\pi}{t}r^{-3}y_\gamma^2 n^2} \\ &\qquad + \sum_{\gamma\in P_2\backslash SL(d)}\sum_{n\in\mathbb{N}}{}'\sum_{p\in\mathbb{Z}^{d-2}} e^{-\frac{\pi}{t}(r^{-3}y_\gamma^{\frac{2}{d-2}}|v_\gamma^{-\intercal}(p+a_\gamma n)|^2+r^{6-d}y_\gamma^2 n^2)} \\ &\qquad + \sum_{\gamma\in P_5\backslash SL(d)}\sum_{n\in\mathbb{N}}{}'\sum_{\substack{q\in\mathbb{Z}^{10} \\ p\in\mathbb{Z}^5 \\ q\wedge q=np}}\sum_{m\in\mathbb{Z}^{d-5}} e^{-\frac{\pi}{t}|Z_\gamma(n,q,p,m)|^2}\Bigg) \end{aligned} \tag{D.24}$$

where the last line vanishes for $3 \leq d \leq 4$ and

$$\begin{aligned} |Z(n,p,q,m)|^2 = r^{-3}\Big(&y_\gamma^{\frac{2}{d-5}}|v(m+\bar{a}q+(\bar{a}a+b)n+c(p+2a\wedge q+a\wedge an))|^2 \\ &+ y_\gamma^{-\frac{2}{5}}|v(p+2a\wedge q+a\wedge an))|^2\Big) + r^{6-d}y_\gamma^{\frac{4}{5}}|v(q+an)|^2 + r^{15-2d}y_\gamma^2 n^2 \ , \end{aligned} \tag{D.25}$$

and y_γ is the P_k multiplicative character in each Poincaré sum over $P_k\backslash SL(d)$, and we avoid writing P_k explicitly for brevity. Also in this last equation, all the moduli but r should have a label γ since their are different for each representative $\gamma \in P_5\backslash SL(d)$, except for $d = 5$ in which case the set is empty, and $y_\gamma = 1$. The second Poincaré sum is straightforward, but let us explain the third in some details. We set $N \in \wedge^5\mathbb{Z}^d$ equal $n \in \mathbf{1}^{(5(d-10))}$ using $\gamma^{-1} \in P_5\backslash SL(d)$. The constraint $Q \times N = 0$ in $(\mathbb{Z}^5 \otimes \mathbb{Z}^{d-5})^{(6d-35)}$ implies that $Q \in \mathbf{10}^{(d-10)}$ that we name q. The constraint $Q \wedge Q = N \cdot p$ implies that the component $p \in \overline{\mathbb{Z}}^5$ of $\overline{\mathbb{Z}}^d$ satisfies $q \wedge q = np$ while its component $m \in \overline{\mathbb{Z}}^{d-5}$ is unconstrained. The remaining constraints are then automatically satisfied.

We write the M2 axions in $(\wedge^3\overline{\mathbb{R}}^d)^{(9-d)}$ as $a \in \mathbf{10}^{(15-3d)}$, $\bar{a} \in (\overline{\mathbb{R}}^{d-5} \otimes \overline{\mathbf{10}})^{(15-2d)}$ and the other components do not appear. We write b in the $(\overline{\mathbb{R}}^{d-5})^{(30-5d)}$, the axion coupled to the M5-brane that contributes. Because two components of the anti-fundamental representation appear we also define the torus metric axion $c \in (\overline{\mathbb{R}}^{d-5} \otimes \mathbf{5})^{(d)}$ of $SL(d)$.

The first Poincaré sum gives by definition $r^{3s}E^{SL(d)}_{s\Lambda_{d-1}}$. The second can be computed by Poisson resummation over $p \in \mathbb{Z}^{d-2}$. One can then exchange the Poincaré sum over $P_2\backslash SL(d)$ and the sum over 3-forms $n\tilde{p} \in (\mathbb{Z}^{d-2})^{(2d-6)}$ for the Poincaré sum over $P_3\backslash SL(d)$ and the sum over primitive

2-forms in $(\overline{\mathbb{Z}}^3)^{(2d-6)}$, such that

$$\begin{aligned}
&\int_0^\infty \frac{dt}{t^{1+s}} \sum_{\gamma\in P_2\backslash SL(d)} {\sum_{n\in\mathbb{N}}}' \sum_{p\in\mathbb{Z}^{d-2}} e^{-\frac{\pi}{t}(r^{-3}y_\gamma^{\frac{2}{d-2}}|v_\gamma^{-\intercal}(p+a_\gamma n)|^2+r^{6-d}y_\gamma^2 n^2)} \\
&= \xi(2s-d+2)r^{(d-6)s+\frac{(d-2)(9-d)}{2}} E^{SL(d)}_{(s-\frac{d-3}{2})\Lambda_2} \\
&\quad +2r^{\frac{d-3}{2}s+\frac{(d-2)(9-d)}{4}} {\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}}' \frac{\sigma_{d-2-2s}(N)}{\gcd N^{\frac{3}{2}(d-3-2s)}} E^{SL(3)}_{(s-\frac{d-3}{2})\Lambda_2}(v_N)\frac{K_{s-\frac{d-2}{2}}(2\pi r^{\frac{9-d}{2}}|Z(N)|)}{|Z(N)|^{2s-d+\frac{7}{2}}} e^{2\pi i(N,a)}
\end{aligned} \tag{D.26}$$

where $N\times N$ is the projection to $\wedge^{5,1}\mathbb{Z}^d$.

For the third and last Poincaré sums in (D.24) one carries out the Poisson resummation over $m\in\mathbb{Z}^{d-5}$ and the change of variable $t\to y_\gamma^{\frac{1}{2}} r^{-\frac{3(d-5)}{4}} t$ that gives the Poincaré sum over $P_5\backslash SL(d)$ of

$$\begin{aligned}
&\frac{r^{\frac{3(d-5)}{8}(2s-d+9)} y_\gamma^{\frac{d-9-2s}{4}}}{\xi(2s)} {\sum_{n\in\mathbb{N}}}' \sum_{\substack{q\in\mathbb{Z}^{10}\\ p\in\mathbb{Z}^5\\ q\wedge q=np}} \int_0^\infty \frac{dt}{t^{1+s-\frac{d-5}{2}}}\Psi(n,q,p) \\
&\qquad\qquad \times e^{-\frac{\pi}{t}\left((r^{\frac{9-d}{4}}y_\gamma^{\frac{3}{10}})^{-3}|(v^{-\intercal}(p+2a\wedge q+a\wedge an)|^2+(r^{\frac{9-d}{4}}y_\gamma^{\frac{3}{10}})|v(p+an))|^2+(r^{\frac{9-d}{4}}y_\gamma^{\frac{3}{10}})^5 n^2\right)}
\end{aligned} \tag{D.27}$$

with

$$\Psi(n,\chi,q)=\sum_{m\in\mathbb{Z}^{d-6}} e^{-\pi t r^{-\frac{3(d-5)}{4}} y_\gamma^{\frac{1}{2}-\frac{2}{d-6}}|v^{-\intercal}(m)|^2+2\pi i m(\bar{a}q+(b+\bar{a}a)n+c(p+2a\wedge q+a\wedge an))}\ . \tag{D.28}$$

In this form mn is manifestly the Fourier coefficient of $e^{2\pi imnb}$, *i.e.* the M5-brane instanton charge. To compute the abelian Fourier coefficient we use $\int db\,\Psi(n,q,p)=1$. The constrained lattice sum is then the same for all $5\le d\le 7$ up to the shift in s and the redefinition of r that are absent in $d=5$.

