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Hypocoercivité, régularité et limites hydrodynamiques de quelques équations cinétiques

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**HABILITATION À DIRIGER
DES RECHERCHES**

DE L'UNIVERSITÉ PSL

Présentée à l'École normale supérieure

**Hypocoercivité, régularité et limites hydrodynamiques
de quelques équations cinétiques**

Présentation des travaux par

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Le 02 juillet 2024

Discipline

Mathématiques



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Avant-propos

Ce mémoire constitue une synthèse de mes travaux de recherche effectués depuis la fin de mon post-doctorat en septembre 2016. Ces travaux concernent différentes équations de la théorie cinétique des gaz et plus précisément, mes résultats s'articulent autour de deux grands axes :

- développer une théorie de Cauchy dans un cadre perturbatif de solutions fortes et étudier le comportement en temps grand des solutions,
- étudier des problèmes de limites hydrodynamiques.

Qu'elles s'inscrivent dans l'un ou l'autre de ces deux axes, les études menées reposent systématiquement sur l'analyse de problèmes linéaires au moyen de différentes théories plus ou moins récentes : théorie des semi-groupes, théorie spectrale, hypocoercivité, élargissement d'espace, argument de perturbation, hypoellipticité etc... Ainsi, nous appliquons et/ou étendons ces techniques à de nouvelles situations : EDP de nouvelle nature, équations discrètes, nouveau cadre fonctionnel notamment. Dans ce préambule, nous expliquons comment s'articulent les différents chapitres de ce mémoire.

Introduction

Chapitre 1. Dans le Chapitre 1, nous présentons la théorie cinétique des gaz de manière générale ainsi que les problèmes de dérivation d'équations fluides à partir d'équations cinétiques. Il s'agit d'introduire des modèles centraux d'un point de vue historique et mathématique, les enjeux qui y sont associés ainsi que les questions auxquelles nous essayons de répondre dans la suite du mémoire.

Chapitre 2. Dans le Chapitre 2, nous présentons des méthodes de preuve qui sont essentielles dans l'argumentation développée dans les chapitres suivants.

- Il s'agit de méthodes qui s'appliquent à des problèmes linéaires (ou éventuellement non linéaires pour des données petites) permettant d'étudier le comportement asymptotique des solutions de l'équation étudiée (dans la Section 2.1), ou bien leur régularité (dans la Section 2.3). Différents cadres fonctionnels peuvent être envisagés grâce à la méthode d'élargissement présentée dans la Section 2.2. Pour illustrer ces méthodes, nous les appliquons à une équation "simple", l'équation de Fokker-Planck cinétique classique (ou de Kolmogorov), et dans un cadre fonctionnel le plus naturel possible.
- Dans la Section 2.4, nous expliquons comment exploiter les résultats obtenus au niveau linéarisé lorsque l'équation de départ étudiée est non linéaire.
- Enfin, pour terminer ce chapitre, dans la Section 2.5, nous présentons quelques difficultés qui apparaissent lorsque l'on veut utiliser ces méthodes pour des problèmes faisant intervenir un petit paramètre, ce qui se produit typiquement dans l'étude de problèmes de limites hydrodynamiques.

Problèmes cinétiques

Chapitre 3. Dans le Chapitre 3, nous étudions des équations de type Fokker-Planck fractionnaire. Dans un cadre homogène en espace, les équations de Fokker-Planck sont des équations qui font intervenir un terme diffusif ainsi qu'un terme de drift, elles modélisent l'interaction entre un système de particules et un milieu ambiant. Précisons que l'équation de Fokker-Planck peut être reliée à l'équation de la chaleur via un changement de variable auto-similaire. L'étude du comportement en temps long des solutions de l'équation de Fokker-Planck peut se faire à l'aide de méthodes robustes qui permettent d'obtenir des informations précises et quantitatives sur les taux de convergence.

- Dans ce mémoire, nous nous penchons sur des équations de Fokker-Planck fractionnaire, ce qui signifie que le terme diffusif de notre équation est donné par un laplacien fractionnaire, de plus, nous nous plaçons dans un cadre cinétique en considérant des distributions de particules non homogènes en espace. Notre but est d’obtenir des résultats quantitatifs de convergence de solutions vers l’équilibre de notre équation. Pour cela, nous utilisons une méthode d’hypocoercivité H^1 présentée dans la Section 2.1. L’enjeu principal est de comprendre comment adapter cette méthode – qui est tout à fait adaptée au cas non fractionnaire (puisque dans ce cas, l’équation fournit des gains de dérivées entières) – au cas fractionnaire. Ceci constitue l’objet de la Section 3.1.
- Dans un deuxième temps, dans la Section 3.2, nous étudions une version discrète de l’équation de Fokker-Planck fractionnaire (d’où l’importance d’avoir une preuve la plus lisible possible dans le cas continu). Plusieurs difficultés apparaissent, nous devons adapter des outils d’analyse fonctionnelle à un cadre discret et fractionnaire et le choix de la discrétisation est essentiel dans le but de nous permettre de mener à bien nos calculs d’hypocoercivité.

Chapitre 4. Dans le Chapitre 4, nous nous intéressons à des équations cinétiques possédant des propriétés diffusives de nature fractionnaire : l’équation de Kolmogorov fractionnaire et l’équation de Boltzmann sans troncature angulaire qui est un modèle physique dans lequel se produisent des interactions binaires entre particules selon un potentiel de type loi de puissance.

- Nous commençons par établir des estimations de régularisation en temps court pour l’équation de Kolmogorov fractionnaire. Il s’agit d’adapter la technique présentée en Section 2.3 à ce cadre fractionnaire, l’enjeu principal est de trouver la bonne fonctionnelle d’entropie et contrairement au résultat d’hypocoercivité fractionnaire présenté dans le Chapitre 3, l’introduction de dérivées fractionnaires semble inévitable.
- Dans la Section 4.2, nous développons une théorie de Cauchy pour une équation de Boltzmann sans troncature angulaire dans un cadre proche de l’équilibre et pour des données à décroissance polynomiale. Dans ce type de modèle sans troncature angulaire, les collisions rasantes sont prises en compte et sont responsables d’une singularité de la partie angulaire du noyau de collision de l’opérateur de Boltzmann. Ceci a pour effet de ne pas pouvoir séparer l’opérateur de collisions en deux parties (gain et perte) mais est aussi à l’origine d’un comportement diffusif de l’opérateur de collisions. En effet, nous verrons au travers de notre étude que l’opérateur linéarisé fournit un gain de régularité anisotropique de dérivée fractionnaire. Pour cela, nous utilisons des outils raffinés de calcul pseudo-différentiel, nos opérateurs ne pouvant rentrer dans un cadre d’étude classique. Grâce à l’analyse de ces propriétés de régularisation en temps court, nous sommes en mesure d’utiliser un argument d’élargissement tel que présenté dans la Section 2.2 et de développer notre théorie de Cauchy pour l’équation non linéaire dans un large cadre fonctionnel.

Chapitre 5. Dans le Chapitre 5, nous établissons un résultat d’hypocoercivité pour une large classe d’équations cinétiques (incluant les équations de Boltzmann et Landau linéarisées) dans des domaines bornés à bord. Ce résultat a pour corollaire immédiat l’obtention d’estimations de décroissance en temps long des solutions des équations étudiées vers l’équilibre associé à la donnée initiale. La spécificité de ce travail est que notre preuve est assez robuste pour pouvoir traiter des conditions au bord de Maxwell générales (incluant la réflexion spéculaire pure) dans un cadre unifié. Il s’agit d’un résultat d’hypocoercivité de type L^2 telle que présentée dans la Section 2.1. L’enjeu principal est ainsi de trouver une bonne fonctionnelle qui pourra être de Lyapunov pour l’équation considérée en se basant sur le fait que ces dernières possèdent naturellement des propriétés de dissipativité au niveau microscopique. Il s’agit ensuite de récupérer de la dissipativité pour la partie macroscopique de la solution et ceci repose sur l’analyse de problèmes elliptiques bien choisis. Nous étudions des équations de Poisson avec des conditions au bord de Robin ou de Neumann mais aussi un système de type Lamé pour lesquels nous établissons des résultats d’existence et de régularité elliptique au moyen d’inégalités de type Poincaré ou Korn.

Problèmes de limites hydrodynamiques

Chapitre 6. Dans le Chapitre 6, nous travaillons sur le lien entre les équations de Boltzmann et Landau et l'équation de Navier-Stokes incompressible.

- Plus précisément, dans la Section 6.1, nous considérons l'équation de Boltzmann avec des interactions de type sphères dures. Pour justifier le lien entre le modèle cinétique et le modèle fluide, nous adoptons un point de vue peu habituel puisque nous prouvons que tant que l'équation fluide a une solution, alors l'équation de Boltzmann aussi. Ainsi, notre approche repose sur l'étude de l'équation de Boltzmann remise à l'échelle mais exploite aussi des propriétés de l'équation limite (de décroissance en temps par exemple). La preuve consiste en le développement d'un argument de type point fixe sur la différence des solutions des équations de Boltzmann et Navier-Stokes. Pour cela, nous donnons une "formulation cinétique" et intégrale de l'équation de Navier-Stokes basée sur une analyse spectrale en Fourier (dans la variable d'espace) de l'équation de Boltzmann. Notre approche permet de traiter dans un cadre unifié les cas des dimensions 2 et 3 dans le tore et l'espace entier, tout cela sans condition de taille sur nos données. Notre preuve fournit un résultat de convergence forte avec taux de convergence explicite (selon la régularité en la variable x de nos données initiales) vers le modèle limite.
- Dans la Section 6.2, nous étudions le lien entre les équations de Landau et Navier-Stokes incompressible. Notre approche est en lien avec celle de la Section 6.1, nous obtenons également un résultat de convergence forte des solutions de l'équation de Landau vers celles de l'équation de Navier-Stokes incompressible avec taux explicite. En revanche, nous ne sommes pas en mesure de développer un argument de type point fixe sur la différence des solutions cinétique et fluide pour des raisons de stabilité d'espaces. En effet, la nature de l'opérateur intégro-différentiel de Landau induit des difficultés techniques supplémentaires. Pour contourner ces difficultés et tout de même obtenir un résultat de convergence forte sans perte de régularité en vitesse, nous établissons des estimations de régularisation en temps court (du type de celles présentées en Section 2.3) pour le problème cinétique linéarisé mais aussi pour le problème non linéaire proche de l'équilibre. Ces estimations ont un intérêt indépendamment du problème de limite hydrodynamique que nous traitons, ce sont les premières du genre dans un cadre non linéaire sur un problème avec petit paramètre, notons aussi que nous obtenons exactement le gain de régularité anisotrope attendu.

Chapitre 7. Dans le Chapitre 7, nous étudions également un problème de limite hydrodynamique mais pour une équation de Boltzmann inélastique. Une des principales caractéristiques de ce modèle est qu'une perte d'énergie s'opère au moment des collisions. Ainsi, notre modèle conserve la masse et la quantité de mouvement au cours du temps mais pas l'énergie (comme cela était le cas pour le modèle élastique). Notons aussi que nous ne disposons pas d'une entropie naturelle comme dans le cas élastique (en particulier, il n'y a pas d'équivalent du théorème-H de Boltzmann pour cette équation et il n'existe donc pas de théorie de type solutions renormalisées). Enfin, un autre point notable est que l'équilibre de notre équation (après changement de variable auto-similaire) n'est pas explicite et est à queue lourde, nous ne disposons pas non plus d'un espace fonctionnel "canonique" dans lequel l'opérateur linéarisé posséderait de bonnes propriétés de coercivité. Une manière de contourner ces problèmes est de se placer dans un cadre perturbatif autour de l'élastique afin de pouvoir exploiter les connaissances bien plus importantes que nous possédons dans ce cas. Dans le Chapitre 7, nous présentons un résultat de limite hydrodynamique de l'équation de Boltzmann inélastique vers un système de Navier-Stokes-Fourier incompressible modifié. Pour cela, nous développons une théorie de Cauchy perturbative pour l'équation remise à l'échelle avec estimations uniformes en le nombre de Knudsen. Une des principales difficultés est d'identifier le bon régime de faible inélasticité qui nous permette à la fois d'obtenir de bonnes estimations uniformes en le nombre de Knudsen et de garder trace de l'inélasticité de l'équation de départ dans le système fluide.

Foreword

This memoir is a synthesis of my research work carried out since the end of my post-doctoral position in September 2016. This work concerns various equations of kinetic theory, and more specifically, my results are structured around two main axes:

- develop a Cauchy theory in a perturbative framework of strong solutions and study the large-time behavior of solutions,
- study hydrodynamic limit problems.

Whether they fall into one or the other of these two areas, the studies carried out are systematically based on the analysis of linear problems using a variety of more or less recent theories: semi-group theory, spectral theory, hypocoercivity, space enlargement, perturbative argument, hypoellipticity... In this way, we apply and/or extend these techniques to new situations: PDEs of new nature, discrete equations, new functional frameworks and so on. In this introduction, we explain how the various chapters of this dissertation fit together.

Introduction

Chapter 1. In Chapter 1, we present kinetic theory in a general way, as well as the problem of deriving fluid equations from kinetic ones. The aim is to introduce central models from a historical and mathematical point of view, the issues associated with them, and the questions we aim to answer in the remainder of the memoir.

Chapter 2. In Chapter 2, we present methods of proof that are essential to the argumentation developed in the following chapters.

- These are methods that apply to linear problems (or possibly nonlinear problems in a framework close to equilibrium), enabling us to study the asymptotic behavior of solutions to the equation under study (in Section 2.1), or their regularity (in Section 2.3). Different functional frameworks can be considered thanks to the method of enlargement presented in Section 2.2. To illustrate these methods, we apply them to a “simple” equation, the classical Fokker-Planck (or Kolmogorov) equation, in the most natural functional setting.
- In Section 2.4, we explain how to exploit the results obtained at the linearized level when the original equation being studied is nonlinear.
- Finally, to conclude this chapter, in Section 2.5, we present some difficulties that arise when we want to use these methods for problems involving a small parameter, which typically occurs in the study of hydrodynamic limit problems.

Kinetic problems

Chapter 3. In Chapter 3, we study Fokker-Planck equations. In a spatially homogeneous framework, Fokker-Planck equations are equations involving a diffusive term as well as a drift term, and model the interaction between a system of particles and a background medium. Let us mention that the Fokker-Planck equation can be related to the heat equation via a self-similar change of variables. The long-time behavior of solutions to the Fokker-Planck equation can be studied using robust methods that provide precise and quantitative information on convergence rates.

- In this memoir, we focus on fractional Fokker-Planck equations, meaning that the diffusive term of our equation is given by a fractional Laplacian, and we place ourselves in a kinetic framework

by considering spatially inhomogeneous distributions of particles. Our aim is to obtain results of convergence of solutions towards equilibrium of our equation. To this end, we use the so-called H^1 -hypo-coercivity method presented in Section 2.1. The main issue is to understand how to adapt this method – which is perfectly adapted to the non-fractional case (since in this case, the equation provides gains of full derivatives) – to the fractional case. This is the subject of Section 3.1.

- Secondly, in Section 3.2, we study a discrete version of the fractional Fokker-Planck equation (hence the importance of having a proof as readable as possible in the continuous case). Several difficulties arise: we have to adapt functional analysis tools to a discrete and fractional framework, and the choice of discretization is essential to enable us to carry out our hypo-coercivity computations.

Chapter 4. In Chapter 4, we focus on kinetic equations with diffusive properties of a fractional nature: the fractional Kolmogorov equation and the Boltzmann equation without angular cut-off, which is a physical model in which binary interactions between particles occur according to a power law potential.

- We begin by establishing short-time regularization estimates for the fractional Kolmogorov equation. The aim is to adapt the technique introduced in Section 2.3 to our fractional framework. The main challenge is to find an adapted entropy functional and unlike in the result presented in Chapter 3, the introduction of fractional derivatives seems unavoidable.
- In Section 4.2, we develop a Cauchy theory for the Boltzmann equation without angular cut-off in a framework close to equilibrium and for data with polynomial decay. In this type of model without angular cut-off, grazing collisions are taken into account and are responsible for a singularity in the angular part of the collision kernel of the Boltzmann operator. This has the effect of not being able to separate the collision operator into two parts (gain and loss), but is also at the origin of a diffusive behavior of the collision operator. In fact, we'll see from our study that the linearized operator provides a gain of anisotropic regularity of fractional derivatives. To achieve this, we use refined pseudo-differential tools, as our operators cannot fit into a classical study framework. Thanks to the analysis of these short-time regularization properties, we are able to use an extension argument as presented in Section 2.2, and to develop our Cauchy theory for the nonlinear equation in a broad functional framework.

Chapter 5. In Chapter 5, we establish a hypo-coercivity result for a large class of kinetic equations (including the Boltzmann and Landau equations) in bounded domains. This result has the immediate corollary of obtaining estimates on the long-time decay of the solutions of the equations under study towards the equilibrium associated with the initial data. The specificity of this work is that our proof is robust enough to handle general Maxwell boundary conditions (including pure specular reflection) in a unified framework. This involves a hypo-coercivity result of L^2 -type as presented in Section 2.1. The main challenge is thus to find a good Lyapunov functional for the considered equation, based on the fact that the latter naturally enjoys dissipative properties at the microscopic level. The next step is to get dissipativity on the macroscopic part of the solution and this relies on the analysis of well-chosen elliptic problems. We study Poisson equations with Robin or Neumann boundary conditions, as well as a Lamé-type system, for which we establish existence and elliptic regularity results by means of Poincaré or Korn-type inequalities.

Hydrodynamic limit problems

Chapter 6. In Chapter 6, we focus on the link between the Boltzmann and Landau equations with the incompressible Navier-Stokes equation.

- More precisely, in Section 6.1, we consider the Boltzmann equation for hard spheres interactions. To justify the link between the kinetic model and the fluid model, we adopt an unusual point of view, proving that as long as the fluid equation has a solution, so does the Boltzmann equation. Thus, our

approach is based on the study of the rescaled Boltzmann equation, but also exploits properties of the limit equation (of time decay, for example). The proof consists in the development of a fixed-point argument on the difference between the solutions of the Boltzmann and Navier-Stokes equations. To this end, we give a kinetic and integral formulation of the Navier-Stokes equation using a Fourier spectral analysis (in the space variable) of the Boltzmann equation. Our approach allows us to treat in a unified framework the 2 and 3 dimensional cases in the torus and the whole space, all without any size condition on our data. Our proof provides a strong convergence result with explicit convergence rate (depending on the regularity in the variable x of our initial data) to the limit model.

- In Section 6.2, we study the link between the Landau and incompressible Navier-Stokes equations. Our approach is related to that of Section 6.1, we also obtain a result of strong convergence of the solutions of the Landau equation to those of the incompressible Navier-Stokes equation with explicit rate. However, we are unable to develop a fixed-point argument on the difference between kinetic and fluid solutions for reasons of space stability. Indeed, the nature of Landau’s integro-differential operator induces additional technical difficulties. To circumvent these difficulties and still obtain a strong convergence result without loss of velocity regularity, we establish short-time regularization estimates (of the type presented in Section 2.3) for both the linearized kinetic problem and the nonlinear problem close to equilibrium. These estimates are of interest independently of the hydrodynamic limit problem we are dealing with, and are the first of their kind in a nonlinear framework on a problem involving a small parameter. Note also that we obtain exactly the expected anisotropic gain of regularity.

Chapter 7. In Chapter 7, we also study a hydrodynamic limit problem, but for an inelastic Boltzmann equation. One of the main features of this model is that energy is lost during collisions. As a result, our model preserves mass and momentum over time, but not energy (as was the case with the elastic model). Note also that we don’t have a natural entropy as in the elastic case (in particular, there is no equivalent of Boltzmann’s H-theorem and thus no renormalized solutions theory for our equation). Finally, another noteworthy point is that the equilibrium of our equation (after a self-similar change of variables) is not explicit and is heavy-tailed, nor do we have a canonical function space in which the linearized operator has nice coercivity properties. A way to circumvent these difficulties is to place ourselves in a perturbative framework around the elastic case in order to exploit the much greater knowledge we possess in this case. In Chapter 7, we present a hydrodynamic limit result for the inelastic Boltzmann equation to a modified incompressible Navier-Stokes-Fourier system. To this end, we develop a perturbative Cauchy theory for the rescaled equation with uniform estimates in the Knudsen number. One of the main difficulties is to identify the right weak inelasticity regime that will allow us both to obtain uniform estimates in the Knudsen number and to keep track of the inelasticity of the starting equation in the fluid system.

List of publications

Here is a list of publications of the author, they are presented in anti-chronological order of publication, and the reference numbers are the same as in the general bibliography presented at the end of the memoir.

Papers presented in this memoir

- [15] ALONSO, R., LODS, B., AND TRISTANI, I. Fluid dynamic limit of Boltzmann equation for granular hard-spheres in a nearly elastic regime. *arXiv preprint arXiv:2008.05173*, to appear in *Mém. Soc. Math. Fr.* (2024).
- [33] BERNOU, A., CARRAPATOSO, K., MISCHLER, S., AND TRISTANI, I. Hypocoercivity for kinetic linear equations in bounded domains with general Maxwell boundary condition. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 40, 2 (2023), 287–338.
- [25] AYI, N., HERDA, M., HIVERT, H., AND TRISTANI, I. On a structure-preserving numerical method for fractional Fokker-Planck equations. *Math. Comp.* 92, 340 (2023), 635–693.
- [59] CARRAPATOSO, K., RACHID, M., AND TRISTANI, I. Regularization estimates and hydrodynamical limit for the Landau equation. *J. Math. Pures Appl.* (9) 163 (2022), 334–432.
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- [103] GALLAGHER, I., AND TRISTANI, I. On the convergence of smooth solutions from Boltzmann to Navier-Stokes. *Ann. H. Lebesgue* 3 (2020), 561–614.
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- [136] HÉRAU, F., TONON, D., AND TRISTANI, I. Short time diffusion properties of inhomogeneous kinetic equations with fractional collision kernel. *arXiv preprint arXiv:1709.09943*, (2018).

Other papers not presented in this memoir

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Introduction

General introduction

1.1 Kinetic theory

1.1.1 General introduction

The purpose of kinetic theory is to describe a system made of a large number of indistinguishable particles such as a gas or a plasma. Such a system can be described through different ways: the *microscopic description* intends to analyze the trajectories of each particle (such a description is not reachable in practice due to the large number of particles that is considered), the *macroscopic description* intends to describe the evolution of macroscopic or observable quantities such as the mass, the mean velocity or the temperature, finally the *mesoscopic description* is an intermediary description which analyzes the typical behavior of particles. Kinetic equations lie at this level of description and the state of the system is described through the evolution of the density of particles $F = F(t, x, v) \geq 0$ that at time $t \in \mathbf{R}^+$ and position $x \in \Omega \subseteq \mathbf{R}^d$ have the velocity $v \in \mathbf{R}^d$.

The general form of a kinetic equation is the following:

$$\partial_t F + v \cdot \nabla_x F + \mathcal{F}[F] \cdot \nabla_v F + \mathcal{F}_{\text{ext}} \cdot \nabla_v F = \mathcal{C}(F).$$

In this equation, there is the free transport operator $v \cdot \nabla_x$, the term $\mathcal{F}[F] = \mathcal{F}[F](t, x)$ represents the fields that are generated by the distribution F itself and the term \mathcal{F}_{ext} represents an exterior force field. In this memoir, we won't take into account the self-induced force fields neither the exterior ones. The typical equation that we will study will be of the following form:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F). \quad (1.1.1)$$

The term $\mathcal{C}(F)$ represents the collisions that can occur between particles or with a background medium and we suppose that the operator \mathcal{C} only acts on the velocity variable. The Boltzmann, Landau and Fokker-Planck equations that will be studied in this memoir fit into this framework. For the sake of simplicity, we shall only consider the case $\Omega = \mathbf{T}^d$ in the whole introduction (Chapter 1). For a thorough introduction to the subject, we refer to Cercignani's and Villani's books [61, 205].

1.1.2 The Boltzmann equation

The spatially inhomogeneous Boltzmann equation derived by Boltzmann [37] after pioneer works by Maxwell [167] describes the evolution of a particle density $F = F(t, x, v)$ at time $t \in \mathbf{R}^+$, position $x \in \Omega$ and velocity $v \in \mathbf{R}^d$ and reads

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q_B(F, F) & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ F(0) = F_{\text{in}} & \text{in } \Omega \times \mathbf{R}^d. \end{cases} \quad (1.1.2)$$

If Ω is a more general bounded domain than the torus, the equation has to be complemented by boundary conditions (see Chapter 5).

The Boltzmann operator. In this introduction, we only present the elastic Boltzmann collision operator (the inelastic one will be introduced in Chapter 7) which is an integral operator defined as

$$Q_B(g, f) := \int_{\mathbf{R}^d \times \mathcal{S}^{d-1}} B(v - v_*, \sigma) (g'_* f' - g_* f) \, d\sigma \, dv_*. \quad (1.1.3)$$

Here and below, we are using the shorthand notations $f = f(v)$, $g_* = g(v_*)$, $f' = f(v')$ and $g'_* = g(v'_*)$. In this expression, v , v_* and v' , v'_* are the velocities of a pair of particles after and before collision. We make a choice of parametrization of the set of solutions to the conservation of momentum and energy (physical laws of elastic collisions):

$$\begin{aligned} v + v_* &= v' + v'_* \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2 \end{aligned} \quad (1.1.4)$$

so that the pre-collisional velocities are given by:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in \mathbf{S}^{d-1}.$$

The Boltzmann collision kernel $B = B(v - v_*, \sigma)$ only depends on the relative velocity $|v - v_*|$ and on the deviation angle ϑ through $\cos \vartheta = \langle v - v_*, \sigma \rangle / |v - v_*|$ where $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbf{R}^d . By a symmetry argument, one can always reduce to the case where $B(v - v_*, \sigma)$ is supported on $\langle v - v_*, \sigma \rangle \geq 0$ i.e. $0 \leq \vartheta \leq \pi/2$. So, without loss of generality, we shall make this assumption throughout this memoir.

The form of the collision kernel depends on the type of collisions that occur between particles. In dimension $d = 3$, in the case where particles behave such as billiard balls, the hard-spheres case, the collision kernel is proportional to the norm of the relative velocity, namely

$$B(v - v_*, \sigma) = C|v - v_*|, \quad C > 0,$$

we will suppose C equal to 1 in Section 6.1 for the sake of simplicity. When particles interact through inverse power law potentials of type

$$\phi(r) = r^{-(p-1)} \quad \text{with} \quad p \in (2, +\infty), \quad (1.1.5)$$

the collision kernel cannot be computed explicitly but Maxwell [167] has shown that the collision kernel can be computed in terms of the interaction potential ϕ . More precisely, in dimension $d = 3$, the kernel B satisfies the following properties.

– It takes product form in its arguments as

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos \vartheta). \quad (1.1.6)$$

– The angular function b is locally smooth, and has a nonintegrable singularity for $\vartheta \rightarrow 0$: it satisfies for some $c_b > 0$ and any $\vartheta \in (0, \pi/2]$,

$$\frac{c_b}{\vartheta^{1+2s}} \leq \sin \vartheta b(\cos \vartheta) \leq \frac{1}{c_b \vartheta^{1+2s}} \quad \text{with} \quad s := \frac{1}{p-1} \in (0, 1). \quad (1.1.7)$$

– The kinetic factor Φ satisfies

$$\Phi(|v - v_*|) = |v - v_*|^\gamma \quad \text{with} \quad \gamma := \frac{p-5}{p-1} \in (-3, 1). \quad (1.1.8)$$

One traditionally calls *hard potentials* the case $p > 5$ (for which $0 < \gamma < 1$), *Maxwell molecules* the case $p = 5$ (for which $\gamma = 0$), *moderately soft potentials* the case where $3 \leq p < 5$ (for which $-2s \leq \gamma < 0$) and *very soft potentials* the case $2 < p < 3$ (for which $-3 < \gamma < -2s$).

The *Grad's cut-off assumption* consists in additionally supposing that the angular kernel b is locally integrable by removing its singularity for small deviation angles ϑ (see (1.1.7)). The corresponding models have been widely studied and one can remark that a first simple consequence of this assumption is that one can split the Boltzmann collision operator defined in (1.1.3) into two parts, the gain one and the loss one.

It is worth mentioning that in the case of Coulomb potential ($s = 2$ and thus $\gamma = -3$), the Boltzmann operator does not make sense (see [205] for example). The Boltzmann operator has then to be replaced by the Landau one which can be obtained in the so-called grazing collision limit after having made a cut-off on Coulomb interaction. This process intuitively corresponds to perform the limit $b \rightarrow \delta_{g=0}$ under some suitable scaling. This limit has been extensively studied, among other works, we can refer to [83, 113, 204, 128] by Desvillettes, Goudon, Villani, He and also to [10] by Alexandre and Villani in which the authors provide the first study of the problem in a spatially inhomogeneous setting. We also mention the more recent work [213] by Yang and Zhou in which the authors give a new point of view of the grazing collision limit by relying on the limit $s \rightarrow 1$ (where s is the parameter of angular singularity defined in (1.1.7)). We refer to Subsection 1.1.3 for more details on the Landau operator.

To end this part, we give a weak formulation of the collision operator Q_B . Using symmetry arguments and the pre-post collisional change of variables, for any suitable test function $\varphi = \varphi(v)$, we have:

$$\begin{aligned} & \int_{\mathbf{R}^d} Q_B(g, g)(v) \varphi(v) dv \\ &= \frac{1}{4} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{S}^{d-1}} B(v - v_*, \sigma) (g'_* g' - g_* g) (\varphi + \varphi_* - \varphi' - \varphi'_*) d\sigma dv_* dv. \end{aligned} \quad (1.1.9)$$

From this formula, one can deduce two of the main features of the Boltzmann collision operator which hold at least formally: the preservation of mass, momentum and energy and the Boltzmann H -theorem.

Conservation laws. The weak formulation given in (1.1.9) and the microscopic laws (1.1.4) imply:

$$\int_{\mathbf{R}^d} Q_B(g, g)(v) \varphi(v) dv = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2, \quad (1.1.10)$$

from which we deduce that a solution F to equation (1.1.2) is conservative, meaning that for any $t \geq 0$,

$$\int_{\Omega \times \mathbf{R}^d} F(t, x, v) \varphi(v) dv dx = \int_{\Omega \times \mathbf{R}^d} F_{\text{in}}(x, v) \varphi(v) dv dx \quad \text{for} \quad \varphi(v) = 1, v, |v|^2.$$

Boltzmann's H-theorem. Let us introduce the Boltzmann entropy

$$\mathcal{H}(g) := \int_{\Omega \times \mathbf{R}^d} g \log(g) dv dx \quad (1.1.11)$$

as well as the entropy production \mathcal{D}_B defined through:

$$\mathcal{D}_B(g) := \frac{1}{4} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{S}^{d-1}} B(v - v_*, \sigma) (g'_* g' - gg_*) \log \frac{g'_* g'}{gg_*} d\sigma dv_* dv. \quad (1.1.12)$$

Boltzmann's H -theorem asserts that the entropy of a solution to the Boltzmann equation is non-increasing along time. More precisely, if F is a solution to (1.1.2), then thanks to (1.1.9), we have

$$\frac{d}{dt} \mathcal{H}(F) = - \int_{\Omega} \mathcal{D}_B(F) dx \leq 0 \quad (1.1.13)$$

so that

$$\mathcal{H}(F)(t) \leq \mathcal{H}(F_{\text{in}}), \quad \forall t \geq 0.$$

The second part of the theorem states that any distribution minimizing the entropy is a local Maxwellian distribution in velocity. More precisely,

$$\mathcal{D}_B(g) = 0 \quad \Leftrightarrow \quad Q_B(g, g) = 0 \quad \Leftrightarrow \quad g = \mu_{\rho, u, \theta}$$

with

$$\mu_{\rho,u,\theta}(t,x,v) := \frac{\rho(t,x)}{(2\pi\theta(t,x))^{\frac{d}{2}}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right) \quad (1.1.14)$$

where $\rho > 0$, $u \in \mathbf{R}^d$ and $\theta > 0$ are respectively the mass, mean velocity and temperature associated to $\mu_{\rho,u,\theta}$. Moreover, if Ω is the torus, global equilibria of (1.1.2) are global Maxwellian distributions that are independent of time t and position x . In the rest of this memoir, we shall denote the global Maxwellian $\mu_{1,0,1}$ by μ :

$$\mu(v) := \frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{|v|^2}{2}\right), \quad \forall v \in \mathbf{R}^d. \quad (1.1.15)$$

Let us mention that Boltzmann's H-theorem does not hold in the inelastic case and moreover, global equilibria of the equation (after a self-similar change of variables) are not explicit, we refer to Chapter 7 for more details on this subject.

1.1.3 The Landau equation

The Landau operator which has been defined in 1936 by Landau [155] (independently of the Boltzmann operator) is used in plasma physics and is an integro-differential operator that is defined as

$$Q_L(g,f)(v) := \partial_{v_i} \int_{\mathbf{R}^d} a_{ij}(v-v_*) \left(g(v_*) \partial_{v_j} f(v) - f(v) \partial_{v_j} g(v_*) \right) dv_*, \quad (1.1.16)$$

where we use the convention of summation of repeated indices. The matrix a_{ij} is symmetric, semi-positive and is given by

$$a_{ij}(v) := |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad -3 \leq \gamma \leq 1. \quad (1.1.17)$$

The Landau equation reads

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q_L(F,F) & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ F(0) = F_{\text{in}} & \text{in } \Omega \times \mathbf{R}^d. \end{cases} \quad (1.1.18)$$

Similarly as for the Boltzmann equation, we have the following classification according to the values of γ : interactions are referred to as *hard potentials* if $\gamma \in (0, 1]$, *Maxwellian molecules* if $\gamma = 0$, *moderately soft potentials* if $\gamma \in [-2, 0)$, *very soft potentials* if $\gamma \in (-3, -2)$ and *Coulomb potential* if $\gamma = -3$. We mention that only the case $\gamma = -3$ is relevant from a physical viewpoint and is the one that has been derived by Landau in [155]. The other cases of γ correspond actually “only” to mathematical models but are interesting for a mathematical point of view since they can be derived rigorously from the corresponding Boltzmann equations.

As for the Boltzmann operator, one can provide a weak formulation of the Landau operator: for any suitable test function $\varphi = \varphi(v)$, we have

$$\begin{aligned} & \int_{\mathbf{R}^d} Q_L(g,g)(v) \varphi(v) dv \\ &= -\frac{1}{2} \int_{\mathbf{R}^d \times \mathbf{R}^d} a_{ij}(v-v_*) g(v) g(v_*) \left(\frac{\partial_{v_i} g(v)}{g(v)} - \frac{\partial_{v_i} g(v_*)}{g(v_*)} \right) \left(\partial_{v_j} \varphi(v) - \partial_{v_j} \varphi(v_*) \right) dv_* dv, \end{aligned}$$

from which we deduce that if F is solution to (1.1.18), then

$$\frac{d}{dt} \int_{\Omega \times \mathbf{R}^d} F(t,x,v) \varphi(v) dv dx = 0 \quad \text{for } \varphi(v) = 1, v, |v|^2.$$

Moreover, the Landau version of the Boltzmann H -theorem asserts that the entropy \mathcal{H} introduced in (1.1.11) is non-increasing along the solutions of (1.1.18). Indeed, at least formally, since a_{ij} is nonnegative, we have the following inequality for the entropy dissipation \mathcal{D}_L :

$$\mathcal{D}_L(g) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} a_{ij}(v - v_*) g(v) g(v_*) \left(\frac{\partial_{v_i} g(v)}{g(v)} - \frac{\partial_{v_i} g(v_*)}{g(v_*)} \right) \left(\frac{\partial_{v_j} g(v)}{g(v)} - \frac{\partial_{v_j} g(v_*)}{g(v_*)} \right) dv_* dv \geq 0.$$

The second part of the H -theorem concerning equilibria of the equation is exactly the same as in the Boltzmann case.

1.1.4 Cauchy theory and trend to equilibrium

One can handle the problem of Cauchy theory for kinetic equations in different frameworks and we only talk here about the elastic Boltzmann equation in order to lighten the presentation, but similar results hold for the Landau equation. We also mention a few results on the inelastic Boltzmann equation for which the literature is more scarce.

- The first theory that has been developed is the *spatially homogeneous* one, meaning that we make the assumption that the density of particles do not depend on the position x that is $F = F(t, v)$. The Boltzmann equation thus reduces to $\partial_t F = Q_B(F, F)$. Another way to see this spatially homogeneous framework is to consider initial data that do not depend on the variable x , the property of being homogeneous in space is then propagated by the Boltzmann equation and it thus amounts to study the equation $\partial_t F = Q_B(F, F)$. The first existence theorem in this setting has been obtained by Carleman [52, 53] for the hard-spheres model. It has then been improved and generalized to other cases by many authors (see [187, 20] respectively by Povzner and Arkeryd for example). Another framework that is linked to this one is the one of *weakly inhomogeneous* solutions initially introduced by Arkeryd, Esposito and Pulvirenti [21]. Notice that such a theory has also been developed for the inelastic Boltzmann equation in [199] by the author of the memoir.
- A theory which is of major importance is the so-called theory of *renormalized solutions* initiated and developed by Di Perna and Lions [88, 89, 163] in the case of cut-off potentials and extended to non cut-off potentials by Alexandre and Villani [9]. This theory allows to construct very weak solutions to the Boltzmann equation in a very general framework. Indeed, the data that are considered are inhomogeneous, only satisfy physical assumptions (namely finite mass, energy and entropy) and no smallness assumption is required. Let us recall the fact that such a theory has not been developed yet for the inelastic Boltzmann equation due to the lack of entropy estimates.
- The Boltzmann equation has also been studied in a *close to vacuum* regime by Illner and Shinbrot [141] in the whole space. We also mention the work by Alonso [11] concerning the inelastic Boltzmann equation in such a regime.
- Finally, let us talk about the *close to equilibrium* framework which is the one on which we will focus in this memoir. As already seen, Boltzmann's H -theorem make us expect the solutions to the Boltzmann equation to converge towards equilibrium. It is thus natural to study the equation with data that are close to equilibrium. This has been the subject of many works. Initially, this theory relied on the spectral study of the linearized operator around equilibrium (this goes back to Hilbert [138], Grad [115] and Carleman [52]). It allowed Ukai [201] to build the first global solutions to Boltzmann equation as well as to prove an exponential stability result. Let us also mention that Guo [119] was then able to deal with many collision operators still in a close to equilibrium framework but by using a different approach, a “nonlinear energy method”. This method is based on the micro-macro decomposition of the particle density and this approach is actually reminiscent of the L^2 -hypocoercivity method that will be presented below in Section 2.1.

As already mentioned, Boltzmann's H -theorem predicts that solutions to the Boltzmann equation converge towards equilibrium as time goes to infinity. Several methods have been used to prove rigorously this convergence that we briefly recall here.

- *Nonlinear entropy methods.* The first approach, initiated at the beginning of the nineties, is to quantify the entropy production at the nonlinear level thanks to suitable functional inequalities that link the entropy and dissipation of entropy functionals respectively defined in (1.1.11) and (1.1.12) for the Boltzmann equation, see for example the works [54] by Carlen and Carvalho, [82] by Desvillettes, [196] by Toscani and Villani, [84, 86] by Desvillettes and Villani, [49] by Caceres, Carrillo and Goudon.
- *Linear methods.* They rely on the study of the linearized operator around the Maxwellian equilibrium from a spectral viewpoint and/or at the level of semigroups. This analysis can be carried out thanks to non constructive methods, by using compactness perturbation arguments (see for example [201] by Ukai) but also thanks to quantitative arguments such as hypocoercivity (see the next Subsection for more details).

Notice that for a nonlinear equation, the linear strategy above mentioned is only relevant in a close to equilibrium regime. It is thus natural to wonder if one can connect the nonlinear and the linear approaches in order to get a “complete” result of convergence towards equilibrium (by using the nonlinear theory far from equilibrium and the linearized one close to equilibrium). This is actually a major challenge because these studies are not compatible in terms of functional frameworks. The enlargement theory (presented in Section 2.2) initiated by Mouhot in [180] and developed in [117] by Gualdani, Mischler and Mouhot allows to make this connection.

1.1.5 Close to equilibrium study

In this memoir, we will mainly focus on linear equations or nonlinear ones in a close to equilibrium framework. In those cases, our strategy will be to study the associated linearized problem. We thus introduce the collisional linearized operator \mathcal{L} defined as

$$\mathcal{L} := Q(\mu, \cdot) + Q(\cdot, \mu) \quad (1.1.19)$$

where Q stands for the elastic Boltzmann or the Landau operator and μ is the Maxwellian distribution defined in (1.1.15). In those two cases (see for example [61]), we have

$$\text{Ker } \mathcal{L} = \text{Span} \{ \mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu \}$$

and the projection onto the kernel of \mathcal{L} is given by

$$\pi_{\mathcal{L}} g(v) = \left(\int_{\mathbf{R}^d} g(w) dw + \left(\int_{\mathbf{R}^d} g(w) w dw \right) \cdot v + \left(\int_{\mathbf{R}^d} g(w) \frac{|w|^2 - d}{d} dw \right) \frac{|v|^2 - d}{2} \right) \mu(v). \quad (1.1.20)$$

We then introduce the microscopic and macroscopic parts of a kinetic distribution, this will be particularly useful when we will develop the L^2 -hypocoercivity method (in Chapter 5) as well as in the study of hydrodynamical problems (in Chapters 6 and 7). For a given kinetic distribution $g = g(x, v)$, we denote by g^\perp its microscopic part, namely

$$g^\perp := (\text{Id} - \pi_{\mathcal{L}})g \quad (1.1.21)$$

and by $(\rho[g], u[g], \theta[g])$ its first macroscopic quantities defined through

$$\rho[g](x) := \int_{\mathbf{R}^d} g(x, v) dv, \quad (1.1.22)$$

$$u[g](x) := \int_{\mathbf{R}^d} v g(x, v) dv, \quad (1.1.23)$$

and

$$\theta[g](x) := \int_{\mathbf{R}^d} \frac{|v|^2 - d}{d} g(x, v) dv, \quad (1.1.24)$$

so that

$$\pi_{\mathcal{L}}g(x, v) = \left(\rho[g](x) + u[g](x) \cdot v + \theta[g](x) \frac{|v|^2 - d}{2} \right) \mu(v). \quad (1.1.25)$$

When one linearizes equations (1.1.2) and (1.1.18) around the equilibrium μ , using that $Q(\mu, \mu) = 0$, by writing $F = \mu + f$, the equation on the fluctuation f under study is

$$\partial_t f = \Lambda f + Q(f, f), \quad (1.1.26)$$

where Λ is the complete linearized operator given by

$$\Lambda := -v \cdot \nabla_x + \mathcal{L}. \quad (1.1.27)$$

The kernel of Λ is given by (see [61] for example)

$$\text{Ker } \Lambda = \text{Span} \{ \mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu \}$$

and the projector π_{Λ} onto $\text{Ker } \Lambda$ is defined by

$$\pi_{\Lambda}g(v) = \left(\int_{\Omega} \rho[g](x) dx + \int_{\Omega} u[g](x) dx \cdot v + \int_{\Omega} \theta[g] dx \frac{|v|^2 - d}{2} \right) \mu(v). \quad (1.1.28)$$

Notice that $\pi_{\Lambda}g(v) = \int_{\Omega} \pi_{\mathcal{L}}g(x, v) dx$ for any $v \in \mathbf{R}^d$.

In many cases that we shall consider (namely for not too soft potentials in the case of Boltzmann and Landau equations), there exists a Hilbert space H in which the operator \mathcal{L} defined in (1.1.19) is self-adjoint and nonpositive and also satisfies a coercivity estimate of the following type:

$$\langle \mathcal{L}g, g \rangle_H \lesssim -\|g\|_H^2, \quad \forall g \in (\text{Ker } \mathcal{L})^{\perp}. \quad (1.1.29)$$

This Hilbert space is typically given by $H = L_v^2(\mu^{-1/2})$ (where μ is the Maxwellian introduced in (1.1.15), see also Section 1.3 for the definition of weighted spaces) and is the space of linearization of the entropy \mathcal{H} defined in (1.1.11). For the Boltzmann equation, we refer to the works [138, 53, 115] respectively by Hilbert, Carleman and Grad for the first results on the subject and to [79] by Degond and Lemou for the Landau equation. This inequality provides dissipativity in the velocity variable, the question is then to understand if we can also recover some dissipation in the position variable for solutions to equation (1.1.26): this is the object of the *hypocoercivity* theory presented in Section 2.1. Let us underline the fact that we do not have such a coercivity result at hand in the case of the inelastic Boltzmann equation.

Another important preoccupation of our works is to get our results of existence of perturbative solutions and convergence to equilibrium in the largest possible functional spaces. More precisely, a relevant functional space for our data from a physical viewpoint would be $L^1(1 + |v|^2)$ (data with finite mass and energy). We won't be able to reach this goal but we can get some of our results in spaces of type $L^2(1 + |v|^k)$ (even in $L^1(1 + |v|^k)$ in one case) for k large enough. This is much more satisfactory than working in spaces of type $L^2(\mu^{-1/2})$ in which the inequality (1.1.29) is naturally obtained. This type of question will be handled thanks to the *enlargement theory* presented in Section 2.2. Let us point out that this theory of enlargement is crucial to obtain results for the inelastic Boltzmann equation. Indeed, the equilibria of this equation are heavy-tailed so that we cannot study it in spaces of type $L^2(\mu^{-1/2})$. As a consequence, being able to study the elastic Boltzmann equation in large functional spaces allows to consider a perturbative argument around the elastic case in order to study the inelastic Boltzmann equation.

1.2 Hydrodynamical problems

1.2.1 General introduction

In the second part of the memoir (Chapters 6 and 7), we will focus on hydrodynamical problems. More precisely, the goal is to obtain rigorous results of convergence of solutions to kinetic equations towards solutions to fluid equations. This problem can be seen as a part of the program initiated by the 6th problem of Hilbert in 1900 at the International Congress of Mathematicians. Indeed, the question is to understand the link between microscopic and macroscopic descriptions of a fluid and deriving macroscopic equations from mesoscopic ones can be seen as an intermediate step of this program. It is also interesting in practice to identify the regimes in which the macroscopic description is a good approximation of the mesoscopic one. The type of fluid equations that one obtains from kinetic equations depends on the relative sizes of the various physical parameters, the main features of the macroscopic flow thus mainly depends on the balance between the transport and the collision terms. We refer to the book by Saint-Raymond [191] and lecture notes by Golse [110] for a detailed presentation of the subject and the mathematical results in the field.

In this memoir, at the kinetic level, we shall only consider Boltzmann or Landau types collision operators so that in this introduction, the operator Q stands for the Boltzmann operator or the Landau operator that have been defined above (notice that in Chapter 7, we will deal with the *inelastic* Boltzmann equation which is not presented in this introduction and additional terms will appear in the equation due to a self-similar change of variables). The dimensionless version of our kinetic equation reads:

$$\text{St } \partial_t F + v \cdot \nabla_x F = \frac{1}{\text{Kn}} Q(F, F),$$

where the Strouhal number St and the Knudsen number Kn are dimensionless parameters that are natural in the kinetic problems. Moreover, all hydrodynamic models leading to incompressible models are based on a regime in which both the Strouhal and the Knudsen numbers are small. In order to reach the incompressible Navier-Stokes equation, we shall work with $\text{St} = \text{Kn} = \varepsilon \ll 1$ (see for example [30]). Our kinetic equation then reads

$$\begin{cases} \partial_t F^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x F^\varepsilon = \varepsilon^{-2} Q(F^\varepsilon, F^\varepsilon) & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ F^\varepsilon(0) = F_{\text{in}} & \text{in } \Omega \times \mathbf{R}^d. \end{cases} \quad (1.2.1)$$

The Knudsen number is actually proportional to the inverse of the average number of collisions for each particle per unit of time. Taking ε small has thus the effect of enhancing the role of collisions and thus when ε goes to 0, in view of the above mentioned Boltzmann H -theorem, the solution will look more and more like a local Maxwellian as defined in (1.1.14). To relate the Boltzmann equation to the incompressible Navier-Stokes equation, we then look at equation (1.2.1) under the following linearization of order ε :

$$F^\varepsilon(t, x, v) = \mu(v) + \varepsilon f^\varepsilon(t, x, v),$$

where we recall that μ is defined in (1.1.15). In what follows, we shall use the following notations:

$$\mathcal{L} = Q(\cdot, \mu) + Q(\mu, \cdot) \quad \text{and} \quad \Lambda^\varepsilon := -\frac{1}{\varepsilon} v \cdot \nabla_x + \frac{1}{\varepsilon^2} \mathcal{L}. \quad (1.2.2)$$

The equation we are going to study on f^ε is thus the following:

$$\begin{cases} \partial_t f^\varepsilon = \Lambda^\varepsilon f^\varepsilon + \varepsilon^{-1} Q(f^\varepsilon, f^\varepsilon) & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ f^\varepsilon(0) = f_{\text{in}}^\varepsilon := \varepsilon^{-1} (F_{\text{in}} - \mu) & \text{in } \Omega \times \mathbf{R}^d. \end{cases} \quad (1.2.3)$$

We first recall that $\pi_{\mathcal{L}}$ is defined in (1.1.20) and that we also have (see for example [191, 110]):

$$\text{Ker } \Lambda^\varepsilon = \text{Span} \{ \mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu \}$$

and the projector $\pi_{\Lambda^\varepsilon}$ onto $\text{Ker } \Lambda^\varepsilon$ is given by

$$\pi_{\Lambda^\varepsilon} g(v) = \left(\int_{\Omega} \rho[g](x) dx + \int_{\Omega} u[g](x) dx \cdot v + \int_{\Omega} \theta[g](x) dx \frac{|v|^2 - d}{2} \right) \mu(v)$$

where we recall that $\rho[\cdot]$, $u[\cdot]$ and $\theta[\cdot]$ are defined in (1.1.22)-(1.1.23)-(1.1.24). Notice that $\text{Ker } \Lambda^\varepsilon$ and $\pi_{\Lambda^\varepsilon}$ are independent of ε and as previously for π_{Λ} , we have $\pi_{\Lambda^\varepsilon} g(v) = \int_{\Omega} \pi_{\mathcal{L}} g(x, v) dx$ for any $v \in \mathbf{R}^d$.

