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Analyses de modèles et de mécanismes incitatifs pour la régulation financière et le suivi des populations

Thibaut Mastrolia

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Discipline : Mathématiques

par

Thibaut MASTROLIA

Analyses de modèles et de mécanismes incitatifs
pour la régulation financière et le suivi des populations

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List of articles¹

Published papers

- **Regulation of renewable resource exploitation**, with I. Kharroubi et T. Lim. *SIAM Journal of Control and Optimization*, 58(1), 551-579. 2020.
See Chapter 9.
- **Moral Hazard in a VUCA world**, with N. Hernandez-Santibanez. *SIAM Journal of Control and Optimization*, 57(4), 3072-3100. 2019.
See Chapter 5.
- **A tale of a Principal and many many Agents**, with R. Elie and D. Possamaï. *Mathematics of Operations Research*, 58(1), 551-579. 2018.
See Chapter 3.
- **Principal-Agent with common Agency without Communication**, with Z. Ren. *SIAM Journal of Mathematical Finance*, Vol. 9, No. 2, pp. 775-799. 2018.
See Chapter 4.
- **Moral hazard under Ambiguity**, with D. Possamaï. *Journal of Optimization Theory and Applications*, Vol 179, Issue 2, pp 452-500. 2018.
- **Density analysis of non-Markovian BSDEs and applications to biology and finance**. *Stochastic Processes and Applications*, Vol. 128, No. 3, 897-938. 2018.
- **On the Malliavin differentiability of BSDEs**, with D. Possamaï and A. Réveillac. *Annales de l'Institut Henri Poincaré*, Vol. 53, No. 1, 464-492. 2017.
- **A note on the Malliavin-Sobolev space**, with P. Imkeller, D. Possamaï and A. Réveillac. *Statistics and Probability Letters*, 109:45-53. 2016.
- **Density analysis of BSDEs**, with D. Possamaï and A. Réveillac. *The Annals of Probability*, 44(4):2817-2857, 2016.
- **Utility maximization with random horizon: a BSDE approach**, with M. Jeanblanc, D. Possamaï and A. Réveillac. *International Journal of Theoretical and Applied Finance*, Vol. 18, No. 7, 2015.

Submitted papers

- **Mean-field moral hazard for optimal energy demand response management**,² with R. Elie, E. Hubert and D. Possamaï. arXiv:1902.10405. See Chapter 3 Section 3.2.
- **Optimal make-take fees for market making regulation**,³ with O. El Euch, M. Rosenbaum and N. Touzi. arXiv:1805.02741. See Chapter 6.
- **Optimal auction duration: A price formation viewpoint**,⁴ with P. Jusselin et M. Rosenbaum. arXiv:1906.01713. See Chapter 7 Section 7.1.
- **Market making and incentives design in the presence of a dark pool: a deep reinforcement learning approach**, with B. Baldacci, I. Manziuk et M. Rosenbaum. arXiv:1912.01129. See Chapter 8.
- **Scaling limit for stochastic control problems in population dynamics**, with P. Jusselin. arXiv:1911.00672. See Chapter 11.
- **Incentives, lockdown, and testing: from Thucydides's analysis to the COVID-19 pandemic**, with E. Hubert, D. Possamaï and X. Warin. arXiv:2009.00484. See Chapter 10.
- **AHEAD: Ad Hoc Electronic Auction Design**, with J. Derchu, P. Guillot and M. Rosenbaum. arXiv:2010.02827. See Chapter 7 Section 7.2.

¹Only underlined papers will be outlined in this habilitation thesis. The other papers are parts of my PhD thesis.

²This paper has been accepted for publication in *Mathematical Finance* in September 2020.

³This paper has been accepted for publication in *Mathematical Finance* in October 2020.

⁴This paper has been accepted for publication in *Operations Research* in November 2020.

Résumé: Ce rapport de synthèse est divisé en trois parties : la première traite de résultats généraux en théorie des contrats et jeux différentiels stochastiques. Partant d'un modèle avec N –agents employés par un Principal pour gérer un projet risqué, l'étude est étendue à un champ moyen d'Agents employés par un Principal, appliquée à la tarification de l'électricité. Puis, le cas de N –Principaux employant un Agent commun est analysé. La seconde partie de ce mémoire porte sur la microstructure des marchés et la régulation financière. Utilisant des outils développés dans la première partie, une politique optimale de commissions d'exécutions sur les transactions financières est proposée par une place boursière, étendue ensuite à un oligopole de places boursières en compétition. Puis, nous modélisons un marché de type enchères dans lequel nous cherchons la durée optimale de ce genre de mécanisme assurant un grand nombre de transactions sur ce marché. L'étude est ensuite étendue à des enchères *ad hoc*, dans lesquels les investisseurs déclenchent une enchère lorsque le prix proposé par un fournisseur de liquidités n'est plus satisfaisant. La pertinence de tous ces mécanismes est quantitativement et qualitativement analysée. Enfin, la troisième partie de ce rapport se focalise sur le suivi des population et la gestion des ressources. Nous considérons tout d'abord une ressource naturelle renouvelable gérée par un exploitant lui-même supervisé par une institution publique pour assurer la durabilité de la ressource. Puis nous regardons une politique optimale d'isolement, de détection et de taxation d'une population en période de pandémie afin de réduire la propagation de celle-ci.

Mots clés : Contrôle stochastique, Problèmes Principal-Agent, Alea Moral, Jeux à champs moyens et jeux différentiels stochastiques, Microstructure des marchés, Régulation financière, Suivi des populations, Gestion des ressources.

Abstract: This synthesis report is divided into three parts: the first deals with general results in the theory of stochastic contracts and differential stochastic games. Starting from a model with N –agents employed by a Principal to manage a risky project, the study is extended to an average field of Agents employed by a Principal, applied to electricity pricing. Then, the case of N –Principals employing a common Agent is analyzed. The second part of this paper deals with market microstructure and financial regulation. Using tools developed in the first part, an optimal policy of execution commissions on financial transactions is proposed by a stock exchange, then extended to an oligopoly of competing stock exchanges. Then, we model an auction-type market in which we seek the optimal duration of this type of mechanism ensuring a large number of transactions on this market. The study is then extended to ad hoc auctions, in which investors trigger an auction when the price offered by a trader is no longer satisfactory. The relevance of all these mechanisms is quantitatively and qualitatively analyzed. Finally, the third part of this report focuses on population monitoring and resource management. We consider first of all a renewable natural resource managed by an operator himself supervised by a public institution to ensure the sustainability of the resource. Then we look at an optimal policy of isolation, detection and taxation of a population in times of pandemic in order to reduce its spread.

Key words: Stochastic control, Principal-Agent problems, Moral Hazard, Mean field Games and stochastic differential games, market microstructure, financial regulation, population monitoring, resource management.

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Introduction

1 Version française

Commençons avec trois analyses de trois études de cas.

Analyse A. “Dans quelle mesure *mon* utilité est impactée par l’ajout d’un animal à mon troupeau dans un pré partagé avec d’autres éleveurs ?” se demande un éleveur de bétail. “D’un côté, l’ajout d’un animal dans notre pré commun augmente mes recettes. Disons que je gagne +1. D’un autre côté, ajouter un animal au pré conduit à un surpâturage de celui-ci mais cet effet est partagé avec mes autres collègues. Je perds donc une fraction de 1.” L’éleveur rationnel conclue ainsi qu’il serait mieux pour son propre revenu d’ajouter de plus en plus d’animaux à son troupeau dans le pré partagé. Sans aucune coordination, nous nous attendons à ce que tous les éleveurs raisonnent ainsi jusqu’à l’extinction inévitable du pré. Cette stratégie bien que rationnelle mène donc à un désastre écologique et économique. Cet exemple est extrait des travaux de Garret Hardin dans [Hardin, 1968] et constitue l’exemple de base de la *tragédie des biens communs*. En 1990, Elinor Ostrom a proposé dans [Ostrom, 1990] la mise en place de politiques de suivi d’une ressource commune en incitant un système d’exploitants de cette ressource à agir contre leurs intérêts personnels au profit de la viabilité de celle-ci, et ainsi améliorer l’état du système général. Ces politiques permettent ainsi de préserver la ressource tout en garantissant un revenu aux exploitants de celle-ci.

Analyse B. Les consommateurs d’énergie gagnent de l’utilité en consommant de l’électricité, du gaz, du pétrole, ou toute autre énergie renouvelable ou non-renouvelable. Cependant, les producteurs d’énergie font face à une flexibilité limitée de la production et aimeraient réduire (en moyenne et en variance) la déviation de la consommation énergétique d’un consommateur par rapport à une consommation habituelle référante. Le producteur renforce la réactivité des consommateurs avec des prix incitatifs de l’énergie, concevant optimalement un mécanisme pour fixer les prix de celle-ci en développant des réseaux informatifs plus intelligents. Dans [Aïd et al., 2018], les auteurs ont proposé un modèle de tarification énergétique indexé sur la consommation en énergie représentative de consommateurs en maintenant raisonnablement la valeur de la consommation moyenne et sa variance.

Analyse C. Considérons maintenant un marché financier avec des investisseurs et des fournisseurs de liquidité (teneurs de marché). Ces derniers fixent les prix d’achat et de vente d’actifs risqués. La fragmentation des marchés financiers a entraîné une compétition au sein des places financières où ont lieu les échanges de produits financiers. Les places boursières traditionnelles sont maintenant soumises à des arrivées d’ordres alternant d’une plateforme financière à une autre. Elles doivent donc trouver des moyens innovants pour attirer de la liquidité sur leurs plateformes. En effet, les teneurs de marchés s’assurent un revenu en augmentant la distance entre prix d’achat et prix de vente (le *spread*). À l’inverse, les investisseurs ont tendance à fuir les marchés avec un *spread* trop grand. Une solution répandue pour attirer les flux financiers consiste à utiliser une politique de taxe des ordres émis et reçus, asymétriquement entre les fournisseurs de liquidité et les investisseurs, voir par exemple [Foucault, 1999]. Ce mécanisme est le plus classique et est utilisé

par beaucoup de places boursières (comme par exemple Nasdaq, Euronext, BATS Chi-X...). Elle consiste à compenser les fournisseurs de liquidités avec des politiques d'incitations, et de prendre une commission sur les ordres émis par les investisseurs (c.-à-d. un coût de transaction par opération).

Ces trois analyses ont en commun la théorie des incitations : politique de suivi des activités d'éleveurs/exploitants d'une ressource (cf *Analyse A*), tarification de l'énergie pour les consommateurs (cf *Analyse B*) et régulation des activités financières des participants à un marché donné (cf *Analyse C*). Elles ont pour but de répondre à la question suivante :

Comment inciter optimalement plusieurs agents à défendre des intérêts plus grands que les leurs ?

Techniquement, ces problèmes de types Principal-Agent peuvent être modélisés comme suit : initialement, un Principal conçoit optimalement un contrat (une taxe ou une rémunération) pour un agent (ou plusieurs agents) pour gérer un projet potentiellement risqué.