We don't know how to compute this sum directly, but using the exact expansion of $E^{E_5}_{s\Lambda_5}$ in P_5 computed in [202] and the Langlands functional relation [33]

$$E^{E_5}_{s\Lambda_5}=\frac{\xi(2s-5)\xi(2s-7)}{\xi(2s)\xi(2s-2)}E^{E_5}_{(4-s)\Lambda_2} \tag{D.29}$$

[33] Or in Bourbaki convention $E^{D_5}_{s\Lambda_5}=\frac{\xi(2s-5)\xi(2s-7)}{\xi(2s)\xi(2s-2)}E^{D_5}_{(4-s)\Lambda_4}$.

altogether with the partial expansion already carried out in this section for $d=5$, one obtains that

$$\begin{aligned}
&\sum_{n\in\mathbb{N}}{}' \sum_{\substack{q\in\mathbb{Z}^{10}\\ p\in\mathbb{Z}^5\\ q\times q=np}} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}\left(r^{-3}|(v^{-\intercal}(p+2a\wedge q+a\wedge an)|^2+r|v(p+an))|^2+r^5n^2\right)} \\
&= \frac{r^{10}}{\xi(2s-2)}\Bigg(\xi(2s-5)\xi(2s-7)r^{10-5s} \qquad\qquad (D.30)\\
&\quad +2\xi(2s-5)r^{3-3s}\sum_{\substack{N\in\wedge^3\mathbb{Z}^5\\ N\times N=0}}\sigma_{7-2s}(N)\frac{K_{s-\frac{7}{2}}(2\pi r^2|Z(N)|)}{|Z(N)|^{\frac{7-2s}{2}}}e^{2\pi i(N,a)}\\
&\quad +2r^{-s}\sum_{\substack{N\in\wedge^3\mathbb{Z}^5\\ N\times N\neq 0}}\sum_{d|N} d^{7-2s}\sigma_{5-2s}(\tfrac{N\times N}{d^2})\frac{B_{1,s-\frac{5}{2}}(r^4|Z(N)|^2,r^4|z(N\times N)|)}{|z(N\times N)|^{\frac{5-2s}{2}}}e^{2\pi i(N,a)}\Bigg)\,.
\end{aligned}$$

Note that this gives the expected overal factor of $\frac{1}{\xi(2s-2)}$ that exhibits that this contribution vanishes for the minimal series at $s=1$. Including this result in (D.27) one finds more generally that the abelian component of this sum vanishes for $s=\frac{d-3}{2}$. This is not the case for the full series (D.27) including the non-abelian Fourier coefficients since the minimal series also carries non-abelian Fourier coefficients. One obtains then for $d=6,7,8$

$$\begin{aligned}
&r^{\frac{3(d-5)(2s-d+9)}{8}}y_\gamma^{\frac{d-9-2s}{4}}\sum_{n\in\mathbb{N}}{}' \sum_{\substack{q\in\mathbb{Z}^{10}\\ p\in\mathbb{Z}^5\\ q\wedge q=np}} \int_0^\infty \frac{dt}{t^{1+s-\frac{d-5}{2}}} e^{-\frac{\pi}{t}\left((r^{\frac{9-d}{4}}y_\gamma^{\frac{3}{10}})^{-3}|(v^{-\intercal}(p+2a\wedge q+a\wedge an)|^2+(r^{\frac{9-d}{4}}y_\gamma^{\frac{3}{10}})|v(p+an))|^2+(r^{\frac{9-d}{4}}y_\gamma^{\frac{3}{10}})^5n^2\right)}\\
&= \frac{V^{\frac{6}{9-d}s}}{\xi(2s-d+3)}\sum_{\gamma\in P_5\backslash SL(d)}\Bigg(\xi(2s-d)\xi(2s-d-2)V^{2d-4s}y_\gamma^{-2s+d} \qquad\qquad (D.31)\\
&\quad +2\xi(2s-d)V^{\frac{3d-2}{2}-3s}y_\gamma^{-\frac{7}{5}s+\frac{7d-6}{10}}\sum_{\substack{q\in\mathbb{Z}^{10}_*\\ q\wedge q=0}}\sigma_{d+2-2s}(q)\frac{K_{s-\frac{d+2}{2}}(2\pi V y_\gamma^{\frac{3}{5}}|v_\gamma(q)|)}{|v_\gamma(q)|^{\frac{d+2}{2}-s}}e^{2\pi i(q,a_\gamma)}\\
&\quad +2V^{d-2s}y_\gamma^{\frac{2(d-2s)}{5}}\sum_{\substack{q\in\mathbb{Z}^{10}\\ q\wedge q\neq 0}}\sum_{n|q} n^{d+2-2s}\sigma_{d-2s}(\tfrac{q\wedge q}{n^2})\frac{B_{\frac{3}{2},s-\frac{d}{2}}(V^2y_\gamma^{\frac{6}{5}}|v_\gamma(q)|^2,V^2y_\gamma^{\frac{6}{5}}|v_\gamma(q\wedge q)|)}{|v_\gamma(q\wedge q)|^{\frac{d}{2}-s}}e^{2\pi i(q,a_\gamma)}\Bigg)
\end{aligned}$$

The first Poincaré sum gives by definition the Eisenstein series $E^{SL(d)}_{(s-\frac{d}{2})\Lambda_5}$. For the other terms one identifies the character y_γ as the norm for a primitive 5-form $\omega\in\wedge^5\mathbb{Z}^d$ with $\omega\times\omega=0$ and N as a 3-form in $\wedge^3\mathbb{Z}^d$ satisfying that $\omega\times N\in\wedge^{6,2}\mathbb{Z}^d$ vanishes. The sum over such ω and N indeed reproduces the Poincaré sum with $\omega\in\mathbf{1}^{(2)}$ and $N=q\in\overline{\mathbf{10}}^{(1)}$. One can rewrite the second sum, for which $N\times N=0$, as a Poincaré sum over $P_3\backslash SL(d)$ with $N\in\mathbf{1}^{(2)}$ and $\omega\in(\wedge^2\mathbb{Z}^{d-3})^{(2\frac{d-5}{d-3})}$. One gets therefore the change of variables

$$y_\gamma^{\frac{3}{5}}|v_\gamma(q)|=|Z(N)|\,,\qquad y_\gamma=\left|\frac{Z(N)}{\gcd N}\right|^{\frac{d-5}{d-3}}y_{\Lambda_2}\,, \qquad (D.32)$$

with the right-hand-sides defined in the parabolic P_3, and y_{Λ_2} the character of the parabolic subgroup P_2 of the Levi stabilizer $SL(d-3)$ of $N \in \wedge^3\mathbb{Z}^d$.

For the last sum over rank 2 M2 instanton charge N one considers instead the further decomposition of P_5 with respect to the P_1 parabolic of the Levi subgroup $SL(5)$ associated to the vector $q \wedge q \in \mathbb{Z}^5$. This gives the non-maximal parabolic $P_{1,5}$ decomposition

$$\begin{aligned}
\mathfrak{sl}_d &\cong \cdots \oplus (\mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_{d-5} \oplus \mathfrak{sl}_4)^{(0)} \oplus (\overline{\mathbf{4}}_{15-3d} \oplus (\mathbf{4} \otimes \overline{\mathbb{Z}}^{d-5})_{d-3})^{(d)} \oplus (\overline{\mathbb{Z}}^{d-5}_{12-2s})^{(2d)} \\
\mathbb{Z}^d &\cong (\mathbb{Z}^{d-5}_{-2})^{(-6)} \oplus (\mathbf{4}_{d-5})^{(d-6)} \oplus (\mathbf{1}_{10-2d})^{(2d-6)} \\
\wedge^3\mathbb{Z}^d &\cong \cdots \oplus (\overline{\mathbf{4}}_{3d-15} \oplus (\mathbf{4} \otimes \mathbb{Z}^{d-5})_{3-d})^{(3d-18)} \oplus \mathbf{6}^{(4d-18)} \\
\wedge^{5,1}\mathbb{Z}^d &\cong \cdots \oplus (\mathbf{4}_{3d-15} \oplus (\overline{\mathbf{4}} \otimes \mathbb{Z}^{d-5})_{3-d})^{(7d-36)} \oplus \mathbf{1}^{(8d-36)} \\
\wedge^5\mathbb{Z}^d &\cong \cdots \oplus (\overline{\mathbf{4}} \otimes \mathbb{Z}^{d-5}{}_{d-7})^{(5d-30)} \oplus (\mathbf{1}_{2d-10})^{(6d-30)} \\
\wedge^{5,2}\mathbb{Z}^d &\cong \cdots \oplus (\mathbf{4} \otimes \mathbb{Z}^{d-5}{}_{d-7})^{(9d-48)}
\end{aligned} \tag{D.33}$$

where the underscript label gives the weight with respect to the $GL(1)$ stabilizer of $N \in \mathbf{6}^{(2d-18)}$. This decomposition determines the stabilizer of a rank 2 charge N as $(GL(d-5) \times Sp(4,\mathbb{R})) \ltimes \mathbb{R}^{4+4\times(d-5)+(d-5)}$ in the parabolic $P_{1,5}$. The redefinition of the $GL(1)$ factors gives

$$y_\gamma^{\frac{3}{5}}|v(q)| = |Z(N)|\,, \quad y_\gamma^{\frac{6}{5}}|v(q\wedge q)| = |z(N\times N)|\,, \quad y_\gamma = \Big|\frac{2z(N\times N)}{\gcd(N\times N)}\Big|^{\frac{3}{4}\frac{d-5}{d-9}}\nu_N\,, \tag{D.34}$$

where ν_N is the character of $P_{1,5}$ that stabilizes N (to the power $\frac{1}{d-5}$ with respect to the grading displayed above).

