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ECOLE DOCTORALE
CARNOT - PASTEUR

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE
UMR 5584 CNRS
UNIVERSITÉ DE BOURGOGNE

HABILITATION À DIRIGER DES RECHERCHES

Méthodes liées aux systèmes dynamiques

Spécialité: Mathématiques

présentée par

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“The only person you should try to be better than, is the person you were yesterday.” *Matty Mullins*

*à mes parents, à mes enfants et à Luc,
à mes amis et à tous ceux qui m'ont toujours soutenue tout au long de ma carrière
et à toutes les femmes victimes de discrimination*

Remerciements

Ceux qui me connaissent bien savent que remercier est une tâche facile pour moi. La vraie difficulté est cependant de n'oublier personne ! Je commencerai donc par demander pardon à ceux, qui par distraction de ma part, ne font pas partie de cette liste et qui méritaient cependant d'y être.

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Je ne pourrais pas oublier mes parents Santina et Casemiro qui n'ont aucune idée de ce qu'une thèse ou une HDR représentent, mais qui m'ont toujours donné l'essentiel : la liberté que j'aie le découvrir ! A eux s'ajoute ma famille brésilienne, qui même de l'autre côté de l'océan, est toujours très proche de moi et m'apporte le soleil quotidien. Mille mercis !

Finalement, je remercie mes enfants, Valentine et Enzo, qui représentent le parfum, la couleur et le rythme de ma vie ! Avec eux les systèmes sont sur-dynamiques et aucune variété n'est vide ! Mes pensées vont aussi à Luc, qui me supporte depuis une vingtaine d'années et qui a été capable d'essayer chacune de mes larmes durant quelques moments difficiles de ma carrière.



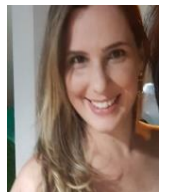
Un petit hommage...

* * *

"L'admission des femmes à l'égalité parfaite serait la marque la plus sûre de la civilisation, et elle doublerait les forces intellectuelles du genre humain." Stendhal



En dépit de mon engagement dans un domaine scientifique qui reste largement masculin, ce texte n'abordera pas les pressions qui m'ont souvent contrainte à adopter un rôle dit "de femme". Le plus important pour moi aujourd'hui est de rendre hommage à toutes celles qui m'ont inspirée et encouragée à poursuivre la lutte pour une reconnaissance en tant qu'être humain au sein de la société.



Résumé

Ce manuscrit résume mes travaux de recherche depuis ma thèse de doctorat (voir la liste des publications dans la bibliographie).

Ma recherche principale porte sur les systèmes dynamiques continus. Un objet combinatoire très important, appelé graphe de Lyapunov, étiqueté avec des invariants topologiques et dynamiques est utilisé pour prouver divers résultats de classification. Les graphes de Lyapunov ont été introduits pour la première fois par Franks et se sont révélés être un excellent moyen de conserver les informations dynamiques et topologiques du flot et de son espace de phase. Tout d'abord, nous nous intéressons à une classe particulière de graphes de Lyapunov, ceux qui sont liés aux flots de Morse. Dans ce cas, l'ensemble invariant isolé correspond à des singularités non dégénérées. Nous avons également considéré une autre classe spéciale de graphes de Lyapunov, à savoir ceux qui peuvent être liés à des flots de Morse-Smale non singuliers sur des variétés (avec ou sans bord), qui sont appelés graphes de Lyapunov de type Smale. Dans ce cas, les ensembles invariants isolés correspondent à des orbites fermées. Nous avons prouvé un résultat de continuation pour les graphes de Lyapunov dans les deux cas. Le nombre minimal de singularités non dégénérées qui peuvent être réalisées sur une variété quelconque avec bord non vide est déterminé uniquement en termes d'informations homologiques abstraites du bord. Un algorithme qui calcule un nombre minimal d'orbites périodiques qui peuvent être réalisées sur une variété est décrit en termes d'informations homologiques abstraites du bord. L'interaction entre la théorie des systèmes dynamiques topologique et la théorie des flux de réseaux a été explorée pour obtenir des résultats combinatoires pour la continuation et la morsification des graphes de Lyapunov. Cette interaction s'est avérée très fructueuse et a donné lieu à de nouvelles techniques. Les matrices de connexion et l'analyse des séquences spectrales d'un complexe de Morse filtrés sont utilisés pour étudier les résultats de continuation globale pour les flots sur les surfaces.

Nous considérons également des systèmes dynamiques discontinus, mais afin de présenter un document concis et cohérent, j'ai choisi de mentionner ces travaux dans l'annexe. Dans ce cas, nous nous intéressons à l'étude des équations différentielles dépendant de façon non régulière sur le temps d'intégration. Dans cette partie on s'intéresse à la existence de solutions périodiques de telles équations.

Abstract

This manuscript summarizes my research since my doctoral thesis (see the publication list in the Bibliography).

My main research considers continuous dynamical systems. A very important combinatorial object called a Lyapunov graph labelled with topological and dynamical invariants is used to prove various classification results. Lyapunov graphs were first introduced by Franks and have proven to be an excellent bookkeeping device of dynamical and topological information of a flow and its phase space. First, we focus in a special class of Lyapunov graphs, those which are related to Morse flows. In this case an isolating invariant set corresponds to nondegenerate singularities. Then we consider another special class of Lyapunov graphs, namely those that can be related to nonsingular Morse-Smale flows on manifolds (with or without boundary), which are called Lyapunov graphs of Smale type. In this case, an isolating invariant set corresponds to closed orbits. Continuation results for Lyapunov graphs are proven in both cases. The minimal number of nondegenerate singularities that can be realized on some manifold with non-empty boundary is determined solely in terms of abstract homological boundary information. An algorithm which computes a minimal number of periodic orbits that can be realized on a manifold is described in terms of abstract homological boundary information. The interplay between topological dynamical systems theory and network flow theory was explored to obtain combinatorial results for the continuation and morsification of Lyapunov graphs. This interaction has proven to be very fruitful, giving rise to novel techniques. Connection matrix theory and a spectral sequence analysis of a filtered Morse chain complex are used to study global continuation results for flows on surfaces.

We also consider discontinuous dynamical systems, but in order to present a concise and coherent document, I have chosen to mention these works in the Appendix. In this case, we are interested in studying differential equations depending nonsmoothly on the integration time. This part of the work deals with the existence of periodic solutions to such equations.

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How it started

"It always seems impossible until it is done." Nelson Mandela

As with every Brazilian, everything started with a soccer ball 😊. This ball was not like the ball used by “King Pelé” in the glory days of winning three of the five stars that made my beloved Brazil shine in the four corners of the globe!¹ It was a slightly deflated ball, as in Figure 1.



Figure 1: Slightly deflated ball.

Prof. Ketty de Rezende, assisted by Prof. Oziride Manzoli, enthusiastically showed me the beauty of **Morse theory**, which can be seen as the investigation of the relationship between functions defined on a manifold and the shape of the manifold itself. She emphasized that the main characteristic of Morse theory is that the information on the topology of the manifold is derived from the information about the critical points of real functions defined on the manifold.

We started our discussion using the slightly deflated ball above. In fact, this ball is a well known deformed version of the 2-sphere S^2 , and if we consider a Morse function $f : S^2 \rightarrow \mathbb{R}$ on it, we know that for each scalar value $a \in \mathbb{R}$, the level set $f^{-1}(a) = \{x \in S^2 \mid f(x) = a\}$ may have multiple connected components. More details about this theory will be given later but, for now, recall that a smooth function $f : S^2 \rightarrow \mathbb{R}$ is a **Morse function** if all of its critical points are nondegenerate, in the sense of having a nondegenerate Hessian matrix of second partial derivatives. For each critical point p , the **Morse index** of p is defined as the number of negative eigenvalues of f 's Hessian at p . One of the main ideas of Morse theory is to associate the topological changes of the sublevel sets, as a varies, with the critical points of f , i.e., the sublevel sets of a Morse function change precisely when

¹Forgive me, experts, but I couldn't resist making this tribute to him.

passing through a critical value. Moreover, this change is completely characterized topologically by the index of the critical point. The gradient of f is a vector field $\nabla f : S^2 \rightarrow TS^2$ consisting of vectors in the direction of the steepest ascent of f , and is formally dual to the differential df , where TS^2 is the tangent space of S^2 . The singularities of ∇f coincide with the critical points of f , and hence are isolated and finite. Hence, the gradient of a Morse function induces a vector field that describes the direction in which the function increases. Away from the critical points, the gradient induces a **smooth flow** ϕ on the manifold. The negative gradient, $-\nabla f$, indicates the orientability of the flow.

By choosing, as the Morse function, the height function f , for example, we can then define a **Morse graph** G as in Figure 2. This graph is obtained by continuously identifying every connected component in a level set to a single point. In other words, G is the image of a continuous surjective map $\pi : S^2 \rightarrow G$, where $\pi(x) = \pi(y)$ if, and only if, x and y come from the same connected component of a level set of f . The Morse graph is the quotient space S^2 / \sim endowed with the quotient topology. The orientation of its edges are induced by that of the flow ϕ . Its vertices correspond to the critical level sets $f^{-1}(c_i)$, where c_i is a critical point of f .

The intersection of edges at the vertices reflects the change in topology of the level set $f^{-1}(t)$ as t passes through the critical value c_i . For example, if c_i is a minimum or a maximum of f , a component is created or destroyed thereat; consequently, an edge is created or terminated at the corresponding vertex, which has degree² one. If c_i is a saddle point of index 1 and two components of $f^{-1}(t)$ merge at $t = c_i$ as t increases, the corresponding vertex of the Morse graph has degree 3 and looks like the letter Y ; the same occurs if a component of $f^{-1}(c_i)$ splits into two. In this case, the Morse graph looks like an inverted letter Y , i.e., “ λ ”. Figure 2 illustrates the effect of applying the height function to the points in the 2-sphere on the left-hand-side of the figure. The real line depicted in the middle of the figure shows the range of f with respect to this domain. Here, f has six singularities: two minima c_1 and c_2 (both of index 0), two saddles c_3 and c_4 (both of index 1) and two maxima c_5 and c_6 (both of index 2). For all $i = 1, \dots, 6$, we have $f(c_i) = i$.

If x is between 1 and 2, the regular level $f^{-1}(x)$ is homeomorphic to the circle S^1 . The same occurs for x between 3 and 4, and for x between 5 and 6. For x in $]2, 3[$ $f^{-1}(x)$ consists of two copies of S^1 . The same for x in $]4, 5[$. We choose to enrich the information contained in a Morse graph by adding the following labels to the vertices and edges of the graph:

- $h_j = 1$ for any vertex associated with a critical point of index j ;
- $\beta_0, \dots, \beta_{n-1}$ for any edge whose points correspond to a regular level, which represent the Betti numbers of the level set. Note that by Poincaré duality $\beta_j = \beta_{n-1-j}$ for all $j = 0, \dots, n-1$.

Observe that, since in dimension two we have only $\beta_0 = \beta_1 = 1$, which represents the number of connected components, we choose not to add this information to the graph in this particular case. But, in higher dimension each edge is labelled with a vector of β_j 's, $j = 1, \dots, n-2$. The label $h_j = 1$ represents the rank of the Conley homology index, that will be described later. The information of the gradient flow of the Morse function is not stored in the graph.

At this point I also learned that what was interesting to observe was that, if we start with the Morse graph G , we could easily reconstruct the deformed sphere S^2 by gluing **handles** as in Figure 3. To effect this reconstruction, we start at the vertices with zero outdegree, that is, vertices that have no outgoing edges therefrom. In our example, these would be the bottommost vertices of Figure 2. Starting from these vertices, as we traverse the graph in the direction opposite to the orientation of the edges, we attach a handle with an index given by the label on the vertex. Recall that in dimension 2 a handle of index 0 or 2 is a disc and a handle of index 1 is a rectangle.

²The degree of a vertex of a graph is the number of edges that are incident to the vertex.

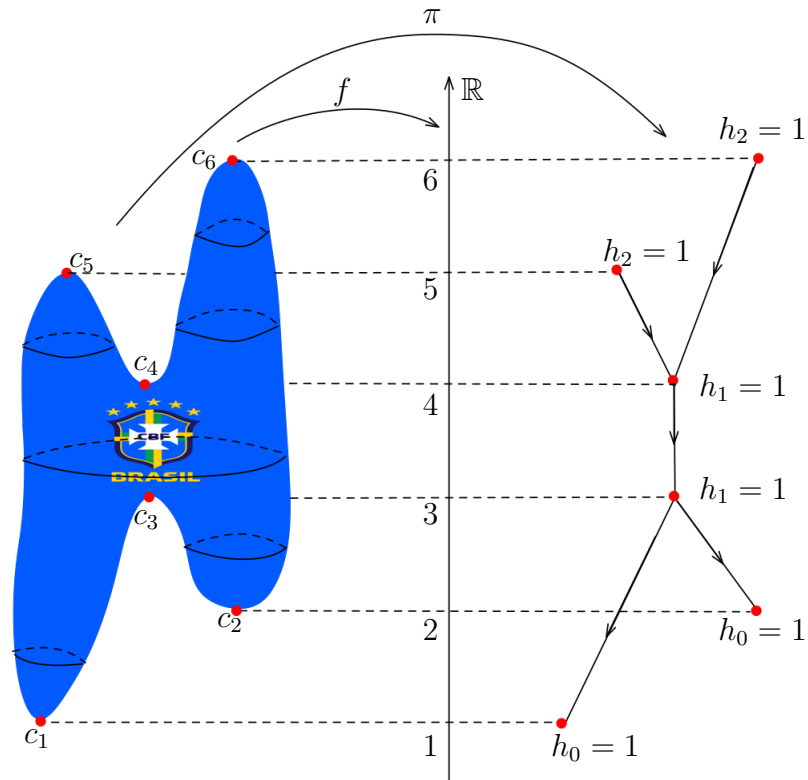


Figure 2: Morse function and Morse graph on the 2-sphere.

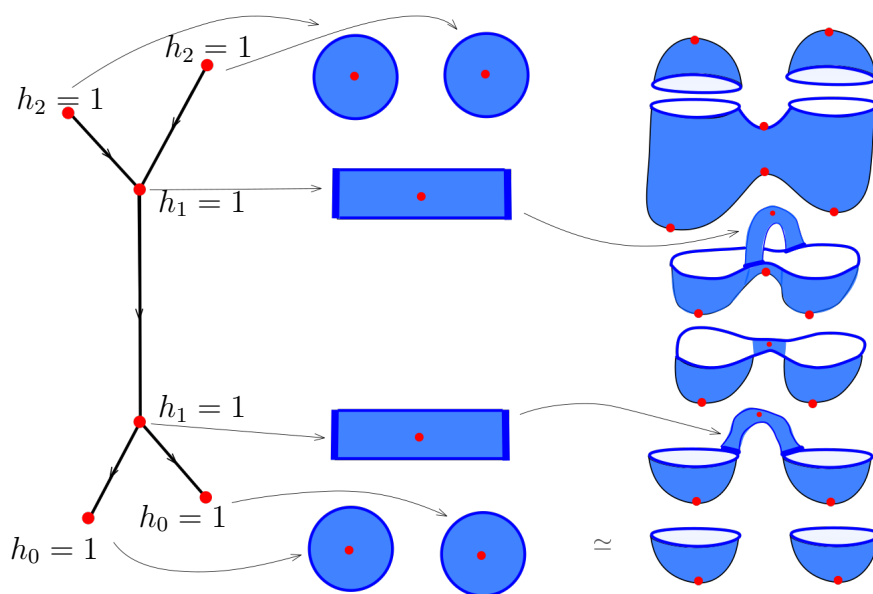


Figure 3: Reconstruction of the 2-sphere by attaching handles.

Further on the **Conley theory** was presented as a generalization of the Morse theory: instead of considering a Morse function, we would take a **Lyapunov function**, the nondegenerate critical points would be replaced by **isolated invariant sets** and in the same way as before we could define a **Lyapunov graph**. I didn't know it at the time, but the main object of my research was presented: a Lyapunov graph. Objects in this class, introduced by Franks, carry dynamical and topological information of the flow and of the manifold and that can be enriched with dynamical and topological invariants in order to provide more knowledge about the n -manifold M . Studying a continuous flow on a closed manifold M with respect to a Lyapunov function $f : M \rightarrow \mathbb{R}$ becomes similar to studying a gradient flow associated with a Morse function. A theorem of Conley guarantees the existence of a Lyapunov function $f : M \rightarrow \mathbb{R}$ of continuous flows $\phi_t : M \rightarrow M$ having a finite number of recurrent components.

Motivated by this similarity we refer to continuous flows admitting a Lyapunov function f as **gradient-like flows**, since f decreases along the orbits of the flow outside the chain recurrent set³ \mathcal{R} and it is constant on each connected component of \mathcal{R} .

As in the Morse case, we define an equivalence relation on M by $x \sim_f y$ if x and y belong to the same connected component of a level set of the Lyapunov function f . We call the quotient M / \sim_f a **Lyapunov graph**. Its **vertices** are the equivalence classes associated with the connected components of the chain recurrent set \mathcal{R} , hence vertices can be labelled with dynamical invariants. An **edge point** is the equivalence class of a regular level set, and points on the same edge correspond to homeomorphic level sets, hence edges can be labelled with topological invariants of the associated level set. The graph is oriented with the orientation inherited from the flow. More precisely, edges are labelled with the Betti numbers of the associated level set (as in the Morse case), while vertices are labelled in a more technical way, based on the Conley theory, which I briefly summarize some techniques now. More details will be given later.

Given R a component of the chain recurrent set, if f is a Lyapunov function associated with a flow and $c = f(R)$ then, for ε sufficiently small, the component of $f^{-1}[c - \varepsilon, c + \varepsilon]$ containing R is an **isolating block** for R . By taking $N^- = f^{-1}(c - \varepsilon)$ the outgoing boundary of the flow, $(N, N^-) = (f^{-1}[c - \varepsilon, c + \varepsilon], f^{-1}(c - \varepsilon))$ is an **index pair**⁴ for R . The relative homology of (N, N^-) is an invariant of R (it does not depend on the index pair chosen). We label the vertices of a Lyapunov graph with the non-zero **ranks of the homology groups** $H_j(N, N^-)$.

Using again a deformation of the soccer ball, Figure 4 shows the result of this procedure. In this example, all points of the equator of S^2 are singularities of $-\nabla f$ and it has the following neighborhood $N = f^{-1}([1.5, 2.5])$, which is a cylinder. The exit boundary of the flow $N^- = f^{-1}(1.5) \cup f^{-1}(2.5)$ consists of two circles S^1 . The homology $H_*(N, N^-)$ coincides with the the homology $H_*(N/N^-)$. Since the quotient space N/N^- is here a pinched torus, we have

$$h_0 = \text{rank}H_0(N/N^-) = 0, \quad h_1 = \text{rank}H_1(N/N^-) = 1, \quad h_2 = \text{rank}H_2(N/N^-) = 1.$$

and we set the label $h_1 = 1, h_2 = 1$. Remark that the ranks of these homology groups generalize the concept of Morse index because, if R is a Morse singularity of Morse index j , then for any index pair (N, N^-) of R we have $\text{rank} H_j(N, N^-) = 1$ and $\text{rank} H_k(N, N^-) = 0$ for all $k \neq j$.

³A point $x \in M$ is chain recurrent if given $\varepsilon > 0$ there exists an ε -chain from x to itself, i.e., there exists points $x = x_1, x_2, \dots, x_{n-1}, x_n = x$ and $t(i) \geq 1$ such that $d(\phi_{t(i)}(x_i), x_{i+1}) < \varepsilon \forall 1 \leq i < n$. A set of such points will be denoted by \mathcal{R} and is called a **chain recurrent set**.

⁴A pair of compact spaces (N, N^-) is an **index pair** for R if:

- $\text{cl}(N - N^-)$ is an isolating neighborhood for R ;
- N^- is positively invariant in N , i.e., if $x \in N^-$ and $\phi_{[0, T]}(x) \subset N$ then $\phi_{[0, T]}(x) \subset N^-$;
- N^- is an exit set for the flow, i.e., if $x \in N$ and $\phi_{[0, \infty)}(x) \not\subset N$ then there exists $T > 0$ such that $\phi_{[0, T]}(x) \subset N$ and $\phi_T(x) \in N^-$.

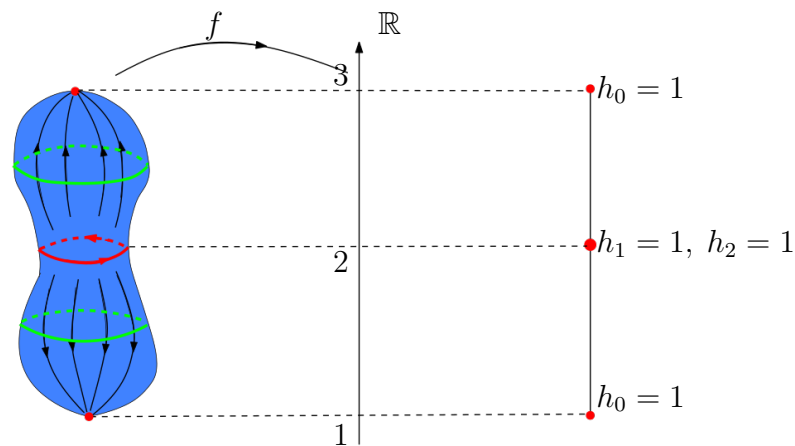


Figure 4: Gradient flow and Lyapunov graph on S^2 .

The discussion continued with a lot of other examples and information about problems of this kind and what type of questions could arise. Again, by using the soccer ball above (Figure 1), we put two flows on it as illustrated in Figure 5. These two flows are non equivalent, nevertheless we'd end up with the same Lyapunov graph as we can see in the middle picture of Figure 5. Hence, an interesting question was how to keep the dynamic information in the graph so as to distinguish these two cases.

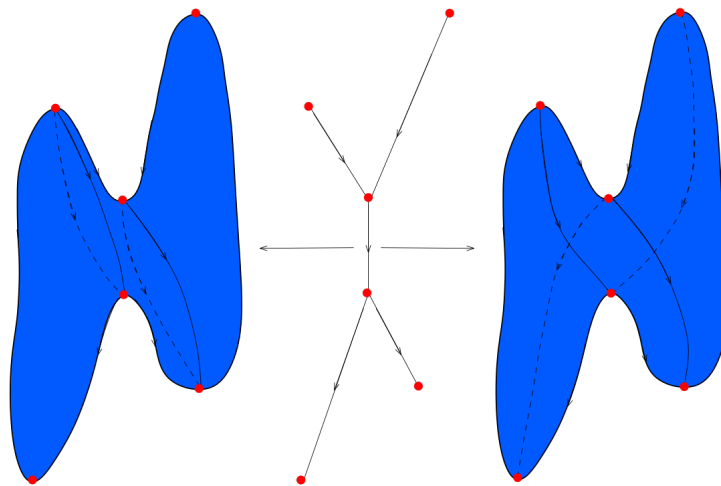


Figure 5: Non equivalent flows realizing the same graph.

I have to tell you that I've never seen a soccer ball in such an interesting way before! 😊

But even before embarking on dynamic detection, we wondered what kind of conditions should be imposed on any abstract Lyapunov graph so that it could be realized by a flow on a manifold, while retaining the information initially given. By “abstract”, we mean a graph without reference to a manifold. Paralleling what happens in the case where the graph is associated with a manifold, we define an **abstract Lyapunov graph** (without reference to a manifold) as a graph which has its vertices labelled with $(n + 1)$ nonnegative integers (h_0, \dots, h_n) and its edges labelled with n nonnegative integers $(1, \beta_1, \dots, \beta_{n-2}, 1)$ such that $\beta_j = \beta_{n-j}$.

Of course, our main goal was to know what abstract Lyapunov graphs can be realized by a flow on some manifold, but this question of realization was of course too ambitious at this moment, so we decided to address it in a sequence of steps.

Since the behavior of a graph coming from a Morse flow was easiest to understand, we asked ourselves when an abstract Lyapunov graph could be transformed to an abstract Lyapunov graph of Morse type. We could then try to produce **isolating blocks** in order to construct a manifold which could be represented by this initial graph.

By analyzing a long exact sequence of a pair (N, N^-) containing an isolated invariant set, we got a collection of inequalities, called **Poincaré–Hopf inequalities**, in order to ensure that given an isolating neighborhood containing a set with possibly complicated dynamical behavior, this neighborhood could be replaced by a neighborhood related to a Morse flow, i.e., where the singularities of the flow are nondegenerate.

We then began to tackle the problem of **continuation of graphs** and a rigorous algorithm was created. Each step of the algorithm imposed a series of restrictions described by linear equations. Thus, a **linear system** of equations was obtained whose solution was the number of different types of vertices that are needed to substitute in the continuation while keeping all the data on the incident edges unaltered.

Everything was great! The solutions to this system would represent the number of possibilities that an abstract Lyapunov graph could be continued to an abstract Lyapunov graph of Morse type. But...since there is always a “ BUT ” we couldn’t be sure that this system had a positive integer solution. Weeks of intensive work went by, and unfortunately we couldn’t figure out how to show the existence of a positive integer solution... 😞

Ketty and I were already starting to discuss the possibility of leaving this problem aside and tackling a new one when, to take my mind off things, I went to see the master’s defense in applied mathematics of one of my friends. Among the many things I heard that day, and that I had no idea what they meant, I saw concrete applications of **network flow theory**. It was magical! My friend presented lots of small linear systems, which under certain conditions had at least one positive integer solution. I realized then that this theory could solve my problem. At the end of the presentation I went to discuss with her advisor. I don’t think he understood what I told him, and I am sure that I wasn’t taken seriously...I really had the impression that he thought that topological dynamical systems had nothing to do with network flow theory and that he was wasting his time...At the end of this conversation, he gave me a book reference of about five hundred pages, and he reassured me that everything I needed to know about network flow theory was in this book. I was very excited to have found a “very small light at the end of the tunnel”, but at the same time very worried about how to learn this theory.

Coming back to Unicamp a week after, I described to Ketty what had happened. I also showed her a very small example that I tried to develop by reading the book cited above. I also added that I had no idea how to generalize this example. After some reflection, she said: "I know someone who can help us."

It was the beginning of my scientific story and also the start of a long collaboration with Prof. Margarida P. Mello. Of course, at the very beginning, things were not so easy. Even though we were both fluent in Portuguese, sometimes we had the impression that we weren’t speaking the same language! But Margarida is a very open minded person and with a lot of work and patience, we established a very nice cooperation. This interaction proved to be very fruitful and gave rise to new techniques.

Thanks to the techniques of network flow theory, we were able to solve our linear system (4.1)-(4.6) and establish conditions under which at least one positive integer solution was guaranteed. It turns out that these feasibility conditions were precisely the Poincaré–Hopf inequalities obtained

previously (3.1)-(3.6) involving the input data. In terms of graphs, we proved that every abstract Lyapunov graph that satisfies the Poincaré–Hopf inequalities at each vertex can be continued to an abstract Lyapunov graph of Morse type. Several other papers were published by combining techniques from both homological Conley index and network flow theories.

At the end of my PhD I had the opportunity to discuss with Prof. John Franks at the Northwestern University and also with Prof. Peter Zvengrowski at the University of Calgary. I also had the opportunity to present my work in their respective departments. Presenting my research work to Franks was a real challenge: a "pope" of dynamical systems and the one who defined the main object of my research. I also discussed with several members and students during the period I visited their universities.

During my first post-doctoral position at Unicamp, I got to know Prof. Gioia Vago and Prof. Alain Jacquemard. We also established a long scientific collaboration. The works carried out with Gioia had a greater topological emphasis. The works with Alain deal with **discontinuous dynamical systems**. Thanks to these collaborations, I was able to apply for a post-doctoral position at the *Institut de Mathématiques de Bourgogne (IMB)*. I was granted a fellowship from *Ministère de la Recherche Français* under supervision of Prof. Christian Bonatti. At the same time, I was appointed to a permanent position at Unicamp, following a public competition. Since the Head of the Department as well as the whole mathematics department realized the importance of such a postdoctoral position in France, they accepted to delay my commencement at Unicamp for one year, so that I could stay in France. With this postdoctoral position I was able to enlarge my scientific knowledge and network of international collaboration.

Coming back to Brazil, I resumed my position at Unicamp, with the usual activities thereof: teaching, research and administrative activities as well. In particular, I was officially co-advisor of Prof. Mariana da Silveira, Prof. Rogerio Casagrande and Prof. Hernan Montufar, and I worked informally with several of Ketty's Ph.D. students. I was part of a very dynamic research group at Unicamp. But as Helen Keller would say: "Life is either a daring adventure or nothing at all." In 2007, for personal reasons, I decided to leave Brazil definitely. So I resigned from my position at Unicamp and moved to Germany in the same year.

Without professional contacts, I undertook a real marathon of job search. In fifteen years, I have had ten different jobs, in three different countries, with different research focuses and with a heavy teaching load. For example, I worked in a Bioinformatics Department of the Institute of Functional Genomics at Regensburg University, in a Financial Mathematics group at the University of Salzburg, at the High School for the Gifted at Neckargemünd, in engineering schools or even in priority-education in middle, high schools in the Paris region, etc... I had to struggle to keep on doing scientific research, during evening, week-ends and holydays.

Why I am telling you this now? I am telling you this so that you can understand how happy I am to be able to come back to the scientific world and to have the opportunity of writing the very first pages of my "French Habilitation" (*Habilitation à diriger des recherches (HDR)*).



An inspirational walk into the past

"Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only."

Henri Poincaré

Being in France and interested in the interplay between topology and dynamics, it is imperative to mention the great French mathematician Henri Poincaré, known as the founder of the field of Algebraic Topology. His innovative and deep contributions to Topology and Dynamics also make



Henri Poincaré

him the founder of the field of Dynamical Systems. The famous Poincaré–Hopf Theorem is a beautiful icon of the interplay between topology and dynamics. This theorem states that under reasonable conditions the sum of the indices of a vector field equals the Euler characteristic of the manifold. Poincaré established a more combinatorial definition of homology via a simplicial decomposition of a manifold M , which was made possible by Whitehead's theorem that guarantees that every smooth manifold has a privileged class of triangulations. Thanks to

Whitehead's theorem, every smooth manifold has a unique piecewise linear compatible manifold structure. This structure makes it possible to calculate the homology of a manifold using polyhedral homology. This algebraization as a chain complex is absolutely brilliant. This algebra developed into what is now known as the **homology groups** of a manifold and permitted the establishment of a homotopical invariant of M . In this sense it captures "holes" that remain in M under deformation.



Marston Morse en 1965 (collection du MFO).

In the early 1920s, inspired by Poincaré, [Morse, 1925] related the topology of a closed manifold M of dimension n to its dynamical data by a collection of inequalities. Morse undertook this work by studying differentiable functions and their critical points. He took interests in how level sets change as one passes critical levels. To study M , instead of taking a triangulation of M he used a real valued function $f : M \rightarrow \mathbb{R}$. Using the critical points of f , one can construct the homology of a chain complex (Morse complex) that will turn out not to depend on f , but only on M . This invariant is called Morse homology and from it one can recover

the Euler characteristic. These **Morse inequalities** constitute a classical result which establishes relations between the number of nondegenerate critical points c_i of Morse index i of a smooth real valued function $f : M \rightarrow \mathbb{R}$ and the Betti numbers of M , $\gamma_i(M)$. The function $f : M \rightarrow \mathbb{R}$ is called a Morse function and its gradient determines a smooth flow which we refer to as a Morse flow. In this sense, the inequalities can be viewed as relations between the number c_i of singularities of index i of the Morse flow and the Betti numbers of the phase space M .

The Morse inequalities provide a relationship between the number of critical points of a Morse function f on a smooth n -dimensional manifold M (without boundary and having a finite number of

critical points) and the homology groups of the manifold, providing a powerful tool for understanding the topology.

The **homology group** $H_\lambda(M)$ is defined as the quotient group of λ -cycles mod λ -boundaries¹ and is finitely generated. Intuitively, it measures the number of λ -dimensional holes in M . A non-zero element in $H_\lambda(M)$ represents a λ -dimensional cycle that cannot be expressed as the boundary of a $(\lambda + 1)$ -dimensional chain, and thus corresponds to a λ -dimensional hole in M . Homology groups are abelian groups, but not free abelian groups in general. Like all finitely-generated abelian groups, each homology group is isomorphic to a product of cyclic groups

$$H_\lambda(M) \simeq \mathbb{Z}^{\gamma_\lambda(M)} \oplus \bigoplus_i (\mathbb{Z}/d_i\mathbb{Z})$$

for some integers γ_λ and $1 \leq d_1 \leq d_2 \leq \dots \leq d_m$, where each integer d_i is a divisor of its successor d_{i+1} . The rank γ_λ of the free component of $H_\lambda(M)$ is called the **λ -th Betti number of M** . The components $(\mathbb{Z}, d_i\mathbb{Z})$ are called **torsion subgroups**, denoted by $\text{Tor}(H_\lambda(M))$. One denotes by t_λ the minimal number of generators of $\text{Tor}(H_\lambda(M))$, which is called the **torsion rank** of $H_\lambda(M)$.

The Morse inequalities relate then the number c_λ of critical points of f with Morse index λ to these ranks, and have the form:

$$\begin{aligned} \gamma_\lambda + t_\lambda + t_{\lambda-1} &\leq c_\lambda, \quad \lambda = 0 \dots n; \\ \sum_{i=0}^{\lambda} (-1)^{\lambda-i} \gamma_i &\leq \sum_{i=0}^{\lambda} (-1)^{\lambda-i} c_i, \quad \lambda = 0 \dots n. \end{aligned}$$

For $\lambda = n$ the last Morse inequality is always an equality, so that

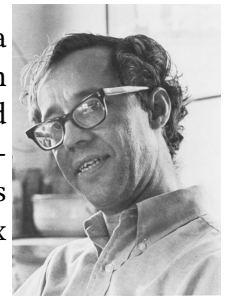
$$\sum_{i=0}^n (-1)^i c_i = \chi(M),$$

where $\chi(M)$ is the **Euler characteristic** of M .

The Morse inequalities also hold for Morse functions of a triple (M, N^-, N^+) , on replace the groups $H_\lambda(M)$ by the relative homology groups $H_\lambda(M, N^-)$.

According to the Morse inequalities, a manifold having “large” homology groups does not admit a Morse function with a small number of critical points.

In the early 70's, [Conley, 1978], Conley generalized these results to a theory with a more topological flavor and independent of the differentiable nature of the flow. In [Conley, 1978] the existence of a Lyapunov function associated to a flow on a manifold is proved. With respect to this function the flow maintains an underlying gradient-like behavior. However, in this setting, the dynamics is much richer and singularities can be exchanged for richer invariant sets (isolated invariant sets) for which the index introduced by Conley can be computed.



Conley in [Conley, 1978] proves that the following **generalized Morse inequalities** are valid, where γ_i is the i -th Betti number of M and h_i is the dimension of the i -th Conley homology index (see Section 2.3 for details about the Conley index theory).

¹Let X be a finite simplicial complex and let X_λ denote the set of oriented λ -dimensional simplices in X . A **λ -chain** over X is a function $\alpha : X_\lambda \rightarrow \mathbb{Z}$, such that $\alpha(-\sigma) = -\alpha(\sigma)$. The set of all λ -chains form an abelian group, called the **λ -th chain group** $C_\lambda(X)$. A **λ -cycle** is a λ -chain α such that $\partial_\lambda \alpha = 0$, where $\partial_\lambda : C_\lambda(X) \rightarrow C_{\lambda-1}(X)$ is the λ -th boundary homomorphism. A **λ -boundary** is a λ -chain α such that $\alpha = \partial_{\lambda+1} \beta$ for some $(\lambda + 1)$ -chain β . The **λ -th cycle group** is the group of λ -dimensional cycles in M , which are formal linear combinations of λ -dimensional simplices in M with integer coefficients. The **λ -th boundary group** is the subgroup of the λ -th cycle group consisting of the λ -dimensional chains that are boundaries of $(\lambda + 1)$ -dimensional chains.

$$\begin{aligned}
\gamma_n - \gamma_{n-1} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &= h_n - h_{n-1} + - \dots \pm h_2 \pm h_1 \pm h_0 & (n) \\
\gamma_{n-1} - \gamma_{n-2} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &\leq h_{n-1} - h_{n-2} + - \dots \pm h_2 \pm h_1 \pm h_0 & (n-1) \\
&\vdots & \vdots \\
\gamma_j - \gamma_{j-1} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &\leq h_j - h_{j-1} + - \dots \pm h_2 \pm h_1 \pm h_0 & (j) \\
\gamma_{j-1} - \gamma_{j-2} + - \dots \pm \gamma_2 \pm \gamma_1 \pm \gamma_0 &\leq h_{j-1} - h_{j-2} + - \dots \pm h_2 \pm h_1 \pm h_0 & (j-1) \\
&\vdots & \vdots \\
\gamma_2 - \gamma_1 + \gamma_0 &\leq h_2 - h_1 + h_0 & (2) \\
\gamma_1 - \gamma_0 &\leq h_1 - h_0 & (1) \\
\gamma_0 &\leq h_0 & (0)
\end{aligned} \tag{1.1}$$

This collection of inequalities is presented in [Morse, 1925] where one should read $h_i = c_i$ as the number of critical points of index i .

Morse-Conley index theory and Lyapunov graphs

"Mathematics are the result of mysterious powers which no one understands, and which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty's sake and pulls it down to earth." Marston Morse

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In this chapter we will have a closer look at a number of fundamental notions about Morse-Conley theory, handle theory and Lyapunov graphs necessary to understand next chapters. Some new notions concerning my research will also be mentioned. In what follows we consider the following notation:

- M is an **oriented manifold** of dimension n , with or without boundary;
- $\phi_t : M \rightarrow M$ is a **continuous flow** on M ;
- N is an **n -dimensional compact connected oriented submanifold** $N \subset M$ with boundary $\partial N = N^+ \cup N^-$, where N^+ and N^- represents the **entering** and **exit boundary** with respect to the flow ϕ_t .

2.1 Morse theory

Intuitively speaking, Morse theory will allow us to:

- decompose a (smooth) manifold M into elementary pieces (isolating blocks);
- conversely, take these elementary pieces and use them to construct manifolds.
- construct an invariant of the manifold called Morse homology.

