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# Optimal control methods for systemic risk

Leila Bassou

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# Optimal control methods for systemic risk

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à École polytechnique

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(EDMH)

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Thèse présentée et soutenue à Palaiseau, le 27/06/2024, par

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*À la mémoire de mon frère  
Noomann*



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# Chapter 1

## Introduction

La modélisation et l'étude des interactions financières entre les entités économiques est cruciale pour éviter le risque systémique ou du moins en limiter les effets. Ce chapitre est dédié à l'introduction des différents modèles considérés dans ce manuscrit et à la présentation des résultats principaux, en se concentrant sur leurs motivations, leurs interprétations et leurs énoncés, et non sur leur preuves.

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## 1.1 Contexte général

Le risque systémique est le risque de faillite d'un système financier en raison des interactions et de l'interdépendance entre les différentes entités du système. Si l'on considère  $N$  agents économiques qui sont interdépendants sur le marché financier, et si l'un d'entre eux subit une perte de ses actifs, alors tous ceux qui interagissent avec lui peuvent être fortement impactés. Par conséquent, un effet domino peut s'ensuivre, et des pertes en cascade peuvent se produire sur les actions de nombreux autres agents.

Comme en témoigne par exemple la crise des *Subprimes de 2008*, le risque systémique est un phénomène complexe à multiples facettes qui doit être abordé sous différents angles. Il existe actuellement une littérature considérable sur ce sujet, à la fois sur le plan théorique et numérique, voir notamment D. Aikman et al [1], S. Alogoskoufis et al [2], H. Amini, R. Cont et A. Minca [3], R. Carmona, J.-P. Fouque et L.-H. Sun [11] ou H. Elsinger [23] et les références qui y figurent, où divers modèles ont été étudiés. Nous renvoyons également au manuel de J.-P. Fouque et J.-A. Langsam [31] pour des développements récents sur le risque systémique avec de nombreux points de vue : statistiques, mathématique financière,...

Pour explorer l'interdépendance entre les institutions financières, un modèle récent appelé **détentions croisées** ou **mutuelles** a été créé par C. Bertucci et N. Touzi puis étudié pour la première fois dans les articles de M-F. Djete et N. Touzi [19] et M-F. Djete, G. Guo et N. Touzi [20]. Dans ces travaux, les auteurs modélisent la dynamique des portefeuilles d'un nombre fini d'agents économiques en tenant compte, d'une part, des différents investissements réalisés directement entre eux à travers la détention de parts, et d'autre part, de la dynamique de leurs revenus qui peuvent être corrélés.

Durant ma thèse, j'ai étudié les situations optimales dans le cadre de ce modèle. Ici, le terme "optimalité" fait référence au fait que chaque agent a une fonction d'utilité spécifique à optimiser en choisissant la quantité de parts des autres agents qu'il doit détenir. Sans aucune collaboration entre les joueurs, c'est-à-dire en supposant que chaque agent se concentre sur l'optimisation de son seul critère, on peut se référer à l'étude d'un équilibre de Nash, qui se définit comme étant un  $N$ -uplet de stratégies où la stratégie de chaque joueur (agent) est la meilleure réponse face aux stratégies des autres joueurs. Par meilleure réponse, on fait allusion au fait que, tout en tenant compte des stratégies jouées par les autres, le joueur ne pourra pas faire mieux en répondant par une autre stratégie pour optimiser son critère. Cette notion d'équilibre, qui porte le nom de John Forbes Nash, a été introduite dans le papier fondateur [51] en 1951, et depuis elle a été largement étudiée dans la théorie des jeux, et a fait l'objet de nombreux travaux. Dans la littérature de gestion des risques en temps continu, on peut citer entre autres l'article de R. Carmona, J.-P. Fouque et L.-H. Sun [15] ou ceux de G.-E. Espinosa et N. Touzi [25], O. Féron, P. Tankov et L. Tinsi [27], D. Lacker et T. Zariphopoulou [46], D. Lacker et A. Soret [45] et les références qui y figurent.

Comme on le verra plus loin, considérer un jeu avec un nombre fini d'agents  $N$  revient à résoudre un problème très complexe. En effet, il faut prendre en considération le problème d'optimisation de chaque agent, tout en tenant compte des caractéristiques des joueurs

et de leur interdépendance, ce qui conduit naturellement à résoudre un système couplé d'équations non linéaires dont la taille augmente avec le nombre d'agents  $N$ , et qui devient difficile à étudier dès que le nombre d'agents  $N$  dépasse 2.

Lorsqu'on considère un très grand nombre d'agents  $N$ , il est possible de simplifier l'étude de ce problème en utilisant **la théorie des jeux à champ moyen**. Cette récente théorie qui a été introduite par M. Huang, P. Caines et R. Malhame [36]-[37]-[38] et J-M. Lasry et P-L. Lions [48]-[49]-[47], a été à l'origine d'une littérature abondante en économie et en mathématique financière, mais aussi dans de nombreux autres domaines tels que la biologie ou l'épidémiologie. L'étude d'un modèle champ moyen repose sur des hypothèses raisonnables afin de simplifier considérablement la complexité du modèle fini associé. Ces hypothèses consistent en l'anonymat des joueurs et la négligeabilité de leur impact sur la dynamique des populations puisque leur nombre est infini. De ce fait, chacun des acteurs modélise le comportement d'un agent représentatif faisant face à l'environnement constitué de la distribution statistique de la population anonyme. Le système d'équations étudié est alors considérablement réduit, et l'enjeu revient à étudier des équations aux dérivées partielles non linéaires, statiques ou dépendantes du temps, impliquant (ou non) un aléa affectant tous les joueurs, avec (ou sans) aspects stochastiques,... Pour une présentation générale de ce sujet, nous renvoyons aux livres de R. Carmona et F. Delarue [11]-[12].

Certains travaux dans le domaine des jeux à champ moyen constituent un pont entre les problèmes concernant les interactions entre un nombre fini de joueurs et ceux correspondant à un nombre infini de joueurs à travers, par exemple, la théorie de la propagation du chaos. Cela permet d'utiliser les solutions des jeux de champ moyen pour approcher les équilibres de Nash dans le jeu fini.

Par exemple, dans le cadre du modèle de détentions croisées étudié dans cette thèse, on peut citer le récent travail de M-F. Djete et N. Touzi [19], où d'abord les auteurs ont étudié le jeu à champ moyen, en l'absence de bruit commun (ce qui signifie que les risques idiosyncratiques des agents sont non corrélés) et ensuite, en tirant profit de la solution explicite qu'ils ont obtenue, ils ont approché l'équilibre de Nash du jeux fini correspondant.

Avant de détailler le contenu de ma thèse, je voudrais présenter brièvement les trois parties qui la constituent.

1. La première partie est consacrée à l'étude du modèle **de détentions croisées** avec un nombre fini d'agents économiques. Chaque agent s'intéresse à la maximisation de sa propre fonction d'utilité exponentielle en choisissant les parts à détenir des autres agents. Nous avons étudié l'équilibre de Nash dans les cas statiques et dynamiques sous une dynamique de type Bachelier gaussienne.

Comme on le verra par la suite, dans le cas statique, ce problème est équivalent à la résolution d'un système couplé d'équations non linéaires dont la taille augmente avec le nombre d'agents économiques. Dans le cas à  $N = 2$  agents, on a résolu **explicitement** le problème, mais dès que le nombre d'agents  $N \geq 3$ , le système d'équations devient plus compliqué. Cependant, on a été capable de caractériser l'ensemble des équilibres de Nash (pouvant être vide), mais sans les

expliciter. L'étude du cas statique est une étape cruciale pour résoudre le problème analogue dans le cas du modèle en temps continu à coefficients déterministes. En fait, comme le critère de chaque agent est une fonction d'utilité exponentielle, le problème d'optimisation dans le cas à temps continu est équivalent à l'analyse d'un système d'équations différentielles stochastiques rétrogrades (EDSR). Rappelons que les EDSR qui ont été introduites en 1973 par J.-M. Bismut [5], se sont révélées pertinentes dans plusieurs domaines, notamment en mathématiques financières. Par exemple, elles peuvent être utilisées dans des problèmes de couverture d'options et de pricing (voir le papier de N. El Karoui, S. Peng et M-C. Quenez [22]). Ce lien entre les problèmes d'optimisation de la fonction d'utilité exponentielle et les EDSR a été mis en lumière par N. Karoui et R. Rouge [21] (voir aussi les papiers Y. Hu et P. Imkeller [33] et J. Sekine [55]). Il ressort de notre étude que, dans le cas des coefficients déterministes, le problème d'optimisation est, à chaque instant, équivalent à celui étudié dans le cas statique. Cependant, quand les coefficients sont stochastiques, l'étude est plus complexe et nécessite l'analyse des EDSR quadratiques multidimensionnelles. Ce type d'EDSR n'est pas toujours bien posée, comme il l'a été démontré dans l'article [30] de C. Frei et G. Dos, Reis, où les auteurs ont exhibé des exemples d'EDSR quadratiques multidimensionnelles mal posées.

2. La seconde partie de ce document est dédiée à l'étude des équilibres du modèle champ moyen **de détentions croisées** ( lorsque  $N \rightarrow \infty$ ), **avec un bruit commun** (ce qui signifie que les risques idiosyncratiques des agents sont corrélés). Pour simplifier l'étude du problème, on s'est limité dans cette partie au cas du modèle à une période.

Quelques résultats surprenants ont été révélés durant cette partie:

Dans la formulation du modèle champ moyen, on a remarqué qu'on peut extraire une notion de **non arbitrage** (NA). Alors, on a cherché des équilibres sous lesquels cette condition NA est vérifiée. En étudiant ce problème, on a découvert que cette condition, à l'équilibre, impose une structure sur la dynamique du modèle champ moyen qui se traduit par le fait que le drift et le terme de diffusion du bruit commun, qui sont présents dans la dynamique du risque idiosyncratique, sont proportionnels. Par analogie, ceci est similaire à la restriction drift-volatilité du modèle HJM dans [32].

Rappelons que le papier de M-F Djete et N. Touzi [19] traite le problème du modèle champ moyen, en l'absence de bruit commun (les risques idiosyncratiques sont indépendants), où l'agent représentatif est amené à appliquer son contrôle sur le drift de son compétiteur.

Dans ce travail, la présence du bruit commun change la situation. En effet, vu que les risques idiosyncratiques sont corrélés, le contrôle est appliqué sur le drift et le terme de diffusion du bruit commun du compétiteur, ce qui amène à considérer le compromis moyenne-variance. On considère donc le problème d'optimisation du critère moyenne-variance de l'agent représentatif. Sous certaines hypothèses sur les stratégies d'investissements, on trouve qu'il existe des équilibres au jeu à champ

moyen si et seulement si une condition de proportionnalité est vérifiée entre le drift et le terme de diffusion du bruit commun, ce qui est en accord avec la condition de non arbitrage.

Dans le cadre du modèle à population finie, la caractérisation de la condition de non arbitrage reste un problème ouvert. Cependant, on a rajouté une structure au modèle fini dans laquelle les coefficients satisfont la condition de non arbitrage du modèle champ moyen. En étudiant ce modèle, on trouve qu'en rajoutant des conditions sur le coefficient de proportionnalité entre le drift et le terme de diffusion du bruit commun, il existe alors des équilibres de Nash.

3. Dans la troisième partie, on a étendu l'étude du modèle champ moyen à une période au modèle temps continu.

Comme dans la deuxième partie, la formulation de la dynamique du modèle champ moyen révèle une notion faible de non arbitrage connue sous le nom de **No-Increasing Profit** (NIP). Nous renvoyons à C. Fontana dans [28] pour une présentation comparative des différentes notions de non arbitrage en temps continu.

La condition NIP impose une structure sur le modèle champ moyen à l'équilibre: le drift et le terme de diffusion du bruit commun, qui sont à la fois présents dans la dynamique des risques idiosynchroniques et dans la dynamique des fonds propres de l'agent représentatif, sont colinéaires à chaque instant. Grâce à la caractérisation de la condition NIP, l'analyse des équilibres au jeu à champ moyen se ramène à celle du problème classique d'optimisation d'un portefeuille avec des dotations aléatoires, qui a été largement étudié dans la littérature. On peut voir par exemple l'article de J. Cvitanic et H. Wang [16] ou celui de J. Hugonnier et D. Kramkov [39].

On a conclu cette partie par des exemples concrets, pour lesquels on a déterminé explicitement les équilibres et leur conditions d'existence.

Soulignons que dans cette thèse, nous nous sommes concentrés uniquement sur l'étude des équilibres pour le modèle de détentions croisées. Nous n'avons pas traité la question du risque systémique qui peut être introduite comme étant la probabilité de défaut des agents. Notre travail peut être considéré comme une première étape pour explorer ce type de question. Par exemple, dans le papier de M-F. Djete, G. Guo et N. Touzi [20], les auteurs ont traité cette question, mais sans bruit commun. Plus précisément, ils ont fourni une solution explicite des équilibres au jeu champ moyen en mettant à profit la solution obtenue dans leur précédent travail [19], et ils ont fourni la dynamique de la probabilité de défaut des agents.

## Notations

On introduit ici quelques notations et conventions utilisées dans le reste de cette introduction.

- Soit  $N \in \mathbb{N}^*$ , la base canonique de  $\mathbb{R}^N$  est notée  $(e_i)_{1 \leq i \leq N}$ .  $|\cdot|$  représentera la norme euclidienne de  $\mathbb{R}^N$ . Pour un scalaire  $\alpha \in \mathbb{R}$ , le vecteur  $\alpha_{\mathbb{R}^N}$  désignera le vecteur de  $\mathbb{R}^N$  dont toutes les coordonnées sont égales à  $\alpha$ .



- L'ensemble des matrices d'ordre  $N$  est notée  $\mathcal{M}_N(\mathbb{R})$  et  $\mathcal{M}_N(\mathbb{R}_+)$  désignera le sous-ensemble de  $\mathcal{M}_N(\mathbb{R})$  à coefficients positifs et nuls sur la diagonale. On note  $\mathcal{S}_N(\mathbb{R})$  l'ensemble de matrices symétriques de  $\mathcal{M}_N(\mathbb{R})$  et  $\mathcal{S}_N^+(\mathbb{R})$  le sous-ensemble de matrices symétriques semi-définies positives.  $\mathcal{D}_N(\mathbb{R})$  désignera l'ensemble des matrices diagonales  $N \times N$ , et  $\mathcal{O}_N(\mathbb{R})$  l'ensemble des matrices orthogonales.
- Pour un vecteur  $v \in \mathbb{R}^N$ , on pose  $\bar{v}_N = \sum_{i=1}^N \frac{v_i}{N}$  où  $v_i = v \cdot e_i$  pour tout  $i \in \{1, \dots, N\}$ . On note  $\text{diag}(v) = \sum_{i=1}^N v_i e_i e_i^\top \in \mathcal{M}_N(\mathbb{R})$ , c'est à dire la matrice diagonale dont le  $i$ -ème élément sur la diagonale est  $v_i$ .
- Pour tout ensemble  $A \subset \mathbb{R}^N$ , on note par  $A^\perp$  l'espace orthogonal à  $A$ . Pour deux vecteurs  $u, v \in \mathbb{R}^N$ , on note  $P_{v^\perp}(u) = u - P_v(u)$  la projection orthogonale de  $u$  sur l'espace  $\text{span}(v)^\perp$ .
- Pour un espace de probabilité filtré  $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ ,  $\mathcal{P}$  désignera la  $\sigma$ -algèbre des sous ensembles prévisibles de  $[0, T] \times \Omega$ . On note par  $\mathbb{L}^2$  l'espace des variables aléatoires  $F$ ,  $\mathcal{F}_T$ -mesurables et vérifiant  $\mathbb{E}[|F|^2] < \infty$ . Pour  $N \in \mathbb{N}^*$ , on note  $\mathcal{B}(\mathbb{R}^N)$  la collection des boréliens de  $\mathbb{R}^N$ ,  $\mathcal{S}^\infty(\mathbb{R}^N)$  désignera l'ensemble des processus adaptés et càdlàg bornés à valeurs réelles,  $\mathbb{H}^2(\mathbb{R}^N)$  représentera l'ensemble des processus stochastiques  $S$  à valeurs dans  $\mathbb{R}^N$  qui sont  $\mathbb{F}$ -prévisibles et tels que

$$\|S\|_{\mathbb{H}^2(\mathbb{R}^N)} = \mathbb{E} \left[ \int_0^T |S_t|^2 dt \right]^{\frac{1}{2}} < \infty.$$

- Pour une mesure finie  $\nu$ , le support est le plus petit sous-ensemble fermé de mesure pleine. L'ensemble  $\mathbb{L}^0(\nu)$  (respectivement  $\mathbb{L}_+^0(\nu)$ ) désignera l'ensemble des fonctions mesurables  $f : \text{supp}(\nu) \rightarrow \mathbb{R}$ . (respectivement  $f : \text{supp}(\nu) \rightarrow \mathbb{R}_+$ ). Pour tout  $p > 0$ , l'ensemble  $\mathbb{L}^p(\nu)$  (respectivement  $\mathbb{L}_+^p(\nu)$ ) désignera l'ensemble des fonctions mesurable  $\mathbb{L}^0(\nu)$  (respectivement,  $\mathbb{L}_+^0(\nu)$ ) dont la norme  $\mathbb{L}^p$  est finie, soit

$$\mathbb{L}^p(\nu) = \left\{ f \in \mathbb{L}^0(\nu) : \int |f(y)|^p \nu(dy) < \infty \right\}.$$

- Pour tout  $p > 0$ , on désigne par  $\mathcal{P}_p(\mathbb{R})$  la collection des mesures de probabilités avec un moment d'ordre  $p$  fini.

## 1.2 L'équilibre de Nash du modèle de détentions croisées pour une population finie

### 1.2.1 Formulation du problème

L'objectif de cette partie est d'étudier les stratégies optimales d'un nombre fini  $N$  d'agents économiques qui sont interdépendants sur le marché financier, à travers le point de vue d'équilibre de Nash. Pour ce faire, on note

$$\pi = \{ \pi^{i,j}, \quad 1 \leq i \neq j \leq N \},$$

## 1.2. L'ÉQUILIBRE DE NASH DU MODÈLE DE DÉTENTIONS CROISÉES POUR UNE POPULATION FINIE

l'ensemble des stratégies des  $N$  joueurs où  $\pi^{i,j}$  représente la part des fonds propres de l'agent  $j$  détenus par l'agent  $i$ . Ils appartiennent à l'ensemble des stratégies admissibles  $\mathcal{A}^N$  qui désigne la collection des processus  $\pi$  qui sont  $\mathbb{F}$ -prévisibles à valeurs dans  $\mathcal{M}_N(\mathbb{R}_+)$ . Donc l'ensemble des stratégies admissibles pour le  $i$ -ème agent est  $\mathcal{A}^{i,N}$  qui désigne tous les processus à valeurs dans  $\mathbb{R}_+^N$  dont la  $i$ -ème composante est nulle.

Pour  $N \in \mathbb{N}^*$ , le modèle de détentions croisées introduit dans le papier de M-F. Djete et N. Touzi [19], décrit la dynamique de la valeur des fonds propres  $X^{i,\pi}$  (pour  $1 \leq i \leq N$ ) du  $i$ -ème agent, en fonction des paramètres intrinsèques (la volatilité et le drift) et les interactions avec les autres agents à travers les investissements  $\pi$ .

Pour simplifier, on suppose que les taux d'intérêt sont ramenés à zéro, donc la condition standard d'auto-financement implique que les dynamiques des  $X^{i,\pi}$  sont données par le système d'interaction suivant:

$$dX_t^{i,\pi} = \sum_{j \neq i}^N \pi_t^{i,j} dX_t^{j,\pi} - \sum_{j \neq i}^N \pi_t^{j,i} dX_t^{i,\pi} + dP_t^i, \quad t \in [0, T], \quad i = 1, \dots, N, \quad (1.1)$$

où  $P^i$  représente le risque idiosyncratique du  $i$ -ème agent. Dans la forme vectorielle  $P_t = (P_t^1, \dots, P_t^N)$ , on suppose que  $P$  satisfait l'équation différentielle stochastique (EDS) suivante:

$$dP_t = \Gamma_t(\gamma_t dt + dW_t), \quad \text{avec la prime de risque } \gamma_t = \Gamma_t^{-1} b_t, \quad t \in [0, T], \quad (1.2)$$

où  $W_t = (W_t^1, \dots, W_t^N)^\top$  est un vecteur de mouvements browniens indépendants sur l'espace de probabilité filtré  $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ . Ici, la matrice de volatilité  $\Gamma$ , dont  $\Gamma^2$  est la matrice de variance-covariance induisant l'interdépendance entre les  $N$  agents, et le drift  $b$  sont des processus bornés  $\mathbb{F}$ -prévisibles respectivement à valeurs dans  $\mathcal{S}_N^+(\mathbb{R})$  et  $\mathbb{R}^N$  et  $\Gamma$  est inversible  $\text{Leb} \otimes \mathbb{P}$ -p.s.

L'objectif de chaque agent est d'optimiser la fonction d'utilité exponentielle suivante en choisissant les stratégies optimales  $\pi^{i,\cdot} \in \mathcal{A}^{i,N}$  qu'il doit investir

$$J^i(\pi) = J^i(\pi^{1,\cdot}, \dots, \pi^{N,\cdot}) = \mathbb{E}\left[-e^{-\frac{X_T^{i,\pi}}{q_i}}\right], \quad i = \{1, \dots, N\}, \quad \pi \in \mathcal{A}^N, \quad (1.3)$$

où le paramètre de risque tolérance  $q_i > 0$  est spécifique pour chaque agent,  $0 \leq T < \infty$ . Le point de vue qu'on adopte ici est celui où la stratégie de chaque agent est la meilleure réponse (pour maximiser sa fonction d'utilité) face aux stratégies des autres joueurs, ce qui nous conduit à l'étude d'équilibres de Nash, à savoir  $\hat{\pi} \in \mathcal{A}^N$  satisfaisant

$$J^i(\hat{\pi}^{1,\cdot}, \dots, \hat{\pi}^{i-1,\cdot}, \pi^{i,\cdot}, \hat{\pi}^{i+1,\cdot}, \hat{\pi}^{N,\cdot}) \leq J^i(\hat{\pi}), \quad \text{pour tout } \pi^{i,\cdot} \in \mathcal{A}^{i,N}. \quad (1.4)$$

Pour éviter des problèmes de bord, on va plutôt étudier les **équilibres de Nash intérieurs**, c'est à dire  $\hat{\pi}$  de sorte que  $\hat{\pi}^{i,j}$  prenne ses valeurs<sup>1</sup> dans  $(0, +\infty)$ , pour tout  $i \neq j$ .

<sup>1</sup>ce qui signifie que chaque agent investit dans tous les autres, ce qui ne correspond pas toujours à des situations usuelles.

### 1.2.2 Le cas statique

Dans ce paragraphe, les paramètres internes sont constants et les stratégies  $\pi$  (respectivement  $\pi^{i,\cdot}$ ) appartiennent à  $\mathcal{A}_{\text{const}}^N$  (respectivement  $\mathcal{A}_{\text{const}}^{i,N}$ ) qui est la collection de stratégies constantes incluses dans  $\mathcal{A}^N$  (respectivement  $\mathcal{A}^{i,N}$ ). Pour simplifier, on considère ici le cas où la maturité  $T = 1$ .

On peut réécrire l'EDS (1.1) sous la forme vectorielle suivante, pour  $\pi \in \mathcal{A}_{\text{const}}^N$ <sup>2</sup>

$$X_1^{i,\pi} = u(\pi)e_i \cdot (\gamma + W_1), \quad 1 \leq i \leq N.$$

$$\text{où } u(\pi) = \Gamma(M(\pi)^{-1})^\top, \text{ avec } M(\pi) = I_N + \sum_{i=1}^N \sum_{j \neq i}^N \pi^{i,j} (e_j - e_i)e_j^\top. \quad (1.5)$$

En analysant la condition de premier ordre liée à (1.4), on montre par un calcul direct que l'étude des équilibres de Nash intérieurs  $\hat{\pi} \in \mathcal{A}_{\text{const}}^N$  est équivalent à la résolution d'un système couplé d'équations non linéaires dont les inconnues sont les bases  $(\hat{u}^1, \dots, \hat{u}^N)$  où  $\hat{u}^i = u(\hat{\pi})e_i$ .

Le système en question est facile à étudier dans le cas où  $b = 0_{\mathbb{R}^N}$ : les solutions (sans la contrainte de positivité des contrôles) sont données par:

$$\hat{u}^i = (\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot R e_i) R e_i, \quad i = 1, \dots, N,$$

où  $R \in \mathcal{O}_N(\mathbb{R})$  et tel que  $\text{diag}(R^\top \Gamma \mathbf{1}_{\mathbb{R}^N})$  soit inversible. Bien-sûr, pour caractériser les équilibres de Nash intérieurs, il faut en plus imposer la contrainte de positivité des contrôles.

Dans la suite, on considère le cas où  $b \neq 0_{\mathbb{R}^N}$ , en commençant par l'étude analytique du système dans le cas le plus simple correspondant à  $N = 2$ , pour lequel on a pu résoudre **explicitement** le problème. C'est l'objet de la proposition suivante:

**Proposition 1.1.** *Soit  $N = 2$  et supposons que  $b \neq 0_{\mathbb{R}^2}$  et  $\Gamma$  sont constants. Il existe un équilibre de Nash intérieur si et seulement si*

$$\bar{b}_2 \neq 0, \quad A := |\Gamma \mathbf{1}_{\mathbb{R}^2}|^2 - \left( \frac{\bar{b}_2(q_1 - q_2)}{|\gamma| \bar{q}_2} \right)^2 - \frac{4q_1 q_2 \bar{b}_2}{\bar{q}_2} \geq 0, \quad \text{et } \alpha = \alpha_+ \text{ ou } \alpha_- \text{ satisfait}$$

$$\frac{1}{\alpha} < -\frac{b_i}{\bar{b}_2 q_i \det(\Gamma)}, \quad i = 1, 2 \quad \text{où } \alpha_{\pm} = \frac{\bar{b}_2}{|\gamma|} \left( \frac{\det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})(q_2 - q_1)}{2|\gamma|} \pm \bar{q}_2 \sqrt{A} \right).$$

Donc, il existe au plus deux équilibres de Nash intérieurs stationnaires  $(\hat{\pi}_{\pm}^{1,2}, \hat{\pi}_{\pm}^{2,1})$  donnés par  $\hat{\pi}_{\pm}^{i,j} = -\frac{1}{2} \left( \frac{q_i \det(\Gamma)}{\alpha_{\pm}} + \frac{b_i}{\bar{b}_2} \right)$ .

**Remark 1.2** (Interprétation et unicité des équilibres pour  $N = 2$ ). Dans le cas où  $b \neq 0_{\mathbb{R}^2}$ , on peut extraire des conditions sous lesquelles les équilibres sont uniques. Par exemple, si  $q_1 = q_2 = q$  et  $\bar{b}_2 > 0$ , alors il existe un unique équilibre de Nash intérieur statique  $\hat{\pi}$  si et seulement si  $\min(b_1, b_2) > 0$ ,  $|\Gamma \mathbf{1}_{\mathbb{R}^2}| > 4q\bar{b}_2$  et  $\frac{|\gamma| \det(\Gamma)}{\sqrt{|\Gamma \mathbf{1}_{\mathbb{R}^2}| - 4q\bar{b}_2}} > \max(b_1, b_2)$ .

<sup>2</sup>la matrice  $M(\pi)$  est inversible et  $(M(\pi)^{-1})^\top$  est une matrice stochastique (voir Lemme 2.1 dans Chapitre 1).

## 1.2. L'ÉQUILIBRE DE NASH DU MODÈLE DE DÉTENTIONS CROISÉES POUR UNE POPULATION FINIE

Si  $b_i < 0$ , pour  $i = 1$  ou  $i = 2$ , alors l'agent  $j \neq i$  a un drift  $b_j > 0$  vu que  $\bar{b}_2 > 0$ . Donc, il ne souhaite pas détenir des parts de l'agent  $i$ , ce qui veut dire que  $\hat{\pi}^{j,i} = 0$ , induisant une stratégie sur le bord.

En faisant évoluer les paramètres, on a simulé les investissements  $\hat{\pi}^{i,j}$  pour interpréter les résultats. Il s'avère de nos simulations que pour minimiser le risque de perte, les agents évitent d'investir dans des portefeuilles corrélés aux leurs ou très volatiles. Par ailleurs, si l'un des agents a une dynamique de fonds propres avec une tendance à la hausse, alors il est plus performant et son compétiteur va donc investir d'avantage sur lui.

Quand  $N$  croit, la taille du système considéré pour la condition de premier ordre (1.4) croit naturellement et l'analyse devient très compliquée. Cependant, on a pu caractériser les équilibres de Nash ainsi que leurs conditions d'existence. Par ailleurs, on a noté que, pour  $N \geq 3$ , il y a dans la plupart des cas *un nombre infini de solutions*. La méthodologie principale qu'on a utilisée est la suivante: en notant  $\eta_i = \frac{q_i}{N\bar{q}_N}$ , on a décomposé les vecteurs  $\hat{u}^i$  en fonctions des paramètres <sup>3</sup>

$$\hat{u}^i = N\eta_i\bar{b}_N \frac{\gamma}{|\gamma|^2} + c_i P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) + w^i, \quad i = 1, \dots, N. \quad (1.6)$$

Conformément à (1.6), en introduisant le vecteur  $c = \sum_{i=1}^N c_i e_i \in \mathbb{R}^N$  et la matrice  $W(c) \in \mathcal{M}_N(\mathbb{R})$  dont la  $i$ -ème colonne est le vecteur  $w^i$ <sup>4</sup>, on trouve que dans le cas général où le paramètre  $\beta_0 = N^2\bar{b}_N \left( \bar{q}_N - \frac{\bar{b}_N}{|\gamma|^2} \right) \neq 0$ <sup>5</sup>, la condition de premier ordre liée à (1.4) est équivalente à avoir

$$W^\top(c)W(c) = A(c) = D(c) - (|P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N})|^2 cc^\top - \beta_0 \eta \eta^\top) \quad (1.7)$$

où  $c \in \Theta = \left\{ v \in \mathbb{R}^N : v \cdot \mathbf{1}_{\mathbb{R}^N} = 1, \det D(v) \neq 0, \beta_0 \eta^\top D(v)^{-1} \eta = -1, \text{ et } A(v) \geq 0 \right\}$ ,

avec  $D(c) = \text{diag}(|P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N})|^2 c - \beta_0 \eta)$ . Grâce au théorème spectral, la matrice  $A(c)$  définie ci-dessus admet la décomposition suivante qui sera utile pour décrire l'ensemble des équilibres de Nash: il existe des matrices  $P(c) \in \mathcal{O}_N(\mathbb{R})$  et  $S(c) \in \mathcal{M}_N(\mathbb{R})$  telles que

$$A(c) = P(c)S(c)(P(c)S(c))^\top, \quad (1.8)$$

avec  $S(c)S(c)^\top \in \mathcal{D}_N(\mathbb{R})$  et  $\text{Ker}(S(c)) = \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N})$ .

Le résultat principal de cette partie est le théorème suivant qui fournit la description de l'ensemble des équilibres de Nash et de leurs conditions d'existence dans le cas stationnaire général:

**Théorème 1.3.** *Avec les notations précédentes, supposons que  $b \neq 0_{\mathbb{R}^N}$  et  $\beta_0 \neq 0$ . Alors, il existe un équilibre de Nash intérieur stationnaire  $\hat{\pi} \in \mathcal{A}_{\text{const}}^N$  si et seulement si  $\Theta \neq \emptyset$  et la matrice suivante*

$$M^P = \left( -\frac{\beta_0}{N\bar{b}_N} b \eta^\top + \Gamma P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) c^\top + \Gamma (P(c)S(c))^\top \right) D^{-1}(c), \quad (1.9)$$

<sup>3</sup>Le système résultant de la condition de premier ordre implique que  $\hat{u}^i \cdot \gamma = N\eta_i\bar{b}_N$ .

<sup>4</sup>Au vu de la condition du premier ordre et (1.6), la matrice  $W$  dépend de  $c$ .

<sup>5</sup>Le cas particulier  $\beta_0 = 0$  a été traité de manière similaire à celui où  $b = 0_{\mathbb{R}^N}$ .

a des coefficients négatifs en dehors de la diagonale, pour un vecteur  $c \in \Theta$  avec des matrices  $S(c), P(c)$  vérifiant la décomposition (1.8).

Dans ce cas,  $\hat{\pi}^{i,j} = -M_{i,j}^P$  pour tout  $1 \leq i \neq j \leq N$ , définit un équilibre de Nash intérieur stationnaire.

**Remark 1.4.** Comme  $\hat{\pi}^{i,j} = -M_{i,j}^P$ , il découle de (1.9) que dans le cas où  $q_i = q$ , pour tout  $i \in \{1, \dots, N\}$ , alors on a

$$Y^j = \sum_{i \neq j}^N (\hat{\pi}^{j,i} - \hat{\pi}^{i,j}) = 1 - \frac{b_j}{\bar{b}_N}, \quad j = 1, \dots, N. \quad (1.10)$$

La variable  $Y^j$  représente le niveau net de détention du  $j$ -ème agent. Si  $\bar{b}_N > 0$ , ceux qui ont un drift  $b_j$  avec une valeur supérieure à la moyenne  $\bar{b}_N$ , sont en détention nette négative, d'autant plus négative que le drift est grand. Cela signifie que les agents ont tendance à détenir des parts plus importantes parmi ceux qui ont les meilleurs drifts.

Lorsque le nombre d'agents économiques est  $N \geq 3$ , on a montré que, dans le cas générique où  $|\eta|_\infty \leq \frac{1}{2}$ ,

$$\Theta \neq \emptyset \Leftrightarrow \beta_0 < \beta_1 \Leftrightarrow N^2 \bar{b}_N \bar{q}_N < |\Gamma \mathbf{1}_{\mathbb{R}^N}|^2. \quad (1.11)$$

Intuitivement, cette condition peut s'interpréter comme suit : lorsque la condition n'est pas satisfaite, c'est à dire  $|\Gamma \mathbf{1}_{\mathbb{R}^N}|^2 \leq N^2 \bar{q}_N \bar{b}_N$ , cela signifie que les investissements des agents ne génèrent pas de revenus suffisants en fonction de leur prise de risque, puisque la volatilité n'a pas une valeur suffisamment élevée.

### 1.2.3 Le modèle en temps continu à coefficients déterministes

Chaque agent  $i \in \{1, \dots, N\}$  est concerné par la maximisation de sa fonction d'utilité exponentielle (1.3), et comme on peut le voir dans la vaste littérature sur ce sujet, ce problème d'optimisation (dans le cas à temps continu) peut être formulé en fonction des équations différentielles stochastiques rétrogrades (EDSR) quadratiques (voir par exemple les articles de N. El Karoui et R. Rouge [21], Y. Hu et P. Imkeller [33] ou G-E. Espinosa et N. Touzi [25]).

En ce qui concerne notre problème, par un argument de vérification, nous nous attendons à ce que le critère de chaque agent  $i \in \{1, \dots, N\}$  prenne la forme suivante, sous l'existence d'un équilibre de Nash intérieur  $\hat{\pi} \in \mathcal{A}^N$ ,

$$J^i(\hat{\pi}) = -e^{-\frac{x_i + Y_0^{\hat{\pi}} \cdot e_i}{q_i}}, \quad i = 1, \dots, N$$

où  $(Y_t^{\hat{\pi}}, Z_t)_{t \in [0, T]}$  est un couple de processus  $(\mathcal{F}_t)$ -progressivement mesurables à valeurs dans  $\mathbb{R}^N \times \mathbb{R}^{N \times N}$  qui satisfont une EDSR telle que  $Y_T^{\hat{\pi}} = \sum_{i=1}^N (\frac{q_i}{2} \int_0^T |\gamma_t|^2 dt) e_i$  et dont le générateur  $Q_t^{\hat{\pi}}$  est mesurable par rapport à  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{N \times N})$  et satisfait

$$\begin{aligned} Q^{\hat{\pi}} : [0, T] \times \Omega \times \mathbb{R}^{N \times N} &\rightarrow \mathbb{R}^N \\ (t, \omega, Z) &\mapsto \sum_{i=1}^N \frac{1}{2q_i} |u_t(\hat{\pi}_t) e_i - \zeta_t^i|^2 e_i, \end{aligned}$$

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<sup>6</sup>Sinon, il existe un agent  $i$  qui est plus tolérant au risque que tous les autres agents pris ensemble.

### 1.3. LE JEU À CHAMP MOYEN DU MODÈLE DE DÉTENTIONS CROISÉES À UNE PÉRIODE AVEC BRUIT COMMUN

avec  $\zeta^i = q_i \gamma - Z e_i$  et l'équilibre de Nash  $\hat{\pi} \in \mathcal{A}^N$  vérifie pour tout  $t \in [0, T]$  et tout  $i \in \{1, \dots, N\}$ ,

$$\left| u_t(\hat{\pi}_t) e_i - \zeta_t^i \right|^2 \leq \left| u_t(\hat{\pi}_t^{1\cdot}, \dots, \hat{\pi}_t^{i-1\cdot}, \pi_t^{i\cdot}, \hat{\pi}_t^{i+1\cdot}, \dots, \hat{\pi}_t^{N\cdot}) e_i - \zeta_t^i \right|^2, \quad \pi^{i\cdot} \in \mathcal{A}^{i,N}. \quad (1.12)$$

Dans le cas où les paramètres  $\gamma, \Gamma$  sont déterministes et les stratégies  $\hat{\pi} \in \mathcal{A}_{\text{det}}^N$  (où  $\mathcal{A}_{\text{det}}^N$  est la collection de stratégies déterministes incluses dans  $\mathcal{A}^N$ ), on a  $Z = 0_{\mathbb{R}^{N \times N}}$ ,  $\mathbb{P}$ -p.s et le problème d'optimisation (1.12) est équivalent à celui étudié au cas stationnaire, pour chaque instant  $t \in [0, T]$ . C'est énoncé dans le théorème suivant que nous avons démontré

**Theorem 1.5.** *Supposons que  $\gamma$  et  $\Gamma$  sont des processus déterministes. Si  $\hat{\pi} \in \mathcal{A}_{\text{det}}^N$  vérifie (1.12) avec  $Z = 0_{\mathbb{R}^{N \times N}}$ ,  $\mathbb{P}$ -p.s, alors c'est un équilibre de Nash intérieur déterministe.*

## 1.3 Le jeu à champ moyen du modèle de détentions croisées à une période avec bruit commun

### 1.3.1 Formulation du problème

Dans cette partie, on considère un modèle légèrement modifié de celui étudié dans la première partie afin d'être en mesure de passer au modèle champ moyen. On suppose que le processus  $X^i$  de la valeur des fonds propres du  $i$ -ème agent,  $i \in \{1, \dots, N\}$ , satisfait la variation suivante:

$$\Delta X^i = \frac{1}{N} \sum_{j \neq i}^N \beta(X_0^i, X_0^j) \Delta X^j - \frac{1}{N} \sum_{j \neq i}^N \pi(X_0^j, X_0^i) \Delta X^i + \Delta P^i \quad (1.13)$$

avec  $\Delta P^i = b(X_0^i) + \sigma(X_0^i) \varepsilon_i + \sigma^0(X_0^i) \varepsilon^0$ ,

où pour tout processus  $S$ , on note la variation  $\Delta S = S_1 - S_0$ . On a noté respectivement par  $\{\beta(X_0^i, X_0^j)\}_{j \neq i}$  et  $\{\pi(X_0^j, X_0^i)\}_{j \neq i}$  les variables de décision du  $i$ -ème agent considéré et des autres agents  $j \neq i$ .

On fait les hypothèses suivantes :

- les variables aléatoires  $(X_0^i)_{i=1, \dots, N}$  sont identiquement distribuées, avec  $X_0^i \stackrel{L}{\sim} \mu_0$ ,
- les variables aléatoires centrées et réduites  $\varepsilon^0$  et  $(\varepsilon_1, \dots, \varepsilon_N)$  sont iid, avec  $\varepsilon^0 \stackrel{L}{\sim} \rho$ ,
- les variables aléatoires  $\varepsilon^0$ ,  $(\varepsilon_i)_{i=1, \dots, N}$  et  $(X_0^i)_{i=1, \dots, N}$  sont indépendantes,
- les fonctions mesurables  $\sigma, \sigma^0, b$  sont bornées, et les termes de diffusion ne s'annulent pas, c'est-à-dire  $\sigma^0(X_0) \neq 0$  et  $\sigma(X_0) \neq 0$ ,  $\mu_0$  - p.s.

Par passage à la limite dans (1.13), la variation des fonds propres de l'agent représentatif prend la forme suivante :

$$\frac{\Delta X^{\mu, \pi, \beta}(X_0)}{m^\pi(X_0)} = \widehat{\mathbb{E}}^{\mu_0} [\beta(X_0, \widehat{X}_0) F^\mu(\widehat{X}_0, \varepsilon^0)] + \Delta P, \quad \text{avec } F^\mu(\widehat{X}_0, \varepsilon^0) = \widehat{\mathbb{E}}^\mu[\Delta \widehat{X} | \widehat{X}_0, \varepsilon^0], \quad (1.14)$$

où  $\Delta X^{\mu,\pi,\beta}(X_0) = X_1^{\mu,\pi,\beta} - X_0$ ,  $\mu$  est la loi du couple  $(X_0, X_1)$ , conditionnée par rapport au bruit commun  $\varepsilon^0$  et  $\frac{1}{m^\pi(X_0)} = 1 + \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0, X_0)]$ .