Using these formulas and manipulating the Poincaré sums as explain above, one obtains eventually

$$\begin{aligned}
\int_{[0,1]} db E^{E_d}_{s\Lambda_d} = V^{\frac{6}{9-d}s}\Bigg(& E^{SL(d)}_{s\Lambda_{d-1}} + \frac{\xi(2s-d+2)}{\xi(2s)} V^{d-2-2s} E^{SL(d)}_{(s-\frac{d-3}{2})\Lambda_2} \\
& + \frac{\xi(2s-d)\xi(2s-d-2)}{\xi(2s)\xi(2s-d+3)} V^{2d-4s} E^{SL(d)}_{(s-\frac{d}{2})\Lambda_5} \\
& + 2\frac{V^{\frac{d-2}{2}-s}}{\xi(2s)} {\sum_{\substack{N\in\wedge^3\mathbb{Z}^d \\ N\times N=0}}}' \frac{\sigma_{d-2-2s}(N)}{\gcd N^{\frac{2}{3}(d-3-2s)}} E^{SL(3)}_{(s-\frac{d-3}{2})\Lambda_2}(v_N) \frac{K_{s-\frac{d-2}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{2s-d+6}{6}}} e^{2\pi i(N,a)} \\
& + 2\frac{\xi(2s-d)V^{\frac{3d-2}{2}-3s}}{\xi(2s)\xi(2s-d+3)} {\sum_{\substack{N\in\wedge^3\mathbb{Z}^d \\ N\times N=0}}}' \frac{\sigma_{d+2-2s}(N)}{\gcd N^{\frac{d-5}{d-3}(d-2s)}} E^{SL(d-3)}_{(s-\frac{d}{2})\Lambda_2}(v_N) \frac{K_{s-\frac{d+2}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{(d-7)s+\frac{d(9-d)}{2}-3}{d-3}}} e^{2\pi i(N,a)} \\
& + 2\tfrac{V^{d-2s}}{\xi(2s)\xi(2s-d+3)} {\sum_{\substack{N\in\wedge^3\mathbb{Z}^d \\ N\times N\neq 0}}}' \sum_{n|N} \frac{n^{d+2-2s}\sigma_{d-2s}(\frac{N\times N}{n^2})}{\gcd(N\times N)^{\frac{d-5}{d-9}\frac{3(d-2s)}{4}}} \frac{B_{\frac{3}{2},s-\frac{d}{2}}(V^2|Z(N)|^2, V^2|z(N\times N)|)}{\nu_N^{2s-d}|z(N\times N)|^{\frac{d+3}{9-d}\frac{d-2s}{4}}} e^{2\pi i(N,a)} \Bigg)
\end{aligned} \tag{D.35}$$

This expression is complete for $d = 5, 6$, but for the particular values $s = \frac{d-3}{2}$ and $s = \frac{d+1}{2}$ corresponding to the minimal and the next-to-minimal representation one expects that it is also complete for $d = 7$ and 8. For $s = \frac{d+1}{2}$ this gives

$$\xi(4)\xi(d+1)\int db E^{E_d}_{\frac{d+1}{2}\Lambda_d} = V^{3\frac{d+1}{9-d}}\Bigg(\xi(4)\xi(d+1)E^{SL(d)}_{\frac{d+1}{2}\Lambda_{d-1}} + \frac{\xi(2)}{V^2}\xi(5)E^{SL(d)}_{\frac{5}{2}\Lambda_1} + \frac{\xi(3)}{V^3}\xi(4)E^{SL(d)}_{2\Lambda_2}$$
$$+ \frac{2}{V^{\frac{3}{2}}}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}{}' \frac{\sigma_3(N)}{\gcd N^{\frac{1}{3}}}\xi(4)E^{SL(3)}_{2\Lambda_2}(v_N)\frac{K_{\frac{3}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{7}{6}}}e^{2\pi i(N,a)}$$
$$+ \frac{2}{V^{\frac{5}{2}}}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}{}' \frac{\sigma_1(N)}{\gcd N^{\frac{5-d}{d-3}}}\xi(2)E^{SL(d-3)}_{\Lambda_1}(v_N)\frac{K_{\frac{1}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{3d-13}{2d-6}}}e^{2\pi i(N,a)}$$
$$+ \frac{2}{V}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N\neq 0}}{}' \sum_{n|N} n\sigma_{-1}(\tfrac{N\times N}{n^2})\frac{K_1(2\pi V\sqrt{|Z(N)|^2+2|z(N\times N)|})}{\nu_N|\frac{2z(N\times N)}{\gcd(N\times N)}|^{\frac{3}{4}\frac{5-d}{9-d}}\sqrt{|Z(N)|^2+2|z(N\times N)|}}e^{2\pi i(N,a)}\Bigg) \quad \text{(D.36)}$$

that indeed agrees with the constant terms $\int da\int db E^{E_d}_{\frac{5}{2}\Lambda_1}$ [76].

For $d=3$ or 4 the formula simplifies to the three terms

$$\xi(4)\xi(d+1)E^{E_d}_{\frac{d+1}{2}\Lambda_d} = V^{3\frac{d+1}{9-d}}\Bigg(\xi(4)\xi(d+1)E^{SL(d)}_{\frac{d+1}{2}\Lambda_{d-1}} + \frac{\xi(3)}{V^3}\xi(4)E^{SL(d)}_{2\Lambda_2}$$
$$+ \frac{2}{V^{\frac{3}{2}}}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}{}' \frac{\sigma_3(N)}{\gcd N^{\frac{1}{3}}}\xi(4)E^{SL(3)}_{2\Lambda_2}(v_N)\frac{K_{\frac{3}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{7}{6}}}e^{2\pi i(N,a)}\Bigg) \quad \text{(D.37)}$$

But for $d\leq 5$ one must also include the Eisenstein series $\zeta(5)E^{E_d}_{\frac{5}{2}\Lambda_1}$ to obtain the non-pertubative coupling $\mathcal{E}_{(1,0)}$. It simply reduces to $\zeta(5)E^{SL(3)}_{\frac{5}{2}\Lambda_1}$ for $d=3$ and can be written for $d=4,5$ as

$$\xi(2s)E^{E_d}_{s\Lambda_1} = V^{\frac{12}{9-d}s}\Bigg(\xi(2s)V^{-2s}E^{SL(d)}_{s\Lambda_1} + \xi(2s-4)V^{4-4s}E^{SL(d)}_{(s-\frac{3}{2})\Lambda_4}$$
$$+ 2V^{2-3s}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}{}' E^{SL(d-3)}_{(s-\frac{3}{2})\Lambda_1}(v'_N)\frac{\sigma_{4-2s}(N)}{\gcd(N)^{\frac{3}{2}-s}}\frac{K_{s-2}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{1}{2}}}e^{2\pi i(N,a)}\Bigg) \quad \text{(D.38)}$$

provided one defines $E^{SL(1)}_{s\Lambda_1} = E^{SL(4)}_{s\Lambda_4} = 1$. Summing terms up for $d=3,4$ one obtains