The importance of a smooth manifold is that we can talk about smooth functions and not just homeomorphisms. We want to use functions to slice our manifold along level sets. For this purpose, one

should consider functions having only isolated critical points. A critical point of a differentiable function f is a point at which the derivative is zero.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. A critical point p of f is said to be **nondegenerate** or of **Morse type** if the Hessian

$$H_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right), \text{ for } i = 1, \dots, n, j = 1, \dots, n.$$

is nondegenerate (i.e., has non-zero determinant).

A function $f : M \rightarrow \mathbb{R}$ is said to be a **Morse function** if all its critical points are nondegenerate. The **Morse index** of a critical point p is defined as the number of negative eigenvalues of the Hessian matrix of f at p .

Example: Let T^2 be a 2-dimensional torus. Let us stand T^2 vertically and let $f : T^2 \rightarrow \mathbb{R}$ be the height function, that is for $p = (x, y, z) \in T^2$ one has $f(p) = z$, as described in the left side of Figure 2.1. In this case f is a Morse function. Nevertheless, by lying T^2 flat as on the right side of Figure 2.1, the height function f is not Morse.

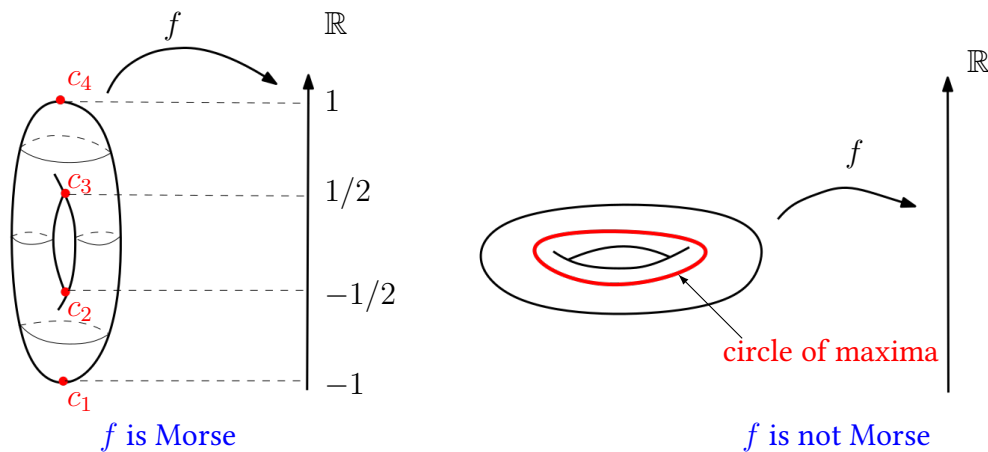


Figure 2.1: Morse and non Morse function on T^2 .

Let us consider the left side of Figure 2.1 and extract what this function tells us about the topology of the manifold M . There are again four critical points: a **minimum** c_1 , a **maximum** c_4 , and two **saddle** points c_2, c_3 . Equation (2.1) illustrates the sublevel sets:

$$f^{-1}((-\infty, z]) = \left\{ \begin{array}{lll} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \bullet \end{array} & \begin{array}{l} T^2 \\ T^2 \text{ minus a} \\ \text{disc at the top} \\ \text{a cylinder} \\ \text{with its} \\ \text{boundaries} \\ \text{pinched} \\ \text{together} \\ \text{a cylinder} \\ \text{a disc pinched} \\ \text{at the bound-} \\ \text{ary} \\ \text{a disc} \\ \{c_1\} \\ \emptyset \end{array} & \begin{array}{l} \text{if } z \geq 1 \\ \text{if } 1/2 < z < 1 \\ \text{if } z = 1/2 \\ \text{if } -1/2 < z < \\ 1/2 \\ \text{if } z = -1/2 \\ \text{if } -1 < z < \\ -1/2 \\ \text{if } z = -1 \\ \text{if } z < -1 \end{array} \end{array} \right. \quad (2.1)$$

The Morse Lemma states that all functions can be expressed in a standard form by just a simple change of coordinates. In fact, this lemma asserts that near a nondegenerate critical point of f , the function f can be approximated locally by a quadratic form whose behavior is determined by the eigenvalues of the Hessian matrix.

Note that the topology changes as we pass through a critical point (and conversely, how it doesn't change when you don't!). The function f provides a "movie" for the surface of interest by cutting it up in level sets $f^{-1}(z)$. The movie effectively decomposes the surface into a collection of elementary pieces (surfaces with boundary). Over each piece the height function has at most one critical point. The relationship between the handle that is added and the critical point is through the "index" of the critical point, determined by its quadratic form. We'll explain the handle theory in more detail in the next section. The critical points on the torus T^2 looks (more or less) locally like:

- $f_{c_1}(x, y) = x^2 + y^2$ at the critical point corresponding to -1 .
- $f_{c_2}(x, y) = x^2 - y^2$ at the critical point corresponding to $-1/2$
- $f_{c_3}(x, y) = y^2 - x^2$ at the critical point corresponding to $1/2$
- $f_{c_4}(x, y) = -x^2 - y^2$ at the critical point corresponding to 1 .

For these local forms, the origin is their only critical point and their Hessians are:

$$H_0(f_{c_1}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad H_0(f_{c_2}) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

$$H_0(f_{c_3}) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, H_0(f_{c_4}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

They correspond to a minimum, two saddles, and a maximum, respectively.

From the Morse Lemma, one obtains that a nondegenerate critical point is isolated and that a Morse function defined on a compact manifold admits only finitely many critical points.

Suppose that M is a closed Riemannian manifold equipped with a Riemannian metric g and that $f : M \rightarrow \mathbb{R}$ is a smooth function. The **gradient vector field** $\nabla f : M \rightarrow TM$ is defined as the unique vector field satisfying the identity $g(\nabla f, \bullet) = df(\bullet)$. The map ϕ associated to the vector field $-\nabla f$ is called the **negative gradient flow** associated to f , and the map ϕ_t solves the ordinary differential equation:

$$\frac{d}{dt}\phi_t(p) = -\nabla f(\phi_t(p)).$$

In other words, ∇f is the dual of the differential df with respect to the Riemannian structure.

This map is smooth because the solution of an ordinary differential equation with smooth coefficients is smooth. The points q such that $\phi_t(p) = q$, for some $t \in \mathbb{R}$, are called the flowline associated to p ; they are the points you can reach by pushing p forwards or backwards in time in the t variable. A basic result of Morse theory states that as you push a point p along its flowline, you end up in a critical point.

A result of Morse theory establishes that almost all functions are Morse functions.

What is important to keep in mind is that given a Morse function $f : M \rightarrow \mathbb{R}$, the negative gradient flow ϕ associated to the gradient vector field, $-\nabla f$, has important properties, such as:

- the singularities of ϕ are exactly the critical points of f ;
- the Morse function f decreases along nonsingular orbits of ϕ ;
- ϕ does not admit periodic orbits except for singularities.
- the Morse index of a critical point p coincides with the dimension of the unstable manifold of p with respect to the negative gradient flow of a Morse function.

2.2 Handle theory

We present a Morse theoretic approach, used in [Bertolim et al., 2003a], for analyzing the changes to a manifold as handles are attached. This topological handle theory approach is necessary for the description of the continuation algorithm presented in Section 4.1.1.

By definition, an n -dimensional **handle** of index ℓ is a product of disks centered at the origin $\mathbf{D}^\ell \times \mathbf{D}^{n-\ell}$. The **core** is $\mathbf{D}^\ell \times 0$ and the **cocore** is $0 \times \mathbf{D}^{n-\ell}$. We define a flow on the handle by considering a nondegenerate index ℓ singularity at the origin and identifying the core and cocore with its unstable and stable manifold respectively. The **attaching region** of the handle is $\mathbf{S}^{\ell-1} \times \mathbf{D}^{n-\ell}$ and corresponds to the part of the boundary of the handle through which the flow exits. Similarly, the **belt region** $\mathbf{D}^\ell \times \mathbf{S}^{n-\ell-1}$ corresponds to the part of the boundary of the handle through which the flow enters. Figure 2.2 illustrates handles in dimension 3.

Morse-Smale flows on a smooth n -dimensional manifold M were considered together with handle decompositions associated with a Lyapunov function in [Cruz and de Rezende, 1999]. Thus, after the attachment to a collar of a closed $(n-1)$ -manifold N^- of an n -handle H of index j , corresponding

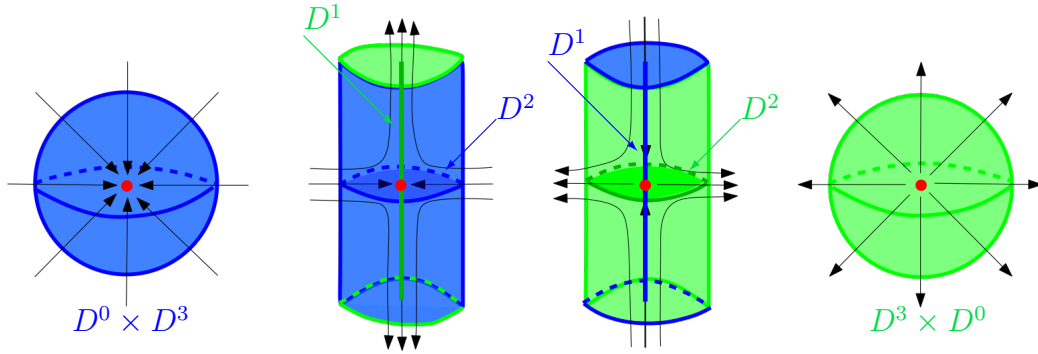


Figure 2.2: Handles in dimension $n = 3$ containing singularities of indices 0, 1, 2 and 3 from left to right.

to a nondegenerate singularity of Morse index j , one can consider the effect on the Betti numbers of the newly formed boundary $N^+ = \partial(H \cup N^- \times I)$.

In other words, attaching an n -handle of index j , for $j = 1 \dots n - 1$, to a collar of N^- can produce one of the following effects if

- (H1) the j -th Betti number of the boundary N^+ is the corresponding number of N^- increased by 1 (or by 2, if $n = 2j + 1$), and the handle will be said of type j -d (d standing for disconnecting);
- (H2) the $(j - 1)$ -th Betti number of the boundary N^+ is the corresponding number of N^- decreased by 1 (or by 2, if $n = 2j + 1$), and the handle will be said of type $(j - 1)$ -c (c standing for connecting);
- (H3) if $n = 4k$ and $j = 2k$ all the Betti numbers of N^+ are the same as those of N^- , and the handle will be said of type β -i (i standing for invariant).

In [Bertolim et al., 2007] we generalize results in [Cruz and de Rezende, 1999] by completely describing the effect that attaching a handle has on the Betti numbers of the boundary, when the homology coefficients are chosen among the most standard ones, that is, \mathbb{Z} , \mathbb{Q} , \mathbb{R} or $\mathbb{Z}/p\mathbb{Z}$, with p prime. Since such a description is technical, we state it in a simpler way and refer to [Bertolim et al., 2007] for the detailed version.

Theorem 1. *Let N be an n -dimensional manifold with compact orientable boundary $\partial N = N^+ \sqcup N^-$, endowed with a Morse flow entering through the regular level set N^+ , exiting through the regular level set N^- and containing a unique singularity of index j inside N . Let the homology coefficients be chosen in \mathbb{Z} , \mathbb{Q} , \mathbb{R} or $\mathbb{Z}/p\mathbb{Z}$, with p prime. Then the Betti numbers of N^+ and N^- are the same except for both β_j and β_{n-1-j} or both β_{j-1} and β_{n-j} , for which the behavior is classified.*

Roughly speaking, up to few exceptions, attaching a handle of index j can either increase by 1 the j -th Betti number and its dual (i.e., $\beta_j(N^+) = \beta_j(N^-) + 1$ and $\beta_{n-1-j}(N^+) = \beta_{n-1-j}(N^-) + 1$) or can decrease by 1 the $(j - 1)$ -st Betti number of N^+ and its dual. The most significant exception is given in the case $n = 4i$ by $j = 2i$ for which there is also the possibility for all the Betti numbers to stay unchanged.

Our generalization implies that all the results and machinery using Conley Index Theory and continuation of Lyapunov graphs developed before this generalisation in [Bertolim et al., 2003a, 2005b, 2006b,a] are still true, independently of the homology coefficients, provided that they be chosen in \mathbb{Z} , \mathbb{Q} , \mathbb{R} or $\mathbb{Z}/p\mathbb{Z}$, with p prime.

2.3 Conley theory

The dynamical behavior of continuous flows $\phi_t : M \rightarrow M$ on their chain recurrent set \mathcal{R} can be very rich. Nevertheless, studying a continuous flow on a closed manifold M with respect to a Lyapunov function $f : M \rightarrow \mathbb{R}$ becomes similar to studying a gradient flow (for which the chain recurrent components are singletons) associated with a Morse function as described in Section 2.1.

A point $x \in M$ is called **chain recurrent** if for every $\varepsilon > 0$ there is an ε -chain from x to itself, i.e., there exists points $x = x_1, x_2, \dots, x_{n-1}, x_n = x$ and reals $t_i \geq 1$ such that

$$d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon \quad \forall 1 \leq i < n.$$

Figure 2.3 illustrates an ε -chain.

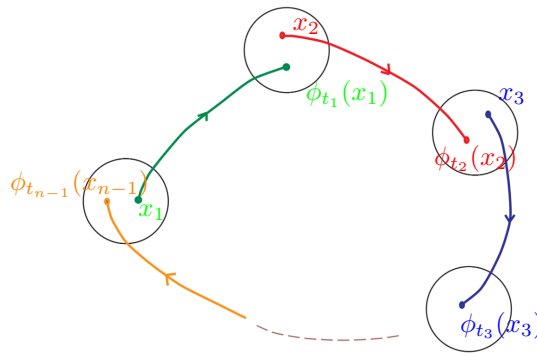


Figure 2.3: ε -Chain.

The set \mathcal{R} of chain recurrent points is called **the chain recurrent set** of ϕ_t . We work under the hypothesis that \mathcal{R} is the finite union of such components.

A set $S \subset M$ is an **invariant set** of a flow ϕ_t if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. The set \mathcal{R} , of the flow ϕ_t , is an example of an invariant set. A compact set $N \subset M$ is an **isolating neighborhood** if $\text{inv}(N, \phi) = \{x \in N : \phi_t(x) \subset N, \forall t \in \mathbb{R}\} \subset \text{int } N$.

Given a smooth flow $\phi_t : M \rightarrow M$, a Conley theorem asserts that there exists a smooth function $f : M \rightarrow \mathbb{R}$ associated to this flow with the property that it decreases along orbits outside the chain recurrent set \mathcal{R} , that is, if $x \notin \mathcal{R}$ then $f(\phi_t(x)) < f(\phi_s(x))$ given that $t > s$ and is constant on connected components of \mathcal{R} . This function is defined as a **Lyapunov function**. We refer to ϕ_t as a **gradient-like** flow with respect to f because of the properties above.

The homotopy Conley index is the homotopy type of a space defined from a topological pair determined by the flow, which is called index pair. A pair of compact spaces (N, N^-) is an **index pair** for S if:

- $\text{cl}(N - N^-)$ is an isolating neighborhood for S ;
- N^- is positively invariant in N , i.e., if $x \in N^-$ and $\phi_{[0,T]}(x) \subset N$ then $\phi_{[0,T]}(x) \subset N^-$;

- N^- is an exit set for the flow, i.e., if $x \in N$ and $\phi_{[0,\infty)}(x) \not\subset N$ then there exists $T > 0$ such that $\phi_{[0,T]}(x) \subset N$ and $\phi_T(x) \in N^-$.

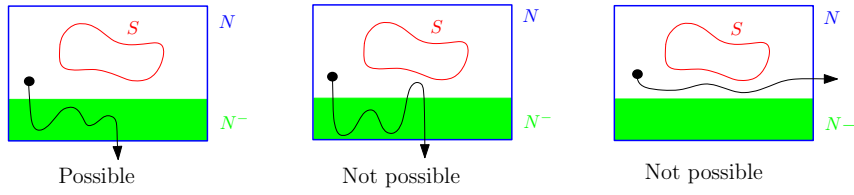


Figure 2.4 illustrates examples of an index pair for an attracting singularity (a sink), a saddle singularity, a repelling singularity (a source), an attracting periodic orbit, and a repelling periodic orbit.

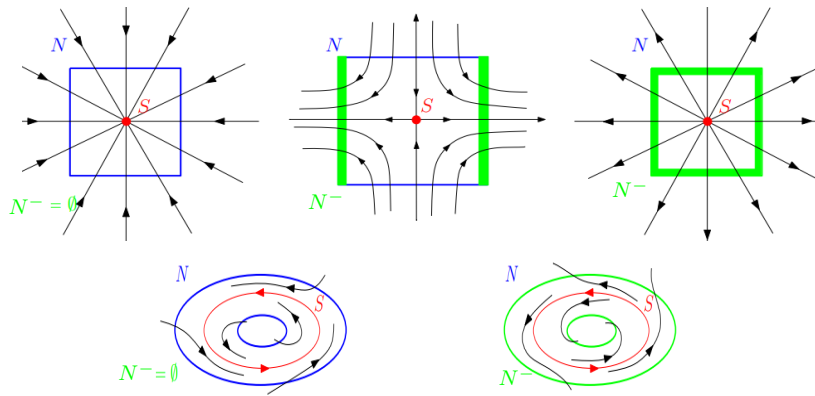


Figure 2.4: Examples of an index pair.

A special index pair which we call **isolating block** can be determined via the Lyapunov function. A compact set N is an **isolating block** if $N^- = \{x \in N : \phi_{[0,t)}(x) \not\subset N, \forall t > 0\}$ is closed and $\text{inv}(N, \phi) \subset \text{int } N$. In other words, the flow exits N^- immediately in the positive flow direction. For example, let S be a saddle singularity as in Figure 2.5. The left side of Figure 2.5 depicts an index pair and on the right side an isolating block for S .

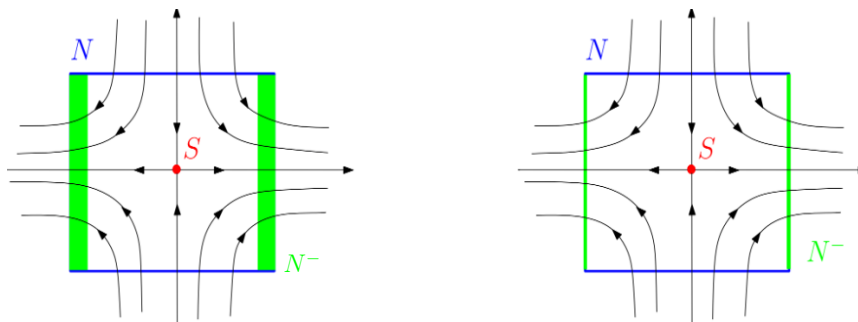


Figure 2.5: On the left, an index pair. On the right, an isolating block.

An invariant set S is called an **isolated invariant set** if it is a maximal invariant set in some isolating neighborhood N , that is, $S = \text{inv}(N, \phi)$. For example, consider a saddle point S at the origin of the plane as in the left side of Figure 2.6. The square $N = [-1, 1] \times [-1, 1]$ is an isolating neighborhood for S and S is an isolated invariant set. Observe that there exist sets which are invariant under the flow but are not isolated. For example, consider the flow on \mathbb{R}^2 associated to $\dot{x}_1 = x_2, \dot{x}_2 = -x_1$ as in the right side of Figure 2.6. The origin $(0, 0)$ is a rest point and $S = \{(0, 0)\}$ is an invariant set, however, S is not isolated.

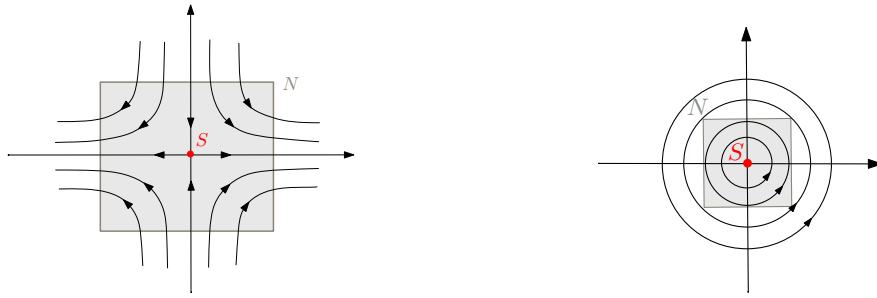


Figure 2.6: On the left side S is an isolated invariant set and on the right side not.

If f is a Lyapunov function associated with a flow and $c = f(R)$ then, for ε sufficiently small, the component of $f^{-1}[c - \varepsilon, c + \varepsilon]$ containing R is an isolating block for R . Take $(N, N^-) = (f^{-1}[c - \varepsilon, c + \varepsilon], f^{-1}(c - \varepsilon))$ as an index pair for R . The **homotopy Conley index** is defined as the homotopy type of N/N^- and it is denoted by $\mathcal{H}(R)$.

Now, if N^- is a neighborhood deformation retract in N , then the homology of the index space N/N^- agrees with the homology of the pair (N, N^-) . The j -th homology of N/N^- is the **j -th Conley homology index** and is denoted by $CH_j(R)$. We denote the dimension of $CH_j(R)$ by $h_j = \dim CH_j(R)$ (see [Conley, 1978] for further details). In Figure 2.7 we can find two examples of the computation of the homotopy Conley index. In both cases we have a wedge of 1-spheres. Hence, $h_1 = 2$.

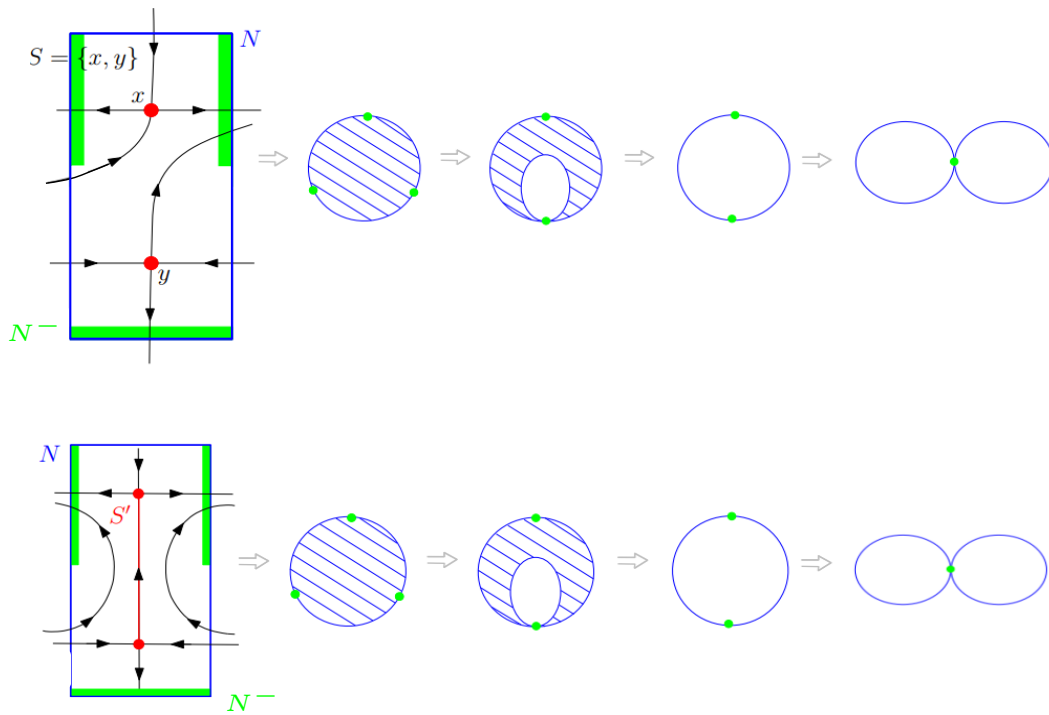


Figure 2.7: Examples of the computation of the homotopy Conley index.

2.4 Lyapunov graphs

The qualitative description of the dynamical behavior of a continuous flow is generally divided in two parts (see [Franks, 1979, 1982]):

- the “gradient-like” behavior, which is reflected in the existence of a Lyapunov function,
- and the “chain recurrent” behavior.

In order to study a class of dynamical systems it is important to understand how these two aspects of the dynamical behavior interact. On the one hand, by taking a closer look one should look for topological-dynamical invariants for the chain recurrent dynamics. On the other hand, understanding global connections and their behavior under continuation is a goal in this type of investigation.



Labelled Lyapunov graphs are combinatorial objects introduced by Franks in [Franks, 1985] in order to enlighten the interactions between the local and global features of a same dynamical system. In particular, the choice of the labels of a Lyapunov graph depends on the nature of the interaction we want to study. As for the graph itself, let us define an equivalence relation on M by $x \sim_f y$ if x and y belong to the same connected component of a level set of the Lyapunov function f . We call the quotient M / \sim_f a **Lyapunov graph**. Its **vertices** are the equivalence classes associated with the connected components¹ of the chain recurrent set \mathcal{R} , hence, vertices can be labelled with dynamical invariants. An **edge point** is the equivalence class of a regular level set, and points on the same edge correspond to homeomorphic level sets, hence edges can be labelled with topological invariants of the associated level set. The graph is oriented with the orientation inherited from the flow. Figure 2.8 illustrates the procedure of obtention of a Lyapunov graph.

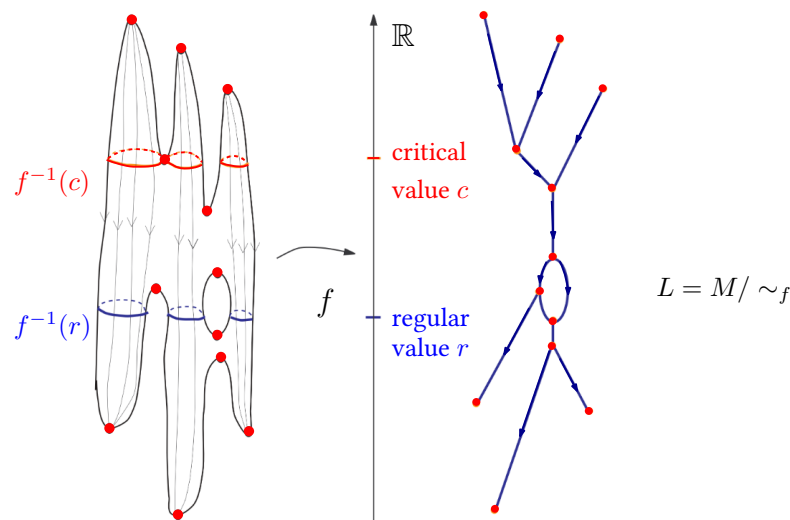


Figure 2.8: A Lyapunov graph.

Of course, one can take the opposite point of view by defining a Lyapunov graph in an abstract way. An **abstract Lyapunov graph of dimension n** is a finite connected oriented graph, without oriented cycles, with labelled vertices and edges. Each vertex can be labelled with a chain recurrent flow on a compact space of dimension n or with some dynamical invariants of such a flow. Each edge can be labelled with topological invariants of a closed manifold of dimension $(n - 1)$.

Note that studying the interactions between the properties recorded in a Lyapunov graph and its labels means knowing when an abstract Lyapunov graph can be realized by a flow on a manifold. In

¹Note that, in the case of a discrete-time dynamics, we cannot associate the vertices with the connected components, but only with the equivalence classes of the chain recurrent set \mathcal{R} .

general, some restrictions appear. For instance, the topology of the underlying manifold constrains the topology of any associated Lyapunov graph. In [Cruz and de Rezende, 1998] it is shown that the cycle rank² κ of a Lyapunov graph is a lower bound of the **Cornea genus**³ $g(M)$ of the manifold M , which is always less than or equal to the first Betti number of M , $\kappa \leq g(M) \leq \gamma_1(M)$. This generalizes a theorem of Franks [Franks, 1985] which asserts that if M is simply connected then $\kappa = 0$ (i.e., the Lyapunov graph is a tree).

In our work, Lyapunov graphs keep track of some local and global homological information of the flow and the phase space M , where M is an oriented manifold of dimension n , with or without boundary. More precisely, edges are labelled by the Betti numbers of the associated level set, while vertices are labelled in a more technical way based on the Conley index theory, as described in Section 2.3. Recall that each vertex of the graph represents a component R of the chain recurrent set \mathcal{R} of the flow ϕ_t and that we work under the hypothesis that \mathcal{R} is the finite union of such components. A vertex of a Lyapunov graph can therefore be labelled with the dimensions of the Conley homology indices of the component of the chain recurrent set associated with the vertex. As a consequence, an abstract Lyapunov graph has its vertices labelled with $(n + 1)$ nonnegative integers (h_0, \dots, h_n) and its edges labelled with n nonnegative integers $(1, \beta_1, \dots, \beta_{n-2}, 1)$ such that $\beta_j = \beta_{n-j}$.

Of course, our main goal was to know what abstract Lyapunov graphs can be realized by a flow on some manifold. We answered this natural question and other related ones through various steps. Some of them, will be described in the next chapters.

Using the three possible topological effects described in Section 2.1, we determined in [Bertolim et al., 2003a] necessary and sufficient conditions for abstract Lyapunov graph morsification. These conditions will form a collection of inequalities called the Poincaré–Hopf inequalities which we discuss in the next section.

Observe that the three handle effects described above, (H1)-(H3), can be viewed in terms of graphs in the following way. A handle containing a singularity of index ℓ corresponds to a vertex on the graph L labeled with $h_\ell = 1$, containing one incoming edge L^+ and one outgoing edge L^- , which can produce the three possible algebraic effects:

- (G1) a vertex $h_\ell = 1$ on L is called ℓ -disconnecting, in short, ℓ -d, if the $\beta_\ell(L^+)$ label on the incoming edge L^+ is the corresponding number of the outgoing edge L^- increased by 1 (or by 2, if $n = 2j + 1$).
- (G2) a vertex $h_\ell = 1$ on L is called $(\ell - 1)$ -connecting, in short, $(\ell - 1)$ -c, if the $\beta_{\ell-1}(L^+)$ label on the incoming edge L^+ is the corresponding number of the outgoing edge L^- decreased by 1 (or by 2, if $n = 2j + 1$).
- (G3) In the case $n = 2i = 0 \pmod{4}$, a vertex on L labeled with $h_i = 1$ is called β -invariant, in short, β -i, if all $\beta_\ell(L^+)$ label, $\ell = 1, \dots, n - 1$, on the incoming edge L^+ are the same as those of the outgoing edge L^- .

See the corresponding graphs in Figure 2.9.

Using these three algebraic effects we define an **abstract Lyapunov graph of Morse type** as an abstract Lyapunov graph that satisfies the following:

1. every vertex is labelled with $h_j = 1$ for some $j = 0, \dots, n$ and the cycle number of each vertex equal to zero.

²The **cycle rank** of a graph is the maximum number of edges that can be removed without disconnecting the graph.

³This genus of a closed orientable manifold is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect the smooth closed manifold M . See [Cornea, 1989].

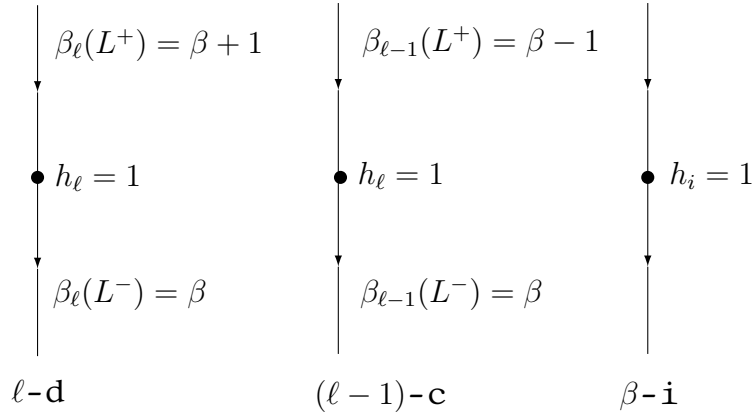


Figure 2.9: The three possible algebraic effects.

2. the number of incoming edges, e^+ , and the number of outgoing edges, e^- , of a vertex labelled with $h_j = 1$, must satisfy:
 - (a) $j \neq 0, 1, n-1, n$ then $e^+ = 1$ and $e^- = 1$;
 - (b) if $j = 1$ then $e^+ = 1$ and $e^- = 1$ or $e^- = 2$; if $j = n-1$ then $e^- = 1$ and $e^+ = 1$ or $e^+ = 2$;
 - (c) if $j = 0$ then $e^- = 0$ and $e^+ = 1$; if $j = n$ then $e^+ = 0$ and $e^- = 1$.
3. every vertex labelled with $h_\ell = 1$ must be of type ℓ -d or $(\ell-1)$ -c. Furthermore if $n = 2i \equiv 0 \pmod{4}$ and $h_i = 1$ then v may be labelled with β -i.

Figure 2.10 illustrates the local conditions of an abstract Lyapunov graph of Morse type.

We describe now a special class of Lyapunov graphs, namely those that can be related to nonsingular Morse-Smale flows on manifolds (with or without boundary). In this case the isolating invariant sets corresponds to the closed orbits. Therefore, given a nonsingular Morse-Smale flow on an isolating neighborhood N , one can associate with it a Lyapunov graph in the same way as described previously, which will be called a **Lyapunov graph of Smale type**. A result of Franks in [Franks, 1982] shows that one can think of a hyperbolic periodic orbit of index k , A_k , as a suitable joining of two hyperbolic singularities p and q of adjacent indices k and $k+1$ respectively, in such a way that a neighborhood of A_k can be built by a suitable attachment of handles of adjacent indices k and $k+1$ respectively. Hence, following Frank's idea, we define an **abstract Lyapunov graph of Smale type** as an abstract Lyapunov graph of Morse type such that each vertex labelled with a hyperbolic singularity of index k can be joined with a vertex labelled with a hyperbolic singularity of adjacent index $k+1$. This means that the collection of labels (h_0, \dots, h_n) on the vertices of the Lyapunov graph can be decomposed in an union $(\sum_{j=0}^n h_j)/2$ of disjoint couples of the form $A_\ell = (h_\ell, h_{\ell+1})$ with $\ell \in \{0, \dots, n\}$. Results concerning this type of graph will be addressed in Section 5.2 and can be found in [Bertolim et al., 2023a].

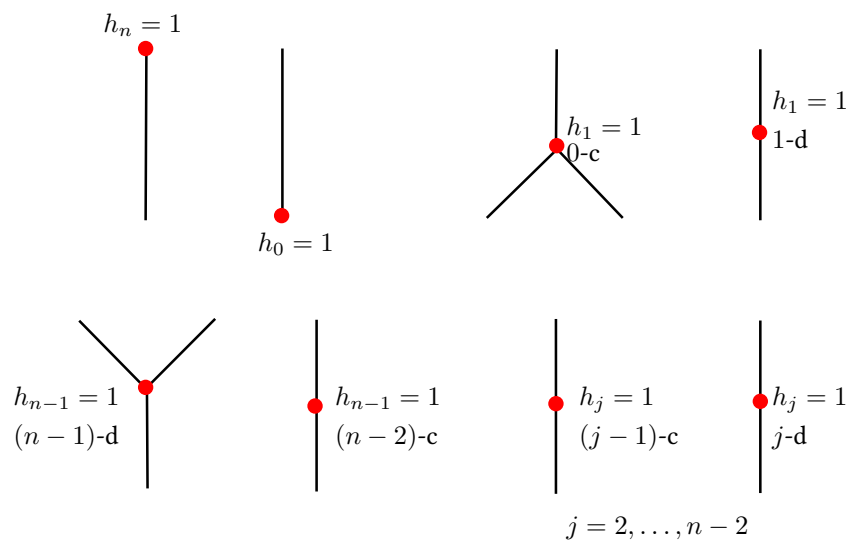


Figure 2.10: Local conditions of an abstract Lyapunov graph of Morse type.

Poincaré–Hopf Inequalities

"Mathematical discoveries, small or great, are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labour, both conscious and subconscious."
Henri Poincaré

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In this chapter, we present seven versions of a collection of inequalities called the Poincaré–Hopf inequalities (shortly PHI). These inequalities generalize the classical Morse inequalities described in Chapter 1. The first version was developed in [Bertolim et al., 2003a] for continuous flows on isolating blocks in order to ensure that given an isolating neighborhood containing a singularity with possibly complicated dynamical behavior, this neighborhood can be replaced by a neighborhood related to a Morse flow, i.e., where the singularities of the flow are nondegenerate. The second version, developed in [Bertolim et al., 2005b], is also a version for isolating blocks, but it includes a connectivity parameter κ which will play the role of the genus, $g(M)$, as defined in the previous chapter which we now refer to as the Cornea genus. The third version, developed in [Bertolim et al., 2005a], is a particular case of the first version by considering closed manifolds instead of isolating blocks. The fourth one is an adaptation of the first version for nonsingular Morse-Smale flows on isolating blocks. The fifth version provides two inequalities, referred to as Poincaré–Hopf inequalities for periodic orbits, which impose constraints on the dynamics of periodic orbits without reference to the Betti numbers of the manifold M . The sixth one, called componentwise Poincaré–Hopf inequalities, was created to ensure a componentwise Lyapunov graph morsification. The last one is a generalization of the Poincaré–Hopf inequalities to a finite-dimensional noncompact isolating block N .

We use the following notation (for definitions see Sections 2.3, 2.4):

- $S \subset M$ is an **isolated invariant set** of a flow ϕ_t .
- $N \subset M$ is an **isolating block** for S .
- \mathcal{R} is a **chain recurrent set** of the flow ϕ_t , which we impose to be a finite union of **isolated invariant sets** R_i .

- f is a **Lyapunov function**.
- $f^{-1}[c - \varepsilon, c + \varepsilon]$ that contains R is an **isolating neighborhood** for R .
- $(N, N^-) = (f^{-1}[c - \varepsilon, c + \varepsilon], f^{-1}(c - \varepsilon))$ as an **index pair** for R .
- h_* is the **rank of the homology Conley index**.
- κ is a connectivity parameter playing the role of the Cornea genus.

The six first versions of these inequalities hold for flows whose inverse flow satisfies the following duality condition on the homology Conley indices. Given a flow ϕ_t and an isolated invariant set $S \in N^n$, it will be assumed that the inverse flow ϕ_{-t} has an isolated invariant set S' with the property that

$$h_i(S) = \dim CH_i(S) = \dim CH_{n-i}(S') = h_{n-i}(S').$$

In other words, we assume that the **Conley duality condition**¹ on the indices holds. That is, the isolated invariant sets S and S' with index pairs (N, N^-) and (N, N^+) have the property that $\text{rank } H_j(N, N^-) = h_j$ and $\text{rank } H_j(N, N^+) = \bar{h}_j = h_{n-j}$.