Coming back to equation (1.2.3) and multiplying it by ε^2 , one immediately sees that formally, f^ε converges towards some distribution $f^0 \in \text{Ker } \mathcal{L}$. Moreover, performing an energy estimate based on the fact that \mathcal{L} satisfies a coercivity estimate of type (1.1.29) in some suitable functional space and on hypocoercivity results (see Section 2.1), then we also get that the microscopic part of the solution f^ε , namely $(f^\varepsilon)^\perp$ is going to tend to 0 as $\varepsilon \rightarrow 0$ in some suitable topology. The macroscopic part of f^ε , namely $\pi_{\mathcal{L}} f^\varepsilon$ is thus going to tend to some distribution f^0 of the form

$$f^0(t, x, v) = \left(\rho[f^0](t, x) + u[f^0](t, x) \cdot v + \theta[f^0](t, x) \frac{|v|^2 - d}{2} \right) \mu(v)$$

where we recall that $\rho[\cdot]$, $u[\cdot]$ and $\theta[\cdot]$ are defined in (1.1.22)-(1.1.23)-(1.1.24). Let us underline the fact that such a formal analysis does not hold in the case of the inelastic Boltzmann equation due to the lack of coercivity estimates of type (1.1.29).

1.2.2 Deriving fluid equations from kinetic ones

We here give a short overview of the existing literature on the problem of deriving fluid equations from kinetic ones (see [191] for more details).

- The first justifications of the link between kinetic and fluid equations were formal and based on *asymptotic expansions* by Hilbert; Chapman, Enskog and Grad (see [138, 67, 114]). The first rigorous convergence proofs based also on asymptotic expansions were given by Caflisch [50] (see also [153] and [76]). In those papers, the limit is justified up to the first singular time for the fluid equation. By using his nonlinear energy method, Guo [121] has justified the limit towards the Navier-Stokes equation and beyond in Hilbert's expansion from Boltzmann and Landau equations.
- There has also been some convergence proofs based on *spectral analysis* in the framework of strong solutions close to equilibrium introduced by Grad [115] and Ukai [201] for the Boltzmann equation. In this respect, we refer to the works by Nishida; Bardos and Ukai [184, 31]. These results use the description of the spectrum of the linearized Boltzmann equation in Fourier space in the space variable performed in [183, 62, 98] by respectively Nicolaenko; Cercignani, Illner and Pulvirenti; Ellis and Pinsky. Our approaches in Chapter 6 are reminiscent of these ones and in Section 6.2, it relies on the generalization of the paper [98] to several kinetic equations (including the Landau one) made in [211] by Yang and Yu. Notice also that such a spectral result has recently been obtained in [107] by Gervais for the hard-spheres Boltzmann equation in a larger class of Sobolev spaces and also more recently, for a large class of kinetic equations in [109] by Gervais and Lods in a large functional framework.
- Finally, let us mention that this problem has been extensively studied in the *framework of weak solutions*, the goal being to obtain solutions for the fluid models from renormalized solutions introduced by DiPerna and Lions in [88] for the Boltzmann equation. We shall not make an extensive presentation of this program as it is out of the realm of this memoir, but let us mention that it was started by Bardos, Golse and Levermore at the beginning of the nineties in [30, 29] and was continued by those authors, Saint-Raymond, Masmoudi, Lions among others. We mention here a (non exhaustive) list of papers which are part of this program [111, 112, 161, 166, 191].
- More recently, some *uniform in ε estimates* on kinetic equations have allowed to prove (at least) weak convergence towards the Navier-Stokes equation. Let us mention [148, 188] in which the cases of Boltzmann equation without cut-off and the Landau equations are treated by Jiang, Xu and Zhao on the

one hand and by Rachid on the other hand. In [43, 46], Briant and Briant, Merino-Aceituno and Mouhot have obtained convergence to equilibrium results for the rescaled Boltzmann equation (and also the Landau equation in [43]) uniformly in the rescaling parameter using respectively hypocoercivity and enlargement methods. In [46], the authors are able to weaken the assumptions on the data down to Sobolev spaces with polynomial weights. We also refer to Chapter 7 in which a similar approach combined with perturbative arguments has been used to derive a fluid system from the inelastic Boltzmann equation. Notice that Briant [43] has combined this with Ellis and Pinsky result [98] to recover strong convergence in the case of the elastic Boltzmann equation.

1.2.3 The incompressible Navier-Stokes-Fourier system

As mentioned above, in this memoir, we are interested in deriving the incompressible Navier-Stokes equation from kinetic models such as Boltzmann and Landau equations. More precisely, the fluid model that we are going to reach in the limit $\varepsilon \rightarrow 0$ is the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation (or a variant of it in the case of the inelastic Boltzmann equation in Chapter 7) which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u - \nu \Delta_x u = -\nabla_x p \\ \partial_t \theta + u \cdot \nabla_x \theta - \kappa \Delta_x \theta = 0 \\ \operatorname{div}_x u = 0 \\ \nabla_x (\rho + \theta) = 0. \end{cases} \quad (1.2.4)$$

In this system, the temperature θ , the density ρ and the pressure p are scalar unknowns, whereas the velocity u is an unknown vector field. The pressure can actually be eliminated from the equations by applying to the momentum equation the Leray projector \mathbb{P} onto the space of divergence-free vector fields (see Section 1.3). To define the viscosity coefficient ν and the heat conductivity κ (see for example [30]), let us introduce the two unique functions Φ (which is a matrix-valued function) and Ψ (which is a vector-valued function) orthogonal to $\operatorname{Ker} \mathcal{L}$ such that

$$\frac{1}{\mu} \mathcal{L}(\mu\Phi) = \frac{|v|^2}{d} \operatorname{Id} - v \otimes v \quad \text{and} \quad \frac{1}{\mu} \mathcal{L}(\mu\Psi) = \left(\frac{d+2}{2} - \frac{|v|^2}{2} \right) v$$

where we recall that \mathcal{L} has been defined in (1.2.2). The viscosity and the heat conductivity coefficients are then defined by

$$\nu := \frac{1}{(d-1)(d+2)} \int_{\mathbb{R}^d} \Phi : \mathcal{L}(\mu\Phi) \, dv \quad \text{and} \quad \kappa := \frac{2}{d(d+2)} \int_{\mathbb{R}^d} \Psi \cdot \mathcal{L}(\mu\Psi) \, dv. \quad (1.2.5)$$

In what follows, we call *well-prepared data* the class of functions $g \in \operatorname{Ker} \mathcal{L}$ that write

$$g(x, v) = \mu(v) \left(\rho[g](x) + u[g](x) \cdot v + \frac{|v|^2 - 3}{2} \theta[g](x) \right) \\ \text{with} \quad \nabla_x \cdot u[g] = 0 \quad \text{and} \quad \rho[g] + \theta[g] = 0 \quad (1.2.6)$$

where we recall that $\rho[\cdot]$, $u[\cdot]$ and $\theta[\cdot]$ are defined in (1.1.22), (1.1.23), (1.1.24). In other words, a well-prepared data is a macroscopic distribution whose associated macroscopic quantities satisfy the Boussinesq relation and the incompressibility condition. On the one hand, the fact that these data belong to the kernel of \mathcal{L} ensures that they already are in a state of local thermodynamic equilibrium and therefore, there is no relaxation layer to be expected; on the other hand, the conditions on the macroscopic quantities of these data allow to avoid the appearance of acoustic waves in the system that could induce an initial layer.

1.3 Notations

- Let X_1, X_2 be Banach spaces and consider a linear operator $\mathcal{P} : X_1 \rightarrow X_2$. When defined, we shall denote by $(S_{\mathcal{P}}(t))_{t \geq 0} = (e^{t\mathcal{P}})_{t \geq 0}$ the semigroup generated by \mathcal{P} . Moreover, we denote by $\mathcal{B}(X_1, X_2)$ the space of bounded linear operators from X_1 to X_2 and by $\|\cdot\|_{\mathcal{B}(X_1, X_2)}$ its norm operator and we shall use the usual simplification $\mathcal{B}(X_1) = \mathcal{B}(X_1, X_1)$.
- If \mathcal{P} is an unbounded linear operator, we denote by $D(\mathcal{P})$ its domain.
- If X_1 and X_2 are two function spaces, we say that a function g belongs to $X_1 + X_2$ if there are $g_1 \in X_1$ and $g_2 \in X_2$ such that $g = g_1 + g_2$ and we define

$$\|g\|_{X_1+X_2} := \min_{\substack{g = g_1 + g_2 \\ g_i \in X_i}} (\|g_1\|_{X_1} + \|g_2\|_{X_2}).$$

- Let us now introduce the notations that we will use for our weighted Lebesgue and Sobolev functional spaces. For any nonnegative measurable weight function $m : \mathbf{R}^d \rightarrow \mathbf{R}^+$ (notice that all the weights we consider in this memoir will depend only on velocity, i.e. $m = m(v)$), we define the Lebesgue space $L_v^q L_x^p(m)$ for $1 \leq p, q \leq +\infty$ through its norm

$$\|g\|_{L_v^q L_x^p(m)} := \left\| \|g(\cdot, v)\|_{L_x^p} m(v) \right\|_{L_v^q}. \quad (1.3.1)$$

- We also consider the standard higher-order Sobolev generalizations $W_v^{\sigma, q} W_x^{s, p}(m)$ for any $\sigma, s \in \mathbf{N}$ defined by the norm

$$\|g\|_{W_v^{\sigma, q} W_x^{s, p}(m)} := \sum_{\substack{0 \leq s' \leq s, 0 \leq \sigma' \leq \sigma, \\ s' + \sigma' \leq \max(s, \sigma)}} \left\| \|\nabla_x^{s'} \nabla_v^{\sigma'} g(\cdot, v)\|_{L_x^p} m(v) \right\|_{L_v^q}. \quad (1.3.2)$$

This definition reduces to the usual weighted Sobolev space $W_{x, v}^{s, p}(m)$ when $q = p$ and $\sigma = s$. For $m \equiv 1$, we simply denote the associated spaces by $L_v^q L_x^p$ and $W_v^{\sigma, q} W_x^{s, p}$. When $p = 2$, we use the classical notation $W^{s, 2} = H^s$.

- We also introduce fractional Sobolev spaces. For $s \in (0, 1)$, the homogeneous space \dot{H}_v^s is defined by the norm

$$\|g\|_{\dot{H}_v^s}^2 := \int_{\mathbf{R}^d \times \mathbf{R}^d} \frac{|g(v) - g(w)|^2}{|v - w|^{d+2s}} dw dv \quad (1.3.3)$$

and the space H_v^s is defined by the norm $\|\cdot\|_{H_v^s} := \|\cdot\|_{L_v^2} + \|\cdot\|_{\dot{H}_v^s}$.

- We introduce the notation $\langle z \rangle := (1 + |z|^2)^{1/2}$ for any $z \in \mathbf{R}^d$.
- For a suitable kinetic distribution $g = g(x, v)$, we denote by $\langle g \rangle$ its local mass, namely

$$\langle g \rangle(x) := \int_{\mathbf{R}^d} g(x, v) dv, \quad \forall x \in \Omega$$

and by $\langle\langle g \rangle\rangle$ its global mass:

$$\langle\langle g \rangle\rangle := \int_{\Omega \times \mathbf{R}^d} g(x, v) dv dx.$$

- We use the notation Div_z for the divergence of a matrix which is defined as follows: for a $d \times d$ -matrix $B = B(z) = (B_{i, j}(z))_{1 \leq i, j \leq d} \in M_{d \times d}(\mathbf{R})$, the vector $\text{Div}_z B \in \mathbf{R}^d$ is defined by

$$(\text{Div}_z B)_i := \sum_{1 \leq j \leq d} \partial_{z_j} B_{i, j}, \quad 1 \leq i \leq d.$$

- For a vector field $b = (b_i)_{1 \leq i \leq d} : \Omega \rightarrow \mathbf{R}^d$, we define its symmetric gradient through

$$\nabla_x^s b := \frac{1}{2} (\partial_j b_i + \partial_i b_j)_{1 \leq i, j \leq d}. \quad (1.3.4)$$

We also introduce its skew-symmetric gradient by

$$\nabla_x^a b := \frac{1}{2} (\partial_j b_i - \partial_i b_j)_{1 \leq i, j \leq d}. \quad (1.3.5)$$

- We denote by \mathbb{P} the Leray projector onto the space of divergence free vector fields. To define it on the torus or the whole space, we consider a vector field $b = (b_i)_{1 \leq i \leq d} : \Omega \rightarrow \mathbf{R}^d$ with $\Omega = \mathbf{T}^d$ or \mathbf{R}^d . Then $\mathbb{P}b$ is defined as follows: for $\xi \in \mathbf{Z}^d$ (if $\Omega = \mathbf{T}^d$) or $\xi \in \mathbf{R}^d$ (if $\Omega = \mathbf{R}^d$),

$$\mathcal{F}_x(\mathbb{P}b)_j(\xi) = \sum_{k=1}^d \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \mathcal{F}_x(b_k)(\xi), \quad \forall 1 \leq j \leq d,$$

where \mathcal{F}_x is the Fourier transform in the variable $x \in \Omega$. Recall that the projector \mathbb{P} is bounded over H_x^ℓ for all ℓ , and in L_x^p for all $1 < p < \infty$ (see for example [26]).

- We introduce a truncation function $\chi \in \mathcal{D}(\mathbf{R})$ which satisfies $\mathbf{1}_{[-1,1]} \leq \chi \leq \mathbf{1}_{[-2,2]}$ and we denote $\chi_a(\cdot) := \chi(\cdot/a)$ for $a > 0$.
- We shall use the same notation C for positive constants depending only on fixed numbers or abbreviate “ $\leq C$ ” by “ \lesssim ”.

Presentation of useful methods

In the three first sections of this chapter, we present linear methods that will be very useful in the following chapters. More precisely,

- some of our works consist in the development of these methods in new frameworks;
- in some cases, we use these methods or variants of them to prove intermediate results;
- some of our works are based on results that have been obtained in previous works thanks to these methods.

We will illustrate these methods on simple cases, the (classical) Fokker-Planck or Kolmogorov equations in convenient functional spaces. In the last two subsections, we explain how to go back to nonlinear problems when one has used these linear methods to handle a linearized problem and finally, we explain how to adapt these methods for equations with a small parameter (this will be the case when we study hydrodynamical problems).

2.1 Hypocoercivity

In this subsection, we consider the equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{L}f & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ f(0) = f_{\text{in}} & \text{in } \Omega \times \mathbf{R}^d, \end{cases} \quad (2.1.1)$$

where \mathcal{L} is a linear operator that only acts on the velocity variable. Moreover, we suppose that there exists a Hilbert space H in which \mathcal{L} is self-adjoint and nonpositive and also satisfies a coercivity estimate of the following type:

$$\langle \mathcal{L}g, g \rangle_H \lesssim -\|g\|_H^2, \quad \forall g \in (\text{Ker } \mathcal{L})^\perp. \quad (2.1.2)$$

In other words, there is a natural Lyapunov functional associated to the operator \mathcal{L} but the set of its extrema is bigger than the set of global equilibria of equation (2.1.1). This type of inequality only provides dissipation in the velocity variable. Hypocoercivity methods allow to recover dissipativity in the whole phase space and rely on the understanding of the interplay between the transport operator $-v \cdot \nabla_x$ which is conservative and the linear collision operator \mathcal{L} which is dissipative in the velocity variable (see (2.1.2)). The goal is to construct a Lyapunov functional for the whole problem whose extrema are exactly the global equilibria of our equation. Note that the transport and the collision operators do not commute, which gives non trivial mixing properties, and that the transport operator does not preserve collision invariants. The macroscopic quantities are thus mixed inside the spatial domain by the transport operator.

To simplify the discussion in what follows, let us say that the above mentioned Hilbert space in (2.1.2) is the weighted L_v^2 -space $H := L_v^2(\mu^{-1/2})$ (which will be the case for the Fokker-Planck equation and also for the linearized Boltzmann and Landau equations with not too soft potentials).

The terminology *hypocoercivity* comes from a parallel with the notion of *hypoellipticity* by Hörmander [139]. The structure of the equation (2.1.1) where $-v \cdot \nabla_x$ is a skew-adjoint and conservative transport operator and the collision operator \mathcal{L} is dissipative in velocity reminds of another type of situation where the operator \mathcal{L} is elliptic instead of dissipative in velocity. In this case, the question is to understand the regularization effect of the equation in the whole phase space that is to say, we wonder if the equation has some hypoelliptic properties (see Section 2.3 for this type of question). The methods of hypocoercivity in kinetic theory were first applied for hypoelliptic equations but also generalized to

equations that do not enjoy hypoelliptic properties. In this introduction, we are going to present two hypocoercivity methods, the H^1 and the L^2 ones.

To end this part, let us point out that other approaches closely linked to the results presented in this memoir exist to tackle the problem of convergence to equilibrium in a linear framework.

- Some results with a probabilistic point of view have been obtained based on the Doeblin/Harris theorem (see the paper [126] by Hairer and Mattingly for a detailed exposition of this theorem) on kinetic equations in [47, 48, 99] by Cañizo et al.; Cañizo and Mischler; Evans and Moyano for example.
- In a very recent paper [87] by Dietert et al., in order to deal with the problem of exponential stability for linear kinetic equations with degenerate thermalization i.e. when the collision operator vanishes on some part of the spatial domain, a trajectorial approach to hypocoercivity is introduced, the idea being to transfer the control on the microscopic part of the equation to the rest of the domain (or the good part of the domain in case of degeneracy) by following characteristics. This approach differs completely from the ones presented in this introduction in the sense that it is based on the integral formulation of the problem rather than energy estimates.

2.1.1 The H^1 -hypocoercivity method

The H^1 -hypocoercivity approach has been first introduced for hypoelliptic operators by Hérau and Nier [134] and Eckmann and Hairer [97], further developed by Nier and Helffer [130] and Villani [205] and extended to more general kinetic operators (with more collisional invariants) in Villani [207], Mouhot and Neumann [181] and for equations with heavy-tail equilibria by Bouin et al. [39, 40] and by Ayi et al. [24] (this work is presented in Chapter 3). It is also reminiscent of the works by Desvillettes and Villani on the trend to global equilibrium for spatially inhomogeneous kinetic systems in [84, 86], and of the high order Sobolev energy method developed by Guo in several works starting from [120]. In summary, the idea consists in endowing the H^1 space with a new scalar product which makes coercive the considered operator and whose associated norm is equivalent to the usual H^1 -norm.

More precisely, the functional space that we consider is $\mathcal{H} := H_{x,v}^1(\mu^{-1/2})$ with associated norm defined in (1.3.2). We also introduce the notation $\mathcal{H} := L_{x,v}^2(\mu^{-1/2})$. We then define a new norm on the functional space \mathcal{H} which is intended to be equivalent to the usual one and to dissipate along the dynamics of (2.1.1):

$$\|g\|_{\mathcal{H}} := \|g\|_{\mathcal{H}}^2 + a \|\nabla_v g\|_{\mathcal{H}}^2 + 2b \langle \nabla_x g, \nabla_v g \rangle_{\mathcal{H}} + c \|\nabla_x g\|_{\mathcal{H}}^2 \quad (2.1.3)$$

for well-chosen positive constants a , b and c . A cornerstone of the proof of this type of approach is that the computations are handled thanks to commutators and the key identity which allows the mixing of variables is the following:

$$[\nabla_v, v \cdot \nabla_x] = \nabla_x.$$

Application to the kinetic Fokker-Planck equation. To illustrate this method, in this part, we consider equation (2.1.1) with \mathcal{L} defined by $\mathcal{L} := \Delta_v + \nabla_v \cdot (v \cdot)$ and $\Omega = \mathbf{T}^d$. We wish to apply the H^1 -hypocoercivity method presented above. In particular, we are going to prove that a solution f to (2.1.1) with associated initial data $f_{\text{in}} \in \mathcal{H}$ satisfies the following exponential decay estimate: there exists a constant $\bar{\lambda} > 0$ such that

$$\|f(t) - \langle f_{\text{in}} \rangle \mu\|_{\mathcal{H}} \lesssim e^{-\bar{\lambda}t} \|f_{\text{in}} - \langle f_{\text{in}} \rangle \mu\|_{\mathcal{H}}, \quad \forall t \geq 0, \quad (2.1.4)$$

where we recall that $\langle f_{\text{in}} \rangle$ is the global mass of f_{in} as defined in Section 1.3.

Proof of (2.1.4). We recall that $\|\cdot\|_{\mathcal{H}}$ is defined in (2.1.3). First, we notice that by choosing a , b , c such that $b^2 < ac$, we have thanks to Cauchy-Schwarz inequality that

$$\|\cdot\|_{\mathcal{H}} \lesssim \|\cdot\|_{\mathcal{H}} \lesssim \|\cdot\|_{\mathcal{H}}.$$

We now consider f solution to (2.1.1) with $\mathcal{L} = \Delta_v + \nabla_v \cdot (v \cdot)$ and $f_{\text{in}} \in \mathcal{H}$ with vanishing global mass. Our equation preserves global mass, there is thus no loss of generality to suppose that $\langle\langle f(t) \rangle\rangle = 0$ for any $t \geq 0$. In this part, the projection $\pi_{\mathcal{L}}$ onto $\text{Ker } \mathcal{L}$ is given by $\pi_{\mathcal{L}}g(x, v) := \mu(v) \int_{\mathbb{R}^d} g(x, w) dw$. We also use the notation $g^\perp := g - \pi_{\mathcal{L}}g$.

Before starting the computations, we make a few remarks on the operator \mathcal{L} itself and on its associated Dirichlet form \mathcal{D} defined in (2.1.5). We first notice that the operator \mathcal{L} can be written as follows:

$$\mathcal{L}g = \nabla_v \cdot (\mu \nabla_v (g\mu^{-1})), \quad \forall g \in D(\mathcal{L}).$$

Using this, one can also remark that

$$-\mathcal{D}(g) := \langle \mathcal{L}g, g \rangle_{\mathcal{H}} = - \int_{\mathbb{T}^d \times \mathbb{R}^d} \mu |\nabla_v (g\mu^{-1})|^2 dv dx \leq 0, \quad \forall g \in D(\mathcal{L}). \quad (2.1.5)$$

Using these two points and the Cauchy-Schwarz inequality, we in particular get that

$$|\langle \mathcal{L}g_1, g_2 \rangle_{\mathcal{H}}| \lesssim \sqrt{\mathcal{D}(g_1)} \sqrt{\mathcal{D}(g_2)}, \quad \forall g_1, g_2 \in D(\mathcal{L}). \quad (2.1.6)$$

We also point out the fact that after integrating in the space variable a weighted Poincaré inequality in velocity coming from [35] (see also [27]), we know that there exists $C_P > 0$ such that

$$\langle \mathcal{L}g, g \rangle_{\mathcal{H}} = -\mathcal{D}(g) \leq -C_P \|g^\perp\|_{\mathcal{H}}^2, \quad \forall g \in D(\mathcal{L}). \quad (2.1.7)$$

Finally, before starting the derivation in time of the triple norm defined above, we remark that we can give different formulations of the Dirichlet form defined in (2.1.5). Indeed, performing an integration by parts in velocity, we have that

$$-\mathcal{D}(g) = -\|\nabla_v g\|_{\mathcal{H}}^2 + d\|g\|_{\mathcal{H}}^2, \quad \forall g \in D(\mathcal{L}). \quad (2.1.8)$$

Noticing that $\mathcal{D}(g) = \mathcal{D}(g^\perp)$, an appropriate convex combination of (2.1.7) and (2.1.8) gives that

$$-\mathcal{D}(g) \lesssim -\|g^\perp\|_{\mathcal{H}}^2 - \|\nabla_v g^\perp\|_{\mathcal{H}}^2, \quad \forall g \in D(\mathcal{L}). \quad (2.1.9)$$

We now compute the derivative in time of each term of the triple norm of f a smooth enough solution to (2.1.1). Using the above reminders, we have the following equality:

$$\frac{1}{2} \frac{d}{dt} \|f\|_{\mathcal{H}}^2 = \langle \mathcal{L}f, f \rangle_{\mathcal{H}} = -\mathcal{D}(f).$$

Similarly, and using the fact that the commutators $[\nabla_x, v \cdot \nabla_x]$ and $[\nabla_x, \mathcal{L}]$ vanish, we have

$$\frac{c}{2} \frac{d}{dt} \|\nabla_x f\|_{\mathcal{H}}^2 = c \langle \mathcal{L} \nabla_x f, \nabla_x f \rangle_{\mathcal{H}} = -c \mathcal{D}(\nabla_x f).$$

Then, using moreover that $[\nabla_v, v \cdot \nabla_x] = \nabla_x$ and $[\nabla_v, \mathcal{L}] = \nabla_v$, we have

$$\begin{aligned} \frac{a}{2} \frac{d}{dt} \|\nabla_v f\|_{\mathcal{H}}^2 &= a \langle \mathcal{L}(\nabla_v f), \nabla_v f \rangle_{\mathcal{H}} + a \|\nabla_v f\|_{\mathcal{H}}^2 - a \langle \nabla_v f, (\nabla_x f)^\perp \rangle_{\mathcal{H}} \\ &\leq -a \mathcal{D}(\nabla_v f) + a \|\nabla_v f\|_{\mathcal{H}}^2 + a \|\nabla_v f\|_{\mathcal{H}} \|(\nabla_x f)^\perp\|_{\mathcal{H}} \end{aligned}$$

where we used the fact that $\int_{\mathbb{R}^d} \nabla_v f dv = 0$ to obtain that

$$\langle \nabla_v f, \nabla_x f \rangle_{\mathcal{H}} = \langle \nabla_v f, (\nabla_x f)^\perp \rangle_{\mathcal{H}}. \quad (2.1.10)$$

Using now Young's inequality and (2.1.7), we get

$$\frac{a}{2} \frac{d}{dt} \|\nabla_v f\|_{\mathcal{H}}^2 \leq -a \mathcal{D}(\nabla_v f) + a c \mathcal{D}(\nabla_x f) + \frac{3a}{2} \|\nabla_v f\|_{\mathcal{H}}^2.$$

Finally, performing an integration by parts in the space variable to simplify the terms coming from the transport operator and the commutators equalities recalled above, we have

$$b \frac{d}{dt} \langle \nabla_v f, \nabla_x f \rangle_{\mathcal{H}} = b \langle \mathcal{L} \nabla_v f, \nabla_x f \rangle_{\mathcal{H}} + b \langle \mathcal{L} \nabla_x f, \nabla_v f \rangle_{\mathcal{H}} - b \|\nabla_x f\|_{\mathcal{H}}^2 + b \langle \nabla_v f, \nabla_x f \rangle_{\mathcal{H}}.$$

Using (2.1.6), (2.1.10) and Cauchy-Schwarz inequality, the previous equality implies that

$$b \frac{d}{dt} \langle \nabla_v f, \nabla_x f \rangle_{\mathcal{H}} \leq -b \|\nabla_x f\|_{\mathcal{H}}^2 + b C \sqrt{\mathcal{D}(\nabla_v f)} \sqrt{\mathcal{D}(\nabla_x f)} + b \|\nabla_v f\|_{\mathcal{H}} \|(\nabla_x f)^\perp\|_{\mathcal{H}}.$$

Young's inequality and (2.1.7) then imply that

$$\begin{aligned} b \frac{d}{dt} \langle \nabla_v f, \nabla_x f \rangle_{\mathcal{H}} \\ \leq -b \|\nabla_x f\|_{\mathcal{H}}^2 + \frac{a}{2} \mathcal{D}(\nabla_v f) + \frac{a}{2} \|\nabla_v f\|_{\mathcal{H}}^2 + C \frac{b^2}{a} \mathcal{D}(\nabla_x f). \end{aligned}$$

Before gathering all the previous inequalities, we give a bound of $\|\nabla_v f\|_{\mathcal{H}}^2$ by decomposing f into two parts, its fluid and kinetic ones:

$$\|\nabla_v f\|_{\mathcal{H}}^2 \lesssim \|\nabla_v f^\perp\|_{\mathcal{H}}^2 + \|\pi_{\mathcal{L}} f\|_{\mathcal{H}}^2$$

where we used that $\|\nabla_v \pi_{\mathcal{L}} f\|_{\mathcal{H}} \lesssim \|\pi_{\mathcal{L}} f\|_{\mathcal{H}}$ which can be obtained from a direct computation. Moreover, from the Poincaré inequality applied in the torus, since f is mean-free, we have $\|\pi_{\mathcal{L}} f\|_{\mathcal{H}} \lesssim \|\nabla_x f\|_{\mathcal{H}}$. From this and (2.1.9), we deduce that

$$\|\nabla_v f\|_{\mathcal{H}}^2 \lesssim \mathcal{D}(f) + \|\nabla_x f\|_{\mathcal{H}}^2. \quad (2.1.11)$$

Finally, we obtain

$$\frac{1}{2} \frac{d}{dt} \|f\|_{\mathcal{H}}^2 \leq (1 - Ca) \mathcal{D}(f) - \left(c - Ca - C \frac{b^2}{a} \right) \mathcal{D}(\nabla_x f) - \frac{a}{2} \mathcal{D}(\nabla_v f) - (b - Ca) \|\nabla_x f\|^2.$$

Fixing then $a := \eta^3$, $b := \eta^{5/2}$ and $c := \eta$ with $\eta > 0$ small enough, we obtain that the right-hand side of our inequality is non-positive and even that it can be bounded by a constant multiplied by $\|f\|_{\mathcal{H}}^2$, which provides the wanted exponential decay of f in the space \mathcal{H} stated in (2.1.4). \square

2.1.2 The L^2 -hypocoercivity method

The L^2 -hypocoercivity technique for one dimensional space of collisional invariants has been introduced by Hérau [132] and by Dolbeault, Mouhot and Schmeiser in [90, 91]. The L^2 -hypocoercivity technique for a space of collisional invariants of dimension larger than one (including the Boltzmann and Landau cases) has been introduced by Guo in [122], and developed further mainly by Guo, collaborators and students. In [33], we deal with the L^2 -hypocoercivity method for a general class of kinetic equations in bounded domains with general boundary conditions in a unified framework (see Chapter 5). Again, the idea consists in endowing the L^2 -space with a new scalar product which makes coercive the considered operator and whose associated norm is equivalent to the usual L^2 -norm.

The L^2 -approach heavily relies on the micro-macro decomposition of the solution of the equation (2.1.1): $f = f^\perp + \pi_{\mathcal{L}} f$, where f^\perp denotes the microscopic part and $\pi_{\mathcal{L}} f$ the macroscopic part which is the projection of f onto the kernel of \mathcal{L} . The coercivity estimate (2.1.2) on the collision operator \mathcal{L} already gives a control on f^\perp but not on the macroscopic part $\pi_{\mathcal{L}} f$. Then, in order to control the macroscopic part, we construct a new norm on $\mathcal{H} = L^2_{x,v}(\mu^{-1/2})$ by adding, step by step, new terms in order to control the missing terms appearing on the macroscopic part $\pi_{\mathcal{L}} f$. Roughly speaking, the norm that is cooked up takes the following form:

$$\|f\|_{\mathcal{H}}^2 := \|f\|_{\mathcal{H}}^2 + \eta \langle \tilde{\pi} f, \nabla_x \Delta_x^{-1} \pi_{\mathcal{L}} f \rangle_{L^2_x}, \quad (2.1.12)$$

choosing $\eta > 0$ small enough, and where the moments operator $\tilde{\pi} : L_{x,v}^2(\mu^{-1/2}) \rightarrow (L_x^2)^d$ and the inverse Laplacian type operator Δ_x^{-1} have to be suitably defined. Let us mention that there are several variants of this approach, it can in particular be adapted to more general geometries than the H^1 -approach (see Chapter 5) and is well suited for problems of hydrodynamical limits (see Chapter 5, Chapter 6-Section 6.2 and Chapter 7).

Application to the kinetic Fokker-Planck equation. As before, we illustrate the L^2 -hypo-coercivity method by applying it on equation (2.1.1) with \mathcal{L} defined by $\mathcal{L} := \Delta_v + \nabla_v \cdot (v \cdot)$ and $\Omega = \mathbf{T}^d$. We wish to apply the L^2 -hypo-coercivity method presented above. In particular, we are going to prove that a solution f to (2.1.1) with associated initial data $f_{\text{in}} \in \mathcal{H}$ satisfies the following exponential decay estimate: there exists a constant $\tilde{\lambda} > 0$ such that

$$\|f(t) - \langle\langle f_{\text{in}} \rangle\rangle \mu\|_{\mathcal{H}} \lesssim e^{-\tilde{\lambda}t} \|f_{\text{in}} - \langle\langle f_{\text{in}} \rangle\rangle \mu\|_{\mathcal{H}}, \quad \forall t \geq 0. \quad (2.1.13)$$

Proof of (2.1.13). We consider f solution to (2.1.1) with $\mathcal{L} = \Delta_v + \nabla_v \cdot (v \cdot)$ and $f_{\text{in}} \in \mathcal{H}$ with vanishing global mass. We recall that due to the fact that our equation preserves global mass, we can suppose that $\langle\langle f(t) \rangle\rangle = 0$ for any $t \geq 0$. We still denote by $\pi_{\mathcal{L}}$ the projection onto $\text{Ker } \mathcal{L}$ and also use the notation $g^\perp = g - \pi_{\mathcal{L}}g$. We also recall that the two first macroscopic quantities $\rho[g]$ and $u[g]$ associated to a kinetic distribution $g = g(x, v)$ that will naturally arise in our computations are defined in (1.1.22) and (1.1.23). Notice that in particular, we have that $\pi_{\mathcal{L}}g = \rho[g]\mu$. Let us now define a new norm on \mathcal{H} which is intended to be a Lyapunov functional for our equation: for a smooth enough function $g = g(x, v)$ with vanishing global mass, we define

$$\| \| g \| \|_{\mathcal{H}}^2 := \|g\|_{\mathcal{H}}^2 + \eta \langle \nabla_x (-\Delta_x)^{-1} \rho[g], u[g] \rangle_{L_x^2} \quad (2.1.14)$$

with $\eta > 0$ to be chosen later. Notice that due to the fact that g has vanishing global mass, $\rho[g]$ is mean-free on the torus so that the quantity $(-\Delta_x)^{-1} \rho[g]$ is well-defined. Moreover, using classical results on elliptic regularity and Cauchy-Schwarz inequality (successively in the variables x and v), we have

$$|\langle \nabla_x (-\Delta_x)^{-1} \rho[g], u[g] \rangle_{L_x^2}| \lesssim \|\rho[g]\|_{L_x^2} \|u[g]\|_{L_x^2} \lesssim \|g\|_{\mathcal{H}}^2.$$

From this, we can deduce that for η small enough, the triple norm $\| \| \cdot \| \|_{\mathcal{H}}$ is equivalent to the usual norm $\| \cdot \|_{\mathcal{H}}$.

Let us now take the derivative in time of the two terms of $\| \| f \| \|_{\mathcal{H}}^2$. First, we have from (2.1.7):

$$\frac{1}{2} \frac{d}{dt} \|f\|_{\mathcal{H}}^2 \leq -C_P \|f^\perp\|_{\mathcal{H}}^2. \quad (2.1.15)$$

In order to deal with the second term, we need to compute the evolution of the macroscopic quantities $\rho[f]$ and $u[f]$. We have:

$$\partial_t \rho[f] + \nabla_x \cdot u[f] = 0 \quad (2.1.16)$$

and

$$\partial_t u[f] + \text{Div}_x \langle v \otimes v f \rangle = -u[f] \quad (2.1.17)$$

where we recall that the definition of the divergence of a matrix is given in Section 1.3 and that $\langle g \rangle$ is the local mass of a distribution g (see also Section 1.3). Using the micro-macro decomposition of $f = \rho[f]\mu + f^\perp$ and computing the first moments of the gaussian μ , we have that $u[f] = u[f^\perp]$ and

$$\langle v_i v_j f \rangle = \delta_{ij} \rho[f] + \langle v_i v_j f^\perp \rangle, \quad \forall i, j = 1, \dots, d,$$

from which we deduce that we can rewrite equation (2.1.17) as

$$\partial_t u[f] + \text{Div}_x \langle v \otimes v f^\perp \rangle + \nabla_x \rho[f] = -u[f^\perp]. \quad (2.1.18)$$

Let us now study the derivative in time of the second term in (2.1.14). Using (2.1.16) and (2.1.18), we have:

$$\begin{aligned}
& \frac{d}{dt} \langle \partial_{x_i} (-\Delta_x)^{-1} \rho[f], u_i[f] \rangle_{L_x^2} \\
&= - \langle \partial_{x_i} (-\Delta_x)^{-1} \partial_{x_j} u_j[f^\perp], u_i[f^\perp] \rangle_{L_x^2} - \langle \partial_{x_i} (-\Delta_x)^{-1} \rho[f], \partial_{x_i} \rho[f] \rangle_{L_x^2} \\
&\quad - \langle \partial_{x_i} (-\Delta_x)^{-1} \rho[f], \partial_{x_j} \langle v_i v_j f^\perp \rangle \rangle_{L_x^2} - \langle \partial_{x_i} (-\Delta_x)^{-1} \rho[f], u_i[f^\perp] \rangle_{L_x^2} \\
&= - \langle \partial_{x_i} (-\Delta_x)^{-1} \partial_{x_j} u_j[f^\perp], u_i[f^\perp] \rangle_{L_x^2} - \|\rho[f]\|_{L_x^2}^2 \\
&\quad + \langle \partial_{x_j} \partial_{x_i} (-\Delta_x)^{-1} \rho[f], \langle v_i v_j f^\perp \rangle \rangle_{L_x^2} - \langle \partial_{x_i} (-\Delta_x)^{-1} \rho[f], u_i[f^\perp] \rangle_{L_x^2}
\end{aligned} \tag{2.1.19}$$

where we performed an integration by parts in the second and third terms and where we used the convention of summation of repeated indices. In order to bound the first term, we remark that performing successively two integrations by parts, we obtain

$$\begin{aligned}
\|\nabla_x (-\Delta_x)^{-1} \nabla_x \cdot u[f^\perp]\|_{L_x^2}^2 &= \int_{\mathbb{T}^d} (\nabla_x \cdot u[f^\perp]) (-\Delta_x)^{-1} \nabla_x \cdot u[f^\perp] \, dx \\
&= - \int_{\mathbb{T}^d} u[f^\perp] \nabla_x (-\Delta_x)^{-1} \nabla_x \cdot u[f^\perp] \, dx.
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality applied successively in x and v , we deduce that

$$\|\nabla_x (-\Delta_x)^{-1} \nabla_x \cdot u[f^\perp]\|_{L_x^2} \leq \|u[f^\perp]\|_{L_x^2} \lesssim \|f^\perp\|_{\mathcal{H}}.$$

Using classical results on elliptic regularity, we have that

$$\|(-\Delta_x)^{-1} \rho[f]\|_{H_x^2} \lesssim \|\rho[f]\|_{L_x^2}.$$

Coming back to (2.1.19), the two previous inequalities combined with the Cauchy-Schwarz inequality in velocity imply that there exists a positive constant C such that

$$\frac{d}{dt} \langle \partial_{x_i} (-\Delta_x)^{-1} \rho[f], u_i[f] \rangle_{L_x^2} \leq -\|\rho[f]\|_{L_x^2}^2 + C\|f^\perp\|_{\mathcal{H}}^2 + C\|\rho[f]\|_{L_x^2} \|f^\perp\|_{\mathcal{H}}.$$

Using now Young's inequality, we obtain that

$$\frac{d}{dt} \langle (-\Delta_x)^{-1} \partial_{x_i} \rho[f], u_i[f] \rangle_{L_x^2} \leq -\frac{1}{2} \|\rho[f]\|_{L_x^2}^2 + C\|f^\perp\|_{\mathcal{H}}^2. \tag{2.1.20}$$

We now recall the definition of the triple norm in (2.1.14). Estimates (2.1.15) and (2.1.20) yield

$$\frac{d}{dt} \| \|f\|_{\mathcal{H}}^2 \leq -C_P \|f^\perp\|_{\mathcal{H}}^2 - \frac{\eta}{2} \|\rho[f]\|_{L_x^2}^2 + C\eta \|f^\perp\|_{\mathcal{H}}^2,$$

so that by choosing $\eta > 0$ small enough, we obtain that

$$\frac{d}{dt} \| \|f\|_{\mathcal{H}}^2 \leq -\frac{C_P}{2} \|f^\perp\|_{\mathcal{H}}^2 - \frac{\eta}{2} \|\rho[f]\|_{L_x^2}^2 \leq -\tilde{\lambda} \| \|f\|_{\mathcal{H}}^2$$

for some positive constant $\tilde{\lambda}$ because

$$\|f^\perp\|_{\mathcal{H}}^2 + \|\rho[f]\|_{L_x^2}^2 = \|f\|_{\mathcal{H}}^2 \gtrsim \| \|f\|_{\mathcal{H}}^2,$$

which ends the proof of (2.1.13). \square

2.2 Enlargement method

In this part, we present the so-called enlargement or factorization method which is a linear method that has been introduced by Mouhot in [180] and developed by Gualdani, Mischler and Mouhot in [117] and Mischler and Mouhot in [175]. The main goal is to exploit knowledge on a linear operator in a “small” functional space of reference, say E , to deduce similar properties in a larger functional space, say \mathcal{E} . Typically, E is a weighted Hilbert space of type $L^2(\mu^{-1/2})$ with μ defined in (1.1.15) and \mathcal{E} is of type $L^p(\langle v \rangle^k)$ with $k \geq 0$ (more details about the functional spaces at stake will be given in each case where we use this method). The said properties are about the spectrum of the linear operator and also the semigroup generated by this linear operator. We will only talk about the semigroup ones in order to lighten the presentation and also because we shall use only these ones in what follows. Moreover, we give a version of the enlargement theorem in a very simple framework (many refinements can actually be made on the assumptions, see [117, 175]).

Theorem 2.2.1. *Let us consider two Banach spaces E and \mathcal{E} such that $E \hookrightarrow \mathcal{E}$ with dense and continuous embedding, a linear operator P which is the generator of a C_0 -semigroup $(S_P(t))_{t \geq 0}$ on E and \mathcal{P} the generator of a C_0 -semigroup $(S_{\mathcal{P}}(t))_{t \geq 0}$ on \mathcal{E} with $\mathcal{P}|_E = P$. We suppose that*

(H1) *There exists $\lambda_1 > 0$ such that the semigroup $(S_P(t))_{t \geq 0}$ satisfies*

$$\|S_P(t)(\text{Id} - \pi_P)\|_{\mathcal{B}(E)} \lesssim e^{-\lambda_1 t}, \quad \forall t \geq 0$$

where π_P is the projection onto the kernel of P . Moreover, we suppose that \mathcal{P} splits as $\mathcal{P} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \in \mathcal{B}(E)$, $\mathcal{A} \in \mathcal{B}(\mathcal{E})$ and \mathcal{B} is a closed unbounded operator on \mathcal{E} with dense domain satisfying

(H2) *There exists $\lambda_2 > 0$ such that the semigroup $(S_{\mathcal{B}}(t))_{t \geq 0}$ satisfies*

$$\|S_{\mathcal{B}}(t)\|_{\mathcal{B}(\mathcal{E})} \lesssim e^{-\lambda_2 t}, \quad \forall t \geq 0.$$

(H3) *There exists $\lambda_3 > 0$ and $n \in \mathbf{N}^*$ such that the operator $\mathcal{A}S_{\mathcal{B}}(t)$ satisfies*

$$\|(\mathcal{A}S_{\mathcal{B}})^{*n}(t)\|_{\mathcal{B}(\mathcal{E}, E)} \lesssim e^{-\lambda_3 t}, \quad \forall t \geq 0,$$

where the convolution product is defined by $S_1 * S_2(t) := \int_0^t S_1(t-s)S_2(s) ds$, $S^{*0} = \text{Id}$, $S^{*n} = S^{*(n-1)}$ for $n \in \mathbf{N}^*$.

Then, for any $\lambda < \min(\lambda_1, \lambda_2, \lambda_3)$, we have

$$\|S_{\mathcal{P}}(t)(\text{Id} - \pi_{\mathcal{P}})\|_{\mathcal{B}(\mathcal{E})} \lesssim e^{-\lambda t}, \quad \forall t \geq 0.$$

Notice that it is part of the result that \mathcal{B} generates a semigroup on \mathcal{E} (see [117, Proof of Theorem 2.13]). Notice also that we gave here above a simplified version of the original theorem stated in [117] which allows to simpler proofs.

Proof with Duhamel formula. We only present the proof in the case $n = 1$, higher values of n can be treated by reiterating Duhamel formula. Consider $f(t) := S_{\mathcal{P}}(t)f_{\text{in}}$ for $t \geq 0$ with $f_{\text{in}} \in \mathcal{E}$. By Duhamel formula, we have

$$(\text{Id} - \pi_{\mathcal{P}})f(t) = (\text{Id} - \pi_{\mathcal{P}})S_{\mathcal{B}}(t)f_{\text{in}} + \int_0^t (\text{Id} - \pi_{\mathcal{P}})S_{\mathcal{P}}(t-s)\mathcal{A}S_{\mathcal{B}}(s)f_{\text{in}} ds, \quad \forall t \geq 0.$$

We implicitly used hypothesis (H3) (recall that $n = 1$ so that $\mathcal{A}S_{\mathcal{B}}(s) \in \mathcal{B}(\mathcal{E}, E)$) to replace the operator $(\text{Id} - \pi_{\mathcal{P}})S_{\mathcal{P}}(t-s)$ by $(\text{Id} - \pi_P)S_P(t-s)$ in the above integral term of Duhamel formula. Using then assumption (H2) and the fact that $\pi_{\mathcal{P}}$ is bounded on \mathcal{E} , we have

$$\|(\text{Id} - \pi_{\mathcal{P}})S_{\mathcal{B}}(t)f_{\text{in}}\|_{\mathcal{E}} \lesssim \|S_{\mathcal{B}}(t)f_{\text{in}}\|_{\mathcal{E}} \lesssim e^{-\lambda_2 t} \|f_{\text{in}}\|_{\mathcal{E}}.$$

Moreover, assumptions (H1) and (H3) and the fact that $E \hookrightarrow \mathcal{E}$ imply that

$$\begin{aligned} \left\| \int_0^t (\text{Id} - \pi_{\mathcal{P}}) S_{\mathcal{P}}(t-s) \mathcal{A} S_{\mathcal{B}}(s) f_{\text{in}} \, ds \right\|_{\mathcal{E}} &\lesssim \int_0^t \|(\text{Id} - \pi_{\mathcal{P}}) S_{\mathcal{P}}(t-s) \mathcal{A} S_{\mathcal{B}}(s) f_{\text{in}}\|_E \, ds \\ &\lesssim \int_0^t e^{-\lambda_1(t-s)} e^{-\lambda_3 s} \, ds \|f_{\text{in}}\|_{\mathcal{E}} \lesssim t e^{-\min(\lambda_1, \lambda_3)t} \|f_{\text{in}}\|_{\mathcal{E}}. \end{aligned}$$

We obtain the wanted result by gathering the two previous estimates. \square

We now give here an alternative proof taken from [175] (which is also straightforward in the simplified framework we are working in) because this type of strategy will be of great use in Chapter 7 in a nonlinear setting.

Proof with the “splitting” method. We only deal with the case $n = 1$, the proof being similar for higher values of n by introducing a hierarchy of $n + 1$ equations instead of 2 equations. Let us consider some positive constant λ such that $\lambda < \min(\lambda_1, \lambda_2, \lambda_3)$. We are going to rewrite our problem into an equivalent system of two equations. More precisely, we introduce $f(t) := S_{\mathcal{P}}(t) f_{\text{in}}$ for $t \geq 0$ and $f_{\text{in}} \in \mathcal{E}$. We then decompose $f = \pi_{\mathcal{P}} f + f_1 + f_2$ with (f_1, f_2) solution to

$$\begin{cases} \partial_t f_1 = \mathcal{B} f_1, & f_1(0) = f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}} \in \mathcal{E} \\ \partial_t f_2 = \mathcal{P} f_2 + \mathcal{A} f_1, & f_2(0) = 0 \in E. \end{cases}$$

From hypothesis (H2), one clearly has that

$$\|f_1(t)\|_{\mathcal{E}} \lesssim e^{-\lambda_2 t} \|f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}}\|_{\mathcal{E}}, \quad \forall t \geq 0.$$

Then, using that $f_2(0) = 0 \in E$ and that $\pi_{\mathcal{P}}$ commute with \mathcal{P} , we have

$$(\text{Id} - \pi_{\mathcal{P}}) f_2(t) = \int_0^t S_{\mathcal{P}}(t-s) (\text{Id} - \pi_{\mathcal{P}}) \mathcal{A} f_1(s) \, ds = \int_0^t S_{\mathcal{P}}(t-s) (\text{Id} - \pi_{\mathcal{P}}) \mathcal{A} S_{\mathcal{B}}(s) (f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}}) \, ds.$$

From assumptions (H1) and (H3), the continuous embedding $E \hookrightarrow \mathcal{E}$ and the fact that $\lambda < \min(\lambda_1, \lambda_3)$, we have:

$$\begin{aligned} \|(\text{Id} - \pi_{\mathcal{P}}) f_2(t)\|_{\mathcal{E}} &\lesssim \|(\text{Id} - \pi_{\mathcal{P}}) f_2(t)\|_E \lesssim \int_0^t e^{-\lambda_1(t-s)} \|\mathcal{A} S_{\mathcal{B}}(s) (f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}})\|_E \, ds \\ &\lesssim \int_0^t e^{-\lambda_1(t-s)} e^{-\lambda_3 s} \, ds \|f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}}\|_{\mathcal{E}} \lesssim e^{-\lambda t} \|f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}}\|_{\mathcal{E}}. \end{aligned}$$

We also have that $\pi_{\mathcal{P}} f_2 = -\pi_{\mathcal{P}} f_1$. Using the continuity of $\pi_{\mathcal{P}}$ on \mathcal{E} , this implies that

$$\|\pi_{\mathcal{P}} f_2(t)\|_{\mathcal{E}} \lesssim \|\pi_{\mathcal{P}} f_1(t)\|_{\mathcal{E}} \lesssim e^{-\lambda_2 t} \|f_{\text{in}} - \pi_{\mathcal{P}} f_{\text{in}}\|_{\mathcal{E}}.$$

The three previous estimates yield the wanted result. \square

This enlargement method has been widely used in recent years in order to enlarge the space in which perturbative Cauchy theories can be developed for kinetic equations: the Boltzmann equation for hard spheres [117] by Gualdani, Mischler and Mouhot, without cut-off [197] by the author of this memoir in an homogeneous setting and then by Hérau, Tonon and the author of this memoir [137]; Alonso et al. [17] in an inhomogeneous framework, the inelastic Boltzmann equation [199] (combined with a perturbative argument) by the author of this memoir, the Landau equation [55, 60, 57] by Carrapatoso; Carrapatoso, the author of this memoir and Wu; Carrapatoso and Mischler, the kinetic (fractional) Fokker-Planck equation [175, 198, 154] by Mischler and Mouhot; Tristani and Lafleche etc... More recently, it has also been used in order to deal with hydrodynamical problems [46, 107, 108, 15] by Briant, Merino-Aceituno and Mouhot; Gervais; Alonso, Lods and the author of this memoir in more general functional frameworks.