$$\mathcal{E}_{(1,0)\,\epsilon} = 8\pi\Big(\xi(4)\xi(d+1+2\epsilon)E^{E_d}_{\frac{d+1+2\epsilon}{2}\Lambda_d} + \xi(2)\xi(5-2\epsilon)E^{E_d}_{(\frac{5}{2}-\epsilon)\Lambda_1}\Big) \quad \text{(D.39)}$$
$$= 8\pi V^{3\frac{d+1+2\epsilon}{9-d}}\Bigg(\xi(4)\xi(d+1)E^{SL(d)}_{\frac{d+1}{2}\Lambda_{d-1}} + \frac{\xi(2)}{V^2}\xi(5)E^{SL(d)}_{\frac{5}{2}\Lambda_1}$$
$$+V^{-3-2\epsilon}\Big(\xi(4)\xi(3)E^{SL(d)}_{(2+\epsilon)\Lambda_2} + \xi(2)\xi(1-2\epsilon)V^{\frac{12\epsilon}{5}}\delta_{d,4}\Big)$$
$$+2\xi(2)V^{-\frac{5}{2}}\sum_{N\in\wedge^3\mathbb{Z}^d}{}' \sigma_1(N)\frac{K_{\frac{1}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{1}{2}}}e^{2\pi i(N,a)}$$
$$+2\xi(4)V^{-\frac{3}{2}}\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}{}' \frac{\sigma_3(N)}{\gcd N^{\frac{1}{3}}}E^{SL(3)}_{2\Lambda_2}(v_N)\frac{K_{\frac{3}{2}}(2\pi V|Z(N)|)}{|Z(N)|^{\frac{7}{6}}}e^{2\pi i(N,a)}\Bigg),$$

and for $d=5$

$$\begin{aligned}
\mathcal{E}_{(1,0)\,\epsilon} &\underset{d=5}{=} 8\pi\Big(\xi(4)\xi(6+2\epsilon)E^{E_5}_{(3+\epsilon)\Lambda_5}+\xi(2)\xi(5-2\epsilon)E^{E_5}_{(\frac{5}{2}-\epsilon)\Lambda_1}\Big)\\
&= 8\pi V^{3\frac{6+2\epsilon}{4}}\Bigg(\xi(4)\xi(6)E^{SL(5)}_{(3+\epsilon)\Lambda_4}+V^{-3-2\epsilon}\big(\xi(4)\xi(3)E^{SL(5)}_{(2+\epsilon)\Lambda_2}+\xi(2)\xi(1-2\epsilon)V^{\frac{3\epsilon}{2}}E^{SL(5)}_{(1-\epsilon)\Lambda_4}\big)\\
&\qquad\qquad+\frac{\xi(2)}{V^{2+4\epsilon}}\big(\xi(5-\epsilon)V^{\frac{3\epsilon}{2}}E^{SL(5)}_{(\frac{5}{2}-\epsilon)\Lambda_1}+\xi(1+2\epsilon)\big)\\
&+2V^{-\frac{5}{2}}{\sum_{N\in\wedge^3\mathbb{Z}^d}}'\Big(\xi(2)V^{\frac{7\epsilon}{2}}E^{SL(2)}_{(1-\epsilon)\Lambda_1}\tfrac{\sigma_{1+2\epsilon}(N)}{\gcd(N)^{2\epsilon}}\tfrac{K_{\frac12+\epsilon}(2\pi V|Z(N)|)}{|Z(N)|^{\frac12}}+\xi(1+2\epsilon)V^{-6\epsilon}\sigma_{1-2\epsilon}(N)\tfrac{K_{\frac12-\epsilon}(2\pi V|Z(N)|)}{|Z(N)|^{\frac12-\epsilon}}\Big)e^{2\pi i(N,a)}\\
&\qquad+\frac{2}{V^{\frac32}}{\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N=0}}}'\frac{\sigma_3(N)}{\gcd N^{\frac13}}\xi(4)E^{SL(3)}_{2\Lambda_2}(v_N)\frac{K_{\frac32}(2\pi V|Z(N)|)}{|Z(N)|^{\frac76}}e^{2\pi i(N,a)}\\
&\qquad+\frac{2}{V}{\sum_{\substack{N\in\wedge^3\mathbb{Z}^d\\ N\times N\neq0}}}'\sum_{n|N}\frac{n^3\sigma_1(\frac{N\times N}{n^2})}{\gcd(N\times N)}\frac{K_1(2\pi V\sqrt{|Z(N)|^2+2|z(N\times N)|})}{\sqrt{|Z(N)|^2+2|z(N\times N)|}}e^{2\pi i(N,a)}\Bigg)\,. \qquad \text{(D.40)}
\end{aligned}$$

One checks that the constant terms that are power-low in V are consistent with the Langlands constant term formula [38].

E Expansion of the 2-loop exceptional field theory amplitude

In this Appendix we complete the computation explained in Section 5.3. We have already carried out the sum over pairs of charges Γ_i for which all brane charges vanish and for which only the M2-brane charges are non-zero but linearly dependent. They correspond to the first and second layers of charges. We shall now consider the other layers of charges.

E.1 Third layer

The third layer of charges includes M2 charges that are linearly independent. The antisymmetric product representation is $[\wedge^2\mathbb{R}^d]\wedge[\wedge^2\mathbb{R}^d]\cong\wedge^{3,1}\mathbb{R}^d$ so one needs a non-maximal parabolic subgroup Poincaré sum over $P_{1,3}\subset SL(d)$, where

$$P_{1,3}=S(GL(1)\times GL(2)\times GL(d-3))\ltimes\mathbb{Z}^{2\times(d-3)+d-3+2} \qquad \text{(E.1)}$$

and

$$\begin{aligned}
\theta_d^{(3)} &= {\sum_{\substack{n_i^{IJ}\in\wedge^2\mathbb{Z}^d\\ n_i^{[IJ}n_j^{KL]}=0\\ n_{[i}^{IJ}n_{j]}^{KL}\neq 0}}}' \sum_{\substack{m_{iI}\in\mathbb{Z}^d\\ n_i{}^{IJ}m_{jJ}=0}} e^{-\pi\Omega_2^{ij}\left(r^{-3}U^{-1IJ}(m_{iI}+1/2a_{IKL}n_i^{KL})(m_{jJ}+1/2a_{JPQ}n_j^{PQ})+1/2r^{6-d}U_{IK}U_{JL}n_i^{IJ}n_j^{KL}\right)} \\
&= \sum_{\gamma\in P_{1,3}\backslash SL(d)} \sum_{\substack{n_i{}^{\hat{\jmath}}\in\mathbb{Z}^{2\times 2}\\ \det n\neq 0}} \sum_{\substack{q\in\mathbb{Z}\\ p_i\in\mathbb{Z}^{2\times(d-3)}}} e^{-\pi\Omega_2^{ij}r^{6-d}y_1y_3\upsilon_{i\hat{\jmath}}n_i{}^{\hat{\imath}}n_j{}^{\hat{\jmath}}} \\
&\quad \times e^{-\pi\Omega_2^{ij}\left(r^{-3}y_3^{\frac{2}{d-3}}u^{-1}(p_i+a_in_i{}^{\hat{\imath}}+\hat{n}_i{}^{\hat{\imath}}c_i(q+an),p_j+a_{\hat{\jmath}}n_j{}^{\hat{\jmath}}+\hat{n}_j{}^{\hat{\jmath}}c_{\hat{\jmath}}(q+an))+r^{-3}y_1y_3^{-1}\upsilon_{i\hat{\jmath}}\hat{n}_i{}^{\hat{\imath}}\hat{n}_j{}^{\hat{\jmath}}(q+an)^2\right)}\Big|_\gamma \\
&= \sum_{\gamma\in P_{1,3}\backslash SL(d)} \sum_{\substack{n_i{}^{\hat{\jmath}}\in\mathbb{Z}^{2\times 2}\\ \det n\neq 0}} \sum_{\substack{q\in\mathbb{Z}\\ p^i\in\mathbb{Z}^{2\times(d-3)}}} \frac{r^{3d-\frac{15}{2}}y_1^{-\frac{1}{2}}y_3^{-\frac{2}{3}}}{\det\Omega_2^{\frac{d-3}{2}}\sqrt{\Omega_2^{ij}\upsilon_{i\hat{\jmath}}\hat{n}_i{}^{\hat{\imath}}\hat{n}_j{}^{\hat{\jmath}}}}e^{2\pi i(n_i{}^{\hat{\jmath}}p^i,a_i)+2\pi inqa} \\
&\quad \times e^{-\pi\Omega_2^{ij}r^{6-d}y_1y_3\upsilon_{i\hat{\jmath}}n_i{}^{\hat{\imath}}n_j{}^{\hat{\jmath}}-\pi\Omega_{2ij}^{-1}r^3y_3^{-\frac{2}{d-3}}u(p^i,p^j)-\frac{\pi}{\Omega_2^{ij}\upsilon_{i\hat{\jmath}}\hat{n}_i{}^{\hat{\imath}}\hat{n}_j{}^{\hat{\jmath}}}r^3y_1^{-1}y_3(q+c_i\hat{n}_i{}^{\hat{\imath}}p^i)^2}\Big|_\gamma .
\end{aligned} \tag{E.2}$$