3.1 PHI for isolating blocks

In this section, we consider the connectivity parameter $\kappa = 0$.

Let N be any compact manifold of dimension n such that $\partial N = \partial N^+ \cup \partial N^-$, with ∂N^+ and ∂N^- non-empty where $\partial N^+(\partial N^-)$ is the disjoint union of $e^+(e^-)$ components of ∂N , and denote it by $\partial N^\pm = \bigcup_{i=1}^{e^\pm} N_i^\pm$. Also, consider the sum of the Betti numbers, $\beta_j(N_i^\pm)$, of these components, i.e., $B_j^\pm = \sum_{i=1}^{e^\pm} \beta_j(N_i^\pm)$ where $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$.

The Poincaré–Hopf inequalities for an isolated invariant set S in an isolating block N with entering set for the flow N^+ and exiting set for the flow N^- , are obtained by analysis of long exact sequences of (N, N^+) and (N, N^-) . This analysis can be found in [Bertolim et al., 2003a] in a more detailed exposition.

Note that (N, N^-) is an index pair for S and (N, N^+) is an index pair for the isolated invariant set of the reverse flow, S' .

Consider the long exact sequences for the pairs (N, N^-) and (N, N^+) , denoted by LES– and LES+, respectively:

$$\begin{aligned} 0 \rightarrow H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N, N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \rightarrow \\ \xrightarrow{i_{n-1}} H_{n-1}(N) \xrightarrow{p_{n-1}} H_{n-1}(N, N^-) \xrightarrow{\partial_{n-1}} H_{n-1}(N^-) \rightarrow \dots \\ \dots \xrightarrow{\partial_2} H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} H_1(N, N^-) \rightarrow \\ \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N, N^-) \rightarrow 0 \end{aligned}$$

$$\begin{aligned} 0 \rightarrow H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N, N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \rightarrow \\ \xrightarrow{i'_{n-1}} H_{n-1}(N) \xrightarrow{p'_{n-1}} H_{n-1}(N, N^+) \xrightarrow{\partial'_{n-1}} \dots \\ \dots \xrightarrow{\partial'_2} H_1(N^+) \xrightarrow{i'_1} H_1(N) \xrightarrow{p'_1} H_1(N, N^+) \rightarrow \\ \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N, N^+) \rightarrow 0 \end{aligned}$$

¹Remark that Morse–Smale flows, as well as Smale flows, satisfy this duality condition.

Since we are assuming the Conley duality condition on the indices, the isolated invariant sets S and S' have the property that $\text{rank } H_j(N, N^-) = h_j$ and $\text{rank } H_j(N, N^+) = \bar{h}_j = h_{n-j}$. Observe that $\text{rank } H_0(N^-) = e^-$, $\text{rank } H_0(N^+) = e^+$ and $\text{rank } (H_j(N^\pm)) = B_j^\pm$.

By simultaneously analyzing the following pairs of maps

$$\{[(p_i, \partial'_i), (p'_i, \partial_i)], \dots [(p_2, \partial'_2), (p'_2, \partial_2)]\}$$

and analyzing p_1 and p'_1 we obtain the Poincaré–Hopf inequalities, (3.1)–(3.6), in all its generality:

$$\left\{ \begin{array}{l} h_j \geq \sum_{k=1}^{j-1} (-1)^{k+j} (B_k^+ - B_k^-) + \sum_{k=0}^{j-1} (-1)^{k+j} (h_{n-k} - h_k) \\ \quad + (-1)^{j+1} (e^- - e^+), \quad j = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor \quad (3.1) \\ \\ h_{n-j} \geq \sum_{k=1}^{j-1} (-1)^{k+j+1} (B_k^+ - B_k^-) + \sum_{k=0}^{j-1} (-1)^{k+j+1} (h_{n-k} - h_k) \\ \quad + (-1)^j (e^- - e^+), \quad j = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor \quad (3.2) \\ \\ h_1 \geq h_0 - 1 + e^- \quad (3.3) \\ \\ h_{n-1} \geq h_n - 1 + e^+ \quad (3.4) \\ \\ n = 2i + 1, i \geq 1 \left\{ \sum_{k=1}^{i-1} (-1)^k (B_k^+ - B_k^-) + (-1)^i \left(\frac{B_i^+ - B_i^-}{2} \right) - \sum_{k=0}^n (-1)^k h_k - (e^- - e^+) = 0 \right. \quad (3.5) \\ \\ n = 2i, i \text{ odd}, i \geq 3 \left\{ h_i - \sum_{k=1}^{i-1} (-1)^k (B_k^+ - B_k^-) - \sum_{k=0}^{i-1} (-1)^k (h_{n-k} - h_k) + (e^- - e^+) \equiv 0 \pmod{2} \right. \quad (3.6) \end{array} \right.$$

Using the effects (H1), (H2) and (H3) described in Section 2.2, the Poincaré–Hopf inequalities were used in [Bertolim et al., 2003a] in order to ensure that, given an isolating neighborhood containing a singularity with possibly complicated dynamical behavior, this neighborhood can be replaced by a neighborhood related to a Morse flow, i.e., where the singularities of the flow are nondegenerate. More precisely, in [Bertolim et al., 2003a], we prove the following result:

Theorem 2. *Suppose that N is an isolating neighborhood with boundary consisting of $(e^+ + e^-)$ connected components, endowed with a continuous flow entering N through e^+ boundary components N_k^+ , $k = 1, \dots, e^+$ and exiting through the remaining e^- boundary components N_k^- , $k = 1, \dots, e^-$, containing a singularity with dimensions of the Conley homology indices equal to (h_0, \dots, h_n) . For all $j = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor$, if $\beta_j(N)$, represents the j -th Betti number of N , we denote $B_j^+ - B_j^- = \sum_{k=1}^{e^+} \beta_j(N_k^+) - \beta_j(N_k^-)$. If the following **abstract homological data** $\left\{ e^+, e^-, \left(B_j^+ - B_j^- \right)_{j=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor}, (h_j)_{j=0}^n \right\}$ satisfies the Poincaré–Hopf inequalities (3.1)–(3.6) then N can be replaced by a neighborhood endowed with a Morse flow entering N through e^+ boundary components N_k^+ , $k = 1, \dots, e^+$ and exiting through the remaining e^- boundary components N_k^- , $k = 1, \dots, e^-$, where for all $j = 0 \dots n$ we have $h_j = 1$, j representing the dimension of the unstable manifold of this isolated singularity.*

These inequalities were also used to obtain results concerning the continuation of abstract Lyapunov graphs. This will be discussed in the next chapter.

3.2 PHI with κ for Isolating Blocks

Let us continue to try to understand how global and local topological aspects of the dynamical behavior interact. Note that studying these interactions means knowing how a flow imposes topo-

logical constraints on its phase space manifold. In general, some restrictions appear. For example, the Cornea genus of a smooth closed manifold M , $g(M)$ is always less than or equal to the first Betti number of M , $g(M) \leq \gamma_1(M)$. This generalizes a theorem of Franks [Franks, 1985] which asserts that if M is simply connected then $\kappa = 0^2$.

Morse–Conley inequalities (1.1) imply that $\gamma_1(M) - \gamma_0(M) \leq h_1 - h_0$. It follows that $g(M) \leq \gamma_1(M) \leq h_1 - h_0 + \gamma_0(M)$. Therefore, one has a weaker version of the Morse inequalities with the Cornea genus: $h_1 \geq h_0 - 1 + g(M)$. We wish to analyse these inequalities for different manifolds and Cornea genera. In order to do this more freely, without reference to a specific manifold, we introduced a parameter κ which will play the role of the Cornea genus.

As discussed in the previous section, in [Bertolim et al., 2003a] we consider the Poincaré–Hopf inequalities (3.1)–(3.6), in the case $\kappa = 0$ for an isolated invariant set Λ with isolating block N , with entering set for the flow N^+ and exiting set for the flow N^- , under the hypothesis that the flow satisfies the Conley index duality condition on components of the chain recurrent set.

In [Bertolim et al., 2003b], these inequalities were treated by considering the presence of a connectivity parameter κ as described above for an isolating block N . Thus, the Poincaré–Hopf inequalities for isolating blocks with this parameter κ will be the same collection as before except for inequalities (3.3)–(3.4) which are replaced with:

$$\begin{cases} h_1 \geq h_0 - 1 + e^- + \kappa, & (3.7) \\ h_{n-1} \geq h_n - 1 + e^+ + \kappa. & (3.8) \end{cases}$$

3.3 PHI for Closed Manifolds

In [Bertolim et al., 2005a], we consider a particular case of the Poincaré–Hopf inequalities for isolating blocks (3.1)–(3.6). The novelty in these inequalities is that they provide constraints on the dynamics without involving the topology of the manifold M , in other words without reference to the Betti numbers of M . These inequalities, (3.9)–(3.11), called Poincaré–Hopf inequalities for closed manifolds, form a set of upper bounds on the number of singularities of index j in terms of the alternating sum of singularities of index $k < j$ and their duals of index $(n - k)$. As mentioned in the previous section, the parameter κ will play the role of the Cornea genus.

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} n=2i+1 \left\{ -h_i \leq (h_{i+2} - h_{i-1}) - (h_{i+3} - h_{i-2}) + \dots \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \leq h_{i+1} \\ n=2i \left\{ -h_i \leq (h_{i+1} - h_{i-1}) - (h_{i+2} - h_{i-2}) + \dots \pm (h_{2i-2} - h_2) \pm (h_{2i} - h_0) \leq h_i \end{array} \right. \\ \left\{ \begin{array}{l} -h_j \leq (h_{n-(j-1)} - h_{j-1}) - (h_{n-(j-2)} - h_{j-2}) + \dots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \leq h_{n-j} \\ \vdots \\ -h_2 \leq (h_{n-1} - h_1) - (h_n - h_0) \leq h_{n-2} \quad (2) \end{array} \right. \\ \left\{ \begin{array}{l} h_1 \geq h_0 - 1 + \kappa \\ h_{n-1} \geq h_n - 1 + \kappa \end{array} \right. \quad (1) \end{array} \right. \quad (3.9)$$

²In terms of graphs, $\kappa = 0$ means that the Lyapunov graph is a tree.

In the case $n = 2i + 1$ we have

$$\sum_{j=0}^{2i+1} (-1)^j h_j = 0 \quad (3.10)$$

and in the case $n = 2i \equiv 2 \pmod{4}$ we have the additional constraint that

$$h_i - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) \text{ be even.} \quad (3.11)$$

Surprisingly, it turns out that, whenever these inequalities are satisfied for a pre-assigned dynamical data (h_0, \dots, h_n) , it can be shown that there exists a collection of Betti numbers that satisfy the Morse inequalities, (1.1), with this same data. Conversely, if the Morse inequalities are satisfied for (h_0, \dots, h_n) and $(\gamma_0, \dots, \gamma_n)$ then (h_0, \dots, h_n) satisfies the Poincaré–Hopf inequalities. This is stated in Theorem 3.

A nonnegative integral vector $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$ satisfying $\gamma_{n-k} = \gamma_k$, for $k = 0, \dots, n$, $\gamma_0 = \gamma_n = 1$ (and in some cases it will be required that $\gamma_{n/2}$ be even if n is even), is called a Betti number vector.

Theorem 3. *A set of nonnegative numbers (h_0, h_1, \dots, h_n) satisfies the Poincaré–Hopf inequalities in (3.9) if and only if it satisfies the Morse inequalities (1.1) for some Betti number vector $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$.*

This result is not merely a change of inequalities. One should note that the Morse inequalities involve (h_0, \dots, h_n) and $(\gamma_0, \dots, \gamma_n)$ whereas the Poincaré–Hopf inequalities only involve (h_0, \dots, h_n) .

This theorem has many applications, in particular, it can be used to obtain partial answers to the question of realizability of abstract Lyapunov graphs $L(h_0, \dots, h_n)$ as flows on closed manifolds. Note that, abstract Lyapunov graphs carry dynamical data and local topological invariants of level sets but no global topological information of the manifold on which it can be realized. Hence, one cannot verify the Morse inequalities for abstract Lyapunov graphs, however, we can verify the Poincaré–Hopf inequalities. For more details see [Bertolim et al., 2003a] and [Bertolim et al., 2005b].

In some sense the Poincaré–Hopf inequalities pre-process admissible data, that is, if (h_0, \dots, h_n) does not satisfy the Poincaré–Hopf inequalities, there is no closed n -manifold which admits (h_0, \dots, h_n) as its dynamical data. This follows from Theorem 3 and from the classical results of Morse [Morse, 1925].

The Poincaré–Hopf inequalities can also be used to prove the existence of critical points of index k from a priori knowledge of the existence of critical points of lower index and their duals. That is, these inequalities can also be used to give bounds on the numbers h_j with respect to alternating sums of h_s with $s < j$ and their duals h_{n-s} . In the case of Morse flows these inequalities provide bounds on the number of singularities c_j of Morse index j with respect to alternating sums of c_s with $s < j$ and their duals c_{n-s} .

Afterwards, in [Bertolim et al., 2005b] we give a topological interpretation of the Poincaré–Hopf inequalities by associating with any vertex label (h_0, h_1, \dots, h_n) a collection of Betti numbers vectors $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$.

In fact, we compute algorithmically all the possible Betti numbers satisfying the Morse inequalities for any given pre-assigned index data (h_0, h_1, \dots, h_n) . This collection of abstract Betti numbers vectors constitutes a polytope which we called Morse polytope and which is completely characterized in [Bertolim et al., 2005b]. The parameter κ played an important role in determining the Morse polytope, that is the convex hull of all the Betti numbers $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$ that satisfy the Morse inequalities for a given dynamical-topological data $(h_0, h_1, \dots, h_n, \kappa)$ that satisfy the Poincaré–Hopf

inequalities for closed manifolds. Among other interesting geometrical properties, as κ increases the Morse polytopes become smaller, the polytope associated with a higher value of κ being contained in the polytope associated with a lower value.

In [Bertolim, 2012a] a new proof of the equivalence result presented in [Bertolim et al., 2005b] is provided by using elementary techniques, where related results and further applications of these inequalities were explored.

3.4 PHI for nonsingular Morse-Smale flows

Morse-Smale flows³ on smooth compact manifolds are roughly those flows which exhibit only two types of recurrent behavior: periodic closed orbits and rest points which are of **hyperbolic type**⁴. Let us emphasize here that the flow Φ is nonsingular Morse-Smale, that is, all of its recurrent sets are *closed periodic orbits* and lie in the interior of M . In the sequel, we shall simply call them periodic orbits because all the flows we shall consider are nonsingular Morse-Smale.

For odd n , let us denote by N any n -dimensional compact connected oriented manifold with boundary ∂N consisting of $(e^+ + e^-)$ connected components, endowed with a nonsingular Morse-Smale flow Φ transversally entering N through e^+ boundary components $N_i^+, i = 1, \dots, e^+$, and transversally exiting through the remaining e^- boundary components $N_i^-, i = 1, \dots, e^-$.

Recall that in [Franks, 1982] Franks proved that a hyperbolic periodic orbit of index j , can be viewed as the joining of two hyperbolic singularities p and q of adjacent indices j and $j + 1$ respectively. Given a nondegenerate singularity of index j , one can associate with it the dimensions of the Conley homology indices, $h_j = 1$ and $h_k = 0$ for all $k \neq j$. Let $A_j = (h_j, h_{j+1})$ be the number of periodic orbits of index j , where h_j be the number of singularities of index j .

The Poincaré–Hopf inequalities for nonsingular Morse-Smale flows on isolating blocks are an adaptation of the Poincaré–Hopf inequalities above described. They differ only in inequalities (3.3) and (3.4). These adaptation should be done in order to ensure the continuation of an abstract Lyapunov graph L_N to an abstract Lyapunov graph of Smale type, in which any vertex v is labeled by $A_j = 1$ if it corresponds to a periodic orbit of index j as defined in Section 2.4.

The difference in inequalities (3.3) and (3.4) comes from the fact that in the case of the Poincaré–Hopf inequalities one treats h_0 by imposing, as necessary and sufficient condition, that $h_1 \geq h_0 + e^- - 1$, i.e., inequality (3.3). This inequality adjusts the problem of connectivity, that is, it ensures that the continued graph – as well as the corresponding isolating neighborhood – are connected. By this inequality we guarantee the possibility of having $h_0 + e^- - 1$ singularities of index 1 of type 0-c, h_1^c , which means connecting. Hence,

$$h_1^c \geq h_0 + e^- - 1. \quad (3.12)$$

It is important to observe that the singularities h_1 of type 0-c are responsible for connecting the outgoing boundary components. For example, if we have three outgoing boundary components, representing by $h_0 = 3$ we need two h_1 of type 0-c for producing a connected neighborhood. In order to better understand, consider the example in three-dimension presented in Figure 3.1. In this example we have a vertex v_k which is labeled by $h_0 = 1$ and $h_1 = 1$. Since these data satisfy inequality (3.3), we can replace the vertex as in the right side of Figure 3.1: by two vertices v_{k_1} and

³A smooth flow ϕ_t on M is called **Morse-Smale** if the chain recurrent set \mathcal{R} of ϕ_t consists of a finite number of hyperbolic closed orbits and hyperbolic rest points, and the unstable manifold of any closed orbit or rest point has transversal intersection with the stable manifold of any closed orbit or rest point.

⁴A subset of a manifold is said to have **hyperbolic structure** with respect to a map f , when its tangent bundle may be split into two invariant subbundles, one of which is contracting, and the other expanding with respect to f .

v_{k_2} respectively labeled with $h_0 = 1$ and $h_1 = 1$. It is important to note that the left and the right side of Figure 3.1 have the same number of incoming and outgoing edges and moreover they are both connected.

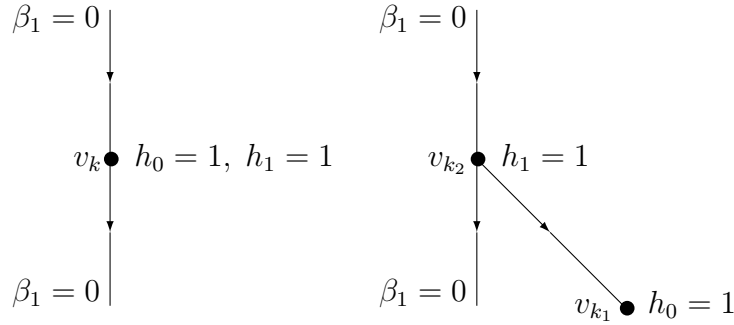


Figure 3.1: Connectivity of the outgoing edges.

In the case of the Poincaré–Hopf inequalities for nonsingular Morse-Smale flows, inequality (3.3) should be modified, because the presence of $h_0 \neq 0$ implies the existence of periodic orbits of index 0. Each one is obtained by joining a singularity of index 0 with a singularity of index 1 of type 1-d, h_1^d (observe that all singularities h_1 of type 0-c, h_1^c , were already used for solving the connectivity problem as explained above). Hence, we have that

$$h_1^d \geq h_0. \tag{3.13}$$

For this reason, since $h_1 = h_1^c + h_1^d$ inequalities (3.12) and (3.13) imply that we have the following inequality replacing (3.3)

$$h_1 \geq 2h_0 - 1 + e^- \tag{3.14}$$

Each of the remaining singularities of index 1 is to be coupled with a singularity of index 2, thus creating a periodic orbit of index 1. Therefore, the left side of inequalities (3.1) remains the same for all $j \neq 1$.

Observe that the graph presented in Figure 3.1 cannot be replaced by a connected graph containing a periodic orbit of index 0: if it were the case we would use $h_1 = 1$ and $h_0 = 1$ to have a vertex labeled with $A_0 = 1$, with one incoming edge and no outgoing edges, and we would be in the situation of the right side of Figure 3.2, that is, a non connected graph. It is also important to observe that inequality (3.14) is not satisfied.

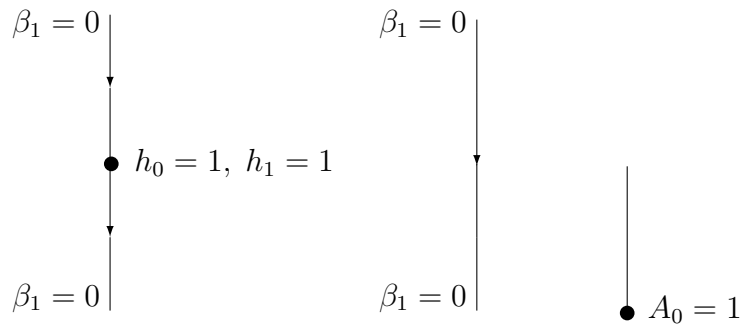


Figure 3.2: Impossibility of continuation to a connected graph containing a periodic orbit of index 0.

Figure 3.3 gives an example in dimension three where inequality (3.14) is satisfied and hence we can

replace the initial vertex to a connected one containing only periodic orbits. It is important to note that the left and the right side of Figure 3.3 have the same number of incoming and outgoing edges and moreover they are both connected.

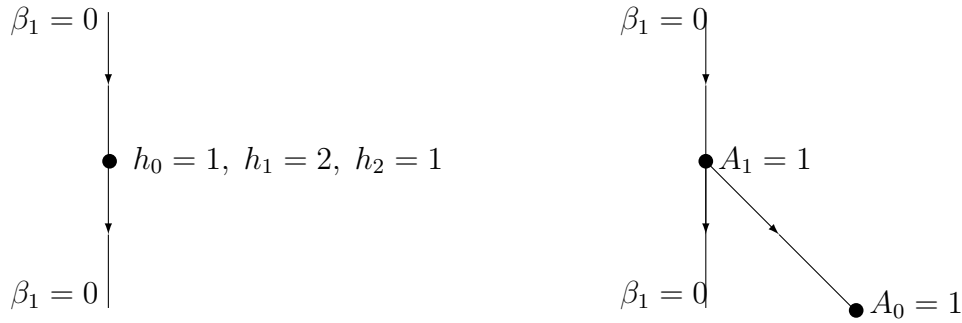


Figure 3.3: Continuation to a connected graph containing only periodic orbits.

Analogously, the necessary and sufficient condition when one treats h_n is $h_{n-1} \geq h_n + e^+ - 1$, i.e., inequality (3.4), which assures the existence of $h_n + e^+ - 1$ singularities of type $(n-1)$ -d, which means disconnecting. Hence,

$$h_{n-1}^d \geq h_n + e^+ - 1. \quad (3.15)$$

In the case of the Poincaré–Hopf inequalities for nonsingular Morse–Smale flows, the presence of $h_n \neq 0$ implies the existence of periodic orbits of index $n-1$. Each one is obtained by joining a singularity of index n with a singularity of index $n-1$ of type $(n-1)$ -c, h_{n-1}^c . Hence, we have that

$$h_{n-1}^c \geq h_n. \quad (3.16)$$

For this reason, since $h_{n-1} = h_{n-1}^c + h_{n-1}^d$ inequalities (3.15) and (3.16) imply that we have the following inequality replacing (3.4)

$$h_{n-1} \geq 2h_n - 1 + e^+ \quad (3.17)$$

Each of the remaining singularities of index $n-1$ is to be coupled with a singularity of index $n-2$, thus creating a periodic orbit of index $n-2$. Therefore, the left side of inequalities (3.1) remains the same for all $j \neq n-1$.

These inequalities were also used to compute, in terms of a given homological information, a number p_{min} such that any nonsingular Morse–Smale flow ϕ on any manifold M satisfying the given abstract homological data must have at least p_{min} closed periodic orbits. This will be discussed in Section 5.2.

3.5 Poincaré–Hopf inequalities for periodic orbits

As discussed in Section 3.3, in [Bertolim et al., 2005b] a collection of inequalities called Poincaré–Hopf inequalities for closed manifolds were presented, providing constraints on the dynamics without involving the topology of the manifold M . In the same paper, it was proved that a pre-assigned dynamical data (h_0, \dots, h_n) satisfies the Poincaré–Hopf inequalities if and only if one of the corresponding Betti numbers vector $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$ satisfies the Morse inequalities.

In [Asimov, 1975], Asimov showed the existence of **nonsingular Morse–Smale flows**⁵ on essentially arbitrary (smooth, connected and compact) manifolds. In [Smale, 1983], Smale investigates

⁵By a **nonsingular Morse–Smale flow** on M^n we mean a flow ϕ_t satisfying the following conditions:

the interplay between the periodic structure of a nonsingular Morse-Smale flow, i.e., the number of closed orbits of each index, and the topology of the manifold on which the flow occurs. Franks in [Franks, 1979] strengthens the necessary conditions presented in Smale [Smale, 1983] to obtain conditions which are both necessary and sufficient. More precisely, Franks in [Franks, 1979] proves the following: Let M be a manifold endowed with a nonsingular Morse-Smale flow and let A_k be the number of closed orbits having orientable unstable manifolds of dimension $k + 1$. It is shown that if $\dim M = n$, $\gamma_k = \dim H_k(M; \mathbb{Q})$, then

$$\left\{ \begin{array}{l} A_k \geq \gamma_k - \gamma_{k-1} + \cdots \pm \gamma_0, \quad \text{for all } k \\ A_1 \geq A_0 - 1 \quad \text{and} \quad A_{n-2} \geq A_{n-1} - 1 \\ \text{if } A_{k-1} = A_{k+1} = 0 \quad \text{and} \quad \gamma_k - \gamma_{k-1} + \cdots \pm \gamma_0 \leq 0, \quad \text{then } A_k = 0 \end{array} \right. \quad (3.18)$$

$$(3.19)$$

$$(3.20)$$

This result gives the necessary conditions for abstract data to be realized as a nonsingular Morse-Smale flow on a manifold. Inequality (3.18) is essentially the Morse inequalities of Smale [Smale, 1983], the only difference being that Franks [Franks, 1979] considers A_k the number of closed orbits of index k having orientable unstable manifolds while Smale [Smale, 1983] considers A_k all orbits of index k . Franks also proved in the same paper, [Franks, 1979], that if M is a simply connected compact manifold, $\dim M \geq 5$, the Euler characteristic of M vanishes and $H_*(M; \mathbb{Z})$ is torsion free, then to any set of nonnegative integers A_k satisfying (3.18)–(3.20) there corresponds a nonsingular Morse-Smale flow with A_k closed orbits having orientable unstable manifolds of dimension $k + 1$ and no closed orbits having nonorientable unstable manifolds.

Equations (3.18), (3.19) and (3.20) will be called **Morse-Smale inequalities** in this work.

In order to get results involving periodic orbits similar to the results presented in [Bertolim et al., 2005b] for singularities, in [Bertolim, 2012b] we introduced two inequalities providing constraints on the dynamics for periodic orbits without involving the topology of the manifold M . One refers to these inequalities, (3.21), as the **Poincaré–Hopf inequalities for periodic orbits**.

$$A_1 \geq A_0 - 1 \quad \text{and} \quad A_{n-2} \geq A_{n-1} - 1. \quad (3.21)$$

It turns out that, whenever these inequalities (3.21) are satisfied for a pre-assigned dynamical data (A_0, \dots, A_{n-1}) , it can be shown that there exists a collection of Betti numbers that satisfy the Morse-Smale inequalities, (3.18), (3.19) and (3.20), with this same data. Conversely, if the Morse-Smale inequalities are satisfied for (A_0, \dots, A_{n-1}) and $(\gamma_0, \dots, \gamma_n)$ then (A_0, \dots, A_{n-1}) satisfies the Poincaré–Hopf inequalities for periodic orbits. This is stated in Theorem 4.

Theorem 4. *A set of nonnegative numbers $(A_0, A_1, \dots, A_{n-1})$, satisfies the Poincaré–Hopf inequalities for periodic orbits (3.21) if and only if it satisfies the Morse-Smale inequalities (3.18), (3.19) and (3.20) for some Betti number vector⁶ $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$.*

(i) the chain-recurrent set \mathcal{R} of ϕ_t consists of finitely many hyperbolic closed orbits;

(ii) for each pair of closed orbits of ϕ_t , the intersection of their stable and unstable manifolds is transversal.

⁶A *Betti number vector* in dimension n is a list of nonnegative integers $(\gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n)$ that satisfy Poincaré duality ($\gamma_{n-k} = \gamma_k$ for all k), and the connectivity, or boundary, conditions $\gamma_0 = \gamma_n = 1$. Furthermore, if $n = 2i$, i odd, we have the additional condition that γ_i be even.

It is important to observe that once again this result is not purely a change of inequalities. The Morse-Smale inequalities involve (A_0, \dots, A_{n-1}) and $(\gamma_0, \dots, \gamma_n)$ whereas the Poincaré–Hopf inequalities for periodic orbits only involve (A_0, \dots, A_{n-1}) .

Here are some applications of these results. Theorem 4 together with results of Franks [Franks, 1979] imply that if (A_0, \dots, A_{n-1}) does not satisfy the Poincaré–Hopf inequalities for periodic orbits, there is no closed n -manifold which admits (A_0, \dots, A_{n-1}) as its dynamical data. In this sense the Poincaré–Hopf inequalities for periodic orbits filter admissible data.

One can also use Theorem 4 to obtain partial answers to the realizability question of abstract Lyapunov graphs. Since an abstract Lyapunov graph carries dynamical data and local topological invariants of level sets, but no global topological information of the manifold on which they can be realized, the Morse-Smale inequalities cannot be verified. However the Poincaré–Hopf inequalities for periodic orbits can be easily verified for Lyapunov graphs.

By considering a closed manifold endowed with a nonsingular Morse-Smale flow, it was also shown in this same paper, [Bertolim, 2012b], that a particular case of the Poincaré–Hopf inequalities for nonsingular Morse-Smale flows on isolating blocks (discussed in Section 3.4), are the inequalities (3.22)-(3.26). One refers to these inequalities (3.22)-(3.26) as the **Poincaré–Hopf inequalities for nonsingular Morse-Smale flows on closed manifolds**, where h_j represents the number of singularities of index j .

$$\left\{ \begin{array}{l} -h_j \leq \sum_{k=0}^{j-1} (-1)^{k+j+1} (h_{n-k} - h_k) \leq h_{n-j}, j = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor \quad (3.22) \\ h_1 \geq 2h_0 - 1 \quad (3.23) \\ h_{n-1} \geq 2h_n - 1 \quad (3.24) \\ n = 2i + 1 \left\{ \begin{array}{l} \sum_{k=0}^n (-1)^k h_k = 0 \quad (3.25) \\ h_i - \sum_{k=0}^{i-1} (-1)^k (h_{n-k} - h_k) \equiv 0 \pmod{2} \quad (3.26) \end{array} \right. \\ n = 2i, i \text{ odd}, i \geq 3 \left\{ \begin{array}{l} h_i - \sum_{k=0}^{i-1} (-1)^k (h_{n-k} - h_k) \equiv 0 \pmod{2} \quad (3.26) \end{array} \right. \end{array} \right.$$

These inequalities (3.22)-(3.26) play important role in the proof of Theorem 4.

We must highlight that the results of [Bertolim, 2012b] differ from results in [Bertolim et al., 2005b] dynamically since periodic orbits are considered instead of singularities. Although we borrow from [Bertolim et al., 2005b] some techniques, the results in [Bertolim, 2012b] are proven by means of more specific techniques for periodic orbits while the equivalent results in [Bertolim et al., 2005b] make use of elaborate techniques from network flow theory.

3.6 Componentwise Poincaré–Hopf inequalities

In [Bertolim et al., 2003a] a morsification algorithm, which will be described in Section 4.1.1, was translated into a system of linear equations whose feasibility was equivalent to that of a set of inequalities involving the input data, given by the Poincaré–Hopf inequalities described in Section 3.1. In fact, much work has been done in this direction combinatorially, for example in [Bertolim et al., 2003a, 2005a,b]. The novelty in the tools employed therein was the use of network flow theory. This application of network flow theory to morsification problems turned out to be a powerful tool in providing constructive proofs, and recipes for the generation of all possible nonnegative integral solutions associated with this linear system of equations. A gateway was open so that one could

now ask questions in both realms allowing for the exploration of many topological problems that may be modeled as a network flow theoretical question. This marked a turning point in this type of investigation which we explore in this work. In an approach evocative of the one employed in [Bertolim et al., 2003a] and that will be described in Section 4.1.1, an algorithm which describes the componentwise morsification was presented in [Bertolim et al., 2023b]. This algorithm determines a system of linear equations. The equivalence between the existence of nonnegative integral solutions to this system of linear equations and the feasibility of a new system of linear inequalities, the **componentwise Poincaré–Hopf inequalities**, involving the topological-dynamical input data set, is established.

Let N be an n -dimensional compact connected oriented manifold with boundary $\partial N = N^+ \cup N^-$ consisting of $(e^+ + e^-)$ connected boundary components, $n \geq 3$, n odd. N is endowed with a flow ϕ which has S as its maximal isolated invariant set with Conley homology indices $(h_0, h_1, \dots, h_{n-1}, h_n)$. Moreover, ϕ enters transversally through e^+ boundary components N_k^+ , $k = 1, \dots, e^+$, and exits transversally through the remaining e^- boundary components N_k^- , $k = 1, \dots, e^-$. $\beta_{j,k}$ corresponds to the j -th Betti number of the component k . The **componentwise Poincaré–Hopf inequalities** are the following:

If ℓ is odd

$$0 \leq -(e^- - 1) + \sum_{j=1}^{(\ell-1)/2} (h_{2j-1} + h_{n-2j}) + h_\ell$$

$$- \sum_{j=1}^{(\ell-1)/2} \left(\sum_{k=1}^{e^+} \beta_{2j-1,k}^+ + \sum_{k=1}^{e^-} \beta_{2j,k}^- \right) \leq \sum_{k=1}^{e^+} \beta_{\ell k}^+, \quad \text{for } 1 \leq \ell < i \quad (3.27)$$

$$0 \leq -(e^- - 1) + \sum_{j=1}^{(n-\ell)/2} (h_{2j-1} + h_{n-2j}) - \sum_{j=1}^{(n-\ell)/2-1} \left(\sum_{k=1}^{e^+} \beta_{2j-1,k}^+ + \sum_{k=1}^{e^-} \beta_{2j,k}^- \right)$$

$$- \sum_{k=1}^{e^+} \beta_{n-\ell-1,k}^+ \leq \sum_{k=1}^{e^-} \beta_{n-\ell,k}^-, \quad \text{for } i+1 < \ell \leq n-1 \quad (3.28)$$

If ℓ is even

$$0 \leq -(e^+ - 1) + \sum_{j=1}^{\ell/2} (h_{2j} + h_{n-2j+1})$$

$$- \sum_{j=1}^{\ell/2-1} \left(\sum_{k=1}^{e^-} \beta_{2j-1,k}^- + \sum_{k=1}^{e^+} \beta_{2j,k}^+ \right) - \sum_{k=1}^{e^-} \beta_{\ell-1,k}^- \leq \sum_{k=1}^{e^+} \beta_{\ell k}^+, \quad \text{for } 1 \leq \ell < i \quad (3.29)$$

$$0 \leq -(e^+ - 1) + \sum_{j=1}^{(n-\ell-1)/2} (h_{2j} + h_{n-2j+1}) + h_\ell$$

$$- \sum_{j=1}^{(n-\ell-1)/2} \left(\sum_{k=1}^{e^-} \beta_{2j-1,k}^- + \sum_{k=1}^{e^+} \beta_{2j,k}^+ \right) \leq \sum_{k=1}^{e^-} \beta_{n-\ell,k}^-, \quad \text{for } i+1 < \ell \leq n-1 \quad (3.30)$$

If i is odd

$$\begin{aligned} \frac{\sum_{k=1}^{e^+} \beta_{ik}^+}{2} &= -(e^- - 1) + \sum_{j=1}^{(i-1)/2} (h_{2j-1} + h_{n-2j}) + h_i \\ &\quad - \sum_{j=1}^{(i-1)/2} \left(\sum_{k=1}^{e^+} \beta_{2j-1,k}^+ + \sum_{k=1}^{e^-} \beta_{2j,k}^- \right), \end{aligned} \quad (3.31)$$

$$\begin{aligned} \frac{\sum_{k=1}^{e^-} \beta_{ik}^-}{2} &= -(e^+ - 1) + \sum_{j=1}^{(i-1)/2} (h_{2j} + h_{n-2j+1}) + h_{i+1} \\ &\quad - \sum_{j=1}^{(i-1)/2} \left(\sum_{k=1}^{e^-} \beta_{2j-1,k}^- + \sum_{k=1}^{e^+} \beta_{2j,k}^+ \right), \end{aligned} \quad (3.32)$$

If i is even

$$\begin{aligned} \frac{\sum_{k=1}^{e^+} \beta_{ik}^+}{2} &= -(e^+ - 1) + \sum_{j=1}^{i/2} (h_{2j} + h_{n-2j+1}) \\ &\quad - \sum_{j=1}^{i/2-1} \left(\sum_{k=1}^{e^-} \beta_{2j-1,k}^- + \sum_{k=1}^{e^+} \beta_{2j,k}^+ \right) - \sum_{k=1}^{e^-} \beta_{i-1,k}^-, \end{aligned} \quad (3.33)$$

$$\begin{aligned} \frac{\sum_{k=1}^{e^-} \beta_{ik}^-}{2} &= -(e^- - 1) + \sum_{j=1}^{i/2} (h_{2j-1} + h_{n-2j}) \\ &\quad - \sum_{j=1}^{i/2-1} \left(\sum_{k=1}^{e^+} \beta_{2j-1,k}^+ + \sum_{k=1}^{e^-} \beta_{2j,k}^- \right) - \sum_{k=1}^{e^+} \beta_{i,k}^+ \end{aligned} \quad (3.34)$$

When $n = 3$, the componentwise Poincaré–Hopf inequalities reduce to the following equations:

$$h_1 - (e^- - 1) - \sum_{k=1}^{e^+} \frac{\beta_{1k}^+}{2} = 0, \quad (3.35)$$

$$h_2 - (e^+ - 1) - \sum_{k=1}^{e^-} \frac{\beta_{1k}^-}{2} = 0, \quad (3.36)$$

which are in fact satisfied by the graphs in Figures 4.13 and 4.14.

3.7 Generalized Poincaré–Hopf inequalities

Our research continued motivated by the possibility of generalizing the Poincaré–Hopf inequalities to a finite-dimensional noncompact manifold N , which was not previously defined in this setting.

