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Rana Badreddine

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Une équation de Schrödinger non-linéaire issue
du système de Calogero-Sutherland-Moser.
*On a DNLS equation related to the
Calogero-Sutherland-Moser Hamiltonian system*

Thèse de doctorat de l'Université Paris-Saclay

École doctorale n°574, mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques fondamentales
Graduate School : Mathématiques, Référent : Faculté des sciences
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Thèse préparée dans l'unité de recherche **Laboratoire de mathématiques
d'Orsay (Université Paris-Saclay, CNRS)**, sous la direction de **Patrick
Gérard**, Professeur.

Thèse soutenue à Paris-Saclay, le 21 juin 2024, par

Rana Badreddine

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Chapter 1

Introduction (in English)

ABSTRACT.

In this introduction, we provide an overview of the results obtained in this thesis concerning a newly investigated nonlocal nonlinear Schrödinger equation, derived as a continuum limit of the classical *Calogero–Sutherland–Moser system*. The main approach employed to prove these results involves utilizing the complete integrability properties concealed within this nonlinear PDE. In what follows, we describe in some words the concept of completely integrable equations, present the results obtained for this equation, and draw comparisons with previously established results for other integrable equations, notably the Szegő and Benjamin–Ono equation.

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Imagine observing a wave and capturing its distribution at one moment. This leads to a natural question : how does the spatial distribution of this wave evolve over time under specific conditions ? Any mathematicians or physicists would tell you that this evolution can be aptly described using a powerful and rigorous tool known as the Partial Differential Equation (PDE). Yet, the challenge resides in explicitly solving this PDE, i.e. finding an explicit formula that describes exactly, at each moment, the wave’s spatial distribution. Unfortunately, this often proves to be an insurmountable task for PDEs describing real-world phenomena (the nonlinear PDE). Nevertheless,

there are certain types of nonlinear PDEs, specifically completely integrable ones, for which explicit solutions can sometimes be found.

1.1 What is a completely integrable PDE?

The specialists appear to lack consensus on a universal definition of integrable infinite dimensional systems. This divergence of opinions may be due to the richness of the structure exhibited by such PDEs, allowing each expert to approach the subject from different perspectives. For some, the immediate association with the term “completely integrable” is linked to the existence of a **Lax pair**—a pair of operators intimately connected to the PDE and playing a crucial role in the spectral study of the problem (We refer to Section 1.1.1 for the definition and more details). Others, however, may perceive integrability through the existence of **Birkhoff coordinates**—a system of coordinates where this nonlinear PDE can be “diagonalized”, enabling the study of the solution’s behavior in the canonical structure (See Section 1.1.2). Another perspective argues that the concept of integrability extends beyond these structures, emphasizing the system’s ability to be integrated and explicitly solved, thus obtaining an **explicit formula** for the solution (See Section 1.1.3)...

Nevertheless, there is an inclination among scholars that completely integrable PDEs possess an **infinite number of conserved quantities**, which play a pivotal role in obtaining spectacularly precise results compared to other types of PDEs. Additionally, the symmetry hidden behind these infinite conservation laws enables the construction of exact solutions for the integrable PDE by using techniques such as the inverse scattering transform or spectral theory. As an example, one could mention the existence of **solitons**, which are particular solutions of traveling waves typically expressed through elementary functions (See Section 1.1.4).

A prime example of an integrable PDE was suggested in 1872 and is known to this day as the Korteweg-de Vries equation [Bou72, KV95]

$$\partial_t u + \partial_x^3 u - 6u\partial_x u = 0. \quad (\text{KdV})$$

This equation serves as a mathematical model describing the propagation of small-amplitude waves in long surface waves of water in a narrow and shallow channel. The integrability property of the KdV equation traces back to the work of Gardner, Greene, Kruskal, and Miura [GGKM67], who introduced the inverse scattering transform to solve explicitly the KdV equation and recover the existence of solitary waves. Specifically, these waves are obtained by solving the scattering problem associated with the linear Schrödinger equation with a potential term related to the solution of the (KdV) equation. The solution of this scattering problem encodes information about the soliton solutions of the KdV equation.

Rather than delving deeper into the complexities of inverse scattering theory [Pal97], our attention shifts towards Peter Lax’s subsequent work [Lax68], published

one year after the work of [GGKM67], which reformulates the concepts presented in the earlier work.

1.1.1 The Lax Pair

In his famous paper [Lax68], Peter Lax demonstrates the existence of two linear operators associated with the (KdV) equation, commonly known now as the Lax pair operators. These operators, denoted as L_u and B_u , are defined as follows :

$$L_u = -\partial_x^2 + u, \quad B_u = 4\partial_x^3 - 6u\partial_x - 3\partial_x u,$$

and they satisfy the Lax equation introduced in (1.1.1) when u satisfies the (KdV)–equation. A remarkable observation is that by considering the eigenfunction equation of L_u , i.e. $L_u f = \lambda f$, one recalls the Schrödinger equation described in the previous work of [GGKM67] which uses the scattering theory. The power of considering these operators lies in examining the (KdV)–equation from a different perspective by studying a spectral problem that offers valuable information about the dynamics and, more importantly, endorses the ability to construct from the Lax operator L_u an infinite number of integrals of motion (i.e. conserved quantities) related to this PDE. To elaborate on the latter point further, we will describe it in the most general setting, that is, in the case where we consider any PDE enjoying a Lax pair structure.

The integrability in terms of Lax operators can be described as follows : Given a Cauchy problem with initial data u_0 , there exists two linear operators (L_u, B_u) depending on the dynamical variables $(u, \partial_x u, \dots, \partial_x^n u)$ of the Cauchy problem, such that if u satisfies the PDE, then the two operators (L_u, B_u) satisfies the Lax equation

$$\partial_t L_u = [L_u, B_u], \quad (1.1.1)$$

where $[L_u, B_u] = L_u B_u - B_u L_u$. For instance, let us consider the Cauchy problem associated with

— **The Benjamin–Ono equation** on \mathbb{R} and $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ [Ben67, Ono75]

$$\partial_t u = \partial_x (|D|u - u^2), \quad D = -i\partial_x, \quad |\widehat{D}|(\xi) = |\xi|. \quad (\text{BO})$$

This equation characterizes the one-dimensional internal wave propagation within a stratified fluid [Sau19]. It is also known to possess a Lax pair structure [Nak79, BK79]

$$L_u = D - T_u, \quad B_u = i(T_{|D|u} - T_u T_u), \quad (1.1.2)$$

where T_u is the Toeplitz operator of symbol $u \in L^\infty$ defined for any $f \in L^2$, as $T_u(f) := \Pi(uf)$, and Π is the Riesz–Szegő projector on the non–negative Fourier modes, defined on \mathbb{R} as $\widehat{\Pi}u(\xi) = \mathbb{1}_{\xi \geq 0} \widehat{u}(\xi)$, and on \mathbb{T} as

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) = \sum_{n \geq 0} \widehat{u}(n) e^{inx}. \quad (1.1.3)$$

From the Lax operator L_u , one infers an infinite number of conservation laws given by $\langle L_u^k \mathbb{1} | \mathbb{1} \rangle$, $k \in \mathbb{N}_{\geq 0}$. [GK21, Sun21].

- **The cubic Szegő equation** introduced in [GG08a, GG10, Poc11a, Poc11b] as a toy model for a totally non-dispersive equation

$$i\partial_t u = \Pi(|u|^2 u), \quad (\text{Sz})$$

where Π is defined in (1.1.3). This equation admits also a Lax Pair structure on \mathbb{R} as well on \mathbb{T} , [GG10, GG12a, Poc11a, Poc11b]

$$\frac{d}{dt} H_u = [B_u, H_u], \quad (1.1.4)$$

where H_u is the *Hankel operator* of symbol $u \in L^\infty$ defined for any $f \in L^2$, as $H_u(f) = \Pi(u\bar{f}) = T_u(\bar{f}) = T_{\bar{f}}(u)$. And B_u is the skew-adjoint operator

$$B_u := -iT_{|u|^2} + \frac{i}{2}H_u^2.$$

More remarkably, this equation enjoys on the torus \mathbb{T} a second pair of Lax operators [GG10, GG12a]

$$\frac{d}{dt} K_u = [C_u, K_u], \quad (1.1.5)$$

where the operator K_u is given by $K_u := H_u S$, with S is the Shift operator $Sf(x) = e^{ix}f(x)$, and C_u is the skew-adjoint operators

$$C_u := -iT_{|u|^2} + \frac{i}{2}K_u^2$$

From the Lax operator H_u , one infers an infinite number of conservation laws given by $\langle u | H_u^{2k-2} u \rangle$, $k \in \mathbb{N}_{\geq 1}$ [GG10].

Furthermore, it is important to mention that whenever the second operator B_u in the Lax pair structure (1.1.1) is skew-adjoint¹, then the *isospectral deformation property* is satisfied, i.e.

$$U(t)^* L_{u(t)} U(t) = L_{u_0}, \quad (1.1.6)$$

where the family of operators $U(t)$ is solution to the Cauchy problem

$$\begin{cases} \frac{d}{dt} U(t) = B_{u(t)} U(t) \\ U(0) = \text{Id} \end{cases}.$$

Indeed, as B_u is a skew-adjoint operator, then the solution $U(t)$ of the previous Cauchy problem is a unitary operator as $\frac{d}{dt}(U(t)U^*(t)) = \frac{d}{dt}(U^*(t)U(t)) = 0$. Therefore, by computing $\frac{d}{dt}(U(t)^* L_{u(t)} U(t))$ and using the Lax equation (1.1.1), one finds

$$\frac{d}{dt}(U(t)^* L_{u(t)} U(t)) = 0.$$

Consequently, (1.1.6) implies that the spectrum of L_u is conserved along the flow, providing a new infinite family of conservation laws for the integrable PDE.

1. One can demonstrate that, even in cases where B_u is not a skew-adjoint operator, we still have $U(t)^{-1} L_{u(t)} U(t) = L_{u_0}$.

1.1.2 Action–Angle variables/Birkhoff coordinates

An important feature of nonlinear integrable PDEs is the fact that they exhibit a remarkable interplay between geometry and analysis. They can hide a significant amount of symmetries within the problem that allows it to be diagonalized. But before delving deeper into this point, we must recall the idea behind a widely recognized tool from the linear PDE domain : the linear Fourier transform.

Consider a linear PDE ; the main goal behind an equation is to solve it, or at least to capture more information hidden within it. That is why powerful techniques have been developed over the centuries. One of the most remarkable and pivotal moments in the analysis of PDEs was the discovery of the Fourier transform. It allows the decomposition of any function into its frequency components and transforms any spatial domain into the frequency domain. Another significant aspect of this method is its ability to diagonalize any linear operator in the PDE, enabling us to analyze phenomena such as dispersion, decay rates, and stability... Now, coming back to the nonlinear integrable PDEs, when faced with one, the aim is to mimic the idea used for linear PDEs by seeking a transform capable of diagonalizing the problem. This allows us to view the problem from different settings/coordinates, thereby extracting much more information about the solution.

For instance, suppose we are dealing with a linear PDE on the torus ($x \in \mathbb{T}$). The linear Fourier transform establishes a bijection between the solution u of the linear PDE and its Fourier coefficients $(\langle u | e^{ikx} \rangle)_{k \in \mathbb{N}_{\geq 0}}$. For a nonlinear integrable PDE, one needs to search for a nonlinear Fourier transform, i.e. a bijection between the solution u of this nonlinear PDE and the nonlinear Fourier coefficients $(\langle u | f_n^t \rangle)_{n \in \mathbb{N}_{\geq 0}}$, where the $(f_n^t)_{n \in \mathbb{N}_{\geq 0}}$ are an adequate orthonormal basis depending on the PDE. These coordinates $(\langle u | f_n^t \rangle)_{n \in \mathbb{N}_{\geq 0}}$, referred to as the **Birkhoff coordinates**, allow us to approach the problem from different perspectives : instead of studying a nonlinear PDE, the problem can be reduced through this change of coordinates to a system of linear ODEs in the variables t (the temporal variable) and of unknown $\langle u, f_n^t \rangle$ for the n^{th} - equation of this linear system. Note that in the literature, one may encounter the term “**action–angle variables**”² instead of Birkhoff coordinates ; they are the same as polar coordinates for a complex point $z \in \mathbb{C}^*$. The action variables can describe a complete set of integrals of motion (conservation laws) in involution³, known as Liouville integrability [KP13].

As we will see in the following two examples, the Liouville integrability will be the main instrument to uncover properties of the PDE’s dynamics, such as GWP results, periodic orbits, norm inflation...

2. [Arn13, KP13]

3. That is, the conservation laws commute in the Poisson brackets $\{\cdot, \cdot\}$ associated with the Hamiltonian of the equation. In other words, the Poisson bracket of these conservation laws vanishes.

1.1.2.1 Application to the Benjamin–Ono equation

We recall the Benjamin–Ono equation

$$\partial_t u = \partial_x (|D|u - u^2), \quad u(0) = u_0, \quad D = -i\partial_x, \quad |\widehat{D}|(\xi) = |\xi|. \quad (\text{BO})$$

It is a completely integrable PDE in the Liouville sense, as described in [GK21] : If we denote by $L_{r,0}^2(\mathbb{T})$ the space of real functions in $L^2(\mathbb{T})$ with zero mean, and

$$h_+^r = \left\{ (\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} ; \|\zeta\|_{h_+^r}^2 = \sum_{n=1}^{+\infty} n^{2r} |\zeta_n|^2 < +\infty \right\}; \quad r \in \mathbb{R},$$

and if we denote for all $t \in \mathbb{R}$, $n \in \mathbb{N}_{\geq 1}$,

$$\zeta_n(u(t)) := \frac{\langle u(t) | f_n^t \rangle}{\kappa_n}$$

where the (f_n^t) are the eigenfunctions of the Lax operator $L_{u(t)}$ defined in (1.1.2) and κ_n is a certain normalization independent of the time t [GK21], then the *nonlinear Fourier transform* for the (BO)–equation is described by the following homeomorphism⁴

$$\Phi : u \in L_{r,0}^2(\mathbb{T}) \longmapsto (\zeta_n(u))_{n \geq 1} \in h_+^{\frac{1}{2}}. \quad (1.1.7)$$

The $(\zeta_n(u))$ represent the Birkhoff coordinates. And each $\zeta_n(u)$ satisfies the following evolution

$$\zeta_n(u(t)) = e^{it\omega_n(u_0)} \zeta_n(u_0), \quad \omega_n(u_0) = n^2 - 2 \sum_{k \geq 1} \min(k, n) |\zeta_k(u_0)|^2.$$

In other words, one can understand the dynamics of the Benjamin–Ono (BO) equation by examining the behavior of the coordinates $(\zeta_n(u))$ evolving nicely in time. In the work [GKT23], the authors generalized this nonlinear Fourier transform to any $H_{r,0}^s(\mathbb{T})$ spaces (i.e., the space of real functions in $H^s(\mathbb{T})$ with zero mean), where $s > -\frac{1}{2}$. Therefore, the following extension holds

$$\Phi : u \in H_{r,0}^s(\mathbb{T}) \longmapsto (\zeta_n(u))_{n \geq 1} \in h_+^{\frac{1}{2}+s}$$

for any $s > -\frac{1}{2}$ and remains a homeomorphism.

As a consequence of this construction, one has the following :

1. **Application to the GWP of (BO) :** The first global well-posedness result of the Benjamin–Ono equation traces back to the work of [Sau79b, JI86, ABFS89, Pon91], who established the GWP of (BO) in $H_r^s \equiv H^s(\mathbb{T} \text{ or } \mathbb{R} ; \mathbb{R})$ with $s > \frac{3}{2}$. Subsequently, inspired by the Cole–Hopf transformation [Col51, Hop50], Tao introduced what is now referred to as Tao’s gauge transform, allowing

4. The shift of 1/2 between the input and output spaces is linked to the fact that the mapping Φ is not only a homeomorphism but also a symplectomorphism. [GK21]

him to establish the global well-posedness of (BO) in $H^1(\mathbb{R})$ [Tao04]. Then, Molinet, using Tao's gauge transform, extended the flow from high regularity spaces up to L_r^2 [Mol08]. Later, in a series of recent papers, [GKT23, GT23] proved via the nonlinear Fourier transform, the global well-posedness of (BO) up to

$$H_r^{-\frac{1}{2}, \sqrt{\log}}(\mathbb{T}) := \left\{ u \in H_r^{-\frac{1}{2}} ; \sum_{n \in \mathbb{Z}} \langle n \rangle^{-1} \log(\langle n \rangle + 1) |\widehat{u}(n)|^2 < \infty \right\} \subsetneq H_r^{-\frac{1}{2}}(\mathbb{T}), \quad (1.1.8)$$

which is a subspace of $H_r^{-\frac{1}{2}}(\mathbb{T})$ containing all the $H_r^\sigma(\mathbb{T})$ spaces for $\sigma > -\frac{1}{2}$. In addition, they showed the ill-posedness of (BO) in $H_r^{-\frac{1}{2}}(\mathbb{T})$ by using complex analysis arguments. For the real line⁵, the sharp well-posedness in $H^s(\mathbb{R})$, $s > -\frac{1}{2}$ has been proved in [KLV23b] by using the method of commuting flows introduced in [KV19] and developed in subsequent papers [BKV21, HGKNV22a, HGKV20, KNV21, Lau21, Lau23, Nte22].

2. **Study of the third equation of the Hierarchy of (BO)** : Using the nonlinear Fourier transform (1.1.7), [Gas21] established the global well-posedness⁶ in $H_{r,0}^s(\mathbb{T})$, $s \geq 0$, of the third equation in the hierarchy of (BO), i.e. when $k = 3$ in

$$\partial_t u = \partial_x (\nabla \mathcal{H}_{k+1}(u)), \quad \mathcal{H}_{k+1}(u) := \langle L_u^{k+1} 1 | 1 \rangle,$$

where L_u is the Lax operator of (BO) defined in (1.1.2). Moreover, she proved the ill-posedness⁷ of the same equation in $H_{r,0}^s(\mathbb{T})$ for $-\frac{1}{2} < s < 0$.

3. **Qualitative properties of the flow** : In [GK21, GKT22, GKT23], the authors proved that the flow $t \in \mathbb{R} \mapsto u(t) \in H_{r,0}^s(\mathbb{T})$, $s > -\frac{1}{2}$, of (BO) is almost periodic in the sense of Bochner's characterization, that is the set $\{u_\tau : t \mapsto u(t + \tau) ; \tau \in \mathbb{R}\}$ is relatively compact in $\mathcal{C}_b(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$. As a consequence, they infer that the flow satisfies the Poincaré recurrence : $\exists(t_n)$ such that $u(t_n) \rightarrow u_0$ in $H^s(\mathbb{T})$ as $t_n \rightarrow \infty$. Furthermore, in [GKT22], they construct on \mathbb{T} large families of periodic or quasi-periodic solutions, which are not \mathcal{C}^∞ smooth.

On the other hand, on the real line $x \in \mathbb{R}$, [Sun21] constructed the nonlinear Fourier transform on the multisolitons manifold, representing a first step toward solving the soliton resolution conjecture.

5. Actually, the method adopted enables the well-posedness on the torus $x \in \mathbb{T}$ as on the real line $x \in \mathbb{R}$.

6. in the sense of extension of the flow defined initially for finite gap potentials, i.e., the initial data such that for all $n \geq N$, $\zeta_n(u_0) = 0$.

7. in the sense that the flow defined on the finite gap potentials fails to be extended continuously to $H_{r,0}^s(\mathbb{T})$, $-\frac{1}{2} < s < 0$ for all $t > 0$.

1.1.2.2 Application to the Szegő equation

We recall the Szegő equation

$$i\partial_t u = \Pi(|u|^2 u), \quad (\text{Sz})$$

where Π is the Szegő projector defined in (1.1.3). In Section 1.1.1, we saw that the (Sz)–equation enjoys a Lax pair structure with a Lax operator $H_u(\cdot) := \Pi(\bar{u} \cdot)$ both on \mathbb{R} and an additional one on \mathbb{T} with a second Lax operator $K_u := H_u S = \Pi(\bar{u} e^{ix} \cdot)$ for the second Lax pair structure (see (1.1.4) and (1.1.5)). More powerfully, the (Sz)–equation is a *completely integrable equation in the Liouville sense* when $x \in \mathbb{R}$ or \mathbb{T} . On the torus, the complete integrability was initially established on a finite-dimensional manifold [GG12b] in the following sense : For

$$M := \left\{ u = \frac{A(z)}{\prod_{k=1}^N (1 - p_k z)}, A \in \mathbb{C}_{\leq N-1}[z], |p_k| < 1 \right\},$$

there exists an open subset M_{gen} of the finite dimension manifold M , whose complement is of Lebesgue measure 0, such that

$$\Phi : u \in M_{\text{gen}} \longmapsto \underbrace{(2\lambda_1^2, \dots, 2\lambda_N^2, 2\mu_1^2, \dots, 2\mu_N^2)}_{\text{action variables}}, \underbrace{(\varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N)}_{\text{angle variables}} \in \Omega_N \times \mathbb{T}^{2N}$$

is a diffeomorphism where

$$H_u(e_j) = \lambda_j e_j, \quad K_u(f_j) = \mu_j f_j,$$

and

$$\begin{aligned} \varphi_j &:= \arg \langle 1 | e_j \rangle^2, & j = 1, \dots, N, \\ \theta_j &:= \arg \langle u | f_j \rangle^2, & j = 1, \dots, N, \end{aligned}$$

and

$$\Omega_N := \{(\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N) \in \mathbb{R}^{2N} \mid \lambda_1 > \mu_1 > \dots > \lambda_N > \mu_N > 0\}.$$

Later, [GG17] extended Φ to any function u in $H_+^{\frac{1}{2}}(\mathbb{T}) := H^{\frac{1}{2}} \cap L_+^2(\mathbb{T})$, where

$$L_+^2(\mathbb{T}) := \{u \in L^2(\mathbb{T}), \widehat{u}(n) = 0, \forall n \in \mathbb{Z} \setminus \mathbb{N}_{\geq 0}\}. \quad (1.1.9)$$

As a consequence, one deduces [GG17] :

1. The **almost periodicity** of the flow in $H_+^{\frac{1}{2}}(\mathbb{T})$.
2. The **phenomena of norm inflation** : There exists a G_δ set of initial data in $L_+^2 \cap C^\infty(\mathbb{T})$ such that, for any $s > \frac{1}{2}$, the solution of (Sz) starting at u_0 satisfies

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\|u(t)\|_{H^s}}{|t|^M} &\rightarrow \infty, & \forall M \in \mathbb{N}, \\ \liminf_{t \rightarrow \infty} \|u(t)\|_{H^s} &< \infty. \end{aligned}$$

On the real line, the absence of the second Lax structure renders the problem somewhat more delicate. However, [Poc11a] succeed in establishing the complete integrability of (Sz) on only an open subset of the finite-dimensional manifold

$$\tilde{M} := \left\{ u = \frac{A(z)}{B(z)}, A \in \mathbb{C}_{\leq N-1}[z], B \in \mathbb{C}_N[z], \deg(B) = N, \text{ and if } B(z) = 0 \text{ then } z \in \mathbb{C}_- \right\}.$$

To describe her result, we need to introduce the T operator. Let $u \in \tilde{M}$, then there exists g such that $H_u(g) = u$ ⁸. Therefore, we define the operator

$$Tf(x) = xf(x) - (1 - g(x)) \lim_{x \rightarrow \infty} xf(x).$$

And the complete integrability of (Sz) on the subset M_{gen} ⁹ of \tilde{M} , is described as follows : The application

$$\Phi : u \in M_{\text{gen}} \longmapsto (2(\lambda_1 \nu_1)^2, \dots, 2(\lambda_N \nu_N)^2, 4\pi\lambda_1^2, \dots, 4\pi\lambda_N^2, 2\varphi_1, \dots, 2\varphi_N, \gamma_1, \dots, \gamma_N)$$

is a diffeomorphism, where $H_u(e_j) = \lambda_j e_j$ for all $j = 1, \dots, N$,

$$\nu_j = |\langle g | e_j \rangle|, \quad \varphi_j = \arg \langle g | e_j \rangle, \quad \gamma_j = \text{Re} \langle Te_j | e_j \rangle.$$

As a consequence, Gérard–Pushnitski [GP24] established the **existence** of a family of **turbulent solutions** for the (Sz)–equation, larger than the one found by Pocovnicu [Poc11a] on the line. Precisely, they demonstrated the existence of a dense subset of rational initial data solutions, with Sobolev norms increasing to infinity as $t \rightarrow \infty$. This contrasts with the earlier result established on the circle [GG15], where solutions with rational initial data were shown to be quasi-periodic. The latter result directly stems from the existence of the *explicit formula* for the Szegő equation, which is discussed in the following section.

1.1.3 Explicit formula

An evident, yet not always easy to discover, route to establishing that a PDE is completely integrable is by demonstrating the capacity to integrate or solve the equation explicitly. This was achieved in the instances of the Benjamin–Ono [Gér23a, KLV23b] and Szegő equation [GG15, GP23a], yielding remarkable consequences for these equations, whether in the context of global well-posedness or pertaining to the qualitative properties of the flow. Moreover, this explicit solvability has served as the key to uncovering more facets behind the problem of the zero-dispersion limit (i.e., semiclassical limit) of the concerned PDE.

8. Kronecker-type theorem [Poc11b, Remark 2.2]

9. It consists of the functions u such that the eigenvalues of H_u^2 are simple $0 < \lambda_1^2 < \dots < \lambda_N^2$ and $\langle u | e_j \rangle \neq 0$ where $H_u(e_j) = \lambda_j e_j$ for all $j = 1, \dots, N$.

1.1.3.1 Application to the GWP

The complete integrability of the Szegő equation manifests, on \mathbb{R} and on \mathbb{T} , through the derivation of an explicit formula for the solution $u(t)$ of the Cauchy problem starting from an initial data u_0 [GG15, Poc11a, GP23a, GP23b]. These formulas depend specifically on the Lax operators H_{u_0} and K_{u_0} defined in (1.1.4) and in (1.1.5) at $t = 0$. On \mathbb{T} , if we denote by S the shift operator defined as $Sh(x) = e^{ix} h(x)$, then for any $u_0 \in L_+^2$, where L_+^2 is the Hardy space¹⁰ defined at (1.1.9), and S^* is the adjoint of S in $L_+^2(\mathbb{T})$, then the explicit formula of (Sz)

$$u(t, z) = \left\langle (\text{Id} - z e^{-itH_{u_0}^2} e^{itK_{u_0}^2} S^*)^{-1} e^{-itH_{u_0}^2} u_0 \mid 1 \right\rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}, \quad z \in \mathbb{D};$$

while on \mathbb{R} , we have for any $z \in \mathbb{C}_+ = \{z \in \mathbb{C}; \text{Im}(z) > 0\}$,

$$u(t, z) = \frac{1}{2\pi i} I_+ \left[\left(X^* + \frac{1}{2\pi} \int_0^t \left\langle \cdot \mid e^{-isH_{u_0}^2} u_0 \right\rangle e^{-isH_{u_0}^2} u_0 ds - z \text{Id} \right)^{-1} e^{-itH_{u_0}^2} u_0 \right],$$

where $I_+(f) = \widehat{f}(0^+)$ and $X^* f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+)$.

As a *consequence* of these explicit formulas, [GP23a, GP23b] **extended the flow continuously** of (Sz) from high regularity spaces up to $L_+^2(\mathbb{T})$ and $L_+^2(\mathbb{R})$. Moreover, these formulas yielded the following outcomes :

1. On the torus, [GG15] proved that the trajectories starting from rational initial data are **quasiperiodic**, i.e., there exists a smooth function F defined on \mathbb{T}^n and $\exists \omega_1, \dots, \omega_n \in \mathbb{R}$, such that the solution $u(t) = F(\omega_1 t, \dots, \omega_n t)$, when u_0 is a rational function. In particular, the trajectories are bounded in the H^s -norms.
2. On the real line, [Poc11a] established that the soliton resolution for generic rational functions⁹. Typically, such functions can be expressed as a superposition of solitons, with a remainder going to 0 in the H^s -norms when $t \rightarrow \pm\infty$.

1.1.3.2 Application to the zero dispersion limit

In [Gér23a], Gérard showed that the solution of the Benjamin–Ono equation (BO) can be expressed explicitly through an explicit formula. Thus, on \mathbb{T} , the solution $u(t)$ starting at $t = 0$ from $u_0 \in H_r^{-\frac{1}{2}, \sqrt{\log}}(\mathbb{T})$, where $H_r^{-\frac{1}{2}, \sqrt{\log}}(\mathbb{T})$ is defined in (1.1.8), and is written as follows

$$u(t) = \Pi u(t) + \overline{\Pi u(t)} - \langle u_0 \mid 1 \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})},$$

where Π is the Szegő projector (1.1.3), and

$$\Pi u(t, z) = \left\langle (\text{Id} - z e^{it(\text{Id} + 2L_{u_0})} S^*)^{-1} \Pi u_0 \mid 1 \right\rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}, \quad z \in \mathbb{D}; \quad (1.1.10)$$

10. We recall that any function in the Hardy space L_+^2 defined in (1.1.9) can be seen as a holomorphic function on \mathbb{D} when considering $L_+^2(\mathbb{T})$, or on \mathbb{C}_+ when considering $L_+^2(\mathbb{R})$ whose trace on $\partial\mathbb{D}$ respectively on \mathbb{R} is in L^2 .

with L_{u_0} is the Lax operator (1.1.2) at $t = 0$, S is the shift operator and S^* is its adjoint in $L^2_+(\mathbb{T})$. In addition, he proved that on the real line, whenever $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have

$$u(t) = \Pi u(t) + \overline{\Pi u(t)},$$

where, for any $z \in \mathbb{C}_+ = \{z \in \mathbb{C} ; \text{Im}(z) > 0\}$,

$$\Pi u(t, z) = \frac{1}{2\pi i} I_+ \left[(X^* - 2tL_{u_0} - z \text{Id})^{-1} \Pi u_0 \right],$$

with $I_+(f) = \widehat{f}(0^+)$ and $X^* f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+)$. Recently, Chen removed the requirement of $L^\infty(\mathbb{R})$ for u_0 , thereby extending the explicit formula to any $u_0 \in L^2_r(\mathbb{R})$ [Che24]. Besides, Killip–Laurens–Visan [KLV23b] provided a generalization of the explicit formula on \mathbb{R} to all the hierarchy of (BO).

As a consequence, by using these explicit formulas, some progress has been made towards understanding the **zero dispersion limit of the Benjamin–Ono equation**. Specifically, considering the Cauchy problem

$$\begin{cases} \partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon |D| \partial_x u^\varepsilon \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad (\text{BO-eps})$$

the aim is to comprehend the behavior of the solution u^ε as the dispersion tends to be small, i.e., as $\varepsilon \rightarrow 0$. This phenomenon of neglecting the dispersion component in the equation is commonly recognized in the literature as the “zero-dispersion limit” or “semiclassical limit”. It’s noteworthy that for $\varepsilon = 0$, the (BO-eps) reduces to the inviscid Burgers equation, renowned for its propensity to exhibit shock waves. In [MX11, MW16], Miller and his collaborators investigate the weak zero dispersion limit in $L^2(\mathbb{R})$, showcasing that for certain specific initial data, the solution u^ε converges weakly to an alternate sum expressed in terms of the branches of the multivalued solution of the Burgers equation. On the torus $x \in \mathbb{T}$, using the nonlinear Fourier transform, Gassot [Gas23a] demonstrated that given a single well initial data¹¹ u_0 , there exists a family u_0^ε approximating u_0 in L^2 such that the solution to (BO-eps) with initial data u_0^ε weakly converges to the same alternate sum found in [MX11]. Then, using the explicit formula, she extends the result [Gas23b] by proving that for any bell shaped initial data u_0 , the solution u^ε of (BO-eps) starting from u_0 converges weakly to the alternate sum. More recently, [Gér23b] generalized the result of [MX11] on \mathbb{R} to any initial data $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by underlining that for such general initial data, the zero dispersion limit solution of (BO-eps) is given by the alternate sum of the branches of the multivalued solution of the Burgers equation. Moreover, he highlights that this sum does not constitute an evolution semi-group. Finally, he deduces that the zero–dispersion limit solution $\text{ZD}[u_0](t)$ of (BO-eps) satisfies the following maximum principle $\|\text{ZD}[u_0](t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$, for all $t \in \mathbb{R}$. Lately, [Che24] substitute the condition of $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of [Gér23b] by $u_0 \in L^2_r(\mathbb{R}) \cap L^\infty_{\text{loc}}(\mathbb{R})$ with $\lim_{x \rightarrow \infty} |u_0(x)|/|x| = 0$, and finds an integral representation of $\text{ZD}[u_0](t)$.

11. See [Gas23b, Definition 1.1] for the definition of bell shaped initial data.

1.1.4 Traveling waves and Solitons

One of the prime motivators and historical milestones in the study of integrable PDEs is the discovery of solitons. The story begins in 1834 with the Scottish civil engineer and naval architect Russell’s observation of a solitary water wave at the Union Canal, setting the stage for the modern exploration of nonlinear integrable phenomena. By 1844, Russell reported his observation to the British Association for the Advancement of Science [Rus45]. Some years later, J. Boussinesq developed a non-linear PDE [Bou72] describing waves in shallow water and admitting solitonic solutions. After that, there was the rediscovery of the KdV equation by Korteweg and de Vries (KdV) in 1895 [KV95], which describes long waves in shallow water and is now one of the famous and well-known integrable PDE. Numerically, the first observation of solitons for the (KdV) equation was reported in a spatially periodic domain by Norman Zabusky and Martin Kruskal in 1965 [ZK65].

Nowadays, it is known that many nonlinear PDEs exhibit these types of waves [Che04, CB13, Pav09]. Thus, how can we describe a soliton? And what aspects make them so intriguing for researchers in various branches of applied and theoretical sciences? Their importance resides as they are **explicit solutions** for nonlinear PDEs, providing thereby, information regarding the dynamics of the equation. In addition, they are special **traveling waves**. These latter are represented in their simplest form as

$$u(t, x) = U(x - ct), \quad (1.1.11)$$

where the profile U is a function of one variable and c is a non-zero real constant. When $c > 0$, (respectively $c < 0$) the wave propagates in the x -direction to the right (resp. to the left). A natural example of a linear equation enjoying such type of solution is the *wave* equation, also known as the d’Alembert’s equation¹²:

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \quad (1.1.12)$$

from which the traveling *wave* derives its name. One of the most spectacular properties that can be noticed regarding (1.1.11), is that this particular solution travels over time at the same speed $|c|$, carrying energy from one point to another, and remains in the same shape defined by the profile U . Hence, we cannot expect to find a linear dispersive PDE to admit a traveling wave solution since the dispersive part of the equation tends to “disperse” the *plane waves*¹³ constituting the solution, and which shall propagate at different velocities depending on their frequencies. When we transit to the nonlinear setup, the situation undergoes a change. In fact, the nonlinear part of the equation can, in some cases, prevent the dispersion

12. in tribute to Jean le Rond d’Alembert, who derived the equation in 1747 as a solution to the problem of a vibrating string.

13. A plane wave is of the form $Ce^{i(\xi \cdot x - \tau t)}$, where $C > 0$ is the amplitude, $\xi \in \mathbb{R}$, $\tau \in \mathbb{R}$ in the non-periodic setting ($x \in \mathbb{R}$); or $Ce^{i(kx - \omega(k)t)}$, $k \in \mathbb{Z}$, in the periodic setting ($x \in \mathbb{T}$). They are mainly used in the theory of linear PDEs to express, under some assumptions, a solution as a superposition of these simple plane waves (think of the Fourier inversion formula).

effects. Therefore, nonlinear dispersive or dissipative PDEs can exhibit wave behavior [SCM73, Tao09]. This is the case, for instance, for very well-known PDEs as the KdV equation, the NLS equation, Sine–Gordon equation, the intermediate long wave equation, Benjamin–Ono equation, and others... In different regards, traveling waves can be interpreted in the nonlinear context as follows : consider, for example, any nonlinear PDE that admits traveling wave solutions ; this means that the PDE acts on some initial data $u_0(x) = U(x)$ as the wave equation (1.1.12).

In contrast with the traveling waves, we have the *standing waves*. This particular type of solution is expressed in the following form

$$u(t, x) := U(x),$$

where the profile U is a function of one variable. Unlike the traveling wave, the standing wave appears to be stationary and does not travel in any direction. Historically, the earliest observation of standing waves can be traced back to 1831 when M. Faraday noticed them on the surface of a liquid in a vibrating container.¹⁴ An example of situation¹⁵ that could create standing waves is when two waves, propagating in opposite directions and having identical frequencies and amplitudes, collide.

Returning to the traveling waves, one can select three specific and important types among these solutions :

- The *periodic traveling wave* $u(t, x) = u(t, x + L)$, $L > 0$, $x \in \mathbb{R}$, propagates through a medium and maintains its periodic shape over time.
- The *solitary waves*, which are localized traveling waves. To the best of the author’s knowledge, there is no universal mathematical definition of the word localized. Generally, it is referred to consider the asymptotic limits of the profile U as $x \rightarrow \pm\infty$, –in the context of unbounded domain– and see if the limit exists. In the case where the limit

$$\lim_{x \rightarrow -\infty} U(x) = \lim_{x \rightarrow +\infty} U(x),$$

then the solution’s trajectory is called *homoclinic*, otherwise, it is referred to be a *heteroclinic* orbit [MB21, AC91, Pav09]. Note that according to some authors, the definition of localized is restricted to the fact that the asymptotic limits of the profile U vanishes at $\pm\infty$ [BBR⁺04].

- The *solitons* are solitary waves with the additional property : they retain their original shape and speed even after colliding with another soliton¹⁶. Namely, they don’t lose energy and don’t spread out. This remarkable phenomenon is a consequence of the perfect balance, between two different tendencies of waves ; those caused by the dispersion and those resulting from the PDE’s nonlinearity.

14. <https://www.surfertoday.com/surfing/what-is-a-standing-wave>.

15. This is not the only physical situation when standing waves could occur.

16. In mathematics, it is common to use the terms “solitons” and “solitary waves” interchangeably, without always considering whether the soliton remains in its original form even after interacting with another soliton.

Usually, solitons arise in integrable models, where the existence of a consequent number of symmetries is behind this “particle–behavior” [Pav09]. Sometimes, a sufficient number of conservation laws can lead to the existence of these stable waves [BBR⁺04].

In fact, to derive these special solutions, various methods have been employed. For instance, for some PDEs, such as the Korteweg–de Vries equation, traveling waves can be obtained –under some assumptions of decay on the solution and its derivatives– by substituting $u(t, x) = U(x - ct)$ into the PDE and integrating the corresponding ordinary differential equation [DJ89]. In [Ben67], Benjamin proved the existence of a periodic traveling wave for the Benjamin–Ono equation (BO), with a profile expressed in terms of elementary functions [PN08, Sau19]

$$u_c(x) = \frac{4\pi}{L} \frac{2 \sinh \phi}{\cosh \phi - \cos(2\pi x/L)}, \quad (1.1.13)$$

where $\phi > 0$ such that $\tanh \phi = \frac{2\pi}{cL}$, implying that speed $c > 2\pi/L$. Additionally, he found that the profile of the algebraically decaying solitary wave

$$u_{c_0}(x) = \frac{2c_0}{1 + c_0^2 x^2},$$

can be obtained by taking a limit in the parameter of (1.1.13). Remarkably, by using the maximum principle for linear elliptic equations, estimates on a Green function and the Cauchy–Riemann equation, Amick and Toland showed [AT91] that the solitary waves obtained by Benjamin [Ben67] are unique.

Another common way to derive the traveling waves is through variational approach, where, the set of these waves contain the set of the constrained minimizers of the energy functional [BL83, NLP20]. In particular, these variational problems are addressed via the Concentration–Compactness Method developed by Lions [Lio84a, Lio84b], as is the case for certain NLS type equation [GH12, Wu14] or the generalized KP equation [DBS97]. Besides, spectral methods relying on the spectral property of the Lax operator can also be used to obtain traveling wave solutions, like for the Szegő equation, where Pocovnicu [Poc11b] characterizes the solution of the form $u(t, x) = e^{-i\omega t} u_0(x - ct)$ when $x \in \mathbb{R}$, and proves that they are of the form

$$u(t, x) = \frac{C e^{-i\omega t}}{x - ct - p}, \quad C, p \in \mathbb{C}, \operatorname{Im}(p) < 0,$$

with $c = c(C, p)$, $\omega = \omega(C, p) \in \mathbb{R}$. In the periodic settings, i.e. on \mathbb{T} , [GG10] found that traveling waves start from the initial data

$$u_0(x) = \frac{C e^{i\ell x}}{1 - p^N e^{iNx}}, \quad C \in \mathbb{C},$$

where $N \in \mathbb{N}_{\geq 0}$, $\ell \in \{0, \dots, N - 1\}$, and $p \in \mathbb{D}$. Moreover, the stationary waves are interior functions (that is the L^2_+ –functions having a modulus equal to 1 almost everywhere on \mathbb{T}).

An example of a completely integrable wave equation in which we are interested is the Calogero–Sutherland–Moser derivative nonlinear Schrödinger equation.

1.2 The Calogero–Sutherland–Moser DNLS equation

1.2.1 The model

We consider a nonlocal nonlinear Schrödinger equation, called the *Calogero–Sutherland–Moser derivative nonlinear Schrödinger equation*

$$i\partial_t u + \partial_x^2 u \pm 2u \cdot D\Pi(|u|^2) = 0, \quad x \in \mathbb{R}, x \in \mathbb{T}, \quad (\text{CSM})$$

where $D = -i\partial_x$, and Π denotes the **Riesz–Szegő projector** onto functions with nonnegative frequencies, defined on $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ as

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) := \sum_{n \geq 0} \widehat{u}(n) e^{inx}, \quad (1.2.1)$$

and on \mathbb{R} as

$$\widehat{\Pi}u(\xi) = \mathbf{1}_{[0, +\infty)}(\xi) \widehat{u}(\xi).$$

The operator Π is an orthogonal projector from L^2 onto the **Hardy space** denoted by

$$\begin{aligned} L_+^2(\mathbb{T}) &:= \{u \in L^2(\mathbb{T}) \mid \widehat{u}(n) = 0, \forall n \in \mathbb{Z}_{\leq -1}\} \\ &\cong \{u \in \text{Hol}(\mathbb{D}_+), \sup_{0 < r < 1} \int_0^{2\pi} |u(re^{i\theta})|^2 d\theta < +\infty\}, \end{aligned} \quad (1.2.2)$$

or

$$\begin{aligned} L_+^2(\mathbb{R}) &:= \{u \in L^2(\mathbb{R}), \text{supp } \widehat{u} \subseteq [0, +\infty[\} \\ &\cong \{u \in \text{Hol}(\mathbb{C}_+), \sup_{y > 0} \int_{\mathbb{R}} |u(x + iy)|^2 dx < +\infty\}, \end{aligned} \quad (1.2.3)$$

where $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$ and $\mathbb{C}_+ := \{z \in \mathbb{C}; \text{Im}(z) > 0\}$.

This equation comes in two flavors : the **focusing type**, distinguished by the “+” **sign** preceding the nonlinearity, aims to concentrate localized waves over time ; the second variant, characterized by a “−” **sign** in front of the nonlinearity and termed by the **defocusing type**, tends to spread out the interaction between waves. It is not surprising that the focusing equation leads to more complex and richer dynamics ; and as we shall see, the Calogero–Sutherland–Moser DNLS equation (CSM) is no exception to this rule. In the following, the \pm and \mp symbols will be interchanged based on the following rule : the upper sign will correspond to the focusing case and the lower sign to the defocusing case.

1.2.1.1 Physical Background of (CSM).

In physics, the focusing (CSM) equation arises as a continuum limit of the Calogero–Sutherland–Moser system introduced in the late 1960s–early 1970s [Cal69, Cal71, Sut71, Sut72, CM74, Mos76]. The latter model corresponds to a N -body problem, describing the pairwise interactions of N identical particles [Hal23]. On the real line, this interaction is given by an inverse square potential

$$V = \sum_{j < k} \frac{1}{(x_j - x_k)^2},$$

while on \mathbb{T} , it is an inverse sin square potential. Abanov–Bettelheim–Wiegmann show in [ABW09] that taking the thermodynamic limit of the discrete model, and applying a change of variables lead to the focusing (CSM)-equation. On the other hand, the defocusing (CSM) has been obtained as a formal limit ($\delta \rightarrow \infty$) of the *intermediate nonlinear Schrödinger equation* introduced by Pelinovsky [Pel95],

$$i\partial_t u = \partial_x^2 u + (i - T)\partial_x(|u|^2)u, \quad (\text{INS})$$

where T is the integral operator

$$Tu(x) = \frac{1}{2\delta} \text{p.v.} \int_{-\infty}^{+\infty} \coth\left(\frac{\pi(x-y)}{2\delta}\right) u(y) dy.$$

The complex function u in (INS) represents the envelope of the fluid, and δ denotes its total depth. By passing to the limit $\delta \rightarrow \infty$, one obtains the same equation as (INS) but with the Hilbert transform $\mathcal{H} = -i \operatorname{sgn}(D)$, $D = -i\partial_x$, instead of T . Hence, the (CSM)-equation can also be interpreted as a model describing the envelope of the interfacial monochromatic wave packet in a deep stratified fluid. Please note that from this perspective, (CSM) shares similarities with the Benjamin–Ono equation (BO), which describes the propagation of interfacial waves in similar settings. However, (CSM) is more focused on studying the envelope.

In physics, the Calogero–Sutherland–Moser DNLS equation (CSM) has garnered significant attention from physicists and engineers. In particular, we cite the works of Tutiya [Tut09], Berntson–Fagerlund [BF23], Stone–Anduaga–Xing [SAX08], Polychronakos [Pol95b, Pol95a] and Matsuno [Mat00, Mat01b, Mat01a, Mat02b, Mat02a, Mat03, Mat04, Mat23]...

Given the physical context with Calogero and Moser’s primary investigation of the discrete model on the real line and Sutherland’s work on the torus, we will refer to (CSM) as the *Calogero–Moser DNLS equation* when $x \in \mathbb{R}$, and as the *Calogero–Sutherland DNLS equation* when (CSM) is examined on the torus $x \in \mathbb{T}$. Another designation for this equation, appearing in the recent works [KLV23a, HK24] as "Continuum Calogero–Moser models" in the case when $x \in \mathbb{R}$.

1.2.1.2 LWP of (CSM) on \mathbb{R} and \mathbb{T} .

To the best of the author's knowledge, the first local well-posedness result for (CSM) equation traced back to De Moura [dM07] who addressed the local well-posedness in $H^s(\mathbb{R})$ with $s \geq 1$, for small initial data, of a family of nonlocal nonlinear Schrödinger equation [PG96], including the Calogero–Moser DNLS equation (CSM). His idea is based on using a fixed-point argument. In addition, he extends his LWP's result to a global well-posedness result, by means of the gauge transformation. Subsequently, Barros–DeMoura–Santos present in [BdMS19] the LWP of (CSM) for small initial data in the Besov space $B_2^{\frac{1}{2},1}(\mathbb{R})$. In the context of the Hardy Sobolev spaces

$$H_+^s := H^s \cap L_+^2, \quad s \geq 0, \quad (1.2.4)$$

where H^s denotes the Sobolev space and L_+^2 is the Hardy space introduced in (1.2.2) or (1.2.3), equipped with the Sobolev norm

$$\|u\|_{H^s} = \|\langle D \rangle^s u\|_{L^2}, \quad \langle D \rangle^s = (1 + |D|^2)^{s/2},$$

the local well-posedness of (CSM) has been established in H_+^s on \mathbb{R} and \mathbb{T} for $s > \frac{3}{2}$, by using iterative schemes à la Kato and energy estimates [GL24, Proposition 2.1]. A more detailed proof of their proposition will be presented in Chapter 3. In the same paper, they extend the LWP of the Calogero–Moser derivative NLS equation to $H_+^s(\mathbb{R})$ with $s > \frac{1}{2}$ by following arguments from [dMP10].

In what follows, we are interested in the (CSM)–equation where u is a function of complex value, belonging to the Hardy Sobolev space H_+^s . Moreover, we denote by

$$\mathcal{S}^\pm(t) : u_0 \mapsto u(t)$$

the flow of (CSM).

1.2.1.3 Integrability of (CSM) : Existence of a Lax pair .

An important feature of the Calogero–Sutherland–Moser DNLS equation (CSM) is its *integrability*. This is not surprising, as it emerges as a continuum limit of the discrete Calogero–Sutherland–Moser model, or of the intermediate nonlinear Schrödinger equation, both of which are known to be integrable [Mos75, Pel95, PG95]. In fact, in [GL24, Bad24a] the authors derived on \mathbb{R} and on \mathbb{T} a Lax pair structure for the continuum limit (CSM), with a Lax operator L_u reminiscent of that of the Benjamin–Ono equation (1.1.2). Thus, if we denote by T_v the Toeplitz operator of symbol $v \in L^\infty$, defined as

$$T_v f = \Pi(vf), \quad \forall f \in L_+^2, \quad (1.2.5)$$

where Π is the Szegő projector (1.2.1), then the Lax operators of (CSM) are given by the following theorem.

Theorem 1.2.1 (The Lax pair structure of (CSM) [GL24, Bad24a]). *For any $s > \frac{3}{2}$, let $u \in \mathcal{C}([-T, T], H_+^s)$ be a solution of the (CSM)-equation. Then, there exist two operators*

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2, \quad (1.2.6)$$

in the focusing case, and

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2. \quad (1.2.7)$$

in the defocusing case, satisfying the Lax equation (1.1.1), where T_u is the Toeplitz operator defined in (1.2.5).

Regardless of whether x belongs to \mathbb{R} or \mathbb{T} , the Lax operators L_u and \tilde{L}_u are unbounded self-adjoint operators of domain H_+^1 [GL24, Bad24a, Proposition 2.1], whereas B_u and \tilde{B}_u are bounded skew-adjoint operators. However, a notable difference between the cases of x being in \mathbb{R} and \mathbb{T} , lies in the nature of the spectrum¹⁷ of L_u and \tilde{L}_u . On \mathbb{T} , the semi-bounded self-adjoint operators L_u and \tilde{L}_u have compact resolvents [Bad24a, Proposition 2.3], leading to spectra consisting of sequences of eigenvalues bounded from below and going to $+\infty$,

$$\begin{aligned} \sigma(L_u) &:= \{\nu_0(u) \leq \dots \leq \nu_n(u) \leq \dots\}, & \nu_0(u) &\geq -\|u\|_{L^\infty}^2, \\ \sigma(\tilde{L}_u) &:= \{\lambda_0(u) \leq \dots \leq \lambda_n(u) \leq \dots\}, & \lambda_0(u) &\geq 0. \end{aligned} \quad (1.2.8)$$

In contrast, on \mathbb{R} the spectrum of L_u is more complicated [GL24, KLV23a], comprising an essential spectrum $\sigma_{\text{ess}}(L_u) = [0, +\infty)$ and a finite number N of simple eigenvalues, controlled via the sharp bound [GL24]

$$N \leq \frac{\|u\|_{L^2}^2}{2\pi}.$$

As we will see in Section 1.2.2.1, this Lax pair structure will serve as a foundation for constructing conservation laws for (CSM). These laws will enable us to manage the growth of Sobolev norms, thereby establishing the global well-posedness of (CSM) in the H_+^s -spaces.

1.2.2 Explicit Formula for (CSM)

In addition to the presence of the Lax operators for the Calogero–Sutherland–Moser DNLS equation, the complete integrability of (CSM) is further demonstrated by a more powerful tool : the existence of an explicit formula of the solution $u(t)$ of (CSM) starting from an initial data $u_0 \in H_+^\infty$.

Let's begin with the case on \mathbb{T} . For that, we recall one of the most important operators in Hardy's space, the shift operator, defined on $L_+^2(\mathbb{T})$ as the isometric map

$$S: h \in L_+^2(\mathbb{T}) \mapsto e^{ix} h \in L_+^2(\mathbb{T}), \quad (1.2.9)$$

17. We recall that we are particularly interested in the spectrum of the Lax operators as it remains invariant under the flow. We refer to Section 1.1.1 for this property.

and its adjoint in $L_+^2(\mathbb{T})$

$$S^* : h \in L_+^2(\mathbb{T}) \longmapsto S^*h = T_{e^{-ix}}h = \Pi(e^{-ix}h) \in L_+^2(\mathbb{T}) .$$

By means of the Lax operators, one can reformulate the focusing and defocusing (CSM) equations as follows [Bad24a]

$$\partial_t u = B_u u - iL_u^2 u, \quad \text{and} \quad \partial_t u = \tilde{B}_u u - i\tilde{L}_u^2 u .$$

And using the commutator identities

$$\begin{aligned} [S^*, L_u] &= S^* - \langle \cdot | u \rangle S^* u, \\ [S^*, B_u] &= i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right), \end{aligned}$$

–and their analogues in the defocusing case–, and recalling that any function in the Hardy space $L_+^2(\mathbb{T})$ can be viewed as a *holomorphic function* on \mathbb{D}_+ , whose trace on the torus is in $L^2(\mathbb{T})$, one infers the following theorem.

Theorem 1.2.2 ([Bad24a]). *Given $u_0 \in H_+^\infty(\mathbb{T})$, the solution of the focusing Calogero–Sutherland DNLS equation (CSM) is given by*

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (1.2.10)$$

Moreover, the solution to the defocusing equation is obtained by replacing L_{u_0} with \tilde{L}_{u_0} in (1.2.10).

On the **real line** $x \in \mathbb{R}$, the shift operator (1.2.9) must be substituted by the contraction semigroup

$$S(\eta)h(x) = \Pi(e^{ix\eta} h(x)), \quad \eta > 0.$$

And if we denote by X its infinitesimal generator

$$Xh(x) = -i \frac{d}{d\eta} \Big|_{\eta=0} (S(\eta)h(x)) = xh(x),$$

of domain

$$\begin{aligned} \text{Dom}(X) &= \{h \in L_+^2(\mathbb{R}); xh \in L^2(\mathbb{R})\} \\ &= \{h \in L_+^2(\mathbb{R}); \hat{h} \in H^1([0, +\infty)), \hat{h}(0) = 0\}, \end{aligned}$$

and X^* its adjoint of domain

$$\begin{aligned} \text{Dom}(X^*) &= \{f \in L_+^2(\mathbb{R}); \exists c > 0, \forall h \in \text{Dom}(X), |\langle f | Xh \rangle| \leq c \|h\|_{L^2}\} \\ &= \{f \in L_+^2(\mathbb{R}); \hat{f}|_{(0, +\infty)} \in H^1((0, +\infty))\}, \end{aligned}$$

defined for all $\xi > 0$ as

$$\widehat{X^* f}(\xi) = i\partial_\xi \widehat{f}(\xi),$$

or, for all $f \in \text{Dom}(X^*)$, as

$$X^* f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+), \quad (1.2.11)$$

we have the following explicit formula for (CSM) when $x \in \mathbb{R}$.

Theorem 1.2.3 ([KLV23a]). *Given $u_0 \in L_+^2(\mathbb{R})$, the solution of the focusing Calogero–Moser DNLS equation is given, for all $z \in \mathbb{C}_+$ by*

$$u(t, z) = \frac{1}{2\pi i} I_+((X^* + 2tL_{u_0} - z)^{-1}u_0), \quad (1.2.12)$$

where I_+ denotes

$$I_+(f) := \widehat{f}(0^+), \quad \forall f \in \text{Dom}(X^*).$$

In addition, the solution of the defocusing equation is obtained by replacing L_{u_0} with \tilde{L}_{u_0} in (1.2.12).

Remark 1.2.1. By defining the Lax operator L_{u_0} for $u_0 \in L_+^2$, which can be viewed as either an infinitesimal form-bounded perturbation of $L_0 \equiv D$ [GL24, Appendix A], or interpreted as a relatively compact perturbation of $L_0 \equiv D$ [KLV23a]; and by using approximation methods, the explicit formulas (1.2.10) and (1.2.12) of (CSM) remain valid for any $u_0 \in L_+^2$.

Now, armed with the Lax pair structure and the two explicit formulas (1.2.10) and (1.2.12) of (CSM), we present, in what follows, two applications :

1. The **Global Well-Posedness** of (CSM) in the Hardy-Sobolev spaces H_+^s for $s > \frac{3}{2}$ [GL24, Bad24a]. And the **extension of the flow** up to the scaling criticality of this equation, namely to L_+^2 [Bad24a, KLV23a].
2. **Zero-dispersion of (CSM)** We delve into the semi-classical limit (i.e., zero dispersion limit) of the (CSM)-equation

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon \partial_x^2 u^\varepsilon \pm 2u^\varepsilon D\Pi(|u^\varepsilon|^2) = 0 \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad \varepsilon \rightarrow 0. \quad (\text{CSM-eps})$$

We prove, for $x \in \mathbb{R}$ and $x \in \mathbb{T}$, the existence of a weak limit for (CSM-eps) as $\varepsilon \rightarrow 0$. In addition, we characterize, when $x \in \mathbb{R}$, the weak limit in $L_+^2(\mathbb{R})$ in terms of the branches of the multivalued solution of the Burgers equation [Bad24b].

1.2.2.1 GWP on \mathbb{T} and on \mathbb{R}

As announced earlier, exploiting the incredible structure of the Lax operators, one constructs, for any $u \in \mathcal{C}_t H_+^{\max\{s,r\}}$, $s > \frac{3}{2}$, an infinite hierarchy of conservation laws for (CSM),

$$\left(\langle (L_u + \lambda)^r u \mid u \rangle \right)_{r \geq 0} \quad \text{and} \quad \left(\langle (\tilde{L}_u + \lambda)^r u \mid u \rangle \right)_{r \geq 0}, \quad (1.2.13)$$

where $\lambda \gg 0$ ¹⁸. These conservation laws effectively control the growth of all the Sobolev norms. As a result, we obtain the following.

Theorem 1.2.4 (GWP on \mathbb{R} [GL24]). *For any integer $n \in \mathbb{N}_{\geq 1}$, the Calogero–Moser DNLS equation (CSM) is globally well-posed*

- in $H_+^n(\mathbb{R})$ for the defocusing equation.
- in $H_+^n(\mathbb{R}) \cap \mathcal{B}_{L_+^2}(\sqrt{2\pi})$ for the focusing equation, where $\mathcal{B}_{L_+^2}(\sqrt{2\pi})$ denotes the open ball of radius $\sqrt{2\pi}$ in $L_+^2(\mathbb{R})$.

Moreover, the following a priori bound holds,

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^n(\mathbb{R})} < +\infty.$$

Theorem 1.2.5 (GWP on \mathbb{T} [Bad24a]). *For any $s > \frac{3}{2}$, the Calogero–Sutherland DNLS equation (CSM) is globally well-posed*

- in $H_+^s(\mathbb{T})$ for the defocusing equation.
- in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ for the focusing equation (with sign +), where $\mathcal{B}_{L_+^2}(1)$ denotes the open unit ball in $L_+^2(\mathbb{T})$.

Moreover, the following a priori bound holds,

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s(\mathbb{T})} < +\infty.$$

Remark 1.2.2.

- (1) The emergence of the subcritical L^2 -mass observed in the focusing case arises from controlling the growth of the Sobolev norms $\|u\|_{H^s}$ by the conservation laws when using a sharp inequality saturated by stationary waves for (CSM). By changing the sign in front of the nonlinearity of (CSM), the Lax operator for the defocusing equation undergoes a change of sign and appears to be a non-negative operator, allowing the conservation laws to control the $\|u(t)\|_{H^s}$ without requiring any condition on the initial data in this case.
- (2) The distinction between the condition $\|u_0\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi}$ on \mathbb{R} , and $\|u_0\|_{L^2(\mathbb{T})} \leq 1$ on \mathbb{T} originates from the variance in normalization selected by the authors. Indeed,

$$\langle u \mid v \rangle_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} = \int_{\mathbb{R}} u \bar{v} \, dx, \quad \langle u \mid v \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} = \int_0^{2\pi} u \bar{v} \frac{dx}{2\pi}.$$

18. λ is uniform with respect to t as the spectrum of L_u and \tilde{L}_u is conserved by the flow.

On \mathbb{R} , the investigation has been more thorough. From [GL24, Section 3], we have the nonexistence of minimal mass blowup in $H_+^1(\mathbb{R})$. Additionally, by the work of [HK24] we know that the conservation laws (1.2.13) of (CSM) fail to control the growth of the Sobolev norms of $u(t)$ for solutions with mass above 2π . More precisely, Hogan and Kowalski [HK24] proved the following.

Theorem 1.2.6 (Turbulent behavior [HK24]). *For any sufficiently small $\varepsilon > 0$, there exist initial data $u_0 \in H_+^\infty(\mathbb{R})$ with $\|u_0\|_{L^2}^2 = 2\pi + \varepsilon$, a time $T \in (0, \infty]$, and a maximal lifespan solution $u(t)$ to the Calogero–Moser DNLS equation (CSM) such that for all $s > 0$,*

$$\lim_{t \nearrow T} \|u(t)\|_{H^s} = +\infty.$$

In particular, if $T = \infty$, then we have the bounds

$$\|u(t)\|_{H^s} \gtrsim t^s.$$

This result aligns with the dynamics of the **multisolitons** of (CSM), where [GL24] established that, for $N \geq 2$, the N -soliton solutions, which are solutions with $2\pi N$ mass, demonstrate a growth of Sobolev norms characterized by $\|u(t)\|_{H^s} \sim |t|^{2s}$ for large $|t|$. Lately, Kim–Kim–Kwon [KKK24] underlined by using modulation analysis that the time of blow-up solution when $x \in \mathbb{R}$ is **finite**. Moreover, they provided a sharp description of the blow-up phenomena.

Extension of the flow of (CSM). In view of Theorem 1.2.4 and 1.2.5, we infer that the flow

$$\mathcal{S}^\pm(t) : u_0 \in H_+^s \longmapsto u(t) \in H_+^s, \quad s > \frac{3}{2},$$

of (CSM) is globally well-defined on H_+^s for $s > \frac{3}{2}$, (with small initial data in the focusing case). The next step is to extend this flow continuously to the scaling critical regularity $L_+^2(\mathbb{T})$. Indeed, one notices that (CSM) is invariant by the scaling

$$u(t, x) \longmapsto \lambda^{1/2} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Now to extend the flow from H_+^s , $s > \frac{3}{2}$, down to L_+^2 , we use an approximation method as detailed in the next theorem. In addition, note that for $u \in L_+^2$, the equation is still defined in the *distribution sense* since **the product of two functions with nonnegative frequencies is well-defined and continuous**.

Theorem 1.2.7 (Extension of the flow of (CSM) to $L_+^2(\mathbb{T})$ [Bad24a]). *Let $u_0 \in L_+^2(\mathbb{T})$, ($\|u_0\|_{L^2(\mathbb{T})} < 1$ in the focusing case). There exists a unique potential $u \in \mathcal{C}(\mathbb{R}; L_+^2(\mathbb{T}))$ such that for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ where $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, the following convergence holds : for all $T > 0$,*

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (1.2.14)$$

Moreover, the L^2 -norm of the limit potential u is conserved

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

As a consequence, Theorem 1.2.7 leads to the global well-posedness of the (CSM) problem in $L_+^2(\mathbb{T})$ in the following sense : There exists a unique continuous extension of the flow defined on $H_+^2(\mathbb{T})$, to $L_+^2(\mathbb{T})$, generating a unique continuous map

$$u_0 \in \mathcal{B}_{L_+^2}(1) \longmapsto u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T})).$$

in the focusing case, and

$$u_0 \in L_+^2(\mathbb{T}) \longmapsto u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T})),$$

in the defocusing case.

The proof is based on three steps : proving the *existence*, the *uniqueness* and the *strong convergence* in L^2 .

- The existence is the easiest part, and can be achieved by applying the Banach theorem to the sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, satisfying for all $\varepsilon > 0$,

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \lesssim \|u_0\|_{L^2}.$$

- The uniqueness can be handled by using the explicit formula of Theorem 1.2.2, which highlights that the sequence $(u^\varepsilon(t))$ converges to a unique limit $u(t)$. And this latter limit is independent of the specific selection of the sequence (u_0^ε) that approximate u_0 in $L_+^2(\mathbb{T})$.
- Finally, to prove the strong convergence in L^2 , and the conservation of the L^2 -norm of the limit potential u , we rely on the integrable property of the (CSM)-equation. We construct an orthonormal basis (f_n^t) of $L_+^2(\mathbb{T})$ made up of the eigenfunctions of the Lax operator $L_{u(t)}$, where the solution $u(t)$ evolves as

$$|\langle u(t) | f_n^t \rangle| = |\langle u_0 | f_n^0 \rangle|, \quad \forall t \in \mathbb{R}. \quad (1.2.15)$$

□

As a consequence of Theorem 1.2.7, combining functional analysis and integrable techniques, we deduce the following.

Corollary 1.2.8 ([Bad24a]). *For all $s \geq 0$, the Calogero–Sutherland DNLS equation (CSM) is globally well-posed in $H_+^s(\mathbb{T})$ (respectively $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ in the focusing case). Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} < \infty.$$

*Furthermore, the orbit $\{u(t), t \in \mathbb{R}\}$ is **relatively compact** in $H_+^s(\mathbb{T})$ (respectively $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ in the focusing case).*



Moving to the **real line**, one can attempt to employ a similar approach as in the proof of Theorem 1.2.7 for \mathbb{T} , using an approximation method and the explicit formula (1.2.12) for (CSM) on \mathbb{R} , to extend the flow down to $L^2_+(\mathbb{R})$ [KLV23a]. However, it becomes apparent that while the problem of existence and uniqueness of the limit solution u can be handled similarly to the torus case, resulting in a weak convergence in (1.2.14); establishing the strong convergence following the footsteps of the ideas on the torus proves to be impossible. As a reminder, a key aspect of the proof of Theorem 1.2.7 is to construct an orthonormal basis comprising the eigenfunctions of $L_{u(t)}$ such that (1.2.15) is satisfied. However, on \mathbb{R} , this approach fails since the spectrum of L_u on \mathbb{R} , is not solely composed of eigenvalues as on \mathbb{T} (as seen in Section 1.2.1.3). Consequently, Killip–Laurens–Viřan [KLV23a] utilized the equicontinuity property (see Definition 1.2.9) to preclude this loss of mass and ensure strong convergence in $L^2(\mathbb{R})$.

Definition 1.2.9 (L^2 Equicontinuity). Let $s \in \mathbb{R}$. A bounded set $U \subset H^s_+(\mathbb{R})$ is said to be equicontinuous in the $H^s(\mathbb{R})$ topology if

$$\limsup_{\delta \rightarrow 0} \sup_{u \in U} \sup_{|y| < \delta} \|u(\cdot + y) - u(\cdot)\|_{H^s} = 0,$$

or equivalently,

$$\limsup_{\kappa \rightarrow \infty} \sup_{u \in U} \int_{|\xi| \geq \kappa} (1 + |\xi|)^{2s} |\widehat{u}(\xi)|^2 d\xi = 0.$$

Theorem 1.2.10 (Extension of the flow of (CSM) to $H^s_+(\mathbb{R})$, $0 \leq s < 1$ [KLV23a]). Let $u_0 \in L^2_+(\mathbb{R})$ (such that $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case). There exists a unique global solution $u \in \mathcal{C}_t L^2_+(\mathbb{R})$ of the Calogero–Moser DNLS equation, such that for any $(u_n^0) \subseteq H^\infty_+(\mathbb{R})$, $(xu_n^0) \subseteq L^2$, $u_n^0 \rightarrow u_0$ in L^2 , we have for all $T > 0$,

$$u_n \longrightarrow u \quad \text{in } \mathcal{C}_t L^2_+([-T, T], \mathbb{R}).$$

Furthermore, for any $0 < s < 1$, the (CSM)–equation is globally well-defined in $H^s_+(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case) in the sense of extension of the flow.

Open problems and Perspectives.

1. In light of the findings presented by [HK24, KKK24] in the case of the real line (cf. Theorem 1.2.6), a natural question arises : does a similar result exist for the torus? Specifically, to establish their result on the real line, Hogan and Kowalski [HK24] demonstrate an orbital stability statement for solitons of (CSM), which are known on \mathbb{R} to have a mass of 2π [GL24]. However, on \mathbb{T} , [Bad23] established the existence of a broader family of traveling waves¹⁹ than those on the real line, and demonstrated the presence of traveling waves with arbitrarily small L^2 -norm for this L^2 -critical equation, with a mass less

19. We refer to Section 1.2.3 for the characterization of the traveling waves for (CSM).

than the critical mass 1. Consequently, one might inquire whether there exists an analogue of their results for \mathbb{T} , or whether the global well-posedness result holds in the focusing case when $\|u_0\|_{L^2(\mathbb{T})} \geq 1$.

2. Another natural question, given the scaling invariance of the equation, is to demonstrate the ill-posedness of (CSM) in H_+^s with $s \leq 0$.
3. Given the existence of the flow of (CSM) in H_+^s , one could explore the intriguing qualitative properties regarding the dynamics of (CSM). We already know that the orbit of (CSM) is relatively compact in $H_+^s(\mathbb{T})$, $s \geq 0$ [Bad24a]. However, questions regarding almost-periodicity, quasi-periodicity, and others remain unanswered. Furthermore, do the same results hold in \mathbb{R} ? Or the (CSM) exhibit more turbulence when $x \in \mathbb{R}$?
4. The construction of the nonlinear Fourier transform²⁰: We expect the coordinates $(\langle u(t) | f_n^t \rangle)_{n \in \mathbb{N}_{\geq 0}}$ introduced in (1.2.15) to be the Birkhoff coordinates associated with the Calogero–Sutherland DNLS equation (CSM). For this purpose, we need to construct a one-by-one Birkhoff map $u \longleftrightarrow (\langle u(t) | f_n^t \rangle)$. This construction can unlock several significant outcomes regarding the equation’s dynamics.
5. Finally, let us mention that Sun [Sun23], inspired by the (CSM)–equation, introduced the intertwined derivative Schrödinger system of the Calogero–Moser–Sutherland type. This system consists of two equations with a pair of solutions (U, V) of matrix value derived from the (CSM)–equation. For this system, he derived the explicit formula of its solutions. However, much remains to be understood about this system.

1.2.2.2 The zero–dispersion limit

Another application of the explicit formulas (1.2.10) and (1.2.12) of (CSM) is to examine the *zero–dispersion* limit or *semiclassical limit* of the Calogero–Sutherland–Moser DNLS equation (CSM). This involves letting $\varepsilon \rightarrow 0$ in the following rescaled version of (CSM),

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon \partial_x^2 u^\varepsilon \pm 2u^\varepsilon D\Pi(|u^\varepsilon|^2) = 0 \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad (\text{CSM-eps})$$

where the initial data u_0 is independent of ε . Thus, what happens to the solution of (CSM-eps) when we neglect the dispersion term of the equation? Moreover, does the solution u^ε converges, and if so, in which sense? In what follows, we provide a partial answer to this question by establishing the existence of a weak limit of u^ε in L^2 . Additionally, we refer to this limit as the “zero-dispersion limit of (CSM)”, denoted by $ZD_\pm[u_0]$, where $ZD_+[u_0]$ represents the weak zero-dispersion limit solution for the focusing (CSM), and $ZD_-[u_0]$ corresponds to the one in the defocusing case.

²⁰. For the definition of Birkhoff coordinates and nonlinear Fourier transform, we refer to Section 1.1.2.

Theorem 1.2.11 ([Bad24b]). *Given an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), the weak (in L^2 -space) zero-dispersion limit of (CSM) exists, and is characterized via the following explicit formula*

$$ZD_\pm[u_0](t, z) = \left(\text{Id} \mp 2tT_{u_0}T_{\bar{u}_0}(X^* - z)^{-1} \right)^{-1} u_0(z), \quad t \in \mathbb{R}, z \in \mathbb{C}_+, \quad (1.2.16)$$

where the operators T_v and X^* are defined respectively at (1.2.5) and (1.2.11). In addition, we have

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2}.$$

Furthermore, if $u_0^n \rightarrow u_0$ strongly in L^2 as $n \rightarrow \infty$, with $\sup_n \|u_0^n\|_{L^\infty} < +\infty$, then for all $T > 0$,

$$\sup_{t \in [-T, T]} |ZD_\pm[u_0^n](t) - ZD_\pm[u_0](t)| \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}).$$

Remark 1.2.3. An analogous result can be established in the case when $x \in \mathbb{T}$, where one obtains that for $u_0 \in L^2_+(\mathbb{T}) \cap L^\infty(\mathbb{T})$, the zero-dispersion limit of (CSM) is characterized via

$$ZD_\pm[u_0](t, z) = \langle (\text{Id} - z e^{-it} e^{\pm 2itT_{u_0}T_{\bar{u}_0}} S^*)^{-1} u_0 | 1 \rangle, \quad z \in \mathbb{D}. \quad (1.2.17)$$

Usually, when considering the scenario of zero-dispersion limit, the emergence of shocks can be observed. These shocks manifest as we begin to neglect dispersive effects, allowing the nonlinear term to dominate. With the existence of the weak zero-dispersion limit established in (1.2.16), our objective in the following theorem is to highlight these shocks, by addressing the connection between this zero-dispersion limit solution of (CSM) and the branches of the multivalued solution of the inviscid Burgers equation, which is known for its tendency to exhibit shock formations. However, a natural question arises, why is the zero-dispersion limit of (CSM) related to the inviscid Burgers–Hopf equation? For this purpose, observe that when formally taking $\varepsilon \rightarrow 0$, the (CSM-eps) becomes

$$i\partial_t u \pm 2uD\Pi(|u|^2) = 0. \quad (\text{CM-zero})$$

Consequently, if u solves the previous equation, then $\mathbf{v} = |u|^2$ solves the Burgers equation

$$\partial_t \mathbf{v} = \pm 2\mathbf{v} \partial_x \mathbf{v} \quad (1.2.18)$$

as

$$\begin{aligned} \partial_t \mathbf{v} &= 2 \text{Re}(\partial_t u \bar{u}) \\ &= \pm 4 \text{Re}(\partial_x \Pi(|u|^2) |u|^2) \\ &= \pm 2 (\partial_x \Pi(|u|^2) + \overline{\partial_x \Pi(|u|^2)}) |u|^2 \\ &= \pm \partial_x |u|^4 = \pm \partial_x \mathbf{v}^2 \\ &= \pm 2\mathbf{v} \partial_x \mathbf{v}. \end{aligned}$$

Hence, armed with formula (1.2.16), we prove that the weak zero-dispersion limit of (CSM), is written in terms of the branches of the multi-valued solution of the Burgers equation.

Theorem 1.2.12 ([Bad24b]). *Let $u_0 \in L^2_+(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), such that u_0 is a C^1 function tending to 0 at infinity, with a bounded derivative in $L^\infty(\mathbb{R})$.²¹ Then, for every time $t \in \mathbb{R}$, and for almost every $x \in \mathbb{R}$, the equation*

$$y \mp 2t|u_0(y)|^2 = x$$

has an odd number of simple real solutions $y_0 := y_0(t, x) < \dots < y_{2\ell} := y_{2\ell}(t, x)$, and the zero–dispersion limit of (CSM) is given by

$$ZD_\pm[u_0](t, x) = e^{i\varphi(t, x)} \left(\mp i \frac{|t|}{t} \right)^\ell \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k}, \quad (1.2.19)$$

where

$$\varphi(t, x) = \arg(u_0(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log \left(\frac{s \mp 2t|u_0(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k)}{-s \mp 2t|u_0(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k)} \right) ds.$$

As a consequence,

1. One infers that the zero–dispersion limit of (CSM) satisfies, for any u_0 regular enough,

$$|ZD_\pm[u_0](t, x)| = \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k},$$

where the $(y_k)_{k=0, \dots, 2\ell}$ represent the branches of the multivalued solution of the Burgers equation (1.2.18) at a time t beyond the shock time, and at a position x . Moreover, by applying the logarithm to the previous identity, one infers

$$\log |ZD_\pm[u_0](t, x)|^2 = \sum_{k=0}^{2\ell} (-1)^k \log |u_0(y_k)|^2,$$

a result reminiscent of a similar one found for the Benjamin-Ono equation [Gér23b], where the zero-dispersion limit in the latter case, is described as

$$ZD_{(\text{BO})}[u_0](t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k^{BO}), \quad (1.2.20)$$

where, here, the $(y_k^{BO})_{0, \dots, 2\ell}$, are the real solutions for the equation

$$y + 2tu_0(y) = x.$$

2. In addition, it is worth noting that via (1.2.19), one infers that if u_0 is a rational initial data, then the weak zero–dispersion limit $ZD_\pm[u_0]$ of (CSM) in the focusing and defocusing case, is also a rational function.
3. Finally, we infer the existence of the following maximum Principle.