The constant terms come from the contribution at $p^i = q = 0$ and one obtains

$$\begin{aligned}
&8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\varphi_{\mathrm{KZ}}^{\mathrm{tr}}\,\theta_d^{(3)} \qquad\qquad (E.3)\\
&= 8\pi\frac{\xi(4\epsilon-1)}{\xi(4\epsilon)}r^{\frac{5d-9}{2}+2(d-6)\epsilon}\sum_{\gamma\in P_{1,3}\backslash SL(d)}y_1^{-2\epsilon}y_3^{-1-2\epsilon}\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{2-\epsilon}}\varphi_{\mathrm{KZ}}^{\mathrm{tr}}\sum_{\substack{n_i{}^{\hat{\jmath}}\in\mathbb{Z}^{2\times 2}\\ \det n\neq 0}}e^{-\pi\Omega_2^{ij}\upsilon_{i\hat{\jmath}}n_i{}^{\hat{\imath}}n_j{}^{\hat{\jmath}}}
\end{aligned}$$

This contribution disappears at $\epsilon \to 0$ in the renormalised coupling (5.31). These are all the contributions for $3 \leq d \leq 4$.

E.2 Fourth layer

The fourth layer includes linearly dependent M5 brane charges n_i^{IJKLP}, that can be rotated to a basis in which n_i^{IJKLP} are integer n_i with $\gamma \in P_5 \subset SL(d)$. For $d = 5$ this is already the case and

the Poincaré sum drops out. The corresponding sum of charges gives

$$\begin{aligned}\theta_d^{(4)} &= \sum_{\gamma\in P_5\backslash SL(d)}\sum_{n\in\mathbb{Z}}{}'\sum_{\substack{q_i\in\wedge^2\mathbb{Z}^5\\ p_i\in\mathbb{Z}^5\\ n_{(i}p_{j)}=q_i\times q_j\\ q_{(i}\cdot p_{j)}=0}}\sum_{m_i\in\mathbb{Z}^{d-5}} e^{-\pi\Omega_2^{ij}r^{-3}y^{\frac{2}{d-5}}\tilde{u}(m_i+bn_i+\ldots,m_j+bn_j+\ldots)}\\ &\times e^{-\pi\Omega_2^{ij}\left(r^{-3}y^{-\frac{2}{5}}u^{-1}(p_i+aq_i+\frac{1}{2}a\times an_i,p_j+aq_j+\frac{1}{2}a\times an_j)+r^{6-d}y^{\frac{4}{5}}u(q_i+an_i,q_j+an_j)+r^{15-2d}y^2n_in_j\right)}\\ &= \sum_{\gamma\in P_5\backslash SL(d)}\frac{r^{3(d-5)}y^{-2}}{\det\Omega_2^{\frac{d-5}{2}}}\sum_{n\in\mathbb{Z}}{}'\sum_{\substack{q_i\in\wedge^2\mathbb{Z}^5\\ p_i\in\mathbb{Z}^5\\ n_{(i}p_{j)}=q_i\times q_j\\ q_{(i}\cdot p_{j)}=0}}\sum_{m^i\in\mathbb{Z}^{d-5}} e^{-\pi\Omega_{2ij}^{-1}r^3y^{-\frac{2}{d-5}}\tilde{u}^{-1}(m^i,m^j)+2\pi i m^i(n_ib+\ldots)}\\ &\times e^{-\pi\Omega_2^{ij}\left(r^{-3}y^{-\frac{2}{5}}u^{-1}(p_i+aq_i+\frac{1}{2}a\times an_i,p_j+aq_j+\frac{1}{2}a\times an_j)+r^{6-d}y^{\frac{4}{5}}u(q_i+an_i,q_j+an_j)+r^{15-2d}y^2n_in_j\right)}\end{aligned} \tag{E.4}$$

so that the contributions from $m^i\neq 0$ are non-abelian. To compute the constant term we can therefore set $m^i=0$ in this formula. This gives the Poincaré sum of the contribution for $d=5$:

$$8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\varphi^{\mathrm{tr}}_{\mathrm{KZ}}\,\theta_d^{(4\mathrm{ab})}=8\pi r^{\frac{3}{2}(d-7)-\frac{3}{2}(d-5)\epsilon}\sum_{\gamma\in P_5\backslash SL(d)}y^{\epsilon-1}\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{1-\epsilon}}\varphi^{\mathrm{tr}}_{\mathrm{KZ}}\,\theta_5^{(4)}(r\to r^{\frac{9-d}{4}}y^{\frac{3}{10}})\,, \tag{E.5}$$

The computation of the large T^5 volume is identical to the large circle limit for $d=5$. This computation was done in [27] and the entire expansion is reproduced by the term computed above from the first three layers. We conclude that this term should vanish at $\epsilon\to 0$. The Poincaré sum could in principle brings an additional pole. The computation of [27] also exhibits that for high layers the computation gives the same result as for the Eisenstein series in the Heisenberg parabolic coming from the three-loop exceptional field theory amplitude. We check using Langlands constant term formula that these contribution indeed vanish for $d=5,6,7,8$, which strongly suggests that the Poincaré sum does not bring in an additional pole that could give a finite contribution at $\epsilon\to 0$.

E.3 Fifth layer

For $d\geq 6$ one considers then linearly independent M5 brane charges n_i^{IJKLP}, that can be rotated to non-degenerate two by two matrices $n_i{}^{\hat{j}}$ under $P_{4,6}\backslash SL(d)$. One obtains the solution

$$n_i^{IJKLP} = (\ldots,0,n_i{}^{\hat{j}})\in\cdots\oplus(\wedge^3\mathbb{Z}^4)^{(-\frac{1}{2},2)}\oplus(\mathbb{Z}^2)^{(1,1)} \tag{E.6}$$

$$n_i^{IJ} = (\ldots,0,n_i{}^{\hat{j}}\tfrac{\chi}{k},q_i)\in\cdots\oplus(\mathbb{Z})^{(-2,2)}\oplus(\mathbb{Z}^{2\times 4})^{(-\frac{1}{2},1)}\oplus(\wedge^2\mathbb{Z}^4)^{(1,0)} \tag{E.7}$$

$$m_{iI} = (q_i\cdot\tfrac{\chi}{k},m_i{}^{\hat{j}},p_i)\in(\overline{\mathbb{Z}}^4)^{(-\frac{1}{2},0)}\oplus(\mathbb{Z}^2)^{(1,-1)}\oplus(\overline{\mathbb{Z}}^{d-6})^{(0,\frac{2}{d-6})} \tag{E.8}$$

where k is coprime to $\chi_a\in\mathbb{Z}^4$ and k divides $n_i{}^{\hat{j}}$ and $q_i^{ab}\chi_b$. The only remaining constraint is

$$2\varepsilon_{i\hat{j}}n_{(i}{}^{\hat{i}}m_{j)}{}^{\hat{j}}+\frac{1}{4}\varepsilon_{abcd}q_i^{ab}q_j^{cd}=0 \tag{E.9}$$

that is recognised as the vanishing norm condition for vectors $Q_i=(m_i{}^{\hat{i}},q_i^{ab},n_i{}^{\hat{i}})\in I\!I_{5,5}$. Similarly as in [27] we shall interpret the sum over k and χ_a as the principal layer of a Poincaré sum over

$P_1\backslash SL(5)$. The sum over Q_i will be computed using the orbit method for the $SO(5,5)$ Narain theta series.