Application to the kinetic Fokker-Planck equation. Let us now apply the enlargement argument to the already seen simple situation. We consider equation (2.1.1) with $\mathcal{L} := \Delta_v + \nabla_v \cdot (v \cdot)$ and $\Omega = \mathbf{T}^d$. In the previous subsection, by denoting $\Lambda := -v \cdot \nabla_x + \mathcal{L}$ and $E := L^2_{x,v}(\mu^{-1/2})$, we have proved that there exists $\lambda_1 > 0$ such that (see (2.1.13))

$$\|S_\Lambda(t)(\text{Id} - \Pi_\Lambda)\|_{\mathcal{B}(E)} \lesssim e^{-\lambda_1 t}, \quad \forall t \geq 0.$$

In other words, the L^2 -hypo-coercivity method presented above allows us to check that Assumption (H1) of Theorem 2.2.1 is satisfied. Our goal is now to obtain a similar result of decay in larger functional space, namely in $\mathcal{E} := L^2_{x,v}(\langle v \rangle^k)$ with $k > d/2$, our goal is to prove that for some $\lambda > 0$, one has

$$\|S_\Lambda(t)(\text{Id} - \Pi_\Lambda)\|_{\mathcal{B}(\mathcal{E})} \lesssim e^{-\lambda t}, \quad \forall t \geq 0. \quad (2.2.1)$$

Notice that if one wants to reach even larger functional spaces such as Lebesgue spaces of type L^1 , other tools are necessary to check assumptions of the enlargement theorem (see [117]) but we do not enter into these considerations here. We thus only want to have a less restrictive condition on the decay of our data (polynomial instead of exponential).

Proof of (2.2.1). For $M, R > 0$ to be chosen later on, we define the following operators:

$$\mathcal{A} := M\chi_R \quad \text{and} \quad \mathcal{B} := \Lambda - M\chi_R$$

where we recall that the truncation function χ_R is defined in Section 1.3. It is clear that \mathcal{A} is bounded on E and \mathcal{E} . We have to check that \mathcal{B} enjoys dissipativity properties in \mathcal{E} for suitable constants M and R . For $g \in D(\mathcal{B})$, since the transport operator is skew-adjoint in \mathcal{E} , we have:

$$\langle \mathcal{B}g, g \rangle_{\mathcal{E}} = \langle \mathcal{L}g, g \rangle_{\mathcal{E}}.$$

Performing integrations by parts in velocity, we get:

$$\langle \mathcal{B}g, g \rangle_{\mathcal{E}} = - \int_{\mathbf{T}^d \times \mathbf{R}^d} |\nabla_v g|^2 \langle v \rangle^{2k} dv dx + \int_{\mathbf{T}^d \times \mathbf{R}^d} g^2 \langle v \rangle^{2k} (\psi_k - M\chi_R) dv dx$$

where $\psi_k = \psi_k(v)$ is a continuous function in velocity such that $\psi_k(v) \xrightarrow{|v| \rightarrow \infty} d/2 - k < 0$. As a consequence, for $0 < \lambda_2 \leq k - d/2$, we can choose M and R large enough so that

$$\psi_k(v) - M\chi_R(v) \leq -\lambda_2, \quad \forall v \in \mathbf{R}^d.$$

It gives us the wanted dissipativity result:

$$\langle \mathcal{B}g, g \rangle_{\mathcal{E}} \leq -\lambda_2 \|g\|_{\mathcal{E}}^2,$$

which allows us to check Assumption (H2) of Theorem 2.2.1. It is then clear that the operator \mathcal{A} provides the wanted regularization property, namely $\mathcal{A} \in \mathcal{B}(\mathcal{E}, E)$, which implies that Assumption (H3) is satisfied with $n = 1$ and $\lambda_3 = \lambda_2$. This ends the proof of (2.2.1) for $\lambda < \min(\lambda_1, \lambda_2)$. \square

2.3 Short time regularization estimates

In this subsection, we consider the equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{L}f & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ f(0) = f_{\text{in}} & \text{in } \Omega \times \mathbf{R}^d, \end{cases} \quad (2.3.1)$$

where \mathcal{L} is a linear operator that only acts on the velocity variable, that is symmetric in $L^2_{x,v}$ and that has some regularizing effect in this variable (we will precise this in the example that we will treat). This part is connected to the two previous ones for different reasons.

- First, as already mentioned, there is a narrow relation between the notions of hypocoercivity and hypoellipticity and the short time regularization estimates that we will obtain will be due to the hypoellipticity of the considered equation. Moreover, the way in which we are going to obtain those estimates is very close to the H^1 -hypocoercivity method.
- On the other hand, when one wants to perform an extension argument, one has to understand how to go from the small space to the larger one. Sometimes, in the splitting $\mathcal{L} - v \cdot \nabla_x = \mathcal{A} + \mathcal{B}$, the operator \mathcal{A} will regularize enough to easily check assumption (H3) (see the example about the kinetic Fokker-Planck equation given in the previous Subsection). But in some cases, the semigroup $(S_{\mathcal{B}}(t))_{t \geq 0}$ is the one which is going to provide regularization (see Chapter 4 for example) and we shall use the method presented below to check that assumption (H3) of Theorem 2.2.1 is satisfied.

Here, we are not interested in the long-time behavior of the solutions of equation (2.3.1) but only on the regularization effect of the equation that occurs instantaneously and that is then propagated over time. It is thus enough to study our equation for small times, namely for $t \in (0, 1]$ and to work in flat spaces (in the previous Sections 2.1 and 2.2, the focus was on the long-time behavior of the solutions and thus required to work in weighted spaces). The presentation here is thus lighter but can of course be adapted to a framework with weighted spaces and where the collisional linear operator \mathcal{L} has some coercivity properties if one wants to tackle simultaneously the problem of short-time regularization and long-time behavior.

If our equation naturally provides a gain of one derivative in velocity in L^2 energy estimates, namely if one has an inequality of type

$$\langle \mathcal{L}g, g \rangle_{L^2_{x,v}} \leq -\|\nabla_v g\|_{L^2_{x,v}}^2, \quad \forall g \in D(\mathcal{L}),$$

the idea is to introduce a functional which involves derivatives in velocity and in space variables:

$$\mathcal{F}(g, t) := \|g\|_{L^2_{x,v}}^2 + a t \|\nabla_v g\|_{L^2_{x,v}}^2 + 2b t^2 \langle \nabla_v g, \nabla_x g \rangle_{L^2_{x,v}} + c t^3 \|\nabla_x g\|_{L^2_{x,v}}^2. \quad (2.3.2)$$

As in the H^1 -hypocoercivity method, the goal is to understand the interaction between the effects of the transport operator and the collisional operator \mathcal{L} and the mixing between variables occurs thanks to the crucial following commutator identity

$$[\nabla_v, v \cdot \nabla_x] = \nabla_x. \quad (2.3.3)$$

We also remark that this functional is very close to the one introduced in the H^1 -hypocoercive approach in (2.1.3) except from the weights which depend now on time. The idea behind these weights is that for $t = 0$, we only have the L^2 -norm which appears in the functional. Moreover, choosing the constants a, b and c so that the functional \mathcal{F} controls the $H^1_{x,v}$ norm ($b^2 < ac$), more precisely so that for any $t \in [0, 1]$,

$$\mathcal{F}(g, t) \gtrsim t \|g\|_{L^2_{x,v}}^2 \quad \text{and} \quad \mathcal{F}(g, t) \gtrsim t^3 \|g\|_{H^1_{x,v}}^2, \quad (2.3.4)$$

then, one can easily obtain regularization estimates for $t > 0$ if \mathcal{F} is indeed a Lyapunov functional for our equation for small times.

As mentioned above, those type of regularization estimates have been obtained for many kinetic equations and are of great use for enlargement arguments presented in Section 2.2 (see [175, 60] and Sections 4.1 and 4.2 for respectively the kinetic Fokker-Planck, the linearized Landau, the fractional Kolmogorov and the linearized Boltzmann without cut-off equations). Let us also mention the work [19] by Alphonse and Bernier in which similar results are obtained for Fokker-Planck type operators in a more general framework with pseudo-differential tools.

Two remarks are in order before treating the example of the Kolmogorov equation.

- This type of method requires an adaptation when the collision operator provides a gain of a fractional derivative in velocity, as it will be the case in Chapter 4 for the fractional Kolmogorov and linearized Boltzmann without cut-off equations. The main difficulty in those cases is to find a good entropy function.

- It is also worth pointing out that this type of method can be also used for nonlinear equations in a perturbative framework, as will be done in Section 6.2 for the Landau equation.

Application to the Kolmogorov equation. We suppose that $\mathcal{L} = \Delta_v$ in (2.3.1) (as in [133]) and $\Omega = \mathbf{T}^d$. We are going to prove the following result: for f a solution to (2.3.1) with $f_{\text{in}} \in L_{x,v}^2$, we have:

$$\|f(t)\|_{L_x^2 H_v^1} \lesssim \frac{1}{\sqrt{t}} \|f_{\text{in}}\|_{L_{x,v}^2} \quad \text{and} \quad \|f(t)\|_{H_{x,v}^1} \lesssim \frac{1}{t^{\frac{3}{2}}} \|f_{\text{in}}\|_{L_{x,v}^2}, \quad \forall t \in (0, 1]. \quad (2.3.5)$$

Proof of (2.3.5). We are going to compute the derivative in time of $\mathcal{F}(f(t), t)$ with \mathcal{F} defined in (2.3.2) and f a solution to (2.3.1) associated to $f_{\text{in}} \in L_{x,v}^2$. First, we have:

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L_{x,v}^2}^2 = -\|\nabla_v f\|_{L_{x,v}^2}^2.$$

We then fix $1 \leq i \leq d$. Since derivatives in x commute with the transport operator and \mathcal{L} , we have:

$$\frac{c}{2} \frac{d}{dt} \left[t^3 \|\partial_{x_i} f\|_{L_{x,v}^2}^2 \right] = -c t^3 \|\nabla_v \partial_{x_i} f\|_{L_{x,v}^2}^2 + \frac{3}{2} c t^2 \|\partial_{x_i} f\|_{L_{x,v}^2}^2.$$

Using the commutator identity (2.3.3) and an integration by parts in x to simplify terms coming from the transport operator, we have

$$b \frac{d}{dt} \left[t^2 \langle \partial_{x_i} f, \partial_{v_i} f \rangle_{L_{x,v}^2} \right] = -b t^2 \|\partial_{x_i} f\|_{L_{x,v}^2}^2 - 2b t^2 \langle \nabla_v \partial_{x_i} f, \nabla_v \partial_{v_i} f \rangle_{L_{x,v}^2} + 2b t \langle \partial_{x_i} f, \partial_{v_i} f \rangle_{L_{x,v}^2}.$$

Then, using the Cauchy-Schwarz and the Young inequalities, we estimate

$$b \frac{d}{dt} \left[t^2 \langle \partial_{x_i} f, \partial_{v_i} f \rangle_{L_{x,v}^2} \right] \leq -\frac{b}{2} t^2 \|\partial_{x_i} f\|_{L_{x,v}^2}^2 + \frac{a}{2} t \|\nabla_v \partial_{v_i} f\|_{L_{x,v}^2}^2 + \frac{2b^2}{a} t^3 \|\nabla_v \partial_{x_i} f\|_{L_{x,v}^2}^2 + 2b \|\partial_{v_i} f\|_{L_{x,v}^2}^2.$$

Finally, performing similar computations, we get

$$\begin{aligned} \frac{a}{2} \frac{d}{dt} \left[t \|\partial_{v_i} f\|_{L_{x,v}^2}^2 \right] &= -a t \|\nabla_v \partial_{v_i} f\|_{L_{x,v}^2}^2 - a t \langle \partial_{v_i} f, \partial_{x_i} f \rangle_{L_{x,v}^2} + \frac{a}{2} \|\partial_{v_i} f\|_{L_{x,v}^2}^2 \\ &\leq -a t \|\nabla_v \partial_{v_i} f\|_{L_{x,v}^2}^2 + \left(\frac{a}{2} + \frac{a^2}{b} \right) \|\partial_{v_i} f\|_{L_{x,v}^2}^2 + \frac{b}{4} t^2 \|\partial_{x_i} f\|_{L_{x,v}^2}^2. \end{aligned} \quad (2.3.6)$$

Gathering all the previous estimates and summing over $1 \leq i \leq d$, we obtain:

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(f(t), t) &\leq - \left(1 - \frac{a}{2} - 2b - \frac{a^2}{b} \right) \|\nabla_v f\|_{L_{x,v}^2}^2 - \left(c - \frac{2b^2}{a} \right) t^3 \|\nabla_v \nabla_x f\|_{L_{x,v}^2}^2 \\ &\quad - \left(\frac{b}{4} - \frac{3c}{2} \right) t^2 \|\nabla_x f\|_{L_{x,v}^2}^2 - \frac{a}{2} t \|\nabla_v \nabla_v f\|_{L_{x,v}^2}^2 \end{aligned}$$

where we used the following notation: $\|\nabla_w \nabla_z f\|^2 := \sum_{1 \leq i \leq d} \|\nabla_w \partial_{z_i} f\|^2$ for $w, z \in \{x, v\}$. By choosing then $a := \eta^2$, $b := \eta^3$ and $c := \eta^{7/2}$ with $\eta > 0$ small enough, we deduce that the right-hand side of our inequality is non-positive. This fact combined with (2.3.4) yields the wanted result (2.3.5). \square

2.4 Coming back to nonlinear problems

In this subsection, we consider the nonlinear equation (1.1.1) with $\mathcal{C} = Q$ the Boltzmann or the Landau collision operator. In those cases, our equation admits a global equilibrium μ defined in (1.1.15) and we linearize the equation around this equilibrium. Considering then the linearized collisional operator $\mathcal{L} = Q(\cdot, \mu) + Q(\mu, \cdot)$, we focus on the equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{L}f + Q(f, f) & \text{in } \mathbf{R}^+ \times \Omega \times \mathbf{R}^d \\ f(0) = f_{\text{in}} & \text{in } \Omega \times \mathbf{R}^d. \end{cases} \quad (2.4.1)$$

We also use the notation $\Lambda = -v \cdot \nabla_x + \mathcal{L}$ introduced in (1.1.27). When one wants to study this whole nonlinear problem in a perturbative framework (as it is the case in Chapters 4, 6 and 7), assuming that we have obtained nice estimates on the linear part of the equation thanks to the methods presented above, it remains to get nonlinear estimates on the collision operator Q . They will allow us to make the link between the gain coming from the linearized part of the equation and the loss coming from the nonlinear one in our energy estimates. Thanks to the nonlinear estimates (that have to be sharp), one will obtain that in some suitable neighborhood of the equilibrium, the linearized part of the equation will dictate the dynamics of the equation. The standard differential inequality that one can obtain (as in Chapter 4) performing energy estimates on the nonlinear problem is the following: for f a solution to (2.4.1),

$$\frac{d}{dt} \|f(t)\|^2 \lesssim -\|f(t)\|_*^2 (1 - C\|f(t)\|) \quad (2.4.2)$$

where C is a positive constant, $\|\cdot\|$ is the norm of the space in which we study our equation and $\|\cdot\|_*$ is a stronger norm that corresponds to the gain provided by the linearized operator \mathcal{L} . Indeed, in the cases that we are going to study, the linearized collisional operator has a spectral gap and provides some additional gain, more precisely, it satisfies a coercivity estimate of the same type as (1.1.29) and even a stronger inequality in the sense that it also provides a gain (of regularity and/or a gain of weight). This additional gain is crucial in order to counterbalance the loss induced by the nonlinear part of the equation and to close our estimates.

The coercivity estimate that we just mentioned is generally satisfied in some particular Hilbert space in which the linearized operator \mathcal{L} has very nice properties. When one wants to work in a larger functional space, one loses these special properties and we will rely on the enlargement argument presented in Section 2.2 in this case. In this context, when one performs an enlargement argument, the gain above mentioned in the coercivity estimate can also be seen in energy estimates for the dissipative part of the linearized operator \mathcal{B} meaning that \mathcal{B} is actually “better than dissipative”. Despite this nice fact, in the case of “large” functional spaces (such as polynomially weighted Sobolev spaces), some additional difficulties appear.

- First, the fact that we do not have nice dissipativity estimates for the whole linearized operator as presented in (1.1.29) but only for the “ \mathcal{B} -part” of our linearized operator is an obstacle when wants to get nice energy estimates for the nonlinear problem of type (2.4.2). To circumvent this difficulty, Gualdani et al. [117] have introduced a norm which is actually dissipative for the whole linearized operator. More precisely, if $\Lambda = -v \cdot \nabla_x + \mathcal{L}$ satisfies assumptions of Theorem 2.2.1 (we only treat the simple case where both the small space E and the larger one \mathcal{E} that are Hilbert spaces), the new norm will be of the following type:

$$\|f\|_{\mathcal{E}}^2 := \eta \|f\|_E^2 + \int_0^\infty \|S_\Lambda(\tau)f\|_{\mathcal{E}}^2 d\tau, \quad \forall f \in (\text{Ker } \Lambda)^\perp \quad (2.4.3)$$

with $\eta > 0$ small enough. This trick allows to get nice dissipativity estimates in the “large” functional space by using the splitting of Λ into two parts, one which is bounded (or better) and the other one which is dissipative (as presented in Theorem 2.2.1). Indeed, if $f(t) = S_\Lambda(t)f_{\text{in}}$, then we can compute

the evolution of f in the following way:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f(t)\|_{\mathcal{E}}^2 &= \eta \langle \Lambda f(t), f(t) \rangle_{\mathcal{E}} + \int_0^\infty \frac{d}{dt} \|S_\Lambda(t+\tau) f_{\text{in}}\|_{\mathcal{E}}^2 d\tau \\ &= \eta \langle \mathcal{A}f(t), f(t) \rangle_{\mathcal{E}} + \eta \langle \mathcal{B}f(t), f(t) \rangle_{\mathcal{E}} + \int_0^\infty \frac{d}{d\tau} \|S_\Lambda(t+\tau) f_{\text{in}}\|_{\mathcal{E}}^2 d\tau \\ &\lesssim \eta C \|f(t)\|_{\mathcal{E}}^2 - \eta \|f(t)\|_{\mathcal{E}}^2 - \|f(t)\|_{\mathcal{E}}^2 \end{aligned}$$

where C is a positive constant and where we used the decay estimate on $S_\Lambda(t)$ in \mathcal{E} coming from Theorem 2.2.1. Taking η small enough allows to recover dissipativity for Λ in this new norm.

- Moreover, nonlinear estimates have to be treated carefully, particularly in polynomially weighted Lebesgue spaces in which the study of the collision operator is less convenient than in the case of gaussian weights.

2.5 Adapting these methods for hydrodynamical problems

Understanding the role of ε in equation (1.2.3) in order to be able to adapt the methods presented in Sections 2.1, 2.2 and 2.3 is a significant challenge. A crucial point is that in the study of hydrodynamical problems, we have to keep track of the difference of behaviors between the microscopic and the macroscopic parts of the solution to (1.2.3). Indeed, as already explained, we expect that the microscopic part is going to vanish in the limit $\varepsilon \rightarrow 0$ while the macroscopic one is going to tend to some kinetic distribution that only depends on time t and position x through macroscopic quantities that are solutions to a suitable fluid system.

- The adaptation of the L^2 -*hypocoercivity* method presented in Section 2.1 to this framework is quite straightforward because it is already based on the micro-macro decomposition of the solution of the kinetic equation (see Chapters 5, 6 and 7).
- On the contrary, the adaptation of the H^1 -*hypocoercivity* method and the method that allows to get *short time regularization estimates* that are presented in Sections 2.1 and 2.3 is not immediate. One has to modify the functional (intended to be an entropy) in order to be able to properly distinguish microscopic and macroscopic behaviors at the level of derivatives. For example, we remark that one can replace the norm of $\nabla_v g$ by that of $\nabla_v g^\perp$ in the definition of the entropy functional (2.3.2) without changing estimates of type (2.3.4). Of course, the computations are more involved and have to be handled more carefully. We refer to Section 6.2 for more details.
- Finally, concerning the *enlargement* method presented in Section 2.2, let us show that one can still get some nice estimates uniformly in ε on the semigroup in “large” functional spaces. Consider the problem

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon.$$

We suppose that the operator \mathcal{L} splits as $\mathcal{L} = A + B$ and that by defining $\mathcal{A} := A$ and $\mathcal{B} := -v \cdot \nabla_x + B$, the operator $-v \cdot \nabla_x + \mathcal{L}$ satisfies assumptions of Theorem 2.2.1. To simplify the discussion, we suppose that \mathcal{A} is bounded from \mathcal{E} into E . Then, we define

$$\mathcal{A}^\varepsilon := \frac{1}{\varepsilon^2} A \quad \text{and} \quad \mathcal{B}^\varepsilon := -\frac{1}{\varepsilon} v \cdot \nabla_x + \frac{1}{\varepsilon^2} B.$$

The goal is now to check that we can get uniform in ε estimates on the semigroup generated by the full linearized operator $\Lambda^\varepsilon = -\varepsilon^{-1} v \cdot \nabla_x + \varepsilon^{-2} \mathcal{L}$ in the large functional space \mathcal{E} . Proceeding as in the proof of Theorem 2.2.1 based on Duhamel formula, for ε small enough, we have

$$\|(\text{Id} - \pi_{\Lambda^\varepsilon}) S_{\Lambda^\varepsilon}(t) g\|_{\mathcal{E}} \lesssim \|S_{\mathcal{B}^\varepsilon}(t) g\|_{\mathcal{E}} + \frac{1}{\varepsilon^2} \int_0^t \|(\text{Id} - \pi_{\Lambda^\varepsilon}) S_{\Lambda^\varepsilon}(t-s) \mathcal{A} S_{\mathcal{B}^\varepsilon}(s) g\|_E ds$$

and thus

$$\begin{aligned} \|(\text{Id} - \pi_{\Lambda^\varepsilon})S_{\Lambda^\varepsilon}(t)g\|_\varepsilon &\lesssim e^{-\lambda_2 \frac{t}{\varepsilon^2}} \|g\|_\varepsilon + \frac{1}{\varepsilon^2} \int_0^t e^{-\lambda_1(t-s)} e^{-\lambda_2 \frac{s}{\varepsilon^2}} ds \|g\|_\varepsilon \\ &\lesssim \left(e^{-\lambda_2 \frac{t}{\varepsilon^2}} + e^{-\lambda_1 t} \right) \|g\|_\varepsilon \end{aligned}$$

for ε small enough, this provides a uniform exponential decay in ε .

In practice, the operator \mathcal{A} only provides regularization or gain of integrability in the velocity variable so that it won't be possible to apply such a proof when one wants to enlarge the functional space in the variable x (for example, if we want to go from L_x^2 to L_x^1 in the torus). The gain in the variable x will come from $\mathcal{A}^\varepsilon S_{\mathcal{B}^\varepsilon}(t)$ and in all the cases considered, it will occur with bad rates in ε . It is thus not possible to recover uniform in ε estimates on the semigroup in such a case.

Moreover, we are not able to adapt the method developed in [117] in order to handle the study of the whole nonlinear problem by performing energy estimates (which is based on the introduction of a norm which is equivalent to the usual one and that is dissipative in the large functional space for the whole linearized operator, see (2.4.3)). For now, we are not able to define such a norm by also keeping track of the fact that the microscopic part is supposed to vanish in the limit $\varepsilon \rightarrow 0$. We refer to Chapter 7 where our study is based on an enlargement argument (combined with a perturbative argument) and where in order to avoid this difficulty, we split our nonlinear equation into two equations in order to get nice a priori estimates. This splitting is done in the spirit of the one presented in the proof of Theorem 2.2.1 for a linear equation.

Kinetic problems

Continuous and discrete fractional Fokker-Planck equations

Summary

In this chapter, we present hypocoercivity results for continuous and discrete fractional Fokker-Planck equations established in two articles [24] and [25] written in collaboration with N. Ayi, M. Herda and H. Hivert. In the continuous case, we aim at understanding how to use the H^1 -hypocoercivity method described in Section 2.1 while our equations involve fractional derivatives. In the discrete case, the challenge is twofold.

- The discretizations of the continuous operators have to be designed to preserve the main features of the continuous model that are conservation of mass, heavy-tailed equilibrium and (hypo)coercivity properties.
- To perform our analysis of the numerical scheme and show exponential stability, we have to introduce new tools of discrete functional analysis, such as discrete nonlocal Poincaré and interpolation inequalities adapted to fractional diffusion.

3.1 The continuous case

3.1.1 The model

In this part, we study the following fractional Fokker-Planck equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{L}_s(f) =: -(-\Delta_v)^s f + \nabla_v f \cdot (vf) & \text{in } \mathbf{R}^+ \times \mathbf{T}^d \times \mathbf{R}^d \\ f(0) = f_{\text{in}} & \text{in } \mathbf{T}^d \times \mathbf{R}^d. \end{cases} \quad (3.1.1)$$

We suppose that $s \in (0, 1)$ and the fractional Laplacian satisfies the following: for any Schwartz function $g : \mathbf{R}^d \rightarrow \mathbf{R}$,

$$\mathcal{F}((-\Delta_v)^s g)(\xi) = |\xi|^{2s} \mathcal{F}(g)(\xi), \quad \forall \xi \in \mathbf{R}^d \quad (3.1.2)$$

where $\mathcal{F}(\cdot)$ is the Fourier transform on \mathbf{R}^d . There are many equivalent definitions of the fractional Laplacian (see [152]). Among them, we shall use (here, P.V. stands for the principal value)

$$(-\Delta_v)^s g(v) = C_{d,s} \text{P.V.} \int_{\mathbf{R}^d} \frac{g(v) - g(w)}{|v - w|^{d+2s}} dw, \quad (3.1.3)$$

where the constant $C_{d,s}$ is given by $C_{d,s} = 2^{2s} \Gamma(\frac{d+2s}{2}) / (\pi^{d/2} |\Gamma(-s)|)$ where $\Gamma(\cdot)$ is the Gamma function.

Going into Fourier variable, one can prove that our equation has an equilibrium μ_s defined by

$$\mu_s := Z_{d,s} \mathcal{F}^{-1} \left(e^{-\frac{|\cdot|^{2s}}{2s}} \right) \quad (3.1.4)$$

where $Z_{d,s}$ is a positive constant that depends on dimension d and s and satisfies $\int_{\mathbf{R}^d} \mu_s(v) dv = 1$. It is worth mentioning that because of the singularity of $|\cdot|^{2s}$ at the origin for $s \in (0, 1)$, μ_s is a heavy tailed distribution. More precisely, we can prove that (see [25, Appendix A])

$$\langle v \rangle^{-d-2s} \lesssim \mu_s(v) \lesssim \langle v \rangle^{-d-2s}, \quad \forall v \in \mathbf{R}^d.$$

One can show that the nullspace of \mathcal{L}_s is exactly given by $\mathbf{R}\mu_s$. The projection $\pi_{\mathcal{L}_s}$ onto the nullspace of \mathcal{L}_s is then given by

$$(\pi_{\mathcal{L}_s}g)(v) = \rho[g]\mu_s(v), \quad \forall v \in \mathbf{R}^d, \quad (3.1.5)$$

where we recall that $\rho[\cdot]$ is defined in (1.1.22).

3.1.2 Notations

We recall that the functional spaces $L_{x,v}^2(\mu_s^{-1/2})$ and $H_{x,v}^1(\mu_s^{-1/2})$ are defined in (1.3.1) and (1.3.2). To lighten the notations, in this Section, we shall use the following notations:

$$H_s := L_v^2(\mu_s^{-1/2}), \quad \mathcal{H}_s := L_{x,v}^2(\mu_s^{-1/2}) \quad \text{and} \quad \mathcal{H}_s := H_{x,v}^1(\mu_s^{-1/2}). \quad (3.1.6)$$

3.1.3 Main result

The main result that we obtain is the following.

Theorem 3.1.1 ([24], Theorem 1.1). *Let f be a solution to the kinetic Lévy-Fokker-Planck equation (3.1.1) with initial data $f_{\text{in}} \in \mathcal{H}_s$. Then, for all $t \geq 0$, one has*

$$\|f(t) - \langle\langle f_{\text{in}} \rangle\rangle \mu_s\|_{\mathcal{H}_s} \lesssim \|f_{\text{in}} - \langle\langle f_{\text{in}} \rangle\rangle \mu_s\|_{\mathcal{H}_s} e^{-\lambda t}$$

for some rate $\lambda > 0$ depending only on d and s and where we recall that $\langle\langle f_{\text{in}} \rangle\rangle$ is the global mass of f_{in} as defined in Section 1.3.

Before going into the description of the strategy of our proof, let us recall that results on large time behavior of solutions to the homogeneous version of (3.1.1) which writes $\partial_t f(t, v) = \mathcal{L}_s f(t, v)$, have been obtained in [105] by Gentil and Imbert in spaces of type $L_v^2(\mu_s^{-1/2})$ (among others) and later in [198] by the author of this memoir in larger Lebesgue spaces. Notice that the presence of the transport operator in our equation (3.1.1) makes the analysis more intricate and requires the use of hypocoercivity techniques and more precisely, we shall use the H^1 -type hypocoercivity technique that has been presented in Section 2.1. Note also that fractional hypocoercivity has also been studied recently in [39] by Bouin, Dolbeault and Lafleche. However, the framework is quite different from ours since a L^2 -hypocoercivity approach is developed.

In the present study, one of the key points is to have a good understanding of the structure of the operator \mathcal{L}_s since we endeavour to carry out our computations as simply as possible in order to adapt our analysis to a discrete framework, as it will be presented in Section 3.2. In particular, let us point out that we do not need fractional derivatives in our Lyapunov functional (as it will be the case in Section 4.1 where regularization properties of the fractional Kolmogorov equation are studied) and our proof does not rely on Fourier transform. In this sense, the method that we use to handle our problem differs completely from that of [137] (which is presented in Section 4.1) and the more recent [39] in which a mode by mode analysis is developed in Fourier space in spatial variable.

Remark 3.1.2. In [24], we also get a similar result for a heavy-tailed BGK equation. We do not present this result here but mention that its proof follows the same lines as the one of Theorem 3.1.1 and is even simpler.

3.1.4 Strategy of proof

Compared to the non-fractional case, we here have a lack of symmetry of our operator \mathcal{L}_s in H_s where we recall that H_s is defined in (3.1.6). To circumvent this difficulty, we introduce the following decomposition (3.1.7) that will be very helpful to simplify the computations in our hypocoercive analysis. Moreover, in the non-fractional case, some simplifications occur in the computations due to the particular form of the gradient of the Gaussian equilibrium. Even though we no longer have such properties in

our case, we are still able to close our estimates thanks to the introduction of the following splitting: for suitable distributions f and g ,

$$-\langle \mathcal{L}_s f, g \rangle_{\mathcal{H}_s} = \mathcal{S}_v(f, g) + \mathcal{A}_v(f, g), \quad (3.1.7)$$

where \mathcal{S}_v and \mathcal{A}_v are bilinear forms that are respectively symmetric and skew-symmetric.

We then use the strategy described in Section 2.1, the H^1 -hypocoercivity method. We thus introduce a new norm $\|\cdot\|_{\mathcal{H}_s}$ on the space \mathcal{H}_s as defined in (2.1.3) by just replacing μ by μ_s . The goal is then to estimate the derivative in time of each term of $\|f(t)\|_{\mathcal{H}_s}^2$ with f solution to (3.1.1) and $f_{\text{in}} \in \mathcal{H}_s$ with vanishing global mass. We notice that, as in the classical case, there is no loss of generality to make this assumption due to the properties of mass conservation of our equation so that we can suppose that $\langle\langle f(t) \rangle\rangle = 0$ for any $t \geq 0$. To do that, we first note that the commutators $[\nabla_x, v \cdot \nabla_x]$ and $[\nabla_x, \mathcal{L}_s]$ vanish while we have $[\nabla_v, v \cdot \nabla_x] = \nabla_x$ and $[\nabla_v, \mathcal{L}_s] = \nabla_v$. Also observe that $v \cdot \nabla_x$ is skew-symmetric in \mathcal{H}_s .

By denoting $\mathcal{S}_{x,v}$ the integral of \mathcal{S}_v in the x variable, one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|f\|_{\mathcal{H}_s}^2 &= -\mathcal{S}_{x,v}(f, f), \\ \frac{c}{2} \frac{d}{dt} \|\nabla_x f\|_{\mathcal{H}_s}^2 &= -c \mathcal{S}_{x,v}(\nabla_x f, \nabla_x f), \\ \frac{a}{2} \frac{d}{dt} \|\nabla_v f\|_{\mathcal{H}_s}^2 &= -a \mathcal{S}_{x,v}(\nabla_v f, \nabla_v f) + a \|\nabla_v f\|_{\mathcal{H}_s}^2 - a \langle \nabla_x f, \nabla_v f \rangle_{\mathcal{H}_s}, \\ b \frac{d}{dt} \langle \nabla_x f, \nabla_v f \rangle_{\mathcal{H}_s} &= -b \|\nabla_x f\|_{\mathcal{H}_s}^2 - 2b \mathcal{S}_{x,v}(\nabla_x f, \nabla_v f) + b \langle \nabla_x f, \nabla_v f \rangle_{\mathcal{H}_s}. \end{aligned} \quad (3.1.8)$$

Notice here that the keystone of the proof of the last equality is the splitting given in (3.1.7) and the Hilbertian setting. Indeed, given any $g(t) = S_{\mathcal{L}_s}(t)g_0$ and operators P and R , one has formally that

$$\frac{d}{dt} \langle Pg, Rg \rangle_{\mathcal{H}_s} = \langle [P, \mathcal{L}_s]g, Rg \rangle_{\mathcal{H}_s} + \langle Pg, [R, \mathcal{L}_s]g \rangle_{\mathcal{H}_s} - 2\mathcal{S}_{x,v}(Pg, Rg).$$

Therefore the skew symmetric part of the operator \mathcal{L}_s only appears in commutators. This observation enables us to avoid loss of moments in velocity which one would face with bad rearrangements of the terms.

As presented in Section 2.1 in the case of the classical Fokker-Planck equation, the goal is then to exploit the nonpositive terms in (3.1.8) in order to control the nonnegative ones. To this end, we first clarify what the terms of the form $\mathcal{S}_{x,v}(g, g)$ (which is the equivalent of $\mathcal{D}(g)$ defined in (2.1.5) in the classical case) allow to control. Combining existing results on coercivity (see [66, 105, 208] by Chafaï; Gentil and Imbert; Wang) and some regularization estimate [24, Lemma 3.2], one can prove that (see [24, Proposition 3.3]) for suitable g ,

$$\mathcal{S}_v(g, g) \gtrsim \left\| (g - \pi_{\mathcal{L}_s} g) \mu_s^{-\frac{1}{2}} \right\|_{H_v^s}^2. \quad (3.1.9)$$

It is worth noticing that in the classical case, from inequality (2.1.11), the Dirichlet form provided a gain of a full derivative in velocity while we only have a gain of fractional derivative here. From (3.1.8), we thus obtain a gain of a fractional derivative in velocity on f , $\nabla_x f$ and $\nabla_v f$. Still in (3.1.8), as in the classical case, we see that there are terms that induce a loss of one derivative in velocity. To handle these terms, we are going to use the following interpolation inequality ([24, Proposition 4.1]): for any $\delta > 0$ and suitable function g ,

$$\|\nabla_v g\|_{\mathcal{H}_s}^2 \leq K(\delta) \left(\mathcal{S}_{x,v}(g, g) + \|\nabla_x f\|_{\mathcal{H}_s}^2 \right) + \delta \mathcal{S}_{x,v}(\nabla_v g, \nabla_v g) \quad (3.1.10)$$

with $K(\delta) > 0$ and where similarly as in the classical case, we used Poincaré inequality in the torus to get that $\|\pi_{\mathcal{L}_s} f\|_{\mathcal{H}_s} \lesssim \|\nabla_x f\|_{\mathcal{H}_s}$ since f is mean-free (recall that $\pi_{\mathcal{L}_s}$ has been defined in (3.1.5)). With

this at hand, we are able to circumvent the difficulty coming from the fact that we only have a fractional gain of derivatives while we loose full derivatives. Gathering all estimates, we obtain that

$$\begin{aligned} \frac{d}{dt} \|f\|_{\mathcal{H}_s}^2 \leq & -\mathcal{S}_{x,v}(f, f)(1 - 2aK(\delta)) - \mathcal{S}_{x,v}(\nabla_x f, \nabla_x f) \left(c - Ca - C\frac{b^2}{a} \right) \\ & - \mathcal{S}_{x,v}(\nabla_v f, \nabla_v f) a \left(\frac{1}{2} - 2\delta \right) - \|\nabla_x f\|_{\mathcal{H}_s}^2 (b - 2aK(\delta)) \end{aligned}$$

for some positive constant $C > 0$. We then conclude as in the classical case by fixing first $\delta = 1/8$ and then $a := \eta^3$, $b := \eta^{5/2}$ and $c := \eta$ for η small enough, we obtain that the right-hand side of our inequality is non-positive and even that it can be bounded by a constant multiplied by $\|f\|_{\mathcal{H}_s}^2$, which provides the wanted exponential decay stated in Theorem 3.1.1.

3.2 The discrete case

3.2.1 The model

In this part, we are interested in a discretized version of (3.1.1). Our main goal is to design a consistent, stable and structure preserving numerical method for our discrete equation in dimension $d = 1$ (i.e. two-dimensional in the phase-space), as well as its analysis, implementation and simulation.

The keystone is the discretization of the operator \mathcal{L}_s and we first treat the case of the unbounded velocity domain \mathbf{R} . It is discretized by a regular subdivision $(v_j = j\Delta v)_{j \in \mathbf{Z}}$ with $\Delta v > 0$ a given step size. For a velocity distribution $f : \mathbf{R} \rightarrow \mathbf{R}$, f_j denotes an approximation of $f(v_j)$ and with a slight abuse of notation, we write $f = (f_j)_{j \in \mathbf{Z}}$. In order to lighten the notations, when the velocity variable is the only one considered, we will use the notation $h = \Delta v$ for the step of our subdivision in velocity.

Discretization of the fractional Laplacian. In order to discretize the fractional Laplacian operator in dimension one, we follow the finite difference-quadrature approach of Huang and Oberman [140]. We just mention that it is based on the integral formulation of the fractional Laplace operator given in (3.1.3) that can be symmetrized and split into the sum of a singular part and a tail part. The discrete fractional Laplace operator (see [25] for details) is given by

$$(\Lambda_{s,h} f)_j = \sum_{k=1}^{\infty} \beta_k^h (f_{j+k} + f_{j-k} - 2f_j) h = \sum_{k \in \mathbf{Z}} \beta_k^h (f_{j-k} - f_j) h, \quad \forall j \in \mathbf{Z}, \quad (3.2.1)$$

with weights β_k^h satisfying

$$\frac{1}{|kh|^{1+2s}} \lesssim \beta_k^h \lesssim \frac{1}{|kh|^{1+2s}}, \quad \forall k \in \mathbf{Z} \setminus \{0\}.$$

The discretization of the fractional Laplace operator is consistent with the usual centered finite difference approximation of the Laplacian, namely, using that $C_{1,s} \sim_{s \rightarrow 1^-} 2(1-s)$ (recall that $C_{d,s}$ is given after (3.1.3)), one can prove that

$$\lim_{s \rightarrow 1^-} h \beta_k^h = \begin{cases} h^{-2} & \text{if } |k| = 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.2.2)$$

and thus one recovers the usual discrete Laplacian in the limit $s \rightarrow 1^-$:

$$\lim_{s \rightarrow 1^-} (\Lambda_{s,h} u)_j = \frac{u_{j+1} + u_{j-1} - 2u_j}{h^2}, \quad \forall j \in \mathbf{Z}.$$

It is also easy to check that the discrete fractional Laplacian then satisfies the conservation of mass and it is self-adjoint in the space of square summable sequences.

Discretization of the drift term. We now turn to the discretization of the full fractional Fokker-Planck operator \mathcal{L}_s . We shall denote by $\Gamma_{s,h}$ the discrete equivalent of $\partial_v(v \cdot)$ so that

$$\mathcal{L}_{s,h} = \Gamma_{s,h} + \Lambda_{s,h}.$$

The goal is to define a consistent approximation that preserves exactly the discrete equilibrium $(M_{s,j})_{j \in \mathbf{Z}}$ defined by

$$M_{s,j} := \frac{\mu_s(v_j)}{\sum_{i \in \mathbf{Z}} \mu_s(v_i) h}, \quad \forall j \in \mathbf{Z}. \quad (3.2.3)$$

The design relies on the identities $\partial_v(v f) = \partial_v(v \mu_s f / \mu_s)$ and the identity

$$v \mu_s(v) := \frac{1}{2} \int_{-v}^v (-\Delta_w)^s \mu_s(w) dw,$$

which is easily obtained using that $\mathcal{L}_s \mu_s = 0$ and that μ_s is symmetric. The non-local reformulation of $v \mu_s(v)$ will be used in the design of the scheme to allow for the exact preservation of equilibrium. In terms of numerical analysis, the drawback of this choice is that (hypo)coercivity estimates in Sobolev norms lead to new technical challenges in the discrete setting and do not rely on a mere adaptation of the continuous case.

In order to preserve the divergence structure and conserve mass, the operator $\Gamma_{s,h}$ is discretized in the finite volume fashion

$$(\Gamma_{s,h} f)_j := \frac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}}{h}, \quad \forall j \in \mathbf{Z},$$

with the numerical flux defined by the centered approximation

$$\mathcal{F}_{j+\frac{1}{2}} := (VM_s)_{j+\frac{1}{2}} \left(\frac{f_j}{2M_{s,j}} + \frac{f_{j+1}}{2M_{s,j+1}} \right), \quad \forall j \in \mathbf{Z},$$

and

$$(VM_s)_{j+\frac{1}{2}} = -(VM_s)_{-j-\frac{1}{2}} := -\frac{1}{2} \sum_{k=-j}^j (\Lambda_{s,h} M_s)_k h, \quad \forall j \geq 0.$$

Basic properties of the discretization

Notice that

$$(VM_s)_{j+1/2} \xrightarrow{s \rightarrow 1^-} \frac{M_{s,j} - M_{s,j+1}}{h},$$

and

$$\mathcal{L}_{s,h} f \xrightarrow{s \rightarrow 1^-} \frac{\mathcal{G}_{j+1/2} - \mathcal{G}_{j-1/2}}{h} \quad \text{with} \quad \mathcal{G}_{j+1/2} := \frac{M_{s,j} + M_{s,j+1}}{2h} \left(\frac{f_{j+1}}{M_{s,j+1}} - \frac{f_j}{M_{s,j}} \right).$$

We recall that in the case $s = 1$, the equilibrium μ_1 is a standard Gaussian. It satisfies $v \mu_1(v) = -\mu_1'(v)$ whose discrete counterpart is given by the first limit above. When $s \rightarrow 1^-$, the scheme degenerates in a conservative finite difference/finite volume scheme which is clearly consistent with the equivalent reformulation

$$\mathcal{L}_1 f = \partial_v \left(\mu_1 \partial_v \left(\frac{f}{\mu_1} \right) \right).$$

Notice also that with such a discretization, the equation preserves mass:

$$\sum_{j \in \mathbf{Z}} (\mathcal{L}_{s,h} f)_j h = 0,$$

as well as the local equilibrium:

$$(\mathcal{L}_{s,h} M_s)_j = 0, \quad \forall j \in \mathbf{Z},$$

and we have the following consistency estimate: for any regular enough function g ,

$$\sup_{j \in \mathbf{Z}} |(\mathcal{L}_s g)(hj) - (\mathcal{L}_{s,h} g)_j| \lesssim \|g\|_{C_b^4(\mathbf{R})} h^{\min(3-2s, 2)}.$$

It is also worth mentioning that similarly as in the continuous case (see (3.1.7)), we have the following decomposition: there exists two bilinear forms \mathcal{S}_h and \mathcal{A}_h that are respectively symmetric and skew-symmetric such that (see [25, Proposition 2.6] for the details)

$$-\sum_{j \in \mathbf{Z}} (\mathcal{L}_{s,h} f)_j g_j M_{s,j}^{-1} h = \mathcal{S}_h(f, g) + \mathcal{A}_h(f, g). \quad (3.2.4)$$

We can also deduce from this decomposition that $\text{Ker}(\mathcal{L}_{s,h}) = \text{Span}\{(M_j)_{j \in \mathbf{Z}}\}$ and that the projection onto the kernel of $\mathcal{L}_{s,h}$ is given by

$$(\pi_{\mathcal{L}_{s,h}} g)_j = M_{s,j} \sum_{k \in \mathbf{Z}} g_k h, \quad \forall j \in \mathbf{Z}. \quad (3.2.5)$$

Discretization of the whole inhomogeneous equation. We now want to end the discretization of the equation (3.1.1) set in the phase space $\mathbf{T} \times \mathbf{R}$ (where we have denoted by \mathbf{T} the one-dimensional torus). The space and velocity step are $\Delta x = N_x^{-1}$ with N_x an odd positive integer, and Δv respectively (instead of h). We write $x_i = i\Delta x$ and $v_j = j\Delta v$ for any $(i, j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}$. The equation that we shall consider is semi-discretized (we here present a version where the time variable is not discretized):

$$\begin{cases} \partial_t f_{i,j} + (T_{\Delta x} f)_{i,j} = (\mathcal{L}_{s,\Delta v} f)_{i,j}, & \forall (i, j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}, \\ f_{i,j}(0) = (f_{\text{in}})_{i,j}, & \forall (i, j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}. \end{cases} \quad (3.2.6)$$

The discrete transport operator writes

$$(T_{\Delta x} f)_{i,j} = v_j \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}, \quad \forall (i, j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}.$$

With this discretization, we have that equilibria of our equation are multiples of $(M_j)_j$ and that the total mass is preserved by the equation. There is thus no loss of generality to consider an initial data f_{in} that has vanishing mass so that

$$\sum_{(i,j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}} f_{i,j}(t) \Delta x \Delta v = 0, \quad \forall t \geq 0.$$

Remark 3.2.1. It is worth mentioning that we only present the analysis of the above semi-discretized equation, the analysis of the full discretized equation can be found in [25, Section 5], this choice is made in order to lighten the presentation and also because the major difficulties are already present in the semi-discretized case.

Remark 3.2.2. We do not enter into details on the case of truncated velocity domain because we only carry out the analysis of the discrete fractional Fokker-Planck equation in the unbounded velocity domain case in [25]. Nevertheless, we mention that for practical computations, the discrete velocity domain has to be reduced to a finite number of points. Despite the truncation, the discretization still needs to approximate the Fokker-Planck operator on the whole real line. Moreover, we also want to preserve the structural properties of the discretization, such as preservation of mass and equilibrium. The truncation of the velocity domain we propose is strongly linked to the expected decay of the solution at infinity. As soon as the initial data decays as fast or faster μ_s , say if f^{in}/μ_s is bounded, the solution $f \equiv f(t, v)$ to the Lévy-Fokker-Planck equation $\partial_t f = \mathcal{L}_s f$ decays exactly like $O(|v|^{-1-2s})$ at infinity for any positive time $t > 0$ (and not only close to equilibrium). It is seen by expressing the solution with the Green function of the equation. The same property holds for the kinetic version. Therefore, for “most” of the

initial data in $L^2(\mu_s^{-1/2})$, the extension of the solution by an algebraically decaying with exponent $1 + 2s$ is the accurate choice. However, for solutions with a slower algebraic decay, which is possible in the class $L^2(\mu_s^{-1/2})$, this choice should be modified. At the continuous level, the same analysis as above shows that the slower decay between that of μ_s and the initial data will be that of the solution for any positive time. In these cases, the definition of the truncated and discrete operators has to be modified. We refer to [25, Section 3] for more details.

3.2.2 Notations

- In order to prove our hypocoercivity estimates for the equation (3.2.6), we introduce the Hilbert spaces $H_s := \ell_{\Delta v}^2(M_s^{-1/2})$ and $\mathcal{H}_s := \ell_{\Delta x, \Delta v}^2(M_s^{-1/2})$ (recall that M_s is defined in (3.2.3)) associated to the scalar products

$$\langle f, g \rangle_{H_s} = \sum_{j \in \mathbf{Z}} \frac{f_j g_j}{M_{s,j}} \Delta v,$$

for $f = (f_j)_{j \in \mathbf{Z}}$ and $g = (g_j)_{j \in \mathbf{Z}}$ and

$$\langle f, g \rangle_{\mathcal{H}_s} = \sum_{i \in \mathbf{Z}/N_x \mathbf{Z}} \sum_{j \in \mathbf{Z}} \frac{f_{i,j} g_{i,j}}{M_{s,j}} \Delta x \Delta v,$$

for $f = (f_{i,j})_{(i,j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}}$ and $g = (g_{i,j})_{(i,j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}}$. The corresponding norms are respectively denoted by $\|\cdot\|_{H_s}$ and $\|\cdot\|_{\mathcal{H}_s}$.

- We also introduce a discrete H^1 weighted norm $\mathcal{H}_s := H_{\Delta x, \Delta v}^1(M_s^{-1/2})$ (in which we will prove that a solution f to (3.2.6) has exponential decay to equilibrium) which is defined by

$$\|g\|_{\mathcal{H}_s} := \|g\|_{H_s} + \|D_{\Delta x} g\|_{H_s} + \|D_{\Delta v} g\|_{H_s}, \quad (3.2.7)$$

where $D_{\Delta x}$ and $D_{\Delta v}$ stand for centered finite differences in space and velocity

$$(D_{\Delta x} g)_{i,j} := \frac{g_{i+1,j} - g_{i-1,j}}{2\Delta x}, \quad (D_{\Delta v} g)_{i,j} = \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta v}, \quad \forall (i,j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}.$$

- We then introduce the following fractional Sobolev spaces in velocity: the homogeneous one $\dot{H}_{\Delta v}^s$ defined through its norm

$$|g|_{\dot{H}_{\Delta v}^s}^2 = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{(g_j - g_{j+k})^2}{|k \Delta v|^{1+2s}} (\Delta v)^2, \quad \forall s \in (0, 1) \quad (3.2.8)$$

and the non homogeneous one $H_{\Delta v}^s$ defined through its norm

$$\|g\|_{H_{\Delta v}^s} = \|g\|_{\ell_{\Delta v}^2} + |g|_{\dot{H}_{\Delta v}^s}, \quad \forall s \in (0, 1). \quad (3.2.9)$$

- Finally, given a summable function $g = (g_{i,j})_{(i,j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}}$, mimicking the continuous notation, we shall denote its global mass by $\langle\langle g \rangle\rangle_{\Delta x, \Delta v} := \sum_{(i,j) \in \mathbf{Z}/N_x \mathbf{Z} \times \mathbf{Z}} g_{i,j} \Delta x \Delta v$.