Dans les équations ci-dessus,  $\widehat{\mathbb{E}}^{\mu_0}[\psi(X_0, \widehat{X}_0, \varepsilon^0)] = \int \psi(X_0, \hat{x}_0, \varepsilon^0) \mu_0(d\hat{x}_0)$ <sup>7</sup>, et  $\mu, \pi$  définissent respectivement l'environnement aléatoire représentant la loi conditionnelle des fonds propres par rapport au bruit commun et la stratégie de détentions mutuelles de la population environnante. Remarquons que  $\mu$  n'influence les dynamiques de  $(X_0, X_1)$  qu'à travers l'espérance conditionnelle  $F^\mu \in \mathbb{L}^1(\mu_0 \otimes \rho)$ . Par conséquent, l'environnement aléatoire se réduit au couple  $(F^\mu, \pi)$ .

Notre objectif principal dans cette partie est de trouver un équilibre de Nash dans le modèle champ moyen où l'agent représentatif cherche à maximiser son critère

$$J(X_0, \beta; F^\mu, \pi) := \mathcal{U}(\mathcal{L}(X_1^{\mu,\pi,\beta}(X_0)|X_0)),$$

pour une certaine fonction  $\mathcal{U} : \mathcal{P}_p(\mathbb{R}) \rightarrow \mathbb{R}$ , avec  $p > 0$ . Bien sûr, ceci exige de restreindre l'ensemble des stratégies admissibles à la collection  $\mathcal{A}_p(F^\mu, \pi)$  de toutes les stratégies de détentions croisées  $\beta$  qui assurent la  $\mathbb{L}^p$ -intégrabilité de la variable  $X_1^{\mu,\pi,\beta}$ <sup>8</sup>.

**Définition 1.1** (Jeu à champ moyen du modèle détentions croisées à une période). *Un environnement aléatoire  $(F^\mu, \pi)$  est un équilibre du jeu à champ moyen du modèle détentions croisées si  $F^\mu \in \mathbb{L}^1(\mu_0 \otimes \rho)$  et  $\pi \in \mathcal{A}_p(F^\mu, \pi)$  vérifient*

$$(i) \quad J(X_0, \pi; F^\mu, \pi) = \sup_{\beta \in \mathcal{A}_p(F^\mu, \pi)} J(X_0, \beta; F^\mu, \pi),$$

$$(ii) \quad \mathbb{E}[\Delta X^{\mu,\pi,\pi}(X_0)|X_0, \varepsilon^0] = F^\mu(X_0, \varepsilon^0).$$

### 1.3.2 Notion de non-arbitrage (NA)

Dans le modèle (1.14), on remarque que les fonds propres de l'agent  $X_1^{\mu,\pi,\beta}$  croissent dès que  $G_\mu^\beta(X_0, \varepsilon^0) := \widehat{\mathbb{E}}^{\mu_0}[\beta(X_0, \widehat{X}_0)F^\mu(\widehat{X}_0, \varepsilon^0)]$  croît. On introduit alors la notion suivante de **non-arbitrage** NA:

**Définition 1.2** (NA). *Soit  $(F^\mu, \pi) \in \mathbb{L}^1(\mu_0 \otimes \rho) \times \mathbb{L}^1(\mu_0 \otimes \mu_0)$ . On dit que  $(F^\mu, \pi)$  satisfait la condition de non arbitrage NA si pour tout  $\beta \in \mathcal{A}_1(F^\mu, \pi)$ , on a :*

$$G_\mu^\beta(X_0, \varepsilon^0) \geq 0, \mu_0 \otimes \rho - p.s \implies G_\mu^\beta(X_0, \varepsilon^0) = 0, \mu_0 \otimes \rho - p.s. \quad (1.15)$$

Le théorème suivant permet de caractériser cette condition:

**Théorème 1.6** (NA). *Les deux assertions suivantes sont équivalentes:*

1.  $(F^\mu, \pi)$  satisfait la condition NA;
2. Il existe  $Z$  qui est  $\varepsilon^0$ -mesurable tel que

$$Z > 0, \rho - p.s, \mathbb{E}^\rho[Z] = 1 \text{ et } \mathbb{E}^\rho[Z F^\mu(X_0, \varepsilon^0)] = 0, \mu_0 - p.s. \quad (1.16)$$

<sup>7</sup>de même  $\mathbb{E}^\rho[\psi(X_0, \widehat{X}_0, \varepsilon^0)] = \int \psi(X_0, \hat{x}_0, \varepsilon^0) \rho(d\varepsilon^0)$ .

<sup>8</sup>sous les hypothèses, on obtient que  $\frac{1}{m^\pi(X_0)} \neq 0, \mu_0 - a.s.$

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*Si de plus  $(\mu, \pi)$  est un équilibre champs moyen, avec la variable aléatoire  $Z$  du (ii), la dernière condition est équivalente à*

$$b(X_0) + \mathbb{E}^\rho[Z\varepsilon^0]\sigma^0(X_0) = 0, \quad \mu_0 - p.s. \quad (1.17)$$

Sous un équilibre, la condition NA doit être vérifiée. Par conséquent, il suffit de chercher des équilibres qui la satisfont. De ce fait, la propriété (1.17) met en évidence que, sous un équilibre, la relation de proportionnalité entre le drift  $b$  et le terme de diffusion  $\sigma^0$  implique la condition NA<sup>9</sup>. Donc, on va chercher des équilibres tels que cette relation de proportionnalité (1.17) soit vérifiée.

#### 1.3.3 Critère moyenne-variance

Dans ce paragraphe, on étudie le jeu à champ moyen correspondant au critère moyenne-variance suivant:

$$\sup_{\beta \in \mathcal{A}_2(F^\mu, \pi)} \mathbb{M}\mathbb{V}_q[X_1^{\mu, \pi, \beta} | X_0], \quad \text{où } \mathbb{M}\mathbb{V}_q[\cdot | X_0] := \mathbb{E}[\cdot | X_0] - \frac{1}{2q} \mathbb{V}[\cdot | X_0]. \quad (1.18)$$

Le choix d'un tel critère est basé sur le fait que si l'on suppose que la variable aléatoire  $\widehat{X}_1^{\mu, \pi, \pi}$  soit de carré intégrable satisfaisant, pour  $\pi \in \mathbb{L}^2(\mu_0 \otimes \mu_0)$ ,

$$\Delta \widehat{X}^{\mu, \pi, \pi}(X_0) = B(\widehat{X}_0) + \Sigma^0(\widehat{X}_0)\varepsilon^0 + \Sigma(\widehat{X}_0)\varepsilon, \quad (1.19)$$

alors, en injectant (1.19) dans (1.14) où on rappelle que  $\varepsilon$  est une variable aléatoire centrée indépendante de  $\varepsilon^0$ , on obtient

$$\frac{\Delta X^{\mu, \pi, \beta}(X_0)}{m^\pi(X_0)} = \widehat{\mathbb{E}}^{\mu_0} \left[ \beta(X_0, \widehat{X}_0) (B(\widehat{X}_0) + \Sigma^0(\widehat{X}_0)\varepsilon^0) \right] + \Delta P. \quad (1.20)$$

En l'absence de bruit commun, pour accroître ses gains, l'agent représentatif est amené à considérer un problème d'optimisation dans lequel il applique son contrôle uniquement sur le drift du compétiteur. C'est ce qui a été étudié dans le papier de M-F. Djete et N. Touzi [19], en temps continu.

Dans notre cas, comme le bruit commun ne s'annule pas, l'agent est amené à appliquer son contrôle à la fois sur le drift et sur le terme de diffusion du compétiteur, ce qui amène à faire un compromis entre la moyenne et la variance.

On souligne que vu que le critère moyenne-variance n'est pas monotone, il n'est pas clair que la condition de non-arbitrage soit une condition nécessaire pour l'existence d'un équilibre. Pour cette raison, on a cherché des équilibres sans imposer la contrainte (NA) sur les coefficients. En étudiant le problème, on se rend compte que la propriété (ii) dans la définition 1.1 est équivalente au fait que, sous un équilibre, l'espérance conditionnelle  $F^\mu$  vérifie l'équation suivante,

$$\frac{F^\mu(X_0, \varepsilon^0)}{m^\pi(X_0)} = b(X_0) + \sigma^0(X_0)\varepsilon^0 + \widehat{\mathbb{E}}^{\mu_0} [\pi(X_0, \widehat{X}_0) F^\mu(\widehat{X}_0, \varepsilon^0)], \quad \mu_0 \otimes \rho - p.s. \quad (1.21)$$

<sup>9</sup>Par analogie, ceci est similaire à la restriction drift-volatilité du modèle HJM [32].



Il s'agit d'une équation de Fredholm de seconde espèce (voir le papier de I. Fredholm [29]). De manière remarquable, on trouve que si elle admet une unique solution  $F^\mu$ , alors nécessairement l'équilibre associé  $(F^\mu, \pi)$  satisfait la condition NA à travers la condition de proportionnalité (1.17). C'est l'objet du théorème suivant:

**Theorem 1.7.** *Il existe des équilibres au jeu à champs moyen  $(F^\mu, \pi) \in \mathbb{L}^2(\mu_0 \otimes \rho) \times \mathbb{L}^2(\mu_0 \otimes \mu_0)$  avec une unique solution de Fredholm (1.21) si et seulement si*

$$\mathbb{E}[b(X_0)] > 0 \text{ and } \sigma^0 = \lambda b \text{ with } \lambda = \pm \sqrt{\frac{q}{\mathbb{E}[b(X_0)]}},$$

et les stratégies  $\pi \in \mathbb{L}^2(\mu_0 \otimes \mu_0)$  induisent l'unicité pour l'équation de Fredholm (1.21) et satisfont

$$0 \neq \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)] + \frac{b(X_0)}{\mathbb{E}[b(X_0)]} = 1 + \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0, X_0)], \mu_0 - p.s. \quad (1.22)$$

Dans ce cas,  $F^\mu$  est explicitement donné par

$$F^\mu(X_0, \varepsilon^0) = \mathbb{E}[b(X_0)](1 + \lambda \varepsilon^0), \mu_0 \otimes \rho - p.s. \quad (1.23)$$

**Remark 1.8.** On peut réécrire l'équation (1.22) comme suit:

$$Y(X_0) = \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)] - \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0, X_0)] = 1 - \frac{b(X_0)}{\mathbb{E}[b(X_0)]}, \mu_0 - p.s.$$

Cette équation est similaire à celle qu'on a obtenue dans le premier chapitre pour les équilibres de Nash intérieurs statiques,  $\hat{\pi} \in \mathcal{A}_{\text{const}}^N$  (1.10), dans le cas où tous les agents ont le même paramètre de tolérance au risque  $q_i = q$ . C'est naturel, car nous y avons étudié le jeu à population finie où chaque agent souhaite maximiser sa fonction d'utilité exponentielle avec l'hypothèse que les variables aléatoires indépendantes  $\varepsilon^0$  et  $(\varepsilon_1, \dots, \varepsilon_N)$  sont des gaussiennes centrées et réduites. Par un simple calcul, ce problème est équivalent au jeu où chaque agent souhaite maximiser son critère moyenne-variance.

Vu que l'on considère des variables de contrôles  $\pi \in \mathbb{L}^2(\mu_0 \otimes \mu_0)$ , l'unicité de (1.21) n'est pas exigée. Comme on le verra dans le Chapitre 3 concernant le modèle à temps continu, l'unicité sera assurée par un choix plus restrictif de l'ensemble des stratégies admissibles.

Cependant l'ensemble des solutions  $\pi \in \mathbb{L}^2(\mu_0 \otimes \mu_0)$  qui satisfont l'équation (1.22) et qui assurent l'unicité pour l'équation de Fredholm (1.21) n'est pas vide. On fournit, en exemple, les stratégies d'équilibres suivantes  $\pi(x, \hat{x})$  qui dépendent de l'argument  $\hat{x}$  et d'une constante  $c$ :

$$\pi(x, \hat{x}) = \frac{b(\hat{x})}{\mathbb{E}[b(X_0)]} - c \text{ tel que } (c - 1)\mathbb{E}[b(X_0)] \notin b(\text{supp}(\mu_0)).$$

Sous ces équilibres, on conclut que les agents tendent plus à investir parmi ceux qui ont une meilleur tendance  $b$ .

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#### 1.3.4 Retour au jeu à population finie

On rappelle que par le théorème 1.6, la condition de non arbitrage (NA), à l'équilibre, se réduit à la condition de proportionnalité entre le drift  $b$  et le terme de diffusion  $\sigma^0$ .

Dans ce paragraphe, on va explorer cette condition de proportionnalité dans le jeu à population finie. On suppose donc qu'il  $\ell \in \mathbb{R}^*$  tel que

$$\sigma^0 = \ell b. \quad (1.24)$$

Puis on suppose que les variables aléatoires  $(\varepsilon^0, \varepsilon_1, \dots, \varepsilon_N)$  sont des gaussiennes centrées, réduites et iid, avec  $\varepsilon^0 \stackrel{L}{\sim} \rho$ . Pour tout  $i \in \{1, \dots, N\}$ , les fonds propres  $X_1^{i,\pi}$  satisfont (1.13) où les investissements  $\pi(X_0^{\cdot,N}) = (\pi(X_0^i, X_0^j))_{1 \leq i \neq j \leq N}$  ne sont plus d'ordre  $\frac{1}{N}$ , comme il l'a été considéré dans le jeu fini, et ils appartiennent à l'ensemble suivant des stratégies admissibles:

$$\mathcal{B}^N = \left\{ \pi(X_0^{\cdot,N}) \in \mathcal{M}_N(\mathbb{R}) : M(\pi) \text{ non singulière et } X_1^{\cdot,\pi,N} \in \mathbb{L}^2(\mu_0^{\otimes N} \otimes \rho^{\otimes N}) \right\},$$

où l'on a noté  $X_t^{\cdot,\pi,N} = \sum_{i=1}^N X_t^{i,\pi} e_i$  pour  $t = 0, 1$  et où  $M(\pi)$  est définie par (1.5). Cet ensemble de stratégies  $\mathcal{B}^N$  diffère de celui utilisé au premier chapitre, car on ne tient pas compte des contraintes de positivité des contrôles. On notera par la suite  $\mathcal{B}^{i,N}$ , le nouvel ensemble des stratégies admissibles du  $i$ -ème agent, à savoir

$$\mathcal{B}^{i,N} = \{ \pi(X_0^{\cdot,N})_{i,\cdot} \in \mathbb{R}^N : \pi \in \mathcal{B}^N \}.$$

Pour  $i \in \{1, \dots, N\}$ , chaque agent  $i$  cherche à maximiser son critère mean variance pour un  $q^i > 0$  donné

$$\sup_{\pi(X_0^{\cdot,N})_{i,\cdot} \in \mathcal{B}^{i,N}} \text{MV}_{q^i}(X_1^{i,\pi,N} | X_0^{\cdot,N}) \text{ où } \text{MV}_{q^i}(\cdot | X_0^{\cdot,N}) = \mathbb{E}[\cdot | X_0^{\cdot,N}] - \frac{1}{2q^i} \mathbb{V}(\cdot | X_0^{\cdot,N}). \quad (1.25)$$

**Remarque 1.9** (Lien avec le chapitre 1). Rappelons que dans la première partie, chaque agent maximise sa fonction d'utilité exponentielle (1.3). Dans le cas statique et comme souligné dans la remarque 1.8, ce problème est équivalent à celui de la moyenne-variance (1.25), où  $q^i$  est le paramètre de tolérance au risque de l'agent  $i$ . Donc, les coefficients  $q^i$  auront la même interprétation, et on fera appel aux résultats obtenus dans le premier chapitre.

Rappelons aussi que dans le premier chapitre, sans tenir compte de la contrainte de positivité des variables de contrôle, on trouve des résultats d'existence explicites correspondants au problème (1.3), sous l'hypothèse <sup>10</sup>

$$q^i \leq \sum_{j \neq i}^N q^j, \text{ pour tout } i \in \{1, \dots, N\}. \quad (1.26)$$

Dans ce paragraphe, on va imposer les mêmes hypothèses sur les paramètres  $q^i$ .

<sup>10</sup>Cela signifie qu'il n'existe pas d'agent  $i$  qui soit plus tolérant au risque que tous les autres agents  $j \neq i$  pris ensemble.

Ensuite, on trouve que sous des conditions suffisantes imposées sur le coefficient  $\ell$  dans (1.24), il existe bien des équilibres, comme l'atteste le théorème suivant:

**Théorème 1.10.** *Sous les hypothèses (1.24) et (1.26), il existe un équilibre de Nash  $\hat{\pi} \in \mathcal{B}^N$  correspondant au critère moyenne-variance (1.25) si la condition suivante est satisfaite*

$$\bar{b}_N(X_0^{\cdot N}) \neq 0, \mu_0^{\otimes N} - p.s \text{ et } \ell^2 > \frac{1}{\bar{b}_N(X_0^{\cdot N})} \left( \bar{q}_N - \frac{\bar{\sigma}_N^2(X_0^{\cdot N})}{\bar{b}_N(X_0^{\cdot N})} \right), \mu_0^{\otimes N} - p.s. \quad (1.27)$$

**Remarque 1.11.** On note que si  $\bar{b}_N(X_0^{\cdot N}) \neq 0, \mu_0^{\otimes N} - p.s$  et  $\bar{b}_N(X_0^{\cdot N})\bar{q}_N < \bar{\sigma}_N^2(X_0^{\cdot N}), \mu_0^{\otimes N} - p.s$ , alors il existe un équilibre de Nash  $\hat{\pi} \in \mathcal{B}^N$ . Cette condition est similaire à celle trouvée dans le chapitre 1 (voir remarque 3.10 du chapitre 1), sans prendre en compte la contrainte de signe des contrôles.

## 1.4 Le jeu à champ moyen du modèle de détentions croisées avec bruit commun en temps continu

Dans cette partie, on étend l'étude du modèle champ moyen mutual holding avec bruit commun au cadre du temps continu. Tout d'abord, nous devons définir des notions sur l'espace des mesures et des processus qui seront utiles par la suite:

(i) Pour un espace Polonais  $E$ , on note  $\mathcal{P}_2(E)$  la collection de toutes les mesures de probabilité, avec un second moment fini où,  $E$  sera ou bien  $\mathbb{R}^d$ , ou bien l'ensemble  $\mathcal{C}$  de toutes les applications continues de  $[0, T]$  dans  $\mathbb{R}$ . Le processus canonique sur  $\mathcal{C}$  est  $\hat{X} : \mathcal{C} \rightarrow \mathbb{R}$  défini par  $\hat{X}_t(\hat{\omega}) = \hat{\omega}(t)$  pour tout  $t \in [0, T]$  et  $\hat{\omega} \in \mathcal{C}$ .

(ii) Soit  $(W, W^0)$  un mouvement brownien sur un espace de probabilité  $(\Omega, \mathbb{F}, \mathbb{P})$ , on note  $\mathbb{F}^0 := (\mathcal{F}_t^0)_{t \in [0, T]}$  le  $\mathbb{P}$ -complété de la filtration canonique engendrée par  $W^0$ . Pour  $\mu_0$  fixée dans  $\mathcal{P}_2(\mathbb{R})$ , on note

- $\mathcal{S}_2$  la collection de tous les processus d'Itô de carré intégrable  $S$  qui sont  $\mathbb{F}$ -adaptés avec loi initiale  $\mathcal{L}(S_0) = \mu_0$ , et satisfaisant

$$dS_t = B_t(S_t)dt + \Sigma_t(S_t)dW_t + \Sigma_t^0(S_t)dW_t^0, \mathbb{P}\text{-p.s.} \quad (1.28)$$

avec  $\mathcal{L}(S_{t \wedge \cdot} | \mathcal{F}_T^0) = \mathcal{L}(S_{t \wedge \cdot} | \mathcal{F}_t^0) dt \otimes d\mathbb{P}\text{-p.p.}$ ,

- et  $\mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  la collection de toutes les mesures qui sont  $\mathcal{F}_T^0$ -mesurables et telles que  $\mu = \mathcal{L}(S | \mathcal{F}_T^0)$  pour  $S \in \mathcal{S}_2$ . Pour un tel  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$ , on notera  $\mu_t = \mathcal{L}(S_t | \mathcal{F}_t^0) = \mu \circ (\hat{X}_t)^{-1}$  et  $\hat{\mathbb{E}}^\mu[\phi(\hat{X})] = \int \phi(\hat{\omega})\mu(d\hat{\omega})$ , pour toute application  $\phi : \Omega \times \mathcal{C} \rightarrow \mathbb{R}$  telle que  $\phi(\omega, \cdot)$  est  $\mu$ -intégrable,  $\mathbb{P}$ -p.p.  $\omega$ .

### 1.4.1 Formulation du problème

La dynamique des portefeuilles de l'agent représentatif est donnée par

$$dX_t = dP_t + \hat{\mathbb{E}}^\mu[\beta_t(X_t, \hat{X}_t)d\hat{X}_t] - \hat{\mathbb{E}}^\mu[\pi_t(X_t, \hat{X}_t)]dX_t, \mathbb{P} - p.s, \quad (1.29)$$

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où le processus de risque idiosyncratique  $P$  est défini par

$$dP_t = b_t(X_t, \mu_t)dt + \sigma_s(X_t, \mu_t)dW_t + \sigma_t^0(X_t, \mu_t)dW_t^0, \quad \mathbb{P}\text{-p.s.} \quad (1.30)$$

Ci-dessus,  $\widehat{\mathbb{E}}^\mu$  désigne l'espérance conditionnelle sur l'espace des trajectoires continues des copies de  $\widehat{X}$  à l'équilibre, par rapport à la loi  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$ , conditionnée par  $W^0$ . Pour mieux comprendre l'espérance  $\widehat{\mathbb{E}}^\mu[\beta_t(X_t, \widehat{X}_t)d\widehat{X}_t]$  qui apparaît dans l'équation (1.29), on rappelle que comme  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  et  $\mu = \mathcal{L}(\widehat{X}|\mathcal{F}_T^0)$ , alors  $\widehat{X} \in \mathcal{S}_2$  est un processus d'Itô satisfaisant (1.28). Donc, en notant  $(B, \Sigma, \Sigma^0)$  comme le triplet des coefficients correspondant à (1.28), l'espérance  $\widehat{\mathbb{E}}^\mu[\beta_t(X_t, \widehat{X}_t)d\widehat{X}_t]$  s'écrit sous la forme suivante en utilisant la dynamique de  $\widehat{X}$  dans (1.28)

$$\begin{aligned} \widehat{\mathbb{E}}^\mu[\beta_t(X_t, \widehat{X}_t)d\widehat{X}_t] &= \widehat{\mathbb{E}}^\mu[\beta_t(X_t, \widehat{X}_t)B_t(\widehat{X}_t)]dt + \widehat{\mathbb{E}}^\mu[\beta_t(X_t, \widehat{X}_t)\Sigma_t^0(\widehat{X}_t)]dW_t^0 \\ &= \int \beta_t(X_t, \hat{x})B_t(\hat{x})\mu_t(\hat{x})dt + \int \beta_t(X_t, \hat{x})\Sigma_t^0(\hat{x})\mu_t(\hat{x})dW_t^0, \quad dt \otimes d\mathbb{P} - \text{p.s.} \end{aligned}$$

L'équation différentielle stochastique (1.29)-(1.30) doit être bien posée en tenant compte que  $\widehat{X} \in \mathcal{S}_2$ . Pour cela, les coefficients  $b, \sigma, \sigma^0$  satisfont certaines conditions (Lipschitz et à croissance linéaire). Puis, on introduit l'ensemble  $\mathcal{A}$  des applications  $(\mathcal{F}_t^0 \otimes \mathcal{B}_{\mathbb{R}^2})_{t \in [0, T]^-}$  progressivement mesurables de  $[0, T] \times \Omega \times \mathbb{R}^2$  dans  $\mathbb{R}$ , et on désigne aussi par  $\mathcal{A}_b^{\mu, \pi}$  le sous-ensemble de  $\mathcal{A}$  des  $\beta$  bornées pour lesquelles l'application suivante de  $\mathbb{R}$  dans  $\mathbb{R} \times \mathbb{R}$ :

$$x \mapsto m_t^{\mu, \pi}(x)\widehat{\mathbb{E}}^\mu[\beta_t(x, \widehat{X}_t)(B_t, \Sigma_t^0)(\widehat{X}_t)] \text{ est continue, } dt \otimes d\mathbb{P} - \text{p.s.}$$

où  $m_t^{\mu, \pi}(x) = \frac{1}{1 + \int_{\mathbb{R}} \pi_t(\hat{x}, x)\mu_t(d\hat{x})}$ , pour tout  $t \in [0, T]$  et  $x \in \mathbb{R}$ .

On travaillera avec des environnements aléatoires  $(\mu, \pi)$  et des stratégies  $\beta$  admissibles pour le problème où l'admissibilité est définie comme suit en notant

$$\begin{pmatrix} \overline{B}_t^{\mu, \pi, \beta} \\ \overline{\Sigma}_t^{\mu, \pi, \beta} \end{pmatrix}(x) = \int_{\mathbb{R}} \beta_t(x, \hat{x}) \begin{pmatrix} B_t \\ \Sigma_t^0 \end{pmatrix}(\hat{x})\mu_t(d\hat{x}) + \begin{pmatrix} b_t \\ \sigma_t^0 \end{pmatrix}(x, \mu_t).$$

- Un environnement aléatoire  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  est dit admissible si  $0 \neq \frac{1}{m_t^{\mu, \pi}(x)}$   $dt \otimes d\mathbb{P}$ -p.p, pour tout  $x \in \mathbb{R}$  et si l'EDS définie par la donnée initiale  $X_0$  avec loi  $\mu_0$  et

$$dX_t = m_t^{\mu, \pi}(X_t) \left( \overline{B}_t^{\mu, \pi, \beta}(X_t)dt + \overline{\Sigma}_t^{\mu, \pi, \beta}(X_t)dW_t^0 + \sigma_t(X_t, \mu_t)dW_t \right), \quad (1.31)$$

a une solution faible, pour toute stratégie  $\beta \in \mathcal{A}_b^{\mu, \pi}$ , et satisfait

$$\begin{aligned} \widehat{\mathbb{E}}^\mu \left[ \int_0^T \beta_t(X_t, \widehat{X}_t)^2 (|B_t|^2 + |\Sigma_t|^2 + |\Sigma_t^0|^2)(\widehat{X}_t)dt \right] &< \infty, \quad \mathbb{P}\text{-p.p,} \\ \mathbb{E} \left[ \int_0^T m_t^{\mu, \pi}(X_t)^2 (\overline{B}_t^{\mu, \pi}(X_t)^2 + \overline{\Sigma}_t^{\mu, \pi}(X_t)^2 + \sigma_t(X_t, \mu_t)^2)dt \right] &< \infty. \end{aligned}$$

- Une stratégie  $\beta \in \mathcal{A}$  est dite  $(\mu, \pi)$ -admissible, et on écrit  $\beta \in \mathcal{A}(\mu, \pi)$ , si elle satisfait les trois conditions ci-dessus.

Ensuite, pour  $\beta \in \mathcal{A}(\mu, \pi)$ , on note  $X^{\mu, \pi, \beta}$  une solution arbitraire de (1.31), et on introduit la fonction gain

$$J_{\mu, \pi}(\beta) := \mathbb{E} \left[ U(X_T^{\mu, \pi, \beta}) \right],$$

où  $U : \mathbb{R} \rightarrow \mathbb{R}$  est une fonction d'utilité croissante vérifiant  $\lim_{x \rightarrow \infty} U(x) = \infty$ .

**Définition 1.3** (Équilibre MFG). *Un environnement aléatoire admissible  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  est un équilibre MFG du modèle de détentions croisées si:*

- (i)  $J_{\mu, \pi}(\beta) \leq J_{\mu, \pi}(\pi) < \infty$ , pour tout  $\beta \in \mathcal{A}(\mu, \pi)$ ,
- (ii)  $\pi \in \mathcal{A}(\mu, \pi)$ ,  $\mathcal{L}(X^{\mu, \pi, \pi} | \mathcal{F}_T^0) = \mu$ ,  $\mathbb{P}$ -p.p. et  $dt \otimes \mu_t(dx)$ -p.p.

$$\Sigma_t(x) = m_t^{\mu, \pi}(x) \sigma_t(x), \quad B_t(x) = m_t^{\mu, \pi}(x) \overline{B}_t^{\mu, \pi, \beta}(x), \quad \Sigma_t^0(x) = m_t^{\mu, \pi}(x) \overline{\Sigma}_t^0{}^{\mu, \pi, \beta}(x).$$

## 1.4.2 Environnement aléatoire sans opportunité d'arbitrage

Dans le modèle (1.29), on note qu'à chaque instant  $t \in [0, T]$ , le processus  $X$  des fonds propres de l'agent représentatif croit dès que  $G_t^{\mu, \pi}(\beta)$  croit où

$$G_t^{\mu, \pi}(\beta) := \widehat{\mathbb{E}}^\mu \left[ \int_0^t \beta_s(X_s^{\mu, \pi, \beta}, \widehat{X}_s) m^{\mu, \pi}(X_s^{\mu, \pi, \beta}) d\widehat{X}_s \right], \quad \text{pour tout } \beta \in \mathcal{A}(\mu, \pi).$$

Ceci justifie la notion de **No-increasing profit** (NIP), dont la terminologie<sup>11</sup> est due à C. Fontana dans [28], qui est un concept plus faible que la condition de non arbitrage (NA). Comme on le verra ci-dessous dans la définition 1.4, la condition (NIP) est une condition de proportionnalité entre le drift et la volatilité. Notons qu'elle a déjà joué un rôle clef dans la théorie d'arbitrage dans des travaux précédents celui de C. Fontana [28], voir par exemple F. Delbaen et W. Schachermayer [17].

Comme pour NA, la condition NIP est nécessaire pour l'existence des équilibres.

**Définition 1.4** (No-increasing profit). *On dit qu'un environnement aléatoire admissible  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  vérifie la condition (NIP) si pour toutes les stratégies  $\beta \in \mathcal{A}_b^{\mu, \pi}$  avec  $t \mapsto G_t^{\mu, \pi}(\beta)$  croissante,  $\mathbb{P}$ -p.s. on a  $G_T^{\mu, \pi}(\beta) = 0$ ,  $\mathbb{P}$ -p.s.*

Le résultat principal de ce paragraphe est le suivant:

**Theorem 1.12.** *Un environnement aléatoire admissible  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  vérifie la condition (NIP) si et seulement les coefficients  $B$  et  $\Sigma^0$  satisfont la condition de proportionnalité suivante:*

$$B_t(x) = \Sigma_t^0(x) \lambda_t, \quad dt \otimes \mu_t(dx)\text{-p.p. } \mathbb{P}\text{-p.s.}$$

<sup>11</sup>Dans la théorie d'arbitrage, il existe d'autres types de conditions comme la condition de **No Unbounded Profit with Bounded Risk** (NUPBR) utilisée dans le papier de I. Karatzas et C. Kardaras [42] ou la condition de **No free lunch with vanishing risk** (NFLVR) introduite dans [17].

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pour un certain processus scalaire  $(\lambda_t)_{t \in [0, T]}$  qui est  $\mathbb{F}^0$ -progressivement mesurable. De plus si  $(\mu, \pi)$  un équilibre MFG du modèle de détentions croisées, alors

$$b_t(x, \mu_t) = \lambda_t \sigma_t^0(x, \mu_t), \quad dt \otimes \mu_t(dx) - p.p. \quad \mathbb{P} - p.s.$$

**Remark 1.13** (Comparaison avec le modèle à une période). Notons que la dernière condition de proportionnalité dans le théorème 1.12 est la même que celle qu'on a obtenue dans le modèle à une période (voir la condition 1.17 du théorème 1.6). Par contre, la première condition du théorème 1.12 n'est pas tout le temps satisfaite dans le modèle à une période. En effet, en rappelant l'assertion (ii) du théorème 1.6, l'espérance conditionnelle  $F^\mu$  n'est pas nécessairement linéaire par rapport au bruit commun  $\varepsilon^0$ , sauf si l'équation de Fredholm (1.21) admet une unique solution.

### 1.4.3 Le jeu à champ moyen réduit

Dans ce paragraphe, on fait une réduction importante du jeu à champ moyen du modèle de détentions croisées grâce à la condition NIP. Le théorème 1.12 entraîne la relation de proportionnalité suivante entre le drift et la volatilité du bruit commun :

$$B_t(x) = \lambda_t \Sigma_t^0(x) \quad \text{et} \quad b_t(x) = \lambda_t \sigma_t^0(x), \quad dt \otimes \mu_t(dx) - p.s. \quad \mathbb{P} - p.s.,$$

pour un processus  $(\lambda_t)_{t \in [0, T]}$ ,  $\mathbb{F}^0$ -progressivement mesurable.<sup>12</sup> En insérant ceci dans la dynamique des fonds propres (1.31), on se ramène à considérer le critère réduit de performance suivant:

$$\mathcal{J}_{\mu, \pi}(\alpha) := \mathbb{E}[U(\mathbb{X}_T^{\mu, \pi, \alpha})],$$

où  $\mathbb{X}$  est défini par

$$d\mathbb{X}_t^{\mu, \pi, \alpha} = \alpha_t(\mathbb{X}_t^{\mu, \pi, \alpha})(\lambda_t dt + dW_t^0) + v_t^{\mu, \pi}(\mathbb{X}_t^{\mu, \pi, \alpha})dW_t, \quad (1.32)$$

avec  $v_t^{\mu, \pi}(x) = m_t^{\mu, \pi}(x)\sigma_t(x)$  et  $\alpha_t(x) = m_t^{\mu, \pi}(x)\overline{\Sigma}_t^{0, \mu, \pi, \beta}(x)$ . La dépendance du contrôle  $\alpha$  de  $\beta$  a été supprimée afin de considérer directement le jeu réduit, pour lequel le problème d'optimisation individuel devient  $\sup_\alpha \mathcal{J}_{\mu, \pi}(\alpha)$  où le processus<sup>13</sup>  $\alpha : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ , est  $\mathbb{F}^0 \otimes \mathcal{B}_{\mathbb{R}}$  mesurable tel que (1.32) admette une solution forte et

$$\mathbb{E} \left[ \int_0^T \alpha_t(\mathbb{X}_t^{\mu, \pi, \alpha})^2 (1 + |\lambda_t|^2) + m_t^{\mu, \pi}(\mathbb{X}_t^{\mu, \pi, \alpha})^2 \sigma_t(\mathbb{X}_t^{\mu, \pi, \alpha})^2 dt \right] < \infty.$$

Ensuite, nous montrons que le jeu d'origine (en considérant  $\sup_{\beta \in \mathcal{A}(\mu, \pi)} J_{\mu, \pi}(\beta)$ ) et le jeu réduit sont équivalents. Ce changement de variable permet de réduire le problème initial en un problème classique d'optimisation de portefeuille avec des dotations aléatoires qui a été largement étudié: voir par exemple les travaux de Cvitanić, Schachermayer et Wang [16], Hugonnier et Kramkov [39], Karatzas et Zitković [41], Zitković [61] et Mostovyi [50].

<sup>12</sup>on a omis la dépendance des coefficients de la loi marginale  $\mu_t$ , car ceci ne joue aucun rôle.

<sup>13</sup>Pour que le contrôle  $\alpha$  soit admissible.

On termine ce paragraphe par l'exemple suivant pour lequel on a analysé le jeu réduit pour déterminer les équilibres: On suppose que les processus  $X$  et  $P$  satisfont

$$\begin{aligned} dP_t &= X_t [\sigma_t^0 (\lambda_t dt + dW_t^0) + \sigma_t dW_t], \quad t \in [0, T], \\ \frac{dX_t}{X_t} &= \sigma_t^0 (\lambda_t dt + dW_t^0) + \frac{\sigma_t}{1 + \kappa_t} dW_t, \quad \mathbb{P} - \text{p.s.} \end{aligned} \quad (1.33)$$

où  $\kappa$  est un processus arbitraire  $\mathbb{F}^0$ -progressivement mesurable tel que  $1 + \kappa_t > 0$ , les processus  $\sigma^0$  et  $\sigma$  sont bornés positifs  $\mathbb{F}^0$ -progressivement mesurables, et on a déjà pris en compte la condition de proportionnalité due à NIP:  $b = \lambda \sigma^0$ , pour un processus  $\lambda$ ,  $\mathbb{F}^0$ -progressivement mesurable telle que:

$$Z_T := e^{\int_0^T \lambda_t dW_t^0 - \frac{1}{2} \int_0^T \lambda_t^2 dt} \quad \text{satisfait} \quad \mathbb{E}[Z_T] = 1 \text{ et } \mathbb{E}[Z_T^2] < \infty.$$

On va considérer ici la fonction d'utilité logarithmique suivante  $U(x) = \log(x)$  pour tout  $x \in \mathbb{R}$  avec la convention  $U = -\infty$  sur  $(-\infty, 0]$ .

On obtient le résultat suivant:

**Proposition 1.14.** *Sous les hypothèses précédentes, en notant  $\mu^\kappa = \mathcal{L}_{X|W^0}$  la loi conditionnelle (log-gaussienne) de  $X$  par rapport à  $\mathbb{F}^{W^0}$ , on a*

- (i) *il existe un équilibre du jeu à champ moyen avec une loi  $\mu^\kappa$  si et seulement si  $\lambda = \sigma^0$ ;*
- (ii) *et dans ce cas toute stratégie d'équilibre  $\pi^\kappa \in \mathcal{A}(\mu, \pi^\kappa)$  satisfait*

$$\widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t^\kappa(X_t, \widehat{X}_t) \widehat{X}_t \right] = \kappa_t X_t, \quad \text{et} \quad \widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t^\kappa(\widehat{X}_t, X_t) \right] = \kappa_t, \quad \mathbb{P} - \text{p.s.} \quad (1.34)$$

On peut remarquer que la stratégie suivante:

$$\pi_t(x, \widehat{x}) = \frac{\kappa_t x}{\widehat{\mathbb{E}}^{\mu^\kappa}[\widehat{X}_t]}, \quad \text{pour tout } x, \widehat{x} \in \mathbb{R},$$

satisfait (1.34). L'une des interprétations de cette stratégie consiste à acheter ou à vendre un nombre proportionnel au ratio de la position actuelle du joueur par la moyenne des positions de tous les joueurs. Donc, chaque joueur a connaissance de son environnement optimal en comparant sa position à la position moyenne des joueurs.

#### 1.4.4 Perspectives

Dans la lignée de mes travaux, il reste bien sûr beaucoup de questions ouvertes à explorer. La recherche continue ! Ci-dessous quelques questions avec quelques idées.

- Partie 1:
  - Dans les cas statique et celui à temps continu déterministe, on a fourni les conditions générales d'existence d'équilibre de Nash, sans tenir compte des contraintes de positivité des variables de contrôle. On a vu que dans le cas de  $N = 2$  agents,

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ces contraintes de positivité exigent des conditions supplémentaires (voir Proposition 3.1 dans Chapitre 1). On a pu déterminer ces conditions supplémentaires dans le cas  $N = 2$ , mais le cas général de  $N$  agents reste à analyser.

- Le cas stochastique n'est toujours pas résolu: il faut étudier le caractère bien posé de l'EDSR quadratique qui intervient dans l'analyse de l'équilibre de Nash tout en explorant le nouveau système d'équations correspondant.

- Partie 2:

Dans cette partie, on a extrait pour le jeu à champ moyen une notion de non arbitrage qu'on a réussi à caractériser, à l'équilibre. Pour le jeu fini correspondant, il reste à formuler correctement cette notion et à l'analyser: la difficulté principale ici réside dans le fait que dès que l'agent individuel dévie de l'équilibre, la dynamique des compétiteurs change et ils ne sont plus à l'équilibre.

- Parties 3 et 4:

- Dans le jeu à champ moyen, on a déterminé explicitement (sous certaines conditions) des équilibres champ moyen. Il reste à approcher par ces solutions les équilibres de Nash pour le jeu fini correspondant (si bien-sûr ils approchent ces équilibres).

- Il reste à la fois dans le cadre du jeu fini et celui du champ moyen à explorer la question du risque systémique à travers l'introduction d'une probabilité de défaut des agents, et trouver un critère raisonnable qui permettrait de sélectionner un unique équilibre de Nash.





# Chapter 2

## Finite population Nash equilibrium of cross-holding game

*In this chapter, we characterize the Nash equilibria and their existence conditions for the cross-holding model, that has been introduced by Djete & Touzi [19], in the finite population setting where each agent aims to maximize his exponential utility function.*

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## 2.1 Introduction

In this work, we consider the so called cross-holding (or mutual holding) model that was first introduced in the paper of M-F. Djete and N. Touzi [19]. This model describes the equity dynamic values of  $N \geq 1$  economic agents by taking into account on the one hand, their intrinsic parameters through the dynamic value of their revenues, that can be correlated, and on the other hand, their interactions with the other agents through the cross-holding strategies  $\pi$  where

$$\pi = \{\pi^{i,j}, \quad 1 \leq i \neq j \leq N\},$$

with  $\pi^{i,j}$ , the proportion of equity of the agent  $j$  detained by the agent  $i$ .

If we consider an optimal situation when each agent  $i \in \{1, \dots, N\}$  seeks to optimize a specific criterion by choosing the optimal shares  $\hat{\pi}^{i,\cdot} = (\hat{\pi}^{i,j})_{1 \leq i \neq j \leq N}$  that he should detain from the other agents  $j \neq i$ , we can rely on the study of Nash equilibrium which is, without any collaboration between the agents, an  $N$ -tuple of strategies  $\hat{\pi} = (\hat{\pi}^{1,\cdot}, \dots, \hat{\pi}^{i,\cdot}, \dots, \hat{\pi}^{N,\cdot})$  where each player's strategy is a best response to the strategies of the other players. Equivalently, no player  $i$  can profitably deviate by playing  $\pi^{i,\cdot} \neq \hat{\pi}^{i,\cdot}$ , given the actions of the other players  $(\hat{\pi}^{1,\cdot}, \dots, \hat{\pi}^{i-1,\cdot}, \hat{\pi}^{i+1,\cdot}, \dots, \hat{\pi}^{N,\cdot})$ . This notion of equilibrium in game theory, named after John Forbes Nash, was introduced in the famous paper [51] in 1951, and since that time it has been extensively studied in the literature of risk management in continuous-time models. We refer for instance to the papers of R. Buckdahn, P. Cardaliaguet and C. Rainer [8], G-E. Espinosa and N. Touzi [25], R. Carmona, J-P. Fouque and L-H. Sun [15], D. Lacker and T. Zariphopoulou [46]. For a general introduction to the subject, one can consult the book of M-J. Osborne and A. Rubinstein [52].