La première partie de ce rapport est consacrée aux problèmes principal-agent et aux jeux différentiels stochastiques. La théorie des contrats a débuté dans les années 70, voir par exemple [Mirrlees, 1976], et est une illustration de problèmes Principal-Agent. Plus précisément, le cadre classique considéré est le suivant : un Principal (*elle*) propose à un agent (*il*) un contrat. L'agent peut accepter ou refuser le contrat proposé par le Principal. Sous condition d'acceptation, il fournira un travail pour gérer un projet garantissant une richesse au Principal. La situation dans laquelle le Principal conçoit à la fois la rémunération de l'agent et gère parfaitement son effort coïncide avec un modèle de partage des risques appelé *first-best case*. Cette situation a été très étudiée dans plusieurs ouvrages en utilisant le principe du maximum et les systèmes d'EDS *forward-backward* couplés. Nous renvoyons au livre [Cvitanić and Zhang, 2012] pour plus de détails à ce sujet. Cependant, le Principal est souvent imparfaitement informé des actions de l'agent, en observant seulement les résultats des efforts de celui-ci et non les efforts de ce dernier. Le Principal conçoit donc un contrat qui maximise sa propre utilité en tenant compte de cette asymétrie d'information (*second-best case*), étant donné que l'utilité de l'Agent est tenue à un niveau donné (son utilité de réservation). D'un point de vue de la théorie des jeux, cette classe de problèmes peut être assimilée à un jeu de Stackelberg entre le Principal (le meneur) et l'agent (le suiveur). Dans ce cas, le Principal anticipe l'effort de meilleure réponse de l'agent étant donné un contrat fixé et le prend ensuite en compte pour maximiser sa propre utilité. L'aléa moral dans la théorie des contrats, c'est-à-dire le fait que le Principal n'a pas accès au travail de son agent, a été développé dans les années 80 et a été étudié dans un cadre de temps continu particulier par Holmström et Milgrom dans [Holmström and Milgrom, 1987]. Plus récemment, Sannikov dans [Sannikov, 2008] a étudié un problème de temps d'arrêt optimal en théorie des contrats, où le Principal rémunère l'Agent jusqu'à un horizon aléatoire qu'il choisit optimalement. Cet article souligne l'impact fondamental de la fonction de valeur du problème de l'Agent pour résoudre le problème du Principal. Ce problème a ensuite été formalisé mathématiquement dans [Cvitanić et al., 2019] en proposant une procédure générale pour résoudre un large panel de problèmes de type Principal-Agent sous aléa moral. Les auteurs ont montré que l'équilibre de Stackelberg entre l'agent et le Principal se réduit à un problème d'optimisation de type bi-niveaux de la façon suivante : d'abord, étant donné une compensation proposée par le Principal, l'agent maximise son utilité en trouvant le meilleur effort associé à la compensation proposée. Ensuite, le problème du Principal peut être réduit à résoudre un problème de contrôle stochastique avec comme variables d'états du problème : la valeur du projet géré par l'agent et la fonction de valeur de l'agent, en utilisant l'équation Hamilton-Jacobi-Bellman qui lui est associée et un théorème de vérification.

Le chapitre 2 fournit un exemple illustratif et un mode opératoire (*modus operandi*) général pour résoudre des problèmes de type Principal/(plusieurs) Agent(s). En particulier, nous mettons en évidence les

principales questions techniques et les subtilités permettant de résoudre ces problèmes, voir la section 2.2.2. Ensuite, nous nous concentrons sur une foule d'agents ayant des interactions de type "champ moyen" supervisée par un Principal, voir le chapitre 3. Faisant écho à l'*Analyse B.* précédente, nous appliquons cette étude dans la section 3.2 aux contrats optimaux de réponse à la demande énergétique d'une multitude de consommateurs avec des termes de champ moyen. Enfin, nous étudions un modèle avec un agent engagé par plusieurs Principaux, voir le chapitre 4.

Dans le chapitre 5, nous donnons quelques extensions possibles de tous les résultats de cette première partie en cours d'étude.

La deuxième partie de ce rapport rappelle l'*Analyse C.* ci-dessus. Elle se concentre sur la réglementation financière *via* des politiques de type commissions d'exécutions *make-take fees* et plus généralement sur la microstructure des marchés financiers. Les modèles de carnets d'ordres sont des systèmes transparents qui permettent de faire correspondre les ordres des clients sur la base d'une priorité prix/temps, selon le principe du premier entré, premier sorti de la file d'attente. Comme l'a déclaré la *Securities and Exchanges Commission* (traduit de l'anglais en français) "les systèmes d'échange hautement automatisés ont contribué à établir un modèle commercial pour un nouveau type de fournisseur professionnel de liquidité qui se distingue du spécialiste d'échange plus traditionnel et du teneur de marché de gré à gré". L'inquiétude autour de ces nouveaux fournisseurs de liquidité, les traders à haute fréquence, est double. Premièrement, leur présence implique que les fournisseurs plus lents n'ont plus accès au carnet d'ordres, ou seulement en cas de situations défavorables pour les traders à haute fréquence ne souhaitant pas soutenir la liquidité. Cela conduit à une deuxième critique classique à l'encontre des teneurs de marché à haute fréquence : ils ont tendance à quitter les marchés en période de tensions. Il faut donc élaborer une politique d'incitation pour maintenir les principaux fournisseurs de liquidité sur le marché en chargeant asymétriquement fournisseurs de liquidité et investisseurs. D'un point de vue académique, les études sur ce type de politiques et leurs impacts sur la qualité des marchés ont été essentiellement empiriques. Une théorie intéressante, suggérée dans [Angel et al., 2011] et développée dans [Colliard and Foucault, 2012], est que ces politiques de *make-take fees* n'ont en fait aucun impact sur les coûts de négociation, montrant que l'écart entre les cours acheteur et vendeur ne devrait pas dépendre de ces politiques de commissions d'exécution, en cohérence avec les résultats empiriques. Néanmoins, les résultats ci-dessus sont obtenus au moyen de modèles en temps discret simples mais maniabiles, que l'on pourrait vouloir réexaminer pour se rapprocher de la réalité du marché. Dans le chapitre 6, nous étudions le cadre continu en considérant un teneur de marché incité de façon optimale par une place boursière à améliorer la liquidité et nous étendons l'étude à un oligopole de places boursières symétriques.

Au-delà de cela, il a été avancé que certains défauts mécaniques sont inhérents aux modèles de carnets d'ordres, en particulier dans le contexte du trading à haute fréquence. Le débat a commencé dans la littérature universitaire notamment avec [Budish et al., 2015] et également [Farmer and Skouras, 2012]. Dans ces articles, les auteurs expliquent que les carnets d'ordres conduisent à un arbitrage mécanique évident et génèrent une concurrence en termes de vitesse plutôt qu'en termes de prix entre les teneurs de marché haute fréquence, au détriment des investisseurs. Ils montrent de manière convaincante que les enchères séquentielles pourraient être un moyen de remédier à ces défauts. Bien entendu, la discussion dans ce rapport ne porte pas sur les possibilités d'arbitrage à haute fréquence, mais plutôt sur le fait que les enchères pourraient être bénéfiques pour le processus de formation des prix en permettant aux investisseurs de négocier directement entre eux. Dans [Budish et al., 2015], les auteurs fournissent un ordre de grandeur de la limite inférieure pour la durée d'une enchère conduisant à l'élimination des défauts à haute fréquence des carnets d'ordres (environ 100 millisecondes). Cependant, la littérature mentionnée plus haut suggère que des périodes d'enchères plus longues pourraient convenir, sans donner d'ordre de grandeur. C'est pourquoi, citant [Budish et al., 2015], (traduit de l'anglais en français) "développer une

meilleure compréhension des coûts liés à l’allongement de la période entre les enchères est un sujet important”. C’est exactement ce que nous faisons dans le chapitre 7. Nous fournissons une analyse quantitative et opérationnelle de la durée optimale des enchères sur un marché financier. Nous travaillons avec un critère basé sur la qualité du processus de formation des prix comme dans [Madhavan, 1992], mais dans le contexte des marchés modernes à haute fréquence comme dans [Budish et al., 2015]. Nous étendons ensuite l’étude à de nouveaux modèles hybrides dans la section 7.2, appelé enchères *ad hoc*. Dans ce modèle, les investisseurs traitent en permanence avec un teneur de marché et ont la possibilité de déclencher une enchère si nécessaire. Nous comparons l’efficacité de ces trois modèles (carnet d’ordre, enchères séquentielles et enchères *ad hoc*). Tout ce chapitre vise à promouvoir l’efficacité et la stabilité des mécanismes de marché. Nous prouvons en particulier qu’il n’y a pas de méta-modèle et que, selon l’actif, certains modèles de marché sont plus adaptés que d’autres.

Dans le chapitre 8 nous donnons quelques extensions possibles de tous les résultats de cette seconde partie actuellement étudiées.

La troisième partie fait écho à l’Analyse A. et à la tragédie des biens communs. En 2015, l’Assemblée générale des Nations unies a fixé 17 objectifs mondiaux “qui doivent servir de modèles pour assurer à tous un avenir meilleur et plus durable”, devant être atteints d’ici 2030. L’objectif de développement durable 14¹, appelé "La vie sous l’eau" est l’un d’entre eux et aspire à “conserver et utiliser durablement les océans, les mers et les ressources marines pour un développement durable”. Cet objectif est divisé en dix cibles pour conserver et utiliser durablement les océans. La cible 14.2 met l’accent sur la nécessité de protéger et de restaurer les écosystèmes. La cible 14.4 vise à réglementer l’exploitation et à mettre fin à la surpêche d’ici 2020. La cible 14.7 vise à accroître les avantages économiques découlant de l’utilisation durable des ressources marines. Au-delà de cet exemple, l’ONUAA (Organisation des Nations unies pour l’alimentation et l’agriculture) propose, par le biais du "Services des écosystèmes et de la biodiversité", de réguler les services de contrôle biologique (contrôle de la population de prédateurs et de parasites dans les écosystèmes par exemple).

La gestion des activités de récolte est fondamentale pour la durabilité des ressources et donc pour la survie et le développement de la population humaine en croissance permanente. Cependant, les ressources naturelles sont limitées car elles sont soit non renouvelables (par exemple, le pétrole, le gaz et le charbon), ce qui limite la quantité disponible, soit renouvelables (*e.g.* nourriture, eau et forêts). Dans ce dernier cas, la ressource naturelle est limitée par sa capacité à se renouveler. En particulier, une exploitation excessive d’une ressource pourrait conduire à son extinction et affecter les économies qui en dépendent avec, par exemple, des augmentations de prix et une plus grande incertitude quant à l’avenir. Le gestionnaire de ressources naturelles est donc confronté à un dilemme : l’exploitation intensive de la ressource augmente ses revenus mais, en tenant compte des externalités négatives potentielles induites par son activité, cette surexploitation a un impact sur son avenir et sur la capacité à exploiter la ressource. Une grande partie de la littérature universitaire a traité de ces questions. Par exemple, [Reed, 1979, Clark and Kirkwood, 1986, May et al., 1978, Regnier and De Lara, 2015, Tromer and Doyen, 2019] ont étudié l’exploitation d’une ressource naturelle avec incertitude quant à son évolution dans des modèles discrets de plusieurs périodes. Nous renvoyons également au livre [De Lara and Doyen, 2008] pour des modèles stochastiques et déterministes explorant de tels problèmes. Il a néanmoins été souligné dans [Clark, 1973] que dans certains cas il est optimal pour le gestionnaire d’une ressource naturelle de récolter jusqu’à l’extinction de celle-ci. Cette stratégie de récolte optimale entraîne donc des coûts pour le système environnemental global. Par conséquent, la gestion et le suivi de l’exploitation des ressources naturelles consistent en un équilibre entre une récolte optimale pour le gestionnaire de ressources naturelles et les implications écologiques pour les organismes publics et l’écologie. Par exemple, l’article

¹Voir <https://www.globalgoals.org/14-life-below-water>

[Béné et al., 2001] étudie un problème de gestion d'une ressource naturelle, modélisée par l'équation de la logistique déterministe, par un exploitant de celle-ci sous contrainte écologique imposé par le gouvernement pour assurer la viabilité de cette ressource. Dans le chapitre 9, nous examinons une politique de suivi d'une ressource naturelle inspirée des "remèdes" proposés par Elinor Ostrom pour résoudre la tragédie des biens communs. Dans ce chapitre, une ressource naturelle renouvelable est gérée par un agent (appelé *gestionnaire de la ressource naturelle*). Un institut public ou un gouvernement (appelé *le régulateur*) incite le gestionnaire de la ressource à assurer la durabilité de celle-ci. Mathématiquement parlant, ce chapitre est particulièrement intéressant puisqu'il fournit une méthode adaptable à d'autres modèles pour trouver des contrôles ε -optimaux à des équations d'HJB non-linéaires, en solutionnant les problèmes techniques mis en avant dans la section 2.2.2.