We sought the relationship of the homology of the flow and its inverse flow in a noncompact setting in order to prove a generalized version of the Poincaré–Hopf inequalities. We investigated the relationship of the homology of the flow and its inverse flow in a noncompact setting and the following duality result was proved: $\check{H}^p(N^*, (N^+)^*) \simeq H_{n-p}(N, N^-)$ for an n -manifold N with boundary $\partial N = N^+ \sqcup N^-$ a disjoint union, where N , N^+ and N^- need not be compact, N^+ and N^- are chosen arbitrarily in ∂N , N^* represents the Alexandroff one-point compactification of N and $\check{H}^*(\cdot)$ represents the Čech cohomology. Observe that the duality results described above imply that $h_j(N, N^+) = \text{rank } H_j(N, N^+) = \text{rank } H^{n-j}(N^*, (N^-)^*) = h_{n-j}^*(N^*, (N^-)^*)$. Using this duality result, we prove in [Bertolim et al., 2011a] a generalized version of the Poincaré–Hopf inequalities. More precisely, we prove the following result:

Theorem 5. *Let N be a connected n -manifold possibly noncompact endowed with a flow satisfying $\check{H}^p(N^*, (N^+)^*) \simeq H_{n-p}(N, N^-)$ and with boundary $\partial N = N^+ \sqcup N^-$, the disjoint union of closed spaces N^+ and N^- chosen arbitrarily in ∂N . Let $h_j = \text{rank } H_j(N, N^-)$, $h_j^* = \text{rank } H_j(N^*, (N^-)^*)$ and $\text{rank}(H_j(N^\pm)) = B_j^\pm$. The Poincaré–Hopf inequalities for $n = 2i$ and $n = 2i + 1$ are:*

$$\left. \begin{array}{l}
 \left\{ \begin{array}{l}
 -(B_{n-1}^+ - B_{n-1}^-) + (B_{n-2}^+ - B_{n-2}^-) - \dots \pm \dots \pm (B_0^+ - B_0^-) = \sum_{j=0}^n (-1)^{j+1} (h_j^* - h_j) \\
 h_1^* \geq -(B_{n-1}^+ - B_{n-1}^-) - (h_n - h_0^*) \\
 h_{n-1} \geq -[-(B_{n-1}^+ - B_{n-1}^-) - (h_n - h_0^*)] \\
 \vdots \\
 h_j^* \geq (-1)^j (B_{n-1}^+ - B_{n-1}^-) \pm (B_{n-2}^+ - B_{n-2}^-) \pm \dots \pm (B_{n-j}^+ - B_{n-j}^-) \\
 \quad - (h_{n-(j-1)} - h_{j-1}^* + (h_{n-(j-2)} - h_{j-2}^*) + \dots \pm (h_{n-1} - h_1^*) \pm (h_n - h_0^*) \\
 h_{n-j} \geq -[(-1)^j (B_{n-1}^+ - B_{n-1}^-) \pm (B_{n-2}^+ - B_{n-2}^-) \pm \dots \pm (B_{n-j}^+ - B_{n-j}^-) \\
 \quad - (h_{n-(j-1)} - h_{j-1}^* + (h_{n-(j-2)} - h_{j-2}^*) + \dots \pm (h_{n-1} - h_1^*) \pm (h_n - h_0^*)] \\
 \vdots \\
 h_j \geq -(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \pm (B_0^+ - B_0^-) \\
 \quad - (h_{n-(j-1)}^* - h_{j-1}) + (h_{n-(j-2)}^* - h_{j-2}) + \dots \pm (h_{n-1}^* - h_1) \pm (h_n^* - h_0) \\
 h_{n-j}^* \geq -[-(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \pm (B_0^+ - B_0^-) \\
 \quad - (h_{n-(j-1)}^* - h_{j-1}) + (h_{n-(j-2)}^* - h_{j-2}) + \dots \pm (h_{n-1}^* - h_1) \pm (h_n^* - h_0)] \\
 \vdots \\
 h_1 \geq -(B_0^+ - B_0^-) - (h_n^* - h_0) \\
 h_{n-1}^* \geq -[-(B_0^+ - B_0^-) - (h_n^* - h_0)]
 \end{array} \right\}
 \end{array} \right\} \quad (3.37)$$

$$\left. \begin{array}{l}
\left\{ \begin{array}{l}
-(B_{n-1}^+ - B_{n-1}^-) + (B_{n-2}^+ - B_{n-2}^-) - \dots \pm \dots \pm (B_0^+ - B_0^-) = \sum_{j=0}^{2i+1} (-1)^{j+1} (h_j + h_j^*) \\
h_1^* \geq -(B_{n-1}^+ - B_{n-1}^-) - (h_n - h_0^*) \\
h_{n-1} \geq -[-(B_{n-1}^+ - B_{n-1}^-) - (h_n - h_0^*)] \\
\vdots \\
h_j^* \geq -(B_{n-1}^+ - B_{n-1}^-) + (B_{n-2}^+ - B_{n-2}^-) + \dots \pm (B_{n-j}^+ - B_{n-j}^-) \\
\quad - (h_{n-(j-1)} - h_{j-1}^*) + (h_{n-(j-2)} - h_{j-2}^*) + \dots \pm (h_{n-1} - h_1^*) \pm (h_n - h_0^*) \\
h_{n-j} \geq -[-(B_{n-1}^+ - B_{n-1}^-) + (B_{n-2}^+ - B_{n-2}^-) + \dots \pm (B_{n-j}^+ - B_{n-j}^-) \\
\quad - (h_{n-(j-1)} - h_{j-1}^*) + (h_{n-(j-2)} - h_{j-2}^*) + \dots \pm (h_{n-1} - h_1^*) \pm (h_n - h_0^*)] \\
\vdots \\
h_j \geq -(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \pm (B_0^+ - B_0^-) \\
\quad - (h_{n-(j-1)}^* - h_{j-1}) + (h_{n-(j-2)}^* - h_{j-2}) + \dots \pm (h_{n-1}^* - h_1) \pm (h_n^* - h_0) \\
h_{n-j}^* \geq -[-(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + \dots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \pm (B_0^+ - B_0^-) \\
\quad - (h_{n-(j-1)}^* - h_{j-1}) + (h_{n-(j-2)}^* - h_{j-2}) + \dots \pm (h_{n-1}^* - h_1) \pm (h_n^* - h_0)] \\
\vdots \\
h_1 \geq -(B_0^+ - B_0^-) - (h_n^* - h_0) \\
h_{n-1}^* \geq -[-(B_0^+ - B_0^-) - (h_n^* - h_0)]
\end{array} \right\} \\
\end{array} \right\} \quad (3.38)$$

Lyapunov graphs continuation

"Knowledge is indivisible. When people grow wise in one direction, they are sure to make it easier for themselves to grow wise in other directions as well. On the other hand, when they split up knowledge, concentrate on their own field, and scorn and ignore other fields, they grow less wise — even in their own field."

Isaac Asimov

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It is well known that the interaction among different areas very often produces interesting results. In our case, the use of network flow theory techniques to obtain topological dynamical system results for Lyapunov graphs, have demonstrated to be very fruitful. This is what will be described in the next sections.

4.1 Lyapunov graphs morsification

First, let us say that an important property of the Conley index is that it is invariant under **continuation**. A parameterized family of flows on M is a collection of flows $\{\phi_t^\lambda : \lambda \in I\}$ indexed by $I = [0, 1]$ such that $\Phi_t(x, \lambda) = (\phi_t^\lambda x, \lambda)$ is a smooth flow on $M \times I$. We say that S^0 , an invariant set for ϕ_t^0 , and S^1 , an invariant set for ϕ_t^1 , are related by continuation if there exists an isolated invariant set $S \subset M \times I$ for Φ_t such that $S^0 = S \cap \{(x, 0)\}$ and $S^1 = S \cap \{(x, 1)\}$. If N is an isolating neighborhood for ϕ_t^λ for all $\lambda \in I$, let $S = \text{Inv}(N \times I)$ in Φ_t . Then S defines a continuation from $S \cap \{(x, 0)\}$ to $S \cap \{(x, 1)\}$. That means the index pair (N, N^-) is the same but the dynamics within N changes significantly. For example, consider the following one parameter family of differential equations: $\dot{x} = x(1 - x^2) - \lambda$, where λ is the parameter. The critical points of this equation is given by the zero set of the function $f(x, \lambda) = x(1 - x^2) - \lambda$, which is represented in Figure 4.1. Each horizontal line corresponds to the phase space of the equation. The fixed points of these equations are isolated invariant sets. An interval in Figure 4.1 is marked which is in fact an isolating neighborhood for the maximal invariant set $S^\lambda = \text{Inv}(N, \phi^\lambda)$ contained in its interior, for each corresponding parameter λ . Therefore, $S^0, S^{\lambda_1}, S^{\lambda_2}$, are related by continuation.

Inspired by Conley's idea of continuation of an isolated invariant set to a simpler one, Reineck proved a continuation result showing that any isolated invariant set on a manifold can be continued to an

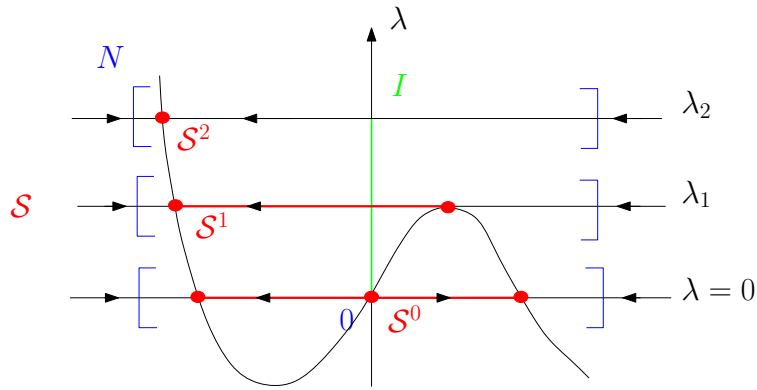


Figure 4.1: Continuation property – 1-parameter family of flows on \mathbb{R} .

isolated invariant set of a gradient flow ([Reineck, 1991, 1992]). One inquires whether (N, N^+, N^-) which has numerical Conley homology indices equal to $(h_0, h_1, \dots, h_{n-1}, h_n)$ admits a gradient-like flow φ_t having precisely h_j nondegenerate singularities of Morse index j , for $j = 0, \dots, n$, such that N^+ and N^- are precisely its entering and exiting sets. As discussed in the previous sections, these gradient flows possessing only nondegenerate singularities are called **Morse flows**. The process of passing from φ to ϕ is called a **morsification of φ** . This is precisely what Figure 4.2 depicts. The picture on its left side represents an isolated invariant set S , labelled with $h_1 = 3$ in (N, N^+, N^-) and endowed with the flow ϕ , whereas the pictures on the right side represent Morse flows ϕ' and ϕ having precisely 3 nondegenerate singularities of Morse index 1 in (N, N^+, N^-) .

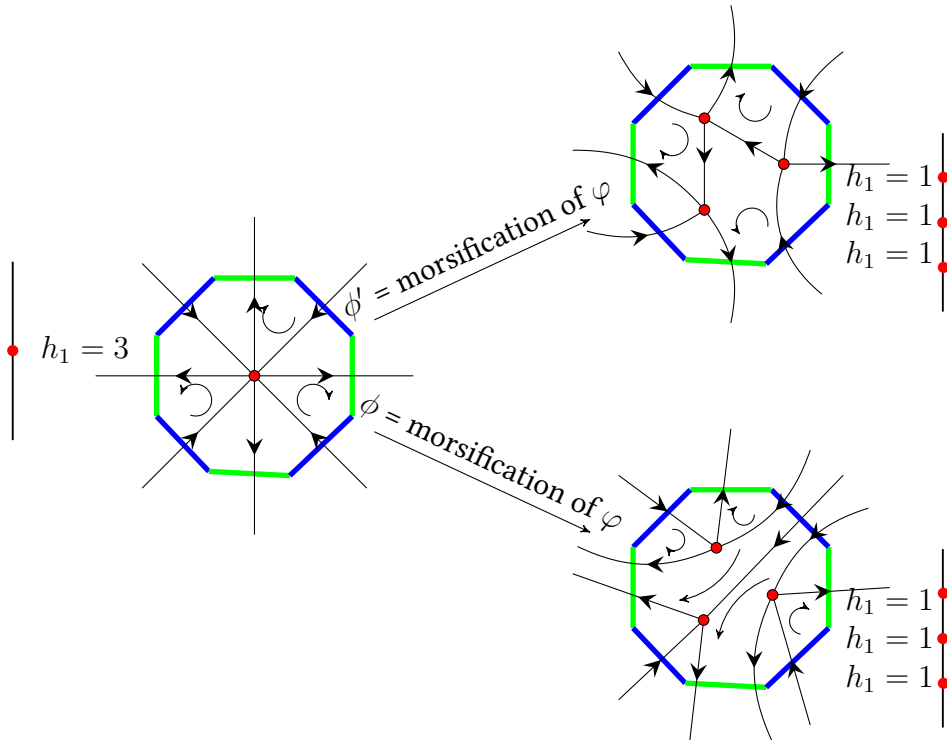


Figure 4.2: Morsification of a degenerate singularity of φ with numerical Conley index $h_1 = 3$.

Since our main question – what abstract Lyapunov graphs L are realizable – is general, we asked how exactly this continuation result could be interpreted in the Lyapunov graphs setting. First we investigate under what conditions L can be continued to an abstract Lyapunov graph of Morse type. We should mention that in [Bertolim et al., 2003a, 2005a, 2003b], we treated the continuation results

of abstract Lyapunov graphs to Lyapunov graphs of Morse type. Since the writing of those articles, the expression **continuation to a graph of Morse type** has evolved to a pithier terminology, **Lyapunov graph morsification**. See [Ledesma et al., 2023]. However, the expression **Lyapunov graph continuation** is still of significance and remains in use whenever the end result is not a graph of Morse type.

Since each vertex should represent a component of the chain recurrent set which we assumed to be an isolated invariant set inside the isolating block, it is natural to treat first the local morsification question. The morsification of L should be done at each vertex v . This consists in replacing v by an abstract graph of Morse type L_v respecting the labels of the incident edges of v and keeping the dimensions of the homological indices of v in the sum of the homological indices of the vertices of L_v as in Figure 4.3. In [Bertolim et al., 2003a], we prove that:

Theorem 6. *Every abstract Lyapunov graph that satisfies the Poincaré–Hopf inequalities (for isolating blocks) at each vertex can be morsified to an abstract Lyapunov graph of Morse type.*

Moreover, all the possible morsifications of a vertex to a Lyapunov graph of Morse type can be computed algorithmically.

As a consequence, an abstract Lyapunov graph which does not admit a morsification cannot be realized by any flow on any manifold. In [Bertolim et al., 2005a] and [Bertolim et al., 2005b] we obtain global conditions for the morsification of abstract Lyapunov graphs.

Figure 4.3 presents an example of an abstract Lyapunov morsification in dimension 7.

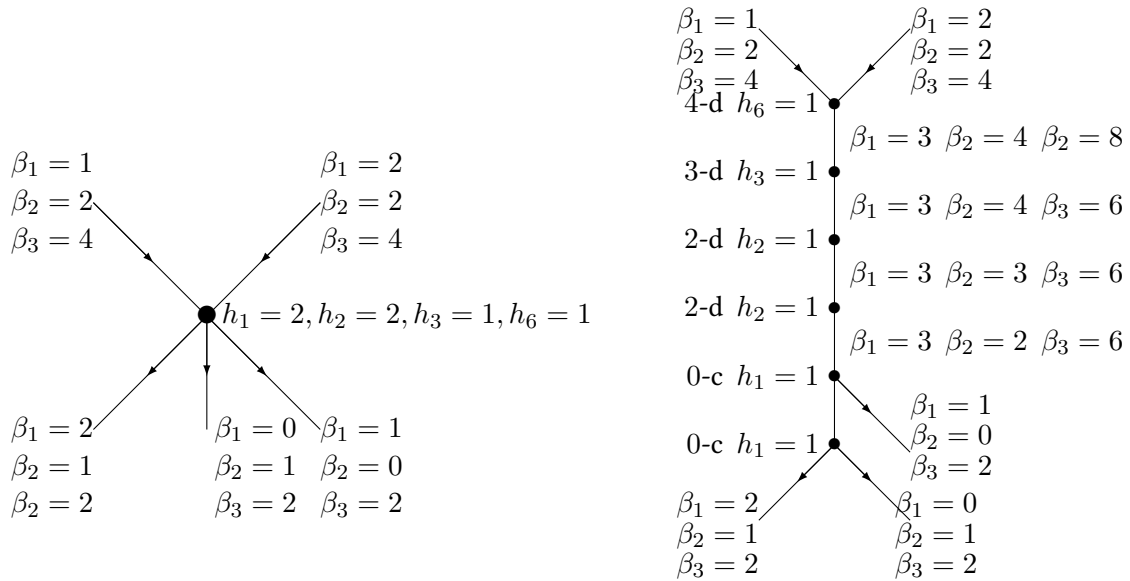


Figure 4.3: A morsification of an abstract Lyapunov.

In [Bertolim et al., 2005b] we also characterize Lyapunov graphs with cycle number κ and we extend the continuation results of [Bertolim et al., 2003a] and the main result of [Bertolim et al., 2005a]. Moreover, we show a more general version of the Poincaré–Hopf inequalities and their relation with the Morse inequalities. We also prove that the first Betti number must be greater than or equal to κ . This work gives a partial answer to the realization problem for Lyapunov graphs on n -manifolds.

The homological techniques used in [Bertolim et al., 2003a,b, 2005a,b] generalize the techniques used in [Cruz and de Rezende, 1999] where Morse–Smale Lyapunov graphs were considered. In

particular, we borrow from it the classification of singularities of index k (into connecting, disconnecting or invariant), based on the homological effect of the singularity on the host manifold's level sets, as discussed in Section 2.4. However, several new techniques were introduced in these works. In [Bertolim et al., 2003a] we introduced the Poincaré–Hopf inequalities for isolating blocks as necessary and sufficient conditions for the morsification. The novelty in the proofs is the use of singularities of index k of type k -disconnecting and $(k - 1)$ -connecting as unknowns of a linear system. Such a linear system is generated by the morsification algorithm which we will discuss in Section 4.1.1. This linear system can be interpreted as a network flow problem. All the solutions to this problem are obtained by means of a particular solution and elementary circulations in the network. From these solutions, we establish a method for constructing the Betti numbers satisfying the generalized Morse inequalities and the relation $\gamma_1 \geq \kappa$ where κ is related to the Cornea genus.

4.1.1 Morsification algorithm

The goal of this algorithm is to give the conditions for an abstract Lyapunov semigraph to be morsified. This was introduced in [Bertolim et al., 2003a]. Let L_v be a Lyapunov semigraph consisting of a vertex v labelled with a singularity whose numerical Conley homology indices are given by $(h_0, h_1, h_2, \dots, h_{n-1}, h_n)$ and incoming edges labelled with $\{(\beta_0^+, \dots, \beta_{n-1}^+)_{i=1}^{e^+}\}$ and outgoing edges labelled with $\{(\beta_0^-, \dots, \beta_{n-1}^-)_{i=1}^{e^-}\}$, where i denotes the edge. Let $B_j^+ = \sum_{i=1}^{e^+} (\beta_j^+)_{i=1}$ and $B_j^- = \sum_{i=1}^{e^-} (\beta_j^-)_{i=1}$. We first consider a Lyapunov semigraph morsification algorithm for a **saddle type vertex** v , that is $h_0 = h_n = 0$, $n \geq 2$ and h_j , $0 < j < n$ nonnegative integers. See Figure 4.4. Observe that $B_0^- = e^-$ and $B_0^+ = e^+$.

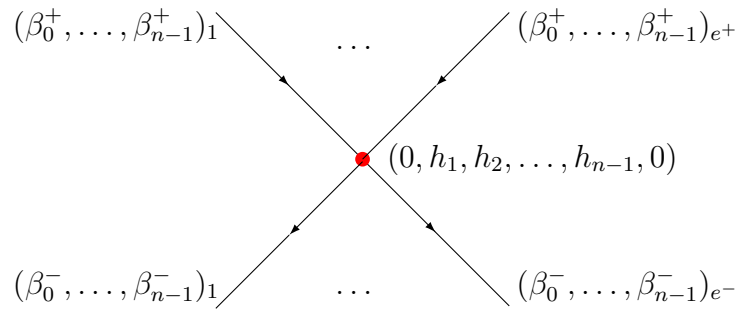


Figure 4.4: Vertex to be morsified.

Using the algebraic effects developed in Section 2.4, the Lyapunov semigraph **morsification algorithm** has four basic steps, which are summarized below. For details see [Bertolim et al., 2003a].

Step 1: adjustment of the incident edges which will defines G^+ and G^- . This is done by choosing $h_1^c = e^- - 1$ and $h_{n-1}^d = e^+ - 1$. This is possible by inequality (3.3). By choosing this number of vertices labelled with 1-singularities, G^- is formed with e^- outgoing edges and one incoming edge. Singularities of type 0-c do not alter the β_j with $0 < j < n - 1$. This type of singularity decreases β_0 and by duality β_{n-1} . Hence, the incoming edge of G^- has $B_0^- = B_{n-1}^- = 1$ and $B_j^- = \sum_{i=1}^{e^-} (\beta_j^-)_{i=1}$ with $j \in \{1, \dots, n - 2\}$. Similarly, the graph G^+ is formed by choosing $e^+ - 1$ vertices labelled with $h_{n-1} = 1$ of type $n - 1$ -d. Figure 4.5 illustrates G^- .

Step 2: insertion of cycles according to the parameter κ . An **elementary cycle** is a pair of (h_1^c, h_{n-1}^d) with one edge labelled with $(1, 0, \dots, 0, 1)$ and the other edge labelled with $(1, \beta_1, \dots, \beta_{n-2}, 1)$. Attach to G^- , κ elementary cycles where $(1, \beta_1, \dots, \beta_{n-2}, 1) = (1, B_1^-, \dots, B_{n-2}^-, 1)$. Of course

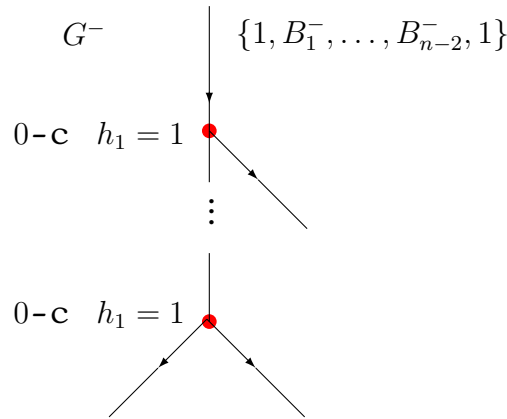


Figure 4.5: Outgoing edges morsified.

this attachment can also be done to G^+ . It is clear that once κ cycles are inserted the number of vertices labelled with $h_1 = 1$ of type 0-c is greater or equal to κ . Similarly, the number of vertices labelled with $h_{n-1} = 1$ of type $n - 1$ -d is greater or equal to κ . Hence all together we have inserted $h_1^c = \kappa + e^- - 1$ vertices labelled with $h_1 = 1$ of type 0-c. Similarly, we have inserted $h_{n-1}^d = \kappa + e^+ - 1$ vertices labelled with $h_{n-1} = 1$ of type $(n - 1)$ -d. Figure 4.6 illustrates the insertion of cycles.

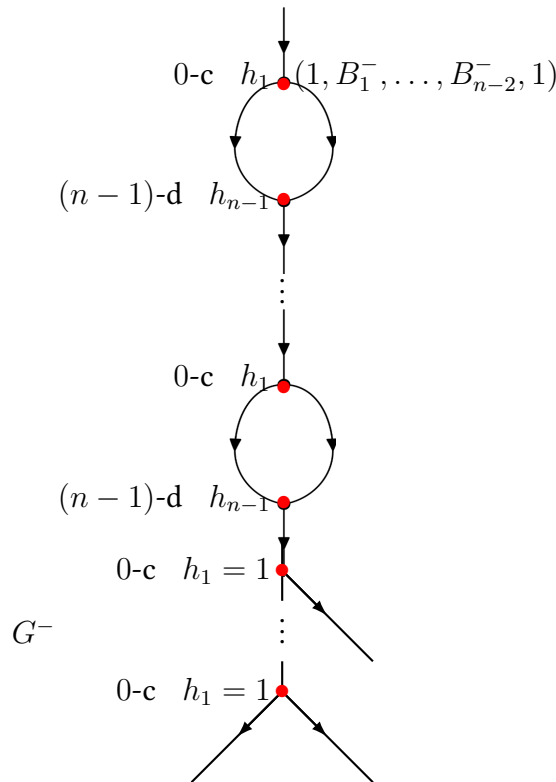


Figure 4.6: Morsification with cycles inserted.

Step 1 and step 2 produces equations (4.1) and (4.2).

Step 3: the linear morsification without middle dimensions which defines $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$ and $G^+ \cup \bigcup_{j=1}^{i-1} L_j^+$. This is done by inserting h_1^d vertices $h_1 = 1$ of type 1-d to G^- and h_{n-1}^c vertices $h_{n-1} = 1$

of type $(n - 2)$ -c. Since the insertion of any other type of vertex will not alter the first and the $(n - 2)$ -th Betti number it is necessary that

$$B_1^- + h_1^d - h_2^c = B_1^+ - h_{n-2}^d + h_{n-1}^c.$$

Figure 4.7 illustrates this procedure.

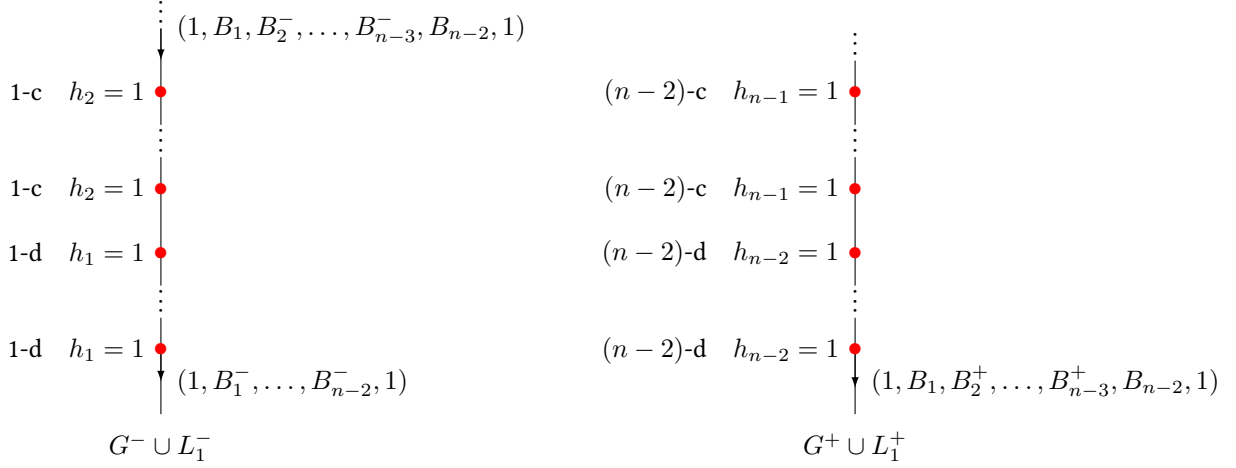


Figure 4.7: Linear morsification: L^- and L^+ .

This step will be repeated until $\lfloor \frac{n-2}{2} \rfloor$, which will produce the equations (4.5) in the linear system.

Step 4: middle dimension morsification where we should consider n odd, $n = 0 \pmod{4}$, $n = 2 \pmod{4}$.

For $n = 2i + 1$: insert h_i^d vertices $h_i = 1$ of type i -d to the incoming edge of $G^- \cup \bigcup_{j=1}^{i-1} L_j^-$ and h_{i+1}^c

vertices $h_{i+1} = 1$ of type i -c to the outgoing edge of $G^+ \cup \bigcup_{j=1}^{i-1} L_j^+$. It is necessary that

$$B_i^- + 2h_i^d = B_i^+ + 2h_{i+1}^c.$$

This produces equation (4.6).

Since, the labels on the outgoing edge of one of the graphs now coincides entirely with the labels on the incoming edge of the other graph, they can be joined to form a connected graph.

For $n = 2i$, there are two cases to consider:

Case 1: $n = 0 \pmod{4}$. Observe that in this case the variation of the middle dimensional Betti number B_{i-1}^- and its dual B_i^- will vary by 0 if the vertex is of β -i type or by 1 otherwise. Insert

h_{i-1}^d vertices $h_{i-1} = 1$ of type $(i - 1)$ -d to the incoming edge of $G^- \cup \bigcup_{j=1}^{i-2} L_j^-$ and insert h_{i+1}^c vertices

$h_{i+1} = 1$ of type i -c to the outgoing edge of $G^+ \cup \bigcup_{j=1}^{i-2} L_j^+$. This explains equation (4.4).

Case 2: $n = 2 \pmod{4}$. The adjustments in this case are identical to the previous case except for the fact that there are no vertices of type β -i.

Hence, at the end of steps 1 to step 4, the following linear h_{κ}^{cd} -system must be solved for $\{h_1^c, h_1^d, \dots, h_{2i}^c, h_{2i}^d\}$, in order for the algorithm to work. We are looking for nonnegative integer solutions to this system.

$$\left\{ \begin{array}{l} h_1^c = e^- - 1 + \kappa, \quad (4.1) \\ h_{n-1}^d = e^+ - 1 + \kappa, \quad (4.2) \\ h_j = h_j^c + h_j^d, \quad j = 1, \dots, n-1 \text{ and } j \neq \left\lfloor \frac{n}{2} \right\rfloor, \quad (4.3) \\ h_{\lfloor \frac{n}{2} \rfloor} = h_{\lfloor \frac{n}{2} \rfloor}^c + h_{\lfloor \frac{n}{2} \rfloor}^d + \beta, \text{ with } \beta = 0 \text{ if } n \not\equiv 0 \pmod{4}, \quad (4.4) \\ h_j^d - h_{j+1}^c - h_{n-j}^c + h_{n-j-1}^d = B_j^+ - B_j^-, \quad j = 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor, \quad (4.5) \\ h_i^d - h_{i+1}^c = \frac{B_i^+ - B_i^-}{2}, \quad \text{if } n = 2i + 1. \quad (4.6) \end{array} \right.$$

In these works, the general case was treated by including h_0 and h_n . This was done by adjusting the edges incident to the vertex v before applying algorithm described above. In this case, we start by constructing G_0^+ with $h_n = 1$ and h_{n-1} vertices labelled with h_{n-1} of type $(n-1)$ -d which has one outgoing edge which connects to v labelled $(1, 0, \dots, 0, 1)$. We also construct G_0^- with $h_0 = 1$ and h_0 vertices labelled with h_0 of type 0-c which has one incoming edge which connects to v labelled $(1, 0, \dots, 0, 1)$. By joining G_0^+ to an edge positively incident to v and G_0^- to an edge negatively incident to v we get a Lyapunov semigraph consisting of a vertex v labelled with a singularity whose numerical Conley homology indices are given by $\{h_0 = 0, h_1 - (h_0 - 1), h_2, \dots, h_{n-1} - (h_n - 1), h_n = 0\}$, which has $e^+ + 1$ positively incident edges to v and $e^- + 1$ negatively incident edges to v . We can then apply the algorithm described above.

We also proved in [Bertolim et al., 2003a] that :

Theorem 7. *The h_κ^{cd} -system (4.1)–(4.6) has nonnegative integral solutions if and only if the Poincaré–Hopf inequalities developed in Section 3.2, for isolating blocks are satisfied. Moreover, the set of all solutions to the h_κ^{cd} -system may be obtained as sums of the complementary solution and multiples of the elementary circulations of the network.*

4.1.2 How network flow theory was used

In this section I present the ideas on how the network flow theory was used to prove our results. In order to facilitate the comprehension, I decided to restrict our considerations to the dimension $n = 7$, $h_0 = h_n = 0$ and $\kappa = 0$. In this case the linear system (4.1)-(4.6) can be written as follows:

$$\left\{ \begin{array}{l} h_1^c = e^- - 1, \quad (4.7) \\ h_6^d = e^+ - 1, \quad (4.8) \\ h_j = h_j^c + h_j^d, \quad j = 1, \dots, 6 \quad (4.9) \\ h_j^d - h_{j+1}^c - h_{7-j}^c + h_{6-j}^d = B_j^+ - B_j^-, \quad j = 1, 2 \quad (4.10) \\ h_3^d - h_4^c = \frac{B_3^+ - B_3^-}{2}. \quad (4.11) \end{array} \right.$$

and our aim is to find a nonnegative integral solution thereto.

The augmented matrix obtained by appending right-hand-side of the system to its coefficient matrix is

$$\left(\begin{array}{cccccccccccc|c} h_1^c & h_1^d & h_2^c & h_2^d & h_3^c & h_3^d & h_4^c & h_4^d & h_5^c & h_5^d & h_6^c & h_6^d & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (e^- - 1) \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & h_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & h_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & h_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & e^+ - 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -(B_1^+ - B_1^-) \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & B_2^+ - B_2^- \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{-(B_3^+ - B_3^-)}{2} \end{array} \right)$$

By a suitable change of sign of half of the equations with right-hand-side h_j (namely, the equations with even j), one can easily assert that the coefficient matrix is totally unimodular¹:

$$\left(\begin{array}{cccccccccccc|c} h_1^c & h_1^d & h_2^c & h_2^d & h_3^c & h_3^d & h_4^c & h_4^d & h_5^c & h_5^d & h_6^c & h_6^d & \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(e^- - 1) \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -h_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & h_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -h_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & e^+ - 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -(B_1^+ - B_1^-) \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & B_2^+ - B_2^- \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{-(B_3^+ - B_3^-)}{2} \end{array} \right) \quad (4.12)$$

As remarked in [Bertolim et al., 2003a], the system may be interpreted as a network flow problem. The corresponding network is shown in Figure 4.8, where $\mathcal{B}_0^+ = e^+ - 1$, $\mathcal{B}_0^- = e^- - 1$, $\mathcal{B}_1 = (B_1^+ - B_1^-)$, $\mathcal{B}_2 = (B_2^+ - B_2^-)$ and $\mathcal{B}_3 = \frac{(B_3^+ - B_3^-)}{2}$. The green arrow in Figure 4.8 is associated with the green coefficients in (4.12). The h^{cd} variables are interpreted as flows along the arcs of the network and associated with each node there is a flow balance equation that may be read as “flow into a node – flow out of a node = node constant”. In the planar embedding adopted in this picture, the zig-zag shape of the directed graph component of the network resembles the lateral structure of a clotheshorse. Arcs corresponding to flow variables $(h_1^c, h_1^d, h_2^c, h_2^d, \dots, h_6^c, h_6^d)$, in this order, form an Eulerian nonoriented path covering the whole directed graph.

This network contains a chain of 2 cycles of length four and the arc sequence associated with $(h_1^c, h_1^d, h_6^c, h_6^d)$ forms a nonoriented path that is adjacent to the first cycle. In the general odd case, when $n = 2i + 1$, the network has i cycles of length four, the arcs in the j -th cycle are associated

¹These matrices were considered by Poincaré [Poincaré, 1900], as early as 1900. A unimodular matrix M is a square integer matrix having determinant $+1$ or -1 . A totally unimodular matrix is a matrix for which every square nonsingular submatrix is unimodular. Equivalently, every square submatrix has determinant $0, +1$ or -1 . A totally unimodular matrix need not be square itself. From the definition, it follows that any submatrix of a totally unimodular matrix is itself totally unimodular (TU). Furthermore, it follows that any TU matrix has only $0, +1$ or -1 entries. The converse is not true, i.e., a matrix with only $0, +1$ or -1 entries is not necessarily totally unimodular. A matrix is TU if and only if its transpose is TU.

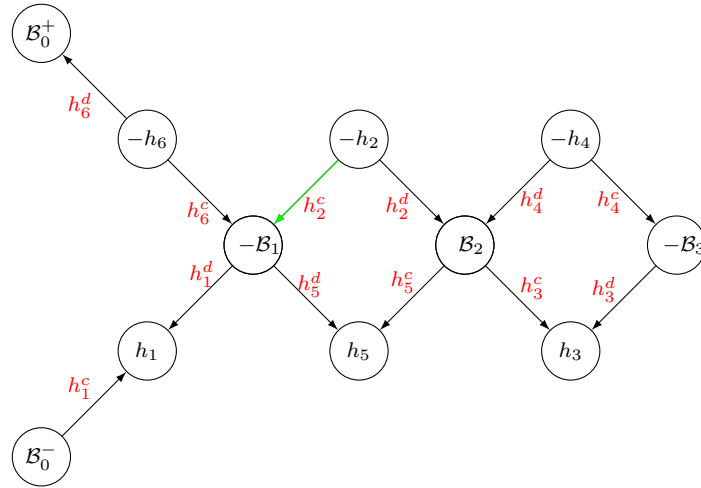


Figure 4.8: Network in dimension 7.

with variables h_{j+1}^d , h_{6-j}^c , h_{6-j}^d and h_{j+1}^c , and the orientation of the first two arcs in the cycle is opposite to the orientation of the last two, with respect to an arbitrary orientation thereof.

The trick to establishing the conditions under which the network flow problem admits solutions, in the general case, is to split the network flow problem into independent subproblems. This splitting occurs at the nodes with degree four. Figure 4.9 shows the decomposition for the problem depicted in Figure 4.8, where $A = -(h_1 - h_6 + (e^+ - 1) - (e^- - 1))$, $B = -(B_1^+ - B_1^-) - A$, $C = -(B - h_2 + h_5)$, $D = (B_2^+ - B_2^-) - C$ and $E = \frac{-(B_3^+ - B_3^-)}{2}$. In the case $n = 2i + 1 = 7$, there will be three independent subproblems, in the general case, there will be i .

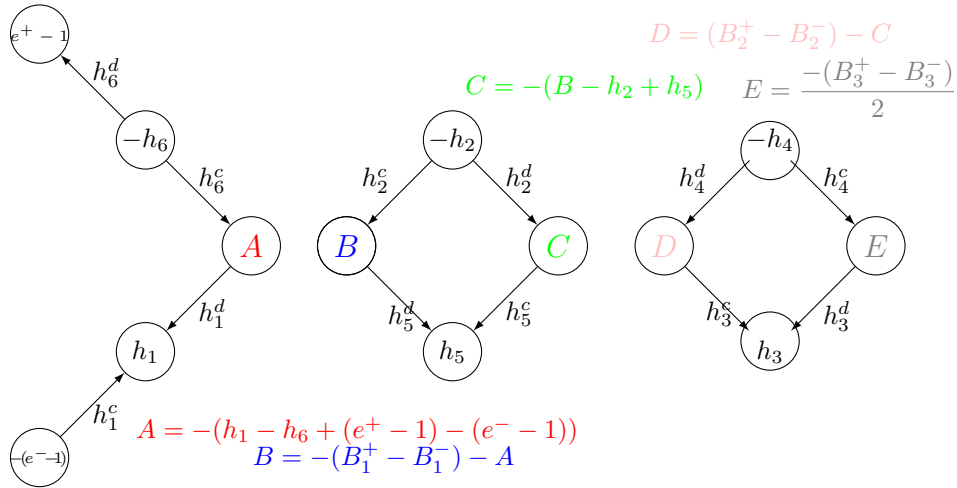


Figure 4.9: Splitting the problem in three.