Corollary 1.2.13 (A Maximum Principle [Bad24b]). *Let $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case). For all $t \in \mathbb{R}$,*

$$\|ZD_\pm[u_0]\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

21. Note that any function in $H^s_+(\mathbb{R}) := H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$, $s > \frac{3}{2}$ satisfies these conditions.

Open problems and Perspectives.

1. A pertinent question is whether the weak zero–dispersion limit $ZD_{\pm}[u_0]$ is indeed the strong limit of u^{ε} in L^2_+ as $\varepsilon \rightarrow 0$. Alternatively, does the fact of neglecting the dispersion component of the (CSM)–equation create pronounced oscillations in the limit solution, similar to what is observed in the case of the Benjamin–Ono equation [MX11]? This could potentially prevent $ZD_{\pm}[u_0]$ from being a strong limit in L^2 .
2. Does there exist an analogue to Theorem 1.2.12 for the periodic settings (i.e., when $x \in \mathbb{T}$)?

1.2.3 Traveling waves, solitons and multi–solitons

As highlighted in Section 1.2.3, a fascinating implication of the integrability of a PDE is the emergence of traveling waves, notably solitons. Here, we provide an overview of what has been uncovered thus far regarding these waves for the (CSM) equation. Our aim is to investigate the existence and potential discovery of explicit formulas for these “exact solutions”, which are simpler than those presented in equations (1.2.10) and (1.2.12), and can be expressed using elementary functions. Consequently, these explicit solutions may also serve as valuable test functions for numerical evaluation, aiding in comprehending the dynamics of this PDE.

It turns out that the non–trivial²² traveling waves of this L^2 –critical equation (CSM), are rational functions. While on the real line (i.e. when $x \in \mathbb{R}$), the traveling waves have been fully characterized in [GL24] as waves satisfying $\|u\|_{L^2} = 2\pi$ for the focusing case, it turns out that in the periodic case (i.e. $x \in \mathbb{T}$), we discovered a diverse class of traveling waves, enjoying a wide range of L^2 –norms, varying from arbitrarily small to arbitrarily large values for the focusing and defocusing equation [Bad23]. We summarize the results in the following table. (Table 1.1).

	Focusing (CSM) on \mathbb{R}	Defocusing (CSM) on \mathbb{R}
Stationary waves	✓	
Traveling waves	✓	
Wave speed	$c \in \mathbb{R}$	
L^2 –norm of the traveling waves	$\ u\ _{L^2} = \sqrt{2\pi}$	
	Focusing (CSM) on \mathbb{T}	Defocusing (CSM) on \mathbb{T}
Non–trivial stationary waves	✓	✗
Traveling waves	✓	✓
Wave speed	$c \in \mathbb{R}$	$c \geq N$
L^2 –norm of the non–trivial traveling waves	$\ u\ _{L^2} \in (0, +\infty)$	$\ u\ _{L^2} \in (0, +\infty)$

TABLE 1.1 – Where $N \in \mathbb{N}$ is the denominator’s degree of a traveling wave appearing in Theorem 1.2.17. Moreover, by non–trivial stationary waves, we mean the solution $u(t, x) = u_0(x)$ that are not constant functions.

²². We mean by non–trivial traveling waves, those that are neither constant nor plane waves $C e^{iN(x-Nt)}$, $C > 0$, $N \in \mathbb{N}_{\geq 1}$.

1.2.3.1 On \mathbb{R}

In order to characterize the traveling waves of (CSM) on \mathbb{R} (Theorem 1.2.14), Gérard–Lenzmann [GL24, Section 4.] characterized the ground states of the Calogero–Moser DNLS equation as the minimizers of the Hamiltonian²³

$$\mathcal{H}(u) := \frac{1}{2} \|L_u u\|_{L^2(\mathbb{R})}^2$$

associated with the non–standard symplectic form defined on $H_+^s(\mathbb{R})$, $s \geq 0$,

$$\omega_u^\sharp(h_1, h_2) := \operatorname{Im} \langle h_1, h_2 \rangle_{L^2} + \iint_{\mathbb{R} \times \mathbb{R}} \operatorname{Re}(\bar{u}h_1)(x) \operatorname{Re}(\bar{u}h_2)(y) \operatorname{sgn}(x - y) \, dx dy.$$

Therefore, they obtained that

$$u(x) = e^{i\theta} \lambda^{1/2} \mathcal{R}(\lambda x + \mu)$$

where $\theta \in [0, 2\pi)$, $\lambda > 0$, $\mu \in \mathbb{R}$, and

$$\mathcal{R}(x) = e^{i\theta} \frac{\sqrt{2 \operatorname{Im} p}}{x + p} \in H_+^1(\mathbb{R}), \quad p \in \mathbb{C}_+, \quad \theta \in \mathbb{R} \quad (1.2.21)$$

is the unique minimizer of $\|L_u u\|_{L^2(\mathbb{R})} = 0$. As a consequence, they infer the following characterization of traveling waves.

Theorem 1.2.14 ([GL24]). *Given \mathcal{R} defined in (1.2.21), we have that every traveling wave for the focusing (CSM) in $H_+^1(\mathbb{R})$ is of the form*

$$u(t, x) = e^{i\theta} e^{iv(x-vt)} \lambda^{1/2} \mathcal{R}(\lambda(x - 2vt) + y), \quad \lambda > 0, \quad y \in \mathbb{R}, \quad \theta \in \mathbb{T}, \quad v \in \mathbb{R}.$$

In particular, they satisfy $\|u\|_{L^2} = \sqrt{2\pi}$.

As a second result, they established the existence of rational solutions of the focusing Calogero–Moser DNLS equation (CSM)

$$u(x) = \frac{P(x)}{Q(x)}, \quad P\bar{P} = i(Q'\bar{Q} - \bar{Q}'Q), \quad \deg(P) < \deg(Q) =: N,$$

completely supported in the **pure point spectrum of the Lax operator**, with a mass

$$\|u\|_{L^2(\mathbb{R})}^2 = 2\pi N,$$

exhibiting frequency cascades as $t \rightarrow \pm\infty$. These solutions, termed “multisolitons” in [GL24], are so named because when $\deg(Q) = 1$, u coincides with \mathcal{R} up to symmetries.

Theorem 1.2.15 ([GL24]). *For every $N \geq 2$, every multisoliton u for the focusing (CSM) exists for all times $t \in \mathbb{R}$ and it exhibits growth of Sobolev norms*

$$\|u(t)\|_{H^s} \sim |t|^{2s}$$

for any real number $s > 0$.

23. Obtained after applying a Gauge transformation to the equation.

1.2.3.2 On \mathbb{T}

The diversity between the traveling waves when $x \in \mathbb{R}$ and $x \in \mathbb{T}$ stems from the **spectral properties inherent in the Lax operators** in each setting. To characterize these waves on the torus, we rely on the spectral theory of L_u and \tilde{L}_u . Thus, we recall that their spectra consist solely of eigenvalues

$$\begin{aligned}\sigma(L_u) &:= \{\nu_0 \leq \dots \leq \nu_n \leq \dots\}, & \nu_0 &\geq -\|u\|_{L^\infty}^2, \\ \sigma(\tilde{L}_u) &:= \{\lambda_0 \leq \dots \leq \lambda_n \leq \dots\}, & \lambda_0 &\geq 0.\end{aligned}$$

These eigenvalues satisfy in the defocusing case [Bad23, Proposition 2.1]

$$\lambda_{n+1} \geq \lambda_n + 1, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (1.2.22)$$

However, in the focusing case, the spectrum of the Lax operator L_u is more intricate. In fact, there is a slight difference in the gap between consecutive eigenvalues of L_u depending on whether we consider a potential u with L^2 -mass less than, or greater than 1, reminding us of the condition $\|u_0\|_{L^2} < 1$ where we know that the **global well-posedness of the focusing (CSM) equation** holds in $H_+^s(\mathbb{T})$ for all $s \geq 0$. Specifically, we have [Bad23, Proposition 2.1]

— If $\|u\|_{L^2} < 1$, then all the eigenvalues of L_u are simple

$$\nu_{n+1} > \nu_n, \quad \forall n \in \mathbb{N}_{\geq 0}.$$

— If $\|u\|_{L^2} \geq 1$, then some eigenvalues may have multiplicity two²⁴, but they are of finite number, since

$$\liminf_{n \rightarrow \infty} \nu_{n+1} - \nu_n \geq 1. \quad (1.2.23)$$

Additionally, none eigenvalues is of multiplicity higher the two, as

$$\nu_{n+2} \geq \nu_n + 1, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (1.2.24)$$

Moreover, apart from the inequalities concerning the eigenvalues of the Lax operators, we will also use some identities induced by the eigenfunctions, to identify the traveling waves. Thus, if we denote by $(f_n^{u_0})$ the eigenfunctions of the Lax operator L_{u_0} , and if we examine the evolution of these vectors as solutions to

$$\begin{cases} \partial_t f_n^t &= B_{u(t)} f_n^t \\ f_n^t|_{t=0} &= f_n^{u_0} \end{cases}, \quad \forall n \in \mathbb{N}_{\geq 0},$$

one deduces the following identities

$$\begin{aligned}\langle 1 | f_n^t \rangle &= \langle 1 | f_n^{u_0} \rangle e^{-i\nu_n^2 \cdot t} \\ \langle u | f_n^t \rangle &= \langle u_0 | f_n^{u_0} \rangle e^{-i\nu_n^2 \cdot t} \\ \langle S f_p^t | u \rangle &= \langle S f_p^{u_0} | u_0 \rangle e^{i(\nu_p+1)^2 \cdot t} \\ \langle S f_p^t | f_n^t \rangle &= \langle S f_p^{u_0} | f_n^{u_0} \rangle e^{i((\nu_p+1)^2 - \nu_n^2) \cdot t},\end{aligned} \quad (1.2.25)$$

24. Take, for instance $u = e^{ix}$, then as $L_u = D - T_u T_{\bar{u}} = D - u\Pi(\bar{u} \cdot)$, one checks $L_u 1 = L_u e^{ix} = 0$.

where S denotes the shift operator (1.2.9). And the analogous identities hold in the defocusing case as well.

Combining the inequalities (1.2.22), (1.2.23) and (1.2.24) related to the eigenvalues, along with the previous identities (1.2.25) related to the eigenfunctions, and the fact that [Bad23, Corollary 2.6]

— in the defocusing case

$$\lambda_n = \lambda_{n-1} + 1 \iff \langle u_0 | f_n^{u_0} \rangle = 0. \quad (1.2.26)$$

— in the focusing case²⁵

$$\nu_n = \nu_{n-1} + 1, \forall n \geq N_1 \iff \langle u_0 | f_n^{u_0} \rangle = 0, \forall n \geq N_2. \quad (1.2.27)$$

one deduces, at a first stage, a spectral characterization of the traveling waves $u(t, x) = u_0(x - ct)$ for (CSM).

Theorem 1.2.16 (Spectral characterization of the traveling waves – $x \in \mathbb{T}$ [Bad23]). *Let $u(t, x) := u_0(x - ct)$ be a traveling wave to the Calogero–Sutherland DNLS equation (CSM). Then*

► **Defocusing equation.** *There exists at most one $N \in \mathbb{N}_{\geq 1}$ such that*

$$\langle u_0 | f_N^{u_0} \rangle \neq 0.$$

► **Focusing equation.** *There exists $N \in \mathbb{N}_{\geq 1}$ such that for all $n \geq N$,*

$$\langle u_0 | f_n^{u_0} \rangle = 0.$$

Armed with these spectral characterizations of the traveling waves, and the fact that any function in the Hardy space can be written as [GK21, GMR16]

$$u(t, z) = \langle (\text{Id} - zM)^{-1} X | Y \rangle, \quad z \in \mathbb{D},$$

where X, Y are infinite column vectors and M is the infinite matrix representation of S^* ,

$$X := (\langle u | f_n^t \rangle), \quad Y := (\langle 1 | f_n^t \rangle), \quad M := (\langle f_m^t | S f_n^t \rangle),$$

and by using the relations

$$\begin{aligned} \langle 1 | u \rangle \langle u | f_n \rangle &= \mp \lambda_n \langle 1 | f_n \rangle, \\ (\lambda_n - \lambda_p - 1) \langle S f_p | f_n \rangle &= \mp \langle S f_p | u \rangle \langle u | f_n \rangle. \end{aligned}$$

where $\langle \cdot | \cdot \rangle$ denotes the inner product in $L^2(\mathbb{T})$, we infer the following explicit formulas for the traveling waves.

²⁵. We refer to the Appendix 2 for an example that shows that N_2 is not necessarily equal to N_1 .

Theorem 1.2.17 ([Bad23]). *The functions $u(t, x) = C e^{in(x-nt)}$, $C \in \mathbb{R}$, $n \in \mathbb{N}_{\geq 0}$, and*

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, \quad (1.2.28)$$

where $N \in \mathbb{N}$, $c := \pm N \left(1 + \frac{2\alpha}{\beta}\right)$, and $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = \pm N, \quad (1.2.29)$$

are traveling waves for the focusing and defocusing (CSM)–equation on \mathbb{T} . Furthermore, in the defocusing case, the above solutions are the only traveling waves $u(t, x) := u_0(x - ct)$. However, in the focusing case, there exist additional traveling waves, such as²⁶

$$u(t, x) = e^{i\theta} e^{im(x-mt)} \left(\alpha + \frac{\beta}{1 - p e^{i(x-mt)}} \right) \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, m \in \mathbb{N}_{\geq 1},$$

where (α, β) satisfies

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = 1, \quad \beta(m - 1) = 2\alpha.$$

Remark 1.2.4. [Bad23]

1. **The dynamics of the focusing (CSM) are richer.** It is worth noting that the condition on (α, β) appearing in (1.2.29) for the focusing case enables the derivation of a *broader range* of traveling waves compared to the defocusing case. For instance,

$$u(t, x) = e^{i\theta} \frac{\sqrt{N(1 - |p|^2)}}{1 - p e^{iN(x+Nt)}} \quad \text{and} \quad u(t, x) = e^{i\theta} \frac{\sqrt{N(1 - |p|^2)} e^{iN(x-Nt)}}{1 - p e^{iN(x-Nt)}}$$

are traveling waves for the focusing (CSM). While there is no traveling waves $u(t, x) := u_0(x - ct)$ with a profile

$$u_0(x) := \frac{\beta}{1 - p e^{iNx}} \quad \text{or} \quad u_0(x) := \frac{\alpha e^{iNx}}{1 - p e^{iNx}}$$

for the defocusing equation.

2. **Stationary solutions.** The family

$$u(t, x) := e^{i\theta} \sqrt{\frac{N(1 - |p|^2)}{2(1 + |p|^2)}} \left(1 - \frac{2}{1 - p e^{iNx}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, N \in \mathbb{N}_{\geq 1}.$$

constitutes non-trivial (i.e., non-constant) stationary wave solutions²⁷ No stationary solutions exist for the defocusing case.

26. and they may not be the only one.

27. i.e. $u(t, x) = u_0(x)$.

3. Correlation between the speed and the L^2 -norm of a traveling wave.

While the focusing and defocusing (CSM) equations exhibit traveling waves with arbitrarily small and large L^2 -norms for this L^2 -critical equation²⁸, the correlation between the speed and L^2 -norm of the traveling waves varies between the focusing and defocusing cases. In the defocusing case, one observes that the nontrivial traveling wave (1.2.28) propagates to the right with a speed $c > N$, where N is the degree appearing in the denominator of (1.2.28). Additionally, when $\|u\|_{L^2} \rightarrow +\infty$, we have $c \rightarrow +\infty$, and when $\|u\|_{L^2} \rightarrow 0$ then $c \rightarrow N$. However, in the focusing case, there is no correlation between the speed and the size of the L^2 -norm, and the traveling wave can propagate to the right or to the left at any speed.

Note that the traveling waves are part of a larger family of rational solutions of (CSM), known as the **finite gap potentials**. These are defined as potentials u for which, starting from a certain rank, two consecutive eigenvalues are spaced by 1. In other words, in view of (1.2.26) and (1.2.27) they are described as potentials u for which, from a certain rank onward, all $\langle u | f_n \rangle = 0$ vanish. This means that u is supported on a finite number of eigenspaces of L_u and \tilde{L}_u , effectively providing an analogue to the *multisolitons* observed on the real line. These finite gap potentials are characterized explicitly as follows :

Theorem 1.2.18 (Characterization in the state space of the (CSM)'s finite gap potentials [Bad23]). *The finite gap potentials of (CSM) are either the functions $u(x) = C e^{iNx}$, $C \in \mathbb{C}^*$, $N \in \mathbb{N}_{\geq 0}$, or the rational function*

$$u(x) = e^{im_0 x} \prod_{j=1}^r \left(\frac{e^{ix} - \bar{p}_j}{1 - p_j e^{ix}} \right)^{m_j - 1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j e^{ix}} \right), \quad p_j \in \mathbb{D}^*, \quad p_k \neq p_j, \quad k \neq j,$$

where, for $N \in \mathbb{N}_{\geq 1}$, $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$ such that $m_0 + \sum_{j=1}^r m_j = N$, and $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfy for all $j = 1, \dots, r$,

(i) In the defocusing case,

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = -m_j,$$

(ii) In the focusing case,

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = m_j,$$

with $a \neq 0$ if $m_0 \neq 0$. Moreover, these finite gap potentials can be regrouped into sets that remain invariant under the evolution of (CSM).

²⁸. More precisely, for any $r > 0$, there exists a non-trivial traveling wave $u(t, x) := u_0(x - ct)$ of (CSM) where

$$\|u_0\|_{L^2} = r.$$

Open problems and Perspectives.

1. An open question is to understand the dynamics of these **finite gap potentials**. Do solutions starting from such initial data remain bounded in all the H^s norms, or do they, akin to the scenario observed on the real line, manifest frequency cascades?
2. Here, we have characterized all the traveling waves of the form $u(t, x) := u_0(x - ct)$ for the defocusing (CSM)–equation, and we have obtained a far-reaching family of traveling waves for the focusing equation. However, the full characterization of the traveling waves in the focusing case remains an open problem.
3. In our study [Bad23], we have treated the case where the traveling waves of the Calogero–Sutherland DNLS equation (CSM) are of the form

$$u(t, x) := u_0(x - ct), \quad c \in \mathbb{R}.$$

But, one may wonder if there exist traveling wave solutions with a phase factor, such as

$$u(t, x) := e^{i\varphi(t)} u_0(x - ct), \quad \varphi(t), c \in \mathbb{R}. \quad (1.2.30)$$

However, let us underline the following fact : observe that the mean $\langle u | 1 \rangle$ is conserved along the flow of the Calogero–Sutherland DNLS equation (CSM), for any solution u in the Hardy space of the circle \mathbb{T} . Indeed, by doing an integration by parts and since u is in the Hardy space, then

$$i\partial_t \langle u | 1 \rangle = -\langle \partial_x^2 u | 1 \rangle \pm 2 \langle D\Pi(|u|^2) | \bar{u} \rangle = 0.$$

Therefore,

- If $\langle u_0 | 1 \rangle \neq 0$, then $\varphi(t)$ in (1.2.30) must be a constant in time.
- If $\langle u_0 | 1 \rangle = 0$, the question of the existence of traveling waves of (CSM) of the form (1.2.30) remains an open problem. However, one can easily prove that $(\varphi(t), c)$ are related via the following identity

$$\varphi'(t) - Nc = -N^2, \quad (1.2.31)$$

where N is the positive integer appearing after rewriting u_0 as $u_0 = S^N v_0$ with $\langle v_0 | 1 \rangle \neq 0$. Indeed, by substituting the solution $u(t, x)$

$$u(t, x) = e^{i\varphi(t)} u_0(x - ct) = e^{i\varphi(t)} e^{iN(x-ct)} v_0(x - ct),$$

in (CSM), we obtain

$$\begin{cases} -(\varphi'(t) - Nc) v_0 - N^2 v_0 + P(\partial_x v_0, \partial_x^2 v_0) \pm 2iD\Pi(|v_0|^2)v_0 = 0. \\ P(w, \tilde{w}) := (2N - c)iw + \tilde{w} \end{cases}$$

Therefore, we infer (1.2.31) by taking the inner product of the last identity with 1.

4. The study of the **orbital stability** of the traveling waves, knowing that on the real line, Hogan–Kowalski showed the orbital stability of solitons on \mathbb{R} for a restricted class of initial data [HK24, Theorem 2.1].

Chapter 2

Introduction (en Français)

RÉSUMÉ.

Dans cette introduction, nous donnons un aperçu des résultats obtenus dans cette thèse concernant une équation du type Schrödinger non-linéaire non-locale, dérivée comme une limite continue du système classique de *Calogero-Sutherland-Moser*. L'approche principale utilisée pour prouver ces résultats implique l'utilisation des propriétés d'intégrabilité complète de cette EDP. Dans ce qui suit, nous décrivons en quelques mots le concept d'équations complètement intégrables, présentons les résultats obtenus pour cette équation et établissons des comparaisons avec les résultats précédemment établis pour d'autres équations intégrables, notamment l'équation de Szegő et de Benjamin-Ono.

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Imaginez que vous observez une onde et vous capturez sa distribution à un moment donné. Cela soulève une question naturelle : comment évolue la distribution spatiale de cette onde dans le temps sous des conditions spécifiques ? Tout mathématicien ou physicien vous dirait que cette évolution peut être décrite de manière adéquate à l'aide d'un outil puissant et rigoureux connu sous le nom d'équation aux dérivées partielles (EDP). Cependant, le défi réside dans la résolution explicite de cette EDP, c'est-à-dire trouver une formule explicite qui décrit exactement, à chaque instant, la

distribution spatiale de l'onde. Malheureusement, cela s'avère souvent être une tâche insurmontable pour les EDP décrivant des phénomènes du monde réel (les EDP non linéaires). Néanmoins, il existe certains types d'EDP non linéaires, spécifiquement les EDP complètement intégrables, pour lesquelles des solutions explicites peuvent parfois être trouvées.

2.1 Qu'est-ce qu'une EDP complètement intégrable ?

Les spécialistes semblent manquer de consensus sur une définition universelle des systèmes intégrables en dimension infinie. Cette divergence d'opinions peut être due à la richesse de la structure exhibée par de telles EDP, permettant à chaque expert d'aborder le sujet sous différents angles. Pour certains, l'association immédiate avec le terme "complètement intégrable" est liée à l'existence d'une **paire de Lax** - une paire d'opérateurs intimement liés à l'EDP et jouant un rôle crucial dans l'étude spectrale du problème (Nous renvoyons à la Section 2.1.1 pour la définition et plus de détails). D'autres, cependant, peuvent percevoir l'intégrabilité à travers l'existence de **coordonnées de Birkhoff** - un système de coordonnées où cette EDP non linéaire peut être "diagonalisée", permettant l'étude du comportement de la solution dans la structure canonique (Voir Section 2.1.2). Une autre perspective soutient que le concept d'intégrabilité dépasse ces structures, en mettant l'accent sur la capacité du système à être intégré et résolu explicitement, permettant ainsi d'obtenir une **formule explicite** pour la solution (Voir Section 2.1.3)...

Néanmoins, il existe une inclination parmi les chercheurs selon laquelle les EDP complètement intégrables possèdent un **nombre infini de quantités conservées**, qui jouent un rôle crucial dans l'obtention de résultats spectaculairement précis par rapport à d'autres types d'EDP. De plus, la symétrie cachée derrière ces lois de conservation infinies permet la construction de solutions exactes pour l'EDP intégrable en utilisant des techniques telles que la transformée de scattering inverse ou la théorie spectrale. À titre d'exemple, on pourrait mentionner l'existence de **solitons**, qui sont des solutions particulières d'ondes progressives, s'exprimant généralement à travers des fonctions élémentaires (Voir Section 2.1.4).

Un exemple typique d'EDP intégrable a été suggéré en 1872 et est connu de nos jours sous le nom d'équation de Korteweg-de Vries [Bou72, KV95]

$$\partial_t u + \partial_x^3 u - 6u\partial_x u = 0. \quad (\text{KdV})$$

Cette équation sert de modèle mathématique décrivant la propagation de petites (en amplitude) ondes dans de longues surface d'eau dans un canal étroit et peu profond. La propriété d'intégrabilité de l'équation KdV remonte au travail de Gardner, Greene, Kruskal et Miura [GGKM67], qui ont introduit la transformée de scattering inverse pour résoudre l'équation de KdV explicitement et retrouver les ondes solitaires. Plus précisément, ces ondes sont obtenues en résolvant le problème de scattering associé à l'équation de Schrödinger linéaire avec un terme de potentiel lié

à la solution de l'équation (KdV). La solution de ce problème de scattering code des informations sur les solitons de l'équation KdV.

Plutôt que de nous plonger plus en profondeur dans les complexités de la théorie de scattering inverse [Pal97], notre attention se tourne vers le travail ultérieur de Peter Lax [Lax68], publié un an après le travail de [GGKM67], qui reformule les concepts présentés dans le travail antérieur.

2.1.1 La paire de Lax

Dans son célèbre article [Lax68], Peter Lax démontre l'existence de deux opérateurs linéaires associés à l'équation (KdV), connus maintenant sous le nom d'opérateurs de la paire de Lax. Ces opérateurs, notés L_u et B_u , sont définis comme suit :

$$L_u = -\partial_x^2 + u, \quad B_u = 4\partial_x^3 - 6u\partial_x - 3\partial_x u,$$

et ils satisfont l'équation de Lax introduite dans (2.1.1) lorsque u satisfait l'équation (KdV). Une observation remarquable est que, en considérant l'équation des fonctions propres de L_u , c'est-à-dire $L_u f = \lambda f$, on se souvient de l'équation de Schrödinger décrite dans le travail précédent de [GGKM67] qui repose sur la théorie du scattering. Le pouvoir de considérer ces opérateurs réside dans l'examen de l'équation (KdV) d'un point de vue différent en étudiant un problème spectral qui offre des informations précieuses sur la dynamique et, plus important encore, qui offre la capacité de construire à partir de l'opérateur de Lax L_u un nombre infini d'intégrales de mouvement (c'est-à-dire des quantités conservées) liées à cette EDP. Pour développer ce dernier point plus en détail, nous le décrivons dans le cadre le plus général, c'est-à-dire dans le cas où nous considérons n'importe quelle EDP bénéficiant d'une structure de paire de Lax.

L'intégrabilité en termes d'opérateurs de Lax peut être décrite comme suit : Étant donné un problème de Cauchy avec des données initiales u_0 , il existe deux opérateurs linéaires (L_u, B_u) dépendant des variables dynamiques $(u, \partial_x u, \dots, \partial_x^n u)$ du problème de Cauchy, tels que si u satisfait l'EDP, alors les deux opérateurs (L_u, B_u) satisfont l'équation de Lax

$$\partial_t L_u = [L_u, B_u], \quad (2.1.1)$$

où $[L_u, B_u] = L_u B_u - B_u L_u$. Par exemple, considérons le problème de Cauchy associé à

— **l'équation de Benjamin-Ono** sur \mathbb{R} et $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ [Ben67, Ono75]

$$\partial_t u = \partial_x (|D|u - u^2), \quad D = -i\partial_x, \quad \widehat{|D|}(\xi) = |\xi|. \quad (\text{BO})$$

Cette équation caractérise la propagation des ondes internes unidimensionnelles au sein d'un fluide stratifié [Sau19]. Elle est également connue pour posséder une structure de paire de Lax [Nak79, BK79]

$$L_u = D - T_u, \quad B_u = i(T_{|D|u} - T_u T_u), \quad (2.1.2)$$

où T_u est l'opérateur de Toeplitz de symbole $u \in L^\infty$ défini pour tout $f \in L^2$, comme $T_u(f) := \Pi(uf)$, et Π est le projecteur de Riesz–Szegő sur les modes de Fourier non négatifs, défini sur \mathbb{R} comme $\widehat{\Pi u}(\xi) = \mathbf{1}_{\xi \geq 0} \widehat{u}(\xi)$, et sur \mathbb{T} comme

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) = \sum_{n \geq 0} \widehat{u}(n) e^{inx} . \quad (2.1.3)$$

À partir de l'opérateur de Lax L_u , on obtient un nombre infini de lois de conservation données par $\langle L_u^k 1 \mid 1 \rangle$, $k \in \mathbb{N}_{\geq 0}$. [GK21, Sun21].

- **L'équation cubique de Szegő** introduite dans [GG08a, GG10, Poc11a, Poc11b] comme un modèle jouet pour une équation totalement non dispersive

$$i\partial_t u = \Pi(|u|^2 u), \quad (\text{Sz})$$

où Π est défini dans (2.1.3). Cette équation possède également une structure de paire de Lax sur \mathbb{R} ainsi que sur \mathbb{T} , [GG10, GG12a, Poc11a, Poc11b]

$$\frac{d}{dt} H_u = [B_u, H_u], \quad (2.1.4)$$

où H_u est l'opérateur de Hankel de symbole $u \in L^\infty$ défini pour tout $f \in L^2$, comme $H_u(f) = \Pi(u\bar{f}) = T_u(\bar{f}) = T_{\bar{f}}(u)$. Et B_u est l'opérateur antisymétrique

$$B_u := -iT_{|u|^2} + \frac{i}{2} H_u^2.$$

Plus remarquablement, cette équation possède sur le tore \mathbb{T} une deuxième paire d'opérateurs de Lax [GG10, GG12a]

$$\frac{d}{dt} K_u = [C_u, K_u], \quad (2.1.5)$$

où l'opérateur K_u est donné par $K_u := H_u S$, avec S l'opérateur de décalage $Sf(x) = e^{ix} f(x)$, et C_u est l'opérateur antisymétrique

$$C_u := -iT_{|u|^2} + \frac{i}{2} K_u^2$$

À partir de l'opérateur de Lax H_u , on déduit un nombre infini de lois de conservation données par $\langle u \mid H_u^{2k-2} u \rangle$, $k \in \mathbb{N}_{\geq 1}$ [GG10].

De plus, il est important de mentionner que chaque fois que le deuxième opérateur B_u dans la structure de la paire de Lax (2.1.1) est antisymétrique¹, alors la *propriété de déformation isospectrale* est satisfaite, c'est-à-dire

$$U(t)^* L_{u(t)} U(t) = L_{u_0}, \quad (2.1.6)$$

1. On peut démontrer que, même dans les cas où B_u n'est pas un opérateur antisymétrique, nous avons toujours $U(t)^{-1} L_{u(t)} U(t) = L_{u_0}$.

où la famille d'opérateurs $U(t)$ est la solution du problème de Cauchy

$$\begin{cases} \frac{d}{dt}U(t) = B_{u(t)}U(t) \\ U(0) = \text{Id} \end{cases} .$$

En effet, comme B_u est un opérateur antisymétrique, alors la solution $U(t)$ du problème de Cauchy précédent est un opérateur unitaire car $\frac{d}{dt}(U(t)U^*(t)) = \frac{d}{dt}(U^*(t)U(t)) = 0$. Par conséquent, en calculant $\frac{d}{dt}(U(t)^*L_{u(t)}U(t))$ et en utilisant l'équation de Lax (2.1.1), on obtient

$$\frac{d}{dt}(U(t)^*L_{u(t)}U(t)) = 0 .$$

En conséquence, (2.1.6) implique que le spectre de L_u est conservé le long du flot, fournissant une nouvelle famille infinie de lois de conservation pour l'EDP intégrable.

2.1.2 Variables action–angle/Coordonnées de Birkhoff

Une caractéristique importante des EDP non linéaires intégrables est le fait qu'elles présentent une interaction remarquable entre la géométrie et l'analyse. Elles peuvent cacher une quantité significative de symétries dans le problème qui permet de le diagonaliser. Mais avant d'approfondir ce point, rappelons l'idée derrière un outil largement reconnu dans le domaine des EDP linéaires : la transformée de Fourier linéaire.

Considérons une EDP linéaire ; le principal objectif derrière une équation est de la résoudre, ou du moins de capturer le plus d'informations. C'est pourquoi de puissantes techniques ont été développées au fil des siècles. L'un des moments les plus remarquables dans l'analyse des EDP a été la découverte de la transformée de Fourier. Il s'agit d'un outil bien connu qui permet de décomposer toute fonction en ses composantes fréquentielles et de transformer tout domaine spatial en domaine fréquentiel. Un autre aspect significatif de cette méthode est sa capacité à diagonaliser tout opérateur linéaire dans l'EDP, nous permettant d'analyser des phénomènes tels que la dispersion, les taux de décroissance et la stabilité... Maintenant, revenons aux EDP non linéaires intégrables, lorsqu'on est confronté à l'une d'elles, le but est de mimer l'idée utilisée pour les EDP linéaires en cherchant une transformation capable de diagonaliser le problème. Cela nous permet de voir le problème sous différents angles/coordonnées, en extrayant ainsi beaucoup plus d'informations sur la solution.

Par exemple, supposons que nous traitons une EDP linéaire sur le tore ($x \in \mathbb{T}$). La transformée de Fourier linéaire établit une bijection entre la solution u de l'EDP linéaire et ses coefficients de Fourier $(\langle u | e^{ikx} \rangle)_{k \in \mathbb{N}_{\geq 0}}$. Pour une EDP non linéaire intégrable, il faut chercher une transformée de Fourier non linéaire, c'est-à-dire une bijection entre la solution u de cette EDP non linéaire et les coefficients de Fourier non linéaires $(\langle u | f_n^t \rangle)_{n \in \mathbb{N}_{\geq 0}}$, où les $(f_n^t)_{n \in \mathbb{N}_{\geq 0}}$ sont une base orthonormée adéquate dépendant de l'EDP. Ces coordonnées $(\langle u | f_n^t \rangle)_{n \in \mathbb{N}_{\geq 0}}$, appelées **coordonnées de Birkhoff**, nous permettent d'aborder le problème sous différents angles : au lieu

d'étudier une EDP non linéaire, le problème peut être réduit grâce à ce changement de coordonnées à un système d'ODE linéaires dans les variables t (la variable temporelle) et d'inconnues $\langle u, f_n^t \rangle$ pour la $n^{\text{ème}}$ équation de ce système linéaire. Notez que dans la littérature, on peut rencontrer le terme “**variables action–angle**”² au lieu de coordonnées de Birkhoff ; ils sont les mêmes que les coordonnées polaires pour un point complexe $z \in \mathbb{C}^*$. Les variables d'action peuvent décrire un ensemble complet d'intégrales de mouvement (lois de conservation) en involution³, connu sous le nom d'intégrabilité de Liouville [KP13].

Comme nous le verrons dans les deux exemples suivants, l'intégrabilité de Liouville sera l'instrument principal pour découvrir les propriétés de la dynamique de l'EDP, telles que les résultats GWP, les orbites périodiques, les phénomènes de turbulences...

2.1.2.1 Application à l'équation de Benjamin–Ono

Nous rappelons l'équation de Benjamin–Ono

$$\partial_t u = \partial_x (|D|u - u^2), \quad u(0) = u_0, \quad D = -i\partial_x, \quad |\widehat{D}|(\xi) = |\xi|. \quad (\text{BO})$$

Il s'agit d'une EDP complètement intégrable au sens de Liouville, comme décrit dans [GK21] : Si nous notons par $L_{r,0}^2(\mathbb{T})$ l'espace des fonctions réelles dans $L^2(\mathbb{T})$ avec une moyenne nulle, et

$$h_+^r = \left\{ (\zeta_n)_{n \geq 1} \in \mathbb{C}^{\mathbb{N}} ; \|\zeta\|_{h_+^r}^2 = \sum_{n=1}^{+\infty} n^{2r} |\zeta_n|^2 < +\infty \right\}; \quad r \in \mathbb{R},$$

et si nous notons pour tout $t \in \mathbb{R}$, $n \in \mathbb{N}_{\geq 1}$,

$$\zeta_n(u(t)) := \frac{\langle u(t) | f_n^t \rangle}{\kappa_n}$$

où les (f_n^t) sont les fonctions propres de l'opérateur Lax $L_{u(t)}$ défini dans (2.1.2) et κ_n est une certaine normalisation indépendante du temps t [GK21], alors la *transformée de Fourier non linéaire* pour l'équation (BO) est décrite par l'homéomorphisme suivant⁴

$$\Phi : u \in L_{r,0}^2(\mathbb{T}) \longmapsto (\zeta_n(u))_{n \geq 1} \in h_+^{\frac{1}{2}}. \quad (2.1.7)$$

Les $(\zeta_n(u))$ représentent les coordonnées de Birkhoff. Et chaque $\zeta_n(u)$ satisfait l'évolution suivante

$$\zeta_n(u(t)) = e^{it\omega_n(u_0)} \zeta_n(u_0), \quad \omega_n(u_0) = n^2 - 2 \sum_{k \geq 1} \min(k, n) |\zeta_k(u_0)|^2.$$

2. [Arn13, KP13]

3. C'est-à-dire, les lois de conservation commutent dans les crochets de Poisson $\{\cdot, \cdot\}$ associés au hamiltonien de l'équation. En d'autres termes, le crochet de Poisson de ces lois de conservation s'annule.

4. Le décalage de 1/2 entre les espaces d'entrée et de sortie est lié au fait que l'application Φ est non seulement un homéomorphisme mais aussi un symplectomorphisme. [GK21]

En d'autres termes, on peut comprendre la dynamique de l'équation de Benjamin–Ono (BO) en examinant le comportement des coordonnées $(\zeta_n(u))$ qui évoluent dans le temps. Dans le travail [GKT23], les auteurs généralisent cette transformée de Fourier non linéaire à tout espace $H_{r,0}^s(\mathbb{T})$ (c'est-à-dire l'espace des fonctions réelles dans $H^s(\mathbb{T})$ avec moyenne nulle), où $s > -\frac{1}{2}$. Par conséquent, l'extension suivante est valable

$$\Phi : u \in H_{r,0}^s(\mathbb{T}) \longmapsto (\zeta_n(u))_{n \geq 1} \in h_+^{\frac{1}{2}+s}$$

pour tout $s > -\frac{1}{2}$ et reste un homéomorphisme.

En conséquence de cette construction, on a :

1. **Application au GWP de (BO) :** Le premier résultat concernant le problème que l'équation de Benjamin–Ono (BO) est globalement bien-posé dans $H_r^s \equiv H^s(\mathbb{T} \text{ ou } \mathbb{R} ; \mathbb{R})$ avec $s > \frac{3}{2}$ remonte aux travaux de [Sau79b, JI86, ABFS89, Pon91]. Ensuite, inspiré par la transformation de Cole–Hopf [Col51, Hop50], Tao introduit ce qui est maintenant appelé la transformation de jauge de Tao, lui permettant d'établir le caractère de globalement bien-posé de (BO) dans $H^1(\mathbb{R})$ [Tao04]. Puis, Molinet, en utilisant la transformation de jauge de Tao, a étendu le flot à partir d'espaces de régularité élevée jusqu'à L_r^2 [Mol08]. Plus tard, dans une série de travaux récents, [GKT23, GT23] ont prouvé via la transformée de Fourier non linéaire, le caractère de globalement bien-posé de (BO) jusqu'à

$$H_r^{-\frac{1}{2}, \sqrt{\log}}(\mathbb{T}) := \left\{ u \in H^{-\frac{1}{2}} ; \sum_{n \in \mathbb{Z}} \langle n \rangle^{-1} \log(\langle n \rangle + 1) |\widehat{u}(n)|^2 < \infty \right\} \subsetneq H_r^{-\frac{1}{2}}(\mathbb{T}), \quad (2.1.8)$$

qui est un sous-espace de $H_r^{-1/2}(\mathbb{T})$ contenant tous les espaces $H_r^\sigma(\mathbb{T})$ pour $\sigma > -\frac{1}{2}$. De plus, ils ont montré que l'équation (BO) est mal-posé dans $H_r^{-1/2}(\mathbb{T})$, en utilisant des arguments d'analyse complexe. Pour la droite réelle⁵, le caractère de globalement bien-posé de l'équation en $H^s(\mathbb{R})$, $s > -\frac{1}{2}$ a été prouvé dans [KLV23b] en utilisant la méthode des flots commutatifs introduite dans [KV19] et développée dans les travaux ultérieurs [BKV21, HGKNV22a, HGKV20, KNV21, Lau21, Lau23, Nte22].

2. **Étude de la troisième équation de la hiérarchie de (BO) :** En utilisant la transformée de Fourier non linéaire (2.1.7), [Gas21] a établi que la troisième équation dans la hiérarchie de (BO), c'est-à-dire lorsque $k = 3$ dans

$$\partial_t u = \partial_x (\nabla \mathcal{H}_{k+1}(u)), \quad \mathcal{H}_{k+1}(u) := \langle L_u^{k+1} 1 \mid 1 \rangle,$$

où L_u est l'opérateur de Lax de (BO) défini dans (2.1.2) est globalement bien-posé⁶ dans $H_{r,0}^s(\mathbb{T})$, $s \geq 0$. De plus, elle a démontré que cette équation est

5. En fait, la méthode adoptée permet de montrer que l'équation de (BO) est bien-posé sur le tore $x \in \mathbb{T}$ ainsi que sur la droite réelle $x \in \mathbb{R}$.

6. au sens de l'extension du flot défini initialement pour des potentiels avec un nombre fini de modes, c'est-à-dire des données initiales telles que pour tout $n \geq N$, $\zeta_n(u_0) = 0$.

mal-posé⁷ de la même équation dans $H_{r,0}^s(\mathbb{T})$ pour $-\frac{1}{2} < s < 0$.

3. **Propriétés qualitatives du flot** : Dans [GK21, GKT22, GKT23], les auteurs ont prouvé que le flot $t \in \mathbb{R} \mapsto u(t) \in H_{r,0}^s(\mathbb{T})$, $s > -\frac{1}{2}$, de (BO) est presque périodique au sens de la caractérisation de Bochner, i.e, l'ensemble $\{u_\tau : t \mapsto u(t + \tau) ; \tau \in \mathbb{R}\}$ est relativement compact dans $\mathcal{C}_b(\mathbb{R}, H_{r,0}^s(\mathbb{T}))$. Par conséquent, ils en déduisent que le flot satisfait la récurrence de Poincaré : $\exists(t_n)$ tel que $u(t_n) \rightarrow u_0$ dans $H^s(\mathbb{T})$ lorsque $t_n \rightarrow \infty$. En outre, dans [GKT22], ils montrent l'existence sur \mathbb{T} de grandes familles de solutions périodiques ou quasi-périodiques, qui ne sont pas \mathcal{C}^∞ lisses.

D'autre part, sur la droite réelle $x \in \mathbb{R}$, [Sun21] a construit la transformée de Fourier non linéaire sur le sous-espace des multisolitons, représentant ainsi une première étape vers la résolution de la conjecture de résolution des solitons.

2.1.2.2 Application à l'équation de Szegő

Considérons l'équation de Szegő

$$i\partial_t u = \Pi(|u|^2 u), \quad (\text{Sz})$$

où Π est le projecteur Szegő défini dans (2.1.3). Dans la Section 2.1.1, nous avons vu que l'équation (Sz) possède une structure de paires de Lax avec un opérateur Lax $H_u(\cdot) := \Pi(\bar{u} \cdot)$ à la fois sur \mathbb{R} et sur \mathbb{T} , et admet l'opérateur $K_u := H_u S = \Pi(\bar{u} e^{ix} \cdot)$ comme un deuxième opérateur de Lax sur \mathbb{T} (voir (2.1.4) et (2.1.5)). Plus marquant, l'équation (Sz) est une *équation complètement intégrable dans le sens de Liouville* pour $x \in \mathbb{R}$ ou \mathbb{T} . Sur le tore, l'intégrabilité complète a été initialement établie sur une variété de dimension finie [GG12b] dans le sens suivant : Pour

$$M := \left\{ u = \frac{A(z)}{\prod_{k=1}^N (1 - p_k z)}, A \in \mathbb{C}_{\leq N-1}[z], |p_k| < 1 \right\},$$

il existe un sous-ensemble ouvert M_{gen} de la variété de dimension finie M , dont le complément a une mesure de Lebesgue 0, tel que

$$\Phi : u \in M_{\text{gen}} \longmapsto \left(\underbrace{2\lambda_1^2, \dots, 2\lambda_N^2, 2\mu_1^2, \dots, 2\mu_N^2}_{\text{variables d'action}}, \underbrace{\varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N}_{\text{variables d'angle}} \right) \in \Omega_N \times \mathbb{T}^{2N}$$

est un difféomorphisme où

$$H_u(e_j) = \lambda_j e_j, \quad K_u(f_j) = \mu_j f_j,$$

et

$$\begin{aligned} \varphi_j &:= \arg \langle 1 | e_j \rangle^2, & j &= 1, \dots, N, \\ \theta_j &:= \arg \langle u | f_j \rangle^2, & j &= 1, \dots, N, \end{aligned}$$

7. au sens où le flot défini sur les potentiels avec un nombre fini de modes, ne peut être étendu de manière continue à $H_{r,0}^s(\mathbb{T})$, pour $-\frac{1}{2} < s < 0$ pour tout $t > 0$.

et

$$\Omega_N := \{(\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N) \in \mathbb{R}^{2N} \mid \lambda_1 > \mu_1 > \dots > \lambda_N > \mu_N > 0\}.$$

Plus tard, [GG17] a étendu Φ à n'importe quelle fonction u dans $H_+^{\frac{1}{2}}(\mathbb{T}) := H^{\frac{1}{2}} \cap L_+^2(\mathbb{T})$, où

$$L_+^2(\mathbb{T}) := \{u \in L^2(\mathbb{T}), \widehat{u}(n) = 0, \forall n \in \mathbb{Z} \setminus \mathbb{N}_{\geq 0}\}. \quad (2.1.9)$$

En conséquence, on déduit [GG17] :

1. La **presque-périodicité** du flot dans $H_+^{\frac{1}{2}}(\mathbb{T})$.
2. Les **phénomènes de turbulences** : Il existe un ensemble G_δ de données initiales dans $L_+^2 \cap C^\infty(\mathbb{T})$ tel que, pour tout $s > \frac{1}{2}$, la solution de (Sz) partant de u_0 satisfait

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\|u(t)\|_{H^s}}{|t|^M} &\rightarrow \infty, & \forall M \in \mathbb{N}, \\ \liminf_{t \rightarrow \infty} \|u(t)\|_{H^s} &< \infty. \end{aligned}$$

Sur la droite réelle, l'absence de la deuxième structure Lax rend le problème quelque peu plus délicat. Cependant, [Poc11a] a réussi à établir l'intégrabilité complète de (Sz) sur seulement un sous-ensemble ouvert de la variété de dimension finie

$$\tilde{M} := \left\{ u = \frac{A(z)}{B(z)}, A \in \mathbb{C}_{\leq N-1}[z], B \in \mathbb{C}_N[z], \deg(B) = N, \text{ et si } B(z) = 0 \text{ alors } z \in \mathbb{C}_- \right\}.$$

Pour décrire son résultat, nous devons introduire l'opérateur T . Soit $u \in \tilde{M}$, alors il existe g tel que $H_u(g) = u$ ⁸. Par conséquent, nous définissons l'opérateur

$$Tf(x) = xf(x) - (1 - g(x)) \lim_{x \rightarrow \infty} xf(x).$$

Et l'intégrabilité complète de (Sz) sur le sous-ensemble M_{gen} ⁹ de \tilde{M} , est décrite comme suit : l'application

$$\Phi : u \in M_{\text{gen}} \longmapsto (2(\lambda_1 \nu_1)^2, \dots, 2(\lambda_N \nu_N)^2, 4\pi\lambda_1^2, \dots, 4\pi\lambda_N^2, 2\varphi_1, \dots, 2\varphi_N, \gamma_1, \dots, \gamma_N)$$

est un difféomorphisme, où $H_u(e_j) = \lambda_j e_j$ pour tout $j = 1, \dots, N$,

$$\nu_j = |\langle g | e_j \rangle|, \quad \varphi_j = \arg \langle g | e_j \rangle, \quad \gamma_j = \text{Re} \langle T e_j | e_j \rangle.$$

En conséquence, Gérard–Pushnitski [GP24] ont établi l'**existence** d'une famille **de solutions turbulentes** pour l'équation (Sz), plus large que celle trouvée par

8. Théorème de type Kronecker [Poc11b, Remarque 2.2]

9. Il est composé des fonctions u telles que les valeurs propres de H_u^2 sont simples $0 < \lambda_1^2 < \dots < \lambda_N^2$ et $\langle u | e_j \rangle \neq 0$ où $H_u(e_j) = \lambda_j e_j$ pour tout $j = 1, \dots, N$.

Pocovnicu [Poc11a] sur la droite. Précisément, ils ont démontré l'existence d'un sous-ensemble dense de solutions initiales rationnelles, avec des normes de Sobolev croissantes à l'infini lorsque $t \rightarrow \infty$. Cela contraste avec le résultat antérieur établi sur le cercle [GG15], où il a été montré que les solutions avec des données initiales rationnelles étaient quasi-périodiques. Ce dernier résultat découle directement de l'existence de la *formule explicite* pour l'équation de Szegő, qui est discutée dans la section suivante.

2.1.3 Formule explicite

Une route évidente, mais pas toujours facile à découvrir, pour établir qu'une EDP est complètement intégrable consiste à démontrer sa capacité à être intégrée ou résolue explicitement. Cela a été réalisé dans les cas de l'équation de Benjamin–Ono [Gér23a, KLV23b] et de l'équation de Szegő [GG15, GP23a], et a des conséquences remarquables, que ce soit dans le contexte de l'existence globale des solutions ou concernant les propriétés qualitatives du flot. De plus, cette résolubilité explicite est la clé pour découvrir davantage de facettes cachées derrière le problème de la limite à faible dispersion (c'est-à-dire la limite semiclassique) de l'EDP concernée.

2.1.3.1 Application à l'existence globale du flot (GWP)

La complète intégrabilité de l'équation de Szegő se manifeste, sur \mathbb{R} et sur \mathbb{T} , par la dérivation d'une formule explicite pour la solution $u(t)$ du problème de Cauchy à partir de données initiales u_0 [GG15, Poc11a, GP23a, GP23b]. Ces formules dépendent spécifiquement des opérateurs de Lax H_{u_0} et K_{u_0} définis dans (2.1.4) et dans (2.1.5) à $t = 0$. Sur \mathbb{T} , si l'on note S l'opérateur de décalage défini par $Sh(x) = e^{ix} h(x)$, alors pour tout $u_0 \in L_+^2$, où L_+^2 est l'espace de Hardy¹⁰ défini à (2.1.9), et S^* est l'adjoint de S dans $L_+^2(\mathbb{T})$, on a que la formule explicite pour (Sz)

$$u(t, z) = \left\langle (\text{Id} - z e^{-itH_{u_0}^2} e^{itK_{u_0}^2} S^*)^{-1} e^{-itH_{u_0}^2} u_0 \mid 1 \right\rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}, \quad z \in \mathbb{D};$$

tandis que sur \mathbb{R} , nous avons pour tout $z \in \mathbb{C}_+ = \{z \in \mathbb{C}; \text{Im}(z) > 0\}$,

$$u(t, z) = \frac{1}{2\pi i} I_+ \left[\left(X^* + \frac{1}{2\pi} \int_0^t \left\langle \cdot \mid e^{-isH_{u_0}^2} u_0 \right\rangle e^{-isH_{u_0}^2} u_0 ds - z \text{Id} \right)^{-1} e^{-itH_{u_0}^2} u_0 \right],$$

où $I_+(f) = \widehat{f}(0^+)$ et $X^* f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+)$.

En conséquence de ces formules explicites, [GP23a, GP23b] **ont étendu le flot de manière continue** de l'équation (Sz) défini sur des espaces de régularité élevée jusqu'à $L_+^2(\mathbb{T})$ et $L_+^2(\mathbb{R})$. De plus, ces formules ont fournis les résultats suivants :

10. Nous rappelons que toute fonction dans l'espace de Hardy L_+^2 défini en (2.1.9) peut être vue comme une fonction holomorphe sur \mathbb{D} lorsqu'on considère $L_+^2(\mathbb{T})$, ou sur \mathbb{C}_+ lorsqu'on considère $L_+^2(\mathbb{R})$ dont la trace sur $\partial\mathbb{D}$ respectivement sur \mathbb{R} est dans L^2 .

1. Sur le tore, [GG15] a prouvé que les trajectoires partant de données initiales rationnelles sont **quasi-périodiques**, c'est-à-dire qu'il existe une fonction lisse F définie sur \mathbb{T}^n et $\exists \omega_1, \dots, \omega_n \in \mathbb{R}$, telle que la solution $u(t) = F(\omega_1 t, \dots, \omega_n t)$, lorsque u_0 est une fonction rationnelle. En particulier, les trajectoires sont bornées dans les normes H^s .
2. Sur la droite réelle, [Poc11a] a établi que la résolution des solitons pour les fonctions rationnelles génériques⁹. Typiquement, de telles fonctions peuvent être exprimées comme une superposition de solitons, avec un reste allant vers 0 dans les normes H^s lorsque $t \rightarrow \pm\infty$.

2.1.3.2 Application à la limite semiclassique

Dans [Gér23a], Gérard a montré que la solution de l'équation de Benjamin-Ono (BO) peut être exprimée explicitement à travers une formule explicite. Ainsi, sur \mathbb{T} , la solution $u(t)$ commençant à $t = 0$ à partir de $u_0 \in H_r^{-\frac{1}{2}, \sqrt{\log}}(\mathbb{T})$, où $H_r^{-\frac{1}{2}, \sqrt{\log}}(\mathbb{T})$ est bien défini dans (2.1.8), et est écrite comme suit

$$u(t) = \Pi u(t) + \overline{\Pi u(t)} - \langle u_0 | 1 \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})},$$

où Π est le projecteur de Szegő (2.1.3), et

$$\Pi u(t, z) = \langle (\text{Id} - z e^{it(\text{Id} + 2L_{u_0})} S^*)^{-1} \Pi u_0 | 1 \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}, \quad z \in \mathbb{D}; \quad (2.1.10)$$

avec L_{u_0} est l'opérateur de Lax (2.1.2) à $t = 0$, S est l'opérateur de décalage et S^* est son adjoint dans $L_+^2(\mathbb{T})$. De plus, sur \mathbb{R} , chaque fois que $u_0 \in L_r^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, nous avons

$$u(t) = \Pi u(t) + \overline{\Pi u(t)},$$

où, pour tout $z \in \mathbb{C}_+ = \{z \in \mathbb{C}; \text{Im}(z) > 0\}$,

$$\Pi u(t, z) = \frac{1}{2\pi i} I_+ \left[(X^* - 2tL_{u_0} - z \text{Id})^{-1} \Pi u_0 \right],$$

avec $I_+(f) = \widehat{f}(0^+)$ et $X^* f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+)$. Récemment, Chen a supprimé l'exigence de $L^\infty(\mathbb{R})$ pour u_0 , étendant ainsi la formule explicite à tout $u_0 \in L_r^2(\mathbb{R})$ [Che24]. De plus, Killip–Laurens–Visan [KLV23b] ont fourni une généralisation de la formule explicite sur \mathbb{R} à toute la hiérarchie de (BO).

En conséquence, en utilisant ces formules explicites, des progrès ont été réalisés dans la compréhension de la **limite semiclassique de l'équation de Benjamin-Ono**. Plus précisément, en considérant le problème de Cauchy

$$\begin{cases} \partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon |D| \partial_x u^\varepsilon \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad (\text{BO-eps})$$

le but est de comprendre le comportement de la solution u^ε lorsque la dispersion tend à être faible, c'est-à-dire lorsque $\varepsilon \rightarrow 0$. Ce phénomène de négligence de la

composante de dispersion dans l'équation est communément reconnu dans la littérature comme la "limite à faible dispersion" ou "limite semi-classique". Il est à noter que pour $\varepsilon = 0$, l'équation (BO-eps) se réduit à l'équation de Burgers inviscide, réputée pour sa propension à exhiber des ondes de choc. Dans [MX11, MW16], Miller et ses collaborateurs étudient la limite semiclassique dans $L^2(\mathbb{R})$, montrant que pour certaines données initiales spécifiques, la solution u^ε converge faiblement vers une somme alternée exprimée en termes des branches de la solution multivaluée de l'équation de Burgers. Sur le tore $x \in \mathbb{T}$, en utilisant la transformée de Fourier non linéaire, Gassot [Gas23a] a démontré qu'étant donné une donnée initiale sous forme particulière¹¹ u_0 , il existe une famille u_0^ε approximant u_0 dans L^2 telle que la solution de (BO-eps) avec donnée initiale u_0^ε converge faiblement vers la même somme alternative trouvée dans [MX11]. Ensuite, en utilisant la formule explicite, elle étend le résultat de [Gas23b] en prouvant que pour toute donnée initiale en forme de cloche u_0 , la solution u^ε de (BO-eps) partant de u_0 converge faiblement vers la somme alternative. Plus récemment, [Gér23b] a généralisé le résultat de [MX11] sur \mathbb{R} à toute donnée initiale $u_0 \in L_r^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ en montrant que pour de telles données initiales générales, la solution de la limite faible semi-classique de (BO-eps) est donnée par la somme alternative des branches de la solution multivaluée de l'équation de Burgers. De plus, il souligne que cette somme ne constitue pas un semi-groupe d'évolution. Enfin, il en déduit que la solution faible de la limite semiclassique $\text{ZD}[u_0](t)$ de (BO-eps) satisfait le principe du maximum suivant $\|\text{ZD}[u_0](t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$, pour tout $t \in \mathbb{R}$, bien que l'équation de Benjmain–Ono est considéré avec faible dispersion sous cet angle d'étude. Récemment, [Che24] substitue la condition de $u_0 \in L_r^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ de [Gér23b] par $u_0 \in L_r^2(\mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R})$ avec $\lim_{x \rightarrow \infty} |u_0(x)|/|x| = 0$, et trouve une représentation intégrale de $\text{ZD}[u_0](t)$.

2.1.4 Ondes progressives et solitons

L'une des principales motivations et jalons historiques dans l'étude des EDP intégrables est la découverte des solitons. L'histoire commence en 1834 par l'observation de l'ingénieur civil et architecte naval Russell d'une onde solitaire sur The Union canal, posant ainsi les bases de l'exploration moderne des phénomènes d'intégrabilité non linéaires. En 1844, Russell a rapporté son observation à "l'Association britannique pour l'avancement de la science" [Rus45]. Quelques années plus tard, J. Boussinesq a développé une EDP non linéaire [Bou72] décrivant les vagues dans l'eau peu profonde et admettant des solutions solitoniques. Par la suite, il y a eu la redécouverte de l'équation de KdV par Korteweg et de Vries (KdV) en 1895 [KV95], qui décrit les longues vagues dans l'eau peu profonde et est désormais l'une des EDP intégrables les plus célèbres et les plus connues. Numériquement, la première observation de solitons pour l'équation (KdV) a été rapportée dans un domaine périodique en espace par Norman Zabusky et Martin Kruskal en 1965 [ZK65].

De nos jours, on sait que de nombreuses EDP non linéaires présentent ces types

11. Voir [Gas23b, Définition 1.1] pour la définition de donnée initiale en forme de cloche.

d'ondes [Che04, CB13, Pav09]. Ainsi, comment pouvons-nous décrire un soliton ? Et quels aspects les rendent-ils si intrigants pour les chercheurs dans diverses branches des sciences appliquées et théoriques ? Leur importance réside dans le fait qu'ils sont des **solutions explicites** pour les EDP non linéaires, fournissant ainsi des informations sur la dynamique de l'équation. De plus, ce sont des **ondes progressives** spéciales. Ces dernières sont représentées dans leur forme la plus simple comme

$$u(t, x) = U(x - ct), \quad (2.1.11)$$

où le profil U est une fonction d'une variable et c est une constante réelle non nulle. Lorsque $c > 0$, (respectivement $c < 0$) l'onde se propage dans la direction x vers la droite (resp. vers la gauche). Un exemple naturel d'une équation linéaire bénéficiant de ce type de solution est l'équation d'onde, également connue sous le nom d'équation de d'Alembert ¹² :

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \quad (2.1.12)$$

à partir duquel l'onde progressive tire son nom. L'une des propriétés les plus spectaculaires que l'on peut remarquer concernant (2.1.11), est que cette solution particulière se déplace dans le temps à la même vitesse $|c|$, transportant ainsi l'énergie d'un point à un autre, et reste dans la même forme définie par le profil U . Par conséquent, on ne peut pas s'attendre à ce qu'une EDP dispersive linéaire admette une solution d'onde progressive puisque la partie dispersive de l'équation tend à "disperser" les ondes planes ¹³ constituant la solution, et qui se propageront à des vitesses différentes selon leurs fréquences. Lorsque nous passons au contexte non linéaire, la situation change. En fait, la partie non linéaire de l'équation peut, dans certains cas, empêcher les effets de dispersion. Par conséquent, les EDP dispersive ou dissipative non linéaires peuvent présenter un comportement d'onde [SCM73, Tao09]. C'est le cas, par exemple, pour des EDP très connues comme l'équation de KdV, l'équation NLS, l'équation de Sine–Gordon, l'équation des ondes longues intermédiaires, l'équation de Benjamin–Ono, et d'autres... À différents égards, les ondes progressives peuvent être interprétées dans le contexte non linéaire comme suit : considérez, par exemple, une EDP non linéaire qui admet des solutions d'ondes progressives ; cela signifie que l'EDP agit sur certaines données initiales $u_0(x) = U(x)$ comme l'équation d'onde (2.1.12).

Contrairement aux ondes progressives, nous avons les *ondes stationnaires*. Ce type particulier de solution est exprimé sous la forme suivante

$$u(t, x) := U(x),$$

12. en hommage à Jean le Rond d'Alembert, qui a dérivé l'équation en 1747 comme solution au problème d'une corde vibrante.

13. Une onde plane est de la forme $Ce^{i(\xi \cdot x - \tau t)}$, où $C > 0$ est l'amplitude, $\xi \in \mathbb{R}$, $\tau \in \mathbb{R}$ dans le cadre non périodique ($x \in \mathbb{R}$) ; ou $Ce^{i(kx - \omega(k)t)}$, $k \in \mathbb{Z}$, dans le cadre périodique ($x \in \mathbb{T}$). Elles sont principalement utilisées dans la théorie des EDP linéaires pour exprimer, sous certaines hypothèses, une solution comme une superposition de ces simples ondes planes (pensez à la formule d'inversion de Fourier).

où le profil U est une fonction d'une variable. Contrairement à l'onde progressive, l'onde stationnaire semble être immobile et ne se déplace dans aucune direction. Historiquement, la première observation d'ondes stationnaires remonte à 1831 lorsque M. Faraday les a remarquées à la surface d'un liquide dans un récipient vibrant.¹⁴ Un exemple de situation¹⁵ qui pourrait créer des ondes stationnaires est lorsque deux ondes, se propageant dans des directions opposées et ayant des fréquences et des amplitudes identiques, entrent en collision.

Revenant aux ondes progressives, on peut sélectionner trois types spécifiques et importants parmi ces solutions :

- L'onde progressive périodique $u(t, x) = u(t, x + L)$, $L > 0$, $x \in \mathbb{R}$, se propage à travers un milieu et conserve sa forme périodique au fil du temps.
- Les ondes solitaires, qui sont des ondes progressives localisées. À la connaissance de l'auteur, il n'existe pas de définition mathématique universelle du mot localisé. En général, il est conseillé de considérer les limites asymptotiques du profil U lorsque $x \rightarrow \pm\infty$, –dans le contexte d'un domaine non borné– et de voir si la limite existe. Dans le cas où la limite

$$\lim_{x \rightarrow -\infty} U(x) = \lim_{x \rightarrow +\infty} U(x),$$

alors la trajectoire de la solution est appelée *homocline*, sinon, elle est appelée orbite *hétérocline* [MB21, AC91, Pav09]. Notez que selon certains auteurs, la définition de localisé est restreinte au fait que les limites asymptotiques du profil U s'annulent en $\pm\infty$ [BBR⁺04].

- Les solitons sont des ondes solitaires ayant la propriété supplémentaire : ils conservent leur forme et leur vitesse initiales même après avoir été en collision avec un autre soliton¹⁶. Autrement dit, ils ne perdent pas d'énergie et ne se dispersent pas. Ce phénomène remarquable est une conséquence de l'équilibre parfait, entre deux tendances différentes des ondes ; celles causées par la dispersion et celles résultant de la non-linéarité de l'EDP. Généralement, les solitons apparaissent dans des modèles intégrables, où l'existence d'un nombre conséquent de symétries est à l'origine de ce "comportement particulière" [Pav09]. Parfois, un nombre suffisant de lois de conservation peut conduire à l'existence de ces ondes stables [BBR⁺04].

En fait, pour dériver ces solutions spéciales, diverses méthodes ont été utilisées. Par exemple, pour certaines EDP, telles que l'équation de Korteweg-de Vries, les ondes progressives peuvent être obtenues – sous certaines hypothèses de décroissance sur la solution et ses dérivées – en substituant $u(t, x) = U(x - ct)$ dans l'EDP et en intégrant l'équation différentielle ordinaire correspondante [DJ89].

14. <https://www.surfertoday.com/surfing/what-is-a-standing-wave>.

15. Ce n'est pas la seule situation physique où des ondes stationnaires pourraient se produire.

16. En mathématiques, il est courant d'utiliser les termes "solitons" et "ondes solitaires" de manière interchangeable, sans toujours considérer si le soliton reste dans sa forme originale même après avoir interagi avec un autre soliton.

Dans [Ben67], Benjamin a prouvé l'existence d'une onde progressive périodique pour l'équation Benjamin-Ono (BO), avec un profil exprimé en termes de fonctions élémentaires [PN08, Sau19]

$$u_c(x) = \frac{4\pi}{L} \frac{2 \sinh \phi}{\cosh \phi - \cos(2\pi x/L)}, \quad (2.1.13)$$

où $\phi > 0$ tel que $\tanh \phi = \frac{2\pi}{cL}$, impliquant que la vitesse $c > 2\pi/L$. De plus, il a trouvé que le profil de l'onde solitaire décroissante algébriquement

$$u_{c_0}(x) = \frac{2c_0}{1 + c_0^2 x^2},$$

peut être obtenu en prenant une limite dans le paramètre de (2.1.13). Remarquablement, en utilisant le principe du maximum pour les équations elliptiques linéaires, des estimations sur une fonction de Green et l'équation de Cauchy-Riemann, Amick et Toland ont montré [AT91] que les ondes solitaires obtenues par Benjamin [Ben67] sont uniques.

Une autre façon courante de dériver les ondes progressives est par une approche variationnelle, où l'ensemble de ces ondes contient l'ensemble des minimiseurs contraints de la fonctionnelle d'énergie [BL83, NLP20]. En particulier, ces problèmes variationnels sont abordés via la Méthode de Concentration-Compacité développée par Lions [Lio84a, Lio84b], comme c'est le cas pour certaines équations de type NLS [GH12, Wu14] ou l'équation KP généralisée [DBS97]. En outre, des méthodes spectrales reposant sur la propriété spectrale de l'opérateur de Lax peuvent également être utilisées pour obtenir des solutions d'ondes progressives, comme pour l'équation de Szegő, où Pocovnicu [Poc11b] caractérise la solution de la forme $u(t, x) = e^{-i\omega t} u_0(x - ct)$ lorsque $x \in \mathbb{R}$, et prouve qu'elles sont de la forme

$$u(t, x) = \frac{C e^{-i\omega t}}{x - ct - p}, \quad C, p \in \mathbb{C}, \operatorname{Im}(p) < 0,$$

avec $c = c(C, p)$, $\omega = \omega(C, p) \in \mathbb{R}$. Dans les configurations périodiques, c'est-à-dire sur \mathbb{T} , [GG10] a trouvé que les ondes progressives partent des données initiales

$$u_0(x) = \frac{C e^{i\ell x}}{1 - p^N e^{iNx}}, \quad C \in \mathbb{C},$$

où $N \in \mathbb{N}_{\geq 0}$, $\ell \in \{0, \dots, N-1\}$, et $p \in \mathbb{D}$. De plus, les ondes stationnaires sont des fonctions intérieures (c'est-à-dire les fonctions L_+^2 ayant un module égal à 1 presque partout sur \mathbb{T}).

Un exemple d'équation des ondes complètement intégrable qui nous intéresse est l'équation de Schrödinger non linéaire dérivée du système de Calogero-Sutherland-Moser.

2.2 L'équation de Calogero-Sutherland-Moser DNLS

2.2.1 Le modèle

Nous considérons une équation du type Schrödinger non linéaire non locale, appelée l'*équation de Calogero-Sutherland-Moser DNLS*

$$i\partial_t u + \partial_x^2 u \pm 2u \cdot D\Pi(|u|^2) = 0, \quad x \in \mathbb{R}, x \in \mathbb{T}, \quad (\text{CSM})$$

où $D = -i\partial_x$, et Π désigne le **projecteur de Riesz-Szegő** à valeur dans l'ensemble des fonctions à fréquences positives, défini sur $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ comme

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) := \sum_{n \geq 0} \widehat{u}(n) e^{inx}, \quad (2.2.1)$$

et sur \mathbb{R} comme

$$\widehat{\Pi u}(\xi) = \mathbf{1}_{[0, +\infty)}(\xi) \widehat{u}(\xi).$$

L'opérateur Π est un projecteur orthogonal de L^2 sur l'**espace de Hardy** dénoté par

$$\begin{aligned} L_+^2(\mathbb{T}) &:= \{u \in L^2(\mathbb{T}) \mid \widehat{u}(n) = 0, \forall n \in \mathbb{Z}_{\leq -1}\} \\ &\cong \{u \in \text{Hol}(\mathbb{D}_+), \sup_{0 < r < 1} \int_0^{2\pi} |u(re^{i\theta})|^2 d\theta < +\infty\}, \end{aligned} \quad (2.2.2)$$

ou

$$\begin{aligned} L_+^2(\mathbb{R}) &:= \{u \in L^2(\mathbb{R}), \text{supp } \widehat{u} \subseteq [0, +\infty[\} \\ &\cong \{u \in \text{Hol}(\mathbb{C}_+), \sup_{y > 0} \int_{\mathbb{R}} |u(x + iy)|^2 dx < +\infty\}, \end{aligned} \quad (2.2.3)$$

où $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$ et $\mathbb{C}_+ := \{z \in \mathbb{C}; \text{Im}(z) > 0\}$.

Cette équation se décline en deux variantes : le type **focalisant**, caractérisé par le **signe “+”** précédant la non-linéarité, vise à concentrer les ondes localisées au fil du temps ; la seconde variante, caractérisée par un **signe “−”** devant la non-linéarité et appelée l'équation **défocalisante**, tend à étaler l'interaction entre les ondes. Il n'est pas surprenant que l'équation focalisante entraîne une dynamique plus complexe et plus riche ; et comme nous le verrons, l'équation de Calogero–Sutherland–Moser DNLS (CSM) ne fait pas exception à cette règle. Dans la suite, les symboles \pm et \mp seront interchangeés en fonction de la règle suivante : le signe supérieur correspondra au cas focalisant et le signe inférieur au cas défocalisant.

2.2.1.1 Contexte physique de (CSM).

En physique, l'équation de (CSM) focalisante apparaît comme une limite continue du système de Calogero–Sutherland–Moser introduit à la fin des années 1960–début des années 1970 [Cal69, Cal71, Sut71, Sut72, CM74, Mos76]. Ce dernier modèle correspond à un problème à N corps, décrivant les interactions par paires de N

particules identiques [Hal23]. Sur la droite réelle, cette interaction est donnée par un potentiel en inverse du carré

$$V = \sum_{j < k} \frac{1}{(x_j - x_k)^2},$$

tandis que sur \mathbb{T} , il s'agit d'un potentiel en inverse du sinus carré. Abanov, Bettelheim et Wiegmann montrent dans [ABW09] qu'en prenant la limite thermodynamique du modèle discret, et en appliquant un changement de variables, on obtient l'équation focalisante (CSM). D'autre part, l'équation défocalisante (CSM) a été obtenue comme limite formelle ($\delta \rightarrow \infty$) de l'équation de Schrödinger non linéaire intermédiaire introduite par Pelinovsky [Pel95],

$$i\partial_t u = \partial_x^2 u + (i - T)\partial_x(|u|^2)u, \quad (\text{INS})$$

où T est l'opérateur intégral

$$Tu(x) = \frac{1}{2\delta} \text{p.v.} \int_{-\infty}^{+\infty} \coth\left(\frac{\pi(x-y)}{2\delta}\right) u(y) dy.$$

La fonction complexe u dans (INS) représente l'enveloppe du fluide, et δ représente sa profondeur totale. En prenant la limite $\delta \rightarrow \infty$, on obtient la même équation que (INS) mais avec la transformation de Hilbert $\mathcal{H} = -i \operatorname{sgn}(D)$, $D = -i\partial_x$, au lieu de T . Par conséquent, l'équation (CSM) peut également être interprétée comme un modèle décrivant l'enveloppe du paquet d'ondes monochromatiques interfaciales dans un fluide stratifié en profondeur. Veuillez noter qu'en adoptant cette perspective, on remarque que l'équation (CSM) présente des similitudes avec l'équation de Benjamin-Ono (BO), qui décrit la propagation des ondes interfaciales dans des configurations similaires. Cependant, (CSM) se concentre davantage sur l'étude de l'enveloppe.

En physique, l'équation de Calogero-Sutherland-Moser DNLS (CSM) a suscité un intérêt considérable de la part des physiciens et des ingénieurs. En particulier, nous citons les travaux de Tutiya [Tut09], Berntson-Fagerlund [BF23], Stone-Anduaga-Xing [SAX08], Polychronakos [Pol95b, Pol95a] et Matsuno [Mat00, Mat01b, Mat01a, Mat02b, Mat02a, Mat03, Mat04, Mat23]...

Étant donné le contexte physique avec les investigations primaires de Calogero et Moser sur le modèle discret sur la droite réelle et le travail de Sutherland sur le tore, nous désignerons (CSM) comme l'équation de Calogero-Moser DNLS lorsque $x \in \mathbb{R}$, et comme l'équation de Calogero-Sutherland DNLS lorsque (CSM) est examinée sur le tore $x \in \mathbb{T}$. Une autre désignation pour cette équation, apparaissant dans les travaux récents [KLV23a, HK24] comme le "Modèles continus de Calogero-Moser" dans le cas où $x \in \mathbb{R}$.

2.2.1.2 Le caractère localement bien posée de (CSM) sur \mathbb{R} et \mathbb{T} . (LWP)

À la connaissance de l’auteur, le premier résultat de LWP pour l’équation (CSM) remonte à De Moura [dM07] qui a traité la caractère de LWP d’une famille d’équations de Schrödinger non linéaires non locales [PG96], incluant l’équation de Calogero–Moser DNLS (CSM), dans $H^s(\mathbb{R})$ avec $s \geq 1$, pour des petites données initiales. Son idée repose sur l’utilisation d’un argument de point fixe. De plus, il étend son résultat en montrant que l’équation est globalement bien posée, par le biais de la transformation de jauge. Ensuite, Barros–DeMoura–Santos présentent dans [BdMS19] le caractère de LWP de (CSM) pour des petites données initiales dans l’espace de Besov $B_2^{1/2,1}(\mathbb{R})$. Dans le contexte des espaces Hardy–Sobolev

$$H_+^s := H^s \cap L_+^2, \quad s \geq 0,$$

où H^s désigne l’espace de Sobolev et L_+^2 est l’espace de Hardy introduit dans (2.2.2) ou (2.2.3), équipé de la norme de Sobolev

$$\|u\|_{H^s} = \|\langle D \rangle^s u\|_{L^2}, \quad \langle D \rangle^s = (1 + |D|^2)^{s/2},$$

le caractère de LWP de (CSM) a été établie dans H_+^s sur \mathbb{R} et \mathbb{T} pour $s > \frac{3}{2}$, en utilisant des schémas itératifs à la Kato et des estimations d’énergie [GL24, Proposition 2.1]. Une preuve plus détaillée de leur proposition sera présentée dans le chapitre 3. Dans le même article, ils étendent caractère de LWP de l’équation de DNLS de Calogero–Moser à $H_+^s(\mathbb{R})$ avec $s > \frac{1}{2}$ en suivant les arguments de [dMP10].

Dans ce qui suit, nous nous intéressons à l’équation (CSM) où u est une fonction de valeur complexe, appartenant à l’espace de Hardy–Sobolev H_+^s . De plus, nous désignons par

$$\mathcal{S}^\pm(t) : u_0 \longmapsto u(t)$$

le flot de (CSM).