Le chapitre 9 se concentre sur la durabilité d'une ressource et sa gestion par les activités humaines sans tenir compte des interactions entre les espèces ou au sein d'une espèce. *L'homme est par nature un animal social*, disait Aristote de sorte que les interactions sont au cœur des activités humaines. Cependant, les interactions humaines entraînent également des inconvénients biologiques. La contagion est définie par la communication de maladies d'une personne ou d'un organisme à un autre par un contact étroit. Dans une phase pandémique les individus n'ont pas accès à suffisamment d'informations, par rapport par exemple aux institutions publiques, pour leur permettre d'évaluer s'ils agissent réellement d'une manière qui est réellement bénéfique pour la population dans son ensemble (bien qu'il semble raisonnable de supposer que certains individus, ayant peur de tomber malades, vont naturellement diminuer leur taux d'interaction). C'est l'une des raisons pour lesquelles les gouvernements peuvent en outre introduire des mesures de quarantaine et de confinement afin de contribuer à ralentir une pandémie, alors qu'aucun remède ni vaccin n'ont été mis au point et que les établissements médicaux risquent d'être débordés par un afflux important de patients. Comme il fallait s'y attendre, une part importante de la littérature récente sur la pandémie COVID-19 a adopté ce point de vue, et ces mesures ainsi que leurs impacts médicaux, sociétaux et économiques sont actuellement étudiés. Un exemple éloquent est le rapport de l'Imperial College London [Ferguson et al., 2020] qui évalue l'impact des interventions non pharmaceutiques visant à réduire le taux de contact au sein d'une population pour la pandémie de COVID-19. Dans le chapitre 10, nous étudions l'impact d'une politique de confinement sur une population en phase de pandémie. Nous considérons un système compartimental stochastique d'EDS de type SIR pour modéliser la population et la propagation de la maladie. Nous considérons un gouvernement incitant une population à réduire son taux d'interaction pour ralentir la pandémie. En plus de cela, nous modélisons également la détection et l'isolement des personnes infectées dans le modèle SIR considéré, de sorte que le gouvernement doit trouver une politique de test optimale pour réduire la propagation de la maladie.

Dans le chapitre 11, nous donnons quelques extensions possibles de tous ces résultats de cette troisième partie actuellement étudiées.

2 English version

Let us begin to introduce three analyses.

Analysis A. "What is the utility to *me* of adding one more animal to my herd in a meadow shared with other herdsman?" asks him/herself the herdsman. "On the one hand, adding an animal in the common meadow is beneficial for me by receiving all the proceeds of the animal. Let say, I win +1. On the other hand, adding one more animal can lead to overgrazing effects but this negative utility is shared by all my herdsman colleagues. Let say that I will loose a fraction of 1." The rational herdsman concludes that it would be better for his own income to add more

and more animals in the meadow. Without any communication, we expect that all the herdsmen would do the same thing until exhaustion of the meadow. This rational strategy would thus lead to ecological and economical negative impacts. This example has been emphasized by Garrett Hardin in [Hardin, 1968] and is the seminal example of *the tragedy of commons*. In 1990, Elinor Ostrom has proposed to monitor a common resource by incentivizing a system of resource managers to act against their own interests for the sustainability of the resource and so for the well-fare of the system. See [Ostrom, 1990] for more details.

Analysis B. Energy consumers win utility by consuming electricity, gas, oil or other renewable or non-renewable resource. However, the energy producer faces limited production flexibility and would like to reduce (in average and in variance) the deviation of a consumer's energy consumption from a baseline consumption. He enhances the responsiveness consumers with incentive prices. This required to design optimally a mechanism for setting energy prices and to develop more deeply smart grids. In [Aïd et al., 2018], the authors have proposed an energy pricing indexed on the baseline consumption of consumers, maintaining reasonable average and volatility of consumption.

Analysis C. Consider a financial market with investors (market takers) and liquidity providers (market makers). Bid/ask prices of risky assets are fixed by the latter. Due to the fragmentation of financial markets, exchanges are nowadays in competition. The traditional international exchanges are challenged by alternative trading venues. Consequently, they have to find innovative ways to attract liquidity on their platforms. On the one hand, market makers earn by increasing the bid/ask spread. On the other hand, investors tend to leave a market with a high spread. One solution to attract liquidity is to use a maker-taker fees system, *i.e.* a rule enabling exchanges to charge in an asymmetric way liquidity provision and liquidity consumption, see for instance [Foucault, 1999]. The most classical setting, used by many exchanges (such as Nasdaq, Euronext, BATS Chi-X...), is of course to subsidize the former with incentivizing policies while taxing the latter. In practice, this means associating a fee rebate to executed limit orders and applying a transaction cost for market orders.

These three analyses have in common the theory of incentives: monitoring policy for the herdsmen in *Analysis A*, pricing policy for the energy consumers in *Analysis B* and regulation of financial activities for market participants in *Analysis C*. These analyses aims at responding to the following question

How incentivize optimally multiple agents for higher interests than their owns?

Technically speaking, the so-called Principal-Agent problem can be modelled by the following: initially a Principal designs optimally a contract (compensation or tax) for an agent (or several agents) to manage a (potentially risky) project.

The first part of this report is devoted to Principal-Agent problems and stochastic differential games. Contract theory started in the 70's, see for instance [Mirrlees, 1976], and is an illustration of Principal-Agent problems. More exactly, the classical framework considered is the following: a Principal (*she*) aims at proposing to an Agent (*he*) a contract. The Agent can accept or reject the contract proposed by the Principal and under acceptance conditions, he will provide a work to manage the wealth of the Principal. The situation where the Principal designs both the compensation given to the Agent and manages perfectly his effort coincides with a risk-sharing model named the *first-best case*. This situation was deeply investigated in several works by using the maximum principle and systems of coupled Forward Backward SDE for instance. We refer to the book [Cvitanic and Zhang, 2012] for more details on it. However, the Principal is often imperfectly informed about the actions of the Agent, by observing only the outcomes of the Agent's efforts and not directly the efforts. The Principal thus designs a contract

which maximizes her own utility by considering this asymmetry of information, given that the utility of the Agent is held to a given level (his reservation utility). From a game theory point of view, this class of problems can be identified with a Stackelberg game between the Principal (the leader) and the Agent (the follower). In this case, the Principal anticipates the best-reaction effort of the Agent and then takes it into account to maximize her own utility. Moral hazard in contracting theory, *i.e.* the Principal has no access on the work of her Agent, has been developed during the 80's and was investigated in a particular continuous time framework by Holmström and Milgrom in [Holmström and Milgrom, 1987]. More recently, the noticeable work [Sannikov, 2008] investigates a stopping time problem in contract theory by emphasizing the fundamental impact of the value function of the Agent's problem to solve the problem of the Principal. This was then mathematically formalized in [Cvitanić et al., 2019] by proposing a handleable procedure to solve a large panel of Principal–Agents problems under moral hazard. The authors have showed that the Stackelberg equilibrium between the Agent and the Principal is reduced to a bi-level optimization problem: first, given a fixed compensation proposed by the Principal, the Agent aims at maximizing his utility by finding the best reaction effort associated with the proposed compensation. Then, the problem of the Principal can be reduced to solve a stochastic control problem with state variables of the problem the output and the value function of the Agent, by using the HJB equations associated with it and verification theorems.

Chapter 2 provides an illustrative example and a general *modus operandi* to solve a Principal/(many) Agent(s) problems. In particular, we highlight the main technical issues and subtleties to solve these problems, see Section 2.2.2. Then, we focus on a crowd of Agents with mean–field type interactions monitored by on Principal, see Chapter 3. Echoing *Analysis B.* above, we apply this study in Section 3.2 to demand-response optimal contracts for energy consumers with mean–field terms. Finally we study a model with an Agent hired by several Principals, see Chapter 4.

In Chapter 5 we gives some possible extensions of these results currently investigated.

The second part of this report echoes *Analysis C.* and focuses on financial regulation *via* make–take fees policy and market microstructure. Limit Order Book (LOB) is a transparent system that matches customer orders on a 'price/time priority' basis, as a first-in-first-out (FIFO) priority execution. As stated by the Securities and Exchanges commission “Highly automated exchange systems and liquidity rebates have helped establish a business model for a new type of professional liquidity provider that is distinct from the more traditional exchange specialist and over-the-counter market maker.” The concern with high frequency traders becoming the new liquidity providers is two-fold. First, their presence implies that slower traders no longer have access to the limit order book, or only in unfavorable situations when high frequency traders do not wish to support liquidity. This leads to the second classical criticism against high frequency market makers: they tend to leave the market in time of stress. This requires to develop incentives policy to keep the main liquidity providers on the market. From an academic viewpoint, studies of make–take fees structures and their impact on the welfare of the markets have been mostly empirical, or carried out in rather stylized models. An interesting theory, suggested in [Angel et al., 2011] and developed in [Colliard and Foucault, 2012] is that make–take fees have actually no impact on trading costs in the sense that the *cum fee* bid-ask spread should not depend on the make–take fees policy. This result is consistent with the empirical findings. Nevertheless, the results above are obtained in tractable but rather simple discrete-time models that one may want to revisit to be closer to market reality. In Chapter 6 we study the continuous framework by considering one market-maker optimally incentivized by one exchange to improve the liquidity and we extend it to an oligopoly of symmetrical exchanges.

Beyond this, it has been argued that some mechanical flaws are inherent to CLOBs, particularly in the context of high frequency trading. The debate started in the academic literature notably with the very

influential paper [Budish et al., 2015], see also [Farmer and Skouras, 2012]. In this work, the authors explain that CLOBs lead to obvious mechanical arbitrage and generate a competition in speed rather than in price between high frequency market makers, to the detriment of final investors. They convincingly show that sequential auctions could be a way to remedy these flaws. Of course the discussion in this report is not about high frequency arbitrage opportunities, but rather on the fact that auctions could be beneficial for the price formation process by enabling investors to trade directly between each others, avoiding to pay spread costs to market makers. In [Budish et al., 2015], the authors provide the order of magnitude of a lower bound for auction period leading to elimination of the high frequency flaws of CLOBs (about 100 milliseconds). However, the mentioned earlier literature suggests that longer auction times could be suitable, but usually without giving figures. This is why, quoting [Budish et al., 2015], “developing a richer understanding of the costs of lengthening the time between auctions is an important topic”. This is exactly what we do in Chapter 7. We provide a sound and operational quantitative analysis of the optimal auction duration on a financial market. We work with a criterion based on quality of the price formation process as in [Madhavan, 1992], but in the context of modern high frequency markets as in [Budish et al., 2015]. We extend then the study to a new hybrid model in Section 7.2, named *ad hoc* auctions. In this model investors deal continuously with a market maker and have the opportunity to trigger an auction if necessary. We compare the efficiency of these three models (CLOB, sequential auctions and *ad hoc* auctions). All this chapter promotes efficiency and stability in market mechanisms. We prove in particular that “one size does not fit all” and depending on the asset some market designs are more suitable than others.

In Chapter 8 we give some possible extensions of these results currently investigated.

The third part echoes *Analysis A.* and tragedy of commons. In 2015, the United Nation General Assembly set 17 global goals “to be a blueprint to achieve a better and more sustainable future for all” intended to be achieved by 2030. The Sustainable Development Goal 14², called “Life below water” is one of them and yearns to “Conserve and sustainably use the oceans, seas and marine resources for sustainable development” divided into ten targets to create action to conserve and sustainably use the oceans. Target 14.2 emphasizes the need for protecting and restoring ecosystems. Target 14.4 aims at regulating harvesting and end overfishing by 2020. Target 14.7 aims at increasing the economic benefits from sustainable use of marine resources. Beyond this example, the FAO (Food and Agriculture Organization) of the UN through the Ecosystem Services & Biodiversity (ESB) service proposes regulating services for biological control (control the population of predators and parasites in ecosystems).