Algebraically, the original linear system in the $2(n - 1)$ variables of the h^{cd} vector, where $n = 2i + 1$, is split into i independent (and smaller) linear systems. Any solution of the original system can be split into solutions of the individual smaller systems, and, conversely, any set of solutions, one for each smaller system, can be composed to form a solution of the larger original system (see [Bertolim et al., 2003a] for a detailed proof of this result). The leftmost network in Figure 4.9 corresponds to a determinate system and has a unique solution. The subsequent subnetworks correspond to indeterminate linear systems, and the all have the same cycle structure. The existence of nonnegative solutions is equivalent to the feasibility of a set of inequalities. This follows from the fact that the

the general solution to a linear system is the sum of a particular solution and a solution to the homogeneous version of the system, that is, with the right-hand-side replaced with the zero vector. This is illustrated in Figure 4.10 for the subnetwork in the middle of Figure 4.9.

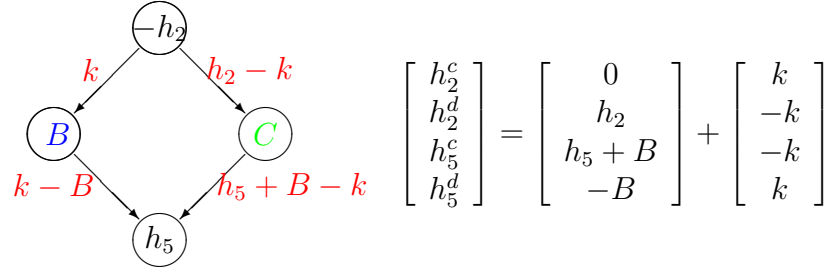


Figure 4.10: General solution of subnetwork in Figure 4.9.

A necessary and sufficient condition for the indeterminate system corresponding to the network flow problem in Figure 4.10 to have a solution is that $-h_2 + B + C + h_5 = 0$. This is automatically satisfied by the definition of C . Nonnegative solutions exist if and only if there exists a value of k for which the corresponding general solution shown in the figure is nonnegative. This is embodied in the inequalities (4.13)-(4.16) below.

$$k \geq 0, \quad (4.13)$$

$$h_2 \geq k, \quad (4.14)$$

$$k \geq B, \quad (4.15)$$

$$h_5 + B \geq k. \quad (4.16)$$

Applying the Fourier-Motzkin method described in [Bachem and Grötschel, 1982] to the previous system, we may eliminate k to obtain the following set of inequalities.

$$h_2 \geq B, \quad (4.17)$$

$$h_5 \geq B. \quad (4.18)$$

By replacing the expression for $B = -(B_1^+ - B_1^-) - A = -(B_1^+ - B_1^-) - [-(h_1 - h_6 + (e^+ - 1) - (e^- - 1))]$ in (4.17)-(4.18), we finally arrive at the following inequalities.

$$\begin{cases} h_2 \geq -(B_1^+ - B_1^-) - (h_6 - h_1) + (e^+ - e^-), \\ h_5 \geq -\left[-(B_1^+ - B_1^-) - (h_6 - h_1) + (e^+ - e^-)\right]. \end{cases}$$

These inequalities are part of the Poincaré–Hopf inequalities presented in Section 3.1. Other are obtained in the same fashion.

The number of continuations, i.e., the number of nonnegative integral flows, has been calculated in [Bertolim et al., 2003a] for $\kappa = 0$. It represents the number of admissible multiples of elementary circulations of the network. Since the values of h_1^c , h_1^d , h_6^c and h_6^d are uniquely determined, this is the number of nonnegative integral flows of the smaller network obtained after the elimination of these four variables. Thus, the total number of continuations is just the number of possible values of $\kappa (1 + \min\{h_1 - (h_0 - 1), h_6 - (h_7 - 1)\})$ times the number of continuations for $\kappa = 0$.

4.2 Ordered continuation

In this section we present ordered morsifications of an abstract Lyapunov semigraph L and define the Ogasa number for ordered continuations of L . We can ask ourselves why this kind of result is relevant. As discussed in the previous sections, abstract Lyapunov semigraphs of Morse type L_M can be viewed as a handle decomposition (h, h^{cd}, β) , where h represents the distribution of the indices of the handles, and h^{cd} the distribution of the types, i.e., connecting or disconnecting. Since the morsification results presented in [Bertolim et al., 2003a] imply that the continuation of an abstract Lyapunov graph is not unique, one natural question in this direction is: are all graphs, generated by the continuation procedure, realizable? If so, on what manifolds? These questions were answered in [Bertolim et al., 2006a] by building Morse flows on n -dimensional isolating blocks, together with a special and restricted class of gluing maps of handles, and in [Cruz et al., 2005] for closed manifolds. This will be discussed in Section 6, but for now just keep in mind that an important step in the construction presented in [Bertolim et al., 2006a] was a handle type decomposition theorem which, together with a special class of gluing maps, insures that this construction not only preserves the given ranks of the homology Conley indices, but it is also optimal in the sense that no other Morse flow can preserve this index with fewer singularities. Another important fact is that, for some abstract Lyapunov graphs, we cannot fix an ordering without producing negative labels on their edges. Since the edges are labeled with the Betti numbers of level sets, this does not make sense in realizability questions. Hence, the need to define ordered continuations.

Let us consider the following example. Suppose that we have an abstract Lyapunov semigraph in dimension $n = 5$ containing one vertex v labeled with $h = (0, 2, 1, 0, 1, 0)$, v has two outgoing edges, $e^- = 2$, and two incoming edges, $e^+ = 2$, as illustrated by the left side of Figure 4.11. Let $B_j^\pm = \sum_{i=1}^{e^\pm} \beta_j$ where $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$. Note that since $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ satisfy the Poincaré duality and since β_0 and β_4 represent the number of boundary components, we have put only (β_1, β_2) in Figure 4.11. Using the continuation results presented in [Bertolim et al., 2003a], we can have two continuations illustrated by the middle and right side of Figure 4.11. Observe that the continuation represented by the middle graph of Figure 4.11 has an edge labeled with a negative number, $B_1 = -1$. This continuation is not admissible in our procedure of ordered continuation, since only nonnegative labels on the edges will be allowed.

An invariant associated with handle decompositions of a smooth connected compact n -manifold with boundary M^n was introduced in [Ogasa, 2005]. The author considers a way of measuring the simplicity of a Morse flow by computing, for each regular level, the sum of its Betti numbers, and then taking the maximum of the obtained values. Given an M^n , its **Ogasa invariant** is then the minimum, over all Morse flows, of these maxima. In other words, a Morse flow realizing the Ogasa invariant of M^n is one for which the **maximum of the sums of the Betti numbers of each regular level is the smallest** possible. This invariant was also studied in [Bertolim et al., 2011b].

Definition 8. (Ogasa invariant, [Ogasa, 2005]) Let (M, f) be the pair consisting of a smooth connected compact n -manifold with boundary and a Morse function f defined on M . Let R denote the ring of the homology coefficients, which can be chosen among $\frac{\mathbb{Z}}{p\mathbb{Z}}$ (p prime), \mathbb{Z} , \mathbb{Q} or \mathbb{R} . For any ordered handle decomposition \mathcal{H} of M , with total numbers $\eta_{\mathcal{H}}$ of handles, let

$$\nu_{\mathcal{H}}(M; R) = \max_{l=1, \dots, \eta_{\mathcal{H}}} \left(\sum_{i=0}^{n-1} \beta_i(N_l; R) \right)$$

where $\beta_i(N_l; R)$ denotes the i -th Betti number of the $(n-1)$ -dimensional manifold $N_l = \partial M_l$, which is the boundary of the manifold obtained after attaching the first l handles of \mathcal{H} . The **Ogasa invariant** of the manifold M (relative to R) is the number defined by:

$$\nu(M; R) = \min_{\mathcal{H}} \nu_{\mathcal{H}}(M; R)$$

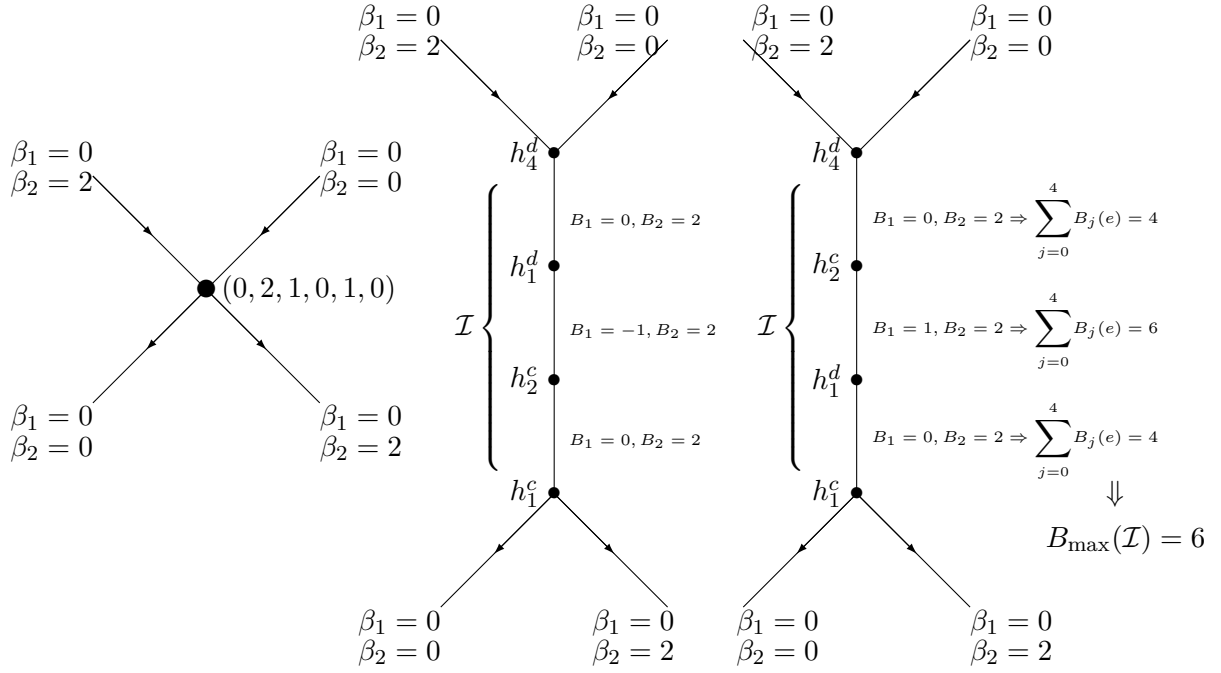


Figure 4.11: The leftmost figure represents an abstract Lyapunov semigraph L while the middle and rightmost figures represent respectively an unfeasible and a feasible morsification of L .

where the minimum is taken over all of the ordered handle decompositions of M .

Given that the realization of ordered continuations of a Lyapunov graph L on M can be viewed as an ordered handle decomposition of M , we study a similar invariant for ordered continuations of L . We denote an ordered continuation of L by $L(h, h^{cd}, \beta)$ where h^{cd} is an ordered list of the types, i.e., ℓ -d and ℓ -c vertices in L .

We now proceed to define the Ogasa number for ordered continuations of an abstract Lyapunov semigraph L . Let \mathcal{I} be the semi-subgraph of an ordered continuation $L(h, h^{cd}, \beta)$ of L obtained by deleting from $L(h, h^{cd}, \beta)$ all the h_1^c vertices together with their outgoing edges and all the h_{n-1}^d vertices together with their incoming edges that do not belong to cycles in the continued semigraph. The continuation algorithm in [Bertolim et al., 2003a] assures that for each ordered continuation, \mathcal{I} is unique and has exactly one semi-edge² (dangling edge) of the form (∞, v) and exactly one semi-edge of the form (v, ∞) , which we denote by $e_{\mathcal{I}}^+$ and $e_{\mathcal{I}}^-$ respectively. Without loss of generality, the ordered continuation $L(h, h^{cd}, \beta)$ of L can be represented by \mathcal{I} . Denote by B_j^+ the j -th Betti number on $e_{\mathcal{I}}^+$ and by B_j^- the j -th Betti number on $e_{\mathcal{I}}^-$.

Now, for each ordered continuation of L , define the sum $\sum_{j=0}^{n-1} B_j(e)$ for each edge e in \mathcal{I} .

Definition 9. Given an ordered continuation for \mathcal{L} , let

$$B_{\max}(\mathcal{I}) = \max_{e \in \mathcal{I}} \sum_{j=0}^{n-1} B_j(e).$$

Let $\mathcal{I}_{\mathcal{L}}$ be the set of all ordered continuations $\mathcal{I} \subset L(h, h^{cd}, \beta)$ of \mathcal{L} . The **Ogasa number of \mathcal{L}** is

$$\min_{\mathcal{I} \in \mathcal{I}_{\mathcal{L}}} B_{\max}(\mathcal{I}).$$

²Given a finite set V we define a *directed semi-graph* $G' = (V', E')$ as a pair of sets $V' = V \cup \{\infty\}$, $E' \subset V' \times V'$. As usual, we call the elements of V' vertices and since we regard the elements of E' as ordered pairs, these are called *directed edges*. Furthermore, the edges of the form (∞, v) and (v, ∞) are called *semi-edges* (or *dangling edges* as in [de Rezende, 1987]).

Our main result in [Bertolim et al., 2013] gives lower and upper bounds for the Ogasa number.

Theorem 10. *Given $B^+ = \sum_{j=0}^{n-1} B_j^+$, $B^- = \sum_{j=0}^{n-1} B_j^-$ and (h, h^{cd}, β) satisfying the Poincaré–Hopf inequalities,*

$$\max \{B^+, B^-\} \leq \min_{\mathcal{I} \in \mathcal{I}_{\mathcal{L}}} B_{\max}(\mathcal{I}) \leq 2 + \max \{B^+, B^-\}.$$

The Ogasa number for ordered continuations of Lyapunov graphs may, in some cases, coincide with the Ogasa invariant for ordered handle decompositions.

In order to strengthen the connection between the Ogasa invariant for ordered handle decompositions of a manifold M and the Ogasa number of ordered continuations of a Lyapunov graph \mathcal{L} , it is important to study ordered continuations under restrictions on the number of consecutive d-type labels that must appear in the continuation. Intervals \mathcal{I} containing at least r consecutive d-type labels are denoted by \mathcal{I}_r .

Corollary 4.2.0.1. *Let (h, h^{cd}, β) be a solution of the Poincaré–Hopf inequalities, with at least $r (\geq 1)$ consecutive d-type elements. Then*

$$\max \{B^+, B^-\} \leq \min_{\mathcal{I}_r \in \mathcal{I}_{\mathcal{L}}} B_{\max}(\mathcal{I}_r) \leq 2r + \max \{B^+, B^-\}.$$

In the example illustrated in Figure 4.11, $B_{\max} = \sum_{j=0}^{n-1} B_j = 6$, and since the middle continuation can not be considered, the minimal of B_{\max} over all continuations is 6. Hence, the Ogasa number for this graph is 6.

4.3 Lyapunov graph continuation to graphs of Smale type

As discussed in Section 2.4, inspired by a result in [Franks, 1982] which shows that one can think of a hyperbolic periodic orbit of index k , A_k , as a suitable joining of two hyperbolic singularities p and q of adjacent indices k and $k + 1$ respectively, we define **an abstract Lyapunov graph of Smale type** as an abstract Lyapunov graph of Morse type such that each vertex labelled with a hyperbolic singularity of index k can be joined with a vertex labelled with a hyperbolic singularity of adjacent index $k + 1$. This means that the collection of labels (h_0, \dots, h_n) on the vertices of the Lyapunov graph can be decomposed in an union $(\sum_{j=0}^n h_j)/2$ of disjoint couples of the form $A_\ell = (h_\ell, h_{\ell+1})$, with $\ell \in \{0, \dots, n\}$.

Motivated by the ideas developed in [Bertolim et al., 2003a], in [Bertolim et al., 2023a] the problem of continuation of an abstract Lyapunov graph to an abstract Lyapunov graph of Smale type was considered. The Poincaré–Hopf inequalities for nonsingular Morse–Smale flows described in 3.4 were introduced. They give necessary and sufficient conditions for the continuation of an abstract Lyapunov graph to an abstract Lyapunov graph of Smale type to occur. More precisely, we prove that:

Theorem 11. *A (generalized) vertex of a Lyapunov graph associated with the homological $\left\{n, e^+, e^-, (-1)^j \{B_j^+ - B_j^-\}_{j=1}^{\frac{n-1}{2}}\right\}$ can be continued to a Lyapunov graph of Smale type if and only if a set of explicit inequalities (Poincaré–Hopf inequalities (3.1), (3.2), (3.5), (3.14) and (3.17) + coupling inequalities(4.19)-(4.21)) are satisfied by such data.*

The **Coupling inequalities** are the following:

$$(-1)^k \sum_{j=0}^k (-1)^j h_j \geq 0, \quad k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (4.19)$$

$$(-1)^k \sum_{j=0}^k (-1)^j h_{n-j} \geq 0, \quad k = 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (4.20)$$

$$\sum_{j=0}^n (-1)^j h_j = 0. \quad (4.21)$$

Coming back to the prior example in dimension 7 developed in Section 4.1 and given in Figure 4.3, we remark that the homological data given initially in this example satisfies the Poincaré–Hopf inequalities (3.1), (3.2), (3.5), (3.14) and (3.17), but doesn't satisfy the coupling inequalities (4.19)–(4.21). In order to be able to continue the abstract Lyapunov graph on the left side of Figure 4.3 to an abstract Lyapunov graph of Smale type, one should increase h_2 and h_5 by 1. Figure 4.12 illustrates a continuation of an abstract Lyapunov graph to an abstract Lyapunov graph of Smale type in dimension 7, where each orbit of index j combinatorially corresponds to a couple of consecutive singularities (h_j, h_{j+1}) . We have two couples (h_1, h_2) , one couple (h_2, h_3) and one couple (h_5, h_6) .

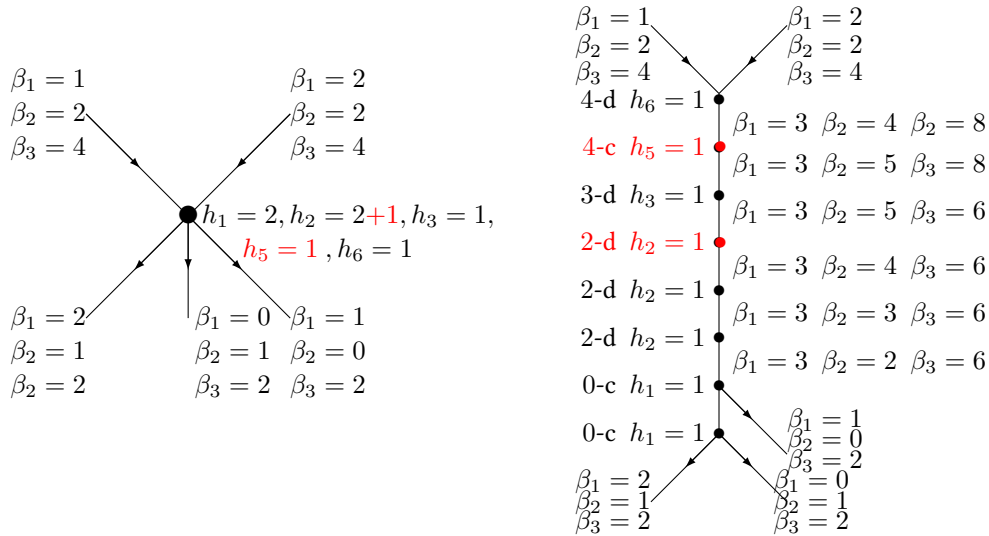


Figure 4.12: A continuation of an abstract Lyapunov graph to an abstract Lyapunov graph of Smale type.

4.4 Componentwise Morsification

In Section 4.1, one easily verifies that the morsification algorithm takes into account exclusively the differences

$$B_j^+ - B_j^- = \sum_{k=1}^{e^+} \beta_j(N_k^+) - \sum_{k=1}^{e^-} \beta_j(N_k^-),$$

for all $j = 1, \dots, (n-1)/2$. This means that, for any choice of $\beta_j(N_k^+)$, on the incoming edges, as well as $\beta_j(N_k^-)$, on the outgoing edges, that respect the above differences will produce the exact same solutions given by the network flow. See Figures 4.13 and 4.14.

In fact, Lyapunov graph morsifications treated in [Bertolim et al., 2003a, 2005a, 2003b] depend only on the global Betti number information of the labels on the incoming and outgoing edges, more

specifically on the differences $B_j^+ - B_j^-$. Hence, this is the reason we are not able to distinguish in terms of the morsified data between the two situations depicted in Figures 4.13 and 4.14. The morsifications of both graphs are identical although the individual Betti numbers on the incident edges are not.

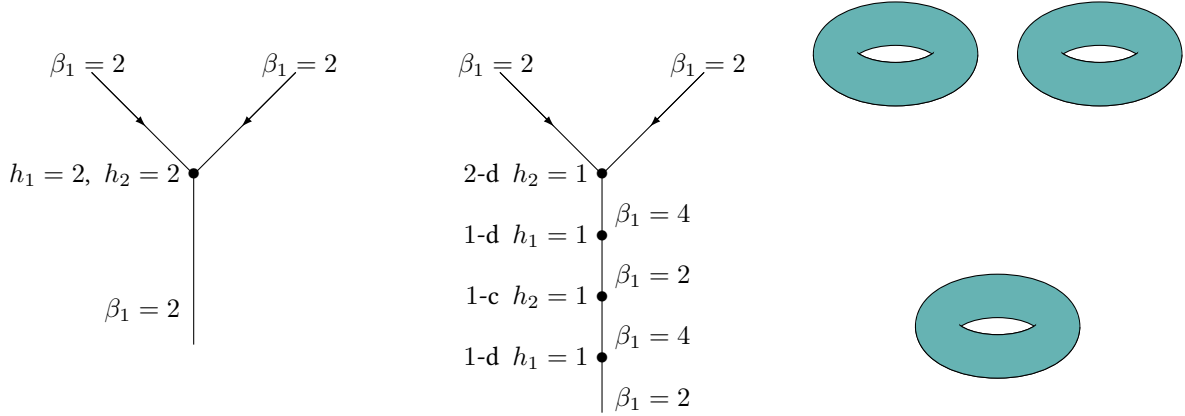


Figure 4.13: An abstract Lyapunov graph satisfying the classical Poincaré–Hopf inequalities on the left and its morsification on the right.

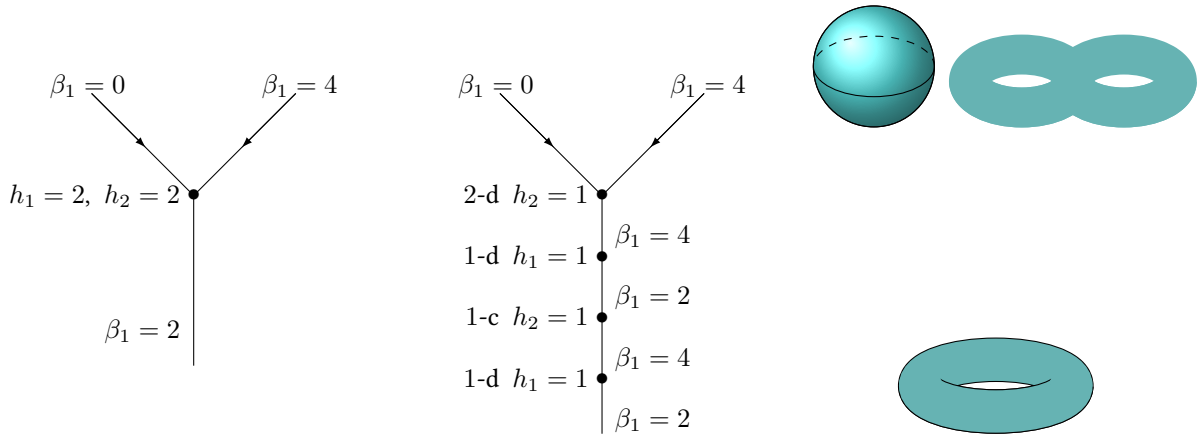


Figure 4.14: An abstract Lyapunov graph satisfying the classical Poincaré–Hopf inequalities on the left and its morsification on the right.

The componentwise morsification developed in [Bertolim et al., 2023b] distinguishes the examples in Figures 4.13 and 4.14. By retaining the Betti number information on each incident edge at each step of the morsification algorithm, one keeps track of the variations of the Betti numbers componentwise. This was done by means of a componentwise morsification algorithm which is similar to the algorithm developed in Section 4.1.1. Nevertheless, one should care about the morsification for each edge and the ground level which we want to attain. We choose to provide elucidating examples rather than explicate the componentwise morsification algorithm that due to the high number of variables involved can be a bit cumbersome. Details on this algorithm can be found in [Bertolim et al., 2023b]. But what is important to mention here is that in the same way as in the morsification developed in Section 4.1, the componentwise morsification algorithm produces a new linear system and in order to find nonnegative solutions to this system, we use the network flow theory once again.

With the aim of keeping a finer control on the changes in Betti numbers during the morsification, a new collection of inequalities, the **Poincaré–Hopf inequalities for componentwise morsification (in short, componentwise Poincaré–Hopf inequalities)** as discussed in Section 3.6 was introduced. They give necessary and sufficient conditions for the componentwise morsification of Lyapunov graphs.

It can be verified that both examples in Figures 4.13 and 4.14 satisfy the componentwise Poincaré–Hopf inequalities and thus have componentwise morsifications as illustrated in Figures 4.15 and 4.16. For $n = 3$, these inequalities reduce to the two equations (3.35) and (3.36). It is easy to see in Figures 4.15 and 4.16 that the componentwise morsifications are different, i.e., the level sets in the morsification are not the same, although the number of singularities of type c and d are identical.

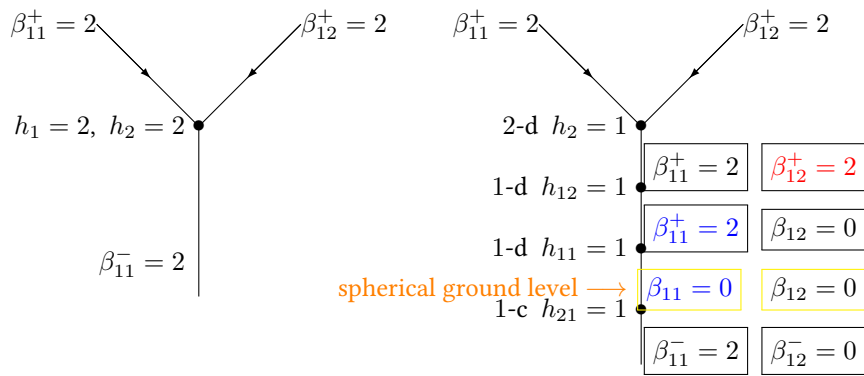


Figure 4.15: An abstract Lyapunov graph satisfying the componentwise Poincaré–Hopf inequalities and its componentwise morsification.

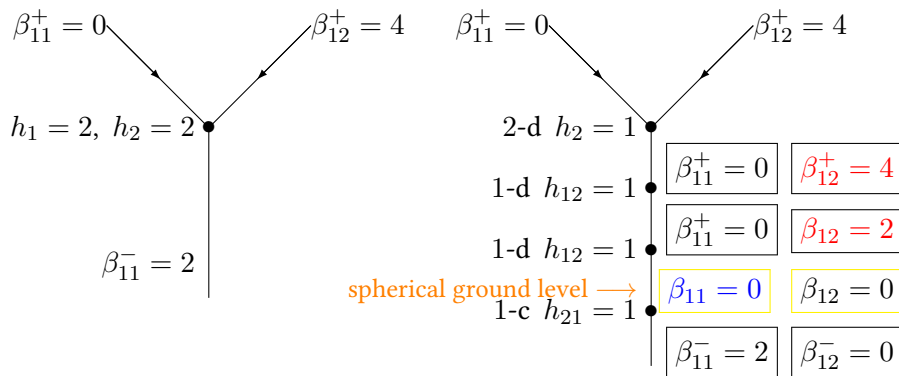


Figure 4.16: An abstract Lyapunov graph satisfying the componentwise Poincaré–Hopf inequalities and its componentwise morsification.

We prove in [Bertolim et al., 2023b] the following main results.

Theorem 12. *The topological-dynamical data that satisfies the componentwise Poincaré–Hopf inequalities (3.29)–(3.34) also satisfies the classical Poincaré–Hopf inequalities (3.1)–(3.5). In other words, every abstract Lyapunov graph which satisfies the componentwise Poincaré–Hopf inequalities also satisfies the classical Poincaré–Hopf inequalities.*

Theorem 13. *A Lyapunov semigraph L_v with the homological data*

$$\{n, h_\ell, e^+, e^-, \beta_{jk}^+, \beta_{jk}^-, k = 1, \dots, e^\pm, j = 1, \dots, \lfloor (n-1)/2 \rfloor, \ell = 1, \dots, n-1\}$$

can be componentwise morsified if and only if the componentwise Poincaré–Hopf inequalities are satisfied by such data.

Topologically speaking, this means that, in case of realization, the isolating block associated with this semigraph may contain a level which is an $(n-1)$ -sphere.

Of course, if each vertex of an abstract Lyapunov graph satisfies the componentwise Poincaré–Hopf inequalities, Theorem 13 can be generalized in the following way.

Theorem 14. *Every abstract Lyapunov graph L that satisfies the componentwise Poincaré–Hopf inequalities at each vertex can be componentwise morsified.*

By no means is componentwise morsification unique. It is precisely the network flow circulations that will exhibit all possibilities.

In order to better understand the morsification algorithm, what it entails and what it produces, consider the following example in dimension 5. For didactic purposes, let L_v be a Lyapunov semigraph consisting of a vertex labelled with a singularity whose numerical Conley homology indices are given by $h_0 = 0, h_1 = 4, h_2 = 4, h_3 = 4, h_4 = 4, h_5 = 0$ and its incident edges. Each incoming and outgoing edge is labeled with Betti numbers $(\beta_{0k}^\pm, \beta_{1k}^\pm, \beta_{2k}^\pm, \beta_{3k}^\pm, \beta_{4k}^\pm)_{k=1}^{e^\pm}$, as depicted in Figure 4.17, in which we assign concrete values to these numbers. Note that since $(\beta_{0k}^\pm, \beta_{1k}^\pm, \beta_{2k}^\pm, \beta_{3k}^\pm, \beta_{4k}^\pm)_{k=1}^{e^\pm}$ satisfy the Poincaré duality and since $\sum_{k=1}^{e^\pm} \beta_0^\pm$ and $\sum_{k=1}^{e^\pm} \beta_4^\pm$ represent the number of boundary components, we need only represent $(\beta_{1k}^\pm, \beta_{2k}^\pm)_{k=1}^{e^\pm}$ in Figure 4.17.

In dimension 5, the componentwise Poincaré–Hopf inequalities, described in Section 3.6, are the following:

$$h_1 \geq e^- - 1 \quad (4.22)$$

$$\sum_{k=1}^{e^+} \beta_{1k}^+ \geq h_1 - (e^- - 1) \quad (4.23)$$

$$\sum_{k=1}^{e^-} \frac{\beta_{2k}^-}{2} = h_3 - \sum_{k=1}^{e^+} \beta_{1k}^+ + h_1 - (e^- - 1) \quad (4.24)$$

$$h_4 \geq e^+ - 1 \quad (4.25)$$

$$\sum_{k=1}^{e^-} \beta_{1k}^- \geq h_4 - (e^+ - 1) \quad (4.26)$$

$$\sum_{k=1}^{e^+} \frac{\beta_{2k}^+}{2} = h_2 - \sum_{k=1}^{e^-} \beta_{1k}^- + h_4 - (e^+ - 1). \quad (4.27)$$

It is easy to verify that the abstract Lyapunov graph in Figure 4.17 satisfies the inequalities (4.22)–(4.27).

Figure 4.18 depicts the network-flow that models the system of equations produced by the componentwise morsification algorithm for the Lyapunov semigraph in Figure 4.17. Notice that it is composed of two disjoint and independent sub-networks. The equations from the linear system produced by the algorithm can be obtained from the network. The labels on the arcs represent the variables (interpreted as flows along the arcs) associated therewith. Each node represents an equation, the label on each node is the constant that appears in the corresponding equation. Flow going

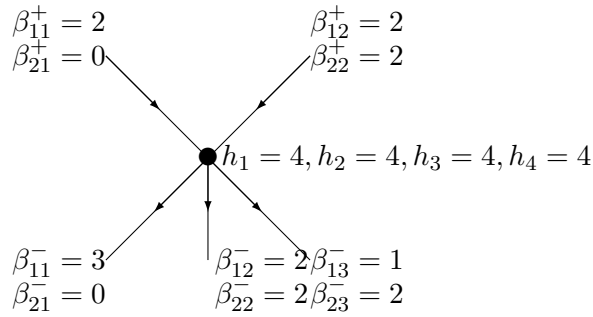


Figure 4.17: A Lyapunov semigraph to be morsified.

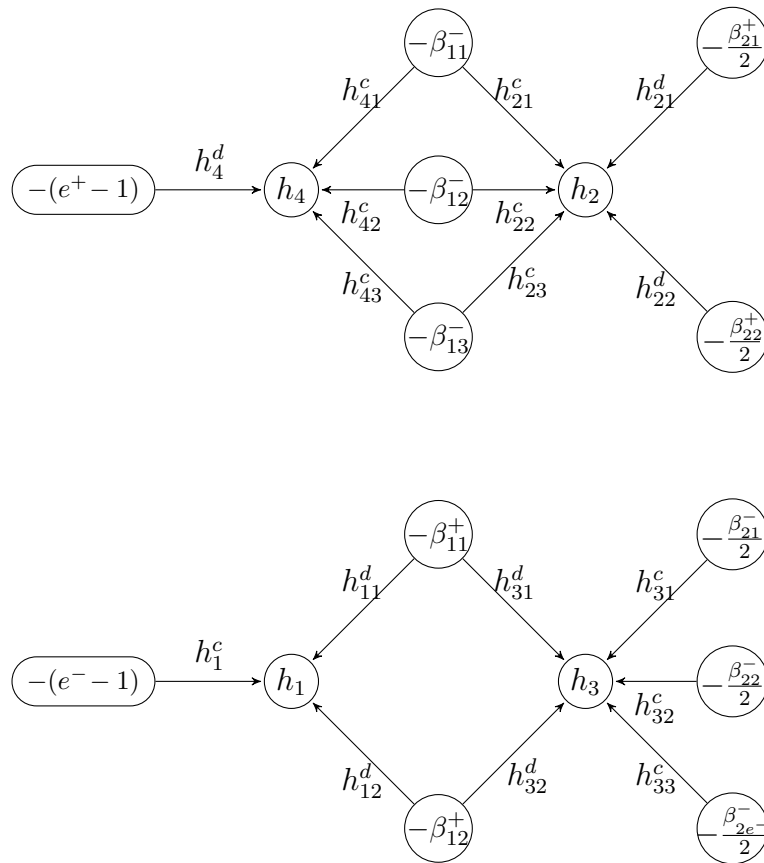


Figure 4.18: Network modeling the system of equations obtained by applying the morsification algorithm to the Lyapunov semigraph in Figure 4.17.

into (resp., out of) a node is the sum of flows along arcs whose arrows point into (resp., out of) the node. The difference of these quantities must equal the constant associated with the node. For instance, the equation associated with node h_1 is $h_1 = h_1^c + h_{11}^d + h_{12}^d$.

Figures 4.19 and 4.20 give two (out of a total of 18 possible) componentwise morsifications of the Lyapunov semigraph in Figure 4.17.

The morsification given in Figure 4.19 of the Lyapunov semigraph in Figure 4.17 is depicted in red, in Figure 4.21, as flows on the network presented in Figure 4.18.

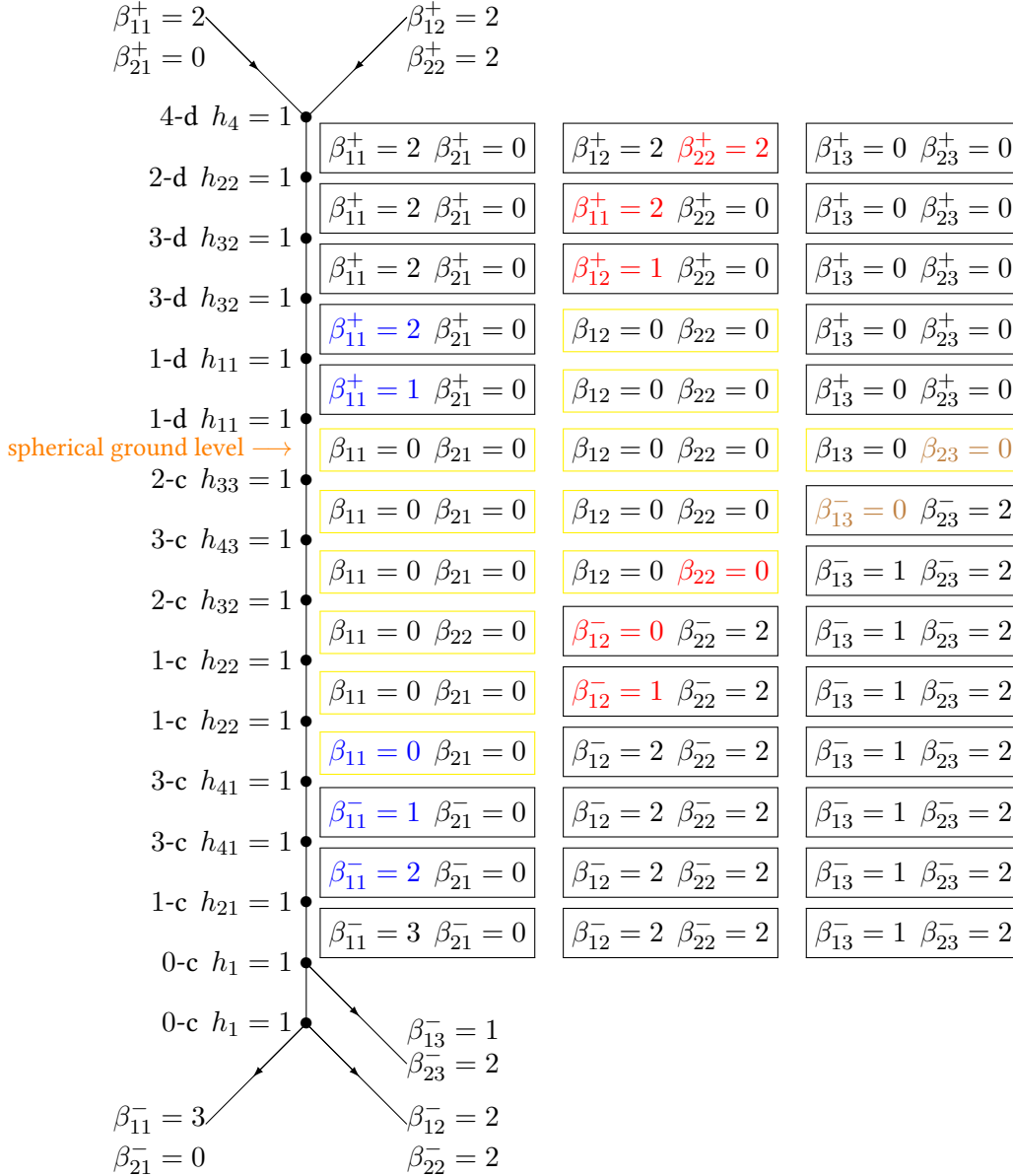


Figure 4.19: A Lyapunov graph componentwise morsification of the semigraph in Figure 4.17.