Here, the criterion of each agent  $i \in \{1, \dots, N\}$  is an exponential utility function, and we investigate the involved model through a Nash equilibrium in the static and dynamic settings under gaussian Bachelier type dynamic. As it will be seen thereafter, the study of this issue amounts to analyze a nonlinear system of coupled equations on the unknown variables  $\pi^{i,j}$ .

For the static model, we shall start with the case of two agents  $N = 2$ , for which we provide a complete analysis and derive explicitly the optimal cross-holding strategies  $\hat{\pi}^{i,j}$ . Next, we focus on the general case of  $N$  agents that is more involved than the case of 2 agents. However, we were able to characterize the Nash equilibria and their existence conditions.

Following the static case, we undertake the study of Nash equilibria between  $N$  agents within the framework of the continuous time model with deterministic coefficients. In this regard, we shall make use of the Backward stochastic differential equations (BSDE): these type of differential equations have been introduced in 1973 by J-M. Bismut [5] and since that time, they received enduring attention and have been extensively studied either for the wellposedness or the uniqueness issues. There is by now a considerable literature dealing with these equations: see the papers of E. Pardoux and S. Peng [53], M. Kobylanski [44], P. Briand and Y. Hu [7] and the references therein.

More specifically, the BSDE plays a key role in the control theory of non-Markovian dynamic stochastic systems. For instance, they are widely used in the option hedging and

pricing problems (see N. El Karoui, S. Peng and M-C. Quenez [22] for an introduction to the BSDE and some examples of their applications in Finance). In this work, as we deal with exponential utilities, our optimization problem can be formulated by the resolution of quadratic BSDE as it has been introduced by N. El Karoui and R. Rouge [21] and thereafter developed by Y. Hu and P. Imkeller [33] or J. Sekine [55].

When the coefficients of the continuous time model are deterministic, it appears that the study of Nash equilibria amounts to an optimization problem equivalent to the one studied in the static model. For the general case when these parameters are stochastic, the corresponding problem is reduced to an optimization problem which is slightly different and more involved than that of the deterministic case.

Let us emphasize that the case of an infinite number of agents which is based on mean fields game theory has been recently undertaken in the works of M-F. Djete and N. Touzi [19] and M-F. Djete, G. Guo and N. Touzi [20].

The paper is organized as follows. Section 2.2 is devoted to the problem formulation of the cross-holding model for  $N$  agents. In Section 2.3, we address the case of the static model. Section 2.4 is devoted to continuous time model with deterministic coefficients.

**Notations.** Throughout this chapter, for  $N \in \mathbb{N}^*$ , we denote by  $\mathcal{M}_N(\mathbb{R})$ , the set of square matrices of order  $N$ , and by  $\mathcal{M}_N(\mathbb{R}_+)$ , the subset of  $\mathcal{M}_N(\mathbb{R})$  with non-negative entries. We also denote by  $\mathcal{S}_N(\mathbb{R})$  the subset of symmetric matrices in  $\mathcal{M}_N(\mathbb{R})$ ,  $\mathcal{S}_N^+(\mathbb{R})$  the collection of all non-negative symmetric matrices,  $\mathcal{D}_N(\mathbb{R})$  the set of diagonal  $N \times N$  matrices, and  $\mathcal{O}_N(\mathbb{R})$  the set of orthogonal matrices. We shall work under the following norm for the matrices  $M \in \mathcal{M}_N(\mathbb{R})$ ,  $|M| := \sup_{|x| \leq 1} |Mx|$ .

Throughout this chapter, the vectors  $(e_i)_{1 \leq i \leq N}$  represent the canonical basis of  $\mathbb{R}^N$ . We shall denote  $\bar{v}_N = \sum_{i=1}^N \frac{v_i}{N}$ , where  $v_i = v \cdot e_i$  for all  $i \in \{1, \dots, N\}$ , and set  $\text{diag}(v)$  the diagonal matrix in  $\mathcal{D}_N(\mathbb{R})$  with diagonal elements defined by the entries of  $v$ . The Euclidean norm (respectively the infinity norm) of  $\mathbb{R}^N$  is denoted by  $|\cdot|$  (respectively  $|\cdot|_\infty$ ). For a scalar  $\alpha \in \mathbb{R}$ , the vector  $\alpha_{\mathbb{R}^N}$  denotes the vector of  $\mathbb{R}^N$  whose coordinates are equal to  $\alpha$ .

For any subset  $A \subset \mathbb{R}^N$ ,  $A^\perp$  denotes the orthogonal space to  $A$ , and for  $v \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$ , we shall denote by  $P_v(u) := \frac{u \cdot v}{|v|^2} v$  the orthogonal projection of the vector  $u \in \mathbb{R}^N$  on the space  $\text{span}(v)$ , and  $P_{v^\perp}(u) := u - P_v(u)$ , the orthogonal projection of the vector  $u \in \mathbb{R}^N$  on the space  $\text{span}(v)^\perp$ .

For a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ , we denote by  $\mathcal{P}$  the sigma-field of predictable subsets of  $[0, T] \times \Omega$ . The expectation with respect to the probability  $\mathbb{P}$  is denoted  $\mathbb{E}^\mathbb{P}[\cdot]$ . The space  $\mathbb{L}^2$  is the set of all  $\mathcal{F}_T$ -measurable random variables  $F$  such that  $\mathbb{E}^\mathbb{P}[|F|^2] < \infty$ . For  $N \in \mathbb{N}^*$ , we denote by  $\mathcal{B}(\mathbb{R}^N)$ , the collection of all Borel subsets of  $\mathbb{R}^N$ , by  $\mathcal{S}^\infty(\mathbb{R}^N)$  the set of real-valued, adapted and càdlàg bounded processes, and by  $\mathbb{H}^2(\mathbb{R}^N)$  the set of all  $\mathbb{R}^N$ -valued stochastic processes  $\nu$  which are  $\mathbb{F}$ -predictable and such that

$$\|\nu\|_{\mathbb{H}^2(\mathbb{R}^N)} = \mathbb{E}^\mathbb{P} \left[ \int_0^T |\nu_t|^2 dt \right]^{\frac{1}{2}} < \infty.$$

Finally, we set for all  $x \in \mathbb{R}$ ,  $x^+ = \max(0, x)$  and  $\text{sgn}(x) = \mathbf{1}_{x \geq 0} - \mathbf{1}_{x < 0}$ .

## 2.2 Problem formulation

Let  $T \geq 0$  be the time horizon, and  $W_t = (W_t^1, W_t^2, \dots, W_t^N)^\top$ ,  $t \in [0, T]$ , a vector of independent Brownian motions on the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ .

We consider  $N$  economic actors defined by their idiosyncratic risks processes  $P = (P^1, P^2, \dots, P^N)^\top$  with dynamics

$$dP_t = \Gamma_t(\gamma_t dt + dW_t), \text{ with the risk premium } \gamma_t = \Gamma_t^{-1} b_t \text{ } t \in [0, T], \quad (2.1)$$

for some  $\mathbb{F}$ -predictable bounded processes  $b$  and  $\Gamma$  with, respectively, values in  $\mathbb{R}^N$  and  $\mathcal{S}_N^+(\mathbb{R})$ , and such that  $\Gamma$  is invertible  $\text{Leb} \otimes \mathbb{P}$ -a.s. In practice, the idiosyncratic risks represent the revenues of the different economic actors.

Motivated by its own risk diversification, each agent,  $i = 1, \dots, N$ , chooses the amount of holding shares  $\pi^{i,j}$  of its  $j$ -th competitor with  $j \neq i$ . The cross-holding strategies  $\pi^{i,\cdot}$  (with  $\pi^{i,\cdot} = (\pi^{i,1}, \dots, \pi^{i,N})^\top$ ) belongs to the collection  $\mathcal{A}^{i,N}$  of all processes with values in  $\mathbb{R}_+^N$  and zero  $i$ -th component. We will denote by  $\mathcal{A}^N$  the set of admissible strategies which consists of the collection of  $\mathbb{F}$ -predictable processes  $\pi = (\pi^{1,\cdot}, \dots, \pi^{N,\cdot})$  with values in  $\mathcal{M}_N(\mathbb{R}_+)$ , and with zero diagonal elements<sup>1</sup>.

Denoting by  $X^{i,\pi}$  the equity value process of agent  $i = 1, \dots, N$ , induced by the strategy  $\pi \in \mathcal{A}^N$ , and assuming zero interest rates for simplicity, it follows from the standard self-financing condition that the dynamics of  $X^{i,\pi}$  are given by<sup>2</sup>

$$dX_t^{i,\pi} = \sum_{j \neq i}^N \pi_t^{i,j} dX_t^{j,\pi} - \sum_{j \neq i}^N \pi_t^{j,i} dX_t^{i,\pi} + dP_t^i, \text{ } t \in [0, T]. \quad (2.2)$$

In order to rewrite this interacting system in vector notations, we introduce the matrix

$$M(p) := I_N + \sum_{i=1}^N \sum_{j \neq i}^N p^{i,j} (e_j e_j^\top - e_i e_i^\top), \text{ for all } p \in \mathcal{M}_N(\mathbb{R}_+), \quad (2.3)$$

which has the following properties.

**Lemma 2.1.** *For all  $p \in \mathcal{M}_N(\mathbb{R}_+)$ , the matrix  $M(p)$  is invertible, and  $(M(p)^{-1})^\top$  is a stochastic matrix.*

*Proof.* Since  $p$  has non-negative entries, the matrix  $M(p)$  is non-singular as a strictly diagonally dominant matrix. Moreover  $M(p)^\top \mathbf{1}_{\mathbb{R}^N} = \mathbf{1}_{\mathbb{R}^N}$ , which implies that

$$\mathbf{1}_{\mathbb{R}^N} = (M(p)(M(p)^{-1})^\top)^\top \mathbf{1}_{\mathbb{R}^N} = (M(p)^{-1})^\top M^\top(p) \mathbf{1}_{\mathbb{R}^N} = (M(p)^{-1})^\top \mathbf{1}_{\mathbb{R}^N}.$$

This shows that  $\mathbf{1}_{\mathbb{R}^N}$  is an eigenvector of  $(M(p)^{-1})^\top$  with eigenvalue 1. It remains to check that  $(M(p)^{-1})^\top$  has non-negative entries. For that purpose, we rewrite  $M^\top(p) = D(I_N - D^{-1}p^\top)$  where  $D = \sum_{i=1}^N (1 + \sum_{j \neq i}^N p^{j,i}) e_i e_i^\top$  is a diagonal non-singular matrix

<sup>1</sup>Clearly  $\pi \in \mathcal{A}^N$  if and only if the rows  $\pi^{i,\cdot} = (\pi^{i,1}, \dots, \pi^{i,N})^\top$  are in  $\mathcal{A}^{i,N}$ , for all  $i = 1, \dots, N$ .

<sup>2</sup>This model, recently introduced in the paper of M-F. Djete and N. Touzi [19], is referred as the cross-holding (or mutual holding) model.

as  $p^{i,j} \geq 0$  for all  $i \neq j$ . Then, we remark that the spectral radius  $\rho(pD^{-1}) < 1$  as  $D^{-1}p^\top$  is a sub-stochastic matrix. Indeed, we have

$$\sum_{j=1}^N (D^{-1}p^\top)_{i,j} = \frac{\sum_{j \neq i}^N p^{j,i}}{1 + \sum_{j \neq i}^N p^{j,i}} < 1, \quad i = 1, \dots, N.$$

Therefore, we deduce that  $(M(p)^{-1})^\top = \sum_{k=0}^{\infty} (D^{-1}p^\top)^k D^{-1}$  and  $(M(p)^{-1})^\top$  has positive coefficients as it is the result of product and sum of matrices with positive coefficients.  $\square$

Lemma 2.1 enables us to rewrite (2.3) as follows

$$dX_t^\pi = u_t(\pi_t)^\top (\gamma_t dt + dW_t), \quad \text{where } u_t(\pi_t) := \Gamma_t(M(\pi_t)^{-1})^\top, \quad t \in [0, T]. \quad (2.4)$$

Since for all  $\pi \in \mathcal{A}^N$ , the process  $u_t(\pi_t)$  inherits the boundedness of  $\Gamma$  and the stochastic matrix process  $(M(\pi_t)^{-1})^\top$ , the equity value process  $X^\pi$  is well-defined by standard stochastic integration. Moreover, it follows from the stochasticity of the matrix  $(M(\pi_t)^{-1})^\top$ :

$$u_t(\pi_t) \mathbf{1}_{\mathbb{R}^N} = \Gamma_t \mathbf{1}_{\mathbb{R}^N}. \quad (2.5)$$

Our objective is to study the interior Nash equilibria of the non-zero sum stochastic differential game defined by the following performance criteria of the economic actors:

$$J^i(\pi) = J^i(\pi^{1\cdot}, \dots, \pi^{N\cdot}) := \mathbb{E}^\mathbb{P} \left[ -e^{-\frac{1}{q_i} X_T^{i,\pi}} \right], \quad \pi \in \mathcal{A}^N, \quad i = 1, \dots, N, \quad (2.6)$$

where  $q_i > 0$  is the risk tolerance parameter.

**Definition 2.2.** *A Nash equilibrium of the  $N$ -agent's cross-holding problem is an admissible strategy  $\hat{\pi} \in \mathcal{A}^N$  such that, for all  $i \in \{1, \dots, N\}$ ,*

$$J^i(\hat{\pi}^{1\cdot}, \dots, \hat{\pi}^{i-1\cdot}, \pi^{i\cdot}, \hat{\pi}^{i+1\cdot}, \hat{\pi}^{N\cdot}) \leq J^i(\hat{\pi}), \quad \text{for all } \pi^{i\cdot} \in \mathcal{A}^{i,N}. \quad (2.7)$$

*A Nash equilibrium  $\hat{\pi}$  is said to be interior if  $\hat{\pi}^{i,j}$  takes values in  $(0, +\infty)$ , for all  $i \neq j$ .*

## 2.3 Interior static Nash equilibria

For simplicity, we set  $T = 1$ , and restrict the analysis to the case of constant parameters  $b$  and  $\Gamma$ . The cross-holding strategies are then constant over the time. We thus consider  $\mathcal{A}_{\text{const}}^{i,N} \subset \mathcal{A}^{i,N}$  and  $\mathcal{A}_{\text{const}}^N \subset \mathcal{A}^N$  the collection of such constant strategies. An (interior) Nash equilibrium with constant strategies  $\pi \in \mathcal{A}_{\text{const}}^N$  will be called static (interior) Nash equilibrium.

As the coefficients are constant, we can compute explicitly the performance criterion (2.6) of each player  $i = 1, \dots, N$ , resorting to the Laplace transform:

$$J^i(\pi) = -e^{-\left(\frac{X_0^i}{q_i} + \frac{|\gamma|^2}{2}\right) + \frac{1}{q_i^2} F^i(\pi)}, \quad \text{with } F^i(\pi) = \frac{1}{2} |u^i(\pi) - q_i \gamma|^2, \quad \pi \in \mathcal{A}_{\text{const}}^N, \quad (2.8)$$

where  $u^i(\pi) = u(\pi)e_i$ , with  $u(\pi)$  defined in (2.4).

### 2.3.1 The case of $N = 2$ agents

We study the interior Nash equilibria for  $N = 2$  agents. We solve explicitly this issue in the general case as it is stated in the following proposition, where  $\Delta$  denotes the open simplex

$$\Delta := \{(\delta_1, \delta_2) \in (0, +\infty)^2 : \delta_1 + \delta_2 < 1\}. \quad (2.9)$$

**Proposition 2.3.** *Let  $N = 2$  and assume that  $b, \Gamma$  are constant.*

(i) *If  $b = 0_{\mathbb{R}^2}$ , then there exists a static interior Nash equilibrium if and only if there exists an orthogonal basis  $(o_1, o_2)$  of  $\mathbb{R}^2$  such that*

$$(\hat{\delta}_1, \hat{\delta}_2) \in \Delta \quad \text{where } \hat{\delta}_1 = (\Gamma \mathbf{1}_{\mathbb{R}^2} \cdot o_1)(\Gamma^{-1} e_2 \cdot o_1), \quad \hat{\delta}_2 = (\Gamma \mathbf{1}_{\mathbb{R}^2} \cdot o_2)(\Gamma^{-1} e_1 \cdot o_2). \quad (2.10)$$

*In this case, the Nash equilibria are given by  $(\hat{\pi}^{1,2}, \hat{\pi}^{2,1})$  with  $\hat{\pi}^{i,j} := \frac{\hat{\delta}_i}{1 - (\hat{\delta}_1 + \hat{\delta}_2)}$ .*

(ii) *If  $b \neq 0_{\mathbb{R}^2}$ , there exists a static interior Nash equilibrium if and only if*

$$\begin{aligned} \bar{b}_2 \neq 0, \quad A := |\Gamma \mathbf{1}_{\mathbb{R}^2}|^2 - \left(\frac{\bar{b}_2(q_1 - q_2)}{|\gamma| \bar{q}_2}\right)^2 - \frac{4q_1 q_2 \bar{b}_2}{\bar{q}_2} \geq 0, \quad \text{and } \alpha = \alpha_+ \text{ or } \alpha_- \text{ satisfies} \\ \frac{1}{\alpha} < -\frac{b_i}{\bar{b}_2 q_i \det(\Gamma)}, \quad i = 1, 2 \quad \text{where } \alpha_{\pm} := \frac{\bar{b}_2}{|\gamma|} \left( \frac{\det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})(q_2 - q_1)}{2|\gamma|} \pm \bar{q}_2 \sqrt{A} \right). \end{aligned} \quad (2.11)$$

*Then, there are at most two static interior Nash equilibria  $(\hat{\pi}_{\pm}^{1,2}, \hat{\pi}_{\pm}^{2,1})$  given by*

$$\hat{\pi}_{\pm}^{i,j} := -\frac{1}{2} \left( \frac{q_i \det(\Gamma)}{\alpha_{\pm}} + \frac{b_i}{\bar{b}_2} \right).$$

**Example 2.4** (Example of unique Nash equilibrium for  $N = 2$ ). It follows from the above proposition that when  $\bar{b}_2 > 0$  and  $q_1 = q_2 = q$ , there exists a unique static interior Nash equilibrium  $\hat{\pi}$  if and only if  $\min(b_1, b_2) > 0$ ,  $|\Gamma \mathbf{1}_{\mathbb{R}^2}| > 4q\bar{b}_2$  and  $\frac{|\gamma| \det(\Gamma)}{\sqrt{|\Gamma \mathbf{1}_{\mathbb{R}^2}|^2 - 4q\bar{b}_2}} > \max(b_1, b_2)$ .

If  $b_i < 0$ , for  $i = 1$  or  $i = 2$ , then agent  $j \neq i$  has a drift  $b_j > 0$  since  $\bar{b}_2 > 0$ . Then, he does not intend to hold shares from agent  $i$ , which means  $\hat{\pi}^{j,i} = 0$ , inducing a strategy on the boundary.

Under these assumptions, the equilibrium is given by  $\hat{\pi}^{i,j} = -\frac{1}{2b_2} \left( b_i - \frac{|\gamma| \det(\Gamma)}{\sqrt{|\Gamma \mathbf{1}_{\mathbb{R}^2}|^2 - 4q\bar{b}_2}} \right)$ .

In Figure 2.1 below, we plot the investments  $\hat{\pi}^{i,j}$  when the 2 agents have the same risk tolerance parameter ( $q_1 = q_2 = q$ ) in order to see how  $\hat{\pi}^{i,j}$  evolves by varying the parameters. We set the variance covariance matrix  $\Gamma^2 = \begin{pmatrix} \Gamma_1^2 & r\Gamma_1\Gamma_2 \\ r\Gamma_1\Gamma_2 & \Gamma_2^2 \end{pmatrix}$ , fix  $\Gamma_1 = 1$ ,  $\Gamma_2 = 2$ ,  $r = 0.3$ ,  $b = [0.8, 0.6]$ ,  $q = 1$  and vary one of these parameters leaving the others fixed for each example.

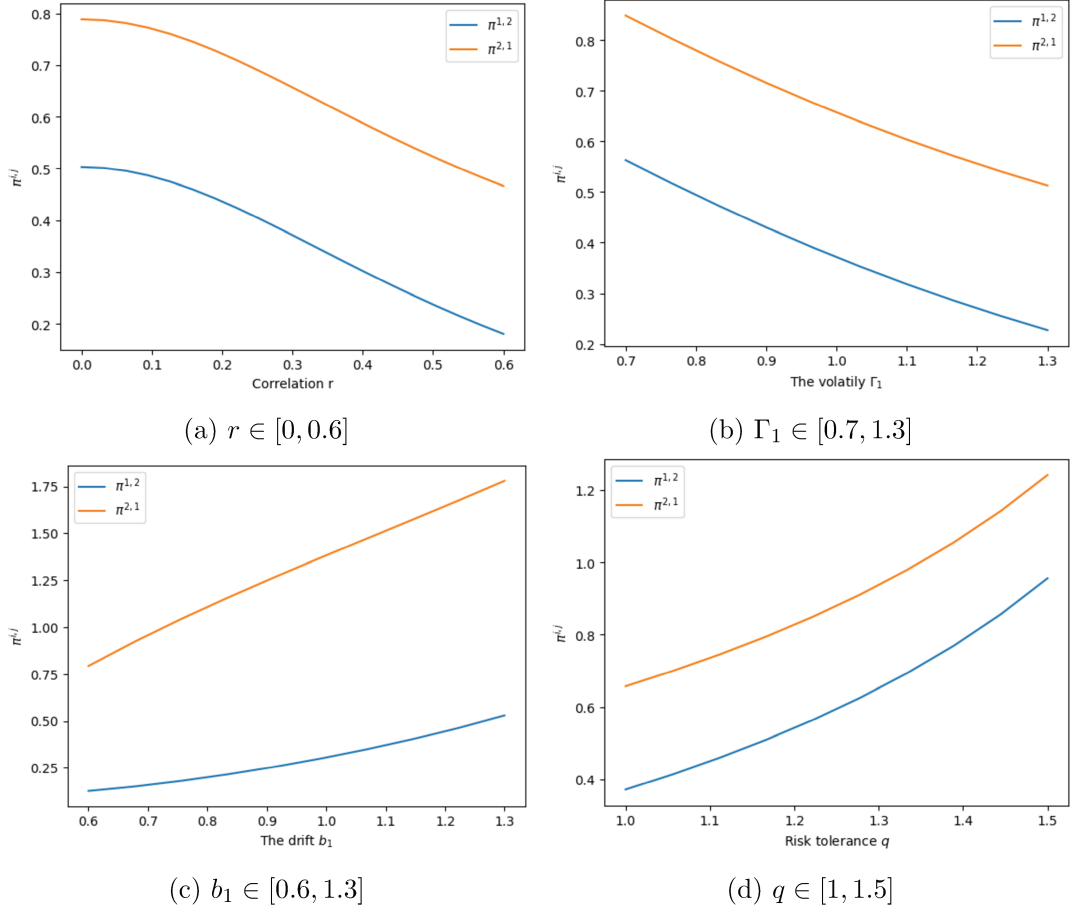


Figure 2.1: dependence of the investments  $(\hat{\pi}^{1,2}, \hat{\pi}^{2,1})$  on the parameters

Figures 2.1a-2.1b show that when the correlation  $r$  or the volatility of one of the agents (for our example  $\Gamma_1$ ) increases, the investment  $\hat{\pi}^{i,j}$  decreases. This is due to the fact that the agents tend to diversify their portfolio or do not want to invest in a highly volatile portfolio in order to minimize their risk of loss. However, in the case when an agent has a portfolio with an upward trend ( $b_1$  increases for our example),  $\hat{\pi}^{2,1}$  increases significantly as shown in Figure 2.1c which is natural as the agent 2 tend to invest more if the portfolio of agent 1 perform well. The last Figure 2.1d reveals that the agents invest more in each other if they are more risk tolerant ( $q$  increases).

*Proof.* We proceed in two steps

*Step 1.* According to (2.8), the individual optimization problem is reduced to the minimization of the functional  $F^i$  defined in (2.8). Straightforward computations provide the first order condition:

$$\frac{\partial F^i(\hat{\pi})}{\partial \pi^{i,j}} = \frac{1}{(1 + \hat{\pi}^{1,2} + \hat{\pi}^{2,1})} (\hat{u}^i - q_i \gamma) \cdot \hat{u}^{-i} = 0, \quad \text{with } \hat{u}^i = u(\hat{\pi})e_i, \quad i = 1, 2, \quad (2.12)$$

where  $(1 + \hat{\pi}^{1,2} + \hat{\pi}^{2,1}) = \det(M(\hat{\pi})) \neq 0$ . Moreover, using the first order condition (2.12),



we obtain the following second order condition for a strict minimizer:

$$\frac{\partial^2 F^k(\hat{\pi})}{\partial(\pi^{i,j})^2} = \frac{|\hat{u}^i|^2}{(1 + \hat{\pi}^{1,2} + \hat{\pi}^{2,1})^2} > 0,$$

which shows that the first order condition characterizes the global minimizers of  $F^i$  as  $\hat{\pi}$  belongs to a convex subset.

*Step 2.* In view of the first step, the study of static interior Nash equilibria reduces to the first order condition (2.12), which forms a system of equations on the unknown vector basis  $(\hat{u}^1, \hat{u}^2)$  of  $\mathbb{R}^2$  (by the non singularity of  $M(\pi)$  and  $\Gamma$ ), together with (2.5) and the constraints of positivity of  $\hat{\pi}$  (i.e.  $(\hat{\pi}^{1,2}, \hat{\pi}^{2,1}) \in (0, +\infty)^2$ ). The latter constraint is equivalent to the following condition

$$\hat{\pi}^{i,j} = \frac{\Gamma^{-1}\hat{u}^i \cdot e_j}{1 - \Gamma^{-1}\hat{u}^1 \cdot e_2 - \Gamma^{-1}\hat{u}^2 \cdot e_1} > 0, \quad 1 \leq i \neq j \leq 2. \quad (2.13)$$

Now, we distinguish two separated cases:

(i)  $b = 0_{\mathbb{R}^2}$ : the system reduces for  $i \neq j \in \{1, 2\}$  to  $\hat{u}^i \cdot \hat{u}^j = 0$  under the relation (2.5). In that case, as the matrix  $\Gamma$  is non-singular, we have  $\Gamma \mathbf{1}_{\mathbb{R}^2} \neq 0_{\mathbb{R}^2}$ . Then, we can find an infinite number of orthogonal basis  $(o_1, o_2)$  of  $\mathbb{R}^2$  such that for  $i \in \{1, 2\}$ , we have  $\Gamma \mathbf{1}_{\mathbb{R}^2} \cdot o_i = r_i \neq 0$  and the solutions of the system (2.12)-(2.5) are described by:  $\hat{u}^i = r_i o_i$ ,  $i = 1, 2$  with  $\hat{u}^i$  satisfying the constraints (2.13) which is equivalent to (2.10).

(ii)  $b \neq 0_{\mathbb{R}^2}$ : the system (2.12) is equivalent to the following one on the unknown vector basis  $(\hat{u}^1, \hat{u}^2)$  under the relation (2.5):

$$|2\hat{u}^1 - (\Gamma \mathbf{1}_{\mathbb{R}^2} + q_1 \gamma)| = |\Gamma \mathbf{1}_{\mathbb{R}^2} - q_1 \gamma| \quad \text{and} \quad \gamma \cdot (2\hat{u}^1 - \frac{q_1}{q_2} \Gamma \mathbf{1}_{\mathbb{R}^2}) = 0. \quad (2.14)$$

Notice that  $\bar{b}_2 \neq 0$  is a necessary condition for the existence of the solutions  $(\hat{u}^1, \hat{u}^2)$  to the latter system. Indeed, if we assume that  $\bar{b}_2 = 0$ , then the last equality in (2.14) ensures that  $\hat{u}^1 \in \text{Span}(\gamma)^\perp$  as  $\gamma \cdot \Gamma \mathbf{1}_{\mathbb{R}^2} = 2\bar{b}_2$ . Therefore, we get  $\hat{u}^2 \in \text{Span}(\gamma)^\perp$  by using the relation (2.5) which is in contradiction with the fact that  $(\hat{u}^1, \hat{u}^2)$  constitutes a basis of  $\mathbb{R}^2$ . Now, assume that  $\bar{b}_2 \neq 0$ . From the latter system,  $(2\hat{u}^1 - \frac{q_1}{q_2} \Gamma \mathbf{1}_{\mathbb{R}^2}) \in \text{Span}(\gamma)^\perp$ . Subsequently,  $\hat{u}^1$  has the following decomposition

$$2\bar{q}_2 \hat{u}^1 = q_1 \Gamma \mathbf{1}_{\mathbb{R}^2} + \frac{\alpha}{\bar{b}_2} \gamma^\perp \quad \text{with} \quad \gamma^\perp = -(\gamma_2) e_1 + (\gamma_1) e_2. \quad (2.15)$$

By observing that  $\Gamma \mathbf{1}_{\mathbb{R}^2} \cdot \gamma^\perp = -\det(\Gamma \mathbf{1}_{\mathbb{R}^2}, \gamma)$  and by injecting (2.15) in the first equality of (2.14), we deduce that  $\alpha \in \mathbb{R}$  can take two possible values  $\alpha_\pm$  defined in (2.11) such that

$$\left( \frac{\det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})}{|\gamma|^2} \left( \frac{q_1 - q_2}{2} \right) + \frac{\alpha}{\bar{b}_2} \right)^2 = \frac{\bar{q}_2^2 A}{|\gamma|^2},$$

where  $A \geq 0$  is defined in (2.11).

We remark from equation (2.15) that  $\alpha \neq 0$ . Indeed, if we assume that  $\hat{u}^1 = \frac{\Gamma \mathbf{1}_{\mathbb{R}^2}}{2}$ . Therefore  $\hat{u}^2 = \frac{\Gamma \mathbf{1}_{\mathbb{R}^2}}{2} = \hat{u}^1$ , since (2.5) is satisfied, which is in contradiction with the fact that  $(\hat{u}^1, \hat{u}^2)$  is a basis of  $\mathbb{R}^2$ .

Now, we check that  $(\hat{u}^1, \hat{u}^2)$  is a basis of  $\mathbb{R}^2$  under the assumption  $\bar{b}_2 \neq 0$  and  $\alpha \neq 0$ . For this, we suppose to the contrary that there exists  $c \in \mathbb{R}^*$  such that  $\hat{u}^1 = c\hat{u}^2$ , then using the relation (2.5) and the formula (2.15), we directly get

$$(q_1 - cq_2)\Gamma\mathbf{1}_{\mathbb{R}^2} + \frac{\alpha}{b_2}(1+c)\gamma^\perp = 0_{\mathbb{R}^2}.$$

The last formula is satisfied if and only if  $\Gamma\mathbf{1}_{\mathbb{R}^2} \in \text{Span}(\gamma^\perp)$  which contradict the fact that  $0 \neq 2\bar{b}_2 = \gamma \cdot \Gamma\mathbf{1}_{\mathbb{R}^2}$  or  $c = -1 = \frac{q_1}{q_2}$  which is impossible as  $q_i > 0$  for  $i = 1, 2$ .

It remains to investigate the constraints of positivity of the coefficients  $\hat{\pi}^{i,j}$ . Noticing that  $\Gamma^{-1}\gamma^\perp = \frac{1}{\det(\Gamma)}(-b_2e_1 + b_1e_2)$  and using relation (2.5) and equation (2.15), we get  $\hat{\pi}_\pm^{i,j} = -\frac{1}{2}\left(\frac{q_i \det(\Gamma)}{\alpha_\pm} + \frac{b_i}{b_2}\right)$ . Therefore,  $\hat{\pi}_\pm^{i,j} > 0$  if and only if  $\alpha_\pm$  satisfies the inequality (2.11). □

### 2.3.2 The case of $N$ finite agents

#### Statements of the results

We have seen in Subsection 2.3.1 that without the constraints of positivity of  $\hat{\pi}$  (that is to say  $\hat{\pi}^{i,j} \in (0, +\infty)$ ,  $\forall i \neq j$ ), the study of static Nash equilibria reduces to the first order condition, for all  $i \neq j$ ,

$$\frac{\partial F^i(\hat{\pi})}{\partial \pi^{i,j}} = 0, \quad (2.16)$$

with  $\hat{u}^i = u(\hat{\pi})e_i$ , for all  $i \in \{1, \dots, N\}$ , where  $u(\hat{\pi})$  is defined in (2.3). It turns out that the first order condition (2.16) is equivalent to the nonlinear system given in the following lemma, whose proof is postponed to Subsection 2.3.2.

**Lemma 2.5.** *Let  $b$  and  $\Gamma$  be constant. The study of the static interior Nash equilibrium  $\hat{\pi} \in \mathcal{A}_{\text{const}}^N$  is equivalent to solving the following coupled nonlinear system on the unknown basis of vector functions  $(\hat{u}^1, \dots, \hat{u}^N)$  together with the relation (2.5)*

$$(\hat{u}^i - q_i\gamma) \cdot \hat{u}^j = 0, \quad 1 \leq i \neq j \leq N, \quad (2.17)$$

**Remark 2.6.** When we take into account the positivity conditions on the cross-holding strategies ( $\hat{\pi}^{i,j} > 0$ , for all  $i \neq j$ ), we have to solve the system (2.17) under the following constraints on the coefficients of the matrix  $\hat{u}^{-1}\Gamma = M^\top(\hat{\pi})$

$$(\hat{u}^{-1}\Gamma)_{i,j} < 0, \quad 1 \leq i \neq j \leq N. \quad (2.18)$$

In the case when  $\gamma \neq 0_{\mathbb{R}^N}$ , taking advantage of the symmetry properties in the system of equations (2.17) under the relation (2.5), we get  $\hat{u}^i \cdot \gamma = N\eta_i \bar{b}_N$  for all  $i \in \{1, \dots, N\}$ , where  $\eta := \frac{1}{N\bar{q}_N} \sum_{i=1}^N q_i e_i$ . Thus, splitting  $\hat{u}^i$  in two terms as follows

$$\hat{u}^i = N\eta_i \bar{b}_N \frac{\gamma}{|\gamma|^2} + P_{\gamma^\perp}(\hat{u}^i), \quad (2.19)$$

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one can easily check that the system under consideration is equivalent to the following one:

$$\begin{cases} P_{\gamma^\perp}(\hat{u}^i) \cdot P_{\gamma^\perp}(\hat{u}^j) = \eta_i \eta_j \beta_0, & 1 \leq i < j \leq N, \\ \sum_{i=1}^N P_{\gamma^\perp}(\hat{u}^i) = P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}), \end{cases} \quad (2.20)$$

where  $\beta_0 = N^2 \bar{b}_N \left( \bar{q}_N - \frac{\bar{b}_N}{|\gamma|^2} \right)$ . Finally, we decompose  $P_{\gamma^\perp}(\hat{u}^i)$  as follows

$$P_{\gamma^\perp}(\hat{u}^i) = c_i P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) + w^i, \quad (2.21)$$

with  $c_i \in \mathbb{R}$  and  $w^i$  a vector orthogonal to  $\gamma$  and  $P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N})$ .

Introducing the matrix<sup>3</sup>  $W(c) \in \mathcal{M}_N(\mathbb{R})$  whose  $i$ -th column  $W(c)e_i = w^i$ , it turns out that, in the general case when  $\beta_0 \neq 0$  (as it will be seen in the proof of Theorem 2.8),  $P_{\gamma^\perp}(\hat{u}^i)$  satisfies (2.20) and (2.17) if and only if  $W(c)^\top W(c) = A(c)$  for some  $c$  in  $\Theta$ , where

$$\begin{aligned} A(c) &= \beta_1 (\text{diag}(c) - cc^\top) - \beta_0 (\text{diag}(\eta) - \eta\eta^\top), \\ \Theta &:= \left\{ c \in \mathbb{R}^N : c \cdot \mathbf{1}_{\mathbb{R}^N} = 1, \det D(c) \neq 0, \beta_0 \eta^\top D(c)^{-1} \eta = -1, \text{ and } A(c) \geq 0 \right\}, \end{aligned}$$

with  $\beta_1 = |P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N})|^2$  and  $D(c) = \text{diag}(\beta_1 c - \beta_0 \eta)$ . Moreover, one has the following lemma, the proof of which will be given in Subsection 2.3.2.

**Lemma 2.7.** *Assume that  $\beta_0 \neq 0$ . Let  $c \in \mathbb{R}^N$  such that  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$  and  $D(c)$  non-singular. We have  $\dim \text{Ker}(A(c)) = \dim \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N})$  if and only if  $\beta_0 \eta^\top D(c)^{-1} \eta = -1$ . In this case, the matrix  $A(c)$  satisfies the following decomposition*

$$\begin{aligned} A(c) &= P(c)S(c)(P(c)S(c))^\top, \text{ for some } P(c) \in \mathcal{O}_N(\mathbb{R}), \text{ and } S(c) \in \mathcal{M}_N(\mathbb{R}) \\ &\text{with } S(c)S(c)^\top \in \mathcal{D}_N(\mathbb{R}) \text{ and } \text{Ker}(S(c)) = \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N}). \end{aligned} \quad (2.22)$$

Our main result is the following:

**Theorem 2.8.** *Let  $b$  and  $\Gamma$  be constant.*

1. *If  $b = 0_{\mathbb{R}^N}$ , then there exist static interior Nash equilibria if and only if there exists  $P \in \mathcal{O}_N(\mathbb{R})$  such that*

$$M^P := \Gamma P \text{diag}(P^\top \Gamma \mathbf{1}_{\mathbb{R}^N})^{-1} \text{ satisfies } M_{i,j}^P < 0, \quad 1 \leq i \neq j \leq N. \quad (2.23)$$

2. *If  $b \neq 0_{\mathbb{R}^N}$  and  $\bar{b}_N = |\gamma|^2 \bar{q}_N \neq 0$  ( $\beta_0 = 0$ ), then there exist static interior Nash equilibria if and only if  $\gamma$  and  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  are linearly independent, and for some  $P \in \mathcal{O}_N(\mathbb{R})$  with  $k$ -th column vector  $Pe_k = \frac{\gamma}{|\gamma|}$ , for some  $k = k^P$ , the following matrix is well-defined with  $M_{i,j}^P < 0$ , for all  $i \neq j$*

$$M^P := \Gamma P \text{diag}(P^\top \Gamma \mathbf{1}_{\mathbb{R}^N})^{-1} \left( I_N + \sum_{i \neq k} \frac{q_i}{q_k} (e_k e_k^\top - e_i e_i^\top) \right). \quad (2.24)$$

<sup>3</sup>for which we add the dependence on the vector  $c$ , since the relation (2.17) is satisfied.

3. Otherwise, if  $b \neq 0_{\mathbb{R}^N}$  and  $\bar{b}_N \neq |\gamma|^2 \bar{q}_N$ , then there exist static interior Nash equilibria if and only if  $\bar{b}_N \neq 0^4$  and the matrix

$$M^P := \left( -\frac{\beta_0}{N\bar{b}_N} b\eta^\top + \Gamma P_{\gamma^\perp} (\Gamma \mathbf{1}_{\mathbb{R}^N}) c^\top + \Gamma ((P(c)S(c))^\top) \right) D^{-1}(c), \quad (2.25)$$

has negative off-diagonal entries for some  $c \in \Theta$  and  $(P(c), S(c))$  satisfying the decomposition (2.22) of  $A(c)$ .

In all of the above cases,  $\hat{\pi}_{i,j} = -M_{i,j}^P$ ,  $1 \leq i \neq j \leq N$ , defines a static interior Nash equilibrium, and the value function of each agent  $i \in \{1, \dots, N\}$  is given by

$$J^i(\hat{\pi}) = -e^{-\left(\frac{x_0^i}{q_i} + \frac{|\gamma|^2}{2}\right) + \frac{1}{q_i} F^i(\hat{\pi})},$$

with

$$2F^i(\hat{\pi}) = \begin{cases} (\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_i)^2 & \text{for } P \text{ satisfying (2.23) if } b = 0_{\mathbb{R}^N}, \\ \mathbf{1}_{i \neq k} (\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_i)^2, & \text{for } P \text{ satisfying (2.24) if } \bar{b}_N = |\gamma|^2 \bar{q}_N. \\ \eta_i^2 \beta_0 \left( \frac{1}{N^2 \bar{b}_N^2} \beta_0 |\gamma|^2 + 1 \right) + \beta_1 c_i - \beta_0 \eta_i & \text{otherwise.} \end{cases} \quad (2.26)$$

**Remark 2.9.** In Appendix 2.5, we check that Theorem 2.8 specialized to  $N = 2$  is in agreement with Proposition 2.3.

**Remark 2.10.** Let  $b \neq 0_{\mathbb{R}^N}$  and  $\hat{\pi} \in \mathcal{A}_{\text{const}}^N$  be an interior static Nash equilibrium. If all the agents have the same risk tolerance parameter  $q_i = q$ , for all  $i \in \{1, \dots, N\}$ , then it follows from the equations (2.24)-(2.25) that  $\hat{\pi}$  satisfies the following property, the proof of which is postponed to Appendix 2.6.

$$Y^j = \sum_{i \neq j}^N \hat{\pi}^{j,i} - \sum_{i \neq j}^N \hat{\pi}^{i,j} = 1 - \frac{b_j}{\bar{b}_N}, \quad j = 1, \dots, N. \quad (2.27)$$

The variable  $Y^j$  represents the  $j$ -th agent's net detention level. Under the assumption that  $\bar{b}_N > 0$ , if the drift of the  $j$ -th agent  $b_j$  is greater than the average population drift  $\bar{b}_N$ , then the agent has a negative net detention. In addition, greater is his drift value, less is his net detention holding. This means that the agents tend to hold more shares among those with greater drift values.