Managing harvesting activities is fundamental for the sustainability of resources and so for the survival and development of the growing human population. However, natural resources are limited since they are either non renewable (*e.g.* minerals, oil, gas and coal) so that the available quantity is limited, or renewable (*e.g.* food, water and forests) and in this case the natural resource is limited by its ability to renew itself. In particular, an excessive exploitation of such resources could lead to their extinctions and so could affect the depending economies with, for instance, high increases of prices and higher uncertainty on the future. Thus, the natural resource manager faces with a dilemma: harvesting intensively the resource increases her incomes, but by taking into account the potential externalities induced by an overexploitation of the resource, it impacts her future ability to harvest the resource. A large part of academic literature has dealt with such issues. For example, [Reed, 1979, Clark and Kirkwood, 1986, May et al., 1978, Regnier and De Lara, 2015, Tromer and Doyen, 2019] have studied the exploitation of a natural resource without or with uncertainty on its evolution in multi-period models. We also refer to the book [De Lara and Doyen, 2008] for stochastic and deterministic models exploring this kind of problems. It has been emphasized in [Clark, 1973] that in

²See <https://www.globalgoals.org/14-life-below-water>

some cases it is optimal for natural resource manager to harvest until the extinction of the resource. This optimal harvesting strategy thus leads to costs for the global welfare related to the environment degradation. Therefore, the management and the monitoring of the exploitation of natural resources are a balance between optimal harvest for the natural resource manager and ecological implications for public organizations. For example, in the article [Béné et al., 2001] the authors model a natural resource as a solution to the deterministic logistic equation and they study a natural resource management problems of a resource manager under an ecological constraint imposed by a government to ensure the sustainability of the resource. In Chapter 9 we investigate a monitoring policy inspired by the “cures” proposed by Elinor Ostrom to fix tragedy of commons. In this chapter, a renewable natural resource is managed by an agent (called *natural resource manager*). A public institute or a government (named *the regulator*) incentivizes the natural resource manager to ensure the sustainability of the resource. Mathematically speaking, this chapter is particularly interesting since it provides a tractable method to find ε -optimal controls to non-linear HJB equations, fixing technical issues put forward in Section 2.2.2.

Chapter 9 focuses on sustainability of a resource and its management by human activities without taking into account inter/extra-species interactions. *Man is a by nature a social animal*, said Aristotle so that interactions are at the heart of human activities. However, interactions lead also to biological inconveniences. Contagion is defined by the communication of diseases from one person or organism to another by close contact. Under a pandemic phase and though it seems reasonable to assume that some individuals, being afraid of getting sick, will naturally decrease their interaction rates, individuals have not access to enough information, compared for instance to public institutions, for them to assess whether they are really acting in a way which is truly beneficial to the population as a whole. This is one of the reasons why quarantine and lockdown measures can be in addition introduced by governments, in order to help slow down a pandemic, when no cure nor vaccine have been developed, and there is a risk for medical facilities to be overwhelmed by a large influx of patients. As should be expected, a significant part of the recent literature on the COVID-19 pandemic has adopted this point of view, and such measures as well as their medical, societal, and economical impacts are currently investigated. A telling example is the report of the Imperial College London [Ferguson et al., 2020] which assesses the impact of non-pharmaceutical interventions to reduce the contact rate within a population for the COVID-19 pandemic. In Chapter 10 we investigate the impact of a lockdown policy on a population under a pandemic phase. We consider a stochastic SIR compartmental system of SDEs to model the population and the spread of the disease. We consider a government incentivizing a population to reduce its interaction rate to slow the pandemic. In addition to that, we also model the detection and the isolation of infected people in the SIR model considered so that the government has to find an optimal testing policy to reduce the spread of the disease.

In Chapter 11 we gives some possible extensions of these results currently investigated.

Notations

Let \mathbb{N}^* be the set of positive integers and \mathbb{R} be the set of reals. Let m and n be in \mathbb{N}^* . We denote by $\mathcal{M}_{m,n}(\mathbb{R})$ the set of matrices with m rows and n columns, and simplify the notations when $m = n$, by using $\mathcal{M}_n(\mathbb{R}) := \mathcal{M}_{n,n}(\mathbb{R})$. We denote by $\text{Id}_n \in \mathcal{M}_n(\mathbb{R})$ the identity matrix of order n . For any $M \in \mathcal{M}_{m,n}(\mathbb{R})$, we define $M^\top \in \mathcal{M}_{n,m}(\mathbb{R})$ as the usual transpose of the matrix M . We will always identify \mathbb{R}^n with $\mathcal{M}_{m,1}(\mathbb{R})$. For any matrix $M \in \mathcal{M}_n(\mathbb{R})$ and for any $1 \leq i \leq n$ we denote by $M^{\cdot i}$ its i th column and similarly we denote by M^i its i th row. Besides, for any $x \in \mathbb{R}^n$, we denote its coordinates by x^1, \dots, x^n . We denote by $\|\cdot\|$ the Euclidian norm on \mathbb{R}^n , when there is no ambiguity on the dimension and we will simply write $|\cdot|$ for $n = 1$. The associated inner product between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is denoted by $x \cdot y$. We also denote by $\mathbf{1}_n$ the n -dimensional vector $(1, \dots, 1)^\top$. Similarly, for any $x \in \mathbb{R}^n$, we define for any $i = 1, \dots, n$, $x^{-i} \in \mathbb{R}^{n-1}$ as the vector x without its i th component, that is to say $x^{-i} := (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)^\top$. For any $(a, \tilde{a}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and any $i = 1, \dots, n$, we define the following n -dimensional vector $a \otimes_i \tilde{a} := (\tilde{a}^1, \dots, \tilde{a}^{i-1}, a, \tilde{a}^i, \dots, \tilde{a}^{n-1})$.

Let $T > 0$ and let (Ω, \mathcal{F}, E) be a probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. For any finite dimensional normed space $(E, N^\circ \cdot_E)$ and for any $p \geq 1$ we define the following space of processes

$$\begin{aligned} \mathbb{S}^p(E) &:= \left\{ Y, \mathbb{F}\text{-adapted, } E\text{-valued and càdlàg, such that } \|Y\|_{\mathbb{S}^p(E)}^p := \mathbb{E} \left[\sup_{t \in [0, T]} \|Y_t\|_E^p \right] < +\infty \right\}, \\ \mathbb{H}^p(E) &:= \left\{ Z, \mathbb{F}\text{-predictable and } E\text{-valued, such that } \|Z\|_{\mathbb{H}^p}^p := \mathbb{E} \left[\left(\int_0^T \|Z_t\|_E^2 dt \right)^{p/2} \right] < +\infty \right\}. \end{aligned}$$

For any measurable space $(\mathcal{S}, \mathcal{F}_\mathcal{S})$, we will denote by $\mathcal{P}(\mathcal{S})$ the set of probability measures on \mathcal{S} and

$$\mathfrak{P}(\mathcal{S}) := \{q : [0, T] \longrightarrow \mathcal{P}(\mathcal{S}), \text{ measurable}\}.$$

For any E -valued and \mathcal{F} -measurable random variable F on Ω , we denote by $\mathcal{L}(F) := \mathbb{P} \circ F^{-1} \in \mathcal{P}(\Omega)$ the law of F under \mathbb{P} . For any $p > 0$, we set

$$L^p(\Omega, \mathcal{F}, E) := \{F : \Omega \longrightarrow E, \mathcal{F}\text{-measurable, s.t. } \mathbb{E}[\|F\|_E^p] < +\infty\}.$$

Without any ambiguity on the probability set, \mathfrak{C}^n denotes the set of \mathbb{R}^n -valued and \mathcal{F}_T -measurable random variables and for $n = 1$ we simplify the notation by writing \mathfrak{C} instead of \mathfrak{C}^1 .

L -differentiability. We refer to the original ideas of [Lions, 2006], the lecture notes [Cardaliaguet, 2010] to define a derivative on the space of measures. See [Carmona and Delarue, 2018, Section 5.2.1.] for more details. Let n be a positive integer, $(\Omega, \mathcal{F}_T, \mathbb{P})$ a probability space and let $u : \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathbb{R}$. We define $\tilde{u} : L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \longrightarrow \mathbb{R}$ by $\tilde{u}(\eta) := u(\mathcal{L}(\eta))$ for any $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$. We then say that u is differentiable on $\mathcal{P}(\mathbb{R}^n)$ if \tilde{u} is Fréchet differentiable on $L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ and we denote by $[D\tilde{u}](\eta)$ its Fréchet derivative in the direction of η , which can be identified as a linear operator from $L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ into \mathbb{R} . According to Riesz's Theorem, for any $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ there exists $D\tilde{u}(\eta) \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ defined for any $Y \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ by $[D\tilde{u}](\eta)(Y) = \mathbb{E}[D\tilde{u}(\eta) \cdot Y]$. We then introduce the notation $\partial_\rho u(\mathcal{L}(\eta)) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as follows, $D\tilde{u}(\eta) =: \partial_\rho u(\mathcal{L}(\eta))(\eta)$, and call the latter the derivative of u at $\rho = \mathcal{L}(\eta)$. We now say that u is partially twice differentiable on $\mathcal{P}(\mathbb{R}^n)$ if u is differentiable in the above sense, and such that the mapping $(\rho, x) \in \mathcal{P}(\mathbb{R}^n) \times \mathbb{R}^n \longmapsto \partial_\rho u(\rho)(x)$ is continuous at any point (ρ, x) , with x being in the support of ρ , and if for any $\rho \in \mathcal{P}(\mathbb{R}^n)$, the map $x \in \mathbb{R}^n \longmapsto \partial_\rho u(\rho)(x)$ is differentiable. We denote the gradient of $\partial_\rho u(\rho)(x)$ by $\partial_x \partial_\rho u(\rho)(x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Part I

Principal(s)-Agent(s) problems and stochastic differential games

An illustrative example

In this chapter we highlight the general procedure to solve a Principal–Agent problem. We focus on an illustrative example dealing with one Principal and N –Agents with $N \in \mathbb{N}^*$ inspired by [Élie and Possamai, 2019] extended to separably linear value functions for the agents and depending on empirical distributions of the project and the agents’ efforts. Most of the result of this chapter are extracted from [Élie et al., 2018, Section 5.3].

2.1 Formulation of the Principal/ N –Agents problem

We consider one Principal hiring N –Agents to manage an N –dimensional¹ risky project denoted by X . We voluntarily omit a lot of technical assumptions since the goal of this section is to set the usual general procedure to solve the problem. We refer to [Élie et al., 2018, Section 5.3] for the technical details.

2.1.1 The model

2.1.1.1 The output process

Our first task is to set the dynamic of the considered risky project denoted by X^N . We start by defining rigorously the different parameters leading to this dynamic. We assume that the project is driven by an initial condition $X_0 \in \mathbb{R}^N$ together with a fixed noise denoted by W^N . More rigorously, we set $(\Omega^N, \mathcal{F}_T, \mathbb{P})$ a probability space carrying an N –dimensional Brownian motion denoted by W^N . For any $t \in [0, T]$, we define \mathcal{F}_t as the \mathbb{P} –augmentation of the σ –field $\sigma((W_s^N)_{s \in [0, t]})$, as well as the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$.

We now introduce² the volatility function $\Sigma : [0, T] \times \mathbb{R}^N \mapsto \mathcal{M}_N(\mathbb{R})$ such that

ASSUMPTION (Σ). *The map Σ is bounded, measurable, and for every $(t, x) \in [0, T] \times \mathbb{R}^N$, $\Sigma(t, x)$ is invertible with inverse bounded by some positive constant M . Moreover, Σ is such that the following stochastic differential equation admits a unique strong solution*

$$X_t^N = X_0 + \int_0^t \Sigma(s, X_s^N) dW_s^N, \quad t \in [0, T], \quad \mathbb{P} - a.s. \quad (2.1.1)$$

We consider a risky project X^N given by the unique solution of (2.1.1).