2.2.1.3 Intégrabilité de (CSM) : Existence d’une paire de Lax.

Une caractéristique importante de l’équation de Calogero–Sutherland–Moser DNLS (CSM) est son *intégrabilité*. Ce n’est pas surprenant, car elle émerge en tant que limite continue du modèle discret de Calogero–Sutherland–Moser, ou de l’équation de Schrödinger non linéaire intermédiaire, toutes deux connues pour être intégrables [Mos75, Pel95, PG95]. Dans [GL24, Bad24a], les auteurs ont dérivé sur \mathbb{R} et sur \mathbb{T} une structure de paire de Lax pour la limite continue (CSM), avec un opérateur de Lax L_u rappelant celui de l’équation de Benjamin–Ono (2.1.2). Ainsi, si nous désignons par T_v l’opérateur de Toeplitz de symbole $v \in L^\infty$, défini comme

$$T_v f = \Pi(vf), \quad \forall f \in L_+^2, \quad (2.2.4)$$

où Π est le projecteur de Szegő (2.2.1), alors les opérateurs de Lax de (CSM) sont donnés par le théorème suivant.

Theorem 2.2.1 (Structure de la paire de Lax de (CSM) [GL24, Bad24a]). *Pour tout $s > \frac{3}{2}$, soit $u \in \mathcal{C}([-T, T], H_+^s)$ une solution de l'équation (CSM). Il existe deux opérateurs*

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2,$$

dans le cas focalisant, et

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2,$$

dans le cas défocalisant, satisfaisant l'équation de Lax (2.1.1), où T_u est l'opérateur de Toeplitz défini dans (2.2.4).

Indépendamment du fait que x appartienne à \mathbb{R} ou à \mathbb{T} , les opérateurs de Lax L_u et \tilde{L}_u sont des opérateurs auto-adjoints non bornés du domaine H_+^1 [GL24, Bad24a, Proposition 2.1], tandis que B_u et \tilde{B}_u sont des opérateurs anti-symétriques bornés. Cependant, une différence notable entre les cas où x est dans \mathbb{R} et \mathbb{T} , réside dans la nature du spectre¹⁷ de L_u et \tilde{L}_u . Sur \mathbb{T} , les opérateurs auto-adjoints semi-bornés L_u et \tilde{L}_u ont des résolvantes compactes [Bad24a, Proposition 2.3], conduisant à des spectres constitués de suites de valeurs propres bornées par le bas et allant vers $+\infty$,

$$\begin{aligned} \sigma(L_u) &:= \{\nu_0(u) \leq \dots \leq \nu_n(u) \leq \dots\}, & \nu_0(u) &\geq -\|u\|_{L^\infty}^2, \\ \sigma(\tilde{L}_u) &:= \{\lambda_0(u) \leq \dots \leq \lambda_n(u) \leq \dots\}, & \lambda_0(u) &\geq 0. \end{aligned}$$

En revanche, sur \mathbb{R} , le spectre de L_u est plus compliqué [GL24, KLV23a], comprenant un spectre essentiel $\sigma_{\text{ess}}(L_u) = [0, +\infty)$ et un nombre fini N de valeurs propres simples, contrôlées via l'inégalité optimale [GL24]

$$N \leq \frac{\|u\|_{L^2}^2}{2\pi}.$$

Comme nous le verrons dans la Section 2.2.2.1, cette structure de paire de Lax servira de fondement pour construire des lois de conservation pour (CSM). Ces lois nous permettront de contrôler la croissance des normes de Sobolev, établissant ainsi le caractère de globalement bien posé de (CSM) dans les espaces H_+^s .

2.2.2 Formule explicite pour (CSM)

En plus de la présence des opérateurs de Lax pour l'équation de Calogero–Sutherland–Moser DNLS, l'intégrabilité complète de (CSM) est démontrée en plus par un outil plus puissant : l'existence d'une formule explicite de la solution $u(t)$ de (CSM) à partir d'une donnée initiale $u_0 \in H_+^\infty$.

17. Nous rappelons que nous nous intéressons particulièrement au spectre des opérateurs de Lax car il reste invariant sous le flot. Nous renvoyons à la Section 2.1.1 pour cette propriété.

Commençons par le **cas sur \mathbb{T}** . Pour cela, rappelons l'un des opérateurs les plus importants dans l'espace de Hardy, *l'opérateur de translation*, défini sur $L_+^2(\mathbb{T})$ comme l'application isométrique

$$S : h \in L_+^2(\mathbb{T}) \longmapsto e^{ix}h \in L_+^2(\mathbb{T}), \quad (2.2.5)$$

et son adjoint dans $L_+^2(\mathbb{T})$

$$S^* : h \in L_+^2(\mathbb{T}) \longmapsto S^*h = T_{e^{-ix}}h = \Pi(e^{-ix}h) \in L_+^2(\mathbb{T}).$$

En utilisant les opérateurs de Lax, on peut reformuler les équations de (CSM) de la manière suivante [Bad24a]

$$\partial_t u = B_u u - iL_u^2 u, \quad \text{et} \quad \partial_t u = \tilde{B}_u u - i\tilde{L}_u^2 u.$$

Et en utilisant les identités du commutateur

$$\begin{aligned} [S^*, L_u] &= S^* - \langle \cdot | u \rangle S^* u, \\ [S^*, B_u] &= i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right), \end{aligned}$$

– et leurs analogues dans le cas défocalisant –, et en rappelant que toute fonction dans l'espace de Hardy $L_+^2(\mathbb{T})$ peut être vue comme une *fonction holomorphe* sur \mathbb{D}_+ , dont la trace sur le tore est dans $L^2(\mathbb{T})$, on obtient le théorème suivant.

Theorem 2.2.2 ([Bad24a]). *Étant donné $u_0 \in H_+^\infty(\mathbb{T})$, la solution de l'équation focalisante de Calogero–Sutherland DNLS (CSM) est donnée par*

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (2.2.6)$$

De plus, la solution de l'équation défocalisante est obtenue en remplaçant L_{u_0} par \tilde{L}_{u_0} dans (2.2.6).

Sur la **droite réelle** $x \in \mathbb{R}$, l'opérateur de translation (2.2.5) doit être remplacé par le semi-groupe de contraction

$$S(\eta)h(x) = \Pi(e^{ix\eta} h(x)), \quad \eta > 0.$$

Et si nous désignons par X son générateur infinitésimal

$$Xh(x) = -i \frac{d}{d\eta} \Big|_{\eta=0} (S(\eta)h(x)) = xh(x),$$

de domaine

$$\begin{aligned} \text{Dom}(X) &= \{h \in L_+^2(\mathbb{R}); xh \in L^2(\mathbb{R})\} \\ &= \{h \in L_+^2(\mathbb{R}); \hat{h} \in H^1([0, +\infty)), \hat{h}(0) = 0\}, \end{aligned}$$

et X^* son adjoint de domaine

$$\begin{aligned} \text{Dom}(X^*) &= \{f \in L_+^2(\mathbb{R}) ; \exists c > 0, \forall h \in \text{Dom}(X), |\langle f | Xh \rangle| \leq c \|h\|_{L^2}\} \\ &= \{f \in L_+^2(\mathbb{R}) ; \widehat{f}|_{(0,+\infty)} \in H^1((0,+\infty))\}, \end{aligned}$$

défini pour tout $\xi > 0$ comme

$$\widehat{X^*f}(\xi) = i\partial_\xi \widehat{f}(\xi),$$

ou, pour tout $f \in \text{Dom}(X^*)$, comme

$$X^*f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+), \quad (2.2.7)$$

nous avons la formule explicite suivante pour (CSM) lorsque $x \in \mathbb{R}$.

Theorem 2.2.3 ([KLV23a]). *Étant donné $u_0 \in L_+^2(\mathbb{R})$, la solution de l'équation de Calogero-Moser DNLS focalisante est donnée, pour tout $z \in \mathbb{C}_+$ par*

$$u(t, z) = \frac{1}{2\pi i} I_+((X^* + 2tL_{u_0} - z)^{-1}u_0), \quad (2.2.8)$$

où I_+ désigne

$$I_+(f) := \widehat{f}(0^+), \quad \forall f \in \text{Dom}(X^*).$$

De plus, la solution de l'équation de défocalisant est obtenue en remplaçant L_{u_0} par \tilde{L}_{u_0} dans (2.2.8).

Remark 2.2.1. En définissant l'opérateur de Lax L_{u_0} pour $u_0 \in L_+^2$, qui peut être considéré comme une perturbation bornée d'une forme infinitésimale de $L_0 \equiv D$ [GL24, Appendice A], ou interprété comme une perturbation relativement compacte de $L_0 \equiv D$ [KLV23a]; et en utilisant des méthodes d'approximation, les formules explicites (2.2.6) et (2.2.8) de (CSM) restent valables pour tout $u_0 \in L_+^2$.

Maintenant, armés de la structure de paire de Lax et des deux formules explicites (2.2.6) et (2.2.8) de (CSM), nous présentons, dans ce qui suit, deux applications :

1. **Le caractère de globalement bien-posé** de (CSM) dans les espaces de Hardy-Sobolev H_+^s pour $s > \frac{3}{2}$ [GL24, Bad24a]. Et l'**extension du flot** jusqu'à la régularité critique de cette équation, à savoir à L_+^2 [Bad24a, KLV23a].
2. **Limite semiclassique de (CSM)** Nous nous intéressons à la limite semiclassique (c'est-à-dire, la limite à faible dispersion) de l'équation (CSM)

$$\begin{cases} i\partial_t + \varepsilon \partial_x^2 u^\varepsilon \pm 2u^\varepsilon \cdot D\Pi(|u^\varepsilon|^2) = 0 \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad \varepsilon \rightarrow 0. \quad (\text{CSM-eps})$$

Nous montrons, que pour $x \in \mathbb{R}$ et $x \in \mathbb{T}$, l'existence d'une limite faible de la solution pour (CSM-eps) lorsque $\varepsilon \rightarrow 0$. De plus, nous caractérisons, lorsque $x \in \mathbb{R}$, la solution limite en termes des branches de la solution multivaluée de l'équation de Burgers [Bad24b].

2.2.2.1 Existence globale du flot (GWP) sur \mathbb{T} et sur \mathbb{R}

Comme annoncé précédemment, en exploitant la structure incroyable des opérateurs de Lax, on construit, pour tout $u \in \mathcal{C}_t H_+^{\max\{s,r\}}$, $s > \frac{3}{2}$, une hiérarchie infinie de lois de conservation pour (CSM),

$$\left(\langle (L_u + \lambda)^r u \mid u \rangle \right)_{r \geq 0} \quad \text{et} \quad \left(\langle (\tilde{L}_u + \lambda)^r u \mid u \rangle \right)_{r \geq 0}, \quad (2.2.9)$$

où $\lambda \gg 0$ ¹⁸. Ces lois de conservation contrôlent efficacement la croissance de toutes les normes de Sobolev. Par conséquent, nous obtenons les résultats suivants.

Theorem 2.2.4 (GWP sur \mathbb{R} [GL24]). *Pour tout entier $n \in \mathbb{N}_{\geq 1}$, l'équation Calogero–Moser DNLS (CSM) est globalement bien-posée*

- dans $H_+^n(\mathbb{R})$ pour l'équation défocalisante.
- dans $H_+^n(\mathbb{R}) \cap \mathcal{B}_{L_+^2}(\sqrt{2\pi})$ pour l'équation focalisante, où $\mathcal{B}_{L_+^2}(\sqrt{2\pi})$ représente la boule ouverte de rayon $\sqrt{2\pi}$ dans $L_+^2(\mathbb{R})$.

De plus, l'inégalité suivante est vérifiée,

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^n(\mathbb{R})} < +\infty.$$

Theorem 2.2.5 (GWP sur \mathbb{T} [Bad24a]). *Pour tout $s > \frac{3}{2}$, l'équation Calogero–Sutherland DNLS (CSM) est globalement bien-posée*

- dans $H_+^s(\mathbb{T})$ pour l'équation défocalisante.
- dans $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ pour l'équation focalisante (avec signe +), où $\mathcal{B}_{L_+^2}(1)$ représente la boule unitaire ouverte dans $L_+^2(\mathbb{T})$.

De plus, l'inégalité suivante est vérifiée,

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s(\mathbb{T})} < +\infty.$$

Remark 2.2.2.

- (1) L'émergence de la masse sous-critique L^2 observée dans le cas focalisant découle du contrôle de la croissance des normes de Sobolev $\|u\|_{H^s}$ par les lois de conservation lors de l'utilisation d'une inégalité optimale saturée par les ondes stationnaires pour (CSM). En changeant le signe devant la non-linéarité de (CSM), l'opérateur de Lax pour l'équation défocalisante subit un changement de signe et devient un opérateur positif, permettant aux lois de conservation de contrôler les $\|u(t)\|_{H^s}$ sans nécessiter de condition sur les données initiales dans ce cas.
- (2) La distinction entre la condition $\|u_0\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi}$ sur \mathbb{R} , et $\|u_0\|_{L^2(\mathbb{T})} \leq 1$ sur \mathbb{T} provient de la différence de normalisation choisie par les auteurs. En effet,

$$\langle u \mid v \rangle_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} = \int_{\mathbb{R}} u \bar{v} \, dx, \quad \langle u \mid v \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} = \int_0^{2\pi} u \bar{v} \frac{dx}{2\pi}.$$

18. λ est uniforme par rapport à t car le spectre de L_u et \tilde{L}_u est conservé par le flot.

Sur \mathbb{R} , l'investigation a été plus approfondie. À partir de [GL24, Section 3], nous avons la non-existence d'un phénomène de “Blow-up” de la masse dans $H_+^1(\mathbb{R})$. De plus, par le travail de [HK24] nous savons que les lois de conservation (2.2.9) de (CSM) échouent à contrôler la croissance des normes de Sobolev de $u(t)$ pour des solutions avec une masse supérieure à 2π . Plus précisément, Hogan et Kowalski [HK24] ont montré ce qui suit.

Theorem 2.2.6 (Comportement turbulent [HK24]). *Pour tout $\varepsilon > 0$ suffisamment petit, il existe des données initiales $u_0 \in H_+^\infty(\mathbb{R})$ avec $\|u_0\|_{L^2}^2 = 2\pi + \varepsilon$, un temps $T \in (0, \infty]$, et une solution maximale $u(t)$ de l'équation de Calogero–Moser DNLS (CSM) telle que pour tout $s > 0$,*

$$\lim_{t \nearrow T} \|u(t)\|_{H^s} = +\infty.$$

En particulier, si $T = \infty$, alors nous avons les bornes

$$\|u(t)\|_{H^s} \gtrsim t^s.$$

Ce résultat est conforme à la dynamique des multisolitons de (CSM), où [GL24] a établi que, pour $N \geq 2$, les solutions N -solitons, qui en fait sont de masse $2\pi N$, présentent une croissance des normes de Sobolev caractérisée par $\|u(t)\|_{H^s} \sim |t|^{2s}$ pour $|t|$ grand. Pour plus d'informations sur les ondes progressives, les solitons et les multisolitons concernant (CSM), nous renvoyons à la Section 2.2.3.

Plus récemment, Kim-Kim-Kwon [KKK24] souligne en utilisant des méthodes d'analyse de modulation qu'il existe des solutions quand $x \in \mathbb{R}$ qui **explose en temps fini**. De plus, ils ont décrit le phénomène de l'explosion avec un temps d'explosion de l'ordre de $\frac{1}{\sqrt{(t-T)^2}}$.

Extension du flot de (CSM). En vue des théorèmes 2.2.4 et 2.2.5, nous en déduisons que le flot

$$\mathcal{S}^\pm(t) : u_0 \in H_+^s \longmapsto u(t) \in H_+^s, \quad s > \frac{3}{2},$$

de (CSM) est globalement bien défini sur H_+^s pour $s > \frac{3}{2}$, (avec de petites données initiales dans le cas focalisant). La prochaine étape est d'étendre ce flot de manière continue jusqu'à la régularité critique de mise à l'échelle $L_+^2(\mathbb{T})$. En effet, on remarque que (CSM) est invariante par la mise à l'échelle

$$u(t, x) \longmapsto \lambda^{1/2} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Maintenant, pour étendre le flot de H_+^s , $s > \frac{3}{2}$, jusqu'à L_+^2 , nous utilisons une méthode d'approximation comme détaillé dans le théorème suivant. De plus, notons que pour $u \in L_+^2$, l'équation est toujours définie au sens de la *distribution* puisque le produit de deux fonctions à fréquences positives est bien défini et continu.

Theorem 2.2.7 (Extension du flot de (CSM) à $L_+^2(\mathbb{T})$ [Bad24a]). *Soit $u_0 \in L_+^2(\mathbb{T})$, ($\|u_0\|_{L^2(\mathbb{T})} < 1$ dans le cas focalisant). Il existe un unique $u \in \mathcal{C}(\mathbb{R}; L_+^2(\mathbb{T}))$ tel que pour toute suite $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ où $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, la convergence suivante est vérifiée : pour tout $T > 0$,*

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (2.2.10)$$

De plus, la norme L^2 de la limite u est conservée

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \text{pour tout } t \in \mathbb{R}.$$

Par conséquent, le théorème 2.2.7 fournit le résultat de GWP du problème (CSM) dans $L_+^2(\mathbb{T})$ de la manière suivante : il existe une extension continue unique du flot défini sur $H_+^2(\mathbb{T})$, à $L_+^2(\mathbb{T})$, générant une application continue unique

$$u_0 \in \mathcal{B}_{L_+^2}(1) \mapsto u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T})).$$

dans le cas de focalisant, et

$$u_0 \in L_+^2(\mathbb{T}) \mapsto u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T})),$$

dans le cas de défocalisant.

La preuve repose sur trois étapes : l'existence, l'unicité et la forte convergence dans L^2 .

- L'existence est la partie la plus facile, et peut être réalisée en appliquant le théorème de Banach à la suite $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, satisfaisant pour tout $\varepsilon > 0$,

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \lesssim \|u_0\|_{L^2}.$$

- L'unicité peut être traitée en utilisant la formule explicite du Théorème 2.2.2, qui met en évidence que la suite $(u^\varepsilon(t))$ converge vers une limite unique $u(t)$. Et cette dernière limite est indépendante de la sélection spécifique de la suite (u_0^ε) qui approxime u_0 dans $L_+^2(\mathbb{T})$.
- Enfin, pour montrer la forte convergence dans L^2 , et la conservation de la norme L^2 de la solution limite u , nous nous appuyons sur la propriété intégrable de l'équation (CSM). Nous construisons une base orthonormale (f_n^t) de $L_+^2(\mathbb{T})$ constituée des fonctions propres de l'opérateur Lax $L_{u(t)}$, où la solution $u(t)$ évolue comme

$$|\langle u(t) | f_n^t \rangle| = |\langle u_0 | f_n^0 \rangle|, \quad \forall t \in \mathbb{R}. \quad (2.2.11)$$

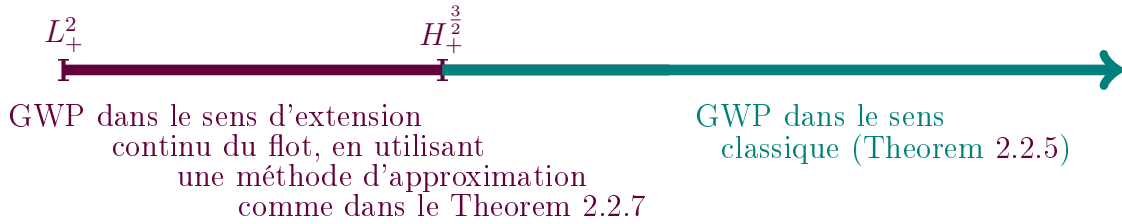
□

En conséquence du Théorème 2.2.7, en combinant l'analyse fonctionnelle et les techniques intégrables, nous déduisons ce qui suit.

Corollary 2.2.8 ([Bad24a]). *Pour tout $s \geq 0$, l'équation de Calogero–Sutherland DNLS (CSM) est globalement bien posée dans $H_+^s(\mathbb{T})$ (respectivement $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ dans le cas de la focalisant). De plus, la borne a priori suivante est vérifiée,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} < \infty.$$

*En outre, l'orbite $\{u(t), t \in \mathbb{R}\}$ est **relativement compacte** dans $H_+^s(\mathbb{T})$ (respectivement $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ dans le cas focalisant).*



En passant à la **droite réelle**, on peut tenter d'employer une approche similaire à celle de la preuve du Théorème 2.2.7 pour \mathbb{T} , en utilisant une méthode d'approximation et la formule explicite (2.2.8) pour (CSM) sur \mathbb{R} , afin d'étendre le flot jusqu'à $L_+^2(\mathbb{R})$ [KLV23a]. Cependant, il devient évident que tandis que le problème de l'existence et de l'unicité de la solution limite u peut être traité de manière similaire au cas du tore, aboutissant à une convergence faible dans (2.2.10); mais établir la convergence forte en suivant les idées sur le tore s'avère impossible. Pour rappel, un aspect clé de la preuve du Théorème 2.2.7 est de construire une base orthonormale comprenant les fonctions propres de $L_{u(t)}$ de telle sorte que (2.2.11) soit satisfaite. Cependant, sur \mathbb{R} , cette approche échoue puisque le spectre de L_u sur \mathbb{R} , n'est pas uniquement composé de valeurs propres comme sur \mathbb{T} (comme on l'a vu dans la Section 2.2.1.3). Par conséquent, Killip–Laurens–Vişan [KLV23a] ont utilisé la propriété d'équicontinuité (voir Définition 2.2.9) pour empêcher la perte en masse lors du passage à la limite et ainsi assurer une convergence forte dans $L^2(\mathbb{R})$.

Definition 2.2.9 (Équicontinuité en L^2). Soit $s \in \mathbb{R}$. Un ensemble borné $U \subset H_+^s(\mathbb{R})$ est dit être équicontinu dans la topologie de $H^s(\mathbb{R})$ si

$$\limsup_{\delta \rightarrow 0} \sup_{u \in U} \sup_{|y| < \delta} \|u(\cdot + y) - u(\cdot)\|_{H^s} = 0,$$

ou de manière équivalente,

$$\limsup_{\kappa \rightarrow \infty} \sup_{u \in U} \int_{|\xi| \geq \kappa} (1 + |\xi|)^{2s} |\widehat{u}(\xi)|^2 d\xi = 0.$$

Theorem 2.2.10 (Extension du flot de (CSM) à $H_+^s(\mathbb{R})$, $0 \leq s < 1$ [KLV23a]). *Soit $u_0 \in L_+^2(\mathbb{R})$ (tel que $\|u_0\|_{L^2} < \sqrt{2\pi}$ dans le cas focalisant). Il existe une solution*

globale unique $u \in \mathcal{C}_t L_+^2(\mathbb{R})$ de l'équation de Calogero–Moser DNLS, telle que pour toute $(u_n^0) \subseteq H_+^\infty(\mathbb{R})$, $(xu_n^0) \subseteq L^2$, $u_n^0 \rightarrow u_0$ dans L^2 , nous avons pour tout $T > 0$,

$$u_n \longrightarrow u \quad \text{dans } \mathcal{C}_t L_+^2([-T, T], \mathbb{R}).$$

De plus, pour tout $0 < s < 1$, l'équation (CSM) est globalement bien définie dans $H_+^s(\mathbb{R})$ (avec $\|u_0\|_{L^2} < \sqrt{2\pi}$ dans le cas focalisant) au sens de l'extension du flot.

Problèmes ouverts et perspectives.

1. À la lumière des résultats présentés par [HK24, KKK24] dans le cas de la droite réelle (cf. Théorème 2.2.6), une question naturelle se pose : un résultat similaire existe-t-il pour le tore ? Plus précisément, pour établir leur résultat sur la droite réelle, Hogan et Kowalski [HK24] démontrent la stabilité orbitale pour une famille de solitons de (CSM), qui sont connus sur \mathbb{R} pour avoir une masse de 2π [GL24]. Cependant, sur \mathbb{T} , [Bad23] a établi l'existence d'une famille plus large d'ondes progressives¹⁹ que celles sur la droite réelle, et a démontré la présence d'ondes progressives avec une norme L^2 arbitrairement petite pour cette équation critique en L^2 , avec une masse inférieure à la masse critique 1. Par conséquent, on pourrait se demander s'il existe un analogue de leurs résultats pour \mathbb{T} , ou si le résultat de GWP est valable dans le cas focalisant lorsque $\|u_0\|_{L^2(\mathbb{T})} \geq 1$.
2. Une autre question naturelle, étant donné l'invariance par changement d'échelle de l'équation, est de démontrer que l'équation (CSM) est mal-posée dans H_+^s avec $s \leq 0$.
3. Étant donné l'existence du flot de (CSM) dans H_+^s , on pourrait explorer les propriétés qualitatives intrigantes concernant la dynamique de (CSM). Nous savons déjà que l'orbite de (CSM) est relativement compacte dans $H_+^s(\mathbb{T})$, $s \geq 0$ [Bad24a]. Cependant, des questions concernant la quasi-périodicité, la presque-périodicité, et d'autres restent sans réponse. De plus, les mêmes résultats s'appliquent-ils dans \mathbb{R} ? Ou bien l'équation (CSM) présente-t-elle plus de turbulence lorsque $x \in \mathbb{R}$?
4. La construction de la transformée de Fourier non linéaire²⁰ : Nous nous attendons à ce que les coordonnées $(\langle u(t) | f_n^t \rangle)_{n \in \mathbb{N}_{\geq 0}}$ introduites dans (2.2.11) soient les coordonnées de Birkhoff associées à l'équation de Calogero–Sutherland DNLS (CSM). À cette fin, nous devons construire une bijection entre $u \longleftrightarrow (\langle u(t) | f_n^t \rangle)$. Cette construction peut débloquent plusieurs résultats significatifs concernant la dynamique de l'équation.
5. Enfin, mentionnons que Sun [Sun23], inspiré par l'équation (CSM), a introduit un système d'équations de Schrödinger : “the intertwined derivative Schrödinger system of the Calogero–Moser–Sutherland type”. Ce système se compose

19. Nous renvoyons à la Section 2.2.3 pour la caractérisation des ondes progressives pour (CSM).

20. Pour la définition des coordonnées de Birkhoff et de la transformée de Fourier non linéaire, nous renvoyons à la Section 2.1.2.

de deux équations avec une paire de solutions (U, V) de valeur matricielle dérivée de l'équation (CSM). Pour ce système, il a dérivé la formule explicite de ses solutions. Cependant, beaucoup reste à comprendre sur ce système.

2.2.2.2 La limite semiclassique

Une autre application des formules explicites (2.2.6) et (2.2.8) de (CSM) consiste à examiner la limite à faible dispersion ou la *limite semiclassique* de l'équation de Calogero–Sutherland–Moser DNLS (CSM). Cela implique de prendre $\varepsilon \rightarrow 0$ dans la version “rescalée” suivante de (CSM),

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon \partial_x^2 u^\varepsilon \pm 2u^\varepsilon D\Pi(|u^\varepsilon|^2) = 0 \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad (\text{CSM-eps})$$

où les données initiales u_0 sont indépendantes de ε . Ainsi, que se passe-t-il avec la solution de (CSM-eps) lorsque nous négligeons le terme de dispersion de l'équation? De plus, la solution u^ε converge-t-elle, et si oui, en quel sens? Dans ce qui suit, nous fournissons une réponse partielle à cette question en établissant l'existence d'une limite faible de u^ε dans L^2 . De plus, nous désignons cette limite par la "limite semiclassique de (CSM)", notée $ZD_\pm[u_0]$, où $ZD_+[u_0]$ représente la solution limite faible semiclassique pour (CSM) focalisante, et $ZD_-[u_0]$ correspond à celle dans le cas défocalisant.

Theorem 2.2.11 ([Bad24b]). *Étant donné des données initiales $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (avec $\|u_0\|_{L^2} < \sqrt{2\pi}$ dans le cas focalisant), la limite semiclassique faible (dans l'espace L^2) de (CSM) existe, et est caractérisée par la formule explicite suivante*

$$ZD_\pm[u_0](t, z) = \left(\text{Id} \mp 2tT_{u_0}T_{\bar{u}_0}(X^* - z)^{-1} \right)^{-1} u_0(z), \quad t \in \mathbb{R}, z \in \mathbb{C}_+, \quad (2.2.12)$$

où les opérateurs T_v et X^* sont définis respectivement en (2.2.4) et (2.2.7). De plus, nous avons

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2}.$$

En outre, si $u_0^n \rightarrow u_0$ fortement dans L^2 lorsque $n \rightarrow \infty$, avec $\sup_n \|u_0^n\|_{L^\infty} < +\infty$, alors pour tout $T > 0$,

$$\sup_{t \in [-T, T]} |ZD_\pm[u_0^n](t) - ZD_\pm[u_0](t)| \xrightarrow{n \rightarrow \infty} 0 \text{ dans } L^2(\mathbb{R}).$$

Remark 2.2.3. Un résultat analogue peut être établi dans le cas où $x \in \mathbb{T}$, où l'on obtient que pour $u_0 \in L^2_+(\mathbb{T}) \cap L^\infty(\mathbb{T})$, la limite semiclassique de (CSM) est caractérisée par

$$ZD_\pm[u_0](t, z) = \langle (\text{Id} - z e^{-it} e^{\pm 2itT_{u_0}T_{\bar{u}_0}} S^*)^{-1} u_0 | 1 \rangle, \quad z \in \mathbb{D}.$$

Généralement, lorsqu'on considère le scénario de la limite semiclassique, l'émergence de chocs peut être observée. Ces chocs se manifestent lorsque nous commençons à négliger les effets dispersifs, permettant au terme non linéaire de dominer. Avec l'existence de la limite faible semiclassique établie dans (2.2.12), notre objectif dans le théorème suivant est de mettre en évidence ces chocs, en abordant l'existence d'une connexion entre cette limite faible semiclassique et les branches de la solution multivaluée de l'équation de Burgers-Hopf inviscide, connue pour sa tendance à présenter des formations de chocs. Cependant, une question naturelle se pose, pourquoi la limite semiclassique de (CSM) est-elle liée à l'équation de Burgers-Hopf inviscide? À cette fin, observez que lorsque l'on prend formellement $\varepsilon \rightarrow 0$, l'équation (CSM-eps) devient

$$i\partial_t u \pm 2u \cdot D\Pi(|u|^2) = 0. \quad (\text{CM-zero})$$

Par conséquent, si u résout l'équation précédente, alors $\mathbf{v} = |u|^2$ résout l'équation de Burgers

$$\partial_t \mathbf{v} = \pm 2\mathbf{v} \partial_x \mathbf{v} \quad (2.2.13)$$

comme

$$\begin{aligned} \partial_t \mathbf{v} &= 2 \operatorname{Re}(\partial_t u \bar{u}) \\ &= \pm 4 \operatorname{Re}(\partial_x \Pi(|u|^2) |u|^2) \\ &= \pm 2 (\partial_x \Pi(|u|^2) + \overline{\partial_x \Pi(|u|^2)}) |u|^2 \\ &= \pm \partial_x |u|^4 = \pm \partial_x \mathbf{v}^2 \\ &= \pm 2\mathbf{v} \partial_x \mathbf{v}. \end{aligned}$$

Ainsi, armé de la formule (2.2.12), nous montrons que la limite faible semiclassique de (CSM), est écrite en termes des branches de la solution multivaluée de l'équation de Burgers.

Theorem 2.2.12 ([Bad24b]). *Soit $u_0 \in L^2_+(\mathbb{R})$ (avec $\|u_0\|_{L^2} < \sqrt{2\pi}$ dans le cas focalisant), tel que u_0 soit une fonction C^1 tendant vers 0 à l'infini, avec une dérivée bornée dans $L^\infty(\mathbb{R})$.²¹ Alors, pour chaque temps $t \in \mathbb{R}$, et pour presque tout $x \in \mathbb{R}$, l'équation*

$$y \mp 2t|u_0(y)|^2 = x$$

a un nombre impair de solutions réelles simples $y_0 := y_0(t, x) < \dots < y_{2\ell} := y_{2\ell}(t, x)$, et la limite semiclassique de (CSM) est donnée par

$$ZD_\pm[u_0](t, x) = e^{i\varphi(t, x)} \left(\mp i \frac{|t|}{t} \right)^\ell \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k}, \quad (2.2.14)$$

où

$$\varphi(t, x) = \arg(u_0(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log \left(\frac{s \mp 2t|u_0(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k)}{-s \mp 2t|u_0(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k)} \right) ds.$$

21. Notez que toute fonction dans $H^s_+(\mathbb{R}) := H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$, $s > \frac{3}{2}$ satisfait ces conditions.

Par conséquent,

1. On en déduit que la limite semiclassique de (CSM) satisfait, pour tout u_0 suffisamment régulier,

$$|ZD_{\pm}[u_0](t, x)| = \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k},$$

où les $(y_k)_{k=0, \dots, 2\ell}$ représentent les branches de la solution multivaluée de l'équation de Burgers (2.2.13) à un temps t au-delà du temps de choc, et à une position x . De plus, en appliquant le logarithme à l'identité précédente, on obtient

$$\log|ZD_{\pm}[u_0](t, x)|^2 = \sum_{k=0}^{2\ell} (-1)^k \log|u_0(y_k)|^2,$$

un résultat rappelant un résultat similaire trouvé pour l'équation de Benjamin-Ono [Gér23b], où la limite semiclassique dans ce dernier cas, est décrite comme

$$ZD_{(\text{BO})}[u_0](t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k^{BO}),$$

où, ici, les $(y_k^{BO})_{0, \dots, 2\ell}$, sont les solutions réelles pour l'équation

$$y + 2tu_0(y) = x.$$

2. De plus, il convient de noter que via (2.2.14), on en déduit que si u_0 est une donnée initiale rationnelle, alors la limite faible semiclassique $ZD_{\pm}[u_0]$ de (CSM) dans le cas focalisant et défocalisant, est également une fonction rationnelle.
3. Enfin, nous déduisons l'existence d'un principe du maximum qui est satisfait bien que l'on considère l'équation avec faible dispersion :

Corollary 2.2.13 (Principe du maximum [Bad24b]). *Soit $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (avec $\|u_0\|_{L^2} < \sqrt{2\pi}$ dans le cas focalisant). Pour tout $t \in \mathbb{R}$,*

$$\|ZD_{\pm}[u_0]\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

Problèmes ouverts et perspectives.

1. Une question pertinente est de savoir si la limite semiclassique $ZD_{\pm}[u_0]$ est effectivement la limite forte de u^ε dans L^2_+ lorsque $\varepsilon \rightarrow 0$. En d'autres termes, le fait de négliger la composante de dispersion de l'équation (CSM) crée-t-il des oscillations prononcées dans la solution limite, similaires à ce qui est observé dans le cas de l'équation Benjamin-Ono [MX11] ? Cela pourrait potentiellement empêcher $ZD_{\pm}[u_0]$ d'être une limite forte dans L^2 .
2. Existe-t-il un analogue du Théorème 2.2.12 pour les configurations périodiques (c'est-à-dire, lorsque $x \in \mathbb{T}$) ?

2.2.3 Ondes progressives, solitons et multi-solitons

Comme souligné dans la Section 2.2.3, une implication fascinante de l'intégrabilité d'une EDP est l'émergence d'ondes progressives, notamment les solitons. Ici, nous fournissons un aperçu de ce qui a été découvert jusqu'à présent concernant ces ondes pour l'équation (CSM). Notre objectif est d'étudier l'existence et la découverte éventuelle de formules explicites pour ces "solutions exactes", qui sont plus simples que celles présentées dans les équations (2.2.6) et (2.2.8), et peuvent être exprimées à l'aide de fonctions élémentaires. Par conséquent, ces solutions explicites peuvent également servir de fonctions de test précieuses pour l'évaluation numérique, aidant à comprendre la dynamique de cette EDP.

Il s'avère que les ondes progressives non triviales²² de cette équation L^2 -critique (CSM) sont des fonctions rationnelles. Alors que sur la droite réelle (c'est-à-dire lorsque $x \in \mathbb{R}$), les ondes progressives ont été entièrement caractérisées dans [GL24] comme des ondes satisfaisant $\|u\|_{L^2} = 2\pi$ pour le cas focalisant, il s'avère que dans le cas périodique (c'est-à-dire $x \in \mathbb{T}$), nous avons découvert une classe plus importante d'ondes progressives, jouissant d'une large gamme de normes L^2 , prenant des valeurs arbitrairement petites à arbitrairement grandes pour l'équation focalisante et défocalisante [Bad23]. Nous résumons les résultats dans le tableau suivant. (Tableau 2.1).

	(CSM) focalisante sur \mathbb{R}	(CSM) défocalisante sur \mathbb{R}
Ondes stationnaires	✓	
Ondes progressives	✓	
Vitesse d'onde	$c \in \mathbb{R}$	
Norme L^2 des ondes progressives	$\ u\ _{L^2} = \sqrt{2\pi}$	
	(CSM) focalisante sur \mathbb{T}	(CSM) défocalisante sur \mathbb{T}
Ondes stationnaires non triviales	✓	✗
Ondes progressives	✓	✓
Vitesse d'onde	$c \in \mathbb{R}$	$c \geq N$
Norme L^2 des ondes progressives non triviales	$\ u\ _{L^2} \in (0, +\infty)$	$\ u\ _{L^2} \in (0, +\infty)$

TABLE 2.1 – Où $N \in \mathbb{N}$ est le degré du dénominateur d'une onde progressive apparaissant dans le Théorème 2.2.17. De plus, par ondes stationnaires non triviales, nous entendons la solution $u(t, x) = u_0(x)$ qui n'est pas une fonction constante.

2.2.3.1 Sur \mathbb{R}

Afin de caractériser les ondes progressives de (CSM) (cf. Théorème 2.2.14), Gérard–Lenzmann [GL24, Section 4.] ont caractérisé les "ground states" de l'équation

22. Nous entendons par ondes progressives non triviales celles qui ne sont ni constantes ni des ondes planes $C e^{iN(x-Nt)}$, $C > 0$, $N \in \mathbb{N}_{\geq 1}$.

Calogero–Moser DNLS comme étant les minimiseurs de l'Hamiltonien²³

$$\mathcal{H}(u) := \frac{1}{2} \|L_u u\|_{L^2(\mathbb{R})}^2$$

associé à la forme symplectique non standard définie sur $H_+^s(\mathbb{R})$, $s \geq 0$,

$$\omega_u^\sharp(h_1, h_2) := \operatorname{Im} \langle h_1, h_2 \rangle_{L^2} + \iint_{\mathbb{R} \times \mathbb{R}} \operatorname{Re}(\bar{u}h_1)(x) \operatorname{Re}(\bar{u}h_2)(y) \operatorname{sgn}(x - y) \, dx dy.$$

Par conséquent, ils ont obtenu que

$$u(x) = e^{i\theta} \lambda^{1/2} \mathcal{R}(\lambda x + \mu)$$

où $\theta \in [0, 2\pi)$, $\lambda > 0$, $\mu \in \mathbb{R}$, et

$$\mathcal{R}(x) = e^{i\theta} \frac{\sqrt{2 \operatorname{Im} p}}{x + p} \in H_+^1(\mathbb{R}), \quad p \in \mathbb{C}_+, \quad \theta \in \mathbb{R} \quad (2.2.15)$$

est l'unique minimiseur de $\|L_u u\|_{L^2(\mathbb{R})} = 0$. Par conséquent, ils déduisent la caractérisation suivante des ondes progressives.

Theorem 2.2.14 ([GL24]). *Étant donné \mathcal{R} défini dans (2.2.15), nous avons que chaque onde progressive de l'équation focalisante (CSM) dans $H_+^1(\mathbb{R})$ est de la forme*

$$u(t, x) = e^{i\theta} e^{iv(x-vt)} \lambda^{1/2} \mathcal{R}(\lambda(x - 2vt) + y), \quad \lambda > 0, \quad y \in \mathbb{R}, \quad \theta \in \mathbb{T}, \quad v \in \mathbb{R}.$$

En particulier, elles satisfont $\|u\|_{L^2} = \sqrt{2\pi}$.

Comme deuxième résultat, ils ont établi l'existence de solutions rationnelles de l'équation Calogero–Moser DNLS focalisante (CSM)

$$u(x) = \frac{P(x)}{Q(x)}, \quad P\bar{P} = i(Q'\bar{Q} - \bar{Q}'Q), \quad \deg(P) < \deg(Q) =: N,$$

complètement supportées dans le spectre pur de l'opérateur Lax, avec une masse

$$\|u\|_{L^2(\mathbb{R})}^2 = 2\pi N,$$

exhibant des cascades de fréquences lorsque $t \rightarrow \pm\infty$. Ces solutions, appelées “multisolitons” dans [GL24], sont ainsi nommées car lorsque $\deg(Q) = 1$, u coïncide avec \mathcal{R} à symétries près.

Theorem 2.2.15 ([GL24]). *Pour chaque $N \geq 2$, chaque multisoliton u pour l'équation focalisante (CSM) existe pour tout $t \in \mathbb{R}$ et elle présente une croissance des normes de Sobolev*

$$\|u(t)\|_{H^s} \sim |t|^{2s}$$

pour tout nombre réel $s > 0$.

23. Obtenu après application d'une transformation de jauge à l'équation.

2.2.3.2 Sur \mathbb{T}

La diversité entre les ondes progressives lorsque $x \in \mathbb{R}$ et $x \in \mathbb{T}$ provient des propriétés spectrales inhérentes aux opérateurs de Lax dans chaque configuration. Pour caractériser ces ondes sur le tore, nous nous appuyons sur la théorie spectrale de L_u et \tilde{L}_u . Ainsi, nous rappelons que leurs spectres ne consistent qu'en des valeurs propres

$$\begin{aligned}\sigma(L_u) &:= \{\nu_0 \leq \dots \leq \nu_n \leq \dots\}, & \nu_0 &\geq -\|u\|_{L^\infty}^2, \\ \sigma(\tilde{L}_u) &:= \{\lambda_0 \leq \dots \leq \lambda_n \leq \dots\}, & \lambda_0 &\geq 0.\end{aligned}$$

Ces valeurs propres satisfont dans le cas défocalisant [Bad23, Proposition 2.1]

$$\lambda_{n+1} \geq \lambda_n + 1, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (2.2.16)$$

Cependant, dans le cas focalisant, le spectre de l'opérateur de Lax L_u est plus complexe. En fait, il existe une légère différence dans l'écart entre les valeurs propres consécutives de L_u selon que l'on considère un potentiel u avec une masse L^2 inférieure ou supérieure à 1, (nous rappelant que pour $\|u_0\|_{L^2} < 1$ le caractère de GWP de l'équation focalisante (CSM) est valide dans $H_+^s(\mathbb{T})$ pour tout $s \geq 0$. Plus précisément, nous avons [Bad23, Proposition 2.1]

— Si $\|u\|_{L^2} < 1$, alors toutes les valeurs propres de L_u sont simples

$$\nu_{n+1} > \nu_n, \quad \forall n \in \mathbb{N}_{\geq 0}.$$

— Si $\|u\|_{L^2} \geq 1$, alors certaines valeurs propres peuvent avoir une multiplicité égale à deux²⁴, mais elles sont en nombre fini, car

$$\liminf_{n \rightarrow \infty} \nu_{n+1} - \nu_n \geq 1. \quad (2.2.17)$$

De plus, aucune valeur propre n'a une multiplicité supérieure à deux, car

$$\nu_{n+2} \geq \nu_n + 1, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (2.2.18)$$

De plus, outre les inégalités concernant les valeurs propres des opérateurs de Lax, nous utiliserons également certaines identités induites par les fonctions propres, pour identifier les ondes progressives. Ainsi, si nous désignons par $(f_n^{u_0})$ les fonctions propres de l'opérateur de Lax L_{u_0} , et si nous examinons l'évolution de ces vecteurs en tant que solutions à

$$\begin{cases} \partial_t f_n^t &= B_{u(t)} f_n^t, \\ f_n^t|_{t=0} &= f_n^{u_0}, \end{cases} \quad \forall n \in \mathbb{N}_{\geq 0},$$

24. Prenez, par exemple, $u = e^{ix}$, alors comme $L_u = D - T_u T_{\bar{u}} = D - u\Pi(\bar{u}\cdot)$, on vérifie $L_u 1 = L_u e^{ix} = 0$.

on déduit les identités suivantes

$$\begin{aligned}
\langle 1 | f_n^t \rangle &= \langle 1 | f_n^{u_0} \rangle e^{-i\nu_n^2 \cdot t} \\
\langle u | f_n^t \rangle &= \langle u_0 | f_n^{u_0} \rangle e^{-i\nu_n^2 \cdot t} \\
\langle S f_p^t | u \rangle &= \langle S f_p^{u_0} | u_0 \rangle e^{i(\nu_p+1)^2 \cdot t} \\
\langle S f_p^t | f_n^t \rangle &= \langle S f_p^{u_0} | f_n^{u_0} \rangle e^{i((\nu_p+1)^2 - \nu_n^2) \cdot t},
\end{aligned} \tag{2.2.19}$$

où S désigne l'opérateur de décalage (2.2.5). Et les identités analogues sont vérifiées dans le cas défocalisant.

En combinant les inégalités (2.2.16), (2.2.17) et (2.2.18) liées aux valeurs propres, ainsi que les identités précédentes (2.2.19) liées aux fonctions propres, et le fait que [Bad23, Corollaire 2.6]

— dans le cas défocalisant

$$\lambda_n = \lambda_{n-1} + 1 \iff \langle u_0 | f_n^{u_0} \rangle = 0. \tag{2.2.20}$$

— dans le cas focalisant²⁵

$$\nu_n = \nu_{n-1} + 1, \forall n \geq N_1 \iff \langle u_0 | f_n^{u_0} \rangle = 0, \forall n \geq N_2. \tag{2.2.21}$$

on déduit, dans un premier temps, une caractérisation spectrale des ondes progressives $u(t, x) = u_0(x - ct)$ pour (CSM).

Theorem 2.2.16 (Caractérisation spectrale des ondes progressives – $x \in \mathbb{T}$ [Bad23]). Soit $u(t, x) := u_0(x - ct)$ une onde progressive pour l'équation de Calogero–Sutherland DNLS (CSM). Alors

► **Équation défocalisante.** Il existe au plus un $N \in \mathbb{N}_{\geq 1}$ tel que

$$\langle u_0 | f_N^{u_0} \rangle \neq 0.$$

► **Équation focalisante.** Il existe $N \in \mathbb{N}_{\geq 1}$ tel que pour tout $n \geq N$,

$$\langle u_0 | f_n^{u_0} \rangle = 0.$$

Armé de ces caractérisations spectrales des ondes progressives, et du fait que toute fonction dans l'espace de Hardy peut être écrite sous la forme [GK21, GMR16]

$$u(t, z) = \langle (\text{Id} - zM)^{-1} X | Y \rangle, \quad z \in \mathbb{D},$$

où X, Y sont des vecteurs colonnes infinis et M est la représentation matricielle infinie de S^* ,

$$X := (\langle u | f_n^t \rangle), \quad Y := (\langle 1 | f_n^t \rangle), \quad M := (\langle f_m^t | S f_n^t \rangle),$$

²⁵. Nous renvoyons à l'Annexe 2 pour un exemple montrant que N_2 n'est pas nécessairement égal à N_1 .

et en utilisant les relations

$$\begin{aligned} \langle 1 | u \rangle \langle u | f_n \rangle &= \mp \lambda_n \langle 1 | f_n \rangle, \\ (\lambda_n - \lambda_p - 1) \langle Sf_p | f_n \rangle &= \mp \langle Sf_p | u \rangle \langle u | f_n \rangle. \end{aligned}$$

où $\langle \cdot | \cdot \rangle$ désigne le produit scalaire dans $L^2(\mathbb{T})$, nous déduisons les formules explicites suivantes pour les ondes progressives.

Theorem 2.2.17 ([Bad23]). *Les fonctions $u(t, x) = C e^{in(x-nt)}$, $C \in \mathbb{R}$, $n \in \mathbb{N}_{\geq 0}$, et*

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, \quad (2.2.22)$$

où $N \in \mathbb{N}$, $c := \pm N \left(1 + \frac{2\alpha}{\beta} \right)$, et $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ tel que

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = \pm N, \quad (2.2.23)$$

sont des ondes progressives pour l'équation focalisante et défocalisante (CSM) sur \mathbb{T} . De plus, dans le cas défocalisant, les solutions ci-dessus sont les seules ondes progressives $u(t, x) := u_0(x-ct)$. Cependant, dans le cas focalisant, il existe d'autres ondes progressives, telles que²⁶

$$u(t, x) = e^{i\theta} e^{im(x-mt)} \left(\alpha + \frac{\beta}{1 - p e^{i(x-mt)}} \right) \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, m \in \mathbb{N}_{\geq 1},$$

où (α, β) satisfait

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = 1, \quad \beta(m-1) = 2\alpha.$$

Remark 2.2.4. [Bad23]

1. **La dynamique de l'équation focalisante (CSM) est plus riche.** Il convient de noter que la condition sur (α, β) apparaissant dans (2.2.23) pour le cas focalisant permet de dériver une *gamme plus large* d'ondes progressives par rapport au cas défocalisant. Par exemple,

$$u(t, x) = e^{i\theta} \frac{\sqrt{N(1-|p|^2)}}{1 - p e^{iN(x+Nt)}} \quad \text{et} \quad u(t, x) = e^{i\theta} \frac{\sqrt{N(1-|p|^2)}}{1 - p e^{iN(x-Nt)}} e^{iN(x-Nt)}$$

sont des ondes progressives pour l'équation focalisante (CSM). Alors qu'il n'existe aucune onde progressive $u(t, x) := u_0(x-ct)$ avec un profil

$$u_0(x) := \frac{\beta}{1 - p e^{iNx}} \quad \text{ou} \quad u_0(x) := \frac{\alpha e^{iNx}}{1 - p e^{iNx}}$$

pour l'équation défocalisante.

26. et elles ne sont peut-être pas les seules.

2. **Solutions stationnaires.** La famille

$$u(t, x) := e^{i\theta} \sqrt{\frac{N(1 - |p|^2)}{2(1 + |p|^2)}} \left(1 - \frac{2}{1 - p e^{iNx}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, N \in \mathbb{N}_{\geq 1}.$$

constitue des solutions d'onde stationnaire non triviales (c'est-à-dire non constantes)²⁷. Aucune solution stationnaire n'existe pour le cas défocalisant.

3. **Corrélation entre la vitesse et la norme L^2 d'une onde progressive.**

Alors que les équations focalisante et défocalisante (CSM) présentent des ondes progressives avec des normes L^2 arbitrairement petites et grandes pour cette équation L^2 -critique²⁸, la corrélation entre la vitesse et la norme L^2 des ondes progressives varie entre les cas focalisant et défocalisant. Dans le cas défocalisant, on observe que l'onde progressive (2.2.22) se propage vers la droite avec une vitesse $c > N$, où N est le degré apparaissant dans le dénominateur de (2.2.22). De plus, lorsque $\|u\|_{L^2} \rightarrow +\infty$, nous avons $c \rightarrow +\infty$, et lorsque $\|u\|_{L^2} \rightarrow 0$ alors $c \rightarrow N$. Cependant, dans le cas focalisant, il n'y a aucune corrélation entre la vitesse et la taille de la norme L^2 , et l'onde progressive peut se propager vers la droite ou vers la gauche à n'importe quelle vitesse.

Notez que les ondes progressives font partie d'une famille plus large de solutions rationnelles de (CSM), connues sous le nom de **finite gap potentials**. Celles-ci sont définies comme des potentiels u pour lesquels, à partir d'un certain rang, deux valeurs propres consécutives sont espacées de 1. En d'autres termes, en vue de (2.2.20) et (2.2.21), elles sont décrites comme des potentiels u pour lesquels, à partir d'un certain rang, tous les $\langle u | f_n \rangle = 0$ s'annulent. Cela signifie que u est supporté sur un nombre fini d'espaces propres de L_u et \tilde{L}_u , fournissant ainsi un analogue aux *multisolitons* observés sur la droite réelle. Ces finite gap potentials sont caractérisés explicitement comme suit :

Theorem 2.2.18 (Caractérisation des finite gap potentials dans l'espace d'état de (CSM) [Bad23]). *Les finite gap potentials de (CSM) sont soit les fonctions $u(x) = C e^{iNx}$, $C \in \mathbb{C}^*$, $N \in \mathbb{N}_{\geq 0}$, soit les fonctions rationnelles*

$$u(x) = e^{im_0 x} \prod_{j=1}^r \left(\frac{e^{ix} - \bar{p}_j}{1 - p_j e^{ix}} \right)^{m_j-1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j e^{ix}} \right), \quad p_j \in \mathbb{D}^*, p_k \neq p_j, k \neq j,$$

où, pour $N \in \mathbb{N}_{\geq 1}$, $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$ tels que $m_0 + \sum_{j=1}^r m_j = N$, et $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfont pour tout $j = 1, \dots, r$,

27. c'est-à-dire $u(t, x) = u_0(x)$.

28. Plus précisément, pour tout $r > 0$, il existe une onde progressive non triviale $u(t, x) := u_0(x - ct)$ de (CSM) où

$$\|u_0\|_{L^2} = r.$$

(i) Dans le cas défocalisant,

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = -m_j,$$

(ii) Dans le cas focalisant,

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = m_j,$$

avec $a \neq 0$ si $m_0 \neq 0$. De plus, ces finite gap potentials peuvent être regroupés en ensembles qui restent invariants sous l'évolution de (CSM).

Problèmes ouverts et perspectives.

1. Une question ouverte est de comprendre la dynamique de ces **finite gap potentials**. Les solutions partant de telles données initiales restent-elles bornées dans toutes les normes H^s , ou bien, à l'instar du scénario observé sur la droite réelle, manifestent-elles des cascades de fréquences ?
2. Ici, nous avons caractérisé toutes les ondes progressives de la forme $u(t, x) := u_0(x - ct)$ pour l'équation défocalisante (CSM), et nous avons obtenu une famille d'ondes progressives étendue pour l'équation focalisante. Cependant, la caractérisation complète des ondes progressives dans le cas focalisant reste un problème ouvert.
3. Dans notre étude [Bad23], nous avons traité le cas où les ondes progressives de l'équation DNLS de Calogero–Sutherland (CSM) sont de la forme

$$u(t, x) := u_0(x - ct), \quad c \in \mathbb{R}.$$

Mais on peut se demander s'il existe des solutions d'ondes progressives avec un facteur de phase, telles que

$$u(t, x) := e^{i\varphi(t)} u_0(x - ct), \quad \varphi(t), c \in \mathbb{R}. \quad (2.2.24)$$

Cependant, soulignons le fait suivant : notons que la moyenne $\langle u | 1 \rangle$ est conservée le long du flot de l'équation de Calogero–Sutherland DNLS (CSM), pour toute solution u dans l'espace de Hardy du cercle \mathbb{T} . En effet, en effectuant une intégration par parties et puisque u est dans l'espace de Hardy, alors

$$i\partial_t \langle u | 1 \rangle = -\langle \partial_x^2 u | 1 \rangle \pm 2 \langle D\Pi(|u|^2) | \bar{u} \rangle = 0.$$

Par conséquent,

- Si $\langle u_0 | 1 \rangle \neq 0$, alors $\varphi(t)$ dans (2.2.24) doit être une constante dans le temps.

- Si $\langle u_0 | 1 \rangle = 0$, la question de l'existence de solutions d'ondes progressives de la forme (2.2.24) pour l'équation (CSM) reste un problème ouvert. Cependant, on peut facilement prouver que $(\varphi(t), c)$ sont liés par l'identité suivante

$$\varphi'(t) - Nc = -N^2, \quad (2.2.25)$$

où N est l'entier positif apparaissant après réécriture de u_0 sous la forme $u_0 = S^N v_0$ avec $\langle v_0 | 1 \rangle \neq 0$. En effet, en substituant la solution $u(t, x)$

$$u(t, x) = e^{i\varphi(t)} u_0(x - ct) = e^{i\varphi(t)} e^{iN(x-ct)} v_0(x - ct),$$

dans (CSM), nous obtenons

$$\begin{cases} -(\varphi'(t) - Nc) v_0 - N^2 v_0 + P(\partial_x v_0, \partial_x^2 v_0) \pm 2iD\Pi(|v_0|^2)v_0 = 0. \\ P(w, \tilde{w}) := (2N - c)i w + \tilde{w} \end{cases}$$

Par conséquent, nous déduisons (2.2.25) en prenant le produit scalaire de la dernière identité avec 1.

4. L'étude de la **stabilité orbitale** des ondes progressives, sachant que sur la droite réelle, Hogan–Kowalski ont montré la stabilité orbitale des solitons sur \mathbb{R} pour une classe restreinte de données initiales [HK24, Theorem 2.1].

Chapter 3

Local well-posedness of the Calogero–Sutherland DNLS equation

ABSTRACT.

In this section, we detail the proof of [GL24, Proposition 2.1] which pertains to the local well-posedness of the Calogero–Sutherland DNLS equation in the Hardy–Sobolev space $H_+^2(\mathbb{T})$

$$\begin{cases} i\partial_t u + \partial_x^2 u + 2uD\Pi(|u|^2) = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (\text{CS})$$

where $D = -i\partial_x$ and Π is the Szegő projector (1.2.1) onto functions with non-negative frequencies.

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3.1 The (CS)–equation

We consider the Calogero–Sutherland DNLS equation

$$\begin{cases} i\partial_t u + \partial_x^2 u + 2u \cdot D\Pi(|u|^2) = 0 \\ u(0, x) = u_0(x) \end{cases}, \quad x \in \mathbb{T}, \quad (\text{CS})$$

where $D = -i\partial_x$ and Π is the Szegő projector (1.2.1) defined, from $L^2(\mathbb{T})$ onto the Hardy space $L_+^2(\mathbb{T})$. The aim of this section is to prove the LWP of the (CS)–equation in the Hardy–Sobolev space $H_+^2(\mathbb{T}) := H^2(\mathbb{T}) \cap L_+^2(\mathbb{T})$, where we recall $H^2(\mathbb{T})$ denotes the Sobolev space, $L_+^2(\mathbb{T})$ is the Hardy space defined in (1.2.2),

For that, we rewrite (CS) after expanding the term

$$D\Pi(|u|^2) \cdot u = -i\Pi(\bar{u}\partial_x u + u\partial_x \bar{u})u = -iT_u T_{\bar{u}} \partial_x u - iu H_u \partial_x u;$$

as

$$\begin{cases} \partial_t u = i\partial_x^2 u + 2T_u T_{\bar{u}} \partial_x u + 2u H_u \partial_x u \\ u(0, x) = u_0(x), \end{cases} \quad (\text{CS}')$$

where T_u is the Toeplitz operator of symbol u defined in (1.2.5), and H is the \mathbb{C} -antilinear *Hankel operator* $H_h : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$, defined on $L_+^2(\mathbb{T})$ as [P+03] : For any $h \in L^\infty(\mathbb{T})$,

$$H_h(u) = T_h \bar{u} = \Pi(h\bar{u}) = T_{\bar{u}} h, \quad u \in L_+^2(\mathbb{T}). \quad (3.1.1)$$

Strategy. To prove the local well-posedness of (CS), we will employ iterative schemes inspired by Kato's method,

$$\begin{cases} \partial_t u^{k+1} = i\partial_x^2 u^{k+1} + 2T_{u^k} T_{\bar{u}^k} \partial_x u^{k+1} + 2u^k H_{u^k} \partial_x u^k \\ u^{k+1}(0, x) = u_0(x) \end{cases} \quad (3.1.2)$$

We initially demonstrate that the operator $u H_u \partial_x$ is of order 0, enabling us to analyze the local well-posedness of the linear equation

$$\partial_t w = i\partial_x^2 w + 2T_u T_{\bar{u}} \partial_x w + f \quad (3.1.3)$$

in $H_+^2(\mathbb{T})$, for $f \in L^1([-T, T], H_+^2(\mathbb{T}))$. Simultaneously, we derive energy estimates for w , providing an upper bound for w when w_0 , u , and f are bounded. Finally, we use (3.1.2) to infer the LWP of (3.1.3) and (CS) in $H_+^2(\mathbb{T})$.

Lemma 3.1.1 ($H_u \partial_x$ is of order 0). *Let $u \in H_+^{3/2}(\mathbb{T})$, then $H_u \partial_x : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$ with*

$$\|H_u \partial_x f\|_{L^2(\mathbb{T})} \leq C \|u\|_{H^{3/2}(\mathbb{T})} \|f\|_{L^2(\mathbb{T})}, \quad \forall f \in L_+^2(\mathbb{T}).$$

Moreover, if $u \in H_+^2(\mathbb{T})$, then $H_u \partial_x : H_+^k(\mathbb{T}) \rightarrow H_+^k(\mathbb{T})$, $k = 0, 1, 2$ with

$$\|H_u \partial_x f\|_{H^k(\mathbb{T})} \leq C \|u\|_{H^2(\mathbb{T})} \|f\|_{H^k(\mathbb{T})}, \quad \forall f \in H_+^k(\mathbb{T}).$$

Proof. Let $u \in H_+^{3/2}(\mathbb{T})$, $f \in L_+^2(\mathbb{T})$. By Parseval's identity,

$$\|H_u \partial_x f\|_{L^2(\mathbb{T})}^2 = \sum_{n \geq 0} \left| \widehat{H_u(\partial_x f)}(n) \right|^2$$

where in terms of Fourier coefficients, the Hankel operator (3.1.1) reads

$$\widehat{H_h(u)}(n) = \sum_{p \geq 0} \widehat{h}(n+p) \overline{\widehat{u}(p)}.$$

Therefore,

$$\widehat{H_u(\partial_x f)}(n) = -i \sum_{k \geq 0} k \widehat{u}(n+k) \overline{\widehat{f}(k)}.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$|\widehat{H_u(\partial_x f)}(n)|^2 \leq \sum_{k \geq 0} k^2 |\widehat{u}(n+k)|^2 \sum_{k \geq 0} |\widehat{f}(k)|^2$$

Hence,

$$\|H_u \partial_x f\|_{L^2(\mathbb{T})}^2 \leq \sum_{n \geq 0} \sum_{k \geq 0} k^2 |\widehat{u}(n+k)|^2 \|f\|_{L^2(\mathbb{T})}^2$$

Set $p = n + k$, then

$$\|H_u \partial_x f\|_{L^2(\mathbb{T})}^2 \leq \sum_{p \geq 0} |\widehat{u}(p)|^2 \sum_{k=0}^p k^2 \cdot \|f\|_{L^2(\mathbb{T})}^2.$$

Using the identity $\sum_{k=0}^p k^2 = \frac{p(p+1)(2p+1)}{6}$, we infer

$$\|H_u \partial_x f\|_{L^2(\mathbb{T})}^2 \leq \frac{1}{6} \sum_{p \geq 0} p(p+1)(2p+1) |\widehat{u}(p)|^2 \|f\|_{L^2(\mathbb{T})}^2 \leq C \|u\|_{H^{3/2}(\mathbb{T})}^2 \|f\|_{L^2(\mathbb{T})}^2,$$

which proves the first point. For the second point, let u be a potential with a bit more regularity, namely $u \in H_+^2(\mathbb{T})$. Writing

$$\begin{aligned} \partial_x(H_u \partial_x f) &= H_{\partial_x u}(\partial_x f) + H_u \partial_x(\partial_x f), \\ \partial_x^2(H_u \partial_x f) &= H_{\partial_x^2 u} \partial_x f + 2H_{\partial_x u} \partial_x f + H_u \partial_x(\partial_x^2 f), \end{aligned}$$

we deduce by the Sobolev embedding that $\partial_x u \in L^\infty(\mathbb{T})$, and $\partial_x f \in L^\infty(\mathbb{T})$ if $f \in H_+^2(\mathbb{T})$, hence

$$\begin{aligned} \|H_{\partial_x u}(\partial_x f)\|_{L^2(\mathbb{T})} &\leq \|\partial_x u\|_{L^\infty} \|\partial_x f\|_{L^2(\mathbb{T})} \leq C \|\partial_x u\|_{H^1} \|\partial_x f\|_{L^2(\mathbb{T})} \leq C \|u\|_{H^2} \|f\|_{H^1}, \\ \|H_{\partial_x^2 u}(\partial_x f)\|_{L^2(\mathbb{T})} &\leq \|\partial_x f\|_{L^\infty} \|\partial_x^2 u\|_{L^2(\mathbb{T})} \leq C \|\partial_x f\|_{H^1} \|\partial_x^2 u\|_{L^2(\mathbb{T})} \leq C \|u\|_{H^2} \|f\|_{H^2}. \end{aligned}$$

In addition, using the first statement, we infer that

$$\begin{aligned} \|H_u \partial_x(\partial_x f)\|_{L^2(\mathbb{T})} &\leq C \|u\|_{H^2} \|f\|_{H^1} \\ \|H_u \partial_x(\partial_x^2 f)\|_{L^2(\mathbb{T})} &\leq C \|u\|_{H^2} \|f\|_{H^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|H_u \partial_x f\|_{H^1(\mathbb{T})} &\leq C \|u\|_{H^2(\mathbb{T})} \|f\|_{H^1(\mathbb{T})}, \\ \|H_u \partial_x f\|_{H^2(\mathbb{T})} &\leq C \|u\|_{H^2(\mathbb{T})} \|f\|_{H^2(\mathbb{T})}. \end{aligned}$$

□

3.1.1 Energy estimates.

In this section, we introduce two lemmas addressing energy estimates. The first lemma will be essential for proving the uniqueness of the solution of the linearized equation (3.1.3) in Proposition (3.1.4), and enables inferring the energy estimate : For all $p = 0, 1, 2$,

$$\sup_{|t| \leq T} \|w(t)\|_{H^p} \lesssim e^{\int_{-T}^T \|u(t)\|_{H^2}^2 dt} (\|w_0\|_{H^p} + \|f\|_{L_t^1 H^p}), \quad (3.1.4)$$

while the second lemma is useful to achieve the existence of a solution for (3.1.3).

Lemma 3.1.2 (Control by A^*). *Let $A := -\partial_t + i\partial_x^2 + 2\partial_x(T_u T_{\bar{u}} \cdot)$. For all $v \in \mathcal{C}^1([-T, T], H_+^{k+1}(\mathbb{T}))$, $k = 0, 1, 2$,*

$$\sup_{|t| \leq T} \|v(t)\|_{H^k} \lesssim \exp\left(\int_{-T}^T \|u(s)\|_{H^2}^2 ds\right) [\|v(0, \cdot)\|_{H^k} + \|A^*v\|_{L_t^1 H_x^k}] \quad (3.1.5)$$

Proof. For all $k = 0, 1, 2$,

$$\begin{aligned} \frac{d}{dt} \|v\|_{H^k}^2 &= \frac{d}{dt} \int_0^{2\pi} (1 + i\partial_x)^{2k} v \bar{v} \frac{dx}{2\pi} \\ &= 2 \operatorname{Re} \int_0^{2\pi} (1 + i\partial_x)^{2k} \partial_t v \bar{v} \frac{dx}{2\pi}, \\ &= 2 \operatorname{Re} \int_0^{2\pi} (1 + i\partial_x)^{2k} (i\partial_x^2 v + 2T_u T_{\bar{u}} \partial_x v + A^*v) \bar{v} \frac{dx}{2\pi}. \end{aligned}$$

First, note that the term

$$\operatorname{Re} \left(i \int_0^{2\pi} (1 + i\partial_x)^{2k} \partial_x^2 v \bar{v} \frac{dx}{2\pi} \right) = \operatorname{Re} \left(-i \langle (1 + i\partial_x)^k \partial_x v | (1 + i\partial_x)^k \partial_x v \rangle \right) = 0.$$

Second, applying Cauchy–Schwarz inequality on the third term

$$\int_0^{2\pi} (1 + i\partial_x)^{2k} A^*v \bar{v} \frac{dx}{2\pi} = \langle (1 + i\partial_x)^k A^*v | (1 + i\partial_x)^k v \rangle,$$

leads to

$$\frac{d}{dt} \|v\|_{H^k}^2 \lesssim \operatorname{Re} \langle (1 + i\partial_x)^k T_u T_{\bar{u}} \partial_x v | (1 + i\partial_x)^k v \rangle + \|A^*v\|_{H^k} \|v\|_{H^k}.$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \|v\|_{H^k}^2 &\lesssim \operatorname{Re} \langle T_u T_{\bar{u}} \partial_x (1 + i\partial_x)^k v | (1 + i\partial_x)^k v \rangle + \|A^*v\|_{H^k} \|v\|_{H^k} \\ &\quad + \operatorname{Re} \langle [(1 + i\partial_x)^k, T_u T_{\bar{u}}] \partial_x v | (1 + i\partial_x)^k v \rangle, \end{aligned} \quad (3.1.6)$$

where $[\cdot, \cdot]$ denotes the commutator. Rewriting the first term of the latter inequality using $\operatorname{Re}(z) = \frac{z+\bar{z}}{2}$ and since $T_u T_{\bar{u}} \partial_x$ is a skew-symmetric operator, we infer

$$\begin{aligned} 2 \operatorname{Re} \langle T_u T_{\bar{u}} \partial_x (1 + i\partial_x)^k v \mid (1 + i\partial_x)^k v \rangle &= \langle T_u T_{\bar{u}} \partial_x (1 + i\partial_x)^k v \mid (1 + i\partial_x)^k v \rangle \\ &\quad - \langle \partial_x (T_u T_{\bar{u}} (1 + i\partial_x)^k v) \mid (1 + i\partial_x)^k v \rangle. \end{aligned}$$

By Leibniz's rule,

$$\begin{aligned} 2 \operatorname{Re} \langle T_u T_{\bar{u}} \partial_x (1 + i\partial_x)^k v \mid (1 + i\partial_x)^k v \rangle &= - \langle \partial_x (T_u T_{\bar{u}}) (1 + i\partial_x)^k v \mid (1 + i\partial_x)^k v \rangle \\ &\lesssim \|u\|_{H^2}^2 \|v\|_{H^k}^2. \end{aligned}$$

Hence, (3.1.6) becomes

$$\frac{d}{dt} \|v\|_{H^k}^2 \lesssim \|u\|_{H^2}^2 \|v\|_{H^k}^2 + \|A^* v\|_{H^k} \|v\|_{H^k} + \operatorname{Re} \langle [(1 + i\partial_x)^k, T_u T_{\bar{u}}] \partial_x v \mid (1 + i\partial_x)^k v \rangle \quad (3.1.7)$$

If $k = 0$ The last term $\operatorname{Re} \langle [(1 + i\partial_x)^k, T_u T_{\bar{u}}] \partial_x v \mid (1 + i\partial_x)^k v \rangle$ vanishes, and we have

$$\frac{d}{dt} \|v\|_{L^2}^2 \lesssim \|u\|_{H^2}^2 \|v\|_{L^2}^2 + \|A^* v\|_{L^2}^2 + \|v\|_{L^2}^2. \quad (3.1.8)$$

If $k = 1$ then $[(1 + i\partial_x)^k, T_u T_{\bar{u}}] \partial_x v = i\partial_x (T_u T_{\bar{u}}) \partial_x v$. Thus, applying Cauchy-Schwarz's inequality,

$$\begin{aligned} \operatorname{Re} \langle [(1 + i\partial_x), T_u T_{\bar{u}}] \partial_x v \mid (1 + i\partial_x) v \rangle &= \operatorname{Re} \langle i\partial_x (T_u T_{\bar{u}}) \partial_x v \mid (1 + i\partial_x) v \rangle \\ &\leq \|T_{\partial_x u} T_{\bar{u}} \partial_x v + T_u T_{\partial_x \bar{u}} \partial_x v\|_{L^2} \|v\|_{H^1} \\ &\lesssim \|u\|_{H^2}^2 \|v\|_{H^1}^2. \end{aligned}$$

If $k = 2$ then $[(1 + i\partial_x)^k, T_u T_{\bar{u}}] \partial_x v = i\partial_x (T_u T_{\bar{u}}) \partial_x v - \partial_x^2 (T_u T_{\bar{u}}) \partial_x v - \partial_x (T_u T_{\bar{u}}) \partial_x^2 v$. Applying again Cauchy-Schwarz's inequality,

$$\begin{aligned} \operatorname{Re} \langle [(1 + i\partial_x)^2, T_u T_{\bar{u}}] \partial_x v \mid (1 + i\partial_x)^2 v \rangle &\leq \|\partial_x (T_u T_{\bar{u}}) \partial_x v\|_{L^2} \|v\|_{H^2} \\ &\quad + \|\partial_x^2 (T_u T_{\bar{u}}) \partial_x v\|_{L^2} \|v\|_{H^2} \\ &\quad + \|\partial_x (T_u T_{\bar{u}}) \partial_x^2 v\|_{L^2} \|v\|_{H^2} \\ &\lesssim \|u\|_{H^2}^2 \|v\|_{H^2}^2. \end{aligned}$$

In sum, for all $k = 0, 1, 2$, we deduce from (3.1.7)

$$\frac{d}{dt} \|v\|_{H^k}^2 \lesssim \|u\|_{H^2}^2 \|v\|_{H^k}^2 + \|A^* v\|_{H^k} \|v\|_{H^k}.$$

Dividing by $\|v\|_{H^k}$, (note that if $v = 0$ the estimate (3.1.5) is trivial) and integrating between 0 and t , we infer

$$\|v(t, \cdot)\|_{H^k} \lesssim \|v(0, \cdot)\|_{H^k} + \|A^* v\|_{L_t^1 H_x^k} + \int_{-T}^T \|u(s, \cdot)\|_{H^2}^2 \|v(s, \cdot)\|_{H^k} ds. \quad (3.1.9)$$

Finally, by applying Gronwall's inequality, for all $k = -1, 0, 1, 2$,

$$\sup_{|t| \leq T} \|v(t, \cdot)\|_{H^k} \lesssim \exp \left(\int_{-T}^T \|u(s, \cdot)\|_{H^2}^2 ds \right) [\|v(0, \cdot)\|_{H^k} + \|A^* v\|_{L_t^1 H_x^k}].$$

□

Lemma 3.1.3 (Control by A). *Let $A := -\partial_t + i\partial_x^2 + 2\partial_x(T_u T_{\bar{u}} \cdot)$. The following energy estimates hold : For all $v \in \mathcal{C}^1([-T, T], H_+^{k+1}(\mathbb{T}))$, $k = -1, 0$,*

$$\sup_{|t| \leq T} \|v(t, \cdot)\|_{H^k} \lesssim \exp \left(\int_{-T}^T \|u(s, \cdot)\|_{H^2}^2 ds \right) [\|v(0, \cdot)\|_{H^k} + \|Av\|_{L_t^1 H_x^k}]. \quad (3.1.10)$$

Proof. The proof is essentially based on the same ideas outlined in Lemma 3.1.2. For all $v \in \mathcal{C}^1([-T, T], H_+^k(\mathbb{T}))$, $k = -1, 0$,

$$\begin{aligned} \frac{d}{dt} \|v\|_{H^k}^2 &= \frac{d}{dt} \int_0^{2\pi} (1 + i\partial_x)^{2k} v \bar{v} \frac{dx}{2\pi} \\ &= 2 \operatorname{Re} \int_0^{2\pi} (1 + i\partial_x)^{2k} \partial_t v \bar{v} \frac{dx}{2\pi}, \\ &= 2 \operatorname{Re} \int_0^{2\pi} (1 + i\partial_x)^{2k} (i\partial_x^2 v + 2\partial_x(T_u T_{\bar{u}} v) - Av) \bar{v} \frac{dx}{2\pi}. \end{aligned}$$

Similarly, we prove

$$\frac{d}{dt} \|v\|_{H^k}^2 \lesssim \operatorname{Re} \langle (1 + i\partial_x)^k \partial_x(T_u T_{\bar{u}} v) | (1 + i\partial_x)^k v \rangle + \|Av\|_{H^k} \|v\|_{H^k}.$$

If $k = 0$. Observe that $\operatorname{Re} \langle \partial_x(T_u T_{\bar{u}} v) | v \rangle = -\operatorname{Re} \langle v | T_u T_{\bar{u}} \partial_x v \rangle$, then by (3.1.7) and (3.1.8) we conclude that

$$\frac{d}{dt} \|v\|_{L^2}^2 \lesssim \|u\|_{H^2}^2 \|v\|_{L^2}^2 + \|Av\|_{L^2}^2 + \|v\|_{L^2}^2.$$

and hence, estimate (3.1.10) holds after dividing by $\|v\|_{L^2}$, integrating and using Gronwall's inequality.