3.2.3 Main result

The main hypocoercivity result that we obtain on equation (3.2.6) is the following:

Theorem 3.2.3 ([25], Theorem 5.3). *Suppose that N_x is odd. There exists $\Delta v_0 > 0$ such that if f is solution to the semi-discrete fractional Fokker-Planck equation (3.2.6) with initial data $((f_{\text{in}})_{i,j})_{i,j} \in \mathcal{H}_s$ then, for all $\Delta v < \Delta v_0$ and for all $t \geq 0$, one has*

$$\|f(t) - \langle\langle f_{\text{in}} \rangle\rangle_{\Delta x, \Delta v} M_s\|_{\mathcal{H}_s} \lesssim \|f_{\text{in}} - \langle\langle f_{\text{in}} \rangle\rangle_{\Delta x, \Delta v} M_s\|_{\mathcal{H}_s} e^{-\bar{\lambda} t},$$

with $\bar{\lambda} > 0$ depending only on s .

In the recent years, there has been several works around the numerical analysis of hypocoercivity properties for discretizations of kinetic equations. The first contribution [186] by Porretta and Zuazua concerns a finite difference discretization of the Kolmogorov equation. For the same model, hypocoercivity properties for other types of schemes were studied in [100] by Foster, Lohéac and Tran and in [106] by Georgoulis. Concerning the classical kinetic Fokker-Planck equation, there has been to our knowledge two main contributions [95, 34] by Hérau, Dujardin and Lafitte on the one hand and by Bessemoulin-Chatard, Herda and Rey on the other hand dealing respectively with H^1 and L^2 hypocoercivity methods. In the present contribution, the range of models is extended to the fractional Fokker-Planck case. Notice also that using a perturbative argument, in [177], Mischler and the author of this memoir prove the exponential stability of a model which in a sense is akin to the discretized fractional Fokker-Planck equation.

For our equation (3.2.6) (and also for the fully discretized fractional Fokker-Planck equation in [25]), we rigorously show coercivity and hypocoercivity properties leading to exponential stability of the discrete solution. In order to prove these results we need to adapt many of the continuous properties to the (more intricate) discrete setting. Apart from the exponential stability results, many intermediate results of discrete functional analysis have their own importance, they include a discrete version of nonlocal Poincaré inequalities as well as many new interpolation and embedding inequalities involving discrete fractional operators and norms.

Compared to the continuous setting, there are two main challenges concerning the coercivity and hypocoercivity analysis in the discrete setting. The first one is that Fourier analysis is not tractable anymore for easily proving intermediate functional inequalities (interpolation, embeddings...). The second challenge concerns commutators between discrete operators, which are essential in hypocoercivity estimates. They contain remainder terms which vanish when the mesh size goes to 0 but need to be dealt with carefully in order to close estimates.

3.2.4 Strategy of proof

Roughly speaking, we follow the same strategy as for the proof of Theorem 3.1.1. Our proof is thus going to rely on the H^1 -hypocoercivity method as presented in Section 2.1. We thus introduce the following norm on \mathcal{H}_s :

$$\|g\|_{\mathcal{H}_s}^2 := \|g\|_{\mathcal{H}_s}^2 + a \|D_{\Delta v} g\|_{\mathcal{H}_s}^2 + 2b \langle D_{\Delta x} g, D_{\Delta v} g \rangle_{\mathcal{H}_s} + c \|D_{\Delta v} g\|_{\mathcal{H}_s}^2 \quad (3.2.10)$$

which is equivalent to the usual one defined in (3.2.7) as soon as $b^2 < ac$. The goal is then to compute the derivative in time of $\|f(t)\|_{\mathcal{H}_s}$ with f a solution to (3.2.6).

As in the continuous case, the decomposition (3.2.4) is crucial to handle the proof. However, the computations are much more tedious than in the continuous case. One of the main reasons is that some commutators that vanish in the continuous case do not vanish anymore in the discrete one. For example, in the continuous case, it is straightforward to prove that $[\partial_v, \mathcal{L}_s] = \partial_v$. In the discrete case which is much more intricate, we are only able to obtain nice estimates on the scalar product $\langle [D_{\Delta v}, \mathcal{L}_{s,\Delta v}]f, g \rangle_{H_s}$. Namely, we can prove that (see [25, Lemma 5.7])

$$\left| \langle [D_{\Delta v}, \mathcal{L}_{s,\Delta v}]f, g \rangle_{H_s} \right| \lesssim \|f\|_{H_s} \|g\|_{H_s} + \|D_{\Delta v} f\|_{H_s} \|g\|_{H_s}.$$

The proof is tricky and relies on a nice rearrangement of terms and estimates on the decay of M_s , (VM_s) and their derivatives that are not immediate to obtain as in the continuous case where everything is explicit (see [25, Appendix A]). Then, when we compute the derivative in time of $\|f(t)\|_{\mathcal{H}_s}$ with f a solution to (3.2.6), the goal is the same as in the continuous case, we have to control the loss terms that are nonnegative by the gain ones and this rely on the use of functional inequalities of Poincaré and interpolation types (as (3.1.9) and (3.1.10) in the continuous case).

Concerning the Poincaré type inequality, adapting arguments of Wang [208] and exploiting the regularization effect of our equation as in the continuous case, it is quite easy to obtain that we have

$$\mathcal{S}_{\Delta v}(g, g) \gtrsim \|(g - \pi_{\mathcal{L}_{s,\Delta v}} g) M_s^{-1/2}\|_{H_{\Delta v}^s}^2$$

where $\pi_{\mathcal{L}_{s,\Delta v}}$ is defined by (3.2.5) and the $H_{\Delta v}^s$ -norm is defined in (3.2.8)-(3.2.9).

However, to derive interpolation inequalities between the previously introduced discrete fractional Sobolev spaces is much more complicated. Indeed, our proof of such inequalities is only based on nonlocal estimates on fractional Sobolev norms whereas in the continuous case, Fourier transform and Young inequality straightforwardly give this type of interpolation inequalities. Our proof is thus completely different from the standard ones that are usually developed the continuous case. The main interpolation inequality that we obtain is the following: for any $s \in (0, 1)$, we have for suitable g ,

$$\|D_{\Delta v} g\|_{\ell_{\Delta v}^2}^2 \lesssim |g|_{\dot{H}_{\Delta v}^{1-s}} |D_{\Delta v} g|_{\dot{H}_{\Delta v}^s},$$

where the fractional Sobolev norms are defined in (3.2.8). Roughly speaking, thanks to this inequality, we are only able to prove “symmetric interpolation inequalities” (we interpolate H^1 between H^{1-s} and H^{1+s}). We are then able to extend this result to a “non symmetric” framework (by interpolating H^1 between H^s and H^{1+s}) thanks to a sort of iteration argument, which is, to our knowledge, not standard in this context. More precisely, we prove that for any $s \in (0, 1)$, there is $\eta_s > 0$ such that for any $\eta \in (0, \eta_s)$, there is $K(\eta) > 0$ such that for suitable g ,

$$\|D_{\Delta v} g\|_{\ell_{\Delta v}^2}^2 \leq \eta |D_{\Delta v} g|_{\dot{H}_{\Delta v}^s}^2 + K(\eta) \|g\|_{H_{\Delta v}^s}^2.$$

From this, we are able to prove an inequality which is similar to (3.1.10) and which is of crucial use in view of proving that the triple norm defined in (3.2.10) is decreasing along the solutions of (3.2.6): there exist $\eta_0 > 0$ and $\Delta v_0 > 0$ such that for any $\Delta v \in (0, \Delta v_0)$ and for any $\eta \in (0, \eta_0)$, there is $K(\eta) > 0$ such that for any suitable g ,

$$\|g\|_{H_s} \|D_{\Delta v} g\|_{H_s} + \|D_{\Delta v} g\|_{H_s}^2 \leq K(\eta) \left(\mathcal{S}_{\Delta v}(g, g) + \|\pi_{\mathcal{L}_{s,\Delta v}} g\|_{H_s}^2 \right) + \eta \mathcal{S}_{\Delta v}(D_{\Delta v} g, D_{\Delta v} g)$$

where we recall that $\pi_{\mathcal{L}_{s,\Delta v}}$ is defined in (3.2.5). With all those elements and intricate computations, we are able to conclude the proof of Theorem 3.2.3 by making a suitable choice of constants a , b and c in (3.2.10).

3.3 Perspectives

We end this Chapter with some questions that could be investigated for future work on the subject.

3.3.1 Extending other hypocoercivity methods to a discrete framework

An interesting question would be to see if the new hypocoercivity method developed in [87] would be adaptable to a discrete setting. One could of course consider the simplest framework to which this method can be applied by considering that particles lie in the torus and that the uniform geometric condition is satisfied (or even that the collision operator never vanishes in the spatial domain). At first sight, it seems to be a very challenging question because the method developed in [87] is based on integral formulation of the problem and the transport equation is solved by the characteristics method. Basically, one of the ideas is to follow the characteristics to be able to transfer the control that we already have on the microscopic part of the solutions. As a consequence, Eulerian methods as developed in [25] do not seem to be adapted to this kind of problem. An idea would thus be to try to use splitting methods to treat this kind of problem.

3.3.2 Asymptotic preserving schemes for equations with fractional diffusion properties

Vlasov-Lévy-Fokker-Planck models (see [64, 63] by Cesbron, Mellet and Trivisa; Cesbron, Mellet and Puel) such as (3.1.1) as well as other kinetic models with heavy tailed local equilibrium (see [168, 169, 32]

by Mellet; Mellet, Mischler and Mouhot; Ben Abdallah, Mellet and Puel) have been studied extensively in the recent years because of their asymptotic properties. Because of the slow decay in velocity, the macroscopic diffusion limits of these kinetic equations are fractional diffusion equations in the space variable. These asymptotics are obtained after an anomalous rescaling of the kinetic equation. The design of appropriate asymptotic preserving numerical schemes in these limits has been investigated in [73, 74, 209] by Crouseilles, Hivert and Lemou; Wang and Yan. The difficulties are twofold: the stiffness in ε and the stiffness coming from the importance of large velocities coming from the slow decay in velocity at infinity of the equilibrium.

The method developed in [41] by Bouin and Mouhot extends to the fractional diffusive limit the approach from [183, 98] which is for example presented in Chapter 6 – Section 6.1.4. It is based on the study of an eigenvalue problem which arises when one takes the Fourier transform in the spatial variable. When the Fourier variable in x vanishes, the problem boils down to the study of the homogeneous collision operator, which is well-known from a spectral point of view. When the Fourier variable in x is small, we obtain a small perturbation of the spectrum and one can then construct the branches of fluid eigenvalues. It is worth mentioning that the paper [41] is inspired by a one-dimensional result obtained in [156] by Lebeau and Puel. An interesting question would be to extend these types of methods to discrete equations by using discrete Fourier transform. Obtaining Taylor expansions of the eigenvalues which are relevant in particular asymptotic regimes would require more attention than in the continuous case because other parameters, namely the mesh sizes, would arise in the eigenvalue problem.

Short time regularization for fractional diffusive equations and Cauchy theory for the Boltzmann equation without angular cut-off

Summary

This chapter focuses on the Boltzmann equation without cut-off and also regularization properties of fractional diffusive linear equations. The results that are presented here come from two papers written in collaboration with F. Hérau and D. Tonon [137, 136]. In this chapter, we study the Boltzmann equation without cut-off with a moderate angular singularity in an inhomogeneous framework. The main result obtained in [136] is about the Cauchy theory of this equation in a close to equilibrium setting and in a “large” functional framework. Among other arguments, the proof relies on an enlargement argument as presented in Section 2.2 which can be developed thanks to short time regularization estimates (as presented in Section 2.3). These estimates are obtained for the linearized Boltzmann operator (which, roughly speaking, behaves as a fractional Laplacian in velocity with anisotropic features) in [136] but also for a simpler model which is the fractional Kolmogorov equation in [137]. These regularization estimates for the linearized Boltzmann equation are crucial in view of carrying out our Cauchy theory. More generally, regularization estimates for equations with fractional diffusive properties are interesting on their own and a large part of the chapter is devoted to this problematic (see Section 4.1 for the fractional Kolmogorov case and Subsection 4.2.4 for the Boltzmann one).

4.1 Regularization for the fractional Kolmogorov equation

4.1.1 The model

In this part, we consider $s \in (0, 1)$ and we focus on the fractional Kolmogorov equation that reads:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = -(-\Delta_v)^s f & \text{in } \mathbf{R}^+ \times \mathbf{T}^d \times \mathbf{R}^d \\ f(0) = f_{\text{in}} & \text{in } \mathbf{T}^d \times \mathbf{R}^d, \end{cases} \quad (4.1.1)$$

where the fractional Laplacian $(-\Delta_v)^s$ has been defined in (3.1.2). It is worth recalling that contrary to the previous Chapter where we were interested in large time behavior of solutions, in this part, we are only interested in obtaining short time regularization estimates. As a consequence, we allow ourselves to work with a collision operator that do not enjoy coercivity properties but only elliptic ones, namely the fractional Laplacian.

4.1.2 Notations

For convenience, we introduce the following strictly positive operators

$$\mathcal{P}_v^2 := \text{Id} - \Delta_v \quad \text{and} \quad \mathcal{P}_x^2 := \text{Id} - \Delta_x \quad (4.1.2)$$

as well as the associated family of Fourier multipliers: for $\alpha, \beta \in \mathbf{R}$,

$$\mathcal{P}_v^\alpha := (\text{Id} - \Delta_v)^{\frac{\alpha}{2}} \quad \text{and} \quad \mathcal{P}_x^\beta := (\text{Id} - \Delta_x)^{\frac{\beta}{2}}$$

which act on Schwartz functions as follows: for $h \in \mathcal{S}(\mathbf{T}^d \times \mathbf{R}^d)$,

$$\widehat{\mathcal{P}_v^\alpha h}(\xi, \eta) = \langle \eta \rangle^\alpha \widehat{h}(\xi, \eta) \quad \text{and} \quad \widehat{\mathcal{P}_x^\beta h}(\xi, \eta) = \langle \xi \rangle^\beta \widehat{h}(\xi, \eta)$$

where the hat corresponds to the Fourier transform in both x (with corresponding variable $\xi \in \mathbf{Z}^d$) and v (with corresponding variable $\eta \in \mathbf{R}^d$). We also introduce the corresponding Sobolev spaces $H_v^\alpha H_x^\beta$ defined through their corresponding norms

$$\|h\|_{H_v^\alpha H_x^\beta}^2 := \sum_{\xi \in \mathbf{Z}^d} \int_{\eta \in \mathbf{R}^d} \left(\langle \eta \rangle^{2\alpha} + \langle \xi \rangle^{2\beta} \right) \widehat{h}(\xi, \eta)^2 d\eta.$$

Remark 4.1.1. When $\alpha, \beta \in \mathbf{N}$, it defines an equivalent norm to the one introduced in (1.3.2), we shall use the same notation for both norms. Similarly, when $\alpha \in (0, 1)$, it defines an equivalent norm to the one introduced in (1.3.3) and we shall use the same notation for both norms.

4.1.3 Main result

As explained in Section 2.3, a natural question is whether a solution f to (4.1.1) benefits from some regularization induced by the elliptic properties of the fractional Laplacian $-(-\Delta_v)^s$ in both variables x and v . The main result in that respect that we obtain is the following one:

Theorem 4.1.2 ([136], Theorem 1.1). *Let f be a solution to (4.1.1) with $f_{\text{in}} \in L_{x,v}^2$. Then, the following estimates hold: for any $t \in (0, 1]$,*

$$\|f(t)\|_{L_x^2 H_v^s} \lesssim \frac{1}{\sqrt{t}} \|f_{\text{in}}\|_{L_{x,v}^2} \quad \text{and} \quad \|f(t)\|_{H_{x,v}^s} \lesssim \frac{1}{t^{\frac{1}{2}+s}} \|f_{\text{in}}\|_{L_{x,v}^2}. \quad (4.1.3)$$

As already mentioned, this result has already been proved in the case $s = 1$ in [133] by Hérau and we give here a result concerning the cases $s \in (0, 1)$ following essentially the same core of the method which has been explained in Section 2.3 (notice that by taking $s = 1$ in (4.1.3), one recovers the estimates stated in (2.3.5)): we want to find a weighted in time functional which shall be a Lyapunov functional for our equation for small times. Let us emphasize that the main difficulty for the fractional case is to find this good entropy function (it is of course not the same as in the non fractional case $s = 1$ defined in (2.3.2)). In the homogeneous case, regularization properties of the fractional Fokker-Planck equation have also been used in [198] by the author of this memoir (thanks to a fractional Nash inequality) in order to study the large-time behavior of solutions to this equation in a large functional framework.

Remark 4.1.3. Let us mention that the result presented in Chapter 3 is reminiscent of this one in terms of method of proof. Recall that it was about H^1 -hypo-coercivity for a fractional Fokker-Planck equation (which can be linked to the fractional Kolmogorov one by a self-similar change of variables) and the proof relies on the introduction of a nice functional which is a Lyapunov one for the equation at stake. Note that in the case of hypo-coercivity, using interpolation tricks, one can perform the study without introducing fractional derivatives in this functional. On the contrary, when one wants to obtain short time regularization estimates with the method presented below, the introduction of fractional derivatives seem to be necessary.

4.1.4 Strategy of proof

First, remark that without loss of generality, we can replace f by $e^{-t} f$ with f solution to (4.1.1) and thus suppose that f is solution to

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = -(-\Delta_v)^s f - f & \text{in } \mathbf{R}^+ \times \mathbf{T}^d \times \mathbf{R}^d \\ f(0) = f_{\text{in}} & \text{in } \mathbf{T}^d \times \mathbf{R}^d. \end{cases} \quad (4.1.4)$$

Indeed, proving short time regularization estimates on f solution to (4.1.1) or (4.1.4) is equivalent since $1 \lesssim e^t \lesssim 1$ for any $t \in [0, 1]$. In the rest of this part, we thus suppose that f is solution to (4.1.4). This trick will allow us to work with the operators Λ_v and Λ_x defined in (4.1.2) (whose inverse are easily defined) in our functional.

We shall follow the lines of the proof given in [133] which has been recalled in Section 2.3 and introduce an adapted entropy functional defined for all $t \in [0, 1]$ by

$$\mathcal{F}(g, t) := \|g\|_{L_{x,v}^2}^2 + a t \|\mathcal{P}_v^{s-1} \nabla_v g\|_{L_{x,v}^2}^2 + 2b t^{1+s} \Re \langle \mathcal{P}_v^{s-1} \nabla_v g, \mathcal{P}_x^{s-1} \nabla_x g \rangle_{L_{x,v}^2} + c t^{1+2s} \|\mathcal{P}_x^{s-1} \nabla_x g\|_{L_{x,v}^2}^2 \quad (4.1.5)$$

for well-chosen constants a, b, c . We also have denoted for $w \in \{x, v\}$:

$$\|\mathcal{P}_w^{s-1} \nabla_w h\|_{L_{x,v}^2}^2 := \sum_{k=1}^d \|\mathcal{P}_w^{s-1} \partial_{w_k} h\|_{L_{x,v}^2}^2$$

and

$$\langle \mathcal{P}_v^{s-1} \nabla_v h, \mathcal{P}_x^{s-1} \nabla_x h \rangle_{L_{x,v}^2} := \sum_{k=1}^d \langle \mathcal{P}_v^{s-1} \partial_{v_k} h, \mathcal{P}_x^{s-1} \partial_{x_k} h \rangle_{L_{x,v}^2}.$$

First, using Cauchy-Schwarz inequality and choosing a, b and c such that $b^2 < ac$, we remark that this functional controls the $L_x^2 H_v^s$ and $H_{x,v}^s$ norms. More precisely, for any $t \in [0, 1]$, we have:

$$\mathcal{F}(g, t) \gtrsim \|g\|_{L_{x,v}^2}^2 + t \|\mathcal{P}_v^{s-1} \nabla_v g\|_{L_{x,v}^2}^2 + t^{1+2s} \|\mathcal{P}_x^{s-1} \nabla_x g\|_{L_{x,v}^2}^2$$

so that for any $t \in [0, 1]$,

$$\mathcal{F}(g, t) \gtrsim t \|g\|_{L_x^2 H_v^s} \quad \text{and} \quad \mathcal{F}(g, t) \gtrsim t^{1+2s} \|g\|_{H_{x,v}^s}.$$

Then, the proof consists in estimating the derivative in time of each term of the functional $\mathcal{F}(f(t), t)$ with f solution to (4.1.4) by using Fourier transform. We do not enter into details here but we give a few key elements of the proof.

- First, since the transport operator is skew-adjoint, for the first term, we have

$$\frac{1}{2} \frac{d}{dt} \|f\|_{L_{x,v}^2}^2 = \langle -(-\Delta_v)^s f - f, f \rangle_{L_{x,v}^2} \lesssim - \left\langle (1 + |\eta|^{2s}) \widehat{f}, \widehat{f} \right\rangle_{\ell_\xi^2 L_\eta^2} \lesssim - \|\langle \eta \rangle^s \widehat{f}\|_{\ell_\xi^2 L_\eta^2}^2.$$

- The other nonnegative terms that naturally arise in the computations are given by $-t \|\langle \eta \rangle^{2s} \widehat{f}\|_{\ell_\xi^2 L_\eta^2}^2$ and $-t^{1+2s} \|\langle \eta \rangle^s \langle \xi \rangle^s \widehat{f}\|_{\ell_\xi^2 L_\eta^2}^2$ which respectively come from the derivative in time of the second and fourth terms of the functional.
- Note that as in Section 2.3, in the computations of the derivative in time of the second term, since the fractional Laplacian in velocity do not commute with the transport operator, we have to handle nonnegative terms. In particular, we have to bound the following remainder:

$$t \left\langle \langle \eta \rangle^{2s-1} \langle \xi \rangle \widehat{f}, \widehat{f} \right\rangle_{\ell_\xi^2 L_\eta^2}.$$

In the classical case, a direct use of Cauchy-Schwarz inequality was enough to compensate the corresponding nonnegative term by nonpositive ones (see (2.3.6)). Here, we have to use a refined Young inequality as follows: for $\delta > 0$,

$$t \langle \eta \rangle^{2s-1} \langle \xi \rangle \lesssim \delta^s t^{1+s} \langle \eta \rangle^{s-1} \langle \xi \rangle^{s+1} + \delta^{-1} \langle \eta \rangle^{2s} \quad (4.1.6)$$

and choose $\delta > 0$ small enough at some point.

- In Section 2.3, the key commutator equality was given by (2.3.3). Here, the equivalent fundamental commutator computation is the following one: for $1 \leq k \leq d$,

$$[\mathcal{P}_v^{s-1} \partial_{v_k}, v \cdot \nabla_x] = \mathcal{P}_v^{s-1} \partial_{x_k} + (1-s) \partial_{v_k} \sum_{j=1}^d \partial_{v_j} \mathcal{P}_v^{s-3} \partial_{x_j}, \quad (4.1.7)$$

which can be proven by writing the symbol of $[\cdot, \cdot]$ thanks to the Poisson bracket.

Thanks to this, we are able to get another nonnegative term: $-t^{1+s} \left\| \langle \eta \rangle^{\frac{s-1}{2}} \langle \xi \rangle^{\frac{s+1}{2}} \widehat{f} \right\|_{\ell_\xi^2 L_\eta^2}^2$. The nonnegative remainders that appear in the computations have to be treated carefully thanks to Young's inequality in the spirit of (4.1.6).

We are then able to make an appropriate choice of the constants $0 < c \ll b \ll a \ll 1$ (so that the condition $b^2 < ac$ is satisfied) in order to be able to control all the nonnegative terms by the nonpositive ones.

4.2 Perturbative Cauchy theory for the Boltzmann equation

4.2.1 The model

In this part, we consider equation (1.1.1) in dimension $d = 3$ in the torus with $Q = Q_B$ which is the Boltzmann operator defined in (1.1.3) satisfying the properties (1.1.6), (1.1.7) and (1.1.8) with the additional assumptions that $\gamma \in (0, 1)$ and $s \in (0, 1/2)$ (moderate angular singularity). Notice that these assumptions cover the physical case of hard potentials. We study this equation in a close to equilibrium framework, we thus rewrite our equation in terms of the perturbation $f := F - \mu$ where μ is defined in (1.1.15). The equation that we shall consider is thus:

$$\begin{cases} \partial_t f = \Lambda f + Q_B(f, f) & \text{in } \mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3 \\ f(0) = f_{\text{in}} := F_{\text{in}} - \mu & \text{in } \mathbf{T}^3 \times \mathbf{R}^3, \end{cases} \quad (4.2.1)$$

where the linearized operator Λ has been defined in (1.1.27) and \mathcal{L} is given by (1.1.19) with $Q = Q_B$.

4.2.2 Notations

We start by defining the functional spaces that we shall use more specifically in this part. Following works from Alexandre et al. (see [4, 6, 7, 8, 5]), we introduce an anisotropic norm that we denote $\|\cdot\|_{\dot{H}_v^{s,*}}$ and which is defined through

$$\|g\|_{\dot{H}_v^{s,*}}^2 := \int_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathcal{S}^2} b_\delta(\cos \theta) \mu_* \langle v_* \rangle^{-\gamma} \left(g' \langle v' \rangle^{\frac{\gamma}{2}} - g \langle v \rangle^{\frac{\gamma}{2}} \right)^2 d\sigma dv_* dv. \quad (4.2.2)$$

In this definition, γ is the power of the kinetic factor in (1.1.8) and μ is given by (1.1.15) with $d = 3$. Moreover, we recall that b is the angular function of the Boltzmann kernel which satisfies (1.1.7) and we define b_δ as the following truncation of b : $b_\delta(\cos \theta) := \chi_\delta(\theta) b(\cos \theta)$ with δ fixed (recall that $\chi_\delta \in \mathcal{D}(\mathbf{R})$ is a truncation function that have been introduced in Section 1.3) so that estimate (4.2.8) holds. Since the constant δ is fixed, we do not mention the dependency of the norm defined above with respect to δ . For a weight function $m = m(v)$, we also introduce the space $H_v^{s,*}(m)$ associated with the norm

$$\|g\|_{H_v^{s,*}(m)} := \|g\|_{L_v^2(\langle v \rangle^{\frac{\gamma}{2}} m)} + \|gm\|_{\dot{H}_v^{s,*}}. \quad (4.2.3)$$

The notation $\dot{H}_v^{s,*}$ is explained by the following estimate [137, Lemma 2.1]: for $m(v) = \langle v \rangle^k$ with $k \geq 0$ and $g \in H_v^s(\langle v \rangle^{\gamma/2+s} m)$, the following estimate holds

$$\delta^{2-2s} \|g\|_{H_v^s(\langle v \rangle^{\frac{\gamma}{2}} m)} \lesssim \|g\|_{H_v^{s,*}(m)} \lesssim \|g\|_{H_v^s(\langle v \rangle^{\frac{\gamma}{2}+s} m)}.$$

Moreover, we introduce the space $\mathcal{H}_x^n \mathcal{H}_v^\ell(m)$, $(n, \ell) \in \mathbb{N}^2$ which is associated to the following norm:

$$\|h\|_{\mathcal{H}_x^n \mathcal{H}_v^\ell(m)} := \sum_{|\alpha| \leq \ell, |\beta| \leq n, |\alpha| + |\beta| \leq \max(\ell, n)} \left\| \partial_v^\alpha \partial_x^\beta h \right\|_{L_{x,v}^2(m \langle v \rangle^{-2|\alpha|s-2|\beta|s})}. \quad (4.2.4)$$

Notice that, compared to classical weighted Sobolev spaces, we have introduced polynomial weights whose order decreases when the order of derivation increases. In the v variable, this is necessary to obtain nice dissipativity properties at the linearized level while in the x variable, it is necessary to close our nonlinear estimates. Finally, for $n \in \mathbb{N}$, we also define the space $\mathcal{H}_{x,v}^{n,s,*}(m)$ associated with the norm

$$\|f\|_{\mathcal{H}_{x,v}^{n,s,*}(m)}^2 := \sum_{0 \leq j \leq n} \int_{\mathbb{T}^3} \|\nabla_x^j f\|_{H_v^{s,*}(\langle v \rangle^{-2js} m)}^2 dx$$

In what follows, the weight m will be fixed and more precisely, we set $m(v) := \langle v \rangle^k$ for any $v \in \mathbb{R}^d$ with $k > 21/2 + \gamma + 22s$ (with γ and s introduced in (1.1.7) and (1.1.8)). The functional spaces

$$X := \mathcal{H}_x^3 L_v^2(m) \quad \text{and} \quad Y := \mathcal{H}_{x,v}^{3,s,*}(m) \quad (4.2.5)$$

will be crucial in the development of our Cauchy theory.

4.2.3 Main result

The main theorem that we obtain is the following:

Theorem 4.2.1 ([137], Theorem 1.1). *Consider the Banach spaces X and Y defined in (4.2.5). There exists a constant $\eta_0 > 0$ such that if $f_{\text{in}} \in X$ has vanishing mass, momentum, energy and satisfies $\|f_{\text{in}}\|_X \leq \eta_0$, then there exists a unique global solution $f \in L^\infty(\mathbb{R}^+, X) \cap L^2(\mathbb{R}^+, Y)$ to the Boltzmann equation (4.2.1) which moreover satisfies*

$$\sup_{t \geq 0} e^{2\lambda t} \|f(t)\|_X^2 + \int_0^\infty e^{2\lambda t} \|f(t)\|_Y^2 dt \lesssim \|f_{\text{in}}\|_X^2,$$

for any $0 < \lambda < \lambda_0$ where $\lambda_0 > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator in X by (4.2.6).

We first give a short state of the art on the Cauchy theory for the equation under study. For general large data, we refer to the paper of DiPerna and Lions [88] for global existence of the so-called renormalized solutions in the case of the Boltzmann equation with cut-off. This notion of solution has been extended to the case of long-range interactions by Villani [203] and Alexandre, Villani [9] (thanks to compactness results established by Lions [164]) where they construct global renormalized solutions with a defect measure. We also mention the work of Desvillettes and Villani [86] that proves the convergence to equilibrium of a priori smooth solutions for both Boltzmann and Landau equations for large initial data. Let us point out the fact that a consequence of our result combined with the one of Desvillettes and Villani is a proof of the exponential H-theorem: we can show exponential decay in time of solutions to the fully nonlinear Boltzmann equation, conditionally to some regularity and moment bounds (the assumption on the exponential lower bound can be removed thanks to the work of Mouhot [178]). As noticed in [117] by Gualdani, Mischler and Mouhot for example in Theorem 5.19, the result of Desvillettes and Villani which is expressed in terms of relative entropy can be translated into stronger norms. This fact allows to do the link between their result and ours.

In a close-to-equilibrium framework, Gressman and Strain [116] in parallel with Alexandre et al. [7] have developed a Cauchy theory in spaces of type $H_x^n H_v^\ell(\mu^{-1/2})$. One of the famous difficulty of the Boltzmann equation without cut-off is to well understand coercivity estimates. In both papers [7] and [116], the gain induced is seen and understood through a non-isotropic norm reminiscent of our $H_v^{s,*}$ -norm defined in (4.2.2)-(4.2.3). Our strategy uses this type of approach but we also exploit the

fact that the linearized Boltzmann operator can be seen as a pseudo-differential operator in order to understand the gain of regularity induced by the linearized operator. Following the strategy presented in Section 2.3, it allows us to obtain regularization estimates (quantified in time) on the semigroup associated to the linearized operator. To end this brief review, we also mention a series of papers by Alexandre et al. [4, 5, 6, 7, 8] in which the Boltzmann equation without cut-off is studied in various aspects (different type of collision kernels, Cauchy theory in exponentially weighted spaces, regularity of the solutions etc...).

Let us underline the fact that Theorem 4.2.1 improves previous results on the Cauchy theory associated to the Boltzmann equation without cut-off for hard potentials in a perturbative setting. Indeed, we have enlarged the space in which the Cauchy theory has been developed in the sense that the weight of our space is much less restrictive (it is polynomial instead of the inverse Maxwellian equilibrium) and we also require few assumptions on the derivatives, in particular no derivatives in the velocity variable. However, we need integrability on three derivatives in the space variable (Gressman and Strain only require integrability on two derivatives in x in [116]): this is the counterpart of the gain in weight we have obtained. Indeed, our framework is less favorable and needs more attention due to the lack of symmetry of the operator in our spaces to obtain nonlinear estimates on the Boltzmann collision operator. And thus, to close our estimates, we require integrability on three derivatives in x .

A challenging question is to obtain a perturbative Cauchy theory in L^∞ type spaces. In a series of works [142, 193, 143, 144, 145] by Imbert, Mouhot and Silvestre, the authors have developed a general approach concerning estimates in L^∞ spaces, which are naturally associated to the standard macroscopic quantities (mass, energy and entropy). More recently, some new results mixing tools used in the above-mentioned works and close-to-equilibrium analysis have been proven in [18, 194] respectively by Alonso et al. and by Silvestre and Snelson. In the latter, the authors show how to combine a short-time existence theorem, the conditional regularity estimates of Imbert and Silvestre [145], and the global trend-to-equilibrium result of Desvillettes and Villani [86] to obtain a proof of existence in a L^∞ setting of global smooth solutions for the non cut-off Boltzmann equation when the initial data is close to equilibrium. The result has been extended to very soft potentials more recently by Henderson, Snelson and Tarfulea [131].

4.2.4 Strategy of proof

Linear study. Since we are working in a perturbative framework, an important part of our proof is devoted to the study of the linearized operator $\Lambda = -v \cdot \nabla_x + \mathcal{L} = -v \cdot \nabla_x + Q_B(\cdot, \mu) + Q_B(\mu, \cdot)$. In order to develop our Cauchy theory in a “large” polynomially weighted functional space, we are first going to establish nice decay estimates on the semigroup $(S_\Lambda(t))_{t \geq 0}$ in this type of spaces. More precisely, we want to prove estimates of type

$$\|S_\Lambda(t)(\text{Id} - \pi_\Lambda)\|_X \lesssim e^{-\lambda_0 t}, \quad \forall t \geq 0, \quad (4.2.6)$$

where we recall that π_Λ is defined in (1.1.28).

To this end, we are going to use the enlargement method presented in Section 2.2. The small space of reference will be $E := H_{x,v}^3(\mu^{-1/2})$ in which some hypocoercivity results on Λ have been established by Mouhot and Neumann [181] so that hypothesis (H1) of Theorem 2.2.1 is satisfied in E . Then, we have to exhibit an appropriate splitting of Λ into two operators \mathcal{A} and \mathcal{B} that satisfy the other assumptions of Theorem 2.2.1. We recall that $\chi_R \in \mathcal{D}(\mathbf{R})$ for $R > 0$ is a truncation function that has been introduced in Section 1.3. We then define

$$\mathcal{A}g := M\chi_R g \quad \text{and} \quad \mathcal{B}g := \Lambda g - \mathcal{A}g = -v \cdot \nabla_x g + \mathcal{L}g - \mathcal{A}g \quad (4.2.7)$$

for some positive constants M and R to be chosen carefully.

Dissipation estimates. In order to check assumption (H2) of Theorem 2.2.1, the first step is to establish a coercivity-type inequality of the following form: for k large enough and for $\delta > 0$ small enough,

$$\langle \mathcal{L}g, g \rangle_{L_v^2(m)} \leq -c_0 \delta^{2-2s} \|g\|_{\dot{H}_v^{s,*}(m)}^2 - c_0 \delta^{-2s} \|g\|_{L_v^2(\langle v \rangle^{\frac{\gamma}{2}} m)}^2 + C_\delta \|g\|_{L_v^2}^2 \quad (4.2.8)$$

where c_0 is a universal positive constant, C_δ is a positive constant depending on δ and the norm $\|\cdot\|_{\dot{H}_v^{s,*}(m)}$ is defined in (4.2.3). We do not give the proof of this estimate which is quite technical but mention that it relies on a splitting of the collision kernel into two parts (which allow us to isolate the grazing collisions) and on the cancellation lemma coming from [2] by Alexandre et al.. Notice that the grazing collisions part is the one which provide the gain of regularity while the gain of decay comes from the cut-off part of the operator.

Then, choosing suitable constants M and R , we can deduce that our operator \mathcal{B} is indeed dissipative in $L_{x,v}^2(m)$ and that it provides a gain of regularity and decay coming from the nonpositive terms of the inequality (4.2.8).

Then, we are able to generalize previous dissipativity results to higher order derivatives spaces of type $\mathcal{H}_x^n \mathcal{H}_v^\ell(m)$ defined through its norm in (4.2.4). Let us highlight the fact that the main difficulties come from the homogeneous case that corresponds to estimate (4.2.8). To go from there to the inhomogeneous case just consists in introducing an equivalent norm to the usual one in inhomogeneous Sobolev spaces and is thus relatively simpler.

Regularization estimates. In order to check that hypothesis (H3) of Theorem 2.2.1 holds true, we need to get short time regularization estimates on the semigroup $S_{\mathcal{B}}(t)$. Indeed, since our operator \mathcal{A} will only provide a gain of weight, the only way to recover a gain of regularity is to exploit the fact that the operator \mathcal{B} induces a gain of regularity. We first remark that in assumption (H3), we actually want to obtain nice regularization estimates for the operator $\mathcal{A}S_{\mathcal{B}}(t)$. Since \mathcal{A} provides any wanted gain in weight, the only thing that matters for $S_{\mathcal{B}}(t)$ is the regularity. Using Duhamel formula, it is thus enough to study the regularization induced by $S_\Lambda(t)$. This will be important since we will be able to rely on results obtained in [3] by Alexandre, Hérau and Li where the operator Λ has been studied from a pseudodifferential point of view. The main result that we obtain is the following:

Theorem 4.2.2 ([137], Theorem 1.2). *Consider $q \geq 0$. The following estimates hold for any $t \in (0, 1]$:*

$$\|S_\Lambda(t)\|_{\mathcal{B}(L_{x,v}^2(\langle v \rangle^{q'}), L_x^2 H_v^s(\langle v \rangle^q))} \lesssim \frac{1}{\sqrt{t}} \quad \text{and} \quad \|S_\Lambda(t)\|_{\mathcal{B}(L_{x,v}^2(\langle v \rangle^{q'}), H_{x,v}^s(\langle v \rangle^q))} \lesssim \frac{1}{t^{\frac{1}{2}+s}} \quad (4.2.9)$$

where $q' \gg q$.

Remark 4.2.3. Notice that there is a loss of weight in those estimates but as mentioned above, we authorize ourselves not to be optimal in our estimates in terms of weights because we have some leeway due to the presence of the operator \mathcal{A} in Duhamel formula.

Let us underline that Theorem 4.2.2 is the first result of regularization quantified in time on the Boltzmann equation without cut-off. It is well-known that the singularity of the Boltzmann kernel in the non cut-off case implies that the Boltzmann operator without cut-off roughly behaves as a fractional Laplacian in velocity:

$$Q(g, f) \approx -C_g(-\Delta_v)^s f + \text{lower order terms}$$

with C_g depending only on the physical properties of g . This type of result has already been studied in the homogeneous and non-homogeneous cases. As mentioned above, the gain in velocity is quite obvious to observe even if it is complicated to understand it precisely: up to now, the most common way to understand it is through an anisotropic norm (see [116] by Gressman and Strain and [6] by Alexandre et al.). It is then natural to expect that the transport term allows to transfer the gain in velocity to the space variable. Let us mention that the paper [3] by Alexandre, Hérau and Li is the first one in which the hypoellipticity features of the operator have been deeply analyzed from a pseudo-differential point of view. The paper [129] by He also provides a deep and quite general understanding of the Boltzmann collisional operator without cut-off, it is robust enough to also capture the same kind of properties for the Landau operator in the grazing collision limit. It in particular gives results that were already known on the linearized operator such as sharp coercivity estimates.

Remark 4.2.4. We can notice that the time rates of regularization in Theorem 4.2.2 are the same as the ones obtained for the fractional Kolmogorov equation in Theorem 4.1.2. This is something that we expected because of the above-mentioned fractional diffusion features of the linearized Boltzmann operator.

To prove Theorem 4.2.2, we follow the same strategy as the one presented in Section 2.3 and more precisely the one described in Section 4.1 for the fractional Kolmogorov equation. The study is actually much harder in the case of the Boltzmann equation without cut-off since the gain in regularity is less clear and consists in an anisotropic gain of fractional derivatives: we have to exploit the fact that one can write a part the Boltzmann linearized operator as a pseudo-differential operator, in the spirit of what has been done in [3]. Indeed, we adapt here some ideas from there allowing to do computations for operators - including the Boltzmann one - whose symbols are in an adapted class. The computations are done using the Wick quantization, widely studied in particular by Lerner (see [159] and [160]), which has very nice positivity properties. This allows to adapt to the Boltzmann case the Lyapunov strategy already presented in Section 4.1 for the fractional Kolmogorov equation. Let us give a few more details on our strategy.

As mentioned above, it is enough to study regularization properties of only a well-chosen part of the operator Λ . More precisely, to prove Theorem 4.2.2, we define the operator Λ_1 by

$$\begin{aligned}\Lambda_1 g &:= -K \langle v \rangle^{Y+2s} g - v \cdot \nabla_x g + \int_{\mathbf{R}^3 \times \mathbf{S}^2} \widetilde{B}_1(v - v_*, \sigma) \mu'_*(g' - g) d\sigma dv_* \\ &=: -K \langle v \rangle^{Y+2s} g - v \cdot \nabla_x g + \Lambda_1^{\text{collision}} g,\end{aligned}$$

for some positive constant K and where the kernel \widetilde{B}_1 is defined by

$$\widetilde{B}_1(v - v_*, \sigma) := \chi(|v' - v|) |v - v_*|^Y b(\cos \theta).$$

It is worth mentioning that truncating $|v' - v|$ is a way to truncate grazing collisions since there holds $|v' - v| = \sin(\theta/2) |v - v_*|$. The operator $\Lambda_2 := \Lambda - \Lambda_1$ is then defined accordingly and one of its important properties is that it does not induce a loss of regularity in velocity when one estimates its Sobolev norms (there is only a loss of weight). Then, in order to work in flat spaces, we introduce the operator Λ_1^m defined as $\Lambda_1^m := m \Lambda_1 (m^{-1} \cdot)$. As previously, we can decompose it into several parts as $\Lambda_1^m = -K \langle v \rangle^{Y+2s} - v \cdot \nabla_x + \Lambda_1^{m, \text{collision}}$.

Remark 4.2.5. In the case $m = 1$, the operator $\Lambda_1^{m, \text{collision}}$ is actually the main one studied in [3]:

$$\Lambda_1^{1, \text{collision}} =: -\tilde{a}_0(v, D_v),$$

where \tilde{a}_0 is a real symbol in (v, η) defined through

$$\tilde{a}_0(v, \eta) := \int_{\mathbf{R}_\beta^3} \frac{d\beta}{|\beta|^{3+2s}} \int_{E_{0, \beta}} d\alpha \widetilde{b}(\alpha, \beta) \mathbf{1}_{|\alpha| \geq |\beta|} \chi(\beta) \mu(\alpha + v) |\alpha + \beta|^{Y+1+2s} (1 - \cos(\eta \cdot \beta))$$

thanks to Carleman representation (see for example [137, Lemma B.2]). An important estimate on this symbol proven in [3] is the following: there exists $c > 0$ such that

$$-c \langle v \rangle^{Y+2s} + \langle v \rangle^Y (1 + |\eta|^2 + |v \wedge \eta|^2)^s \lesssim \tilde{a}_0(v, \eta) \lesssim \langle v \rangle^Y (1 + |\eta|^2 + |v \wedge \eta|^2)^s.$$

This estimate makes clear the anisotropic and fractional behavior of the Boltzmann collisional operator.

We also introduce a_0 the Weyl symbol of operator $\tilde{a}_0(v, D_v)$, so that

$$a_0^w = \tilde{a}_0(v, D_v).$$

We refer to [137, Appendix C] for the definitions of quantizations that we use. Note that a_0 is not real anymore but shall conserve good ellipticity properties. Denoting then

$$\bar{a}(v, \eta) := (m \# a_0 \# m^{-1})(v, \eta) + K \langle v \rangle^{Y+2s},$$

where $\#$ denotes the usual Weyl composition and we omit the dependency of a with respect to K in our notation, we have $\Lambda_1^m = -\bar{a}^w - v \cdot \nabla_x$.

Before defining our functional that is intended to be a Lyapunov functional for our equation, we introduce some weights that will play the role of the classical derivation symbols whose corresponding operators had been introduced in (4.1.5). We define for $(v, \eta) \in \mathbf{R}^6$:

$$\lambda_v^2(v, \eta) := \langle \eta \rangle^2 + \langle v \wedge \eta \rangle^2 + \langle v \rangle^2 \quad \text{and} \quad p(v, \eta) = \langle v \rangle^Y \lambda_v^{2s} + K \langle v \rangle^{Y+2s}.$$

We shall need their counterparts in the ξ variable (considered as a parameter) instead of η and thus also introduce

$$\lambda_x^2(v, \eta) := \langle \xi \rangle^2 + \langle v \wedge \xi \rangle^2 + \langle v \rangle^2 \quad \text{and} \quad q(v, \eta) := \langle v \rangle^Y \lambda_x^{2s} + K \langle v \rangle^{Y+2s}.$$

We eventually introduce a mixed symbol

$$\omega(v, \eta) := \langle v \rangle^Y \lambda_x^{s-1} \lambda_v^{s-1} (\eta \cdot \xi + (v \wedge \eta) \cdot (v \wedge \xi))$$

which will be crucial in the analysis. The main idea in the proof of the regularization result is to use the positivity preserving property of the Wick quantization. The crucial point is to be able to “compare” our operator \bar{a}^w with quantizations of the simpler symbols p and q we just introduced for K large enough. For this, we refer to [137, Lemmas 3.9 to 3.12]. For example, we are able to prove that for K large enough,

$$\Re \langle \bar{a}^w g, g \rangle_{L_{x,v}^2} \gtrsim \langle p^{\text{Wick}} g, g \rangle_{L_{x,v}^2}.$$

We finally define our functional

$$\mathcal{F}(\varphi, t) := \|\varphi\|_{\ell_\xi^2 L_v^2}^2 + at \langle p^{\text{Wick}} \varphi, \varphi \rangle_{\ell_\xi^2 L_v^2} + 2b t^{1+s} \langle \omega^{\text{Wick}} \varphi, \varphi \rangle_{\ell_\xi^2 L_v^2} + ct^{1+2s} \langle q^{\text{Wick}} \varphi, \varphi \rangle_{\ell_\xi^2 L_v^2}.$$

We can prove (see [137, Lemma 3.15]) that if φ is solution to $\partial_t \varphi = -\bar{a}^w \varphi - iv \cdot \xi \varphi$, then for well-chosen constants a, b and c ,

$$\frac{d}{dt} \mathcal{F}(\varphi(t), t) \leq 0, \quad \forall t \in [0, 1].$$

To conclude the proof, it remains to compare the bounds provided by this inequality with classical derivation operators. We refer to [137, Proof of Proposition 3.1] for this but we mention for example that we have

$$\langle p^{\text{Wick}} g, g \rangle_{L_{x,v}^2} \gtrsim \left\| \langle v \rangle^{\frac{Y}{2}} \langle D_v \rangle^s g \right\|_{L_{x,v}^2}.$$

Remark 4.2.6. We did not entered into theoretical details here in order to lighten the presentation. It is however important to mention a few key elements that allow us to perform the above mentioned computations. The first point is to verify that the introduced symbols and weights are indeed in a suitable symbolic calculus with large parameter K uniformly in the parameter ξ . Moreover, working with K large enough allows to get invertibility of operators and define powers of operators easily when they are associated to symbols that have suitable properties, essentially independently of the quantization. We also refer to [189] where Rachid provides a more robust framework in Appendix A: a general statement for Weyl Hörmander tools with a large parameter K is given.

Nonlinear study. In order to come back to the nonlinear problem, we follow the strategy used in [117] by Gualdani, Mischler and Mouhot for the hard spheres Boltzmann equation. More precisely, the first step is to introduce a norm which is equivalent to the usual one in the space X and which is dissipative for the whole operator Λ , namely, for $g \in (\text{Ker } \Lambda)^\perp$, we define

$$\|g\|_X^2 := \eta \|g\|_X^2 + \int_0^\infty \|S_\Lambda(t)e^{\lambda_1 t} g\|_{X_1}^2 dt$$

where $X_1 := \mathcal{H}_x^3 L_v^2(\langle v \rangle^{\tilde{k}})$ with $\tilde{k} \ll k$ and $0 < \lambda_1 < \lambda_0$ with λ_0 the optimal rate of decay of $S_\Lambda(t)$ in X (see (4.2.6)). Then, we have the following estimate: for $g \in (\text{Ker } \Lambda)^\perp$, there exists $C_0 > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \|S_\Lambda(t)g\|_X^2 \leq -\lambda_1 \|S_\Lambda(t)g\|_X^2 - C_0 \|S_\Lambda(t)h\|_Y^2$$

where we recall that Y is defined in (4.2.5). Then, using the short time regularization estimates (we actually use a dual version of Theorem 4.2.2) and sharp nonlinear estimates proven in [137, Lemma 2.4], any solution f to (4.2.1) satisfies the following differential inequality: for any $0 < \lambda_1 < \lambda_0$, there holds

$$\frac{1}{2} \frac{d}{dt} \|f(t)\|_X^2 \leq -\lambda_1 \|f(t)\|_X^2 - (C_0 - C \|f(t)\|_X) \|f(t)\|_Y^2$$

for some positive constant C . Thanks to this a priori estimate, one can end the proof of Theorem 4.2.1 by introducing an appropriate iterative scheme.

4.3 Perspective

To conclude this Chapter, we present a problem in the study of which the tools for short time regularization estimates we presented might be used.