The bilinear form decomposes as

$$\begin{aligned} G(\Gamma_i,\Gamma_j) &= r^{-3}y_6^{\frac{2}{d-6}}\tilde{u}(p_i+b_i{n_i}^{\hat{i}}+\dots,p_j+b_j{n_j}^{\hat{j}}+\dots) \\ &\quad +r^{6-d}y_4\Big(y_6(r^{9-d}+y_4^{-\frac{3}{2}}u(\tfrac{\chi}{k}+a,\tfrac{\chi}{k}+a))\upsilon_{\hat{i}\hat{j}}{n_i}^{\hat{i}}{n_j}^{\hat{j}} \\ &\qquad +(u_{ac}u_{bd}+r^{d-9}y_4^{-\frac{3}{4}}u_{ac}(\tfrac{\chi_b}{k}+a_b)(\tfrac{\chi_d}{k}+a_d))(q_i^{ab}+\tilde{a}_{\hat{j}}^{ab}{n_i}^{\hat{j}})(q_j^{ab}+\tilde{a}_{\hat{j}}^{cd}{n_j}^{\hat{j}}) \\ &\quad +r^{d-9}\upsilon_{\hat{i}\hat{j}}({m_i}^{\hat{i}}+\tfrac{1}{2}\tilde{a}_{ab}^{\hat{i}}q_i^{ab}+\tfrac{1}{4}\tilde{a}_{ab}^{\hat{i}}\tilde{a}_{\hat{k}}^{ab}{n_i}^{\hat{k}}+b{n_i}^{\hat{i}},{m_j}^{\hat{j}}+\tfrac{1}{2}\tilde{a}_{cd}^{\hat{j}}q_j^{cd}+\tfrac{1}{4}\tilde{a}_{cd}^{\hat{j}}\tilde{a}_{\hat{l}}^{cd}{n_j}^{\hat{l}}+b{n_j}^{\hat{j}})\Big) \end{aligned} \tag{E.10}$$

where

$$\tilde{a}_{ab}^{\hat{i}}=\frac{1}{2}\varepsilon_{abcd}\varepsilon^{\hat{i}\hat{j}}\tilde{a}_{\hat{j}}^{cd}=a_{ab}^{\hat{i}}+c_a^{\hat{i}}\tfrac{\chi_b}{k}-c_b^{\hat{i}}\tfrac{\chi_a}{k}\ . \tag{E.11}$$

One first perform the Poisson summation over p_i, as

$$\begin{aligned} &\sum_{p_i\in\mathbb{Z}^{d-6}} e^{-\pi\Omega_2^{ij}r^{-3}y_6^{\frac{2}{d-6}}\tilde{u}(p_i+b_i{n_i}^{\hat{i}}+\dots,p_j+b_j{n_j}^{\hat{j}}+\dots)} \\ &= \frac{r^{3(d-6)}y_6^{-2}}{\det\Omega_2^{\frac{d-6}{2}}}\sum_{p^i\in\mathbb{Z}^{d-6}} e^{-\pi\Omega_{2ij}^{-1}r^3y_6^{-\frac{2}{d-6}}\tilde{u}^{-1}(p^i,p^j)+2\pi ip^i({n_i}^{\hat{i}}b_i+\dots)}\ . \end{aligned} \tag{E.12}$$

The terms with $p^i\neq 0$ only contribute to non-abelian Fourier coefficients. Concentrating on abelian Fourier coefficients one obtains

$$\begin{aligned} &8\pi\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}}\varphi_{\mathrm{KZ}}^{\mathrm{tr}}\,\theta_d^{(5\mathrm{ab})} \\ &= 8\pi r^{(6+2\epsilon)(d-6)}\sum_{\gamma\in P_{4,6}\backslash SL(d)}\sum_{(k,\chi)}\frac{y_4^{-3-2\epsilon}y_6^{-2}}{\big(1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}\big)^{\frac{3}{2}+\epsilon}}\int_{\mathcal{G}_2}\frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{1}{2}-\epsilon}}\varphi_{\mathrm{KZ}}^{\mathrm{tr}}\sum_{\substack{Q_i\in I\!I_{5,5}\\ \det n\neq 0\\ (Q_i,Q_j)=0}} e^{-\pi\Omega_2^{ij}g(Q_i,Q_j)}\,, \end{aligned} \tag{E.13}$$

with

$$\begin{aligned} g(Q_i,Q_j) &= r^{9-d}y_6\sqrt{1+\tfrac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}}\upsilon_{\hat{i}\hat{j}}n^{\hat{i}}{n_j}^{\hat{j}} \\ &\quad +\frac{u^{-1}\times u^{-1}+r^{9-d}y_4^{-\frac{3}{2}}u\times(\frac{\chi}{k}+a)^2}{\sqrt{1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}}}(q_i+\tilde{a}_i{n_i}^{\hat{i}},q_j+\tilde{a}_{\hat{j}}{n_j}^{\hat{j}}) \\ &\quad +\frac{r^{d-9}y_6^{-1}}{\sqrt{1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}}}\upsilon_{\hat{i}\hat{j}}({m_i}^{\hat{i}}+(\tilde{a}^{\hat{i}},q_i)+\tfrac{1}{2}(\tilde{a}^{\hat{i}},\tilde{a}_{\hat{k}})n_i^{\hat{k}}+b{n_i}^{\hat{i}})({m_j}^{\hat{j}}+(\tilde{a}^{\hat{j}},q_j)+\tfrac{1}{2}(\tilde{a}^{\hat{j}},\tilde{a}_{\hat{l}})n_i^{\hat{l}}+b{n_j}^{\hat{j}})\ . \end{aligned} \tag{E.14}$$

Using the orbit method as in [27] one obtains

$$
\begin{aligned}
& 8\pi \int_{\mathcal{G}_2} \frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}} \varphi_{\mathrm{KZ}}^{\mathrm{tr}}\, \theta_d^{(5\mathrm{ab})} \qquad\qquad (E.15)\\
&= 8\pi r^{(6+2\epsilon)(d-6)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} \sum_{(k,\chi)} \frac{y_4^{-3-2\epsilon} r^{2(9-d)}}{\left(1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}\right)^{\frac{1}{2}+\epsilon}} \int_{\mathcal{G}_2} \frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{3}{2}}} \sum_{\substack{\delta\in P_2\backslash Sp(4,\mathbb{Z})\\ \det C\neq 0}} \left(\det\Omega_2^{\epsilon}\varphi_{\mathrm{KZ}}^{\mathrm{tr}}\right)\big|_{\delta}\\
&\times \sum_{\substack{m^i{}_{\hat{\imath}}\in\mathbb{Z}^2\\ \det m\neq 0\\ q_i\in\Pi_{3,3}}} e^{-\pi\Omega_{2ij}^{-1} r^{9-d} y_6 \sqrt{1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}} \upsilon^{\hat{\imath}\hat{\jmath}} m^i{}_{\hat{\imath}} m^j{}_{\hat{\jmath}} - \pi\Omega_2^{ij} \frac{u^{-1}u^{-1}+r^{9-d}y_4^{-\frac{3}{2}}u^{-1}(\frac{\chi}{k}+a)^2}{\sqrt{1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}}} (q_i+\tilde{a}_i n_i{}^{\hat{\imath}}, q_j+\tilde{a}_j n_j{}^{\hat{\jmath}}) + 2\pi i m^i{}_{\hat{\imath}} (q_i,\tilde{a}^{\hat{\imath}})}\\
&\sim 20\xi(3) r^{(6+2\epsilon)(d-6)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} \sum_{(k,\chi)} \frac{y_4^{-3-2\epsilon} y_6^{-2}}{\left(1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}\right)^{\frac{3}{2}+\epsilon}} \int_{\mathcal{G}_2} \frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{5}{2}}} E^{SL(2)}_{2\epsilon\Lambda_1}(\tau) \sum_{\substack{m^i{}_{\hat{\imath}}\in\mathbb{Z}^2\\ \det m\neq 0}} e^{-\pi\Omega_{2ij}^{-1}\upsilon^{\hat{\imath}\hat{\jmath}} m^i{}_{\hat{\imath}} m^j{}_{\hat{\jmath}}}\\
&= \frac{40\xi(3)\xi(1+2\epsilon)\xi(2-2\epsilon)}{\xi(3+2\epsilon)} r^{(6+2\epsilon)(d-6)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} \sum_{\substack{k\geq 1\\ \chi\in\mathbb{Z}^4}} y_4^{-3-2\epsilon} y_6^{-2} E^{SL(2)}_{2\epsilon\Lambda_1}(\upsilon) \int_0^\infty \frac{dt}{t^{\frac{5}{2}+\epsilon}} e^{-\frac{\pi}{t}\left(k^2+\frac{r^{d-9}}{y_4^{\frac{3}{2}}}u(\chi+ak,\chi+ak)\right)}\\
&\sim \frac{40\xi(3)\xi(1+2\epsilon)\xi(2-2\epsilon)}{\xi(3+2\epsilon)} r^{(6+2\epsilon)(d-6)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} y_4^{-3-2\epsilon} y_6^{-2} E^{SL(2)}_{2\epsilon\Lambda_1}(\upsilon) \xi(-1+2\epsilon) y_4^3 r^{2(9-d)}\\
&\sim 40\xi(1+2\epsilon)\xi(2)^2 r^{(6+2\epsilon)(d-6)+2(9-d)} E^{SL(d)}_{2\epsilon\Lambda_5+(1-\epsilon)\Lambda_6}
\end{aligned}
$$