In the following proposition, we state the existence conditions of the set  $\Theta$ , depending on the sign of  $\beta_0$ .

**Proposition 2.11.** *Let  $N \geq 2$ .*

- (i) *If  $\beta_0 < 0$ , then  $\Theta \neq \emptyset$ . In addition if  $\beta_1 > 0$ , then the set  $\Theta$  is an  $N - 2$  smooth manifold of  $\mathbb{R}^N$ .*
- (ii) *If  $\beta_0 > 0$  and  $|\eta|_\infty \leq \frac{1}{2}$ , then  $\Theta \neq \emptyset$  if and only if  $\beta_0 < \beta_1$ .*

<sup>4</sup>then,  $\beta_0 \neq 0$  as  $\bar{b}_N \notin \{0, |\gamma|^2 \bar{q}_N\}$ .

**Remark 2.12.** In the general case when  $|\eta|_\infty < \frac{1}{2}$ <sup>5</sup> and without taking into account the constraints of positivity of  $\hat{\pi}^{i,j}$ , it follows from Theorem 2.8 that the existence of static Nash equilibria is ensured by the condition  $N^2\bar{b}_N\bar{q}_N < |\Gamma\mathbf{1}_{\mathbb{R}^N}|^2$ .

Intuitively when  $\bar{b}_N > 0$ , saying that  $N^2\bar{b}_N\bar{q}_N \geq |\Gamma\mathbf{1}_{\mathbb{R}^N}|^2$  means that the investments of the agents do not generate sufficient incomes based on their risk-taking, since the volatility does not have a high enough value, which prevents to attain an equilibrium.

### Proof of the results

*Proof of Theorem 2.8.* According to Lemma 2.5 and Remark 2.6, the proof of Theorem 2.8 reduces to the study of the nonlinear system (2.17). under the relation (2.5) and the positivity constraints (2.18). To this end, we shall treat separately the case  $b = 0_{\mathbb{R}^N}$  from the case  $b \neq 0_{\mathbb{R}^N}$ .

1) If  $b = 0_{\mathbb{R}^N}$ , then the system (2.17) takes the following form:  $\hat{u}^i \cdot \hat{u}^j = 0$ , for  $1 \leq i \neq j \leq N$ . Under the relation (2.5), we are reduced to look for an orthogonal matrix  $R$  such that, for all  $i \in \{1, \dots, N\}$ ,

$$\hat{u}^i = r_i Re_i \text{ with } r_i \neq 0 \text{ and } \sum_{i=1}^N \hat{u}^i = \Gamma\mathbf{1}_{\mathbb{R}^N} = \sum_{i=1}^N (\Gamma\mathbf{1}_{\mathbb{R}^N} \cdot Re_i) Re_i.$$

By identification, we infer that the solutions are given by  $\hat{u}^i = (\Gamma\mathbf{1}_{\mathbb{R}^N} \cdot Re_i) Re_i$ , and then, with the previous notations,  $\Gamma(M(\hat{\pi})^{-1})^\top = \hat{u} = R\text{diag}(R^\top\Gamma\mathbf{1}_{\mathbb{R}^N})$ . Since the matrix  $\text{diag}(R^\top\Gamma\mathbf{1}_{\mathbb{R}^N})$  is non singular, we deduce that

$$M(\hat{\pi}) = \Gamma R\text{diag}(R^\top\Gamma\mathbf{1}_{\mathbb{R}^N})^{-1}.$$

Finally the investments being assumed positive, we require  $\hat{\pi}^{i,j} = -M(\hat{\pi})_{i,j} > 0$ , for all  $i \neq j$ . This completes the proof in the first case when  $b = 0_{\mathbb{R}^N}$ .

2) Consider now the case when  $b \neq 0_{\mathbb{R}^N}$ . As already mentioned above, the system (2.17) on the unknown basis of functions  $(\hat{u}^1, \dots, \hat{u}^N)$  is, under the relation (2.5), equivalent to the system (2.20), where the unknown  $P_{\gamma^\perp}(\hat{u}^i)$  are given by (2.19). To go further, we shall distinguish the case  $\beta_0 = 0$  from the case  $\beta_0 \neq 0$ .

Consider first the case when  $\beta_0 = N^2\bar{b}_N\left(\bar{q}_N - \frac{\bar{b}_N}{|\gamma|^2}\right) = 0$

- Let us start by investigating the case 2 when  $\bar{b}_N = |\gamma|^2\bar{q}_N \neq 0$ . According to (2.19)-(2.20), we are reduced to investigate the following system

$$\begin{cases} P_{\gamma^\perp}(\hat{u}^i) \cdot P_{\gamma^\perp}(\hat{u}^j) = 0, & 1 \leq i < j \leq N, \\ \sum_{i=1}^N P_{\gamma^\perp}(\hat{u}^i) = P_{\gamma^\perp}(\Gamma\mathbf{1}_{\mathbb{R}^N}). \end{cases} \quad (2.28)$$

We claim that there exists a unique  $1 \leq k \leq N$  such that  $P_{\gamma^\perp}(\hat{u}^k) = 0_{\mathbb{R}^N}$ . Indeed, on the one hand  $(P_{\gamma^\perp}(\hat{u}^1), \dots, P_{\gamma^\perp}(\hat{u}^N))$  are  $N$  orthogonal vectors in the  $(N-1)$ -dimensional space  $\text{Span}(\gamma)^\perp$ , then at least one of them is necessarily the zero vector.

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<sup>5</sup>this means that there is no agent which is more tolerant to risk than all the other agents taken together ( $q_i \leq \sum_{j \neq i} q_j$  for all  $i$ ).

On the other hand, if there exist two distinct vectors  $P_{\gamma^\perp}(\hat{u}^k) = P_{\gamma^\perp}(\hat{u}^j) = 0_{\mathbb{R}^N}$ , then it follows from (2.19) that  $\frac{\hat{u}^i}{q_i} = \frac{\hat{u}^k}{q_k} = \gamma$ , which contradicts the fact that the family  $(\hat{u}^1, \dots, \hat{u}^N)$  constitutes a basis of  $\mathbb{R}^N$ .

Now let us establish that the system (2.28) admits solutions if and only if  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  and  $\gamma$  are non-collinear. First, if  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  is collinear to  $\gamma$ , then the second equation in (2.28) implies that  $\sum_{i \neq k} P_{\gamma^\perp}(\hat{u}^i) = 0_{\mathbb{R}^N}$ , which is in contradiction with the fact that  $(P_{\gamma^\perp}(\hat{u}^i))_{i \neq k}$  constitutes an orthogonal basis of the hyperplane  $\text{Span}(\gamma)^\perp$ . Next, assuming that  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  and  $\gamma$  are non-collinear, let us establish that the system (2.28) admits solutions under the form (2.24). For that purpose, let us start by recalling that we have established the existence of a unique  $k \in \{1, \dots, N\}$  such that

$$\hat{u}^k = q_k |\gamma| \frac{\gamma}{|\gamma|} \text{ and } \hat{u}^i = q_i |\gamma| \frac{\gamma}{|\gamma|} + P_{\gamma^\perp}(\hat{u}^i), \quad i \neq k = 1, \dots, N,$$

where, according to the second equation in (2.28)  $\sum_{i \neq k} P_{\gamma^\perp}(\hat{u}^i) = P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N})$ . Therefore, if  $O \in \mathcal{O}_N(\mathbb{R})$  such that  $Oe_k = \frac{\gamma}{|\gamma|}$ , combining the fact that the  $(P_{\gamma^\perp}(\hat{u}^i))_{i \neq k}$  are pairwise orthogonal along with the decomposition  $P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) = \sum_{i \neq k} (P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot Oe_i) Oe_i$ , we readily gather that the solutions of the system (2.28) can be described as follows:

$$\hat{u}^k = q_k |\gamma| Oe_k \text{ and } \hat{u}^i = q_i |\gamma| Oe_k + (P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot Oe_i) Oe_i, \quad i \neq k, \quad (2.29)$$

where  $P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot Oe_i \neq 0$ , for all  $i \neq k$ , since  $P_{\gamma^\perp}(\hat{u}^i) \neq 0$  for all  $i \neq k$ .

In order to determine explicitly the Nash equilibria  $\hat{\pi}^{i,j}$ , consider the matrix  $\hat{U}$  whose  $i$ -th columns  $\hat{U}e_i$  are given by the vector functions  $\hat{u}^i$  satisfying (2.29). By construction,  $\hat{U} = A + B$  with  $A = O \text{diag}(v)$  and  $B = |\gamma| Oe_k (q - q_k e_k)^\top$ , where  $q = \sum_{i=1}^N q_i e_i$  and

$$v = q_k |\gamma| e_k + \sum_{i \neq k} (P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot Oe_i) e_i = q_k |\gamma| e_k + \sum_{i \neq k} (\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot Oe_i) e_i.$$

Since  $v_i \neq 0$ , for all  $1 \leq i \leq N$ ,  $A$  is a non-singular matrix. Moreover  $B$  is a rank-one matrix satisfying  $1 + (q - q_k e_k)^\top A^{-1} Oe_k = 1$ . Then, applying the Sherman-Morrison formula [56] on the constructed matrix  $\hat{U}$ , we easily get  $\hat{U}^{-1} = A^{-1} - A^{-1} B A^{-1}$ . Since  $A^{-1} B = \sum_{i \neq k} \frac{q_i}{q_k} e_k e_i^\top$ , we deduce that

$$\hat{U}^{-1} = (I_N - \sum_{i \neq k} \frac{q_i}{q_k} e_k e_i^\top) \text{diag}(v)^{-1} O^\top.$$

Recalling that  $M = \Gamma(\hat{U}^{-1})^\top$  by definition in (2.4), we deduce that the associated matrix  $M$  takes the following form

$$\begin{aligned} M &= \Gamma O \text{diag}(v)^{-1} (I_N - \sum_{i \neq k} \frac{q_i}{q_k} e_i e_k^\top) \\ &= \Gamma O \text{diag}(O^\top \Gamma \mathbf{1}_{\mathbb{R}^N})^{-1} \left( I_N + \left( \frac{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot Oe_k}{|\gamma| q_k} - 1 \right) e_k e_k^\top - \sum_{i \neq k} \frac{q_i}{q_k} e_i e_k^\top \right). \end{aligned}$$

Since in that case  $\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot Oe_k = \frac{N\bar{b}_N}{|\gamma|} = N|\gamma|\bar{q}_N$ , we infer that

$$\begin{aligned} M &= \Gamma O \text{diag}(O^\top \gamma \mathbf{1}_{\mathbb{R}^N})^{-1} \left( I_N + \left( \frac{\sum_{1 \leq i \leq N} q_i}{q_k} - 1 \right) e_k e_k^\top - \sum_{i \neq k} \frac{q_i}{q_k} e_i e_k^\top \right) \\ &= \Gamma O \text{diag}(O^\top \Gamma \mathbf{1}_N)^{-1} \left( I_N + \sum_{i \neq k} \frac{q_i}{q_k} (e_k - e_i) e_k^\top \right), \end{aligned}$$

which proves Formula (2.24).

Finally, there exists an interior Nash equilibrium  $\hat{\pi}$  if the constraints of positivity on the investments  $\hat{\pi}^{i,j} = -M(\hat{\pi})_{i,j}$  are satisfied for all  $i \neq j$ .

- In the case when  $\bar{b}_N = 0$  (with  $b \neq 0_{\mathbb{R}^N}$ ), there is no solution. Indeed, assume that  $\bar{b}_N = 0$ , then in view of (2.28), the vectors  $P_{\gamma^\perp}(\hat{u}^i)_{i=1, \dots, N}$  are orthogonal with  $P_{\gamma^\perp}(\hat{u}^i) \cdot \gamma = 0$  for all  $i$ . Since  $\gamma \neq 0_{\mathbb{R}^N}$ , this can happen if and only if there exists  $i \in \{1, \dots, N\}$  such that  $\hat{u}^i = 0_{\mathbb{R}^N}$  which is in contradiction with the fact that  $(\hat{u}^i)_{i=1, \dots, N}$  is a basis vectors of  $\mathbb{R}^N$ .

Finally, consider the general case 3 when  $\beta_0 \neq 0$ . It follows from the decomposition of  $P_{\gamma^\perp}(\hat{u}^i)$  in (2.21) that the system (2.20) reduces to the following one

$$\sum_{i=1}^N c_i = 1, \quad \sum_{i=1}^N w^i = 0 \quad \text{and} \quad w^i \cdot w^j = \eta_i \eta_j \beta_0 - c_i c_j \beta_1, \quad \forall i \neq j, \quad (2.30)$$

Then, by considering the matrix  $W(c) \in \mathcal{M}_N(\mathbb{R})$  whose its  $i$ -th column  $W(c)e_i = w^i$ , it results that (2.30) is equivalent to the following matrix equations

$$W(c)^\top W(c) = A(c), \quad \text{Ker}(W(c)^\top) = \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N}) \quad \text{and} \quad c \cdot \mathbf{1}_{\mathbb{R}^N} = 1, \quad (2.31)$$

Note that since  $\eta \cdot \mathbf{1}_{\mathbb{R}^N} = 1$  and  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ , then  $A(c)\mathbf{1}_{\mathbb{R}^N} = 0_{\mathbb{R}^N}$  and  $W(c)\mathbf{1}_{\mathbb{R}^N} = 0_{\mathbb{R}^N}$  as  $\text{Im}(W(c)) = \text{Ker}(W(c)^\top)^\perp = \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N})^\perp$ .

Next, consider  $(c, W) \in \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R})$  satisfying (2.31), and define the following matrix  $\hat{U}(c, W)$  whose its  $i$ -th column  $\hat{U}(c, W)e_i = \hat{u}^i$

$$\hat{U}(c, W) = \frac{N\bar{b}_N}{|\gamma|^2} \gamma \eta^\top + P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) c^\top + W(c). \quad (2.32)$$

This allows to rewrite the system (2.17)-(2.5) in matrix form which gives rise to the following equations together with the relation (2.5)

$$\hat{U}^\top(c, W) \left( \hat{U}(c, W) - N\bar{q}_N \gamma \eta^\top \right) e_i \cdot e_j = 0, \quad 1 \leq i \neq j \leq N,$$

which is equivalent to

$$\hat{U}^\top(c, W) C(c, W) = D(c), \quad (2.33)$$

where  $D(c) = \text{diag}(\beta_1 c - \beta_0 \eta)$  and  $C(c, W) = -\frac{\beta_0}{N\bar{b}_N} \gamma \eta^\top + P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) c^\top + W(c)$ . Under the assumption  $\beta_0 \neq 0$  and since  $\text{Im}(W(c)) = \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N})^\perp$ , the matrices  $\hat{U}(c, W)$  and  $C(c, W)$  are non-singular, which is equivalent to the non-singularity of  $D(c)$  by (2.33).

Then, the matrix equation (2.31) admits a solution  $(c, W) \in \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R})$  if and only if  $c \in \Theta$ . Indeed, assume that there exists  $(c, W) \in \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R})$  satisfying (2.31) and such that  $D(c)$  is non-singular, then by construction this implies that the matrix  $A(c)$  involved in (2.31) satisfies

$$\begin{aligned} \dim \overline{\text{Ker}}(A(c)) &= \dim \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N}), \quad A(c) \geq 0, \\ c \cdot \mathbf{1}_{\mathbb{R}^N} &= 1 \text{ and } \text{Det}(D(c)) \neq 0, \end{aligned} \quad (2.34)$$

which by virtue of Lemma 2.7 is equivalent to say that  $c$  belongs to the set  $\Theta$ . Conversely, let  $c \in \Theta$ , then by virtue of Lemma 2.7, there exist  $S(c) \in \mathcal{M}_N(\mathbb{R})$  and  $P(c) \in \mathcal{O}_N(\mathbb{R})$  that satisfy (2.22). Therefore considering  $W(c) = (P(c)S(c))^\top$ , one can easily check that  $(c, W)$  satisfy (2.31) under the non singularity of  $D(c)$ .

Now, let  $c \in \Theta$ . Then in view of the above computations, the system (2.17) under the relation (2.5) admits a solution  $(\hat{u}^1, \dots, \hat{u}^N)$  such that the associated matrix  $\hat{U}$  defined by  $\hat{U}e_i = \hat{u}^i$  is non singular and satisfies (2.32)-(2.33) with  $W(c) = (P(c)S(c))^\top$ , for  $(P(c), S(c))$  satisfying (2.22). Since by definition, one has  $M(\hat{\pi}) = \Gamma(\hat{U}(c, (PS)^\top)^{-1})^\top = \Gamma C(c, (PS)^\top)D(c)^{-1}$ , by using (2.33), this completes the proof of (2.25). Finally, there exists an interior Nash equilibrium  $\hat{\pi}$  if the constraints of positivity are satisfied on the investments  $\hat{\pi}^{i,j} = -M(\hat{\pi})_{i,j}$  for all  $i \neq j$ .  $\square$

*Proof of Lemma 2.5.* Let  $k \in \{1, \dots, N\}$ . According to (2.8), the individual optimization problem for the static interior Nash equilibrium  $\hat{\pi}$  is reduced to the minimization of the functional  $F^k$  introduced in (2.8). Using the following formula

$$\frac{\partial \hat{\alpha}}{\partial \pi^{k,i}} = \hat{\alpha} e_i (e_k^\top - e_i^\top) \hat{\alpha}, \quad i \neq k = 1, \dots, N \text{ where } \hat{\alpha} = \Gamma^{-1} \hat{u} = (M(\pi)^{-1})^\top, \quad (2.35)$$

we infer that  $\frac{\partial \hat{u}^j}{\partial \pi^{k,i}} = \Gamma \frac{\partial \hat{\alpha}}{\partial \pi^{k,i}} e_j = \hat{u}^i (\hat{\alpha}_{k,j} - \hat{\alpha}_{i,j})$  with  $\hat{\alpha}_{\ell,j} := e_\ell^\top \hat{\alpha} e_j$  and then, for all  $i \neq k \in \{1, \dots, N\}$  and all  $j \in \{1, \dots, N\}$ ,

$$\frac{\partial F^k(\hat{\pi})}{\partial \pi^{k,i}} = (\hat{u}^k - q_k \gamma) \cdot \frac{\partial \hat{u}^k}{\partial \pi^{k,i}} = (\hat{\alpha}_{k,k} - \hat{\alpha}_{i,k}) (\hat{u}^k - q_k \gamma) \cdot \hat{u}^i = 0. \quad (2.36)$$

We deduce that the entries of the matrix  $A^k = \left( \frac{\partial^2 F^k}{\partial \pi^{k,i} \partial \pi^{k,j}} \right) (\hat{\pi})$  satisfies the following formulas

$$\begin{aligned} (A^k)_{i,j} &= (\hat{\alpha}_{k,k} - \hat{\alpha}_{j,k}) (\hat{\alpha}_{k,k} - \hat{\alpha}_{i,k}) \hat{u}^j \cdot \hat{u}^i \\ &+ (\hat{\alpha}_{k,i} - \hat{\alpha}_{j,i}) (\hat{\alpha}_{k,k} - \hat{\alpha}_{i,k}) (\hat{u}^k - q_k \gamma) \cdot \hat{u}^j \\ &+ (\hat{\alpha}_{k,j} - \hat{\alpha}_{i,j}) (\hat{\alpha}_{k,k} - \hat{\alpha}_{j,k}) (\hat{u}^k - q_k \gamma) \cdot \hat{u}^i. \end{aligned} \quad (2.37)$$

Our task now is to show that, for all  $i \neq k \in \{1, \dots, N\}$ , the constraints  $\hat{\alpha}_{i,k} \neq \hat{\alpha}_{k,k}$  are satisfied when  $\hat{\pi}$  is an interior static Nash equilibrium and then the minimization problem on  $F^k$  is equivalent to the following system:

$$(\hat{u}^k - q_k \gamma) \cdot \hat{u}^j = 0, \quad (2.38)$$

under the following constraints (2.18) when we take into account the positivity conditions:  $\hat{\pi}^{i,j} > 0$ , for all  $i \neq j$ .



CHAPTER 2. FINITE POPULATION NASH EQUILIBRIUM OF CROSS-HOLDING GAME

Now let us establish that if  $\hat{\pi}^{i,j} > 0$ , then we have, for all  $i \neq k$ ,  $\hat{\alpha}_{i,k} \neq \hat{\alpha}_{k,k}$ . Indeed, by symmetry, it suffices to show that  $\hat{\alpha}_{1,1} \neq \hat{\alpha}_{2,1}$ . Then using the relation of the co-factors to determine the coefficients of the inverse  $(M(\hat{\pi})^{-1})^\top$ , we get

$$\det\left((M(\hat{\pi})^{-1})^\top\right)(\hat{\alpha}_{1,1} - \hat{\alpha}_{2,1}) = \det \underbrace{\begin{pmatrix} 1 + \sum_{j \neq 2} \hat{\pi}^{j,2} - \hat{\pi}^{1,2} & \cdots & -\hat{\pi}^{2,N} - \hat{\pi}^{1,N} \\ -\hat{\pi}^{3,2} & \cdots & -\hat{\pi}^{3,N} \\ \vdots & \ddots & \vdots \\ -\hat{\pi}^{N,2} & \cdots & 1 + \sum_{j \neq N} \hat{\pi}^{j,N} \end{pmatrix}}_C.$$

As  $\hat{\pi}^{i,j} > 0$ ,  $C^\top$  is a strictly diagonally dominant matrix and thus it is non-singular which implies that  $\hat{\alpha}_{1,1} \neq \hat{\alpha}_{2,1}$ .

Finally, assuming that (2.38) holds, it follows from (2.37) that  $Be_i = (\hat{\alpha}_{k,k} - \hat{\alpha}_{i,k})\hat{u}^i$  for all  $i \neq k$  where  $B \in \mathcal{M}_N(\mathbb{R})$  such that  $A^k = B^\top B$ . The fact that  $\hat{u}$  is non-singular ensures that  $\dim(\text{Span}((\hat{u}^i)_{i \neq k})) = N - 1$ . Moreover, in view of (2.38),  $\hat{\alpha}_{k,k} \neq \hat{\alpha}_{i,k}$  for all  $i \neq k$ , which implies that  $\text{Span}(((\hat{\alpha}_{k,k} - \hat{\alpha}_{i,k})\hat{u}^i)_{i \neq k}) = \text{Span}((\hat{u}^i)_{i \neq k})$ . Consequently,  $\text{rank}(B) = N - 1$ , which entails that  $A^k$  is semi definite positive with rank  $N - 1$ . This shows that (2.38) characterizes global minimizers  $\hat{\pi}^k$  of  $F^k$ , since  $\hat{\pi}^k$  belongs to a convex subset. This completes the proof of the result.

In turn, solving the system (2.17) under the constraints (2.5) and (2.18) is equivalent to study a static interior Nash equilibrium  $\hat{\pi}$ .  $\square$

*Proof of Lemma 2.7.* Assume  $\beta_0 \neq 0$ . Let  $c \in \mathbb{R}^N$  such that  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$  and  $D(c)$  is non singular. We have

$$x \in \text{Ker}(A(c)) \iff x = -\beta_0 \alpha_1 D(c)^{-1} \eta + \beta_1 \alpha_2 D(c)^{-1} c. \quad (2.39)$$

where  $\alpha_1 = \eta \cdot x$  and  $\alpha_2 = c \cdot x$ .

(i)  $\beta_1 = 0$ , which means that  $\gamma$  and  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  are collinear, and  $x = -\beta_0 \alpha_1 D(c)^{-1} \eta = \alpha_1 \mathbf{1}_{\mathbb{R}^N}$ . As  $\eta \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ , this shows that  $\text{Ker}(A(c)) = \text{Span}(\mathbf{1}_{\mathbb{R}^N})$ . Notice that in this case, we already have  $-\beta_0 \eta^\top D(c)^{-1} \eta = 1$ .

(ii) Otherwise,  $\beta_1 \neq 0$ , and therefore the vectors  $\gamma$  and  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  are not collinear. In this case, by equation (2.39), we have  $\dim \text{Ker}(A(c)) = 2$  if and only if  $D(c)^{-1} c$  is not collinear to  $D(c)^{-1} \eta$  and the following system is satisfied

$$\begin{aligned} (1 + \beta_0 \eta^\top D(c)^{-1} \eta) \alpha_1 - \beta_1 \eta^\top D(c)^{-1} c \alpha_2 &= 0, \\ (1 - \beta_1 c^\top D(c)^{-1} c) \alpha_2 + \beta_0 c^\top D(c)^{-1} \eta \alpha_1 &= 0. \end{aligned}$$

As  $c \cdot \mathbf{1}_{\mathbb{R}^N} = \eta \cdot \mathbf{1}_{\mathbb{R}^N} = 1$  and  $D$  is non-singular, the two vectors  $D(c)^{-1} c$  and  $D(c)^{-1} \eta$  are not collinear if and only if  $c \neq \eta$ . Concerning the last system of equations, using again that  $c \cdot \mathbf{1}_{\mathbb{R}^N} = \eta \cdot \mathbf{1}_{\mathbb{R}^N}$ , we get

$$\frac{\beta_0}{\beta_1} (1 + \beta_0 \eta^\top D(c)^{-1} \eta) = \beta_0 \eta^\top D(c)^{-1} c = 1 - \beta_1 c^\top D(c)^{-1} c.$$

Then, there exists a solution to the above system of equations if and only if  $\alpha_1 = \alpha_2$  or  $(1 + \beta_0 \eta^\top D(c)^{-1} \eta) = 0$ . If  $\alpha_1 = \alpha_2$ , this implies a unique solution  $x = \mathbf{1}_{\mathbb{R}^N}$  since equation (2.39) is satisfied. Thus, as  $\dim \ker(A(c)) = 2$ , we also have  $\beta_0 \eta^\top D(c)^{-1} \eta = -1$ . Since  $\eta \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ , if  $c = \eta$ , then we get  $\beta_0 \eta^\top D(\eta)^{-1} \eta = -1$  if and only if  $\beta_1 = 0$ , which is a contradiction.

Now, it remains to prove that there exists  $S(c) \in \mathcal{M}_N(\mathbb{R})$  and  $P(c) \in \mathcal{O}_N(\mathbb{R})$  such that (2.22) is satisfied. To this end, by applying the spectral theorem on the symmetric matrix  $A(c)$ , it suffices to take  $P(c) \in \mathcal{O}_N(\mathbb{R})$  with its columns  $(Pe_i)_{i=1, \dots, N}$  constituting the eigenvectors of  $A(c)$  and  $S(c)S(c)^\top \in \mathcal{D}_N(\mathbb{R})$  is the diagonal matrix such that  $(S(c)S(c)^\top)_{i,i} = a_i$  for all  $i \in \{1, \dots, N\}$  with  $a_i \in \mathbb{R}_+$  such that  $A(c)P(c)e_i = a_i P(c)e_i$ . The equation (2.22) admits solutions  $S(c) \in \mathcal{M}_N(\mathbb{R})$  such that  $S(c)^\top e_i = 0_{\mathbb{R}^N}$  for  $i \in J^P(c) = \{1 \leq i \leq N : A(c)Pe_i = 0_{\mathbb{R}^N}\}$  and  $(S(c)^\top e_j)_{j \notin J^P(c)}$  constitutes an orthogonal basis of dimension  $N - \dim \text{Ker}(A(c)) = N - \dim \text{Span}(\Gamma \mathbf{1}_{\mathbb{R}^N}, \gamma)$  with  $|S(c)^\top e_j|^2 = a_j$  for all  $j \notin J^P(c)$ . Moreover,  $\text{Ker}(S(c)) = \text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N})$  is equivalent to the fact that  $(S(c)^\top e_j)_{j \notin J^P(c)}$  constitute an orthogonal basis of  $\text{Span}(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^N})^\perp$ .  $\square$

*Proof of Proposition 2.11.* We will study the cases (i) and (ii) separately.

Case (i) First, assume that  $\Gamma \mathbf{1}_{\mathbb{R}^N}$  and  $\gamma$  are collinear ( $\beta_1 = 0$ ). Under this assumption, we have  $A(c) = -\beta_0(\text{diag}(\eta) - \eta \eta^\top)$  which turns out to be semi definite positive ( $\mathbf{1}_{\mathbb{R}^N}$  belongs to the kernel) if and only if  $\beta_0 < 0$ . Indeed, as  $\eta_i > 0$ , for all  $1 \leq i \leq N$ , and  $\sum_{i=1}^N \eta_i = 1$ , it follows from Cauchy-Schwarz inequality that, for all  $x \in \mathbb{R}^N$ ,

$$\left( \sum_{i=1}^N x_i \eta_i \right)^2 \leq \sum_{i=1}^N x_i^2 \eta_i \sum_{i=1}^N \eta_i = \sum_{i=1}^N x_i^2 \eta_i.$$

Next, let  $\beta_0 < 0 < \beta_1$ . Then, the set  $\Theta$  can be defined as follows:

$$\Theta = \Theta_0 := \left\{ c \in \mathbb{R}^N : c \cdot \mathbf{1}_{\mathbb{R}^N} = 1, D(c) > 0 \text{ and } \beta_0 \eta^\top D(c)^{-1} \eta = -1 \right\}. \quad (2.40)$$

Indeed, assume that  $c \in \Theta$ , then it follows from the positivity of the matrix  $A(c)$  that

$$e_i^\top A(c) e_i = \beta_1 c_i (1 - c_i) - \beta_0 \eta_i (1 - \eta_i) \geq 0, \quad i = 1, \dots, N.$$

We deduce that

$$\beta_1 c_i \geq \beta_1 c_i (1 - c_i) \geq \beta_0 \eta_i (1 - \eta_i) > \beta_0 \eta_i, \quad i = 1, \dots, N,$$

where the first inequality follows from the fact that  $\beta_1 > 0$  and the last one is deduced from the condition  $\beta_0 < 0$  and the positivity of the  $\eta_i$ , for  $i = 1, \dots, N$ . This ensures that  $c \in \Theta_0$ .

Conversely assume that  $c \in \Theta_0$ . To establish that  $c$  belongs to  $\Theta$ , we have to show that  $A(c) \geq 0$ , which according to the hypothesis  $D(c) > 0$ , is equivalent to the condition  $D^{-\frac{1}{2}}(c) A(c) D^{-\frac{1}{2}}(c) \geq 0$ . To go further, let us start by emphasizing that if  $c \in \mathbb{R}^N$  satisfies  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$  and  $\det D(c) \neq 0$ , then

$$\beta_0 \eta^\top D(c)^{-1} \eta = -1 \iff \eta^\top D(c)^{-1} c = 0 \iff \beta_1 b^\top D(c)^{-1} b = 1. \quad (2.41)$$

Indeed since  $\eta \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ , we infer that

$$\beta_0 \eta^\top D(c)^{-1} \eta = -1 \iff \beta_0 \eta^\top D(c)^{-1} \eta + \sum_{i=1}^N \eta_i = 0 \iff \beta_1 \eta^\top D(c)^{-1} c = 0.$$

Similarly using that  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ , we get

$$\beta_0 \eta^\top D(c)^{-1} \eta = -1 \iff \beta_0 \eta^\top D(c)^{-1} \eta + \sum_{i=1}^N c_i = 0 \iff \beta_1 b^\top D(c)^{-1} c = 1.$$

We deduce that  $o_1 := |\beta_0|^{1/2} D^{-\frac{1}{2}}(c) \eta$  and  $o_2 := |\beta_1|^{1/2} D^{-\frac{1}{2}}(c) c$  are two orthonormal vectors of  $\mathbb{R}^N$ . Recalling that  $A(c) = D(c) - |\beta_0| \eta \eta^\top - \beta_1 c c^\top$ , we readily gather that, for all  $x \in \mathbb{R}^N$ ,

$$x^\top D^{-\frac{1}{2}}(c) A(c) D^{-\frac{1}{2}}(c) x = |x|^2 - |P_{o_1}(x)|^2 - |P_{o_2}(x)|^2 \geq 0,$$

which ensures that  $c \in \Theta$ , which shows that when  $\beta_0 < 0 < \beta_1$ , we have  $\Theta = \Theta_0$ .

To complete the proof of Proposition 2.11 in the case (i), it left to prove that the set  $\Theta_0$  is an  $N - 2$  dimensional smooth manifold of  $\mathbb{R}^N$ . For that purpose, first observe that a vector  $\vec{c}$  belongs to  $\Theta_0$  if and only the vector  $\vec{d}$ , with positive coordinates  $d_i = \beta_1 c_i - \beta_0 \eta_i$ , for  $i = 1, \dots, N$  satisfies

$$\begin{cases} \sum_{i=1}^N d_i = \beta_1 - \beta_0, \\ \sum_{i=1}^N \frac{\eta_i^2}{d_i} = -\frac{1}{\beta_0}, \end{cases} \quad (2.42)$$

that is to say that the unknown vector  $\vec{d}$  (with positive coordinates) belongs to the intersection of the hypersurfaces  $M_1$  and  $M_2$ , where

$$\begin{aligned} M_1 &= \left\{ \vec{d} \in \mathbb{R}^N, f(\vec{d}) = \sum_{i=1}^N d_i = \beta_1 - \beta_0 \right\}, \\ M_2 &= \left\{ \vec{d} \in \mathbb{R}^N, g(\vec{d}) = \sum_{i=1}^N \frac{\eta_i^2}{d_i} = -\frac{1}{\beta_0} \right\}. \end{aligned}$$

In order to show that  $M := M_1 \cap M_2 \cap (\mathbb{R}_+^*)^N \neq \emptyset$ , let us start by investigating the minimum value of the function  $g$  (defined by  $g(\vec{d}) = \sum_{i=1}^N \frac{\eta_i^2}{d_i}$  which can attain arbitrarily large values on  $M_1 \cap (\mathbb{R}_+^*)^N$  by taking one of the component  $d_i$  small) on  $M_1 \cap \mathbb{R}_+^N$ . Invoking Lagrange multipliers, we claim that this minimum is achieved at  $\vec{d}$  where  $\nabla f(\vec{d}) = \mathbf{1}_{\mathbb{R}^N}$  and  $\nabla g(\vec{d}) = -\sum_{i=1}^N \frac{\eta_i^2}{d_i^2} e_i$  are collinear. Then, the minimum happens by definition when

$$d_i = h \eta_i, \quad i = 1, \dots, N.$$

So it is achieved on the only point of  $M_1 \cap \mathbb{R}_+^N$  given by

$$d_i = (\beta_1 - \beta_0) \eta_i,$$

and therefore the minimum value of the function  $g$  on  $M_1 \cap \mathbb{R}_+^N$  is

$$\sum_{i=1}^N \frac{\eta_i}{\beta_1 - \beta_0} = \frac{1}{\beta_1 - \beta_0}.$$

Our purpose is to find  $\vec{d} \in \Sigma_1 \cap \mathbb{R}_+^N$  so that  $g(\vec{d}) = -\frac{1}{\beta_0}$ . Since  $\beta_0 < 0 < \beta_1$ , we have  $-\frac{1}{\beta_0} > \frac{1}{\beta_1 - \beta_0}$  and so there is a solution.

Finally,  $M$  is a smooth  $N - 2$  dimensional manifold since  $(\nabla f, \nabla g)$  are linearly independent on  $M$ , which completes the proof of the proposition in the case (i), since  $\Theta_0$  and  $M$  are diffeomorphic.

Case (ii) To prove the result in that case, we first establish the following lemma:

**Lemma 2.13.** *Let  $N \geq 2$  and  $\eta \in \mathbb{R}^N$  with coordinates  $0 < \eta_i < 1$ ,  $1 \leq i \leq N$ , and such that  $\eta \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ . Assume that  $\beta_0 > 0$ , and let  $c \in \mathbb{R}^N$  such that  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ . Then  $A(c) \geq 0$  if and only if*

$$4\eta_i(1 - \eta_i)\beta_0 \leq \beta_1, \quad \forall 1 \leq i \leq N, \quad \text{and } N - 2 \leq \sum_{i=1}^N \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}. \quad (2.43)$$

*Proof.* We proceed in two steps:

Step 1: We start by proving that

$$A(c) \geq 0 \Leftrightarrow \beta_1 c_i(1 - c_i) - \beta_0 \eta_i(1 - \eta_i) \geq 0, \quad \forall i = 1, \dots, N. \quad (2.44)$$

Indeed, observe that if  $A(c) \geq 0$  then

$$e_i^\top A(c) e_i = \beta_1 c_i(1 - c_i) - \beta_0 \eta_i(1 - \eta_i) \geq 0, \quad i = 1, \dots, N.$$

Now, assuming (under the hypothesis  $c \cdot \mathbf{1}_{\mathbb{R}^N} = 1$ ) that

$$\beta_1 c_i(1 - c_i) - \beta_0 \eta_i(1 - \eta_i) \geq 0, \quad i = 1, \dots, N, \quad (2.45)$$

let us show that  $A(c) \geq 0$ . To this end, let us start by noticing that since  $\beta_0 > 0$  and  $0 < \eta_i < 1$ , (2.45) implies that

$$\beta_1 c_i \geq \beta_1 c_i^2 + \beta_0 \eta_i(1 - \eta_i) > 0, \quad i = 1, \dots, N.$$

Therefore  $\beta_1 > 0$  and  $c_i > 0$ , for each  $i \in \{1, \dots, N\}$ , and one may then consider the unitary vector  $e(c) = (\sqrt{c_1}, \dots, \sqrt{c_N})$ . Therefore

$$A(c) \geq 0 \iff \beta_1^{-1} \text{diag}(e(c))^{-1} A(c) \text{diag}(e(c))^{-1} \geq 0, \quad (2.46)$$

where by definition

$$\beta_1^{-1} \text{diag}(e(c))^{-1} A(c) \text{diag}(e(c))^{-1} = I_N - e(c)e(c)^\top - \frac{\beta_0}{\beta_1} N(c), \quad (2.47)$$

with  $N(c) = \text{diag}(e(c))^{-1} (\text{diag}(\eta) - \eta^\top \eta) \text{diag}(e(c))^{-1}$ , which as mentioned above is a semi-definite matrix. We deduce that

$$A(c) \geq 0 \iff x^\top \frac{\beta_0}{\beta_1} N(c) x \leq x^\top H(e(c)) x, \quad \forall x \in \mathbb{R}^N, \quad (2.48)$$

where  $H(e(c)) = I_N - e(c)e(c)^\top$ .

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To go further, let us point out that  $e(c)$  belongs to the kernel of both  $N(c)$  and  $H(e(c))$ . Indeed, on the one hand  $\mathbf{1}_{\mathbb{R}^N}$  belongs to the kernel of the matrix  $\text{diag}(\eta) - \eta\eta^\top$ , which readily ensures that  $N(c)e(c) = 0_{\mathbb{R}^N}$ . On the other hand, recalling that  $e(c)$  is a unitary vector, we get  $H(e(c))e(c) = 0_{\mathbb{R}^N}$ .

Moreover all the other eigenvalues of  $H(e(c))$  are equal to 1. Indeed, let  $\zeta$  an eigenvalue of  $H(e(c))$  and  $V$  its associated eigenvector (which is orthogonal to  $e(c)$ ). Then, by definition,

$$\zeta V = H(e(c))V = V + e(c)e(c)^\top V = V,$$

since  $e(c)^\top V = 0$ , which shows that  $\zeta = 1$ .