On the Vlasov-Poisson fractional Fokker-Planck equation

The problem presented here is about the Vlasov-Poisson fractional Fokker-Planck equation in $\mathbf{R}^+ \times \mathbf{T}^d \times \mathbf{R}^d$ with $d = 2, 3$ which reads

$$\begin{cases} \partial_t F + v \cdot \nabla_x F + E \cdot \nabla_v F = -(-\Delta_v)^s F + \nabla_v \cdot (vF), \\ E(t, x) := -\nabla_x \phi \quad \text{with} \quad -\Delta_x \phi = \rho[F] - 1, \\ F(0, x, v) = F_{\text{in}}(x, v), \end{cases}$$

with an initial data of global mass 1 and where we recall that $\rho[F]$ is the local mass of F defined in (1.1.22). We would like to develop a Cauchy theory close to the global equilibrium μ_s defined in (3.1.4) and study the large-time behavior of the constructed solutions. To this end, we introduce the perturbation $f := F - \mu_s$ and thus look at the equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu_s + E \cdot \nabla_v f = -(-\Delta_v)^s f + \nabla_v \cdot (vf), \\ E(t, x) = -\nabla_x \phi \quad \text{with} \quad -\Delta_x \phi = \rho[f], \\ f(0, x, v) = f_{\text{in}}(x, v) := F_{\text{in}}(x, v) - \mu_s. \end{cases} \quad (4.3.1)$$

A first goal would be to look for the lowest regularity to impose on the initial data f_{in} to prove existence and uniqueness of mild solutions of (4.3.1) using a fixed point strategy in the spirit of the paper [135] by Hérau and Thomann. This is strongly related to the regularization properties of the linearized Vlasov-Poisson Fokker-Planck operator with fractional diffusion. One can hope to rely on results presented in Section 4.1 by slightly adapting them. More precisely, it seems that the linearized

Poisson term introduce a simple bounded perturbation and moreover, that one can get rid of the drift term by using a suitable self-similar change of variables.

A second goal would be to study the long-time behavior of solutions by hypocoercivity methods. Notice that the framework of [135] is a bit different since in the latter paper, the authors were working in the whole space and they also had a small parameter in front of the Poisson term. We would like to work in a close to equilibrium setting, which is not completely equivalent to adding this small parameter. Indeed, at the linearized level, when one wants to study exponential stability via hypocoercivity, to handle the Poisson term $E \cdot \nabla_v \mu_s$, the presence of a small parameter helps a lot since this term becomes a small bounded perturbation whereas in our framework of small initial data, the Poisson term cannot be seen as a perturbation term and thus requires a new approach for hypocoercivity. Let us mention that hypocoercivity results have already been obtained in in the non fractional diffusion case in [1] by Addala et al. for a Vlasov-Poisson Fokker-Planck equation.

Let us end this part by mentioning that getting hypocoercivity results for fractional diffusive equations is a question of interest even without considering self-induced fields by the particules as in the Vlasov-Poisson equation. We refer to Subsection 5.5.2 for a more detailed discussion on this topic.

Hypocoercivity for linear kinetic equations in bounded domains

Summary

This chapter presents results of hypocoercivity for various linear collisional kinetic equations in bounded domains with general Maxwell boundary condition established in an article [33] written in collaboration with A. Bernou, K. Carrapatoso and S. Mischler. It extends the L^2 -hypocoercivity method presented in Section 2.1 to general geometries of bounded domains. It in particular provides results of convergence to equilibrium for several equations with physical local conservation laws (including the linearized Boltzmann and Landau equations). Our approach allows us to treat general domains with all type of boundary conditions in a unified framework. It is worth mentioning that our result includes the case of vanishing accommodation coefficient and thus the specific case of the specular reflection boundary condition.

5.1 The model

In this chapter, we study a linear collisional kinetic equation in a bounded domain with general Maxwell boundary condition. More precisely, we consider a smooth enough bounded domain $\Omega \subseteq \mathbf{R}^d$, $d \geq 2$, we denote by $O := \Omega \times \mathbf{R}^d$ the interior set of phase space and $\Sigma := \partial\Omega \times \mathbf{R}^d$ the boundary set of phase space. For a (variation of a) density function $f = f(t, x, v)$, $t \geq 0$, $x \in \Omega$, $v \in \mathbf{R}^d$, we then look at the following equation

$$\begin{cases} \partial_t f &= \Lambda f := -v \cdot \nabla_x f + \mathcal{L}f & \text{in } \mathbf{R}^+ \times O \\ \gamma_- f &= \mathcal{R} \gamma_+ f & \text{on } \mathbf{R}^+ \times \Sigma, \end{cases} \quad (5.1.1)$$

where $\gamma_{\pm} f$ denote the trace of f at the boundary set (γ_+ corresponding to outgoing velocities and γ_- to incoming velocities) and where \mathcal{L} and \mathcal{R} stand for two linear collisional operators, all of which will be described in details below. Our goal is to investigate the long-time behavior of solutions to this linear equation. In order to do so, we will prove an hypocoercivity result using a general and robust approach inspired by previous works on L^2 -hypocoercivity, the method presented in Section 2.1.

The motivation to study this problem directly comes from kinetic equations such as the Boltzmann and the Landau ones and more precisely from their study in a close to equilibrium regime since, as we have already seen, the form of equation (5.1.1) is exactly the form of the linearized Boltzmann and Landau equations.

The assumptions (A1)-(A2)-(A3) made below on the collisional operator \mathcal{L} are met by the linearized Boltzmann and Landau equations for the so-called hard and moderately soft potentials and thus include the Boltzmann hard spheres case.

Remark 5.1.1. Let us point out that our analysis extends to more general assumptions that allow us to deal with linearized Boltzmann and Landau operators corresponding to softer potentials (for which there is no spectral gap). We won't give details on this fact in this memoir and refer to [33] where the proof is complete, we just mention that our result provides a sub-exponential rate of decay for these equations.

Remark 5.1.2. It is also worth noting that by a straightforward adaptation of our method, we can also treat linear operators preserving only mass such as the Fokker-Planck operator or the relaxation operator. We believe that our analysis is also new in this setting.

The boundary condition. Let us now describe the boundary condition in (5.1.1). We assume that the boundary $\partial\Omega$ is smooth enough so that the outward unit normal vector $n(x)$ at $x \in \partial\Omega$ is well-defined as well as $d\sigma_x$ the Lebesgue surface measure on $\partial\Omega$ (we refer to [33, Section 1] for more details on the assumptions made on $\partial\Omega$). We then define $\Sigma_{\pm}^x := \{v \in \mathbf{R}^d; \pm v \cdot n(x) > 0\}$ the sets of outgoing (Σ_+^x) and incoming (Σ_-^x) velocities at the point $x \in \partial\Omega$ as well as

$$\Sigma_{\pm} := \left\{ (x, v) \in \Sigma; \pm n(x) \cdot v > 0 \right\} = \left\{ (x, v); x \in \partial\Omega, v \in \Sigma_{\pm}^x \right\}.$$

We denote by γf the trace of f on Σ , and by $\gamma_{\pm} f = \mathbf{1}_{\Sigma_{\pm}} \gamma f$ the traces on Σ_{\pm} . The boundary condition in (5.1.1) thus takes into account how particles are reflected by the wall and takes the form of a balance between the values of the trace γf on the outgoing and incoming velocities subsets of the boundary. It is worth emphasizing that because $f \in \text{Dom}(\Lambda)$, the trace functions $\gamma_{\pm} f$ are well defined. We refer the interested reader to [28, 65] by Bardos and Cessenat for the classical definition of the trace of a solution to a transport equation as well as to [171, 170] by Mischler and [42] by Boyer for a more modern approach.

We assume that the reflection operator acts locally in time and position, namely

$$(\mathcal{R}\gamma_+ f)(t, x, v) = \mathcal{R}_x(\gamma_+ f(t, x, \cdot))(v) \quad (5.1.2)$$

and more specifically it is a possibly position dependent Maxwell boundary condition operator

$$\mathcal{R}_x(g(x, \cdot))(v) := (1 - \alpha(x))g(x, R_x v) + \alpha(x)Dg(x, v), \quad (5.1.3)$$

for any $(x, v) \in \Sigma_-$ and for any function $g : \Sigma_+ \rightarrow \mathbf{R}$. Here $\alpha : \partial\Omega \rightarrow [0, 1]$ is a Lipschitz function, called the accommodation coefficient, R_x is the specular reflection operator

$$R_x v := v - 2n(x)(n(x) \cdot v),$$

and D is the diffusive operator

$$Dg(x, v) := c_{\mu}\mu(v)\widetilde{g}(x) \quad \text{with} \quad \widetilde{g}(x) := \int_{\Sigma_+^x} g(x, w) n(x) \cdot w \, dw, \quad (5.1.4)$$

where the constant $c_{\mu} := (2\pi)^{1/2}$ is such that $c_{\mu}\widetilde{\mu} = 1$ and we recall that μ stands for the standard Maxwellian (1.1.15). The boundary condition (5.1.3) corresponds to the *pure specular reflection* boundary condition when $\alpha \equiv 0$ and it corresponds to the *pure diffusive* boundary condition when $\alpha \equiv 1$.

It is worth emphasizing that when γf satisfies the boundary condition in (5.1.1) with \mathcal{R} defined in (5.1.3), for any test function $\varphi = \varphi(v)$ and any $x \in \partial\Omega$,

$$\int_{\mathbf{R}^d} \gamma f \varphi n(x) \cdot v \, dv = \int_{\Sigma_+^x} \gamma_+ f n(x) \cdot v \left[\varphi - (1 - \alpha(x))\varphi \circ R_x - \alpha(x)c_{\mu}\overline{\varphi \circ R_x \mu} \right] \, dv. \quad (5.1.5)$$

Consequently, whatever is the accommodation coefficient α , taking $\varphi = 1$ so that $\varphi \circ R_x = c_{\mu}\overline{\varphi \circ R_x \mu} = 1$, we get

$$\int_{\mathbf{R}^d} \gamma f n(x) \cdot v \, dv = 0, \quad (5.1.6)$$

which means that there is no flux of mass at the boundary (no particle goes out nor enters in the domain). Assuming now $\alpha \equiv 0$, making the choice $\varphi(v) = |v|^2$ and observing that $|R_x v|^2 = |v|^2$, we get

$$\int_{\mathbf{R}^d} \gamma f |v|^2 n(x) \cdot v \, dv = 0, \quad (5.1.7)$$

which means that there is no flux of energy at the boundary in the case of the pure specular reflection boundary condition.

The collisional operator. Let us now describe the hypotheses made on the collisional linear operator \mathcal{L} involved in the linear evolution equation (5.1.1). We assume that the operator acts locally in time and position, namely

$$(\mathcal{L}f)(t, x, v) = \mathcal{L}(f(t, x, \cdot))(v),$$

that the operator has mass, velocity and energy conservation laws, namely

$$\int_{\mathbf{R}^d} (\mathcal{L}g)(v) \varphi(v) dv = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2, \quad (5.1.8)$$

and for any nice enough function g , and that the operator has a spectral gap in the classical Hilbert space $L_v^2(\mu^{-1/2})$.

We assume that the operator \mathcal{L} is a closed operator with dense domain $\text{Dom}(\mathcal{L})$ in $L_v^2(\mu^{-1/2})$ which satisfies the following assumptions.

(A1) Its kernel is given by

$$\text{Ker}(\mathcal{L}) = \text{Span} \left\{ \mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu \right\},$$

and the projection $\pi_{\mathcal{L}}$ onto $\text{Ker}(\mathcal{L})$ is given by (1.1.20).

(A2) The operator is self-adjoint on $L_v^2(\mu^{-1/2})$ and negative $\langle \mathcal{L}f, f \rangle_{L_v^2(\mu^{-1/2})} \leq 0$, so that its spectrum is included in \mathbf{R}^- , and (5.1.8) holds true for any $g \in \text{Dom}(\mathcal{L})$. We assume furthermore that \mathcal{L} satisfies a coercivity estimate, more precisely that there is a positive constant $\lambda > 0$ such that for any $f \in \text{Dom}(\mathcal{L})$, one has

$$\langle \mathcal{L}f, f \rangle_{L_v^2(\mu^{-\frac{1}{2}})} \leq -\lambda \|f^\perp\|_{L_v^2(\mu^{-\frac{1}{2}})}^2, \quad (5.1.9)$$

where we recall that $f^\perp = f - \pi_{\mathcal{L}}f$.

(A3) For any polynomial function $\phi = \phi(v) : \mathbf{R}^d \rightarrow \mathbf{R}$ of degree less than 4, there holds $\mu\phi \in \text{Dom}(\mathcal{L})$, so that there exists a constant $C_\phi > 0$ such that

$$\|\mathcal{L}(\phi\mu)\|_{L_v^2(\mu^{-\frac{1}{2}})} \leq C_\phi.$$

Conservation laws. Without loss of generality, we shall assume hereafter that the domain Ω verifies

$$|\Omega| = \int_{\Omega} dx = 1 \quad \text{and} \quad \int_{\Omega} x dx = 0. \quad (5.1.10)$$

One easily obtains from (5.1.8), the Stokes theorem and (5.1.6) that any solution f to equation (5.1.1) satisfies the conservation of mass

$$\frac{d}{dt} \int_{\Omega} f dv dx = \int_{\Omega} (\mathcal{L}f - v \cdot \nabla_x f) dv dx = 0.$$

In the case of the specular reflection boundary condition, that is $\alpha \equiv 0$ in (5.1.3), some additional conservation laws appear. On the one hand, one also has the conservation of energy

$$\frac{d}{dt} \int_{\Omega} |v|^2 f dv dx = \int_{\Omega} |v|^2 (\mathcal{L}f - v \cdot \nabla_x f) dv dx = 0,$$

because of (5.1.8), the Stokes theorem again and (5.1.7). On the other hand, if the domain Ω possesses rotational symmetry, we also have the conservation of the corresponding angular momentum. More precisely, we define the set of all infinitesimal rigid displacement fields

$$\mathcal{R} := \left\{ x \in \Omega \mapsto Ax + b \in \mathbf{R}^d; A \in M_{d \times d}^a(\mathbf{R}), b \in \mathbf{R}^d \right\}, \quad (5.1.11)$$

where $M_{d \times d}^a(\mathbf{R})$ denotes the set of skew-symmetric $d \times d$ -matrices with real coefficients, as well as the linear manifold of *centered* infinitesimal rigid displacement fields preserving Ω

$$\mathcal{R}_\Omega := \left\{ R \in \mathcal{R} \mid b = 0, R(x) \cdot n(x) = 0, \forall x \in \partial\Omega \right\}. \quad (5.1.12)$$

We observe here that, thanks to the assumption (5.1.10), we can work only with *centered* infinitesimal rigid displacement fields preserving Ω . Indeed, if R is an infinitesimal rigid displacement field preserving Ω , that is, $R(x) = Ax + b \in \mathcal{R}$ is such that $R(x) \cdot n(x) = 0$ on $\partial\Omega$, then

$$\begin{aligned} |b|^2 &= \int_\Omega \nabla(b \cdot x) \cdot (Ax + b) \, dx \\ &= - \int_\Omega (b \cdot x) \operatorname{div}(Ax + b) \, dx + \int_{\partial\Omega} (b \cdot x)(Ax + b) \cdot n(x) \, d\sigma_x = 0, \end{aligned}$$

and thus $b = 0$. When the set \mathcal{R}_Ω is not reduced to $\{0\}$, that is when Ω has rotational symmetries, then one deduces the conservation of angular momentum

$$\frac{d}{dt} \int_O R(x) \cdot v f \, dv \, dx = 0, \quad \forall R \in \mathcal{R}_\Omega.$$

Indeed if $R \in \mathcal{R}_\Omega$, there exists $A \in M_{d \times d}^a(\mathbf{R})$ such that $R(x) = Ax$ for any $x \in \Omega$. We then compute, using integration by parts,

$$\begin{aligned} \frac{d}{dt} \int_O R(x) \cdot v f \, dv \, dx &= \int_O Ax \cdot v (-v \cdot \nabla_x f + \mathcal{L}f) \, dv \, dx \\ &= \int_O \partial_{x_k} (Ax \cdot v) v_k f \, dv \, dx - \int_\Sigma Ax \cdot v \gamma f n(x) \cdot v \, dv \, d\sigma_x \\ &= - \int_\Sigma Ax \cdot v \gamma f n(x) \cdot v \, dv \, d\sigma_x, \end{aligned}$$

thanks to the velocity conservation law (5.1.8) and the fact that A is skew-symmetric. For the boundary term, using (5.1.5) with $\varphi(x, v) := Ax \cdot v$ and $\alpha \equiv 0$, we get

$$\begin{aligned} \int_\Sigma Ax \cdot v \gamma f n(x) \cdot v \, dv \, d\sigma_x &= \int_{\Sigma_+} Ax \cdot (v - R_x v) \gamma_+ f |n(x) \cdot v| \, dv \, d\sigma_x \\ &= 2 \int_{\Sigma_+} (Ax \cdot n(x)) \gamma_+ f |n(x) \cdot v|^2 \, dv \, d\sigma_x = 0, \end{aligned}$$

because $v - R_x v = 2(n(x) \cdot v)n(x)$ and $R \in \mathcal{R}_\Omega$.

5.2 Notations

We introduce the following functional space:

$$\mathcal{H} := L_{x,v}^2(\mu^{-\frac{1}{2}}). \quad (5.2.1)$$

For simplicity, we also introduce the following notations: for a suitable distribution g , we write $g_\pm := \gamma_\pm g$, we introduce the operator $D^\perp := \operatorname{Id} - D$, where D is given by (5.1.4) and the following space

$$\partial\mathcal{H}_+ := L^2(\Sigma_+; \mu^{-1}(v)n(x) \cdot v).$$

For $g \in \mathcal{H}$, we also introduce the following conditions:

$$\int_O g \, dv \, dx = 0, \quad (\text{C1})$$

$$\int_{\mathcal{O}} |v|^2 g \, dv \, dx = 0, \quad (\text{C2})$$

$$\int_{\mathcal{O}} R(x) \cdot v g \, dv \, dx = 0, \quad \forall R \in \mathcal{R}_{\Omega}. \quad (\text{C3})$$

Our proof relies on the study of elliptic problems in the position variable x and some of them shall involve the symmetric gradient of a vector field, we recall that it is defined in Section 1.3 where the skew-symmetric gradient of a vector field is also defined.

5.3 Main result

The main result of this chapter is the following:

Theorem 5.3.1 ([33], Theorem 1.1). *There exists a scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{H}}$ on the space \mathcal{H} defined in (5.2.1) so that the associated norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the usual norm $\|\cdot\|_{\mathcal{H}}$, and for which the linear operator Λ satisfies the following coercivity estimate: there is a positive constant $\tilde{\lambda} > 0$ such that*

$$\langle\langle -\Lambda f, f \rangle\rangle_{\mathcal{H}} \geq \tilde{\lambda} \|f\|_{\mathcal{H}}^2$$

for any $f \in \text{Dom}(\Lambda)$ satisfying the boundary condition in (5.1.1) with \mathcal{R} defined in (5.1.2)-(5.1.3), assumption (C1) and furthermore assumptions (C2)-(C3) in the specular reflection case ($\alpha \equiv 0$ in (5.1.3)).

Let us now mention that this hypocoercivity result enables us to deduce an exponential stability result for our equation (5.1.1) supplemented with the boundary condition (5.1.3).

Theorem 5.3.2 ([33], Theorem 1.2). *Let $f_{\text{in}} \in \mathcal{H}$ satisfying assumption (C1) and furthermore assumptions (C2) and (C3) in the specular reflection case ($\alpha \equiv 0$ in (5.1.3)). There exist positive constants $\tilde{\lambda}, C > 0$ such that for any solution f to (5.1.1) associated to the initial data f_{in} , there holds*

$$\|f(t)\|_{\mathcal{H}} \leq C e^{-\tilde{\lambda}t} \|f_{\text{in}}\|_{\mathcal{H}}, \quad \forall t \geq 0.$$

Remark 5.3.3. As already mentioned, in [33, Section 4], we broaden our study to the case where the linearized operator only enjoys a weak coercivity estimate to obtain results of weak hypocoercivity and sub-exponential stability, we do not present this result here.

Remark 5.3.4. In [33, Section 5], we extend our study to a rescaled version of (5.1.1) which naturally arises in the analysis of diffusion limit problems, we obtain hypocoercivity and stability results uniformly with respect to the rescaling parameter, we neither present it here but we mention that in Section 6.2, a similar result is obtained and presented in the simpler case of the torus, the treatment of the rescaling parameter is completely similar to the one that is made in [33, Section 5] (see (6.2.19) for the definition of the new norm involving the rescaling parameter).

This result improves existing results regarding hypocoercivity in a bounded domain for the linearized Boltzmann and Landau equations (and consequently for their long-time stability) in three regards.

- We consider a general, smooth enough, convex or non-convex domain.
- The L^2 -estimates that we establish are constructive, which means that they depend constructively of some collisional constants (that appear in the estimates (A2)-(A3) satisfied by the collisional operator \mathcal{L}) and some geometrical constants depending on the domain Ω (that appear in some Poincaré and Korn inequalities which can be made explicit, at least for a domain with simple geometry);
- Our method encompasses the three boundary conditions (pure diffusive, specular reflection and Maxwell) in a single treatment. In particular, we can solve the Maxwell boundary condition in the case where the accommodation coefficient α vanishes everywhere or on some subset of the boundary.

We here briefly mention some similar coercivity estimates or exponential stability results established in the last decade for linear kinetic equations (mainly for the linearized Boltzmann equation) in a bounded domain. These ones have then been used for proving global existence of solutions to nonlinear equation in a close-to-equilibrium regime and convergence to the equilibrium in the long-time asymptotic. As already mentioned, Guo [122] has first proved a $L^2_{x,v}$ -coercivity estimate for the cut-off Boltzmann equation with hard potentials or hard-spheres by using non-constructive technique in two cases: the specular reflection boundary condition with strictly convex and analytic domains Ω and the pure diffusive boundary condition assuming the domain Ω is smooth and convex. These results have been generalized by Briant and Guo [45] who derived constructive exponential stability estimates in $L^2_{x,v}$ for any positive and constant accommodation coefficient $\alpha \in (0, 1)$, with no more convexity assumptions on Ω . For the same equation endowed with specular reflection boundary condition, a still non-constructive L^2 estimate was derived in the convex setting, without analyticity assumptions on the domain, by Kim and Lee [150]. The authors then extended their results to periodic cylindrical domain with non-convex analytic cross-section [151].

Furthermore, the only results we are aware of in the case of long-range interaction, that is, for non-cut-off Boltzmann and Landau collision operators in a bounded domain, are the recent works of Guo et al. [123] (see also [124]) and [92] by Dong, Guo and Ouyang for respectively the Landau equation and the Vlasov-Poisson-Landau equation with specular reflection boundary condition, and Duan et al. [94] for non-cut-off Boltzmann and Landau equations in a finite channel with inflow or specular reflection boundary conditions. Let us also mention the more recent work by Deng [81] in which the nonlinear Boltzmann equation without cutoff is studied in a bounded domain with Maxwell boundary conditions except for pure specular reflection. To end this short review of the existing results, we mention the work [185] in which Ouyang and Silvestre provide conditional L^∞ -estimates for the non-cutoff Boltzmann equation with in-flow, bounce-back, specular-reflection and diffuse-reflection boundary conditions.

In Subsection 1.1.4, we briefly presented the DiPerna and Lions existence theory of global weak (renormalized) solutions of arbitrary amplitude to the Boltzmann equation for initial data satisfying only the physically natural condition that the total mass, energy and entropy are finite. The extension to the case of a bounded domain with reflection conditions (including specular reflection, pure diffusive reflection and Maxwell reflection) has been then obtained in [127, 22, 170, 172] by Hamdache; Arkeryd and Maslova; and Mischler. We must emphasize that our treatment of boundary terms bears some similarity with the analysis made in [172] in order to take advantage of the information provided by Darrozès and Guiraud inequality [75].

5.4 Strategy of proof

As already mentioned, the strategy of our proof is based on the L^2 -hypocoercivity method presented in Section 2.1. We thus define a new scalar product on \mathcal{H} of type (2.1.12). The construction of the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ on the space \mathcal{H} begins with the usual scalar product, which gives us a control of the microscopic part f^\perp defined in (1.1.21), and after that, step by step, new terms are added to it in order to control all components of the macroscopic part $\pi_{\mathcal{L}}f$ defined in (1.1.20). Let us however point out that our proof is a variant of previous proofs and differs from them by several aspects.

- The order between the ∇ operator and the Δ^{-1} operator is the one from Guo’s approach [122, 45] rather than the one from Dolbeault, Mouhot and Schmeiser’s approach [90, 91]. That is important in order to handle the rather singular operator involved by the boundary condition.
- The choice of the mean operator $\tilde{\pi}f$ differs from the one used in [122, 45, 44, 151] but looks very much like the one in [93, 94, 56] by Duan, Duan et al. and Carrapatoso et al.. It allows to deal with general Maxwell boundary condition (and the possibility that α vanishes somewhere or everywhere) but leads to a first natural control of the symmetric gradient of the momentum component of the macroscopic part $\nabla^s u$ instead of the full derivative ∇u as in Guo’s approach.

- The definition of the Δ^{-1} operator has to be chosen wisely in order to handle the general Maxwell boundary condition and the mean operator $\tilde{\pi}f$. We thus need to establish natural $H^{-1} \rightarrow H^1$ and $L^2 \rightarrow H^2$ regularity estimates for some classical elliptic problems but associated with somehow unusual boundary conditions.

In what follows, we start by giving a few more details about the third point above.

Elliptic problems

Poincaré inequalities and Poisson equation. First, we shall introduce an auxiliary *Poisson equation with Robin or Neumann boundary conditions*, which is devised in order to control *mass* and *energy* terms of $\pi_{\mathcal{L}}f$:

$$\begin{cases} -\Delta \mathbf{u} = \xi & \text{in } \Omega, \\ (2 - \alpha) \nabla \mathbf{u} \cdot \mathbf{n} + \alpha \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.4.1)$$

for a scalar source term $\xi : \Omega \rightarrow \mathbf{R}$. Remark that when $\alpha \equiv 0$ then (5.4.1) corresponds to the Poisson equation with homogeneous Neumann boundary condition. Otherwise, (5.4.1) corresponds to the Poisson equation with homogeneous Robin (or mixed) boundary condition.

We define the Hilbert spaces

$$V_1 := H^1(\Omega) \quad \text{and} \quad V_0 := \left\{ w \in H^1(\Omega); \int_{\Omega} w \, dx = 0 \right\}$$

endowed with the $H^1(\Omega)$ -norm, and next

$$V_{\alpha} := \begin{cases} V_1 & \text{if } \alpha \neq 0 \\ V_0 & \text{if } \alpha \equiv 0. \end{cases}$$

On V_{α} , we define the bilinear form

$$a_{\alpha}(w_1, w_2) := \int_{\Omega} \nabla w_1 \cdot \nabla w_2 \, dx + \int_{\partial\Omega} \frac{\alpha}{2 - \alpha} w_1 w_2 \, d\sigma_x.$$

We start by a result on Poincaré-type inequalities:

Proposition 5.4.1. *There hold*

$$\|w\| \lesssim \|\nabla w\|, \quad \forall w \in V_0, \quad (5.4.2)$$

and

$$\|w\|^2 \lesssim a_{\alpha}(w, w), \quad \forall w \in V_1. \quad (5.4.3)$$

The first inequality is nothing but the classical Poincaré-Wirtinger inequality. The second inequality (which is probably also classical) can be obtained thanks to a non constructive proof by contradiction based on a compactness argument. We can now state a result on the existence, uniqueness and regularity of solutions to (5.4.1).

Theorem 5.4.2 ([33], Theorem 2.2). *For any given $\xi \in L^2(\Omega)$, there exists a unique $\mathbf{u} \in V_{\alpha}$ solution to the variational problem*

$$a_{\alpha}(\mathbf{u}, w) = (\xi, w), \quad \forall w \in V_{\alpha}. \quad (5.4.4)$$

Assuming furthermore that $\langle \xi \rangle = 0$ when $\alpha \equiv 0$, there holds $\mathbf{u} \in H^2(\Omega)$, \mathbf{u} verifies the elliptic equation (5.4.1) almost everywhere and

$$\|\mathbf{u}\|_{H^2(\Omega)} \lesssim \|\xi\|. \quad (5.4.5)$$

The proof of the first part of the result relies on Lax-Milgram theorem (which can be applied thanks to Poincaré inequalities recalled in Proposition 5.4.1). The result of elliptic regularity can be obtained by proving directional regularity estimates, we refer to [33, Section 2] for more details.

Korn inequalities and the associated elliptic problem. We shall introduce a tailored *Lamé-type system with mixed Robin-type boundary conditions* in order to deal with the *momentum* component of the macroscopic part $\pi_{\mathcal{L}}f$. In what follows, we will write ∇^s for ∇_x^s , and ∇^a for ∇_x^a where we recall that the symmetric and skew-symmetric gradients are defined in (1.3.4)-(1.3.5). We consider the system of equations

$$\begin{cases} -\operatorname{div}(\nabla^s \mathbf{U}) = \Xi & \text{in } \Omega, \\ \mathbf{U} \cdot n = 0 & \text{on } \partial\Omega, \\ (2 - \alpha) [\nabla^s \mathbf{U} n - (\nabla^s \mathbf{U} : n \otimes n)n] + \alpha \mathbf{U} = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.4.6)$$

for a vector-field source term $\Xi : \Omega \rightarrow \mathbf{R}^d$. Because

$$\operatorname{div}(\nabla^s \mathbf{U}) = \Delta \mathbf{U} + \nabla \operatorname{div} \mathbf{U},$$

we see that (5.4.6) is nothing but a Lamé-type system with a kind of homogeneous Robin (or mixed) boundary condition.

We define the Hilbert spaces

$$\mathcal{V}_1 := \{W : \Omega \rightarrow \mathbf{R}^d \mid W \in H^1(\Omega), W \cdot n(x) = 0 \text{ on } \partial\Omega\}$$

and

$$\mathcal{V}_0 := \{W : \Omega \rightarrow \mathbf{R}^d \mid W \in H^1(\Omega), W \cdot n(x) = 0 \text{ on } \partial\Omega, P_\Omega \langle \nabla^a W \rangle = 0\},$$

where P_Ω denotes the orthogonal projection onto the set $\mathcal{A}_\Omega = \{A \in M_{d \times d}^a(\mathbf{R}); Ax \in \mathcal{R}_\Omega\}$ of all skew-symmetric matrices giving rise to a centered infinitesimal rigid displacement field preserving Ω (see (5.1.12) for the definition of \mathcal{R}_Ω). Both spaces are endowed with the $H^1(\Omega)$ -norm. We then denote

$$\mathcal{V}_\alpha := \begin{cases} \mathcal{V}_1 & \text{if } \alpha \neq 0 \\ \mathcal{V}_0 & \text{if } \alpha \equiv 0. \end{cases}$$

We also define on \mathcal{V}_α the bilinear form

$$A_\alpha(W_1, W_2) := \int_\Omega \nabla^s W_1 : \nabla^s W_2 \, dx + \int_{\partial\Omega} \frac{\alpha(x)}{2 - \alpha(x)} W_1 \cdot W_2 \, d\sigma_x,$$

where $M : N := \sum_{ij} m_{ij} n_{ij}$ for two matrices $M = (m_{ij})$, $N = (n_{ij})$.

The coercivity of the bilinear form A_α is related to Korn-type inequalities that we present briefly. The fundamental results about Korn inequalities are the ones by Duvaut and Lions [96] and Desvillettes and Villani [85]. A more recent treatment of Korn's inequality has also been developed by Ciarlet and Ciarlet in [72]. We here state two Korn-type inequalities that are useful for our analysis and whose proofs (that are based on the results above mentioned) can be found in [33, Section 2].

Proposition 5.4.3. *Suppose that $\alpha \neq 0$. For any $W \in H^1(\Omega)$ such that $W(x) \cdot n(x) = 0$ on $\partial\Omega$, there holds*

$$\|W\|_{H^1(\Omega)}^2 \lesssim \|\nabla^s W\|^2 + \left\| \sqrt{\frac{\alpha}{2 - \alpha}} W \right\|_{L^2(\partial\Omega)}^2. \quad (5.4.7)$$

Proposition 5.4.4. *For any vector-field $W \in H^1(\Omega)$ such that $W \cdot n(x) = 0$ on $\partial\Omega$, there holds*

$$\|W\|_{H^1(\Omega)}^2 \lesssim \|\nabla^s W\|^2 + |P_\Omega \langle \nabla^a W \rangle|^2, \quad (5.4.8)$$

where we recall that P_Ω is the orthogonal projection onto $\mathcal{A}_\Omega = \{A \in M_{d \times d}^a(\mathbf{R}); Ax \in \mathcal{R}_\Omega\}$.

We can now state our result concerning the existence, uniqueness and regularity of solutions to the elliptic system (5.4.6).

Theorem 5.4.5 ([33], Theorem 2.11). *For any given $\Xi \in L^2(\Omega)$, there exists a unique solution $\mathfrak{U} \in \mathcal{V}_\alpha$ to the variational problem associated to (5.4.6), namely*

$$A_\alpha(\mathfrak{U}, W) = (\Xi, W), \quad \forall W \in \mathcal{V}_\alpha. \quad (5.4.9)$$

If furthermore Ξ satisfies the condition $\langle \Xi, Ax \rangle = 0$ for any $Ax \in \mathcal{R}_\Omega$ when $\alpha \equiv 0$, then the variational solution \mathfrak{U} to (5.4.6) satisfies $\mathfrak{U} \in H^2(\Omega)$ with

$$\|\mathfrak{U}\|_{H^2(\Omega)} \lesssim \|\Xi\|,$$

and moreover \mathfrak{U} verifies (5.4.6) almost everywhere.

The proof of Theorem 5.4.5 follows the same steps as the proof of Theorem 5.4.2. We don't give details here, we just mention that we obtain the coercivity of A_α in \mathcal{V}_α thanks to the inequalities (5.4.7) for the case $\alpha \neq 0$ and (5.4.8) for the case $\alpha \equiv 0$.

Proof of Theorem 5.3.1

We consider hereafter f satisfying the conditions of Theorem 5.3.1. To lighten the notations, we will write $\rho := \rho[f]$, $u := u[f]$ and $\theta := \theta[f]$ where we recall that these macroscopic quantities are defined in (1.1.22), (1.1.23) and (1.1.24). We thus consider $f \in \text{Dom}(\Lambda)$ satisfying the boundary condition in (5.1.1), so that in particular (5.1.6) holds, which translates into

$$u(x) \cdot n(x) = 0, \quad \forall x \in \partial\Omega. \quad (5.4.10)$$

The assumption made on the global mass of f translates into $\int_\Omega \rho \, dx = 0$. In the specular reflection case ($\alpha \equiv 0$ in (5.1.3)), the additional assumptions correspond to

$$\int_\Omega \theta \, dx = 0 \quad \text{and} \quad \int_\Omega R \cdot u \, dx = 0, \quad \forall R \in \mathcal{R}_\Omega. \quad (5.4.11)$$

Microscopic part. We start with the following result, giving a control of the microscopic part f^\perp and a boundary term.

Lemma 5.4.6. *We have the following inequality:*

$$\langle -\Lambda f, f \rangle_{\mathcal{H}} \geq \lambda \|f^\perp\|_{\mathcal{H}}^2 + \frac{1}{2} \left\| \sqrt{\alpha(2-\alpha)} D^+ f_+ \right\|_{\partial\mathcal{H}_+}^2$$

where $\lambda > 0$ is defined in (5.1.9).

We give the proof of this estimate because it is informative of the way in which we treat boundary terms coming from the transport operator.

Proof. We write

$$\langle -\Lambda f, f \rangle_{\mathcal{H}} = \langle -\mathcal{L}f, f \rangle_{\mathcal{H}} + \langle v \cdot \nabla_x f, f \rangle_{\mathcal{H}}.$$

Thanks to (5.1.9), one has

$$\langle -\mathcal{L}f, f \rangle_{\mathcal{H}} \gtrsim \|f^\perp\|_{\mathcal{H}}^2.$$

For the second term, we first get thanks to an integration by parts

$$\langle v \cdot \nabla_x f, f \rangle_{\mathcal{H}} = \int_O (v \cdot \nabla_x f) f \mu^{-1} \, dx \, dv = \frac{1}{2} \int_\Sigma \gamma f^2 \mu^{-1} n(x) \cdot v \, d\sigma_x \, dv.$$

Writing $\gamma f^2 = f_+^2 \mathbf{1}_{\Sigma_+} + f_-^2 \mathbf{1}_{\Sigma_-}$ and using the boundary condition in (5.1.1), we thus obtain

$$\langle v \cdot \nabla_x f, f \rangle_{\mathcal{H}} = \frac{1}{2} \int_{\Sigma_+} f_+^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv - \frac{1}{2} \int_{\Sigma_-} f_-^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv$$

$$\begin{aligned}
&= \frac{1}{2} \int_{\Sigma_+} f_+^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv \\
&\quad - \frac{1}{2} \int_{\Sigma_-} \left\{ (1 - \alpha(x)) f_+(x, R_x v) + \alpha(x) Df_+(x, v) \right\}^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv.
\end{aligned}$$

We apply the change of variables $v \mapsto R_x v$, so that Σ_- transforms into Σ_+ , which yields

$$\begin{aligned}
\langle v \cdot \nabla_x f, f \rangle_{\mathcal{H}} &= \frac{1}{2} \int_{\Sigma_+} f_+^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv \\
&\quad - \frac{1}{2} \int_{\Sigma_+} \left\{ (1 - \alpha(x)) f_+ + \alpha(x) Df_+ \right\}^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv,
\end{aligned}$$

since $Df_+(x, R_x v) = Df_+(x, v)$ and $|n(x) \cdot R_x v| = |n(x) \cdot v|$. Writing $f_+ = D^\perp f_+ + Df_+$ (where we recall that $D^\perp = \text{Id} - D$), one has

$$\int_{\Sigma_+} f_+^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv = \int_{\Sigma_+} (Df_+)^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv + \int_{\Sigma_+} (D^\perp f_+)^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv,$$

since $Df_+ \perp D^\perp f_+$ in $\partial\mathcal{H}_+$. All together, we conclude to

$$\begin{aligned}
&\langle v \cdot \nabla_x f, f \rangle_{\mathcal{H}} \\
&= \frac{1}{2} \int_{\Sigma_+} \left\{ (Df_+)^2 + (D^\perp f_+)^2 - [(1 - \alpha(x)) D^\perp f_+ + Df_+]^2 \right\} \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv \\
&= \frac{1}{2} \int_{\Sigma_+} \left\{ [1 - (1 - \alpha(x))^2] (D^\perp f_+)^2 - 2(1 - \alpha(x)) Df_+ D^\perp f_+ \right\} \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv \\
&= \frac{1}{2} \int_{\Sigma_+} \alpha(x) (2 - \alpha(x)) (D^\perp f_+)^2 \mu^{-1} |n(x) \cdot v| \, d\sigma_x \, dv.
\end{aligned}$$

We finish the proof by gathering previous estimates. \square

Energy. The goal of this part is to construct a functional in order to control the energy component of the macroscopic part $\pi_{\mathcal{L}} f$. We recall that for a kinetic distribution g , $\theta[g]$ is defined in (1.1.24) so that $\theta = \theta[f]$. We define $\mathbf{u}[\theta]$ as the solution to the elliptic equation (5.4.1) associated to $\xi = \theta \in L_x^2(\Omega)$ given by Theorem 5.4.2, in particular

$$\|\mathbf{u}[\theta]\|_{H_x^2(\Omega)} \lesssim \|\theta\|_{L_x^2(\Omega)}. \quad (5.4.12)$$

It is worth noticing that in the specular reflection case (when $\alpha \equiv 0$ in (5.1.1)), we have $\int_{\Omega} \theta(x) \, dx = 0$ from (5.4.11), so that the solution $\mathbf{u}[\theta]$ to the Poisson equation with Neumann boundary condition is well-defined.

We also introduce the vector $p = (p_i)_{1 \leq i \leq d}$ defined by

$$p_i(v) := v_i \frac{|v|^2 - d - 2}{d},$$

and the associated moment functional $M_p[g] = (M_{p_i}[g])_{1 \leq i \leq d}$ given by

$$M_{p_i}[g] := \int_{\mathbb{R}^d} v_i \frac{|v|^2 - d - 2}{d} g \, dv. \quad (5.4.13)$$

We are able to prove the following result, which gives us a control of the energy θ .

Lemma 5.4.7. *There are constants $\kappa_1, C > 0$ such that*

$$\begin{aligned}
&\left\langle -\nabla_x \mathbf{u}[\theta], M_p[\Lambda f] \right\rangle_{L_x^2(\Omega)} + \left\langle -\nabla_x \mathbf{u}[\theta[\Lambda f]], M_p[f] \right\rangle_{L_x^2(\Omega)} \\
&\quad \geq \kappa_1 \|\theta\|_{L_x^2(\Omega)}^2 - C \|u\|_{L_x^2(\Omega)} \|f^\perp\|_{\mathcal{H}} - C \|f^\perp\|_{\mathcal{H}}^2 - C \left\| \sqrt{\alpha(2 - \alpha)} D^\perp f_+ \right\|_{\partial\mathcal{H}_+}^2.
\end{aligned}$$

We do not give the proof of this lemma which only consists in computations and estimates that are similar to the ones already performed above (especially concerning the boundary terms).

Momentum. In this part, we construct a functional that is devised to control the momentum component of the macroscopic part $\pi_{\mathcal{L}}f$. We recall that for a kinetic distribution g , $u[g]$ is defined in (1.1.23) so that $u = u[f]$. We define $\mathfrak{U}[u]$ as the solution to the elliptic equation (5.4.6) associated to $\Xi = u \in L_x^2(\Omega)$ given by Theorem 5.4.5, whence

$$\|\mathfrak{U}[u]\|_{H_x^2(\Omega)} \lesssim \|u\|_{L_x^2(\Omega)}. \quad (5.4.14)$$

It is worth noting that in the specular reflection case ($\alpha \equiv 0$ in (5.1.1)-(5.1.3)), the condition (5.4.11) holds, and therefore the solution $\mathfrak{U}[u]$ is indeed well-defined.

Considering the matrix $q = (q_{ij})_{1 \leq i, j \leq d}$ given by

$$q_{ij}(v) := v_i v_j - \delta_{ij},$$

we define the associated moment functional $M_q[g] = (M_{q_{ij}}[g])_{1 \leq i, j \leq d}$ as

$$M_{q_{ij}}[g] := \int_{\mathbf{R}^d} (v_i v_j - \delta_{ij}) g \, dv. \quad (5.4.15)$$

Performing a similar analysis as in the previous part for the energy, we are able to prove the following lemma:

Lemma 5.4.8. *There are constants $\kappa_2, C > 0$ such that*

$$\begin{aligned} & \langle -\nabla_x^s \mathfrak{U}[u], M_q[\Lambda f] \rangle_{L_x^2(\Omega)} + \langle -\nabla_x^s \mathfrak{U}[u[\Lambda f]], M_q[f] \rangle_{L_x^2(\Omega)} \\ & \geq \kappa_2 \|u\|_{L_x^2(\Omega)}^2 - C \|f^\perp\|_{\mathcal{H}} \|\rho\|_{L_x^2(\Omega)} - C \|\theta\|_{L_x^2(\Omega)} \|\rho\|_{L_x^2(\Omega)} \\ & \quad - C \|\theta\|_{L_x^2(\Omega)}^2 - C \|f^\perp\|_{\mathcal{H}}^2 - C \left\| \sqrt{\alpha(2-\alpha)} D^\perp f_+ \right\|_{\partial \mathcal{H}_+}^2. \end{aligned}$$

Mass. In this part, we introduce the last functional, which is built in order to control the mass component of the macroscopic part $\pi_{\mathcal{L}}f$. We recall that for a kinetic distribution g , $\rho[g]$ is defined in (1.1.22) so that $\rho = \rho[f]$. We consider $u_N[\rho]$ the solution to the Poisson equation (5.4.1) with Neumann boundary condition associated to $\xi = \rho \in L_x^2(\Omega)$ constructed in Theorem 5.4.2, namely $u_N[\rho]$ satisfies a.e.

$$\begin{cases} -\Delta_x u_N[\rho] = \rho & \text{in } \Omega, \\ \nabla_x u_N[\rho] \cdot n(x) = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.4.16)$$

which is indeed well-defined since $\int_{\Omega} \rho(x) \, dx = 0$. In particular, we have

$$\|u_N[\rho]\|_{H_x^2(\Omega)} \lesssim \|\rho\|_{L_x^2(\Omega)}. \quad (5.4.17)$$

Similarly as for the energy and momentum, one can prove the following result:

Lemma 5.4.9. *There are constants $\kappa_3, C > 0$ such that*

$$\begin{aligned} & \langle -\nabla_x u_N[\rho], u[\Lambda f] \rangle_{L_x^2(\Omega)} + \langle -\nabla_x u_N[\rho[\Lambda f]], u[f] \rangle_{L_x^2(\Omega)} \\ & \geq \kappa_3 \|\rho\|_{L_x^2(\Omega)}^2 - C \left(\|u\|_{L_x^2(\Omega)}^2 + \|\theta\|_{L_x^2(\Omega)}^2 + \|f^\perp\|_{\mathcal{H}}^2 \right) \\ & \quad - C \left\| \sqrt{\alpha(2-\alpha)} D^\perp f_+ \right\|_{\partial \mathcal{H}_+}^2. \end{aligned}$$

End of the proof of Theorem 5.3.1. We define the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{H}}$ on \mathcal{H} by

$$\begin{aligned} \langle\langle f, g \rangle\rangle_{\mathcal{H}} &:= \langle f, g \rangle_{\mathcal{H}} \\ &+ \eta_1 \langle -\nabla_x \mathbf{u}[\theta[f]], M_p[g] \rangle_{L_x^2(\Omega)} + \eta_1 \langle -\nabla_x \mathbf{u}[\theta[g]], M_p[f] \rangle_{L_x^2(\Omega)} \\ &+ \eta_2 \langle -\nabla_x^s \mathbf{u}[u[f]], M_q[g] \rangle_{L_x^2(\Omega)} + \eta_2 \langle -\nabla_x^s \mathbf{u}[u[g]], M_q[f] \rangle_{L_x^2(\Omega)} \\ &+ \eta_3 \langle -\nabla_x \mathbf{u}_N[\rho[f]], u[g] \rangle_{L_x^2(\Omega)} + \eta_3 \langle -\nabla_x \mathbf{u}_N[\rho[g]], u[f] \rangle_{L_x^2(\Omega)} \end{aligned}$$

with $0 \ll \eta_3 \ll \eta_2 \ll \eta_1 \ll 1$, and where we recall that the moments M_p and M_q are defined respectively in (5.4.13) and (5.4.15); $\mathbf{u}[\theta[f]]$ is the solution of the Poisson equation (5.4.1) with data $\theta[f]$; $\mathbf{u}[u[f]]$ is the solution to the elliptic system (5.4.6) with data $u[f]$; $\mathbf{u}_N[\rho[f]]$ is the solution to the Poisson equation with homogeneous Neumann boundary condition (5.4.16) with data $\rho[f]$, and similarly for the terms depending on g . We denote by $\|\cdot\|_{\mathcal{H}}$ the norm associated to the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{H}}$, and we observe that

$$\|f\|_{\mathcal{H}} \lesssim \|\!|f|\!\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}.$$

To end the proof of Theorem 5.3.1, we consider f satisfying the assumptions of Theorem 5.3.1 and estimate $\langle\langle -\Lambda f, f \rangle\rangle_{\mathcal{H}}$. Then, we gather results from Lemmas 5.4.6, 5.4.7, 5.4.8 and 5.4.9, make an appropriate use of Young's inequality and choose suitably the constants η_i for $i = 1, 2, 3$. In the end, we obtain

$$\langle\langle -\Lambda f, f \rangle\rangle_{\mathcal{H}} \gtrsim \|f^\perp\|_{\mathcal{H}}^2 + \|\rho\|_{L_x^2(\Omega)}^2 + \|u\|_{L_x^2(\Omega)}^2 + \|\theta\|_{L_x^2(\Omega)}^2 + \left\| \sqrt{\alpha(2-\alpha)} D^+ f_+ \right\|_{\partial\mathcal{H}_+}^2.$$

This yields the conclusion of Theorem 5.3.1 because

$$\|f^\perp\|_{\mathcal{H}}^2 + \|\rho\|_{L_x^2(\Omega)}^2 + \|u\|_{L_x^2(\Omega)}^2 + \|\theta\|_{L_x^2(\Omega)}^2 = \|f\|_{\mathcal{H}}^2$$

and $\|\cdot\|_{\mathcal{H}}$ is equivalent to $\|\!|\cdot|\!\|_{\mathcal{H}}$. □

5.5 Perspectives

Before starting the next Chapter, we present some closely related problems which might be of interest for future work.

5.5.1 Nonlinear equations in bounded domains

The results presented in this chapter can be seen as a first step towards the global existence and the study of the long-time behavior of solutions to the nonlinear problem (2.4.1) with a boundary condition defined as in (5.1.1)-(5.1.3) in a close-to-equilibrium regime. The goal would be to improve results by Guo [122] who considered the pure diffusive case and by Briant and Guo [45] who only treated the case of "large" enough constant accommodation coefficient: $\alpha(x) \equiv \alpha > 2/3$. It still remains major obstacles coming from the nonlinearity of the collision operator. Nice estimates on the latter are available in spaces that have an algebra structure in the spatial variable x such as L_x^∞ . A first step towards this difficult problem would thus be to get nice linear estimates in L^∞ spaces on the linearized problem. A semigroup approach taking into account the boundary condition but that does not follow characteristics as in [122, 45] seems to be convenient in order to deal with this type of problems in view of applying an argument of enlargement reminiscent of the one presented in Section 2.2.

5.5.2 Equations with heavy tailed equilibria

It would be interesting to adapt our approach to problems whose global equilibria of the equation are heavy tailed. A typical example would be a fractional Fokker-Planck equation as the one studied in Chapter 3. The problem of developing a L^2 -hypo-coercivity approach for this type of equations is related

to the study of fractional diffusion limits. To this matter, we mention a non exhaustive list of papers on this topic [168] by Mellet, [169] by Mellet, Mischler and Mouhot, [156] by Lebeau and Puel, [63] by Cesbron, Mellet and Puel, [41] by Bouin and Mouhot and [77, 78] by Dechicha and Puel.

Let us mention that a L^2 -hypocoercivity approach has already been developed for this type of models in [39] by Bouin, Dolbeault and Lafleche but the general framework in which the authors work does not include bounded domains with boundary conditions. It seems difficult to adapt their method to the case of a bounded domain with general boundary conditions because their analysis is based on a mode by mode analysis in Fourier space for the spatial variable.