The morsification described in Figure 4.20 differs from the one in Figure 4.19 in only four elements. These are the flows depicted in blue in Figure 4.21. The change from the red to the blue/red solution is effected by means of passing a circulation along the cycle containing arcs $h_{41}^c, h_{42}^c, h_{22}^c, h_{21}^c$.

Concerning linear graphs (corresponding to $e^- = e^+ = 1$), if the homological data satisfies the componentwise Poincaré–Hopf inequalities we proved in [Bertolim et al., 2023b] that the morsification is unique. Figure 4.22 furnishes an example of a linear abstract Lyapunov graph. It corresponds to the following homological data: $h_1 = 3, h_2 = 4, h_3 = 1, \beta_{11}^- = 2, \beta_{21}^- = 2, \beta_{11}^+ = 3, \beta_{21}^+ = 4$. On the right of this figure we present the only morsification available to this linear graph.

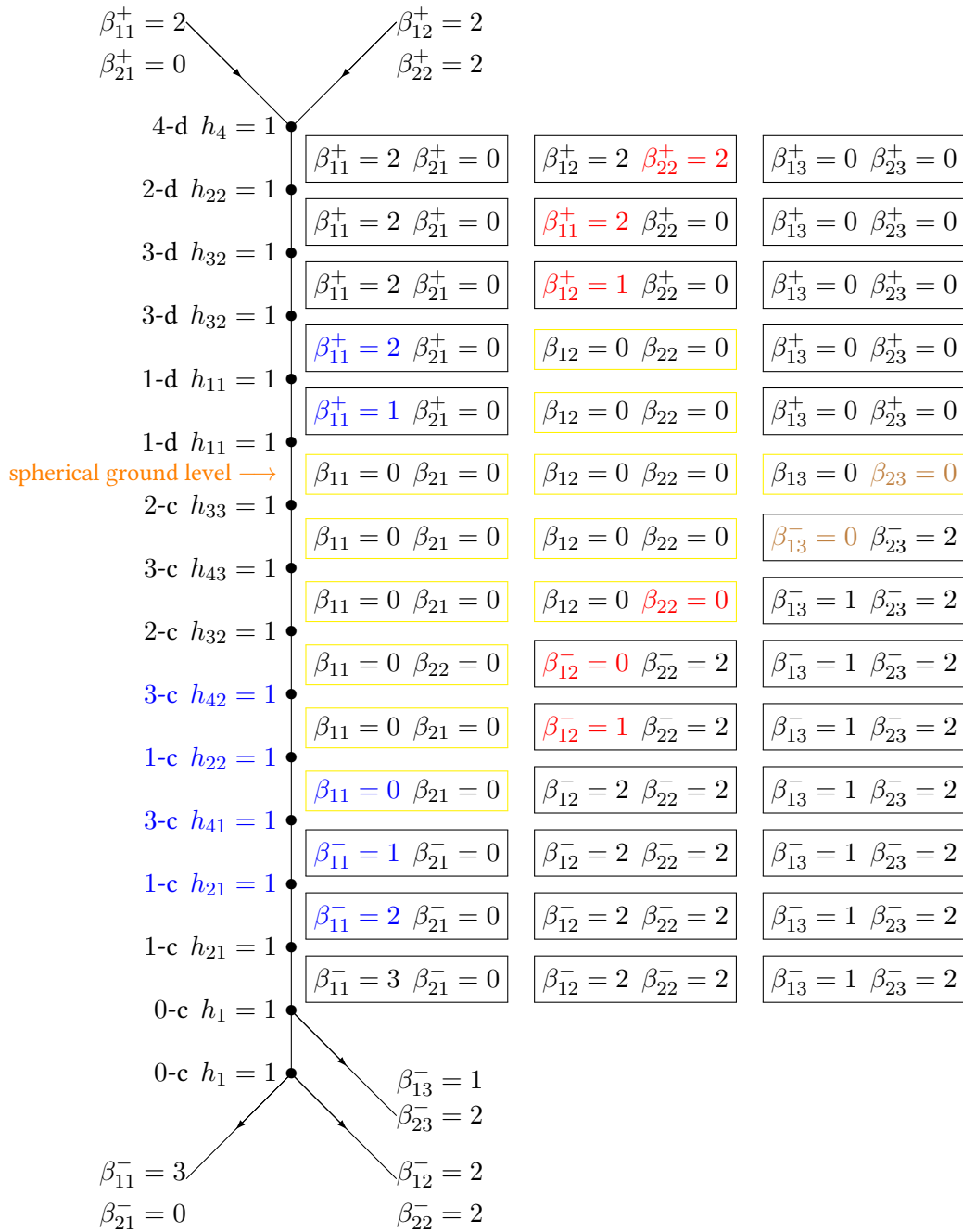


Figure 4.20: Another Lyapunov graph componentwise morsification of the semigraph in Figure 4.17.

As mentioned before, it is important to observe that thus far, componentwise morsification produces a level where $\beta_{jk} = 0$, for all $j = 1, \dots, n - 2, k = 1, \dots, e^\pm$. In the event of realization, this can be interpreted as a level containing at least one even-dimensional sphere.

The underlying idea of this work is to describe under which conditions an abstract Lyapunov graph can be componentwise morsified as in Figure 4.19 or Figure 4.20. These conditions form a new collection of inequalities, the componentwise Poincaré–Hopf inequalities, described in Section 3.6, which are shown to be necessary and sufficient for the componentwise morsification to occur.

The proofs in [Bertolim et al., 2023b] were done for zero Betti number ground level set because the exposition is clearer and more didactic with this simplification. But in the same work the proofs were presented in the general case, i.e., the restriction that the Betti numbers of the ground level set are all zero is removed and ground level sets with arbitrary Betti numbers are considered.

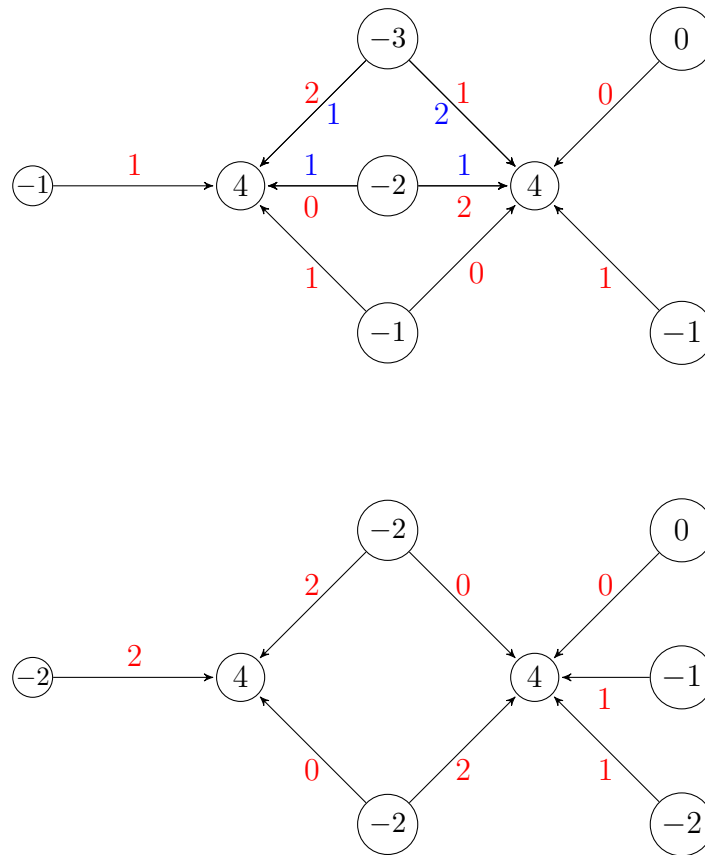


Figure 4.21: Flows corresponding to the morsifications in Figure 4.19 (red) and Figure 4.20 (blue and red).

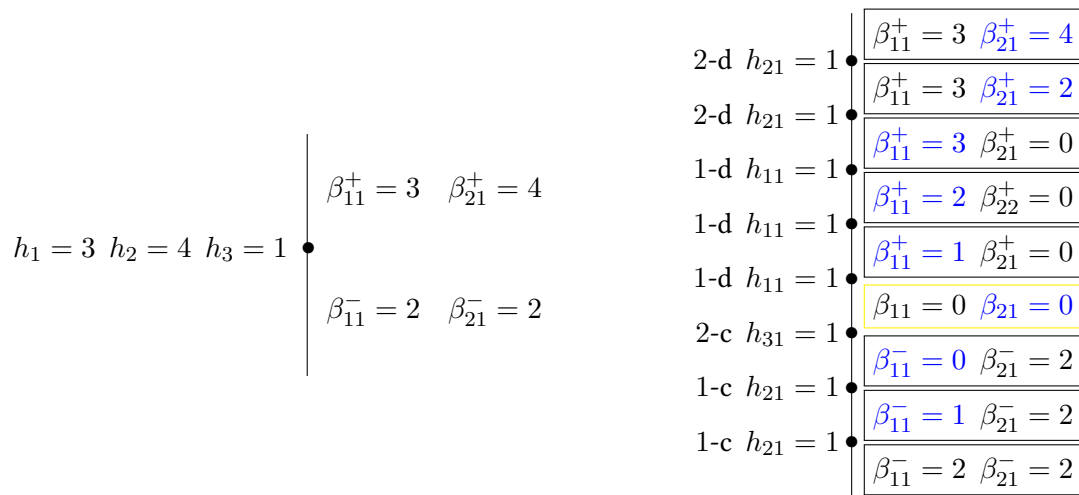


Figure 4.22: Linear abstract Lyapunov graph componentwise morsification.

Minimal number of singularities and periodic orbits

"Nothing takes place in the world whose meaning is not that of some maximum or minimum."

Leonhard Euler.

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In this chapter we use the following notation:

- N is a compact manifold of dimension n such that $\partial N = \partial N^+ \cup \partial N^-$,
- ∂N^+ and ∂N^- non-empty,
- $\partial N^+(\partial N^-)$ is the disjoint union of $e^+(e^-)$ components of ∂N , i.e., $\partial N^\pm = \bigcup_{i=1}^{e^\pm} N_i^\pm$,
- $B_j^\pm = \sum_{i=1}^{e^\pm} \beta_j(N_i^\pm)$ where $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ is the sum of the Betti numbers, $\beta_j(N_i^\pm)$, of these components.

5.1 Minimal number of singularities for Morse flows

In [Bertolim et al., 2006b] we introduce a notion of minimal Morse flows on compact manifolds. Classically, a Morse flow on a compact manifold M with total number of singularities h is minimal if there exists no other flow realizable on M with fewer singularities than h . In [Reineck, 1991] techniques are developed to continue a gradient flow to one with the minimal number of critical points. However, the approach is quite distinct from ours and our results are of a different nature.

Consider that one has abstract information on an isolating block N , the number of entering N^+ and exiting N^- boundaries of N are given by positive integers e^+, e^- and integers corresponding to the differences (of the j -th Betti numbers) $B_j^+ - B_j^-$ of N^+ and N^- . A **minimum number of singularities** h_{\min} can be determined depending only on e^+, e^- and the differences $B_j^+ - B_j^-$ where $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$. This minimum number of singularities h_{\min} has a topological-dynamical meaning. Given any compact manifold N with $e^+ + e^-$ boundary components, e^\pm components $N_i^\pm, i = 1, \dots, e^\pm$ labelled with $\{\beta_j(N_i^\pm); j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$, there exists no Morse flow realizable on N entering through the (N_i^+) 's and exiting through the (N_i^-) 's with fewer singularities than h_{\min} . Hence, in this sense, a flow realizing h_{\min} is a minimal flow on some compact manifold N respecting the given homological restrictions on the entering and exiting boundaries for the flow. Of course, there may be many such compact manifolds realizing this minimal flow. On the other

hand, there are many compact manifolds with the same boundary specification given above which possess minimal Morse flows with total number of singularities greater than h_{\min} .

The simple example in Figure 5.1, on compact 2-manifolds illustrates this point, i.e., given one entering boundary component and two exiting boundary components, (in this case these components must be circles) $h_{\min} = 1$ hence $h_1 = 1$. However, there are other minimal Morse flows on other compact 2-manifolds with the same homological boundary specification which possess a greater number of singularities than h_{\min} . Of course, in dimension two, the number of boundary components, $e^+ + e^-$ and the genus g completely determines h_{\min} , i.e., the number of singularities of index one, h_1 , by the formula $2 - 2g - (e^+ + e^-) = h_1$. If we define topological complexity in terms of the genus, note that h_{\min} is realized on the manifold of lowest complexity. However, in higher dimensions we can also measure topological complexity in terms of the presence of dual pairs and hence a similar phenomena is observed.

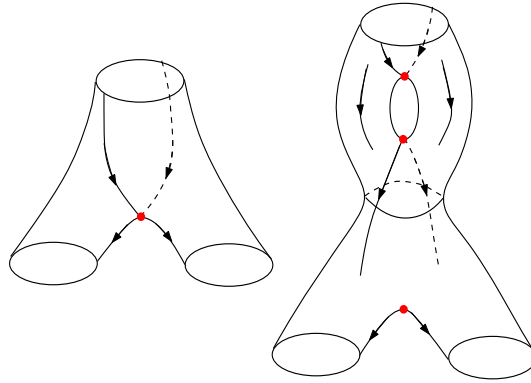


Figure 5.1: Minimal Morse flows and topological complexity.

The main result in [Bertolim et al., 2006b], Theorem 15, asserts that a minimum number of singularities h_{\min} can be determined depending only on e^+ , e^- and the differences (of Betti numbers) $B_j^+ - B_j^-$.

Theorem 15. *Consider some abstract homological information, namely, positive integers e^+ and e^- and integers $(B_j^+ - B_j^-)_{j=0}^{n-1}$ such that $B_j^+ - B_j^- = B_{n-j-1}^+ - B_{n-j-1}^-$. Then*

1. *there exists a number h_{\min} which is the lower bound on the number of singularities of any Morse flow realizable on any compact manifold with e^+ entering boundaries and e^- exiting boundaries with Betti numbers satisfying the given differences $B_j^+ - B_j^-$.*
2. *this number h_{\min} can be computed by a simple formula and it depends only on e^+ , e^- and on the differences $B_j^+ - B_j^-$;*
3. *the set $\mathcal{H} = \{(h_1, \dots, h_{n-1}) : \sum_{i=1}^{n-1} h_i = h_{\min} : \text{Poincaré-Hopf inequalities are satisfied}\}$ is completely determined and each element in \mathcal{H} determines a set $\mathcal{H}^{cd}(h_1, \dots, h_{n-1}) = \{(h_1^c, h_1^d, \dots, h_{n-1}^c, h_{n-1}^d) : \text{the } h^{cd}\text{-system}^1 \text{ is satisfied}\}$ which is a singleton.*
4. *the range of the Euler characteristics, $\chi_{\min}(M, \partial M)$ of the compact manifolds M realizing the minimal Morse flows is obtained.*
5. *given $\mathcal{H}^{cd}(h_1, \dots, h_n)$ the corresponding family of Lyapunov semigraphs of Morse type is completely determined; all graphs are explicitly described and possess lowest topological complexity.*

¹This system was defined in Section 4.1

Example in dimension 5

In the case that $n = 2i + 1$, we use the following formula for computing h_{\min} .

$$h_{\min} = e^- + e^+ - 2 + \sum_{j=1}^{i-1} |B_j^+ - B_j^-| + \left| \frac{B_i^+ - B_i^-}{2} \right|$$

Consider the following homological boundary information in dimension 5:

$$\{e^+ = 2, e^- = 3, B_1^+ - B_1^- = -2, B_2^+ - B_2^- = -2\}$$

We have in this case $h_{\min} = 6$. As for the distribution of the six singularities we have $\underline{h} = (h_1, h_2, h_3, h_4)$ satisfying the Poincaré–Hopf inequalities and realizing h_{\min} are

$$\{(2, 0, 1, 3), (2, 1, 1, 2), (2, 2, 1, 1)\}$$

Concerning their types of singularities, which are uniquely determined by \underline{h} , we have

$$\begin{aligned} \mathcal{H}^{cd}((2, 0, 1, 3)) &= \{(h_1^c = 2, h_1^d = 0, h_2^c = 0, h_2^d = 0, h_3^c = 1, h_3^d = 0, h_4^c = 2, h_4^d = 1)\} \\ \mathcal{H}^{cd}((2, 1, 1, 2)) &= \{(h_1^c = 2, h_1^d = 0, h_2^c = 1, h_2^d = 0, h_3^c = 1, h_3^d = 0, h_4^c = 1, h_4^d = 1)\} \\ \mathcal{H}^{cd}((2, 2, 1, 1)) &= \{(h_1^c = 2, h_1^d = 0, h_2^c = 2, h_2^d = 0, h_3^c = 1, h_3^d = 0, h_4^c = 0, h_4^d = 1)\} \end{aligned}$$

as we can obtain either from applying the algorithm of [Bertolim et al., 2003a].

Now we fix the labels of the edges satisfying our initial data, for instance as in the example of Figure 5.2.

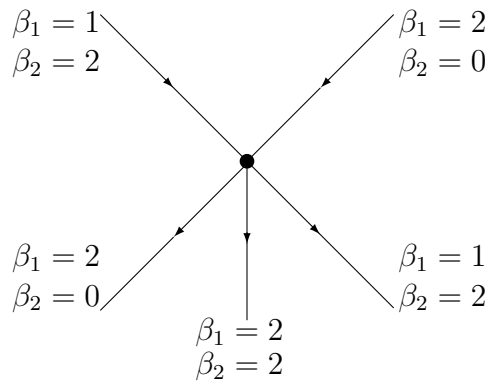


Figure 5.2: Labels respecting $\{e^+ = 2, e^- = 3, B_1^+ - B_1^- = -2, B_2^+ - B_2^- = -2\}$

We have three abstract Lyapunov semigraphs, one for each \underline{h} realizing h_{\min} (Figure 5.3). Hence, we have three abstract Lyapunov semigraphs of Morse type (Figure 5.4) corresponding to each graph of Figure 5.3. Observe that we can obtain one abstract Lyapunov semigraph of Morse type from another one by replacing singularities of type 3-c with singularities of type 1-c, both having the algebraic effect of decreasing β_1 . Furthermore, note that in these Lyapunov linear semigraphs for each j , β_j is strictly decreasing or increasing as one walks on the graph following the opposite orientation of the directed edges. We can easily see that this implies that these Lyapunov linear semigraphs possess neither dual pairs nor null pairs. Hence these semigraphs possess the lowest topological complexity.

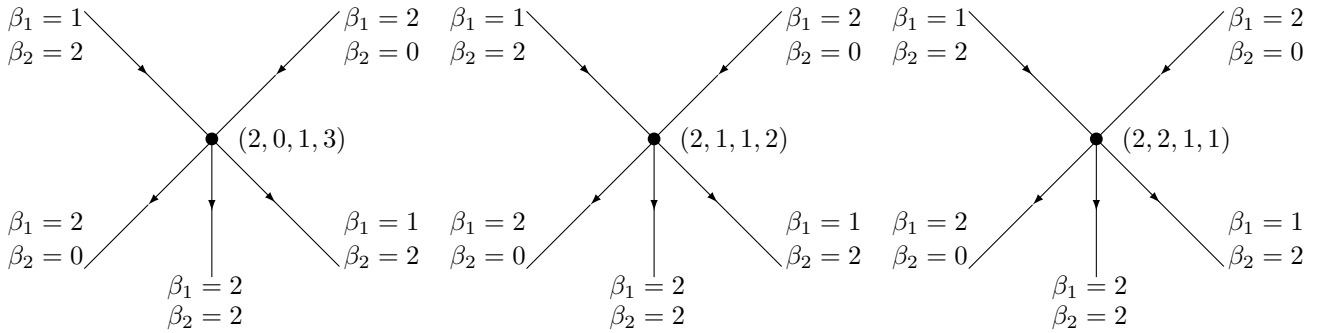


Figure 5.3: The three abstract Lyapunov semigraphs realizing h_{\min} .

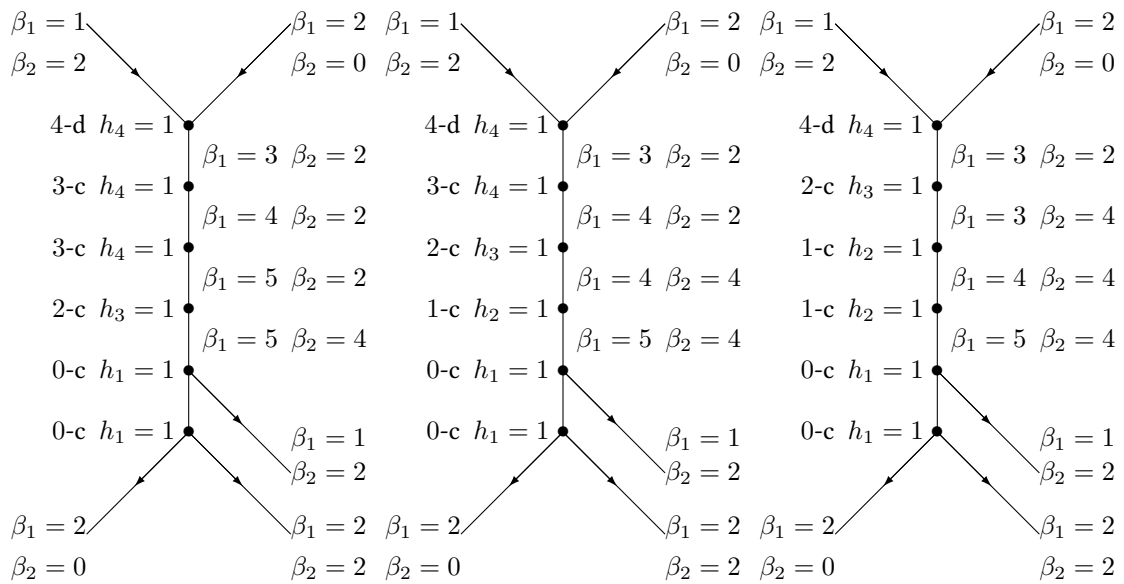


Figure 5.4: The three abstract Lyapunov semigraphs of Morse type corresponding to each graph of Figure 5.3.

5.2 Minimal number of periodic orbits

As mentioned in the previous section, in [Bertolim et al., 2006b] we give a complete description of the “simplest” Lyapunov graphs that can be associated with any given initial homological data. This was done by computing the minimal number of nondegenerate singularities that can be realized on some manifold with non-empty boundary in terms only of abstract homological boundary information.

In the setting of nonsingular Morse-Smale flows on compact n -dimensional manifolds, a natural question is to understand how the topology of the manifold M forces the number and indices of the hyperbolic closed periodic orbits of any possible nonsingular Morse-Smale flow over M . Of course it is not reasonable to try to answer precisely this question in its entire generality. However, it is possible to answer the same question in a wider context and still get some interesting dynamical information, which is optimal in the new more relaxed setting.

In [Bertolim et al., 2023a] we got similar results to those in [Bertolim et al., 2006b], i.e., a constructive algorithm which computes a minimal number of periodic orbits that can be realized on M in terms only of abstract homological boundary information. As a consequence, this algorithm produces a minimal number of periodic orbits such that an abstract Lyapunov graph can be continued to an abstract Lyapunov graph of Smale type (definition can be found in Section 2.4 and the description about the continuation results in this case can be found in Section 4.3).

Our main result consists of an algorithm for computing a lower bound p_{min} of the number of periodic orbits of any nonsingular Morse-Smale flow on any manifold with boundary satisfying some homological boundary conditions. Such a lower bound p_{min} can be computed in terms of the Betti numbers of the boundary components, regardless of the topology of the underlying manifold, and there exists a manifold and a nonsingular Morse-Smale flow for which this lower bound is optimal. However, despite the analogy of the statement, some relevant differences appears both in the techniques and in the nature of the result, especially in large dimension and for large initial homological information.

We work with some partial abstract homological information, without any reference to a specific manifold or flow, which we refer to **abstract homological data**. This information consists of

- an odd integer n , $n \geq 3$;
- two positive integers e^+ and e^- ;
- $\frac{n-1}{2}$ integers, denoted by the expressions $B_j^+ - B_j^-$, for $j = 1, \dots, \frac{n-1}{2}$.

For any odd n , let us denote by M any n -dimensional compact connected oriented manifold with boundary ∂M consisting of $(e^+ + e^-)$ connected components, endowed with a nonsingular Morse-Smale flow Φ transversally entering M through e^+ boundary components N_i^+ , $i = 1, \dots, e^+$, and transversally exiting through the remaining e^- boundary components N_i^- , $i = 1, \dots, e^-$. For all $j = 1, \dots, \frac{n-1}{2}$, if $\beta_j(N)$ represents the j -th Betti number of N , then we denote by $B_j^+ - B_j^-$ the sum:

$$B_j^+ - B_j^- = \sum_{k=1}^{e^+} \beta_j(N_k^+) - \sum_{k=1}^{e^-} \beta_j(N_k^-)$$

Within this notation, we say that a manifold M and a nonsingular Morse-Smale flow Φ on M as in our context satisfy the **abstract homological data**

$$\left\{ n, e^+, e^-, \{B_j^+ - B_j^-\}_{j=1}^{\frac{n-1}{2}} \right\}.$$

Let us emphasize here that the flow Φ is nonsingular Morse-Smale, that is, all of its recurrent sets are **closed periodic orbits** and lie in the interior of M . We shall simply call them periodic orbits because all the flows we shall consider are nonsingular Morse-Smale.

We proved the following theorem.

Theorem 16. *Let us be given the following abstract homological data $\left\{n, e^+, e^-, \{B_j^+ - B_j^-\}_{j=1}^{\frac{n-1}{2}}\right\}$ satisfying (n odd and) $e^+ - e^- + \sum_{j=1}^{\frac{n-1}{2}} (-1)^j (B_j^+ - B_j^-) = 0$.*

Then the following conclusions hold true.

1. *One provides an explicit algorithm computing a number p_{min} associated with the given homological data, such that any nonsingular Morse-Smale flow on any manifold satisfying these given abstract homological data must have at least p_{min} closed periodic orbits.*
2. *There exists a manifold M and a nonsingular Morse-Smale flow Φ with exactly p_{min} periodic orbits satisfying the given abstract homological data.*

Let's emphasize some points of this theorem:

- The assumption $e^+ - e^- + \sum_{j=1}^{\frac{n-1}{2}} (-1)^j (B_j^+ - B_j^-) = 0$ means that the Euler characteristic of any manifold admitting a nonsingular Morse-Smale flow is necessarily zero. It is known that this is required for having nonsingular Morse-Smale flows. Hence this assumption on the abstract homological data is natural because of their interpretation in case of realization. This condition is also crucial in one of the main ingredients of the proof: the use of attaching handles vs. attaching round handles. On the one hand, in Morse theory, where each singularity of index j corresponds to a handle of index j , we know the effect of attaching one handle of index j of the Betti numbers of the boundary. This tells us that the list $\underline{h} = (h_0, \dots, h_n)$ of the number of singularities h_j of index j compatible with the initial homological data are of the form $\underline{h} = \underline{h}_{min} + \underline{h}_{consecutive} + \underline{h}_{dual}$ (see [Bertolim et al., 2006a]), where

\underline{h}_{min} is one of the labels associated with h_{min} and the boundary conditions;

$\underline{h}_{consecutive}$ is a vector corresponding to a collection of couples (h_j, h_{j+1}) with adjacent indices (necessarily of types $(j-d, j-c)$);

\underline{h}_{dual} is a vector corresponding to a collection of couples (h_j, h_{n-j}) with dual indices (either of types $((j-1)-c, (n-j)-d)$ or of types $(j-d, (n-j-1)-c)$).

On the other hand, in nonsingular Morse-Smale theory ([Asimov, 1975]), where each orbit of index j corresponds to a round handle of index j , replacing it by two handles of indices j and $j+1$ (see [Franks, 1979]) implies that among all compatible vectors \underline{h} as above, there must be some \underline{h}' of the form $\underline{h}' = \underline{h}'_{consecutive}$. Therefore, the assumption on the Euler characteristic, ensures that each vector \underline{h}_{min} can be completed with some $\underline{h}_{consecutive} + \underline{h}_{dual}$ in order to obtain a vector \underline{h}' of the form $\underline{h}' = \underline{h}'_{consecutive}$ which is also compatible with the given homological data.

- *Item 1* of the theorem says that some loose information on the homology of any compatible underlying manifold M together with the knowledge of the entry and exit boundary of the underlying flow Φ allow us to guarantee the existence of a lower bound for the number of the periodic orbits of Φ . However, if we wish to work with a restricted class of manifolds admitting nonsingular Morse-Smale flows, such a bound may not be sharp. Consider for instance the data $\{n = 3, e^+ = e^- = 1, B_1^+ - B_1^- = 0\}$. For these fixed data one has $p_{min} = 0$. The

manifold $M = \mathbb{T}^2 \times I$ endowed with the trivial flow Φ_M entering $\mathbb{T}^2 \times \{1\}$ and exiting $\mathbb{T}^2 \times \{0\}$ is an example of manifold and flow satisfying the homological data for which the computed p_{min} coincides with its minimal number of periodic orbits, denoted by $A_{min}(M, \partial)$. If we consider M' obtained by attaching a round handle to M as in Figure 5.5 below, we can see that the boundary of M' is also made of two copies of \mathbb{T}^2 . The homological data restriction implies

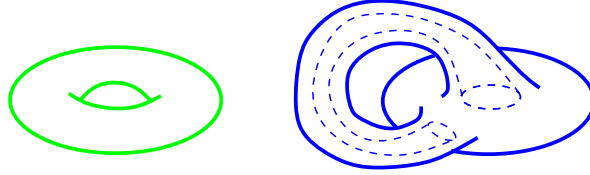


Figure 5.5: Exit and entry boundary of the nontrivial manifold M'

that the nonsingular Morse-Smale flows we shall consider must enter through one torus component and exit through the other. The computation of the Conley index of the manifold and the way we constructed it show that $A_{min}(M', \partial) = 1$ which is strictly greater than p_{min} . The situation here is analogous to the one discussed in the previous section (see[Bertolim et al., 2006b]) where the minimal number of singularities of Morse flows on manifolds satisfying some abstract homological data is discussed. In our present context we can also consider that the manifolds and flows for which p_{min} is sharp (that is, $A_{min}(M, \partial) = p_{min}$) are in some sense the "simplest", topologically speaking, among those admitting nonsingular Morse-Smale flows and satisfying the given abstract homological information.

- Let us also highlight the fact that the abstract homological information only deals with the difference of the Betti numbers of the entry and exit boundary. For instance, in dimension 3, the couple $(N^+, N^-) = (\mathbb{S}^2 \sqcup \mathbb{S}^2, \mathbb{S}^2 \sqcup \mathbb{S}^2)$, the couple $(N^+, N^-) = (\mathbb{S}^2 \sqcup \mathbb{T}^2, \mathbb{S}^2 \sqcup \mathbb{T}^2)$ and the couple $(N^+, N^-) = (\mathbb{S}^2 \sqcup \mathbb{T}^2 \# \mathbb{T}^2, \mathbb{T}^2 \sqcup \mathbb{T}^2)$ all correspond to the same abstract homological data $\{n = 3, e^+ = e^- = 2, B_1^+ - B_1^- = 0\}$
- The description of the algorithm of *Item 2* is very technical. I present some guidelines. The conditions ensuring the existence of a solution starting from the abstract data are expressed in terms of a semi-algebraic system with integer coefficients. We are looking for positive solutions to this system whose sum is minimal. We are therefore dealing with an optimization problem. The initial pairing problem is hence reformulated as a minimum cost flow (MCF) problem. Then the interpretation of the specific MCF problem as a transportation problem yields the wanted algorithm.
- *Item 2* is constructive and explicit once we have run its algorithm. In fact, as a result, such an algorithm gives not only the value of p_{min} but also a list of abstract round handles $R_1, \dots, R_{p_{min}}$, of given index and type, associated with p_{min} . The type contains the information of the effect of the corresponding round handle on the Betti number of the boundary. Using the realizations described in [Bertolim et al., 2007] one can easily conclude: for any R_k given by the algorithm, let (U_k, ϕ_k) be the isolating neighborhood of the corresponding index and type built as in [Bertolim et al., 2007]. A connected sum along the boundary of these $U_k, k = 1, \dots, p_{min}$ is a manifold M endowed with a nonsingular Morse-Smale flow Φ (conjugate to ϕ_k on each U_k) such that (M, Φ) satisfies the given homological data. By construction, Φ has p_{min} periodic orbits.

It is important to observe that, especially in large dimension and when the difference of the Betti numbers is also large and arbitrarily distributed, we cannot expect to compute the number p_{min} by any naive strategy. Let us take for instance the homological data

$$\{n = 17, e^+ = 7, e^- = 6, B_1^+ - B_1^- = 3, B_2^+ - B_2^- = -1, B_3^+ - B_3^- = -2,$$

$$\left. B_4^+ - B_4^- = -7, B_5^+ - B_5^- = -6, B_6^+ - B_6^- = 8, B_7^+ - B_7^- = -4, \frac{B_8^+ - B_8^-}{2} = -10 \right\}$$

Our algorithm gives us (among other information) the solution

- $p_{min} = 32$;
- the sequence $\underline{h}' = [h'_1, \dots, h'_{16}] = [5, 5, 1, 1, 2, 2, 0, 0, 10, 10, 0, 6, 6, 2, 8, 6]$, which can be translated into the sequence

$$[R_1, \dots, R_{15}] = [5, 0, 1, 0, 2, 0, 0, 0, 10, 0, 0, 6, 0, 2, 6]$$

where R_j would correspond to the number of round handles (or equivalently, periodic orbits) of index j ;

- \underline{h}' has been obtained by completing $\underline{h}_{min} = [5, 0, 0, 0, 1, 2, 0, 0, 10, 10, 0, 6, 6, 2, 4, 6]$ by $\underline{h}_{consecutive} + \underline{h}_{dual} = [0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] + [0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0]$

If one try to complete \underline{h}_{min} in any other naive way (by adding couples of handles of consecutive indices from left to right, or from right to left, or by adding couples of handles of dual indices from the extremities of \underline{h}_{min}) gives strict upper bounds of p_{min} and makes one understand that a more sophisticated and complex strategy is unavoidable.

This is a main difference with the Morse setting described in the previous section. There, the value h_{min} can be computed by hand directly from the homological data. Moreover, all the possible sequences of the indices of the Morse singularities associated with h_{min} can be listed explicitly. Here, in contrast, the computation of p_{min} is intrinsically more laborious and our algorithm gives us just one way of realizing it as a sequence of round handles.

Let us underline that a Python program associated with this algorithm is freely available on the page <https://github.com/MargaridaMello/OddMinimumPairingProblem/>

As a counterpart of the complexity of the algorithm discussed above, we discover that, in contrast with the Morse setting, here the indices of the appearing periodic orbits cannot be always foreseen. For instance, for the homological data

$$\left\{ n = 15, e^+ = 1, e^- = 1, B_1^+ - B_1^- = B_6^+ - B_6^- = 1, B_j^+ - B_j^- = 0, \forall j = 1 \dots 7, j \neq 1, 6 \right\}$$

we have ($h_{min} = 2$ and) $p_{min} = 3$. The algorithm outputs an abstract realization of p_{min} with an orbit of index 3, an orbit of index 5 and an orbit of index 12 appearing from the combinatorics of the algorithm. The index 5 orbit is of course related to the fact that the 6-th (and the 8-th) Betti number vary; the index 12 orbit is related to the variation of the 1st (and 13-th) Betti number of the boundary. The index 3 orbit is somehow hidden in the data even though the difference $B_j^+ - B_j^-$ of the Betti numbers of indices 2, 3 and 4, as well as 10, 11 and 12, are zero). For the sake of completeness, let us mention that there are three other solutions (each with three orbits of indices respectively (1, 3, 5), (5, 10, 12) and (8, 10, 12) for which the analogous observations hold. Note that in this case only three of the four possible \underline{h}_{min} can be completed to a \underline{h}'_{min} combinatorially realizing $p_{min} = 3$ and one can be completed in two different ways².

² \underline{h}_{min} such that $h_1 = 1, h_6 = 1$ yields the (1, 3, 5) triple; \underline{h}_{min} such that $h_6 = 1, h_{13} = 1$ yield the (3, 5, 12) and the (5, 10, 12) triples; \underline{h}_{min} such that $h_8 = 1, h_{13} = 1$ yields the (8, 10, 12) triple and \underline{h}_{min} such that $h_1 = 1, h_8 = 1$ cannot be completed in order to realize $p_{min} = 3$.