For the complete integrand in (5.31) one gets instead the contribution

$$
\begin{aligned}
&\sim 20\xi(3-2\epsilon) r^{6(d-6)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} \sum_{(k,\chi)} \frac{y_4^{-3} y_6^{-2+2\epsilon} r^{-2\epsilon(d-9)}}{\left(1+\frac{u(\frac{\chi}{k}+a,\frac{\chi}{k}+a)}{r^{9-d}y_4^{\frac{3}{2}}}\right)^{\frac{3}{2}-\epsilon}} \int_{\mathcal{G}_2} \frac{\mathrm{d}^3\Omega_2}{\det\Omega_2^{\frac{5}{2}-\epsilon}} \sum_{\substack{m^i{}_{\hat{\imath}}\in\mathbb{Z}^2\\ \det m\neq 0}} e^{-\pi\Omega_{2ij}^{-1}\upsilon^{\hat{\imath}\hat{\jmath}} m^i{}_{\hat{\imath}} m^j{}_{\hat{\jmath}}}\\
&= 40\xi(1-2\epsilon)\xi(2-2\epsilon) r^{6(d-6)-2\epsilon(d-9)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} \sum_{\substack{k\geq 1\\ \chi\in\mathbb{Z}^4}} y_4^{-3} y_6^{-2+2\epsilon} \int_0^\infty \frac{dt}{t^{\frac{5}{2}-\epsilon}} e^{-\frac{\pi}{t}\left(k^2+\frac{r^{d-9}}{y_4^{\frac{3}{2}}}u(\chi+ak,\chi+ak)\right)}\\
&\sim 40\xi(1-2\epsilon)\xi(2-2\epsilon) r^{6(d-6)-2\epsilon(d-9)} \sum_{\gamma\in P_{4,6}\backslash SL(d)} y_4^{-3} y_6^{-2+2\epsilon} \xi(-1-2\epsilon) y_4^3 r^{2(9-d)}\\
&\sim 40\xi(1-2\epsilon)\xi(2-2\epsilon)\xi(2+2\epsilon) r^{6(d-6)+2(1+\epsilon)(9-d)} E^{SL(d)}_{(1-\epsilon)\Lambda_6}
\end{aligned}
$$

where we have [34]

$$
\begin{aligned}
E^{SL(6)}_{(1-\epsilon)\Lambda_6} &\equiv 1\\
\xi(2-2\epsilon) E^{SL(7)}_{(1-\epsilon)\Lambda_6} &= \xi(5+2\epsilon) E^{SL(7)}_{(\frac{5}{2}+\epsilon)\Lambda_1}\\
\xi(1-2\epsilon)\xi(2-2\epsilon) E^{SL(8)}_{(1-\epsilon)\Lambda_6} &= \xi(6+2\epsilon)\xi(5+2\epsilon) E^{SL(8)}_{(3+\epsilon)\Lambda_2}\,. \qquad (E.16)
\end{aligned}
$$

[34] For $SL(6)$ the Poincaré sum is ill-defined as one gets $\sum_{\gamma\in P_4\backslash SL(d)} 1$, so one needs to introduce an additional analytic continuation to compute $\sum_{\gamma\in P_4\backslash SL(d)} y_4^{-2\delta} = 1+\mathcal{O}(\delta)$. This can be done using $\det\Omega_2^{\delta} E^{SL(2)}_{-3+2\epsilon}$ and taking first the limit $\delta\to 0$ and then $\epsilon\to 0$.

E.4 Additional layers

For $d = 7$ and $d = 8$ there are additional layers: two more for $d = 7$ and eight more for $d = 8$. The Langlands consitant term formula gives no more constant terms for the corresponding E_7 Eisenstein series $E^{E_7}_{(6+\epsilon)\Lambda_1}$ so one expects that the two additional layers give vanishing contributions to the abelian Fourier coefficients at $\epsilon \to 0$. For $d = 8$ the E_8 function must include one additional contribution that is missing at this level, probably from the ninth layer.

F Some integrals

Here we collect some additional integrals that have been used.

F.1 Infrared regularised integrals at two-loop

Using the parametrisation

$$\Omega_2 = \begin{pmatrix} \rho_2 & \rho_2 u \\ \rho_2 u & t + \rho_2 u^2 \end{pmatrix} , \tag{F.1}$$

the integral can be unfolded to Schwinger parameter space domain

$$\begin{aligned} I(d) \;&=\; \frac{2\pi^2}{9} \int_0^\infty \frac{d\rho_2}{\rho_2^{\frac{5-d}{2}}} \int_0^1 du \int_{\rho_2 u(1-u)}^\infty \frac{dt}{t^{\frac{5-d}{2}}} \Big(1 + \frac{\rho_2}{t}\big((1-6u(1-u)\big) + \frac{5\rho_2^2}{t^2} u^2 (1-u)^2\Big) e^{-\pi t \mu_1^2 - \pi \rho_2 \mu_2^2} \\ &= \frac{64\pi^2}{9} \frac{d(d-2)\Gamma(d-3)\Gamma(\frac{d-1}{2})\Gamma(\frac{d+3}{2})}{(d-3)(7-d)\Gamma(d+2)} (\pi\mu_2^2)^{3-d} + \frac{2\pi^2}{9} \Gamma(\tfrac{d-3}{2})^2 (\pi\mu_1\mu_2)^{\frac{3-d}{2}} \\ &\qquad + \frac{\pi^2}{27} \Gamma(\tfrac{d-7}{2}) (\pi\mu_1^2)^{\frac{7-d}{2}} \Gamma(\tfrac{d+1}{2}) (\pi\mu_2^2)^{-\frac{d+1}{2}} + \mathcal{O}(\mu_1^2) \; . \end{aligned} \tag{F.2}$$

The double pole divergence occurs at $d = 3 + 2\epsilon$ and using the prescription $\mu_1 = \mu_2 = \mu$ after having eliminated the second ligne that vanishes at $\mu_1 \to 0$, one obtains

$$I(3+2\epsilon) = \frac{2\pi^2}{9} \frac{3-\epsilon}{2-\epsilon} \frac{\Gamma(\epsilon)^2}{(\pi\mu^2)^{2\epsilon}} \; . \tag{F.3}$$

The double pole is universal, but the single pole depends on the ratio $\frac{\mu_2}{\mu_1}$. One fixes the prescription such that the amplitude is finite.