When one wants to directly adapt the method presented in Section 2.1, the fact that the equilibrium has polynomial decay at infinity immediately brings a problem. Indeed, when one computes the evolution of the macroscopic quantities $\rho[f]$ and $u[f]$ as done in (2.1.16)-(2.1.17) for the non fractional case, the term $\text{Div}_x \int_{\mathbf{R}^d} v \otimes v \pi_{\mathcal{L}_s} f \, dv$ is not well-defined since $\int_{\mathbf{R}^d} v \otimes v \mu_s(v) \, dv = \infty$. As a consequence, we cannot rely on the micro-macro decomposition as in the non fractional case to obtain the same type of equation as (2.1.18) which makes appear the quantity $\nabla_x \rho[f]$.

An alternative approach would be to consider twisted macroscopic quantities that involve weights in velocity that would allow to circumvent the difficulty coming from the lack of integrability of the equilibrium μ_s . This idea is reminiscent of the one used in [39] and in [41] to study problems of fractional diffusion limit.

Hydrodynamic limit problems

Hydrodynamic limit for the Boltzmann and Landau equations

Summary

This chapter is dedicated to the study of the link between strong solutions of the Boltzmann and Landau equations and the incompressible Navier-Stokes equation.

The first section is dedicated to the study of the Boltzmann equation for hard spheres collisions and presents results established with I. Gallagher in [103]. We prove that the life span of the solutions to the rescaled Boltzmann equation is bounded from below by that of the Navier-Stokes equation. The main feature of our proof is that we perform a fixed point argument by using information on the limit system since we start from solutions of the Navier-Stokes system (which is not the most common viewpoint). We are able to treat general initial data in the whole space in dimensions 2 and 3 and also well-prepared data in the torus, all of this in a unified framework.

The second section focuses on results established in a paper written in collaboration with K. Carrapatoso and M. Rachid [59], these results deal with the Landau equation under the Navier-Stokes scaling in the torus for hard and moderately soft potentials (this only includes the cases in which the linearized Landau operator has a spectral gap). We investigate the Cauchy theory in a perturbative framework and establish some new short time regularization estimates for our rescaled nonlinear Landau equation in the spirit of what has been presented in Section 2.3. Even if the method and the result are close to the ones of Section 2.3, we would like to point out that the method needs some adaptations for several reasons: we deal with the nonlinear Landau equation, the instantaneous expected gain of regularity is anisotropic (see [189] by Rachid for the corresponding hypoelliptic estimates on the linearized Landau collision operator) and we want our estimates giving the gain of regularity in the velocity variable to be uniform in the Knudsen number. Following ideas introduced in [103], we then intertwine these new estimates on the Landau equation with estimates on the Navier-Stokes-Fourier system in order to obtain a result of strong convergence towards the incompressible Navier-Stokes equation.

6.1 Hydrodynamical limit for the Boltzmann equation

6.1.1 The model

In this part, we study equation (1.2.3) in $\mathbf{R}^+ \times \Omega \times \mathbf{R}^d$ with $\Omega = \mathbf{R}^d$ or \mathbf{T}^d in 2 or 3 dimensions that is $d = 2, 3$. We only focus on the case where Q_B is defined by (1.1.3) with $B(v - v_*, \sigma) = |v - v_*|$. The goal is to study the connection between f^ε solution to (1.2.3) and $f \in \text{Ker } \mathcal{L}$ of the form (6.1.5) with (ρ, u, θ) solution to the system (1.2.4).

6.1.2 Notations

The following functional space will be of constant use in this section:

$$X := L_v^\infty H_x^\ell(\langle v \rangle^k) \quad \text{with} \quad \ell > \frac{d}{2} \quad \text{and} \quad k > \frac{d}{2} + 1 \quad (6.1.1)$$

and we recall that the associated norm is given by

$$\|f\|_X := \sup_{v \in \mathbf{R}^d} \langle v \rangle^k \|f(\cdot, v)\|_{H_x^\ell}.$$

Our proof is going to rely on a fixed point argument, we will thus need to find a functional space in which to apply this strategy. It turns out that in the \mathbf{R}^2 -case, to reach this goal, we cannot work directly in $L_t^\infty(X)$ which is the most natural space to consider and we have to introduce a weight in time (note that in the references mentioned below in this context, only the three-dimensional case is treated, in which case it is not necessary to introduce that weight). We thus introduce a function $\chi_\Omega = \chi_\Omega(t)$ defined by

$$\chi_\Omega(t) := \begin{cases} 1 & \text{if } \Omega = \mathbf{T}^d, d = 2, 3, \text{ or } \mathbf{R}^3, \\ \langle t \rangle^{\frac{1}{4}} & \text{if } \Omega = \mathbf{R}^2, \end{cases} \quad \forall t \geq 0.$$

For a given $T > 0$, we then define the associate weighted in time space

$$\mathcal{X}_T := L^\infty(\mathbf{1}_{[0,T]}(t)\chi_\Omega(t), X) \quad (6.1.2)$$

endowed with the norm

$$\|f\|_{\mathcal{X}_T} := \sup_{t \in [0, T]} \chi_\Omega(t) \|f(t)\|_X.$$

6.1.3 Main result

First, we recall that the Leray projector \mathbb{P} has been introduced in Section 1.3 as well as the sum of two functional spaces. Here is our main result:

Theorem 6.1.1 ([103], Theorem 1). *Consider $(\rho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$ in $H^\ell(\Omega)$ if $\Omega \neq \mathbf{R}^2$ and in $H^\ell(\Omega) \cap L^1(\Omega)$ if $\Omega = \mathbf{R}^2$. If $\Omega = \mathbf{T}^d$, we furthermore assume that $\rho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}}$ are mean free. Define*

$$\rho_0 := \frac{2}{d+2}\rho_{\text{in}} - \frac{d}{d+2}\theta_{\text{in}}, \quad u_0 := \mathbb{P}u_{\text{in}}, \quad \theta_0 := -\rho_0. \quad (6.1.3)$$

Let (ρ, u, θ) be the unique solution to (1.2.4) associated with the initial data (ρ_0, u_0, θ_0) on a time interval $[0, T]$. Set

$$f_0(x, v) := \mu(v) \left(\rho_0(x) + u_0(x) \cdot v + \frac{|v|^2 - d}{2} \theta_0(x) \right), \quad (6.1.4)$$

and define on $[0, T] \times \Omega \times \mathbf{R}^d$

$$f(t, x, v) := \mu(v) \left(\rho(t, x) + u(t, x) \cdot v + \frac{|v|^2 - d}{2} \theta(t, x) \right). \quad (6.1.5)$$

The well prepared case. *Assume $\Omega = \mathbf{T}^d$ or \mathbf{R}^d , $d = 2, 3$. There is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, there is a unique solution f^ε to (1.2.3) in $L^\infty([0, T], X)$ with initial data f_0 , and it satisfies*

$$\|f^\varepsilon - f\|_{L^\infty([0, T], X)} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (6.1.6)$$

Moreover, if the solution (ρ, u, θ) to (1.2.4) is defined on \mathbf{R}^+ , then ε_0 depends only on the initial data and not on T and there holds

$$\|f^\varepsilon - f\|_{L^\infty(\mathbf{R}^+, X)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The ill prepared case. *Assume $\Omega = \mathbf{R}^d$, $d = 2, 3$. For all initial data f_{in} in X satisfying*

$$\rho[f_{\text{in}}] = \rho_{\text{in}}, \quad u[f_{\text{in}}] = u_{\text{in}} \quad \text{and} \quad \theta[f_{\text{in}}] = \theta_{\text{in}}$$

(where $\rho[\cdot], u[\cdot]$ and $\theta[\cdot]$ are defined in (1.1.22)-(1.1.23)-(1.1.24)), there is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, there is a unique solution f^ε to (1.2.3) in $L^\infty([0, T], X)$ with initial data f_{in} . It satisfies for all $p > 2/(d-1)$

$$\|f^\varepsilon - f\|_{L^\infty([0, T], X) + L^p([0, T], L_v^\infty(W_x^{\ell, \infty} + H_x^\ell)(\langle v \rangle^k))} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (6.1.7)$$

Moreover, if the solution (ρ, u, θ) to (1.2.4) is defined on \mathbf{R}^+ , then ε_0 depends only on the initial data and not on T and there holds

$$\|f^\varepsilon - f\|_{L^\infty(\mathbf{R}^+, X) + L^p(\mathbf{R}^+, L_v^\infty(W_x^{\ell, \infty} + H_x^\ell)(\langle v \rangle^k))} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Remark 6.1.2. Let us mention that if we work with smooth data, we can obtain a rate of convergence of $\sqrt{\varepsilon}$ in (6.1.6) and (6.1.7) (we did not look for optimality for this rate).

Remark 6.1.3. Notice that f_0 given by (6.1.4) with (ρ_0, u_0, θ_0) defined in (6.1.3) is well prepared in the sense given in (1.2.6).

Remark 6.1.4. Still about the initial data, notice that the last assumption (that the solution (ρ, u, θ) to (1.2.4) is defined on \mathbf{R}^+) always holds when $d = 2$ and is also known to hold for small data in dimension 3 (see [101, 68] by Fujita and Kato; Chemin) or without any smallness assumption in some cases (see examples in [69] in the periodic case, [70] in the whole space for instance by Chemin and Gallagher).

Let us now explain the differences between our result and the ones established in previous works on the topic. Roughly speaking, in the paper by De Masi, Esposito and Lebowitz [76], it is proved that in the torus, if the Navier-Stokes equation has a smooth solution on some interval $[0, T]$, then there also exists a solution to the rescaled Boltzmann equation on this interval of time. Our main theorem is actually reminiscent of this type of result, we try to use information on the limit system (for instance the fact that the Navier-Stokes equations are globally wellposed in two space dimensions) to obtain results on the life span of solutions to the rescaled Boltzmann equation. We would like to emphasize here that in our result, if the solution to the limit equation is global (regardless of its size), then, we are able to construct a global solution to the Boltzmann equation, which is not the case in the aforementioned result [76]. Moreover, we treat both the cases of the torus and the whole space.

In [31], Bardos and Ukai treat the case of the whole space, with a smallness assumption on the initial data which allows them to work with global solutions in time. Their result uses the description of the spectrum of the linearized Boltzmann equation performed by Ellis and Pinsky in [98]. In our result, no smallness assumption is needed and we can thus treat the case of non global in time solutions to the Navier-Stokes equation. We would also like to emphasize that Bardos and Ukai also deal with the case of ill-prepared data but their result is not formulated in terms of strong convergence up to $t = 0$ contrary to the present work (where the strong convergence holds in an averaged sense in time).

Finally, let us mention that more recently, Gervais [108] has extended the functional framework in which our result holds. He has proven a similar result in polynomially weighted space, his strategy is a combination of ours and of the one used in [46] by Briant, Merino-Aceituno and Mouhot in order to get uniform in ε estimates on solutions to (1.2.3) in polynomially weighted spaces. We also point out that our strategy is at the basis of several more recent results of hydrodynamical limit results concerning the Landau equation [59] (the result is presented in Section 6.2), the Boltzmann equation without cut-off [51] by Cao and Carrapatoso, a unified framework which encompasses a large class of kinetic equations (including in particular our result) has also been provided in [109] by Gervais and Lods. We also refer to Subsection 6.3.1 where more details are given on the papers [51, 109].

6.1.4 Strategy of proof

The proof of the theorem mainly relies on a fixed point argument, which enables us to prove that the equation satisfied by the difference g^ε between the solution f^ε of the Boltzmann equation and its expected limit f does have a solution (which is arbitrarily small) as long as f exists. In order to develop this fixed point argument, we have to filter the unknown g^ε by some well chosen exponential function which depends on the solution of the Navier-Stokes-Fourier equation. This enables us to obtain a contraction estimate. Let us also point out that the analysis of the operators that appear in the equation on g^ε is akin to the one made by Bardos and Ukai [31] and it heavily relies on the Ellis and Pinsky decomposition [98] (see (6.1.10)).

In the case of ill-prepared data, the fixed point argument needs some adjusting. Indeed, the linear propagator consists in two classes of operators, one of which vanishes identically when applied to well-prepared case, and in general decays to zero in an averaged sense in time due to dispersive properties. Consequently, we choose to apply the fixed point theorem not to g^ε but to the difference between g^ε

and those dispersive-type remainder terms. This induces some additional terms to estimate, which turn out to be harmless thanks to their dispersive nature.

Before entering into more details, we first mention that in order to apply our fixed point strategy, we are going to use an integral formulation of (1.2.3). To this end, we use the following notations:

$$U^\varepsilon(t) := S_{\Lambda^\varepsilon}(t) \quad \text{and} \quad \Psi^\varepsilon(t)(f, g) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) Q_B(f, g)(\tau) d\tau \quad (6.1.8)$$

where we recall that Λ^ε has been defined in (1.2.2) with $Q = Q_B$. Then, the integral reformulation of (1.2.3) is given by

$$f^\varepsilon(t) = U^\varepsilon(t)f_{\text{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon). \quad (6.1.9)$$

Overview of the spectral results from [98] and their consequences

We start by explaining quickly the spectral results obtained in [98] since they are crucial in our proof. We denote by \mathcal{F}_x the Fourier transform in $x \in \mathbf{R}^d$ with $\xi \in \mathbf{R}^d$ its dual variable. Since in this part, we will only be working with Fourier transform in x , we will also interchangeably use the classical “hat” notation. Moreover, to lighten the reading, for any operator that acts only on velocity, with a little abuse of notation, we will omit the “hat” in the notation for its x -wise Fourier transform. In this part, we are going to look at the Fourier transform in $x \in \mathbf{R}^d$ of the operator Λ^1 :

$$\widehat{\Lambda}^1(\xi) = -i v \cdot \xi + \mathcal{L}$$

and study the spectrum of $\widehat{\Lambda}^1(\xi)$ for $\xi \in \mathbf{R}^d$. This type of analysis was initiated in [183, 62, 98] by Nicolaenko; Cercignani, Illner and Pulvirenti; Ellis and Pinsky for the Boltzmann equation for hard spheres and then with hard cut-off potentials (see also [202] by Ukai and Yang). In [211, 212], Yang and Yu were then able to adapt it to more general kinetic equations including the linearized Landau one for hard and moderately soft potentials (which will be useful in Section 6.2). It is important to notice that such studies allow to describe the spectrum of $\widehat{\Lambda}^1(\xi)$ at all scales of frequencies which let emerge the kinetic, hydrodynamic and dispersive regimes of description.

Roughly speaking, for small frequencies, the spectrum of $\widehat{\Lambda}^1(\xi)$ is a perturbation of the one of the homogeneous collision operator \mathcal{L} (which acts only on velocity). In the case of hard-spheres collisions, the operator \mathcal{L} has a spectral gap and in [98], the authors prove that for small frequencies ξ , the spectrum of $\widehat{\Lambda}^1(\xi)$ is made of “small” eigenvalues around 0 in the right part of the plane that we denote by $\lambda_\star(\xi)$ for $\star = \nu, \kappa, \pm\text{wave}$. They also provide Taylor expansions of those eigenvalues as well as for their associated projectors denoted by $P_\star(\xi)$ for $\star = \nu, \kappa, \pm\text{wave}$. For large frequencies, they prove that the operator $\widehat{\Lambda}^1(\xi)$ has no spectrum in some suitable right part of the plane. All those spectral results provide a decomposition of the semigroup $\widehat{U}^1(t) = \mathcal{F}_x U^1(t) \mathcal{F}_x^{-1}$ as

$$\widehat{U}^1(t, \xi) = \underbrace{e^{t\lambda_\nu(\xi)} P_\nu(\xi) + e^{t\lambda_\kappa(\xi)} P_\kappa(\xi)}_{=: \widehat{U}_{\text{NS}}^1(t, \xi)} + \underbrace{e^{t\lambda_{+\text{wave}}(\xi)} P_{+\text{wave}}(\xi) + e^{t\lambda_{-\text{wave}}(\xi)} P_{-\text{wave}}(\xi)}_{=: \widehat{U}_{\text{wave}}^1(t, \xi)} + \widehat{U}_{\text{kin}}^1(t, \xi) \quad (6.1.10)$$

with $\widehat{U}_{\text{kin}}^1(t, \xi)$ which is the kinetic part of the semigroup that enjoys exponential decay at infinity. The eigenvalues $\lambda_\star(\xi)$ have a Taylor expansion around 0 of the following form:

$$\lambda_\star(\xi) = i\alpha_\star |\xi| - \beta_\star |\xi|^2 + O(|\xi|^3), \quad (6.1.11)$$

with $\beta_\star > 0$ for all \star while some of the α_\star vanish. Let us give a more precise description of these eigenvalues. Here and after, the notations ν and κ correspond to the viscosity and the heat conductivity coefficients in (1.2.4) that are defined in (1.2.5). The first two eigenvalues $\lambda_\nu(\xi)$ and $\lambda_\kappa(\xi)$ with vanishing first order terms are precisely the ones that dictate the hydrodynamic limit while the two other ones $\lambda_{\pm\text{wave}}$ contains the acoustic waves (that will have dispersive effect in the hydrodynamic limit). More precisely, we have the following description:

- The eigenvalue $\lambda_\nu(\xi)$ is $(d-1)$ -dimensional and the first order term of its Taylor expansion vanishes:

$$\lambda_\nu(\xi) = -\nu|\xi|^2 + O(|\xi|^3).$$

- The eigenvalue $\lambda_\kappa(\xi)$ is simple and the first order term of its Taylor expansion also vanishes:

$$\lambda_\kappa(\xi) = -\kappa|\xi|^2 + O(|\xi|^3).$$

- The two other eigenvalues $\lambda_{\pm\text{wave}}$ are simple and have a non-vanishing first order term, their Taylor expansions are given by

$$\lambda_{\pm\text{wave}}(\xi) = \pm ic|\xi| - \beta|\xi|^2 + O(|\xi|^3)$$

where $c > 0$ is the speed of sound and β is a positive constant whose explicit formula can be given in terms of ν and κ .

Using the scaling property $\widehat{U}^\varepsilon(t, \xi) = \widehat{U}^1\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right)$, we obtain that $U^\varepsilon(t)$ satisfies a decomposition of the following type:

$$U^\varepsilon(t) = U_{\text{NS}}^\varepsilon(t) + U_{\text{wave}}^\varepsilon(t) + U_{\text{kin}}^\varepsilon(t) = [U_{\text{NS}}(t) + V^\varepsilon(t)] + U_{\text{wave}}^\varepsilon(t) + U_{\text{kin}}^\varepsilon(t) \quad (6.1.12)$$

where

- $U_{\text{NS}}(t)$ contains the terms independent of ε (it is the leading order term in the expansion of $U_{\text{NS}}^\varepsilon(t)$ and corresponds to the linearized Navier-Stokes semigroup),
- $V^\varepsilon(t)$ contains the ones that depend on ε and that will tend to 0 in the limit $\varepsilon \rightarrow 0$ for any data,
- $U_{\text{wave}}^\varepsilon(t)$ contains the ones that will induce dispersive effects (that disappear for well-prepared data),
- $U_{\text{kin}}^\varepsilon(t)$ the ones that will be exponentially small in $\varepsilon^{-2}t$ (and small in L_t^∞ for hydrodynamic data and thus for well-prepared data).

The parts $U_{\text{wave}}^\varepsilon(t)$ and $U_{\text{kin}}^\varepsilon(t)$ are the ones responsible of an initial layer and a kinetic relaxation (respectively corresponding to the waves and to the purely kinetic part of the solution) when the considered data are not well-prepared.

Among other properties, thanks to this decomposition, one can obtain the following estimate for the semigroup $(U^\varepsilon(t))_{t \geq 0}$:

$$\|U^\varepsilon(t)(\text{Id} - \pi_{\mathcal{L}})\|_{\mathcal{B}(L_{x,v}^2(\mu^{-\frac{1}{2}}))} \lesssim \varepsilon, \quad (6.1.13)$$

where we recall that $\pi_{\mathcal{L}}$ is the projector onto $\text{Ker } \mathcal{L}$ defined in (1.1.20). One can actually obtain decay estimates in time, the rates of decay depending on the spatial domain that is considered. This implies that even if there is $1/\varepsilon$ in the definition of $\Psi^\varepsilon(t)$ in (6.1.8), there is not really a singularity in ε because of estimate (6.1.13). Indeed, the semigroup $U^\varepsilon(t)$ acts on a microscopic quantity since $\pi_{\mathcal{L}}Q_B = 0$ from the conservation properties of Q_B stated in (1.1.10).

The decomposition of $U^\varepsilon(t)$ given above also induces a decomposition of $\Psi^\varepsilon(t)$ in which some of the terms are also going to be independent of ε and will be denoted by $\Psi(t)$. With all this at hand, we can provide an integral formulation for (1.2.4). More precisely, if f is given by (6.1.5) as defined in Theorem 6.1.1, then

$$f(t) = U(t)f_0 + \Psi(t)(f, f) \quad (6.1.14)$$

with f_0 defined in (6.1.4).

The fixed point strategy

The fixed-point lemma that we shall use is the following:

Lemma 6.1.5. *Let \mathcal{E} be a Banach space, let L be a continuous linear map from \mathcal{E} to \mathcal{E} , and let B be a bilinear map from $\mathcal{E} \times \mathcal{E}$ to \mathcal{E} . Let us define*

$$\|L\| := \sup_{\|x\|_{\mathcal{E}}=1} \|Lx\|_{\mathcal{E}} \quad \text{and} \quad \|B\| := \sup_{\|x\|_{\mathcal{E}}=\|y\|_{\mathcal{E}}=1} \|B(x, y)\|_{\mathcal{E}}.$$

If $\|L\| < 1$, then for any x_0 in \mathcal{E} such that $\|x_0\|_{\mathcal{E}} < (1 - \|L\|)^2(4\|B\|)^{-1}$, the equation $x = x_0 + Lx + B(x, x)$ has a unique solution in the ball of center 0 and radius $(1 - \|L\|)(2\|B\|)^{-1}$ and there exists a positive constant C_0 such that $\|x\|_{\mathcal{E}} \leq C_0\|x_0\|_{\mathcal{E}}$.

The goal is now to reformulate our problem in order to fall within this framework.

The case of well-prepared data

Rewriting our problem. In this case, the initial data that we consider for the Boltzmann equation is f_0 defined in (6.1.4). Thanks to (6.1.9) and (6.1.14), we can reformulate our problem onto an equation on the difference $g^\varepsilon = f^\varepsilon - f$:

$$g^\varepsilon(t) = \underbrace{(U^\varepsilon(t) - U(t))f_0 + (\Psi^\varepsilon(t) - \Psi(t))(f, f)}_{\text{source term}} + \underbrace{\Psi^\varepsilon(t)(g^\varepsilon, f) + \Psi^\varepsilon(t)(f, g^\varepsilon)}_{\text{linear part}} + \underbrace{\Psi^\varepsilon(t)(g^\varepsilon, g^\varepsilon)}_{\text{quadratic part}}. \quad (6.1.15)$$

This equation has a suitable form in view of applying the above fixed point lemma. Nevertheless, there is no reason for the linear part of the equation to be a contraction. In order to overcome this problem, we introduce a new unknown g_λ^ε defined as

$$g_\lambda^\varepsilon(t) := g^\varepsilon(t) \exp\left(-\lambda \int_0^t \|f(\tau)\|_X^r d\tau\right), \quad r > 4.$$

The choice of $r > 4$ is explained by the fact that if f is defined globally in time, then in all the considered cases of Ω , we will have $f \in L^r(\mathbf{R}^+, X)$. If f is not globally defined in time but on an interval $[0, T]$, we will have that $f \in L^\infty([0, T], X)$ in any cases so in particular we will also have $f \in L^r([0, T], X)$. It will then remain to choose λ large enough so that the linear part of the equation will be a contraction. We rewrite our equation in terms of our new variable g_λ^ε :

$$g_\lambda^\varepsilon(t) = S_\lambda^\varepsilon(t) + L_\lambda^\varepsilon(t)g_\lambda^\varepsilon + \Psi_\lambda^\varepsilon(t)(g_\lambda^\varepsilon, g_\lambda^\varepsilon) \quad (6.1.16)$$

where the source term $S_\lambda^\varepsilon(t)$ is given by

$$S_\lambda^\varepsilon(t) := \exp\left(-\lambda \int_0^t \|f(\tau)\|_X^r d\tau\right) \left((U^\varepsilon(t) - U(t))f_0 + (\Psi^\varepsilon(t) - \Psi(t))(f, f) \right),$$

the linear part of the equation by

$$L_\lambda^\varepsilon(t)g := \frac{2}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \exp\left(-\lambda \int_\tau^t \|f(s)\|_X^r ds\right) \tilde{Q}_B(f, g)(\tau) d\tau,$$

where we have introduced the symmetrized version of the collision operator Q_B defined by

$$\tilde{Q}_B(g_1, g_2) := \frac{1}{2}(Q_B(g_1, g_2) + Q_B(g_2, g_1)),$$

and the quadratic part of the equation is given by

$$\Psi_\lambda^\varepsilon(t)(g, g) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \exp\left(-\lambda \int_\tau^t \|f(s)\|_X^r ds\right) Q_B(g, g)(\tau) d\tau.$$

Let us now explain why we can apply Lemma 6.1.5 to equation (6.1.16).

The source terms. The fact that the source terms converge towards 0 as $\varepsilon \rightarrow 0$ can be proven thanks to the results of [98], we refer to [103, Lemmas 3.5 and 4.1] in which complete proofs are given. It is worth pointing out that the proof of the convergence of the second source term, namely $(\Psi^\varepsilon(t) - \Psi(t))(f, f)$ towards 0 heavily relies on estimates on the limit solution f and more precisely on its macroscopic quantities ρ, u, θ . We use that from [101, 68, 71] by Fujita and Kato; Chemin; Chemin and Lerner,

$$(\rho, u, \theta) \in \widetilde{L}_t^\infty H_x^\ell \quad \text{and} \quad (\nabla \rho, \nabla u, \nabla \theta) \in L_t^2 H_x^\ell$$

where the spaces $\widetilde{L}^\infty([0, T], H^\ell(\mathbf{R}^d))$ and $\widetilde{L}^\infty([0, T], H^\ell(\mathbf{T}^d))$ are defined through their norms (see [71])

$$\|f\|_{\widetilde{L}^\infty([0, T], H^\ell(\mathbf{R}^d))}^2 := \int_{\mathbf{R}^d} \langle \xi \rangle^{2\ell} \sup_{t \in [0, T]} |\widehat{f}(t, \xi)|^2 d\xi$$

and

$$\|f\|_{\widetilde{L}^\infty([0, T], H^\ell(\mathbf{T}^d))}^2 := \sum_{\xi \in \mathbf{Z}^d} \langle \xi \rangle^{2\ell} \sup_{t \in [0, T]} |\widehat{f}(t, \xi)|^2.$$

Moreover, we also rely on precise informations on the rate of decay of solutions to the Navier-Stokes-Fourier system. More precisely, we use that in the periodic case, the mean-free assumption implies that the solution (ρ, u, θ) has an exponential decay. When $\Omega = \mathbf{R}^2$ the solution to (1.2.4) is defined globally in time and if the initial data lies in $(H^\ell \cap L^1(\Omega))^4$ then, from [210, 192] by Wiegner and Schonbek, we have that for $i = 1, 2$ and for any $t > 0$,

$$\|(\rho, u_i, \theta)(t)\|_{L^q(\Omega)} \lesssim \frac{1}{\langle t \rangle^{1-\frac{1}{q}}}, \quad \forall 2 \leq q < \infty$$

and for $i = 1, 2$ and for any $t > 0$,

$$\|(D^\alpha \rho, D^\alpha u_i, D^\alpha \theta)\|_{L^2(\Omega)}^2 \lesssim \frac{1}{\langle t \rangle^{1+|\alpha|}}, \quad \forall \alpha \in \mathbf{N}^2, \quad |\alpha| \leq \ell$$

with $D := \sqrt{-\Delta_x}$. We also use that if $\Omega = \mathbf{R}^3$ and if the solution with initial data in $(H^\ell \cap L^1(\Omega))^5$ is defined globally in time, from [102, 192] by Gallagher, Iftimie and Planchon and Schonbek, we have for any $t > 0$,

$$\|(\rho, u, \theta)(t)\|_{H^\ell(\Omega)} \lesssim \frac{1}{\langle t \rangle^{\frac{1}{4}}}.$$

These estimates allow us to obtain nice estimates on the quantities

$$\left\| \widehat{Q}_B(f, f)(t, \xi) \right\|_{L_v^2(\mu^{-\frac{1}{2}})} \quad \text{and} \quad \left\| \partial_t \widehat{Q}_B(f, f)(t, \xi) \right\|_{L_v^2(\mu^{-\frac{1}{2}})}$$

where we recall that f is given by (6.1.5).

Study of the linear part of the equation. In order to check that L_λ^ε lets \mathcal{X}_T stable (recall that \mathcal{X}_T is defined in (6.1.2)) and is a contraction, we use several things. First, the collision operator Q_B satisfies the following estimate:

$$\|Q_B(g_1, g_2)\|_{L_v^\infty(\langle v \rangle^k \mu^{-\frac{1}{2}})} \lesssim \|g_1\|_{L_v^\infty(\langle v \rangle^{k+1} \mu^{-\frac{1}{2}})} \|g_2\|_{L_v^\infty(\langle v \rangle^k \mu^{-\frac{1}{2}})} + \|g_1\|_{L_v^\infty(\langle v \rangle^k \mu^{-\frac{1}{2}})} \|g_2\|_{L_v^\infty(\langle v \rangle^{k+1} \mu^{-\frac{1}{2}})}.$$

We can thus not check directly the stability of the space \mathcal{X}_T . The argument that allows to get this stability result is based on the Grad's splitting of \mathcal{L} (see [115]):

$$\mathcal{L} = [Q_B(\mu, \cdot) + Q_B^+(\cdot, \mu)] - Q_B^-(\cdot, \mu) =: K - \nu(v) \quad (6.1.17)$$

where we have denoted by Q_B^+ and Q_B^- the gain and loss parts of the collision operator Q_B and $\nu(v)$ is the so-called collision frequency defined through

$$\nu(v) := \int_{\mathbf{R}^d} \mu(v_*) |v - v_*| dv_*, \quad \forall v \in \mathbf{R}^d$$

which satisfies for some positive constants ν_0 and ν_1 :

$$\nu_0 \langle v \rangle \leq \nu(v) \leq \nu_1 \langle v \rangle, \quad \forall v \in \mathbf{R}^d.$$

The operator K has nice properties, namely, it is compact, bounded from the space $L_v^\infty(\langle v \rangle^j \mu^{-1/2})$ into $L_v^\infty(\langle v \rangle^{j+1} \mu^{-1/2})$ for any $j \geq 0$ and also from $L_v^2(\mu^{-1/2})$ into $L_v^\infty(\mu^{-1/2})$.

Remark 6.1.6. In polynomially weighted spaces, the operator K does not enjoy such nice properties. This explains the introduction of a new splitting of \mathcal{L} in [117]. It is a refinement of Grad's splitting, the operator K is split into two parts, one which has some regularization properties and another one which is small so that the addition of it and $-\nu(v)$ keeps nice dissipativity properties:

$$\mathcal{L} = \underbrace{K}_{\text{compact}} - \underbrace{\nu(v)}_{\text{dissipative}} = \underbrace{A}_{\text{regularizing}} + \underbrace{A^c - \nu(v)}_{\text{small + dissipative}} =: \underbrace{A}_{\text{regularizing}} + \underbrace{B}_{\text{dissipative}}$$

where A and B satisfy the assumptions of Theorem 2.2.1 for a large class of functional spaces. This new splitting allows to develop quantitative estimates and is the starting point of [108] in which Gervais proves a result of hydrodynamical limit similar to ours in a larger functional framework, namely in polynomially weighted spaces.

Remark 6.1.7. The drawback of this new splitting is that when one writes the problem in an integral form, the semigroup $S_B(t)$ is less convenient than $e^{-\nu(v)t}$ which is explicit and allows to compensate the loss of weights coming from the nonlinear collision operator in L_v^∞ -spaces. Indeed, one has

$$\left\| \nu(v) \int_0^t e^{-\nu(v)\tau} d\tau \right\|_{L_v^\infty} \leq 1, \quad \forall t \geq 0, \quad (6.1.18)$$

and we are not able to obtain such an estimate on $S_B(t)$. To circumvent this difficulty, in [108], Gervais mixes our approach with the one from [46] in which the equation is split into two parts. This trick will also be used in Chapter 7.

Using Grad's splitting (6.1.17), we have

$$\Lambda^\varepsilon = -\frac{1}{\varepsilon} v \cdot \nabla_x - \frac{1}{\varepsilon^2} \nu(v) + \frac{1}{\varepsilon^2} K =: \widetilde{\Lambda}^\varepsilon + \frac{1}{\varepsilon^2} K.$$

The semigroup generated by $\widetilde{\Lambda}^\varepsilon$ is explicit and given by

$$S_{\widetilde{\Lambda}^\varepsilon}(t)g = e^{-\nu(v)\frac{t}{\varepsilon^2}} g\left(x - v\frac{t}{\varepsilon}, v\right), \quad \forall t \geq 0. \quad (6.1.19)$$

Using this new splitting and Duhamel formula, we obtain that

$$\begin{aligned} L_\lambda^\varepsilon(t)g &= \frac{1}{\varepsilon} \int_0^t S_{\widetilde{\Lambda}^\varepsilon}(t-\tau) \exp\left(-\lambda \int_\tau^t \|f(s)\|_X^r ds\right) \widetilde{Q}(f,g)(\tau) d\tau \\ &\quad + \frac{1}{\varepsilon} \int_0^t \int_0^{t-\tau} S_{\widetilde{\Lambda}^\varepsilon}(t-\tau-\tau') \frac{1}{\varepsilon^2} K U^\varepsilon(\tau') \exp\left(-\lambda \int_\tau^t \|f(s)\|_X^r ds\right) \widetilde{Q}(f,g)(\tau) d\tau' d\tau \\ &=: L_\lambda^{\varepsilon,1}(t)g + L_\lambda^{\varepsilon,2}(t)g. \end{aligned}$$

Performing an adapted change of variable, one can prove that

$$L_\lambda^{\varepsilon,2}(t)g = \frac{1}{\varepsilon^2} \int_0^t S_{\tilde{\lambda}^\varepsilon}(t-\tau) K L_\lambda^\varepsilon(\tau)g \, d\tau.$$

Using this equality and reiterating Duhamel formula, we can prove that

$$\|L_\lambda^\varepsilon(t)g\|_{\mathcal{X}_T} \lesssim \|L_\lambda^{\varepsilon,1}(t)g\|_{\mathcal{X}_T} + \|L_\lambda^\varepsilon(t)g\|_{\mathcal{Y}_T}$$

where the space \mathcal{Y}_T is defined through

$$\mathcal{Y}_T := L^\infty\left(\mathbf{1}_{[0,T]}(t)\chi_\Omega(t), H_x^\ell L_v^2(\mu^{-\frac{1}{2}})\right).$$

The problem thus boils down to get estimates on $L_\lambda^\varepsilon(t)g$ in $H_x^\ell L_v^2(\mu^{-1/2})$ and on $L_\lambda^{\varepsilon,1}(t)g$ in X . In order to estimate $L_\lambda^\varepsilon(t)g$ in $H_x^\ell L_v^2(\mu^{-1/2})$, we use that from the results established in [98], we have

$$\frac{1}{\varepsilon} \|U^\varepsilon(t)(\text{Id} - \pi_{\mathcal{L}})g\|_{H_x^\ell L_v^2(\mu^{-\frac{1}{2}})} \lesssim \tilde{\chi}_\Omega(t) \begin{cases} \|g\|_{H_x^\ell L_v^2(\mu^{-\frac{1}{2}})} & \text{if } \Omega = \mathbf{T}^d, \\ \|g\|_{H_x^\ell L_v^2(\mu^{-\frac{1}{2}})} + \|g\|_{L_v^2 L_x^1(\mu^{-\frac{1}{2}})} & \text{if } \Omega = \mathbf{R}^d. \end{cases}$$

with

$$\tilde{\chi}_\Omega(t) := \begin{cases} \frac{e^{-\sigma t}}{\sqrt{t}} & \text{if } \Omega = \mathbf{T}^d, \\ \frac{1}{\sqrt{t}\langle t \rangle^{\frac{d}{4}}} & \text{if } \Omega = \mathbf{R}^d. \end{cases}$$

Remark 6.1.8. Notice that for $d = 2$, $\tilde{\chi}_\Omega$ is not integrable at infinity, this is why in the case of \mathbf{R}^2 , we develop our argument in a weighted in time space instead of doing it in $L_t^\infty(X)$.

Moreover, we have the following inequality:

$$\|g\|_{H_x^\ell L_v^2(\mu^{-\frac{1}{2}})} \lesssim \|v^{-1}g\|_X,$$

this embedding inequality allows us to compensate the loss of weight coming from the collision operator Q_B . Finally, using Hölder inequality, we obtain that

$$\|L_\lambda^\varepsilon(t)g\|_{\mathcal{Y}_T} \lesssim \frac{1}{\lambda^{\frac{1}{r}}} \|g\|_{\mathcal{X}_T}.$$

Estimating $L_\lambda^{\varepsilon,1}(t)g$ in X is easier and based on the explicit form of $S_{\tilde{\lambda}^\varepsilon}(t)$ given in (6.1.19) and the estimate (6.1.18). Overall, we obtain that

$$\|L_\lambda^\varepsilon(t)g\|_{\mathcal{X}_T} \lesssim \left(\frac{1}{\lambda^{\frac{1}{r}}} + \varepsilon\right) \|g\|_{\mathcal{X}_T}. \quad (6.1.20)$$

This inequality allows us to get a contraction estimate by choosing λ large enough and ε small enough.

Study of the quadratic part of the equation. The fact that $\Psi_\lambda^\varepsilon(t)$ sends $\mathcal{X}_T \times \mathcal{X}_T$ into \mathcal{X}_T follows by using the same arguments that we used to get a contraction estimate on $L_\lambda^\varepsilon(t)$.

Conclusion. Gathering our results on the source terms, the linear and the quadratic parts of the equation, we are able to apply Lemma 6.1.5 and obtain that g^ε exists as long as g exists and thus, the same result holds for f^ε .

The case of ill-prepared data

In this part, we only work in the case $\Omega = \mathbf{R}^d$. The major difficulty in the ill-prepared case is that the linear propagator $U^\varepsilon - U$ acting on the initial data can be decomposed into several operators (see (6.1.12)), some of which vanish in the well-prepared case only, and are dispersive (but not small in the energy space) in the ill-prepared case, namely $U_{\text{wave}}^\varepsilon(t)f_{\text{in}}$ and $U_{\text{kin}}^\varepsilon(t)f_{\text{in}}$. We denote by $U_{\text{disp}}^\varepsilon(t)$ the leading order term of $U_{\text{wave}}^\varepsilon(t)$. The terms $U_{\text{disp}}^\varepsilon(t)f_{\text{in}}$ and $U_{\text{kin}}^\varepsilon(t)f_{\text{in}}$ need to be removed from $g^\varepsilon = f^\varepsilon - f$ if one is to apply the fixed point lemma in the energy space. We actually have the following estimates on these terms: for suitable g ,

$$\|U_{\text{disp}}^\varepsilon(t)g\|_{L_v^\infty W_x^{\ell,\infty}(\langle v \rangle^k)} \lesssim \left(1 \wedge \left(\frac{\varepsilon}{t}\right)^{\frac{d-1}{2}}\right) \left(\|g\|_X + \|g\|_{L_v^2 L_x^1(\mu^{-\frac{1}{2}})}\right) \quad \text{and} \quad \|U_{\text{kin}}^\varepsilon(t)g\|_X \lesssim e^{-\tilde{\sigma}\frac{t}{\varepsilon^2}} \|g\|_X \quad (6.1.21)$$

for some $\tilde{\sigma} > 0$. In particular, we have that

$$\left\| (U_{\text{disp}}^\varepsilon(t) + U_{\text{kin}}^\varepsilon(t))g \right\|_{L^p(\mathbf{R}^+, L_v^\infty(W_x^{\ell,\infty} + H_x^\ell)(\langle v \rangle^k))} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \forall p \in (2/(d-1), \infty).$$

Consequently, we are going to rewrite our problem a bit differently, we won't perform our fixed point theorem on $g^\varepsilon = f^\varepsilon - f$ but on $\tilde{g}^\varepsilon := f^\varepsilon - f - U_{\text{disp}}^\varepsilon(t)f_{\text{in}} - U_{\text{kin}}^\varepsilon(t)f_{\text{in}}$. It induces the appearance of new source terms that have to be treated carefully, using among other things the above mentioned estimates on $U_{\text{disp}}^\varepsilon(t)$ and $U_{\text{kin}}^\varepsilon(t)$.

Remark 6.1.9. Notice that the fact that we are not able to get a better rate of convergence than $\sqrt{\varepsilon}$ when considering smooth initial data comes from the first estimate in (6.1.21) which is based on dispersion.

Remark 6.1.10. To end this part, let us mention that in [103], using the stability of the equations at stake, we work with regular data and associated regular solutions in order to give a fully rigorous proof of our results of convergence without rate. We are of course able to obtain quantitative rates of convergence as explained above.

6.2 Regularization and hydrodynamical limit for the Landau equation

6.2.1 The model

In this part, we study the equation (1.2.3) in $\mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3$ with $Q = Q_L$ the Landau collision operator defined in (1.1.16) with $-2 \leq \gamma \leq 1$ (we only consider hard potentials, Maxwellian molecules and moderately soft potentials). We (re)introduce the following notations:

$$\mathcal{L} := Q_L(\cdot, \mu) + Q_L(\mu, \cdot) \quad \text{and} \quad \Lambda^\varepsilon := -\frac{1}{\varepsilon}v \cdot \nabla_x + \frac{1}{\varepsilon^2}\mathcal{L}.$$

The goal is thus to study the Cauchy theory of the following equation

$$\begin{cases} \partial_t f^\varepsilon = \Lambda^\varepsilon f^\varepsilon + \varepsilon^{-1}Q_L(f^\varepsilon, f^\varepsilon) & \text{in } \mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3 \\ f^\varepsilon(0) = f_{\text{in}} & \text{in } \mathbf{T}^3 \times \mathbf{R}^3. \end{cases} \quad (6.2.1)$$

as well as the link between a solution f^ε to (6.2.1) and $f \in \text{Ker } \mathcal{L}$ of the form (6.1.5) with (ρ, u, θ) solution to the system (1.2.4).

6.2.2 Notations

We introduce the following H^1 -norm in velocity which naturally arises in the study of the Landau equation:

$$\|g\|_{H_v^{1,*}(\mu^{-\frac{1}{2}})} := \|\langle v \rangle^{\frac{Y}{2}+1} g\|_{L_v^2(\mu^{-\frac{1}{2}})} + \|\langle v \rangle^{\frac{Y}{2}} P_v \nabla_v g\|_{L_v^2(\mu^{-\frac{1}{2}})} + \|\langle v \rangle^{\frac{Y}{2}+1} (\text{Id} - P_v) \nabla_v g\|_{L_v^2(\mu^{-\frac{1}{2}})}, \quad (6.2.2)$$

where P_v stands for the projection on v , namely, $P_v w = |v|^{-2} (w \cdot v) v$.

Remark 6.2.1. It is interesting to remark that there is strong link between this norm and the norm $H_v^{s,*}$ introduced in (4.2.2)-(4.2.3). The way in which the latter is formulated does not enlighten this but if one uses a pseudo-differential point of view, it is then clear that at least formally, the $H_v^{1,*}$ norm is equivalent to the limit of the $H_v^{s,*}$ -norm as $s \rightarrow 1^-$. We refer to [3] by Alexandre, Hérau and Li for such a point of view.

We define the weighted Sobolev-type spaces X and Y as the spaces associated to the following norms:

$$\begin{aligned} \|g\|_X^2 := & \|\langle v \rangle^{3(\frac{Y}{2}+1)} g\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 + \|\langle v \rangle^{2(\frac{Y}{2}+1)} \nabla_x g\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 \\ & + \|\langle v \rangle^{\frac{Y}{2}+1} \nabla_x^2 g\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 + \|\nabla_x^3 g\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2, \end{aligned} \quad (6.2.3)$$

and

$$\begin{aligned} \|g\|_Y^2 := & \|\langle v \rangle^{3(\frac{Y}{2}+1)} g\|_{L_x^2(H_{v,*}^1(\mu^{-\frac{1}{2}}))}^2 + \|\langle v \rangle^{2(\frac{Y}{2}+1)} \nabla_x g\|_{L_x^2(H_{v,*}^1(\mu^{-\frac{1}{2}}))}^2 \\ & + \|\langle v \rangle^{\frac{Y}{2}+1} \nabla_x^2 g\|_{L_x^2(H_{v,*}^1(\mu^{-\frac{1}{2}}))}^2 + \|\nabla_x^3 g\|_{L_x^2(H_{v,*}^1(\mu^{-\frac{1}{2}}))}^2. \end{aligned} \quad (6.2.4)$$

Remark that as in Chapter 4, we work with “twisted” Sobolev spaces in which the weights depend on the order of the derivative in x , it will allow us to close our nonlinear estimates. We also need our space X to be included in $H_x^3 L_v^2(\mu^{-1/2})$ in order to be able to use the spectral analysis developed in [211] by Yang and Yu which explains the fact that we work with positive exponents in the polynomial weights (which was not the case in Chapter 4).

For $v \in \mathbf{R}^3$, we define the symmetric matrix $\mathbf{A}(v) = (A_{ij}(v))_{1 \leq i,j \leq 3}$ whose coefficients are given by

$$A_{ij}(v) = (a_{ij} *_v \mu)(v)$$

where the coefficients a_{ij} are defined in (1.1.17) and the Maxwellian μ in (1.1.15). We can decompose $\mathbf{A}(v)$ as $\mathbf{A}(v) = \mathbf{B}^\top(v) \mathbf{B}(v)$, where $\mathbf{B}(v) = (B_{ij}(v))_{1 \leq i,j \leq 3}$ is a matrix with real-valued smooth entries (see [189] by Rachid). Recall also that in [80], Degond and Lemou have proven that for $v \in \mathbf{R}^3 \setminus \{0\}$, the matrix $\mathbf{A}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace v^\perp . Moreover, when $|v| \rightarrow +\infty$, we have

$$\ell_1(v) \sim 2\langle v \rangle^Y \quad \text{and} \quad \ell_2(v) \sim \langle v \rangle^{Y+2}. \quad (6.2.5)$$

As such, one can write that for any $v \in \mathbf{R}^3 \setminus \{0\}$,

$$\mathbf{A}(v) = \ell_1(v) \frac{v}{|v|} \otimes \frac{v}{|v|} + \ell_2(v) \left(\text{I}_d - \frac{v}{|v|} \otimes \frac{v}{|v|} \right),$$

where I_d denotes the identity matrix and

$$\mathbf{B}(v) = \sqrt{\ell_1(v)} \frac{v}{|v|} \otimes \frac{v}{|v|} + \sqrt{\ell_2(v)} \left(\text{I}_d - \frac{v}{|v|} \otimes \frac{v}{|v|} \right). \quad (6.2.6)$$

In what follows, we will use the following differential operators

$$\tilde{\nabla}_v := \mathbf{B}(v) \nabla_v \quad \text{and} \quad \tilde{\nabla}_x := \mathbf{B}(v) \nabla_x. \quad (6.2.7)$$

We notice that the $H_{v,*}^1$ -norm in velocity defined in (6.2.2) satisfies

$$\|g\|_{H_{v,*}^1(\mu^{-\frac{1}{2}})} \lesssim \|\langle v \rangle^{\frac{Y}{2}+1} g\|_{L_v^2(\mu^{-\frac{1}{2}})} + \|\tilde{\nabla}_v g\|_{L_v^2(\mu^{-\frac{1}{2}})} \lesssim \|g\|_{H_{v,*}^1(\mu^{-\frac{1}{2}})}^2.$$

This comes from [80, 119] by Degond and Lemou on the one hand and by Guo on the other hand, more precisely, we use the smoothness of the eigenvalues $\ell_1(v)$ and $\ell_2(v)$ in $v \in \mathbf{R}^3 \setminus \{0\}$, their asymptotic behavior at infinity (6.2.5) and the fact that if $|v| < 1$, they are bounded from below by a positive constant independent of v . We also introduce a space which depend on ε , namely the space Z^ε whose associated norm is given by

$$\|g\|_{Z^\varepsilon}^2 := \|g\|_X^2 + \|g\|_Y^2 + \varepsilon^2 \|\tilde{\nabla}_x g\|_X^2 \quad (6.2.8)$$

where we recall that the X and Y -norms are defined in (6.2.3) and (6.2.4).

To end this part, we (re)introduce the notation $U^\varepsilon(t) := S_{\Lambda^\varepsilon}(t)$ for the semigroup generated by the linear operator $\Lambda^\varepsilon = -\varepsilon^{-1}v \cdot \nabla_x + \varepsilon^{-2} \mathcal{L}$.

6.2.3 Main results

Cauchy theory and regularization estimates. The main result that we obtain on the Cauchy theory and regularization properties of equation (6.2.1) is the following:

Theorem 6.2.2 ([59], Theorem 1.1). *There is $\eta_0 > 0$ small enough such that for any $\varepsilon \in (0, 1)$, if $f_{\text{in}}^\varepsilon \in X$ has vanishing mass, momentum, energy and is such that $\|f_{\text{in}}^\varepsilon\|_X \leq \eta_0$, then the following holds:*

(i) *There is a unique global solution $f^\varepsilon \in L^\infty(\mathbf{R}^+, X) \cap L^2(\mathbf{R}^+, Y)$ to (6.2.1) associated to the initial data $f_{\text{in}}^\varepsilon$ which verifies moreover*

$$\sup_{t \geq 0} e^{2\lambda t} \|f^\varepsilon(t)\|_X^2 + \frac{1}{\varepsilon^2} \int_0^\infty e^{2\lambda t} \|(f^\varepsilon(t))^\perp\|_Y^2 dt + \int_0^\infty e^{2\lambda t} \|f^\varepsilon(t)\|_Y^2 dt \lesssim \|f_{\text{in}}^\varepsilon\|_X^2, \quad (6.2.9)$$

for any $0 < \lambda < \lambda_0$, where λ_0 is the hypocoercivity constant defined in (6.2.18), and where we recall that $(f^\varepsilon)^\perp$ is the microscopic part of f^ε defined in (1.1.21).