Consequently,  $N(c)$  and  $H(e(c))$  can be diagonalized simultaneously in an orthogonal basis of  $\mathbb{R}^N$ , say  $(Pe_1, \dots, Pe_N)$ , where  $P \in \mathcal{O}_N(\mathbb{R})$  so that the matrices  $\tilde{N}(c) := PN(c)P^\top$  and  $\tilde{H}(e(c)) := PH(e(c))P^\top$  are diagonal. Then, in view of (2.48), we have

$$\begin{aligned} A(c) \geq 0 &\iff (Px)^\top \frac{\beta_0}{\beta_1} \tilde{N}(c) Px \leq (Px)^\top \tilde{H}(e(c)) Px, \quad \forall x \in \mathbb{R}^N, \\ &\iff y^\top \frac{\beta_0}{\beta_1} \tilde{N}(c) y \leq y^\top \tilde{H}(e(c)) y, \quad \forall y \in \mathbb{R}^N. \end{aligned} \quad (2.49)$$

Since the common eigenvectors  $(Pe_1, \dots, Pe_N)$  of  $N(c)$  and  $H(e(c))$  form an orthogonal basis of  $\mathbb{R}^N$ , it suffices to check (2.49) on  $(Pe_1, \dots, Pe_N)$ . This completes the proof of (2.44), since

$$e_i^\top A(c) e_i \geq 0 \iff (Pe_i)^\top \frac{\beta_0}{\beta_1} \tilde{N}(c) Pe_i \leq (Pe_i)^\top \tilde{H}(e(c)) Pe_i, \quad i = 1, \dots, N.$$

Step 2. Straightforward computations show that there exists  $c_i$  satisfying  $\beta_1 c_i (1 - c_i) - \beta_0 \eta_i (1 - \eta_i) \geq 0$  if and only if  $\Delta_i := \beta_1 - 4\eta_i (1 - \eta_i) \beta_0 \geq 0$ , and in that case

$$0 < X_i^- = \frac{1}{2} - \frac{\sqrt{1 - 4\eta_i (1 - \eta_i) \frac{\beta_0}{\beta_1}}}{2} \leq c_i \leq X_i^+ = \frac{1}{2} + \frac{\sqrt{1 - 4\eta_i (1 - \eta_i) \frac{\beta_0}{\beta_1}}}{2} < 1. \quad (2.50)$$

Since by hypothesis  $\sum_{i=1}^N c_i = 1$ , we should have

$$\frac{N}{2} - \frac{1}{2} \sum_{i=1}^N \sqrt{1 - 4\eta_i (1 - \eta_i) \frac{\beta_0}{\beta_1}} \leq 1 \leq \frac{N}{2} + \frac{1}{2} \sum_{i=1}^N \sqrt{1 - 4\eta_i (1 - \eta_i) \frac{\beta_0}{\beta_1}}. \quad (2.51)$$

For  $N = 2$ , the estimate (2.51) is always satisfied. But if  $N \geq 3$ , it holds true if and only if

$$N - 2 \leq \sum_{i=1}^N \sqrt{1 - 4\eta_i (1 - \eta_i) \frac{\beta_0}{\beta_1}}. \quad (2.52)$$

According to (2.44), this achieves the proof of the lemma.  $\square$

To end the proof of the proposition in the case (ii), define  $d_i = \beta_1 c_i - \beta_0 \eta_i$ , recalling that we assumed in this case that  $0 < \eta_i \leq \frac{1}{2}$ , for all  $i \in \{1, \dots, N\}$ . When (2.43) is

satisfied, it follows from (2.50) that

$$\begin{aligned} m_i^- &= \frac{\beta_1}{2} - \beta_0 \eta_i - \frac{\sqrt{\beta_1^2 - 4\eta_i(1-\eta_i)\beta_0\beta_1}}{2} \leq d_i \\ &\leq m_i^+ = \frac{\beta_1}{2} - \beta_0 \eta_i + \frac{\sqrt{\beta_1^2 - 4\eta_i(1-\eta_i)\beta_0\beta_1}}{2}. \end{aligned} \quad (2.53)$$

By definition, the set  $\Theta \neq \emptyset$  if (2.43) is satisfied, and one can find  $d \in \mathbb{R}^N$  such that

$$d \cdot \mathbf{1}_{\mathbb{R}^N} = \beta_1 - \beta_0, \quad m_i^- \leq d_i \leq m_i^+, \quad d_i \neq 0 \quad \text{and} \quad \sum_{i=1}^N \frac{\eta_i^2}{d_i} = -\frac{1}{\beta_0}. \quad (2.54)$$

Since  $\beta_0 > 0$ , if (2.54) holds then necessarily there exists  $i_0$  such that  $d_{i_0} < 0$ . Because of (2.53), this requires that  $m_{i_0}^- < 0$ . But

$$m_{i_0}^- < 0 \iff \beta_0 < \beta_1. \quad (2.55)$$

Indeed, under the assumption  $|\eta|_\infty \leq \frac{1}{2}$ , it follows from (2.43) that  $\frac{\beta_1}{2} - \beta_0 \eta_i \geq 0$ , for all  $i \in \{1, \dots, N\}$ . We deduce that

$$m_{i_0}^- < 0 \iff (\beta_1 - 2\beta_0\eta_{i_0})^2 < \beta_1^2 - 4\eta_{i_0}(1-\eta_{i_0})\beta_0\beta_1 \iff \beta_0 < \beta_1.$$

Therefore, under the hypothesis  $|\eta|_\infty \leq \frac{1}{2}$ , we have

$$\Theta \neq \emptyset \Rightarrow \beta_0 < \beta_1.$$

Assume now that  $\beta_0 < \beta_1$  (with  $|\eta|_\infty \leq \frac{1}{2}$ ), and let us show that  $\Theta \neq \emptyset$ . Observing that

$$\sqrt{1 - 4\eta_i(1-\eta_i)\frac{\beta_0}{\beta_1}} = \sqrt{(1-2\eta_i)^2 + 4\eta_i(1-\eta_i)\left(1 - \frac{\beta_0}{\beta_1}\right)},$$

it follows from Lemma 2.13 that when  $\beta_0 < \beta_1$ , we have  $A(c) \geq 0$ . Then, we only have to find  $d \in \mathbb{R}^N$  such that

$$\sum_{i=1}^N d_i = \beta_1 - \beta_0 > 0 \quad \text{and} \quad \sum_{i=1}^N \frac{\eta_i^2}{d_i} = -\frac{1}{\beta_0} < 0, \quad (2.56)$$

under the constraints  $m_i^- \leq d_i \leq m_i^+$  (with  $d_i \neq 0$ ).

Up to a permutation, we can assume that  $\max_{1 \leq i \leq N} \eta_i = \eta_1$ . Then, we set

$$d_1 = \eta_1(-\beta_0 + \alpha^-) \quad \text{and} \quad d_i = \eta_i(-\beta_0 + \alpha^+), \quad \text{for } 2 \leq i \leq N,$$

so that  $d_1 < 0$  and  $d_i > 0$ , for  $2 \leq i \leq N$ . Since  $\eta_i \alpha^\pm = \beta_1 c_i > 0$  and  $c_i > 0$  in view of (2.44), we have  $0 < \alpha^- < \beta_0 < \alpha^+$ . The problem (2.56) then becomes

$$\eta_1 \alpha^- + (1 - \eta_1) \alpha^+ = \beta_1 \quad \text{and} \quad \frac{\eta_1}{-\beta_0 + \alpha^-} + \frac{1 - \eta_1}{-\beta_0 + \alpha^+} = -\frac{1}{\beta_0}, \quad (2.57)$$

under the constraints<sup>6</sup>

$$\begin{aligned} \frac{\beta_1}{2\eta_1} \left(1 - \sqrt{1 - 4\eta_1(1 - \eta_1)\frac{\beta_0}{\beta_1}}\right) &\leq \alpha^- < \beta_0 < \beta_1 < \alpha^+ \\ &\leq \min_{2 \leq i \leq N} \frac{\beta_1}{2\eta_i} \left(1 + \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}\right). \end{aligned} \quad (2.58)$$

A direct computation shows that (2.57) is equivalent to

$$\eta_1\alpha^- + (1 - \eta_1)\alpha^+ = \beta_1 \quad \text{and} \quad \alpha^-\alpha^+ = \beta_0\beta_1. \quad (2.59)$$

Setting  $\alpha^- = \frac{\beta_0}{C}$  and  $\alpha^+ = C\beta_1$ , we deduce that  $(1 - \eta_1)\beta_1C^2 - \beta_1C + \eta_1\beta_0 = 0$ . This implies

$$C = \frac{1}{2(1 - \eta_1)} \left(1 + \sqrt{1 - 4\eta_1(1 - \eta_1)\frac{\beta_0}{\beta_1}}\right). \quad (2.60)$$

When  $\beta_0 < \beta_1$  and  $\eta_1 \leq 1/2$ , we have  $\sqrt{1 - 4\eta_1(1 - \eta_1)\frac{\beta_0}{\beta_1}} > 1 - 2\eta_1$ . This shows that  $C > 1$ , which ensures that  $0 < \alpha^- < \beta_0 < \beta_1\alpha^+$ .

To end the proof, it suffices to check that the constraints (2.58) are satisfied. The left hand side  $\frac{\beta_1}{2\eta_1} \left(1 - \sqrt{1 - 4\eta_1(1 - \eta_1)\frac{\beta_0}{\beta_1}}\right) \leq \alpha^- = \frac{\beta_0}{C}$ , easily follows from (2.60). The right hand side  $C\beta_1 \leq \min_{2 \leq i \leq N} \frac{\beta_1}{2\eta_i} \left(1 + \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}\right)$ , is deduced from the choice of  $\eta_1 = \max_{1 \leq i \leq N} \eta_i$ . Indeed, we get for all  $2 \leq i \leq N$ ,

$$C = \frac{1}{2(1 - \eta_1)} \left(1 + \sqrt{1 - 4\eta_1(1 - \eta_1)\frac{\beta_0}{\beta_1}}\right) \leq \frac{1}{2\eta_i} \left(1 + \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}\right)$$

since we have

$$\begin{aligned} \left(1 + \sqrt{1 - 4\eta_1(1 - \eta_1)\frac{\beta_0}{\beta_1}}\right) &\leq \frac{(1 - \eta_1)}{\eta_i} \left(1 + \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}\right) \\ &\leq \frac{\sum_{j \geq 2}^N \eta_j}{\eta_i} \left(1 + \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}\right) \\ &\leq \left(1 + \frac{\sum_{j \neq \{i, 1\}}^N \eta_j}{\eta_i}\right) \left(1 + \sqrt{1 - 4\eta_i(1 - \eta_i)\frac{\beta_0}{\beta_1}}\right). \end{aligned}$$

This ends the proof of the proposition.  $\square$

<sup>6</sup>Under the hypothesis  $\beta_0 < \beta_1$ , the property (2.57) requires that  $\alpha^+ > \beta_1$ .

## 2.4 Interior deterministic Nash equilibria

In this section, we turn our attention to the continuous time model with a finite investment horizon  $T \in \mathbb{R}^+$  and in the general case of  $N$  finite agents. We shall denote by  $\mathcal{A}_{\text{det}}^{i,N} \subset \mathcal{A}^{i,N}$  and  $\mathcal{A}_{\text{det}}^N \subset \mathcal{A}^N$  the collection of deterministic strategies.

Following the previous works in this setting, we shall make use of the backward stochastic differential equations (BSDE in short). Indeed, as already emphasized, each agent  $i \in \{1, \dots, N\}$  maximizes an exponential utility function (2.6). As it can be seen in the broad literature, this optimization problem can be formulated in terms of quadratic BSDE (see for instance the articles of N. El Karoui and R. Rouge [21], Y. Hu and P. Imkeller [33] or G-E. Espinosa and N. Touzi [25]).

As we shall see later, it turns out that investigating our Nash equilibrium problem in the continuous time model reduces to the static setting which consists in looking for Nash equilibria at the level of the individual agent's Hamiltonians. In the particular case when the coefficients are deterministic, this leads to investigate the system of equations as in the static setting studied in the previous section. However, when the coefficients are stochastics, the minimization problem that arises is slightly different and more challenging than the one involved in the deterministic case.

Let us start by introducing the following multidimensional BSDE: Let  $(Y_t^{\hat{\pi}}, Z_t^{\hat{\pi}})_{t \in [0, T]}$  be a pair of  $(\mathcal{F}_t)$ -progressively measurable processes with values in  $\mathbb{R}^N \times \mathbb{R}^{N \times N}$  satisfying

$$dY_t^{\hat{\pi}} = Q_t^{\hat{\pi}}(Z_t^{\hat{\pi}})dt + Z_t^{\hat{\pi}T}dB_t, \quad t \in [0, T] \quad \text{with} \quad Y_T^{\hat{\pi}} = \sum_{i=1}^N \left( \frac{q_i}{2} \int_0^T |\gamma_t|^2 dt \right) e_i \in \mathbb{R}^N. \quad (2.61)$$

By virtue of the boundedness of  $\gamma$ ,  $B_t = \int_0^t \gamma_s ds + W_t$  is a Brownian motion under  $\mathbb{Q}$  which is an equivalent probability measure via the density with respect to  $\mathbb{P}$ :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T -\gamma_t \cdot dW_t - \frac{1}{2} \int_0^T |\gamma_t|^2 dt} \quad \text{on } \mathcal{F}_T.$$

The generator  $Q_t^{\hat{\pi}}$  is measurable with respect to  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{N \times N})$  and has the following form:

$$\begin{aligned} Q^{\hat{\pi}} : [0, T] \times \Omega \times \mathbb{R}^{N \times N} &\rightarrow \mathbb{R}^N \\ (t, \omega, Z) &\mapsto \sum_{i=1}^N \frac{1}{2q_i} \left| u_t(\hat{\pi}_t) e_i - \zeta_t^i \right|^2 e_i, \end{aligned} \quad (2.62)$$

where  $u_t(\hat{\pi}_t)$  is defined in (2.4), the variable  $\zeta_t^i = q_i \gamma_t - Z_t e_i$  and  $\hat{\pi} \in \mathcal{A}^N$  satisfy, for all  $t \in [0, T]$  and for all  $i \in \{1, \dots, N\}$ ,

$$\left| u_t(\hat{\pi}_t) e_i - \zeta_t^i \right|^2 \leq \left| u_t(\hat{\pi}_t^{1,\dots, \hat{\pi}_t^{i-1,\dots, \pi_t^{i,\dots, \hat{\pi}_t^{i+1,\dots, \hat{\pi}_t^{N,\dots}}})} e_i - \zeta_t^i \right|^2, \quad \pi^{i,\dots} \in \mathcal{A}^{i,N}. \quad (2.63)$$

In our problem, we expect that the criterion of each agent  $1 \leq i \leq N$  takes the following form under the existence of interior Nash equilibrium  $\hat{\pi} \in \mathcal{A}^N$ :

$$J^i(\hat{\pi}) = -e^{-\frac{X_0^i + Y_0^{\hat{\pi}} \cdot e_i}{q_i}}, \quad i = 1, \dots, N. \quad (2.64)$$



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Using the same approach as in the static case (see the proof of Lemma 2.5), the optimization problem (2.63) is equivalent to the fact that, for all  $t \in [0, T]$ , the basis of vector  $(u_t(\hat{\pi}_t)e_1, \dots, u_t(\hat{\pi}_t)e_N)$  satisfies the following coupled nonlinear system

$$\begin{cases} (u_t(\hat{\pi}_t)e_i - \zeta^i) \cdot u_t(\hat{\pi}_t)e_j = 0, & 1 \leq i \neq j \leq N, \\ \sum_{i=1}^N u_t(\hat{\pi}_t)e_i = \Gamma_t \mathbf{1}_{\mathbb{R}^N}, \end{cases} \quad (2.65)$$

under the following constraints<sup>7</sup>

$$(u_t(\hat{\pi}_t)^{-1}\Gamma_t)_{i,j} < 0, \quad 1 \leq i \neq j \leq N. \quad (2.66)$$

**Remark 2.14.** In the case when the coefficients are stochastic, let us point out that the well-posedness of the multidimensional quadratic BSDE (2.61) is still an open question. Indeed, this new system is more involved than the one studied in Section 2.3 concerning the static case, and it is challenging to investigate the properties of the BSDE's generator  $Q^{\hat{\pi}}$ , given by (2.62) (for related issues, one can see the articles of Y. Hu and S. Tang [34] or S. Fang, Y. Hu and S. Tang [26]).

When the coefficients are deterministic, the system (2.65)-(2.66) is at each instant  $t \in [0, T]$  equivalent to the system (2.17)-(2.18) studied in the static case. Indeed, notice that the role of the process  $Z$  in the BSDE (2.61) is to make the process  $Y^{\hat{\pi}}$   $\mathcal{F}_t$ -adapted. Therefore, if the parameters  $\Gamma$ ,  $\gamma$  and the investments  $\pi$  are deterministic, then  $Z = 0_{\mathbb{R}^N \times \mathbb{R}^N}$ ,  $\mathbb{P}$ -a.s. It follows from (2.61) that  $Y$  satisfies the following ODE

$$dY_t^{\hat{\pi}} = Q_t^{\hat{\pi}}(0_{\mathbb{R}^N \times \mathbb{R}^N})dt, \quad t \in [0, T] \quad \text{with} \quad Y_T^{\hat{\pi}} = \sum_{i=1}^N \left( \frac{q_i}{2} \int_0^T |\gamma_t|^2 dt \right) e_i \in \mathbb{R}^N, \quad (2.67)$$

where we recall that  $Q$  is defined in (2.62) and  $\hat{\pi} \in \mathcal{A}_{\text{det}}^N$  satisfies (2.65)-(2.66) with  $Z = 0_{\mathbb{R}^N \times \mathbb{R}^N}$ . Thus, the interior deterministic Nash equilibria  $\hat{\pi} \in \mathcal{A}_{\text{det}}^N$  is given, for all  $t \in [0, T]$ , by  $\hat{\pi}_t$  provided by Theorem 2.8, with parameters  $b_t$  and  $\Gamma_t$ . Hence, we deduce the following theorem in the deterministic case:

**Theorem 2.15.** *Assume that  $\gamma$  and  $\Gamma$  are deterministic functions satisfying the conditions of Theorem 2.8. Then, there exists an interior deterministic Nash equilibria  $\hat{\pi} \in \mathcal{A}_{\text{det}}^N$  that are given by the cross-holding strategies  $\hat{\pi}$  satisfying at each instant  $t \in [0, T]$  the equations (2.65)-(2.66) with  $Z = 0_{\mathbb{R}^N \times \mathbb{R}^N}$ ,  $\mathbb{P}$ -a.s, and the value function of each agent  $i \in \{1, \dots, N\}$  is given by*

$$J^i(\hat{\pi}) = -e^{-\frac{x_i + Y_0^{\hat{\pi}} \cdot e_i}{q_i}},$$

with

$$Y_0^{\hat{\pi}} = \sum_{i=1}^N \left( \int_0^T u_t(\hat{\pi}_t)e_i \cdot \left( \gamma_t - \frac{u_t(\hat{\pi}_t)e_i}{2q_i} \right) dt \right) e_i.$$

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<sup>7</sup>since  $\hat{\pi}_t^{i,j} > 0$  for all  $i \neq j$ .

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*Proof.* Let  $\gamma, \Gamma$  be deterministic processes and let  $\hat{\pi} \in \mathcal{A}_{\text{det}}^N$  be a solution to the BSDE (2.61) with  $Z = 0_{\mathbb{R}^{N \times N}}$ ,  $\mathbb{P}$ -a.s. Setting

$$\tilde{Y}_t^{\hat{\pi}} = Y_t^{\hat{\pi}} - \sum_{i=1}^N \left( \frac{q_i}{2} \int_0^t |\gamma_s|^2 ds \right) e_i, \text{ for all } t \in [0, T],$$

we observe that  $\tilde{Y}^{\hat{\pi}}$  satisfies

$$d\tilde{Y}_t^{\hat{\pi}} = P_t^{\hat{\pi}} dt, \text{ for all } t \in [0, T] \quad \text{with} \quad \tilde{Y}_T^{\hat{\pi}} = 0_{\mathbb{R}^N}, \quad (2.68)$$

where  $P_t^{\hat{\pi}} = \sum_{i=1}^N \left( \frac{1}{2q_i} |u_t(\hat{\pi}_t) e_i|^2 - u_t(\hat{\pi}_t) e_i \cdot \gamma_t \right) e_i$ .

To go further, set the following notation, for all  $i \in \{1, \dots, N\}$ ,

$$(\pi^{i\cdot}, \hat{\pi}^{-i\cdot}) = (\hat{\pi}^{1\cdot}, \dots, \hat{\pi}^{i-1\cdot}, \pi^{i\cdot}, \hat{\pi}^{i+1\cdot}, \dots, \hat{\pi}^{N\cdot}), \text{ for all } \pi^{i\cdot} \in \mathcal{A}_{\text{det}}^{i,N}. \quad (2.69)$$

Our aim is to prove that the family of stochastic processes

$$R_t^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} = -e^{-\frac{X_t^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} + \tilde{Y}_t^{\hat{\pi}} e_i}{q_i}}, \text{ for all } t \in [0, T] \text{ and all } \pi^{i\cdot} \in \mathcal{A}_{\text{det}}^{i,N}, \quad (2.70)$$

satisfy the following properties, for any process  $\pi^{i\cdot} \in \mathcal{A}_{\text{det}}^{i,N}$ :

1.  $R_T^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} = -e^{-\frac{X_T^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}}}{q_i}}$ ,
2.  $R_0^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} = R_0^i$ ,
3.  $R^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}}$  is a  $\mathbb{P}$ -supermartingale, and  $R^{i, \hat{\pi}^{i\cdot}, \hat{\pi}^{-i\cdot}} = R^{i, \hat{\pi}}$  is a  $\mathbb{P}$ -martingale.

Note that, under the above properties, the value function of the  $i$ -th agent is given by  $J^i(\hat{\pi}) = R_0^i$ . Indeed, it follows from these properties that

$$\mathbb{E}[R_T^{i, \hat{\pi}}] = R_0^i \geq \mathbb{E}\left[-e^{-\frac{X_T^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}}}{q_i}}\right], \text{ for all } \pi^{i\cdot} \in \mathcal{A}_{\text{det}}^{i,N}.$$

The first two properties are direct consequences of the definitions of  $\tilde{Y}^{\hat{\pi}}$  and  $R^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}}$ , respectively given by equations (2.68) and (2.70). To show the third property, we start by applying Itô's formula on the process  $R^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}}$ :

$$\begin{aligned} dR_t^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} &= -\frac{1}{2q_i^2} R_t^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} \left( |u(\hat{\pi}_t) e_i - q_i \gamma_t|^2 - |u_t(\pi_t^{i\cdot}, \hat{\pi}_t^{-i\cdot}) e_i - q_i \gamma_t|^2 \right) dt \\ &\quad - \frac{1}{q_i} R_t^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} u_t(\pi_t^{i\cdot}, \hat{\pi}_t^{-i\cdot}) e_i \cdot dW_t. \end{aligned} \quad (2.71)$$

It remains to prove that the process  $R^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}} u(\pi^{i\cdot}, \hat{\pi}^{-i\cdot}) e_i$  belongs to  $\mathbb{H}^2(\mathbb{R}^N)$ , for all  $i \in \{1, \dots, N\}$ . In this case, since  $\hat{\pi}$  satisfies (2.63) with  $Z = 0_{\mathbb{R}^{N \times N}}$ ,  $\mathbb{P}$ -a.s, it follows from (2.71) that the process  $R^{i, \pi^{i\cdot}, \hat{\pi}^{-i\cdot}}$  is a  $\mathbb{P}$ -supermartingale, for all  $\pi^{i\cdot} \in \mathcal{A}_{\text{det}}^{i,N}$ , and is a square integrable  $\mathbb{P}$ -martingale for  $\pi^{i\cdot} = \hat{\pi}^{i\cdot}$ .

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Let  $i \in \{1, \dots, N\}$ . In order to show that  $R^{i, \pi^{i, \cdot}, \hat{\pi}^{-i, \cdot}} u(\pi^{i, \cdot}, \hat{\pi}^{-i, \cdot}) e_i \in \mathbb{H}^2(\mathbb{R}^N)$ , we first observe that there exists a positive constant  $C_N$  such that<sup>8</sup>

$$|u_t(\pi_t) e_i| \leq C_N, \text{ for all } t \in [0, T] \text{ and all } \pi \in \mathcal{A}_{\det}^N. \quad (2.72)$$

Indeed, thanks to Lemma 2.1, the matrix  $(M_t(\pi_t)^{-1})^\top$  is a stochastic matrix, for all  $t \in [0, T]$  and all  $\pi \in \mathcal{A}_{\det}^N$ , which implies that  $\|(M_t(\pi_t)^{-1})^\top\|^2 \leq N$ . Therefore, we get (2.72) according to the boundedness of  $\Gamma_t$ .

Combining (2.68) and (2.72) with the boundedness of  $\gamma_t$ , we infer that there exists a positive constant  $C_N$  such that

$$|\tilde{Y}_t^{\hat{\pi}} \cdot e_i| \leq C_N, \text{ for all } t \in [0, T]. \quad (2.73)$$

Next, in view of (2.4), we have

$$\begin{aligned} \mathbb{E}\left[\left(e^{-\frac{X_t^{i, \pi}}{q_i}}\right)^2\right] &= e^{-2\frac{X_0^i}{q_i}} \mathbb{E}\left[e^{-\frac{2}{q_i} \int_0^t \left(u_s(\pi_s) e_i \cdot \gamma_s ds + u_s(\pi_s) e_i \cdot dW_s\right)}\right] \\ &\leq C_N \mathbb{E}\left[e^{\int_0^t -\frac{2}{q_i} u_s(\pi_s) e_i \cdot dW_s - \frac{1}{2} \int_0^t \frac{2}{q_i} u_s(\pi_s) e_i|^2 ds}\right] = C_N, \end{aligned}$$

where the last equality follows from the fact that  $e^{\int_0^t -\frac{2}{q_i} u_s(\pi_s) e_i \cdot dW_s - \frac{1}{2} \int_0^t \frac{2}{q_i} |u_s(\pi_s) e_i|^2 ds}$  is an  $\mathbb{P}$ -exponential martingale (since  $u_t(\pi_t) e_i$  is bounded in view of (2.72)), and where the constant  $C_N$  satisfies (by the boundedness of  $u_t(\pi_t) e_i$  and  $\gamma_t$ )

$$e^{-2\frac{X_0^i}{q_i}} e^{-\frac{2}{q_i} \int_0^t \left(u_s(\pi_s) e_i \cdot \gamma_s + \frac{1}{2} \frac{2}{q_i} u_s(\pi_s) e_i|^2\right) ds} \leq C_N.$$

Consequently

$$\mathbb{E}\left[\left(e^{-\frac{X_t^{i, \pi}}{q_i}}\right)^2\right] \leq C_N, \text{ for all } t \in [0, T]. \quad (2.74)$$

Invoking the properties (2.72)-(2.74), we deduce that there exists  $C_N > 0$  such that

$$\mathbb{E}\left[\left(R_t^{i, \pi^{i, \cdot}, \hat{\pi}^{-i, \cdot}}\right)^2 |u_t(\pi_t^{i, \cdot}, \hat{\pi}_t^{-i, \cdot}) e_i|^2\right] \leq C_N, \text{ for all } t \in [0, T],$$

which implies that

$$\mathbb{E}\left[\int_0^T |R_s^{i, \pi^{i, \cdot}, \hat{\pi}^{-i, \cdot}} u_s(\pi_s^{i, \cdot}, \hat{\pi}_s^{-i, \cdot}) e_i|^2 ds\right] \leq TC_N.$$

This completes the proof of the theorem.  $\square$

## 2.5 Appendix A: Comparison of Theorem 2.8 with Proposition 2.3

In this section, we show that Theorem 2.8 (corresponding to the general case of  $N \geq 2$  agents) is in agreement with Proposition 2.3 (concerning the case of 2 agents), in the case when  $\beta_1 > 0$ .

<sup>8</sup>In what follows,  $C_N$  will be used to denote constants which may vary from line to line.

First, observe that the existence conditions are the same. In fact the parameter  $A$  defined by (2.11) can be written as follows:  $A = \beta_1 - 4\beta_0\eta_1\eta_2$ . Indeed, we have

$$\begin{aligned}\beta_1 - 4\beta_0\eta_1\eta_2 &= |\Gamma\mathbf{1}_{\mathbb{R}^2}|^2 - \frac{4\bar{b}_2^2}{|\gamma|^2} - 4\bar{b}_2\left(\bar{q}_2 - \frac{\bar{b}_2}{|\gamma|^2}\right)\frac{q_1q_2}{\bar{q}_2^2} \\ &= |\Gamma\mathbf{1}_{\mathbb{R}^2}|^2 - \frac{4q_1q_2}{\bar{q}_2}\bar{b}_2 - \frac{4\bar{b}_2^2}{|\gamma|^2}\left(1 - \frac{q_1q_2}{\bar{q}_2^2}\right) = A\end{aligned}$$

and one can easily check that  $\Theta \neq \emptyset$  if and only if  $A \geq 0$ , with

$$\Theta = \left\{c \in \mathbb{R}^2 : c \cdot \mathbf{1}_{\mathbb{R}^2} = 1 \text{ where } c_1 = \frac{1}{2\sqrt{\beta_1}}(\sqrt{\beta_1} \pm \sqrt{A}), \text{ and } A > 0 \text{ if } \eta_1 = \frac{1}{2}\right\}.$$

Let us also check that the optimal strategies  $\hat{\pi}^{i,j}$  given by Theorem 2.8 and Proposition 2.3 coincide. Since  $P_{\gamma^\perp}(\Gamma\mathbf{1}_{\mathbb{R}^2}) = \frac{\Gamma\mathbf{1}_{\mathbb{R}^2} \cdot \gamma^\perp}{|\gamma^\perp|} \frac{\gamma^\perp}{|\gamma^\perp|} = \frac{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})}{|\gamma|^2} \gamma^\perp$ , Formula (2.25) in Theorem 2.8 takes the following form

$$M = \left(-\frac{\beta_0}{2\bar{b}_2} b\eta^\top + \frac{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})}{|\gamma|^2} \Gamma\gamma^\perp c^\top\right) D^{-1}(c).$$

where

$$D(c)^{-1} = 2 \begin{pmatrix} \frac{1}{\sqrt{\beta_1}(\sqrt{\beta_1} + \sqrt{A}) - 2\beta_0\eta_1} & 0 \\ 0 & \frac{1}{\sqrt{\beta_1}(\sqrt{\beta_1} - \sqrt{A}) - 2\beta_0\eta_2} \end{pmatrix}, \quad (2.75)$$

with  $A = \beta_1 - 4\beta_0\eta_1\eta_2$ . In view of (2.25), we obtain

$$M = \left(-\frac{\beta_0}{2\bar{b}_2} b\eta^\top + \frac{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})}{|\gamma|^2} \Gamma\gamma^\perp \frac{1}{2\sqrt{\beta_1}}(\sqrt{\beta_1} + \sqrt{A}, \sqrt{\beta_1} - \sqrt{A})\right) D^{-1}(c).$$

Since by definition  $\beta_1 = |P_{\gamma^\perp}(\Gamma\mathbf{1}_{\mathbb{R}^2})|^2 = \frac{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})^2}{|\gamma|^2}$ , we infer that

$$M = \left(-\frac{\beta_0}{2\bar{b}_2} b\eta^\top + \sqrt{\beta_1} \Gamma\gamma^\perp \frac{1}{2\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})}(\sqrt{\beta_1} + \sqrt{A}, \sqrt{\beta_1} - \sqrt{A})\right) D^{-1}(c).$$

Invoking (2.75) and recalling that  $\hat{\pi}^{i,j} = -M_{i,j}$ , we deduce that

$$\begin{aligned}-\hat{\pi}^{1,2} &= \frac{2}{\sqrt{\beta_1}(\sqrt{\beta_1} - \sqrt{A}) - 2\beta_0\eta_2} \left(-\frac{\beta_0}{2\bar{b}_2} b_1\eta_2 + \frac{\Gamma\gamma^\perp \cdot e_1}{2\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})} \sqrt{\beta_1}(\sqrt{\beta_1} - \sqrt{A})\right) \\ &= \frac{\Gamma\gamma^\perp \cdot e_1}{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})} + \frac{2\beta_0\eta_2}{\sqrt{\beta_1}(\sqrt{\beta_1} - \sqrt{A}) - 2\beta_0\eta_2} \left(\frac{\Gamma\gamma^\perp \cdot e_1}{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})} - \frac{b_1}{2\bar{b}_2}\right).\end{aligned}$$

Recalling that for any unitary vector  $u \in \mathbb{R}^2$ , one has

$$(u^\perp \cdot \Gamma e_2)u \cdot \Gamma e_1 - (u \cdot \Gamma e_2)u^\perp \cdot \Gamma e_1 = \det(\Gamma),$$

with  $u^\perp = -u_2e_1 + u_1e_2$ , we readily gather that

$$\frac{b_1}{2\bar{b}_2} - \frac{\Gamma\gamma^\perp \cdot e_1}{\det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})} = \frac{|\gamma|^2 \det(\Gamma)}{2\bar{b}_2 \det(\gamma, \Gamma\mathbf{1}_{\mathbb{R}^2})} = \pm \frac{|\gamma| \det(\Gamma)}{2\bar{b}_2 \sqrt{\beta_1}}, \quad (2.76)$$

which implies that

$$-\hat{\pi}^{1,2} = \frac{b_1}{2\bar{b}_2} - \frac{|\gamma|^2 \det(\Gamma)}{2\bar{b}_2 \det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})} \left( \frac{\sqrt{\beta_1}(\sqrt{\beta_1} - \sqrt{A})}{\sqrt{\beta_1}(\sqrt{\beta_1} - \sqrt{A}) - 2\beta_0 \eta_2} \right).$$

Combining (2.76) together with the relations  $A = \beta_1 - 4\beta_0 \eta_1 \eta_2$  and  $\eta_1 + \eta_2 = 1$ , we deduce that

$$\begin{aligned} -\hat{\pi}^{1,2} &= \frac{b_1}{2\bar{b}_2} - \frac{|\gamma|^2 \det(\Gamma)}{2\bar{b}_2 \det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})} \left( \frac{\sqrt{\beta_1}(\beta_1 - A)}{\sqrt{\beta_1}(\beta_1 - A) - 2\beta_0 \eta_2(\sqrt{\beta_1} + \sqrt{A})} \right) \\ &= \frac{b_1}{2\bar{b}_2} - \frac{|\gamma|^2 \det(\Gamma)}{\bar{b}_2 \det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})} \left( \frac{\sqrt{\beta_1} \eta_1}{\sqrt{\beta_1}(2\eta_1 - 1) - \sqrt{A}} \right) \\ &= \frac{b_1}{2\bar{b}_2} - \frac{|\gamma|^2 \det(\Gamma)}{\bar{b}_2 \det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})} \left( \frac{\sqrt{\beta_1} \eta_1}{\sqrt{\beta_1}(\eta_1 - \eta_2) - \sqrt{A}} \right), \end{aligned}$$

which shows that the optimal strategies given by Theorem 2.8 coincide with that of Proposition 2.3 given by

$$\hat{\pi}_{\pm}^{i,j} = -\frac{b_i}{2\bar{b}_2} \pm \frac{|\gamma| q_i \det(\Gamma)}{2\bar{b}_2 \left( \frac{\det(\gamma, \Gamma \mathbf{1}_{\mathbb{R}^2})(q_2 - q_1)}{2|\gamma|} \pm \bar{q}_2 \sqrt{A} \right)}.$$

## 2.6 Appendix B: Proof of Identity (2.27)

Let  $b \neq 0_{\mathbb{R}^N}$  and  $\hat{\pi}^N \in \mathcal{A}_{\text{const}}^N$  be a Nash equilibrium. Recall that  $\bar{b}_N \neq 0$  is a necessary condition to the existence of a Nash equilibrium  $\hat{\pi}$  in this case. We shall prove that  $\hat{\pi}$  satisfies (2.27) in two steps.

(i) Step 1: Let  $\bar{b}_N = |\gamma|^2 \bar{q}_N$ . Using that  $\hat{\pi}^{i,j} = -M_{i,j}^P$ , with  $M^P$  given by (2.24), we shall establish that, for all  $j \in \{1, \dots, N\}$ ,

$$\begin{aligned} \sum_{i \neq j}^N (\hat{\pi}^{j,i} - \hat{\pi}^{i,j}) &= 1 + \sum_{i \neq k} \left( \frac{q_i}{q_k} - 1 \right) \frac{\Gamma e_j \cdot P e_i}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_i} - \left( \sum_{i \neq k} \frac{q_i}{q_k} + 1 \right) \frac{\Gamma e_j \cdot P e_k}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_k} \\ &= 1 + \sum_{i \neq k} \left( \frac{q_i}{q_k} - 1 \right) \frac{\Gamma e_j \cdot P e_i}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_i} - \left( \sum_{i \neq k} \frac{q_i}{q_k} + 1 \right) \frac{b_j}{N \bar{b}_N}, \end{aligned} \quad (2.77)$$

where the last equality is deduced from the fact that  $P e_k = \frac{\gamma}{|\gamma|}$ . Obviously, if  $q_i = q$  for all  $i \in \{1, \dots, N\}$ , then Identity (2.27) follows immediately the relation (2.77).

In view of (2.24), we shall investigate separately the case  $j \neq k$  from the case  $j = k$ .

1. Case  $j \neq k$ : by definition, we have

$$\begin{aligned} -\sum_{i \neq j}^N \hat{\pi}^{i,j} &= \sum_{i \neq j} \frac{\Gamma e_i \cdot P e_j}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_j} = 1 - \frac{\Gamma e_j \cdot P e_j}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_j}, \\ -\sum_{i \neq j}^N \hat{\pi}^{j,i} &= \sum_{i \neq j,k}^N \frac{\Gamma e_j \cdot P e_i}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_i} \left( 1 - \frac{q_i}{q_k} \right) - \frac{q_j}{q_k} \frac{\Gamma e_j \cdot P e_j}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_j} + \left( 1 + \sum_{i \neq k} \frac{q_i}{q_k} \right) \frac{\Gamma e_j \cdot P e_k}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_k}, \end{aligned}$$

which leads to (2.77).

2. Case  $j = k$ : in that case, we get

$$\begin{aligned} -\sum_{j \neq k} \hat{\pi}^{j,k} &= 1 - \left(1 + \sum_{i \neq k} \frac{q_i}{q_k}\right) \frac{\Gamma e_k \cdot P e_k}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_k} + \sum_{i \neq k} \frac{q_i}{q_k} \frac{\Gamma e_k \cdot P e_i}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_i}, \\ -\sum_{j \neq k} \hat{\pi}^{k,j} &= \sum_{j \neq k} \frac{\Gamma e_k \cdot P e_j}{\Gamma \mathbf{1}_{\mathbb{R}^N} \cdot P e_j}, \end{aligned}$$

which gives (2.77).

(ii) Step 2: Let  $\beta_0 \neq 0$ . Then  $\hat{\pi}^{i,j} = -M_{i,j}^P$ , with  $M^P$  defined by (2.25), which implies that

$$\begin{aligned} -\sum_{i \neq j} \hat{\pi}^{i,j} \eta_j &= \frac{1}{D_j(c)} \left( \frac{-\beta_0 \eta_j^2}{N \bar{b}_N} (N \bar{b}_N - b_j) + c_j \eta_j P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot (\Gamma \mathbf{1}_{\mathbb{R}^N} - \Gamma e_j) \right. \\ &\quad \left. + \eta_j P(c)^\top e_j \cdot S(c) \Gamma(\mathbf{1}_{\mathbb{R}^N} - e_j) \right), \end{aligned}$$

where  $D_j(c) = D(c) e_j \cdot e_j$ . Recalling that, in view of (2.22),  $\Gamma \mathbf{1}_{\mathbb{R}^N} \in \text{Ker}(S(c))$ , we deduce that

$$\begin{aligned} -\sum_{i \neq j} \hat{\pi}^{i,j} &= \frac{1}{D_j(c)} \left( -\beta_0 \eta_j^2 + \frac{\beta_0 \eta_j^2}{N \bar{b}_N} b_j + c_j \eta_j \beta_1 - c_j \eta_j P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot \Gamma e_j \right) \\ &\quad - \frac{1}{D_j(c)} \left( \eta_j P(c)^\top e_j \cdot S(c) \Gamma e_j \right) \\ &= \frac{1}{D_j(c)} \left( \frac{\beta_0 \eta_j^2}{N \bar{b}_N} b_j - c_j \eta_j P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot \Gamma e_j - \eta_j P(c)^\top e_j \cdot S(c) \Gamma e_j \right) + \eta_j, \end{aligned} \tag{2.78}$$

where in the last equality, we used the fact that  $-\beta_0 \eta_j + \beta_1 c_j = D_j(c)$ .

Using again (2.25), we get

$$\begin{aligned} -\sum_{i \neq j} \hat{\pi}^{j,i} \eta_i &= -\left( \sum_{i \neq j} \frac{\eta_i^2}{D_i(c)} \right) \frac{\beta_0}{N \bar{b}_N} b_j + \left( \sum_{i \neq j} \frac{c_i \eta_i}{D_i(c)} \right) P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot \Gamma e_j \\ &\quad + P(c)^\top \left( \sum_{i \neq j} \frac{\eta_i e_i}{D_i(c)} \right) \cdot S(c) \Gamma e_j. \end{aligned} \tag{2.79}$$

Then, as in proof of Lemma 2.7, since  $c \in \Theta$ , we have

$$0 = \beta_0 \eta^\top D(c)^{-1} \eta + 1 = \beta_1 c^\top D(c)^{-1} \eta, \tag{2.80}$$

which implies

$$A(c) D(c)^{-1} \eta = \eta - \beta_1 c^\top D(c)^{-1} \eta c + \beta_0 \eta^\top D(c)^{-1} \eta \eta = 0_{\mathbb{R}^N}.$$

Thus, we deduce that  $D(c)^{-1} \eta \in \text{Ker}(A(c))$ , and then  $D(c)^{-1} \eta \in \text{Ker}(P^\top(c))$  by virtue of (2.22). Injecting this last property together with (2.80) in (2.79), we infer that

$$-\sum_{i \neq j} \hat{\pi}^{j,i} \eta_i = -\left( \sum_{i \neq j} \frac{\eta_i^2}{D_i(c)} \right) \frac{\beta_0}{N \bar{b}_N} b_j - c_j \eta_j P_{\gamma^\perp}(\Gamma \mathbf{1}_{\mathbb{R}^N}) \cdot \Gamma e_j - \frac{\eta_j}{D_j(c)} P(c)^\top e_j \cdot S(c) \Gamma e_j. \tag{2.81}$$

Consequently, subtracting (2.81) from (2.78), we get

$$\sum_{i \neq j}^N \hat{\pi}^{j,i} \eta_i - \hat{\pi}^{i,j} \eta_j = \sum_{i=1}^N \frac{\beta_0 \eta_i^2}{N D_i(c)} \frac{b_j}{b_N} + 1.$$

Since  $\sum_{i=1}^N \frac{\beta_0 \eta_i^2}{D_i(c)} = -1$  by (2.80), setting  $q_i = q$ , for all  $i \in \{1, \dots, N\}$ , completes the proof of the result.