Numerical examples

A computer code was built to construct and solve the odd dimensional minimum pairing problem. It was written in Python 3.7.6 and accepts as inputs the parameter i (the instance to be solved is of dimension $n = 2i + 1$) and, optionally, the data $(\mathcal{B}_0^-, \mathcal{B}_0^+, \mathcal{B}_1, \dots, \mathcal{B}_i)$, where $\mathcal{B}_j = B_j^+ - B_j^-$, for $j = 1, \dots, \lfloor (n - 2)/2 \rfloor$, and, since n is considered odd, $\mathcal{B}_j = (B_j^+ - B_j^-)/2$ for $j = (n - 1)/2$, $\mathcal{B}_0^+ = e^+ - 1$ and $\mathcal{B}_0^- = e^- - 1$. If the latter is not supplied, random values are assigned to these constants, satisfying $\mathcal{B}_0^+ - \mathcal{B}_0^- + \sum_{j=1}^{(n-1)/2} (-1)^j \mathcal{B}_j = 0$. The optimal value and various bounds are calculated, as well as a solution. The output is supplied as formatted latex code and a sample is provided below. This program is freely available on the page <https://github.com/MargaridaMello/OddMinimumPairingProblem/>. The relevant file is `ValueAndSolutionTN.py`. Once this code is executed, we may generate and solve a problem of dimension $n = 5$ by entering the command `ValueSolutionMinPairingProblem(2)` at the console window. Alternatively, if we want to solve the specific instance with $n = 5$ and $(\mathcal{B}_0^-, \mathcal{B}_0^+, \mathcal{B}_1, \mathcal{B}_2) = (6, 6, 5, 5)$, the command would be `ValueSolutionMinPairingProblem(2, [6, 6, 5, 5])`.

$$i = 2 \quad n = 5$$

$$(\mathcal{B}_0^-, \mathcal{B}_0^+, \mathcal{B}) = (6, 6, 5, 5)$$

In this case we obtain the following answer :

	g_0	g_1	g_2	θ
f_0	¹ 1		5	6
f_1	5			5
f_2				
δ	6		5	

Optimal value 1

Column lower bound 0

Row lower bound 0

First upper bound 6

Second upper bound 7

j^{cd}	1^c 6	2^d 5	3^d 5	4^d 6	
τ					
\mathfrak{d}	² 1				
s	1 6	⁴ 6			
h^{cd}	h_1^c 6	h_2^d 6	h_3^c 1	h_3^d 5	h_4^d 6
h	h_1 6	h_2 6	h_3 6	h_4 6	

If we are only interested in the minimal number of periodic orbits, we just need to read the last line labelled by "h". In our example, it is the line

$$h \quad | \quad h_1 \ 6 \quad h_2 \ 6 \quad h_3 \ 6 \quad h_4 \ 6$$

From it, we obtain the following information :

- the minimal³ number of periodic orbits, p_{min} , is just the sum of the elements of the line "h" divided by 2. Here $p_{min} = \frac{6 + 6 + 6 + 6}{2} = 12$.
- the indices of the periodic orbits associated to this line, since each orbit of index j combinatorially corresponds to a couple of consecutive (h_j, h_{j+1}) . Here, the line "h" is uniquely

³in the sense of Theorem 16.

partitioned into 6 couples (h_1, h_2) and 6 couples (h_3, h_4) , so that six of the twelve periodic orbits are of index 1 and the other six are of index 3.

If we are interested in realizing these abstract data by attaching round handles, we need further information about the compatible types “ c ” and “ d ” of such round handles (see [Bertolim et al., 2007]). Here, we obtain this information by the penultimate line :

$$h^{cd} \quad \left| \quad \begin{array}{|c|c|c|c|c|c|} \hline h_1^c & 6 & h_2^d & 6 & h_3^c & 1 & h_3^d & 5 & h_4^d & 6 \\ \hline \end{array} \right.,$$

that is, a realization can be obtained by attaching to a suitable boundary six round handles of index 1 and type (c, d) , one round handle of index 3 and type (c, d) and five round handles of index 3 and type (d, d) according to the pattern given in [Bertolim et al., 2007].

The other information created by the algorithm is useful whenever one wishes to follow step by step the algorithm presented in [Bertolim et al., 2023a].

Coming back to the example in dimension 17 developed in the introduction, after entering `ValueSolutionMinPairingProblem(8, [5, 6, 3, -1, -2, -7, -6, 8, -4, -10])` the output is:

$$i = 8 \quad n = 17$$

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	θ
f_0	1 4			1	1	2	2 1	3	3	5
f_1	2			1	1	1	2	2	3	3
f_2				1		1	1	2	2	1
f_3	1						1	1	2	
f_4	1	1				6	1	1	1	7
f_5	2	1	1						1	
f_6	2	2	1	1						
f_7	3	2	2	1	1					
f_8	3	3	2	2	1	1	6	4		10
δ	6			2		6	8	4		

Optimal value 6
 Column lower bound 0
 Row lower bound 0
 First upper bound 33
 Second upper bound 30

j^{cd}	1^c 5	5^c 1	6^d 2	9^c 10	10^c 4	10^d 6	12^c 6	13^c 6	14^c 2	15^c 1	15^d 3	16^d 6			
τ	2 4														
\mathfrak{d}	2 1	4 1													
s	1 5	3 1	5 2	10 10	13 6	15 2	16 6								
h^{cd}	h_1^c 5	h_2^d 5	h_3^c 1	h_4^d 1	h_5^c 2	h_6^d 2	h_9^c 10	h_{10}^c 4	h_{10}^d 6	h_{12}^c 6	h_{13}^c 6	h_{14}^c 2	h_{15}^c 5	h_{15}^d 3	h_{16}^d 6
h	h_1 5	h_2 5	h_3 1	h_4 1	h_5 2	h_6 2	h_9 10	h_{10} 10	h_{12} 6	h_{13} 6	h_{14} 2	h_{15} 8	h_{16} 6		

Hence, for this data $p_{min} = 32$.

Isolating blocks as realization of Lyapunov graphs

"Conley's view of the importance of isolating blocks did not stem from the mathematical elegance of the theory alone, but also from his profound belief that isolating blocks are fundamental to an understanding of natural phenomena. . .

Simply put, he believed that isolating blocks were the only dynamical objects that could be detected in nature and their properties reflected the important properties of natural systems." R. McGehee

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Realizability of Lyapunov graphs in low dimension $n \leq 3$ was considered in [de Rezende, 1987, 1993] and [De Rezende and Franzosa, 1993]. In [Bertolim et al., 2006b], [Bertolim et al., 2006a], [Bertolim et al., 2007] we answer the local and global realizability questions in any dimension by using topological techniques and handle theory one builds models by gluing handles containing Morse singularities and round handles containing periodic orbits.

6.1 Isolating block for Morse flows

An n -dimensional elementary isolating block (N, N^+, N^-) can be constructed by considering an $(n - 1)$ -dimensional manifold N^- and its collar $N^- \times [0, 1]$ with attached handle $h_k = \mathbf{D}^k \times \mathbf{D}^{n-k}$, which contains an index k nondegenerate singularity. Different attachments may produce non-homeomorphic isolating neighbourhoods with non-homeomorphic boundaries. Any such neighbourhood can be schematically represented by a Lyapunov semigraph with outgoing edge(s) corresponding to the connected components of $N^- \times [0, 1]$, a vertex corresponding to the singularity, while the incoming edge(s) would correspond to the connected components of $N^+ \times [0, 1]$, N^+ being the new boundary created by the attachment of h_k to $N^- \times \{1\}$.

For example, it is well known that in dimension 2 a handle of index 0, 1 and 2 is a disk $D^0 \times D^2$, $D^1 \times D^1$ and $D^2 \times D^0$ respectively. In this case, two dimensional isolating blocks for Morse singularities are as in Figure 6.1. If the singularity is a sink, i.e., $h_0 = 1$ then N is a disk with $N^+ = S^1$ and if the singularity is a source, i.e., $h_2 = 1$ then N is a disk with $N^- = S^1$. If the singularity is a saddle, i.e., $h_1 = 1$ then in the orientable case, N is a disk with two disjoint subdisks removed from its interior and in the nonorientable case, N is a Mobius band with a subdisk removed from its interior. The Lyapunov semigraphs containing Morse singularities and their realization as 2-dimensional isolating blocks are depicted in Figure 6.1.

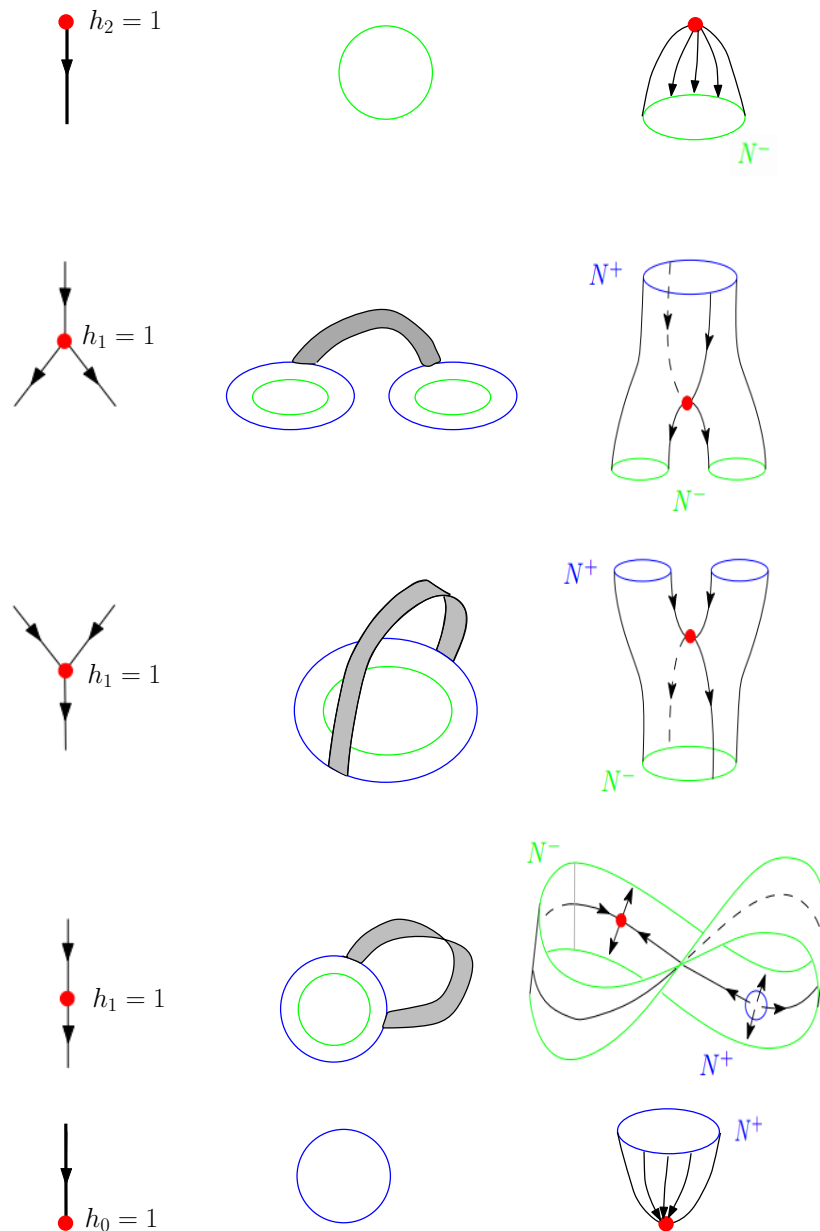


Figure 6.1: Isolating blocks and Lyapunov semigraphs for Morse singularities on surfaces.

As discussed in the previous chapter, by using the Poincaré–Hopf inequalities, we show in [Bertolim et al., 2006b] that a minimal number of nondegenerate singularities can be computed in terms of given abstract homological data. Moreover, this minimal number can be realized by a flow on some manifold with boundary, satisfying the abstract homological information. Such manifolds are shown to have the lowest topological complexity among all manifolds satisfying the given homological data. We describe all the possible indices and types (connecting or disconnecting) of singularities realizing this minimal number.

Then, in [Bertolim et al., 2006a] we study the algebraic structure of the dynamical data recorded a priori in an abstract Lyapunov graph. We establish a general decomposition theorem for handles which is also a consequence of the previous results. We present a constructive general procedure to build Morse flows on n -dimensional isolating blocks respecting the given dynamical and homological boundary data recorded in the abstract Lyapunov graphs. Thanks to the decomposition theorem for handles, these isolating blocks can be obtained by using a special class of classical handles gluings. Moreover, the corresponding flow not only preserves the given ranks of the homology Conley indices, but it is also optimal in the sense that no other Morse flow can preserve this index with

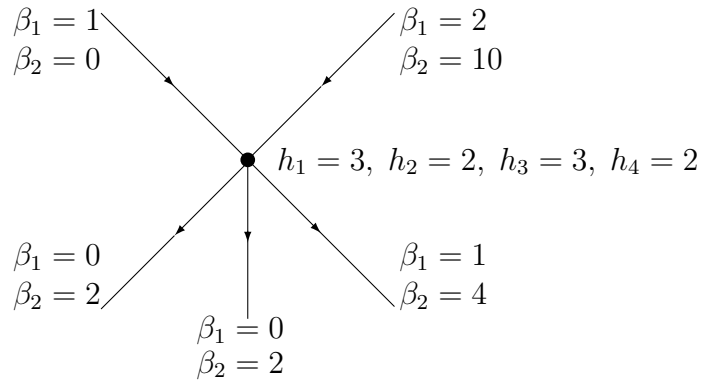


Figure 6.2: A vertex

fewer singularities. The main result in [Bertolim et al., 2006a] is:

Theorem 17 (Decomposition theorem). *Let v be a vertex of a Lyapunov graph. Let \underline{h} denote its label, with the convention that the j -th coordinate corresponds to the value of h_j . Then \underline{h} is compatible with the boundary conditions¹ if and only if it can be decomposed as*

$$\underline{h} = \underline{h}_{\min} + \underline{h}_{\text{consecutive}} + \underline{h}_{\text{dual}} + \underline{h}_{\text{invariant}}, \quad \text{where}$$

\underline{h}_{\min} is one of the labels associated with h_{\min} and the boundary conditions;

$\underline{h}_{\text{consecutive}}$ is a vector corresponding to a collection of couples (h_j, h_{j+1}) with adjacent indices (necessarily of types $(j\text{-disconnecting}, j\text{-connecting})$);

$\underline{h}_{\text{dual}}$ is a vector corresponding to a collection of couples (h_j, h_{n-j}) with dual indices (either of types $((j-1)\text{-connecting}, (n-j)\text{-disconnecting})$ or of types $(j\text{-disconnecting}, (n-j-1)\text{-connecting})$);

$\underline{h}_{\text{invariant}}$ is a vector which may be non-zero only in dimension $n = 4k$, corresponding to a collection of middle dimension h_{2k} 's of type β -invariant.

We remark that the decomposition of Theorem 17 is in general not unique. Consider the following example in dimension $n = 5$ (see Figure 6.2, where it is understood that for every edge we have $\beta_0 = \beta_4 = 1$ and $\beta_3 = \beta_1$). In this case $e^+ = 2$, $e^- = 3$, $(B_1^+ - B_1^-) = (1 + 2) - (0 + 0 + 1) = 2$, $(B_2^+ - B_2^-) = (0 + 10) - (2 + 2 + 4) = 2$ and $\underline{h} = (3, 2, 3, 2)$.

For this data we have $h_{\min} = 6$ and three vectors realizing it:

$$\begin{cases} \underline{h}_{\min}^{(0)} = (2, 1, 2, 1) & \text{corresponding to } \{h_1^c = 2, h_2^d = 1, h_3^d = 2, h_4^d = 1\}; \\ \underline{h}_{\min}^{(1)} = (3, 1, 1, 1) & \text{corresponding to } \{h_1^c = 2, h_1^d = 1, h_2^d = 1, h_3^d = 1, h_4^d = 1\}; \\ \underline{h}_{\min}^{(2)} = (4, 1, 0, 1) & \text{corresponding to } \{h_1^c = 2, h_1^d = 2, h_2^d = 1, h_4^d = 1\}. \end{cases}$$

The semigraph of Morse type shown in Figure 6.3 represents one of the possible morsifications of the vertex we are studying. By considering only this special morsification it is easy to verify that

¹ \underline{h} is compatible with the boundary conditions if \underline{h} satisfies the Poincaré-Hopf inequalities.

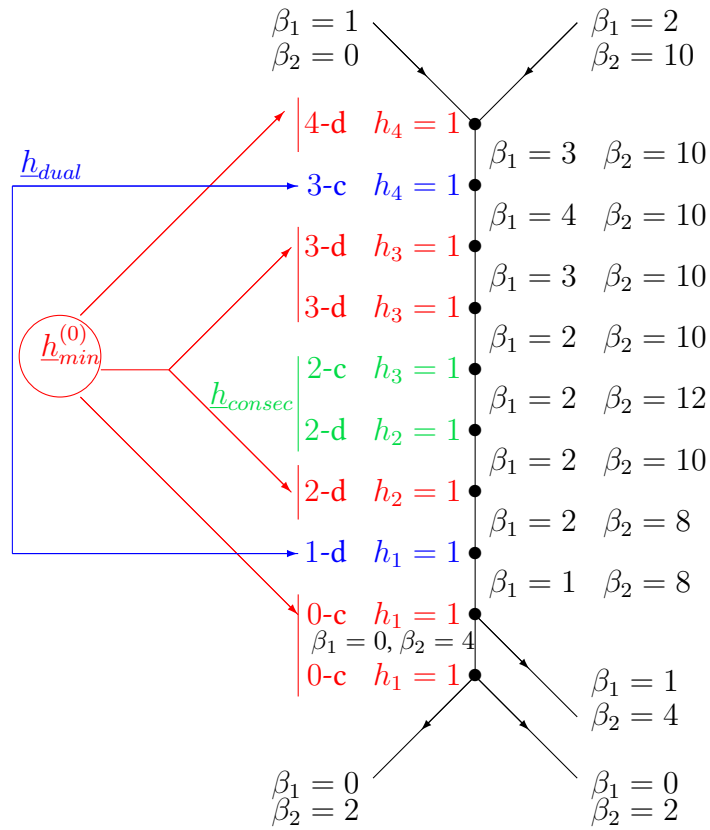


Figure 6.3: A possible morsification of the vertex of Figure 6.2

the label \underline{h} admits at least the three following decompositions:

$$\begin{aligned}
 \underline{h} = (3, 2, 3, 2) &= \underbrace{\overbrace{(2, 1, 2, 1)}^{(0-c, 0-c, 2-d, 3-d, 3-d, 4-d)}}_{\underline{h}_{\min}^{(0)}} + \underbrace{\overbrace{(1, 0, 0, 1)}^{(1-d, 3-c)} + \overbrace{(0, 1, 1, 0)}^{(2-d, 2-c)}}_{\underline{h}_{\text{dual}}} \\
 &= \underbrace{\overbrace{(2, 1, 2, 1)}^{(0-c, 0-c, 2-d, 3-d, 3-d, 4-d)}}_{\underline{h}_{\min}^{(0)}} + \underbrace{\overbrace{(0, 1, 1, 0)}^{(2-d, 2-c)}}_{\underline{h}_{\text{consecutive}}} + \underbrace{\overbrace{(1, 0, 0, 1)}^{(1-d, 3-c)}}_{\underline{h}_{\text{dual}}} \\
 &= \underbrace{\overbrace{(3, 1, 1, 1)}^{(0-c, 0-c, 1-d, 2-d, 3-d, 4-d)}}_{\underline{h}_{\min}^{(1)}} + \underbrace{\overbrace{(0, 1, 1, 0)}^{(2-d, 2-c)} + \overbrace{(0, 0, 1, 1)}^{(3-d, 3-c)}}_{\underline{h}_{\text{consecutive}}}
 \end{aligned}$$

In addition to the structural explanation of labels compatible with the abstract homological data of the edge, the interest of the writing highlighted in Theorem 17 is to be able to constructively guarantee the realizability of abstracts Lyapunov graphs by means of classical elementary gluings whose detailed description can be found in [Bertolim et al., 2006b]. I summarize them below.

Notation: Let h_q be an n -dimensional handle of index q , M_i the n -dimensional manifold obtained after step i and N_i its modified boundary. Recall that an n -dimensional *handle* of index k is a product of disks centered at the origin $\mathbf{D}^k \times \mathbf{D}^{n-k}$. The *core* is $\mathbf{D}^k \times 0$ and the *cocore* is $0 \times \mathbf{D}^{n-k}$. We define a flow on the handle by considering a non-degenerate index k singularity at the origin and identifying the core and cocore with its unstable and stable manifold respectively. The *attaching region* of the handle is $\mathbf{S}^{k-1} \times \mathbf{D}^{n-k}$ and corresponds to the part of the boundary of the handle through which the flow exits. Similarly, the *belt region* $\mathbf{D}^k \times \mathbf{S}^{n-k-1}$ corresponds to the part of the boundary of the handle through which the flow enters.

- **Trivial gluing:** Let us start from M_0 and N_0 . A trivial gluing is a way of attaching a handle of index q to a disc $\mathbf{D}^n = \mathbf{D}^q \times \mathbf{D}^{n-q}$ in order to create a q -handlebody in such way that the effect of the trivial gluing on the Betti numbers of the boundary is that only the q -th Betti number β_q and its dual β_{n-q-1} have changed by being increased by 1. For this reason, the trivial gluing of h_q is of type q -d.
- **Null gluing:** Let us start from M_0 and N_0 . A null gluing concerns two handles of consecutive indices, say, q and $q + 1$ in such way that the effect of the null gluing on the Betti numbers of the boundary is globally null. This is done in two steps : after the first step, only the q -th Betti number β_q and its dual β_{n-q-1} have changed by being increased by 1 (trivial gluing of h_q of type q -d). After the second step the gluing of h_{q+1} decreases by 1 the same Betti numbers β_q and β_{n-q-1} (gluing of h_{q+1} of type q -c). The net effect after both gluings is zero.
- **Dual gluing:** can only be performed by using two handles of complementary indices q and $(n - q)$ in such way that the effect of the dual gluing on the Betti numbers of the boundary is globally null. The first step consists in gluing a q -handle h_q to M_0 via a trivial gluing. After the first step, only the q -th Betti numbers β_q and its dual β_{n-q-1} have changed by being increased by 1. The second step is to attach the $(n - q)$ -handle h_{n-q} by identifying its attaching region $\mathbf{S}^{n-q-1} \times \mathbf{D}^q$ to the belt region of h_q . After the second step the gluing of h_{n-q} decreases by 1 the same Betti numbers β_q and β_{n-q-1} (gluing of h_{n-q} of type $(n - q - 1)$ -c).
- **Invariant gluing:** when the ambient dimension n is of the form $n = 4k$ and the index of the singularity is the middle dimension $2k$, starting from M_0 and N_0 , there is the possibility of gluing the corresponding handle in an invariant way, that is, in such a way that the Betti numbers of the boundary after such a gluing are the same as those of the boundary before the gluing. For this reason, all gluings of a single handle h_{2k} like these are of type β -i. Examples of invariant gluings can be found in the construction of the projective spaces \mathbf{CP}^{2k} , \mathbf{HP}^{2k} and \mathbf{OP}^{2k} . For details, see [Bertolim et al., 2006b].

These elementary gluings allowed us to prove the Decomposition Theorem by realizing step by step the isolating block. Each group of handles appearing in the decomposition theorem (Theorem 17) is treated separately in [Bertolim et al., 2006b]. For example, Figure 6.4 illustrates an isolating block of a consecutive pair of handles.

6.2 Isolating blocks for periodic orbits

In order to realize Morse–Smale isolating blocks for periodic orbits, one must make use of round handles. The space $R_k^n = S^1 \times D^k \times D^{n-k-1}$ or R for short is called an n -round handle of index k .

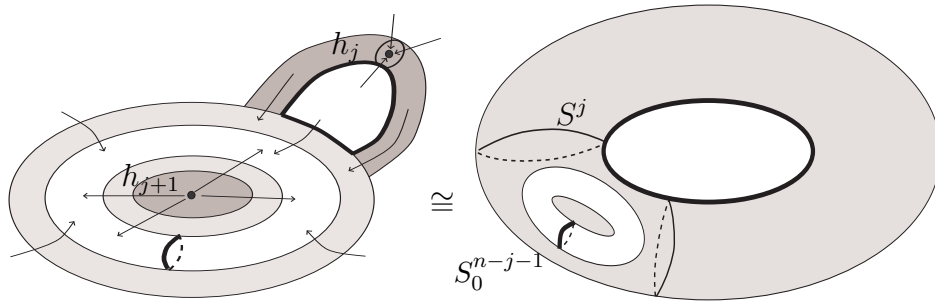


Figure 6.4: Isolating block of a consecutive pair of handles.

Its boundary ∂R is made up of two parts, the attaching region which is $\partial_A R = S^1 \times S^{k-1} \times D^{n-k-1}$ and the belt region which is $\partial_B R = S^1 \times D^k \times S^{n-k-2}$ which intersect in $\partial_{A \cap B} R = \partial_B R \cap \partial_A R = S^1 \times S^{k-1} \times S^{n-k-2}$. Figure 6.5 illustrates round handles in dimension 3.

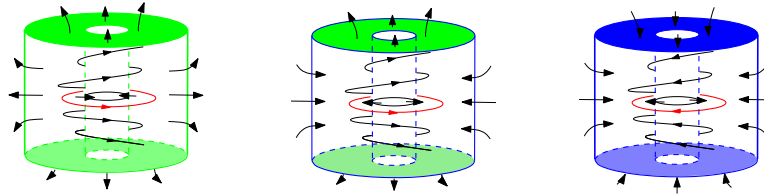


Figure 6.5: A round handle R for repelling periodic orbit, saddle periodic orbit and attracting periodic orbit respectively.

Round handles were introduced in [Asimov, 1975] and it was proved that flow manifolds admit round handle decompositions. In [Franks, 1985], Franks constructed isolating blocks for nonsingular Morse-Smale flows on S^3 . In [Cruz and de Rezende, 1999] Lyapunov graphs were generalized to represent flows on n -manifolds using Conley homology indices. Furthermore, in [Cruz and de Rezende, 1999] a classification is obtained describing the possible homological effects on the boundary of a manifold after attaching handles and round handles and this information is coded in Lyapunov semigraphs. However, isolating blocks realizing these Lyapunov semigraphs had not been constructed in that work.

In [Bertolim et al., 2007] we prove that some Lyapunov graphs associated with periodic orbits are realizable by constructing isolating blocks for periodic orbits of Morse-Smale flows. We analyze the effects on the Betti numbers of a manifold after a round handle operation is performed and a variety of situations are considered. Since we are concerned in showing the existence of some blocks, we keep the complexity of the manifolds in consideration under control by considering essentially manifolds with free homology groups, in particular we consider connected sums of tori manifolds. In [Bertolim et al., 2007] we prove the following theorem:

Theorem 18. *Given an abstract Lyapunov semigraph L labelled with a periodic orbit it can be realized as a Morse-Smale flow on an isolating neighborhood which respects the homological information on L .*

Based on Asimov's result [Asimov, 1975] which asserts roughly that two consecutive singularities p of index $k+1$ and q of index k with $W^u(p) \cap W^s(q) = \emptyset$ can be replaced by a round handle of index k , it is shown in [Cruz and de Rezende, 1999] that a Lyapunov semigraph with a vertex labelled with a periodic orbit of index k is derived from a Lyapunov semigraph consisting of two vertices

labelled with singularities of consecutive indices. Later this notion of derivation was generalized in [Bertolim et al., 2003a] as graph continuation and now called graph morsification.

Combining the possible connectivity types of two consecutive singularities (leaving the β -invariant case aside), i.e., a singularity of index k with connectivity type $(k - 1)$ -c or k -d and a singularity of index $k + 1$ with connectivity type k -c or $k + 1$ -d produces the following table:

singularity p of index k + 1	singularity q of index k	
	type k-d	type (k - 1)-c
type (k + 1)-d	$(k + 1)$ -d; k -d	$(k + 1)$ -d; $(k - 1)$ -c
type k-c	k -c; k -d	k -c; $(k - 1)$ -c

We consider in [Bertolim et al., 2007] periodic orbits classified by their connectivity types and denote them as indicated in the following table:

periodic orbit	connectivity type
R_k -disconnecting	$(k + 1)$ -d; k -d
R_k -disconnecting/connecting	$(k + 1)$ -d; $(k - 1)$ -c
R_k -invariant	k -c; k -d
R_k -connecting	k -c; $(k - 1)$ -c

Given an n -dimensional manifold N^n and N^{n-1} a component of its boundary ∂N , we have that the gluing of R to N is performed by identifying $\partial_A R$ to a correspondent diffeomorphic image of $S^1 \times S^{k-1} \times D^{n-k-1}$ in N^{n-1} and we refer to this operation as **adding an n -round handle of index k to N** . It is essentially defined by the embedding $\varphi : S^1 \times S^{k-1} \times D^{n-k-1} \hookrightarrow N^-$ which defines where the attaching region of R , $\partial_A R$, will be attached to N . This surgery changes the manifold N and its boundary component N^- .

In this work we are interested in describing the changes that occur to the collar of N^- once the handle has been attached in some special situations, namely when certain controlled changes in the homology of N^- is required (as coded in the Lyapunov semigraph). For this purpose it is enough to consider $N = N^- \times [0, 1]$ and do the handle operation in one of the components of its boundary. For this purpose, we defined three types of embedding : trivial embedding, small handle embedding and essential embedding that were used in the construction of the isolating blocks. Since the description of these embedding are technical, I prefer to omit them. Details can be found in [Bertolim et al., 2007].

Since we want to consider the realization of isolating blocks for periodic orbits, we analyze the possible effects on the Betti numbers of N^+ once a round handle R of index k is attached to N^- .

The non trivial homology groups of the various regions of the round handle were used in our analysis together with a Mayer-Vietoris sequence to compute the homology making it possible to build the following four realizations of Lyapunov semigraphs for periodic orbits:

- **Disconnecting case:** in this case we will use the trivial embedding to construct an isolating block with the effect described in the Lyapunov semigraph in Figure 6.6. See Figure 6.7 for a three-dimensional disconnecting isolating block.

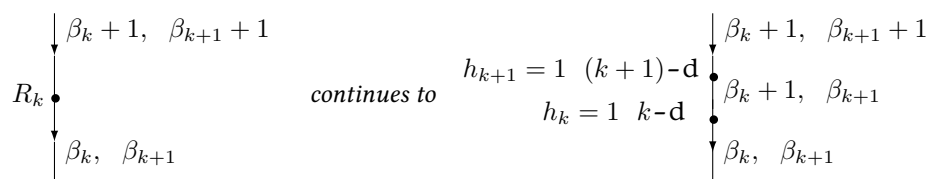


Figure 6.6: k -d and $(k + 1)$ -d, or k and $(k + 1)$ -disconnecting.

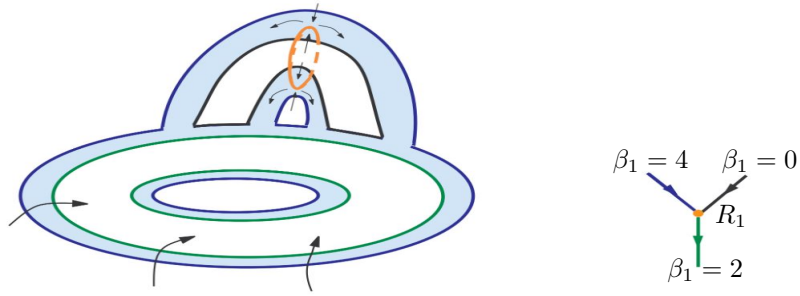


Figure 6.7: Disconnecting isolating block $N^- = T_1^2$ and $N^+ = T_1^2 \# T_2^2 \sqcup S^2$.

- **Invariant case - $N^- = S^k \times S^{n-k-1}$:** in this case we will use the small handle embedding to construct an isolating block with the invariant effect described in the Lyapunov semigraph in Figure 6.8. See Figure 6.9 for a three-dimensional invariant isolating block.

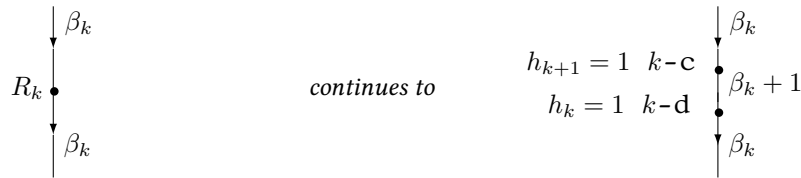


Figure 6.8: Invariant effect.

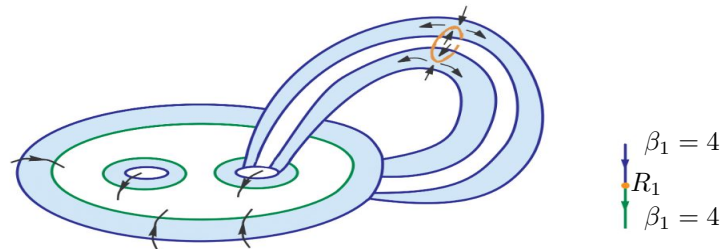


Figure 6.9: Invariant isolating block $N^- = N^+ = \#^2 T^2 = T_1^2 \# T^2$.

- **Invariant case in the middle dimension:** the analysis is slightly more delicate in the middle dimension and we illustrate it with the case $S^{k-1} \times S^k$.

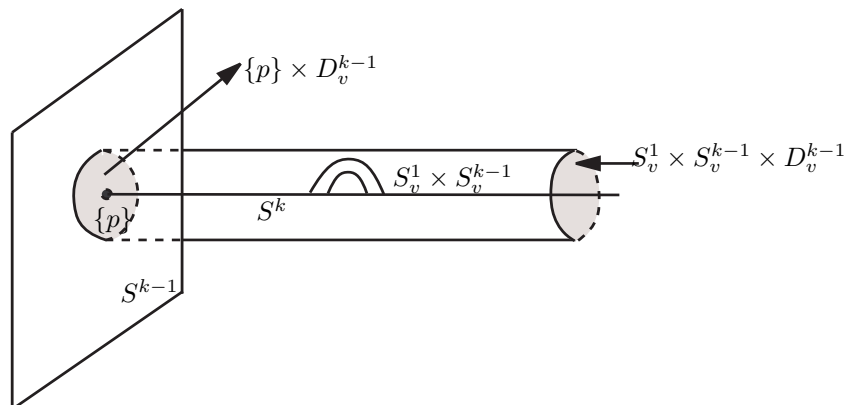


Figure 6.10: Middle-dimensional Case: $S^k \times S^{k-1}$.

- **Connecting-Disconnecting case** - $N^- = S^{n-k} \times S^{k-1}$: in this case the essential embedding will be used to construct an isolating block with the effect described in the Lyapunov semigraph in Figure 6.11.

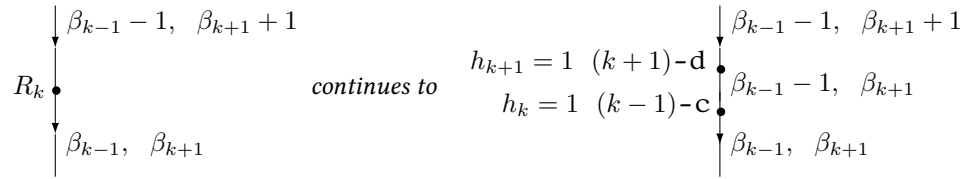


Figure 6.11: $(k - 1)$ -c and $(k + 1)$ -d, or $(k - 1)$ -connecting and $(k + 1)$ -disconnecting.

- **Connecting case**: in this case the essential embedding is used to construct an isolating block with the effect described in the Lyapunov semigraph in Figure 6.12. See Figure 6.13 for a three-dimensional connected isolating block.

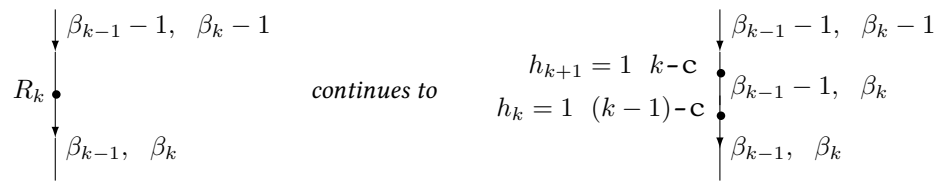


Figure 6.12: $(k - 1)$ -c and k -c, or $(k - 1)$ -connecting and k -connecting.

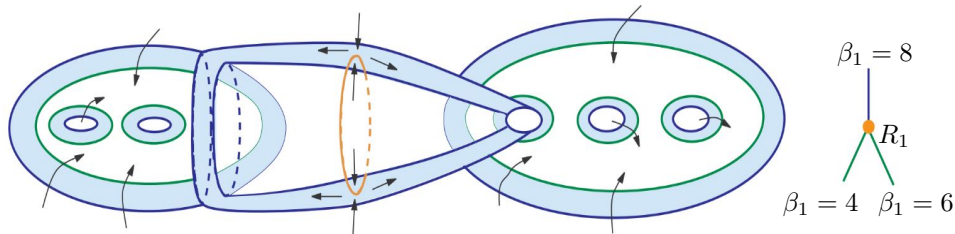


Figure 6.13: Connecting isolating block $N^- = \#^2 T^2 \sqcup \#^3 T^2$ and $N^+ = \#^4 T^2$.

Spectral sequences

"Poetry and code - and mathematics - make us read differently from other forms of writing. Written poetry makes the silent reader read three kinds of pattern at once; code moves the reader from a static to an active, interactive and looped domain; while algebraic topology allows us to read qualitative forms and their transformations."

Stephanie Strickland

In the last chapter we considered isolating blocks. In a certain sense isolating blocks (N, N^+, N^-) are cobordisms between the codimension one closed manifolds which are entering and exiting boundaries, N^+ and N^- respectively, of a flow defined on N . We can also view them as building blocks of a closed manifold. As we saw in Chapter 1, Morse considered slicing a manifold at regular and critical level sets and studied their topological relation up to homotopy. Passing through a critical level of a Morse function was equivalent to adding a handle. Conley generalized this idea in his definition of Morse decompositions which are closely related to isolating blocks, see Figure 7.1. So, locally, isolating blocks are extremely important in the understanding of the dynamics.