If $k = -1$. Using Leibniz's rule, we have

$$\begin{aligned} \operatorname{Re} \langle (1 + i\partial_x)^{-1} \partial_x(T_u T_{\bar{u}} v) | (1 + i\partial_x)^{-1} v \rangle &= \operatorname{Re} \langle (1 + i\partial_x)^{-1} T_{\partial_x u} T_{\bar{u}} v | (1 + i\partial_x)^{-1} v \rangle \\ &\quad + \operatorname{Re} \langle (1 + i\partial_x)^{-1} T_u T_{\partial_x \bar{u}} v | (1 + i\partial_x)^{-1} v \rangle \\ &\quad + \operatorname{Re} \langle (1 + i\partial_x)^{-1} T_u T_{\bar{u}} \partial_x v | (1 + i\partial_x)^{-1} v \rangle \end{aligned}$$

Hence, applying Cauchy–Schwarz's inequality,

$$\begin{aligned} \operatorname{Re} \langle (1 + i\partial_x)^{-1} T_{\partial_x u} T_{\bar{u}} v | (1 + i\partial_x)^{-1} v \rangle &\leq \|T_{\partial_x u} T_{\bar{u}} v\|_{H^{-1}} \|v\|_{H^{-1}} \\ &= \sup_{\|\varphi\|_{H^1} \leq 1} \langle T_{\partial_x u} T_{\bar{u}} v | \varphi \rangle \|v\|_{H^{-1}} \\ &= \sup_{\|\varphi\|_{H^1} \leq 1} \langle v | u \Pi(\partial_x \bar{u} \Pi \varphi) \rangle \|v\|_{H^{-1}} \\ &\lesssim \|u\|_{H^2}^2 \|v\|_{H^{-1}}^2 \end{aligned}$$

Similarly, we prove

$$\operatorname{Re} \langle (1 + i\partial_x)^{-1} T_u T_{\partial_x \bar{u}} v | (1 + i\partial_x)^{-1} v \rangle \lesssim \|u\|_{H^2}^2 \|v\|_{H^{-1}}^2$$

All it remains is to deal with the last term $\operatorname{Re} \langle (1 + i\partial_x)^{-1} T_u T_{\bar{u}} \partial_x v \mid (1 + i\partial_x)^{-1} v \rangle$, which has been handled in the proof of the last lemma. As a consequence,

$$\operatorname{Re} \langle (1 + i\partial_x)^{-1} \partial_x (T_u T_{\bar{u}} v) \mid (1 + i\partial_x)^{-1} v \rangle \lesssim \|u\|_{H^2}^2 \|v\|_{H^{-1}}^2$$

And hence,

$$\frac{d}{dt} \|v\|_{H^{-1}}^2 \lesssim \|Av\|_{H^{-1}} \|v\|_{H^{-1}} + \|u\|_{H^2}^2 \|v\|_{H^{-1}}^2.$$

We conclude by dividing by $\|v\|_{L^2}$, integrating between 0 and t , and applying Gronwall's inequality. \square

3.1.2 LWP of the linear equation 3.1.3.

Given those energy estimates, we can now infer the local well-posedness of the linearized Cauchy problem (3.1.3) in $\mathcal{C}_t H_+^2(\mathbb{T})$.

Proposition 3.1.4. *Let $u \in \mathcal{C}([-T, T]; H_+^2)$, $w_0 \in H_+^2$ and $f \in L^1([-T, T]; H_+^2)$. There exists a unique $w \in \mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$ solution of the linear problem*

$$\begin{cases} \partial_t w = i\partial_x^2 w + 2T_u T_{\bar{u}} \partial_x w + f, \\ w(0, x) = w_0(x), \end{cases} \quad (3.1.11)$$

and satisfying the energy estimates (3.1.4).

Proof. The proof will be outlined in two stages. For $u \in \mathcal{C}([-T, T]; H_+^2)$, we first present the local well-posedness of the linear equation (3.1.11) in $\mathcal{C}_t H_+^1$. This will provide a solution $w \in \mathcal{C}([-T, T]; H_+^1)$ of (3.1.11). Second, we consider $w_0 \in H_+^2$, then $\partial_x w \in \mathcal{C}_t L_+^2$ satisfies the equation

$$\partial_t \tilde{w} = i\partial_x^2 \tilde{w} + 2\partial_x (T_u T_{\bar{u}} \tilde{w}) + \partial_x f, \quad (3.1.12)$$

with an initial datum $\partial_x w_0 \in H_+^1(\mathbb{T})$. We prove that the latter equation is locally well-posed in $H_+^1(\mathbb{T})$. Consequently, we conclude by the uniqueness of the solution of (3.1.12) that $\partial_x w = \tilde{w} \in \mathcal{C}_t H_+^1$, and hence $w \in \mathcal{C}_t H_+^2$.

Step 1 : Local well-posedness of (3.1.11) in $H_+^1(\mathbb{T})$. The proof is inspired by [AG07, Proposition 1.2. II-C]. Consider, for $v \in \mathcal{F} := \{v \in \mathcal{C}^\infty([0, T] \times \mathbb{T}); v(T) = 0\}$,

$$\Psi_+ : \underbrace{-\partial_t v + i\partial_x^2 v + 2\partial_x (T_u T_{\bar{u}} v)}_{=: Av} \mapsto \Psi_+(Av) = \int_0^T \langle f \mid v \rangle dt + \langle w_0 \mid v(0, \cdot) \rangle,$$

where $\langle \cdot \mid \cdot \rangle$ denotes the standard L^2 -inner product on \mathbb{T} . To motivate the choice of Ψ_+ , let w be a solution of the linear equation (3.1.11), then for all $v \in \mathcal{F}$, we have after integrating by parts

$$\langle \partial_t w \mid v \rangle = \langle w \mid -i\partial_x^2 v - 2\partial_x (T_u T_{\bar{u}} v) \rangle + \langle f \mid v \rangle.$$

Integrating in time between 0 and T , we find

$$\int_0^T \langle w | -\partial_t v + i\partial_x^2 v + 2\partial_x(T_u T_{\bar{u}} v) \rangle dt = \int_0^T \langle f | v \rangle dt + \langle w_0 | v(0, \cdot) \rangle.$$

That is, for all $v \in \mathcal{F}$,

$$\int_0^T \langle w | Av \rangle dt = \int_0^T \langle f | v \rangle dt + \langle w_0 | v(0, \cdot) \rangle =: \Psi_+(Av).$$

Our strategy is to prove that the functional Ψ_+ defines a continuous antilinear form so that, it can be represented by a function w , providing a solution to our equation.

By the estimate energy (3.1.10) in $H^{-1}(\mathbb{T})$,

$$\sup_{t \in [0, T]} \|v(t, \cdot)\|_{H^{-1}} \leq C_T \int_0^T \|Av(t, \cdot)\|_{H^{-1}} dt.$$

Therefore, Ψ_+ is well-defined and is continuous on the subspace $A(\mathcal{F})$ of $L^1([0, T]; H_+^{-1})$, since for all $v \in \mathcal{F}$,

$$\begin{aligned} |\Psi_+(Av)| &= \left| \int_0^T \langle f | v \rangle dt + \langle w_0 | v(0, \cdot) \rangle \right| \\ &\leq \sup_{t \in [0, T]} \|v(t, \cdot)\|_{H^{-1}} \int_0^T \|f(t, \cdot)\|_{H^1} dt + \|v(0, \cdot)\|_{H^{-1}} \|w_0\|_{H^1} \\ &\leq C_T \left(\|w_0\|_{H^1} + \int_0^T \|f(t, \cdot)\|_{H^1} dt \right) \int_0^T \|Av(t, \cdot)\|_{H^{-1}} dt. \end{aligned}$$

Hence, applying the Hahn–Banach theorem, we can extend the functional Ψ_+ to the $L^1([0, T]; H_+^{-1})$ -space. We then represent this extension by a vector w_+ in $L^\infty([0, T]; H_+^1)$, the dual space of $L^1([0, T]; H_+^{-1})$, according to the formula

$$\Psi_+(Av) = \int_0^T \langle w_+ | Av \rangle dt, \quad \forall v \in \mathcal{F}.$$

That is,

$$\int_0^T \langle w_+ | Av \rangle dt = \int_0^T \langle f | v \rangle dt + \langle w_0 | v(0, \cdot) \rangle, \quad (3.1.13)$$

for any $v \in \mathcal{F}$. Our goal is to show that such an w_+ is, in fact, solution of the linear equation (3.1.11) for $t \in [0, T]$, and satisfies the initial datum $w_+(0, x) = w_0(x)$.

For this purpose, take any $v \in \mathcal{C}_0^\infty((0, T); \mathcal{C}^\infty(\mathbb{T}))$, namely v equal to zero near at $t = 0, T$. Thus, using the formula (3.1.13), we deduce that w_+ satisfies the equation $A^*w_+ = f$ in the distribution sense, when $0 < t < T$. In other words, w_+ is a solution of the linear equation (3.1.11) in the distribution sense, when $0 < t < T$. Suppose for the moment that f and w_0 are very regular, let us say $w_0 \in \mathcal{C}^\infty$ and $f \in \mathcal{C}_{t,x}^\infty$. Then, from the equation (3.1.11), we have for all $p \geq 0$, $\partial_t w_+ \in L^\infty((0, T); H_+^{p-2})$ in

the distribution sense. Hence, $\int_0^t \partial_t w_+(s, \cdot) ds$ is continuous on $[0, T]$ and therefore $w_+ \in \mathcal{C}([0, T]; H_+^{p-2})$ as

$$w_+(t, \cdot) = \int_0^t \partial_t w_+(s, \cdot) ds + w_+(0, \cdot).$$

Making again the same manipulations, we deduce that $w_+ \in \mathcal{C}^1([0, T]; H_+^{p-4})$. Thus, for such an w , one can integrate by parts in time the term $\int_0^T \langle w_+ | Av \rangle dt$ in the formula (3.1.13) and infer

$$\langle w_0 | v(0, \cdot) \rangle = \langle w_+(0, \cdot) | v(0, \cdot) \rangle, \quad \forall v \in \mathcal{F}.$$

In particular, we have $w_+(0, \cdot) = w_0$. As a consequence, we have established the existence of a solution w_+ of (3.1.11) for $t \in [0, T]$, that satisfies the initial datum w_0 , and has all the regularity necessary to apply the energy estimates (3.1.5) in $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ if f and w_0 are \mathcal{C}^∞ , and so (3.1.4).

Now, to prove the existence of a solution to (3.1.11) for $t \in [-T, 0]$, we proceed in a similar way by considering the antilinear continuous form

$$\Psi_- : -\partial_t v - i\partial_x^2 v + 2\partial_x T_u T_{\bar{u}} v \rightarrow \int_{-T}^0 \langle f | v \rangle dt + \langle w_0 | v(0, \cdot) \rangle.$$

Similarly, we represent the extension of Ψ_- by a function w_- , leading to a solution on $[-T, 0]$, which satisfies the initial datum w_0 and the energy estimates (3.1.4), if f and w_0 are \mathcal{C}^∞ .

At this point of the proof, consider $w_0 \in H_+^1(\mathbb{T})$ and $f \in L^1([-T, T]; H_+^1)$. By an approximation argument, we choose $(w_0^k) \subseteq \mathcal{C}^\infty$ and $(f_k) \subseteq \mathcal{C}_{t,x}^\infty$ such that

$$\|w_0^k - w_0\|_{H^1} \rightarrow 0, \quad \int_0^T \|f_k(t) - f(t)\|_{H^1} dt \rightarrow 0.$$

Based on the facts set out above, there exists w^k solution of (3.1.11), satisfying the energy estimates (3.1.4). Hence,

$$\sup_{|t| \leq T} \|w^k(t) - w^\ell(t)\|_{H^1} \lesssim C_T (\|w^k(0, \cdot) - w^\ell(0, \cdot)\|_{H^1} + \|f_k(t) - f_\ell(t)\|_{L_t^1 H_x^1}).$$

As a result, (w^k) is a Cauchy sequence in $\mathcal{C}^0([-T, T]; H_+^1)$, whose limit $w \in \mathcal{C}^0([-T, T]; H_+^1)$ satisfies the linear equation (3.1.11), and the energy estimates (3.1.4) in $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$.

Note that the uniqueness of the solution is derived from the energy estimate in H^1 . Indeed, if w_1 and w_2 are two solutions of the linear equation (3.1.11), then $w_1 - w_2$ satisfies

$$\partial_t(w_1 - w_2) = i\partial_x^2(w_1 - w_2) + 2T_u T_{\bar{u}} \partial_x(w_1 - w_2),$$

with 0 as initial datum. Therefore, by the energy estimate (3.1.4) in H^1 , $w_1 - w_2 = 0$.

Step 2. Let $w_0 \in H_+^2(\mathbb{T})$ and $f \in L^1([-T, T]; H_+^2(\mathbb{T}))$. Observe that the first step of the proof ensures the existence of $w \in \mathcal{C}([-T, T]; H_+^1(\mathbb{T}))$, a unique solution to the linear equation (3.1.11)

$$\partial_t w = i\partial_x^2 w + 2T_u T_{\bar{u}} \partial_x w + f, \quad w(0, \cdot) = w_0.$$

Deriving the last equation, we find

$$\partial_t \partial_x w = i\partial_x^2 \partial_x w + 2\partial_x(T_u T_{\bar{u}} \partial_x w) + \partial_x f, \quad \tilde{w}(0, \cdot) = \partial_x w_0,$$

which means, $\partial_x w \in \mathcal{C}([-T, T]; L^2(\mathbb{T}))$ is a solution of the following equation

$$\partial_t \tilde{w} = i\partial_x^2 \tilde{w} + 2\partial_x(T_u T_{\bar{u}} \tilde{w}) + \partial_x f, \quad \tilde{w}(0, \cdot) = \partial_x w_0, \quad (3.1.14)$$

where its initial datum $\tilde{w}(0, \cdot) \in H_+^1(\mathbb{T})$ since $w_0 \in H_+^2(\mathbb{T})$, and $\partial_x f \in L_t^1(H_+^1)_x$. In the same vein as step 1, we prove that the last equation (3.1.14) is well-posed in $H_+^1(\mathbb{T})$ and satisfies the energy estimates (3.1.4) in $L_+^2(\mathbb{T})$. Hence, applying this estimate to $\partial_x w - \tilde{w}$, we infer that $\partial_x w = \tilde{w}$. Therefore, $w \in \mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$ since $\tilde{w} \in \mathcal{C}([-T, T]; H_+^1(\mathbb{T}))$.

Finally, it remains to prove the energy estimate in $H_+^2(\mathbb{T})$. For this purpose, we use the same argument as in Step 1 to show that, for f and w_0 regular enough, the energy estimate holds. Then, we pass to the limit using approximation arguments to conclude. \square

3.1.3 LWP of CS.

The next proposition illustrates the local well-posedness of the (CS')-problem, or initially, the (CS)-problem.

Proposition 3.1.5. *Let $R > 0$. For every $u_0 \in H_+^2(\mathbb{T})$, $\|u_0\|_{H^2} \leq R$, there exists $T := T_R > 0$ and $u \in \mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$ a unique solution of*

$$i\partial_t u + \partial_x^2 u + 2D\Pi(|u|^2)u = 0, \quad u(0, x) = u_0(x). \quad (\text{CS})$$

Proof. After rewriting the (CS)-problem as in (CS'), we recall the Kato-type iterative scheme corresponding to this equation :

$$\begin{cases} \partial_t u^{k+1} = i\partial_x^2 u^{k+1} + 2T_{u^k} T_{\bar{u}^k} \partial_x u^{k+1} + 2u^k H_{u^k} \partial_x u^k \\ u^{k+1}(0, x) = u_0(x). \end{cases} \quad (3.1.15)$$

Note that, Proposition 3.1.4 allows us to construct by induction the above sequence $(u^k) \subseteq \mathcal{C}(\mathbb{R}, H_+^2(\mathbb{T}))$ which we initiate by $u^0(t, x) := u_0(x) \in \mathcal{C}(\mathbb{R}, H_+^2(\mathbb{T}))$. Additionally, from the same proposition, we have

$$\sup_{|t| \leq T} \|u^{k+1}(t)\|_{H^2} \lesssim \exp\left(\int_{-T}^T \|u^k(t)\|_{H^2}^2 dt\right) \left[\|u_0\|_{H^2} + \int_{-T}^T \|u^k H_{u^k} \partial_x u^k\|_{H^2} dt\right].$$

Our aim is to show that this sequence converge to a limit in $\mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$, leading to a solution of the (CS)–problem. Applying Lemma 3.1.1 on the last inequality, it follows

$$\sup_{|t| \leq T} \|u^{k+1}(t)\|_{H^2} \leq C \exp\left(\int_{-T}^T \|u^k(t)\|_{H^2}^2 dt\right) \left[\|u_0\|_{H^2} + \int_{-T}^T \|u^k(t)\|_{H^2}^3 dt\right].$$

Assume $\|u_0\|_{H^2} \leq R$ for $R > 0$, and take $R_1 = (1 + C)R$. Since $R_1 > CR$, one can choose T depending on R such that,

$$Ce^{2CTR_1^2} (R + 2TR_1^3) \leq R_1 \quad (3.1.16)$$

Therefore, by induction, we infer that $\sup_{|t| \leq T} \|u^k(t)\|_{H^2} \leq R_1$ for all k , which means that up to a subsequence, it converges weakly star in L_t^∞ and weakly in $[H_+^2(\mathbb{T})]_x$. To characterize the limit for all t , let us check the uniform convergence in L^2 –norm. Observe that $u^{k+1} - u^k$ satisfies

$$\begin{aligned} \partial_t(u^{k+1} - u^k) &= i\partial_x^2(u^{k+1} - u^k) + 2T_{u^k}T_{\bar{u}^k}\partial_x(u^{k+1} - u^k) + \\ &\quad + 2(T_{u^k}T_{\bar{u}^k} - T_{u^{k-1}}T_{\bar{u}^{k-1}})\partial_x u^k + 2u^k H_{u^k}\partial_x u^k - 2u^{k-1}H_{u^{k-1}}\partial_x u^{k-1}. \end{aligned}$$

Thus, by the estimate energy (3.1.4) in $L^2(\mathbb{T})$ applied to $u^{k+1} - u^k$, we deduce

$$\begin{aligned} \sup_{|t| \leq T} \|u^{k+1} - u^k\|_{L^2} &\lesssim \exp\left(\int_{-T}^T \|u(t)\|_{H^2}^2 dt\right) \cdot \\ &\quad \left[\|2(T_{u^k}T_{\bar{u}^k} - T_{u^{k-1}}T_{\bar{u}^{k-1}})\partial_x u^k + 2(u^k H_{u^k}\partial_x u^k - u^{k-1}H_{u^{k-1}}\partial_x u^{k-1})\|_{L_t^1 L_x^2}\right]. \end{aligned}$$

Since $\sup_{|t| \leq T} \|u^k(t)\|_{H^2} \leq R_1$, we infer then by Lemma 3.1.1,

$$\sup_{|t| \leq T} \|u^{k+1}(t) - u^k(t)\|_{L^2} \leq CT \sup_{|t| \leq T} \|u^k(t) - u^{k-1}(t)\|_{L^2}.$$

To conclude, we need an additional condition on T , namely $CT < 1$, in addition to the one already mentioned in (3.1.16), so that the series of general term $\sup_{|t| \leq T(R)} \|u^{k+1}(t) - u^k(t)\|_{L^2}$ is geometrically convergent.

In sum, the sequence (u^k) converges weakly star in L_t^∞ and weakly in $[H_+^2(\mathbb{T})]_x$, and converges strongly uniformly in $\mathcal{C}([-T, T]; L_+^2(\mathbb{T}))$. Hence, the limit is well characterized, we denote it by u . To prove that u belongs to $\mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$, it remains to prove that $\|u(t)\|_{H^2}$ is continuous on $[-T, T]$. For this purpose, let $t \in [-T, T]$, and take any $h > 0$. Following the same lines of the proof of the energy estimate (3.1.4) in $H_+^2(\mathbb{T})$, we prove as in (3.1.9)

$$\|u(t+h)\|_{H^2} - \|u(t)\|_{H^2} \lesssim \int_t^{t+h} \|uH_u\partial_x u(s)\|_{H^2} ds + \int_t^{t+h} \|u(s)\|_{H^2}^3 ds.$$

Applying Lemma 3.1.1 on the latter inequality, and since

$$\|u(s)\|_{H^2} \leq \liminf_{k \rightarrow \infty} \|u^k(s)\|_{H^2} \leq R_1$$

for all $s \in [-T, T]$, we infer

$$\|u(t+h)\|_{H^2} - \|u(t)\|_{H^2} \longrightarrow 0, \quad \text{as } h \rightarrow 0.$$

Hence, $u \in \mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$ is a solution of (CS). This completes the problem of existence. The uniqueness of the initial value problem is established along the same lines as the uniform convergence in L^2 : We suppose that there exists $w \in \mathcal{C}([-T, T]; H_+^2(\mathbb{T}))$ satisfying (CS) or (CS'). Thus,

$$\begin{aligned} \partial_t(u^{k+1} - w^{k+1}) &= i\partial_x^2(u^{k+1} - w^{k+1}) + 2T_{u^k}T_{\bar{u}^k}\partial_x(u^{k+1} - w^{k+1}) + \\ &\quad + 2(T_{u^k}T_{\bar{u}^k} - T_{w^k}T_{\bar{w}^k})\partial_x u^k + 2(u^k H_{u^k}\partial_x u^k - w^k H_{w^k}\partial_x w^k) \end{aligned}$$

And we conclude in the same way as above using (3.1.4) and Lemma 3.1.1. □

Chapter 4

On the global well-posedness of the Calogero–Sutherland Derivative nonlinear Schrödinger equation

This chapter is from the preprint [Bad24a], accepted at Pure and Applied analysis.

ABSTRACT.

We consider the Calogero–Sutherland derivative nonlinear Schrödinger equation in the focusing (with sign $+$) and defocusing case (with sign $-$)

$$i\partial_t u + \partial_x^2 u \pm \frac{2}{i} u \partial_x \Pi(|u|^2) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T},$$

where Π is the Szegő projector $\Pi(\sum_{n \in \mathbb{Z}} \hat{u}(n) e^{inx}) = \sum_{n \geq 0} \hat{u}(n) e^{inx}$. Thanks to a Lax pair formulation, we derive the *explicit solution* to this equation. Furthermore, we prove the *global well-posedness* for this L^2 -critical equation in all the Hardy Sobolev spaces $H_+^s(\mathbb{T})$, $s \geq 0$, with small L^2 -initial data in the focusing case, and for arbitrarily L^2 -data in the defocusing case. In addition, we establish the relative compactness of the trajectories in all $H_+^s(\mathbb{T})$, $s \geq 0$.

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4.1 Introduction

This paper aims to prove the global well-posedness for the Calogero–Sutherland derivative nonlinear Schrödinger equations on the torus ($x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$) :

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm 2D_+(|u|^2)u = 0, \\ u(t=0, x) = u_0, \quad x \in \mathbb{T}, \end{cases} \quad (\text{CS})$$

for small L^2 -initial data u_0 in the focusing case (with sign +), and for arbitrarily L^2 -initial data in the defocusing case (with sign −). The operator D_+ in the nonlinear term of (CS) denotes $D\Pi$, where $D = -i\partial_x$, and Π is the Szegő projector acting on $L^2(\mathbb{T})$ as

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) := \sum_{n \geq 0} \widehat{u}(n) e^{inx}, \quad (4.1.1)$$

with value onto the Hardy space

$$L^2_+(\mathbb{T}) := \{u \in L^2(\mathbb{T}) \mid \widehat{u}(n) = 0, \forall n \in \mathbb{Z}_{\leq -1}\} \equiv \Pi(L^2(\mathbb{T})). \quad (4.1.2)$$

We equip $L^2_+(\mathbb{T})$ with the standard inner product of $L^2(\mathbb{T})$, $\langle u \mid v \rangle = \int_0^{2\pi} u \bar{v} \frac{dx}{2\pi}$. Our interest focuses on studying this equation with an unknown function u taken in the Hardy space of the torus, with a certain regularity. Thus, we denote by $H^s_+(\mathbb{T})$, the subspace of the Sobolev space $H^s(\mathbb{T})$, defined as

$$H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T}), \quad s \geq 0, \quad (4.1.3)$$

and equipped with the Sobolev norm

$$\|u\|_{H^s} = \|\langle D \rangle^s u\|_{L^2}, \quad \langle D \rangle^s = (1 + |D|^2)^{s/2}.$$

In Physics, this dynamical (CS)–equation is derived from the classical Calogero–Sutherland–Moser system (or Toda system) introduced in the end sixties–early seventies [Cal69, Cal71, Sut71, Sut75]. This physical model corresponds to a N -body problem describing the pairwise interactions of N identical particles. Abanov–Bettelheim–Wiegmann show in [ABW09] that taking the thermodynamic limit of such a model, and applying a change of variables leads to the (CS)–equation. In Mathematics, this equation has recently been studied on the real line ($x \in \mathbb{R}$) by [GL24], who referred to the equation as the Calogero–Moser derivative NLS equation. The transition of nomenclature to the Calogero–Sutherland DNLS equation

in the periodic setting ($x \in \mathbb{T}$) is connected to the physicist Sutherland, who has studied the Calogero–Sutherland–Moser system in the case where the N particles lie on the circle and interact with an inverse sin-square potential (trigonometric-type potential). Besides, one can obtain the (CS)–equation formally as a limit of the *intermediate nonlinear Schrödinger equation* introduced by Pelinovsky [Pel95],

$$i\partial_t u = \partial_x^2 u + (i - T)\partial_x(|u|^2)u, \quad (\text{INS})$$

where T is the integral operator

$$Tu(t, x) = \frac{1}{2\delta} \text{p.v.} \int_{-\infty}^{+\infty} \coth\left(\frac{\pi(x-y)}{2\delta}\right) u(t, y) dy,$$

by taking $\delta \rightarrow \infty$. The complex function u in (INS) represents the envelope of the fluid, and δ denotes its total depth. By passing to the limit $\delta \rightarrow \infty$, one obtains the same equation as (INS) but with the Hilbert transform

$$Hu(t, x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{u(t, y)}{x - y} dy,$$

instead of T [PG95]. And since the Szegő projector $\Pi_{\mathbb{T}} = \frac{1}{2}(\text{Id} + iH + \langle \cdot | 1 \rangle)$, then the (CS)–equation can also be interpreted as a model describing the interfacial wave packets in a deep stratified fluid.

It turns out that the Calogero–Sutherland DNLS equation is completely integrable. Thus, what does the word “integrability” mean? In line with the different perspectives developed by various schools, a number of definitions have been raised. If the word “integrable system” means for some researchers the existence of action–angle variables, a coordinate system in which the equation is completely solvable by quadratures, others would say that it refers to the existence of a Lax operator associated with the equation, and satisfying the isospectral property¹. However, a common facet of all these definitions is the presence of infinitely independent *integrals of motion*, or what we can also call *conservation laws*. Naturally, this infinite number of conservation laws plays a crucial role in proving some global well–posedness results.

In our case, Gérard–Lenzmann derived in [GL24, Lemma 5.1], for u sufficient regular, a Lax operator so that the focusing Calogero–Sutherland DNLS equation (CS)⁺ enjoys a Lax pair formulation on the real line \mathbb{R} . i.e., for any $u \in H_+^s(\mathbb{R})$ with s sufficiently large, there exist two operators (L_u, B_u) such that the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u], \quad [B_u, L_u] := B_u L_u - L_u B_u, \quad (4.1.4)$$

is satisfied with

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2. \quad (4.1.5)$$

1. See Remark 4.2.1.

The operator T_u is the Toeplitz operator of symbol u , and is defined for any $u \in L^\infty$ by

$$T_u f = \Pi(uf), \quad \forall f \in L_+^2, \quad (4.1.6)$$

where Π is the Szegő projector given in (4.1.1). In what follows, we check that this Lax equation holds true on the torus \mathbb{T} by retrieving the same Lax operators (L_u, B_u) as on the real line. And, as expected, through this Lax formalism, we derive infinite conservation laws $\langle (L_u + \lambda)^s 1 | 1 \rangle$, $\lambda \gg 0$, $s \geq 0$, in order to control the growth of the Sobolev norms $\|u(t)\|_{\dot{H}^s}$ uniformly for all $t \in \mathbb{R}$.²

Observe, the Calogero–Sutherland DNLS equation is invariant under the scaling

$$u(t, x) \mapsto \lambda^{1/2} u(\lambda^2 t, \lambda x), \quad \lambda \in \mathbb{R}, (t, x) \in I \times \mathbb{R}. \quad (4.1.7)$$

This suggests the L^2 -criticality of (CS) on \mathbb{R} as well as on \mathbb{T} . In [GL24, Theorem 2.1], the local well-posedness of the (CS) equation was achieved in $H_+^s(\mathbb{R})$ for $s > \frac{1}{2}$ by following the analysis of [dMP10]. In particular, for $s > \frac{3}{2}$, Gérard–Lenzmann [GL24, Proposition 2.1] used iterative schemes of Kato’s type and energy estimates to derive the local well-posedness in $H_+^s(\mathbb{R})$ [Sau79a]. On \mathbb{T} , the same proof of iterative schemes holds, and we deduce the local well-posedness in $H_+^s(\mathbb{T})$ for $s > \frac{3}{2}$. Therefore, we denote by $\mathcal{S}^+(t)$ the flow of the focusing Calogero–Sutherland DNLS equation (CS)⁺ and by $\mathcal{S}^-(t)$ the flow of the defocusing equation (CS)[−] : for all $s > \frac{3}{2}$, $t \in I_{\max}$,

$$\begin{aligned} \mathcal{S}^\pm(t) : H_+^s(\mathbb{T}) &\longrightarrow H_+^s(\mathbb{T}) \\ u_0 &\longmapsto u(t) \end{aligned}, \quad (4.1.8)$$

where I_{\max} denotes the maximal interval of the existence of the solution.

4.1.1 Main results

Some notation. In the sequel, we denote for any nonnegative integer a , by $\mathbb{N}_{\geq a}$ the subset of \mathbb{Z} given by $\{k \in \mathbb{Z} \mid k \geq a\}$. Moreover, we denote by $\mathcal{B}_{L_+^2}(r)$ the open ball of $L_+^2(\mathbb{T})$ centered at the origin, with radius $r > 0$.

The goal of the paper is to prove the global well-posedness of the L^2 -critical equation (CS) in all $H_+^s(\mathbb{T})$, $s \geq 0$. As a starting point, we state the results for the more challenging equation, the focusing Calogero–Sutherland DNLS equation

$$i\partial_t u + \partial_x^2 u + 2D_+(|u|^2)u = 0, \quad (\text{CS}^+) \quad (4.1.9)$$

then, we present the results for the defocusing case³

$$i\partial_t u + \partial_x^2 u - 2D_+(|u|^2)u = 0. \quad (\text{CS}^-) \quad (4.1.10)$$

2. In particular, one can see that the usual conservation laws : the average $\langle 1 | u \rangle$, and the L^2 norm $\|u\|_{L^2}$ are conserved for $s = 1$ and 2, since by definition of $L_u = D - T_u T_{\bar{u}}$ we have $L_u 1 = -\langle 1 | u \rangle u$.

3. We refer to the introduction of Weinstein [Wei15] for a mathematical and physical meaning of the terms focusing and defocusing for any dispersive equation.

Theorem 4.1.1. *For all $s > \frac{3}{2}$, the Calogero–Sutherland DNLS focusing equation (CS⁺) is globally well-posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C ,$$

where $C(\|u_0\|_{H^s})$ is a positive constant.

Remark 4.1.1. The restriction of smallness on the L^2 -norm of the initial data, namely $\|u_0\|_{L^2} < 1$, appears after applying a sharp inequality (Lemma 4.2.7) in order to control the growth of the Sobolev norms $\|u(t)\|_{\dot{H}^s}$, $s \geq 0$, by the conservation laws. This sharp inequality is an equality when we take for example

$$u_0(x) = \frac{\sqrt{1 - |q|^2}}{1 - q e^{ix}} , \quad q \in \mathbb{D} ,$$

which correspond to the profile of a (CS⁺)'s traveling wave of L^2 -norm $\|u_0\|_{L^2} = 1$ [Bad23]. More details for an eventual way to avoid this condition are presented in Section 4.6, but so far it is still an open problem.

As a second step, we focus on the main point of this paper : how the flow $\mathcal{S}^+(t)$ defined globally on $H_+^2(\mathbb{T})$ for $u_0 \in \mathcal{B}_{L_+^2}(1)$, can be extended to less regularity spaces for instance $L_+^2(\mathbb{T})$? Recall, as noted in (4.1.7), the Calogero–Sutherland DNLS equation is L^2 -critical. Based on the previous Theorem, and under the notation $u^\varepsilon(t) = \mathcal{S}^+(t)u_0^\varepsilon$, $\varepsilon > 0$, we state the following result.

Theorem 4.1.2. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists a unique potential $u \in \mathcal{C}(\mathbb{R}; L_+^2(\mathbb{T}))$ such that for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ where $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, the following convergence holds : for all $T > 0$,*

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0 , \quad \varepsilon \rightarrow 0 .$$

Moreover, the L^2 -norm of the limit potential u is conserved

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} , \quad \forall t \in \mathbb{R} . \tag{4.1.9}$$

As a consequence, Theorem 4.1.2 leads to the global well-posedness of the (CS⁺) problem in $L_+^2(\mathbb{T})$ in the following sense : There exists a unique continuous extension of the flow defined on $H_+^2(\mathbb{T})$, to $L_+^2(\mathbb{T})$, generating a unique continuous map

$$u_0 \in \mathcal{B}_{L_+^2}(1) \longmapsto u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T})) .$$

The key ingredient of the proof is to obtain $H^{\frac{1}{2}}$ bounds (inequality (4.3.21)) on the eigenfunctions of the Lax operator L_{u^ε} , which also constitute an orthonormal basis of $L_+^2(\mathbb{T})$. Therefore, we deduce the strong convergence of these eigenfunctions in L^2 . Finally, using Parseval's identity, we infer (4.1.9).

We also need to emphasize the important aspect of the uniqueness of the limit potential $u(t)$, obtained independently of the choice of the sequence (u_0^ε) that approximates $u_0 \in L_+^2(\mathbb{T})$. For this purpose, we derive in Proposition 4.2.5, **an explicit formula of the solution** of the focusing (CS⁺) equation. Thus, for any initial data u_0 , the solution of the (CS⁺) focusing equation is given by

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad z \in \mathbb{D} := \{|z| < 1\}, \quad (4.1.10)$$

where S^* denotes the adjoint of the Shift operator $S : h \mapsto zh$ in $L_+^2(\mathbb{T})$, and L_{u_0} is the Lax operator at $t = 0$. We underline two important facts about (4.1.10) :

- I. First, this inversion dynamical formula defined inside the open unit disc consists an explicit solution for the nonlinear PDE (CS⁺). This is not the first time that an explicit solution occurs while dealing with nonlinear integrable PDEs. Indeed, Gérard–Grellier derived in [GG15] an explicit solution for the Szegő equation, and recently Gérard also proves in [Gér23a] that the Benjamin–Ono equation has an explicit solution on \mathbb{R} and on \mathbb{T} . The common point to all these dynamical explicit formulas is that they all rely closely on the structure of the Lax operators induced by these equations.
- II. Beyond the fact that we have an explicit solution, this formula stresses out that the dynamics of the (CS⁺) equation are encoded by the Lax operator L_{u_0} , suggesting thus, that the so-called *actions–angles variables* must be related to the spectral elements of the Lax operators L_u .

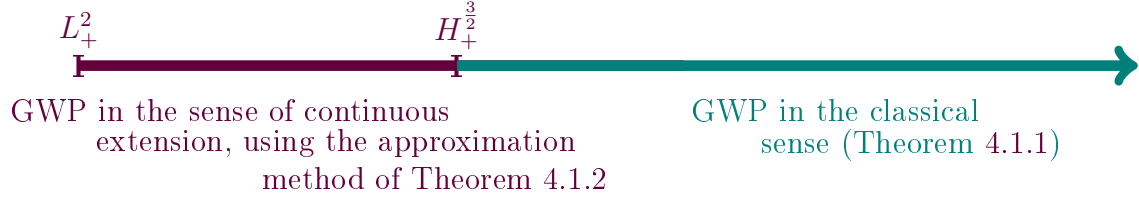
In view of Theorem 4.1.2, we state the third result.

Corollary 4.1.3. *For all $0 \leq s \leq \frac{3}{2}$, the Calogero–Sutherland DNLS focusing equation (CS⁺) is globally well-posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C,$$

where $C = C(\|u_0\|_{H^s}) > 0$ is a positive constant.

Remark 4.1.2. There is a subtlety hidden in the words of “globally well-posed” in the last statement. In fact, it is important to distinguish here the two different aspects of global well-posedness. First, we have the classical definition of GWP used in Theorem 4.1.1 : for any $u_0 \in H_+^s$ there exists a unique solution u defined on \mathbb{R} with value in H_+^s , such that u depends continuously on the initial data u_0 as a map $u_0 \in H_+^s \mapsto u \in \mathcal{C}(\mathbb{R}, H_+^s)$. The second definition is the one described in Theorem 4.1.2 in the sense : we suppose that the equation is defined at least in the distribution sense, then we extend the flow defined on high regularity spaces to low regularity spaces through continuous extension.



In this corollary, the global well-posedness is in the sense used in Theorem 4.1.2. This will become clearer once the proof is established (see Section 4.4). We also expect that, following arguments in [dMP10], one can go down for the global well-posedness in the classical sense to $H_+^s(\mathbb{T})$ with $s > \frac{1}{2}$.

Beyond the global well-posedness results on the Cauchy Problem of (CS^+) , we are interested in some qualitative properties about the flow $\mathcal{S}^+(t)$ of this equation.

Theorem 4.1.4. *Given an initial data $u_0 \in \mathcal{B}_{L_+^2}(1) \cap H_+^s(\mathbb{T})$, $s \geq 0$, the orbit of the solution $\{\mathcal{S}^+(t)u_0; t \in \mathbb{R}\}$ is relatively compact in $H_+^s(\mathbb{T})$.*

The defocusing equation (CS^-) . Moving now to the defocusing case of the Calogero–Sutherland DNLS equation, this latter equation enjoys also a Lax pair structure : for any $u(t) \in H_+^s$, with s large enough, there exist two operators

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2,$$

satisfying the Lax equation

$$\frac{d\tilde{L}_u}{dt} = [\tilde{B}_u, \tilde{L}_u].$$

Therefore, using the same methods as on the focusing case, we prove that the conservation laws $\langle (\tilde{L}_u + \lambda)^s 1 \mid 1 \rangle$, $s \geq 0$, $\lambda > 0$, controls uniformly the growth of the Sobolev norms without requiring any additional condition on the initial data. As a consequence, we obtain similar results in the defocusing case as in the focusing case, *regardless of how large the initial data is in L^2* . To summarize, we have the following.

Theorem 4.1.5. *The Calogero–Sutherland DNLS defocusing equation (CS^-) is globally well-posed in $H_+^s(\mathbb{T})$ for any $s \geq 0$ in the sense of Remark 4.1.2. In addition, for all $u_0 \in H_+^s(\mathbb{T})$,*

$$u(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2it\tilde{L}_{u_0}} S^*)^{-1} u_0 \mid 1 \right\rangle,$$

is the solution to the (CS^-) -defocusing equation. Furthermore, the trajectories

$$\{\mathcal{S}^-(t)u_0; t \in \mathbb{R}\}$$

are relatively compact in $H_+^s(\mathbb{T})$.

4.1.2 Other related equations

As explained in [GL24], the Calogero–Sutherland DNLS equation (CS) can be seen as mass critical version of the Benjamin–Ono equation. We refer to [GK21, GKT23] for a deep study of this latter equation on the torus. Of course, the Calogero–Sutherland DNLS equation (CS) is also considered as part of the nonlinear Schrödinger’s family. Several authors have been interested in different types of NLS–equations over the years. Some of these equations are classified and presented in [Bou99]. Maybe the most closely related to the (CS)–equation are :

(i) **Cubic NLS equation.**

$$i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, \quad (\text{NLS-cubic})$$

which is considered as one of the simplest PDE enjoying complete integrable properties. Zakharov–Shabat have studied this equation in [SZ72] using inverse scattering method. Moreover, global well–posedness results in $L^2(\mathbb{T})$ are presented in Bourgain [Bou93] after he introduced the $X^{s,b}$ –spaces. His proof relies on establishing $L^4(\mathbb{T})$ –Strichartz estimates and using L^2 –conservation norm. Actually, this result of L^2 –well–posedness is known to be sharp, and it is illustrated by various types of ill–posedness results below the regularity $L^2(\mathbb{T})$. Indeed, Burq–Gérard–Tzvetkov proved in [BGT02] that the flow map of (NLS-cubic) fails to be uniformly continuous for Sobolev regularity below L^2 . Christ–Colliander–Tao [CCT03] and Molinet [Mol09] showed the discontinuity of the map solution in $H^s(\mathbb{T})$ for $s < 0$.

For a deep study of (NLS-cubic) using integrable tools, Birkhoff normal form, and some applications, we refer to Kappeler–Lohrmann–Topalov–Zung [KLTZ09], Grébert–Kappeler [GKK14] and Kappeler–Schaad–Topalov [KST17]. For a study on the line \mathbb{R} , we cite [HGKV20]. More references are also provided in [OS12].

(ii) **DNLS equation.**

$$i\partial_t u + \partial_x^2 u + \pm i\partial_x (|u|^2 u) = 0, \quad (\text{DNLS})$$

which is also an integrable equation enjoying infinite conservation laws [KN78]. Using the I-method, Win proved in [Win10] the global well–posed of (DNLS)–equation in $H^s(\mathbb{T})$, $s > \frac{1}{2}$ for small data in $L^2(\mathbb{T})$. More recently, Klaus–Schippa [KS22] presented law regularity a priori estimates of $\|u\|_{H^s}$ for $0 < s < \frac{1}{2}$ upon small L^2 –norm, where $u \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{S}(\mathbb{T}))$ and $\mathcal{S}(\mathbb{T})$ denotes the Schwartz space. Actually, they proved the a priori estimates

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{B_{r,2}^s} \lesssim \|u(0)\|_{B_{r,2}^s}$$

in any Besov space $B_{r,2}^s$, with $r \in [1, \infty]$ and $0 < s < \frac{1}{2}$. For a study on the line \mathbb{R} , we cite [JLPS20, BP22, BLP22, KNV21, HGKV23, HGKNV22b].

4.1.3 Outline of the paper

The paper is organized as follows.

In Section 4.2, we discuss some properties about the Lax operators of the Calogero–Sutherland DNLS focusing equation (CS^+) . We derive the explicit formula of the solution of (CS^+) in the first subsection 4.2.1. Then, we prove in the second subsection 4.2.2, the global well-posedness of the (CS^+) problem in $H_+^s(\mathbb{T})$ for any $s > \frac{3}{2}$.

In Section 4.3, we extend the flow $\mathcal{S}^+(t)$ of (CS^+) continuously from $H_+^2(\mathbb{T})$ to $L_+^2(\mathbb{T}) \equiv H_+^0(\mathbb{T})$. To this end, we use an approximation method, and we characterize in the first subsection 4.3.1.1 the limit potential $u(t)$ for all $t \in \mathbb{R}$. Then, in the second subsection 4.3.2, we make sure that the lack of compactness in $L_+^2(\mathbb{T})$ do not occur while passing to the limit from $H_+^2(\mathbb{T})$ to $L_+^2(\mathbb{T})$. In the same subsection, we derive an orthonormal basis of $L_+^2(\mathbb{T})$ where the coordinates of the solution $u(t)$ have nice evolution in this basis. This evolution suggests that the so-called “Birkhoff coordinates” are the coordinates of $u(t)$ in this basis.

After that, we deal in Section 4.4 with the problem of global well-posedness of (CS^+) in $H_+^s(\mathbb{T})$ for $0 < s \leq \frac{3}{2}$. Moreover, we address the property of relative compactness of the orbits of (CS^+) in $H_+^s(\mathbb{T})$, $s \geq 0$.

Moving to Section 4.5, we present the Lax pair for the defocusing Calogero–Sutherland DNLS equation (CS^-) and we state the analogous results of (CS^+) in the case of (CS^-) .

Finally, in Section 4.6, we discuss some remarks and open problems related to this equation.

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4.2 The Lax pair structure

As noted in the introduction, we first check that the Lax pair defined in (4.1.5) holds the same in the context of the torus \mathbb{T} as on the real line \mathbb{R} , even though on the real line \mathbb{R} , a complex function f is decomposed as

$$f = \Pi f + \overline{\Pi f}, \quad \widehat{\Pi f}(\xi) = \mathbf{1}_{\xi > 0} \widehat{f}(\xi), \quad \xi \in \mathbb{R},$$

while on the torus \mathbb{T} ,

$$f = \Pi f + \overline{\Pi f} - \langle f | 1 \rangle, \quad \Pi \left(\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx} \right) := \sum_{n \in \mathbb{N}_{\geq 0}} \widehat{f}(n) e^{inx}.$$

Proposition 4.2.1 (The Lax pair). *For any $s > \frac{3}{2}$, let $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$ be a solution of the focusing equation (CS^+) . Then, there exist two operators*

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$$

satisfying the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u],$$

where T_u is the Toeplitz operator defined in (4.1.6).

Proof. Let $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, be a solution of (CS⁺) equation. On the one hand, we have by definition of L_u and for all $h \in H_+^1(\mathbb{T})$,

$$\begin{aligned} \frac{dL_u(h)}{dt} &= -T_{\partial_t u} T_{\bar{u}}(h) - T_u T_{\partial_t \bar{u}}(h) \\ &= -T_{i\partial_x^2 u + 2u\partial_x \Pi(|u|^2)} T_{\bar{u}} h - T_u T_{-i\partial_x^2 \bar{u} + 2\bar{u}\partial_x \overline{\Pi(|u|^2)}} h. \end{aligned}$$

Therefore, since u belongs to the Hardy space,

$$\frac{dL_u(h)}{dt} = i[T_u T_{\partial_x^2 \bar{u}} - T_{\partial_x^2 u} T_{\bar{u}}](h) - 2u \left[\partial_x \Pi(|u|^2) \cdot \Pi(\bar{u}h) + \Pi \left(\partial_x \overline{\Pi(|u|^2)} \cdot \bar{u}h \right) \right]. \quad (4.2.1)$$

On the other hand, expanding the commutator $[B_u, L_u](h) = B_u L_u h - L_u B_u h$, we obtain

$$\begin{aligned} &T_u T_{\partial_x \bar{u}} Dh - T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}} h - T_{\partial_x u} T_{\bar{u}} Dh + T_{\partial_x u} T_{\bar{u}} T_u T_{\bar{u}} h + i(T_u T_{\bar{u}})^2 Dh \\ &- D(T_u T_{\partial_x \bar{u}} h) + T_u T_{\bar{u}} T_u T_{\partial_x \bar{u}} h + D(T_{\partial_x u} T_{\bar{u}} h) - T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} h - iD((T_u T_{\bar{u}})^2 h), \end{aligned}$$

where by the Leibniz rule, $D(T_u h) = -iT_{\partial_x u} h + T_u Dh$, so that

$$\begin{aligned} D(T_u T_{\partial_x \bar{u}} \cdot) &= T_u T_{\partial_x \bar{u}} D - iT_{\partial_x u} T_{\partial_x \bar{u}} - iT_u T_{\partial_x^2 \bar{u}}, \\ D(T_{\partial_x u} T_{\bar{u}} \cdot) &= T_{\partial_x u} T_{\bar{u}} D - iT_{\partial_x^2 u} T_{\bar{u}} - iT_{\partial_x u} T_{\partial_x \bar{u}}, \end{aligned}$$

$$D((T_u T_{\bar{u}})^2 \cdot) = -i[T_{\partial_x u} T_{\bar{u}} T_u T_{\bar{u}} + T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}} + T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} + T_u T_{\bar{u}} T_u T_{\partial_x \bar{u}}] + (T_u T_{\bar{u}})^2 D \cdot.$$

As a consequence,

$$\begin{aligned} [B_u, L_u](h) &= iT_u T_{\partial_x^2 \bar{u}} h - iT_{\partial_x^2 u} T_{\bar{u}} h - 2(T_u T_{\bar{u}} T_{\partial_x u} T_{\bar{u}} + T_u T_{\partial_x \bar{u}} T_u T_{\bar{u}})(h) \\ &= i[T_u T_{\partial_x^2 \bar{u}} - T_{\partial_x^2 u} T_{\bar{u}}] - 2u \cdot \Pi(\partial_x |u|^2 \cdot \Pi(\bar{u}h)). \end{aligned} \quad (4.2.2)$$

Comparing (4.2.1) and (4.2.2), it appears that all that remains to be proved is

$$\left[\partial_x \Pi(|u|^2) \cdot \Pi(\bar{u}h) + \Pi(\partial_x \overline{\Pi(|u|^2)} \cdot \bar{u}h) \right] = \Pi(\partial_x |u|^2 \cdot \Pi(\bar{u}h)), \quad h \in H_+^1(\mathbb{T}). \quad (4.2.3)$$

In fact, any complex function $f \in L^2(\mathbb{T})$ can be decomposed as

$$f = \Pi f + \overline{\Pi f} - \langle f | 1 \rangle.$$

In particular, for $f = \bar{u}h$, we have $\Pi(\partial_x \overline{\Pi(|u|^2)} \cdot \bar{u}h)$ equal to

$$\Pi \left(\partial_x \overline{\Pi(|u|^2)} \cdot \Pi(\bar{u}h) \right) + \Pi \left(\partial_x \overline{\Pi(|u|^2)} \cdot \overline{\Pi(\bar{u}h)} \right) - \langle \bar{u}h | 1 \rangle \Pi(\partial_x \overline{\Pi(|u|^2)}),$$

where the last two terms vanishes, since Π is an orthogonal projector into the Hardy space. Therefore, the left-hand side of (4.2.3) coincides with

$$\Pi \left(\partial_x \Pi(|u|^2) \cdot \Pi(\bar{u}h) \right) + \Pi \left(\partial_x \overline{\Pi(|u|^2)} \cdot \Pi(\bar{u}h) \right),$$

which is equal to $\Pi(\partial_x |u|^2 \cdot \Pi(\bar{u}h))$ since $\langle \partial_x (|u|^2) | 1 \rangle = 0$. \square

4.2.1 The explicit formula of the solution

Using this Lax pair structure, we derive in this subsection the explicit formula of the solution of the focusing Calogero–Sutherland DNLS equation (CS⁺). To this end, we also need the shift operator introduced in the following paragraph.

Some Preliminaries. We recall one of the most important operator on Hardy's space, *the shift operator*, defined on $L_+^2(\mathbb{T})$ as the isometric map

$$S: h \in L_+^2(\mathbb{T}) \mapsto e^{ix}h \in L_+^2(\mathbb{T}).$$

Its adjoint in $L_+^2(\mathbb{T})$ is given by

$$S^*: h \in L_+^2(\mathbb{T}) \mapsto S^*h = T_{e^{-ix}}h = \Pi(e^{-ix}h) \in L_+^2(\mathbb{T}).$$

In particular, we have

$$S^*S = \text{Id}, \quad SS^* = \text{Id} - \langle \cdot | 1 \rangle 1, \quad (4.2.4)$$

leading to the fact that the shift map S is injective but not surjective. Pointing out that the Hardy space can be defined with different approaches, for instance,

$$\mathbb{H}_2(\mathbb{D}) := \left\{ u \in \text{Hol}(\mathbb{D}); \sup_{0 \leq r < 1} \int_0^{2\pi} |u(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\},$$

which is equivalent via the isometric isomorphism

$$u(z) = \sum_{k \geq 0} \hat{u}(k) z^k \mapsto u^*(x) := \sum_{k \geq 0} \hat{u}(k) e^{ikx},$$

to the Hardy space $L_+^2(\mathbb{T})$ defined in (4.1.2), then one could read the shift operator acting as multiplication by z . In what follows, we use indifferently u and the boundary function u^* , by making a slight abuse of notation and denoting both by u .

Coming back to the problem, we need some commutator identities to obtain the explicit formula. This is the purpose of the next Lemma.

Lemma 4.2.2. *Let $u \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, then*

$$\begin{aligned} [S^*, L_u] &= S^* - \langle \cdot | u \rangle S^*u, \\ [S^*, B_u] &= i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right). \end{aligned} \quad (4.2.5)$$

Proof. The first identity is a direct consequence of proving

$$L_u S = S L_u + S - \langle \cdot | S^*u \rangle u, \quad (4.2.6)$$

and taking the adjoint of all these operators in $L^2_+(\mathbb{T})$. Recall $L_u = D - T_u T_{\bar{u}}$. On the one hand, we have by the Leibniz rule $D(Sh) = S(\text{Id} + D)h$, for all $h \in H^1_+(\mathbb{T})$. On the other hand, observe for all $f \in L^2(\mathbb{T})$,

$$\Pi(Sf) = S\Pi(f) + \langle Sf | 1 \rangle .$$

In particular, for $f = h\bar{u}$, we infer

$$T_{\bar{u}}(Sh) = ST_{\bar{u}}h + \langle Sh | u \rangle . \quad (4.2.7)$$

Hence, taking into consideration that the operators S and T_u commute, we deduce identity (4.2.6).

Now, to prove the second point of (4.2.5) we use the first point. Recall that $B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$, and by (4.2.7) we have $[T_{\bar{u}}, S] = \langle \cdot | S^* u \rangle$, in other words, $[S^*, T_u] = \langle \cdot | 1 \rangle S^* u$. Thus, after noting that S^* and $T_{\bar{u}}$ commute, we deduce

$$\begin{aligned} [S^*, T_u T_{\partial_x \bar{u}}] &= \langle \cdot | \partial_x u \rangle S^* u . \\ [S^*, T_{\partial_x u} T_{\bar{u}}] &= \langle \cdot | u \rangle S^* \partial_x u , \\ [S^*, (T_u T_{\bar{u}})^2] &= \langle \cdot | T_u T_{\bar{u}} u \rangle S^* u + T_u T_{\bar{u}} (\langle \cdot | u \rangle S^* u) . \end{aligned}$$

As a result,

$$[S^*, B_u] = \langle \cdot | \partial_x u \rangle S^* u - \langle \cdot | u \rangle S^* \partial_x u + i \langle \cdot | T_u T_{\bar{u}} u \rangle S^* u + iT_u T_{\bar{u}} (\langle \cdot | u \rangle S^* u) .$$

Using the adjoint Leibniz rule $S^* D = (D + \text{Id})S^*$ and since $L_u = D - T_u T_{\bar{u}}$, we infer

$$\begin{aligned} [S^*, B_u] &= -i \langle \cdot | L_u u \rangle S^* u - i L_u (\langle \cdot | u \rangle S^* u) - i \langle \cdot | u \rangle S^* u \\ &= -i (\langle \cdot | u \rangle S^* u) L_u - i (L_u + \text{Id}) (\langle \cdot | u \rangle S^* u) . \end{aligned}$$

We conclude by the first identity of (4.2.5) that $-\langle \cdot | u \rangle S^* u = S^* L_u - L_u S^* - S^*$ and hence

$$[S^*, B_u] = i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right) .$$

□

Proposition 4.2.3. *Let $u(t) \in H^s_+(\mathbb{T})$, $s > \frac{3}{2}$. The Lax operator $(L_{u(t)}, H^1_+(\mathbb{T}))$ is a self-adjoint operator with a discrete spectrum bounded from below. Moreover, $B_{u(t)}$ is a skew-symmetric bounded operator on $L^2_+(\mathbb{T})$.*

Proof. The proof is a direct consequence of Kato–Rellich’s theorem. Indeed, the differential operator $(D, H^1_+(\mathbb{T}))$ is a positive self-adjoint operator on the Hardy space $L^2_+(\mathbb{T})$. In addition, $T_u T_{\bar{u}}$ is relatively bounded with respect to D , since for all $h \in H^1_+(\mathbb{T})$,

$$\|T_u T_{\bar{u}} h\|_{L^2} \leq \|u\|_{L^\infty}^2 \|h\|_{L^2} \leq \varepsilon \|Dh\|_{L^2(\mathbb{T})} + \|u\|_{L^\infty}^2 \|h\|_{L^2(\mathbb{T})}, \quad 0 \leq \varepsilon < 1 .$$

Furthermore, the spectrum of L_u is discrete since the resolvent of L_u is compact by the Rellich–Kondrachev theorem. And it is bounded from below as the operator L_u is a semi-bounded operator. Besides, one can easily observe by definition of $B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$, that this operator is a skew-symmetric operator. □

In view of the previous proposition, we denote by $(\lambda_n(u))_{n \geq 0}$ the eigenvalues of L_u ordered by increasing modulus, and taking into account their multiplicity

$$\lambda_0(u) \leq \lambda_1(u) \leq \lambda_2(u) \leq \dots \leq \lambda_n(u) \leq \dots$$

Remark 4.2.1 (Isospectral property). As discovered in the modern theory of integrable systems [GGKM67] and reformulated by [Lax68], the eigenvalues of a Lax operator are integrals of motion of the associated equation. In fact, any Lax operator satisfies the isospectral property, namely, there exists a one-parameter family of unitary operators $U(t)$ such that $U(t)^{-1}L_{u(t)}U(t)$ is independent of t . That is,

$$U(t)^{-1}L_{u(t)}U(t) = L_{u_0}. \quad (4.2.8)$$

This implies, that the eigenvalues $(\lambda_n(u))$ of L_u are all conserved along the flow of (CS^+) . Or in other words, for all $n \in \mathbb{N}_{\geq 0}$, $\lambda_n(u(t)) = \lambda_n(u_0)$ for all $t \in \mathbb{R}$.

The following lemma provides a rewrite of the Calogero–Sutherland DNLS equation focusing on (CS^+) in terms of the Lax operators L_u and B_u . This will certainly be useful during the proof of the dynamical explicit formula.

Lemma 4.2.4. *Given $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, a solution of (CS^+) equation, then*

$$\partial_t u = B_u u - iL_u^2 u.$$

Proof. By definition of $B_u := T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$,

$$\begin{aligned} \partial_t u - B_u u &= i\partial_x^2 u + 2iD_+(|u|^2)u - T_u T_{\partial_x \bar{u}} u + T_{\partial_x u} T_{\bar{u}} u - i(T_u T_{\bar{u}})^2 u \\ &= -i [D^2 u - 2u \cdot D\Pi(|u|^2) + u \cdot \Pi(D\bar{u} \cdot u) - Du \cdot \Pi(|u|^2) + (T_u T_{\bar{u}})^2 u]. \end{aligned}$$

Applying Leibniz's rule on $D(u \cdot \Pi(|u|^2))$, we infer

$$\partial_t u - B_u u = -i [D^2 u - D[\Pi(|u|^2) \cdot u] + u\Pi(D\bar{u} \cdot u) - uD\Pi(|u|^2) + (T_u T_{\bar{u}})^2 u].$$

Again, using Leibniz's rule on the term $D\Pi(|u|^2)$,

$$\partial_t u - B_u u = -i [D^2 u - D(T_u T_{\bar{u}} u) - u\Pi(\bar{u} \cdot Du) + (T_u T_{\bar{u}})^2 u] = -iL_u^2 u.$$

□

Following [Gér23a] and [GG15], we derive the explicit formula for the solution of the Calogero–Sutherland DNLS focusing equation.

Proposition 4.2.5 (The explicit formula). *Given $u_0 \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, the solution of the focusing Calogero–Sutherland DNLS equation (CS^+) is given by*

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}.$$

Proof. Since $u(t, \cdot) \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, for all $t \in [-T, T]$, then for all $z \in \mathbb{D}$

$$u(t, z) = \sum_{k=0}^{\infty} \widehat{u}(t, k) z^k = \sum_{k=0}^{\infty} \langle u(t) | S^k 1 \rangle z^k = \sum_{k=0}^{\infty} \langle (S^*)^k u(t) | 1 \rangle z^k,$$

where by the Neumann series of

$$\sum_{k=0}^{\infty} (zS^*)^k = (\text{Id} - zS^*)^{-1},$$

we infer

$$u(t, z) = \langle (\text{Id} - zS^*)^{-1} u(t) | 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (4.2.9)$$

Now, consider a one-parameter family $U(t)$ solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt} U(t) = B_{u(t)} U(t) \\ U(0) = \text{Id}. \end{cases} \quad (4.2.10)$$

Observe that $U(t)$ is a unitary operator since B_u is skew-adjoint. Moreover, using the Lax pair structure of Proposition 4.2.1, and (4.2.10),

$$\frac{d}{dt} (U(t)^* L_{u(t)} U(t)) = 0,$$

and thus

$$U(t)^* L_{u(t)} U(t) = L_{u_0}. \quad (4.2.11)$$

Hence, by applying $U(t)^*$ to both sides of the inner product (4.2.9),

$$\begin{aligned} u(t, z) &= \langle U(t)^* (\text{Id} - zS^*)^{-1} u(t) | U(t)^* 1 \rangle \\ &= \langle (\text{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* u(t) | U(t)^* 1 \rangle. \end{aligned} \quad (4.2.12)$$

The aim is to express differently $U(t)^* S^* U(t)$, $U(t)^* u(t)$ and $U(t)^* 1$. Using (4.2.10) and since B_u is a skew-adjoint operator (Proposition 4.2.3), we find

$$- \frac{d}{dt} [U(t)^* 1] = -U(t)^* B_{u(t)} 1 = -iU(t)^* L_{u(t)}^2 1$$

$$- \frac{d}{dt} [U(t)^* u(t)] = -U(t)^* B_{u(t)} u(t) + U(t)^* \partial_t u(t) = -iU(t)^* L_{u(t)}^2 u(t) \text{ by Lemma 4.2.4}$$

$$- \frac{d}{dt} [U(t)^* S^* U(t)] = -U(t)^* B_{u(t)} S^* U(t) + U(t)^* S^* B_{u(t)} U(t) = U(t)^* [S^*, B_{u(t)}] U(t),$$

where the third point is equal to

$$\frac{d}{dt} [U(t)^* S^* U(t)] = iU(t)^* \left(S^* L_{u(t)}^2 - (L_{u(t)} + \text{Id})^2 S^* \right) U(t),$$

by Lemma 4.2.2. Therefore, applying the identity $U(t)^* L_{u(t)} = L_{u_0} U(t)^*$ of (4.2.11), we deduce

$$\begin{aligned}
& - \frac{d}{dt}[U(t)^*1] = -iL_{u_0}^2[U(t)^*1] \\
& - \frac{d}{dt}[U(t)^*u(t)] = -iL_{u_0}^2[U(t)^*u(t)] \\
& - \frac{d}{dt}[U(t)^*S^*U(t)] = i\left([U(t)^*S^*U(t)]L_{u_0}^2 - (L_{u_0} + \text{Id})^2[U(t)^*S^*U(t)]\right).
\end{aligned}$$

As a consequence,

$$U(t)^*1 = e^{-itL_{u_0}^2} 1, \quad U(t)^*u(t) = e^{-itL_{u_0}^2} u_0, \quad (4.2.13)$$

and

$$U(t)^*S^*U(t) = e^{-it(L_{u_0} + \text{Id})^2} S^* e^{itL_{u_0}^2}. \quad (4.2.14)$$

Combining (4.2.12), (4.2.13) and (4.2.14), the claimed formula follows. \square

4.2.2 Global well-posedness of (CS^+) in $H_+^s(\mathbb{T})$, $s > \frac{3}{2}$

To prove the global well-posedness of (CS^+) , we need to derive some conservation laws and energy estimates.

Lemma 4.2.6 (Conservation laws). *Let $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $r > \frac{3}{2}$, solution of (CS^+) . For all $\lambda \gg 0$, the family $\{\mathcal{H}_s(u) := \langle (L_u + \lambda)^s u \mid u \rangle$; $0 \leq s \leq 2r\}$ is conserved by the flow of (CS) .*

Remark 4.2.2.

- Using complex interpolation method [Tay81, Chapter I. 4.], one can observe as demonstrate in Proposition 4.2.8, that the $\mathcal{H}_s(u) \leq C\|u\|_{H^{\frac{s}{2}}}^2$.
- The condition $r > \frac{3}{2}$ is to guarantee the existence of $u(t)$. It can be omitted once we prove in Section 4.4 that the flow $u_0 \in \mathcal{B}_{L_+^2}(1) \cap H_+^r(\mathbb{T}) \mapsto u(t) \in H_+^r(\mathbb{T})$ exists for all $r \geq 0$.

Proof. Given $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $r > \frac{3}{2}$, solution of (CS^+) , we consider the unitary operator $U(t)$ defined in (4.2.10). Then, by (4.2.13), we know that $U(t)^*u(t) = e^{-itL_{u_0}^2} u_0$. And, since L_u is a self-adjoint operator by Proposition 4.2.3, we infer by (4.2.11),

$$U(t)^*(L_{u(t)} + \lambda)^s U(t) = (L_{u_0} + \lambda)^s.$$

Therefore, for all $0 \leq s \leq 2r$,

$$\begin{aligned}
\mathcal{H}_s(u(t)) &= \langle U(t)^*(L_{u(t)} + \lambda)^s u(t) \mid U(t)^*u(t) \rangle \\
&= \langle (L_{u_0} + \lambda)^s U(t)^*u(t) \mid U(t)^*u(t) \rangle \\
&= \langle (L_{u_0} + \lambda)^s e^{-itL_{u_0}^2} u_0 \mid e^{-itL_{u_0}^2} u_0 \rangle
\end{aligned}$$

As a consequence, $\mathcal{H}_s(u(t)) = \mathcal{H}_s(u_0)$ as $(L_{u_0} + \lambda)^s$ and $e^{-itL_{u_0}^2}$ commute. \square

Remark 4.2.3. Using the identity $U(t)^*1 = e^{-itL_{u_0}}1$ of (4.2.13) and repeating the same proof of Lemma 4.2.6, one can also deduce for $\lambda \gg 0$, that the quantities $\langle (L_u + \lambda)^q 1 | u \rangle$ and $\langle (L_u + \lambda)^p 1 | 1 \rangle$ are conserved by the flow. Another way to show this, is to observe by definition of $L_u = D - T_u T_{\bar{u}}$ we have $L_u 1 = -\langle 1 | u \rangle u$ and the average $\langle u | 1 \rangle$ is conserved along the evolution, since

$$\partial_t \langle u | 1 \rangle = i \langle \partial_x^2 u | 1 \rangle + 2i \langle D\Pi(|u|^2) | \bar{u} \rangle = 0.$$

To prove the energy estimates and for future requests, we need the following lemma.

Lemma 4.2.7. *Let $h \in H_+^{\frac{1}{2}}(\mathbb{T})$, $u \in L_+^2(\mathbb{T})$,*

$$\|T_{\bar{u}}h\|_{L^2(\mathbb{T})}^2 \leq \left(\langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2 \right) \|u\|_{L^2(\mathbb{T})}^2. \quad (4.2.15)$$

Proof. By Parseval's identity,

$$\|T_{\bar{u}}h\|_{L^2(\mathbb{T})}^2 = \sum_{n \geq 0} |\widehat{T_{\bar{u}}h}(n)|^2,$$

where

$$\widehat{T_{\bar{u}}h}(n) = \widehat{\Pi(h\bar{u})}(n) = \sum_{p \geq 0} \widehat{h}(n+p) \overline{\widehat{u}(p)}. \quad (4.2.16)$$

Applying Cauchy–Schwarz's inequality, we infer

$$\|T_{\bar{u}}h\|_{L^2(\mathbb{T})}^2 \leq \|u\|_{L^2(\mathbb{T})}^2 \sum_{p \geq 0} \sum_{n \geq 0} |\widehat{h}(n+p)|^2.$$

Set $k = n + p$, then

$$\sum_{p \geq 0} \sum_{n \geq 0} |\widehat{h}(n+p)|^2 = \sum_{k \geq 0} (k+1) |\widehat{h}(k)|^2 = \langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2.$$

□

Remark 4.2.4.

1. Recall that the embedding $H^{\frac{1}{2}}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ fails to be true. However, taking the potential u as an element of the Hardy space $L_+^2(\mathbb{T})$, improved the estimate from $\|T_{\bar{u}}h\|_{L^2} \leq \|h\|_{L^\infty} \|u\|_{L^2}$ to (4.2.15).
2. From (4.2.16), one could see that, for all $h \in H_+^{\frac{1}{2}}(\mathbb{T})$, the Hilbert–Schmidt norm of the antilinear operator $u \in L_+^2(\mathbb{T}) \mapsto T_{\bar{u}}h$ is given by

$$\|\Pi(\bar{\cdot}h)\|_{HS}^2 = \sum_{p \geq 0} \sum_{n \geq 0} |\widehat{h}(n+p)|^2 = \sum_{k \geq 0} (k+1) |\widehat{h}(k)|^2 = \langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2.$$

In particular, we have $u \mapsto T_{\bar{u}}h$ is a compact antilinear operator in $L_+^2(\mathbb{T})$.

3. The inequality (4.2.15) of Lemma 4.2.7 is a sharp inequality since its proof relies on a simple application of the Cauchy–Schwarz inequality. In particular, when $h = u$, inequality (4.2.15) is an equality, if and only if

$$u(z) = \frac{c}{1 - qz}, \quad |q| < 1, \quad c \in \mathbb{C}$$

–which corresponds to the profile of a traveling wave of (CS⁺) when $c = \sqrt{1 - |q|^2}$ [Bad23]. Indeed, following arguments used in [GG08b, Lemma 1], one observe that the Cauchy–Schwarz inequality applied to (4.2.16) is an equality, if and only if for all $n \geq 0$, there exists $c_n \in \mathbb{C}$ such that

$$\widehat{u}(n+p) = c_n \widehat{u}(p), \quad \forall p \geq 0. \quad (4.2.17)$$

Hence, if $\widehat{u}(1) \neq 0$ and $\widehat{u}(0) \neq 0$,

$$c_n = \frac{\widehat{u}(n+1)}{\widehat{u}(1)} = \frac{\widehat{u}(n)}{\widehat{u}(0)},$$

leading to,

$$\widehat{u}(n) = \left(\frac{\widehat{u}(1)}{\widehat{u}(0)} \right)^n \widehat{u}(0), \quad \forall n \in \mathbb{N}.$$

Therefore, the sequence $(\widehat{u}(n))$ is a geometric progression with common ratio $q := \frac{\widehat{u}(1)}{\widehat{u}(0)}$, where $0 < |q| < 1$ since $\sum_{n=0}^{\infty} |\widehat{u}(n)|^2 < +\infty$. Hence,

$$u(z) = \sum_{n=0}^{\infty} \widehat{u}(n) z^n = \frac{\widehat{u}(0)}{1 - qz}.$$

Now, if $\widehat{u}(1) = 0$ or $\widehat{u}(0) = 0$ then by (4.2.17) we infer for $p = 0$ or 1 , $u = \widehat{u}(0)$.

We recall that $\mathcal{B}_{L^2_+}(r)$ denotes the open ball of $L^2_+(\mathbb{T})$ centered at the origin, with radius $r > 0$. And we denote for any $s \in \mathbb{R}$ by $\lfloor s \rfloor := \max\{k \in \mathbb{Z}; k < s\}$.

Proposition 4.2.8. *Let $u_0 \in \mathcal{B}_{L^2_+}(1) \cap H^r_+(\mathbb{T})$, $r > \frac{3}{2}$. Then, for all $\lambda \gg 0$, there exists $C = C(\|u_0\|_{H^{\lfloor \frac{1}{2} \rfloor}}, \lambda) > 0$ independent of t , such that for every $f \in H^s_+(\mathbb{T})$,*

$$\frac{1}{C} \|f\|_{H^s} \leq \|(L_{u(t)} + \lambda)^s f\|_{L^2} \leq C \|f\|_{H^s}, \quad \forall 0 \leq s \leq \frac{\lfloor 2r \rfloor + 1}{2}. \quad (4.2.18)$$

Remark 4.2.5. The condition $r > \frac{3}{2}$ is to guarantee the existence of $u(t)$. It can be replaced by $r \geq 0$ once we prove in Section 4.4 that the flow $u_0 \in \mathcal{B}_{L^2_+}(1) \cap H^r_+(\mathbb{T}) \mapsto u(t) \in H^r_+(\mathbb{T})$ exists for all $r \geq 0$.

Proof. The proof is done by induction on every interval of length $1/2$.

Step 1 : $s \in [0, \frac{1}{2}]$. Let $s = \frac{1}{2}$, we have by definition of $L_u = D - T_u T_{\bar{u}}$,

$$\langle (L_u + \lambda)f | f \rangle = \langle Df | f \rangle - \|T_{\bar{u}}f\|_{L^2}^2 + \lambda \|f\|_{L^2}^2. \quad (4.2.19)$$

Applying the sharp inequality of Lemma 4.2.7,

$$\|(L_u + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq (1 - \|u\|_{L^2}^2) \langle Df | f \rangle + (\lambda - \|u\|_{L^2}^2) \|f\|_{L^2}^2. \quad (4.2.20)$$

Thus, we infer since $\|u\|_{L^2} = \|u_0\|_{L^2} < 1$,

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq \frac{1}{C^2} \|f\|_{H^{\frac{1}{2}}}^2,$$

where $C = C(\|u_0\|_{L^2}, \lambda) > 0$ is a positive constant independent of t . On the other hand, using the definition of L_u , it is easy to see that

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2} \leq C \|f\|_{H^{\frac{1}{2}}}.$$

Therefore, by complex interpolation [Tay81, Chapter I. 4], we deduce that inequality (4.2.18) holds true for all $s \in [0, \frac{1}{2}]$.

Step 2 : Uniform bounds on $\|u(t)\|_{L^p}$, $p \in [2, \infty)$. Applying step 1, and using the conservation laws of Lemma 4.2.6, we infer for $f = u$,

$$\frac{1}{C} \|u(t)\|_{H^s} \leq \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2} = \|(L_{u_0} + \lambda)^s u_0\|_{L^2} \leq C \|u_0\|_{H^s},$$

for all $s \in [0, \frac{1}{2}]$. Therefore, $\sup_{t \in \mathbb{R}} \|u(t)\|_{H^{\frac{1}{2}}} \lesssim \|u_0\|_{H^{\frac{1}{2}}}$, and thus by Sobolev embedding

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{L^p} \lesssim \|u_0\|_{H^{\frac{1}{2}}}, \quad \forall p \in [2, \infty).$$

Step 3 : $s \in [\frac{1}{2}, 1]$. As in Step 1, the idea is to prove that it is true for $s = 1$ and then by complex interpolation, infer that it is true for all $s \in [\frac{1}{2}, 1]$. Let $s = 1$, by (4.2.20) of Step 1,

$$\begin{aligned} \|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})}^2 &= \langle (L_{u(t)} + \lambda)f | L_{u(t)}f \rangle + \lambda \langle (L_{u(t)} + \lambda)f | f \rangle \\ &\geq \|L_{u(t)}f\|_{L^2}^2 + \lambda \left((1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - \|u\|_{L^2}^2) \|f\|_{L^2}^2 \right) \\ &\quad + \lambda \left((1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 - \|u\|_{L^2}^2 \|f\|_{L^2}^2 \right) \\ &= \|Df\|_{L^2}^2 + \|uT_{\bar{u}}f\|_{L^2}^2 - 2\operatorname{Re} \langle Df | uT_{\bar{u}}f \rangle \\ &\quad + 2\lambda(1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + \lambda(\lambda - 2\|u\|_{L^2}^2) \|f\|_{L^2}^2. \end{aligned}$$

Using Young's inequality, we deduce

$$\begin{aligned} \|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})}^2 &\geq (1 - \varepsilon) \|Df\|_{L^2}^2 + (1 - C_\varepsilon) \|uT_{\bar{u}}f\|_{L^2}^2 \\ &\quad + 2\lambda(1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + \lambda(\lambda - 2\|u\|_{L^2}^2) \|f\|_{L^2}^2. \end{aligned} \quad (4.2.21)$$

Now, applying Cauchy-Schwarz's inequality on $\|uT_{\bar{u}}f\|_{L^2}$ and since $\|u\|_{L^s}$ and $\|u\|_{L^4}$ are uniformly bounded by Step 2., we infer

$$\|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})}^2 \geq \frac{1}{C^2} \|f\|_{H^1}^2, \quad C = C(\|u_0\|_{H^{\frac{1}{2}}}, \lambda).$$

On the other hand, we have by definition of L_u ,

$$\|(L_{u(t)} + \lambda)f\|_{L^2(\mathbb{T})} \leq C\|f\|_{H^1}.$$

Therefore, inequality (4.2.18) holds for $s = 1$ and thus by complex interpolation [Tay81], it holds for all $s \in [\frac{1}{2}, 1]$.