2.1.1.2 The impact of the system of agents

Our second task is to define how the risky project is modified under the efforts of the system of Agents. We now introduce a drift function, which drives the value of the output. We assume that this function depends on the output process X^N , its empirical distribution, the efforts made by the system of N –agents and their empirical distribution. This dependencies incorporates the following effects.

¹For the sake of simplicity we assume that the dimension of X is N but it could be different without changing significantly the computations.

²In this chapter we only focus on the Markovian case, *i.e.* all the objects introduced depends on X^N at time t only through X_t^N . We could extend it to the path-dependent case by considering the classical Wiener space for Ω^N . We refer to [Élie et al., 2018] for more details.

- (i) The average value of the firm can have a positive or negative impact on its future evolution: when things are going well, they have a tendency to keep doing so, and conversely. We can also consider that large disparities in the distribution of X^N can also negatively impact the future evolution of the outputs. This would then naturally make this drift depending on the variance of X^N .
- (ii) Similarly, if all the other Agents in the firm are working on average very hard, this could have a ripple effect on the whole firm. Hence the drift also depends on the law of the control itself.

In this example, we will assume that the dynamic of the i th component of the project will depend on the project itself only through its i th component and that the i th component is managed only by the i th agent but it can depend on the empirical law of the whole project of the general effort. Technically speaking, the drift b^N is a map from $[0, T] \times \mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^N) \times A^N$ into \mathbb{R}^N such that

$$b^{N,i}(t, x, \mu, q, a) := b(t, x^i, \mu, q, a^i), \quad (t, x, \mu, q, a) \in [0, T] \times \mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^N) \times A^N,$$

for some function $b : [0, T] \times \mathbb{R}^N \times \mathcal{P}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{R}^N) \times A \rightarrow \mathbb{R}$, where A is a closed subset of \mathbb{R} . For any $x \in \mathbb{R}^N$, let $\mu^N(x) \in \mathcal{P}(\mathbb{R}^N)$ be the empirical distribution of x defined by

$$\mu^N(x)(dz^1, \dots, dz^N) := \frac{1}{N} \sum_{i=1}^N \delta_{x^i}(dz^i),$$

where for any $y \in \mathbb{R}$, $\delta_y(dz) \in \mathcal{P}(\mathbb{R})$ denotes the Dirac mass at y . Similarly, for any $a \in \mathbb{R}^N$, we let $q^N(a) \in \mathcal{P}(\mathbb{R}^N)$ the empirical distribution of a defined by

$$q^N(a)(dz^1, \dots, dz^N) := \frac{1}{N} \sum_{i=1}^N \delta_{a^i}(dz^i).$$

We define the set of admissible controls \mathcal{A}^N by all A^N -valued and \mathbb{F} -adapted process α such that

$$\left(\mathcal{E} \left(\int_0^t (\Sigma^{-1}(s, X_s^N))^\top b^N(s, X_s^N, \mu^N(X_s^N), q^N(\alpha_s), \alpha_s) \cdot dW_s^N \right) \right)_{t \in [0, T]} \text{ is an integrable } (\mathbb{P}, \mathbb{F})\text{-martingale.} \quad (2.1.2)$$

This condition is crucial to modify the law of the project under an action α of the system of agents as we will see above. We also define the set of admissible controls for the i th Agent, given some other controls played by the $N - 1$ other Agents, $\alpha^{-i} := (\alpha^1, \dots, \alpha^{i-1}, \alpha^{i+1}, \dots, \alpha^N)$

$$\mathcal{A}_i(\alpha^{-i}) := \{ \alpha, A\text{-valued and } \mathbb{F}\text{-adapted s.t. } \alpha \otimes_i \alpha^{-i} \in \mathcal{A}^N \}.$$

For any $\alpha \in \mathcal{A}^N$ we define a probability measure \mathbb{P}_N^α which is equivalent to \mathbb{P} and from which we derive thanks to Girsanov Theorem the existence of an N -dimensional \mathbb{P}_N^α -Brownian motion $W^{N, \alpha}$ so that

$$X_t^N = X_0 + \int_0^t b^N(s, X_s^N, \mu^N(X_s^N), q^N(\alpha_s), \alpha_s) ds + \int_0^t \Sigma(s, X_s^N) dW_s^{N, \alpha}, \quad t \in [0, T], \quad \mathbb{P} - a.s. \quad (2.1.3)$$

Remark 2.1. *To define rigorously the weak formulation of the problem, we have to write the martingale problem associated with the characteristic triplet (b^N, C^N) , where $C_t^N := \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top(t, X_t^N))$ in the sense of [Jacod and Shiryaev, 1998, II.2]. Hence, the problem can be written as a martingale problem, see [Hausman, 1986, El Karoui et al., 1987, El Karoui and Tan, 2013] (among others) so that it is equivalent to control a probability rule \mathbb{P}^α . From [El Karoui, 1981, El Karoui and Jeanblanc, 1988], it is then equivalent to directly control α thanks to Girsanov formulation when only the drift of X^N is controlled. Nevertheless, for controlled volatility in the output dynamic, we will have to come back to the canonical definition of the weak formulation, see Section [Cvitanić et al., 2019] or [Élie et al., 2019].*

2.1.2 Moral hazard and bi-level optimization

Motivated by the next chapter, we consider a homogeneous model, in the sense that the N Agents are supposed to be similar. Under moral hazard, the Principal does not observe the effort of the Agents. She has to anticipate their best reaction function given a contract. The contracting problem is thus reduced to solve a bi-level type optimization. We first study the game plays by the system of agents then we focus on the optimal the contract design.

2.1.2.1 The Agents' problems: discount factor, cost, utilities and Nash equilibrium

We assume that the payoffs of the agents are impacted by a discount factor k , that is an \mathbb{R} -valued map depending on the time, the output process, the laws of the output and of the efforts. Agents are also impacted by a cost function c that is an \mathbb{R}^+ -valued map depending on the time, the output process, the laws of the output and of the efforts and the efforts themselves.

The discount factor models a possible impatience of the Agents, who would value more having utility now than later. Letting it depend on the law of X^N and the law of the control played by the Agents is again for generality and possible ripple effects. Agents may for instance become more impatient if the firm is currently doing extremely well. The cost function models an exhaustion effect in the management of the risky project. This cost faced by the Agents can be influenced by its law, as well as the law of the control played by the other Agents. It helps to model the fact that Agents may find it hard to work when everyone else is working (the classical free rider problem), or if the situation of the firm is on average extremely good. As for the growth conditions assumed, they are basically here to ensure that the Hamiltonian of the Agent, which will involve both b and c , has at least one maximiser in a , thanks to nice coercivity properties.

The Agents are hired at time 0 by the Principal on a "take it or leave it" basis. The Principal proposes a contract to the Agents, which³ consists in a final payment $\xi \in \mathfrak{C}^N$, which is an \mathbb{R}^N -valued and \mathcal{F}_T -measurable random variable, such that, for $i = 1, \dots, N$, ξ^i represents the amount of money received by Agent i at time T . We assume that the Agents have a similar utility function U_A with respect to this final compensation assumed to be non-decreasing and concave.

Hence, the problem faced by the i th Agent, given a contract $\xi \in \mathfrak{C}^N$ as well as a given A^{N-1} -valued control α^{-i} chosen by the $N - 1$ other Agents, is the following

$$V_0^{A,i}(\xi^i, \alpha^{-i}) := \sup_{\alpha \in \mathcal{A}_i(\alpha^{-i})} v_0^{A,i}(\alpha, \xi^i, \alpha^{-i}),$$

where for any $\alpha \in \mathcal{A}_i(\alpha^{-i})$

$$v_0^{A,i}(\alpha, \xi^i, \alpha^{-i}) := \mathbb{E}^{\mathbb{P}_N^{\alpha \otimes_i \alpha^{-i}}} \left[K_{0,T}^{i, X_s^N, \alpha_s, \alpha_s^{-i}} U_A(\xi^i) - \int_0^T K_{0,s}^{i, X_s^N, \alpha_s, \alpha_s^{-i}} c(s, X_s^N, \mu^N(X_s^N), q^N((\alpha \otimes_i \alpha^{-i})_s), \alpha_s) ds \right]$$

by setting for any $i = 1, \dots, N$, and any $(t, s, x, a, a^{-i}) \in [0, T] \times [t, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{N-1}$

$$K_{t,s}^{i, x, a, a^{-i}} := \exp \left(- \int_t^s k(s, x^i, \mu^N(x), q^N(a \otimes_i a^{-i})) ds \right).$$

We are looking for an equilibrium between the Agents, in the sense of a Nash equilibrium.

³Again for the sake of simplicity, we consider only terminal payments but we can extend it to a stream of instantaneous payments as in [Sannikov, 2008, Possamaï and Touzi, 2020, Hajjej et al., 2017]. See [Élie et al., 2018] for more details.

Definition 2.1 (Nash equilibrium). *Given a contract $\xi \in \mathfrak{C}^N$, a Nash equilibrium for the N Agents is an admissible control process $\alpha^*(\xi) \in \mathcal{A}^N$ satisfying for any $1 \leq i \leq N$*

$$V_0^{A,i}(\xi^i, \alpha^{*, -i}(\xi)) = v_0^{A,i}(\alpha^{*, i}(\xi), \xi^i, \alpha^{*, -i}(\xi)).$$

For a given $\xi \in \mathfrak{C}^N$, we let $\text{NA}(\xi)$ denote the (possibly empty) set of Nash equilibria for the Agents' problem. Of course, in general, when a Nash equilibrium exists between the Agents, there is absolutely no reason for it to be unique. However, since now Agents are indifferent between any Nash equilibrium in $\text{NA}(\xi)$, we follow the usual assumption in the Principal–Agent literature, as stated below, and assume that they let the Principal chooses the equilibrium that suits her the most.

2.1.2.2 The Principal's problem: admissible contracts and bi-level optimization

It is quite reasonable to consider only contracts such that the Principal is able to compute the reaction of the Agents, that is to say the ones for which there is at least one Nash equilibrium. The contracts also have to take into account the fact that all the Agents have a reservation utility R_0 and will never accept a contract which does not guarantee them at least that amount. In the following we will denote by R_0^N the N –dimensional vector of coordinate R_0 . Finally, and again up to integrability assumptions, the set of *admissible* contracts is denoted by Ξ_N , and consists in $\xi \in \mathfrak{C}^N$, such that

- (a) ξ satisfied specific integrability conditions;
- (b) $\text{NA}(\xi) \neq \emptyset$;
- (c) for any $1 \leq i \leq N$ and $\alpha^* \in \text{NA}(\xi)$ we have $V_0^{A,i}(\xi^i, \alpha^{-i,*}) \geq R_0$.

The choice of integrability conditions in (a) will be specified later. Most of the cases, exponential moments with well-chosen orders. Condition (b) is quite natural meaning that the Principal has no interest to proposes a contract without any solution for the agents. Condition (c) means that the reservation utility of the agents is ensured. The Principal's problem is then

$$U_0^{P,N} := \sup_{\xi \in \Xi_N} \sup_{\alpha \in \text{NA}(\xi)} \mathbb{E}^{\mathbb{P}_N^\alpha} \left[-\exp \left(-\frac{R_P}{N} (X_T - \xi) \cdot \mathbf{1}_N \right) \right]. \quad (2.1.4)$$

The risk aversion of the Principal is R_P/N , which means that the more Agents there are, the less risk averse the Principal becomes. We interpret this as a sort of diversification effect. The optimization problem (2.1.4) reduces to solve a bi-level optimization problem:

1. First we need to focus on the constraint $\alpha \in \text{NA}(\xi)$ when ξ is fixed. We thus have to characterize Nash equilibria in the sense of Definition 4.1.
2. Second, we design optimal compensations ξ^* given the characterization of Nash equilibria found in the first step of the bi-level optimization.