# Chapter 3

## One period mutual holding game with common noise

*In this chapter, we consider the mean field mutual holding model that was studied in the papers of M-F. Djete and N. Touzi [19] and M-F. Djete, G. Guo and N. Touzi [20]. We introduce a common noise and we limit ourselves to the one period model. In this framework, the formulation of the dynamic of the mean field model reveals a structure related to a No-arbitrage (NA) condition. We characterize this condition, then we study the mean field game related to a mean-variance optimization problem. Under some appropriate hypothesis on the cross-holding strategies, the existence conditions of the equilibria are in agreement with (NA) condition. This leads us to study the mean field equilibrium associated with this optimization problem for which we derive the system of equations to solve.*

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### 3.1 Introduction

In this paper, we consider the mutual holding or cross-holding model that was introduced in the papers of M-F. Djete and N. Touzi [19] and M-F. Djete, G. Guo and N. Touzi [20]. This model describes the equity dynamic value of a number  $N \geq 1$  of interdependent economics agents by taking into account on the one hand, the different investments made directly between themselves through the holding of shares, and on the other hand, the revenues of the economic entities that can be correlated. These revenues are modeled through the idiosyncratic risk processes of the economic agents.

We have already seen in the first chapter of this thesis that studying the Nash equilibrium<sup>1</sup> in the finite game ( $N < \infty$  agents) is really complex because we have to take into account the optimization problem of each agent and their characteristics, which leads to analyse a system of equations that becomes difficult to solve, as the number  $N$  of agents increases.

When we consider a very large number of agents  $N$ , there is a possibility to reduce the complexity of such a problem by using the mean field games theory. This recent theory which was introduced by M. Huang, P. Caines and R. Malhame [36]-[37]-[38] and J.-M. Lasry and P.-L. Lions [48]-[49]-[47], has been at the origin of a huge literature in economics and mathematical finance, but also in many other areas such as biology and epidemiology. In the study of mean field models, some reasonable assumptions are added in order to considerably simplify the complexity of the associated finite model. These hypotheses consist of the anonymity of the players and the negligibility of their impact on population dynamics, since their number is infinite. In this way, each player models the behavior of a representative agent facing the environment constituted by the statistical distribution of the surrounding anonymous population. The studied system of equations is then considerably reduced, and the issue amounts to investigate nonlinear differential or partial differential equations. For an overview on this topic, we refer to the books of F. Delarue et R. Carmona [13]-[14].

Within the framework of the cross-holding model, we can cite the recent work of M-F. Djete and N. Touzi [19], where the authors studied the mean field game, in absence of common noise (the idiosyncratic risks of the agents are independent), where the representative agent aims to maximize a non-decreasing reward function.

In this chapter, we introduce the common noise (the idiosyncratic risks of the agents are correlated) and we limit ourselves to the one period model, in order to simplify the problem. The continuous time model will be investigated in Chapter 3.

First, we notice that the formulation of the mean field model reveals a notion of No-arbitrage (NA). Since under an equilibrium, the NA condition must hold, we will look then for equilibria under which the model satisfies this condition. The first main result of this chapter reveals that, under the equilibrium, the NA condition imposes a structure on the dynamics of the mean field model: it turns out that the drift and the diffusion term of the common noise, by means of the representative agent's idiosyncratic risk dynamic value, are proportional. By analogy, this is similar to the drift-volatility restriction in the HJM term structure model [32].

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<sup>1</sup>see Definition 2.2 of a Nash equilibrium in Chapter 1 and the references therein.

Next, as in [19], if we consider that each agent is interested in increasing his profits, then the presence of common noise reduces to an optimization problem in which the representative agent apply his control both on the drift and on the diffusion term of the common noise of the competitor. This essentially leads to trade off between mean and variance, and then to analyze the mean field equilibrium related to the mean variance optimization problem.

We derive from our analysis, the second main result of the paper: it appears that under some assumptions on the control strategies, the existence conditions of such equilibria are reduced to a proportionality condition between the drift and the diffusion term of the common noise, which is in agreement with the NA condition.

Finally, the characterization of the No-arbitrage condition in the finite population setting remains an unsolved problem. Nevertheless, we provide additional structure in the game when the coefficients satisfy the mean field no-arbitrage restriction. By investigating this problem, we find that under some hypothesis on the proportionality coefficient between the drift and the diffusion term of the common noise, there exist Nash equilibria.

The paper is organized as follows: Section 3.2 is devoted to the problem formulation of the one-period mean field cross-holding model, in presence of common noise. Section 3.3 is dedicated to the introduction and the characterization of a notion of No-arbitrage (NA) for the considered one-period model. Section 3.4 is devoted to the study of the mean field equilibrium related to the mean variance optimization problem. In Section 3.5, we study the finite game under the characterization of the NA condition defined in Section 3.3.

**Notations:** We introduce the following useful notations that will be used all along the chapter:

- For any process  $S$ , we denote the variation  $\Delta S = S_1 - S_0$ .
- Let  $n \in \mathbb{N}^*$  and  $\nu$  a measure, we denote the product measures

$$\nu^{\otimes n} = \underbrace{\nu \otimes \cdots \otimes \nu}_{n \text{ times}}.$$

- The support of a measure  $\nu$  is the smallest closed subset with full measure.
- The set  $\mathbb{L}^0(\nu)$  (respectively,  $\mathbb{L}_+^0(\nu)$ ) is denoted as the set of measurable functions  $f : \text{supp}(\nu) \rightarrow \mathbb{R}$ . (respectively,  $f : \text{supp}(\nu) \rightarrow \mathbb{R}_+$ ).
- For every  $p > 0$ , the  $\mathbb{L}^p$ -norm of a measurable function  $f$  with respect to the measure  $\nu$  is denoted as

$$\|f\|_{\mathbb{L}^p(\nu)} = \left( \int |f(y)|^p \nu(dy) \right)^{\frac{1}{p}},$$

and the set  $\mathbb{L}^p(\nu)$  (respectively,  $\mathbb{L}_+^p(\nu)$ ) denotes the set of measurable functions  $\mathbb{L}^0(\nu)$  (respectively,  $\mathbb{L}_+^0(\nu)$ ) with finite  $\mathbb{L}^p$ -norm

$$\mathbb{L}^p(\nu) = \{f \in \mathbb{L}^0(\nu) : \|f\|_{\mathbb{L}^p(\nu)} < \infty\}.$$

- For every  $p > 0$ , we denote by  $\mathcal{P}_p(\mathbb{R})$  the collection of all probability measures with finite  $p$ -th moment.
- Let  $N \in \mathbb{N}^*$ , the canonical basis of  $\mathbb{R}^N$  is denoted as  $(e_i)_{1 \leq i \leq N}$  and  $\mathcal{M}_N(\mathbb{R})$  is the set of square  $N \times N$  matrices taking values in  $\mathbb{R}$ . For any vector  $v \in \mathbb{R}^N$ , we have  $\bar{v}_N = \sum_{i=1}^N \frac{v_i}{N}$  where  $v_i$  is the  $i$ -th component of  $v$ . We denote  $\text{diag}(v) = \sum_{i=1}^N v_i e_i e_i^\top \in \mathcal{M}_N(\mathbb{R})$  i.e a diagonal matrix whose  $i$ -th diagonal element is  $v_i$ .

## 3.2 Problem formulation

This work deals with the study of the one-period mean field mutual holding model with common noise. In the case of a finite number  $N$  of agents that are interindependent, the equity process  $X^i$  of the  $i$ -th agent,  $i \in \{1, \dots, N\}$ , satisfy the following stochastic equation

$$\Delta X^i = \frac{1}{N} \sum_{j \neq i}^N \beta(X_0^i, X_0^j) \Delta X^j - \frac{1}{N} \sum_{j \neq i}^N \pi(X_0^j, X_0^i) \Delta X^i + \Delta P^i. \quad (3.1)$$

with  $\Delta P^i = b(X_0^i) + \sigma(X_0^i) \varepsilon^i + \sigma^0(X_0^i) \varepsilon^0$ .

The first two terms on the right hands side of (3.1) represent the returns collected by agent  $i$  from the cross-holding strategies of the other agents, and the last term  $\Delta P^i$  represents the idiosyncratic risk variation of the  $i$ -th agent. The decision variable  $\beta(X_0^i, X_0^j)$  is the quantity of equity the  $i$ -th agent holds from the  $j$ -th agent and  $\pi(X_0^j, X_0^i)$  is similarly the quantity of equity that hold  $j$ -th agent from the individual agent  $i$ .

We make the following assumptions:

- the r.v.  $(X_0^i)_{i=1, \dots, N}$  are identically distributed, with  $X_0^i \stackrel{L}{\sim} \mu_0$ ,
- the reduced centered r.v.  $\varepsilon^0$  and  $(\varepsilon_1, \dots, \varepsilon_N)$  are iid, with  $\varepsilon^0 \stackrel{L}{\sim} \rho$ .
- the r.v.  $\varepsilon^0$ ,  $(\varepsilon_i)_{i=1, \dots, N}$  are independent of the r.v.  $(X_0^i)_{i=1, \dots, N}$ ,
- the functions  $\sigma^0, \sigma$  and  $b$  are measurable and bounded, and the diffusion terms  $\sigma^0$  and  $\sigma$  do not vanish i.e  $\sigma^0(X_0) \neq 0$  and  $\sigma(X_0) \neq 0$ ,  $\mu_0$  - a.s.

Intuitively, By using the law of large numbers, we guess (without proof) that the limiting infinite population problem of (3.1) is given by

$$\frac{\Delta X}{m^\pi(X_0)} = \widehat{\mathbb{E}}^{\mu_0} [\beta(X_0, \widehat{X}_0) F^\mu(\widehat{X}_0, \varepsilon^0)] + \Delta P, \quad (3.2)$$

where  $\widehat{\mathbb{E}}^{\mu_0} [\psi(X_0, \widehat{X}_0, \varepsilon^0)] = \int \psi(X_0, \hat{x}_0, \varepsilon^0) \mu_0(d\hat{x}_0)^2$ , and  $\mu, \pi$  defining respectively the random environment representing the equity value distribution  $(X_0, X_1)$  conditional on

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<sup>2</sup>Similarly, we have  $\mathbb{E}^{\mu_0} [\psi(X_0, \widehat{X}_0, \varepsilon^0)] = \int \psi(x_0, \hat{x}_0, \varepsilon^0) \mu_0(dx_0)$ , and  $\mathbb{E}^\rho [\psi(X_0, \widehat{X}_0, \varepsilon^0)] = \int \psi(X_0, \hat{x}_0, z) \rho(dz)$ .

the common noise and the cross-holding strategy of the surrounding population, and we denoted

$$m^\pi(X_0) = \frac{1}{1 + \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0, X_0)]}, \quad \mu_0 - \text{a.s.}, \quad (3.3)$$

$$F^\mu(\widehat{X}_0, \varepsilon^0) := \widehat{\mathbb{E}}^\mu[\Delta\widehat{X}|\widehat{X}_0, \varepsilon^0], \quad \mu_0 \otimes \rho - \text{a.s.}, \quad (3.4)$$

As we shall see later in Definition 3.1,  $(\widehat{X}_1, \widehat{X}_0)$  will be the copy of  $(X_1, X_0)$  under the equilibrium.

Notice that  $\mu$  impacts the dynamics of  $(X_0, X_1)$  only through the conditional expectation  $F^\mu \in \mathbb{L}^1(\mu_0 \otimes \rho)$ . Consequently, the random environment reduces to the pair  $(F^\mu, \pi)$ . We shall denote by

$$X_1^{\mu, \pi, \beta}(X_0) := X_0 + \Delta X^{\mu, \pi, \beta}(X_0),$$

where  $\Delta X^{\mu, \pi, \beta}(X_0) := \Delta X$  as given by (3.2).

Our main concern is to find a Nash equilibrium in the current mean field context where the representative agent seeks to maximize the criterion

$$J(X_0, \beta; F^\mu, \pi) := \mathcal{U}(\mathcal{L}(X_1^{\mu, \pi, \beta}(X_0)|X_0)),$$

for some function  $\mathcal{U} : \mathcal{P}_p(\mathbb{R}) \rightarrow \mathbb{R}$ , for some  $p > 0$ . Of course, this requires to restrict the admissible strategies to the collection  $\mathcal{A}_p(F^\mu, \pi)$  of all cross-holding strategies  $\beta$  which induce an  $\mathbb{L}^p$ -integrable random variable  $X_1^{\mu, \pi, \beta}(X_0)$ .

**Definition 3.1** (One-period mean field game of cross-holding). *A random environment  $(F^\mu, \pi)$  is an equilibrium solution of the mean field game of cross-holding if  $F^\mu \in \mathbb{L}^1(\mu_0 \otimes \rho)$ ,  $\pi \in \mathcal{A}_p(F^\mu, \pi)$ , and*

- (i)  $J(X_0, \pi; F^\mu, \pi) = \sup_{\beta \in \mathcal{A}_p(F^\mu, \pi)} J(X_0, \beta; F^\mu, \pi)$ ,
- (ii)  $\mathbb{E}[\Delta X^{\mu, \pi, \pi}(X_0)|X_0, \varepsilon^0] = F^\mu(X_0, \varepsilon^0)$ .

### 3.3 No-arbitrage (NA)

In the context of the current one-period model, we introduce the following notion of no-arbitrage which appears naturally as a necessary condition for the individual optimization of Step (i) for Definition 3.1 to have a solution if  $\mathcal{U}$  is increasing. As a consequence, it is sufficient to limit our search of equilibrium solutions of the mean field game of cross-holding to those environments which satisfy the following no-arbitrage condition. Denote:

$$G_\mu^\beta = G_\mu^\beta(X_0, \varepsilon^0) := \widehat{\mathbb{E}}^{\mu_0}[\beta(X_0, \widehat{X}_0)F^\mu(\widehat{X}_0, \varepsilon^0)] = \int \beta(X_0, \hat{x}_0)F^\mu(\hat{x}_0, \varepsilon^0)\mu_0(d\hat{x}_0).$$

**Definition 3.2** (No-arbitrage (NA)). *Let  $(F^\mu, \pi) \in \mathbb{L}^1(\mu_0 \otimes \rho) \times \mathbb{L}^1(\mu_0 \otimes \mu_0)$ . We say that  $(F^\mu, \pi)$  satisfies the no arbitrage condition if for all  $\beta \in \mathcal{A}_1(F^\mu, \pi)$ , we have:*

$$G_\mu^\beta \geq 0, \quad \mu_0 \otimes \rho - \text{a.s.} \implies G_\mu^\beta = 0, \quad \mu_0 \otimes \rho - \text{a.s.} \quad (3.5)$$

As standard, the last condition can be reformulated as

$$K \cap \mathbb{L}_+^1(\mu_0 \otimes \rho) = \{0\}, \quad (3.6)$$

where  $K$  is the collection of all (super-)hedgeable claims

$$K := \left\{ \xi = \xi(X_0, \varepsilon^0) \in \mathbb{L}^1(\mu_0 \otimes \rho) : \xi \leq G_\mu^\beta, \text{ a.s. for some } \beta \in \mathcal{A}_1(F^\mu, \pi) \right\}. \quad (3.7)$$

The following result is proved by adapting the standard methods in the no-arbitrage literature.

**Lemma 3.3.** *Under the no-arbitrage condition (3.6), the set  $K$  is closed in  $\mathbb{L}^1$ .*

*Proof.* First, notice that  $F^\mu = 0$ , a.s implies  $G_\mu^\beta = 0$ , a.s for all  $\beta \in \mathcal{A}_1(F^\mu, \pi)$ , then  $K = \mathbb{L}_-^1(\mu_0 \otimes \rho)$  is a closed subset. Next, let  $F^\mu \in \mathbb{L}^1(\mu_0 \otimes \rho)$  such that

$$(\mu_0 \otimes \rho)[F^\mu \neq 0] > 0. \quad (3.8)$$

Let  $(\xi_n = \xi_n(X_0, \varepsilon^0))_{n \in \mathbb{N}}$  be some sequence in  $K$  converging to some  $\xi = \xi(X_0, \varepsilon^0)$ , i.e

$$\xi_n \leq G_\mu^{\beta_n}, \text{ a.s for some } \beta_n \in \mathcal{A}_1(F^\mu, \pi), \text{ and } \xi_n \longrightarrow \xi \text{ in } \mathbb{L}^1, \quad (3.9)$$

and let us show that  $\xi \leq G_\mu^\beta$ ,  $\mu_0 \otimes \rho$  - a.s, for some  $\beta \in \mathcal{A}_1(F^\mu, \pi)$ . We argue separately the two following alternative cases.

1. If  $\liminf_n \|\beta_n\|_{\mathbb{L}^\infty} < \infty$ , then by Mazur's Lemma, see e.g. Theorem III.7 in Brezis [6], we may find  $\hat{\beta}_n \in \text{Conv}(\beta_k, k \geq n)$  such that  $\hat{\beta}_n \longrightarrow \beta$  in  $\mathbb{L}^\infty$ . In particular,

$$\int |(\hat{\beta}_n - \beta)(x, \hat{x}) F^\mu(\hat{x}, y)| \mu(dx) \mu(d\hat{x}) \rho(dy) \leq \|\hat{\beta}_n - \beta\|_{\mathbb{L}^\infty} \|F^\mu\|_{\mathbb{L}^1} \xrightarrow{n \rightarrow \infty} 0,$$

so that  $G_\mu^{\hat{\beta}_n} \longrightarrow G_\mu^\beta$  in  $\mathbb{L}^1$ . Then, taking the appropriate convex combinations on both sides of the first inequality of (3.9) and passing to the limit  $n \rightarrow \infty$  induces  $\xi \leq G_\mu^\beta$ , as required.

2. In the alternative case when  $\lim_n \|\beta_n\|_{\mathbb{L}^\infty} = \infty$ , we introduce the sequence  $\gamma_n := \frac{1}{2} \frac{\beta_n}{\|\beta_n\|_{\mathbb{L}^\infty}} \mathbf{1}_{\{\|\beta_n\|_{\mathbb{L}^\infty} \neq 0\}}$ ,  $n \geq 1$ , which satisfies  $\sup_n \|\gamma_n\|_{\mathbb{L}^\infty} \leq \frac{1}{2}$ . Similarly to the previous case, we may find a sequence  $\hat{\gamma}_n \in \text{Conv}(\gamma_k, k \geq n)$  such that  $\hat{\gamma}_n \longrightarrow \gamma$  in  $\mathbb{L}^\infty$ , with  $\|\gamma\|_{\mathbb{L}^\infty} \leq \frac{1}{2}$ , and  $G_\mu^{\hat{\gamma}_n} \longrightarrow G_\mu^\gamma$  in  $\mathbb{L}^1$ . Dividing both sides of (3.9) by  $2\|\beta_n\|_{\mathbb{L}^\infty}$  and taking the appropriate convex combinations on both sides of the first inequality, we obtain by sending  $n \rightarrow \infty$  that  $0 \leq G_\mu^\gamma$ , a.s. due to the fact that  $\|\beta_n\|_{\mathbb{L}^\infty} \longrightarrow \infty$ . By the no-arbitrage condition, this implies that  $0 = G_\mu^\gamma = \int \gamma(x, \hat{x}) F^\mu(\hat{x}, \varepsilon^0) \mu(d\hat{x})$ , a.s. Now, since  $\|\gamma\|_{\mathbb{L}^\infty} \leq \frac{1}{2}$ , this Fredholm integral equation has a unique solution  $F^\mu(X_0, \varepsilon^0) = 0$ , a.s (see T. Kato [43] page 153 by considering the operator  $T_\gamma : f \mapsto \int \gamma(x, \hat{x}) f(\hat{x}, \varepsilon^0) \mu_0(d\hat{x})$ ). This contradicts the condition (3.8).

□

We shall next use Yan's theorem (see [59]) that we recall in the following statement.

**Theorem 3.4.** *[(Yan 1980)] Let  $Q$  be a given probability measure, and  $C$  a closed convex cone of  $\mathbb{L}^1(Q)$  satisfying:*

$$\mathbb{L}_-^1(Q) \subset C \quad \text{and} \quad C \cap \mathbb{L}_+^1(Q) = \{0\}.$$

*Then, there exists a random variable  $Z \in \mathbb{L}_+^\infty(Q)$  such that  $Q[Z > 0] = 1$  and*

$$\mathbb{E}^Q[YZ] \leq 0, \quad \text{for all } Y \in C.$$

The following result provides a characterization of the NA condition by means of a proportionality relationship between the drift and the volatility of the common noise which holds at the level of the idiosyncratic risk process  $(b, \sigma^0)$ . This is similar to the drift-volatility restriction in the HJM term structure model. The latter restriction is a consequence of the fact that self-financing strategies in the HJM model are built on the financial market containing an infinite number of assets, namely a zero-coupon bond for all maturity  $T > 0$ . Notice that the analogy with the HJM model is natural as our model involves trading in an infinite number of assets defined as the equity value of each agent of the surrounding population.

**Theorem 3.5.** *For a random environment  $(F^\mu, \pi) \in \mathbb{L}^1(\mu_0 \otimes \rho) \times \mathbb{L}^1(\mu_0 \otimes \mu_0)$ , the following statements are equivalent:*

1.  $(F^\mu, \pi)$  satisfies the NA condition;
2. There exists an  $\varepsilon^0$ -measurable r.v.  $Z$  such that

$$Z > 0, \quad \rho - \text{a.s.}, \quad \mathbb{E}^\rho[Z] = 1 \quad \text{and} \quad \mathbb{E}^\rho[Z F^\mu(X_0, \varepsilon^0)] = 0, \quad \mu_0 - \text{a.s.} \quad (3.10)$$

*If in addition  $(F^\mu, \pi)$  is an equilibrium solution of the MFG, then with the r.v.  $Z$  of (ii), the last conditions are equivalent to*

$$b(X_0) + \mathbb{E}^\rho[Z \varepsilon^0] \sigma^0(X_0) = 0, \quad \mu_0 - \text{a.s.} \quad (3.11)$$

*Proof.* (i) $\implies$ (ii): By Lemma 3.3,  $K$  is a closed convex cone containing  $\mathbb{L}_-^1(\mu_0 \otimes \rho)$  satisfying (3.6). Then, it follows from Yan's theorem [59] that there exists an  $(\varepsilon^0, X_0)$ -measurable random variable  $\tilde{Z} \in \mathbb{L}_+^\infty(\mu_0 \otimes \rho)$  satisfying  $\mathbb{E}^{\mu_0 \otimes \rho}[\tilde{Z}(X_0, \varepsilon^0) G_\mu^\beta(X_0, \varepsilon^0)] \leq 0$ , for all  $\beta \in \mathcal{A}_1(F^\mu, \pi)$ , which we may rewrite by Fubini's Theorem as:

$$\mathbb{E}^{\mu_0 \otimes \rho}[\tilde{Z}(X_0, \varepsilon^0) G_\mu^\beta(X_0, \varepsilon^0)] = \mathbb{E}^{\mu_0^{\otimes 2}}[\beta(X_0, \hat{X}_0) \mathbb{E}^\rho[F^\mu(\hat{X}_0, \varepsilon^0) \tilde{Z}(X_0, \varepsilon^0)]] \leq 0.$$

By the arbitrariness of  $\beta \in \mathcal{A}_1(F^\mu, \pi)$ , this implies  $\mathbb{E}^\rho[Z(\varepsilon^0) F^\mu(X_0, \varepsilon^0)] = 0$ ,  $\mu_0 - \text{a.s.}$ , where  $Z(\varepsilon^0) := \mathbb{E}^{\mu_0}[\tilde{Z}(X_0, \varepsilon^0)]$  satisfies (3.10).

(ii) $\implies$ (i): Under 2, we obtain for all  $\beta \in \mathcal{A}_1(F^\mu, \pi)$  that

$$\mathbb{E}^\rho[Z(\varepsilon^0) G_\mu^\beta(X_0, \varepsilon^0)] = \hat{\mathbb{E}}^{\mu_0}[\beta(X_0, \hat{X}_0) \mathbb{E}^\rho[Z(\varepsilon^0) F^\mu(\hat{X}_0, \varepsilon^0)]] = 0, \quad \mu_0 - \text{a.s.}$$

by the Fubini's theorem and the fact that  $Z$  is  $\varepsilon^0$ -measurable. As  $Z > 0$ , a.s. we deduce that whenever  $G_\mu^\beta \geq 0$  a.s, then  $G_\mu^\beta = 0$ , a.s. Therefore, NA holds 1.

We finally show that (ii) is equivalent to (3.11) for an equilibrium solution  $(F^\mu, \pi)$ . As  $\varepsilon$  is centered and independent of  $\varepsilon^0$  and  $X_0$ , we obtain by taking expectation conditional on  $(X_0, \varepsilon^0)$  in (3.2):

$$\frac{F^\mu(X_0, \varepsilon^0)}{m^\pi(X_0)} = b(X_0) + \sigma^0(X_0)\varepsilon^0 + \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)F^\mu(\widehat{X}_0, \varepsilon^0)], \mu_0 \otimes \rho - \text{a.s.} \quad (3.12)$$

which in turn implies by Fubini's theorem that  $\mu_0 - \text{a.s}$

$$\frac{\mathbb{E}^\rho[Z F^\mu(X_0, \varepsilon^0)]}{m^\pi(X_0)} = \mathbb{E}^\rho[Z]b(X_0) + \mathbb{E}^\rho[Z\varepsilon^0]\sigma^0(X_0) + \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)\mathbb{E}^\rho[Z F^\mu(X_0, \varepsilon^0)]].$$

This provides the required equivalence.  $\square$

### 3.4 The mean-variance criterion

In this section, we discuss the case where the performance of the representative agent is measured by the mean-variance criterion:

$$\sup_{\beta \in \mathcal{A}_2(F^\mu, \pi)} \text{MV}_q[X_1^{\mu, \pi, \beta} | X_0], \text{ where } \text{MV}_q[\cdot | X_0] := \mathbb{E}[\cdot | X_0] - \frac{1}{2q} \mathbb{V}[\cdot | X_0], \quad (3.13)$$

for some given parameter  $q > 0$ . Although, this criterion fails to be nondecreasing, we shall focus on this standard mean-variance problem for computational tractability. Moreover, as standard in the portfolio optimization literature, the mean-variance criterion is often believed to capture the main features of a risk-averse expected utility performance criterion.

We emphasize that due to the failure of the monotonicity of the mean-variance performance criterion, it is not anymore clear that the no-arbitrage condition is a necessary condition of equilibrium. For this reason, we look for an equilibrium solution of the MFG problem of cross-holding without imposing the NA constraint on the coefficient. Remarkably, our results below say that such an equilibrium fails to exist if the coefficients  $b$  and  $\sigma^0$  are not proportional.

Our starting point is the following equation for  $F^\mu$  which follows from taking expectations on both sides of (3.2):

$$(1 + \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0, X_0)])F^\mu(X_0, \varepsilon^0) = b(X_0) + \sigma^0(X_0)\varepsilon^0 + \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)F^\mu(\widehat{X}_0, \varepsilon^0)]. \quad (3.14)$$

This equation expresses the map  $F^\mu$  as a solution of a Fredholm integral equation of the second type whose Kernel depends on the equilibrium strategy  $\pi$  and the corresponding environment distribution  $\mu$ . Our main result provides a complete characterization of equilibrium solutions of our MFG generated by cross-holding strategies under which uniqueness holds for the last Fredholm integral equation. We notice that, under this uniqueness condition, it follows that

$$F^\mu(X_0, \varepsilon^0) := \mathbb{E}^\rho[F^\mu(X_0, \varepsilon^0)] + \varepsilon^0 F_1^\mu(X_0), \quad (3.15)$$

where  $\mathbb{E}^\rho[F^\mu(X_0, \varepsilon^0)]$  satisfies a similar Fredholm equation, by taking expectations conditional to  $X_0$ , and  $F_1^\mu$  satisfies

$$(1 + \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0, X_0)])F_1^\mu(X_0) = \sigma^0(X_0) + \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)F_1^\mu(\widehat{X}_0)], \mu_0 \otimes \rho - \text{a.s.}$$

**Theorem 3.6.** *There exists a mean field equilibrium  $(F^\mu, \pi) \in \mathbb{L}^2(\mu_0 \otimes \rho) \times \mathbb{L}^2(\mu_0^{\otimes 2})$  with unique solution to the Fredholm equation (3.14) if and only if*

$$\mathbb{E}[b(X_0)] > 0 \quad \text{and} \quad \sigma^0 = \lambda b \quad \text{with} \quad \lambda = \pm \sqrt{\frac{q}{\mathbb{E}[b(X_0)]}},$$

and the strategy  $\pi \in \mathbb{L}^2(\mu_0 \otimes \mu_0)$  induces uniqueness for the Fredholm equation (3.14) and satisfies

$$0 \neq \widehat{\mathbb{E}}^{\mu_0} [\pi(X_0, \widehat{X}_0)] + \frac{b(X_0)}{\mathbb{E}[b(X_0)]} = 1 + \widehat{\mathbb{E}}^{\mu_0} [\pi(\widehat{X}_0, X_0)], \quad \mu_0 - a.s. \quad (3.16)$$

In this case,  $F^\mu$  is explicitly given by

$$F^\mu(X_0, \varepsilon^0) = \mathbb{E}[b(X_0)](1 + \lambda \varepsilon^0), \quad \mu_0 \otimes \rho - a.s. \quad (3.17)$$

In other words, there exists a mean field equilibrium  $(F^\mu, \pi)$  with a unique solution to the Fredholm equation (3.14) if and only if one of the two following cases is satisfied:

1. The first case is  $\sigma^0 = \sqrt{\frac{q}{\mathbb{E}[b(X_0)]}}b$ , where necessarily  $F^\mu$  denotes the unique solution of (3.14) given by

$$F^\mu(X_0, \varepsilon^0) = \mathbb{E}[b(X_0)](1 + \sqrt{\frac{q}{\mathbb{E}[b(X_0)]}}b(X_0)\varepsilon^0), \quad \mu_0 \otimes \rho - a.s.$$

2. The second case is the case where  $\sigma^0 = -\sqrt{\frac{q}{\mathbb{E}[b(X_0)]}}b$ , with  $F^\mu$  solving (3.14) and taking the following form

$$F^\mu(X_0, \varepsilon^0) = \mathbb{E}[b(X_0)](1 + \sqrt{\frac{q}{\mathbb{E}[b(X_0)]}}b(X_0)\varepsilon^0), \quad \mu_0 \otimes \rho - a.s.$$

The proof is postponed to the end of this section. The existence of a solution of (3.16) satisfying uniqueness for the Fredholm equation (3.14) will be discussed in Examples 3.7 and 3.8 below. We emphasize again that the last equilibrium characterization imposes a collinearity condition between the drift coefficient of the idiosyncratic risk and its common noise volatility. In particular the mean-variance equilibrium solution of the MFG satisfies the NA condition.

The equilibrium dynamics derived in the last result are defined by

$$X_1^{\mu, \pi, \pi} = X_0 + \mathbb{E}[b(X_0)](1 + \lambda \varepsilon^0) + \frac{\sigma(X_0)\varepsilon}{1 + \widehat{\mathbb{E}}^{\mu_0} [\pi(\widehat{X}_0, X_0)]}. \quad (3.18)$$

In particular, the common noise volatility under equilibrium is dilated by  $\sqrt{q}$ . Moreover, when the mean-variance criterion has large penalty on the variance (i.e. small  $q$ ), the equilibrium common noise is reduced by the factor  $\sqrt{q}$ .

We also notice that equation (3.16) can be written as

$$Y(X_0) := \widehat{\mathbb{E}}^{\mu_0} [\pi(X_0, \widehat{X}_0)] - \widehat{\mathbb{E}}^{\mu_0} [\pi(\widehat{X}_0, X_0)] = 1 - \frac{b(X_0)}{\mathbb{E}[b(X_0)]}, \quad \mu_0 - a.s.,$$



The variable  $Y$  indicates the representative agent's net detention level. As  $\mathbb{E}[b(X_0)] > 0$  at equilibrium, if the drift  $b(X_0)$  of the representative agent is greater than the average population drift  $\mathbb{E}[b(X_0)]$ , then the agent has a negative net detention. In addition, greater is his drift value, less is his net detention holding. This means that the agent tend to hold more shares among those with greater drift values.

For arbitrary strategies  $\pi \in \mathbb{L}^2(\mu_0 \otimes \mu_0)$ , the uniqueness issue of the Fredholm equation (3.14) is not granted, in general. Our restriction to the equilibria generating uniqueness for the Fredholm integral equation imposes more constraints on the set of admissible strategies.

We end this section by the following simplest example of equilibrium strategies satisfying (3.16).

**Example 3.7** (cross-holding depending on the second argument only). By direct substitution in (3.16), we may find equilibrium strategies  $\pi(x, \hat{x})$  depending on the  $\hat{x}$ -argument only, defined up to a constant  $c$  by:

$$\pi(x, \hat{x}) = \frac{b(\hat{x})}{\mathbb{E}[b(X_0)]} - c \text{ such that } (c-1)\mathbb{E}[b(X_0)] \notin b(\text{supp}(\mu_0)). \quad (3.19)$$

Under such equilibrium strategies, the representative agent holds more shares from those competitors with larger expected return.

We next examine the uniqueness for Fredholm equation (3.14) induced with such strategies. As  $\pi$  depends only on the second argument, the equation (3.14) becomes

$$F^\mu(X_0, \varepsilon^0) = \frac{\varphi(X_0, \varepsilon^0) + \alpha(\varepsilon^0)}{1 + \pi(X_0)}, \text{ where } \varphi(X_0, \varepsilon^0) := b(X_0) + \sigma^0(X_0)\varepsilon^0, \text{ a.s.}$$

and  $\alpha(\varepsilon^0) := \widehat{\mathbb{E}}^{\mu_0}[\pi(\widehat{X}_0)F^\mu(\widehat{X}_0, \varepsilon^0)]$ . Thus, uniqueness for  $F^\mu$  reduces to the uniqueness of the map  $\alpha(\varepsilon^0)$ . Substituting the expression of  $F^\mu$ , we see that

$$\alpha(\varepsilon^0) = \frac{\widehat{\mathbb{E}}\left[\varphi(\widehat{X}_0, \varepsilon^0) \frac{b(\widehat{X}_0) - c\mathbb{E}[b(X_0)]}{b(\widehat{X}_0) - (c-1)\mathbb{E}[b(X_0)]}\right]}{\widehat{\mathbb{E}}\left[\frac{\mathbb{E}[b(X_0)]}{b(\widehat{X}_0) - (c-1)\mathbb{E}[b(X_0)]}\right]},$$

provided that the constant  $c$  in (3.19) is chosen so that the denominator of the last expression does not vanish. In particular, notice that  $\alpha(\varepsilon^0)$  is affine in  $\varepsilon^0$  in agreement with (3.15).

**Example 3.8** (Separable form). We may also search for solutions of (3.16) with separable form  $\pi(x, \hat{x}) = \psi(x)\phi(\hat{x})$ . Direct substitution provides:

$$\pi(x, \hat{x}) = \psi(x) \left( c\psi(\hat{x}) + \frac{b(\hat{x})}{\mathbb{E}[b(X_0)]} - 1 \right) \text{ with } \mathbb{E}[\psi(X_0)] = 1 \text{ and} \\ c\psi(X_0) \neq \frac{b(X_0)}{\mathbb{E}[b(X_0)]}, \text{ a.s.} \quad (3.20)$$

We next examine the uniqueness for the Fredholm equation defined by (3.14). Since, we have  $1 + \widehat{\mathbb{E}}[\pi(\widehat{X}_0, X_0) = c\psi(X_0) + \frac{b(X_0)}{\mathbb{E}[b(X_0)]}]$ , it follows that (3.14) reduces to

$$F^\mu(X_0, \varepsilon^0) = \frac{\varphi(X_0, \varepsilon^0) + \alpha(\varepsilon^0)\psi(X_0)}{c\psi(X_0) + \frac{b(X_0)}{\mathbb{E}[b(X_0)]}} \text{ with } \varphi(X_0, \varepsilon^0) := b(X_0) + \sigma^0(X_0)\varepsilon^0, \text{ a.s.}$$

and  $\alpha(\varepsilon^0) := \widehat{\mathbb{E}}^{\mu_0}[(c\psi(X_0) + \frac{b(X_0)}{\mathbb{E}[b(X_0)]})^{-1}F^\mu(\widehat{X}_0, \varepsilon^0)]$ . As in the previous example, the uniqueness for the Fredholm equation is reduced to the uniqueness of the map  $\alpha(\varepsilon^0)$ , and we obtain by direct substitution that

$$\alpha(\varepsilon^0) = \frac{\widehat{\mathbb{E}}^{\mu_0}[\varphi(X_0, \varepsilon^0)(1 - \frac{1}{c\psi(X_0) + \frac{b(X_0)}{\mathbb{E}[b(X_0)]}})]}{\widehat{\mathbb{E}}^{\mu_0}[\frac{\psi(X_0)}{c\psi(X_0) + \frac{b(X_0)}{\mathbb{E}[b(X_0)]}}]},$$

provided that the constant  $c$  in (3.20) is chosen so that the denominator in the last expression does not vanish. We notice again that  $\alpha(\varepsilon^0)$  is affine in  $\varepsilon^0$  in agreement with (3.15).

**Proof of Theorem 3.6** The proof is organized in two steps.

*Step 1:* We first show that, under the uniqueness of the Fredholm equation (3.14), the random environment  $(F^\mu, \pi) \in \mathbb{L}^2(\mu_0 \otimes \rho) \times \mathbb{L}^2(\mu_0 \otimes \mu_0)$  is solution to the MFG problem if and only if the following system is satisfied

$$\begin{aligned} F^\mu(X_0, \varepsilon^0) &= F_0^\mu(X_0) + F_1^\mu(X_0)\varepsilon^0, \quad \mu_0 \otimes \rho - \text{a.s.}, \\ qF_0^\mu(\widehat{X}_0) &= F_1^\mu(X_0)F_1^\mu(\widehat{X}_0) \quad \mu_0 \otimes \mu_0 - \text{a.s.}, \end{aligned} \quad (3.21)$$

where  $F_0^\mu(X_0) = \mathbb{E}^\rho[F^\mu(X_0, \varepsilon^0)]$  and  $F_1^\mu(X_0) \in \mathbb{L}^2(\mu_0)$  satisfy respectively

$$\frac{F_0^\mu(X_0)}{m^\pi(X_0)} = b(X_0) + \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)F_0^\mu(\widehat{X}_0)], \quad \mu_0 - \text{a.s.} \quad (3.22)$$

$$\frac{F_1^\mu(X_0)}{m^\pi(X_0)} = \sigma(X_0) + \widehat{\mathbb{E}}^{\mu_0}[\pi(X_0, \widehat{X}_0)F_1^\mu(\widehat{X}_0)], \quad \mu_0 - \text{a.s.} \quad (3.23)$$

While the first equation in (3.21) was already obtained in (3.15), the second one is equivalent to the individual optimization step (i) of Definition 3.1. To prove it, we first notice that  $\frac{1}{m^\pi(X_0)} \neq 0$ ,  $\mu_0 - \text{a.s.}$  Indeed, arguing by contradiction, assume that  $\frac{1}{m^\pi} = 0$  on some subset  $A \subset \text{supp}(\mu_0)$  with  $\mu_0(A) > 0$ , then by conditioning on  $X_0 = x \in A$  and by recalling the mean field equation (3.2), it follows that

$$\int \pi(x, y)F^\mu(y, \varepsilon^0)\mu_0(dy) + b(x) + \sigma^0(x)\varepsilon^0 + \sigma(x)\varepsilon = 0, \quad \rho - \text{a.s. for all } x \in A.$$

Since  $\varepsilon$  is centered and independent of  $(X_0, \varepsilon^0)$ , this is in contradiction with our non-degeneracy assumption  $\sigma(X_0) \neq 0$ ,  $\mu_0 - \text{a.s.}$

We next turn to the characterization of the representative agent optimization problem of the mean-variance criterion:

$$J(\beta) = X_0 + m^\pi(X_0) \left( b(X_0) + \iint \beta(X_0, y) (F_0^\mu(y) + zF_1^\mu(y)) \mu_0(dy) \rho(dz) \right) - \frac{\sigma(X_0)^2}{2q} - \frac{m^\pi(X_0)^2}{2q} \int \left( \int \beta(X_0, y) z F_1^\mu(y) \mu_0(dy) + \sigma^0(X_0) z \right)^2 \rho(dz).$$

Since  $q > 0$ , it is clear that  $J$  is a concave function, and that the optimality is characterized by the first order condition which follows from direct calculation of the Gâteaux derivative of the functional  $J$ :

$$qF_0^\mu(\widehat{X}_0) - F_1^\mu(\widehat{X}_0)m^\pi(X_0)(\sigma^0(X_0) + \int \pi(X_0, u)F_1^\mu(u)\mu_0(du)) = 0, \text{ a.s.}$$

In view of (3.23), the last condition is equivalent to the second one in (3.21).

*Step 2:* By the symmetry of the second equation of (3.21), we see that  $F_0^\mu(X_0) = c$  is a constant,  $\mu_0$  - a.s., which can be determined by direct substitution in (3.22). This provides  $c(\frac{1}{m^\pi(X_0)} - \int \pi(X_0, y)\mu_0(dy)) = b(X_0)$ , a.s. and therefore  $c = \mathbb{E}[b(X_0)]$  by taking the expectations on both sides. This shows that (3.21) is equivalent to

$$\begin{aligned} F^\mu(X_0, \varepsilon^0) &= \mathbb{E}[b(X_0)] + F_1^\mu(X_0)\varepsilon^0, \quad \mu_0 \otimes \rho - \text{a.s.}, \\ q\mathbb{E}[b(X_0)] &= F_1^\mu(X_0)F_1^\mu(\widehat{X}_0), \quad \mu_0 \otimes \mu_0 - \text{a.s.}, \end{aligned} \tag{3.24}$$

with the restriction (3.16) and with  $F_1^\mu$  satisfying (3.23). As  $X_0$  and  $\widehat{X}_0$  are identically distributed, and by taking the expectation on  $F_1^\mu(X_0)$  and  $F_1^\mu(X_0)^2$ , we see that  $\mathbb{E}^{\mu_0} [F_1^\mu(X_0)]^2 = \mathbb{E}^{\mu_0} [F_1^\mu(X_0)^2] = q\mathbb{E}[b(X_0)]$ . Then  $\mathbb{V}\text{ar}^{\mu_0} [F_1^\mu(X_0)] = 0$ , and we get

$$\text{either } F_1^\mu(X_0) = \sqrt{q\mathbb{E}[b(X_0)]}, \quad \mu_0 - \text{a.s.} \quad \text{or} \quad F_1^\mu(X_0) = -\sqrt{q\mathbb{E}[b(X_0)]}, \quad \mu_0 - \text{a.s.}$$

If  $F_1^\mu(X_0) = \sqrt{q\mathbb{E}[b(X_0)]}$ , a.s. then the Fredholm equation (3.23) is equivalent to

$$\sigma^0(X_0) = \sqrt{q\mathbb{E}[b(X_0)]} \left( \frac{1}{m^\pi(X_0)} - \widehat{\mathbb{E}}^{\mu_0} [\pi(X_0, \widehat{X}_0)] \right), \quad \mu_0 - \text{a.s.},$$

Since  $\sigma^0(X_0) \neq 0$ ,  $\mu_0$  - a.s. and the drift  $b$  satisfies (3.16), the last equation is equivalent to  $\mathbb{E}[b(X_0)] > 0$  and  $\sigma^0 = \sqrt{\frac{q}{\mathbb{E}[b(X_0)]}}b$ .