Step 4 : Uniform bounds on $\|u(t)\|_{H^1}$. Since (4.2.18) holds for $s = 1$ then we infer by repeating the same proof of Step 2 that

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1} < \infty.$$

Step 5 : $s \in [1, \frac{3}{2}]$ Again, we prove that inequality (4.2.18) is true for $s = \frac{3}{2}$, then by complex interpolation, we deduce it for all $s \in [1, \frac{3}{2}]$. Let $s = \frac{3}{2}$. By applying inequality (4.2.21) of Step 3,

$$\begin{aligned} \langle (L_{u(t)} + \lambda)^3 f | f \rangle &= \langle (L_{u(t)} + \lambda)^2 f | L_{u(t)} f \rangle + \lambda \langle (L_{u(t)} + \lambda)^2 f | f \rangle \\ &\geq \langle (D - uT_{\bar{u}} + \lambda)^2 f | (D - uT_{\bar{u}})f \rangle + \lambda \left[(1 - \varepsilon) \|Df\|_{L^2}^2 \right. \\ &\quad \left. + (1 - C_\varepsilon) \|uT_{\bar{u}}f\|_{L^2}^2 + 2\lambda(1 - \|u\|_{L^2}^2) \|f\|_{H^{\frac{1}{2}}}^2 + \lambda(\lambda - 2\|u\|_{L^2}^2) \|f\|_{L^2}^2 \right] \end{aligned}$$

Now, expanding the first inner product on the right-hand side, and using the Cauchy–Schwarz inequality, Young’s inequality, and Sobolev embedding, we obtain

$$\begin{aligned} \langle (L_{u(t)} + \lambda)^3 f | f \rangle &\gtrsim \langle D^3 f | f \rangle + p_1(\|u\|_{H^1}^2, \lambda) \langle D^2 f | f \rangle \\ &\quad + p_2(\|u\|_{H^1}^2, \lambda) \langle Df | f \rangle + p_3(\|u\|_{H^1}^2, \lambda) \|f\|_{L^2}^2 \end{aligned}$$

where for all $j = 1, 2, 3$, $p_j(\|u\|_{H^1}^2, \lambda)$ is a positive polynomial for $\lambda \gg 0$. Therefore, there exists $C = C(\|u_0\|_{H^1}, \lambda) > 0$ such that

$$\|(L_{u(t)} + \lambda)^{\frac{3}{2}} f\|_{L^2(\mathbb{T})} \geq \frac{1}{C} \|f\|_{H^{\frac{3}{2}}}.$$

Finally, by repeating the same previous procedure, we infer that inequality (4.2.18) holds true for all $s \geq 0$. \square

Theorem. 4.1.1. *For all $r > \frac{3}{2}$, the Calogero–Sutherland DNLS focusing equation (CS⁺) is globally well-posed in $H_+^r(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^r} \leq C,$$

where $C = C(\|u_0\|_{H^r}) > 0$ is a positive constant.

Proof. Let $u_0 \in H_+^r(\mathbb{T})$. Recall by (4.1.8), there exists a unique solution $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $s > \frac{3}{2}$, satisfying $u(0, \cdot) = u_0$. In addition, in view of the previous proposition, we infer for $\|u_0\|_{L^2} < 1$,

$$\frac{1}{C} \|u(t)\|_{H^r} \leq \|(L_{u(t)} + \lambda)^r u(t)\|_{L^2} = \|(L_{u_0} + \lambda)^r u_0\|_{L^2} \leq C\|u_0\|_{H^r}.$$

\square

4.3 Extension of the flow of (CS^+) to $L_+^2(\mathbb{T})$

In this section, we establish our main result, which states that, for $\|u_0\|_{L^2} < 1$, the flow of (CS^+)

$$\mathcal{S}^+(t) : \begin{array}{ccc} H_+^2(\mathbb{T}) & \longrightarrow & H_+^2(\mathbb{T}) \\ u_0 & \longmapsto & u(t) \end{array}, \quad (4.3.1)$$

defined globally on $H_+^2(\mathbb{T})$ via Theorem 4.1.1, can be extended continuously to the critical regularity $L_+^2(\mathbb{T})$. For this purpose, consider any initial data $u_0 \in L_+^2(\mathbb{T})$, $\|u_0\|_{L^2} < 1$. Then we approximate u_0 by a sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$. Thus, in view of Theorem 4.1.1, the time evolution $u^\varepsilon(t) := \mathcal{S}^+(t)u_0^\varepsilon$ of u_0^ε is well-defined for all $t \in \mathbb{R}$. Our goal is to prove that the sequence (u^ε) converge to a unique limit u in $\mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T}))$. This limit potential shall be called “solution” to the Cauchy problem (CS^+) . It will be uniquely well-defined, regardless of the chosen approximate sequence (u_0^ε) that approximate $u_0 \in L_+^2(\mathbb{T})$. Moreover, it will satisfies the conservation of the L^2 -norm (i.e. $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ for all $t \in \mathbb{R}$).

$$\begin{array}{ccc} \text{-----} & & \text{-----} \\ L_+^2(\mathbb{T}) : & u_0 \overset{?}{\dashrightarrow} & u(t) \\ \text{-----} & & \text{-----} \\ & \downarrow & \uparrow \\ H_+^2(\mathbb{T}) : & u_0^\varepsilon \longrightarrow & u^\varepsilon(t) \\ \text{-----} & & \text{-----} \end{array}$$

Remark 4.3.1. Note that, due to the presence of the nonlinear term $D\Pi_+(|u|^2)u$ in the equation, it may seem intriguing to say that there exists a solution with L^2 -regularity. Nevertheless, the equation is still well-defined in the distribution sense since the product of two functions with nonnegative frequencies is well-defined and continuous. Indeed, let $\mathcal{D}'_+(\mathbb{T})$ denotes the following distribution space

$$\mathcal{D}'_+(\mathbb{T}) = \left\{ u = \sum_{k \geq 0} \widehat{u}(k) e^{ikx} ; \exists M \in \mathbb{N}, |\widehat{u}(k)| \lesssim (1 + |k|^{2M})^{\frac{1}{2}} \right\},$$

and consider two sequences of smooth functions in the Hardy space $f_n, g_n \in \mathcal{C}_+^\infty(\mathbb{T})$ such that we suppose

$$f_n \longrightarrow f \quad \text{and} \quad g_n \longrightarrow g \quad \text{in } \mathcal{D}'_+(\mathbb{T}).$$

Then, $f_n g_n \longrightarrow f g$ in $\mathcal{D}'_+(\mathbb{T})$ and for all $k \in \mathbb{N}_{\geq 0}$,

$$\widehat{f g}(k) = \sum_{\ell=0}^k \widehat{f}(k-\ell) \widehat{g}(\ell),$$

as $\widehat{f_n g_n}(k) = \sum_{\ell=0}^k \widehat{f_n}(k-\ell) \widehat{g_n}(\ell)$.

4.3.1 Uniqueness of the limit and weak convergence in $L_+^2(\mathbb{T})$

By passing to the limit as $\varepsilon \rightarrow 0$, it is necessary to first prove that the limit potential $u(t)$ is uniquely well-characterized for all $t \in \mathbb{R}$, and is independent of the choice of the sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ that approximate $u_0 \in L_+^2(\mathbb{T})$. The key point is to use the explicit formula of the solution of the focusing Calogero–Sutherland DNLS equation. Thus, for all $(u^\varepsilon(t)) \subseteq H_+^2(\mathbb{T})$, we have by Proposition 4.2.5,

$$u^\varepsilon(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2itL_{u_0^\varepsilon}} S^*)^{-1} u_0^\varepsilon \mid 1 \right\rangle, \quad \forall z \in \mathbb{D}. \quad (4.3.2)$$

Our goal in this subsection is to pass to the limit in this formula. Therefore, we need first to give a meaning to the operator L_{u_0} when $u_0 \in L_+^2(\mathbb{T})$. To handle this, we recall in a few lines the work of Gérard–Lenzmann [GL24, Appendix A] who defined the operator L_u with $u \in L_+^2(\mathbb{R})$ via the standard theory of quadratic form. This new operator will coincide with the former Lax operator $L_u = D - T_u T_{\bar{u}}$ when $u \in H_+^2(\mathbb{R})$. The same proof presented in [GL24, Appendix A], works out on the torus \mathbb{T} , and thus, one can define L_u for $u \in L_+^2(\mathbb{T})$. We recall the main points of the proof :

- (i) For $u \in L_+^2(\mathbb{T})$ and $f, g \in H_+^{\frac{1}{2}}(\mathbb{T})$, consider the quadratic form

$$\mathcal{Q}_u(f, g) = \langle D^{1/2} f \mid D^{1/2} g \rangle - \langle T_{\bar{u}} f \mid T_{\bar{u}} g \rangle .$$

- (ii) Observe that, by decomposing u in high and low frequency

$$u(x) = u_N(x) + R_N(u, x), \quad \begin{cases} u_N(x) := \sum_{n \geq 0}^N \widehat{u}(n) e^{inx} \\ R_N(u, x) := \sum_{n \geq N+1} \widehat{u}(n) e^{inx} \end{cases},$$

and using Lemma 4.2.7, one can prove that for all $\eta > 0$, $\exists N_\eta := N_\eta(u) \in \mathbb{N}_{\geq 0}$ uniform on every compact set of $L_+^2(\mathbb{T})$, such that

$$\|T_{\bar{u}} f\|_{L^2(\mathbb{T})}^2 < 2\eta^2 \left(\langle Df \mid f \rangle + \|f\|_{L^2(\mathbb{T})}^2 \right) + 2N_\eta^2 \|u\|_{L^2}^2 \|f\|_{L^2(\mathbb{T})}^2 . \quad (4.3.3)$$

Therefore,

$$\mathcal{Q}_u(f, f) \geq (1 - 2\eta^2) \|f\|_{H^{1/2}}^2 - 2(N_\eta^2 \|u\|_{L^2}^2 + \eta^2) \|f\|_{L^2(\mathbb{T})}^2 . \quad (4.3.4)$$

- (iii) Now, fixing η small enough, there exists $K := K(u) > 0$ uniform on every compact of $L_+^2(\mathbb{T})$, such that the following positive definite quadratic form

$$\tilde{\mathcal{Q}}_u(f, g) := \mathcal{Q}_u(f, g) + K \langle f \mid g \rangle, \quad f, g \in H_+^{\frac{1}{2}}(\mathbb{T}),$$

define a new inner product on $H_+^{\frac{1}{2}}(\mathbb{T})$.

(iv) Using the theory of quadratic forms (see [Ree72]), we introduce for $u \in L_+^2(\mathbb{T})$,

$$\text{Dom}(L_u) = \left\{ h \in H_+^{\frac{1}{2}}(\mathbb{T}); \exists C > 0, |\tilde{\mathcal{Q}}_u(h, g)| \leq C \|g\|_{L^2(\mathbb{T})}, \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}) \right\}, \quad (4.3.5)$$

and for any $f \in \text{Dom}(L_u)$

$$\langle L_u(f) | g \rangle = \mathcal{Q}_u(f, g), \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}), \quad (4.3.6)$$

and one shows that this new operator L_u is a self-adjoint operator with a dense domain in $H_+^{\frac{1}{2}}(\mathbb{T})$.

4.3.1.1 Spectral properties of L_{u_0} for $u_0 \in L_+^2(\mathbb{T})$

Now that the operator L_{u_0} has been introduced for $u_0 \in L_+^2(\mathbb{T})$, one can examine some of its spectral properties. As noted above, it is a self-adjoint operator with compact resolvent then it has discrete spectrum. Moreover, its quadratic form \mathcal{Q}_{u_0} is bounded from below. therefore,

$$\sigma(L_{u_0}) := \{\lambda_0(u_0) \leq \dots \leq \lambda_n(u_0) \leq \dots\}, \quad \lambda_0 > -\infty. \quad (4.3.7)$$

To characterize this spectrum, we use the following proposition.

Proposition 4.3.1. *For every $n \in \mathbb{N}_{\geq 0}$, the map $u \in L_+^2(\mathbb{T}) \mapsto \lambda_n(u)$ is Lipschitz continuous on compact subsets of $L_+^2(\mathbb{T})$.*

Proof. Let $u \in L_+^2(\mathbb{T})$. The key ingredient is to use the max-min principle,

$$\lambda_n(u) = \max_{\substack{F \subseteq L_+^2 \\ \dim F \leq n}} \min \left\{ \mathcal{Q}_u(h, h); h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}), \|h\|_{L^2} = 1 \right\}.$$

For any $v \in L_+^2(\mathbb{T})$, $h \in H_+^{\frac{1}{2}}(\mathbb{T})$, $\|h\|_{L^2} = 1$,

$$\begin{aligned} |\mathcal{Q}_u(h, h) - \mathcal{Q}_v(h, h)| &= \left| \|T_{\bar{v}}h\|_{L^2(\mathbb{T})}^2 - \|T_{\bar{u}}h\|_{L^2(\mathbb{T})}^2 \right| \\ &\leq \|T_{(\bar{u}-\bar{v})}h\|_{L^2(\mathbb{T})} (\|T_{\bar{v}}h\|_{L^2(\mathbb{T})} + \|T_{\bar{u}}h\|_{L^2(\mathbb{T})}) \\ &\leq \|u - v\|_{L^2(\mathbb{T})} (\|u\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})}) (1 + \langle Dh | h \rangle) \end{aligned}$$

thanks to inequality (4.2.15). Thus,

$$\mathcal{Q}_v(h, h) \leq \mathcal{Q}_u(h, h) + \|u - v\|_{L^2(\mathbb{T})} (\|u\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})}) (1 + \langle Dh | h \rangle). \quad (4.3.8)$$

In particular, considering any subspace F of $L_+^2(\mathbb{T})$ of dimension n , and for any $h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}) \cap \bigoplus_{k=0}^n \ker(L_u - \lambda_k(u) \text{Id})$, the latter inequality holds. In addition,

observe by definition of $L_u = D - T_u T_{\bar{u}}$, and by applying inequality (4.3.3) with $\eta = \frac{1}{2}$,

$$\begin{aligned} \langle Dh | h \rangle &\leq \langle L_u h | h \rangle + \|T_{\bar{u}} h\|_{L^2}^2 \\ &\leq \langle L_u h | h \rangle + \frac{1}{2}(\langle Dh | h \rangle + 1) + 2N^2 \|u\|_{L^2}^2, \end{aligned}$$

where $N \in \mathbb{N}$ is uniform on every compact subset of $L_+^2(\mathbb{T})$. That is,

$$\langle Dh | h \rangle \leq 2\lambda_n(u) + 1 + 4N^2 \|u\|_{L^2}^2, \quad (4.3.9)$$

since $\mathcal{Q}_u(h, h) = \langle L_u h | h \rangle \leq \lambda_n(u)$ when $h \in \bigoplus_{k=0}^n \ker(L_u - \lambda_k(u) \text{Id})$. Furthermore, applying once more the max-min principle,

$$\begin{aligned} \lambda_n(u) &\leq \max_{\substack{F \subseteq L_+^2 \\ \dim F \leq n}} \min \left\{ \langle Dh | h \rangle ; h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}), \|h\|_{L^2} = 1 \right\} \\ &= \min \left\{ \langle Dh | h \rangle ; h \in \{1, \dots, e^{i(n-1)x}\}^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}), \|h\|_{L^2} = 1 \right\} \\ &= n \end{aligned} \quad (4.3.10)$$

Hence, combining (4.3.8), (4.3.9) and (4.3.10), we find for all $h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}) \cap \bigoplus_{k=0}^n \ker(L_u - \lambda_k(u) \text{Id})$, and since $\mathcal{Q}_u(h, h) \leq \lambda_n(u)$,

$$\mathcal{Q}_v(h, h) \leq \lambda_n(u) + 2 \|u - v\|_{L^2(\mathbb{T})} (\|u\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})}) (n + 1 + 2N^2 \|u\|_{L^2}^2).$$

Therefore,

$$\lambda_n(v) \leq \lambda_n(u) + 2(n + 1 + 2N^2 \|u\|_{L^2}^2) \|u - v\|_{L^2(\mathbb{T})} (\|u\|_{L^2(\mathbb{T})} + \|v\|_{L^2(\mathbb{T})}).$$

□

Corollary 4.3.2 (Characterization of the spectrum of L_{u_0}). *Let $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$. The spectrum of L_{u_0} is given by*

$$\sigma(L_{u_0}) = \left\{ \lim_{\varepsilon \rightarrow 0} \lambda_n(u_0^\varepsilon) \mid \lambda_n(u_0^\varepsilon) \in \sigma(L_{u_0^\varepsilon}), n \in \mathbb{N}_{\geq 0} \right\}.$$

Proof. In light of the previous proposition, the result follows directly. □

Proposition 4.3.3. *Let $u_0 \in L_+^2(\mathbb{T})$ and $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$. Then $L_{u_0^\varepsilon} \rightarrow L_{u_0}$ in the strong resolvent sense as $\varepsilon \rightarrow 0$.*

Proof. For all $\varepsilon > 0$, we denote by ϕ_λ^ε the vector $\phi_\lambda^\varepsilon := (L_{u_0^\varepsilon} + \lambda)^{-1} h$, $h \in L_+^2(\mathbb{T})$, where $\lambda \gg 0$. Observe that λ can be chosen uniformly with respect to ε . Indeed, by inequality (4.3.3), and since $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, there exists $N \in \mathbb{N}_{\geq 0}$ uniform for all ε , such that for ε small enough

$$\langle (L_{u_0^\varepsilon} + \lambda)g | g \rangle \geq \frac{1}{2} \|g\|_{\dot{H}^{\frac{1}{2}}}^2 + \left(\lambda - \frac{1}{2} - 2N^2 \|u_0\|_{L^2}^2\right) \|g\|_{L^2}^2. \quad (4.3.11)$$

Then, to apply the Lax–Miligram theorem, we choose in view of the last inequality $\lambda \gg 0$, such that $\langle (L_{u_0^\varepsilon} + \lambda) \cdot | \cdot \rangle$ is coercive and so $(L_{u_0^\varepsilon} + \lambda)$ is invertible for all ε small. Our goal is to prove that ϕ_λ^ε converges in $L_+^2(\mathbb{T})$. For $g = \phi_\lambda^\varepsilon$ in (4.3.11),

$$\langle L_{u_0^\varepsilon} \phi_\lambda^\varepsilon | \phi_\lambda^\varepsilon \rangle + \lambda \|\phi_\lambda^\varepsilon\|_{L^2}^2 \geq \frac{1}{2} \|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2 + \left(\lambda - \frac{1}{2} - 2N^2 \|u_0^\varepsilon\|_{L^2} \right) \|\phi_\lambda^\varepsilon\|_{L^2}^2,$$

which leads, for $\lambda \gg 0$, to

$$\langle L_{u_0^\varepsilon} \phi_\lambda^\varepsilon | \phi_\lambda^\varepsilon \rangle + \lambda \|\phi_\lambda^\varepsilon\|_{L^2}^2 \gtrsim \|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2.$$

Hence, as $\langle L_{u_0^\varepsilon} \phi_\lambda^\varepsilon | \phi_\lambda^\varepsilon \rangle + \lambda \|\phi_\lambda^\varepsilon\|_{L^2}^2 = \langle h | \phi_\lambda^\varepsilon \rangle$,

$$\langle h | \phi_\lambda^\varepsilon \rangle \gtrsim \|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2. \quad (4.3.12)$$

Using the Cauchy–Schwarz’s inequality, we deduce $\|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2 \leq \|h\|_{L^2} \|\phi_\lambda^\varepsilon\|_{L^2}$, where, in view of Corollary 4.3.2 and by (4.3.7), we have for all $\varepsilon > 0$,

$$\|\phi_\lambda^\varepsilon\|_{L^2} \leq \sup_n \frac{1}{|\lambda_n(u_0^\varepsilon) + \lambda|} \|h\|_{L^2} \leq C(\lambda) \|h\|_{L^2}. \quad (4.3.13)$$

Therefore,

$$\|\phi_\lambda^\varepsilon\|_{H^{\frac{1}{2}}}^2 \leq C(\lambda) \|h\|_{L^2}^2, \quad \forall \varepsilon > 0.$$

Thus, there exists $\phi_\lambda \in L_+^2$ such that up to a subsequence,

$$\phi_\lambda^\varepsilon \rightharpoonup \phi_\lambda \text{ in } H_+^2(\mathbb{T}) \quad \text{and} \quad \phi_\lambda^\varepsilon \rightarrow \phi_\lambda \text{ in } L_+^2(\mathbb{T}).$$

It remains to show that $\phi_\lambda = (L_{u_0} + \lambda)^{-1}h$. Indeed, for any $g \in H_+^{\frac{1}{2}}(\mathbb{T})$, we have by definition of ϕ_λ^ε , $\langle (L_{u_0^\varepsilon} + \lambda) \phi_\lambda^\varepsilon | g \rangle = \langle h | g \rangle$. Namely,

$$\langle h | g \rangle = \langle D^{\frac{1}{2}} \phi_\lambda^\varepsilon | D^{\frac{1}{2}} g \rangle - \langle T_{\bar{u}_0^\varepsilon} \phi_\lambda^\varepsilon | T_{\bar{u}_0^\varepsilon} g \rangle + \lambda \langle \phi_\lambda^\varepsilon | g \rangle. \quad (4.3.14)$$

Since

$$\begin{cases} T_{\bar{u}_0^\varepsilon} g \rightarrow T_{\bar{u}_0} g \\ T_{\bar{u}_0^\varepsilon} \phi_\lambda^\varepsilon \rightharpoonup T_{\bar{u}_0} \phi_\lambda \\ D^{\frac{1}{2}} \phi_\lambda^\varepsilon \rightharpoonup D^{\frac{1}{2}} \phi_\lambda \end{cases}$$

in $L_+^2(\mathbb{T})$ as $\varepsilon \rightarrow 0$, then passing to the limit in (4.3.14), we infer for all $g \in H_+^{\frac{1}{2}}(\mathbb{T})$,

$$\langle h | g \rangle = \langle D^{\frac{1}{2}} \phi_\lambda | D^{\frac{1}{2}} g \rangle - \langle T_{\bar{u}_0} \phi_\lambda | T_{\bar{u}_0} g \rangle + \lambda \langle \phi_\lambda | g \rangle =: \langle (L_{u_0} + \lambda) \phi_\lambda | g \rangle.$$

That is, $\phi_\lambda \in \text{Dom}(L_{u_0})$ and $\phi_\lambda = (L_{u_0} + \lambda)^{-1}h$. Therefore, $L_{u_0^\varepsilon} \rightarrow L_{u_0}$ in the strong resolvent sense as $\varepsilon \rightarrow 0$. \square

4.3.1.2 Characterization of the limit $u(t)$

Proposition 4.3.4 (Uniqueness of the limit potential $u(t)$). *Let $u_0 \in L_+^2(\mathbb{T})$. There exists a unique potential $u(t) \in L_+^2(\mathbb{T})$,*

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 \mid 1 \rangle, \quad \forall z \in \mathbb{D}, \quad (4.3.15)$$

such that, for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ with $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, we have

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ in } L_+^2(\mathbb{T}), \quad \forall t \in \mathbb{R}.$$

Proof. By the conservation of the L^2 -norm (Lemma 4.2.6),

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \lesssim \|u_0\|_{L^2}, \quad \forall \varepsilon \ll 1.$$

Then, $\forall t \in \mathbb{R}$, $\exists u_t^* \in L_+^2(\mathbb{T})$ such that

$$u^\varepsilon(t) \rightharpoonup u_t^* \text{ in } L_+^2(\mathbb{T}), \quad \text{and} \quad \|u_t^*\|_{L^2} \lesssim \|u_0\|_{L^2}. \quad (4.3.16)$$

Let

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 \mid 1 \rangle, \quad \forall z \in \mathbb{D},$$

and recall by Proposition 4.2.5,

$$u^\varepsilon(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0^\varepsilon}} S^*)^{-1} u_0^\varepsilon \mid 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (4.3.17)$$

Our goal is to prove that for all $t \in \mathbb{R}$, $z \in \mathbb{D}$, one has $u^\varepsilon(t, z) \rightarrow u(t, z)$, and thus, by the uniqueness of the limit, one can conclude that u_t^* is a well-defined function on $L_+^2(\mathbb{T})$, given as a holomorphic function on \mathbb{D} by $u(t, z)$. Indeed, by Proposition 4.3.3, $L_{u_0^\varepsilon} \rightarrow L_{u_0}$ in the strong resolvent sense as $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$. Thus, for any bounded continuous functions f , we have $f(L_{u_0^\varepsilon}) \rightarrow f(L_{u_0})$ in the strong operator topology [DO08, Proposition 10.1.9]. In particular for all $t \in \mathbb{R}$, and for $f(x) = e^{-2ixt}$,

$$e^{-2itL_{u_0^\varepsilon}} \rightarrow e^{-2itL_{u_0}}$$

strongly as $\varepsilon \rightarrow 0$. Therefore, passing to the limit in (4.3.17), we deduce

$$u^\varepsilon(t, z) \rightarrow u(t, z), \quad \varepsilon \rightarrow 0, \quad \forall z \in \mathbb{D}, t \in \mathbb{R}.$$

□

4.3.2 Strong convergence in $L_+^2(\mathbb{T})$ and conservation of the L^2 -mass

Our aim in this subsection is to prove Theorem 4.1.2. In light of the previous subsection, it remains to have

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad (4.3.18)$$

in order to guarantee the strong convergence of $u^\varepsilon(t) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$, as when $\varepsilon \rightarrow 0$,

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \longrightarrow \|u_0\|_{L^2} .$$

The main idea to prove (4.3.18) is to use Parseval's identity on $u(t)$, where $u(t)$ is written in a suitable evolving L_+^2 -basis (f_n^t) , and satisfying

$$|\langle u(t) | f_n^t \rangle| = |\langle u_0 | f_n^0 \rangle|, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (4.3.19)$$

Definition 4.3.5 (An orthonormal basis of $L_+^2(\mathbb{T})$). For all $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{C}(\mathbb{R}, H_+^2(\mathbb{T}))$. We denote by $(f_n^{\varepsilon,t})$ the evolving orthonormal basis of $L_+^2(\mathbb{T})$ along the curve $t \mapsto u^\varepsilon(t)$ and satisfying the Cauchy problem

$$\begin{cases} \partial_t f_n^{\varepsilon,t} = B_{u^\varepsilon(t)} f_n^{\varepsilon,t} \\ f_n^{\varepsilon,t} |_{t=0} = f_n^{\varepsilon,0} \end{cases},$$

for all n , where $(f_n^{\varepsilon,0})$ is the orthonormal basis of $L_+^2(\mathbb{T})$ constituted from the eigenfunctions of the self-adjoint Lax operator $L_{u_0^\varepsilon}$, and $B_{u^\varepsilon(t)}$ is the skew-adjoint operator defined in (4.1.5).

Remark 4.3.2.

1. Since for all $\varepsilon > 0$, $B_{u^\varepsilon(t)}$ is a skew-adjoint bounded operator (cf. Proposition 4.2.3) then the orthogonality of the $(f_n^{\varepsilon,t})$ is conserved in time. Indeed, for all $t \in \mathbb{R}$,

$$\partial_t \langle f_n^{\varepsilon,t} | f_m^{\varepsilon,t} \rangle = \langle B_u f_n^{\varepsilon,t} | f_m^{\varepsilon,t} \rangle + \langle f_n^{\varepsilon,t} | B_u f_m^{\varepsilon,t} \rangle = 0,$$

2. By [Kuk06, Lemma 4.1], such orthonormal basis is formed by the eigenfunctions of the Lax operator $L_{u^\varepsilon(t)}$. Typically, we have for all $\varepsilon > 0$, for all $t \in \mathbb{R}$,

$$L_{u^\varepsilon(t)} f_n^{\varepsilon,t} = \lambda_n(u_0^\varepsilon) f_n^{\varepsilon,t}.$$

With this choice of $L_+^2(\mathbb{T})$ -basis, we have a nice description of the evolution of the coordinates of $u^\varepsilon(t)$. This is the aim of the next Lemma.

Lemma 4.3.6. For all $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{C}(\mathbb{R}, H_+^2(\mathbb{T}))$ solution of (CS⁺). Under the same notation of Definition 4.3.5, we have for any $n \in \mathbb{N}_{\geq 0}$,

$$\langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle = \langle u_0^\varepsilon | f_n^{\varepsilon,0} \rangle e^{-i\lambda_n(u_0^\varepsilon)^2 t}. \quad (4.3.20)$$

Proof. By Lemma 4.2.4, and since $L_{u^\varepsilon(t)}$ and $B_{u^\varepsilon(t)}$ are respectively self-adjoint and skew-adjoint operators

$$\begin{aligned} \partial_t \langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle &= \langle B_{u^\varepsilon(t)} u^\varepsilon(t) - i L_{u^\varepsilon(t)}^2 u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle + \langle u^\varepsilon(t) | B_{u^\varepsilon(t)} f_n^{\varepsilon,t} \rangle \\ &= -i\lambda_n^2(u_0^\varepsilon) \langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle, \end{aligned}$$

which leads to the statement. □

CONSEQUENCE. From the previous lemma, we infer for all $\varepsilon > 0$, $n \in \mathbb{N}_{\geq 0}$,

$$|\langle u^\varepsilon(t) | f_n^{\varepsilon,t} \rangle| = |\langle u_0^\varepsilon | f_n^{\varepsilon,0} \rangle|.$$

At this stage, we want to take $\varepsilon \rightarrow 0$ in the latter identity in order to deduce (4.3.19). However, one first might ask two questions :

- I. Does the orthonormal basis $(f_n^{\varepsilon,0})$ constituted from the eigenfunctions of the self-adjoint Lax operator $L_{u_0^\varepsilon}$ remains an orthonormal basis of $L_+^2(\mathbb{T})$ under the limit $\varepsilon \rightarrow 0$?
- II. Suppose that the answer to the former question is affirmative, and denote by (f_n^0) this orthonormal basis limit. Based on Definition 4.3.5, could we construct a time-evolving orthonormal basis, coinciding at $t = 0$ with (f_n^0) , and inducing a nice evolution as in Lemma 4.3.6 of the coordinates of u in this basis ? A priori, the operator B_u defined in (4.1.5) is not well-defined when $u \in \mathcal{C}_t[L_+^2(\mathbb{T})]_x$. Therefore, we should find another way to circumvent this problem.

The following proposition aims to answer question I. and to characterize the eigenfunctions of L_{u_0} for $u_0 \in L_+^2(\mathbb{T})$, by finding a uniform bound on the growth of the Sobolev norm $\|f_n^{\varepsilon,0}\|_{H^{\frac{1}{2}}}$. For the second question II., we avoid the problem of defining (f_n^t) via Definition 4.3.5 by using the same strategy done in the previous subsection, that is, we characterize the limit f_n^t , for all $t \in \mathbb{R}$. Therefore, we should derive an explicit formula of $f_n^{\varepsilon,t}$, for all $\varepsilon > 0$ in order to pass to the limit. Unfortunately, we won't directly obtain that the limit (f_n^t) forms an orthonormal basis of $L_+^2(\mathbb{T})$. However, it shall be an orthonormal family in $L_+^2(\mathbb{T})$, which will be sufficient to conclude.

Proposition 4.3.7. *Given $u_0 \in L_+^2(\mathbb{T})$ there exists a sequence $(f_n^0) \subseteq \text{Dom}(L_{u_0})$, such that for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, we have up to a subsequence*

$$\lim_{\varepsilon \rightarrow 0} \|f_n^{\varepsilon,0} - f_n^0\|_{L^2} = 0, \quad \forall n \in \mathbb{N}_{\geq 0}.$$

In addition, for all n ,

$$L_{u_0} f_n^0 = \lambda_n(u_0) f_n^0.$$

Proof. By definition of $L_{u_0^\varepsilon} = D - T_{u_0^\varepsilon} T_{\bar{u}_0^\varepsilon}$, and since $L_{u_0^\varepsilon} f_n^{\varepsilon,0} = \lambda_n(u_0^\varepsilon) f_n^{\varepsilon,0}$, it follows

$$\lambda_n(u_0^\varepsilon) + \|T_{\bar{u}_0^\varepsilon} f_n^{\varepsilon,0}\|_{L^2}^2 = \|f_n^{\varepsilon,0}\|_{\dot{H}_+^{\frac{1}{2}}}^2, \quad \forall n \geq 0.$$

Note that as $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, and by applying inequality (4.3.3), we infer that $\exists N \geq 0$ independent of ε , such that

$$\lambda_n(u_0^\varepsilon) + \frac{1}{2} \|f_n^{\varepsilon,0}\|_{\dot{H}_+^{\frac{1}{2}}}^2 + 2N^2 \|u_0^\varepsilon\|_{L^2} + \frac{1}{2} > \|f_n^{\varepsilon,0}\|_{\dot{H}_+^{\frac{1}{2}}}^2.$$

Hence, by Proposition 4.3.1 and since $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$,

$$\|f_n^{\varepsilon,0}\|_{\dot{H}_+^{1/2}}^2 \lesssim \lambda_n(u_0) + \|u_0\|_{L^2}^2. \quad (4.3.21)$$

Therefore, up to a subsequence, $\exists (f_n^0)$ such that, as $\varepsilon \rightarrow 0$,

$$f_n^{\varepsilon,0} \rightharpoonup f_n^0 \text{ in } H_+^{\frac{1}{2}}(\mathbb{T}) \quad \text{and} \quad f_n^{\varepsilon,0} \rightarrow f_n^0 \text{ in } L_+^2(\mathbb{T}). \quad (4.3.22)$$

At present, for the second part of the proof we show that the (f_n^0) are eigenfunctions of L_{u_0} . Note that by Lemma 4.2.7, one can directly check that $(f_n^0) \subseteq \text{Dom}(L_{u_0})$ where $\text{Dom}(L_{u_0})$ was defined in (4.3.5). Besides, by definition of $L_{u_0^\varepsilon}$, we have for all $g \in H_+^{\frac{1}{2}}(\mathbb{T})$,

$$\langle D^{\frac{1}{2}} f_n^{\varepsilon,0} \mid D^{\frac{1}{2}} g \rangle - \langle T_{\bar{u}_0^\varepsilon} f_n^{\varepsilon,0} \mid T_{\bar{u}_0^\varepsilon} g \rangle = \lambda_n(u_0^\varepsilon) \langle f_n^{\varepsilon,0} \mid g \rangle, \quad (4.3.23)$$

where by Lemma 4.2.7 $T_{\bar{u}_0^\varepsilon} g \rightarrow T_{\bar{u}_0} g$ in $L_+^2(\mathbb{T})$, by Proposition 4.3.1 $\lambda_n(u_0^\varepsilon) \rightarrow \lambda_n(u_0)$, and by (4.3.22) $T_{\bar{u}_0^\varepsilon} f_n^{\varepsilon,0} \rightarrow T_{\bar{u}_0} f_n^0$. Hence, passing to the limit in (4.3.23), we infer

$$\langle L_{u_0} f_n^0 \mid g \rangle = \lambda_n(u_0) \langle f_n^0 \mid g \rangle, \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}),$$

leading to $L_{u_0} f_n^0 = \lambda_n(u_0) f_n^0$ for all $n \geq 0$, where $(\lambda_n(u_0))$ denotes all the spectrum of L_{u_0} by Corollary 4.3.2. □

In the sequel, thanks to Corollary 4.3.2 and Proposition 4.3.7, we denote by (f_n^0) the orthonormal basis of $L_+^2(\mathbb{T})$ made up of the eigenfunctions of L_{u_0} obtained in the previous proposition. The following lemma aims to give an explicit formula to the $(f_n^{\varepsilon,t})$ defined in Definition 4.3.5 in order to characterize at a second stage their limits when $\varepsilon \rightarrow 0$.

Lemma 4.3.8 (The explicit formula of $f_n^{\varepsilon,t}$). *Under the same notation of Definition 4.3.5, we have for all $\varepsilon > 0$, $t \in \mathbb{R}$,*

$$f_n^{\varepsilon,t}(z) = \left\langle \left(\text{Id} - z e^{-it(L_{u_0^\varepsilon} + \text{Id})^2} S^* e^{itL_{u_0^\varepsilon}^2} \right)^{-1} f_n^{\varepsilon,0} \mid e^{-itL_{u_0^\varepsilon}^2} 1 \right\rangle, \quad \forall z \in \mathbb{D}. \quad (4.3.24)$$

Proof. Like the proof of Proposition 4.2.5, we have

$$f_n^{\varepsilon,t}(z) = \langle (\text{Id} - zS^*)^{-1} f_n^{\varepsilon,t} \mid 1 \rangle, \quad \forall z \in \mathbb{D}.$$

Using the unitary operator $U(t)$ introduced in (4.2.10), we deduce

$$\begin{aligned} f_n^{\varepsilon,t}(z) &= \langle U(t)^* (\text{Id} - zS^*)^{-1} f_n^{\varepsilon,t} \mid U(t)^* 1 \rangle \\ &= \langle (\text{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* f_n^{\varepsilon,t} \mid U(t)^* 1 \rangle \\ &= \langle (\text{Id} - zU(t)^* S^* U(t))^{-1} f_n^{\varepsilon,0} \mid U(t)^* 1 \rangle. \end{aligned} \quad (4.3.25)$$

By the formulae of (4.2.13) and (4.2.14), the explicit formula of $f_n^{\varepsilon,t}$ follows. □

Proposition 4.3.9. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. Under the same notation of Definition 4.3.5, there exists an orthonormal family (f_n^t) of $L_+^2(\mathbb{T})$, such that for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$, we have up to a subsequence,*

$$\|f_n^{\varepsilon,t} - f_n^t\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof. This proof is similar to the one done in Proposition 4.3.7. However, it presents two main differences. We will discuss these later in the upcoming remark. Now, coming back to the proof, recall by Proposition 4.3.4, there exists a unique $u(t) \in L_+^2(\mathbb{T})$ such that for any $u_0^\varepsilon \rightarrow u_0$ in $L_+^2(\mathbb{T})$ we have $u^\varepsilon(t) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$ as $\varepsilon \rightarrow 0$. Therefore, by definition of $L_{u^\varepsilon(t)} = D - T_{u^\varepsilon(t)} T_{\bar{u}^\varepsilon(t)}$, and since $L_{u^\varepsilon(t)} f_n^{\varepsilon,t} = \lambda_n(u_0^\varepsilon) f_n^{\varepsilon,t}$ by the second point of Remark 4.3.2,

$$\lambda_n(u_0^\varepsilon) + \|T_{\bar{u}^\varepsilon(t)} f_n^{\varepsilon,t}\|_{L^2}^2 = \|f_n^{\varepsilon,t}\|_{\dot{H}^{\frac{1}{2}}}^2, \quad \forall n \geq 0.$$

Thus, applying Lemma 4.2.7,

$$(1 - \|u^\varepsilon(t)\|_{L^2}^2) \|f_n^{\varepsilon,t}\|_{\dot{H}^{\frac{1}{2}}}^2 \leq \|u^\varepsilon(t)\|_{L^2}^2 + \lambda_n(u_0^\varepsilon), \quad \forall n \geq 0.$$

Taking ε small enough to guarantee $\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} < 1$, we deduce by Proposition 4.3.1, for ε small

$$\|f_n^{\varepsilon,t}\|_{\dot{H}^{1/2}}^2 \lesssim \frac{\lambda_n(u_0) + \|u_0\|_{L^2}^2}{1 - \|u_0\|_{L^2}^2}, \quad \forall n \geq 0. \quad (4.3.26)$$

Hence, up to a subsequence,

$$f_n^\varepsilon(t) \rightharpoonup f_{n,t}^* \text{ in } H_+^{\frac{1}{2}}(\mathbb{T}), \quad f_n^\varepsilon(t) \rightarrow f_{n,t}^* \text{ in } L_+^2(\mathbb{T}).$$

It remains to show that $f_{n,t}^*$ is uniquely characterized for all t . Using the explicit formula of Lemma 4.3.8,

$$f_n^{\varepsilon,t}(z) = \left\langle \left(\text{Id} - z e^{-it(L_{u_0^\varepsilon} + \text{Id})^2} S^* e^{itL_{u_0^\varepsilon}^2} \right)^{-1} f_n^{\varepsilon,0} \mid e^{-itL_{u_0^\varepsilon}^2} 1 \right\rangle, \quad \forall z \in \mathbb{D},$$

and applying Proposition 4.3.3, one can conclude that there exists

$$f_n^t(z) = \left\langle \left(\text{Id} - z e^{-it(L_{u_0} + \text{Id})^2} S^* e^{itL_{u_0}^2} \right)^{-1} f_n^0 \mid e^{-itL_{u_0}^2} 1 \right\rangle, \quad \forall z \in \mathbb{D}, \quad (4.3.27)$$

where (f_n^0) denotes the eigenfunctions of L_{u_0} obtained in Proposition 4.3.7. Therefore, the limit $f_{n,t}^* = f_n^t$ for all t on \mathbb{D} . Finally, observe that since the $(f_n^{\varepsilon,t})$ is an orthonormal basis of $L_+^2(\mathbb{T})$ and as $f_n^{\varepsilon,t} \rightarrow f_n^t$ in $L_+^2(\mathbb{T})$, then (f_n^t) forms an orthonormal family in $L_+^2(\mathbb{T})$. \square

Remark 4.3.3. There are two main differences between the proof of Proposition 4.3.7 and Proposition 4.3.9 :

- (i) First, note that in the last proof, we cannot control the growth of the Sobolev norm $\|f_n^{\varepsilon,t}\|_{\dot{H}^{\frac{1}{2}}}$ uniformly for all t by using the inequality (4.3.3), since the integer N_η in (4.3.3) is not uniform for all $t \in \mathbb{R}$. As an alternative, we rely on Lemma 4.2.7. Consequently, the condition of $\|u^\varepsilon(t)\|_{L^2} < 1$ for ε small enough, is crucial here in order to conclude.

- (ii) Second in the previous proof, we had to give a meaning to the limit $f_{n,t}^*$ by characterizing this limit for all $t \in \mathbb{R}$.

A common feature about these two proofs is to obtain a uniform bounds on the growth of the Sobolev norm $H_+^{\frac{1}{2}}(\mathbb{T})$ of the eigenfunctions $f_n^{\varepsilon,0}$ and $f_n^{\varepsilon,t}$ to be able to conclude.

In view of Proposition 4.3.1, Lemma 4.3.6, Proposition 4.3.7 and Proposition 4.3.9, we infer the following lemma.

Lemma 4.3.10. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists an orthonormal family (f_n^t) of $L_+^2(\mathbb{T})$ such that for all $n \geq 0$,*

$$\langle u(t) | f_n^t \rangle = \langle u_0 | f_n^0 \rangle e^{-it\lambda_n^2(u_0)}, \quad \forall t \in \mathbb{R}. \quad (4.3.28)$$

We are, at this stage, in a position to prove Theorem 4.1.2.

Theorem 4.1.2. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists a unique potential $u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T}))$ such that, for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$, $\|u_0^\varepsilon - u_0\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0$, the following convergence holds : for all $T > 0$,*

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

In addition,

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad \forall z \in \mathbb{D}. \quad (4.3.29)$$

Moreover, the L^2 -norm of the limit potential u is conserved

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

Proof. Let $(t^\varepsilon) \subseteq \mathbb{R}$ such that $t^\varepsilon \rightarrow t$ as $\varepsilon \rightarrow 0$. Since $\|u_0^\varepsilon - u_0\|_{L^2} \rightarrow 0$, then for ε small enough

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \lesssim \|u_0\|_{L^2}.$$

Hence, for any $t^\varepsilon \rightarrow t$, there exists $u_t^* \in L_+^2(\mathbb{T})$ such that up to a subsequence, $u^\varepsilon(t^\varepsilon) \rightharpoonup u_t^*$ in $L_+^2(\mathbb{T})$ and

$$\|u_t^*\|_{L^2} \leq \liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(t^\varepsilon)\|_{L^2} = \liminf_{\varepsilon \rightarrow 0} \|u_0^\varepsilon\|_{L^2} = \|u_0\|_{L^2}. \quad (4.3.30)$$

Our goal is to show that $u^\varepsilon(t^\varepsilon)$ converges strongly in $L_+^2(\mathbb{T})$. As a first step, we stress that the weak limit potential u_t^* is uniquely characterized for all t , and is equal to a unique limit $u(t)$. For that, we repeat the same proof of Proposition 4.3.4 by exchanging t into t^ε with $t^\varepsilon \rightarrow t$, and we obtain $u^\varepsilon(t^\varepsilon) \rightharpoonup u(t)$ in $L_+^2(\mathbb{T})$, where $u(t)$ is defined in equation (4.3.15). Moreover,

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad (4.3.31)$$

by (4.3.30). As a second step, we prove that this weak convergence in $L_+^2(\mathbb{T})$ is actually a strong convergence. This can be achieved by checking

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2(\mathbb{T})} \rightarrow \|u(t)\|_{L^2(\mathbb{T})}, \quad \varepsilon \rightarrow 0.$$

In fact, it is actually sufficient to prove that $\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$ since

$$\|u^\varepsilon(t^\varepsilon)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \rightarrow \|u_0\|_{L^2}, \quad \text{as } \varepsilon \rightarrow 0. \quad (4.3.32)$$

Thanks to (4.3.31), we already have $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$. Now, to prove $\|u(t)\|_{L^2} \geq \|u_0\|_{L^2}$, we use Lemma 4.3.10 to infer the existence of an orthonormal family (f_n^t) of $L_+^2(\mathbb{T})$ such that

$$\sum_{n=0}^{\infty} |\langle u(t) | f_n^t \rangle|^2 = \sum_{n=0}^{\infty} |\langle u_0 | f_n^0 \rangle|^2 = \|u_0\|_{L^2}^2.$$

Hence, by Bessel's inequality

$$\|u(t)\|_{L^2} \geq \|u_0\|_{L^2}.$$

As a conclusion, we have proved for any $t^\varepsilon \rightarrow t$, $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$. This means, $u \in \mathcal{C}(\mathbb{R}, L_+^2(\mathbb{T}))$ and for all $T > 0$,

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

□

In view of the last Theorem, we denote through on, $u(t)$ the solution of (CS^+) in $L_+^2(\mathbb{T})$ starting from an initial datum u_0 that lies inside the open ball $\mathcal{B}_{L_+^2}(1)$ of $L_+^2(\mathbb{T})$.

Corollary 4.3.11. *The spectrum $\sigma(L_{u(t)})$ is invariant under the flow of (CS^+) .*

Proof. Let $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $\|u_0^\varepsilon - u_0\|_{L^2} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since, for all $n \in \mathbb{N}_{\geq 0}$,

$$\lambda_n(u_0^\varepsilon) = \lambda_n(u^\varepsilon(t)), \quad \forall t \in \mathbb{R},$$

then by passing to the limit, we infer by Proposition 4.3.1 and Theorem 4.1.2 that the spectrum of $L_{u(t)}$ is conserved in time.

□

Corollary 4.3.12. *Let $u_0 \in \mathcal{B}_{L_+^2}(1)$. There exists an orthonormal basis (f_n^t) of $L_+^2(\mathbb{T})$ constituted from the eigenfunction of $L_{u(t)}$, such that for all $n \in \mathbb{N}_{\geq 0}$,*

$$\langle u(t) | f_n^t \rangle = \langle u_0 | f_n^0 \rangle e^{-it\lambda_n^2(u_0)}, \quad \forall t \in \mathbb{R}. \quad (4.3.33)$$

Proof. Taking into account Lemma 4.3.10, we only need to prove that the orthonormal family (f_n^t) found in Proposition 4.3.9 as

$$f_n^{\varepsilon,t} \rightharpoonup f_n^t \text{ in } H_+^{\frac{1}{2}}(\mathbb{T}), \quad \text{and} \quad f_n^{\varepsilon,t} \rightarrow f_n^t \text{ in } L_+^2(\mathbb{T}), \quad (4.3.34)$$

is actually an orthonormal basis of $L_+^2(\mathbb{T})$. On the one hand, using (4.3.34) and since $u^\varepsilon(t) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$, one can directly prove that $L_{u^\varepsilon(t)} f_n^{\varepsilon,t} \rightharpoonup L_{u(t)} f_n^t$ in $L_+^2(\mathbb{T})$. On the other hand, using Proposition 4.3.1, we infer that taking $\varepsilon \rightarrow 0$ in

$$\langle L_{u^\varepsilon(t)} f_n^{\varepsilon,t} | g \rangle = \lambda_n(u^\varepsilon(t)) \langle f_n^{\varepsilon,t} | g \rangle, \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}),$$

leads to

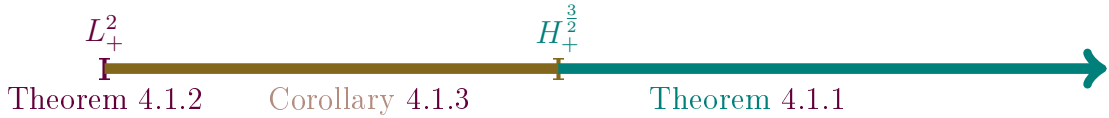
$$\langle L_{u(t)} f_n^t | g \rangle = \lambda_n(u(t)) \langle f_n^t | g \rangle, \quad \forall g \in H_+^{\frac{1}{2}}(\mathbb{T}).$$

As a result, the (f_n^t) describes all the eigenfunctions of the self-adjoint operator $L_{u(t)}$, thanks to Corollary 4.3.11 and Corollary 4.3.2. Hence, they form an orthonormal basis of $L_+^2(\mathbb{T})$. \square

Remark 4.3.4. The nice evolution in (4.3.33) of such coordinates suggests that the so-called ‘‘Birkhoff coordinates’’ of (CS^+) are the $(\langle u(t) | f_n^t \rangle)$. To be sure, we need to construct a one-by-one Birkhoff map $u \longleftrightarrow (\langle u(t) | f_n^t \rangle)$, similar to the remarkable achievement for the Benjamin–Ono equation in [GK21]. This construction can unlock several significant outcomes regarding the equation’s dynamics. In particular, the global well-posedness of the focusing equation when $\|u_0\|_{L^2} \geq 1$.

4.4 Proof of Corollary 3.1.3 and Theorem 3.1.4

To summarize, we have proved the global well-posedness of (CS^+) -equation in $H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, and for $s = 0$ which correspond to $H_+^0(\mathbb{T}) \equiv L_+^2(\mathbb{T})$. The following corollary aims to prove the global well-posedness for $0 < s \leq \frac{3}{2}$.



Corollary. 4.1.3. *For all $0 \leq s \leq \frac{3}{2}$, the Calogero–Sutherland DNLS focusing equation (CS^+) is globally well-posed in $H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$. Moreover, the following a-priori bound holds,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C,$$

where $C = C(\|u_0\|_{H^s}) > 0$ is a positive constant.

Proof. For $s = 0$, we infer by Theorem 4.1.2 the global well-posedness of the problem in $L_+^2(\mathbb{T})$ in the sense of continuous extension of the flow from $H_+^2(\mathbb{T})$ to $L_+^2(\mathbb{T})$. For $0 < s \leq \frac{3}{2}$, let $u_0 \in H_+^s(\mathbb{T}) \cap \mathcal{B}_{L_+^2}(1)$ and consider $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$ such that $u_0^\varepsilon \rightarrow u_0$ in $H_+^s(\mathbb{T})$. Then applying Proposition 4.2.8, and since $u_0^\varepsilon \rightarrow u_0$ in $H_+^s(\mathbb{T})$, there exists $C > 0$ uniform with respect to all small ε , such that

$$\frac{1}{C} \|u^\varepsilon(t^\varepsilon)\|_{H^s} \leq \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon)\|_{L^2} = \|(L_{u_0^\varepsilon} + \lambda)^s u_0^\varepsilon\|_{L^2} \leq C \|u_0^\varepsilon\|_{H^s},$$

thanks to Lemma 4.2.6. Note that λ is also uniform with respect to all ε small for the same reasons presented in the proof of Proposition 4.3.3. Therefore, for all $\varepsilon > 0$ small,

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \leq C \|u_0\|_{H^s}. \quad (4.4.1)$$

Hence, as $t^\varepsilon \rightarrow t$, we have $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $H_+^s(\mathbb{T})$, where u is a characterized function for all t obtained as in Proposition 4.3.4. In particular, we infer $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$ with

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}, \quad (4.4.2)$$

by (4.4.1). As of now, to deduce the strong convergence in $H_+^s(\mathbb{T})$ we use Proposition 4.2.8. Thus, for all $\varepsilon > 0$,

$$\begin{aligned} \|u^\varepsilon(t^\varepsilon) - u(t)\|_{H^s}^2 &\lesssim \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s (u^\varepsilon(t^\varepsilon) - u(t))\|_{L^2}^2 \\ &= \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon)\|_{L^2}^2 + \|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u(t)\|_{L^2}^2 \\ &\quad - 2\operatorname{Re} \langle (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon) | (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u(t) \rangle \end{aligned} \quad (4.4.3)$$

Recall that $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L_+^2(\mathbb{T})$ which leads by Proposition 4.3.3 to $L_{u^\varepsilon(t^\varepsilon)} \rightarrow L_{u(t)}$ in the strong resolvent sense. Thus, by functional calculus, (see the following lemma—Lemma 4.4.1) we infer

$$\begin{cases} (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u(t) \rightarrow (L_{u(t)} + \lambda)^s u(t) \text{ in } L_+^2(\mathbb{T}), \\ (L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon) \rightarrow (L_{u(t)} + \lambda)^s u(t) \text{ in } L_+^2(\mathbb{T}), \end{cases}$$

as $\varepsilon \rightarrow 0$. In addition, for $\varepsilon > 0$, recall by Lemma 4.2.6,

$$\|(L_{u^\varepsilon(t^\varepsilon)} + \lambda)^s u^\varepsilon(t^\varepsilon)\|_{L^2}^2 = \|(L_{u_0^\varepsilon} + \lambda)^s u_0^\varepsilon\|_{L^2}^2. \quad (4.4.4)$$

Therefore, passing to the limit in (4.4.3), and since $u_0^\varepsilon \rightarrow u_0$ in $H_+^s(\mathbb{T})$ combined with Lemma 4.4.1 and Proposition 4.2.8, we deduce for all ε small,

$$\|u^\varepsilon(t^\varepsilon) - u(t)\|_{H^s}^2 \lesssim \|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 - \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2}^2.$$

At this stage, it remains to show that the right-hand side of the previous inequality is vanishing. Indeed, by Corollary 4.3.11,

$$\begin{aligned} \langle (L_{u(t)} + \lambda)^{2s} u(t) | u(t) \rangle &= \sum_{n \geq 0} (\lambda_n(u(t)) + \lambda)^{2s} |\langle u(t) | f_n^t \rangle|^2 \\ &= \sum_{n \geq 0} (\lambda_n(u_0) + \lambda)^{2s} |\langle u_0 | f_n^0 \rangle|^2 = \langle (L_{u_0} + \lambda)^{2s} u_0 | u_0 \rangle, \end{aligned}$$

where (f_n^t) are the orthonormal basis obtained in Corollary 4.3.12 since $u^\varepsilon(t^\varepsilon) \rightarrow u(t)$ in $L^2_+(\mathbb{T})$. As a result, $\|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 = \|(L_{u(t)} + \lambda)^s u(t)\|_{L^2}^2$, and as $\varepsilon \rightarrow 0$,

$$\|u^\varepsilon(t^\varepsilon) - u(t)\|_{H^s}^2 \rightarrow 0.$$

Hence, $u \in \mathcal{C}(\mathbb{R}, H^s_+(\mathbb{T}))$ such that (4.4.2) is satisfied, and for all $T > 0$,

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{H^s} \rightarrow 0.$$

□

To conclude the proof of Corollary 4.1.3, we need to prove the following functional analysis result.

Lemma 4.4.1. *Let (A_ε) be a sequence of positive self-adjoint operators in L^2 . Suppose that $A_\varepsilon \rightarrow A$ in the strong resolvent sense as $\varepsilon \rightarrow 0$, and for all $s \geq 0$,*

$$\text{Dom}(A_\varepsilon^s) = \text{Dom}(A^s) = H^s, \quad \varepsilon > 0.$$

Moreover, assume that for all $u \in H^s$, the $(A_\varepsilon^s u)$ are uniformly bounded with respect to $\varepsilon > 0$ in the following sense $\|A_\varepsilon^s u\| \leq C \|u\|_{H^s}$. Then, for all $s \geq 0$,

$$A_\varepsilon^s u \rightarrow A^s u \text{ in } L^2, \quad \varepsilon \rightarrow 0. \quad (4.4.5)$$

Proof. For all $R > 0$, let $\chi_R \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that $\chi_R \equiv 1$ on $[0, R]$ and $\text{supp}(\chi_R) \subseteq [0, 2R]$. Note that, for all $s \geq 0$, the subset $\{\chi_R(A)u; u \in H^s, R > 0\}$ is dense in H^s . Then, since the $(A_\varepsilon^s u)$ are uniformly bounded with respect to ε , it is sufficient to prove for all $R > 0$,

$$A_\varepsilon^s \chi_R(A)u \rightarrow A^s \chi_R(A)u, \quad \varepsilon \rightarrow 0,$$

to obtain (4.4.5). Toward this end, let $R > 0$, and write for any $s \geq 0$, for all $\varepsilon > 0$,

$$A_\varepsilon^s \chi_R(A)u = A_\varepsilon^s \chi_{\tilde{R}}(A_\varepsilon) \chi_R(A)u + A_\varepsilon^s (1 - \chi_{\tilde{R}}(A_\varepsilon)) \chi_R(A)u, \quad (4.4.6)$$

where $\chi_{\tilde{R}} \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that $\chi_{\tilde{R}} \equiv 1$ on $[0, \tilde{R}]$ and $\text{supp}(\chi_{\tilde{R}}) \subseteq [0, 2\tilde{R}]$, $\tilde{R} \geq 2R$. Notice that,

$$\begin{aligned} \|A_\varepsilon^s (1 - \chi_{\tilde{R}}(A_\varepsilon)) \chi_R(A)u\| &= \|A_\varepsilon^{-s} (1 - \chi_{\tilde{R}}(A_\varepsilon)) A_\varepsilon^{2s} \chi_R(A)u\| \\ &\leq \frac{C}{(2\tilde{R})^s} \|\chi_R(A)u\|_{H^{2s}} \\ &\leq \frac{C}{(2\tilde{R})^s} (\|A^{2s} \chi_R(A)u\| + \|\chi_R(A)u\|) \\ &\leq \frac{C}{(2\tilde{R})^s} (1 + (2R)^s) \|u\|. \end{aligned}$$

Therefore, for all $\eta > 0$, there exists $\tilde{R} \gg 0$, such that for all $\varepsilon > 0$,

$$\|A_\varepsilon^s (1 - \chi_{\tilde{R}}(A_\varepsilon)) \chi_R(A) u\| < \eta,$$

and so, by (4.4.6),

$$\|A_\varepsilon^s \chi_R(A) u - A_\varepsilon^s \chi_{\tilde{R}}(A_\varepsilon) \chi_R(A) u\| < \eta, \quad \forall \varepsilon > 0. \quad (4.4.7)$$

Besides, recall that $A_\varepsilon \rightarrow A$ in the strong resolvent sense as $\varepsilon \rightarrow 0$. Hence, by [DO08, Proposition 10.1.9], $f(A_\varepsilon) \rightarrow f(A)$ as $\varepsilon \rightarrow 0$, in the operator norm for all continuous bounded f . In particular, for $f(x) = x^s \chi_{\tilde{R}}(x)$, we have

$$A_\varepsilon^s \chi_{\tilde{R}}(A_\varepsilon) \chi_R(A) u \rightarrow A^s \chi_R(A) u, \quad \varepsilon \rightarrow 0. \quad (4.4.8)$$

Thus, combining (4.4.7) and (4.4.8), we infer for all $\tilde{\eta} > 0$, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \leq \varepsilon_0$,

$$\|A_\varepsilon^s \chi_R(A) u - A^s \chi_R(A) u\| < \tilde{\eta}.$$

□

Beyond the global well-posedness results of the (CS⁺) Cauchy's Problem, we are interested in some qualitative properties about the flow $\mathcal{S}^+(t)$ of this equation. Therefore, we prove that all weak limit points of the orbit are actually strong limit points.

Theorem. 4.1.4 . *Given an initial data $u_0 \in \mathcal{B}_{L^2_+}(1) \cap H^s_+(\mathbb{T})$, $s \geq 0$, the orbit of the solution $\{\mathcal{S}^+(t)u_0; t \in \mathbb{R}\}$ is relatively compact in $H^s_+(\mathbb{T})$.*

Proof. Let $(t_n) \subseteq \mathbb{R}$ such that $t_n \rightarrow \infty$.

Step 1 : $s = 0$. By Theorem 4.1.2,

$$\|u(t_n)\|_{L^2} = \|u_0\|_{L^2}. \quad (4.4.9)$$

Then, $\exists \tilde{u} \in L^2_+(\mathbb{T})$ such that, up to a subsequence,

$$u(t_n) \rightharpoonup \tilde{u} \text{ in } L^2_+(\mathbb{T}) \quad \text{and} \quad \|\tilde{u}\|_{L^2} \leq \|u_0\|_{L^2}.$$

In order to obtain the strong convergence $u(t_n) \rightarrow \tilde{u}$ in $L^2_+(\mathbb{T})$, all it remains is to show that $\|u(t_n)\|_{L^2} \rightarrow \|\tilde{u}\|_{L^2}$, or by (4.4.9),

$$\|u_0\|_{L^2} = \|\tilde{u}\|_{L^2}.$$

Observe that we already have $\|u_0\|_{L^2} \geq \|\tilde{u}\|_{L^2}$. Now, to prove $\|u_0\|_{L^2} \leq \|\tilde{u}\|_{L^2}$, recall by Corollary 4.3.12,

$$\langle u(t_n) | f_m^{t_n} \rangle = \langle u_0 | f_m^0 \rangle e^{-it_n \lambda_m^2(u_0)}, \quad (4.4.10)$$

where $(f_m^{t_n})$ is the orthonormal basis of $L^2_+(\mathbb{T})$ constituted of the eigenfunctions of $L_{u(t_n)}$. The idea is to pass to the limit as $t_n \rightarrow \infty$ in the above identity and conclude

by using Bessel's identity. First, we have $u(t_n) \rightharpoonup \tilde{u}$ in $L_+^2(\mathbb{T})$. Second, notice that the $(f_m^{t_n})$ converges strongly in $L_+^2(\mathbb{T})$ as $t_n \rightarrow \infty$ to an orthonormal family denoted by (g_m) . Indeed, by definition of $L_{u(t_n)}$,

$$\lambda_m(u_0) + \|T_{u(t_n)}^{-1} f_m^{t_n}\|_{L^2}^2 = \|f_m^{t_n}\|_{\dot{H}^{\frac{1}{2}}}^2,$$

leading to $\|f_m^{t_n}\|_{\dot{H}^{\frac{1}{2}}} \lesssim \frac{\lambda_m(u_0) + \|u_0\|_{L^2}^2}{1 - \|u_0\|_{L^2}^2}$ for all m , thanks to Lemma 4.2.7. Hence, by Rellich–Kondrachov's Theorem, $f_m(t_n) \rightarrow g_m$ in $L_+^2(\mathbb{T})$ as $t_n \rightarrow \infty$. Third, using Cantor diagonalization procedure, one can extract a subsequence $t_n \rightarrow \theta_m \pmod{\frac{2\pi}{\lambda_m^2}}$, as the circle is compact. Hence, by passing to the limit in (4.4.10), we obtain

$$\langle \tilde{u} | g_m \rangle = \langle u_0 | f_m^0 \rangle e^{-i\lambda_m^2(u_0)\theta_m}.$$

As a result, using Bessel's inequality, we conclude

$$\|\tilde{u}\|_{L^2}^2 \geq \sum_{n=0}^{\infty} |\langle \tilde{u} | g_n \rangle|^2 = \sum_{n=0}^{\infty} |\langle u_0 | f_n^0 \rangle|^2 = \|u_0\|_{L^2}^2.$$

Consequently, $\|u(t_n)\|_{L^2} \rightarrow \|\tilde{u}\|_{L^2}$ and thus $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$.

Step 2 : $s > 0$. By inequality (4.4.2) of the proof of Corollary 4.1.3, we have $\|u(t_n)\|_{H^s} \lesssim \|u_0\|_{H^s}$ leading to

$$u(t_n) \rightharpoonup \tilde{u} \text{ in } H_+^s(\mathbb{T}) \quad \text{and} \quad \|\tilde{u}\|_{H^s} \lesssim \|u_0\|_{H^s}.$$

In particular, $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$. Then, in view of Remark 4.2.5,

$$\begin{aligned} \|u(t_n) - \tilde{u}\|_{H^s}^2 &\lesssim \|(L_{u(t_n)} + \lambda \text{Id})^s (u(t_n) - \tilde{u})\|_{L^2}^2 \\ &= \|(L_{u(t_n)} + \lambda \text{Id})^s u(t_n)\|_{L^2}^2 + \|(L_{u(t_n)} + \lambda \text{Id})^s \tilde{u}\|_{L^2}^2 \\ &\quad - 2\text{Re} \langle (L_{u(t_n)} + \lambda)^s u(t_n) | (L_{u(t_n)} + \lambda)^s \tilde{u} \rangle, \end{aligned} \tag{4.4.11}$$

where by the second point of Remark 4.2.2,

$$\|(L_{u(t_n)} + \lambda \text{Id})^s u(t_n)\|_{L^2}^2 = \|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2.$$

Besides, as $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$, then $L_{u(t_n)} \rightarrow L_{\tilde{u}}$ in the strong resolvent sense thanks to Proposition 4.3.3. Hence, by functional calculus (see Lemma 4.4.1), we infer

$$\begin{cases} (L_{u(t_n)} + \lambda)^s \tilde{u} \rightarrow (L_{\tilde{u}} + \lambda)^s \tilde{u} \text{ in } L_+^2(\mathbb{T}), \\ (L_{u(t_n)} + \lambda)^s u(t_n) \rightharpoonup (L_{\tilde{u}} + \lambda)^s \tilde{u} \text{ in } L_+^2(\mathbb{T}), \end{cases}$$

as $n \rightarrow \infty$. Therefore, by passing to the limit in (4.4.11), we deduce

$$\lim_{n \rightarrow +\infty} \|u(t_n) - \tilde{u}\|_{H^s}^2 \lesssim \|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 - \|(L_{\tilde{u}} + \lambda)^s \tilde{u}\|_{L^2}^2,$$

where the right-hand side vanishes. Indeed, by Corollary 4.3.11 and Proposition 4.3.1, $\lambda_n(\tilde{u}) = \lambda_n(u_0)$. Hence,

$$\begin{aligned} \langle (L_{\tilde{u}} + \lambda)^{2s} \tilde{u} \mid \tilde{u} \rangle &= \sum_{n \geq 0} (\lambda_n(u_0) + \lambda)^{2s} |\langle \tilde{u} \mid g_n \rangle|^2 = \sum_{n \geq 0} (\lambda_n(u_0) + \lambda)^{2s} |\langle u_0 \mid f_n^0 \rangle|^2 \\ &= \langle (L_{u_0} + \lambda)^{2s} u_0 \mid u_0 \rangle, \end{aligned}$$

where (g_n) is the orthonormal family of $L_+^2(\mathbb{T})$ found in Step 1. Nevertheless, since $u(t_n) \rightarrow \tilde{u}$ in $L_+^2(\mathbb{T})$, one could show as in Corollary 4.3.12 that this orthonormal family is indeed an orthonormal basis of $L_+^2(\mathbb{T})$ by proving that the (g_n) constitutes all the eigenfunction of the self-adjoint operator $L_{\tilde{u}}$. As a consequence,

$$\|(L_{u_0} + \lambda)^s u_0\|_{L^2}^2 = \|(L_{\tilde{u}} + \lambda)^s \tilde{u}\|_{L^2}^2,$$

and thus $\|u(t_n) - \tilde{u}\|_{H^s}^2 \rightarrow 0$, as $n \rightarrow \infty$. \square

4.5 The Calogero–Sutherland DNLS defocusing equation (CS[−])

In this section, we consider the defocusing equation of (CS)

$$i\partial_t u + \partial_x^2 u - 2D_+(|u|^2)u = 0. \quad (\text{CS}^-)$$

Note that by adapting the argument of [GL24, Proposition 2.1] to the defocusing equation, one can infer the local well-posedness of the (CS[−]) problem in $H_+^s(\mathbb{T})$ for $s > \frac{3}{2}$. And we expect that one can go down to $s > \frac{1}{2}$ by following [dMP10].

Below are a series of lemmas, propositions, and theorems that can be proved similarly to their analogs in the focusing case. Again, the integrable methods are the main ingredients to conclude. The first proposition is to announce that the defocusing equation of (CS[−]) enjoys also a Lax pair formalism.

Proposition 4.5.1 (Lax pair for (CS[−])). *Let $u \in \mathcal{C}([-T, T], H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$, be a solution of (CS[−]). There exist two operators*

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2$$

satisfying the Lax equation

$$\frac{d\tilde{L}_u}{dt} = [\tilde{B}_u, \tilde{L}_u].$$

Lemma 4.5.2. *Given $u \in \mathcal{C}([-T, T], H_+^r(\mathbb{T}))$, $r > \frac{3}{2}$, a solution of (CS[−]) equation, then*

$$\partial_t u = \tilde{B}_u u - i\tilde{L}_u^2 u.$$

As a consequence, the quantities $\tilde{\mathcal{H}}_s(u) := \langle (\tilde{L}_u + \lambda)^s u \mid u \rangle$, $\lambda > 0$, are conserved by the flow $\mathcal{S}^-(t)$ of (CS[−]) for all $0 \leq s \leq 2r$.

Remark 4.5.1. Expanding the conservation laws $\tilde{\mathcal{H}}_k(u)$ for all $k \in \mathbb{N}_{\geq 0}$, we have

$$\begin{aligned}\tilde{\mathcal{H}}_0(u) &= \langle u(t) | u(t) \rangle = \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \\ \tilde{\mathcal{H}}_1(u) &= \langle \tilde{L}_{u(t)}u(t) | u(t) \rangle = \|u(t)\|_{\dot{H}^{1/2}}^2 + \|T_{\bar{u}(t)}u(t)\|_{L^2}^2 \geq \|u(t)\|_{\dot{H}^{1/2}}^2 \\ \tilde{\mathcal{H}}_2(u) &= \|\tilde{L}_{u(t)}u(t)\|_{L^2}^2 \geq (1 - \varepsilon)\|Du(t)\|_{L^2}^2 + (1 - C_\varepsilon)\|T_{u(t)}T_{\bar{u}(t)}u(t)\|_{L^2}^2 \\ &\geq (1 - \varepsilon)\|u(t)\|_{\dot{H}^1}^2 - C(\varepsilon, \|u\|_{\dot{H}^{1/2}}) \\ &\quad \vdots\end{aligned}$$

thanks to Young's and Sobolev's inequalities. Unlike the focusing case, here we deduce the uniform control of the growth of Sobolev norms of the solution u by the conservation laws, without requiring any additional condition of smallness on the initial data u_0 . Therefore, Proposition 4.2.8 holds in the defocusing case for all $u_0 \in H_+^r(\mathbb{T})$, $r > \frac{3}{2}$.

As a result, we state the following theorem which is the analog of Theorem 4.1.1 but for equation (CS⁻).

Theorem 4.5.3. *For all $s > \frac{3}{2}$, let $u_0 \in H_+^s(\mathbb{T})$. There exists a unique global solution $u \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T}))$ of the defocusing equation (CS⁻), satisfying at $t = 0$, $u(0, \cdot) = u_0$. Furthermore, for all $s > \frac{3}{2}$,*

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \leq C,$$

where $C = C(\|u_0\|_{H^s}) > 0$ is a positive constant.

As for the focusing case, the defocusing Calogero–Sutherland DNLS has an explicit solution.

Lemma 4.5.4. *Let $u_0 \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T}))$, $s > \frac{3}{2}$ then the solution of the defocusing Calogero–Sutherland DNLS equation (CS⁻) is given by*

$$u(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2it\tilde{L}_{u_0}} S^*)^{-1} u_0 | 1 \right\rangle.$$

In particular, using this explicit formula, we extend the flow $\mathcal{S}^-(t)$ continuously from $H_+^2(\mathbb{T})$ to $H_+^s(\mathbb{T})$, for $0 \leq s \leq \frac{3}{2}$. Therefore, we have :

Theorem. 4.1.5 . *The Calogero–Sutherland DNLS defocusing equation (CS⁻) is globally well-posed in $H_+^s(\mathbb{T})$ for any $s \geq 0$, in the sense of Remark 4.1.2. In addition, for all $u_0 \in H_+^s(\mathbb{T})$,*

$$u(t, z) = \left\langle (\text{Id} - z e^{-it} e^{-2it\tilde{L}_{u_0}} S^*)^{-1} u_0 | 1 \right\rangle,$$

is solution of the (CS⁻)-defocusing equation. Furthermore, the trajectories

$$\{\mathcal{S}^-(t)u_0; t \in \mathbb{R}\}$$

are relatively compact in $H_+^s(\mathbb{T})$.

4.6 Final remarks and open problems

Let us briefly discuss here some remarks related to the previous sections.

1. One interesting feature about the focusing Calogero–Sutherland DNLS equation is that it admits a rich dynamic in comparison to the defocusing equation. For instance, as we shall see [Bad23], the focusing equation has a wider collection of traveling wave solutions.

2. The problem of global well-posedness of the focusing Calogero–Sutherland DNLS equation (CS^+) without restriction on the initial data is wide open. Nevertheless, we expect that the explicit solution (equation (4.3.29))

$$u(t, z) = \langle (\text{Id} - z e^{-it} e^{-2itL_{u_0}} S^*)^{-1} u_0 | 1 \rangle, \quad (4.6.1)$$

is a key ingredient to answer this question. Indeed, writing for all $t \in \mathbb{R}$, Σ_t the operator $S e^{2itL_{u_0}} e^{it}$, we have by (4.6.1)

$$u(t, z) = \sum_{n \geq 0} \langle u_0 | \Sigma_t^n 1 \rangle z^n. \quad (4.6.2)$$

Observe that, if u_0 belongs to the space \mathcal{J} generated by the orthonormal family $\{\Sigma_t^n 1, n \geq 0\}$, then using Parseval's identity on (4.6.2), we infer

$$\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2}, \quad t \in \mathbb{R}, \quad (4.6.3)$$

leading to say that the set $\{u(t), t \in \mathbb{R}\}$ is relatively compact in $L^2_+(\mathbb{T})$. Hence, the integer N_η set out in inequality (4.3.3) is now independent of t , and thus applying inequality (4.3.3) to (4.2.19), we obtain for all $\eta > 0$,

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq (1 - 2\eta^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - 2\eta^2 - 2N_\eta^2 \|u_0\|_{L^2}) \|f\|_{L^2}^2,$$

instead of having inequality (4.2.20) :

$$\|(L_{u(t)} + \lambda)^{\frac{1}{2}} f\|_{L^2}^2 \geq (1 - \|u\|_{L^2}^2) \|f\|_{\dot{H}^{\frac{1}{2}}}^2 + (\lambda - \|u\|_{L^2}^2) \|f\|_{L^2}^2.$$

Therefore, using (4.3.3), we control the growth of all the Sobolev norm $\|u(t)\|_{H^s}$ for all $s \geq 0$, and we infer the global well-posedness of the focusing (CS^+) in all $H^s_+(\mathbb{T})$, $s > \frac{3}{2}$ for arbitrary initial data. In addition, by the same manner, we deduce also $H^{\frac{1}{2}}$ -bounds on the eigenfunctions ($f_n^{\varepsilon, t}$) –inequality (4.3.26)– implying that the flow $\mathcal{S}^+(t)$ can be extended to $L^2_+(\mathbb{T})$, for arbitrary initial data. Besides, if u_0 does not belong to \mathcal{J} , then we expect blow-up results in finite time T .

Chapter 5

Zero dispersion limit of the Calogero–Moser DNLS equation

This chapter is from the submitted paper [Bad24b].

ABSTRACT.

We study the zero–dispersion limit (i.e. the semi–classical limit) of the Calogero–Moser derivative NLS equation

$$i\partial_t u + \partial_x^2 u \pm 2u \cdot D\Pi(|u|^2) = 0, \quad x \in \mathbb{R}, \quad (\text{CM})$$

where $D = -i\partial_x$, and Π is the Szegő projector defined by $\widehat{\Pi u}(\xi) = \mathbf{1}_{\xi \geq 0} \widehat{u}(\xi)$. We provide an explicit formula to characterize and prove the uniqueness of the weak limit solution when the dispersion term is neglected in the Cauchy problem of (CM) starting from an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Moreover, we identify this limit in terms of the branches of the multivalued solution of the inviscid Burgers–Hopf equation starting from an initial data $|u_0|^2$. Finally, we infer that it satisfies a maximum principle even though we have small dispersion.