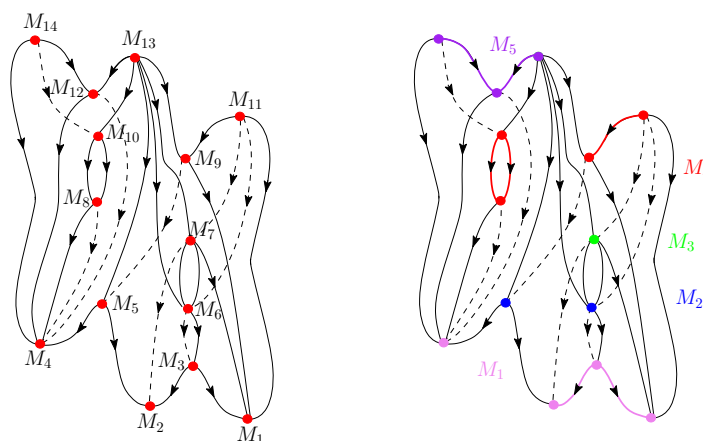
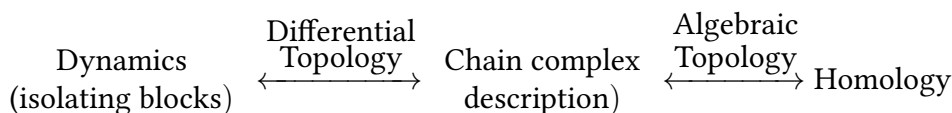


Figure 7.1: Morse Decompositions.

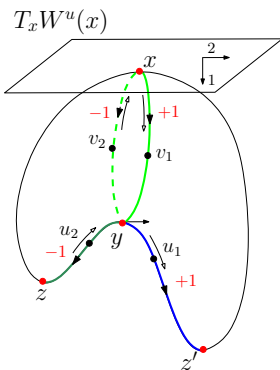
In this chapter, one would like to consider a global view of the dynamics. As mentioned in Chapter 1, Conley's Fundamental Theorem of Dynamical Systems asserts that any continuous flow has a gradient-like behaviour. With this in mind, this chapter aims to explain how these blocks are connected by their stable and unstable manifolds. In this sense we want explore how topological and algebraic invariants are related to these global connections in the dynamics.

Our inspiration is a paradigm established by Smale and Franks.



Here, one uses two main tools: firstly, Differential Topology to describe the behaviour of stable and unstable manifold in terms of a chain complex. Secondly, by using Algebraic Topology, once one has a chain complex the natural thing to do is compute its homology as we showed in Chapter 1 that Poincaré did when he had a simplicial complex. Other authors such as A. Floer, E. Witten and D. Salomon adopted this paradigm and worked with a Morse chain complex. Also, the complex is generated by the critical points of a Morse function and the differential of the complex is determined by the intersection numbers of the unstable manifolds. An important theorem proved and used in their subsequent work is that the Morse homology of this chain complex is isomorphic to the singular homology of the manifold phase space of a Morse gradient flow defined on it.

Consider a gradient flow ϕ of a Morse function $f : M \rightarrow \mathbb{R}$ on a closed Riemannian manifold M . Let $\mathcal{D}(M)$ be a Morse decomposition of M . Given nondegenerate singularities x and y of indices k and $k - 1$, respectively, the set of connecting orbits is finite. By orienting the unstable and stable manifolds, we define the **intersection number** $n(x, y)$ to be the number of connecting orbits counted with orientation, i.e., $n(x, y) = \sum_{u \in \mathcal{M}_y^x} n_u$, where n_u is the characteristic sign.



- $n(x, y) = n_{v_1} + n_{v_2} = 0$
- $n(y, z) = n_{u_2} = -1$
- $n(y, z') = n_{u_1} = 1.$

Let C be the Morse complex associated to f and M . C is the free module generated over \mathbb{Z} by the critical points of f graded by the Morse index, $C_k = \mathbb{Z}\langle \text{Crit}_k(f) \rangle$, that is

$$C_k(f) := \bigoplus_{x \in \text{Crit}_k(f)} \mathbb{Z}\langle x \rangle, \quad k \in \mathbb{Z}.$$

The **connection matrix** $\Delta : C \rightarrow C$ associated with $\mathcal{D}(M)$ is defined to be the differential of the graded Morse chain complex $C = \mathbb{Z}\langle \text{crit } f \rangle$, i.e., it is determined by the maps $\Delta_k(x) : C_k(f) \rightarrow C_{k-1}(f)$ via

$$\Delta_k \langle x \rangle := \sum_{y \in \text{Crit}_{k-1}(f)} n(x, y) \langle y \rangle.$$

We will write the boundary operator ∂ and its matrix Δ interchangeably.

Consider the following example of a Morse Chain Complex on S^2 .

$$\Delta(S^2) = \begin{bmatrix} & z' & z & y & x \\ z' & 0 & 0 & 1 & 0 \\ z & 0 & 0 & -1 & 0 \\ y & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \end{bmatrix}$$

Our idea in [Bertolim et al., 2016a, 2017] is to take this idea a step further. Instead of computing the homology once, one proceeds to compute a sequence of homology complexes as defined by a spectral sequence.

The dynamics given in each Morse isolating block can be connected to a Morse chain complex as shown above. Intuitively, one can look at a spectral sequences as a book made up of pages E^r and each page r there is a chain complex E^r, d^r with a differential d^r is of bi-degree $(-r, r - 1)$. So for example on E^2 , all differentials have bi-degree $(-2, 1)$ which means the arrow connects modules in position (p, q) to modules -2 to the left in a horizontal direction and 1 in a positive vertical direction, i.e., to a module in position $(p - 2, q + 1)$. The turning of the pages corresponds to computing the homology of the previous page, $H_*(E^r) \approx E^{r+1}$. In this sense one produces a sequence of homologies.

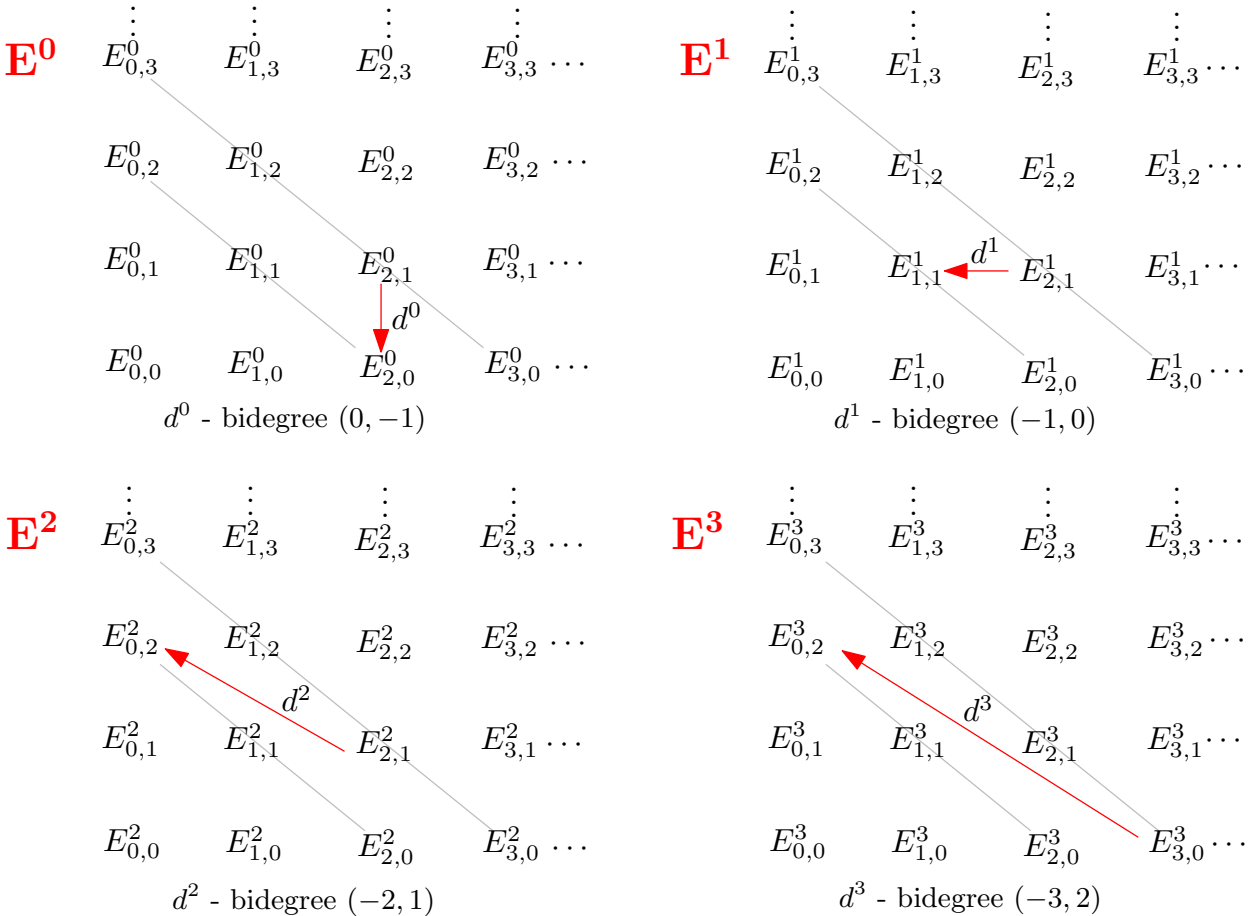
So now let us define a spectral sequence.

A **spectral sequence** $E = \{E^r\}_{r \geq 0}$ is a sequence of chain complexes $\{E^r, d^r\}$, $r \geq 0$, such that

- $E^r_{p,q}$ is **bigraded** and is the set of $p + q$ -chains.
- d^r is a **differential of bidegree** $(-r, r - 1)$, i.e., it is a collection of homomorphisms $d^r : E^r_{p,q} \rightarrow E^r_{p-r, q+r-1}$ for all p and q such that $d^r \circ d^r = 0$.
- For all $r \geq 0$ there exists an isomorphism $H(E^r) \approx E^{r+1}$ where

$$H_{p,q}(E^r) = \frac{\text{Ker}d^r : E^r_{p,q} \rightarrow E^r_{p-r, q+r-1}}{\text{Im}d^r : E^r_{p+r, q-r+1} \rightarrow E^r_{p,q}}$$

(homology module)



Now we want to understand what each module $E_{p,q}^r$ means in terms of the dynamics.

A filtration $F = \{F_p\}$ on a chain complex C is a sequence of subcomplexes $F_p C$, $p \in \mathbb{Z}$, such that $F_p C \subset F_{p+1} C$, for each p . In this work, one considers a specific filtration in a Morse complex induced by the flow ϕ_f . Given a finest Morse decomposition $\{M(p) \mid p \in P = \{1, \dots, m\}, m = \#Crit(f)\}$ such that there are distinct critical values c_p with $f^{-1}(c_p) \supset M(p)$, we can define a filtration on M by

$$\{F_{p-1}\}_{p=1}^m = \{f^{-1}(-\infty, c_p + \epsilon)\}_{p=1}^m.$$

Since for each $p \in P$ there is only one singularity in $F_p \setminus F_{p-1}$ the filtration F is called a finest filtration. The filtration F is **convergent**, i.e., $\bigcap_p F_p C = 0$ and $\bigcup_p F_p C = C$. We define the **filtration length** with respect to a filtration F of the orbit $\mathcal{O}_{h_k, h_{k-1}}$ that connects h_k to h_{k-1} as being the natural number r whenever $h_k \in F_p C$ and $h_{k-1} \in F_{p-r} C$. The number r is also called the **gap** between the singularities h_k to h_{k-1} .

The algebraic formulas for the modules are

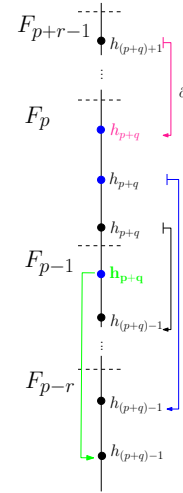
$$E_{p,q}^r = \frac{Z_{p,q}^r}{Z_{p-1,q+1}^{r-1} + \partial Z_{p+r-1,q-r+2}^{r-1}}$$

$$Z_{p,q}^r = \{c \in F_p C_{p+q} : \partial c \in F_{p-r} C_{p+q-1}\}$$

$$Z_{p-1,q+1}^{r-1} = \{c \in F_{p-1} C_{p+q} : \partial c \in F_{p-r} C_{p+q-1}\}$$

$$\partial Z_{p+r-1,q-r+2}^{r-1} = \partial \{c \in F_{p+r-1} C_{p+q+1} : \partial c \in F_p C_{p+q}\}$$

$$E_{p,q}^r = \frac{\{h_{p+q}, h_{p+q}, h_{p+q}\}}{\{h_{p+q}, h_{p+q}\}}$$



In order to understand these formulas, let us be inspired by Poincaré once again in his search for i -cycles, which are not the boundary of some $i+1$ -chain which is the main idea behind the homology computation. In the Morse chain complex, C_i are the i -chains generated by the index i critical points. Once again we wish to find i -cycles, which are not the boundary of some $i+1$ -chain. When one computes the modules of the spectral sequence, the filtration plays an extremely important role. $E_{p,q}^r$ will be generated by index $p+q$ critical points in filtration F_p which are i -cycles whose boundary are critical points of index $p+q-1$ with the additional property that it is in a filtration level of gap r , F_{p-r} . In a certain sense, as one computes the spectral sequence, longer connections are determined.

Note that one quotients out cycles which are boundaries of $p+q+1$ -chains in F_{p+r} , as well as, $p+q$ -cycles which have $p+q-1$ -chains as boundaries which are not in F_{p-r} , i.e are not long enough.

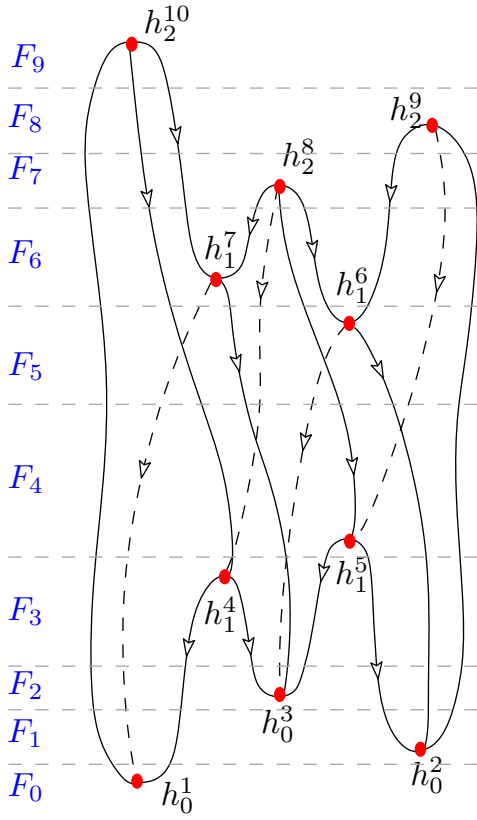
Whenever the filtration considered is the finest filtration F , since each Morse set is a singularity of index k , the only q such that $E_{p,q}^r$ is non-zero is $q = k - p$. Hence, we omit reference to q , i.e., E_p^r is in

fact $E_{p,k-p}^r$. The algebraic formulas above for the modules became: $E_p^r = \frac{Z_p^r}{Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}}$, where

$Z_p^r = \{c \in F_p C_p : \partial c \in F_{p-r} C_p\}$. Each h_k column of the connection matrix Δ represents the connections of an elementary chain h_k of C_k to an elementary chain h_{k-1} of C_{k-1} . The \mathbb{Z} -module $Z_{p,k-p}^{r-1} = \{c \in F_p C_k : \partial c \in F_{p-r} C_{k-1}\}$ is generated by k -chains contained in F_p with boundaries in F_{p-r} . In the matrix Δ , this corresponds to all h_k columns to the left of the $(p+1)$ st column, or linear combinations thereof, such that their boundaries (non-zero entries) are above the $(p-r+1)$ st row. $Z_{p-1,k-(p-1)}^{r-1} = \{c \in F_{p-1} C_k : \partial c \in F_{p-r} C_{k-1}\}$ corresponds to all h_k columns to the left of the p th column in the matrix Δ , or linear combinations thereof, such that their boundaries are above

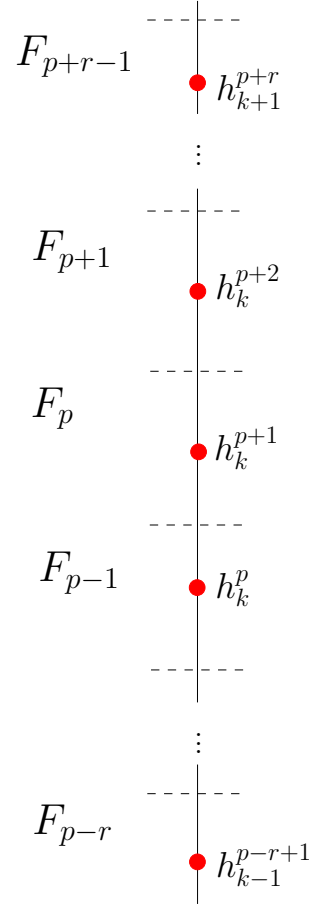
the $(p - r + 1)$ st row and $\partial Z_{p+r-1, (k+1)-(p+r-1)}^{r-1} = \partial\{c \in F_{p+r-1}C_{k+1} : \partial c \in F_p C_k\}$ is the set of all the boundaries of elements in $Z_{p+r-1, (k+1)-(p+r-1)}^{r-1}$, corresponds to all h_k columns to the left of the $(p + 1)$ st column (or, equivalently, all h_k rows above the $(p + 1)$ st row) in the matrix Δ that are boundaries of h_{k+1} columns to the left of the $(p + r)$ th column. The index- k singularity in $Fp \setminus F_{p-1}$ corresponds to the k -chain associated to the $(p + 1)$ st column of Δ .

An example of a finest filtration is given below.



Filtration in M

- If $h_k^{(p+1)} \in F_p C \setminus F_{p-1} C$.
- Simplifying notation:
 $E_p^r = E_{p, k-p}^r$.



We say that two singularities of consecutive Morse index of a smooth flow ϕ on a manifold can be **dynamically cancelled** if there exists a neighborhood U of these singularities that also contains the orbits connecting them and a new flow ϕ' that coincides with ϕ outside of U and is a tubular flow inside U .

A particular case of interest occurs whenever the entry ± 1 in the connection matrix is a primary pivot or change-of-basis pivot. The differentials $d_p^r : E_p^r \rightarrow E_{p-r}^r$ associated to primary pivots are isomorphisms and the ones associated to change of basis pivots always correspond to zero maps. The non-zero differentials are isomorphisms and this implies **algebraic cancellations** in the spectral sequence, i.e., $E_p^r = E_{p-r}^r = 0$. These algebraic cancellations also correspond to **dynamical cancellation** of consecutive index singularities h_k and h_{k-1} in ϕ , i.e., this corresponds dynamically to the death of the singularities and its connecting orbits and the birth of a new connection.

Let us exemplify dynamical cancellations by using Smale's Cancellation Theorems. In the two-dimensional case this is done in the following way: suppose that there is a unique orbit between h_k and h_{k-1} and let U be a neighborhood containing them. The orbit connecting these singularities is the transversal intersection of their unstable and stable manifolds, $W^u(h_k) \cap W^s(h_{k-1})$. By Smale's First Cancellation Theorem, one may cancel h_k and h_{k-1} resulting in a flow which coincides with the initial flow outside U and has no critical points in U .

For higher dimensions (≥ 6) first, we adopt the loose terminology that a critical point h_k^j connects

with a critical point h_{k-1}^i if the moduli space $\mathcal{M}_{h_k^j}^{h_{k-1}^i}$ is non-empty. In fact, even when the intersection number between two singularities on an n -dimensional manifold, $n > 2$, is equal to ± 1 , there may be several connecting orbits between them. Smale’s Second Cancellation Theorem guarantees that whenever the intersection number between two critical points x and y is ± 1 , the flow can be modified locally so that the connection between x and y is a unique flow line.

Hence, if h_k^j and h_{k-1}^{j-r} are two consecutive critical points of a Morse-Smale function f such that $n(h_k^j, h_{k-1}^{j-r}, \phi_f) = \pm 1$, then by Smale’s Second Cancellation Theorem the flow can be modified locally so that the connection between them is a unique flow line and by using Smale’s First Cancellation Theorem these critical points can be cancelled, i.e. there is a gradient flow ϕ' which coincides with ϕ outside a neighborhood of U of $\{h_k^j, h_{k-1}^{j-r}\} \cup \mathcal{O}(u)$, where $\mathcal{M}_{h_k^j}^{h_{k-1}^{j-r}} = \{u\}$. See Figure 7.2.

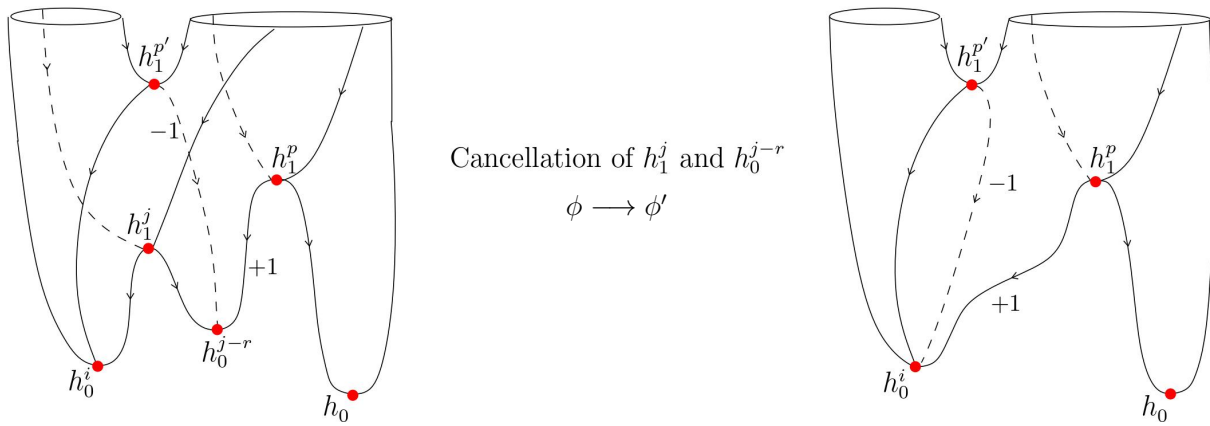


Figure 7.2: Birth and death of connections.

Our main contribution was the following global cancellation result “Smale’s Cancellation Theorem via spectral sequences”:

Theorem 19. *Let (C, Δ) be the Morse chain complex associated to a Morse function f . Let (E^r, d^r) be the associated spectral sequence for the finest filtration $F = \{F_p C\}$ defined by f . The algebraic cancellation of the modules E^r of the spectral sequence are in one-to-one correspondence with dynamical cancellations of critical points of f . Moreover, the order of cancellation occurs as gap r increases.*

This theorem shows us that every time we detect differentials d^r of the sequence in E^r which are isomorphisms, on the next page those modules in those same positions, zero out. This is natural since the kernel of an isomorphism is zero and hence the homology is zero. Our theorem provides a dynamical counterpart to this algebraic phenomena. It corresponds to a connection of an unstable manifold of two consecutive index critical points, one of index p to an index $p - 1$ which cancels due to its intersection number being ± 1 . So algebraic cancellations and dynamical cancellations are closely related. See Figures 7.3 and 7.4 which illustrates this correspondence.

Our motivation throughout this work was to obtain as much algebraic information from the most simple topological setting for a flow, namely, a Morse flow on an orientable surface and its connections. One hopes in the future that given a more complicated dynamical setting such as a smooth flow on a 2-dimensional manifold, or on higher dimensional manifolds, these algebraic invariants will also provide other dynamical information on the nature of the connections as well as bifurcation behaviour, see Franzosa et al. [2014]. This is the central stimulus in studying spectral sequences for Morse chain complexes with coarser filtration since this provides the finest filtration for a more general flows. See Figure 7.5. In Figure 7.6, the Lyapunov graph on the left represents a more general flow with the finest filtration and the Lyapunov graph on the right represents a Morse flow with a coarser filtration.

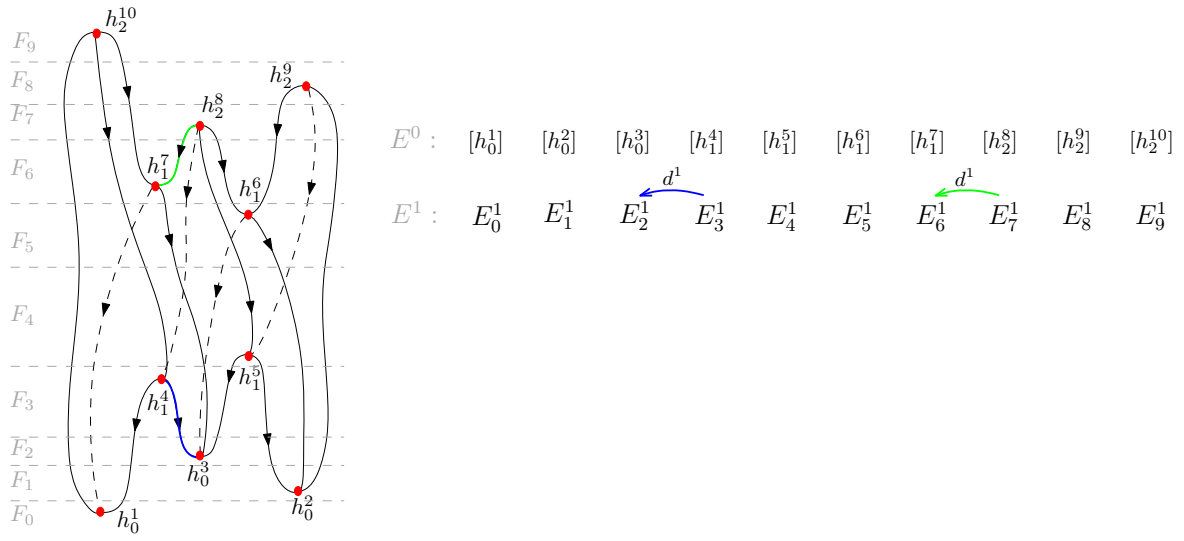


Figure 7.3: Correspondence between algebraic cancellations and dynamical cancellations.

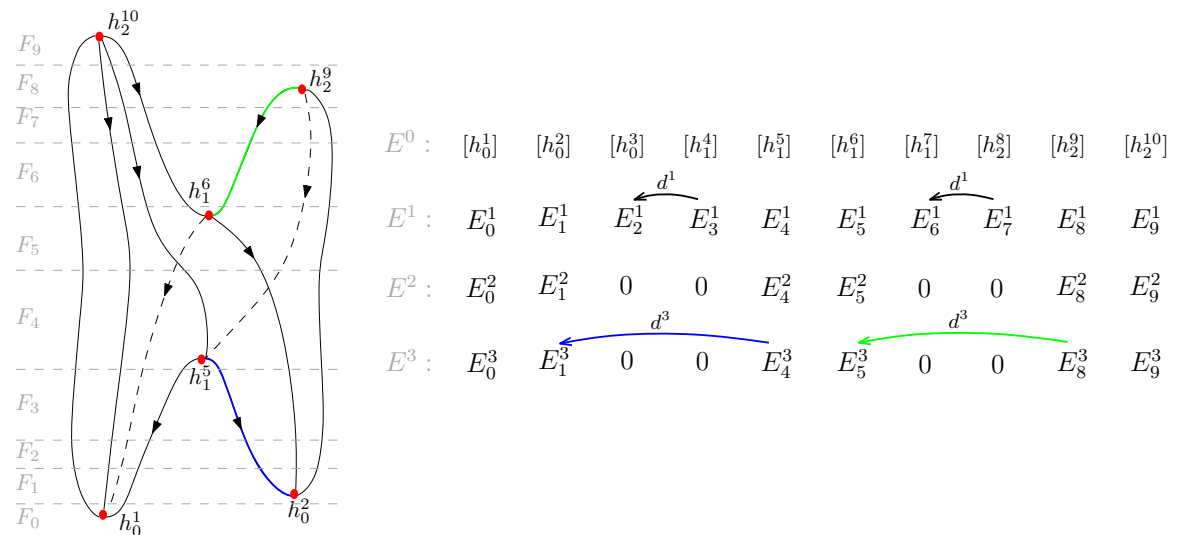


Figure 7.4: Correspondence between algebraic cancellations and dynamical cancellations.

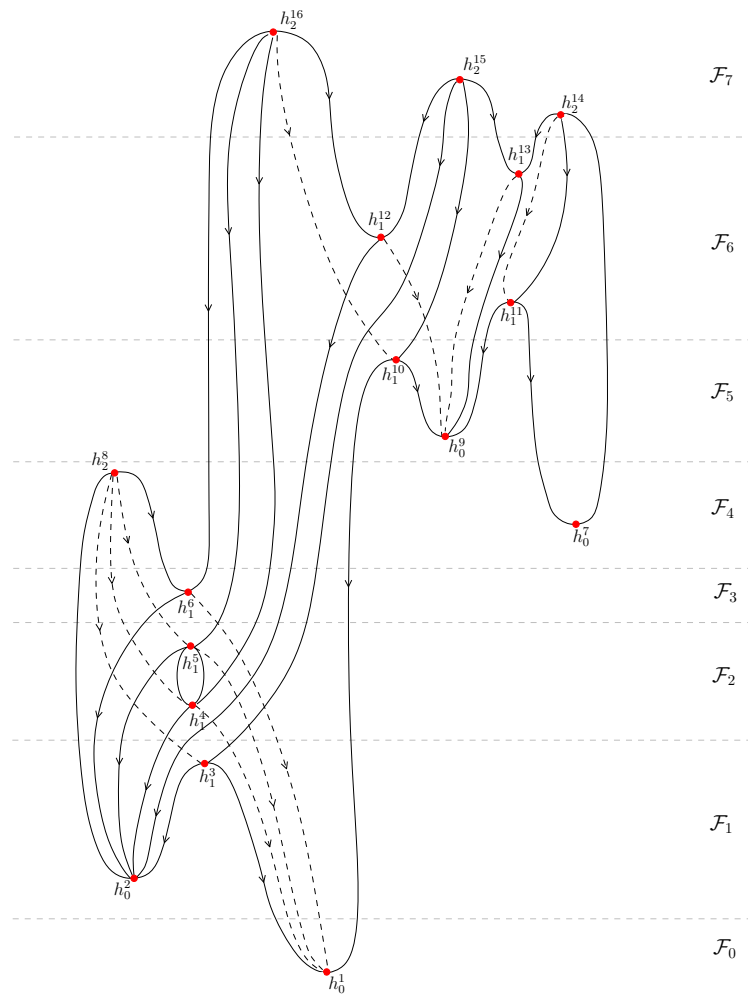


Figure 7.5: Morse flow with coarser filtration.

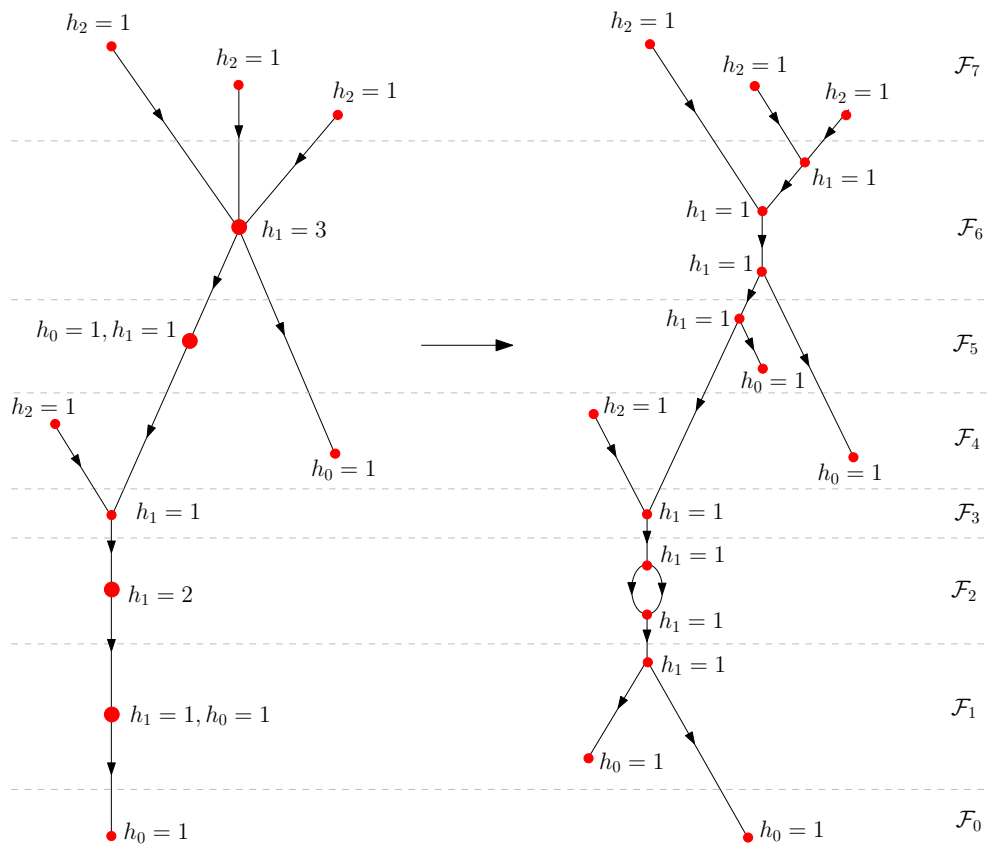


Figure 7.6: Coarser filtration on Morse type Lyapunov graph (on the right).

Perspectives

*"The important thing is not to stop questioning. Curiosity has its own reason for existing."
Albert Einstein*

In the Section 5.2 we described our result developed in [Bertolim et al., 2023a]. In this work we have considered only the odd dimensional case. The proof of our results opens way to the consideration of the even dimensional setting. Even though the guidelines for finding an algorithm are comparable, the combinatorics in the even dimensional setting is richer because of the existence of “invariant” handles¹ and must be treated independently. Moreover, handles of invariant type are difficult to realize in the nonsingular Morse-Smale context, so that the realization of the general abstract solution by a nonsingular Morse-Smale model remains an open question. For the same reason, our results in [Bertolim et al., 2023b], described in Section 4.4 treated only the odd dimensional case and the even case remains an open question. In fact, an important step to solve these open questions is to construct a β -invariant isolating block following the ideas of [Bertolim et al., 2007].

We also intend to generalize the Poincaré-Hopf inequalities by removing the Conley duality condition in order to apply them in a more general context.

Since I am now in an Engineering School, another direction that I intend to explore is Lyapunov graph applications. In fact, Lyapunov graphs can be seen as enriched Reeb graphs. Reeb graphs were introduced in [Reeb, 1946] and have been used in a range of applications in computer graphics and visualization, see for example the survey [Biasotti et al., 2008] and references therein. They also are an important tool in the field of Computational Topology for shape analysis, see for example [Edelsbrunner and Harer, 2010]. Applications also appear in medical imaging and some other areas of science and engineering, see for example [Pepe et al., 2012], [Mestiri, 2012], [Herlem et al., 2013] and [Khoury et al., 2012]. Therefore the use of some of our results transcribed to lower dimension can be useful for applications and may originate new techniques.

There are many unexplored paths one can undertake in this area of research. To consider other dynamical settings and explore the beautiful algebraic connections between topological and dynamical invariants is an arduous yet fascinating endeavour of an isolating block.

¹Handles whose attaching produces no effect on the Betti numbers of the boundary.

Appendix

"We ourselves are co-called non-linear dynamical systems... I don't feel quite so pathetic when I interrupt a project to check on some obscure web site or newsgroup or derive an iota of cheer by getting rid of pocketful of change."
John Allen Paulos

As I mentioned previously, in order to present a coherent document, I have chosen to present the publications [Bertolim and Jacquemard, 2014, Bertolim et al., 2016b] in this appendix.

Indeed, with Alain Jacquemard and Gioia Vago in [Bertolim and Jacquemard, 2014, Bertolim et al., 2016b] we considered **discontinuous dynamical systems**. In this case we are interested in studying differential equations depending nonsmoothly on the integration time. We deal with the existence of periodic solutions to such equations and our goal is to give effective methods and algorithms which explain the behavior of such nonsmooth dynamical systems.

In [Bertolim and Jacquemard, 2014] we discuss differential equations depending nonsmoothly on the integration time of the form

$$y^{(n)} = \operatorname{sgn}(\sigma(t)) + F(t)$$

where $n \in \mathbb{N}$, $n > 0$, and F, σ are piecewise- \mathcal{C}^∞ periodic functions. This work deals with the study of the periodic solutions to piecewise- \mathcal{C}^∞ differential equations. Up to normalization, the periodic solutions to our order n equations are expressed in a very direct way in terms of the n -primitive of F vanishing at 0, the function σ and the Euler polynomial of degree n , E_n , which is a degree n polynomial defined by:

$$E_n(t) = \sum_{k=0}^n \frac{1}{2^k} \left(\sum_{j=0}^k (-1)^j \binom{k}{j} (t+j)^n \right).$$

Note that the interest of this formulation in terms of Euler polynomials is two-fold. On the one hand, although Euler polynomials are well known and extensively studied by several mathematicians, it is the very first time that they appear in nonsmooth dynamical systems theory. On the other hand, they allow us to write explicitly our solutions without any numerical approximation due to the presence of infinite series. In this sense, this approach is conceptually different from the classical one using Fourier series theory.

In [Bertolim et al., 2016b] it is shown that Bernoulli polynomials can be used to construct solutions to some other interesting classes of ordinary differential equations. These equations are beyond the theory of piecewise-smooth differential equations and have to be treated in Distribution Theory, since their expressions involve Dirac combs. Here, too, we wish to emphasize that all the periodic solutions to the equations we deal with are expressed in terms of finite exact formulae. In these formulae, a family of switched Bernoulli polynomials appear. They are defined in a simple way from the Bernoulli polynomials themselves. The intrinsic nature of all the solutions is thus as well understood as the Bernoulli polynomials are.

One remark that even though the problems solved in [Bertolim and Jacquemard, 2014] and those solved in [Bertolim et al., 2016b] are different and independent, they also share a common point, namely, the central role played by the two remarkable families of Appell polynomials (the Bernoulli ones and the Euler ones, respectively). I conclude then this part of my research with the following questions:

- Are these two families exceptional among the Appell polynomials because of their symmetries?
- Or are there other families of natural discontinuous differential equations with direct applications whose periodic solutions can be expressed in terms of other remarkable bases of Appell polynomials?

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