The alternative case  $F_1^\mu(X_0) = -\sqrt{q\mathbb{E}[b(X_0)]}$  is studied similarly. □

### 3.5 Finite population game

We stated in Theorem 3.5 that the No-arbitrage condition for the mean field model, under the equilibrium, is equivalent to the proportionality condition 3.11 between the diffusion term  $\sigma^0$  and the drift  $b$ .

In this section, we consider the finite population problem under this proportionality condition. Therefore, we assume that there exists a constant  $\ell \in \mathbb{R}^*$  such that for all  $i \in \{1, \dots, N\}$

$$\sigma^0 = \ell b. \quad (3.25)$$

In view of the SDE (3.1), we write the equity variations of the  $N$  economic agents in the following way

$$M(\pi, X_0^{\cdot, N}) \Delta X^{\cdot, \pi, N} = b(X_0^{\cdot, N}) + \Gamma(X_0^{\cdot, N}) \varepsilon^{\cdot, N}, \quad \mu_0^{\otimes N} \otimes \rho^{\otimes N} - \text{a.s.}, \quad (3.26)$$

where we used the following notations for the parameters:

- the r.v.  $\varepsilon^{\cdot, N} = \sum_{i=1}^N \varepsilon^i e_i$  and for  $t \in \{0, 1\}$  we set  $X_t^{\cdot, \pi, N} = \sum_{i=1}^N X_t^{i, \pi} e_i$ ,
- for the drift  $b \in \mathbb{R}^N$  (also for  $\sigma^0$  and  $\sigma^2$ ), we set  $b(X_0^{\cdot, N}) = \sum_{i=1}^N b(X_0^{i, \pi}) e_i$ ,
- the volatility matrix  $\Gamma(X_0^{\cdot, N})$  is symmetric definite positive such that

$$\Gamma^2(X_0^{\cdot, N}) = \sigma^0(X_0^{\cdot, N}) \sigma^0(X_0^{\cdot, N})^\top + \text{diag}(\sigma^2(X_0^{\cdot, N})), \quad \mu_0^{\otimes N} - \text{a.s.}, \quad (3.27)$$

- the investments of all the agents refer to  $\pi(X_0^{\cdot, N}) = (\pi(X_0^i, X_0^j))_{1 \leq i, j \leq N}$  and they are no longer of order  $\frac{1}{N}$ , as we consider the finite game,
- the matrix  $M(\pi, X_0^{\cdot, N})$  is given by

$$M(\pi, X_0^{\cdot, N}) = I_N + \sum_{i=1}^N \sum_{j \neq i}^N \pi(X_0^i, X_0^j) (e_j e_j^\top - e_i e_i^\top). \quad \mu_0^{\otimes N} - \text{a.s.} \quad (3.28)$$

We assume that  $(\varepsilon^i)_{i=1, \dots, N}$  are iid reduced centered gaussian variables, and that the cross-holding strategies  $\pi(X_0^{\cdot, N})$  belong to the set of admissible strategies  $\mathcal{B}^N$ , that is given by

$$\mathcal{B}^N = \left\{ \pi(X_0^{\cdot, N}) \in \mathcal{M}_N(\mathbb{R}) : M(\pi, X_0^{\cdot, N}) \text{ is non-singular} \right. \\ \left. \text{and } X_1^{\cdot, \pi, N} \in \mathbb{L}^2(\mu_0^{\otimes N} \otimes \rho^{\otimes N}) \right\},$$

The  $i$ -th agent invests in the following set of strategies

$$\mathcal{B}^{i, N} := \{ \pi(X_0^{\cdot, N})_{i, \cdot} \in \mathbb{R}^N : \pi(X_0^{\cdot, N}) \in \mathcal{B}^N \}.$$

Observe that these last sets differ respectively from the sets of admissible strategies  $\mathcal{A}_{\text{const}}^N$  and  $\mathcal{A}_{\text{const}}^{i, N}$  introduced in the first chapter. Indeed, we did not take into account the positivity constraints of the control variables  $\pi(X_0^i, X_0^j)$ . This is due to the fact that in the mean field model and for simplicity, we relax the sign assumptions on the control strategies by taking  $\beta \in \mathcal{A}_2(F^\mu, \pi)$ .

Let  $\pi \in \mathcal{B}^N$ , the matrix  $M(\pi, X_0^{\cdot, N})$  is non-singular and we get from the SDE (3.29)

$$\Delta X^{\cdot, \pi, N} = U^\top(X_0^{\cdot, N}, \pi) \left( \gamma(X_0^{\cdot, N}) + \varepsilon^{\cdot, N} \right), \quad \mu_0^{\otimes N} \otimes \rho^{\otimes N} - \text{a.s. where} \\ U(X_0^{\cdot, N}, \pi) = \Gamma(X_0^{\cdot, N}) (M(\pi, X_0^{\cdot, N})^{-1})^\top, \quad \mu_0^{\otimes N} - \text{a.s.} \\ \gamma(X_0^{\cdot, N}) = \Gamma^{-1}(X_0^{\cdot, N}) b(X_0^{\cdot, N}), \quad \mu_0^{\otimes N} - \text{a.s.}, \text{ is the risk premium.} \quad (3.29)$$

Let  $i \in \{1, \dots, N\}$ , each agent  $i$  seeks to maximize his mean variance criterion for a given  $q^i > 0$

$$\sup_{\pi(X_0^{\cdot N})_{i, \cdot} \in \mathcal{B}^{i, N}} \text{MV}_{q^i}(X_1^{i, \pi, N} | X_0^{\cdot N}) \text{ where } \text{MV}_{q^i}(\cdot | X_0^{\cdot N}) = \mathbb{E}[\cdot | X_0^{\cdot N}] - \frac{1}{2q^i} \mathbb{V}(\cdot | X_0^{\cdot N}). \quad (3.30)$$

**Remark 3.9 (Link with Chapter 1).** We recall that in Chapter 1, we treated the case where each agent  $i \in \{1, \dots, N\}$  is willing to maximize his exponential utility function

$$\sup_{\pi(X_0^{\cdot N})_{i, \cdot} \in \mathcal{A}_{\text{const}}^{i, N}} \mathbb{E}\left[-e^{-\frac{X_1^{i, \pi, N}}{q^i}} \mid X_0^{\cdot N}\right], \text{ where } q_i > 0 \text{ is the risk tolerance.} \quad (3.31)$$

As  $(\varepsilon^i)_{i=1, \dots, N}$  are gaussians, by simple computations, this optimization problem is equivalent to the mean variance optimization problem (3.30) by replacing the set  $\mathcal{A}_{\text{const}}^{i, N}$  by  $\mathcal{B}^{i, N}$ . Therefore,  $q^i$  represents the risk tolerance of the  $i$ -th agent and we can rely on the results of the first chapter in order to study the finite game in this section.

Moreover, we recall that in the first chapter, without taking into account the positivity condition of the control variables  $\pi(X_0^i, X_0^j)$ , we find general existence conditions of Nash equilibrium related to the optimization problem (3.31), under the following assumption<sup>3</sup>

$$q_i \leq \sum_{j \neq i}^N q_j, \text{ for all } i \in \{1, \dots, N\}. \quad (3.32)$$

In this section, we shall work under the same assumption.

The following theorem provides sufficient conditions for the existence of a Nash equilibrium related to (3.30).

**Theorem 3.10.** *Under the assumptions (3.25) and (3.32), there exists a Nash equilibrium  $\hat{\pi} \in \mathcal{B}^N$  related to the mean variance criterion (3.30) if the following condition is satisfied*

$$\bar{b}_N(X_0^{\cdot N}) \neq 0, \mu_0^{\otimes N} - a.s \text{ and } \ell^2 > \frac{1}{\bar{b}_N(X_0^{\cdot N})} \left( \bar{q}_N - \frac{\bar{\sigma}_N^2(X_0^{\cdot N})}{\bar{b}_N(X_0^{\cdot N})} \right), \mu_0^{\otimes N} - a.s. \quad (3.33)$$

**Remark 3.11.** Notice that if the following condition is satisfied

$$\bar{b}_N(X_0^{\cdot N}) \neq 0, \mu_0^{\otimes N} - a.s \text{ and } \bar{b}_N(X_0^{\cdot N}) \bar{q}_N < \bar{\sigma}_N^2(X_0^{\cdot N}), \mu_0^{\otimes N} - a.s$$

then, there exists a Nash equilibrium  $\hat{\pi} \in \mathcal{B}^N$ . This condition is similar to that found in the first chapter (see Theorem 3.6 and Remark 3.10 in Chapter 1).

*Proof.* Assume that Conditions (3.25)-(3.32) are satisfied. Using the form of the variance matrix (3.27), we infer that Condition (3.25) is equivalent to

$$\Gamma^2(X_0^{\cdot N}) = \ell^2 b(X_0^{\cdot N}) b(X_0^{\cdot N})^\top + \text{diag}(\sigma^2(X_0^{\cdot N})), \mu_0^{\otimes N} - a.s. \quad (3.34)$$

<sup>3</sup>It means that there is no agent agent  $i$  that is more risk tolerant than all the other agents  $j \neq i$  taken together.

As by product of Remark 3.9, the existence conditions of Nash equilibria related to the mean variance criterion (3.30) are equivalent to those related to the optimization problem (3.31), that was studied in the first chapter. Then, as in the proof of Lemma 3.3 in Chapter 1, the first order condition related to a Nash equilibrium  $\hat{\pi} \in \mathcal{B}^N$  is equivalent to solve the following coupled nonlinear system on the unknown basis of vector functions  $(\hat{u}^1(X_0^{\cdot N}), \dots, \hat{u}^N(X_0^{\cdot N}))$ ,

$$(\hat{\alpha}_{k,k}(X_0^{\cdot N}) - \hat{\alpha}_{i,k}(X_0^{\cdot N}))(\hat{u}^k(X_0^{\cdot N}) - q_k \gamma(X_0^{\cdot N})) \cdot \hat{u}^i(X_0^{\cdot N}) = 0, \mu_0^{\otimes N} - \text{a.s.}, i \neq k, \quad (3.35)$$

with the relation<sup>4</sup>

$$\sum_{i=1}^N \hat{u}^i(X_0^{\cdot N}) = \Gamma(X_0^{\cdot N}) \mathbf{1}_{\mathbb{R}^N} \mu_0^{\otimes N} - \text{a.s.}, \quad (3.36)$$

where  $\hat{u}^i(X_0^{\cdot N}) = U(X_0^{\cdot N}, \hat{\pi}) e_i$  and  $\hat{\alpha}(X_0^{\cdot N}) = \Gamma^{-1}(X_0^{\cdot N}) U(X_0^{\cdot N}, \hat{\pi})$  with  $U$  defined in (3.29).

Recall that In Chapter 1, due to the positivity of the control variables, we necessarily have  $\hat{\alpha}_{k,k}(X_0^{\cdot N}) \neq \hat{\alpha}_{i,k}(X_0^{\cdot N})$ ,  $\mu_0^{\otimes N} - \text{a.s.}$  for all  $i \neq k$ . Since  $\hat{\pi} \in \mathcal{B}^N$ , it is not clear anymore that this condition holds. However, it is sufficient to find  $\hat{\pi}$  satisfying the following condition with  $(\hat{u}^1(X_0^{\cdot N}), \dots, \hat{u}^N(X_0^{\cdot N}))$

$$(\hat{u}^k(X_0^{\cdot N}) - q_k \gamma(X_0^{\cdot N})) \cdot \hat{u}^i(X_0^{\cdot N}) = 0, \mu_0^{\otimes N} - \text{a.s.}, i \neq k, \quad (3.37)$$

under relation (3.36) in order to get (3.35)-(3.36)

Next, if  $\hat{\pi}$  satisfies (3.37)-(3.36), then immediately  $X_1^{i, \hat{\pi}, N} \in \mathbb{L}^2(\mu_0^{\otimes N} \otimes \rho^{\otimes N})$ . Indeed, invoking (3.37)-(3.36), we get

$$\begin{aligned} 0 &= (\hat{u}^i(X_0^{\cdot N}) - q_i \gamma(X_0^{\cdot N})) \cdot \sum_{j \neq i} \hat{u}^j(X_0^{\cdot N}) \\ &= (\hat{u}^i(X_0^{\cdot N}) - q_i \gamma(X_0^{\cdot N})) \cdot (\Gamma(X_0^{\cdot N}) \mathbf{1}_{\mathbb{R}^N} - \hat{u}^i(X_0^{\cdot N})), \mu_0^{\otimes N} - \text{a.s.}, \end{aligned}$$

which implies the following property

$$|2\hat{u}^i(X_0^{\cdot N}) - (\Gamma(X_0^{\cdot N}) \mathbf{1}_{\mathbb{R}^N} + \gamma(X_0^{\cdot N}))| = |\Gamma(X_0^{\cdot N}) \mathbf{1}_{\mathbb{R}^N} - \gamma(X_0^{\cdot N})|, \mu_0^{\otimes N} - \text{a.s.}$$

By the boundedness of  $\Gamma$  and  $\gamma$ , we deduce that  $|\hat{u}^i(X_0^{\cdot N})|$  is bounded  $\mu_0^{\otimes N} - \text{a.s.}$  We deduce that  $X_1^{i, \pi, N} \in \mathbb{L}^2(\mu_0^{\otimes N} \otimes \rho^{\otimes N})$  by (3.29).

Finally, notice that since the proportionality condition (3.25) is satisfied and  $\sigma^0$  does not vanish, we have  $b(X_0^{\cdot N}) \neq 0_{\mathbb{R}^N}$ ,  $\mu_0^{\otimes N} - \text{a.s.}$  Then, without taking into account the positivity of the control variables, by using Theorem 3.6 and Remark 3.10 in Chapter 1, there exists a Nash equilibrium  $\hat{\pi} \in \mathcal{B}^N$  related to the optimization problem (3.31) if  $\bar{b}_N(X_0^{\cdot N}) \neq 0$ ,  $\mu_0^{\otimes N} - \text{a.s.}$  and  $N^2 \bar{b}_N(X_0^{\cdot N}) \bar{q}_N < \Gamma^2(X_0^{\cdot N}) \mathbf{1}_{\mathbb{R}^N} \cdot \mathbf{1}_{\mathbb{R}^N}$ ,  $\mu_0^{\otimes N} - \text{a.s.}$  This condition is equivalent to (3.33), according to the form of  $\Gamma^2$  in (3.34).

□

<sup>4</sup>As in the proof of Lemma 2.1 of Chapter 1, recall that we have  $(M(X_0^{\cdot N}, \pi)^{-1})^\top \mathbf{1}_{\mathbb{R}^N} = \mathbf{1}_{\mathbb{R}^N}$ .

CHAPTER 3. ONE PERIOD MUTUAL HOLDING GAME WITH COMMON NOISE

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# Chapter 4

## Mean Field Game of Mutual Holding with common noise

*In this chapter, we consider the mean field game of cross-holding introduced in Djete & Touzi [19] in the context where the equity value dynamics are affected by a common noise. In contrast with [19], the problem exhibits the standard paradigm of mean-variance trade off. Our crucial observation is to search for equilibrium solutions of our mean field game among those models which satisfy an appropriate notion of no-arbitrage. Under this condition, it follows that the representative agent optimization step is reduced to a standard portfolio optimization problem with random endowment.*

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## 4.1 Introduction

Modeling financial interactions between economic entities is a crucial step in order to gain some understanding of the major question of financial stability of the economic sphere. The equity value of an economic entity depends on one hand on the revenues generated by its specific business structure. On the other hand, they also need to access to appropriate funding for their development. In addition to this first source of interaction through debt, economic entities may also interact with each other through equity cross-holding for risk diversification purpose by appropriate sharing of their profits-and-losses.

This paper follows the model introduced in Djete & Touzi [19] and Djete, Guo & Touzi [20] where the interaction through debt is ignored. Similar to these works, we are interested in characterizing the Nash equilibria of the problem of optimal cross-holding. We consider a large population of symmetric agents who seek to optimize their equity value by optimally choosing the level of holding their peers, while undergoing the level of holding its own assets by them. This induces an optimal response function for each agent, and we search for a Nash equilibrium of this cross-holding problem by analyzing the fixed points of the optimal responses of all agents.

Similar to [19], we consider the mean field version of this problem which is more accessible due to the fact that the limiting model erases the impact of the representative agent on the population distribution because one single agent becomes negligible in the infinite population limit.

Our main objective in this part is to allow for correlated revenue structures of the population of firms. We model this by introducing a common factor of risk which affects their idiosyncratic risk process. Our main results are the following:

- The existence of a solution for the mean field game of cross holding is intimately related to an appropriate notion of no-arbitrage which must be satisfied at equilibrium. We obtain an explicit characterization of this condition in terms of the model ingredients, which is in the same spirit as the Heath-Jarrow-Morton [32] restriction on the drift in the interest rates modeling literature.
- Under the no-arbitrage condition, we provide a characterization of the set of equilibrium solutions for the mean field game of cross-holding.
- Explicit examples are obtained in the context of the Black-Scholes model with common-noise depending coefficients.

We consider the continuous time version of the cross-holding problem. The dynamics of the representative equity value process is defined by

$$\begin{aligned} dX_t &= dP_t + \widehat{\mathbb{E}}^\mu [\beta_t(X_t, \widehat{X}_t) d\widehat{X}_t] - \widehat{\mathbb{E}}^\mu [\pi_t(X_t, \widehat{X}_t)] dX_t, \\ dP_t &= b_t(X_t) dt + \sigma_t(X_t) dW_t + \sigma_t^0(X_t) dW_t^0, \end{aligned} \quad (4.1)$$

where  $W^0, W$  are independent scalar Brownian motions, and  $\widehat{\mathbb{E}}^\mu$  denotes now the conditional expectation on the continuous paths space of the copy  $\widehat{X}$  at equilibrium with respect to law  $\mu$ , conditional on the common noise  $W^0$ .

In the no-common noise situation  $\sigma^0 = 0$  studied in [19, 20], the equilibrium distribution  $\mu$  is a deterministic object and the solution of the problem is derived by guessing that the equilibrium dynamics of the environment is of the form  $d\widehat{X}_t = B_t(\widehat{X}_t)dt + \Sigma_t(\widehat{X}_t)d\widehat{W}_t$ . The critical observation is that under this guess, the continuous time dynamics of the controlled representative equity process is

$$dX_t = dP_t + \widehat{\mathbb{E}}^\mu [\beta_t(X_t, \widehat{X}_t)B_t(\widehat{X}_t)]dt - \widehat{\mathbb{E}}^\mu [\pi_t(X_t, \widehat{X}_t)]dX_t,$$

where the “ $d\widehat{W}_t$ ” term vanishes under the  $\widehat{\mathbb{E}}^\mu$ -expectation. We are then reduced to a problem of linear drift control leading to a standard bang-bang type of solution which consists in maximizing the instantaneous return only, and thus does not follow the usual paradigm in portfolio optimization of balancing mean and variance of returns.

In contrast, by allowing for the common noise  $W^0$  in the one period and the continuous time model, respectively, the situation becomes more intriguing due to the randomness of the conditional distribution  $\mu$ . The equilibrium dynamics of the environment are now expected to be of the form  $d\widehat{X}_t = B_t(\widehat{X}_t)dt + \Sigma_t(\widehat{X}_t)d\widehat{W}_t + \Sigma_t^0(\widehat{X}_t)dW_t^0$ , leading to the following continuous time dynamics of the controlled representative equity process:

$$dX_t = dP_t + \widehat{\mathbb{E}}^\mu [\beta_t(X_t, \widehat{X}_t)B_t(\widehat{X}_t)]dt + \widehat{\mathbb{E}}^\mu [\beta_t(X_t, \widehat{X}_t)\Sigma_t^0(\widehat{X}_t)]dW_t^0 - \widehat{\mathbb{E}}^\mu [\pi_t(X_t, \widehat{X}_t)]dX_t.$$

Notice that in the present context, unlike the no-common noise setting, the control  $\beta$  impact both the drift and the diffusion coefficient, and we are therefore back to the usual paradigm of mean-variance tradeoff in portfolio optimization.

The first main result of this paper is that any equilibrium solution of the above mean field game of cross-holding must satisfy an appropriate notion of no-arbitrage, namely the no-increasing profit (NIP) condition as appeared in the previous literature, see e.g. [28]. We shall provide a characterization of the NIP condition by means of a proportionality relation between instantaneous expected mean  $b$  and the common noise volatility  $\sigma^0$ . Finally, by restricting the search of equilibria to those models which satisfy NIP condition, we provide a necessary and sufficient condition for the existence of an equilibrium solution of the mean field game which can be derived explicitly in some examples.

Actually, in our context, the NIP condition reduces the representative agent’s optimization step to a standard portfolio optimization problem for a single agent with random endowment. In particular, the strategic cross-holding activity within the population does not induce a mean field interaction in the equilibrium dynamics. This is another significant difference with the results of [19, 20]. However, we observe that the mean field interaction in the equilibrium dynamics of [19, 20] is fundamentally induced by the constraint on the strategies  $\beta$  to takes values in  $[0, 1]$  due to the bang-bang feature of the problem in the no-common noise setting.

## 4.2 Continuous time MFG of cross-holding

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space supporting an  $\mathbb{R}^2$ -valued Brownian motion  $(W, W^0)$ . We denote by  $\mathbb{F}^0 := (\mathcal{F}_t^0)_{t \in [0, T]}$  the  $\mathbb{P}$ -completion of the canonical filtration generated by  $W^0$ .

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For some Polish space  $E$ , we denote by  $\mathcal{P}_2(E)$  the collection of all probability measures, with finite second moment. Throughout this paper,  $E$  will be either  $\mathbb{R}^d$ , for some integer  $d$ , or the set  $\mathcal{C}$  of all continuous maps from  $[0, T]$  to  $\mathbb{R}$ . The canonical process on  $\mathcal{C}$  is  $\widehat{X} : \mathcal{C} \rightarrow \mathbb{R}$  defined by  $\widehat{X}_t(\widehat{\omega}) := \widehat{\omega}(t)$  for all  $t \in [0, T]$  and  $\widehat{\omega} \in \mathcal{C}$ .

For fixed  $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ , we denote

- by  $\mathcal{S}_2$  the collection of all  $\mathbb{F}$ -adapted Itô processes  $S$  with initial law  $\mathcal{L}(S_0) = \mu_0$ , and satisfying

$$dS_t = B_t(S_t)dt + \Sigma_t(S_t)dW_t + \Sigma_t^0(S_t)dW_t^0, \quad \mathbb{P}\text{-a.s.}$$

with  $\mathcal{L}(S_{t \wedge \cdot} | \mathcal{F}_t^0) = \mathcal{L}(S_{t \wedge \cdot} | \mathcal{F}_t^0) dt \otimes d\mathbb{P}$ -a.e. for some coefficients

$$B, \Sigma, \Sigma^0 : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}, \quad \{\mathcal{F}_t^0 \otimes \mathcal{B}(\mathbb{R})\}_{t \in [0, T]} \text{ - prog. measurable,} \quad (4.2)$$

and satisfying the integrability condition:

$$\mathbb{E} \left[ \int_0^T |B_t(S_t)|^2 + |\Sigma_t(S_t)|^2 + |\Sigma_t^0(S_t)|^2 dt \right] < \infty,$$

in particular, we have  $\mathbb{E}[\sup_{t \in [0, T]} |S_t|^2] < \infty$ ;

- and by  $\mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  the collection of all  $\mathcal{F}_T^0$ -measurable  $\mathcal{P}_2(\mathcal{C})$ -valued random variable  $\mu$  such that  $\mu = \mathcal{L}(S | \mathcal{F}_T^0)$  for some  $S \in \mathcal{S}_2$ .

For such a  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$ , we shall denote  $\mu_t := \mathcal{L}(S_t | \mathcal{F}_t^0) = \mu \circ (\widehat{X}_t)^{-1}$ , and  $\widehat{\mathbb{E}}^\mu[\phi(\widehat{X})] := \int \phi(\widehat{\omega})\mu(d\widehat{\omega})$ , for all map  $\phi : \Omega \times \mathcal{C} \rightarrow \mathbb{R}$  with  $\mu$ -integrable  $\phi(\omega, \cdot)$ ,  $\mathbb{P}$ -a.e.  $\omega$ .

We next introduce the set  $\mathcal{A}$  of scalar  $(\mathcal{F}_t^0 \otimes \mathcal{B}_{\mathbb{R}^2})_{t \in [0, T]}$ -progressively measurable maps from  $[0, T] \times \Omega \times \mathbb{R}^2$  to  $\mathbb{R}$ . For all  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  and  $\pi \in \mathcal{A}$ , we denote

$$m_t^{\mu, \pi}(x) := \frac{1}{1 + \int_{\mathbb{R}} \pi_t(\widehat{x}, x)\mu_t(d\widehat{x})}, \quad t \in [0, T], \quad x \in \mathbb{R},$$

with the convention  $\frac{1}{0} = \infty$  and we define by  $\mathcal{A}_b^{\mu, \pi}$  the subset of  $\mathcal{A}$  consisting of bounded  $\beta \in \mathcal{A}$  such that the map from  $\mathbb{R}$  to  $\mathbb{R} \times \mathbb{R}$ :

$$x \mapsto m_t^{\mu, \pi}(x)\widehat{\mathbb{E}}^\mu \left[ \beta_t(x, \widehat{X}_t)(B_t, \Sigma_t^0)(\widehat{X}_t) \right] \quad \text{is continuous,} \quad dt \otimes d\mathbb{P} \text{ - a.e.}$$

We further define appropriate conditions so that, given  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  and  $\pi \in \mathcal{A}$ , the following cross-holding state dynamics induce a well defined process started from an  $\mathcal{F}_0$ -r.v.  $X_0$  with law  $\mu_0$ :

$$X_t = X_0 + P_t + \widehat{\mathbb{E}}^\mu \left[ \int_0^t \beta_s(X_s, \widehat{X}_s)d\widehat{X}_s \right] - \int_0^t \widehat{\mathbb{E}}^\mu \left[ \pi_s(\widehat{X}_s, X_s) \right] dX_s, \quad \mathbb{P}\text{-a.s.} \quad (4.3)$$

with idiosyncratic risk process  $P$  defined by

$$P_t := \int_0^t b_s(X_s, \mu_s)ds + \int_0^t \sigma_s(X_s, \mu_s)dW_s + \int_0^t \sigma_s^0(X_s, \mu_s)dW_s^0, \quad \mathbb{P}\text{-a.s.} \quad (4.4)$$

for some  $\mathbb{F}^0 \otimes \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathcal{P}_2(\mathbb{R})}$ -progressively measurable coefficients

$$b, \sigma, \sigma^0 : [0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$$

satisfying the following conditions ensuring the existence of a unique square integrable solution for the McKean-Vlasov SDE (4.3) with no interaction  $\beta = 0$ , see the book of R. Carmona and F. Delarue [14], page 76.

**Assumption 4.1.** For  $\varphi = (b, \sigma, \sigma^0)$ , the map  $\varphi : [0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$  is

- (i) Lipschitz in  $(x, m)$  uniformly in  $(t, \omega)$ ,
- (ii) and has uniform linear growth in  $(x, m)$  in the following sense:

$$\sup_{(t, \omega, x, m)} \frac{|\varphi(t, \omega, x, m)|}{1 + |x| + \left(\int_{\mathbb{R}} x^2 m(dx)\right)^{1/2}} < \infty.$$

The dynamics of the state process  $X$  in (4.3) have the following interpretation:

- The probability measure  $\mu$  is the  $W^0$ -conditional law of the process  $\widehat{X}$  which represents the population surrounding the representative agent with state process  $X$ ;
- The strategy  $\pi$  represents the investment of the population in the equity of the representative agent;
- In our definition below of the MFG,  $\widehat{X}$  can be seen as a conditional independent copy of the equilibrium dynamics of  $X$ , with distribution  $\mu$ , and  $\pi$  is an equilibrium cross-holding strategy of the surrounding population.
- For this reason, we say that the pair  $(\mu, \pi)$  is a *random environment*, meaning that it describes the population of equity processes surrounding the representative agent. We then introduce the set of random environments

$$\mathcal{E} := \{(\mu, \pi) : \mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \text{ and } \pi \in \mathcal{A}\}.$$

- The strategy  $\beta$  represents the investment strategy that the representative agent may implement in reaction to the random environment  $(\mu, \pi)$ . We naturally define below the solution of the MFG as a random environment under which the representative agent finds no benefit to deviate from the surrounding population's cross-holding strategy, and the induced state process reproduces the law of the surrounding population equity process.

Before going further, let us recall that given two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a function  $F : (X, d_X) \rightarrow (Y, d_Y)$  is called Lipschitz if there exists a positive constant  $K$  such that

$$d_Y(F(x), F(x')) \leq K d_X(x, x'),$$

for all  $(x, x') \in X^2$ . When one of the argument is a measure, a family of widely used distances are the Wasserstein distances: for a general  $p > 1$ , the  $p$ -Wasserstein distance  $W_p(\mu, \mu')$  is defined on  $\mathcal{P}_p(X)$  (the set of measures of finite moment of order  $p$ ) by:

$$W_p(\mu, \mu') := \inf \left[ \left( \int_{X \times X} |x - y|^p \pi(dx, dy) \right)^{1/p}, \pi \in \mathcal{P}_2(X \times X) \text{ with marginales } \mu \text{ and } \mu' \right].$$

For more details, we refer for instance the interested reader to Chapter 5 in the monograph of F. Santambrogio [54] and the references therein.

In the following definition,  $(B, \Sigma, \Sigma^0)$  is the triplet of coefficients corresponding to the above  $\mu$ , and we denote for all  $\beta \in \mathcal{A}$

$$\begin{pmatrix} \overline{B}_t^{\mu, \pi, \beta} \\ \overline{\Sigma}_t^{\mu, \pi, \beta} \end{pmatrix}(x) := \int_{\mathbb{R}} \beta_t(x, \widehat{x}) \begin{pmatrix} B_t \\ \Sigma_t^0 \end{pmatrix}(\widehat{x}) \mu_t(d\widehat{x}) + \begin{pmatrix} b_t \\ \sigma_t^0 \end{pmatrix}(x, \mu_t). \quad (4.5)$$

**Definition 4.2.** (1) A random environment  $(\mu, \pi) \in \mathcal{E}$  is admissible if

(1-i)  $dt \otimes d\mathbb{P}$ -a.e.,  $0 \neq 1 + \int_{\mathbb{R}} \pi_t(\widehat{x}, x) \mu_t(d\widehat{x}) = \frac{1}{m_t^{\mu, \pi}(x)}$  for all  $x \in \mathbb{R}$ .

(1-ii) The SDE defined by the starting data  $X_0$  with law  $\mu_0$  and

$$dX_t = m_t^{\mu, \pi}(X_t) \left( \overline{B}_t^{\mu, \pi, \beta}(X_t) dt + \overline{\Sigma}_t^{\mu, \pi, \beta}(X_t) dW_t^0 + \sigma_t(X_t, \mu_t) dW_t \right) \quad (4.6)$$

has a weak solution, for all cross-holding strategy  $\beta \in \mathcal{A}_b^{\mu, \pi}$ , and satisfies

$$\widehat{\mathbb{E}}^\mu \left[ \int_0^T \beta_t(X_t, \widehat{X}_t)^2 (|B_t|^2 + |\Sigma_t|^2 + |\Sigma_t^0|^2)(\widehat{X}_t) dt \right] < \infty, \quad \mathbb{P}\text{-a.e.} \quad (4.7)$$

and

$$\mathbb{E} \left[ \int_0^T m_t^{\mu, \pi}(X_t)^2 (\overline{B}_t^{\mu, \pi}(X_t)^2 + \overline{\Sigma}_t^{\mu, \pi}(X_t)^2 + \sigma_t(X_t, \mu_t)^2) dt \right] < \infty. \quad (4.8)$$

(2) A strategy  $\beta \in \mathcal{A}$  is said to be  $(\mu, \pi)$ -admissible, and we write  $\beta \in \mathcal{A}(\mu, \pi)$ , if it satisfies the above (4.6), (4.7) and (4.8).

Notice that  $\mathcal{A}_b^{\mu, \pi} \subset \mathcal{A}(\mu, \pi)$ , and that the dynamics of  $X$  introduced in the last definition exactly reproduces the required cross-holding dynamics in (4.3)-(4.4). We also observe that, although for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , the canonical process  $\widehat{X}$  is not a semi-martingale under  $\mu(\omega, d\widehat{\omega})$ , the quantity  $\mathbb{E}^\mu \left[ \int_0^\cdot \beta_s(X_s, \widehat{X}_s) d\widehat{X}_s \right]$  is always well defined  $\mathbb{P}$ -a.e.

**Example 4.3.** Let  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  and  $\pi \in \mathcal{A}$  be such that  $x \mapsto m_t^{\mu, \pi}(x)$  is continuous  $dt \otimes d\mathbb{P}$ -a.e. and

$$\widehat{\mathbb{E}}^\mu \left[ \int_0^T \sup_{x \in \mathbb{R}} |m_t^{\mu, \pi}(x)|^2 (|B_t(\widehat{X}_t)|^2 + |\Sigma_t(\widehat{X}_t)|^2 + |\Sigma_t^0(\widehat{X}_t)|^2) dt \right] < \infty,$$

then, we can verify that  $(\mu, \pi)$  is admissible.

For  $\beta \in \mathcal{A}(\mu, \pi)$ , we denote by  $X^{\mu, \pi, \beta}$  an arbitrary solution of the SDE (4.6), and we introduce the reward function

$$J_{\mu, \pi}(\beta) := \mathbb{E} \left[ U(X_T^{\mu, \pi, \beta}) \right],$$

where  $U : \mathbb{R} \rightarrow \mathbb{R}$  is a non-decreasing utility function verifying  $\lim_{x \rightarrow \infty} U(x) = \infty$ .

**Definition 4.4.** An admissible random environment  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  is an equilibrium solution of the MFG of cross-holding if:

- (i)  $J_{\mu, \pi}(\beta) \leq J_{\mu, \pi}(\pi) < \infty$ , for all  $\beta \in \mathcal{A}(\mu, \pi)$ ,
- (ii)  $\pi \in \mathcal{A}(\mu, \pi)$ ,  $\mathcal{L}(X^{\pi, \pi} | \mathcal{F}_T^0) = \mu$ ,  $\mathbb{P}$ -a.e. and  $dt \otimes \mu_t(dx)$ -a.e.

$$\Sigma_t(x) = m_t^{\mu, \pi}(x) \sigma_t(x), \quad B_t(x) = m_t^{\mu, \pi}(x) \bar{B}_t^{\mu, \pi, \beta}(x), \quad \Sigma_t^0(x) = m_t^{\mu, \pi}(x) \bar{\Sigma}_t^0{}^{\mu, \pi, \beta}(x),$$

with the notations of (4.5).

The last equation follows from the identification of the equilibrium dynamics with the coefficients  $(B, \Sigma, \Sigma^0)$  of the Itô process  $X^{\mu, \pi, \pi} := X$  under  $\mu$ :

$$\begin{aligned} X_t &= X_0 + \int_0^t m^{\mu, \pi}(X_s) dP_s + \widehat{\mathbb{E}}^\mu \left[ \int_0^t m^{\mu, \pi}(X_s) \pi_s(X_s, \widehat{X}_s) d\widehat{X}_s \right] \\ &= X_0 + \int_0^t m^{\mu, \pi}(X_s) dP_s + \widehat{\mathbb{E}}^\mu \left[ \int_0^t m^{\mu, \pi}(X_s) \pi_s(X_s, \widehat{X}_s) (B_s(\widehat{X}_s) ds + \Sigma_s^0(\widehat{X}_s) dW_s^0) \right]. \end{aligned}$$

### 4.3 Arbitrage free random environment

In this section, we introduce the notion of no-increasing profit, a weaker concept of no-arbitrage introduced in the previous literature, see Fontana [28]. Similar to the standard no-arbitrage condition, the no-increasing profit condition is necessary for the existence of equilibrium in the context where the performance criterion is increasing in terms of the state process  $X$ . We shall denote by

$$G_t^{\mu, \pi}(\beta) := \widehat{\mathbb{E}}^\mu \left[ \int_0^t \beta_s(X_s^{\mu, \pi, \beta}, \widehat{X}_s) m^{\mu, \pi}(X_s^{\mu, \pi, \beta}) d\widehat{X}_s \right], \quad \text{for all } \beta \in \mathcal{A}(\mu, \pi).$$

**Definition 4.5.** An admissible random environment  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  is said to have no increasing profit (NIP) if for all strategy  $\beta \in \mathcal{A}_b^{\mu, \pi}$  with  $t \mapsto G_t^{\mu, \pi}(\beta)$  nondecreasing,  $\mathbb{P}$ -a.s. we have  $G_T^{\mu, \pi}(\beta) = 0$ ,  $\mathbb{P}$ -a.s.

Indeed, if there exists some bounded  $\beta \in \mathcal{A}_b^{\mu, \pi}$  violating the condition of the last definition, then, it follows from Definition 4.2 (1-ii) that  $\lambda\beta \in \mathcal{A}(\mu, \pi)$  for all  $\lambda > 0$ , and this in turn implies that  $\mathbb{P}[\lim_{\lambda \rightarrow \infty} X_T^{\lambda\beta, \mu, \pi} = \infty] > 0$ . Therefore, the existence of an optimal response (i.e. a maximizer) in item (i) of Definition (4.4) fails to hold under any increasing agent's performance criterion.

The main result of this section states that the coefficients  $(B, \Sigma^0)$  of an increasing profit free admissible random environment satisfy a proportionality condition similar to the Heath–Jarrow–Morton [32] restriction in the term structure modeling.

**Theorem 4.6.** An admissible random environment  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  satisfies the (NIP) condition if and only if the corresponding coefficients  $B$  and  $\Sigma^0$  satisfy the following proportionality condition:

$$B_t(x) = \Sigma_t^0(x) \lambda_t, \quad dt \otimes \mu_t(dx)\text{-a.e. } \mathbb{P}\text{-a.s.}$$

for some scalar  $\mathbb{F}^0$ -progressively measurable process  $(\lambda_t)_{t \in [0, T]}$ .

If in addition  $(\mu, \pi)$  is an equilibrium solution of the MFG of cross-holding, then

$$b_t(x, \mu_t) = \lambda_t \sigma_t^0(x, \mu_t), \quad dt \otimes \mu_t(dx)\text{-a.e. } \mathbb{P}\text{-a.s.}$$

*Proof.* (i) We start with the last part of the statement, given the characterization of the (NIP) condition which we prove in (ii)-(iii) below. Since  $(\mu, \pi)$  is an equilibrium solution of the MFG of cross-holding, we know that  $\mathbb{P}$ -a.e.,  $dt \otimes \mu_t(dx)$ -a.e.

$$\begin{pmatrix} B_t \\ \Sigma_t^0 \end{pmatrix}(x) = m_t^{\mu, \pi}(x) \left( \int_{\mathbb{R}} \pi_t(x, \hat{x}) \begin{pmatrix} B_t \\ \Sigma_t^0 \end{pmatrix}(\hat{x}) \mu_t(d\hat{x}) + \begin{pmatrix} b_t \\ \sigma_t^0 \end{pmatrix}(x, \mu_t) \right). \quad (4.9)$$

Using the characterization  $B = \Sigma^0 \lambda$ ,  $dt \otimes \mu_t(dx)$ -a.e.,  $\mathbb{P}$ -a.s. of the NIP condition stated in the first part, it follows from the first equation in (4.9) that

$$\begin{aligned} m_t^{\mu, \pi}(x) b_t(x, \mu_t) &= B_t(x) - m_t^{\mu, \pi}(x) \int_{\mathbb{R}} \pi_t(x, \hat{x}) B_t(\hat{x}) \mu_t(d\hat{x}) \\ &= \lambda_t \left( \Sigma_t^0(x) - m_t^{\mu, \pi}(x) \int_{\mathbb{R}} \pi_t(x, \hat{x}) \Sigma_t^0(\hat{x}) \mu_t(d\hat{x}) \right) = \lambda_t m_t^{\mu, \pi}(x) \sigma_t^0(x, \mu_t), \end{aligned}$$

by the second equation in (4.9). The required result follows from the fact that  $m_t^{\mu, \pi}(x) \neq 0$ , due to the admissibility of the random environment  $(\mu, \pi)$ .