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5.1 Introduction

We consider the nonlocal nonlinear Schrödinger equation called *the Calogero–Moser derivative nonlinear Schrödinger equation*

$$i\partial_t u + \partial_x^2 u \pm 2u D\Pi(|u|^2) = 0, \quad x \in \mathbb{R}, \quad (\text{CM})$$

where $D = -i\partial_x$, and $\Pi \equiv \Pi_+$ is the Szegő projector defined as

$$\widehat{\Pi u}(\xi) = \widehat{u}(\xi)\mathbf{1}_{\xi \geq 0} \quad \text{or as} \quad \Pi u(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{u(y)}{y-z} dy, \quad z \in \mathbb{C}_+, \quad (5.1.1)$$

which is an orthogonal projector from $L^2(\mathbb{R})$ into the Hardy space

$$\begin{aligned} L_+^2(\mathbb{R}) &:= \{u \in L^2(\mathbb{R}), \text{supp } \widehat{u} \subseteq [0, +\infty[\} \\ &\cong \{u \in \text{Hol}(\mathbb{C}_+), \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|^2 dx < +\infty\}, \end{aligned}$$

with $\mathbb{C}_+ := \{z \in \mathbb{C}; \text{Im}(z) > 0\}$. This equation comes in two versions : one with the “+” sign in front of the nonlinearity, referring to the focusing equation, and the other with the “−” sign, referring to the defocusing equation. Through this paper, the \pm and \mp symbols will be interchanged based on the following rule : the upper sign will correspond to the focusing case and the lower sign to the defocusing case.

The Calogero–Moser DNLS (CM) is invariant under the scaling

$$u(t, x) \longmapsto \lambda^{1/2} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \quad (5.1.2)$$

This suggests that the L_+^2 space is the scaling-critical space of (CM) when $x \in \mathbb{R}$ and $x \in \mathbb{T}$. In [GL24], the authors studied the focusing (CM) equation and proved the global well-posedness of (CM) in $H_+^k(\mathbb{R}) := H^k(\mathbb{R}) \cap L_+^2(\mathbb{R})$, $k \in \mathbb{N}_{\geq 1}$, for small initial data $\|u_0\|_{L^2} < \sqrt{2\pi}$. This smallness condition appears after applying a sharp inequality to control the growth of the Sobolev norms $\|u(t)\|_{H^k}$, $k \in \mathbb{N}_{\geq 1}$, over time by the conservation laws. Subsequently, the case on the torus ($x \in \mathbb{T}$) has been investigated by the author under the name of *Calogero–Sutherland DNLS equation* [Bad24a, Bad23], where the global well-posedness of the (CM) equation was obtained in all the Hardy–Sobolev spaces $H_+^s(\mathbb{T})$ for all $s \geq 0$, for both the focusing and defocusing equations, under small initial data in the focusing case. Later on, Killip–Laurens–Vişan [KLV23a] extended the Global well-posedness results for the case of $x \in \mathbb{R}$ to all the Hardy–Sobolev spaces $H_+^s(\mathbb{R})$ for all $s \geq 0$, still under the condition of $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case. Although, thanks to the work in [GL24, Section 3], we know that there is no finite-time minimal mass blow-up solution when $\|u_0\|_{L^2}^2 = 2\pi$, the result of [HK24] emphasizes the importance of this smallness condition for ensuring global solutions. In fact, they proved the existence of $u_0 \in H_+^\infty(\mathbb{R})$ with mass $\|u_0\|_{L^2}^2 = 2\pi + \epsilon$, $\epsilon > 0$, such that there exists $T \in (0, +\infty]$ and

$$\lim_{t \rightarrow T} \|u(t)\|_{H^s} = \infty, \quad s > 0.$$

More recently, Kim–Kim–Kwon underlined in [KKK24] using modulation analysis that one can indeed find blow-up solutions in finite time. Moreover, they provided a sharp description of the blow-up phenomena.

From a physical standpoint, the scenarios described by the Calogero–Moser DNLS equation share notable similarities with the **Benjamin–Ono equation**. In

both cases, they characterize weakly nonlinear dispersive internal waves located at the interface between two fluid layers of different densities, with the lower layer having infinite depth [JBS08, Sau19, Pel95]. In the context of the Benjamin–Ono equation, the solution delineates the progression of these internal waves. On the other hand, concerning the (CM)–equation, it illustrates a model for the envelope of approximately monochromatic waves within the aforementioned settings.

In recent decades, some mathematicians have studied the Benjamin–Ono equation with small dispersion $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon + \partial_x((u^\varepsilon)^2) = \varepsilon |D| \partial_x u^\varepsilon \\ u^\varepsilon|_{t=0} = u_0 \end{cases}, \quad (\text{BO-eps})$$

where they established that the weak limit in L^2 of u^ε , as ε approaches 0, is characterized in terms of the branches of the multivalued solution of the **inviscid Burgers–Hopf equation**. This result was first obtained by Miller et al. [MX11, MW16] for specific initial data, then by [Gas23a, Gas23b] in the case of the torus $x \in \mathbb{T}$ for bell-shaped initial data, and finally by Gérard [Gér23b] for any $u_0 \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ where $|u_0| + |u'_0| \rightarrow 0$ at infinity. Observe that one can discern the emergence of the Burgers equation by formally taking the limit in (BO-eps) as $\varepsilon \rightarrow 0$. The act of neglecting the dispersion component in the equation is commonly acknowledged in the literature as the “zero-dispersion limit” or “semiclassical limit”. In the following, we will use the terminology of “zero-dispersion limit”. Additionally, we will refer to the weak L^2 –limit of u^ε when $\varepsilon \rightarrow 0$, as “*the weak zero-dispersion limit solution*”. It is important to note that the selection of initial data, represented by u_0 , remains independent of ε .

In this paper, we propose to investigate the zero–dispersion limit problem for the Calogero–Moser DNLS equation. Thus, we consider the rescaled version of (CM) with small dispersion $\varepsilon > 0$,

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon \partial_x^2 u^\varepsilon \pm 2u^\varepsilon D\Pi(|u^\varepsilon|^2) = 0 \\ u^\varepsilon|_{t=0} = u_0 \end{cases}. \quad (\text{CM-eps})$$

The aim is to write the weak limit in L^2 of the solution u^ε of (CM-eps), as $\varepsilon \rightarrow 0$, in terms of the branches of the multivalued solution of the Burgers equation. However, here, it is less evident, compared to the Benjamin–Ono case, why the Burgers equation emerges in this context. For this purpose, observe that when formally taking $\varepsilon \rightarrow 0$, the (CM-eps) becomes

$$i\partial_t u \pm 2u \cdot D\Pi(|u|^2) = 0. \quad (\text{CM-zero})$$

Consequently, if u solves the previous equation, then $\mathbf{v} = |u|^2$ solves the Burgers equation

$$\partial_t \mathbf{v} = \pm 2\mathbf{v} \partial_x \mathbf{v}, \quad (5.1.3)$$

since

$$\begin{aligned}
\partial_t \mathbf{v} &= 2 \operatorname{Re}(\partial_t u \bar{u}) \\
&= \pm 4 \operatorname{Re}(\partial_x \Pi(|u|^2) |u|^2) \\
&= \pm 2 (\partial_x \Pi(|u|^2) + \overline{\partial_x \Pi(|u|^2)}) |u|^2 \\
&= \pm \partial_x |u|^4 = \pm \partial_x \mathbf{v}^2 \\
&= \pm 2 \mathbf{v} \partial_x \mathbf{v}.
\end{aligned}$$

But before deriving the relation of the weak zero–dispersion limit of (CM) in terms of the branches of the multivalued solution of the Burgers equation, it is essential to prove the existence of this uniquely well–defined weak limit. The upcoming theorem seeks to address this issue. Moreover, it aims to characterize this L^2 –weak limit explicitly as a unique element of the Hardy space. The notation $ZD_+[u_0]$ represents the weak zero–dispersion limit solution in the focusing case for (CM), and $ZD_-[u_0]$ corresponds to the one in the defocusing case.

Theorem 5.1.1. *Given an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}^4$ in the focusing case), the weak (in L^2 –space) zero–dispersion limit solution $ZD_\pm[u_0]$ of (CM) exists, and is characterized via the following explicit formula*

$$ZD_\pm[u_0](t, z) = \left(\operatorname{Id} \mp 2t T_{u_0} T_{\bar{u}_0} (X^* - z)^{-1} \right)^{-1} u_0(z), \quad t \in \mathbb{R}, z \in \mathbb{C}_+, \quad (5.1.4)$$

where the operators T_v and X^* are defined respectively at (5.2.2) and (5.2.3). In addition, we have

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2}.$$

Furthermore, if $u_0^n \rightarrow u_0$ strongly in L^2 as $n \rightarrow \infty$, with $\sup_n \|u_0^n\|_{L^\infty} < +\infty$, then for all $T > 0$,

$$\sup_{t \in [-T, T]} |ZD_\pm[u_0^n](t) - ZD_\pm[u_0](t)| \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}). \quad (5.1.5)$$

Remark 5.1.1. One can prove easily by relying on the explicit formula of the solution of (CM) on the torus ($x \in \mathbb{T}$) [Bad24a, Proposition 2.5], that the zero dispersion limit of (CM), on the torus case, exists and is also uniquely well–defined in the Hardy space via an explicit formula.

Usually, when considering the scenario of zero–dispersion limit, the emergence of shocks can be observed. These shocks manifest as we begin to neglect dispersive effects, allowing the nonlinear term to dominate. With the existence of the weak zero–dispersion limit established in the previous theorem, our objective in the following theorem is to highlight these shocks, by addressing the connection between this zero–dispersion limit solution of (CM) and the branches of the multivalued solution of the inviscid Burgers equation, which is known for its tendency to exhibit shock formations.

1. The constant $\sqrt{2\pi}$ is to ensure the GWP of (CM) in the focusing case.

Theorem 5.1.2. *Let $u_0 \in L^2_+(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), such that u_0 is a C^1 function tending to 0 at infinity, with a bounded derivative in $L^\infty(\mathbb{R})$.² Then, for every time $t \in \mathbb{R}$, and for almost every $x \in \mathbb{R}$, the equation*

$$y \mp 2t|u_0(y)|^2 = x \quad (5.1.6)$$

has an odd number of simple real solutions $y_0 := y_0(t, x) < \dots < y_{2\ell} := y_{2\ell}(t, x)$, where $\ell := \ell(t, x)$, and the zero–dispersion limit of (CM) is given by

$$ZD_\pm[u_0](t, x) = e^{i\varphi(t, x)} \left(\mp i \frac{|t|}{t} \right)^\ell \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k}, \quad (5.1.7)$$

where

$$\varphi(t, x) = \arg(u_0(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log \left(\frac{s \mp 2t|u_0(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k)}{-s \mp 2t|u_0(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k)} \right) ds.$$

Remark 5.1.2. We refer to the end of the Proof of Theorem 5.1.2 in Section 3 to underline why the previous integral is well-defined.

Remark 5.1.3. When $\ell > 0$, then any solution $y_k := y_k(t, x)$, $k \in \{0, \dots, 2\ell\}$ satisfying the equation (5.1.6), represents a branch of the multivalued solution of the Burgers equation (5.1.3) at a time t beyond the shock time, and at a position x .

Remark 5.1.4. By taking the modulus of (5.1.7), we deduce

$$\log|ZD_\pm[u_0](t, x)|^2 = \sum_{k=0}^{2\ell} (-1)^k \log|u_0(y_k)|^2.$$

This result should be compared to the ones obtained for the (BO)–equation, where we have [Gér23b] : for all $t \in \mathbb{R}$, for almost every $x \in \mathbb{R}$, and under the same condition of smoothness on the initial data of Theorem 5.1.2, the zero–dispersion limit of (BO) is given as

$$ZD_{(\text{BO})}[u_0](t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k^{BO}), \quad (5.1.8)$$

where the $(y_k^{BO})_{0, \dots, 2\ell}$, $\ell = \ell(t, x) \in \mathbb{N}_{\geq 0}$, are real solutions for the equation

$$y + 2tu_0(y) = x.$$

A consequence of the previous Theorem, is the existence of a maximum Principle even though we have small dispersion.

2. Note that any function in $H^s_+(\mathbb{R}) := H^s(\mathbb{R}) \cap L^2_+(\mathbb{R})$, $s > \frac{3}{2}$ satisfies these conditions.

Corollary 5.1.3 (A Maximum Principle). *Let $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case). For all $t \in \mathbb{R}$,*

$$\|ZD_\pm[u_0]\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

Zero dispersion limit of the KdV equation. The problem of zero dispersion limit was first investigated by Lax and Levermore [LDL83] for the KdV equation on the real line

$$\partial_t u - 3\partial_x(u^2) + \varepsilon^2 \partial_x^3 u = 0, \quad u^\varepsilon(0, x) = u_0(x), \quad (\text{KdV})$$

describing, thus, the weak zero dispersion limit for nonpositive initial data decaying sufficiently fast at infinity. In contrast with the Benjamin–Ono equation [MX11, MW16, Gas23a, Gas23b, Gér23b] and the Calogero–Moser DNLS equation, the zero-dispersion limit for the KdV equation is expressed implicitly, as it is characterized by a quadratic minimum problem with constraints. Lax–Levermore’s work initiated a series of papers. One can cite [Ven87, Ven90, GK07, CG09, CG10a, CG10b], where in all these works the inverse scattering theory, the spectral theory of the Lax operator and the associated Riemann–Hilbert problem are the main keys.

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5.2 The explicit formula of the zero dispersion limit of (CM)

This section aims to establish the existence of a weak (in L^2) zero-dispersion limit solution to the Calogero–Moser DNLS equation (CM). Additionally, it seeks to properly characterize this weak limit for all time t , through an explicit formula. However, before proceeding, it is necessary to revisit some properties regarding the (CM) equation.

The Calogero–Moser DNLS equation (CM) is a completely integrable PDE in the following two senses : First, it possesses a Lax Pair structure (L_u, B_u) that satisfies the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u], \quad [B_u, L_u] := B_u L_u - L_u B_u$$

for enough regular u satisfying the (CM)–equation [GL24]. The Lax operators for (CM) are given by

$$L_u = D \mp T_u T_{\bar{u}}, \quad B_u = \pm T_u T_{\partial_x \bar{u}} \mp T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2, \quad (5.2.1)$$

where T_v is the Toeplitz operator of symbol v defined as

$$T_v f = \Pi(vf), \quad f \in L^2_+(\mathbb{R}), \quad (5.2.2)$$

and Π denotes the Szegő projector (5.1.1). Second, the complete integrability manifests through the finding of an explicit formula of the solution of the (CM)–equation [KLV23a]. To introduce this formula, specific notation needs to be presented. Thus, we consider on $L_+^2(\mathbb{R})$, the contraction semigroup

$$S(\eta)h(x) = \Pi(e^{ix\eta} h(x)) = e^{ix\eta} h(x), \quad \eta > 0.$$

And we denote by X its infinitesimal generator

$$Xh(x) = -i \frac{d}{d\eta} \Big|_{\eta=0} (S(\eta)h(x)) = xh(x),$$

of domain

$$\begin{aligned} \text{Dom}(X) &= \{h \in L_+^2(\mathbb{R}); xh \in L^2(\mathbb{R})\} \\ &= \{h \in L_+^2(\mathbb{R}); \widehat{h} \in H^1([0, +\infty)), \widehat{h}(0) = 0\}. \end{aligned}$$

Its adjoint X^* has the following domain

$$\begin{aligned} \text{Dom}(X^*) &= \{f \in L_+^2(\mathbb{R}); \exists c > 0, \forall h \in \text{Dom}(X), |\langle f | Xh \rangle| \leq c \|h\|_{L^2}\} \\ &= \{f \in L_+^2(\mathbb{R}); \widehat{f}|_{(0, +\infty)} \in H^1((0, +\infty))\}, \end{aligned}$$

and is defined for all $\xi > 0$ as

$$\widehat{X^*f}(\xi) = i\partial_\xi \widehat{f}(\xi).$$

That is, for all $f \in \text{Dom}(X^*)$,

$$X^*f(x) = xf + \frac{1}{2\pi i} \widehat{f}(0^+). \quad (5.2.3)$$

The following theorem aims to recall the explicit formula of (CM) defined for any $u_0 \in L_+^2(\mathbb{R})$ [KLV23a].

Theorem 5.2.1 ([KLV23a]). *Let $u_0 \in L_+^2(\mathbb{R})$ (such that $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case). Then there exists a unique global solution $u \in \mathcal{C}_t L_+^2(\mathbb{R})$ of (CM) such that for any $(u_n^0) \subseteq H_+^\infty(\mathbb{R})$, $(xu_n^0) \subseteq L^2$, $u_n^0 \rightarrow u_0$ in L^2 , we have for all $T > 0$,*

$$u_n \rightarrow u \quad \text{in } \mathcal{C}_t L_+^2([-T, T], \mathbb{R}).$$

Additionally, for all $z \in \mathbb{C}_+ := \{z \in \mathbb{C}, \text{Im}(z) > 0\}$,

$$u(t, z) = \frac{1}{2\pi i} I_+((X^* + 2tL_{u_0} - z)^{-1}u_0), \quad (5.2.4)$$

where I_+ denotes

$$I_+(f) := \widehat{f}(0^+), \quad \forall f \in \text{Dom}(X^*). \quad (5.2.5)$$

As a consequence, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Remark 5.2.1. We would like to draw the reader's attention to the fact that a similar explicit formula has already been found in the case of the circle [Bad24a, Proposition 2.5]. This is not the first instance of discovering an explicit formula for a completely integrable PDE; for previous examples, we refer to [GG15, Gér23a, GP23a].

In what follows, we consider the rescaled version of the (CM)–equation with initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$: For all $\varepsilon > 0$,

$$\begin{cases} i\partial_t u^\varepsilon + \varepsilon \partial_x^2 u^\varepsilon \pm 2u^\varepsilon D\Pi(|u^\varepsilon|^2) = 0, \\ u^\varepsilon|_{t=0} = u_0. \end{cases} \quad (\text{CM-eps})$$

Our primary focus is on establishing the existence and the uniqueness of the weak zero dispersion limit for (CM) as ε approaches zero. The following theorem addresses this question. We recall that the considered initial data u_0 is independent of the parameter ε , and that $ZD_+[u_0]$ represents the weak zero-dispersion limit solution in the focusing case, and $ZD_-[u_0]$ corresponds to the one in the defocusing case.

Theorem. 5.1.1. *Given an initial data $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$, (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), the weak (in L^2 -space) zero-dispersion limit solution $ZD_\pm[u_0]$ of (CM) exists and is characterized via the following explicit formula*

$$ZD_\pm[u_0](t, z) = \left(\text{Id} \mp 2tT_{u_0}T_{\bar{u}_0}(X^* - z)^{-1} \right)^{-1} u_0(z), \quad t \in \mathbb{R}, z \in \mathbb{C}_+. \quad (5.2.6)$$

In addition, we have

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2}. \quad (5.2.7)$$

Furthermore, if $u_0^n \rightarrow u_0$ strongly in L^2 as $n \rightarrow \infty$ with $\sup_n \|u_0^n\|_{L^\infty} < +\infty$, then for all $T > 0$,

$$\sup_{t \in [-T, T]} |ZD_\pm[u_0^n](t) - ZD_\pm[u_0](t)| \rightarrow 0 \quad \text{in } L^2(\mathbb{R}). \quad (5.2.8)$$

Proof. In view of Theorem 5.2.1 we have for all $\varepsilon > 0$,

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2},$$

where $u^\varepsilon(t)$ is the solution of (CM-eps). Hence, by Banach's theorem, we deduce that there exists $ZD_\pm[u_0] \in L^2_+(\mathbb{R})$, such that, up to a sequence, $u^\varepsilon(t) \rightharpoonup ZD_\pm[u_0](t)$ in L^2 as $\varepsilon \rightarrow 0$, and

$$\|ZD_\pm[u_0](t)\|_{L^2} \leq \liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}.$$

To characterize $ZD_\pm[u_0]$, we will use the explicit formula of Theorem 5.2.1. However, first observe that when u is a solution of (CM) for an initial data u_0 , then $\sqrt{\varepsilon}u(\varepsilon t, \cdot) \equiv \sqrt{\varepsilon}\mathcal{S}(\varepsilon t)[u_0]$ is a solution to (CM-eps) for an initial data $\sqrt{\varepsilon}u_0$, where $\mathcal{S}(t)$ denotes the flow of (CM). That is

$$u^\varepsilon(t) := \sqrt{\varepsilon}\mathcal{S}(\varepsilon t) \begin{bmatrix} u_0 \\ \sqrt{\varepsilon} \end{bmatrix}$$

is a solution to (CM-eps) for an initial data u_0 . Therefore, starting from an initial data u_0 , one deduces by (5.2.4) and (5.2.1) that the solution of (CM-eps) in the focusing and defocusing case is explicitly given, for all $\varepsilon > 0$, by

$$u^\varepsilon(t, z) := \frac{1}{2\pi i} I_+ ((X^* + 2t\varepsilon D \mp 2tT_{u_0}T_{\bar{u}_0} - z)^{-1} u_0), \quad z \in \mathbb{C}_+. \quad (5.2.9)$$

The next step is to pass to the limit $\varepsilon \rightarrow 0$ in the above formula. For this purpose, we rewrite (5.2.9) as follows

$$u^\varepsilon(t, z) = \frac{1}{2\pi i} \left(\text{Id} \mp 2te^{-i\varepsilon t D^2} T_{u_0} T_{\bar{u}_0} e^{i\varepsilon t D^2} (X^* - z)^{-1} \right)^{-1} e^{-i\varepsilon t D^2} u_0.$$

Indeed, by using the Fourier transform, for all $\xi > 0$,

$$(X^* + \widehat{2t\varepsilon D})f(\xi) = e^{it\varepsilon\xi^2} i\partial_\xi(e^{-it\varepsilon\xi^2} \widehat{f}(\xi)),$$

(5.2.9) becomes

$$\begin{aligned} u^\varepsilon(t, z) &= \frac{1}{2\pi i} I_+ \left((e^{i\varepsilon t D^2} X^* e^{-i\varepsilon t D^2} \mp 2tT_{u_0}T_{\bar{u}_0} - z)^{-1} u_0 \right) \\ &= \frac{1}{2\pi i} I_+ \left(e^{i\varepsilon t D^2} (X^* \mp 2te^{-i\varepsilon t D^2} T_{u_0} T_{\bar{u}_0} e^{i\varepsilon t D^2} - z)^{-1} e^{-i\varepsilon t D^2} u_0 \right). \end{aligned}$$

Thus, by definition of I_+ in (5.2.5), we deduce

$$\begin{aligned} u^\varepsilon(t, z) &= \frac{1}{2\pi i} I_+ \left((X^* \mp 2te^{-i\varepsilon t D^2} T_{u_0} T_{\bar{u}_0} e^{i\varepsilon t D^2} - z)^{-1} e^{-i\varepsilon t D^2} u_0 \right) \\ &= \frac{1}{2\pi i} I_+ \left((X^* - z)^{-1} \cdot (\text{Id} \mp 2te^{-i\varepsilon t D^2} T_{u_0} T_{\bar{u}_0} e^{i\varepsilon t D^2} (X^* - z)^{-1})^{-1} e^{-i\varepsilon t D^2} u_0 \right). \end{aligned}$$

Now, using the fact that [Gér23a]

$$\begin{aligned} I_+ \left((X^* - z)^{-1} f \right) &= \lim_{\varepsilon \rightarrow 0} \left\langle (X^* - z)^{-1} f, \frac{1}{1 - i\varepsilon x} \right\rangle = \lim_{\varepsilon \rightarrow 0} \left\langle f, (X - \bar{z})^{-1} \left(\frac{1}{1 - i\varepsilon x} \right) \right\rangle \\ &= \lim_{\varepsilon \rightarrow 0} \left\langle f, (x - \bar{z})^{-1} \left(\frac{1}{1 - i\varepsilon x} \right) \right\rangle = 2\pi i f(z), \end{aligned} \quad (5.2.10)$$

we infer,

$$u^\varepsilon(t, z) = \left(\text{Id} \mp 2te^{-i\varepsilon t D^2} T_{u_0} T_{\bar{u}_0} e^{i\varepsilon t D^2} (X^* - z)^{-1} \right)^{-1} e^{-i\varepsilon t D^2} u_0(z). \quad (5.2.11)$$

Observing first that $\|e^{-i\varepsilon t D^2} u_0\|_{L^2} = \|u_0\|_{L^2}$, and second, $e^{-i\varepsilon t D^2} T_{u_0} T_{\bar{u}_0} e^{i\varepsilon t D^2} (X^* - z)^{-1}$ is a bounded operator as $u_0 \in L^\infty$, we infer by passing to the limit $\varepsilon \rightarrow 0$ in (5.2.11),

$$ZD_\pm[u_0](t, z) := \left(\text{Id} \mp 2tT_{u_0}T_{\bar{u}_0}(X^* - z)^{-1} \right)^{-1} u_0(z).$$

To prove the final statement (5.2.8) one can employ the same ideas presented in this proof, along with the inequality from [GL24, Lemma A.1]

$$\|T_{\bar{u}} f\|_{L^2} \leq \frac{1}{2\pi} \|u\|_{L^2} \langle Df | f \rangle$$

to establish this weak convergence. \square

5.3 Link with the multivalued solution of the Burgers equation

The aim of this section is to prove Theorem 5.1.2, which describes the weak zero dispersion limit solution of (CM) starting from an initial data $u_0 \in L^2_+(\mathbb{R}) \cap \mathcal{C}^1$ tending to 0 at infinity and satisfying $u'_0 \in L^\infty(\mathbb{R})$, in terms of the branches of the multivalued solution for the Burgers equation (5.1.3). However, before proving this theorem for such initial data u_0 , we first focus on proving it for rational initial data in the Hardy space

$$u_0(y) = \frac{P(y)}{Q(y)}, \quad Q(y) := (y + \bar{p}_0) \cdots (y + \bar{p}_{N-1}), \quad p_k \neq p_j, \quad k \neq j. \quad (5.3.1)$$

where $p_k \in \mathbb{C}$, $\text{Im}(p_k) < 0$ for all $k = 0, \dots, N-1$, and $P(y) = \sum_{n=0}^{N-1} a_n y^n$, $a_n \in \mathbb{C}$.

Proposition 5.3.1. *Let u_0 be a rational function defined in (5.3.1). Then for every time $t \in \mathbb{R}$, and for almost every $x \in \mathbb{R}$, the algebraic equation*

$$y \mp 2t|u_0(y)|^2 = x \quad (5.3.2)$$

has an odd number of simple real solutions $y_0 := y_0(t, x) < \dots < y_{2\ell} := y_{2\ell}(t, x)$, and the zero-dispersion limit of (CM) is given, for almost every $x \in \mathbb{R}$, by

$$ZD_\pm[u_0](t, x) = e^{i\varphi(t, x)} \left(\mp i \frac{|t|}{t} \right)^\ell \prod_{k=0}^{2\ell} |u_0(y_k)|^{(-1)^k} \quad (5.3.3)$$

where

$$\varphi(t, x) = \arg(u_0(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log \left(\frac{s \mp 2t|u_0(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k)}{-s \mp 2t|u_0(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k)} \right) ds.$$

To prove the previous proposition, we split the proof into the following lemmas.

Lemma 5.3.2. *Let $u_0(y) = \frac{P(y)}{Q(y)}$ be a rational function defined as in (5.3.1). Then, for all $t \in \mathbb{R}$, $x \in \mathbb{R} \setminus K_t$, where K_t is a finite set in \mathbb{R} , the algebraic equation (5.3.2) admits an odd number of simple real solutions*

$$y_0 := y_0(t, x) < \dots < y_{2\ell} := y_{2\ell}(t, x).$$

Furthermore, the function $\gamma_t(y) := y \mp 2t|u_0(y)|^2$ is increasing near y_{2k} , $k = 0, \dots, \ell$, and decreasing near y_{2k+1} , $k = 0, \dots, \ell - 1$.

Proof. Given $u_0(y) = \frac{P(y)}{Q(y)}$ as in (5.3.1), we introduce³

$$y \in \mathbb{C} \mapsto v_0(y) := \frac{P(y)\bar{P}(y)}{Q(y)\bar{Q}(y)}, \quad \text{where } \bar{P}(y) = \sum_{n=0}^{N-1} \bar{a}_n y^n, \quad \bar{Q}(y) = (y+p_0) \cdots (y+p_{N-1}).$$

3. Observe that when $y \in \mathbb{R}$, $v_0(y) = |u_0(y)|^2$.

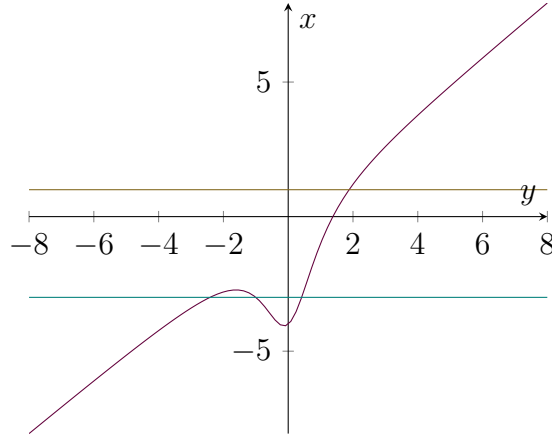


FIGURE 5.1 – For an initial data $u_0(y) := \frac{1}{y+i}$, we have in purple the graph of $\gamma_2(y) := y - 2 \cdot 2|u_0(y)|^2 = y - \frac{2 \cdot 2}{y^2+1}$. In yellow, we have the graph of $x = 1$. The abscissa of the intersection of the axis $x = 1$ with the graph of $\gamma_2(y)$ corresponds to the unique real solution y_0 of $\gamma_2(y) = 1$. In blue, we have the graph of $x = -3$. The abscissas of the intersection of the graph $\gamma_2(y)$ with $x = -3$ correspond to the points $y_0 < y_1 < y_2$ solutions to the algebraic equation $\gamma_2(y) = -3$.

Our goal is to study the *real* solutions of the algebraic equation (5.3.2). Observe that, y is a real solution of (5.3.2), if and only if, y is a real solution of

$$y \mp 2tv_0(y) = x, \quad t \in \mathbb{R},$$

that is, if and only if, y is a real solution of the polynomial equation of degree $2N + 1$,

$$(y - x)Q(y)\overline{Q}(y) \mp 2tP(y)\overline{P}(y) = 0. \tag{5.3.4}$$

Now, focusing on (5.3.4), one notices $y \in \mathbb{C} \setminus \mathbb{R}$ is a solution to (5.3.4), if and only if, its complex conjugate \bar{y} is a solution to (5.3.4). Therefore, the polynomial equation (5.3.4) of degree $2N + 1$ admits an odd number of real solutions

$$y_0 := y_0(t, x) \leq \dots \leq y_{2\ell} := y_{2\ell}(t, x).$$

Discarding a finite set of critical values⁴ x of the function $\gamma_t(y) := y \mp 2t|u_0(y)|^2$ for a given t , one may assume that these real solutions y_k are simple and that the function γ_t is increasing in a neighborhood of the points y_{2k} , $k = 0, \dots, \ell$, and decreasing in a neighborhood of the points y_{2k+1} , $k = 0, \dots, \ell - 1$, as $y \mapsto \gamma_t(y)$ is a continuous function behaving like

$$\gamma_t(y) \underset{y \rightarrow \pm\infty}{\sim} y.$$

□

4. We mean by critical values of a function γ_t , the γ_t -image of the critical points of γ_t , i.e. the γ_t -images of the points where $\gamma'_t(y) = 0$.

Lemma 5.3.3. Let $u_0(y) = \frac{P(y)}{Q(y)}$ be the rational function and $(p_k)_{k=0, \dots, N-1}$ be the complex constants defined in (5.3.1). Moreover, denote, for almost every $x \in \mathbb{R}$, $y_0 := y_0(t, x), \dots, y_{2N} := y_{2N}(t, x)$ the solutions of the equation

$$y \mp 2t \frac{P(y)\overline{P}(y)}{Q(y)\overline{Q}(y)} = x, \quad t \in \mathbb{R}. \quad (5.3.5)$$

Then, the zero-dispersion limit of (CM) is given by

$$ZD_{\pm}[u_0](t, x) = \frac{u_0(y_0)u_0(y_2) \dots u_0(y_{2N}) \begin{vmatrix} 1 & \frac{1}{y_0+p_0} & \frac{1}{y_0+p_1} & \cdots & \frac{1}{y_0+p_{N-1}} \\ 1 & \frac{1}{y_2+p_0} & \frac{1}{y_2+p_1} & \cdots & \frac{1}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{1}{y_{2N}+p_0} & \frac{1}{y_{2N}+p_1} & \cdots & \frac{1}{y_{2N}+p_{N-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{u_0(y_0)}{y_0+p_0} & \frac{u_0(y_0)}{y_0+p_1} & \cdots & \frac{u_0(y_0)}{y_0+p_{N-1}} \\ 1 & \frac{u_0(y_2)}{y_2+p_0} & \frac{u_0(y_2)}{y_2+p_1} & \cdots & \frac{u_0(y_2)}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{u_0(y_{2N})}{y_{2N}+p_0} & \frac{u_0(y_{2N})}{y_{2N}+p_1} & \cdots & \frac{u_0(y_{2N})}{y_{2N}+p_{N-1}} \end{vmatrix}}. \quad (5.3.6)$$

Proof. The main component is to use the explicit formula (5.1.4) of $ZD_{\pm}[u_0]$, which can be reexpressed using (5.2.10) as

$$ZD_{\pm}[u_0](t, z) = \frac{1}{2\pi i} I_+ [(X^* \mp 2tT_{u_0}T_{\overline{u_0}} - z)^{-1}u_0], \quad z \in \mathbb{C}_+, \quad (5.3.7)$$

where I_+ is defined in (5.2.5). The goal is to transform (5.3.7) into (5.3.6). For that, we decompose u_0 in terms of its partial fractional decomposition $u_0(y) = \sum_{k=0}^{N-1} \frac{c_k}{y+p_k}$, $c_k \in \mathbb{C}$, to infer by (5.2.2) and (5.2.3),

$$\begin{aligned} (X^* \mp 2tT_{u_0}T_{\overline{u_0}} - z)f(y) &= (y \mp 2t|u_0(y)|^2 - z)f(y) + \frac{1}{2\pi i} I_+(f) \\ &\quad \pm 2tu_0(y) \sum_{k=0}^{N-1} \frac{\overline{c_k}}{y+p_k} f(-p_k). \end{aligned} \quad (5.3.8)$$

Indeed, for all $f \in L^2_+(\mathbb{R})$,

$$T_{\overline{u_0}}f(y) = \sum_{k=0}^{N-1} \Pi_+ \left(\frac{\overline{c_k}}{y+p_k} f(y) \right) = \overline{u_0}(y)f(y) - \sum_{k=0}^{N-1} \frac{\overline{c_k}}{y+p_k} f(-p_k), \quad \Pi_+ \equiv \Pi.$$

Thus, for all $f \in L^2_+(\mathbb{R})$,

$$T_{u_0}T_{\overline{u_0}}f(y) = |u_0(y)|^2 f(y) - u_0(y) \sum_{k=0}^{N-1} \frac{\overline{c_k}}{y+p_k} f(-p_k).$$

Now, observe since formula (5.3.8) is valid for any $f \in L^2_+(\mathbb{R})$, then one can extend it to any holomorphic function f in \mathbb{C}_+ whose trace on \mathbb{R} is in $L^2(\mathbb{R})$. That is, if we denote by

$$y \in \mathbb{C} \longmapsto v_0(y) := \frac{P(y)\bar{P}(y)}{Q(y)\bar{Q}(y)},$$

then the following identity holds

$$\begin{aligned} (X^* \mp 2tT_{u_0}T_{\bar{u}_0} - z)f(y) &= (y \mp 2tv_0(y) - z)f(y) + \frac{1}{2\pi i}I_+(f) \\ &\quad \pm 2tu_0(y) \sum_{k \geq 0}^{N-1} \frac{\bar{c}_k}{y + p_k} f(-p_k), \end{aligned} \quad (5.3.9)$$

for all $y \in \mathbb{C}_+$, and for all holomorphic function f on \mathbb{C}_+ whose trace is in L^2 . In particular, for $f(y) = f_{t,z}(y) := (X \mp 2tT_{u_0}T_{\bar{u}_0} - z)^{-1}u_0(y) \in L^2_+(\mathbb{R})$, we infer by (5.3.7),

$$u_0(y) = (y \mp 2tv_0(y) - z)f_{t,z}(y) + ZD_{\pm}[u_0](t, z) \pm 2tu_0(y) \sum_{k \geq 0}^{N-1} \frac{\bar{c}_k}{y + p_k} f_{t,z}(-p_k),$$

or

$$f_{t,z}(y) = \frac{u_0(y) - ZD_{\pm}[u_0](t, z) \mp 2tu_0(y) \sum_{k \geq 0}^{N-1} \frac{\bar{c}_k}{y + p_k} f_{t,z}(-p_k)}{y \mp 2tv_0(y) - z}. \quad (5.3.10)$$

However, recall $y \mapsto f_{t,z}(y)$ is a holomorphic function in the upper-half complex plane. This means that the zeros in \mathbb{C}_+ of the denominator of $f_{t,z}$ must cancel its numerator. Therefore, the next step is to find the zeros of the algebraic equation $y \mp 2tv_0(y) = z$ on \mathbb{C}_+ , with the note that, at the end of the day, $z \in \mathbb{C}_+$ will be replaced by $x \in \mathbb{R}$ almost everywhere, as $ZD_{\pm}[u_0]$ belongs to the Hardy space $L^2_+(\mathbb{R})$ thanks to (5.2.7).

Let $x \in \mathbb{R}$. In view of Lemma 5.3.2, the algebraic equation $y \mp 2tv_0(y) = x$ admits an odd number of real solutions⁵

$$y_0 := y_0(t, x) < \dots < y_{2\ell} := y_{2\ell}(t, x).$$

Moreover, we denote by

$$y_{2\ell+1} := y_{2\ell+1}(t, x), \dots, y_{2N} := y_{2N}(t, x),$$

the remaining solutions of $y \mp 2tv_0(y) = x$ belonging to the complex plane, where thanks to (5.3.4) we can impose that $y_{2p-1} = \bar{y}_{2p}$ for all $p = \ell + 1, \dots, N$; and in what follows, we suppose $\text{Im}(y_{2p}) > 0$ for all $p = \ell + 1, \dots, N$.

⁵. We recall that the real solutions y of the equation $y \mp 2tv_0(y) = x$ are the same real solutions of $y \mp 2t|u_0(y)|^2 = x$ as $v_0(y) = |u_0(y)|^2$ when y is real.

By moving $x = z$ slightly up to the upper half-complex plane, one proves by using the implicit function theorem in its holomorphic version, that $z \mapsto y_k(t, z)$ is a holomorphic function and thus satisfies the Cauchy–Riemann equations

$$\frac{\partial \operatorname{Im}(y_k)}{\partial \operatorname{Im}(z)} = \frac{\partial \operatorname{Re}(y_k)}{\partial \operatorname{Re}(z)}.$$

Besides, recall for all $k = 0, \dots, \ell, j = 1, \dots, \ell,$

$$\frac{\partial \operatorname{Re}(y_{2k})}{\partial \operatorname{Re}(z)} > 0, \quad \frac{\partial \operatorname{Re}(y_{2j-1})}{\partial \operatorname{Re}(z)} < 0,$$

since by Lemma 5.3.2 the function $\gamma_t(y) := y \mp 2tv_0(y)$ is increasing near $y_{2k}, k = 0, \dots, \ell,$ and decreasing near $y_{2k+1}, k = 0, \dots, \ell - 1.$ As a result, for all $k = 0, \dots, N, j = 1, \dots, N,$

$$\frac{\partial \operatorname{Im}(y_{2k})}{\partial \operatorname{Im}(z)} > 0, \quad \frac{\partial \operatorname{Im}(y_{2j-1})}{\partial \operatorname{Im}(z)} < 0.$$

That is $(y_{2k})_{k=0, \dots, N} \subseteq \mathbb{C}_+,$ and thus, at these points, the numerator of (5.3.10) must vanish. Consequently, one deduces the following linear system of unknowns $ZD[u_0](t, z),$ and $(f_{t,z}(-p_j))_{j=0, \dots, N-1},$

$$u_0(y_{2k}) = ZD_{\pm}[u_0](t, z) \pm 2tu_0(y_{2k}) \sum_{j \geq 0}^{N-1} \frac{\bar{c}_j}{y_{2k} + p_j} f_{t,z}(-p_j), \quad k = 0, \dots, N.$$

Applying the Cramer rule, one finds for all $z \in \mathbb{C}_+,$

$$ZD_{\pm}[u_0](t, z) = \frac{u_0(y_0)u_0(y_2) \dots u_0(y_{2N}) \begin{vmatrix} 1 & \frac{1}{y_0+p_0} & \frac{1}{y_0+p_1} & \cdots & \frac{1}{y_0+p_{N-1}} \\ 1 & \frac{1}{y_2+p_0} & \frac{1}{y_2+p_1} & \cdots & \frac{1}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{1}{y_{2N}+p_0} & \frac{1}{y_{2N}+p_1} & \cdots & \frac{1}{y_{2N}+p_{N-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{u_0(y_0)}{y_0+p_0} & \frac{u_0(y_0)}{y_0+p_1} & \cdots & \frac{u_0(y_0)}{y_0+p_{N-1}} \\ 1 & \frac{u_0(y_2)}{y_2+p_0} & \frac{u_0(y_2)}{y_2+p_1} & \cdots & \frac{u_0(y_2)}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{u_0(y_{2N})}{y_{2N}+p_0} & \frac{u_0(y_{2N})}{y_{2N}+p_1} & \cdots & \frac{u_0(y_{2N})}{y_{2N}+p_{N-1}} \end{vmatrix}}. \quad (5.3.11)$$

Hence, for almost every $x \in \mathbb{R},$ identity (5.3.6) holds, as $ZD_{\pm}[u_0](t)$ belongs to the Hardy space $L^2_+(\mathbb{T})$ for all $t,$ since it is a holomorphic function, exhibiting a finite trace in $L^2(\mathbb{R})$ thanks to inequality (5.2.7). \square

Lemma 5.3.4 (Solving the determinants of Lemma 5.3.3). *Under the same conditions and notations as in Lemma 5.3.3, the zero–dispersion limit of (CM) associated with $u_0 = \frac{P(y)}{Q(y)}$ defined in (5.3.1), is given for almost every $x \in \mathbb{R}$ by*

$$ZD_{\pm}[u_0](t, x) = \frac{P(x)}{\prod_{k=1}^N (x - y_{2k-1})}. \quad (5.3.12)$$

Proof. We recall from Lemma 5.3.3, for almost every $x \in \mathbb{R}$,

$$ZD_{\pm}[u_0](t, x) = \frac{u_0(y_0)u_0(y_2) \cdots u_0(y_{2N}) \begin{vmatrix} 1 & \frac{1}{y_0+p_0} & \frac{1}{y_0+p_1} & \cdots & \frac{1}{y_0+p_{N-1}} \\ 1 & \frac{1}{y_2+p_0} & \frac{1}{y_2+p_1} & \cdots & \frac{1}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{1}{y_{2N}+p_0} & \frac{1}{y_{2N}+p_1} & \cdots & \frac{1}{y_{2N}+p_{N-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{u_0(y_0)}{y_0+p_0} & \frac{u_0(y_0)}{y_0+p_1} & \cdots & \frac{u_0(y_0)}{y_0+p_{N-1}} \\ 1 & \frac{u_0(y_2)}{y_2+p_0} & \frac{u_0(y_2)}{y_2+p_1} & \cdots & \frac{u_0(y_2)}{y_2+p_{N-1}} \\ \vdots & & & & \\ 1 & \frac{u_0(y_{2N})}{y_{2N}+p_0} & \frac{u_0(y_{2N})}{y_{2N}+p_1} & \cdots & \frac{u_0(y_{2N})}{y_{2N}+p_{N-1}} \end{vmatrix}}.$$

By expanding the determinants, one finds

$$ZD_{\pm}[u_0](t, x) = \frac{\sum_{k=0}^N (-1)^k \Delta_k}{\sum_{k=0}^N (-1)^k \frac{1}{u_0(y_{2k})} \Delta_k}$$

where Δ_k is the minor obtained after removing the first column and the k^{th} -row of the matrix in the numerator of (5.3.12). Therefore, observing that Δ_k is a Cauchy determinant, one infers that

$$ZD_{\pm}[u_0](t, x) = \frac{\sum_{k=0}^N (-1)^k \prod_{j=0}^{N-1} (y_{2k} + p_j) \prod_{\substack{0 \leq j < j' \leq N \\ j, j' \neq k}} (y_{2j'} - y_{2j})}{\sum_{k=0}^N (-1)^k \frac{1}{u_0(y_{2k})} \prod_{j=0}^{N-1} (y_{2k} + p_j) \prod_{\substack{0 \leq s < s' \leq N \\ s, s' \neq k}} (y_{2s'} - y_{2s})} =: \frac{R}{D}. \quad (5.3.13)$$

First, by using the Vandermonde determinant, one can rewrite R in (5.3.13) as

$$R := \sum_{k=0}^N (-1)^k \prod_{j=0}^{N-1} (y_{2k} + p_j) \prod_{\substack{0 \leq j < j' \leq N \\ j, j' \neq k}} (y_{2j'} - y_{2j})$$

$$= \begin{vmatrix} \prod_{j=0}^{N-1} (y_0 + p_j) & 1 & y_0 & \dots & y_0^{N-1} \\ \prod_{j=0}^{N-1} (y_2 + p_j) & 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & & & & \\ \prod_{j=0}^{N-1} (y_{2N} + p_j) & 1 & y_{2N} & \dots & y_{2N}^{N-1} \end{vmatrix}$$

which is equivalent to

$$R = \begin{vmatrix} y_0^N & 1 & y_0 & \dots & y_0^{N-1} \\ y_2^N & 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & & & & \\ y_{N-1}^N & 1 & y_{2N} & \dots & y_{2N}^{N-1} \end{vmatrix}$$

thanks to the multi-linearity of the determinant. Hence, applying once more the Vandermonde determinant, we conclude that

$$R = (-1)^N \prod_{0 \leq m < n \leq N} (y_{2n} - y_{2m}). \quad (5.3.14)$$

Second, moving to the expression of D in (5.3.13). Observe that by definition of the polynomial Q in (5.3.1), we have

$$\frac{1}{u_0(y_{2k})} \prod_{j=0}^{N-1} (y_{2k} + p_j) \equiv \frac{\bar{Q}(y_{2k})}{u_0(y_{2k})}. \quad (5.3.15)$$

In addition, recall that the $(y_{2k})_{k=0, \dots, N}$ are solutions of the algebraic equation (5.3.4). Hence, for all $k = 0, \dots, N$,

$$\frac{\bar{Q}(y_{2k})}{u_0(y_{2k})} = \mp 2t \frac{\bar{P}(y_{2k})}{x - y_{2k}},$$

which can be rewritten as

$$\frac{\bar{Q}(y_{2k})}{u_0(y_{2k})} = \mp 2t \left(\frac{\bar{P}(y_{2k}) - \bar{P}(x)}{x - y_{2k}} + \frac{\bar{P}(x)}{x - y_{2k}} \right),$$

to infer via (5.3.15)

$$\frac{1}{u_0(y_{2k})} \prod_{j=0}^{N-1} (y_{2k} + p_j) = \mp 2t \left(\frac{\bar{P}(y_{2k}) - \bar{P}(x)}{x - y_{2k}} + \frac{\bar{P}(x)}{x - y_{2k}} \right). \quad (5.3.16)$$

Therefore, by observing that $\frac{\bar{P}(y_{2k}) - \bar{P}(x)}{x - y_{2k}}$ is a polynomial in y_{2k} of degree strictly less than $N - 1$, one finds by (5.3.16), that D defined in (5.3.13) is equal to,

$$D = \mp 2t \bar{P}(x) \begin{vmatrix} \frac{1}{x - y_0} & 1 & y_0 & \dots & y_0^{N-1} \\ \frac{1}{x - y_2} & 1 & y_2 & \dots & y_2^{N-1} \\ \vdots & & & & \\ \frac{1}{x - y_{2N}} & 1 & y_{2N} & \dots & y_{2N}^{N-1} \end{vmatrix}.$$

Consequently, by using the Vandermonde determinant,

$$D = \mp 2t \bar{P}(x) \sum_{k=0}^N \frac{(-1)^k}{x - y_{2k}} \prod_{\substack{0 \leq m < n \leq N \\ m, n \neq k}} (y_{2n} - y_{2m}),$$

which can be rewritten as

$$D = \mp 2t (-1)^N \bar{P}(x) \frac{\prod_{0 \leq m < n \leq N} (y_{2n} - y_{2m})}{\prod_{k=0}^N (x - y_{2k})}, \quad (5.3.17)$$

thanks to the partial fractional decomposition of

$$\frac{\prod_{0 \leq m < n \leq N} (y_{2n} - y_{2m})}{\prod_{k=0}^N (x - y_{2k})} = \sum_{k=0}^N \frac{(-1)^{N-k}}{x - y_{2k}} \prod_{\substack{0 \leq m < n \leq N \\ m, n \neq k}} (y_{2n} - y_{2m}).$$

Consequently, substituting (5.3.14), (5.3.17) in (5.3.13), one infers that

$$ZD_{\pm}[u_0](t, x) = \frac{\prod_{k=0}^N (x - y_{2k})}{\mp 2t \bar{P}(x)}. \quad (5.3.18)$$

Furthermore, if we take into account that $(y_k)_{k=0, \dots, 2N}$ are solutions to (5.3.5), thus also to the polynomial equation (5.3.4), we can write

$$\prod_{k=0}^{2N} (y - y_k) = (y - x)Q(y)\bar{Q}(y) \mp 2tP(y)\bar{P}(y), \quad (5.3.19)$$

and when $y = x$ in the above equation, we obtain $\prod_{k=0}^{2N} (x - y_k) = \mp 2t P(x) \overline{P}(x)$, which implies that (5.3.18) can be replaced by

$$ZD_{\pm}[u_0](t, x) = \frac{P(x)}{\prod_{k=1}^N (x - y_{2k-1})}. \quad (5.3.20)$$

□

Now, equipped with Lemma 5.3.4, let us prove Proposition 5.3.1.

Proof of Proposition 5.3.1. We recall from Lemma 5.3.4, that for almost every $x \in \mathbb{R}$,

$$ZD_{\pm}[u_0](t, x) = \frac{P(x)}{\prod_{k=1}^N (x - y_{2k-1})}, \quad (5.3.21)$$

where the $(y_k)_{k=0, \dots, 2\ell}$ are the real solutions of the algebraic equation (5.3.2), and the $(y_p)_{p=2\ell+1, \dots, 2N}$ are the complex solutions of (5.3.5) with $y_{2p-1} = \overline{y_{2p}}$, $\text{Im}(y_{2p}) > 0$ for all $p = \ell + 1, \dots, N$. We rewrite (5.3.21) as

$$\begin{aligned} ZD_{\pm}[u_0](t, x) &= u_0(x) \frac{Q(x)}{\prod_{k=1}^N (x - y_{2k-1})} \\ &= \frac{u_0(x)}{\prod_{k=1}^{\ell} (x - y_{2k-1})} \frac{Q(x)}{\prod_{p=\ell+1}^N (x - y_{2p-1})}. \end{aligned} \quad (5.3.22)$$

The goal is to get rid of $\frac{Q(x)}{\prod_{k=\ell+1}^N (x - y_{2k-1})}$, in order to express $ZD_{\pm}[u_0]$ only in terms of $y_0, \dots, y_{2\ell}$, thereby ensuring that $ZD_{\pm}[u_0]$ can be expressed exclusively in terms of the branches of the multivalued solution of the Burgers equation (5.1.3).

For that, we recall from (5.3.19), for all $y \in \mathbb{C}$,

$$\frac{\prod_{k=0}^{2N} (y - y_k)}{Q(y)\overline{Q}(y)} = y - x \mp 2t \frac{P(y)\overline{P}(y)}{Q(y)\overline{Q}(y)}.$$

In particular, for all $y \in \mathbb{R}$,

$$\frac{y \mp 2t |u_0(y)|^2 - x}{\prod_{k=0}^{2\ell} (y - y_k)} = \frac{\prod_{p=2\ell+1}^{2N} (y - y_p)}{|Q(y)|^2},$$

or, since $y_{2p-1} = \overline{y_{2p}}$ for all $p = \ell + 1, \dots, N$, and as $Q(y) := (y + \overline{p}_0) \cdots (y + \overline{p}_{N-1})$ by (5.3.1),

$$\frac{y \mp 2t|u_0(y)|^2 - x}{\prod_{k=0}^{2\ell} (y - y_k)} = \frac{\prod_{p=\ell+1}^N |y - y_{2p-1}|^2}{\prod_{j=0}^{N-1} |y + \overline{p}_j|^2}. \quad (5.3.23)$$

Let $a > 0$, the next step is to prove that the term we need to get rid of is equal to

$$\frac{Q(x)}{\prod_{p=\ell+1}^N (x - y_{2p-1})} = \frac{(x + ia)^\ell}{\exp\left(\Pi_+\left(\log\left((y^2 + a^2)^\ell g_{t,x}(y)\right)\right)\right)\Big|_{y=x}}, \quad (5.3.24)$$

where

$$g_{t,x}(y) := \frac{\prod_{p=\ell+1}^N |y - y_{2p-1}|^2}{\prod_{j=0}^{N-1} |y + \overline{p}_j|^2}. \quad (5.3.25)$$

Indeed, by definition of $g_{t,x}$,⁶

$$\log\left((y^2 + a^2)^\ell g_{t,x}(y)\right) = \log\left(\frac{(y + ia)^\ell \prod_{p=\ell+1}^N (y - y_{2p-1})}{\prod_{j=0}^{N-1} (y - p_j)}\right) + \log\left(\frac{(y - ia)^\ell \prod_{p=\ell+1}^N (y - \overline{y_{2p-1}})}{\prod_{j=0}^{N-1} (y - \overline{p}_j)}\right),$$

where one observes the first term of the right-hand side belongs to $L_+^2(\mathbb{R})$ as $\text{Im}(p_k) < 0$, while the second term belongs to $L_-^2(\mathbb{R})$ ⁷. Therefore, by the uniqueness of the decomposition of any L^2 -function in $L_+^2(\mathbb{R}) \oplus L_-^2(\mathbb{R})$, we infer for a fixed $x \in \mathbb{R}$,

$$\Pi_+\left(\log\left((y^2 + a^2)^\ell g_{t,x}(y)\right)\right) = \log\left(\frac{(y + ia)^\ell \prod_{k=\ell+1}^N (y - y_{2k-1})}{Q(y)}\right).$$

Applying the exponential function to both sides of the previous identity and setting $y = x$, one deduces (5.3.24). Now, the only task left is to compute the right-hand side of (5.3.24). To do so, we need the following classical lemma, the proof of which will be presented later for the convenience of the reader,

Lemma 5.3.5. *For any $h \in L^2(\mathbb{R})$ of class \mathcal{C}^1 satisfying $\|h'\|_{L^\infty} < \infty$,*

$$\Pi h(x) = \frac{h(x)}{2} - \frac{i}{2\pi} \int_0^{+\infty} \frac{h(x+s) - h(x-s)}{s} ds, \quad x \in \mathbb{R}.$$

6. The multiplication of $g_{t,x}$ by $(y^2 + a^2)^\ell$ aims to ensure that each term on the right-hand side of the following identity is in $L^2(\mathbb{R})$

7. i.e. the space of functions having a trace in $L^2(\mathbb{R})$, such that they can be holomorphically extended to \mathbb{C}_-

Thus, applying this lemma with $h(y) = \log((y^2 + a^2)^\ell g_{t,x}(y))$, one obtains

$$\begin{aligned} \Pi_+ \left(\log((y^2 + a^2)^\ell g_{t,x}(y)) \right) &= \frac{1}{2} \log((y^2 + a^2)^\ell g_{t,x}(y)) - \frac{i\ell}{2\pi} \int_0^{+\infty} \frac{1}{s} \log\left(\frac{(y+s)^2 + a^2}{(y-s)^2 + a^2}\right) ds \\ &\quad - \frac{i}{2\pi} \int_0^{+\infty} \frac{\log(g_{t,x}(y+s)) - \log(g_{t,x}(y-s))}{s} ds, \end{aligned} \quad (5.3.26)$$

where after some computation, one finds

$$\frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log\left(\frac{(y+s)^2 + a^2}{(y-s)^2 + a^2}\right) ds = \frac{\pi}{2} - \arctan\left(\frac{a}{y}\right). \quad (5.3.27)$$

Indeed, let $f(y) := \int_0^{+\infty} \frac{1}{s} \log\left(\frac{(y+s)^2 + a^2}{(y-s)^2 + a^2}\right) ds$. By deriving f , one finds

$$\begin{aligned} f'(y) &= \int_0^{+\infty} \frac{4(a^2 + s^2 - y^2)}{((y+s)^2 + a^2)((y-s)^2 + a^2)} ds \\ &= \int_{-\infty}^{+\infty} \frac{2(a^2 + s^2 - y^2)}{((y+s)^2 + a^2)((y-s)^2 + a^2)} ds \\ &= \lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{2(a^2 + z^2 - y^2)}{((y+z)^2 + a^2)((y-z)^2 + a^2)} dz \end{aligned}$$

where Γ_R is the contour composed of the real axis from $-R$ to R and the upper semicircle. Using Cauchy's residue theorem, we infer $f'(y) = \frac{2\pi a}{y^2 + a^2}$. Therefore, by integrating the previous expression and observing that $f(0) = 0$, and using that $\arctan(\theta) + \arctan(\frac{1}{\theta}) = \frac{\pi}{2}$, one obtains (5.3.27). Consequently, (5.3.26) becomes

$$\begin{aligned} \Pi_+ \left(\log((y^2 + a^2)^\ell g_{t,x}(y)) \right) &= \frac{1}{2} \log((y^2 + a^2)^\ell g_{t,x}(y)) - \frac{i\ell\pi}{2} + i\ell \arctan\left(\frac{a}{y}\right) \\ &\quad - \frac{i}{2\pi} \int_0^{+\infty} \frac{\log(g_{t,x}(y+s)) - \log(g_{t,x}(y-s))}{s} ds, \end{aligned} \quad (5.3.28)$$

and hence,

$$\begin{aligned} \exp \left(\Pi_+ \left(\log((y^2 + a^2)^\ell g_{t,x}(y)) \right) \right) \Big|_{y=x} &= e^{-\frac{i\ell\pi}{2}} \sqrt{(x^2 + a^2)^\ell g_{t,x}(x)} e^{i\ell \arctan(\frac{a}{x})} \\ &\quad \times \exp \left(-\frac{i}{2\pi} \int_0^{+\infty} \frac{\log(g_{t,x}(x+s)) - \log(g_{t,x}(x-s))}{s} ds \right) \end{aligned} \quad (5.3.29)$$

Thus, combining (5.3.22), (5.3.24) and (5.3.29)

$$\begin{aligned}
 ZD_{\pm}[u_0](t, x) &= \frac{u_0(x)}{\prod_{k=1}^{\ell} (x - y_{2k-1})} \frac{(x + ia)^{\ell}}{\sqrt{(x^2 + a^2)^{\ell}} e^{i\ell \arctan(a/x)}} \frac{e^{i\frac{\pi\ell}{2}}}{\sqrt{g_{t,x}(x)}} \\
 &\quad \cdot \exp\left(\frac{i}{2\pi} \int_0^{+\infty} \frac{\log(g_{t,x}(x+s)) - \log(g_{t,x}(x-s))}{s} ds\right) \\
 &= \frac{|u_0(x)|}{\prod_{k=1}^{\ell} (x - y_{2k-1})} \frac{(i)^{\ell}}{\sqrt{g_{t,x}(x)}} e^{i\varphi(t,x)}, \tag{5.3.30}
 \end{aligned}$$

where

$$\varphi(t, x) = \arg(u_0(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{\log(g_{t,x}(x+s)) - \log(g_{t,x}(x-s))}{s} ds. \tag{5.3.31}$$

Substituting $g_{t,x}$ in (5.3.30) by its value in (5.3.23) with $y = x$,

$$ZD_{\pm}[u_0](t, x) = \frac{|u_0(x)|}{\prod_{k=1}^{\ell} (x - y_{2k-1})} \frac{(i)^{\ell}}{\sqrt{\frac{\mp 2t|u_0(x)|^2}{2^{\ell}} \prod_{k=0}^{2\ell} (x - y_k)}} e^{i\varphi(t,x)},$$

and using the fact that the y_k are solutions of the algebraic equation $y_k \mp 2t|u_0(y_k)|^2 = x$, for all $k = 0, \dots, 2\ell$, we conclude

$$\begin{aligned}
 ZD_{\pm}[u_0](t, x) &= \frac{1}{(\mp 2t)^{\ell} \prod_{k=1}^{\ell} |u_0(y_{2k-1})|^2} \frac{(i)^{\ell}}{\sqrt{\frac{\mp 2t}{(\mp 2t)^{2\ell+1} \prod_{k=0}^{2\ell} |u_0(y_k)|^2}}} e^{i\varphi(t,x)} \\
 &= \left(\mp i \frac{|t|}{t}\right)^{\ell} \frac{\prod_{k=0}^{\ell} |u_0(y_{2k})|}{\prod_{k=1}^{\ell} |u_0(y_{2k-1})|} e^{i\varphi(t,x)},
 \end{aligned}$$

where by (5.3.31) and (5.3.23),

$$\varphi(t, x) = \arg(u_0(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log\left(\frac{s \mp 2t|u_0(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k)}{-s \mp 2t|u_0(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k)}\right) ds.$$

□

Proof of Lemma 5.3.5. For all $x \in \mathbb{R}$,

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \Pi h(x + i\delta) &= \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(y)}{y - x - i\delta} dy \\ &= \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\delta h(y)}{(y - x)^2 + \delta^2} dy + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(y)(y - x)}{(y - x)^2 + \delta^2} dy. \end{aligned}$$

Using the Poisson integral formula on the upper half-plane of \mathbb{C} , we infer for all $\delta > 0$,

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\delta h(y)}{(y - x)^2 + \delta^2} dy = \frac{h(x)}{2}.$$

For the second term, observe that,

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}} \frac{h(y)(y - x)}{(y - x)^2 + \delta^2} dy &= \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{x-\varepsilon} \frac{h(y)(y - x)}{(y - x)^2 + \delta^2} dy + \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0} \int_{x+\varepsilon}^{+\infty} \frac{h(y)(y - x)}{(y - x)^2 + \delta^2} dy \\ &= \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0} \left(- \int_{\varepsilon}^{+\infty} \frac{h(x-s)s}{s^2 + \delta^2} ds + \int_{\varepsilon}^{+\infty} \frac{h(x+s)s}{s^2 + \delta^2} ds \right) \\ &= \lim_{\delta \rightarrow 0^+} \int_0^{+\infty} \frac{(h(x+s) - h(x-s))s}{s^2 + \delta^2} dy, \end{aligned}$$

where we can switch lim and integral, as $\int_0^{+\infty} \frac{h(x+s)-h(x-s)}{s} ds$ is well-defined as the principal value of $\frac{h(x)}{x}$ since $\|h'\|_{L^\infty} < \infty$. \square

Now, armed with Proposition 5.3.1, let us proceed to prove the extended version of the statement of this proposition to Theorem 5.1.2.

Proof of Theorem 5.1.2. Let $u_0 \in L^2_+(\mathbb{R}) \cap C^1(\mathbb{R})$, such that u_0 is tending to 0 at infinity and $u'_0 \in L^\infty(\mathbb{R})$. For such u_0 , the C^1 -function $\gamma_t(y) := y \mp 2t|u_0(y)|^2$ is asymptotically equivalent to y at $\pm\infty$. Moreover, since u'_0 is bounded in $L^\infty(\mathbb{R})$, then for all $t \in \mathbb{R}$, and for any $x \in \mathbb{R}$ that is not a critical value of γ_t , the equation $\gamma_t(y) = x$ has a finite number of real solutions

$$y_0(t, x) < \dots < y_{2\ell}(t, x). \quad (5.3.32)$$

Besides, note that by the Sard theorem, the set of critical values of γ_t has zero Lebesgue measure. Moreover, since

$$\gamma'_t(y) = 1 + 2t(|u_0(y)|^2)' \longrightarrow 1, \quad y \rightarrow \pm\infty,$$

then the set of critical points $\{y; \gamma'_t(y) = 0\}$ is compact for a given t , so that its image –the set of critical values of γ_t – is compact, and hence in particular closed. Thus, let Ω be any open connected set (for the x variable) where (5.3.32) is satisfied. The idea, at this stage of the proof, is to deduce the result from Proposition 5.3.1. Thus, by using Runge's approximation theorem, we approximate u_0 in $L^2_+(\mathbb{R}) \cap C^1(\mathbb{R})$ by a sequence of rational functions (u_0^δ) belonging to the Hardy space. Now, take

ω any arbitrary open subset of Ω such that $\bar{\omega}$ is compact. Therefore, for δ small enough,

$$y + 2t|u_0^\delta(y)|^2 = x, \quad x \in \omega,$$

has $2\ell + 1$ solutions

$$y_0^\delta(t, x) < \dots < y_{2\ell}^\delta(t, x).$$

Since u_0^δ is a rational function in the Hardy space, then by Proposition 5.3.1,

$$ZD_\pm[u_0^\delta](t, x) = e^{i\varphi_\delta(x)} \left(\mp i \frac{|t|}{t} \right)^\ell \prod_{k=0}^{2\ell} |u_0^\delta(y_k^\delta)|^{(-1)^k},$$

where

$$\varphi_\delta(x) = \arg(u_0^\delta(x)) + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{s} \log \left(\frac{s \mp 2t|u_0^\delta(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k^\delta)}{-s \mp 2t|u_0^\delta(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k^\delta)} \right) ds.$$

By passing to the limit as $\delta \rightarrow 0$, and using that $u_0^\delta \rightarrow u_0$ in $L^2_+ \cap C^1$ so that $y_k^\delta(t, x) \rightarrow y_k(t, x)$, and by the weak convergence of (5.2.8), we deduce for every $x \in \omega$, formula (5.1.7). Now, since ω is chosen arbitrarily in Ω , this achieves the proof. Finally, note that the integral

$$\int_0^{+\infty} \frac{1}{s} \log \left(\frac{s \mp 2t|u_0(x+s)|^2 \prod_{k=0}^{2\ell} (x-s-y_k)}{-s \mp 2t|u_0(x-s)|^2 \prod_{k=0}^{2\ell} (x+s-y_k)} \right) ds$$

is nothing else than

$$I = \int_0^{+\infty} \frac{\log g_{t,x}(x+s) - \log g_{t,x}(x-s)}{s} ds$$

where we recall by (5.3.23) and (5.3.25)

$$g_{t,x}(y) = \frac{y \mp 2t|u_0(y)|^2 - x}{\prod_{k=0}^{2\ell} (y - y_k)}$$

and $y_0 < \dots < y_{2\ell}$ are all the real solutions of the numerator $y \mp 2t|u_0(y)|^2 - x$. Observe that as $u_0 \in \mathcal{C}^1$ than $y \mapsto \log g_{t,x}(y)$ is so. Moreover, since

$$|(\log(g_{t,x}(y)))'| \lesssim_{y \rightarrow \infty} \frac{1}{y}$$

then by Lemma 5.3.5, the integral I is well-defined. □

An immediate consequence of Theorem 5.1.2 is the following corollary.

Corollary 5.3.6. *Let $u_0 \in L^2_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (with $\|u_0\|_{L^2} < \sqrt{2\pi}$ in the focusing case), then for all $t \in \mathbb{R}$,*

$$\|ZD_\pm[u_0](t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}.$$

Proof. In view of Theorem 5.1.2, we have for any $u_0 \in L^2_+(\mathbb{R}) \cap C^1(\mathbb{R})$, satisfying the property that u_0 is tending to 0 at infinity, with u'_0 bounded,

$$|ZD_\pm[u_0](t, x)| = \frac{\prod_{k=0}^{\ell} |u_0(y_{2k})|}{\prod_{k=1}^{\ell} |u_0(y_{2k-1})|},$$

where $y_0 < \dots < y_{2\ell}$ are solutions for the algebraic equation

$$x - y_k = \mp 2t |u_0(y_k)|^2.$$

Therefore, by the monotonicity of $k \mapsto y_k$ we infer the monotonicity of $k \mapsto |u_0(y_k)|^2$, so that we can deduce

$$|ZD_\pm[u_0](t, x)| \leq \max\{|u_0(y_0)|, |u_0(y_{2\ell})|\} \leq \|u_0\|_{L^\infty}.$$

The general case of $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ follows by Runge's approximation theorem used to approximate u_0 by rational functions like those described to prove Theorem 5.1.2 and by applying property (5.2.8) in Theorem 5.1.1. □

5.4 $ZD_\pm[u_0]$ fails to be a strong limit of (CM-eps) as $\varepsilon \rightarrow 0$

In this paragraph, we underline that the *weak* zero-dispersion limit $ZD_\pm[u_0]$ of Theorem 5.1.1, characterized in Theorem 5.1.2, is only a weak limit solution, and fails to be a strong limit of (CM-eps) as $\varepsilon \rightarrow 0$.

To prove that, one can argue by contradiction, and suppose that every zero-dispersion limit $ZD_\pm[u_0]$ is a strong limit, implying that $|ZD_\pm[u_0]|^2$ is a solution, for all $x \in \mathbb{R}$, to the Burgers Hopf Cauchy problem (5.1.3) starting from an initial data $|u_0|^2$. To extract the contradiction, let us consider an initial data $|u_0|^2$ such that in some neighborhood V of (t, x) , the equation

$$y \mp 2t |u_0(y)|^2 = x$$

has three solutions⁸ $y_0 := y_0(t, x)$, $y_1 := y_1(t, x)$ and $y_2 := y_2(t, x)$, i.e. $\ell = 1$ in Theorem 5.1.2 in V and $\ell = 0$ elsewhere. In view of this theorem, we have that in

8. We refer to Theorem 5.1.2 asserting the existence of an odd number of solutions for this equation

the neighborhood V ,

$$|ZD_{\pm}[u_0](t, x)|^2 = \frac{|u_0(y_0)|^2 |u_0(y_2)|^2}{|u_0(y_1)|^2},$$

satisfy by assumption the Burgers equation

$$\partial_t \left(\frac{|u_0(y_0)|^2 |u_0(y_2)|^2}{|u_0(y_1)|^2} \right) = \pm \partial_x \left(\frac{|u_0(y_0)|^4 |u_0(y_2)|^4}{|u_0(y_1)|^4} \right)$$

which can be rewritten, after dividing both sides by $\frac{|u_0(y_0)|^2 |u_0(y_2)|^2}{|u_0(y_1)|^2}$ as

$$\partial_t \log \left(\frac{|u_0(y_0)|^2 |u_0(y_2)|^2}{|u_0(y_1)|^2} \right) = \pm 2 \partial_x \left(\frac{|u_0(y_0)|^2 |u_0(y_2)|^2}{|u_0(y_1)|^2} \right). \quad (5.4.1)$$

On the other hand, since y_0 , y_1 and y_2 describe the branches of the multivalued solution of the Burgers equation starting from an initial data $|u_0|^2$ then

$$\partial_t |u_0(y_k)|^2 = \pm \partial_x (|u_0(y_k)|^4)$$

which, in turn, can be rewritten by dividing both sides by $|u_0(y_k)|^2$ as

$$\partial_t \log(|u_0(y_k)|^2) = \pm 2 \partial_x (|u_0(y_k)|^2). \quad (5.4.2)$$

Therefore, combining (5.4.1) and (5.4.2), we infer

$$\partial_x (|u_0(y_0)|^2 - |u_0(y_1)|^2 + |u_0(y_2)|^2) = \partial_x \left(\frac{|u_0(y_0)|^2 |u_0(y_2)|^2}{|u_0(y_1)|^2} \right).$$

That is, on V

$$\partial_x \left(\frac{(|u_0(y_0)|^2 - |u_0(y_1)|^2) \cdot (|u_0(y_2)|^2 - |u_0(y_1)|^2)}{|u_0(y_1)|^2} \right) = 0$$

Hence, the function $\frac{(|u_0(y_0)|^2 - |u_0(y_1)|^2) \cdot (|u_0(y_2)|^2 - |u_0(y_1)|^2)}{|u_0(y_1)|^2}$ is independent of x in all V . In particular, when x tends to the boundary of V , this function tends to 0, as $\ell = 0$ outside V . And since we are dealing with holomorphic functions, then one infers

$$\frac{(|u_0(y_0)|^2 - |u_0(y_1)|^2)(|u_0(y_2)|^2 - |u_0(y_1)|^2)}{|u_0(y_1)|^2} = 0$$

which is a contradiction since $y_0 \neq y_1 \neq y_2$ and $|u_0(y_k)|^2$ satisfies $y_k \mp 2t|u_0(y_k)|^2 = x$.

Chapter 6

Traveling waves & finite gap potentials for the Calogero–Sutherland derivative nonlinear Schrödinger equation

This chapter is from the paper [Bad23], accepted at Ann. Inst. Henri Poincaré, Analyse Non Linéaire.

ABSTRACT.

We consider the Calogero–Sutherland derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u \pm \frac{2}{i} u \partial_x \Pi(|u|^2) = 0, \quad x \in \mathbb{T}, \quad (\text{CS})$$

where Π is the Szegő projector

$$\Pi\left(\sum_{n \in \mathbb{Z}} \hat{u}(n) e^{inx}\right) = \sum_{n \geq 0} \hat{u}(n) e^{inx}.$$

First, we characterize the *traveling wave* $u_0(x - ct)$ solutions to the defocusing equation (CS[−]), and prove for the focusing equation (CS⁺), that all the traveling waves must be either the constant functions, or plane waves, or rational functions. A noteworthy observation is that the (CS)–equation, which is an L^2 –critical equation, is one of the fewest nonlinear PDE enjoying nontrivial traveling waves with arbitrary small and large L^2 –norms. Second, we study the *finite gap potentials*, and show that they are also rational functions, containing the traveling waves, and they can be grouped into sets that remain invariant under the evolution of the system.

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6.1 Introduction

In recent decades, the theory of traveling wave solutions has been the subject of intense research in theoretical and numerical analysis. Indeed, many nonlinear PDEs exhibit these type of waves [Che04, CB13, Pav09]. Their importance resides as they are explicit solutions for nonlinear PDEs, and they can sometimes provide information regarding the dynamics of the equation. However, the problem of proving the existence of these waves can be more or less challenging depending on the nonlinear part of the PDE.

In this paper, we consider a type of derivative nonlinear Schrödinger equation with a nonlocal nonlinearity, called the *Calogero–Sutherland derivative nonlinear Schrödinger equation*

$$i\partial_t u + \partial_x^2 u \pm 2D\Pi(|u|^2)u = 0, \quad x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}), \quad (\text{CS})$$

where $D = -i\partial_x$, and Π denotes the Szegő projector

$$\Pi \left(\sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{inx} \right) := \sum_{n \geq 0} \widehat{u}(n) e^{inx}, \quad (6.1.1)$$

which is an orthogonal projector from $L^2(\mathbb{T})$ into the Hardy space

$$L_+^2(\mathbb{T}) := \{u \in L^2(\mathbb{T}) \mid \widehat{u}(n) = 0, \forall n \in \mathbb{Z}_{\leq -1}\}. \quad (6.1.2)$$

We are interested in studying the traveling waves $u_0(x-ct)$ of this equation in the focusing (with sign + in front of the nonlinearity) and defocusing case (with sign−)

in the periodic setting, namely when $x \in \mathbb{T}$. As noted in [Pav09], the presence of the nonlocal operator $D\Pi$ appearing in the nonlinearity can make the problem of the existence of traveling waves more complicated. The approach addressed in this paper to characterize the traveling waves is based on studying them, at a first stage, spectrally i.e. by means of spectral property of the Lax operator related to this equation (see below), before deriving, at a second stage, their **explicit formulas**.¹

6.1.1 Main results

Settings and notation. In the sequel, our study takes place with potentials in the Hardy Sobolev spaces of the torus

$$H_+^s(\mathbb{T}) := H^s(\mathbb{T}) \cap L_+^2(\mathbb{T}), \quad s \geq 0,$$

where $L_+^2(\mathbb{T})$ is defined in (6.1.2) and H^s refers to the Sobolev space. We equip $L_+^2(\mathbb{T})$ with the standard inner product of $L^2(\mathbb{T})$,

$$\langle u | v \rangle = \int_0^{2\pi} u \bar{v} \frac{dx}{2\pi}.$$

We recall also, that via the following isometric isomorphism

$$z \in \mathbb{D}, \quad u(z) = \sum_{k \geq 0} \hat{u}(k) z^k \quad \longmapsto \quad u^*(x) := \sum_{k \geq 0} \hat{u}(k) e^{ikx}, \quad x \in \mathbb{T},$$

$$\sum_{k \geq 0} |\hat{u}(k)|^2 < \infty,$$

one can interpret any element of the Hardy space as an analytic function on the open unit disc \mathbb{D} , whose trace on the boundary $\partial\mathbb{D}$ is in L^2 .² We will frequently utilize this property in various proofs. Furthermore, we denote by \mathbb{D} the open unit disc on \mathbb{C} , $\mathbb{D}^* := \{z \in \mathbb{C}; 0 < |z| < 1\}$. Moreover, $\mathbb{N} \equiv \mathbb{N}_{\geq 1}$ denotes the positive integers $1, 2, 3, \dots$. And for all $a \in \mathbb{N} \cup \{0\}$, $\mathbb{N}_{\geq a}$ refers to the set of integer numbers $\{n \in \mathbb{Z}; n \geq a\}$.