2.2 *Modus operandi* and main technical issues

In this section, we aim at giving the general procedure to solve a Principal/ N –agents problem without going deeply in the technical details.

2.2.1 Characterization of Nash equilibria and verification theorem.

We first focus on the first step of the bi-level optimization to characterize the Nash equilibrium in the sense of Definition 4.1 for the game played by the agents. It is well-known that in a Markovian case, *i.e.* without memory dependancies in the different parameters or payoff considered, the existence of a Nash equilibrium is reduced to find solutions to systems of Hamilton-Jacobi-Bellman equations see for instance [Dockner et al., 2000, Buckdahn et al., 2011]. In the path dependent setting, the existence of Nash equilibria corresponds to the existence of a BSDE system, see for instance [Hamadène et al., 1997, El Karoui and Hamadène, 2003] or more recently [Élie and Possamaï, 2019] adapted to contract theory. Since our setting is extremely similar to the latter, the results of [Élie and Possamaï, 2019] tell us that we should consider the following N -dimensional BSDE, given a contract $\xi \in \mathfrak{C}^N$

$$Y_t^N = U_A(\xi) + \int_t^T g_N^*(s, X_s^N, Y_s^N, Z_s) ds - \int_t^T Z_s^\top \Sigma(s, X_s^N) dW_s, \quad (2.2.1)$$

where $g_N^* : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \longrightarrow \mathbb{R}^N$ is defined by

$$\begin{aligned} & (g_N^*)^i(s, x, y, z) \\ &= \sup_{a \in A} \left\{ \sum_{j=1}^N b(s, x^i, \mu^N(x), q^N(a \otimes_i (a^*)^{-i}(s, x, y, z)), a) Z^{j,i} - c(s, x, \mu^N(x), q^N(a \otimes_i (a^*)^{-i}(s, x, y, z)), a) \right. \\ & \quad \left. - k(s, x^i, \mu^N(x), q^N(a \otimes_i (a^*)^{-i}(s, x, y, z))) y \right\}, \end{aligned}$$

with $a^* : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \longrightarrow \mathbb{R}^N$ is defined for any $1 \leq i \leq N$ by

$$\begin{aligned} (a^*)^i(s, x, y, z) \in \operatorname{argmax}_{a \in A} \left\{ \sum_{j=1}^N b(s, x^i, \mu^N(x), q^N(a \otimes_i (a^*)^{-i}(s, x, y, z)), a) Z^{j,i} \right. \\ \left. - c(s, x, \mu^N(x), q^N(a \otimes_i (a^*)^{-i}(s, x, y, z)), a) \right. \\ \left. - k(s, x^i, \mu^N(x), q^N(a \otimes_i (a^*)^{-i}(s, x, y, z))) y \right\}. \quad (2.2.2) \end{aligned}$$

As in [Élie and Possamaï, 2019], the definition of a^* might happen to be truly circular. We thus consider the following assumption

ASSUMPTION 2.1. *For any $(s, x, y, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R})$, there exists at least one N -dimensional vector $a^*(s, x, y, z)$ satisfying (2.2.2).*

We are now able to provide an *informal* result, based on martingale representation theorem, see for instance [Élie and Possamaï, 2019, Theorem 4.1] or [Élie et al., 2018, Theorem 3.1]:

Proposition 2.1 (Nash equilibrium characterization). *Let $\xi \in \mathfrak{C}^N$. The following results are equivalent*

- (i) ξ satisfies Condition (a) and Condition (b), *i.e.* there exists a Nash equilibrium $\alpha^*(\xi) \in \text{NA}(\xi)$;
- (ii) there exists a solution (Y^N, Z) to BSDE (2.2.1), where Z is a process in a space predictable and \mathbb{R}^N -valued processes with specific integrability properties denoted by $\tilde{\mathcal{Z}}$.

Hence, $Y_0^{N,i} = V_0^{A,i}(\xi^i, \alpha^{*,i}(\xi))$ and $\alpha^*(\xi)$ can be identified with $a^*(\cdot, X, Y^N, Z)$ for any $1 \leq i \leq N$.

We voluntarily remain rather vague in this statement, because it has been proved in [Élie and Possamaï, 2019] (considering exponential utilities) and this result is a small extension of their theorem. Moreover, we provide a rigorous result in the mean-field case (see Chapter 3 or [Élie et al., 2018, Theorem 3.1]) which can be easily adapted to this proposition.

Note that the existence of a solution to a system of BSDEs which may be with a quadratic growth with respect to the Z component may be a huge issue. In [Tevzadze, 2008] a wellposedness result is given in the case of a bounded and sufficiently small terminal condition. Even if in very particular cases, existence of solutions have been proved, see for instance [Cheredito and Nam, 2015, Kramkov and Pulido, 2016, Hu and Tang, 2015, Luo and Tangpi, 2019] and as explained in [Frei and Dos Reis, 2011, Frei, 2011] the existence of global solutions could fail even in simple cases. More recently, [Xing and Žitković, 2018, Kupper et al., 2019] obtained quite general existence and uniqueness results, but in a Markovian frameworks and [Song, 2017] obtained same results but for bounded terminal conditions. All these results cannot be applied in our model if one aims at dealing with the most general possible class of terminal payment ξ . The most close article is [Harter and Richou, 2019] dealing with non-Markovian quadratic growth BSDE but with quite strong assumptions involving with Malliavin calculus. We however circumvent these technical issues by reducing the equivalence in Proposition 2.1 to the existence of a system of coupled SDE.

Notting that $\xi = U_A^{-1}(U_A(\xi))$ with $U_A(\xi) = Y_T$ motivate us to introduce the following state variables

$$(\text{SDE}) \begin{cases} X_t^N = X_0 + \int_0^t b^N(s, X_s^N, \mu^N(X_s^N), q^N(a^*(s, X_s, Y_s^{Y_0^N, Z}, Z_s)), a^*(s, X_s, Y_s^{Y_0^N, Z}, Z_s)) ds \\ \quad + \int_0^t \Sigma(s, X_s^N) dW_s^{N, a^*(\cdot, X, Y^{Y_0^N, Z}, Z)}, \\ Y_t^{Y_0^N, Z} = Y_0^N + \int_0^t Z_s^\top b^N(s, X_s^N, \mu^N(X_s^N), q^N(a^*(s, X_s, Y_s^{Y_0^N, Z}, Z_s)), a^*(s, X_s, Y_s^{Y_0^N, Z}, Z_s)) ds \\ \quad - \int_0^t g_N^*(s, X_s, Y_s^{Y_0^N, Z}, Z_s) ds + \int_0^t Z_s^\top \Sigma(s, X_s^N) dW_s^{N, a^*(\cdot, X, Y^{Y_0^N, Z}, Z)}, \end{cases}$$

controlled by $Y_0 := (Y_0^i)_{1 \leq i \leq N}$ which has to be bigger than R_0^N component-wise and by some process Z having some integrability conditions. As a consequence of Proposition 2.1 we have the corollary below.

Corollary 2.1. *The following characterization holds*

$$\Xi_N \equiv \{U_A^{-1}(Y_T^{Y_0^N, Z}), \text{ such that } Y_0^N \geq R_0^N \text{ and } Z \in \mathcal{Z}\},$$

where \mathcal{Z} is a subspace of $\tilde{\mathcal{Z}}$ so that (SDE) has a solution $(X, Y^{Y_0^N, Z})$ where $U_A^{-1}(Y_T^{Y_0^N, Z})$ satisfies Condition (a). Moreover, $a^*(\cdot, X, Y^{Y_0^N, Z}, Z) \in \text{NA}(\xi)$.

Remark 2.2. *We are able in very particular examples to directly deduce from Condition (a) (as soon as ξ is enough integrable) that Condition (b) holds in the definition of Ξ_N and that any ξ can be represented by $\xi = U_A^{-1}(Y_T^{Y_0^N, Z})$ for some $Z \in \tilde{\mathcal{Z}}$, by solving martingale problems. See for instance [El Euch et al., 2018, Theorem 3.1] for an exemple with $N = 1$ or [Baldacci et al., 2019b, Theorem 3.7].*

As an intermediary conclusion, BSDE theory is not necessary to solve Principal-agents problems (when only the drift of X^N is controlled). If ξ is enough integrable such that there exists a solution to (2.2.1) then we directly have the existence of a Nash equilibrium. Otherwise, the existence of a Nash equilibrium is reduced to enough integrability properties of the Z process appearing in (SDE). The characterization given by Corollary 2.1 makes the problem of the Principal (2.1.4) easier. Indeed, remember that the Principal's problem was to optimize a non-Markovian terminal payoff with respect to ξ . Thus (2.1.4) becomes

$$U_0^{P, N} = \sup_{Y_0^N \geq R_0^N} U_0^{P, N}(Y_0^N),$$

with

$$U_0^{P,N}(Y_0^N) = \sup_{Z \in \mathcal{Z}} \sup_{a^*(\cdot, X, Y_0^N, Z, Z)} \mathbb{E}^{\mathbb{P}_N^{a^*(\cdot, X, Y_0^N, Z, Z)}} \left[-\exp \left(-\frac{R_P}{N} \left(X_T - U_A^{-1}(Y_T^{Y_0^N, Z}) \right) \cdot \mathbf{1}_N \right) \right]. \quad (2.2.3)$$

Remark 2.3. *If $k = 0$, note that neither g_N^* and so nor a^* depend on $Y^{Y_0^N, Z}$. Therefore, $U_0^{P,N} = U_0^{P,N}(R_0^N)$ so that the constraint given by the reservation utility is saturated.*

This new form of the optimization problem of the Principal is now classical in stochastic control and allow use to solve it by using verification arguments. Under uniqueness⁴ of the optimizer a^* , we can write the associated HJB equation as follows

$$\begin{cases} -\partial_t v(t, x, y) - H^N(t, x, y, \partial_x v, \partial_y v, \partial_{xx} v, \partial_{yy} v, \partial_{xy} v) = 0, & (t, x, y) \in [0, T) \times \mathbb{R}^N \times \mathbb{R}^N, \\ v(T, x, y) = -\exp \left(-\frac{R_P}{N} (x - U_A^{-1}(y)) \cdot \mathbf{1}_N \right), & (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \end{cases} \quad (2.2.4)$$

where $H^N : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \times \mathcal{M}_N(\mathbb{R}) \rightarrow \mathbb{R}$ is defined for any $(t, x, y, p_x, p_y, \gamma_x, \gamma_y, \gamma_{xy}) \in [0, T] \times (\mathbb{R}^N)^2 \times (\mathbb{R}^N)^2 \times (\mathcal{M}_N(\mathbb{R}))^3$

$$\begin{aligned} & H^N(t, x, y, p_x, p_y, \gamma_x, \gamma_y, \gamma_{xy}) \\ &= \sup_{z \in \mathcal{M}_N(\mathbb{R})} \left\{ b^N(t, x, \mu^N(x), q^N(a^*(t, x, y, z)), a^*(t, x, y, z)) \cdot p_x + \frac{1}{2} \text{Tr} \left[\Sigma \Sigma^\top(t, x) \gamma_x \right] \right. \\ & \quad + (z^\top b^N(t, x, \mu^N(x), q^N(a^*(t, x, y, z)), a^*(t, x, y, z)) - g_N^*(t, x, y, z)) \cdot p_y \\ & \quad \left. + \frac{1}{2} \text{Tr} \left[z^\top \Sigma \Sigma^\top(t, x) z \gamma_y \right] + \text{Tr} \left[\Sigma \Sigma^\top(t, x) z \gamma_{xy} \right] \right\}. \end{aligned}$$

We have finally the following verification argument (see [Élie and Possamai, 2019, Proposition 4.1]).