(ii) We next prove the necessary condition for the characterization of the NIP condition. Let  $c > 0$  and  $N \geq 1$ , and set

$$H_t^{c, N}(\lambda) := \widehat{\mathbb{E}}^\mu \left[ \left| B_t(\widehat{X}_t) - \lambda \Sigma_t^0(\widehat{X}_t) \right|^2 \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t) \right]$$

where

$$A_t^{c, N} := \{x : |B_t(x)|^2 + |\Sigma_t^0(x)|^2 \leq N, \quad c \leq |\Sigma_t^0(x)|\}.$$

Then, the  $\mathbb{F}^0$ -progressively measurable process

$$\lambda_t^{c, N} := \frac{\widehat{\mathbb{E}}^\mu \left[ \Sigma_t^0(\widehat{X}_t) B_t(\widehat{X}_t) \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t) \right]}{\widehat{\mathbb{E}}^\mu \left[ \Sigma_t^0(\widehat{X}_t)^2 \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t) \right]}, \quad t \in [0, T],$$

is a minimizer of  $H_t^{c, N}$  over the set of  $\mathbb{F}^0$ -progressively measurable processes  $\lambda$ . We next introduce the scalar  $(\mathcal{F}_t^0 \otimes \mathcal{B}(\mathbb{R}))_{t \in [0, T]}$ -progressively measurable process

$$\nu_t^{c, N}(\widehat{X}_t) := \left( B_t(\widehat{X}_t) - \Sigma_t^0(\widehat{X}_t) \lambda_t^{c, N} \right) \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t), \quad (4.10)$$

which satisfies

$$\widehat{\mathbb{E}}^\mu \left[ \Sigma_t^0(\widehat{X}_t) \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t) \hat{\nu}_t^{c, N}(\widehat{X}_t) \right] = 0, \quad \mathbb{P} - \text{a.s. } t \in [0, T]. \quad (4.11)$$

We now claim that the NIP condition implies that

$$\hat{\nu}^{c, N} = 0. \quad (4.12)$$

Before proving this claim, let us show how it induces the required necessary condition. Since  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$ , we have  $\mathbf{1}_{A_t^{c, N}}(x) \rightarrow 1$ , as  $N \nearrow \infty$  and  $c \searrow 0$ ,  $dt \otimes \mu_t(dx)$ -a.e. Moreover, it follows from (4.12) and (4.10) that the process  $\lambda_t := \limsup_{N \rightarrow \infty, c \rightarrow 0} \lambda_t^{c, N}$ , satisfies the required proportionality condition  $B = \Sigma^0 \lambda$ ,  $dt \otimes \mu_t(dx)$ -a.e.,  $d\mathbb{P}$ -a.s.

To complete the proof of the necessary conditions, we prove (4.12) in the two following sub-steps.

- (a) We start by showing that the NIP condition implies that for all bounded  $\mathbb{R}$ -valued  $(\mathcal{F}_t^0 \otimes \mathcal{B}(\mathbb{R}))_{t \in [0, T]}$ -progressively measurable process  $\beta$ :

$$\widehat{\mathbb{E}}^\mu [(\beta_t \Sigma_t^0)(\widehat{X}_t)] = 0, \quad dt \otimes d\mathbb{P}\text{-a.e. implies } \widehat{\mathbb{E}}^\mu [(\beta_t B_t)(\widehat{X}_t)] = 0, \quad dt \otimes d\mathbb{P}\text{-a.e.} \quad (4.13)$$

Indeed, assume to the contrary that  $\widehat{\mathbb{E}}^\mu [\beta_t(\widehat{X}_t)\Sigma_t^0(\widehat{X}_t)] = 0, dt \otimes d\mathbb{P}\text{-a.e.}$  and

$$\int_0^T \int_\Omega \mathbf{1}_A(t, \omega) \mathbb{P}(d\omega) dt > 0 \quad \text{where } A := \left\{ \widehat{\mathbb{E}}^\mu [\beta_t(\widehat{X}_t)B_t(\widehat{X}_t)] \neq 0 \right\}.$$

for some bounded  $\beta$ . We define  $\beta_t^*(x, \widehat{X}_t) := s_t^0(x)\beta_t(\widehat{X}_t)$ , with

$$s_t^0(x) := \text{sgn} \left\{ m_t^{\mu, \pi}(x) \widehat{\mathbb{E}}^\mu [\beta_t(\widehat{X}_t)B(t, \widehat{X}_t)] \right\}.$$

Then  $\beta^* \in \mathcal{A}(\mu, \pi)$  is admissible, and

$$\begin{aligned} G_t^{\mu, \pi}(\beta^*) &= \widehat{\mathbb{E}}^\mu \left[ \int_0^t m_s^{\mu, \pi}(X_s^{\mu, \pi, \beta^*}) \beta_s^*(X_s^{\mu, \pi, \beta^*}, \widehat{X}_s) d\widehat{X}_s \right] \\ &= \widehat{\mathbb{E}}^\mu \left[ \int_0^t m_s^{\mu, \pi}(X_s^{\mu, \pi, \beta^*}) \beta_s^*(X_s^{\mu, \pi, \beta^*}, \widehat{X}_s) B(s, \widehat{X}_s) ds \right] \\ &= \int_0^t \left| m_s^{\mu, \pi}(X_s^{\mu, \pi, \beta^*}) \widehat{\mathbb{E}}^\mu [\beta_s(\widehat{X}_s)B(s, \widehat{X}_s)] \right| ds. \end{aligned}$$

Consequently  $t \mapsto G_t^{\mu, \pi}(\beta^*)$  is non-decreasing,  $\mathbb{P}$ -a.s. Since  $m_t^{\mu, \pi}(X_t^{\mu, \pi, \beta^*}) \neq 0$   $dt \otimes d\mathbb{P}$ -a.e., by definition of the set  $A$ , we deduce that  $\mathbb{P}[G_T^{\mu, \pi}(\beta^*) > 0] > 0$ . This is in contradiction with the non-increasing profit condition (NIP).

- (b) We now show that (4.13) implies that  $\widehat{\nu}_t^{c, N}(\widehat{X}_t) = 0$ . Notice that  $\sup_{(t, \omega)} \lambda_t^{c, N}(\omega) < \infty$ . Then,  $\left( \nu_t^{c, N}(x) \mathbf{1}_{A_t^{c, N}}(x) \right)_{(t, x) \in [0, T] \times \mathbb{R}}$  is a bounded scalar  $(\mathcal{F}_t^0 \otimes \mathcal{B}(\mathbb{R}))_{t \in [0, T]}$ -prog-meas. process. By (4.11), this process satisfies the left hand side of (4.13), and it follows from (a) that the right hand side of (4.13) holds, namely

$$\widehat{\mathbb{E}}^\mu [B_t(\widehat{X}_t) \widehat{\nu}_t^{c, N}(\widehat{X}_t) \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t)] = 0.$$

By the definition of  $\widehat{\nu}^{c, N}$  in (4.10) together with (4.11), we get

$$\widehat{\mathbb{E}}^\mu [\widehat{\nu}_t^{c, N}(\widehat{X}_t)^2 \mathbf{1}_{A_t^{c, N}}(\widehat{X}_t)] = 0.$$

which provides the the required (4.12).

- (iii) We finally prove the sufficient condition for the characterization of the NIP condition. Assume that  $B_t(x) = \Sigma_t^0(x)\lambda_t$ ,  $dt \otimes \mu_t(dx) - \text{a.e.}, \mathbb{P}$ -a.e. for some  $\mathbb{F}^0$ -prog-meas. process  $\lambda$ . then the NIP condition is true. Let  $\beta \in \mathcal{A}(\mu, \pi)$  be such that the process  $(G_t^{\mu, \pi}(\beta))_{t \in [0, T]}$  is non-decreasing, and let us show that the process  $G^{\mu, \pi}(\beta)$  is zero,  $\mathbb{P}$ -a.s.



As a non-increasing process,  $G^{\mu,\pi}(\beta)$  has finite variation, implying that its martingale part is zero:

$$\begin{aligned} 0 = R_t &:= \int_0^t \widehat{\mathbb{E}}^\mu \left[ \beta_s(X_s^{\mu,\pi,\beta}, \widehat{X}_s) m_s^{\mu,\pi}(X_s^{\mu,\pi,\beta}) \Sigma_s^0(\widehat{X}_s) \right] dW_s^0 \\ &= G_t^{\mu,\pi}(\beta) - \int_0^t \widehat{\mathbb{E}}^\mu \left[ \beta_s(X_s^{\mu,\pi,\beta}, \widehat{X}_s) m_s^{\mu,\pi}(X_s^{\mu,\pi,\beta}) B_s(\widehat{X}_s) \right] ds, \quad t \in [0, T]. \end{aligned}$$

As  $\lambda$  is  $\mathbb{F}^0$ -progressively measurable, this can be written equivalently as

$$G_t^{\mu,\pi}(\beta) = \int_0^t \lambda_s \widehat{\mathbb{E}}^\mu \left[ \beta_s(X_s^{\mu,\pi,\beta}, \widehat{X}_s) m_s^{\mu,\pi}(X_s^{\mu,\pi,\beta}) \Sigma_s^0(\widehat{X}_s) \right] ds, \quad \text{for all } t \in [0, T].$$

On the other hand, we have that

$$\int_0^t \lambda_s \mathbf{1}_{|\lambda_s^0| \leq N} \widehat{\mathbb{E}}^\mu \left[ \beta_s(X_s^{\mu,\pi,\beta}, \widehat{X}_s) m_s^{\mu,\pi}(X_s^{\mu,\pi,\beta}) \Sigma_s^0(\widehat{X}_s) \right] ds = \left\langle \int_0^\cdot \lambda_s \mathbf{1}_{|\lambda_s^0| \leq N} dW_s^0, R \right\rangle_t = 0.$$

Moreover, as  $\beta \in \mathcal{A}(\mu, \pi)$  is an admissible strategy, we have

$$\int_0^t \mathbb{E} \widehat{\mathbb{E}}^\mu \left[ \left| \lambda_s \beta_s(X_s^{\mu,\pi,\beta}, \widehat{X}_s) m_s^{\mu,\pi}(X_s^{\mu,\pi,\beta}) \Sigma_s^0(\widehat{X}_s) \right| \right] ds < \infty.$$

We may then take the limit  $N \rightarrow \infty$  in the last equality, and deduce from the dominated convergence Theorem that  $G_t^{\mu,\pi}(\beta) = 0$ ,  $t \in [0, T]$ .  $\square$

## 4.4 The reduced mean field game

In this section, we derive an important reduction of the mean field game of cross-holding. Recall that any admissible random environment  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  which is a solution of the mean field game satisfies the No-Increasing Profit condition. By Theorem 4.6, we have the following proportionality relation between the drift and the volatility of the common noise:

$$B_t(x) = \lambda_t \Sigma_t^0(x) \quad \text{and} \quad b_t(x) = \lambda_t \sigma_t^0(x), \quad dt \otimes \mu_t(dx) - \text{a.e. } \mathbb{P} - \text{a.s.}$$

for some  $\mathbb{F}^0$ -progressively measurable process  $(\lambda_t)_{t \in [0, T]}$ . Here, for notational simplicity, we have omitted the dependence of the coefficients on the marginal law  $\mu_t$  because this is not playing any role. Plugging this into the dynamics of the equity process (4.6), we see that

$$dX_t^{\mu,\pi,\beta} = A_t^{\mu,\pi}(X_t^{\mu,\pi,\beta}, \beta_t) (\lambda_t dt + dW_t^0) + v_t^{\mu,\pi}(X_t^{\mu,\pi,\beta}) dW_t, \quad (4.14)$$

where

$$A_t^{\mu,\pi}(x, \beta_t) := m_t^{\mu,\pi}(x) \overline{\Sigma}_t^{\mu,\pi,\beta}(x), \quad \text{and} \quad v_t^{\mu,\pi}(x) := m_t^{\mu,\pi}(x) \sigma_t(x),$$

where we recall the notation of (4.5):

$$\overline{\Sigma}_t^{\mu,\pi,\beta}(x) = \left[ \sigma_t^0(x) + \int_{\mathbb{R}} \beta_t(x, \widehat{x}) \Sigma_t^0(\widehat{x}) \mu_t(d\widehat{x}) \right].$$

Motivated by this reduced form, we introduce the reduced performance criterion  $\mathcal{J}$  defined by

$$\mathcal{J}_{\mu,\pi}(\alpha) := \mathbb{E}[U(\mathbb{X}_T^{\mu,\pi,\alpha})],$$

where the reduced controlled state  $\mathbb{X}$  is defined by

$$d\mathbb{X}_t^{\mu,\pi,\alpha} = \alpha_t(\mathbb{X}_t^{\mu,\pi,\alpha})(\lambda_t dt + dW_t^0) + v_t^{\mu,\pi}(\mathbb{X}_t^{\mu,\pi,\alpha})dW_t, \quad (4.15)$$

and the reduced control process

$$\alpha : [0, T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}, \mathbb{F}^0 \otimes \mathcal{B}_{\mathbb{R}} - \text{prog. meas. such that} \\ (4.15) \text{ has a strong solution satisfying}$$

$$\mathbb{E} \left[ \int_0^T \alpha_t(\mathbb{X}_t^{\mu,\pi,\alpha})^2 (1 + |\lambda_t|^2) + m_t^{\mu,\pi}(\mathbb{X}_t^{\mu,\pi,\alpha})^2 \sigma_t(\mathbb{X}_t^{\mu,\pi,\alpha})^2 dt \right] < \infty. \quad (4.16)$$

The reduced control  $\alpha$  represents the aggregate instantaneous return from the cross-holding strategy  $\beta$  representing the investment of the representative agent in the surrounding environment.

The main consequence of the NIP condition, which we state below, says that the individual optimization step of any equilibrium solution of the mean field game of cross-holding reduces to the problem  $\sup_{\alpha} \mathcal{J}_{\mu,\pi}(\alpha)$ . Observe that the last problem is nothing but the standard portfolio optimization problem under random endowment represented by the additional term  $v_t^{\mu,\pi}(\mathbb{X}_t^{\mu,\pi,\alpha})dW_t$ . Such problems have been studied extensively in the previous literature, and are well-known to differ from the standard Merton portfolio optimization problem only when the endowment process introduces an additional risk, because otherwise it can be perfectly hedged thus reducing to the classical situation. In contrast, our reduced model is a typical incomplete market situation as the random endowment process is driven by the idiosyncratic noise  $W$ , while the portfolio optimization acts on the common noise  $W^0$ . The existing literature is mainly focused on the convex duality representation of the problem, see e.g. Cvitanic, Schachermayer & Wang [16], Hugonnier & Kramkov [39], and also extensions by Karatzas & Zitković [41], Zitković [61] and Mostovyi [50] when the intermediate consumption is allowed.

**Proposition 4.7.** *Let  $(\mu, \pi) \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C}) \times \mathcal{A}$  be an admissible random environment such that  $\mu_t[\Sigma_t^0 \neq 0] > 0$ ,  $\mathbb{P}$ -a.s. Then for any  $\beta \in \mathcal{A}(\mu, \pi)$ , there is  $\alpha$  satisfying (4.16) such that*

$$\mathbb{X}^{\mu,\pi,\alpha} = X^{\mu,\pi,\beta}, \quad \mathbb{P}\text{-a.e.} \quad (4.17)$$

*Conversely, for any  $\alpha$  satisfying (4.16), there is  $\beta \in \mathcal{A}(\mu, \pi)$  such that (4.17) holds.*

*In addition, if we set  $\alpha_t^{\mu,\pi}(x) := A_t^{\mu,\pi}(x, \pi_t)$  for all  $(t, x) \in [0, T] \times \mathbb{R}$ , the pair  $(\mu, \pi)$  is an equilibrium solution of the MFG of cross-holding if and only if:*

- (i)  $\mathcal{J}_{\mu,\pi}(\alpha) \leq \mathcal{J}_{\mu,\pi}(\alpha^{\mu,\pi}) < \infty$ , for all  $\alpha$  satisfying (4.16),
- (ii)  $\mathcal{L}(\mathbb{X}^{\mu,\pi,\alpha^{\mu,\pi}} | \mathcal{F}_T^0) = \mu$ ,  $\mathbb{P}$ -a.e.

*Proof.* First, let us observe that for any  $\alpha$  satisfying (4.16), there is  $\beta \in \mathcal{A}(\mu, \pi)$  admissible s.t.  $\alpha_t(\mathbb{X}_t^{\mu, \pi, \alpha}) = A_t^{\mu, \pi}(\mathbb{X}_t^{\mu, \pi, \alpha}, \beta_t) dt \otimes d\mathbb{P}$ -a.e. Indeed, by setting  $\beta_t(x, \hat{x}) = f_t(x) \text{sg}[\Sigma_t^0(\hat{x})]$ , with

$$\frac{\alpha_t(x)}{m_t^{\mu, \pi}(x)} = \sigma_t^0(x) + f_t(x) \int_{\mathbb{R}} |\Sigma_t^0(\hat{x})| \mu_t(d\hat{x}),$$

This clearly defines  $f_t(x)$ ,  $(t, x) \in [0, T] \times \mathbb{R}$ , since  $\mu_t[\Sigma_t^0 \neq 0] > 0$ ,  $\mathbb{P}$ -a.s. We can see that  $\mathbb{X}^{\mu, \pi, \alpha} = X^{\mu, \pi, \beta}$   $\mathbb{P}$ -a.e. and check that  $\beta \in \mathcal{A}(\mu, \pi)$ . Conversely, for any  $\beta \in \mathcal{A}(\mu, \pi)$ , by defining  $\alpha_t^{\mu, \pi, \beta}(x) := A_t^{\mu, \pi}(x, \beta_t)$ , we can check that  $\alpha^{\mu, \pi, \beta}$  satisfies (4.16) and  $\mathbb{X}^{\mu, \pi, \alpha^{\mu, \pi, \beta}} = X^{\mu, \pi, \beta}$ ,  $\mathbb{P}$ -a.e.

Let  $(\mu, \pi)$  be an equilibrium solution of the MFG of cross-holding. Then

$$\mathcal{J}_{\mu, \pi}(\alpha) = J_{\mu, \pi}(\beta) \leq J_{\mu, \pi}(\pi) = \mathcal{J}_{\mu, \pi}(\alpha^{\mu, \pi})$$

and  $\mathcal{L}(\mathbb{X}^{\mu, \pi, \alpha^{\mu, \pi}} | \mathcal{F}_T^0) = \mathcal{L}(X^{\mu, \pi, \pi} | \mathcal{F}_T^0) = \mu$ . The converse equivalence follows by using similar arguments.  $\square$

We conclude this section by isolating a simple result which allows to recover a cross-holding strategy from some given aggregate hedging strategy  $\alpha$  which is eligible for an equilibrium solution of the mean field game.

**Proposition 4.8.** *Let  $\mu \in \mathcal{P}_2^{\mathbb{F}^0}(\mathcal{C})$  and  $\alpha$  be a scalar  $(\mathcal{F}_t^0 \otimes \mathcal{B}(\mathbb{R}))_{t \in [0, T]}$ -progressively measurable process satisfying*

$$\mathbb{E} \left[ \widehat{\mathbb{E}}^\mu \int_0^T \alpha_t(\widehat{X}_t)^2 (1 + |\lambda_t|^2) dt \right] < \infty \quad \text{and} \quad \widehat{\mathbb{E}}^\mu \left[ \sigma_t^0(\widehat{X}_t) \right] = \widehat{\mathbb{E}}^\mu \left[ \alpha_t(\widehat{X}_t) \right] \neq 0. \quad (4.18)$$

For an arbitrary scalar  $\mathbb{F}^0$ -prog. meas. process  $\kappa \neq -1$ ,  $\mathbb{P}$ -a.s. let

$$\pi_t(x, y) := \frac{(1 + \kappa_t) \alpha_t(x) - \sigma_t^0(x)}{\widehat{\mathbb{E}}^\mu[\alpha_t(\widehat{X}_t)]} \quad x, \hat{x} \in \mathbb{R}, \quad t \in [0, T].$$

If  $(\mu, \pi)$  is an equilibrium, then  $\pi \in \mathcal{A}(\mu, \pi)$  and

$$\alpha_t(x) = \Sigma_t^0(x) = A_t^{\mu, \pi}(x, \pi_t) \quad \text{and} \quad m_t^{\mu, \pi}(x) = (1 + \kappa_t)^{-1}, \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}.$$

*Proof.* We just need to check the condition with  $\pi$  given in the statement. First, we observe that  $\Sigma_t^0(x) = A_t^{\mu, \pi}(x, \pi_t)$  in (4.14) since  $(\mu, \pi)$  is an equilibrium. Then, using the condition (4.18), it is straightforward that  $\alpha_t(x) = A_t^{\mu, \pi}(x, \pi_t)$  and  $m_t^{\mu, \pi}(x) = (1 + \kappa_t)^{-1}$ . By the integrability condition on  $\alpha$  and  $\mu$ , we check that  $\pi \in \mathcal{A}(\mu, \pi)$ .  $\square$

$\square$

## 4.5 Black–Scholes idiosyncratic dynamics examples

In this section, we examine the case where the idiosyncratic risk process  $P$  of (4.4) is defined by:

$$dP_t = X_t[\sigma_t^0(\lambda_t dt + dW_t^0) + \sigma_t dW_t], \quad t \in [0, T], \quad (4.19)$$

where we have already accounted for the restriction on the drift  $b = \lambda\sigma^0$ , for some  $\mathbb{F}^0$ -progressively measurable process  $\lambda$ , due the NIP condition. Here,  $\sigma^0$  and  $\sigma$  are bounded positive  $\mathbb{F}^0$ -progressively measurable processes, and we assume in addition that

$$Z_T := \mathcal{E}\left(\int_0^T \lambda_t dW_t^0\right)_T \text{ satisfies } \mathbb{E}[Z_T] = 1 \text{ and } \mathbb{E}[Z_T^2] < \infty, \quad (4.20)$$

with  $\mathcal{E}(M)_t := e^{M_t - \frac{1}{2}\langle M \rangle_t}$  denoting the Doléans-Dade exponential for all semimartingale  $M$ .

Our main results in this section provides a class of equilibrium solutions to the MFG of cross-holding under logarithmic and power utility functions.

Observe that we have deliberately omitted here the possible dependence of the coefficients  $\lambda, \sigma, \sigma^0$  on the marginal distribution  $\mu_t$ . We emphasize that there would be no technical additional difficulties added by including the mean field interaction through such a dependence in the coefficients. However, our choice is motivated by the following two important reasons:

- first, the presentation is much simpler as we do not have to carry everywhere the dependence of the coefficients on the marginal law;
- second, and more importantly, the problem reduction of Proposition 4.7 shows that the cross-holding feature of our problem does not induce an additional mean field interaction at the level of the equilibrium dynamics. This is in contrast with the no-common noise setting of Djete & Touzi [?] where the cross-holding equilibrium strategy is the source of a mean field interaction in the equilibrium dynamics. Notice the major difference between our context and [?] that no portfolio constraints are considered in the present problem as we are focusing on the mean-variance tradeoff only.

### 4.5.1 The logarithmic utility case

Let us consider the special case

$$U(x) := \log(x), \quad x > 0, \quad \text{with convention } U = -\infty \text{ on } (-\infty, 0]. \quad (4.21)$$

**Proposition 4.9.** *Under (4.19), (4.20) and (4.21), let  $\kappa$  be an arbitrary  $\mathbb{F}^0$ -progressively measurable process with  $1 + \kappa_t > 0$ , and let  $X$  be an Itô process in  $\mathcal{S}^2$  defined by*

$$\frac{dX_t}{X_t} = \sigma_t^0(\lambda_t dt + dW_t^0) + \frac{\sigma_t}{1 + \kappa_t} dW_t, \quad \mathbb{P} - a.s. \quad (4.22)$$

denote  $\mu^\kappa := \mathcal{L}_{X|W^0}$  the (log-gaussian) conditional law of  $X$  given  $\mathbb{F}^{W^0}$ . Then

(i) *there exists a solution of the mean field game of cross-holding with equilibrium distribution  $\mu^\kappa$  if and only if  $\lambda = \sigma^0$ ;*

(ii) *and in this case any strategy  $\pi^\kappa \in \mathcal{A}(\mu, \pi^\kappa)$  satisfying*

$$\widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t^\kappa(X_t, \widehat{X}_t) \widehat{X}_t \right] = \kappa_t X_t, \quad \text{and} \quad \widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t^\kappa(\widehat{X}_t, X_t) \right] = \kappa_t, \quad \mathbb{P} - a.s. \quad (4.23)$$

*is a corresponding equilibrium cross-holding strategy.*

*Proof.* We use the equivalence in Proposition 4.7 and argue in three steps.

1. Given that the log-utility function is negative infinite on  $(-\infty, 0]$ , we may restrict the representative agent utility maximization step in the MFG definition to those strategies which induce strictly positive equity value at the final time  $T$ . Now, as  $\mathbb{E}[Z_T] = 1$  by (4.20), the equity process  $X$  is a local martingale under the equivalent probability measure  $\mathbb{Q} := Z_T \cdot \mathbb{P}$ . Moreover  $\mathbb{E}^{\mathbb{Q}}[\sup_{t \leq T} |X_t|] \leq \|Z\|_{\mathbb{L}^2} \|\sup_{t \leq T} |X_t|\|_{\mathbb{L}^2} < \infty$  by the second condition in (4.20) and the admissibility condition of any MFG solutions. It follows that the equity process  $X$  is a  $\mathbb{Q}$ -martingale, and consequently

$$X_t = \mathbb{E}^{\mathbb{Q}}[X_T | \mathcal{F}_t] \geq 0 \quad \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.} \quad (4.24)$$

If  $\mathbb{P}[\mathbb{E}^{\mathbb{Q}}[X_T | \mathcal{F}_t] = 0] > 0$ , we have  $X_T = 0$  on  $\{\mathbb{E}^{\mathbb{Q}}[X_T | \mathcal{F}_t] = 0\}$ . Since  $\sup_{t \leq T} |X_t| < \infty$  a.e.,  $U(X_T) < \infty$  a.e., this leads to

$$U(X_T) = U(X_T) \mathbf{1}_{\{\mathbb{E}^{\mathbb{Q}}[X_T | \mathcal{F}_t] = 0\}} + U(X_T) \mathbf{1}_{\Omega \setminus \{\mathbb{E}^{\mathbb{Q}}[X_T | \mathcal{F}_t] = 0\}} = -\infty.$$

Therefore, any MFG solutions must verify: for all  $t \in [0, T]$ ,  $X_t > 0$ ,  $\mathbb{P}$ -a.e. and we may restrict the expected utility maximization to the strategies verifying this condition.

2. Next, identifying the  $dW_t$ -coefficient of SDE (4.22) with that in the dynamics of the equilibrium equity process (4.14), we see that whenever  $(\mu, \pi)$  is an equilibrium solution of the MFG, we have

$$m_t^{\mu, \pi}(X_t^{\mu, \pi, \pi}) = \frac{1}{1 + \kappa_t}, \quad \mathbb{P}\text{-a.s. and then } m_t^{\mu, \pi}(x) = \frac{1}{1 + \kappa_t}, \quad \mathbb{P}\text{-a.s. for a.e. } x > 0.$$

This already shows that the second equation in (4.23) holds true. On the other hand, together with (4.24), this allows to rewrite the SDE of  $\mathbb{X}^\alpha$  as

$$\frac{d\mathbb{X}_t^\alpha}{\mathbb{X}_t^\alpha} = a_t(\mathbb{X}_t^\alpha)(\lambda_t dt + dW_t^0) + v_t dW_t, \quad \text{with } \alpha_t(\mathbb{X}_t^\alpha) = \mathbb{X}_t^\alpha a_t(\mathbb{X}_t^\alpha), \quad v_t := \frac{\sigma_t}{1 + \kappa_t}. \quad (4.25)$$

3. Next, given our admissibility conditions on  $\alpha$ , we compute for the present logarithmic utility context (4.21) that

$$\mathcal{J}_{\mu, \pi}(\alpha) = \log(X_0) + \mathbb{E} \left[ \int_0^T \left( a_t(\mathbb{X}_t^\alpha) \lambda_t - \frac{1}{2} a_t(\mathbb{X}_t^\alpha)^2 - \frac{1}{2} v_t^2 \right) dt \right],$$

which obviously achieves a maximum at  $\hat{\alpha}_t(\mathbb{X}_t^\alpha) = \lambda_t$ . Then,

$$\frac{d\mathbb{X}_t^{\hat{\alpha}}}{\mathbb{X}_t^{\hat{\alpha}}} = \lambda_t(\lambda_t dt + dW_t^0) + v_t dW_t,$$

which compared with (4.22) implies that  $\lambda = \sigma^0$  is necessary for the dynamics of  $\mathbb{X}$  to qualify for an equilibrium solution of the mean field game. Finally, identifying again the last equilibrium dynamics with (4.14), we see that:

$$X_t = \frac{X_t + \widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t(X_t, \widehat{X}_t) \widehat{X}_t \right]}{1 + \kappa_t},$$

□

### 4.5.2 The power utility case

Similar to the previous section, we consider the case of power utility for the representative agent

$$U(x) := \frac{x^p}{p}, \quad x > 0, \quad \text{with convention } U = -\infty \text{ on } (-\infty, 0]. \quad (4.26)$$

**Proposition 4.10.** *Under (4.19), (4.20) and (4.26), let  $\kappa$  be an arbitrary  $\mathbb{F}^0$ -progressively measurable process with  $1 + \kappa_t > 0$ , and*

$$\mathbb{E} \left[ e^{\frac{2-p}{1-p} |\xi^\kappa|} \right] < \infty \quad \text{with} \quad \xi^\kappa := p \int_0^T \left( \frac{\lambda_r^2}{1-p} - \frac{(1-p)\sigma_r^2}{(1+\kappa_r)^2} \right) dr. \quad (4.27)$$

Let  $X$  be an Itô process in  $\mathcal{S}^2$  defined by

$$\frac{dX_t}{X_t} = \sigma_t^0 (\lambda_t dt + dW_t^0) + \frac{\sigma_t}{1+\kappa_t} dW_t, \quad \mathbb{P} - a.s. \quad (4.28)$$

and denote  $\mu^\kappa := \mathcal{L}_{X|W^0}$  the (gaussian) conditional law of  $X$  given  $\mathbb{F}^{W^0}$ . Then

(i) *there exists a solution of the mean field game of cross-holding with equilibrium distribution  $\mu^\kappa$  if and only if  $(1-p)\sigma^0 = \lambda + Z^\kappa$ , where  $(Y^\kappa, Z^\kappa)$  is the unique solution of the quadratic backward SDE*

$$Y_t^\kappa = \frac{1}{2} \int_t^T \left( (Z_r^\kappa)^2 + \frac{p(\lambda_r + Z_r^\kappa)^2}{1-p} - \frac{p(1-p)\sigma_r^2}{(1+\kappa_r)^2} \right) dr - \int_t^T Z_r^\kappa dW_r^0, \quad t \leq T; \quad (4.29)$$

(ii) *In this case any strategy  $\pi^\kappa \in \mathcal{A}(\mu, \pi^\kappa)$  satisfying*

$$\widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t^\kappa(X_t, \widehat{X}_t) \widehat{X}_t \right] = \kappa_t X_t, \quad \text{and} \quad \widehat{\mathbb{E}}^{\mu^\kappa} \left[ \pi_t^\kappa(\widehat{X}_t, X_t) \right] = \kappa_t, \quad \mathbb{P} - a.s. \quad (4.30)$$

*is a corresponding equilibrium cross-holding strategy.*

*Proof.* We first follow exactly the same line of argument as in the proof of Proposition 4.9 to rewrite the reduced state process  $\mathbb{X}$  in the geometric form (4.25).

We next focus on the representative agent optimization step of the MFG problem. By Briand & Hu [7], the quadratic BSDE (4.29) has a solution  $(Y^\kappa, Z^\kappa)$  with

$$\mathbb{E} \left[ \int_0^T (Z_t^\kappa)^2 dt \right] < \infty. \quad (4.31)$$

The uniqueness of such a solution is a consequence of the following argument.

Denote by  $H$  the generator of the quadratic backward SDE (4.29), and set  $v := \frac{\sigma}{1+\kappa}$ . Then, for all  $\alpha$  satisfying the admissibility condition (4.16), it follows from Itô's formula

that the dynamics of process  $V_t^\alpha := U(\mathbb{X}_t^\alpha)e^{Y_t^\kappa}$  are given by:

$$\begin{aligned} dV_t^\alpha &= e^{Y_t^\kappa} \left[ (\mathbb{X}_t^\alpha)^{p-1} d\mathbb{X}_t^\alpha + \frac{1}{2}(p-1)(\mathbb{X}_t^\alpha)^{p-2} d\langle \mathbb{X}^\alpha \rangle_t \right] \\ &\quad + \frac{1}{p} (\mathbb{X}_t^\alpha)^p e^{Y_t^\kappa} [dY_t^\kappa + \frac{1}{2} d\langle Y^\alpha \rangle_t] + (\mathbb{X}_t^\alpha)^{p-1} e^{Y_t^\kappa} d\langle \mathbb{X}^\alpha, Y^\kappa \rangle_t \\ &= e^{Y_t^\kappa} \left[ (\mathbb{X}_t^\alpha)^p (a_t(\lambda_t dt + dW_t^0) + v_t dW_t) + \frac{1}{2}(p-1)(\mathbb{X}_t^\alpha)^p (a_t^2 + v_t^2) dt \right] \\ &\quad + \frac{1}{p} (\mathbb{X}_t^\alpha)^p e^{Y_t^\kappa} \left[ -H_t dt + Z_t^\kappa dW_t^0 + \frac{1}{2}(Z_t^\kappa)^2 dt \right] + (\mathbb{X}_t^\alpha)^p e^{Y_t^\kappa} a_t Z_t^\kappa dt \\ &= V_t^\alpha \left[ \left( -H_t + pa_t \lambda_t + \frac{1}{2} p(p-1)(a_t^2 + v_t^2) + \frac{1}{2} (Z_t^\kappa)^2 + pa_t Z_t^\kappa \right) dt + (pa_t + Z_t^\kappa) dW_t^0 \right]. \end{aligned}$$

By the definition of the generator  $H$ , we see that

- for any control process  $\alpha$ , the drift in the previous dynamics is nonpositive, so that  $V_t^\alpha$  is a nonnegative local supermartingale and therefore a supermartingale, implying that

$$U(\mathbb{X}_0)e^{Y_0^\kappa} = V_0 \geq \mathbb{E}[V_T^\alpha] = \mathbb{E}[U(\mathbb{X}_T^\alpha)], \text{ for all } \alpha;$$

- while for the optimal choice of  $\hat{a} := \frac{\lambda_t + Z_t^\kappa}{1-p}$ , the drift in the previous dynamics is zero, so that  $V_t^\alpha$  is a local martingale, and a martingale due to the integrability condition (4.31) on  $Z^\kappa$  and the expression of  $\hat{a}$ , implying that

$$V_0 = \mathbb{E}[V_T^{\hat{\alpha}}] = \mathbb{E}[U(\mathbb{X}_T^{\hat{\alpha}})], \text{ with } \hat{\alpha}_t(x) := \hat{a}_t x.$$

This shows on one hand that  $V_0 = \sup_\alpha J_{\mu, \pi}(\alpha)$ , and in fact the same argument started at the time origin  $t$  induces also a unique representation of  $V_t$  in terms of the similar control problem started from time  $t$ . Hence the uniqueness of  $Y^\kappa$ , and the uniqueness of  $Z_t^\kappa = \frac{1}{dt} \langle Y^\kappa, W^0 \rangle_t$ ,  $\mathbb{P}$ -a.s. follows immediately.

The last argument also provides as a by-product the optimality of  $\hat{a}$ , and we may now write the dynamics of the state under this optimal policy:

$$\frac{d\mathbb{X}_t^{\hat{\alpha}}}{\mathbb{X}_t^{\hat{\alpha}}} = \frac{\lambda_t + Z_t^\kappa}{1-p} (\lambda_t dt + dW_t^0) + \frac{\sigma_t}{1 + \kappa_t} dW_t,$$

which is consistent with (4.28) if and only if  $(1-p)\sigma^0 = \lambda + Z^\kappa$ . This completes the proof of (i), and (ii) follows exactly the same line of argument as in the proof of Proposition 4.9.  $\square$

**Remark 4.11.** (i) In light of the two previous examples, by using Proposition 4.8, we can see that a cross-holding strategy is given by

$$\pi_t(x, \hat{x}) = \frac{\kappa_t x}{\mathbb{E}^{\mu^\kappa}[\hat{X}_t]}, \text{ for all } x, \hat{x} \in \mathbb{R}.$$

(ii) However, we may search for solutions of the form  $\pi_t(x, \hat{x}) = \sum_{i=1}^N f_t^{(i)}(x) \hat{f}_t^{(i)}(\hat{x})$  for some  $N \in \mathbb{N} \cup \{\infty\}$ . Let us develop here the calculations for the case  $N = 2$ :

$$\pi_t(x, \hat{x}) = f_t(x) \hat{f}_t(\hat{x}) + g_t(x) \hat{g}_t(\hat{x}), \text{ for all } x, \hat{x} \in \mathbb{R}.$$

with the normalization

$$\mathbb{E}^{\mu^\kappa}[f_t(\widehat{X}_t)] = \mathbb{E}^{\mu^\kappa}[g_t(\widehat{X}_t)] = 1.$$

In this case, the second equation of (4.23) says that  $\hat{f}_t(\hat{x}) + \hat{g}_t(\hat{x}) = \kappa_t$ , which induces by injecting in the first equation that

$$(f_t - g_t)(x)\mathbb{E}^{\mu^\kappa}[\hat{f}_t(\widehat{X}_t)\widehat{X}_t] = \kappa_t(x - g(x)\mathbb{E}^{\mu^\kappa}[\widehat{X}_t]).$$

Consequently, we obtain a class of equilibrium cross-holding strategies parameterized by arbitrary random maps  $\hat{f}$  and  $g$  with  $\mathbb{E}^{\mu^\kappa}[g(\widehat{X}_t)] = 1$ :

$$\begin{aligned} \pi_t(x, \hat{x}) &= f_t(x)\hat{f}_t(\hat{x}) + g_t(x)[\kappa_t - \hat{f}_t(\hat{x})] \\ &= \kappa_t g_t(x) + (f_t - g_t)(x)\hat{f}_t(\hat{x}) \\ &= \kappa_t \left[ g_t(x) + (x - g_t(x)\mathbb{E}^{\mu^\kappa}[\widehat{X}_t]) \frac{\hat{f}_t(\hat{x})}{\mathbb{E}^{\mu^\kappa}[\hat{f}_t(\widehat{X}_t)\widehat{X}_t]} \right]. \end{aligned} \quad (4.32)$$

**Remark 4.12.** (Interpretation of the optimal strategy)

(i) The item (i) of the previous remark shows that an optimal strategy consists to buy or sell a number which is proportional to the ratio of the current position of the player by the mean/average of the positions of the population/other players. Thus, each player has an idea of his proximity to the optimal by comparing his position to the average position of the others. (iii) Notice that, by choosing  $\kappa = 0$ , we obtain an optimal strategy by taking the identical null control i.e.  $\pi_t(x, \hat{x}) = 0$ . In other words, we can reach the optimal by doing nothing i.e. no buy no sell. From the initial formulation of our MFG formulation or from the  $n$ -player game formulation, this conclusion seems hard to observe at first sight.





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**Titre :** Méthodes de contrôle optimal pour le risque systémique

**Mots clés :** Jeux différentiels stochastiques, Équilibres de Nash, Jeux à champ moyen, Bruit commun, Non-arbitrage.

**Résumé :** Cette thèse porte sur l'étude des équilibres de Nash du modèle de détentions mutuelles sous divers cadres. Ce modèle, présenté pour la première fois dans le papier de M-F. Djete et N. Touzi en 2020, vise à capturer l'interdépendance entre différents agents économiques, en prenant en compte à la fois les détentions mutuelles de parts entre les entités et leurs revenus potentiellement corrélés. La thèse est structurée en trois parties :

- Dans la première partie, nous avons analysé le jeu à population finie dans le cadre du critère d'utilité exponentielle. Dans les cas statiques et déterministes, sous une dynamique de type Bachelier gaussienne, nous avons obtenu une caractérisation complète des équilibres de Nash ainsi que leurs conditions d'existence.

- La deuxième partie est consacrée à l'analyse du jeu à champ moyen avec bruit commun (où les revenus sont corrélés), selon le critère moyenne-variance sur une période. La résolution de ce problème a révélé une structure associée à une condition de non-arbitrage. Dans ce contexte, nous avons déterminé une caractérisation explicite de cette condition ainsi qu'une caractérisation complète des équilibres de Nash.

- Dans la troisième partie, nous avons étendu le jeu à champ moyen avec bruit commun au cadre du temps continu. Dans ce contexte, une condition de non-arbitrage plus faible a été utilisée. Sa caractérisation permet de simplifier le problème d'optimisation de l'agent représentatif en le transformant en un problème classique d'optimisation de portefeuille avec des dotations aléatoires.

**Title :** Optimal control methods for systemic risk

**Keywords :** Stochastic differential games, Nash equilibria, Mean-field games, Common noise, No-arbitrage.

**Abstract :** This thesis focuses on the study of Nash equilibria in the cross-holding model under various frameworks. This model, first introduced in the paper of M-F. Djete and N. Touzi in 2020, aims to capture the interdependence among different economic agents by considering both the mutual holding of shares between entities and their potentially correlated incomes. The thesis is structured into three parts :

- In the first part, we analyzed the finite population game using the exponential utility criterion. In static and deterministic cases, under a Gaussian Bachelier type dynamic, we obtained a complete characterization of Nash equilibria as well as their conditions for existence.

- The second part focuses on the one-period mean-field game with common noise (where the revenues are correlated), using the mean-variance criterion. The problem formulation reveals a no-arbitrage condition. Within this framework, we explicitly characterized this condition, as well as the mean-field equilibria.

- In the third part, we extended the study of the mean-field game with common noise to a continuous-time setting. Here, we use a weaker notion of the no-arbitrage condition. Characterizing this condition simplifies the representative agent's optimization step, reducing it to the classical problem of optimal portfolio with random endowment.