First, we deal with the defocusing Calogero–Sutherland DNLS equation

$$i\partial_t u + \partial_x^2 u - 2D\Pi(|u|^2)u = 0. \quad (\text{CS}^-)$$

1. It should be noted that the idea of using the spectral theory to derive the traveling waves of (CS), draws inspiration from [GK21, Appendix B], where the authors provide an alternative proof to the characterization of the traveling waves for the Benjamin–Ono equation [AT91, Ben67] by characterizing them first spectrally.

2. For a simple introduction to the different definitions of Hardy space, we refer to [GMR16, Chapter 3.]

We denote by \mathcal{G}_1 the set of the trivial traveling waves, made up from the constant functions and the plane wave solutions

$$\mathcal{G}_1 = \{C e^{iN(x-Nt)} \mid C \in \mathbb{C}, N \in \mathbb{N}_{\geq 0}\}. \quad (6.1.3)$$

Theorem 6.1.1 (Characterization of the traveling waves of (CS[−])). *A potential u is a traveling wave of (CS[−]) if and only if $u \in \mathcal{G}_1$ or*

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, \quad (6.1.4)$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N \left(1 + \frac{2\alpha}{\beta}\right)$, and (α, β) are two real constants satisfying

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = -N. \quad (6.1.5)$$

Remark 6.1.1. The condition (6.1.5) implies that the real constants α and β must be of opposite signs.

Second, we pass to the focusing Calogero–Sutherland DNLS equation

$$i\partial_t u + \partial_x^2 u + 2D\Pi(|u|^2)u = 0. \quad (\text{CS}^+)$$

By changing the sign in front of the nonlinearity, the strategy adopted in the defocusing case to exhibit the traveling waves becomes significantly more complicated. However, we can ensure the existence of a larger set of traveling wave solutions for (CS⁺) comparing to (CS[−]), and that all the non-trivial traveling waves $u(t, x) := u_0(x-ct)$ of (CS⁺) are also rational functions.

Theorem 6.1.2. *The traveling waves $u_0(x-ct)$ of (CS⁺) are either rational functions or trivial waves in \mathcal{G}_1 . In addition, the potentials*

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, N \in \mathbb{N}_{\geq 1},$$

where $c = -N \left(1 + \frac{2\alpha}{\beta}\right)$, $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = N, \quad (6.1.6)$$

and the potentials

$$u(t, x) = e^{i\theta} e^{im(x-mt)} \left(\alpha + \frac{\beta}{1 - p e^{i(x-mt)}} \right) \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, m \in \mathbb{N}_{\geq 1},$$

where $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = 1, \quad \beta(m-1) = 2\alpha,$$

are parts of the set of the traveling waves of (CS⁺).

Remark 6.1.2. It is worth noting that the condition on (α, β) appearing in (6.1.6) for the focusing case, allows to obtain a *larger set* of traveling waves in comparison to the condition (6.1.5) of the defocusing case. Indeed, (6.1.6) enables, for instance α or $\alpha + \beta$ to vanish, which leads respectively to the following traveling waves

$$u(t, x) = e^{i\theta} \frac{\sqrt{N(1 - |p|^2)}}{1 - p e^{iN(x+Nt)}} \quad \text{and} \quad u(t, x) = e^{i\theta} \frac{\sqrt{N(1 - |p|^2)} e^{iN(x-Nt)}}{1 - p e^{iN(x-Nt)}}.$$

Contrary to the focusing case, no traveling waves $u(t, x) := u_0(x - ct)$ with a profile $u_0(x) := \frac{\beta}{1 - p e^{iNx}}$ or $u_0(x) := \frac{\alpha e^{iNx}}{1 - p e^{iNx}}$ can be found for the (CS^-) -equation because otherwise, thanks to (6.1.5),

$$\frac{\beta^2}{1 - |p|^2} = -N \quad \text{or} \quad \frac{|p|^2}{1 - |p|^2} \beta^2 = -N$$

which is clearly impossible for $p \in \mathbb{D}^*$.

Remark 6.1.3. (The L^2 -norm and the speed of the traveling waves of (CS))

1. As will be established in Subsection 6.3.3 for the defocusing equation and in Subsection 6.4.2 for the focusing equation, the L^2 -norm of the non-trivial traveling waves of (CS) can be arbitrarily small or large in $L^2_+(\mathbb{T})$. More rigorously, for any $r > 0$, there exists a non-trivial traveling wave $u(t, x) := u_0(x - ct)$ of (CS) where

$$\|u_0\|_{L^2} = r.$$

2. *For the defocusing (CS^-) -equation.* The nontrivial traveling waves u of the form (6.1.4) propagate to the right with a speed $c > N$, where N is the degree appearing in the denominator of u . In addition, when $\|u\|_{L^2} \rightarrow +\infty$, we have $c \rightarrow +\infty$, and when $\|u\|_{L^2} \rightarrow 0$ then $c \rightarrow N$. (See Remark 6.3.3 and Subsection 6.3.3 for the proofs).

For the focusing (CS^+) -equation. Contrary to the defocusing equation, the (CS^+) 's nontrivial traveling waves do not necessarily propagate at a relatively high speed (i.e. $c \rightarrow \infty$) when $\|u\|_{L^2}^2$ is large (i.e. $\|u\|_{L^2} \rightarrow \infty$). In fact, the speed of the traveling waves in the focusing case is independent of the size of its $L^2_+(\mathbb{T})$ -norm. We refer to Remark 6.4.2 for an example.

In light of the previous remarks, we infer that the Calogero–Sutherland DNLS equation enjoys a significantly richer dynamic in the focusing case. In particular, one can observe that the (CS^+) admits non-trivial *stationary waves* $u(t, x) := u_0(x)$, which is not the case of the defocusing equation. An example of nontrivial stationary waves for (CS^+) is

$$u(t, x) := e^{i\theta} \sqrt{\frac{N(1 - |p|^2)}{2(1 + |p|^2)}} \left(1 - \frac{2}{1 - p e^{iNx}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, N \in \mathbb{N}_{\geq 1}.$$

At a second stage, we study the *finite gap potentials* of the Calogero–Sutherland DNLS equation (CS), i.e. potentials satisfying that, from a certain rank, all the gaps between the consecutive eigenvalues of the Lax operator are equal to 1 (see Subsection 6.1.2.2 for the Lax operator). It turns out that these potentials are *multiphase solutions* containing the stationary and traveling waves of (CS). The following theorem aims to characterize the finite gap potentials on \mathbb{T} in the state space.

Theorem 6.1.3 (Characterization in the state space of the (CS)’s finite gap potentials). *The finite gap potentials of (CS) are either the functions $u(x) = C e^{iNx}$, $C \in \mathbb{C}^*$, $N \in \mathbb{N}_{\geq 0}$, or the rational function*

$$u(x) = e^{im_0 x} \prod_{j=1}^r \left(\frac{e^{ix} - \bar{p}_j}{1 - p_j e^{ix}} \right)^{m_j-1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j e^{ix}} \right), \quad p_j \in \mathbb{D}^*, \quad p_k \neq p_j, \quad k \neq j,$$

where, for $N \in \mathbb{N}_{\geq 1}$, $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$ such that $m_0 + \sum_{j=1}^r m_j = N$, and $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfy for all $j = 1, \dots, r$,

(i) *In the defocusing case,*

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = -m_j, \quad (6.1.7)$$

(ii) *In the focusing case,*

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = m_j, \quad (6.1.8)$$

with $a \neq 0$ if $m_0 \neq 0$. Moreover, these finite gap potentials can be regrouped into sets that remain invariant under the evolution of (CS).

In order to establish the results mentioned above, it is necessary to provide an overview regarding the integrability of the Calogero–Sutherland derivative nonlinear Schrödinger equation (CS).

6.1.2 About the Calogero–Sutherland DNLS equation

The Calogero–Sutherland DNLS equation (CS) has been actively studied by physicists and engineers. In particular, we cite the works of Tutiya [Tut09], Berntson–Fagerlund [BF23], Stone–Anduaga–Xing [SAX08], Polychronakos [Pol95b, Pol95a] and Matsuno [Mat00, Mat01b, Mat01a, Mat02b, Mat02a, Mat03, Mat04, Mat23]...

Mathematically, recent progress has been made with regard to this equation. In this subsection, we provide a brief overview of some established results concerning (CS).

6.1.2.1 Local and global well-posedness results

To the best of the author's knowledge, the first LWP result for (CS) equation traced back to De Moura [dM07] who established the LWP³ of (CS) for small initial data in $H^s(\mathbb{R})$ with $s \geq 1$, and extend his LWP's result to a GWP by means of the gauge transformation.

More recently, Barros–DeMoura–Santos present in [BdMS19] the LWP of (CS) for small initial data in the Besov space $B_2^{\frac{1}{2},1}(\mathbb{R})$.

Besides, observe that the Calogero–Sutherland DNLS equations (CS) is invariant under the scaling

$$u_\lambda(t, x) = \lambda^{\frac{1}{2}} u(\lambda t, \lambda^2 x), \quad \lambda > 0.$$

This suggests that (CS) is L^2 -critical. In the Hardy Sobolev spaces settings, i.e. in $H_+^s := H^s \cap L_+^2$, where recall L_+^2 is the Hardy space defined in (6.1.2) in the periodic case, and as follows in the non-periodic case

$$L_+^2(\mathbb{R}) = \{u \in L^2(\mathbb{R}) ; \text{supp } \hat{u} \subset [0, \infty)\},$$

Gérard–Lenzmann [GL24] obtained the LWP in $H_+^s(\mathbb{R})$ with $s > \frac{1}{2}$, by following the arguments of [dMP10]. Furthermore, by the virtue of a Lax pair structure associated with the Calogero–Sutherland DNLS equation (CS) (see below), they inferred the global well-posedness of the equation in all $H_+^k(\mathbb{R})$, $k \in \mathbb{N}_{\geq 1}$ for small initial data $\|u_0\|_{L^2(\mathbb{R})} < \sqrt{2\pi}$ in the focusing case.

Moving to the periodic setting, i.e. when $x \in \mathbb{T}$, a recent work of the author shows the GWP of (CS) in all $H_+^s(\mathbb{T})$, $s \geq 0$, for small critical initial data in the focusing case, namely when $\|u_0\|_{L^2(\mathbb{T})} < 1$, and for arbitrary initial data in the defocusing case. In particular, the extension of the flow to the critical space $L_+^2(\mathbb{T})$ has been achieved after deriving *the explicit formula* for the solution of the Calogero–Sutherland DNLS equation (CS) [Bad24a, Proposition 2.5]. Moreover, under the same assumptions, the relative compactness of the trajectories has been established in $H_+^s(\mathbb{T})$, for all $s \geq 0$ [Bad24a].

6.1.2.2 Integrability of the (CS)-equation

One of the most remarkable features of the Calogero–Sutherland DNLS equation is its integrability as a PDE on \mathbb{R} and on \mathbb{T} . In fact, it enjoys a *Lax pair structure* in the focusing and defocusing case [GL24, Bad24a] : for any $u \in H_+^s(\mathbb{T})$, $s > \frac{3}{2}$, there exists two operators (L_u, B_u) satisfying the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u], \quad [B_u, L_u] := B_u L_u - L_u B_u,$$

where

3. Actually, they prove the local well-posedness of a family of nonlocal nonlinear Schrödinger equation [PG96] that includes also the (CS)-equation.

(i) In the focusing case,

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2. \quad (6.1.9)$$

(ii) In the defocusing case,

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \tilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2. \quad (6.1.10)$$

The differential operator D is $-i\partial_x$, and T_u is the Toeplitz operator of symbol u defined for any $u \in L^\infty$ by

$$T_u f = \Pi(uf), \quad \forall f \in L^2_+, \quad (6.1.11)$$

where Π is the Szegő projector introduced in (6.1.1). Note that since we are working in the Hardy space, L_u is a semi-bounded operator from below and \tilde{L}_u is a nonnegative operator. In addition, as noted in [Bad24a, Proposition 2.3], the Lax operators L_u and \tilde{L}_u are self-adjoint operators of domain $H^1_+(\mathbb{T})$, and are of compact resolvent. Therefore, their spectra are made up of a sequence of eigenvalues going to $+\infty$,

$$\begin{aligned} \sigma(L_u) &:= \{\nu_0(u) \leq \dots \leq \nu_n(u) \leq \dots\}, & \nu_0(u) &\geq -\|u\|_{L^\infty}^2, \\ \sigma(\tilde{L}_u) &:= \{\lambda_0(u) \leq \dots \leq \lambda_n(u) \leq \dots\}, & \lambda_0(u) &\geq 0. \end{aligned} \quad (6.1.12)$$

Recall that any Lax operator satisfies the isospectral property

$$L_{u_0} = U(t)^{-1} L_{u(t)} U(t), \quad (6.1.13)$$

where u_0 is the initial data, $u(t)$ is the evolution of the solution starting from u_0 , and $U(t)$ is a family of operators solving the Cauchy problem

$$\begin{cases} \frac{d}{dt} U(t) = B_{u(t)} U(t) \\ U(0) = \text{Id} \end{cases}.$$

The identity (6.1.13) implies that the spectrum of $L_{u(t)}$ is invariant by the evolution, i.e. $\nu_n(u(t)) = \nu_n(u_0)$ and $\lambda_n(u(t)) = \lambda_n(u_0)$ for all n . Therefore, in the sequel, we omit the variable u in $\nu_n(u)$ and $\lambda_n(u)$ when it does not make confusion.

Further information regarding the spectrum of the Lax operators will be provided in Section 6.2.

6.1.2.3 Traveling waves on \mathbb{R}

Let us mention that the focusing Calogero–Sutherland DNLS equation (CS^+) also enjoys traveling waves and stationary waves in the *nonperiodic case* (i.e. $x \in \mathbb{R}$). They are of the form

$$u(t, x) = e^{i\theta} e^{iv(x-vt)} \lambda^{\frac{1}{2}} \mathcal{R}(\lambda(x - 2vt) + y), \quad \lambda > 0, \quad y \in \mathbb{R}, \quad \theta \in \mathbb{T}, \quad v \in \mathbb{R},$$

where the profile

$$\mathcal{R}(x) = e^{i\theta} \frac{\sqrt{2\text{Im}p}}{x+p}, \quad p \in \mathbb{C}_+, \theta \in \mathbb{R}, \quad (6.1.14)$$

is obtained as ground states (minimizers) for the energy functional [GL24, Section 4]. Notice that all these waves are of L^2 -norm equal to $\sqrt{2\pi}$. Therefore, this situation differs from the torus \mathbb{T} , where in the latter case, there is no L^2 -threshold that would prevent the existence of small or large traveling waves in $L^2(\mathbb{T})$. Essentially, the main reason that leads to a more diverse class of traveling waves in the periodic setting compared to the non-periodic setting, is due to the spectral property carried by the Lax operator in both cases. Indeed, on \mathbb{R} , the Lax operator has an absolute continuous spectrum and a finite number of eigenvalues [GL24, Section 5]. In contrast with \mathbb{T} , the Lax operator presents only point spectrum formed by eigenvalues [Bad24a, Section 2].

To summarize, we refer to the following table (Table 6.1).

	Focusing (CSM) on \mathbb{R}	Defocusing (CSM) on \mathbb{R}
Stationary waves	✓	
Traveling waves	✓	
Wave speed	$c \in \mathbb{R}$	
L^2 -norm of the traveling waves	$\ u\ _{L^2} = \sqrt{2\pi}$	
	Focusing (CSM) on \mathbb{T}	Defocusing (CSM) on \mathbb{T}
Non-trivial stationary waves	✓	✗
Traveling waves	✓	✓
Wave speed	$c \in \mathbb{R}$	$c \geq N$
L^2 -norm of the non-trivial traveling waves	$\ u\ _{L^2} \in (0, +\infty)$	$\ u\ _{L^2} \in (0, +\infty)$

TABLE 6.1 – Where $N \in \mathbb{N}_{\geq 1}$ is the denominator's degree of a traveling wave of the form (6.1.4). In addition, by a non-trivial traveling wave, we mean the traveling wave that does not belong to \mathcal{G}_1 where \mathcal{G}_1 is the set of trivial traveling waves defined in (6.1.3). Moreover, by non-trivial stationary waves we mean the solution $u(t, x) = u_0(x)$ that are not constant functions.

6.1.3 Outline of the paper

The paper is organized as follows. In Section 6.2, we present some spectral properties concerning the eigenvalues and the eigenfunctions of the Lax operators L_u and \tilde{L}_u . Moving on to Section 6.3, we focus on the traveling waves of the defocusing Calogero-Sutherland DNLS equation (CS⁻). This section follows a two-step process. Subsection 6.3.1 provides a spectral characterization of these waves, while Subsection 6.3.2 derives their explicit formulas. Moreover, Subsection 6.3.3 includes remarks concerning the speed and L^2 -norm of these traveling waves for the defocusing (CS⁻)-equation.

In Section 6.4, we delve into the analysis of traveling waves for the focusing Calogero-Sutherland DNLS equation (CS⁺). Thus, we describe the set of traveling

waves of (CS⁺) in Subsubsection 6.4.1, and we highlight the presence of a larger set of traveling waves in the focusing case comparing to the defocusing case. Similarly to the defocusing case, some remarks related to the speed and the L^2 –norm of the traveling waves of (CS⁺) are discussed in Subsection 6.4.2, and in particular we establish the existence of stationary wave solutions for the focusing Calogero–Sutherland DNLS equation (CS⁺).

Note that in order to describe the traveling waves of (CS⁺), one need to understand the set of finite gap potentials. To this end, Section 6.5 is dedicated to the study of finite gap potentials for the Calogero–Sutherland DNLS equation (CS).

Throughout this paper, we have assumed sufficient regularity on the solutions. However, in Section 6.6, we discuss how the same analysis can be extended to solutions with lower regularity. Lastly, in Section 6.7, we present some open problems for further exploration.

Acknowledgments The author would like to thank her Ph.D. advisor *Patrick Gérard* for proposing this problem and suggesting [GK21, Appendix B] as a useful reference to start the investigation.

6.2 Spectral properties for the Lax operators

As mentioned in the introduction, our aim is to describe the traveling waves of the Calogero–Sutherland DNLS equation (CS). In order to accomplish this goal, our strategy relies on characterizing them first in the state space, by means of some spectral tools of the Lax operators L_u and \tilde{L}_u introduced in (6.1.9) and in (6.1.10), respectively. Therefore, we need to delve deeper into the spectral properties of the Lax operators.

In the sequel, we assume, for more convenience, that u is any function of the state space with enough regularity, for example, $u \in H_+^2(\mathbb{T})$. But, it is worth mentioning that the analysis can be easily extended to potentials with less regularity as well (see Section 6.6). Besides, recall from (6.1.12), that the Lax operators \tilde{L}_u and L_u have point spectra, bounded from below

$$\begin{aligned}\sigma(\tilde{L}_u) &:= \{\lambda_0 \leq \dots \leq \lambda_n \leq \dots\}, & \lambda_0 &\geq 0, \\ \sigma(L_u) &:= \{\nu_0 \leq \dots \leq \nu_n \leq \dots\}, & \nu_0 &\geq -\|u\|_{L^\infty}^2.\end{aligned}$$

The following proposition aims to give more information, regarding the multiplicity of the eigenvalues (ν_n) and (λ_n) . But before, we need to recall two useful commutator identities. We denote by S the *shift operator* defined as

$$S : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T}), \quad Sh(x) = e^{ix} h(x). \quad (6.2.1)$$

Thus, for all $u \in H_+^2(\mathbb{T})$, we have from [Bad24a, Lemma 2.3],

$$\begin{aligned}\tilde{L}_u S &= S \tilde{L}_u + S + \langle \cdot | S^* u \rangle u, \\ L_u S &= S L_u + S - \langle \cdot | S^* u \rangle u,\end{aligned}\tag{6.2.2}$$

where S^* denotes the adjoint operator of S

$$S^* : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T}), \quad S^* h(x) = \Pi(e^{-ix} h(x)),\tag{6.2.3}$$

with Π is the Szegő projector defined in (6.1.1), and L_u and \tilde{L}_u are defined in (6.1.9) and (6.1.10). In addition, we also have from the same lemma [Bad24a, Lemma 2.3],

$$\begin{aligned}[S^*, B_u] &= i \left(S^* L_u^2 - (L_u + \text{Id})^2 S^* \right), \\ [S^*, \tilde{B}_u] &= i \left(S^* \tilde{L}_u^2 - (\tilde{L}_u + \text{Id})^2 S^* \right),\end{aligned}\tag{6.2.4}$$

where $[S^*, B_u]$ denotes the commutator $S^* B_u - B_u S^*$, B_u and \tilde{B}_u are the two skew-adjoint operators of the Lax pairs, defined respectively in (6.1.9) and (6.1.10).

Proposition 6.2.1 (Multiplicity of (λ_n) and (ν_n)).

Defocusing case. The eigenvalues (λ_n) of \tilde{L}_u are all simple. More precisely,

$$\lambda_{n+1} \geq \lambda_n + 1, \quad n \in \mathbb{N}_{\geq 0}.\tag{6.2.5}$$

Focusing case. The eigenvalues (ν_n) of L_u are of multiplicity at most two

$$\nu_{n+2} \geq \nu_n + 1 \quad n \in \mathbb{N}_{\geq 0}.\tag{6.2.6}$$

Moreover, when n is large enough, the eigenvalues of L_u are simple. More precisely,

$$\liminf_{n \rightarrow \infty} \nu_{n+1} - \nu_n \geq 1.\tag{6.2.7}$$

Furthermore, for all $0 \leq \alpha < 1$ such that $\|u\|_{L^2}^2 < 1 - \alpha$, we have for all $n \in \mathbb{N}_{\geq 0}$,

$$\nu_{n+1} > \nu_n + \alpha.\tag{6.2.8}$$

Remark 6.2.1.

1. It should be noted that for any potential u , the eigenvalues (ν_n) of L_u cannot be all simple. For instance, take $u(x) = e^{ix}$, one can easily check that for $L_u = D - T_u T_{\bar{u}}$,

$$L_u 1 = L_u e^{ix} = 0.$$

2. Inequality (6.2.7) implies that as $n \gg 1$, the lower bound of the distance between two consecutive eigenvalues ν_n gets closer to 1.

Proof. All the presented inequalities are a direct consequence of the max–min principle

$$\lambda_n = \max_{\substack{F \subseteq L_+^2 \\ \dim F \leq n}} \min \left\{ \langle \tilde{L}_u h | h \rangle ; h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}), \|h\|_{L^2} = 1 \right\} .$$

$$\nu_n = \max_{\substack{F \subseteq L_+^2 \\ \dim F \leq n}} \min \left\{ \langle L_u h | h \rangle ; h \in F^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}), \|h\|_{L^2} = 1 \right\} .$$

Spectrum of \tilde{L}_u . Let F be any subspace of $L_+^2(\mathbb{T})$ of dimension n , and consider $E := \mathbb{C}1 \oplus S(F)$, where S is the shift operator, then

$$\lambda_{n+1} \geq \min \{ \langle \tilde{L}_u h | h \rangle ; \|h\|_{L^2} = 1, h \in E^\perp \cap H_+^{\frac{1}{2}} \}$$

Observe that $E^\perp = S(F^\perp)$, thus by (6.2.2),

$$\lambda_{n+1} \geq \min \left\{ \langle \tilde{L}_u g | g \rangle + 1 + |\langle Sg | u \rangle|^2 ; \|g\|_{L^2} = 1, g \in F^\perp \cap H_+^{\frac{1}{2}} \right\} .$$

In addition, since $|\langle Sg | u \rangle|^2 \geq 0$, we infer for all $n \in \mathbb{N}_{\geq 0}$,

$$\lambda_{n+1} \geq \lambda_n + 1 .$$

Spectrum of L_u –Inequality (6.2.6). let F be any subspace of $L_+^2(\mathbb{T})$ of dimension n , and take $G := \mathbb{C}1 \oplus S(F) + \mathbb{C}u$. Then,

$$\nu_{n+2}(u) \geq \min \{ \langle L_u h | h \rangle ; \|h\|_{L^2} = 1, h \in G^\perp \cap H_+^{\frac{1}{2}} \} .$$

Since $G^\perp = S(F^\perp \cap (S^*u)^\perp)$, then

$$\nu_{n+2} \geq \min \left\{ \langle L_u Sg | Sg \rangle ; \|g\|_{L^2} = 1, g \in F^\perp \cap (S^*u)^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}) \right\} .$$

Note that $g \perp S^*u$, then by (6.2.2),

$$\nu_{n+2} \geq \min \left\{ \langle L_u g | g \rangle + 1 ; \|g\|_{L^2} = 1, g \in F^\perp \cap (S^*u)^\perp \cap H_+^{\frac{1}{2}}(\mathbb{T}) \right\} ,$$

leading to

$$\nu_{n+2} \geq \nu_n + 1 .$$

Inequality (6.2.7). For any n , let $F_n = \text{span}\{f_0, f_1, \dots, f_{n-1}\}$ be the subspace of $L_+^2(\mathbb{T})$ of dimension n made up of the first n eigenfunctions of L_u . For this choice of F_n ,

$$\min \left\{ \langle L_u h | h \rangle ; \|h\|_{L^2} = 1, h \in F_n^\perp \cap H_+^{\frac{1}{2}} \right\} = \nu_n . \quad (6.2.9)$$

Let us consider the subspace $E := \mathbb{C}1 \oplus S(F_n)$ of $L_+^2(\mathbb{T})$ of dimension $n + 1$, then

$$\nu_{n+1} \geq \min \{ \langle L_u g | g \rangle ; \|g\|_{L^2} = 1, g \in E^\perp \cap H_+^{\frac{1}{2}} \} .$$

Note that $E^\perp \cap H_+^{\frac{1}{2}} = S \left(F_n^\perp \cap H_+^{\frac{1}{2}} \right)$. Therefore, by (6.2.2),

$$\nu_{n+1} \geq \min \{ \langle L_u \phi | \phi \rangle + 1 - |\langle S\varphi | u \rangle|^2 ; \|\varphi\|_{L^2} = 1, \varphi \in F_n^\perp \cap H_+^1 \}.$$

It results, for all $n \in \mathbb{N}_{\geq 0}$,

$$\nu_{n+1} \geq \nu_n + 1 - \sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1 \\ \varphi \in F_n^\perp}} |\langle S\varphi | u \rangle|^2. \quad (6.2.10)$$

To conclude the proof, it remains to prove $\sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1 \\ \varphi \in F_n^\perp}} |\langle S\varphi | u \rangle|^2 \xrightarrow{n \rightarrow \infty} 0$.

Lemma 6.2.2. *Let F_n be the subspace of $L_+^2(\mathbb{T})$ defined as above. Then*

$$\sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1 \\ \varphi \in F_n^\perp}} |\langle S\varphi | u \rangle| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Suppose for the sake of contradiction, that for all $n \in \mathbb{N}_{\geq 0}$,

$$\sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1 \\ \varphi \in F_n^\perp}} |\langle S\varphi | u \rangle| \geq \varepsilon, \quad \varepsilon > 0.$$

Namely, there exists $\varphi_n \in F_n^\perp$, $\|\varphi_n\|_{L^2(\mathbb{T})} = 1$ such that $|\langle S\varphi_n | u \rangle| \geq \frac{\varepsilon}{2}$. Hence, since $\|\varphi_n\|_{L^2(\mathbb{T})} = 1$, then up to a sub-sequence $\varphi_n \rightharpoonup \varphi$ in $L_+^2(\mathbb{T})$ as $n \rightarrow \infty$, which yields to

$$|\langle S\varphi_n | u \rangle| \xrightarrow{n \rightarrow \infty} |\langle S\varphi | u \rangle|,$$

and so $\langle S\varphi | u \rangle \geq \frac{\varepsilon}{2}$. On the other hand, since $\varphi_n \perp F_n$ then

$$\langle \varphi_n | f_p \rangle = 0, \quad \forall 0 \leq p \leq n-1.$$

Taking $n \rightarrow \infty$, we infer

$$\langle \varphi | f_p \rangle = 0, \quad \forall p \in \mathbb{N}_{\geq 0}.$$

Note that the eigenfunctions (f_p) of the self-adjoint operator L_u form an orthonormal basis of $L_+^2(\mathbb{T})$. Therefore, we have $\varphi = 0$, which is a contradiction with $\langle S\varphi | u \rangle \geq \frac{\varepsilon}{2}$. \square

Inequality (6.2.8). It is a consequence of inequality (6.2.10) after applying the Cauchy–Schwarz inequality and considering the fact that $\|u\|_{L^2}^2 < 1 - \alpha$. \square

In what follows, we make a slight abuse of notation by using (f_n) to denote both an orthonormal basis of $L_+^2(\mathbb{T})$ consisting of the eigenfunctions of the self-adjoint operator L_u , and an orthonormal basis of $L_+^2(\mathbb{T})$ consisting of the eigenfunctions of \tilde{L}_u . Nonetheless, we will specify the context in which we are working to avoid confusion and ensure that (f_n) is understood appropriately as either the eigenfunctions of L_u or \tilde{L}_u .

Lemma 6.2.3. *Given $u \in H_+^2(\mathbb{T})$, then for all $n, p \in \mathbb{N}_{\geq 0}$,*

— *Defocusing case.*

$$\begin{aligned} \langle 1 | u \rangle \langle u | f_n \rangle &= \lambda_n \langle 1 | f_n \rangle, \\ (\lambda_n - \lambda_p - 1) \langle Sf_p | f_n \rangle &= \langle Sf_p | u \rangle \langle u | f_n \rangle. \end{aligned}$$

— *Focusing case.*

$$\begin{aligned} \langle 1 | u \rangle \langle u | f_n \rangle &= -\nu_n \langle 1 | f_n \rangle, \\ (\nu_n - \nu_p - 1) \langle Sf_p | f_n \rangle &= -\langle Sf_p | u \rangle \langle u | f_n \rangle. \end{aligned}$$

Proof. We prove first the identities for the *defocusing case*. By definition of $\tilde{L}_u = D + T_u T_{\bar{u}}$, we have

$$\tilde{L}_u 1 = \langle 1 | u \rangle u.$$

Then taking the inner product of both sides with f_n , and using the fact that \tilde{L}_u is a self-adjoint operator, lead to the first identity. For the second one, thanks to the commutator relation between \tilde{L}_u and S

$$\tilde{L}_u S f_p = S \tilde{L}_u f_p + S f_p + \langle S f_p | u \rangle u,$$

of equation (6.2.2), we infer by taking the inner product with f_n the second identity.

Besides, by considering the *focusing case* with $L_u = D - T_u T_{\bar{u}}$ it follows that $L_u 1 = -\langle 1 | u \rangle u$. This explains the sign $-$ appearing in the first statement. As for the second one, since by (6.2.2)

$$L_u S f_p = S L_u f_p + S f_p - \langle S f_p | u \rangle u.$$

than taking once more the inner product with f_n leads to the desired identity. \square

In light of the previous lemma and based on the commutator identities (6.2.2), one can investigate further information regarding the spectral data (i.e. the eigenvalues and the eigenvectors) of L_u and \tilde{L}_u , especially when the quantities $\langle u | f_n \rangle$ vanishes. The following lemmas/propositions aim to achieve this.

For the following, we denote by E_{ν_n} the eigenspace of L_u corresponding to the eigenvalue ν_n . In addition, the notation $f // g$ means that the two vectors f and g are collinear in $L_+^2(\mathbb{T})$.

Proposition 6.2.4. *For all $n \in \mathbb{N}_{\geq 1}$, such that $\nu_n \neq 0$, we have*

$$\nu_n = \nu_{n-1} + 1 \implies [Sf_{n-1} \in E_{\nu_n}] \text{ or } [f_n \in SE_{\nu_{n-1}}],$$

Moreover, for the *defocusing case*,

$$\lambda_n = \lambda_{n-1} + 1 \implies Sf_{n-1} // f_n.$$

Remark 6.2.2.

1. The condition $\nu_n \neq 0$ cannot be omitted. For an example, we refer to the Appendix 1.
2. For the defocusing case, the condition of non-vanishing eigenvalues $\lambda_n \neq 0$ is already satisfied for all $n \in \mathbb{N}_{\geq 1}$, since \tilde{L}_u is a non-negative operator on the Hardy space, and for all $n \in \mathbb{N}_{\geq 1}$, we have by (6.2.5), $\lambda_n \geq \lambda_{n-1} + 1$.

Proof. The key is to use Lemma 6.2.3 and the commutator identities (6.2.2). In view of the second identity of Lemma 6.2.3, we have

$$\langle Sf_{n-1} | u \rangle \langle u | f_n \rangle = 0. \quad (6.2.11)$$

If $\langle u | Sf_{n-1} \rangle = 0$, then by (6.2.2),

$$\begin{aligned} L_u Sf_{n-1} &= SL_u f_{n-1} + Sf_{n-1} \\ &= (\nu_{n-1} + 1) Sf_{n-1} \\ &= \nu_n Sf_{n-1}, \end{aligned}$$

as $\nu_n = \nu_{n-1} + 1$. Namely, $Sf_{n-1} \in E_{\nu_n}$. Let us move to the second case where $\langle u | f_n \rangle = 0$. By the first identity of Lemma 6.2.3,

$$\nu_n \langle f_n | 1 \rangle = 0.$$

Therefore, since $\nu_n \neq 0$, there exists $g_n \in H_+^1(\mathbb{T})$ such that $f_n = Sg_n$. Using again the commutator identity (6.2.2), we have

$$SL_u g_n = (\nu_n - 1) Sg_n.$$

Applying S^* to both sides of the latter identity, and using the fact that $S^*S = \text{Id}$, and as $\nu_n = \nu_{n-1} + 1$, we find

$$L_u g_n = \nu_{n-1} g_n.$$

That is, $g_n \in E_{\nu_{n-1}}$, and so $f_n \in SE_{\nu_{n-1}}$.

Besides, note that for the *defocusing equation*, the vector spaces E_{λ_n} are of dimension one, thanks to Proposition 6.2.1. Consequently, the results $[Sf_{n-1} \in E_{\lambda_n}]$ or $[f_n \in SE_{\lambda_{n-1}}]$, leads to $Sf_{n-1} // f_n$. \square

In the sequel, we denote by $\mathcal{I}(u)$ the set of

$$\mathcal{I}(u) := \{n \in \mathbb{N}_{\geq 1} \mid \langle Sf_{n-1} | f_n \rangle = 0\}. \quad (6.2.12)$$

Lemma 6.2.5. *Defocusing case.* For any $u \in H_+^2(\mathbb{T})$, the set $\mathcal{I}(u)$ is empty.

Focusing case. Given $u \in H_+^2(\mathbb{T})$, let $m \in \mathcal{I}(u)$. Assume that the eigenvalues ν_m and ν_{m-1} are simple, Then, either

$$\nu_{m-1} + 1 = \nu_{m+1}, \quad \text{with} \quad Sf_{m-1} \in E_{\nu_{m+1}}$$

or

$$\nu_{m-2} + 1 = \nu_m, \quad \text{with} \quad S^* f_m \in E_{\nu_{m-2}}$$

or

$$\nu_m = 0, \quad \text{with} \quad f_m // 1.$$

Remark 6.2.3.

(i) Observe that in the focusing case, if $\|u\|_{L^2}^2 < \frac{1}{2}$, then by inequality (6.2.8),

$$\nu_n > \nu_{n-1} + \frac{1}{2}, \quad \forall n \in \mathbb{N}_{\geq 1}.$$

Hence, for such u , if $m \in \mathcal{I}(u)$ then the only possible choice is to have $\nu_m = 0$ with $f_m // 1$. In other words, for $\|u\|_{L^2}^2 < \frac{1}{2}$, we have, either $\mathcal{I}(u) = \emptyset$, or $\mathcal{I}(u) = \{m\}$ and in such case $\nu_m = 0$ and $f_m // 1$.

(ii) For any $u \in H_+^2(\mathbb{T})$, the set $\mathcal{I}(u)$ in the focusing case is of finite cardinal, since by inequality (6.2.7) we have $\nu_n > \nu_{n-1} + \frac{1}{2}$ and $\nu_n \neq 0$ for all n large enough.

Proof. Focusing case. Let $m \in \mathcal{I}(u)$. By the second identity of Lemma 6.2.3,

$$\langle Sf_{m-1} | u \rangle \langle u | f_m \rangle = 0.$$

If $\langle Sf_{m-1} | u \rangle = 0$, then applying the commutator identity (6.2.2),

$$L_u Sf_{m-1} = (\nu_{m-1} + 1) Sf_{m-1}. \quad (6.2.13)$$

Namely, $\nu_{m-1} + 1$ is an eigenvalue of L_u and Sf_{m-1} is a corresponding eigenfunction. Since ν_m is simple, then Sf_{m-1} cannot be collinear to f_m as $\langle Sf_{m-1} | f_m \rangle = 0$ for $m \in \mathcal{I}(u)$. Therefore, by (6.2.6),

$$\nu_{m-1} + 1 = \nu_{m+1}.$$

If $\langle u | f_m \rangle = 0$, then by applying the adjoint of the commutator identity (6.2.2),

$$S^* L_u = L_u S^* + S^* + \langle \cdot | u \rangle S^* u,$$

we infer,

$$L_u S^* f_m = (\nu_m - 1) S^* f_m.$$

That is, if $S^* f_m \neq 0$, then $S^* f_m$ is an eigenfunction of L_u associated with the eigenvalue $\nu_m - 1$. Recall, we have by assumption that ν_{m-1} is simple, and since $S^* f_m$ cannot be collinear to f_{m-1} as $m \in \mathcal{I}(u)$, then

$$\nu_m - 1 = \nu_{m-2},$$

thanks to (6.2.6). It remains to study the case where $S^* f_m = 0$, i.e., $f_m // 1$. For that case, we have thanks to the first identity of Lemma 6.2.3, $\nu_m = 0$ as $\langle u | f_m \rangle = 0$.

The defocusing case. Suppose that there exists $m \in \mathcal{I}(u)$. Then, using the same analysis as in the focusing case, we infer that, either $\lambda_{m-1} + 1 = \lambda_{m+1}$ or $\lambda_{m-2} + 1 = \lambda_m$ or $\lambda_m = 0$. However, recall that $\lambda_n \geq \lambda_{n-1} + 1$ for all $n \in \mathbb{N}_{\geq 1}$ (inequality 6.2.5), thus the first two cases cannot occur. In addition, since \tilde{L}_u is a non negative operator, where all the eigenvalues satisfy the inequality (6.2.5), then $\lambda_m = 0$ implies $m = 0 \notin \mathcal{I}(u)$. \square

Corollary 6.2.6. *For all $n \geq 1$,*

$$\lambda_n = \lambda_{n-1} + 1 \iff \langle u | f_n \rangle = 0.$$

In addition,

$$\nu_n = \nu_{n-1} + 1, \forall n \geq N_1 \iff \langle u | f_n \rangle = 0, \forall n \geq N_2.$$

Remark 6.2.4. We refer to the Appendix 2 for an example that shows that N_2 is not necessarily equal to N_1 .

Proof. For the *defocusing case*. Suppose that $\lambda_n = \lambda_{n-1} + 1$. Then, from one side we have by Proposition 6.2.4, $Sf_{n-1} // f_n$, and from the other hand, we infer by the second identity of Lemma 6.2.3,

$$\langle Sf_{n-1} | u \rangle \langle u | f_n \rangle = 0.$$

That is, $\langle u | f_n \rangle = 0$. The converse is a direct consequence of the second identity of Lemma 6.2.3 and the previous lemma.

For the *focusing case*, the same analysis can be applied. However, it should be noted that, since not all the eigenvalues (ν_n) satisfy $\nu_n > \nu_{n-1} + \frac{1}{2}$, and $\nu_n \neq 0$, for all $n \in \mathbb{N}_{\geq 1}$, but only for large n , thanks to Proposition 6.2.1, then the equivalence holds for n sufficiently large. \square

6.3 Traveling waves for the defocusing (CS⁻)

6.3.1 Spectral Characterization

One way to understand the behavior of a linear PDE's solution is to consider its Fourier transform. Specifically, on the periodic domain \mathbb{T} , this consists of computing the inner product with $\langle \cdot | e^{inx} \rangle$ for all $n \in \mathbb{Z}$. The main idea behind this approach is to “diagonalize” the problem in the (e^{inx}) -basis, which facilitates solving the equation. However, by considering the Calogero–Sutherland DNLS equation (CS⁻), we are dealing with a nonlinear *integrable* PDE, which can also be “diagonalized” in some coordinate system (think about the Birkhoff coordinates). Thus, by imitating the idea of the linear case, we suggest taking the inner product of the (CS⁻)-equation with an appropriate orthonormal basis of $L^2_+(\mathbb{T})$. Before proceeding, observe that

the defocusing Calogero–Sutherland DNLS equation can be rewritten in terms of the Lax pair as [Bad24a, Lemma 2.4, Lemma 5.2]

$$\partial_t u = \tilde{B}_u u - i\tilde{L}_u^2 u. \quad (6.3.1)$$

This motivates the choice of the following orthonormal basis of $L_+^2(\mathbb{T})$.

Definition 6.3.1. Given $u \in \mathcal{C}_t H_+^2(\mathbb{T})_x$, let (g_n^t) be the evolving orthonormal basis of $L_+^2(\mathbb{T})$ defined along the curve $t \mapsto u(t)$ as

$$\begin{cases} \partial_t g_n^t &= \tilde{B}_{u(t)} g_n^t, \\ g_n^t|_{t=0} &= f_n^{u_0} \end{cases}, \quad \forall n \in \mathbb{N}_{\geq 0}, \quad (6.3.2)$$

where $(f_n^{u_0})$ is an orthonormal basis of $L_+^2(\mathbb{T})$ made up of the eigenfunctions of \tilde{L}_{u_0} at $t = 0$, and $\tilde{B}_{u(t)}$ is the skew-adjoint operator defined in (6.1.10).

Remark 6.3.1. Note that the (g_n^t) satisfies for all $n \in \mathbb{N}_{\geq 0}$ [Kuk06, Lemma 4.1],

$$\tilde{L}_{u(t)} g_n^t = \lambda_n g_n^t. \quad (6.3.3)$$

Therefore, as it was established in [Bad24a, Lemma 3.6], by taking the inner product of (6.3.1) with the g_n^t and using that \tilde{L}_u is a self-adjoint operator and \tilde{B}_u is skew-symmetric, we find

$$\partial_t \langle u(t) | g_n^t \rangle = -i\lambda_n^2 \langle u(t) | g_n^t \rangle,$$

or

$$\langle u(t) | g_n^t \rangle = \langle u_0 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t}. \quad (6.3.4)$$

Lemma 6.3.2. For any $u \in \mathcal{C}_t H_+^2(\mathbb{T})_x$ solution of (CS[−]) and for all $n, p \in \mathbb{N}_{\geq 0}$,

$$\begin{aligned} \langle 1 | g_n^t \rangle &= \langle 1 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t}, \\ \langle Sg_p^t | g_n^t \rangle &= \langle Sf_p^{u_0} | f_n^{u_0} \rangle e^{i((\lambda_p+1)^2 - \lambda_n^2)t}. \end{aligned} \quad (6.3.5)$$

Proof. By Definition 6.3.1, and since \tilde{B}_u is skew-symmetric operator,

$$\partial_t \langle 1 | g_n^t \rangle = \langle 1 | \tilde{B}_u g_n^t \rangle = -\langle \tilde{B}_u 1 | g_n^t \rangle,$$

where by (6.1.10),

$$\begin{aligned} \tilde{B}_u 1 &= -T_u T_{\partial_x \bar{u}} 1 + T_{\partial_x u} T_{\bar{u}} 1 + i(T_u T_{\bar{u}})^2 1 \\ &= \langle 1 | u \rangle (\partial_x u + iT_u T_{\bar{u}} u). \end{aligned}$$

Note that $\tilde{L}_u 1 = -i\partial_x 1 + T_u T_{\bar{u}} 1 = \langle 1 | u \rangle u$. Therefore, $\tilde{B}_u 1 = i\tilde{L}_u^2 1$ and

$$\partial_t \langle 1 | g_n^t \rangle = -i \langle L_u^2 1 | g_n^t \rangle = -i\lambda_n^2 \langle 1 | g_n^t \rangle.$$

This achieves the proof of the first point. To prove the second one, we proceed with the same manner. By Definition 6.3.1,

$$\partial_t \langle g_n^t | S g_p^t \rangle = \langle \tilde{B}_u g_n^t | S g_p^t \rangle + \langle g_n^t | S \tilde{B}_u g_p^t \rangle = \langle [S^*, \tilde{B}_u] g_n^t | g_p^t \rangle.$$

Hence, applying the commutator identity (6.2.4), and since \tilde{L}_u is a self-adjoint operator, we infer

$$\begin{aligned} \partial_t \langle g_n^t | S g_p^t \rangle &= i \langle (S^* \tilde{L}_u^2 - (\tilde{L}_u + \text{Id})^2 S^*) g_n^t | g_p^t \rangle \\ &= i (\lambda_n^2 + (\lambda_p + 1)^2) \langle g_n^t | S g_p^t \rangle. \end{aligned}$$

Therefore,

$$\langle g_n^t | S g_p^t \rangle = \langle f_n^{u_0} | S f_p^{u_0} \rangle e^{i(\lambda_n^2 - (\lambda_p + 1)^2)t}.$$

□

Remark 6.3.2. The consideration of the evolution of $\langle u | g_n^t \rangle$, $\langle 1 | g_n^t \rangle$, and $\langle S g_p^t | g_n^t \rangle$ is motivated by the fact that any element u of the Hardy space can be written as follows.

Lemma 6.3.3 ([GK21, GMR16]). *For any $u \in L_+^2(\mathbb{T})$,*

$$u(z) = \langle (\text{Id} - zS^*)^{-1} u | 1 \rangle, \quad z \in \mathbb{D},$$

where S^* is the adjoint operator of S in $L_+^2(\mathbb{T})$.

Therefore, by expressing the operator S^* , and the two vectors u and 1 in their matrix representations with respect to the (g_n^t) -basis, we obtain

$$u(t, z) = \langle (\text{Id} - zM)^{-1} X | Y \rangle, \quad z \in \mathbb{D}, \quad (6.3.6)$$

where X, Y are infinite column vectors and M is the infinite matrix representation :

$$X := (\langle u | g_n^t \rangle), \quad Y := (\langle 1 | g_n^t \rangle), \quad M := (\langle g_m^t | S g_n^t \rangle).$$

Proof of Lemma 6.3.3. ([GK21]) The idea is to observe that any element u of the Hardy space $L_+^2(\mathbb{T})$ can be read as an analytic function on the open unit disc \mathbb{D} , whose trace on the boundary $\partial\mathbb{D}$ is in L^2 ⁴. Thus, for any $z \in \mathbb{D}$,

$$u(z) = \sum_{k \in \mathbb{N}_{\geq 0}} \hat{u}(k) z^k = \sum_{k \in \mathbb{N}_{\geq 0}} \langle u | S^k 1 \rangle z^k = \sum_{k \in \mathbb{N}_{\geq 0}} \langle (S^*)^k u | 1 \rangle z^k.$$

As a result, by Neumann series,

$$u(z) = \langle (\text{Id} - zS^*)^{-1} u | 1 \rangle.$$

□

4. For a simple introduction to the different definitions of Hardy space, we refer to [GMR16, Chapter 3.]

At this stage, we consider $u(t) := u_0(x - ct)$ to be a traveling wave to the Calogero–Sutherland DNLS equation (CS⁻). We denote, for all $c, t \in \mathbb{R}$, by τ_{ct} the isometric linear map

$$\tau_{ct} : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T}), \quad \tau_{ct} u_0(x) = u_0(x - ct).$$

Our aim for this subsection is to prove the following Theorem.

Theorem 6.3.4. *Let $u(t) := \tau_{ct} u_0$ be a traveling wave to the (CS⁻)-equation. Then there exists at most one $N \in \mathbb{N}_{\geq 1}$ such that*

$$\langle u_0 | f_N^{u_0} \rangle \neq 0.$$

Moreover, the speed c is given by

$$c = 1 + \frac{2}{N} \sum_{k=0}^{N-1} \lambda_k.$$

To this end, we shall need two key elements. Firstly, Lemma 6.3.2 and identity (6.3.4). Secondly, we will utilize the existence of a relationship (identity (6.3.7)) connecting the eigenfunctions (g_n^t) of $\tilde{L}_{u(t)}$ introduced in Definition 6.3.1, with the functions $(\tau_{ct} f_n^{u_0})$, where recall $(f_n^{u_0})$ represents the eigenfunctions of \tilde{L}_{u_0} .

To establish this connection, we present the following proposition, which also describes the behavior of the eigenfunctions $(f_n^{u_0})$ of \tilde{L}_{u_0} under the action of the translation map on the spatial variable

$$f_n^{u_0} \longmapsto \tau_{ct} f_n^{u_0}, \quad c, t \in \mathbb{R}.$$

Proposition 6.3.5. *Let $u(t) := \tau_{ct} u_0$ be a solution to (CS⁻). There exists a sequence $(\theta_n(t)) \subseteq \mathbb{R}$, such that*

$$\tau_{ct} f_n^{u_0} = e^{i\theta_n(t)} g_n^t, \quad \forall n \in \mathbb{N}_{\geq 0}. \quad (6.3.7)$$

In other words, the $(\tau_{ct} f_n^{u_0})$ are also eigenfunctions of $\tilde{L}_{u(t)}$.

Proof. By definition of $\tilde{L}_u = D + u \Pi(\bar{u} \cdot)$, and since $u(t) = \tau_{ct} u_0$

$$\begin{aligned} \tilde{L}_{u(t)} \tau_{ct} f_n^{u_0} &= D f_n^{u_0}(x - ct) + u_0(x - ct) \Pi(\bar{u}_0(x - ct) f_n^{u_0}(x - ct)), \\ &= \tau_{ct} \left(\tilde{L}_{u_0} f_n^{u_0} \right) \\ &= \lambda_n(u_0) \tau_{ct} f_n^{u_0}, \end{aligned} \quad \forall n \in \mathbb{N}_{\geq 0}.$$

In other words, $\tau_{ct} f_n^{u_0}$ is an eigenfunction of $\tilde{L}_{u(t)}$ associated with the eigenvalue $\lambda_n(u_0)$. On the other hand, recall that all the eigenvalues $\lambda_n(u_0)$ of \tilde{L}_{u_0} are simple, as stated in Proposition 6.2.1. Additionally, according to Remark 6.3.1, the (g_n^t) are eigenfunctions of $\tilde{L}_{u(t)}$ associated to the eigenvalues $\lambda_n(u_0)$. Therefore, for all

$n \in \mathbb{N}_{\geq 0}$, the two vectors $\tau_{ct}f_n^{u_0}$ and g_n^t are collinear. Since both vectors belong to an orthonormal basis of $L_+^2(\mathbb{T})$, then each one has an L^2 -norm equal to one. Thus, we infer for all $n \in \mathbb{N}_{\geq 0}$, there exists $\theta_n(t) \in \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$\tau_{ct}f_n^{u_0} = e^{i\theta_n(t)} g_n^t.$$

□

Corollary 6.3.6. *For all $n, p \in \mathbb{N}_{\geq 0}$, and for all $t \in \mathbb{R}$, we have*

1. *If $\langle 1 | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = -\lambda_n^2 t.$$

2. *If $\langle u_0 | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = -\lambda_n^2 t.$$

3. *If $\langle Sf_p^{u_0} | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = ((\lambda_p + 1)^2 - \lambda_n^2)t - ct + \theta_p(t),$$

where $\theta_n(t)$ is the angle obtain in (6.3.7).

Proof. By combining identity (6.3.4) and the two identities of Lemma 6.3.2, with identity (6.3.7) of the previous proposition, we infer

$$\begin{cases} e^{i\theta_n(t)} \langle 1 | \tau_{ct}f_n^{u_0} \rangle = \langle 1 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t}, \\ e^{i\theta_n(t)} \langle \tau_{ct}u_0 | \tau_{ct}f_n^{u_0} \rangle = \langle u_0 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t} \\ e^{-i\theta_p(t)} e^{i\theta_n(t)} \langle S\tau_{ct}f_p^{u_0} | \tau_{ct}f_n^{u_0} \rangle = \langle Sf_p^{u_0} | f_n^{u_0} \rangle e^{i((\lambda_p+1)^2 - \lambda_n^2)t}. \end{cases}$$

Note that, $S\tau_{ct}(\cdot) = e^{ict}\tau_{ct}(S\cdot)$, and since we are dealing with periodic functions, we deduce,

$$\begin{cases} e^{i\theta_n(t)} \langle 1 | f_n^{u_0} \rangle = \langle 1 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t}, \\ e^{i\theta_n(t)} e^{i\varphi(t)} \langle u_0 | f_n^{u_0} \rangle = \langle u_0 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t} \\ e^{-i\theta_p(t)} e^{i\theta_n(t)} e^{ict} \langle Sf_p^{u_0} | f_n^{u_0} \rangle = \langle Sf_p^{u_0} | f_n^{u_0} \rangle e^{i((\lambda_p+1)^2 - \lambda_n^2)t}. \end{cases} \quad (6.3.8)$$

leading to the result. □

At this point, we are ready to prove the spectral characterization of the traveling waves for (CS⁻), namely Theorem 6.3.4.

Proof of Theorem 6.3.4. The proof relies on the spectral property of \tilde{L}_u discussed in Section 6.2 and on Corollary 6.3.6. Indeed, observe first by Lemma 6.2.5, we have $\langle Sf_{n-1}^{u_0} | f_n^{u_0} \rangle \neq 0$ for all $n \in \mathbb{N}_{\geq 1}$. Hence, applying the third identity of Corollary 6.3.6 with $p = n - 1$, leads to the recurrence relation

$$\theta_n(t) = ((\lambda_{n-1} + 1)^2 - \lambda_n^2)t - ct + \theta_{n-1}(t), \quad n \geq 1.$$

Taking the sum of all these expressions from $n = 1$ to $n \in \mathbb{N}_{\geq 1}$, we infer

$$\theta_n(t) = \lambda_0^2 t + 2t \sum_{k=0}^{n-1} \lambda_k + nt - \lambda_n^2 t - nct + \theta_0(t). \quad (6.3.9)$$

Our aim is to prove that for all $n \geq 1$, $\langle u_0 | f_n^{u_0} \rangle = 0$ unless at most for one n . For the sake of contradiction, suppose that there exist two integers $1 \leq n_1 < n_2$ such that $\langle u_0 | f_{n_1}^{u_0} \rangle \neq 0$ and $\langle u_0 | f_{n_2}^{u_0} \rangle \neq 0$. Then by Corollary 6.3.6, we infer

$$\begin{aligned} \theta_{n_1}(t) &= -\lambda_{n_1}^2 t \\ \theta_{n_2}(t) &= -\lambda_{n_2}^2 t \end{aligned} \quad (6.3.10)$$

Plugging (6.3.10) in (6.3.9) we obtain

$$\begin{aligned} n_1 ct &= n_1 t + 2t \sum_{k=0}^{n_1-1} \lambda_k + \theta_0(t) + \lambda_0^2 t. \\ n_2 ct &= n_2 t + 2t \sum_{k=0}^{n_2-1} \lambda_k + \theta_0(t) + \lambda_0^2 t. \end{aligned} \quad (6.3.11)$$

Besides, notice that

$$\theta_0(t) = -\lambda_0^2 t. \quad (6.3.12)$$

Indeed, if $\langle u_0 | f_0^{u_0} \rangle \neq 0$ then by the second point of Corollary 6.3.6, we have the claimed identity. Otherwise, if $\langle u_0 | f_0^{u_0} \rangle = 0$ then $\langle 1 | f_0^{u_0} \rangle \neq 0$, since if it is not the case, i.e. if there exists $h \in L_+^2(\mathbb{T})$ such that $f_0^{u_0} = Sh$, then we have by the commutator relation (6.2.2)

$$\lambda_0 Sh = \tilde{L}_{u_0} Sh = S\tilde{L}_{u_0} h + Sh + \langle Sh | u_0 \rangle u_0,$$

implying, as $\langle Sh | u_0 \rangle = \langle f_0^{u_0} | u_0 \rangle = 0$,

$$\tilde{L}_{u_0} h = (\lambda_0 - 1)h.$$

That means, h is an eigenvector of \tilde{L}_{u_0} associated with an eigenvalue strictly less than λ_0 , which is impossible. Therefore $\langle 1 | f_0^{u_0} \rangle \neq 0$, and so by the first identity of Corollary 6.3.6, we infer $\theta_0(t) = -\lambda_0^2 t$. Substituting (6.3.12) in (6.3.11), we obtain

$$\left| \begin{aligned} c &= 1 + \frac{2}{n_1} \sum_{k=0}^{n_1-1} \lambda_k. \\ c &= 1 + \frac{2}{n_2} \sum_{k=0}^{n_2-1} \lambda_k. \end{aligned} \right.$$

That is,

$$n_2 \sum_{k=0}^{n_1-1} \lambda_k = n_1 \sum_{k=0}^{n_2-1} \lambda_k,$$

or

$$(n_2 - n_1) \sum_{k=0}^{n_1-1} \lambda_k = n_1 \sum_{k=n_1}^{n_2-1} \lambda_k$$

But recall by (6.2.5), $\lambda_{n+1} > \lambda_n$, for all n . Combining this fact with the last equality, we conclude

$$n_1(n_2 - n_1) \lambda_{n_1-1} > n_1(n_2 - n_1) \lambda_{n_1},$$

leading to a contradiction. As a consequence, for any traveling wave solution $u(t, x) := u_0(x - ct)$ of (CS⁻), there exists at most one $N \in \mathbb{N}_{\geq 1}$ such that

$$\langle u_0 | f_N^{u_0} \rangle \neq 0,$$

where $(f_n^{u_0})$ is any orthonormal basis of $L_+^2(\mathbb{T})$ consisting of the eigenfunctions of \tilde{L}_{u_0} . Moreover, u travels with the speed

$$c = 1 + \frac{2}{N} \sum_{k=0}^{N-1} \lambda_k. \quad (6.3.13)$$

□

Remark 6.3.3. In view of the previous Theorem and Corollary 6.2.6, it follows that any traveling wave solution u of (CS⁻) propagates with a speed

$$c = N + 2\lambda_0. \quad (6.3.14)$$

Indeed, since $\langle u_0 | f_n^{u_0} \rangle = 0$ for all $1 \leq n < N$, then by Corollary 6.2.6,

$$\lambda_n = \lambda_{n-1} + 1, \quad \forall 1 \leq n < N.$$

leading to the fact that (6.3.13) is equivalent to (6.3.14). Besides, since \tilde{L}_u is a non-negative operator, then $\lambda_0 \geq 0$, which implies that the speed of the traveling wave solution satisfies $c \geq N$. However, as will be observed in Subsection 6.3.3, the speed $c = N$ can only be reached by traveling waves of the form $u(t, x) = e^{iN(x-Nt)}$.

6.3.2 Explicit formulas of the traveling waves

Recall by Remark 6.3.2, any elements of the Hardy space, in particular u_0 , can be written as

$$u_0(z) = \langle (\text{Id} - zM)^{-1} X | Y \rangle, \quad (6.3.15)$$

where X, Y are infinite column vectors, M is an infinite matrix :

$$X := (\langle u_0 | f_n^{u_0} \rangle), \quad Y := (\langle 1 | f_n^{u_0} \rangle), \quad M = \left(\langle f_p^{u_0} | S f_n^{u_0} \rangle \right). \quad (6.3.16)$$

In the following, we denote by \mathcal{G}_1 the set of the *semi-trivial traveling waves*, made up from the constant and the plane wave solutions

$$\mathcal{G}_1 = \{ C e^{iN(x-Nt)} \mid C \in \mathbb{C}, N \in \mathbb{N}_{\geq 0} \}. \quad (6.3.17)$$

Theorem (6.1.1). *The traveling waves $u(t, x) = u_0(x - ct)$ of (CS⁻) are the potentials $u(t, x) \in \mathcal{G}_1$ and*

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T},$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N \left(1 + \frac{2\alpha}{\beta} \right)$, and (α, β) are two real constants satisfying

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = -N. \quad (6.3.18)$$

Proof. The proof is based on the inversion spectral formula

$$u_0(z) = \langle (\text{Id} - zM)^{-1} X | Y \rangle,$$

of (6.3.15), and on the spectral characterization of u_0 described in Theorem 6.3.4. In the sequence, to make the notation less cluttered, we denote $f_n := f_n^{u_0}$.

Let $u(t, x) := u_0(x - ct)$. As a first step, we prove that the infinite matrices X, Y and M reduce to finite matrices in the context of a traveling wave solution. Indeed, by Theorem 6.3.4, there exists at most one $N \in \mathbb{N}_{\geq 1}$, such that $\langle u_0 | f_N \rangle \neq 0$. We focus on the case where such an N exists, that is :

$$\begin{cases} \langle u_0 | f_N \rangle \neq 0 \\ \langle u_0 | f_n \rangle = 0, \forall n \in \mathbb{N}_{\geq 1} \setminus \{N\} \end{cases}. \quad (6.3.19)$$

The case where $\langle u | f_n \rangle = 0$ for all $n \in \mathbb{N}_{\geq 1}$ can be handled similarly, leading also to the reduction of the study to finite matrices. From now on, we suppose (6.3.19) holds. Therefore, it follows by Lemma 6.2.3, that $\lambda_n \langle 1 | f_n \rangle = 0$, implying that

$$\langle 1 | f_n \rangle = 0, \quad \forall n \in \mathbb{N} \setminus \{N\},$$

as the eigenvalues λ_n are all positive for any $n \in \mathbb{N}_{\geq 1}$ since \tilde{L}_u is a non-negative operator. Therefore, the two infinite column vectors X and Y of (6.3.16) reduces to

$$X = \begin{pmatrix} \langle u_0 | f_0 \rangle \\ 0 \\ \vdots \\ 0 \\ \langle u_0 | f_N \rangle \\ 0 \\ \vdots \end{pmatrix}, \quad Y = \begin{pmatrix} \langle 1 | f_0 \rangle \\ 0 \\ \vdots \\ 0 \\ \langle 1 | f_N \rangle \\ 0 \\ \vdots \end{pmatrix}. \quad (6.3.20)$$

On the other hand, since $\langle u_0 | f_n \rangle = 0$, for all $n \in \mathbb{N} \setminus \{N\}$, then by Corollary 6.2.6, we have $\lambda_n = \lambda_{n-1} + 1$ for all $n \in \mathbb{N} \setminus \{N\}$. Whence, $Sf_{n-1} \parallel f_n$ for all $n \in \mathbb{N} \setminus \{N\}$, thanks to Proposition 6.2.4. More specifically,

$$\begin{cases} f_n \parallel S^n f_0, & 1 \leq n \leq N-1 \\ f_n \parallel S^{n-N} f_N, & n \geq N \end{cases}. \quad (6.3.21)$$

As a consequence, the set $\{(S^n f_0)_{n=0, \dots, N-1}, (S^n f_N)_{n \geq 0}\}$ is an orthonormal basis of $L_+^2(\mathbb{T})$ and the matrix $M = (\langle f_p | S f_n \rangle)$ reduces to

$$M = \left(\begin{array}{cccccc|ccc} 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \ddots & 1 & 0 & 0 & & \\ \langle f_0 | S^N f_0 \rangle & 0 & \dots & 0 & \langle f_N | S^N f_0 \rangle & 0 & & \\ 0 & \dots & & & 0 & 1 & 0 & \\ \hline 0 & \dots & & & 0 & 0 & 1 & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots \end{array} \right).$$

Hence, the infinite matrices X , Y and M in formula (6.3.15) can be restrained to finite matrices involving only the first $N+1$ coordinates of X , Y , and M [GK21]. Indeed, denoting $\boldsymbol{\xi} := (\text{Id} - zM)^{-1} X$, we have

$$(\text{Id} - zM) \boldsymbol{\xi} = X.$$

That is,

$$\left(\begin{array}{cccccc|ccc} 1 & -z & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \ddots & -z & 0 & 0 & & \\ -\langle f_0 | S^N f_0 \rangle z & 0 & \dots & 1 & -\langle f_N | S^N f_0 \rangle z & 0 & & \\ 0 & \dots & & & 1 & -z & 0 & \\ \hline 0 & \dots & & & 0 & 1 & -z & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots \end{array} \right) \cdot \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{N-1} \\ \xi_N \\ \xi_{N+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle u_0 | f_0 \rangle \\ 0 \\ \vdots \\ 0 \\ \langle u_0 | f_N \rangle \\ 0 \\ \vdots \end{pmatrix}$$

Thus, for all $n \geq N+1$, the n^{th} coordinate of $\boldsymbol{\xi}$ is $\xi_n = z \xi_{n+1}$, i.e.

$$\xi_{N+1} = z^{n-N-1} \xi_n, \quad \forall n \geq N+1.$$

And since $\sum_{n \geq 0} |\xi_n|^2 < \infty$, then

$$\xi_n = 0, \quad \forall n \geq N+1.$$

As a result,

$$\langle (\text{Id} - z(M_{mn})_{m,n \geq N+1})^{-1} (X_n)_{n \geq N+1} | (Y_m)_{m \geq N+1} \rangle = 0, \quad (6.3.22)$$

and therefore

$$u_0(z) = \langle (\text{Id} - zM_{\leq N})^{-1} X_{\leq N} | Y_{\leq N} \rangle_{\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}},$$

where $M_{\leq N} := (M_{mn})_{0 \leq m, n \leq N}$, $X_{\leq N} := (X_n)_{0 \leq n \leq N}$ and $Y_{\leq N} := (Y_N)_{0 \leq n \leq N}$. Consequently, u_0 is a rational function

$$u_0(z) = \frac{P(z)}{\det(\text{Id} - zM_{\leq N})},$$

where $P(z) = Y_{\leq N}^* \cdot \text{Com}(\text{Id} - zM_{\leq N})^T \cdot X_{\leq N}$. Computing the numerator P and the denominator of u_0 via these finite matrices, we obtain that u_0 is of the form

$$u_0(z) = \frac{az^N + b}{1 - pz^N}, \quad a, b \in \mathbb{C}, \quad (6.3.23)$$

where $p = \langle f_0 | S^N f_0 \rangle$, $|p| < 1$.

If $p = 0$. Namely, if $\langle f_0 | S^N f_0 \rangle = 0$, then

$$S^N f_0 = \sum_{n \geq N} \langle S^N f_0 | f_n \rangle f_n = \langle S^N f_0 | f_N \rangle f_N,$$

since by (6.3.21), the set $\{(S^n f_0)_{n=0, \dots, N-1}, (S^n f_N)_{N \geq 0}\}$ is an orthonormal basis of $L_+^2(\mathbb{T})$. Thus, the two vectors f_N and $S^N f_0$ are collinear, leading to : for all $n \in \mathbb{N}_{\geq 0}$,

$$f_n // S^n f_0,$$

thanks to (6.3.21). Consequently, $\{S^n f_0, n \in \mathbb{N}_{\geq 0}\}$ is an orthonormal basis of $L_+^2(\mathbb{T})$, which means, the vector f_0 is necessarily collinear to 1. Besides, recall from (6.3.20),

$$\begin{aligned} u_0 &= \langle u_0 | f_0 \rangle f_0 + \langle u_0 | f_N \rangle f_N \\ &= \langle u_0 | f_0 \rangle f_0 + \langle u_0 | S^N f_0 \rangle S^N f_0, \\ &= \langle u_0 | 1 \rangle + \langle u_0 | e^{iNx} \rangle e^{iNx} \end{aligned}$$

and, as $p = 0$ i.e. $\langle f_0 | S^N f_0 \rangle = 0$, we have by the second identity of Lemma 6.2.3, either

$$\langle u_0 | f_0 \rangle = 0 \quad \text{or} \quad \langle u_0 | S^N f_0 \rangle = 0.$$

i.e.

$$\langle u_0 | 1 \rangle = 0 \quad \text{or} \quad \langle u_0 | e^{iNx} \rangle = 0.$$

Therefore, either $u_0(x)$ is a complex constant, or $u_0(x) = Ce^{iNx}$, with $C \in \mathbb{C}$, $N \in \mathbb{N}_{\geq 1}$. Taking, $u(t, x) = u_0(x - ct) = Ce^{iN(x-ct)}$, and substituting it in the defocusing Calogero–Sutherland DNLS equation (CS⁻), we infer, since the nonlinearity $D\Pi(|e^{iN(x-ct)}|^2)e^{iN(x-ct)}$ vanishes,

$$Nc e^{iN(x-ct)} - N^2 e^{iN(x-ct)} = 0,$$

and thus $c = N$. As a result, if $p = 0$ then the traveling waves $u(t, x) := u_0(x - ct)$ are

$$u(t, x) = C e^{iN(x-Nt)}, \quad C \in \mathbb{C}, \quad N \in \mathbb{N}_{\geq 0}.$$

Let us move, to the case where $p \neq 0$. The potential u_0 of (6.3.23) can be re-written as

$$u_0 = \alpha + \frac{\beta}{1 - pz^n}, \quad \alpha, \beta \in \mathbb{C}, \quad p, z \in \mathbb{D},$$

In order to find the relation between α, β and obtain the speed c , we substitute $u(t, z) := u_0(e^{-ict} z)$ into the defocusing Calogero–Sutherland DNLS equation (CS⁻). This equation can be rewritten as

$$i\partial_t u - (z\partial_z)^2 u - 2z\partial_z \Pi(|u|^2)u = 0, \quad (6.3.24)$$

after observing that $D = -i\partial_x$ can be expressed as $D \equiv z\partial_z$. Thus, starting from

$$u := u(t, z) = \alpha + \frac{\beta}{1 - p e^{-iNct} z^N},$$

and computing $i\partial_t u$ and $(z\partial_z)^2 u$, we find

$$\begin{aligned} i\partial_t u &= -c\beta N \left(\frac{1}{1 - p e^{-iNct} z^N} - \frac{1}{(1 - p e^{-iNct} z^N)^2} \right), \\ (z\partial_z)^2 u &= \beta N^2 \left(\frac{1}{1 - p e^{-iNct} z^N} - \frac{3}{(1 - p e^{-iNct} z^N)^2} + \frac{2}{(1 - p e^{-iNct} z^N)^3} \right). \end{aligned}$$

For the nonlinear part,

$$|u|^2 = |\alpha|^2 + \alpha\bar{\beta} + \frac{\alpha\bar{\beta} \bar{p} e^{iNct}}{z^N - \bar{p} e^{iNct}} + \frac{\alpha\bar{\beta}}{1 - p e^{-iNct} z^N} + \frac{|\beta|^2 z^N}{(1 - p e^{-iNct} z^N)(z^N - \bar{p} e^{iNct})}.$$

Recall that Π is an orthonormal projector into the Hardy space (in particular to a subspace of the holomorphic functions on \mathbb{D}). Thus, applying Π , it follows

$$\Pi(|u|^2) = |\alpha|^2 + \alpha\bar{\beta} + \frac{\alpha\bar{\beta}}{1 - p e^{-iNct} z^N} + \frac{|\beta|^2}{1 - |p|^2} \frac{1}{1 - p e^{-iNct} z^N}.$$

And hence,

$$z\partial_z \Pi(|u|^2) \cdot u = A \left(\frac{-\alpha}{1 - p e^{-iNct} z^N} + \frac{-\beta + \alpha}{(1 - p e^{-iNct} z^N)^2} + \frac{\beta}{(1 - p e^{-iNct} z^N)^3} \right),$$

where

$$A = N \left(\bar{\alpha}\beta + \frac{|\beta|^2}{1 - |p|^2} \right).$$

Substituting the expressions of $i\partial_t u$, $(z\partial_z)^2 u$ and $z\partial_z \Pi(|u|^2)u$ into (6.3.24), and comparing the terms $\frac{1}{(1 - p e^{-iNct} z^N)^n}$ for $n = 1, 2, 3$, we deduce

— With $n = 3$, $A = -N^2$. That is,

$$\bar{\alpha}\beta + \frac{|\beta|^2}{1 - |p|^2} = -N$$

— With $n = 2$ and $n = 1$,

$$c = -N \left(1 + \frac{2\alpha}{\beta} \right).$$

As a result, for $p \neq 0$,

$$u(t, z) := \alpha + \frac{\beta}{1 - p e^{-iNct} z^N}, \quad p \in \mathbb{D}^*, \theta \in \mathbb{T},$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N\left(1 + \frac{2\alpha}{\beta}\right)$, and $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$ satisfy

$$\alpha\beta + \frac{|\beta|^2}{1 - |p|^2} = -N. \quad (6.3.25)$$

Finally, observe by (6.3.25), the two complex constants (α, β) satisfy $\bar{\alpha}\beta \in \mathbb{R}$. Thus, by making a slight abuse of notation on α and β , we have obtained that the traveling waves of (CS⁻) with $p \neq 0$ are given by

$$u(t, z) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{-iNct} z^N} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T},$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N\left(1 + \frac{2\alpha}{\beta}\right)$, and $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ satisfy

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = -N.$$

□

6.3.3 The L^2 –norm and the speed

In this subsection, we analyze how the traveling waves of (CS⁻) behaves, by providing information regarding their L^2 –norm and their speed c . Recall that the set of traveling wave solutions of the defocusing Calogero–Sutherland DNLS equation are made up by the trivial solutions

$$\mathcal{G}_1 = \{C e^{iN(x-Nt)} \mid C \in \mathbb{C}, N \in \mathbb{N}_{\geq 0}\},$$

and by the set of functions

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, \quad (6.3.26)$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N\left(1 + \frac{2\alpha}{\beta}\right)$, and (α, β) are two real constants satisfying (6.3.18).

For $u \in \mathcal{G}_1$, it is easy to see that the L^2 –norm of the semi–trivial solution can be arbitrarily small or large in $[0, +\infty)$, and its speed c is given as $c = N \in \mathbb{N}_{\geq 0}$. The following proposition aims to provide those for the nontrivial traveling waves of (CS⁻).

Proposition 6.3.7 (L^2 norm of a non–trivial traveling wave and the speed).

(i) For any $r > 0$, there exists a non-trivial traveling waves $u(t, x) := u_0(x - ct)$ for (CS⁻) with

$$\|u_0\|_{L^2} = r.$$

In other words, the traveling waves of (CS⁻) can be arbitrarily small or large in $L^2_+(\mathbb{T})$.

(ii) Let u be a traveling wave for (CS⁻) of the form (6.3.26), then u propagates to the right with a speed $c > N$. In addition, when $\|u\|_{L^2} \rightarrow \infty$ then $c \rightarrow \infty$ and when $\|u\|_{L^2} \rightarrow 0$ then $c \rightarrow N$.

Remark 6.3.4 (Non-existence of stationary solution for (CS⁻)). Since for any traveling wave $u_0(x - ct)$ of the defocusing Calogero–Sutherland DNLS equation (CS⁻) we have $c \geq N$, where N is the numerator's degree of u_0 , then there is no stationary solution (i.e. $u(t, x) = u_0(x)$) for the (CS⁻)–equation. Another way to see this, is by observing that if $c = 0$, which occurs when $\alpha = -\frac{\beta}{2}$ according to Theorem 6.1.1, then we have by (6.3.18)

$$\frac{1 + |p|^2}{1 - |p|^2} \beta^2 = -N,$$

which is impossible as $p \in \mathbb{D}^*$.