(ii) *In addition, the solution satisfies the following regularization estimates, for all $t > 0$ and any $0 < \lambda < \lambda_0$,*

$$\|f^\varepsilon(t)\|_Y \lesssim \frac{e^{-\lambda t}}{\min(1, \sqrt{t})} \|f_{\text{in}}^\varepsilon\|_X \quad \text{and} \quad \varepsilon \|\tilde{\nabla}_x f^\varepsilon(t)\|_X \lesssim \frac{e^{-\lambda t}}{\min(1, t^{\frac{3}{2}})} \|f_{\text{in}}^\varepsilon\|_X, \quad (6.2.10)$$

where $\tilde{\nabla}_x$ is a weighted anisotropic gradient defined in (6.2.7).

Remark 6.2.3. The regularization estimate in the space variable (the second estimate in (6.2.10)) is not uniform in ε . This non-uniformity in ε of such a gain is expected since the transport operator and the linearized collision operator (which gives the gain of regularity in velocity) do not act at the same scale.

Remark 6.2.4. We can obtain the same type of regularization estimates as in (6.2.10) for the semigroup $U^\varepsilon(t)$ in a simpler way following the same strategy of proof.

The Cauchy theory and the large-time behavior of the Landau equation for $\varepsilon = 1$ have been extensively studied. We here give a small sample of the existing literature. Let us mention [9] by Alexandre and Villani for renormalized solutions with defect measure, [86] by Desvillettes and Villani for the convergence to equilibrium for a priori smooth solutions with general initial data, [119, 181, 60, 58] by Guo; Mouhot and Neumann; Carrapatoso, Wu and the author of this memoir; Carrapatoso and Mischler for strong solutions in a perturbative framework. Concerning the well-posedness of our rescaled

equation, it has already been obtained in [121, 189] respectively by Guo and Rachid in Sobolev spaces (respectively in $H_{x,v}^N(\mu^{-1/2})$ with $N \geq 8$ and in $H_x^3 L_v^2(\mu^{-1/2})$) thanks to nonlinear energy methods. In [43], Briant has obtained a similar result in $H_{x,v}^N(\mu^{-1/2})$ with $N \geq 4$ thanks to the so-called H^1 -hypocoercivity method at the linear level.

Remark 6.2.5. We point out that the results obtained in Theorem 6.2.2 could be obtained in larger spaces of the type $\mathcal{E} := H_x^3 L_v^2(\langle v \rangle^k)$ for k large enough. We chose to only present the proof in the functional space X because this functional framework is compatible with the forthcoming result which is about hydrodynamical limit of the Landau equation (6.2.1). Let us though mention that the strategy to perform such an extension of our results from the functional space X to \mathcal{E} is the same as the one used in [46] by Briant, Merino-Aceituno and Mouhot where uniform in ε estimates on solutions to the hard-spheres Boltzmann equation have been obtained, we also refer to Chapter 7 for this type of strategy at the nonlinear level and to the remark made in Section 2.5.

Remark 6.2.6. We also underline that our method should be robust enough to also treat the case of very soft and Coulomb potentials $-3 \leq \gamma < -2$ in which the linearized operator does not have a spectral gap. More precisely, we should be able to obtain a similar result of Theorem 6.2.2 with the exponential time-decay in (6.2.9) being replaced by a sub-exponential one, by combining the arguments developed in this paper together with the study made in the case $\varepsilon = 1$ in [58] by Carrapatoso and Mischler. We also refer to [51] in which Cao and Carrapatoso treat the case of very soft potentials for the Boltzmann equation without cut-off.

Hydrodynamical limit. Mention first that it is known that for mean free $(\rho_0, u_0, \theta_0) \in H_x^3$ small enough and satisfying

$$\nabla_x \cdot u_0 = 0 \quad \text{and} \quad \rho_0 + \theta_0 = 0, \quad (6.2.11)$$

there exists a unique solution $(\rho, u, \theta) \in H_x^3$ to (1.2.4) defined on \mathbf{R}^+ with associated initial data (ρ_0, u_0, θ_0) (see [157, 158] by Lemarié-Rieusset for example). For such an initial data, we also define f_0 a well-prepared data with (ρ_0, u_0, θ_0) as associated first macroscopic quantities as in (6.1.4) with $d = 3$. Notice that from the definition of the functional space X in (6.2.3), we have that $f_0 \in X$ and the mean-free assumption made on (ρ_0, u_0, θ_0) implies that $f_0 \in (\text{Ker } \Lambda^\varepsilon)^\perp$. Notice also that the smallness assumption made on (ρ_0, u_0, θ_0) can be translated into a smallness assumption on f_0 in X . As a consequence, there exists $\eta_1 > 0$ such that if f_0 is of the form (6.1.4) and satisfies $\|f_0\|_X \leq \eta_1$, then there exists $(\rho, u, \theta) \in H_x^3$ defined on \mathbf{R}^+ solution to (1.2.4). Considering then f defined in (6.1.5) with $d = 3$, we also have the following estimate

$$\|f\|_{L_t^\infty(X)} \lesssim C(\|f_0\|_X) \quad (6.2.12)$$

where $C(\|f_0\|_X)$ is a constant only depending on the X -norm of the data f_0 . The aforementioned results on the system (1.2.4) can be found in [103, Appendix B.3] in which more details and references on the subject are given. We also refer to Section 6.1 for more information on the limit system.

Theorem 6.2.7 ([59], Theorem 1.5). *Let $f_{\text{in}}^\varepsilon \in X \cap (\text{Ker } \Lambda^\varepsilon)^\perp$ for $\varepsilon \in (0, 1)$ such that $\|f_{\text{in}}^\varepsilon\|_X \leq \eta_0$ (where η_0 is defined in Theorem 6.2.2) and $f^\varepsilon \in L_t^\infty(X)$ being the associated solutions of (6.2.1) constructed in Theorem 6.2.2. Consider also $f_0 \in X \cap (\text{Ker } \Lambda^\varepsilon)^\perp$ such that $\|f_0\|_X \leq \eta_1$ and f defined respectively as in (6.1.4) and (6.1.5) (where we recall that η_1 has been chosen small enough so that f is defined globally in time).*

There exists $\eta_2 \in (0, \min(\eta_0, \eta_1))$ such that if $\max(\|f_{\text{in}}^\varepsilon\|_X, \|f_0\|_X) \leq \eta_2$ and

$$\|f_{\text{in}}^\varepsilon - f_0\|_X \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (6.2.13)$$

then we have

$$\|f^\varepsilon - f\|_{L_t^\infty(X)} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (6.2.14)$$

If $\max(\|f_{\text{in}}^\varepsilon\|_X, \|f_0\|_X) \leq \eta_2$ and

$$\|\pi_{\mathcal{L}} f_{\text{in}}^\varepsilon - f_0\|_X \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (6.2.15)$$

then we have

$$\|f^\varepsilon - f\|_{L_t^1(Y) + L_t^\infty(X)} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (6.2.16)$$

Remark 6.2.8. One can get a rate of convergence in (6.2.14) and (6.2.16) if we suppose that f_0 has some additional regularity in x , namely a rate of ε^δ if the regularity is supposed to be $H_x^{3+\delta}$ for $\delta \in (0, 1/2]$. If we suppose more regularity on our data, we will still have a rate of $\sqrt{\varepsilon}$ which is not optimal. By refining our estimate on the semigroup generated by Λ^ε , one could hope getting a rate of ε .

Remark 6.2.9. Let us point out that we are not able to perform a fixed point argument as in Section 6.1. This is due to the fact that the structure of the Landau bilinear operator is more complicated than the hard-spheres Boltzmann one. Indeed, there is an anisotropic loss of derivatives and weights in the nonlinear estimates which prevents us from closing a fixed point estimate. To circumvent this difficulty and to obtain a strong convergence without loss of regularity in velocity, we use the new a priori estimates on the solution of the linearized rescaled Landau equation and on the nonlinear rescaled Landau equation that have been obtained in Theorem 6.2.2.

Let us focus on the literature about hydrodynamical limits of the Landau equation which is more scarce than for the Boltzmann equation. As already mentioned, in [121], Guo justifies the Navier-Stokes limit and beyond in the Hilbert's expansion from (the Boltzmann and) the Landau equations (even for the case of very soft potentials) in the torus by using his nonlinear energy method. Our result on the hydrodynamical limit is reminiscent of the one obtained in [121] for the hard and moderately soft potentials in the sense that we work with strong solutions and we prove a strong convergence result. It is however worth noticing that our functional framework is less restrictive (we only work with 3 derivatives in x and no derivative in v whereas in [121], regularity on 8 derivatives in both variables x and v is required). Moreover, there is an important loss of regularity in the estimates of convergence proven in [121] whereas we only lose δ derivatives in x and there is no loss in v to get a rate of convergence of ε^δ from Landau to Navier-Stokes equation (see Remarks 6.2.8 and 6.2.9).

In this chapter and in [121], the fluid initial data are supposed to be well-prepared, namely the divergence free condition and the Boussinesq relation (6.2.11) are supposed to hold. We refer to [147] by Jiang and Xiong for an extension to the case where the fluid part is not supposed to be well-prepared and the creation and propagation of initial layers is studied. In [121, 147], the kinetic initial data is supposed to have a specific form so that there is no creation of kinetic initial layers. Our presentation is slightly different since we do not use Hilbert expansion to study the limit towards the Navier-Stokes equation, our assumption being the following: the projection of the kinetic initial data onto the kernel of the linearized operator L is supposed to converge towards the well-prepared fluid initial data as $\varepsilon \rightarrow 0$.

Finally, in [188], Rachid obtained a result of weak- \star convergence in $L_t^\infty(H_x^3 L_v^2(\mu^{-1/2}))$ towards the incompressible Navier-Stokes-Fourier system, we have thus strengthen this result for the type of initial data that we consider here. One can notice that the assumptions made on the fluid initial data in [188] are a bit less restrictive since the divergence free condition for u_0 and the Boussinesq relation ρ_0 and θ_0 are not supposed to hold. The initial layer that such an initial condition creates is absorbed there in the weak convergence. In our framework, we cannot hope to absorb it in a strong convergence framework because the initial layer is propagated over time in the periodic domain (see [147]). Note also that in Section 6.1, we are able to treat this type of “completely ill-prepared” data only in the case of the whole space since those terms have some dispersive properties in the whole space.

6.2.4 Strategy of proof

Cauchy theory and exponential convergence to equilibrium. Our global strategy to prove Theorem 6.2.2-(i) is standard to study a nonlinear problem in a close to equilibrium regime: it starts with the study of the linearized equation before going back to the analysis of the fully nonlinear problem (we

have already used this strategy in Chapter 4 for example). However, we have to emphasize here that our study is quite involved for several reasons that will be explained below.

Hypocoercivity

Coercivity in L^2 . For the coercivity estimates on the linearized operator \mathcal{L} , we refer to [80] by Degond and Lemou, [119] by Guo, [179] by Mouhot and [182] by Mouhot and Strain. We know that \mathcal{L} has a spectral gap, more precisely, there is $\lambda_L > 0$ such that

$$\langle \mathcal{L}g, g \rangle_{L_v^2(\mu^{-\frac{1}{2}})} \leq -\lambda_L \|g - \pi_{\mathcal{L}}g\|_{H_v^{1,*}(\mu^{-\frac{1}{2}})}^2 \quad (6.2.17)$$

for any $g \in \text{Dom}(\mathcal{L})$, where we recall that $\pi_{\mathcal{L}}$ is the projector onto the kernel of \mathcal{L} defined in (1.1.20) and the $H_v^{1,*}$ -norm is defined in (6.2.2). Notice that in the case $-2 \leq \gamma \leq 1$ which we study in the present chapter, the previous inequality is indeed a coercivity estimate because $\|\cdot\|_{H_v^{1,*}(\mu^{-1/2})} \geq \|\cdot\|_{L_v^2(\mu^{-1/2})}$.

Hypocoercivity in L^2 . At the linear level, our strategy is based on the L^2 -hypocoercivity method presented in Section 2.1 and adapted to our framework which involves the small parameter ε (see also Remark 5.3.4). We are able to define a scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{L_{x,v}^2(\mu^{-1/2})}$ and whose associated norm $\|\!\| \cdot \|\!\|_{L_{x,v}^2(\mu^{-1/2})}$ is equivalent to the usual norm on $L_{x,v}^2(\mu^{-1/2})$ uniformly in ε and is such that there exist constants λ_0 and κ_0 satisfying

$$\langle\langle \Lambda^\varepsilon g, g \rangle\rangle_{L_{x,v}^2(\mu^{-\frac{1}{2}})} \leq -\lambda_0 \|\!\| g \|\!\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 - \frac{\kappa_0}{\varepsilon^2} \|g^\perp\|_{L_x^2(H_v^{1,*}(\mu^{-\frac{1}{2}}))}^2 - \kappa_0 \|g\|_{L_x^2(H_v^{1,*}(\mu^{-\frac{1}{2}}))}^2. \quad (6.2.18)$$

The strategy is close to the one exposed in Chapter 5, it relies on the coercivity estimate (6.2.17) and we here just give the form of the triple scalar product: it roughly writes as

$$\langle\langle f, g \rangle\rangle_{L_{x,v}^2(\mu^{-\frac{1}{2}})} := \langle f, g \rangle_{L_{x,v}^2(\mu^{-\frac{1}{2}})} + \varepsilon \sum_{i=1}^3 \eta_i \left(\langle \partial_{x_i} \Delta_x^{-1} \tilde{\pi}_{\mathcal{L}} f, \tilde{\pi}_i g \rangle_{L_x^2} + \langle \partial_{x_i} \Delta_x^{-1} \tilde{\pi}_{\mathcal{L}} g, \tilde{\pi}_i f \rangle_{L_x^2} \right) \quad (6.2.19)$$

where $\tilde{\pi}_{\mathcal{L}} : L_{x,v}^2(\mu^{-1/2}) \rightarrow L_x^2$ and for $i = 1, 2, 3$, $\tilde{\pi}_i : L_{x,v}^2(\mu^{-1/2}) \rightarrow L_x^2$ are some suitable moment operators that are such that for any $g \in (\text{Ker } \mathcal{L})^\perp$, $\tilde{\pi}_{\mathcal{L}} g = \tilde{\pi}_i g = 0$ and the constants η_i are chosen small enough in order to get the hypocoercivity inequality (6.2.18). Let us just mention that it is simpler in the present framework than in Chapter 5 because our data are periodic and in particular, the inverse Laplacian

$$\begin{aligned} (-\Delta_x)^{-1} : L_x^2(\mathbf{T}^3) \cap \{\langle \cdot \rangle = 0\} &\longrightarrow H_x^2(\mathbf{T}^3) \cap \{\langle \cdot \rangle = 0\} \\ \phi &\longmapsto u \quad \text{such that} \quad -\Delta_x u = \phi \end{aligned} \quad (6.2.20)$$

is well-defined and bounded.

Remark 6.2.10. Due to the fact that derivatives in x commute with Λ^ε , it is easy to deduce a similar result as (6.2.18) in the space $H_x^3 L_v^2(\mu^{-1/2})$. However, it is not an easy matter to recover such an energy estimate in larger or smaller spaces than $H_x^3 L_v^2(\mu^{-1/2})$. Actually, the methods presented in [117, 175] by Galdani, Mischler and Mouhot to develop shrinkage or enlargement arguments at the level of energy estimates is not easily adaptable to rescaled equations if one wants to get uniform estimates in the parameter of rescaling. To develop an enlargement argument, we shall see in Chapter 7 that one can use the trick introduced in [175] of splitting the original equation into several ones. However, we do not have such a method of splitting to perform a “shrinkage” argument.

Nevertheless, for our problem, we are able to exhibit a norm equivalent to the usual one that provides dissipativity for Λ^ε in a smaller space than $H_x^3 L_v^2(\mu^{-1/2})$ (namely in the space X defined in (6.2.3)) and that also preserves the gain of $1/\varepsilon$ on the microscopic part of the solution. Let us explain our strategy to do that.

Hypocoercivity in X . We introduce the following notation for $i = 0, 1, 2$:

$$m_i(v) := \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)}, \quad \forall v \in \mathbf{R}^d.$$

We define the following norm on X :

$$\|g\|_X^2 := \sum_{i=0}^2 \left(\delta \|m_i \nabla_x^i g^\perp\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 + \|\nabla_x^i g\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 \right) + \|\nabla_x^3 g\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 \quad (6.2.21)$$

for some constant $\delta \in (0, 1)$ to be chosen small enough. First, we remark that this norm is equivalent to the usual norm on X because the projector $\pi_{\mathcal{L}}$ is bounded from $L_{x,v}^2(\mu^{-1/2})$ into $L_{x,v}^2(m_i \mu^{-1/2})$ for $i = 0, 1, 2$ and also because the triple norm $\|\cdot\|_{L_{x,v}^2(\mu^{-1/2})}$ is equivalent to the usual $L_{x,v}^2(\mu^{-1/2})$ -norm uniformly in ε . Let us now explain why such a norm is going to provide us coercivity in X . First, by denoting $f(t) = S_{\Lambda^\varepsilon}(t)f_{\text{in}}$ and using (6.2.18), we notice that for $i = 0, 1, 2, 3$, we have

$$\frac{d}{dt} \|\nabla_x^i f\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 \lesssim -\frac{1}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L_x^2(H_v^{1,*}(\mu^{-\frac{1}{2}}))}^2 - \|\nabla_x^i f\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2. \quad (6.2.22)$$

Then, performing tedious estimates (see the proof of [59, Proposition 3.3]), for $i = 0, 1, 2$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|m_i \nabla_x^i f^\perp\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 \\ &= \frac{1}{\varepsilon^2} \langle m_i \mathcal{L} \nabla_x^i f^\perp, m_i \nabla_x^i f^\perp \rangle_{L_{x,v}^2(\mu^{-\frac{1}{2}})} - \frac{1}{\varepsilon} \langle m_i (\text{Id} - \pi_{\mathcal{L}})(v \cdot \nabla_x^{i+1} f), m_i \nabla_x^i f^\perp \rangle_{L_{x,v}^2(\mu^{-\frac{1}{2}})} \\ &\lesssim \frac{C}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 - \frac{1}{\varepsilon^2} \|m_i \nabla_x^i f^\perp\|_{L_x^2 H_v^{1,*}(\mu^{-\frac{1}{2}})}^2 + \frac{C}{\varepsilon} \|\nabla_x^{i+1} f\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})} \|m_i \nabla_x^i f^\perp\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})} \\ &\lesssim -\frac{1}{\varepsilon^2} \|m_i \nabla_x^i f^\perp\|_{L_x^2 H_v^{1,*}(\mu^{-\frac{1}{2}})}^2 + \frac{C}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 + C \|\nabla_x^{i+1} f\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2 \end{aligned}$$

where we used Cauchy-Schwarz inequality to get the last inequality. This computation highlights the fact that we could not have replaced $\nabla_x^i f^\perp$ by $\nabla_x^i f$ in our new norm because we have a loss term coming from the commutator between \mathcal{L} and m_i (the second term in the right-hand side above) which comes with a $1/\varepsilon^2$. In our computation, since it only concerns the microscopic part of $\nabla_x^i f$, it will be counterbalanced by the gain term coming from the derivation of $\|\nabla_x^i f\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2$ by taking δ small enough (see (6.2.22)). The third term in the inequality above is also a loss term and will be compensated by a gain term coming from the derivation in time of $\|\nabla_x^{i+1} f\|_{L_{x,v}^2(\mu^{-\frac{1}{2}})}^2$ (see (6.2.22)) by taking δ small enough. In the end, by taking δ small enough, we obtain that for any $0 < \lambda < \lambda_0$ (where λ_0 is defined in (6.2.18)) and for some $\kappa > 0$,

$$\frac{1}{2} \frac{d}{dt} \|f\|_X^2 = \langle \Lambda^\varepsilon f, f \rangle_X \leq -\lambda \|f\|_X^2 - \kappa \|f\|_Y^2 - \frac{\kappa}{\varepsilon^2} \|f^\perp\|_Y^2 \quad (6.2.23)$$

where we recall that the functional space Y has been defined in (6.2.4).

Remark 6.2.11. Notice that it is also possible to obtain decay estimates directly on the semigroup associated to Λ^ε thanks to Duhamel formula once one has exhibited a nice splitting of Λ^ε as explained in Section 2.5.

Nonlinear problem

We then prove some new and sharp nonlinear estimates on the Landau collision operator to be able to develop our Cauchy theory for the whole nonlinear problem in a close-to-equilibrium framework. It is worth mentioning that to prove good a priori estimates on the nonlinear problem, we use the hypocoercive norm we just introduced and we only perform energy estimates. It is actually crucial to be able to avoid the use of Duhamel formula to obtain nice estimates on the nonlinear problem because of the singularity in ε that is in front of the nonlinear term in (6.2.1). Our strategy is to perform direct energy estimates with the norm $\|\cdot\|_X$ introduced above. For this, we introduce the symmetrized version of the collision operator Q_L defined by

$$\tilde{Q}_L(f, g) := \frac{1}{2} (Q_L(f, g) + Q_L(g, f))$$

and exploit the fact that $\tilde{Q}_L(f, g) = (\tilde{Q}_L(f, g))^\perp$ so that

$$\langle \tilde{Q}_L(f, g), h \rangle_{L^2_{x,v}(\mu^{-1/2})} = \langle \tilde{Q}_L(f, g), h^\perp \rangle_{L^2_{x,v}(\mu^{-1/2})} .$$

From the form of the new scalar product on $L^2_{x,v}(\mu^{-1/2})$ introduced in (6.2.19) (we refer to (A.19) in [59] for the exact definition), we thus have

$$\begin{aligned} \langle \langle \tilde{Q}_L(f, g), h \rangle \rangle_{L^2_{x,v}(\mu^{-1/2})} &= \langle \tilde{Q}_L(f, g), h^\perp \rangle_{L^2_{x,v}(\mu^{-1/2})} \\ &+ \eta_1 \varepsilon \sum_{k=1}^3 \langle \partial_{x_k} (-\Delta_x)^{-1} \theta [h], \psi_k [\tilde{Q}_L(f, g)] \rangle_{L^2_x} + \eta_2 \varepsilon \sum_{k,\ell=1}^3 \langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k [h], \Theta_{k\ell} [\tilde{Q}_L(f, g)] \rangle_{L^2_x} \end{aligned}$$

where ψ_k and $\Theta_{k\ell}$ are suitable moments operators defined respectively in [59, (A.3) and (A.9)]. Recalling that $\delta \in (0, 1)$ is a small enough constant chosen so that inequality (6.2.23) holds and rearranging terms, we then deduce

$$\begin{aligned} &\langle \langle \tilde{Q}_L(f, g), h \rangle \rangle_X \\ &= \delta \langle \tilde{Q}_L(f, g), h^\perp \rangle_X + \sum_{i=0}^2 \langle \nabla_x^i \tilde{Q}_L(f, g), \nabla_x^i h^\perp \rangle_{L^2_{x,v}(\mu^{-1/2})} + (1 - \delta) \langle \nabla_x^3 \tilde{Q}_L(f, g), \nabla_x^3 h^\perp \rangle_{L^2_{x,v}(\mu^{-1/2})} \\ &+ \eta_1 \varepsilon \sum_{i,k=1}^3 \langle \partial_{x_k} (-\Delta_x)^{-1} \theta [\nabla_x^i h], \psi_k [\nabla_x^i \tilde{Q}_L(f, g)] \rangle_{L^2_x} \\ &+ \eta_2 \varepsilon \sum_{i,k,\ell=1}^3 \langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k [\nabla_x^i h], \Theta_{k\ell} [\nabla_x^i \tilde{Q}_L(f, g)] \rangle_{L^2_x} . \end{aligned} \tag{6.2.24}$$

With this at hand, we want to obtain nice a priori estimates for the equation (6.2.1) in order to be able to construct solutions to this equation and prove Theorem 6.2.2-(i). Let $\lambda \in (0, \lambda_0)$ (where λ_0 is defined in (6.2.18)) be fixed and compute

$$\frac{1}{\varepsilon} \frac{d}{dt} \|\| f^\varepsilon \|\|_X^2 = \langle \Lambda^\varepsilon f^\varepsilon, f^\varepsilon \rangle_X + \frac{1}{\varepsilon} \langle Q(f^\varepsilon, f^\varepsilon), f^\varepsilon \rangle_X .$$

For the linear part, estimate (6.2.23) gives

$$\langle \Lambda^\varepsilon f^\varepsilon, f^\varepsilon \rangle_X \leq -\lambda \|\| f^\varepsilon \|\|_X^2 - \kappa \|f^\varepsilon\|_Y^2 - \frac{\kappa}{\varepsilon^2} \|(f^\varepsilon)^\perp\|_Y^2 .$$

For the nonlinear part, we use that for any polynomial $p = p(v)$, thanks to the Cauchy-Schwarz inequality and nonlinear estimates on the Landau operator (see [59, Proposition 4.2] for more details),

$$\left| \int_{\mathbf{R}^3} \tilde{Q}_L(f, g) p(v) dv \right| \lesssim \|f\|_{L^2_v(\mu^{-1/2})} \|g\|_{H_v^{1,*}(\mu^{-1/2})} + \|f\|_{H_v^{1,*}(\mu^{-1/2})} \|g\|_{L^2_v(\mu^{-1/2})}$$

where we recall that the $H_v^{1,*}(\mu^{-1/2})$ -norm is defined in (6.2.2). From this and (6.2.24), we can prove that

$$\langle \langle \tilde{Q}_L(f, g), h \rangle \rangle_X \lesssim (\|f\|_X \|g\|_Y + \|f\|_Y \|g\|_X) (\|h^\perp\|_Y + \varepsilon \|h\|_Y) .$$

As a consequence, recalling that $\|\cdot\|_X$ and $\|\|\cdot\|\|_X$ are equivalent, we obtain that

$$\frac{1}{\varepsilon} \langle Q(f^\varepsilon, f^\varepsilon), f^\varepsilon \rangle_X \leq \frac{C}{\varepsilon} \|\| f^\varepsilon \|\|_X \|f^\varepsilon\|_Y \|(f^\varepsilon)^\perp\|_Y + C \|\| f^\varepsilon \|\|_X \|f^\varepsilon\|_Y^2 \tag{6.2.25}$$

for some constant $C > 0$.

Denoting $e_\lambda : t \mapsto e^{\lambda t}$ and $f_\lambda^\varepsilon = e_\lambda f^\varepsilon$, we therefore obtain

$$\frac{1}{2} \frac{d}{dt} \| \| f_\lambda^\varepsilon \| \|_X^2 \leq -\kappa \| f_\lambda^\varepsilon \|_Y^2 - \frac{\kappa}{\varepsilon^2} \| (f_\lambda^\varepsilon)^\perp \|_Y^2 + \frac{C}{\varepsilon} \| \| f_\lambda^\varepsilon \| \|_X \| f_\lambda^\varepsilon \|_Y \| (f_\lambda^\varepsilon)^\perp \|_Y + C \| \| f_\lambda^\varepsilon \| \|_X \| f_\lambda^\varepsilon \|_Y^2.$$

Thanks to Young's inequality we write

$$\frac{C}{\varepsilon} \| \| f_\lambda^\varepsilon \| \|_X \| f_\lambda^\varepsilon \|_Y \| (f_\lambda^\varepsilon)^\perp \|_Y \leq \frac{C^2}{2\kappa} \| \| f_\lambda^\varepsilon \| \|_X^2 \| f_\lambda^\varepsilon \|_Y^2 + \frac{\kappa}{2\varepsilon^2} \| (f_\lambda^\varepsilon)^\perp \|_Y^2,$$

which then gives the following a priori estimate

$$\frac{1}{2} \frac{d}{dt} \| \| f_\lambda^\varepsilon \| \|_X^2 \leq - \left(\kappa - \frac{C^2}{2\kappa} \| \| f_\lambda^\varepsilon \| \|_X^2 - C \| f^\varepsilon \| \|_X \right) \| f_\lambda^\varepsilon \|_Y^2 - \frac{\kappa}{2\varepsilon^2} \| (f_\lambda^\varepsilon)^\perp \|_Y^2. \quad (6.2.26)$$

Remark 6.2.12. We here see that it is crucial to have been able to keep the microscopic part of f_λ^ε in the nonlinear part of the estimates (this was possible thanks to the remarks leading to (6.2.24)) in order to be able to use Cauchy-Schwarz inequality to handle the $1/\varepsilon$ singularity in front of the nonlinear term.

From this differential inequality, we easily obtain that if $\| \| f_{\text{in}}^\varepsilon \| \|_X$ is small enough then f^ε satisfies the uniform in time estimate (6.2.9). The proof of existence and uniqueness of a solution f^ε to (6.2.1) satisfying (6.2.9) for small data $\| \| f_{\text{in}}^\varepsilon \| \|_X \leq \eta_0$ follows a standard iterative scheme that uses estimate (6.2.26) (see for example [60]).

Regularization estimates

The strategy that we use to prove the regularization estimates in Theorem 6.2.2-(ii) is similar to the one presented in Section 2.3. To our knowledge, it is the first time that such a strategy is used for a nonlinear equation (even in the simpler case $\varepsilon = 1$) and for a rescaled equation (with uniform estimates in the rescaling parameter). The functional that we introduce is the following:

$$\begin{aligned} \mathcal{F}^\varepsilon(g, t) := & \| \| g \| \|_X^2 + a t \left(\| \tilde{\nabla}_v g^\perp \|_X^2 + K \| \langle v \rangle^{\frac{\gamma}{2} + 1} g^\perp \|_X^2 \right) \\ & + 2 b \varepsilon t^2 \left\langle \tilde{\nabla}_v g, \tilde{\nabla}_x g \right\rangle_X + c \varepsilon^2 t^3 \left(\| \tilde{\nabla}_x g \|_X^2 + K \| \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g \|_X^2 \right), \end{aligned} \quad (6.2.27)$$

with constants $K > 0$ large enough and $0 < c \ll b \ll a \ll 1$ so that $b^2 < ac$. We also notice that for any $t \in (0, 1]$, one has the lower bounds

$$\mathcal{F}^\varepsilon(g, t) \geq t \| \| g \| \|_Y^2 \quad \text{and} \quad \mathcal{F}^\varepsilon(g, t) \geq t^3 \| \| g \| \|_{Z^\varepsilon}^2 \quad (6.2.28)$$

where we recall that the spaces Y and Z^ε are defined in (6.2.4) and (6.2.8). We then have to estimate the derivative in time of each term of $\mathcal{F}^\varepsilon(f^\varepsilon(t), t)$ with f^ε solution to (6.2.1). Let us comment a bit the definition of this functional.

- First, the reason that explains the fact that we only have put the gradient in v of g^\perp in the definition of this functional is similar to the one that led us to put only the microscopic part of g in the definition of the triple norm (6.2.21) to get hypocoercivity results and also in the spirit of the H^1 -hypocoercivity theory developed by Briant [43].
- Then, since we want to recover the expected gain of regularity (the corresponding hypoelliptic estimates are provided in [189] by Rachid), our functional has to be defined accordingly. For example, in [60] in which we were not interested in getting the optimal gain of regularity (and in which only the case $\varepsilon = 1$ was treated), only classical derivation operators were involved in the definition of the functional. Here, the definition of the functional is much more intricate, we have to work with the anisotropic operators $\tilde{\nabla}_v$ and $\tilde{\nabla}_x$ defined in (6.2.7) and our functional also involves additional terms which are necessary to close our estimates.

- Finally, the functional \mathcal{F}^ε involves some powers of ε in order to handle the dependencies in ε in the equation (6.2.1). Notice here that if we only wanted to handle the case $\varepsilon = 1$, we could have used the triple norm introduced in [117] (see [60] for the Landau equation) which is dissipative for the whole linearized operator and equivalent to the usual one. Here, to handle the ε -dependencies, we have to use our hypocoercive norm on X (see (6.2.23)) and separate carefully the behaviors of microscopic and macroscopic parts of the solution. Some additional remainder terms coming from the fact that the transport operator and the projector $\pi_{\mathcal{L}}$ onto the kernel of \mathcal{L} (see (1.1.20)) do not commute have to be treated.

We do not enter into details of computations, let us mention that we have to use new and sharp nonlinear estimates on the Landau operator as well as the estimates obtained in Theorem 6.2.2-(i) on the constructed solution f^ε and since we work with small data, we are able to absorb the nonlinearity. Notice finally that by combining these short time regularization estimates and the exponential decay obtained in Theorem 6.2.2-(i), we are able to recover the estimates stated in Theorem 6.2.2-(ii) that also include the exponential decay of solutions in large time.

Hydrodynamical limit. Let us describe into more details our strategy to obtain our strong convergence results stated in Theorem 6.2.7. It is inspired by the ones used in [31, 43] by Bardos and Ukai, and by Briant and in [103] by Gallagher and the author of this memoir (see Section 6.1). Indeed, as in Section 6.1, using the spectral analysis performed in [211] by Yang and Yu, in order to prove our main convergence result, we reformulate the fluid equation in a kinetic fashion and we then study the equation satisfied by the difference satisfied between the kinetic and the fluid solutions. By intertwining the refined and sharp kinetic estimates stated in Theorem 6.2.2 and fluid mechanics ones, we are able to prove a result of strong convergence from the solutions of the Landau equation to the Navier-Stokes one as stated in Theorem 6.2.7. Before starting, let us mention that the results established in [211] about the spectral analysis are an extension of [98] to a larger class of kinetic equations including the linearized Landau one. It is important to keep in mind that basically, all the results from [98] presented in Subsection 6.1.4 extend to the linearized Landau equation.

L_t^∞ -estimate

We first reformulate the problem in order to prove (6.2.14). To this end, we write the relation satisfied by $g^\varepsilon := f^\varepsilon - f$:

$$\begin{aligned} g^\varepsilon &= U^\varepsilon(t)f_{\text{in}}^\varepsilon + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) - U(t)f_0 - \Psi(t)(f, f) \\ &= (U^\varepsilon(t) - U(t))f_0 + U^\varepsilon(t)(f_{\text{in}}^\varepsilon - f_0) + (\Psi^\varepsilon(t) - \Psi(t))(f, f) \\ &\quad + \Psi^\varepsilon(t)((f^\varepsilon)^\perp, (f^\varepsilon)^\perp) + \Psi^\varepsilon(t)((f^\varepsilon)^\perp, \pi_{\mathcal{L}}f^\varepsilon) + \Psi^\varepsilon(t)(\pi_{\mathcal{L}}f^\varepsilon, (f^\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\pi_{\mathcal{L}}g^\varepsilon, \pi_{\mathcal{L}}f^\varepsilon) + \Psi^\varepsilon(t)(f, \pi_{\mathcal{L}}g^\varepsilon). \end{aligned} \tag{6.2.29}$$

The goal is to study each term in the right-hand side of the above equality. Some terms are going to vanish in the limit $\varepsilon \rightarrow 0$ and other ones will be absorbed in the left-hand side under suitable smallness assumptions on the initial data. Let us recall that the singularity in ε in the definition of Ψ^ε is going to be handled thanks to (6.1.13) which provides a gain of ε when the semigroup $U^\varepsilon(t)$ acts on microscopic quantities. Using that $\pi_{\mathcal{L}}Q(f_1, f_2) = 0$ for any suitable functions f_1, f_2 , we are thus going to be able to remove the singularity in ε in the operator Ψ^ε . We then broadly describe how to prove that the terms of the right-hand side of the above equality tend to 0 as $\varepsilon \rightarrow 0$.

- The three first terms tend to 0 as ε go to 0 (similarly as in the proof of Theorem 6.1.1, see Subsection 6.1.4).
- The fourth, fifth and sixth terms tend to 0 as $\varepsilon \rightarrow 0$ because those three terms involve the microscopic part of the kinetic solution f^ε , which provides us some extra smallness in ε in $L_t^2(Y)$ thanks to Theorem 6.2.2. We also use the regularization estimates proven in Theorem 6.2.2-(ii) in order to obtain estimates without loss of regularity in velocity. Indeed, if one performs a naive estimate, one

immediately sees that we cannot get an estimate in X without loss of regularity in the velocity variable due to the presence of the nonlinear Landau operator which induces a loss of two derivatives in velocity. It is thus important to exploit the regularization effects of both the semigroup $(U^\varepsilon(t))_{t \geq 0}$ and the whole nonlinear equation. Indeed, let us look for example at the fourth term which is given by

$$\Psi^\varepsilon(t)((f^\varepsilon)^\perp, (f^\varepsilon)^\perp) = \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-\tau) Q_L((f^\varepsilon)^\perp, (f^\varepsilon)^\perp)(\tau) d\tau.$$

Unfortunately, $U^\varepsilon(t-\tau)$ provides a gain of two derivatives in velocity but with a rate of $(t-\tau)^{-1}$ which is not integrable at t . The same thing occurs for the nonlinear equation. The idea is thus to gain one derivative with $U^\varepsilon(t-\tau)$ and another one with the nonlinear flow which in the end allow us to compensate the loss of two derivatives with a rate of $(t-\tau)^{-1/2}\tau^{-1/2}$ which is an integrable rate between 0 and t . The proof is actually more complicated if one wants to get smallness in ε and is based on a reiteration of Duhamel formula based on a nice splitting of the linearized operator Λ^ε into two parts in the same spirit as what is done to develop an enlargement argument. We refer to [59, Lemma 6.6] for more details.

- The last two terms are bounded by some quantity that involves the norms of the kinetic and fluid initial data multiplied by the $L_t^\infty(X)$ -norm of g^ε . It will thus be absorbed in the left-hand side of the equality if the initial data $f_{\text{in}}^\varepsilon$ and f_0 are chosen to be small enough. Notice that one cannot hope smallness in ε for those terms because they only involve macroscopic quantities and also that there is no problem of regularity in velocity for those terms because of the regularization properties of $\pi_{\mathcal{L}}$.

$L_t^1 + L_t^\infty$ -estimate

We now reformulate the problem in view of proving (6.2.16). To this end, we introduce the quantity $R^\varepsilon(t) := U^\varepsilon(t)(f_{\text{in}}^\varepsilon)^\perp$ and we write that

$$U^\varepsilon(t)f_{\text{in}}^\varepsilon - U(t)f_0 = (U^\varepsilon(t) - U(t))f_0 + U^\varepsilon(t)(\pi_{\mathcal{L}}f_{\text{in}}^\varepsilon - f_0) + R^\varepsilon(t).$$

From the regularization properties of the semigroup $(U^\varepsilon(t))_{t \geq 0}$, we have that for $0 < \lambda < \lambda_0$ (see Remark 6.2.4),

$$\|R^\varepsilon(t)\|_Y \lesssim \frac{\varepsilon}{\sqrt{t}} e^{-\lambda t} \|f_{\text{in}}^\varepsilon\|_X \quad (6.2.30)$$

and thus

$$\|R^\varepsilon\|_{L_t^1(Y)} \lesssim \varepsilon \|f_{\text{in}}^\varepsilon\|_X. \quad (6.2.31)$$

To conclude, we prove that for $\delta \in [0, 1/2]$,

$$\|f^\varepsilon - g - R^\varepsilon\|_{L_t^\infty(X)} \lesssim \varepsilon^\delta C \left(\|f_0\|_{H_x^{3+\delta}L_v^2}, \|f_{\text{in}}^\varepsilon\|_X \right) + \|\pi_{\mathcal{L}}f_{\text{in}}^\varepsilon - f_0\|_X.$$

We then write the relation satisfied by $\tilde{g}^\varepsilon := f^\varepsilon - f - R^\varepsilon$:

$$\begin{aligned} \tilde{g}^\varepsilon &= (U^\varepsilon(t) - U(t))f_0 + U^\varepsilon(t)(\pi_{\mathcal{L}}f_{\text{in}}^\varepsilon - f_0) + (\Psi^\varepsilon(t) - \Psi(t))(f, f) \\ &\quad + \Psi^\varepsilon(t)((f^\varepsilon)^\perp, (f^\varepsilon)^\perp) + \Psi^\varepsilon(t)((f^\varepsilon)^\perp, \pi_{\mathcal{L}}f^\varepsilon) + \Psi^\varepsilon(t)(\pi_{\mathcal{L}}f^\varepsilon, (f^\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\pi_{\mathcal{L}}R^\varepsilon, \pi_{\mathcal{L}}f^\varepsilon) + \Psi^\varepsilon(t)(f, \pi_{\mathcal{L}}R^\varepsilon) \\ &\quad + \Psi^\varepsilon(t)(\pi_{\mathcal{L}}\tilde{g}^\varepsilon, \pi_{\mathcal{L}}f^\varepsilon) + \Psi^\varepsilon(t)(f, \pi_{\mathcal{L}}\tilde{g}^\varepsilon). \end{aligned} \quad (6.2.32)$$

The goal is then to study each term in the right-hand side of this equality, the ideas being the same as the ones explained after (6.2.29). Notice furthermore that thanks to estimate (6.2.30), we are also able to prove that the seventh and eighth terms tend to 0 when $\varepsilon \rightarrow 0$.

6.3 Perspectives

As a conclusion to this Chapter, we introduce some prospective problems to which the methods we presented might be of use.

6.3.1 Equations with softer potentials

It is worth mentioning that several works have already allowed to enlarge the range of applications of our method [108, 109, 51]. The two first papers [108, 109] by Gervais, Gervais and Lods only deal with the case where the linearized collisional operator enjoys coercivity estimates while the last one [51] by Cao and Carrapatoso also deals with the case of soft potentials (for the Boltzmann equation without cut-off using spectral results from [212]). Let us also point out the fact that the paper [109] provide a result of the same type as ours in [103] and the approach is robust enough to deal with many kinetic models in a large functional setting (data with polynomial decay). It in particular improves our result in [103] presented in Section 6.1 (since the considered data are more general) and also the one from [59] presented in Section 6.2 (where we only proved a result of strong convergence for small initial data in the torus with data having a gaussian decay in velocity).

In the line of the above-mentioned papers, the natural next step would be to combine the approaches of [51] and [109] to obtain a general result of the same type as the one obtained in [109] that also include the cases where the linearized operator have no spectral gap.

6.3.2 Working in “kinetic-fluid functional spaces”

A question that naturally arises in the study of problems of hydrodynamical limits is the fact that the spaces in which we perform our analysis are not the classical spaces in which the fluid equations are studied. To sum up, we would like to get a result in which the connection between the kinetic and the fluid equations would be more accurate in terms of functional spaces than what the usual results provide.

More precisely, an option would be to obtain a result in a space which is related to $L_t^\infty H_x^{\ell-1} \cap L_t^2 H_x^\ell$ in the torus with $\ell > d/2$ (we do not mention the spaces in velocity which could be the natural ones in which the Boltzmann equation is studied, the main difficulty comes from the regularity in the x variable). A first obstacle is that it seems unreachable because of the lack of algebra structure of the space $H_x^{\ell-1}$ (if ℓ is close to $d/2$) to study the Boltzmann equation. An alternative would be to prove a result of convergence in a space which writes as the sum of two Banach spaces, one which is “compatible” with the study of the Boltzmann equation and one which is “compatible (in a more accurate way)” with the study of the fluid equation. For example, in the case of the study of the limit towards the incompressible Navier-Stokes equation as in the results presented above, it would be interesting to split the solution into two parts: $f^\varepsilon = f^{\varepsilon,b} + f^{\varepsilon,\#}$ where $f^{\varepsilon,b}$ would correspond to the part of f^ε that provides the fluid equation in the limit $\varepsilon \rightarrow 0$ and $f^{\varepsilon,\#}$ would be the kinetic part that vanishes in the limit $\varepsilon \rightarrow 0$. In the case of hard spheres collisions, the idea would be to study a system of two equations that allow to construct $f^{\varepsilon,b}$ in a space of type $L_t^\infty H_x^{\ell-1} L_v^2(\mu^{-1/2}) \cap L_t^2 H_x^\ell L_v^2(\langle v \rangle^{1/2} \mu^{-1/2})$ and $f^{\varepsilon,\#}$ in $L_t^\infty H_x^\ell L_v^2(\mu^{-1/2}) \cap L_t^2 H_x^\ell L_v^2(\langle v \rangle^{1/2} \mu^{-1/2})$ which is the classical space in which we study the Boltzmann equation.

6.3.3 Hydrodynamical limit for the Vlasov-Poisson-Boltzmann equation

It is a natural and more complicated question to study hydrodynamical limits of equations that involve force fields that are induced by the distribution of particles itself. In particular, one can wonder if the method presented in Chapter 6 could be applied to a Vlasov-Poisson type equation which takes into account an electric field generated by the motion of particles itself as well as collisions that are given by a Boltzmann operator. We refer to the book by Arsénio and Saint-Raymond [23] for results on hydrodynamical limits of the Vlasov-Maxwell-Boltzmann system in a functional framework of weak solutions. A result of weak convergence towards a Navier-Stokes-Fourier-Poisson system for strong solutions has already been obtained in [118] by Guo, Jiang and Lu thanks to a nonlinear energy method.

The rescaled equation that gives in the limit the incompressible Navier-Stokes-Fourier-Poisson

system is the following (see [23]):

$$\begin{cases} \partial_t F + \frac{1}{\varepsilon} v \cdot \nabla_x F + E \cdot \nabla_v F = \frac{1}{\varepsilon^2} Q_B(F, F), \\ E(t, x) := -\nabla_x \phi \quad \text{with} \quad -\Delta_x \phi = \frac{1}{\varepsilon} (\rho_F - 1). \end{cases}$$

After a linearization of order ε around the global equilibrium μ defined in (1.1.15), we are led to study the following system:

$$\begin{cases} \partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f - \frac{1}{\varepsilon} v \mu \cdot \nabla_x \phi + \nabla_x \phi \cdot \nabla_v f = \frac{1}{\varepsilon^2} \mathcal{L}f + \frac{1}{\varepsilon} Q_B(f, f), \\ \Delta_x \phi = \rho_f, \end{cases}$$

where $\mathcal{L} = Q_B(\mu, \cdot) + Q_B(\cdot, \mu)$ is the collisional linearized Boltzmann operator. The expected limit system is the so-called incompressible Navier-Stokes-Fourier-Poisson system that writes in dimension 3

$$\begin{cases} \partial_t u + (u \cdot \nabla_x)u - \nu \Delta_x u = -\nabla_x p + \rho \nabla_x \theta, \\ \operatorname{div}_x u = 0, \\ \partial_t \left(\frac{3}{2} \theta - \rho \right) + u \cdot \nabla_x \left(\frac{3}{2} \theta - \rho \right) - \frac{5}{2} \kappa \Delta_x \theta = 0, \\ \Delta_x (\rho + \theta) = \rho, \end{cases}$$

where the viscosity and heat conductivity coefficients are defined as in (1.2.4).

In [162], Li, Yang and Zhong study the following linear operator:

$$\widehat{B}^1(\xi) \widehat{f} := \mathcal{L} \widehat{f} - i(v \cdot \xi) \widehat{f} - i \frac{v \cdot \xi}{|\xi|^2} \mu \int_{\mathbf{R}^3} \widehat{f} \, dv.$$

By making the appropriate rescaling, we have to study the following linear operator:

$$\widehat{B}^\varepsilon(\xi) \widehat{f} := \mathcal{L} \widehat{f} - i\varepsilon(v \cdot \xi) \widehat{f} - i\varepsilon \frac{v \cdot \xi}{|\xi|^2} \mu \int_{\mathbf{R}^3} \widehat{f} \, dv. \quad (6.3.1)$$

In order to be able to develop an argument similar as the one we used in [103], we have to analyze the spectrum of $\widehat{B}^\varepsilon(\xi)$ for any $\xi \in \mathbf{R}^3$, this will in particular allow us to identify the limit equation. The spectral analysis of this operator seems to be tough. Indeed, it does not seem easy to take advantage of the result of [162] where $\widehat{B}^1(\xi)$ is studied to describe the spectrum of $\widehat{B}^\varepsilon(\xi)$ (in particular, the connection of the spectral analysis for small and large $|\xi|$ turns out to be intricate).

Hydrodynamic limit of the inelastic Boltzmann equation

Summary

In this chapter, we present a result obtained in collaboration with R. Alonso and B. Lods in [15] on a problem of hydrodynamical limit for the *inelastic* Boltzmann equation (for which there is an energy loss during the collision process). We obtain a new system of hydrodynamic equations describing granular flows and prove existence of classical solutions to the aforementioned system. One of the main issue is to identify the correct relation between the restitution coefficient (which quantifies the rate of energy loss at the microscopic level) and the Knudsen number which allows us to obtain non trivial hydrodynamic behavior. In such a regime, we construct strong solutions to the inelastic Boltzmann equation – near thermal equilibrium whose role is played by the so-called homogeneous cooling state – on which we are able to obtain nice estimates that allow us to rigorously derive (in a weak sense) a fluid system. More precisely, the hydrodynamic limit depends on time and space variables only through macroscopic quantities that satisfy a suitable modification of the incompressible Navier-Stokes-Fourier system.

7.1 The model

In this part, we study an equation which is akin to (1.2.3) in $\mathbf{R}^+ \times \Omega \times \mathbf{R}^d$ with $\Omega = \mathbf{T}^d$ in dimension $d \geq 3$. Let us be more precise on the collision process that we consider in this chapter. Our original equation (before doing any rescaling) is the so-called freely cooling Boltzmann equation which provides a statistical description of identical smooth hard spheres suffering binary and inelastic collisions:

$$\partial_t F + v \cdot \nabla_x F = Q_{B,\alpha}(F, F) \quad \text{in } \mathbf{R}^+ \times \mathbf{T}^d \times \mathbf{R}^d. \quad (7.1.1)$$

The parameter α is the restitution coefficient, it belongs to $(0, 1]$ and quantifies the loss of energy during collisions. The collision operator $Q_{B,\alpha}$ is defined in weak form as

$$\int_{\mathbf{R}^d} Q_{B,\alpha}(g, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbf{R}^d \times \mathbf{R}^d} f(v) g(v_*) |v - v_*| \mathcal{A}_\alpha[\psi](v, v_*) dv_* dv, \quad (7.1.2)$$

where

$$\mathcal{A}_\alpha[\psi](v, v_*) := \int_{\mathbf{S}^{d-1}} (\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)) b(\sigma \cdot \widehat{u}) d\sigma, \quad (7.1.3)$$

and the post-collisional velocities (v', v'_*) are given by

$$v' = v + \frac{1+\alpha}{4} (|u|\sigma - u), \quad v'_* = v_* - \frac{1+\alpha}{4} (|u|\sigma - u), \quad \text{where } u := v - v_*, \quad \widehat{u} := \frac{u}{|u|}. \quad (7.1.4)$$

The angular part b of the collision kernel appearing in (7.1.3) is a non-negative measurable mapping integrable over \mathbf{S}^{d-1} and there is no loss of generality assuming

$$\int_{\mathbf{S}^{d-1}} b(\sigma \cdot \widehat{z}) d\sigma = 1, \quad \forall \widehat{z} \in \mathbf{S}^{d-1}.$$

In [15], some additional technical assumptions are made on the angular part b that allow to get nice estimates on the collision operator $Q_{B,\alpha}$ as well as on the difference $Q_{B,\alpha} - Q_{B,1}$. To make the reading

simpler, in this memoir, we only consider the case of hard-spheres collisions in the 3-dimensional case which allows us to suppose that $b = 1$.