The degenerate contribution associated to one massive loop and the other massless corresponds to take $\mu_2^2 = r^{-3} U^{-1}[n]$. The first term is then finite, and can be reabsorbed into the nondegenerate contribution to recombine into the $GL(2,\mathbb{Z})$ invariant sum over all non-zero charges with a infrared regulator that is set to zero before one expand in ϵ.

$$
\begin{aligned}
& 8\pi \int_{\mathcal{G}_2} \frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}} \varphi^{\rm tr}_{\rm KZ}(\Omega_2) \sum_{p_i\in\mathbb{Z}^{2\times 3}}{}' e^{-\pi\Omega_2^{ij} r^{-3} n_i^\intercal U^{-1} n_j - \pi R(\mu)} \\
= \;& \frac{2\pi^2}{9} \int_0^\infty \frac{d\rho_2}{\rho_2^{\frac{5-d}{2}}} \int_0^1 du \int_{\rho_2 u(1-u)}^\infty \frac{dt}{t^{\frac{5-d}{2}}} \Big(1 + \frac{\rho_2}{t}\big((1-6u(1-u)\big) + \frac{5\rho_2^2}{t^2} u^2(1-u)^2\Big) \\
& \times \Bigg(\sum_{\substack{p_i\in\mathbb{Z}^{2\times 3} \\ p_i\neq 0,\, p_1+p_2\neq 0}}{}' e^{-\pi r^{-3}\left(\rho_2 u^{-1}(p_1+up_2, p_1+up_2) + t u^{-1}(p_2,p_2)\right)} + 3 \sum_{n\in\mathbb{Z}^d}{}' e^{-\pi\rho_2 r^{-3} U^{-1}[n] - \pi t\mu^2} + e^{-\pi(\rho_2+t)\mu^2} \Bigg) \\
= \;& 8\pi r^{3(d-3)} \int_{\mathcal{G}_2} \frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}}} \varphi^{\rm tr}_{\rm KZ}(\Omega_2) \sum_{p_i\in\mathbb{Z}^{2\times 3}}{}' e^{-\pi\Omega_2^{ij} n_i^\intercal U^{-1} n_j} + \frac{4\pi^2}{3} \frac{\Gamma(\frac{d-3}{2})}{(\pi\mu^2)^{\frac{d-3}{2}}} \xi(d-3) r^{\frac{3}{2}(d-3)} E^{SL(d)}_{\frac{d-3}{2}\Lambda_{d-1}} \\
& + \frac{2\pi^2}{9} \frac{9-d}{7-d} \frac{\Gamma(\frac{d-3}{2})^2}{(\pi\mu^2)^{d-3}} + \frac{2\pi^2}{9} \frac{\Gamma(\frac{d-7}{2})}{(\pi\mu^2)^{\frac{d-7}{2}}} \xi(d+1) r^{\frac{3}{2}(d+1)} E^{SL(d)}_{\frac{d+1}{2}\Lambda_{d-1}} \,,
\end{aligned}
\tag{F.4}
$$

F.2 Divergences of the supergravity amplitude

One defines

$$
\mathcal{E}^{SL(d)}_{(0,1),\epsilon} = 8\pi \int_{\mathcal{G}_2} \frac{d^3\Omega_2}{\det\Omega_2^{\frac{7-d}{2}-\epsilon}} \varphi^{\rm tr}_{\rm KZ}(\Omega_2) \sum_{n_i\in\mathbb{Z}^d}{}' e^{-\pi\Omega_2^{ij} n_i^\intercal U^{-1} n_j} \,. \tag{F.5}
$$

In the limit P_{d-1} this gives

$$
\begin{aligned}
\mathcal{E}^{SL(d)}_{(0,1),\epsilon} \;=\; & \frac{16\pi^2\xi(d-3+2\epsilon)^2}{(7-d-2\epsilon)(d+2\epsilon)} y^{-2(d-3)-4\epsilon} \\
& + 8\pi y^{-2} \int_{\mathcal{G}_2} \frac{d^3\Omega_2}{\det\Omega_2^{\frac{6-d}{2}-\epsilon}} \varphi^{\rm tr}_{\rm KZ}(\Omega_2) \sum_{p_i\in\mathbb{Z}^{2\times(d-1)}}{}' \sum_{n^i\in\mathbb{Z}^2} e^{-\pi\Omega_2^{ij} y^{-\frac{2}{d-1}} n_i^\intercal U^{-1} n_j - \pi\Omega_{2ij}^{-1} y^{-2} n^i n^j + 2\pi n^i(p_i,a)} \\
\sim \;& \frac{16\pi^2\xi(d-3+2\epsilon)^2}{(7-d-2\epsilon)(d+2\epsilon)} y^{-2(d-3)-4\epsilon} + y^{-2\frac{3-2\epsilon}{d-1}} \mathcal{E}^{SL(d\text{-}1)}_{(0,1),\epsilon} \\
& + \frac{8\pi^2}{3} \xi(d-4+2\epsilon)\xi(d-3+2\epsilon) y^{-d+3-\frac{3+2(d-2)\epsilon}{d-1}} E^{SL(d\text{-}1)}_{(\frac{d-4}{2}+\epsilon)\Lambda_{d-2}} \\
& + \frac{4\pi^2}{9} \xi(d+2\epsilon)\xi(d-7+2\epsilon) E^{SL(d\text{-}1)}_{(\frac{d}{2}+\epsilon)\Lambda_{d-2}} \\
& - \frac{16\pi^2}{21} y^{-d+3-\frac{3+2(d-2)\epsilon}{d-1}} \sum_{n\in\mathbb{Z}^{d-1}}{}' [\sigma_{d-4+2\epsilon}(n)]^2 \frac{B_{d-4+2\epsilon}(2\pi y^{-\frac{d}{d-1}} \sqrt{U^{-1}[n]})}{[U^{-1}[n]]^{\frac{d-4}{2}+\epsilon}} \,.
\end{aligned}
\tag{F.6}
$$

One computes then that it diverges for $d=3$ as

$$
\mathcal{E}^{SL(3)}_{(0,1),\epsilon} = \frac{16\pi^2\xi(2\epsilon)^2}{(4-2\epsilon)(3+2\epsilon)} E^{SL(3)}_{2\epsilon\Lambda_2} + \mathcal{O}(\epsilon^0) \,, \tag{F.7}
$$

and for $d=7$ with

$$
\mathcal{E}^{SL(7)}_{(0,1),\epsilon} = \frac{2\pi^2}{9} \Gamma(\epsilon)\xi(8+2\epsilon) E^{SL(7)}_{(4+\epsilon)\Lambda_6} + \mathcal{O}(\epsilon^0) \,. \tag{F.8}
$$

F.3 Automorphic distribution integrals

We have the integrals

$$\begin{aligned} I_s[\phi] &= \int_{\mathcal{G}} \frac{d^3\Omega}{|\Omega|^{\frac{3-s}{2}}} \phi(\tau) \sum_{n_i\in\mathbb{Z}}{}' e^{-\pi\Omega^{ij}n_i n_j} \\ &= 4\xi(2s) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \phi(\tau) E^{SL(2)}_{s\Lambda_1}(\tau) \end{aligned} \tag{F.9}$$

for automorphic distributions $\phi(\tau)$ satisfying

$$\Delta\phi(\tau) = \lambda\phi(\tau) - \tau_2 \sum_k \lambda_k (\tau_2^k + \tau_2^{-k})\delta(\tau_1) \tag{F.10}$$

so that they behave at $\tau_1 \approx 0$ as

$$\phi(\tau) = \phi(i\tau_2) - \tau_2^{-1} \sum_k \lambda_k (\tau_2^k + \tau_2^{-k})|\tau_1| + \mathcal{O}({\tau_1}^2) \tag{F.11}$$

for $k = k_0, k_0 - 2, \ldots$ and grow as $\tau_2^{k_0+1}$ at large τ_2. The integral can be carried out by integration by part as

$$\begin{aligned} & (s(s-1) - \lambda) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \phi(\tau) E^{SL(2)}_{s\Lambda_1}(\tau) \\ = & -\sum_k \lambda_k \int_1^L \frac{d\tau_2}{\tau_2} (\tau_2^k + \tau_2^{-k}) E^{SL(2)}_{s\Lambda_1}(i\tau_2) \\ = & -\sum_k \frac{2\lambda_k}{\zeta(2s)} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Big(\frac{m^{-2s}}{s+k} {}_2F_1(s, \tfrac{s+k}{2}{+}1, \tfrac{s+k}{2}{+}1, -\frac{n^2}{m^2}) + \tfrac{n^{-2s}}{s-k} {}_2F_1(s, \tfrac{s-k}{2}{+}1, \tfrac{s-k}{2}{+}1, -\tfrac{m^2}{n^2}) \Big) \\ = & -\sum_k \frac{\lambda_k}{\zeta(2s)} \frac{\Gamma(\frac{s-k}{2})\Gamma(\frac{s+k}{2})}{\Gamma(s)} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{-s-k} m^{-s+k} \\ = & -\sum_k \lambda_k \frac{\xi(s-k)\xi(s+k)}{\xi(2s)} \end{aligned} \tag{F.12}$$

from which one concludes that

$$I_s[\phi] = \sum_k \frac{4\lambda_k}{\lambda - s(s-1)} \xi(s+k)\xi(s-k) \,. \tag{F.13}$$

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