Proof. (i) The L^2 –norm of the non-trivial traveling wave can be arbitrarily small or large. Let u be a traveling wave of the form (6.3.26)

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*, N \in \mathbb{N}_{\geq 1},$$

where $(\alpha, \beta) \in \mathbb{R}^2$ satisfies the identity (6.3.18). Recall that any function u in the Hardy space can be seen as an analytic function on the open unit disc \mathbb{D} , whose trace on the boundary $\partial\mathbb{D}$ is in L^2 . Hence,

$$\|u\|_{L^2}^2 = \int_{z \in \mathcal{C}(0,1)} |u(z)|^2 \frac{dz}{2\pi iz},$$

where

$$\begin{aligned} |u(z)|^2 &= \left(\alpha + \frac{\beta}{1 - p e^{-iNct} z^N} \right) \left(\alpha + \frac{\beta z^N}{z^N - \bar{p} e^{iNct}} \right) \\ &= \alpha^2 + \alpha\beta + \frac{\alpha\beta e^{iNct} \bar{p}}{z^N - \bar{p} e^{iNct}} + \frac{\alpha\beta}{1 - p e^{-iNct} z^N} + \frac{\beta^2 z^N}{(1 - p e^{-iNct} z^N)(z^N - \bar{p} e^{iNct})} \end{aligned}$$

Writing

$$\frac{\beta^2 z^N}{(1 - p e^{-iNct} z^N)(z^N - \bar{p} e^{iNct})} = \frac{\beta^2}{1 - |p|^2} \left(\frac{1}{1 - p e^{-iNct} z^N} + \frac{\bar{p} e^{iNct}}{z^N - \bar{p} e^{iNct}} \right),$$

we infer

$$|u(z)|^2 = \alpha^2 + \alpha\beta + \left(\alpha\beta + \frac{\beta^2}{1 - |p|^2} \right) \frac{1}{1 - p e^{-iNct} z^N} + \left(\alpha\beta + \frac{\beta^2}{1 - |p|^2} \right) \frac{e^{iNct} \bar{p}}{z^N - \bar{p} e^{iNct}}.$$

Therefore,

$$\|u\|_{L^2}^2 = \alpha^2 + \alpha\beta + \alpha\beta + \frac{\beta^2}{1 - |p|^2}, \quad (6.3.27)$$

since for $N \in \mathbb{N}_{\geq 1}$,

$$\left\langle 1 \left| \frac{1}{z^N - \bar{p} e^{iNct}} \right. \right\rangle = \int_{z \in \mathcal{C}(0,1)} \frac{z^N}{1 - p e^{-iNct} z^N} \frac{dz}{2\pi i z} = 0.$$

Consequently, by (6.3.18),⁵

$$\|u\|_{L^2}^2 = \alpha^2 + \alpha\beta - N. \quad (6.3.28)$$

In addition, since by (6.3.18)

$$\alpha = -\frac{N}{\beta} - \frac{\beta}{1 - |p|^2},$$

then,

$$\begin{aligned} \|u\|_{L^2}^2 &= \left(-\frac{N}{\beta} - \frac{\beta}{1 - |p|^2} \right)^2 + \left(-\frac{N}{\beta} - \frac{\beta}{1 - |p|^2} \right) \beta - N \\ &= \frac{|p|^2}{1 - |p|^2} \left(\frac{\beta^2}{1 - |p|^2} + 2N \right) + \frac{N^2}{\beta^2}. \end{aligned}$$

Observe that, $\|u\|_{L^2}^2$ is a continuous function of $|p|^2$ and β^2 . Moreover, by taking $\beta \rightarrow 0$ then $\|u\|_{L^2}^2 \rightarrow \infty$. And if we take $|p|^2 \rightarrow 0$ then

$$\|u\|_{L^2}^2 \underset{|p|^2 \rightarrow 0}{\sim} \frac{N^2}{\beta^2}$$

which can be arbitrary small when $\beta \gg 1$.

(ii) **Speed : $c > N$.** By Theorem 6.1.1, the speed of the traveling waves of the form (6.3.26) is given by $c = -N(1 + \frac{2\alpha}{\beta})$. Besides, recall from (6.3.18),

$$\frac{\alpha}{\beta} = -\frac{N}{\beta^2} - \frac{1}{1 - |p|^2}.$$

Substituting the latter identity in the expression of c , it follows

$$c = N \left(\frac{1 + |p|^2}{1 - |p|^2} + \frac{2N}{\beta^2} \right) > N. \quad (6.3.29)$$

It remains to prove that

5. As we shall see in Corollary 6.5.5 of Section 6.5, this corresponds to $\|u\|_{L^2}^2 = \lambda_N - N$ where $\lambda_N > N + \lambda_0 > N$.

- when $\|u\|_{L^2} \rightarrow +\infty$, we have $c \rightarrow +\infty$,
- and when $\|u\|_{L^2} \rightarrow 0$ then $c \rightarrow N$.

Indeed, observe that $\|u\|_{L^2}^2 \rightarrow \infty$ when $\beta^2 \rightarrow 0$ or $|p|^2 \rightarrow 1$, and in both cases

$$c \rightarrow \infty.$$

On the other side, $\|u\|_{L^2}^2$ is arbitrary small when $|p|^2 \rightarrow 0$ and β is big enough. Hence, by passing to the limit $|p|^2 \rightarrow 0$ in (6.3.29), we infer

$$c \underset{|p|^2 \rightarrow 0}{\sim} N \left(1 + \frac{2N}{\beta^2} \right),$$

which can arbitrary close to N as $\|u\|_{L^2}^2$ is arbitrary close to 0. \square

6.4 Traveling waves for the focusing (CS⁺)

6.4.1 Toward the characterization of the traveling waves for (CS⁺)

Recall that to characterize the traveling waves of the defocusing equation (CS⁻), a spectral analysis was initially conducted, followed by the derivation of explicit formulas. Here, we aim to footstep the same strategy. But before proceeding, we shall require some analogous lemmas to the defocusing case.

Lemma 6.4.1 (The analog of Lemma 6.3.2). *Let $u \in C_t H_+^2(\mathbb{T})_x$ solution of (CS⁺). Then, for all $n, p \in \mathbb{N}_{\geq 0}$,*

$$\begin{aligned} \langle 1 | g_n^t \rangle &= \langle 1 | f_n^{u_0} \rangle e^{-i\nu_n^2 t}, \\ \langle u | g_n^t \rangle &= \langle u_0 | f_n^{u_0} \rangle e^{-i\nu_n^2 t} \\ \langle Sg_p^t | u \rangle &= \langle Sf_p^{u_0} | u_0 \rangle e^{i(\nu_p+1)^2 t}, \\ \langle Sg_p^t | g_n^t \rangle &= \langle Sf_p^{u_0} | f_n^{u_0} \rangle e^{i((\nu_p+1)^2 - \nu_n^2) t}, \end{aligned} \tag{6.4.1}$$

where the (g_n^t) denotes the orthonormal basis of $L_+^2(\mathbb{T})$ solution to the Cauchy problem

$$\begin{cases} \partial_t g_n^t &= B_{u(t)} g_n^t, \\ g_n^t|_{t=0} &= f_n^{u_0} \end{cases}, \quad \forall n \in \mathbb{N}_{\geq 0},$$

and $(f_n^{u_0})$ is an orthonormal basis of $L_+^2(\mathbb{T})$ made up of the eigenfunctions of L_{u_0} and $B_{u(t)}$ is the skew-adjoint operator defined in (6.1.10).

Proof. Since the focusing Calogero–Sutherland DNLS equation can also be rewritten in terms of its Lax operators [Bad24a, Lemma 2.4]

$$\partial_t u = B_u u - iL_u^2 u,$$

then one can repeat exactly the same proof of Lemma 6.3.2 and obtain the same results. \square

Lemma 6.4.2 (The analog of Proposition 6.3.5). *Let $u := \tau_{ct}u_0$ be a traveling wave of (CS^+) such that the eigenvalue $\nu_n(u_0)$ is simple. Then, there exists $\theta_n(t) \in \mathbb{R}$ such that*

$$\tau_{ct}f_n^{u_0} = e^{i\theta_n(t)}g_n^t, \quad (6.4.2)$$

where the (g_n^t) denotes the orthonormal basis defined in the previous lemma.

Lemma 6.4.3 (The analog of Corollary 6.3.6 in the focusing case). *Let u_0 be a function such that the eigenvalues $(\nu_n(u_0))$ are simple. Then, for all $n, p \in \mathbb{N}_{\geq 0}$, $t \in \mathbb{R}$, we have*

1. *If $\langle 1 | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = -\nu_n^2 t.$$

2. *If $\langle u_0 | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = -\nu_n^2 t.$$

3. *If $\langle S f_p^{u_0} | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = ((\nu_p + 1)^2 - \nu_n^2)t - ct + \theta_p(t),$$

where $\theta_n(t)$ is the angle obtained in (6.4.2).

At this stage, we are equipped with the necessary tools to footstep the proof of the defocusing equation. However, it is important to emphasize two fundamental differences between the Lax operators L_u and \tilde{L}_u , which ultimately offer a considerably expanded set of traveling waves for (CS^+) in comparison to (CS^-) :

- The gap between the eigenvalues differs between the focusing and the defocusing case (Proposition 6.2.1).
- The fact that the eigenvalues λ_n of \tilde{L}_u are not zero for any $n \in \mathbb{N}_{\geq 1}$.

Indeed, in the defocusing case, since all the eigenvalues satisfy $\lambda_n > \lambda_{n-1} + \frac{1}{2}$ (Inequality (6.2.5)) and $\lambda_n \neq 0$ for all $n \in \mathbb{N}_{\geq 1}$, then we obtained in Lemma 6.2.5,

$$\mathcal{I}(u) = \emptyset, \quad \forall u \in H_+^2(\mathbb{T}),$$

where $\mathcal{I}(u)$ was defined in (6.2.12). As a consequence, we inferred that if $u(t, x) = u_0(x - ct)$ is a traveling wave of (CS^-) , then there exists at most one $N \in \mathbb{N}_{\geq 1}$ such that $\langle u_0 | f_n^{u_0} \rangle = 0$ for all $n \in \mathbb{N} \setminus \{N\}$. Now, for the focusing equation, recall we have previously observed in the second point of Remark 6.2.3, that $\mathcal{I}(u)$ is of finite cardinal for all $u \in H_+^2(\mathbb{T})$. In particular, for $u_0 \in H_+^2(\mathbb{T})$, we denote by m_1, \dots, m_n its elements

$$\mathcal{I}(u_0) = \{m_1, \dots, m_n\}.$$

Theorem 6.4.4 (Toward the Characterization of the traveling waves of (CS^+)). *The traveling waves $u_0(x - ct)$ of (CS^+) are either rational functions or the plane waves $u(t, x) = C e^{iN(x - Nt)}$. In addition, the potentials*

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1 - p e^{iN(x - ct)}} \right), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, N \in \mathbb{N}_{\geq 1}, \quad (6.4.3)$$

where $c = -N \left(1 + \frac{2\alpha}{\beta}\right)$, $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = N, \quad (6.4.4)$$

and the potentials

$$u(t, x) = e^{i\theta} e^{im(x-ct)} \left(\alpha + \frac{\beta}{1 - p e^{i(x-ct)}} \right) \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, m \in \mathbb{N}_{\geq 1}, \quad (6.4.5)$$

where $c = m$, $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = 1, \quad \beta(m - 1) = 2\alpha,$$

are parts of the set of the traveling waves of (CS⁺).

Proof.

Step 1. (Spectral characterization of the traveling waves of (CS⁺)). Let $u(t, x) := u_0(x - ct)$ be a traveling wave for (CS⁺). Our goal is to prove that there exists $N \in \mathbb{N}_{\geq 1}$ such that $\langle u_0 | f_n^{u_0} \rangle = 0$ for all $n \geq N$. Once more, in order to simplify the notation, we denote in the following f_n instead of $f_n^{u_0}$. Recall by the second point of Remark 6.2.3, $\mathcal{I}(u_0)$ is of finite cardinal, that is there exists $m_1 < \dots < m_j \in \mathbb{N}_{\geq 1}$ such that

$$\begin{cases} \langle S f_{n-1} | f_n \rangle = 0, & \forall n \in \{m_1, \dots, m_j\} \\ \langle S f_{n-1} | f_n \rangle \neq 0, & \forall n \in \mathbb{N} \setminus \{m_1, \dots, m_j\} \end{cases}.$$

Suppose that there exists an integer $\ell \gg 1$, $\ell > m_j$ such that $\langle u_0 | f_\ell \rangle \neq 0$. Otherwise, we already have what we claim to prove. Then,

- For all $n \geq \ell + 1$, the quantities $\langle S f_{n-1} | f_n \rangle \neq 0$.
- Since $\ell \gg 1$, then by inequality (6.2.7), the eigenvalues $(\nu_n)_{n \geq \ell+1}$ are simple. This implies that Lemma 6.4.2 holds for $n \geq \ell + 1$.

Therefore, using the third point of Lemma 6.4.3, we obtain, for all $n \geq \ell + 1$,

$$\begin{aligned} \theta_n(t) &= -\nu_n^2 t + (\nu_{n-1} + 1)^2 t - ct + \theta_{n-1}(t) \\ &= -\nu_n^2 t + (\nu_{n-1} + 1)^2 t - \nu_{n-1}^2 t + (\nu_{n-2} + 1)^2 t - 2ct + \theta_{n-2}(t) \\ &= \dots \\ &= -\nu_n^2 t + (n - \ell)t + \nu_\ell^2 t - (n - \ell)ct + 2t \sum_{k=\ell}^{n-1} \nu_k + \theta_\ell(t), \end{aligned}$$

where $\theta_n(t)$ is the angle obtained in Lemma 6.4.2, and $\theta_\ell(t) = -\nu_\ell^2 t$ thanks to the second point of Lemma 6.4.3. Hence, for all $n \geq \ell + 1$,

$$\theta_n(t) = -\nu_n^2 t + (n - \ell)t - (n - \ell)ct + 2t \sum_{k=\ell}^{n-1} \nu_k, \quad (6.4.6)$$

As a consequence, there exists at most one integer $N \geq \ell$ such that

$$\langle u_0 | f_N \rangle \neq 0.$$

Indeed, suppose for the sake of contradiction that there exist $n_2 > n_1 > \ell$, such that $\langle u_0 | f_{n_1} \rangle \neq 0$ and $\langle u_0 | f_{n_2} \rangle \neq 0$. Then, combining the second point of Lemma 6.4.3, and equation (6.4.6), we obtain

$$\begin{cases} c = 1 + \frac{2}{n_1 - \ell} \sum_{k=\ell}^{n_1-1} \nu_k. \\ c = 1 + \frac{2}{n_2 - \ell} \sum_{k=\ell}^{n_2-1} \nu_k. \end{cases}$$

Hence,

$$(n_2 - \ell) \sum_{k=\ell}^{n_1-1} \nu_k = (n_1 - \ell) \sum_{k=\ell}^{n_2-1} \nu_k,$$

or

$$(n_2 - n_1) \sum_{k=\ell}^{n_1-1} \nu_k = (n_1 - \ell) \sum_{k=n_1}^{n_2-1} \nu_k.$$

As a result,

$$(n_1 - \ell)(n_2 - n_1) \nu_{n_1-1} \geq (n_1 - \ell)(n_2 - n_1) \nu_{n_1},$$

leading to a contradiction, since for $k \geq \ell$, we have $\nu_{k+1} > \nu_k$. Therefore, there exists $N \in \mathbb{N}_{\geq 1}$ such that $\langle u_0 | f_n \rangle = 0$ for all $n \geq N$.

Step 2. (They are rational functions or potentials in \mathcal{G}_1) Since $\langle u_0 | f_n \rangle = 0$ for all $n \geq N$, it follows by Corollary 6.2.6 that $\nu_n = \nu_{n-1} + 1$ for all $n \geq N_2$. Note that the potentials satisfying $\nu_n = \nu_{n-1} + 1$ for all $n \geq N_2$, are referred to be “finite gap potentials” for (CS^+) , and are studied deeply in Section 6.5. In particular, Theorem 6.1.3 provides a full characterization of these potentials in the state space. They are either $u(x) = C e^{iNx}$, $C \in \mathbb{R}^*$, $N \in \mathbb{N}_{\geq 0}$, or rational functions

$$u(x) = e^{im_0 x} \prod_{j=1}^r \left(\frac{e^{ix} - \bar{p}_j}{1 - p_j e^{ix}} \right)^{m_j-1} \left(\alpha + \sum_{j=1}^r \frac{\beta_j}{1 - p_j e^{ix}} \right), \quad p_j \in \mathbb{D}^*, p_k \neq p_j, k \neq j, \quad (6.4.7)$$

where, for $N \in \mathbb{N}_{\geq 1}$, $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^r m_j = N$, and $(\alpha, \beta_1, \dots, \beta_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfy for all $j = 1, \dots, r$,

$$\bar{\alpha} \beta_j + \sum_{k=1}^r \frac{\alpha_j \bar{\alpha}_k}{1 - p_j \bar{p}_k} = m_j.$$

It remains to verify that (6.4.3) and (6.4.5) are traveling waves for (CS^+) . To do so, one can simply substitute them into the equation (CS^+) –equation and check that they satisfy the equation. □

Remark 6.4.1. As was observed in the previous proof all the traveling waves $u_0(x-ct)$ of (CS⁺) are either $u(t, x) = C e^{iN(x-Nt)}$ or the rational functions $u(t, x) := u_0(x-ct)$ where u_0 is defined in (6.4.7) and the constants α, β_j and c can be described by substituting u in the (CS⁺)–equation.

6.4.2 The L^2 –norm and the speed

In this subsection, we analyze the L^2 –norm and the speed of the traveling waves of (CS⁺) and establish the existence of stationary solutions for the focusing Calogero–Sutherland DNLS equation (CS⁺).

Proposition 6.4.5.

(i) For any $r > 0$, there exists a non-trivial traveling wave $u(t, x) := u_0(x-ct)$ for (CS⁺) with

$$\|u_0\|_{L^2} = r.$$

(ii) Let u be a traveling wave for (CS⁺) of the form (6.4.3), then u can propagate to the right or to the left with any speed $c \in \mathbb{R}$.

Remark 6.4.2. Contrary to the defocusing case, we do not necessarily have that the traveling wave propagates with a speed $c \rightarrow \infty$ when $\|u\|_{L^2}^2 \rightarrow \infty$. For instance, take

$$u(t, x) := \frac{N}{\beta} - \frac{\beta}{1-|p|^2} + \frac{\beta}{1-p e^{iN(x-ct)}}, \quad \beta^2 := \frac{2N}{\frac{1+|p|^2}{1-|p|^2} - \frac{c}{N}}.$$

The proof of this statement will be achieved in the end of the following proof.

Proof of proposition 6.4.5. The L^2 –norm . Let u be a traveling wave for (CS⁺) of the form (6.4.3),

$$u(t, x) := e^{i\theta} \left(\alpha + \frac{\beta}{1-p e^{iN(x-ct)}} \right), \quad p \in \mathbb{D}^*.$$

Our goal is to prove that the L^2 norm of these traveling waves can be arbitrary small or large. The computation of its L^2 –norm has been performed in the proof of Proposition 6.3.7. Therefore, by identity (6.3.27),

$$\|u\|_{L^2}^2 = \alpha^2 + \alpha\beta + \alpha\beta + \frac{\beta^2}{1-|p|^2}, \quad (6.4.8)$$

where the two reals (α, β) satisfies condition (6.4.4)

$$\alpha\beta + \frac{\beta^2}{1-|p|^2} = N.$$

That is,

$$\|u\|_{L^2}^2 = \frac{|p|^2}{1-|p|^2} \left(\frac{\beta^2}{1-|p|^2} - 2N \right) + \frac{N^2}{\beta^2}. \quad (6.4.9)$$

Like for the defocusing case, $\|u\|_{L^2}^2$ is a continuous function of β^2 and $|p|^2$. And by taking $\beta \rightarrow 0$ one has $\|u\|_{L^2}^2 \rightarrow \infty$. In addition, if $|p| \rightarrow 0$ then

$$\|u\|_{L^2}^2 \underset{|p|^2 \rightarrow 0}{\sim} \frac{N^2}{\beta^2}.$$

Hence, it is sufficient to take β big enough so that $\|u\|_{L^2}$ can be arbitrary small.

Speed : $c \in \mathbb{R}$. By Theorem 6.4.4, there exists traveling waves for (CS⁺) that propagates with a speed $c = -N \left(1 + \frac{2\alpha}{\beta}\right)$ where $N \in \mathbb{N}_{\geq 1}$ and the two reals (α, β) satisfy

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = N, \quad 0 < |p| < 1.$$

That is

$$\begin{aligned} c &= -N \left(1 + \frac{2N}{\beta^2} - \frac{2}{1 - |p|^2}\right) \\ &= -N \left(-\frac{1 + |p|^2}{1 - |p|^2} + \frac{2N}{\beta^2}\right). \end{aligned} \quad (6.4.10)$$

By taking, for example $\beta = |p|$, we infer

$$c = N \frac{|p|^4 + (2N + 1)|p|^2 - 2N}{|p|^2(1 - |p|^2)}$$

Assume that $N = 1$, and by taking $x = |p|^2 \in (0, 1)$, we infer that the continuous function

$$c(x) := \frac{x^2 + 3x - 2}{x(1 - x)},$$

satisfies $\inf_{x \in (0, 1)} c(x) = -\infty$ and $\sup_{x \in (0, 1)} c(x) = +\infty$.

Proof of Remark 6.4.2. For a traveling wave u of the form (6.4.3),

$$u(t, x) := \alpha + \frac{\beta}{1 - p e^{iN(x-ct)}}, \quad \alpha\beta + \frac{\beta^2}{1 - |p|^2} = N,$$

where $N \in \mathbb{N}_{\geq 1}$, one has by (6.4.10), that u propagates with a speed

$$c = -N \left(-\frac{1 + |p|^2}{1 - |p|^2} + \frac{2N}{\beta^2}\right).$$

Thus, for any $N \in \mathbb{N}_{\geq 1}$, let

$$\beta := \sqrt{\frac{2N}{\frac{1 + |p|^2}{1 - |p|^2} - \frac{\lambda}{N}}}, \quad p \in \mathbb{D},$$

where λ is a parameter in \mathbb{R} , and with $|p|^2$ big enough so that β is well-defined. Hence, one computes

$$c = -N \left(-\frac{1+|p|^2}{1-|p|^2} + \frac{2N}{\frac{1+|p|^2}{1-|p|^2} - \frac{\lambda}{N}} \right) = \lambda \in \mathbb{R}.$$

That is, u can propagate with any speed in \mathbb{R} , regardless of the valued attained by the L^2 -norm of u . \square

Corollary 6.4.6. *The potentials*

$$u(t, x) := e^{i\theta} \sqrt{\frac{N(1-|p|^2)}{2(1+|p|^2)}} \left(1 - \frac{2}{1 - p e^{iNx}} \right), \quad p \in \mathbb{D}^*, N \in \mathbb{N}_{\geq 1}, \theta \in \mathbb{T},$$

are stationary solutions for (CS^+) . Conversely, the defocusing (CS^-) equation does not exhibit stationary wave solutions except the complex constant functions.

Proof. Through a straightforward calculation, one can easily check that the obtained waves satisfy the (CS^+) -equation. On the other side, for the defocusing equation, we already established via Remark 6.3.3 or the second point of Proposition 6.3.7, the non-existence of stationary waves $u(t, x) = u_0(x)$ for (CS^-) . \square

6.5 The finite gap potentials

This section aims to examine the finite gap potentials associated with the Calogero–Sutherland DNLS equation (CS) in both the focusing and defocusing cases. Remarkably, these potentials manifest as rational functions containing the traveling and solitary waves of (CS) .

In the following, we adopt a slight abuse of notation, where for all $n \in \mathbb{N}_{\geq 1}$, we denote

$$\gamma_n(u) := \nu_n - \nu_{n-1} - 1, \quad (6.5.1)$$

the *gap* between the consecutive eigenvalues in the focusing context, and

$$\gamma_n(u) := \lambda_n - \lambda_{n-1} - 1,$$

as the gap in the defocusing context. At this point, several observations can be made. First, recall that in the defocusing case, the (λ_n) satisfies inequality (6.2.5), and thus, for all $n \in \mathbb{N}_{\geq 1}$, $\gamma_n(u)$ is non-negative in the defocusing case. Second, notice that since the eigenvalues (ν_n) and (λ_n) of the Lax operators L_u and \tilde{L}_u are invariant by the evolution, then for all $n \in \mathbb{N}_{\geq 1}$,

$$\gamma_n(u(t)) = \gamma_n(u_0), \quad \forall t.$$

Definition 6.5.1 (Finite gap potential). A function $u \in L^2_+(\mathbb{T})$ is said to be a finite gap potential of (CS) if there exists $m \in \mathbb{N}_{\geq 1}$ such that

$$\gamma_n(u) = 0, \quad \forall n \geq m, \quad (6.5.2)$$

where γ_n is defined in (6.5.1).

Recall that any function in the Hardy space $L^2_+(\mathbb{T})$ can be seen as a holomorphic function on the unit disc \mathbb{D} whose trace on the boundary $\partial\mathbb{D}$ is in L^2 . Hence, in what follows, we denote by \mathcal{B}_N the set of finite Blaschke products of degree N :

$$\psi(x) = e^{i\theta} \prod_{k=1}^N \frac{e^{ix} - \bar{p}_k}{1 - p_k e^{ix}}, \quad \theta \in \mathbb{R}, p_k \in \mathbb{D},$$

which can be identified as the set of functions

$$\psi(z) = e^{i\theta} \frac{z^N \bar{Q}(\frac{1}{z})}{Q(z)}, \quad z \in \bar{\mathbb{D}} := \{|z| \leq 1\}, \theta \in \mathbb{R}.$$

where

$$Q(z) := \prod_{j=1}^N (1 - p_j z), \quad p_j \in \mathbb{D}.$$

In other words, $z^N \bar{Q}(\frac{1}{z})$ is a Schur polynomial⁶ of degree N .

Remark 6.5.1. By convention, we suppose that a finite Blaschke product of degree 0 is a constant in \mathbb{C} .

Proposition 6.5.2. *Let u be a finite gap potential of (CS⁺). There exist $(\nu, \psi) \in \mathbb{R} \times \mathcal{B}_n$, $n \in \mathbb{N}_{\geq 0}$ such that*

$$L_u S^k \psi = (\nu + k) S^k \psi, \quad \forall k \in \mathbb{N}_{\geq 0}. \quad (6.5.3)$$

In addition, the same goes for the defocusing Calogero–Sutherland DNLS equation (CS[−]).

Proof. Let u be a finite gap potential, that is $\nu_n = \nu_{n-1} + 1$ for all $n \geq m$. We denote by n_0 the eventual indices where ν_{n_0} may vanish. Then, by Proposition 6.2.4,

$$S f_{n-1} // f_n, \quad \forall n \geq N := \max\{m, n_0\} + 2,$$

as the eigenvalues (ν_n) are simple for $n \geq m + 1$. Therefore, letting $\psi := f_{N-1}$, we have

$$L_u S^k \psi = (\nu_{N-1} + k) S^k \psi \quad \forall k \in \mathbb{N}_{\geq 0}. \quad (6.5.4)$$

6. A polynomial $q(z) = \sum_{k=1}^N a_k z^k$ is called a Schur polynomial if all its roots are in the open unit disc \mathbb{D} .

It remains to prove that ψ is a finite Blaschke product. Observe that, by taking the inner product of both sides of the previous identity with ψ ,

$$\langle S^k \psi | \psi \rangle = 0, \quad \forall k \in \mathbb{N}_{\geq 1}. \quad (6.5.5)$$

That is

$$\langle |\psi|^2 | e^{ikx} \rangle = 0, \quad \forall k \in \mathbb{N}_{\geq 1}.$$

or

$$\langle |\psi|^2 | e^{ikx} \rangle = 0, \quad \forall k \in \mathbb{Z} \setminus \{0\},$$

as $|\psi|^2$ is real value. Consequently, $|\psi|^2$ is a real constant, which can be supposed equal to 1 since we have assumed that the eigenfunctions of L_u constitute an orthonormal basis of $L^2_+(\mathbb{T})$. Thus, $|\psi| = 1$ on \mathbb{T} . In order to conclude, we need the following classical lemma [Bur12][Exercise 6.12] which we prove for the convenience of the reader.

Lemma 6.5.3. *Let ϕ be an analytic function on the open unit ball that extends continuously to an inner function⁷ on the closed unit disc. Then $\phi \in \mathcal{B}_n$.*

Proof. Given a holomorphic function ϕ on the open unit ball that extends continuously to the unit circle while satisfying $|\phi| = 1 > 0$ on \mathbb{T} , we know that its zeros are finite, isolated and all localized inside the open unit disk \mathbb{D} . We denote them by $\overline{p}_1, \dots, \overline{p}_n$. Hence, ϕ can be factorized as

$$\phi(z) = v(z) \cdot \prod_{k=1}^n \frac{z - \overline{p}_k}{1 - p_k z},$$

where v is a holomorphic function without zeros on \mathbb{D} . Therefore, $1/v$ is a holomorphic function on \mathbb{D} , which continuously extends to the unit circle while satisfying $|1/v| = 1$ on \mathbb{T} . Thus, by the maximum principle, we infer that $|1/v| \leq 1$ on \mathbb{D} . Using the same argument on v instead of $1/v$, we deduce that $|v| \leq 1$ on the unit disc. As a consequence, $|v| = 1$ on the close unit disc $\{|z| \leq 1\}$ and so

$$\phi(z) = e^{i\theta} \cdot \prod_{k=1}^n \frac{z - \overline{p}_k}{1 - p_k z}, \quad \theta \in \mathbb{R}.$$

□

Coming back to the proof of Proposition 6.5.2, we denote $\underline{\psi}$ the function obtained by the isometric isomorphism map

$$\underline{\psi}(z) = \sum_{k \geq 0} \widehat{\psi}(k) z^k, \quad z \in \mathbb{D} \quad \longmapsto \quad \psi(x) := \sum_{k \geq 0} \widehat{\psi}(k) e^{ikx}, \quad x \in \mathbb{T},$$

7. A bounded analytic function ψ on \mathbb{D} is said to be *inner* if $|\psi(e^{ix})| = 1$ for almost every $x \in \mathbb{T}$. Note that a Blaschke product is a rational inner function.

In particular, since $\psi \in L_+^2(\mathbb{T})$ then $\underline{\psi} \in \mathbb{H}_2(\mathbb{D})$, where

$$\mathbb{H}_2(\mathbb{D}) := \left\{ u \in \text{Hol}(\mathbb{D}); \sup_{0 \leq r < 1} \int_0^{2\pi} |u(r e^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\},$$

Hence, by [CHA08, Theorem 4.5.3],

$$\underline{\psi}(r e^{ix}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(x - \theta) \psi(e^{i\theta}) dt, \quad 0 \leq r < 1,$$

where P_r denotes the Poisson Kernel

$$P_r(x - \theta) = \frac{1 - r^2}{1 - 2r \cos(x - \theta) + r^2}.$$

Note that the function $\psi \in \text{Dom}(L_u) := H_+^1(\mathbb{T})$ is continuous on \mathbb{T} as

$$D\psi = L_u\psi + u\Pi(\bar{u}\psi) \in L_+^1(\mathbb{T}). \quad (6.5.6)$$

Therefore, the Poisson Theorem [Gir04, Théorème 30] implies that the holomorphic function $\underline{\psi}(r e^{ix})$ extends continuously to $\psi(e^{ix})$ as $r \rightarrow 1$. In addition, recall that $|\psi| = 1$ on \mathbb{T} . Thus, applying the previous lemma, we infer that $\underline{\psi}$ is a Blaschke product and so is ψ . □

At this stage, we aim to characterize the finite gap potentials of (CS). To this end, we regroup them according to the following procedure : for any finite gap potential u of (CS), we denote by $\mathcal{N}(u)$ the non-negative integer

$$\mathcal{N}(u) := \min \{ n \in \mathbb{N}_{\geq 0} \mid \exists \psi \in \mathcal{B}_n, L_u S^k \psi = (\nu + k) S^k \psi, \forall k \geq 0 \}, \quad (6.5.7)$$

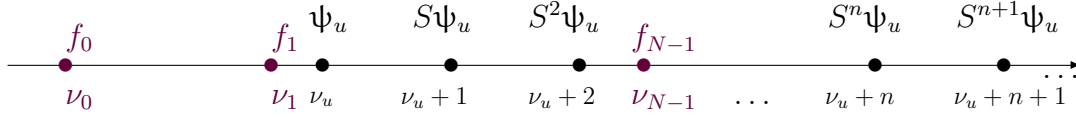
and we define, for $N \in \mathbb{N}_{\geq 0}$, the set

$$\mathcal{U}_N := \{ u \text{ finite gap potential}, \mathcal{N}(u) = N \}.$$

This means that for any $u \in \mathcal{U}_N$, there exists a finite Blaschke product ψ_u of minimal degree N , satisfying

$$L_u S^k \psi_u = (\nu_u + k) S^k \psi_u, \quad \forall k \in \mathbb{N}_{\geq 0}, \quad (6.5.8)$$

where ν_u is the corresponding eigenvalue of ψ_u . That is $\{ S^k \psi_u \mid k \in \mathbb{N}_{\geq 0} \}$ are parts of the orthonormal basis of $L_+^2(\mathbb{T})$. Besides, observe that, since $\deg \psi_u = N$, then there exists N eigenfunctions f_0, \dots, f_{N-1} of L_u that generate the model space $(\psi_u L_+^2)^\perp$ which is of dimension N [GMR16, Corollary 5.18]. We denote ν_0, \dots, ν_{N-1} the associated eigenvalues. Note that the latter N eigenvalues are not necessarily smaller than ν_u . We summarize this discussion by the following diagram. For any $u \in \mathcal{U}_N$,



Of course, the same goes for the defocusing equation with \tilde{L}_u instead of L_u , up to the fact that the remaining N eigenvalues ν_0, \dots, ν_{N-1} are necessarily smaller than ν_u , since the eigenvalues of \tilde{L}_u satisfy the property (6.2.5). Besides, note that by taking the minimum in (6.5.7) we guarantee that :

1. If $u \in \{v \text{ finite gap potential}, L_v S^k \psi = (\nu + k) S^k \psi, \psi \in \mathcal{B}_N\}$, then $u \notin \{v \text{ finite gap potential}, L_v S^k \psi = (\nu + k) S^k \psi, \psi \in \mathcal{B}_{N-1}\}$.
2. The set \mathcal{U}_N is invariant under the evolution of (CS). (See Proposition 6.5.6).

The following theorem aims to characterize the finite gap potentials of the Calogero–Sutherland DNLS (CS) in the state space.

Theorem 6.5.4. *Let $N \in \mathbb{N}_{\geq 1}$. A potential u is in \mathcal{U}_N if and only if $u(x) = C e^{iNx}$, $C \in \mathbb{C}^*$, or u is a rational function*

$$u(x) = e^{im_0 x} \prod_{j=1}^r \left(\frac{e^{ix} - \bar{p}_j}{1 - p_j e^{ix}} \right)^{m_j-1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j e^{ix}} \right), \quad p_j \in \mathbb{D}^*, p_k \neq p_j, k \neq j, \quad (6.5.9)$$

where $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^r m_j = N$, and $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfy for all $j = 1, \dots, r$,

(i) In the focusing case,

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = m_j, \quad (6.5.10)$$

(ii) In the defocusing case,

$$\bar{a} c_j + \sum_{k=1}^r \frac{c_j \bar{c}_k}{1 - p_j \bar{p}_k} = -m_j, \quad (6.5.11)$$

with $a \neq 0$ if $m_0 \in \{1, \dots, N-1\}$. Besides, if $N = 0$, then u is a complex constant function.

Remark 6.5.2. As we shall see in Step 4 of the proof of Theorem 6.5.4, if $u \in \mathcal{U}_N$, then the eigenvalue of L_u associated

— with the Blaschke product $\psi_u = e^{i\theta} e^{iNx}$ if $u = C e^{iNx}$, is given by

- (i) $\nu_u = N - C^2$ in the focusing case.
- (ii) $\lambda_u = N + C^2$ in the defocusing case.

— with the Blaschke product

$$\psi_u = e^{i\theta} e^{im_0 x} \prod_{j=1}^r \left(\frac{e^{ix} - \bar{p}_j}{1 - p_j e^{ix}} \right)^{m_j}, \quad \theta \in \mathbb{R}, p_j \neq p_k, j \neq k,$$

if u is the rational function (6.5.9), is given by

- (i) $\nu_u = m_0 - |a|^2 - \sum_{j=1}^r a \bar{c}_j$ in the focusing case.
- (ii) $\lambda_u = m_0 + |a|^2 + \sum_{j=1}^r a \bar{c}_j$ in the defocusing case.

In order to establish this theorem, we recall a specific case of formula (6.3.15).

Remark 6.5.3. Let (f_n) be an orthonormal basis of $L_+^2(\mathbb{T})$. For any $n \geq 0$,

$$f_n(z) = \langle (\text{Id} - zM)^{-1} \mathbf{1}_n \mid Y \rangle_{\ell^2}, \quad z \in \mathbb{D},$$

where $\mathbf{1}_n$ and Y are the column vectors

$$\mathbf{1}_n := (\delta_{pn})_{p \geq 0}, \quad Y := (\langle 1 \mid f_m \rangle)_{m \geq 0}$$

and M is the matrix representation of the operator S^* in the (f_m) -basis

$$M = (M_{mp})_{mp \geq 0}, \quad M_{mp} = \langle f_p \mid S f_m \rangle.$$

In what follows, we denote by $\mathbb{C}_{\leq N}[X]$ the set of polynomials P in complex coefficients with degree at most N and by $\mathbb{C}_N[X]$ those of degree N .

Proof of Theorem 6.5.4. We present the proof for the focusing case. Note that the same arguments can be performed to deduce the result in the defocusing case. The key ingredient is the inversion spectral formula (6.3.15)

$$u(z) = \langle (\text{Id} - zM)^{-1} X \mid Y \rangle, \quad (6.5.12)$$

where X , Y and M are defined in (6.3.16). The proof will be split in 5 steps.

Let $u \in \mathcal{U}_N$, then by Proposition 6.5.2 there exists a finite Blaschke product

$$\psi_u(z) = e^{i\theta} \frac{e^{iNx} \bar{Q}(1/z)}{Q(z)}; \quad Q(z) := \prod_{j=1}^N (1 - p_j z), \quad p_j \in \mathbb{D}, \quad (6.5.13)$$

such that (6.5.3) is satisfied

$$L_u S^k \psi_u = (\nu_u + k) S^k \psi_u, \quad \forall k \in \mathbb{N}_{\geq 0}.$$

Step 1. As a first step, we prove that any $u \in \mathcal{U}_N$ must be a rational function

$$u(z) = \frac{P(z)}{Q(z)}, \quad P \in \mathbb{C}_{\leq N}[z],$$

where $Q(z)$ is the same denominator of the Blaschke product $\Psi_u(z)$ associated with $u \in \mathcal{U}_N$. Indeed, first observe that combining (6.5.3) with the commutator identity (6.2.2) leads to

$$\langle u | S^k \Psi_u \rangle = 0, \quad \forall k \geq 1. \quad (6.5.14)$$

Hence, we infer thanks to Lemma 6.2.3, that the infinite matrices M , X , and Y of (6.5.12) written in the basis $(f_k)_{k=0}^{N-1} \cup (S^k \Psi_u)_{k \geq 0}$ are of the form

$$M = \left(\begin{array}{cccc|cc} \langle f_0 | S f_0 \rangle & \cdots & \langle f_{n-1} | S f_0 \rangle & \langle \Psi_u | S f_0 \rangle & & \\ \vdots & & \vdots & \vdots & & \\ \langle f_0 | S f_{N-1} \rangle & \cdots & \langle f_{N-1} | S f_{N-1} \rangle & \langle \Psi_u | S f_{N-1} \rangle & & \\ \hline 0 & \cdots & \cdots & 0 & 1 & \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ \vdots & & & & \vdots & \ddots \end{array} \right), \quad n \in \mathbb{N}$$

$$X = \begin{pmatrix} \langle u | f_0 \rangle \\ \vdots \\ \langle u | f_{N-1} \rangle \\ \langle u | \Psi_u \rangle \\ 0 \\ \vdots \end{pmatrix}, \quad Y = \begin{pmatrix} \langle 1 | f_0 \rangle \\ \vdots \\ \langle 1 | f_{N-1} \rangle \\ \langle 1 | \Psi_u \rangle \\ 0 \\ \vdots \end{pmatrix}.$$

Therefore, following the same procedure presented in the proof of Theorem 6.1.1, one can observe that the infinite matrices M , X and Y can be reduced to finite matrices that involve only the first $N + 1$ coordinates of each of these matrices. That is,

$$u(z) = \langle (\text{Id} - z M_{\leq N})^{-1} X_{\leq N} | Y_{\leq N} \rangle_{\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}},$$

where $M_{\leq N} = (M_{mn})_{0 \leq m, n \leq N}$, $X_{\leq N} = (X_n)_{0 \leq n \leq N}$ and $Y_{\leq N} = (Y_n)_{0 \leq n \leq N}$. As a consequence, u is a rational function

$$u(z) = \frac{P(z)}{\det(\text{Id} - z M_{\leq N})}, \quad P \in \mathbb{C}_{\leq N}[z].$$

Note that $\det(\text{Id} - z M_{\leq N})$ coincides with the denominator of the eigenfunction

$$\psi_u = e^{i\theta} \frac{z^N \bar{Q}(1/z)}{Q(z)},$$

since by Remark 6.5.3, ψ_u is also expressed via the inversion spectral formula

$$\psi_u(z) = \langle (\text{Id} - z M_{\leq N})^{-1} \mathbf{1}_N | Y_{\leq N} \rangle_{\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}} = \frac{\langle \text{Com}(\text{Id} - z M_{\leq N})^T \mathbf{1}_N | Y_{\leq N} \rangle}{\det(\text{Id} - z M_{\leq N})},$$

and hence $\det(\text{Id} - z M_{\leq N}) = Q(z)$. Thus,

$$u(z) = \frac{P(z)}{Q(z)}, \quad P \in \mathbb{C}_{\leq N}[z], \quad Q(z) = \prod_{k=1}^N (1 - p_k z), \quad p_k \in \mathbb{D}. \quad (6.5.15)$$

Step 2. In this step, we prove that if $u \in \mathcal{U}_N$ then

$$|u|^2 = z\partial_z \log \psi_u - \nu_u, \quad \text{on } \partial\mathbb{D}. \quad (6.5.16)$$

Indeed, recall that $L_u \psi_u = \nu_u \psi_u$. Then by definition of $L_u = z\partial_z - T_u T_{\bar{u}}$,

$$z\partial_z \psi_u - u\Pi(\bar{u}\psi_u) = \nu_u \psi_u. \quad (6.5.17)$$

On $\partial\mathbb{D}$,

$$\bar{u}\psi_u(z) = e^{i\theta} \frac{z^N \bar{P}(1/z)}{z^N \bar{Q}(1/z)} \cdot \frac{z^N \bar{Q}(1/z)}{Q(z)} = e^{i\theta} \frac{z^N \bar{P}(1/z)}{Q(z)},$$

extends as a holomorphic function on \mathbb{D} . Hence, $\Pi(\bar{u}\psi_u) = \bar{u}\psi_u$, and so identity (6.5.17) can be read as

$$\frac{z\partial_z \psi_u}{\psi_u} = |u|^2 + \nu_u,$$

implying that identity (6.5.16) holds.

Step 3. In this step, we prove that the rational function u obtained in Step 1 can be rewritten either as $u(z) = Cz^N$, $C \in \mathbb{C}^*$, or

$$u(z) = z^{m_0} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z} \right)^{m_j-1} \frac{q(z)}{\prod_{j=1}^r (1 - p_j z)}, \quad p_j \in \mathbb{D}^*, \quad p_k \neq p_j, \quad k \neq j,$$

where $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$,

$$m_0 + \sum_{j=1}^r m_j = N,$$

and such that $\deg(q) = r$ if $m_0 \neq 0$. Indeed, we write (6.5.13) as $\psi_u = e^{i\theta} z^N$ (if all the p_k in (6.5.13) vanish), or

$$\psi_u = e^{i\theta} z^{m_0} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z} \right)^{m_j}, \quad p_j \in \mathbb{D}^*, \quad p_k \neq p_j, \quad k \neq j, \quad \theta \in \mathbb{R} \quad (6.5.18)$$

where $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^r m_j = N$. As a first point, we prove when $m_0 \geq 1$, then the numerator P of u can be factorized as $P(z) = z^{m_0} P_{N-m_0}(z)$ with $P_{N-m_0} \in \mathbb{C}_{N-m_0}[z]$. Let $m_0 \geq 1$ then $\langle u | \psi_u \rangle \neq 0$, because otherwise there exists a Balschke product $\phi_u = S^* \psi_u$ of degree $N-1$, such that by the commutator identity (6.2.2),

$$L_u S^k \phi_u = (\nu_k - 1 + k) S^k \phi_u, \quad k \geq 0,$$

meaning that $u \in \mathcal{U}_{N-1}$, which is a contradiction with the fact that $u \in \mathcal{U}_N$. Hence, $\langle u | \psi_u \rangle \neq 0$. This leads to

- (i) The numerator P of $u(z)$ must be degree N .

(ii) $\langle 1 | u \rangle = 0$.

Indeed, for (i), it is sufficient to note that

$$0 \neq \langle \Psi_u | u \rangle = \int_{z \in \mathcal{C}(0,1)} e^{i\theta} \frac{z^N \bar{Q}(1/z)}{Q(z)} \frac{\bar{P}(1/z)}{\bar{Q}(1/z)} \frac{dz}{2\pi i z} = e^{i\theta} z^N \bar{P}(1/z)|_{z=0}$$

For (ii), observe by Lemma 6.2.3,

$$\langle 1 | u \rangle \underbrace{\langle u | \Psi_u \rangle}_{\neq 0} = -\nu_u \langle 1 | \Psi_u \rangle ,$$

where the right-hand side vanishes since $\Psi_u = S\phi$, $\phi \in L_+^2(\mathbb{T})$ for $m_0 \geq 1$. Therefore, if $m_0 = 1$, then by (i) and (ii),

$$u(z) = \frac{z P_{N-1}(z)}{\prod_{j=1}^r (1 - p_j z)^{m_j}}, \quad P_{N-1} \in \mathbb{C}_{N-1}[z], \quad p_k \in \mathbb{D}^*, \quad p_j \neq p_k .$$

Now, if $m_0 = 2$, we have by (ii) $\langle u | 1 \rangle = 0$, that is $u = Sv$ with $v \in L_+^2(\mathbb{T})$. Thus, by the definition of $L_u = z\partial_z - u\Pi(\bar{u}\cdot)$,

$$L_u z = z - \langle 1 | v \rangle u .$$

Taking the inner product of the latter identity with Ψ_u ,

$$(\nu_u - 1) \langle z | \Psi_u \rangle = -\langle 1 | v \rangle \underbrace{\langle u | \Psi_u \rangle}_{\neq 0} .$$

Note that for $m_0 = 2$ we have $\langle z | \Psi_u \rangle = 0$. This implies that $\langle 1 | v \rangle = 0$ leading to $u = S^2 w$ with $w \in L_+^2(\mathbb{T})$. Therefore, if $m_0 = 2$, then u can be decomposed as

$$u(z) = \frac{z^2 P_{N-2}(z)}{\prod_{j=1}^r (1 - p_j z)^{m_j}}, \quad P_{N-2} \in \mathbb{C}_{N-2}[z], \quad p_k \in \mathbb{D}^*, \quad p_j \neq p_k .$$

Now, if $m_0 = 3$ then by repeating the same above procedure and taking the inner product of

$$L_u z^2 = 2z^2 - \langle 1 | w \rangle u$$

with Ψ_u , one obtains

$$(\nu_u - 2) \underbrace{\langle z^2 | \Psi_u \rangle}_{=0} = -\langle 1 | w \rangle \underbrace{\langle u | \Psi_u \rangle}_{\neq 0} .$$

That is, $\langle 1 | w \rangle = 0$ i.e., $u = S^3 \underline{w}$ with $\underline{w} \in L_+^2(\mathbb{T}) \dots$ Therefore, for all $m \in \{0, \dots, N-1\}$,

$$u(z) = \frac{z^{m_0} P_{N-m_0}(z)}{\prod_{j=1}^r (1 - p_j z)^{m_j}}, \quad p_j \in \mathbb{D}^*, \quad p_j \neq p_k ,$$

where $P_{N-m_0} \in \mathbb{C}_{N-m_0}[z]$ if $m_0 \geq 1$ thanks to (i). And if $m_0 = N$ i.e. all the p_j in (6.5.18) vanish, then $u(z) = Cz^N$, $C \in \mathbb{C}^*$. Finally, it remains to prove that $(z - \bar{p}_j)^{m_j-1}$ divides the numerator of u . Indeed, by identity (6.5.16) of Step 2, $u(z)\bar{u}(\frac{1}{z}) = z\partial_z \log \Psi_u - \nu_u$ where one computes by (6.5.18),

$$z\partial_z \log \Psi_u = m_0 + \sum_{j=1}^r m_j \left(\frac{1}{1-p_j z} + \frac{\bar{p}_j}{z-\bar{p}_j} \right).$$

That is, for all $m_0 \in \{0, \dots, N-1\}$,

$$\frac{P_{N-m_0}(z)}{\prod_{j=1}^r (1-p_j z)^{m_j}} \frac{z^{N-m_0} \overline{P_{N-m_0}(\frac{1}{z})}}{\prod_{j=1}^r (z-\bar{p}_j)^{m_j}} = m_0 - \nu_u + \sum_{j=1}^r m_j \left(\frac{1}{1-p_j z} + \frac{\bar{p}_j}{z-\bar{p}_j} \right),$$

where $p_j \in \mathbb{D}^*$, $p_k \neq p_j$ for $k \neq j$. Observe that in the right-hand side, $\frac{1}{p_j}$ is a pole of multiplicity one. Then, the same should hold for the left-hand side as well. Therefore, if $m_j \geq 2$, $j = 1, \dots, r$, this implies that $\frac{1}{p_j}$ in the left-hand side must be a root of multiplicity $(m_j - 1)$ of $\overline{P_{N-m_0}(\frac{1}{z})}$. That is,

$$P_{N-m_0}(\bar{p}_j) = 0, \dots, P_{N-m_0}^{(m_j-2)}(\bar{p}_j) = 0,$$

where $P_{N-m_0}^{(m)}$ is the m^{th} derivative of P_{N-m_0} . As a result, $(z - \bar{p}_j)^{m_j-1}$ divides $P_{N-m_0}(z)$, and so

$$u(z) = z^{m_0} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z} \right)^{m_j-1} \frac{q(z)}{\prod_{j=1}^r (1 - p_j z)}, \quad p_j \in \mathbb{D}^*, p_k \neq p_j, k \neq j,$$

with $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, $m_0 + \sum_{j=1}^r m_j = N$, and such that $\deg(q) = r$ if $m_0 \neq 0$ thanks to (i).

Step 4. In this step, we write the rational function u obtained in Step 3 on its partial fractional decomposition

$$u(z) = z^{m_0} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z} \right)^{m_j-1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j z} \right), \quad p_j \in \mathbb{D}^*, p_k \neq p_j, k \neq j,$$

where $a \neq 0$ if $m_0 \neq 0$, and we infer by (6.5.16) of Step 2 that, for all $j = 1, \dots, r$,

$$\bar{a}c_j + \sum_{k=1}^r \frac{\bar{c}_k c_j}{1 - \bar{p}_k p_j} = m_j. \quad (6.5.19)$$

Indeed, by applying Π to (6.5.16),

$$\Pi(|u|^2) = \Pi(z\partial_z \log \Psi_u - \nu_u),$$

Observe, on the one hand,

$$\Pi(z\partial_z \log \psi_u - \nu_u) = \sum_{j=1}^r \frac{m_j}{1-p_j z} + m_0 - \nu_u.$$

And on the other hand,

$$\begin{aligned} \Pi(|u|^2) &= \Pi\left(a + \sum_{j=1}^r \frac{c_j}{1-p_j z}\right) \\ &= |a|^2 + \sum_{j=1}^r \bar{a}c_j + \bar{a} \sum_{j=1}^r \frac{c_j}{1-p_j z} + \sum_{j=1}^r \sum_{k=1}^r \frac{c_j \bar{c}_k}{(1-p_j \bar{p}_k)(1-p_j z)}. \end{aligned}$$

Therefore, for all $j = 1, \dots, r$, the request conditions (6.5.19) and

$$\nu_u = m_0 - |a|^2 - \sum_{j=1}^r \bar{a}c_j.$$

Step 5. We prove the converse. For $N \in \mathbb{N}_{\geq 1}$, let $u = Cz^N$, $C \in \mathbb{C}^*$, or

$$u(z) = z^{m_0} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z}\right)^{m_j - 1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j z}\right), \quad p_j \in \mathbb{D}^*, \quad p_k \neq p_j, \quad k \neq j,$$

where $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^r m_j = N$, and $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$, satisfy

$$\bar{a}c_j + \sum_{k=1}^r \frac{\bar{c}_k c_j}{1 - \bar{p}_k p_j} = m_j, \quad (6.5.20)$$

with $a \neq 0$ if $m \neq 0$. Our aim is to prove that $u \in \mathcal{U}_N$, that is

- $\exists \psi \in \mathcal{B}_N$ such that $L_u S^k \psi = (\mu + k)S^k \psi$ for all $k \in \mathbb{N}_{\geq 0}$, where μ is a real constant.
- ψ is of minimal degree, i.e. there does not exist $\phi \in \mathcal{B}_\ell$ with $\ell < N$, such that ϕ satisfies $L_u S^k \phi = (\mu_1 + k)S^k \phi$ for all $k \in \mathbb{N}_{\geq 0}$.

For the moment, let us deal with the more complicated case, i.e. u is a rational function. We start by proving the first point. Let

$$\psi := e^{i\theta} z^{m_0} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z}\right)^{m_j} \in \mathcal{B}_N, \quad \theta \in \mathbb{R}, \quad p_k \in \mathbb{D}^*. \quad (6.5.21)$$

Observe that $\bar{u}\psi$ extends as a holomorphic function on \mathbb{D} as $p_k \in \mathbb{D}$. Then, by definition of L_u ,

$$L_u \psi = z\partial_z \psi - |u|^2 \psi,$$

where

$$z\partial_z\psi = m_0\psi + \sum_{k=1}^r \left(\frac{\bar{p}_k}{z - \bar{p}_k} + \frac{1}{1 - p_k z} \right) \psi,$$

and thanks to (6.5.20),

$$\begin{aligned} |u|^2 &= |a|^2 + \bar{a} \sum_{k=1}^r \frac{c_k}{1 - p_k z} + a \sum_{k=1}^r \frac{\bar{c}_k z}{z - \bar{p}_k} + \sum_{k=1}^r \sum_{j=1}^r \frac{c_k \bar{c}_j z}{(1 - p_k z)(z - \bar{p}_j)} \\ &= |a|^2 + a \sum_{k=1}^r \bar{c}_k + \sum_{k=1}^r \left(\bar{a} c_k + \sum_{j=1}^r \frac{c_k \bar{c}_j}{1 - p_k \bar{p}_j} \right) \frac{1}{1 - p_k z} \\ &\quad + \sum_{j=1}^r \left(\bar{a} c_j + \sum_{k=1}^r \frac{c_k \bar{c}_j}{1 - p_k \bar{p}_j} \right) \frac{\bar{p}_j}{z - \bar{p}_j} \\ &= |a|^2 + a \sum_{k=1}^r \bar{c}_k + \sum_{k=1}^r \frac{m_k}{1 - p_k z} + \sum_{j=1}^r \frac{m_j \bar{p}_j}{z - \bar{p}_j}. \end{aligned} \quad (6.5.22)$$

Therefore,

$$L_u \psi = \left(m_0 - |a|^2 - a \sum_{k=1}^r \bar{c}_k \right) \psi$$

Additionally, observe that for all $k \in \mathbb{N}_{\geq 1}$, $\langle S^k \psi | u \rangle = 0$. Hence, by applying the commutator identity (6.2.2), we deduce

$$L_u S^k \psi = (\mu + k) S^k \psi, \quad \forall k \in \mathbb{N}_{\geq 0}, \quad (6.5.23)$$

where $\mu := m_0 - |a|^2 - a \sum_{k=1}^r \bar{c}_k$.

It remains to prove that ψ is of degree minimal. Suppose for the seek of contradiction that there exists $\phi \in \mathcal{B}_\ell$, with $\ell < N$, such that $L_u S^j \phi = (\mu_1 + k) S^j \phi$ for all $j \in \mathbb{N}_{\geq 0}$. By comparing the latter identity to (6.5.23), and thanks to (6.2.7) we infer that there exists $k', k \in \mathbb{N}_{\geq 1}$ such that $S^k \psi = S^{k'} \phi$, i.e.

$$\phi := e^{i\bar{\theta}} z^{m_0+k-k'} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z} \right)^{m_j}, \quad m_0 + k - k' + \sum_{j=1}^r m_j = \ell < N.$$

Therefore, by repeating Step 1 to Step 4, we infer that u must be of the form

$$u(z) = z^{m_0+k-k'} \prod_{j=1}^r \left(\frac{z - \bar{p}_j}{1 - p_j z} \right)^{m_j-1} \left(a + \sum_{j=1}^r \frac{c_j}{1 - p_j z} \right),$$

which is a contradiction. \square

Corollary 6.5.5. *Given $N \in \mathbb{N}_{\geq 1}$, let $u \in \mathcal{U}_N$. Then,*

(i) *In the focusing case, $\|u\|_{L^2}^2 = N - \nu_u$,*

(ii) In the defocusing case, $\|u\|_{L^2}^2 = \lambda_u - N$,

where ν_u is the eigenvalue introduced in (6.5.8) and λ_u is the corresponding one in the defocusing case.

Remark 6.5.4. Based on the previous statement, one can conclude that for any potential $u \in \mathcal{U}_N$, we have

$$\begin{cases} \nu_u < N \text{ (focusing case)} \\ \lambda_u > N \text{ (defocusing case)} \end{cases} . \quad (6.5.24)$$

Proof. Let $u \in \mathcal{U}_N$. Then in light of the previous theorem, we have either $u = C e^{iNx}$, $C \in \mathbb{C}^*$ or u is the rational function (6.5.9). Thus, if $u = C e^{iNx}$ then the results follow easily by Remark 6.5.2. Now, if u is the rational function (6.5.9), then by computing the L^2 -norm of u in the focusing case, we infer via (6.5.22),

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{z \in \mathcal{C}(0,1)} \left(|a|^2 + a \sum_{k=1}^r \bar{c}_k + \sum_{k=1}^r \frac{1}{1 - p_k z} + \sum_{k=1}^r \frac{\bar{p}_k}{z - \bar{p}_k} \right) \frac{dz}{2\pi i z} \\ &= |a|^2 + a \sum_{k=1}^r \bar{c}_k + N - m, \end{aligned}$$

which is equal to $-\nu_u + N$ by (i) of Remark 6.5.2. For the defocusing case, we shall have

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{z \in \mathcal{C}(0,1)} \left(|a|^2 + a \sum_{k=1}^r \bar{c}_k - \sum_{k=1}^r \frac{1}{1 - p_k z} - \sum_{k=1}^r \frac{\bar{p}_k}{z - \bar{p}_k} \right) \frac{dz}{2\pi i z} \\ &= |a|^2 + a \sum_{k=1}^r \bar{c}_k - N + m, \end{aligned}$$

which is equal to $\lambda_u - N$ by (ii) of Remark 6.5.2. \square

Proposition 6.5.6. *For any $N \in \mathbb{N}_{\geq 0}$, the set of finite gap potential \mathcal{U}_N is conserved along the flow of the (CS)-equation.*

Proof. Let u_0 be a finite gap potential in \mathcal{U}_N , that is there exists $\psi_{u_0} \in \mathcal{B}_N$ of minimal degree N satisfying

$$L_{u_0} S^k \psi_{u_0} = (\nu_{u_0} + k) S^k \psi_{u_0}, \quad \forall k \geq 0. \quad (6.5.25)$$

Our aim is to prove that there exists $\varrho(t) \in \mathcal{B}_N$ of minimal degree⁸ such that

$$L_{u(t)} S^k \varrho(t) = (\nu_{u_0} + k) S^k \varrho(t), \quad \forall k \geq 0.$$

8. In the sense, that there does not exist $\phi(t) \in \mathcal{B}_\ell$ with $\ell < N$, such that $\phi(t)$ satisfies $L_u S^k \phi(t) = (\mu_1 + k) S^k \phi(t)$ for all $k \in \mathbb{N}_{\geq 0}$.

Let $\varrho(t)$ be a solution of the Cauchy problem

$$\begin{cases} \partial_t \varrho(t) = B_{u(t)} \varrho(t) \\ \varrho(0) = \Psi_{u_0} \end{cases} .$$

Hence, by Remark 6.3.1 ,

$$L_{u(t)} \varrho(t) = \nu_{u_0} \varrho(t) . \quad (6.5.26)$$

In addition, recall by Lemma 6.4.1 ,

$$\langle S\varrho(t) | u(t) \rangle = \langle S\Psi_0 | u_0 \rangle e^{-i\nu_{u_0}^2 t} ,$$

where here $\langle S\Psi_0 | u_0 \rangle$ vanishes after combining the commutator identity (6.2.2) and equation (6.5.25) . Therefore, by (6.2.2) ,

$$L_{u(t)} S\varrho(t) = (\nu_{u_0} + 1)S\varrho(t) . \quad (6.5.27)$$

This yields to

$$\begin{cases} \partial_t S\varrho(t) = B_{u(t)} S\varrho(t) \\ S\varrho(0) = S\Psi_{u_0} \end{cases} .$$

Indeed, by the commutator identity (6.2.4) ,

$$\begin{aligned} \partial_t S\varrho(t) &= SB_{u(t)}\varrho(t) \\ &= B_{u(t)}S\varrho(t) - i\left(\tilde{L}_u^2 S - S(\tilde{L}_u + \text{Id})^2\right)\varrho(t) , \end{aligned}$$

which is equal to $\partial_t S\varrho(t) = B_{u(t)}S\varrho(t)$ thanks to (6.5.26) and (6.5.27) . Consequently, by repeating the same procedure, we obtain for all $k \in \mathbb{N}_{\geq 0}$,

$$L_{u(t)} S^k \varrho(t) = (\nu_{u_0} + k)S^k \varrho(t) ,$$

with

$$\begin{cases} \partial_t S^k \varrho(t) = B_{u(t)} S^k \varrho(t) \\ S^k \varrho(0) = S^k \Psi_{u_0} \end{cases} .$$

Besides, observe that $\varrho(t) \in \mathcal{B}_N$. Indeed, by applying Lemma 6.4.1 ,

$$\langle S^k \varrho(t) | \varrho(t) \rangle = \langle S^k \Psi_{u_0} | \Psi_{u_0} \rangle e^{i((\nu_{u_0} + k)^2 - \nu_{u_0}^2)t} = 0 , \quad \forall k \in \mathbb{N}_{\geq 1} ,$$

leading to

$$\langle e^{ikx} | |\varrho(t)|^2 \rangle = 0 , \quad k \in \mathbb{Z} \setminus \{0\} .$$

Thus, following the same lines of the proof of Proposition 6.5.2, we deduce that $\varrho(t)$ is a finite Blaschke product. To infer that the degree of this finite Blaschke product is N , we should notice that each of Ψ_{u_0} and $\varrho(t)$ enjoys an inverse spectral formula (Remark 6.5.3)

$$\begin{aligned} \Psi_{u_0} &= \langle (\text{Id} - zM_{\leq N}(u_0))^{-1} \mathbf{1}_N | Y_{\leq N}(u_0) \rangle , \\ \varrho(t) &= \langle (\text{Id} - zM_{\leq N}(u(t)))^{-1} \mathbf{1}_N | Y_{\leq N}(u(t)) \rangle . \end{aligned}$$

where $M_{\leq N}(u_0)$ and $M_{\leq N}(u(t))$ are the finite matrix of order $(N + 1) \times (N + 1)$ obtained respectively from the representation matrix of S^* in the L^2 basis $(h_k)_{k=0}^{N-1} \cup (S^k \psi_{u_0})_{k \geq 0}$ constituted of the eigenfunctions of L_{u_0} at $t = 0$, and from the eigenfunctions $(e_k(t))_{k=0}^{N-1} \cup (S^k \varrho(t))_{k \geq 0}$ of $L_{u(t)}$ at any time t . Therefore, in view of the fourth identity of Lemma 6.4.1, we infer

$$M_{\leq N}(u(t)) = \text{Diag}(e^{-i(\nu_n+1)^2 t}) M_{\leq N}(u_0) \text{Diag}(e^{-i\nu_n^2 t}).$$

That is,

$$|\det(M_{\leq N}(u_0))| = |\det(M_{\leq N}(u(t)))|$$

and so,

$$\deg(\det(\text{Id} - zM_{\leq N}(u(t)))) = \deg(\det(\text{Id} - zM_{\leq N}(u_0))) = N. \quad (6.5.28)$$

As a result, $u(t) \in \mathcal{U}_n$ with $n \leq N$. It remains to show that $u(t) \notin \mathcal{U}_n$ with $n < N$. Suppose that there exists $\phi(t) \in \mathcal{B}_n$ with $n < N$ such that

$$L_{u(t)} S^k \phi(t) = (\nu_u + k) S^k \phi(t),$$

then applying the same above procedure, we infer that $\phi(0) \in \mathcal{B}_n$ with $n < N$ and

$$L_{u_0} S^k \phi(0) = (\lambda_u + k) S^k \phi(0),$$

leading to $u_0 \in \mathcal{U}_n$, $n < N$ which is a contradiction.

Note that the same proof works in the defocusing case. □

6.6 Remark on the regularity of u

Recall that in the beginning of Section 6.2, we have supposed for more convenience that u is a function with enough regularity, typically in $H_+^2(\mathbb{T})$. However, the same strategy adopted to derive the traveling waves of the Calogero–Sutherland DNLS equation (CS) and to characterize the finite gap potentials can be extended to less regularity spaces. *In this section, we discuss some remarks that allow the extension of the main results to the critical regularity $L_+^2(\mathbb{T})$.*

First, we recall from [Bad24a] the following Theorem.

Theorem ([Bad24a]). *For any $0 \leq s \leq 2$, let $u_0 \in H_+^s(\mathbb{T})$. Then, there exists a unique potential $u \in \mathcal{C}(\mathbb{R}, H_+^s(\mathbb{T}))$ solution of (CS⁻) such that, for any sequence $(u_0^\varepsilon) \subseteq H_+^2(\mathbb{T})$,*

$$\|u_0^\varepsilon - u_0\|_{H^s} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

we have for all $T > 0$,

$$\sup_{t \in [-T, T]} \|u^\varepsilon(t) - u(t)\|_{H^s} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Moreover, the L^2 –norm of the limit potential u is conserved in time :

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

Furthermore, the same holds for (CS^+) under the additional condition $\|u_0\|_{L^2} < 1$.

At a second stage, recall that Lemma 6.3.2, Proposition 6.3.5 and Corollary 6.3.6 have been the keys to characterize the traveling waves for the defocusing equation (CS^-) , and Lemma 6.4.1, Proposition 6.4.2 and Corollary 6.4.3 for the focusing equation (CS^+) . As a result, we need to extend these Proposition/lemma/Corollary to less regular potentials u . Hence, we recall from [Bad24a, Corollary 3.12] the following result.

Corollary (Corollary 3.12 of [Bad24a]). *For any $0 \leq s \leq 2$, let $u_0 \in H_+^s(\mathbb{T})$. There exists an orthonormal basis (g_n^t) of $L_+^2(\mathbb{T})$ constituted from the eigenfunctions of $L_{u(t)}$, such that for all $n \in \mathbb{N}_{\geq 0}$,*

$$\langle u(t) | g_n^t \rangle = \langle u_0 | f_n^{u_0} \rangle e^{-it\lambda_n^2(u_0)}, \quad \forall t \in \mathbb{R},$$

where $u(t)$ is the solution of (CS^-) starting at u_0 at $t = 0$. Furthermore, the same holds for (CS^+) under the additional condition $\|u_0\|_{L^2} < 1$.

Remark 6.6.1. Note that there is a point hidden in the previous corollary, namely, the fact that L_u is well–defined with $u \in L_+^2(\mathbb{T})$. We refer the readers to [GL24, Appendix A] for the construction of this operator and to [Bad24a, Corollary 3.2] for a way to identify its spectrum.

By repeating the same analysis of the proof of [Bad24a, Corollary 3.12], one can establish the existence of an orthonormal basis (g_n^t) of $L_+^2(\mathbb{T})$ satisfying

$$\begin{aligned} \langle 1 | g_n^t \rangle &= \langle 1 | f_n^{u_0} \rangle e^{-i\lambda_n^2 t}, \\ \langle Sg_p^t | g_n^t \rangle &= \langle Sf_p^{u_0} | f_n^{u_0} \rangle e^{i((\lambda_p+1)^2 - \lambda_n^2)t}. \end{aligned}$$

Finally, in Section 6.5, more precisely in (6.5.6), we made use of the fact that the domain of the Lax operator L_u with $u \in H_+^2(\mathbb{T})$ is $H_+^1(\mathbb{T})$ in order to infer that $\Pi(\bar{u}\psi) \in L^2$. However, it should be noted that the lax operator L_u with $u \in L_+^2(\mathbb{T})$ has its domain a subset of $H_+^{\frac{1}{2}}(\mathbb{T})$ [GL24, Appendix A]. Hence, we need the following lemma to infer the result.

Lemma (Lemma 2.7 of [Bad24a]). *Let $h \in H_+^{\frac{1}{2}}(\mathbb{T})$, $u \in L_+^2(\mathbb{T})$,*

$$\|T_{\bar{u}}h\|_{L^2(\mathbb{T})}^2 \leq \left(\langle Dh | h \rangle + \|h\|_{L^2(\mathbb{T})}^2 \right) \|u\|_{L^2(\mathbb{T})}^2,$$

where we recall T_u was defined in (6.1.11).

6.7 Open problems

1. The full characterization of the traveling waves $u_0(x - ct)$ of (CS⁺) is still an open problem.

2. Note that along this paper, we have treated the case where the traveling waves of the Calogero–Sutherland DNLS equation (CS) are of the form

$$u(t, x) := u_0(x - ct), \quad c \in \mathbb{R}.$$

But, one may wonder if there exist traveling wave solutions with a phase factor, such as

$$u(t, x) := e^{i\varphi(t)} u_0(x - ct), \quad \varphi(t), c \in \mathbb{R}. \quad (6.7.1)$$

However, let us underline the following feature : observe that the mean $\langle u | 1 \rangle$ is conserved along the flow of the Calogero–Sutherland DNLS equation (CS), for any solution u in the Hardy space of the circle \mathbb{T} . Indeed, by applying an integration by parts and since u is in the Hardy space, then

$$i\partial_t \langle u | 1 \rangle = - \langle \partial_x^2 u | 1 \rangle \pm 2 \langle D\Pi(|u|^2) | \bar{u} \rangle = 0.$$

Therefore,

- If $\langle u_0 | 1 \rangle \neq 0$, then $\varphi(t)$ in (6.7.1) must be a constant in time.
- In regard to the case where $\langle u_0 | 1 \rangle = 0$, the question of the existence of traveling waves of (CS) of the form (6.7.1) remains an open problem. However, one can easily prove that $(\varphi(t), c)$ are related via the following identity

$$\varphi'(t) - Nc = -N^2,$$

where N is the positive integer appearing after rewriting u_0 as $u_0 = S^N v_0$ with $\langle v_0 | 1 \rangle \neq 0$, as $\langle u_0 | 1 \rangle = 0$. Indeed, by writing the solution $u(t, x)$ as

$$\begin{aligned} u(t, x) &= e^{i\varphi(t)} u_0(x - ct) \\ &= e^{i\varphi(t)} e^{iN(x-ct)} v_0(x - ct), \end{aligned}$$

one observes that if u satisfies (CS), then

$$\begin{cases} -(\varphi'(t) - Nc) v_0 - N^2 v_0 + P(\partial_x v_0, \partial_x^2 v_0) \mp 2i\partial_x \Pi(|v_0|^2) v_0 = 0. \\ P(w, \tilde{w}) := (2N - c) i w + \tilde{w} \end{cases} \quad (6.7.2)$$

We conclude by taking the inner product of the last identity with 1, that

$$\varphi'(t) - Nc = -N^2.$$

Appendix

1. *The following counterexample illustrate the necessity of the condition $\nu_n \neq 0$ in order to obtain the first point in Proposition 6.2.4.*

Consider the 0-gap potential (i.e. a potential satisfying $\gamma_n(u) = 0$ for all $n \in \mathbb{N}_{\geq 0}$, where $\gamma_n(u)$ is defined in (6.5.1))

$$u(z) = \frac{\sqrt{1 - |p|^2}}{1 - pz}, \quad p \in \mathbb{D}.$$

One can easily check that $L_u f_0 = -f_0$ for

$$f_0(z) := \frac{\sqrt{1 - |p|^2}}{1 - pz},$$

and that, for all $k \in \mathbb{N}_{\geq 0}$, $L_u S^k \psi = k S^k \psi$ where

$$\psi(z) := \frac{z - \bar{p}}{1 - pz}.$$

Therefore, the spectrum of L_u is given by

$$\sigma(L_u) = \{-1 < 0 < 1 < 2 < \dots < n < n + 1 < \dots\},$$

where notice $\nu_1 = \nu_0 + 1$ and $S f_0 \neq \psi$.

2. *In this part of the Appendix we prove that the two integers N_1 and N_2 appearing in Corollary 6.2.6 are not necessarily equal.*

Let

$$u(z) := \frac{\sqrt{2(1 - |p|^4)} z}{1 - p^2 z^2}, \quad p \in \mathbb{D}^*.$$

For such u , one can check that

$$\psi_u := \frac{(z - \bar{p})(z + \bar{p})}{(1 - pz)(1 + pz)}$$

is an eigenfunction of L_u associated with the eigenvalue 0. Additionally, for all $k \in \mathbb{N}_{\geq 0}$,

$$\langle S^k \psi_u | u \rangle = \int_{z \in \mathcal{C}(0,1)} \frac{z^k (z^2 - \bar{p}^2)}{1 - p^2 z^2} \frac{\sqrt{2(1 - |p|^4)} z}{z^2 - \bar{p}^2} \frac{dz}{2\pi i z} = 0,$$

leading, by (6.2.2), to $L_u S^k \psi_u = k S^k \psi_u$, for all $k \in \mathbb{N}_{\geq 0}$. Note that $\deg \psi_u = 2$, then it remains to find two eigenvectors of L_u , generating the model space⁹

9. [GMR16, Corollary 5.18]

$(\Psi_u L_+^2(\mathbb{T}))^\perp$. First, we have $L_u 1 = 0$ as $L_u 1 = -\langle 1 | u \rangle 1$ and $\langle u | 1 \rangle = 0$. Second, by taking

$$f_0 = \frac{\sqrt{1 - |p|^4} z}{1 - p^2 z^2},$$

one has, $L_u f_0 = -f_0$. Therefore, by denoting for all $k \in \mathbb{N}_{\geq 0}$, $f_{2+k} := S^k \Psi_u$ and $f_1 := 1$, we have $\langle u | f_n \rangle = 0$ for all $n \geq 1$. But, on the other side, $\nu_2 - \nu_1 - 1 = 0 - 0 - 1 \neq 0$.

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Titre : Une équation de Schrödinger non-linéaire issue du système de Calogero-Sutherland-Moser.

Mots clés : *Equation de Schrödinger non linéaire, Espace de Hardy, Formule explicite, Limite à faible dispersion, Paire de Lax, Ondes progressives, Systèmes de Calogero-Sutherland-Moser, Systèmes intégrables,*

Résumé : Il s'agit d'étudier une EDP obtenue par A. Abanov et al (J. Phys. A, 2009) à partir de la limite hydrodynamique du système hamiltonien de Calogero-Sutherland-Moser. On obtient ainsi une équation intégrable de type Schrödinger non linéaire sur l'espace de Hardy qui se trouve posséder une paire de Lax sur la droite et sur le cercle. Le but de cette thèse est d'utiliser la structure d'intégrabilité afin d'établir que l'équation est globalement bien-posée sur le cercle en allant jusqu'à l'espace

de régularité critique. En second lieu, on s'intéresse à l'existence de solutions particulières sur le tore. Ainsi, on caractérise les ondes progressives de cette équation, ainsi qu'une classe de solutions s'écrivant sous la forme de fractions rationnelles et qui sont définies spectralement à partir de l'opérateur de Lax. En troisième lieu, on étudie la limite à faible-dispersion (semi-classique) de cette équation sur la droite et on caractérise ses solutions grâce à une formule explicite.

Title : On a DNLS equation related to the Calogero-Sutherland-Moser Hamiltonian system.

Keywords : *Nonlinear Schrödinger equation, Hardy space, Explicit formula, Zero-dispersion limit, Lax pair, Traveling waves, Calogero-Sutherland-Moser systems, Integrable system.*

Abstract : This thesis is devoted to a PDE obtained by A. Abanov et al (J. Phys. A, 2009) from the hydrodynamic limit of the Calogero-Sutherland Hamiltonian system. A nonlinear integrable Schrödinger-type equation on the Hardy space is obtained and has a Lax pair structure on the line and on the circle. The goal of this thesis is to establish, by using the integrability structure of this PDE, some global well-posedness results on

the circle, extending down to the critical regularity space. Secondly, we investigate the existence of particular solutions. Thus, we characterize the traveling waves and finite gap potentials of this equation on the circle. Thirdly, we study the zero-dispersion (or semiclassical) limit of this equation on the line and characterize its solutions using an explicit formula.