Theorem 2.1 (Verification Theorem). *Assume that there exists a smooth function $v : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$ continuously differentiable with respect to time and twice differentiable with respect to its space variables solving the PDE (2.2.4), and such that the supremum in the definition of H^N is always attained for at least one $z^*(t, x, y, p_x, p_y, \gamma_x, \gamma_y, \gamma_{xy})$, for any $(t, x, y, p_x, p_y, \gamma_x, \gamma_y, \gamma_{xy}) \in [0, T] \times (\mathbb{R}^N)^2 \times (\mathbb{R}^N)^2 \times (\mathcal{M}_N(\mathbb{R}))^3$. Then, if*

$$z^*(t, X_t^*, Y_t^*, \partial_x v(t, X_t^*, Y_t^*), \partial_y v(t, X_t^*, Y_t^*), \partial_{xx} v(t, X_t^*, Y_t^*), \partial_{yy} v(t, X_t^*, Y_t^*), \partial_{xy} v(t, X_t^*, Y_t^*)) \in \mathcal{Z},$$

where (X^*, Y^*) solves the system (SDE), then

$$U_0^P = \sup_{Y_0^N \in [R_0, \infty)^N} v(0, \psi, Y_0^N).$$

From Remark 2.3 we have the following corollary

Corollary 2.2. *Under the assumptions of the previous theorem and if $k = 0$, we have $U_0^P = v(0, \psi, R_0^N)$ and an optimal contract is given by*

$$\xi^* := U_A^{(-1)} \left(R_0^N - \int_0^T g_N^*(s, X_s^*, Y_s^*, z_s^*) ds + \int_0^T (z_s^*)^\top dX_s^N \right),$$

where the inverse of U_A has to be taken component-wise.

⁴If not, we just have to add a supremum in the Hamiltonian, over all the possible maximisers a^* . Furthermore, in the examples solved in Section 2.3, there will always be uniqueness.

2.2.2 The admissibility of contracts: a major issue

The equivalence in Proposition 2.1 or Corollary 2.1 requires to restrict a little bit the set \mathfrak{C}^N to a set Ξ_N of admissible contracts by adding integrability conditions, specifically Condition (a), on ξ as for instance in [Élie and Possamaï, 2019, Section 4.1.3]. Thus, these integrability conditions are directly translated into the control Z appearing in (SDE) via the space \mathcal{Z} . Note also that z^* is determined as an optimizer of G depending on the solution of (2.2.4).

The procedure uses standard tools of stochastic analysis and control but is very dependent on the final PDE (2.2.4) and more particularly on the gradient of such PDE. Hence, without solving this PDE and without any information on the solution and its gradient in term of regularity or growth we cannot be sure that z^* is admissible in the sense that $\xi^* := U_A^{-1}(Y_T^{Y_0^N, z^*})$ is an admissible contract itself. In addition to that, remember that the optimal control has also to be admissible and so such that (2.1.2) is satisfied. This also asks for regularity of the solution of (2.2.4).

In a nutshell. We would like now to sum up the general procedure to solve the bi-level optimization.

- First, we have to identify Condition (a) on the terminal compensation ξ together with the space $\tilde{\mathcal{Z}}$ required for having Proposition 2.1 or equivalently Corollary 2.1.
- Then, we have to check at the end that the optimal process $Z_t^* := z^*(t, X_t^*, Y_t^*, \partial_x v(t, X_t^*, Y_t^*))$ is enough integrable, *i.e.* in \mathcal{Z} , ensuring that $\xi^* := U_A^{-1}(Y_T^{Y_0^N, Z^*}) \in \Xi_N$, that is it satisfies Condition (a) and (SDE) controlled by z^* admits⁵ a solution.
- Moreover, we have also to check that the corresponding α^* is indeed admissible.

Since all the technical issues are reduced to the integrability of the optimizer z^* and so to the regularity of solutions to (2.2.4), case by case approach seems to be the only way to solve Principal/Agent(s) problem.

2.3 Application and preliminary to Chapter 3

Let $A = \mathbb{R}$. We consider a drift b^N defined for any $(t, x, a) \in [0, T] \times \mathbb{R}^N \times A^N$ by

$$b^N(t, x, \mu^N(x), a) := a + \alpha x + \beta_1 \int_{\mathbb{R}^N} w \mu^N(dw), \text{ with } \alpha, \beta \in \mathbb{R}$$

which can be rewritten $b^N(t, x, \mu^N(x), a) = a + B^N x$, with $B^N := \alpha I_N + \frac{\beta_1}{N} \mathbf{1}_{N,N}$. The volatility of the output is given by $\Sigma := \sigma I_N$, $\sigma > 0$ and the components of the initial condition ψ^N are independent and identically distributed such that $\psi^{N,i} = \psi$ in law for any $i \in \{1, \dots, N\}$. We moreover assume that any Agent is risk neutral, *i.e.* $U_A(x) = x$, with $c(a) := ca^2/2$, $c > 0$ and without any discount factor in the model ($k = 0$). The Principal is also risk neutral, *i.e.* $U_P(x) = x$, $x \in \mathbb{R}$.

In this case, the optimal effort for the Agents should be given by the N -dimensional vector $a^{N,*}(z)$ defined for any $1 \leq i \leq N$ and $z \in \mathcal{M}_N(\mathbb{R})$ by $(a^{N,*}(z))^i = \frac{z^{i,i}}{c}$. We consider the following set of admissible⁶ contracts

⁵Which is the case as soon as neither a^* nor z^* depend on Y for instance.

⁶The exponential condition on ξ is Condition (a). We could weaken this assumption for some $p > 1$ well-chosen but since the optimal z is deterministic at the end, it is without any restriction here. It is natural for $N = 1$ if one aims at using a comparison theorem for quadratic BSDE, see for instance [Briand and Hu, 2006]. This set of contract has been used in [Élie and Possamaï, 2019, Section 4.1.3].

$$\Xi_N := \{\xi \in \mathfrak{C}^N, \mathbb{E}[e^{p|\xi|}] < +\infty, \forall p \geq 1, \text{NA}(\xi) \neq \emptyset, V_0^{A,i}(\xi^i, \alpha^{-i,\star}) \geq R_0, \forall \alpha^\star \in \text{NA}(\xi).\}$$

From Theorem 4.1 in [Élie and Possamaï, 2019] adapted to our setting for any $\xi \in \Xi_N$, there exists some process $Z^i \in \mathbb{H}^p(\mathbb{R}^N)$ for any $p \geq 1$ and any $i \in \{1, \dots, N\}$ such that $\xi = Y_T^{Y_0^N, Z}$. We now define the map $\tilde{g} : \mathcal{M}_N(\mathbb{R}) \rightarrow \mathbb{R}^N$ by $\tilde{g}(z) := \left(\frac{1}{2c} |z^{1,1}|^2, \dots, \frac{1}{2c} |z^{N,N}|^2\right)^\top$. Hence, the Hamiltonian of the HJB equation associated with the problem of the Principal is given by

$$\begin{aligned} & H^N(t, x, y, p_x, p_y, \gamma_x, \gamma_y, \gamma_{xy}) \\ &= \sup_{z \in \mathcal{M}_N(\mathbb{R})} \left\{ a^\star(z) \cdot p_x + \tilde{g}(z) \cdot p_y + \frac{\sigma^2}{2} \text{Tr}[z^\top z \gamma_y] + \sigma^2 \text{Tr}[z \gamma_{xy}] \right\} + B^N x \cdot p_x + \frac{\sigma^2}{2} \text{Tr}[\gamma_x]. \end{aligned}$$

We get an explicit smooth solution to PDE (2.2.4) given by $v(t, x, y) = f(t, x) - \frac{1}{N} y \cdot \mathbf{1}_N$ with

$$f(t, x) = e^{(\alpha + \beta_1)(T-t)} \frac{1}{N} \sum_{i=1}^N x^i + \frac{\gamma^\star \sigma^2}{4(\alpha + \beta_1)} \left(e^{2(\alpha + \beta_1)(T-t)} - 1 \right).$$

Theorem 2.2. *Assume that for any $i = 1, \dots, N$, $(\lambda_0^N)^i = \lambda_0$. We have the following two properties.*

(i) *The optimal effort $a^{N,\star}$ of Agents in the N -players' model is given by*

$$a_t^{N,\star} = \frac{\exp((\alpha + \beta_1)(T-t))}{c} \mathbf{1}_N.$$

(ii) *The optimal contract $\xi^{N,\star}$ proposed by the Principal is*

$$\xi^{N,\star} := R_0^N - \int_0^T \frac{\exp(2(\alpha + \beta_1)(T-t))}{2c} \mathbf{1}_N dt - \int_0^T e^{(\alpha + \beta_1)(T-t)} B_N X_t^N dt + \int_0^T e^{(\alpha + \beta_1)(T-t)} dX_t^N,$$

and belongs to the set Ξ_N of admissible contracts.

Remark 2.4 (On the admissibility of the optimal contract and effort.). *We see that $z^{N,\star}$ is deterministic in this case and consequently $\alpha^{N,\star}$ is deterministic too. Thus, $\alpha^{N,\star}$ indeed satisfies (2.1.2). Moreover, the optimal contract $\xi^{\star,N}$ is given by*

$$\xi^{N,\star} := Y_T^{R_0^N, z^{N,\star}} = R_0^N - \int_0^T \frac{\exp(2(\alpha + \beta_1)(T-t))}{2c} \mathbf{1}_N dt - \int_0^T e^{(\alpha + \beta_1)(T-t)} B_N X_t^N dt + \int_0^T e^{(\alpha + \beta_1)(T-t)} dX_t^N.$$

Since z^\star is deterministic we directly see from Corollary 2.1 that $\xi^{N,\star} \in \Xi_N$ and (SDE) admits a solution with this deterministic control.

One Principal and a continuum of agents: mean field moral hazard

In this chapter, we investigate a moral hazard problem in finite time with lump-sum and continuous payments, involving infinitely many Agents with mean field type interactions, hired by one Principal. We solve completely and explicitly the problem in special cases, going beyond the usual linear-quadratic framework. In passing, we also explain how our approach allows to solve a version of Hölmstrom and Milgom's model where both the Principal and the Agent have mean-variance type criteria. This chapter sums up [Élie et al., 2018]. We provide in particular explicit tractable examples solving the problem, then we turn to the convergence of the N -players' game and finally we apply our framework to electricity demand response management (extended to volatility control), see [Élie et al., 2019].

3.1 Theoretical framework, main results and application

We focus our attention on the situation where the Principal has to hire infinitely many Agents who are supposed to be identical, and who can control the drift of an output process, representing the project that the Agent has to manage on behalf of the Principal. The value of this project is affected through his drift by both its law, and the law of the control, representing both the aggregated impact of the other Agents. This extend directly the previous chapter to the case $N \rightarrow +\infty$.

3.1.1 The model

Let Ω be the space of continuous map from $[0, T]$ into \mathbb{R} . and define the coordinate processes on Ω by $W(\omega) := \omega, \forall \omega \in \Omega$. We denote by \mathbb{P} the Wiener measure defined on $\mathcal{B}(\Omega)$. For any $t \in [0, T]$, we define \mathcal{F}_t as the \mathbb{P} -augmentation of the σ -field $\sigma((W_s)_{s \in [0, t]})$, as well as the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. The filtered probability space we will be interested in is $(\Omega, \mathcal{B}(\Omega), \mathbb{F}, \mathbb{P})$. We first introduce a volatility process $\sigma : [0, T] \times \Omega \rightarrow \mathbb{R} \setminus \{0\}$.

ASSUMPTION (σ). *The map σ is bounded by some positive constant M , \mathbb{F} -optional, and for every $(t, x) \in [0, T] \times \Omega$, $\sigma(t, x)$ is invertible with inverse bounded by some positive constant M . Moreover, σ is such that the following stochastic differential equation admits a unique strong solution*

$$X_t = x_0 + \int_0^t \sigma(s, X) dW_s, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad x_0 \in \mathbb{R}.$$

The idea now is that there is only one (representative) Agent which is interacting with the theoretical distribution of the infinite number of other players. The natural consequence on our notations is that we will enforce $N = 1$ throughout this chapter compared to the previous. He impacts the dynamic of X by adding a drift term $b : \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{R}) \times \mathcal{A} \rightarrow \mathbb{R}$ depending on

- $\mu \in \mathcal{P}(\Omega)$ the "arbitrary distribution of the output managed by the infinitely many other Agents"
- $q : [0, T] \rightarrow \mathcal{P}(\mathbb{R})$ the "arbitrary distribution of infinitely many Agents' actions"
- $\alpha \in \mathcal{A}$, \mathbb{F} -adapted control process with integrability conditions for the representative Agent.