Notice also that one can give a strong formulation of the collision operator $Q_{B,\alpha}$ (see [15, Appendix A]) and that this strong formulation is much simpler in the elastic case ($\alpha = 1$) in which the collision process is reversible at the microscopic scale.

The fundamental distinction between the classical elastic Boltzmann equation and the Boltzmann equation associated to granular gases lies in the role of the parameter $\alpha \in (0, 1)$, the coefficient of restitution that we suppose constant. This coefficient is given by the ratio between the magnitude of the normal component (along the line of separation between the centers of the two spheres at contact) of the relative velocity after and before the collision. The case $\alpha = 1$ corresponds to perfectly elastic collisions where kinetic energy is conserved. However, when $\alpha < 1$, part of the kinetic energy of the relative motion is lost since

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-\alpha^2}{4}|u|^2 (1 - \sigma \cdot \widehat{u}) \leq 0.$$

Notice also that the microscopic description (7.1.4) preserves momentum:

$$v' + v'_* = v + v_*.$$

Using the weak formulation given above, one can prove that our model preserves global mass and momentum while it dissipates energy. More precisely, if F is solution to (7.1.1), then

$$\frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbf{R}^3} F(t, x, v) \varphi(v) dv dx = 0 \quad \text{for } \varphi(v) = 1, v,$$

and on the other hand,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbf{R}^3} F(t, x, v) |v|^2 dv dx &= -(1 - \alpha^2) \mathcal{D}_{\mathcal{E}}(F(t), F(t)) \leq 0 \\ \text{with } \mathcal{D}_{\mathcal{E}}(g, f) &:= \frac{\pi}{4} \int_{\mathbf{R}^3 \times \mathbf{R}^3} g(x, v) f(x, v_*) |v - v_*|^3 dv dv_*. \end{aligned} \quad (7.1.5)$$

Various forcing terms have been added to (7.1.1) depending on the underlying physics. Forcing terms prevent the total cooling of the gas (heated bath, thermal bath, see [206] by Villani for details) since they act as an energy supply source to the system and induce the existence of a non-trivial steady state. These are, however, systems different from the free-cooling Boltzmann equation (7.1.1) that we aim to investigate here. To understand better this free-cooling scenario, it is still possible to introduce an intermediate asymptotics and a steady state to work with. This is done by performing a self-similar change of variables (see [176, 173, 174] by Mischler, Mouhot and Rodriguez Ricard). The new equation that we shall study is the following:

$$\partial_t \widetilde{F} + v \cdot \nabla_x \widetilde{F} = Q_{B,\alpha}(\widetilde{F}, \widetilde{F}) - (1 - \alpha) \nabla_v \cdot (v \widetilde{F}) \quad \text{in } \mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3. \quad (7.1.6)$$

The underlying anti-drift term $-(1 - \alpha) \nabla_v \cdot (v \widetilde{F})$ acts as an energy supply which prevents the total cooling down of the gas. Indeed, it has been shown in [176, 173, 174] that there exists a spatially homogeneous steady state G_α to (7.1.6) which is unique for $\alpha \in (\alpha_0, 1)$ for an explicit threshold value $\alpha_0 \in (0, 1)$. More specifically, for $\alpha \in (\alpha_0, 1)$, there exists a unique solution G_α to the spatially homogeneous steady equation

$$(1 - \alpha) \nabla_v \cdot (v G_\alpha(v)) = Q_{B,\alpha}(G_\alpha, G_\alpha) \quad \text{with} \quad \int_{\mathbf{R}^3} G_\alpha(v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (7.1.7)$$

Moreover, there exists some constant $C > 0$ independent of α such that

$$\|G_\alpha - \mathcal{M}\|_{L^1_v(\langle v \rangle^2)} \leq C(1 - \alpha) \quad (7.1.8)$$

where \mathcal{M} is the Maxwellian distribution

$$\mathcal{M}(v) := (2\pi\vartheta_1)^{-\frac{3}{2}} \exp\left(-\frac{|v|^2}{2\vartheta_1}\right), \quad \forall v \in \mathbf{R}^3, \quad (7.1.9)$$

for some explicit temperature $\vartheta_1 > 0$. The Maxwellian distribution \mathcal{M} is a steady solution for $\alpha = 1$ and its prescribed temperature ϑ_1 ensures (7.1.8) to hold.

We then perform a diffusive scaling as presented in Section 1.2 so that we are led to study the following equation:

$$\begin{cases} \partial_t \widetilde{F}^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x \widetilde{F}^\varepsilon + \varepsilon^{-2} (1 - \alpha) \nabla_v \cdot (v \widetilde{F}^\varepsilon) = \varepsilon^{-2} Q_{B,\alpha}(\widetilde{F}^\varepsilon, \widetilde{F}^\varepsilon) & \text{in } \mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3 \\ \widetilde{F}^\varepsilon(0) = \widetilde{F}_{\text{in}}^\varepsilon & \text{in } \mathbf{T}^3 \times \mathbf{R}^3. \end{cases} \quad (7.1.10)$$

Notice also that our rescaled equation in self-similar variables (7.1.10) preserves mass and *vanishing* momentum. From now on, we will thus always assume that

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} \widetilde{F}_{\text{in}}^\varepsilon(x, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \\ E^\varepsilon \end{pmatrix} \quad \text{with } E^\varepsilon > 0 \quad \text{s.t.} \quad \frac{E^\varepsilon - 3\vartheta_1}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (7.1.11)$$

The choice of prescribing as initial energy some constant $E^\varepsilon > 0$ with such a limit property for our problem is natural because $3\vartheta_1$ is the energy of the Maxwellian \mathcal{M} introduced in (7.1.9) and as we shall see, the restitution coefficient α is intended to tend to 1 as ε goes to 0 in our analysis (see Assumption 7.1.1). Moreover, using assumption (7.1.11), we can suppose that

$$\int_{\mathbf{R}^3 \times \mathbf{T}^3} \widetilde{F}^\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \forall t \geq 0.$$

The central underlying assumption in our work is the following relation between the restitution coefficient and the Knudsen number.

Assumption 7.1.1. *The restitution coefficient $\alpha = \alpha(\varepsilon)$ is a continuously decreasing function of the Knudsen number ε satisfying the scaling behavior*

$$\alpha(\varepsilon) = 1 - \varepsilon^2(\lambda_0 + \eta(\varepsilon)) \quad (7.1.12)$$

with $\lambda_0 \geq 0$ and some function $\eta = \eta(\varepsilon)$ that tends to 0 as ε goes to 0. We also assume that there exists $\varepsilon_0 > 0$ such that $\eta(\varepsilon)$ is positive for $\varepsilon \in (0, \varepsilon_0)$ (which in particular implies that $\varepsilon^{-2}(1 - \alpha(\varepsilon)) > 0$ for $\varepsilon \in (0, \varepsilon_0)$).

Assumption 7.1.1 means that the limit $\varepsilon \rightarrow 0$ produces a model of the cumulative effect of *nearly elastic* collisions in the *hydrodynamic regime*. Two situations are of interest in our analysis.

Case 1: If $\lambda_0 = 0$ the cumulative effect of the inelasticity is too weak in the hydrodynamic scale and the expected model is the classical incompressible Navier-Stokes-Fourier system.

Case 2: If $0 < \lambda_0 < \infty$, the cumulative effect is visible in the hydrodynamic scale and we expect a model different from the classical incompressible Navier-Stokes-Fourier equation accounting for that. As we mentioned, we require λ_0 to be relatively small compared to some explicit quantities completely determined by the mass and energy of the initial datum, say, $0 < \lambda_0 \ll 1$ with some explicit upper bounds on λ_0 .

Mimicking the strategy used to derive the classical incompressible Navier-Stokes-Fourier system (presented in Section 1.2), we perform a linearization of order ε around G_α . More precisely, we write

that $\tilde{F}^\varepsilon = G_\alpha + \varepsilon f^\varepsilon$. Linearizing our equation around G_α enables us to use equation (7.1.7) in order to simplify our study. Our new equation at stake is thus:

$$\begin{cases} \partial_t f^\varepsilon = \Lambda_\alpha^\varepsilon f^\varepsilon + \varepsilon^{-1} Q_{B,\alpha}(f^\varepsilon, f^\varepsilon) & \text{in } \mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3 \\ f^\varepsilon(0) = f_{\text{in}}^\varepsilon := \varepsilon^{-1}(\tilde{F}_{\text{in}}^\varepsilon - G_\alpha) & \text{in } \mathbf{T}^3 \times \mathbf{R}^3, \end{cases} \quad (7.1.13)$$

where

$$\Lambda_\alpha^\varepsilon := -\frac{1}{\varepsilon} v \cdot \nabla_x - \frac{1-\alpha}{\varepsilon^2} \nabla_v \cdot (v \cdot) + \frac{1}{\varepsilon^2} \mathcal{L}_\alpha \quad \text{with} \quad \mathcal{L}_\alpha := Q_{B,\alpha}(G_\alpha, \cdot) + Q_{B,\alpha}(\cdot, G_\alpha). \quad (7.1.14)$$

It is also worth noticing that assumption (7.1.11) and (7.1.7) result in

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} f_{\text{in}}^\varepsilon(x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, equation (7.1.13) also preserves mass and *vanishing* momentum. Consequently, there is no loss of generality assuming that

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} f^\varepsilon(t, x, v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall t \geq 0. \quad (7.1.15)$$

Notice that under this assumption, the hypothesis made on the energy of the initial data in (7.1.11) implies that

$$\int_{\mathbf{T}^3 \times \mathbf{R}^3} f_{\text{in}}^\varepsilon(x, v) |v|^2 dv dx \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (7.1.16)$$

Indeed, using (7.1.11) and Assumption 7.1.1 combined with (7.1.8), we obtain

$$\begin{aligned} \int_{\mathbf{T}^3 \times \mathbf{R}^3} f_{\text{in}}^\varepsilon(x, v) |v|^2 dv dx &= \frac{1}{\varepsilon} \int_{\mathbf{T}^3 \times \mathbf{R}^3} \left(\tilde{F}_{\text{in}}^\varepsilon(x, v) - G_{\alpha(\varepsilon)}(v) \right) |v|^2 dv dx \\ &= \frac{E^\varepsilon - 3\vartheta_1}{\varepsilon} + \frac{1}{\varepsilon} \int_{\mathbf{T}^3 \times \mathbf{R}^3} (\mathcal{M}(v) - G_{\alpha(\varepsilon)}(v)) |v|^2 dv dx \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned} \quad (7.1.17)$$

Remark 7.1.2. The above basic estimate and limit is illustrating the two-levels features of the perturbative strategy we adopt to prove Theorem 7.3.1: $\tilde{F}_{\text{in}}^\varepsilon$ stays close to the equilibrium (close-to-equilibrium perturbation) but this comes from the fact that $G_{\alpha(\varepsilon)}$ is close to \mathcal{M} (nearly elastic regime).

7.2 Notations

The functional spaces that are going to be useful in this chapter are the following ones. First, we introduce two Hilbert spaces:

$$\mathcal{H} := L_v^2 H_x^\ell(\mathcal{M}^{-\frac{1}{2}}) \quad \text{and} \quad \mathcal{H}_1 := L_v^2 H_x^\ell(\mathcal{M}^{-\frac{1}{2}} \langle v \rangle^{\frac{1}{2}}), \quad \ell > \frac{3}{2}, \quad (7.2.1)$$

where we recall that \mathcal{M} has been defined in (7.1.9). Then, we introduce the main Banach spaces in which we are going to carry out our analysis:

$$\mathcal{E} := L_v^1 H_x^\ell(\langle v \rangle^k) \quad \text{and} \quad \mathcal{E}_1 := L_v^1 H_x^\ell(\langle v \rangle^{k+1}), \quad \ell > \frac{3}{2} \quad \text{and} \quad k \geq 3. \quad (7.2.2)$$

Remark 7.2.1. Taking $k \geq 3$ will allow us to control the dissipation of energy $\mathcal{D}_\mathcal{E}$ defined in (7.1.5).

Remark 7.2.2. In [117], Gualdani, Mischler and Mouhot are able to apply their enlargement argument in functional spaces in x of L^p -type with $p \neq 2$. They are able to do that by using velocity averaging lemmas. Here, the same type of arguments would apply but would also make appear bad rates in ε . Since the dependency of the restitution α with respect to ε is dictated by Assumption 7.1.1 (and is the only one that allows to capture an hydrodynamic limit different from the elastic one), we cannot rely on this type of arguments. As a consequence, our results are established in functional spaces of L^2 -type in the x variable.

7.3 Main results

Before stating our main results, we recall that one of the main objects of the presented work is to perform the first rigorous derivation of a suitable modification of incompressible Navier–Stokes equation from the Boltzmann equation for inelastic hard-spheres as the Knudsen number goes to zero. However, it is important to mention that other regimes than ours can be considered depending on the rate at which kinetic energy is dissipated; for example, an interesting regime is the mono-kinetic one which considers the extreme case of infinite energy dissipation rate. In this way, the limit is formally described by enforcing a Dirac mass solution in the kinetic equation yielding the *pressureless Euler system* (corresponding to sticky particles). Such a regime has been rigorously addressed in the one-dimensional framework in [146] by Jabin and Rey. It is an open question to extend such analysis to higher dimensions since the approach of [146] uses the so-called Bony functional which is a tool specifically tailored for 1D kinetic equations.

Cauchy theory

An important part of our work is devoted to the study of the equation (7.1.13). More precisely, we aim at developing a perturbative Cauchy theory and getting nice uniform in ε estimates on it in order to be able to derive a fluid system in the limit $\varepsilon \rightarrow 0$. This approach corresponds to the fourth one described in Subsection 1.2.2.

Theorem 7.3.1 ([15], Theorem 1.2). *Let $r \in (0, 1)$. There exists a triple $(\varepsilon_r, \lambda_r, \eta_0)$ of positive constants that depend on the mass and energy of $\tilde{F}_{\text{in}}^\varepsilon$, ℓ and k (the parameters that define \mathcal{E} in (7.2.2)), ε_r and λ_r additionally depending on r whereas η_0 is not, such that for $\varepsilon \in (0, \varepsilon_r)$, $\lambda_0 \in [0, \lambda_r)$, if $\|f_{\text{in}}^\varepsilon\|_{\mathcal{E}} \leq \eta_0$ then equation (7.1.13) has a unique solution $f^\varepsilon \in L_t^\infty(\mathcal{E}) \cap L_t^1(\mathcal{E}_1)$ satisfying*

$$\|f^\varepsilon(t)\|_{\mathcal{E}} \leq C(r, \eta_0) \exp(-(1-r)\lambda_\varepsilon t),$$

$$\text{and} \quad \int_0^t \|f^\varepsilon(\tau)\|_{\mathcal{E}_1} d\tau \leq C(r, \eta_0) \min\left\{1+t, 1+\frac{1}{\lambda_\varepsilon}\right\}, \quad \forall t \geq 0,$$

for some positive constant $C(r, \eta_0) > 0$ independent of ε and where $\lambda_\varepsilon \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^{-2}(1 - \alpha(\varepsilon))$.

Let us mention that two existence results have been obtained in the spatially inhomogeneous setting for $\varepsilon = 1$. The first one is a work by Alonso [11] that treats the Cauchy problem in the case of near-vacuum data and thus in a completely different setting from ours. The second one is a result by the author of this memoir established in an inelastic “collision regime” (in both cases of constant and non-constant coefficient of inelasticity) in [199]. Our approach is actually reminiscent of the one adopted in [199] in several aspects (perturbative approach based on an enlargement argument [117], splitting of the linearized operator...). However, we do not provide a spectral result as in [199] because of the rescaling parameter. Let us be more precise on this fact. The perturbative argument used to get spectral results developed in [199] used a hierarchy of functional spaces. The operator $\mathcal{A}_\alpha^1 \mathcal{S}_{\mathcal{B}_\alpha^1}(t)$ or at least its iterations (see (7.4.1) for the definition of \mathcal{A}_α^1 and \mathcal{B}_α^1) was regularizing enough to go from a large space to a smaller space of this hierarchy. The same arguments would provide bad rates in ε in our context. For those reasons, we do not give any spectral results on our rescaled linearized operator. Though based on a thorough analysis of the linearized operator, our approach is more nonlinear and is explained below in Subsection 7.4.

Before these works, most of the previous results had been established in an homogeneous framework (see the paper of Gamba, Panferov and Villani [104] and the one of Mischler, Mouhot and Rodriguez-Ricard [176] for homogeneous Cauchy theories). We also precise that a large literature has been devoted to the study (existence, uniqueness and growth properties) of self-similar profiles (resp. stationary solutions) for freely cooling (resp. driven by a thermal bath) inelastic hard spheres with a constant restitution coefficient (see [36] by Bobylev, Gamba, Panferov and the already mentioned papers [104, 173, 174]) and with a non-constant one (see [12, 14] by Alonso and Lods).

The lack of results in the inhomogeneous framework is explained by two facts:

- the fact that the steady state is not explicit and is heavy-tailed brings major difficulties when one wants to get spectral results on the linearized operator, in particular, no natural functional framework is available to develop a perturbative Cauchy theory;
- the fact that in the inelastic case, we do not have nice entropy estimates as in the elastic one and as a consequence, the theory of renormalized solutions is not available in the inelastic case.

Hydrodynamic limit

The estimates on the solution f^ε provided by Theorem 7.3.1 are enough to study the limit of f^ε as $\varepsilon \rightarrow 0$. The result that we obtain is the following:

Theorem 7.3.2 ([15], Theorem 1.4). *We suppose that the assumptions of Theorem 7.3.1 are satisfied. We assume furthermore that there exists $(\rho_0, u_0, \theta_0) \in (H_x^\ell)^5$ such that*

$$\|\pi_{\mathcal{L}_1} f_{\text{in}}^\varepsilon - f_0\|_{\mathcal{E}} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (7.3.1)$$

where we recall that $\pi_{\mathcal{L}_1}$ is the projection onto the kernel of \mathcal{L}_1 defined in (1.1.20) and

$$f_0(x, v) := \left(\rho_0(x) + u_0(x) \cdot v + \frac{1}{2} \theta_0(x) (|v|^2 - 3\vartheta_1) \right) \mathcal{M}(v). \quad (7.3.2)$$

Then, for any $T > 0$, the family of solutions $\{f^\varepsilon\}_\varepsilon$ constructed in Theorem 7.3.1 converges in some weak sense to a limit $f = f(t, x, v)$ which is such that

$$f(t, x, v) = \left(\rho(t, x) + u(t, x) \cdot v + \frac{1}{2} \theta(t, x) (|v|^2 - 3\vartheta_1) \right) \mathcal{M}(v), \quad (7.3.3)$$

where

$$(\rho, u, \theta) \in C([0, T], (H_x^{\ell-1})^5) \cap L^2((0, T), (H_x^\ell)^5)$$

is solution to the following incompressible Navier-Stokes-Fourier system with forcing

$$\begin{cases} \partial_t u - \frac{\nu}{\vartheta_1} \Delta_x u + \vartheta_1 u \cdot \nabla_x u + \nabla_x p = \lambda_0 u, \\ \partial_t \theta - \frac{\kappa}{\vartheta_1^2} \Delta_x \theta + \vartheta_1 u \cdot \nabla_x \theta = \lambda_0 \bar{c} \sqrt{\vartheta_1} \theta, \\ \nabla_x \cdot u = 0, \quad \varrho + \vartheta_1 \theta = 0, \end{cases} \quad (7.3.4)$$

subject to initial conditions $(\rho_{\text{in}}, u_{\text{in}}, \theta_{\text{in}})$ defined by

$$u_{\text{in}} := \mathbb{P}u_0, \quad \theta_{\text{in}} := \frac{3}{5}\theta_0 - \frac{2}{5\vartheta_1}\rho_0, \quad \rho_{\text{in}} := -\vartheta_1\theta_{\text{in}}, \quad (7.3.5)$$

where \mathbb{P} is the Leray projection on divergence-free vector fields and (ρ_0, u_0, θ_0) have been introduced in (7.3.2). The viscosity $\nu > 0$ and heat conductivity $\kappa > 0$ are explicit and $\lambda_0 \geq 0$ is the parameter appearing in Assumption 7.1.1. The parameter $\bar{c} > 0$ is depending on the collision kernel.

Remark 7.3.3. For the sake of simplicity, we keep the same notation of the viscosity and heat conductivity as the one we used for the classical Navier-Stokes-Fourier system (see (1.2.5) for their definition). Note however that the definitions are actually slightly different because they involve the temperature ϑ_1 of the Maxwellian \mathcal{M} defined in (7.1.9).

Remark 7.3.4. The data that we consider here are actually quite general. Indeed, the assumption that we make only tells that the macroscopic projection of $f_{\text{in}}^\varepsilon$ converges towards some macroscopic distribution and we do not make any assumption on the macroscopic quantities of this distribution

(see (7.3.1)). Namely, we do not suppose that the divergence free and the Boussinesq relations are satisfied by (ρ_0, u_0, θ_0) , oscillations induced by acoustic waves that could be created by such a lack of assumption is actually absorbed in our notion of weak convergence, the precise notion of which being very peculiar and strongly related to the a priori estimates used for the proof of Theorem 7.3.1 (see (7.4.8) for more details on the type of convergence).

Remark 7.3.5. We point out that the above incompressible Navier-Stokes-Fourier system (7.3.4) with the self-consistent forcing terms on the right-hand-side is a new system of hydrodynamic equations that, to our knowledge, has never been rigorously derived earlier to describe granular flows. We also notice that the last two identities in (7.3.4) give respectively the incompressibility condition and a strong Boussinesq relation. It is important to point out that in the case $\lambda_0 = 0$, one recovers the classical incompressible Navier-Stokes-Fourier system derived from elastic Boltzmann equation (see Chapter 6). Moreover, in both cases $\lambda_0 = 0$ or $\lambda_0 > 0$, the limiting system (7.3.4) is conservative (as soon as the initial bulk velocity is vanishing) which illustrates the perfect balance of the self-similar scaling in the hydrodynamic limit.

To end this part, we explain why the derivation of hydrodynamic limits for granular gases is a challenging question. Indeed, several hurdles stand on way when trying to adapt the classical approaches described in Subsection 1.2.2.

- The first difficulty lies in the identification of the typical hydrodynamic solution when one wants to develop an approach based on asymptotic expansions. One wonders if the homogeneous cooling state G_α can play the role of the Maxwellian in this approach. This is indeed the case up to first order capturing the fat tails of inelastic distributions, yet surprisingly, a suitable Maxwellian plays the role of the hydrodynamic solution in the ε -order correction. This Gaussian behavior actually emerges in the hydrodynamic limit because of the near elastic regime that we treat here (see Assumption 7.1.1).
- Homogeneous cooling states G_α are not explicit, this is a technical difficulty when adapting the approach of Ellis and Pinsky in [98] for the spectral analysis of the linearized inelastic Boltzmann equation in the spatial Fourier variable. Partial interesting results have been obtained in [190] by Rey (devoted to diffusively heated granular gases) but they do not give a complete asymptotic expansion of eigenvalues and eigenfunctions up to the order leading to the Navier-Stokes asymptotic. See also Subsection 7.5.1 for comments on this.
- The strategy based on renormalized solutions appears for the moment out of reach in the context of granular gases. Renormalized solutions in the context of the inelastic Boltzmann equation (7.1.1) have not been obtained due to the lack of an H -Theorem for granular gases. It is unclear if the classical entropy (or a suitable modification of it) remains bounded in general for granular gases.
- Finally, a major obstacle to adapt energy estimates lies in the choice of functional spaces. While the linearized Boltzmann operator associated to elastic interactions is self-adjoint and coercive in the weighted L^2 -space $L^2_\nu(\mathcal{M}^{-1/2})$, there is no such “self-adjoint” space for the inelastic case. This yields technical difficulties in the study of the spectral analysis of the linearized operator which is still actually missing. In the proof of Theorem 7.3.1 described below, we are able to overcome these difficulties thanks to a splitting method.

Haff’s law

We finally mention that we can obtain quite precise description of the hydrodynamic behavior of the original problem in physical variables, namely

$$\begin{cases} \partial_t F^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x F^\varepsilon = \varepsilon^{-2} Q_{B,\alpha}(F^\varepsilon, F^\varepsilon) & \text{in } \mathbf{R}^+ \times \mathbf{T}^3 \times \mathbf{R}^3 \\ f^\varepsilon(0) = F_{\text{in}}^\varepsilon & \text{in } \mathbf{T}^3 \times \mathbf{R}^3. \end{cases} \quad (7.3.6)$$

In this framework, the aforementioned Case 2 for which $\lambda_0 > 0$ enjoys some special features for which uniform-in-time error estimates can be obtained. Turning back to the original problem (7.3.6) not only gives a precise answer to Haff’s law (with an explicit cooling rate of the granular temperature) but also describes the cooling rate of the local temperature of the gas. Precisely, one can deduce the following:

Theorem 7.3.6 (Haff's law: local and global). *Let Assumption 7.1.1 be in force with $\lambda_0 > 0$. We consider a solution F^ε to (7.3.6) (which can be obtained thanks to a change of variables from the solutions constructed in Theorem 7.3.1) for ε , λ_0 and η_0 sufficiently small and define the local temperature of the gas $T^\varepsilon(t, x)$ and the global temperature $\mathbf{T}^\varepsilon(t)$ as*

$$T^\varepsilon(t, x) = \int_{\mathbf{R}^3} F^\varepsilon(t, x, v) |v|^2 dv, \quad \forall (t, x) \in \mathbf{R}^+ \times \mathbf{T}^3 \quad \text{and} \quad \mathbf{T}^\varepsilon(t) = \int_{\mathbf{T}^3} T^\varepsilon(t, x) dx, \quad \forall t \in \mathbf{R}^+.$$

Then, $\mathbf{T}^\varepsilon(t) \approx \frac{3\vartheta_1}{(1+\lambda_0 t)^2}$ for $t \gg \frac{1}{\lambda_0}$, and there exist $C_0, C_1 > 0$ such that

$$\frac{C_0}{(1 + \lambda_0 t)^2} \leq T^\varepsilon(t, x) \leq \frac{C_1}{(1 + \lambda_0 t)^2}, \quad \forall (t, x) \in \mathbf{R}^+ \times \mathbf{T}^3.$$

We point out that the time-decay of the temperature for granular gases has been deduced, under some heuristic considerations, in [125] by Haff and an algebraic decay of the order $(1 + t)^{-2}$ has been established. This is known as *Haff's law* for the cooling of granular gases. A rigorous proof of such a decay has been established in the spatially homogeneous setting in [173] by Mischler and Mouhot in a regime of relative small inelasticity (i.e. assuming $\alpha \geq \alpha_0$ for some explicit value of $\alpha_0 \in (0, 1)$) and for an initial datum $F_{\text{in}} \in L^p(\mathbf{R}^d)$ ($p > 1$). This result has been extended by Alonso and Lods in [13] to consider initial datum with finite entropy. For the physically relevant model of visco-elastic hard-spheres, Haff's law predicts a different decay and has been proved rigorously in the spatially homogeneous setting in [12] by Alonso and Lods. To our knowledge, the above result is the first result applying to spatially inhomogeneous setting and it shows that the decay of the global temperature predicted by Haff's law is actually accurate also for the *local temperature*: in the perturbative setting considered here, Haff's law occurs somehow *uniformly* with respect to the spatial variable $x \in \mathbf{T}^3$.

7.4 Strategy of proof

Cauchy theory and uniform in ε estimates

As pointed out after (7.1.17), our approach is perturbative in two aspects: first, as already said, we are considering close-to-equilibrium solutions (i.e. fluctuations around the homogeneous cooling states) and second, we consider a nearly elastic regime (i.e. fluctuations around the classical/elastic Boltzmann equation). This implies in particular that we shall enforce in (7.1.10) the elastic Boltzmann operator (at both the linearized and nonlinear levels) and will treat, up to some extent, the various inelastic operators as source terms which can be controlled in the limit $\varepsilon \rightarrow 0$ thanks to Assumption 7.1.1. Let us try to make this basic idea more precise.

Linearized operator. First, our approach requires a very fine analysis of the full linearized operator $\Lambda_\alpha^\varepsilon$ appearing in (7.1.13) but is not directly related to a description of the spectral properties of $\Lambda_\alpha^\varepsilon$. We treat $\Lambda_\alpha^\varepsilon$ as a perturbation of the elastic linearized operator Λ_1^ε . However, such a perturbation does not fall into the realm of the classical perturbation theory of unbounded operators as described by Kato in [149]. Typically, the domain of $\Lambda_\alpha^\varepsilon$ is much smaller than the one of Λ_1^ε (because of the drift term in velocity) and the relative bound between Λ_1^ε and $\Lambda_\alpha^\varepsilon$ does not converges to zero in the elastic limit $\alpha \rightarrow 1$. It is one of the reasons why it is not easy to deduce the spectral properties of $\Lambda_\alpha^\varepsilon$ from those of Λ_1^ε (which are well-understood, see for instance [46] by Briant, Merino-Aceituno and Mouhot) and we rather adopt an alternative approach based upon a combination of the enlargement methods introduced in [117] and the L^2 -hypocoercivity method presented in Section 2.1. Indeed, first borrowing ideas from [117, 199], we can split $\Lambda_\alpha^\varepsilon$ as

$$\Lambda_\alpha^\varepsilon = \mathcal{A}^\varepsilon + \mathcal{B}_\alpha^\varepsilon$$

where \mathcal{A}^ε is a regularizing operator in the velocity variable and $\mathcal{B}_\alpha^\varepsilon$ is a suitable dissipative operator. We recall that in [117], the following splitting is provided (see also Remark 6.1.6):

$$\mathcal{L}_1 = \underbrace{A}_{\text{regularizing}} + \underbrace{B}_{\text{dissipative}},$$

in view of applying Theorem 2.2.1. From this, we can deduce the following splitting:

$$\begin{aligned} \Lambda_\alpha^\varepsilon &= \frac{1}{\varepsilon^2} \mathcal{L}_1 - \frac{1}{\varepsilon} v \cdot \nabla_x + \frac{\mathcal{L}_\alpha - \mathcal{L}_1}{\varepsilon^2} - \frac{1-\alpha}{\varepsilon^2} \nabla_v \cdot (v \cdot) \\ &= \underbrace{\frac{1}{\varepsilon^2} A}_{\text{regularizing}} + \underbrace{\frac{1}{\varepsilon^2} B - \frac{1}{\varepsilon} v \cdot \nabla_x + \frac{\mathcal{L}_\alpha - \mathcal{L}_1}{\varepsilon^2}}_{\text{dissipative}} - \underbrace{\frac{1-\alpha}{\varepsilon^2} \nabla_v \cdot (v \cdot)}_{\text{small as } \varepsilon \rightarrow 0} \\ &= \underbrace{\frac{1}{\varepsilon^2} A}_{=: \mathcal{A}^\varepsilon} + \underbrace{\frac{1}{\varepsilon^2} B - \frac{1}{\varepsilon} v \cdot \nabla_x + \frac{\mathcal{L}_\alpha - \mathcal{L}_1}{\varepsilon^2} - \frac{1-\alpha}{\varepsilon^2} \nabla_v \cdot (v \cdot)}_{=: \mathcal{B}_\alpha^\varepsilon}, \end{aligned} \tag{7.4.1}$$

where we recall that \mathcal{L}_α is defined in (7.1.14). We refer to [15, Section 2] for details and insist on the fact that, to capture the dissipativity properties of the operator $\mathcal{B}_\alpha^\varepsilon$, a very fine analysis of the collision operators $Q_{B,\alpha}$ and \mathcal{L}_α is needed with particular emphasis of the quantification of the elastic limit $Q_{B,\alpha} \rightarrow Q_{B,1}$ and $\mathcal{L}_\alpha \rightarrow \mathcal{L}_1$ as $\alpha \rightarrow 1$ in various functional spaces. Let us highlight the fact that in order to be able to carry out our analysis under Assumption 7.1.1, we need to optimise the rate of convergence of the inelastic collision operator towards the elastic one in the limit $\alpha \rightarrow 1$. Our results optimise the ones obtained in [173] by Mischler and Mouhot in a different functional framework. In particular, we point out that due to the lack of symmetry in the two entries of the Boltzmann collision operator for inelastic collisions, it is not trivial to show that one can choose the entry on which the additional derivative is carried when estimating $Q_{B,1}(g, f) - Q_{B,\alpha}(g, f)$. It is possible in a L_v^2 -framework thanks to a result by Bouchut and Desvillettes [38] (which is available only in the L_v^2 -framework).

Besides this splitting, we also can write

$$\Lambda_\alpha^\varepsilon = \Lambda_1^\varepsilon + (\Lambda_\alpha^\varepsilon - \Lambda_1^\varepsilon)$$

and exploits the hypocoercivity properties of Λ_1^ε , hoping the reminder term $\Lambda_\alpha^\varepsilon - \Lambda_1^\varepsilon$ will not make them degenerate too much. To do so, and because of the additional derivative in the velocity variable (appearing now in the difference $\Lambda_\alpha^\varepsilon - \Lambda_1^\varepsilon$), we use hypocoercivity estimates for Λ_1^ε (see [15, Section 3]) of the same type as the ones developed in Chapter 5 (see also (6.2.18)-(6.2.19) for a rescaled version of hypocoercivity estimates in the case of the Landau equation). We are thus able to build a new norm $\|\cdot\|_{\mathcal{H}}$ with associated scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{H}}$ on the Hilbert space \mathcal{H} such that

$$\langle\langle \Lambda_1^\varepsilon f, f \rangle\rangle_{\mathcal{H}} \lesssim -\frac{1}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_1}^2 - \|f\|_{\mathcal{H}_1}^2 \tag{7.4.2}$$

where the functional spaces \mathcal{H} and \mathcal{H}_1 are defined in (7.2.1).

Remark 7.4.1. Here, we keep the same notation $f^\perp = f - \pi_{\mathcal{L}_1} f$ as in the rest of the memoir. It is an abuse of notations because we actually consider the projection onto $\text{Ker } \mathcal{L}_1$ associated to the equilibrium \mathcal{M} defined in (7.1.9) instead of μ defined in (1.1.15). But we keep the same notations as previously in order to lighten the presentation.

Cauchy theory for the nonlinear problem. With this at hands, in order to prove Theorem 7.3.1, several a priori estimates for the solutions to (7.1.13) are required. The crucial point in the analysis lies in the splitting of (7.1.13) into a system of two equations mimicking the spectral enlargement method presented in the “Proof with the “splitting” method” in Section 2.2 (see [175] for the original argument and [46] for

the adaptation to a nonlinear setting). More precisely, the splitting performed in [15, Section 4 and 5] amounts to look for a solution of (7.1.13) of the form

$$f^\varepsilon(t) = f_0^\varepsilon(t) + f_1^\varepsilon(t)$$

where f_0^ε and f_1^ε are the solutions to the following system of equations (in order to lighten the notations, we omit the dependency on ε for f_0 and f_1):

$$\begin{cases} \partial_t f_0 &= \mathcal{B}_{\alpha(\varepsilon)}^\varepsilon f_0 + \varepsilon^{-2}(\mathcal{L}_{\alpha(\varepsilon)} - \mathcal{L}_1) f_1 + \varepsilon^{-2}(1 - \alpha(\varepsilon))K \langle v \rangle^2 (\text{Id} - \pi_{\Lambda_1}) f_1 \\ &+ \varepsilon^{-1} \left[Q_{B,\alpha(\varepsilon)}(f_0, f_0 + f_1) + Q_{B,\alpha(\varepsilon)}(f_1, f_0) + \pi_{\mathcal{L}_1} Q_{B,\alpha(\varepsilon)}(f_1, f_1) \right], \\ f_0(0, x, v) &= f_{\text{in}}^\varepsilon(x, v) \in \mathcal{E} \end{cases} \quad (7.4.3)$$

and

$$\begin{cases} \partial_t f_1 &= \Lambda_1^\varepsilon f_1 + \varepsilon^{-2}(1 - \alpha(\varepsilon)) \left[-\text{div}_v(v f_1) - K \langle v \rangle^2 (\text{Id} - \pi_{\Lambda_1}) f_1 \right] \\ &+ \varepsilon^{-1} (\text{Id} - \pi_{\mathcal{L}_1}) Q_{B,\alpha(\varepsilon)}(f_1, f_1) + \mathcal{A}^\varepsilon f_0, \\ f_1(0, x, v) &= 0 \in \mathcal{H} \end{cases} \quad (7.4.4)$$

where we recall that $\pi_{\mathcal{L}_1} = \pi_{\mathcal{L}}$ and $\pi_{\Lambda_1} = \pi_{\Lambda}$ are defined in (1.1.20)-(1.1.28) and K is a positive constant which is such that (see [15, Lemma 3.3])

$$- \langle \nabla_v \cdot (vg), g \rangle_{\mathcal{H}} \leq K \langle \langle v \rangle^2 g, g \rangle_{\mathcal{H}} + C\varepsilon \|g\|_{\mathcal{H}}^2. \quad (7.4.5)$$

Remark 7.4.2. It is important to recall that the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is of the form (6.2.19) and thus depends on ε , which explains the fact that there is an ε in the above estimate.

Tailoring the splitting of (7.1.13) into two equations (7.4.3)-(7.4.4) is actually one of the most difficult part of this work and some comments are in order. Let us first mention that our splitting bears similarities with the one used in [46] by Briant, Merino-Aceituno and Mouhot but differ from it on several aspects.

- First, notice that as in [46], we can analyze the equation on f_1 in the most convenient functional space since we have put 0 as initial data for f_1 . It is then natural to study the equation on f_1 in the Hilbert space \mathcal{H} since the elastic linearized operator Λ_1^ε is well-understood in this type of space (see (7.4.2)). It is worth mentioning that in [46], the equation on f_1 is posed in $H_{x,v}^\ell(\mathcal{M}^{-1/2})$ whereas we study it in the space \mathcal{H} with no derivative in velocity. It is made possible by our above L^2 -hypocoercivity results.
- Similarly as in [46], we have put the nice dissipative part of the linearized equation on f_0 in the equation on f_0 (namely $\mathcal{B}_{\alpha(\varepsilon)}^\varepsilon f_0$) while we have put $\mathcal{A}^\varepsilon f_0$ in the equation on f_1 . We are able to do this thanks to the regularizing properties of \mathcal{A}^ε (namely \mathcal{A}^ε regularizes from \mathcal{E} into \mathcal{H}).

We already mentioned a first difference with the study by [46] in the first item concerning the space in which we study the equation on f_1 . If we compare our splitting to the one in [46], it is also much more complicated because of the additional terms coming from the inelasticity of our equation. Let us present and explain those differences.

- First, notice that we can keep the drift term on f_1 in the equation on f_1 because when performing energy estimates in \mathcal{H} , it only induces the appearance of bounded terms in f_1 that are small and terms with a loss of weight (see (7.4.5)) but no terms with a loss of derivative in velocity. This loss of weight is counterbalanced by the introduction of the term $-K \langle v \rangle^2 (\text{Id} - \pi_{\Lambda_1}) f_1$.
- The term $K \langle v \rangle^2 (\text{Id} - \pi_{\Lambda_1}) f_1$ has thus to be added to the equation on f_0 but it turns out to be harmless when estimated in the functional space \mathcal{E} because of the continuous embedding of \mathcal{H} into \mathcal{E}_2 where the functional space \mathcal{E}_2 is defined as

$$\mathcal{E}_2 := L_v^2 H_x^\ell(\langle v \rangle^{k+2\kappa+3}) \quad \text{with} \quad \kappa > \frac{3}{2},$$

and acts as an intermediate space which allows somehow to make the link between the L^1_v -space \mathcal{E} and the L^2_v -space \mathcal{H} .

- Since we only put the elastic linearized collision operator on f_1 in the equation on f_1 , it still remains to deal with the difference $\varepsilon^{-2}(1 - \alpha(\varepsilon))(\mathcal{L}_{\alpha(\varepsilon)} - \mathcal{L}_1)$. Notice that some crucial point is that we are able to obtain estimates on this term with no loss of derivative in velocity in the functional space \mathcal{E} (thanks to [15, Lemma 2.4]) and we can handle this term still thanks to the embedding $\mathcal{H} \hookrightarrow \mathcal{E}_2$.
- Finally, concerning the nonlinear term $\varepsilon^{-1}Q_{B,\alpha(\varepsilon)}(f_1, f_1)$, it is split into two parts. We only keep the microscopic part of this term, namely $\varepsilon^{-1}(\text{Id} - \pi_{\mathcal{L}_1})Q_{\alpha(\varepsilon)}(f_1, f_1)$ in the equation on f_1 because this term can be handled thanks to the nice hypocoercivity estimates (7.4.2) satisfied by Λ_1^ε in \mathcal{H} (as the term $\varepsilon^{-1}Q_{B,1}(f_1, f_1) = \varepsilon^{-1}(\text{Id} - \pi_{\mathcal{L}_1})Q_{B,1}(f_1, f_1)$ was treated in [46]). As a consequence, the remainder $\varepsilon^{-1}\pi_{\mathcal{L}_1}Q_{B,\alpha(\varepsilon)}(f_1, f_1)$ is then added to the equation on f_0 and does not induce any difficulty because the projector $\pi_{\mathcal{L}_1}$ regularizes as necessary in velocity and because $\mathcal{B}_{\alpha(\varepsilon)}^\varepsilon$ is $-\varepsilon^{-2}$ -dissipative in \mathcal{E} .

Remark 7.4.3. The physical meaning of the above decomposition is not clear to us but appears to be a convenient tool to enforce the energy method we adopt here. Notice that, in the framework of strong solutions in the elastic case, some other kinds of splitting have been considered (see [108, 109]). They have a clearer physical meaning since they are related to the decomposition of the solution f^ε to the kinetic equation into several pieces (kinetic, macroscopic, dispersive terms) which all have a precise meaning. This is made possible thanks to a suitable mild reformulation of both the Boltzmann equations and Navier-Stokes-Fourier system and a fine spectral analysis of the linearized operator as presented in Chapter 6.

With all this at hands, we are able to get nice a priori estimates on (f_0, f_1) satisfying (7.4.3)-(7.4.4). Let us point out that getting estimates on f_1 is trickier than in [46], indeed, in the latter paper, the idea was to estimate separately $\pi_{\Lambda_1}f_1$ and $(\text{Id} - \pi_{\Lambda_1})f_1$ where π_{Λ_1} is the projector onto $\text{Ker } \Lambda_1^\varepsilon$ defined by (1.1.28) and thanks to the properties of conservation of mass, momentum and energy of the whole equation, one could write that $\pi_{\Lambda_1}f = 0$ so that $\pi_{\Lambda_1}f_1 = -\pi_{\Lambda_1}f_0$ and directly get an estimate on $\pi_{\Lambda_1}f_1$ from the one on f_0 . In our case, the energy is no longer preserved which induces additional difficulties. However, we keep the same strategy and are able to get nice estimates on the energy of $\Lambda_{\alpha(\varepsilon)}^\varepsilon$ thanks to the dissipation of energy defined in (7.1.5).

In the end, we can obtain the following estimates (which will be useful to derive our hydrodynamic system) on f_0 and f_1 that are respectively solutions to (7.4.3) and (7.4.4):

$$\|f_0\|_{L_t^\infty(\mathcal{E})} + \varepsilon^{-2}\|f_0\|_{L_t^1(\mathcal{E}_1)} \lesssim 1 \quad (7.4.6)$$

as well as

$$\|f_1\|_{L_t^\infty(\mathcal{H})} + \|f_1\|_{L_t^2(\mathcal{H}_1)} \lesssim 1, \quad (7.4.7)$$

where the multiplicative constants only involve quantities related to the initial data of the problem and are independent of ε . We are actually able to get rates of time decay by using a refined Grönwall argument. Thanks to these a priori estimates, we are able to prove Theorem 7.3.1 by introducing an adapted iterative scheme as presented in [15, Section 5].

Hydrodynamic limit

We can actually give a more refined version of the convergence result given in Theorem 7.3.2 thanks to (7.4.6)-(7.4.7) together with hypocoercivity estimates:

$$\begin{cases} \{f_0^\varepsilon\}_\varepsilon \text{ converges to } 0 \text{ strongly in } L^1((0, T), \mathcal{E}_1), \\ \{f_1^\varepsilon\}_\varepsilon \text{ converges to } f \text{ weakly in } L^2((0, T), \mathcal{H}). \end{cases} \quad (7.4.8)$$

To identify the limit system satisfied by the macroscopic quantities (ρ, u, θ) , we follow the same lines as in the elastic case and more precisely the same path of Bardos, Golse, Levermore and Saint-Raymond [30,

[29, 111, 112] exploiting the fact that the mode of convergence given in (7.4.8) is stronger than the one in the above mentioned works. In particular, we can adapt some of the main ideas of [111] regarding the delicate convergence of nonlinear terms (the detailed computations and arguments are included in [15, Section 6]). The regime of weak inelasticity is central in the analysis, basically, when we write the local conservation laws, some terms can be treated exactly in the same way as in the elastic case while the remainders are treated as a source term (still under Assumption 7.1.1) which takes into account the drift term and the dissipation of kinetic energy at the microscopic level. It is worth mentioning that due to the non conservation of energy by the kinetic model, the strong Boussinesq relation cannot be obtained as directly as in the elastic case. We refer to [15, Section 6] for the derivation of the limit system, the proof is classical and based on compactness arguments (Aubin-Lions lemma [195], Arzelà-Ascoli theorem and a compensated compactness argument established in [165] by Lions and Masmoudi).

7.5 Perspectives

In this last Section, we describe a few directions in which we would like to improve the result presented in this Chapter.

7.5.1 Strong convergence towards inelastic hydrodynamic limit

The goal would be to obtain a similar result as the ones presented in Chapter 6 concerning the elastic Boltzmann equation and the Landau equation. The proof of these results are based on the spectral analysis of the linearized operator in Fourier space for the spatial variable. As already mentioned, such an analysis is not available for the inelastic Boltzmann equation. The only result in this direction is given by a paper by Rey [190] and does not give enough precise Taylor expansions of eigenvalues to be able to perform a diffusive limit of the kinetic model. It actually seems difficult to refine enough this approach to obtain hydrodynamic limit results in the inelastic case. Indeed, it relies on a perturbative argument around the elastic case in which the Taylor expansions of eigenvalues are of the form (6.1.11). In this (elastic) case, two eigenvalues have a vanishing first order term and these eigenvalues are precisely the ones that dictate the hydrodynamic behavior in the limit $\varepsilon \rightarrow 0$. It seems that using a perturbative argument on this type of Taylor expansion does not give precise enough information on the eigenvalues to perform an hydrodynamic limit.

Another way to tackle the problem would be to try to develop a perturbative argument at the level of semigroups. Getting a formal result of convergence starting from a mild formulation of solutions seems doable under Assumption 7.1.1. Indeed, we could use the precise analysis on expansions of the collisional operator $Q_{B,\alpha}$ and the steady state G_α around $Q_{B,1}$ and \mathcal{M} respectively performed in [173] by Mischler and Mouhot as well as estimates we established in [15]. However, make this analysis rigorous does not seem to be straightforward in terms of stability of functional spaces (due to the loss of derivatives in the estimates of the drift term for example).

7.5.2 Hydrodynamic limit for viscoelastic hard spheres

The goal is to obtain a similar result as the one presented in this Chapter in the case of viscoelastic hard spheres. It corresponds to a case where the restitution coefficient α is not constant and more precisely, we suppose that $e := 1 - \alpha$ satisfies the following conditions (see [12, 14] by Alonso and Lods and [199] by the author of this memoir):

- the mapping $r \rightarrow e(r)$ from \mathbf{R}^+ to $(0, 1]$ is absolutely continuous and non-increasing;
- the mapping $r \rightarrow r e(r)$ is strictly increasing on \mathbf{R}^+ ;
- there exist $a, b > 0$ and $\bar{\gamma} > \gamma > 0$ such that

$$|e(r) - 1 + ar^\gamma| \leq br^{\bar{\gamma}}, \quad \forall r \geq 0.$$

As in the constant case, we can perform a self-similar change of variables in order to introduce an intermediate Maxwellian asymptotic. After a diffusive scaling, we are led to study an equation of the following form:

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon + \varepsilon^2 \xi_\varepsilon(t) \nabla_v \cdot (v f^\varepsilon) = Q_{e_{\varepsilon,t}}(f^\varepsilon, f^\varepsilon).$$

Such a change of variables induces difficulties, for example, it introduces a coefficient $\xi_\varepsilon(t)$ in our equation which is singular for small times. Indeed, we have

$$\xi_\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{1 + \gamma} \frac{1}{t}.$$

It thus seems unadapted if one wants to perform energy estimates that provide uniform in ε estimates in the limit $\varepsilon \rightarrow 0$. We could try to introduce a less naive change of variables in the sense that it would separate different scales of time but that would still allow to prevent the total cooling down of the gas.

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RÉSUMÉ

Les travaux présentés dans ce mémoire concernent différentes équations de la théorie cinétique des gaz (Boltzmann, Landau, Fokker-Planck) et s'articulent autour de deux grands axes :

- développer une théorie de Cauchy dans un cadre perturbatif de solutions fortes et étudier le comportement en temps grand des solutions,
- étudier des problèmes de limites hydrodynamiques.

Qu'elles s'inscrivent dans l'un ou l'autre de ces deux axes, les études menées reposent sur l'analyse de problèmes linéaires au moyen de différentes théories : théorie des semi-groupes, théorie spectrale, hypocoercivité, élargissement d'espace, argument de perturbation, hypoellipticité etc... Ainsi, nous appliquons et/ou étendons ces techniques à de nouvelles situations : EDP de nouvelle nature, équations discrètes, nouveau cadre fonctionnel notamment.

ABSTRACT

The works presented in this memoir are about various equations of kinetic theory (Boltzmann, Landau, Fokker-Planck) and are structured around two main axes:

- develop a Cauchy theory in a perturbative framework of strong solutions and study the large-time behavior of solutions,
- study hydrodynamic limit problems.

Whether they fall into one or the other of these two areas, the studies carried out are based on the analysis of linear problems using a variety of theories: semi-group theory, spectral theory, hypocoercivity, space enlargement, perturbative argument, hypoellipticity... In this way, we apply and/or extend these techniques to new situations: new types of PDEs, discrete equations, new functional frameworks and so on.