In this case and under suitable integrability assumptions we can define a new probability $\mathbb{P}^{\mu,q,\alpha}$ equivalent to \mathbb{P} and $(\mathbb{P}^{\mu,q,\alpha}, \mathbb{F})$ -Brownian motion $W^{\mu,q,\alpha}$ such that

$$X_t = x_0 + \int_0^t b(s, X, \mu, q_s, \alpha_s) ds + \int_0^t \sigma(s, X) dW_s^{\mu,q,\alpha}, \quad t \in [0, T], \quad \mathbb{P} - a.s.$$

As before, we assume that the payoffs of a representative agent is impacted by a discount factor k and by a cost function c that is an \mathbb{R}^+ -valued map depending on the time, the (paths of) output process, the laws of the output and of the efforts, and the effort itself. The Principal proposes a contract to the representative Agent, which¹ consists in a terminal payment ξ with the same conditions than the previous chapter (taking $N = 1$) in a path dependent setting.

Consequently, for any contract ξ and for any $(\mu, q) \in \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{R})$ the representative agent aims at solving

$$U_0^A(\mu, q, \xi) := \sup_{a \in \mathcal{A}} u_0^A(\mu, q, \xi, a)$$

where

$$u_0^A(\mu, q, \xi, a) := \mathbb{E}^{\mathbb{P}^{\mu,q,\alpha}} \left[K_{0,T}^{X,\mu,q} U_A(\xi) - \int_0^T K_{0,s}^{X,\mu,q} c(s, X, \mu, q_s, \alpha_s) ds \right],$$

with $K_{t,s}^{x,\mu,q} := \exp\left(-\int_t^s k(u, x, \mu, q_u) du\right)$. We now have to focus on the game played by the crowd of agents. Similarly to Definition 4.1 and inspiring by [Carmona and Lacker, 2015], we define the mean-field equilibrium in the following.

Definition 3.1 (Mean-field equilibrium). *A mean-field equilibrium is a triplet $(a^*, \mu^*, q^*) \in \mathcal{A} \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{R})$ such that*

$$(\mathbf{MFG})(\xi) \begin{cases} u_0^A(\mu, q, \xi, a^*) = U_0^A(\mu, q, \xi), \\ \mathbb{P}^{a^*, \mu^*, q^*} \circ (X)^{-1} = \mu^* \\ \mathbb{P}^{a^*, \mu^*, q^*} \circ (a_t^*)^{-1} = q_t^*. \end{cases}$$

Similarly to the definition of Ξ_N for the N -players' model, we denote by Ξ the set of admissible contracts, that is to say the set of $\xi \in \mathcal{C}$ having integrability conditions such that $(\mathbf{MFG})(\xi) \neq \emptyset$ and for any $(\mu^*, q^*, \alpha^*) \in (\mathbf{MFG})(\xi)$ we have $V_0^A(\xi, \mu^*, q^*) \geq R_0$.

Formally, by taking the limit when $N \rightarrow \infty$ in the Principal's problem (2.1.4), we expect the Principal to become risk-neutral. In other words, the mean-field version of (2.1.4) for the Principal is

$$U_0^P := \sup_{\xi \in \Xi} \sup_{(\mu, q, \alpha) \in (\mathbf{MFG})(\xi)} \mathbb{E}^{\mathbb{P}^{\mu,q,\alpha}} [X_T - \xi]. \quad (3.1.1)$$

3.1.2 Solving the bi-level optimization with mean field interactions

The contracting problem being now set in a mean-field framework, we now follow the *modus operandi* introduced in Section 2.2. As in the first chapter and Section 2.2.1, we are able to rewrite this characterization in terms on BSDE with mean-field parameters in the spirit of [Carmona and Delarue, 2013, Buckdahn et al., 2009]

$$\begin{cases} Y_t^*(\xi) = U_A(\xi) + \int_t^T g^*(s, X, Y_s^*(\xi), Z_s^*(\xi), \mu, q_s, \chi_s) ds - \int_t^T Z_s^*(\xi) \sigma_s(X) dW_s, \\ \mathbb{P}^{\mu, q, a^*(\cdot, X, Z^*(\xi), \mu, q)} \circ (X)^{-1} = \mu, \\ \mathbb{P}^{\mu, q, a^*(\cdot, X, Z^*(\xi), \mu, q)} \circ (a^*(s, X, Z_s^*(\xi), \mu, q_s))^{-1} = q_s, \text{ for a.e. } s \in [0, T], \end{cases} \quad (3.1.2)$$

¹Again, for the sake of simplicity, we do not consider instantaneous payments. See [Élie et al., 2018] for such a consideration.

with

$$g(s, x, y, z, \mu, q_s, a) := zb(s, x, \mu, q_s, a) - k(s, x, \mu, q_s)y - c(s, x, \mu, q_s, a), \quad (3.1.3)$$

and

$$g^*(s, x, y, z, \mu, q_s) = \sup_{a \in A} g(s, x, y, z, \mu, q_s, a), \quad a^*(s, x, z, \mu, q_s) \in \arg \max g(s, x, y, z, \mu, q_s, a).$$

This system will provide us the required probabilistic representation of the solution to the mean-field game of the Agent similarly to Proposition 2.1

Proposition 3.1. *Let $\xi \in \Xi$. There is a one-to-one correspondence between a mean-field equilibrium (a^*, μ^*, q^*) , and a solution (Y, Z) to the mean-field BSDE (3.1.2), with Z in an appropriate space with integrability conditions. In this case, the one-to-one correspondence says that a^* can be identified with $a^*(\cdot, X, Z, \mu^*, q^*)$ where (Y, Z, μ^*, q^*) is a solution to the mean-field BSDE (2.2.1).*

Again, this result can be seen quite disappointing since it required solutions to mean-field BSDE. As before, the trick is to rewrite the BSDE in a forward form with control Y_0 and Z . For the sake of simplicity, we introduce the two-dimensional state variable $M^Z := (X, Y^{Y_0, Z})^\top$ controlled by the pair of processes (χ, Z) . This motivates us to introduce for a given maximiser a^* of g^* the following system

$$\text{(MV-SDE)} \quad \begin{cases} M_t^Z = \begin{pmatrix} x_0 \\ Y_0 \end{pmatrix} + \int_0^t C(s, M^Z, \mu, q_s, Z_s) ds + \int_0^t S(s, M^Z, Z_s) \mathbf{1}_2 dW_s^{a^*(M^1, Z, \mu^1, q)}, \\ \mu = \mathbb{P}^{a^*(M^1, Z, \mu^1, q)} \circ (M^Z)^{-1}, \\ q_t = \mathbb{P}^{a^*(M^1, Z, \mu^1, q)} \circ (a^*(t, M^1, Z_t, \mu^1, q_t))^{-1}, \text{ for Lebesgue a.e. } t \in [0, T]. \end{cases}$$

where Y_0 is bigger than R_0 and where Z is a predictable process in a space² \mathcal{Z} , such that

• $C : [0, T] \times \Omega^2 \times \mathcal{P}(\Omega^2) \times \mathfrak{P}(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}^2$ is defined for any $(t, m, \mu, q, z, \chi) \in [0, T] \times \Omega^2 \times \mathcal{P}(\Omega^2) \times \mathfrak{P}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}_+$ by

$$C(t, m, \mu, q, z) := \begin{pmatrix} b(t, m^1, \mu^1, q, a^*(t, m^1, z, \mu^1, q)) \\ k(t, m^1, \mu^1, q)m^2(t) + c(t, m^1, \mu^1, q, a^*(t, m^1, z, \mu^1, q)) \end{pmatrix}.$$

• $S : [0, T] \times \Omega^2 \times \mathbb{R} \rightarrow \mathcal{M}_2(\mathbb{R})$ is defined for any $(t, m, z) \in [0, T] \times \Omega^2 \times \mathbb{R}$ by

$$S(t, m, z) := \begin{pmatrix} \sigma_t(m^1) & 0 \\ \sigma_t(m^1)z & 0 \end{pmatrix},$$

We emphasise that the above does not depend on the choice of the maximiser a^* , in the sense that the drift functions b actually cancel out with the Brownian motion $W^{\mu, q, a^*(\cdot, X, Z, \mu, q)}$. Thus, by setting $G : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $G(m) := m^1 - U_A^{-1}(m^2)$, $m \in \mathbb{R}^2$, the problem of the Principal (3.1.1) becomes

$$U_0^P = \sup_{Y_0 \geq R_0} U_0^P(Y_0),$$

with

$$U_0^P(Y_0) = \sup_{Z \in \mathcal{Z}} \sup_{(\mu, q, \alpha) \in (\mathbf{MFG})} \mathbb{E}^{\mathbb{P}^{\mu, q, \alpha}} \left[U_P \left(G(M_T^Z) \right) \right]. \quad (3.1.4)$$

From now on, we consider the Markovian framework by enforcing the following assumption

²Again, \mathcal{Z} is a set of predictable process ensuring that (MV-SDE) has a solution with $Y_T^{Y_0, Z}$ satisfying the same integrability assumptions than Condition (a) to define Ξ .

ASSUMPTION 3.1 (Markovian Framework). *For any $(t, x, \mu, q, a) \in [0, T] \times \Omega \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}$ we assume that $b(t, x, \mu, q, a) = b(t, x(t), \mu_t, q, a)$, $c(t, x, \mu, q, a) = c(t, x(t), \mu_t, q, a)$, and $k(t, x, \mu, q) = k(t, x(t), \mu_t, q)$.*

Unlike the previous chapter, this new form of the optimization problem of the Principal is now more tractable but nonstandard. Indeed, this coincides with an optimal stochastic control problem with state variables given by the system of McKean-Vlasov SDE (**MV-SDE**).

The approach that we present here to solve (3.1.4) is mainly based on the papers [Bayraktar et al., 2018, Pham and Wei, 2017] or the recent book [Carmona and Delarue, 2018] (see the references therein for earlier results) and consists in using the dynamic programming principle and solving the corresponding Hamilton-Jacobi-Bellman equation in an infinite dimensional space. This requires to define first a notion of L -derivative on this space, see Section Notations. We now follow [Carmona and Delarue, 2018, Chapter 6] or [Pham and Wei, 2018]. We also refer to [Djete et al., 2019] for a general study allowing for the law of the control process to appear in the state dynamics.

We restrict our attention to control processes Z which are actually (partially) Markovian, in the sense that there is a map $\tilde{Z} : [0, T] \times \mathbb{R}^2 \times \mathcal{P}(\mathbb{R}^2)$ such that

$$Z_t = \tilde{Z}(t, M_t^{\tilde{Z}}, \mathcal{L}(M_t^{\tilde{Z}})), \quad t \in [0, T],$$

and such that SDE (**MV-SDE**) admits a solution M^Z . Following [Carmona and Delarue, 2018, Pham and Wei, 2017] or in a very general setting [Djete et al., 2019], we introduce an HJB equation defined on the space of measures:

$$\begin{cases} -\partial_t v(t, \rho) - \sup_{\tilde{z} \in \text{Lip}(\mathbb{R}^2; \mathbb{R})} \tilde{H}(t, \rho, \partial_\rho v, \partial_x \partial_\rho v, \tilde{z}) = 0, & (t, \rho) \in [0, T] \times \mathcal{P}(\mathbb{R}^2), \\ v(T, \rho) = U_P \left(\int_{\mathbb{R}^2} G(x) \rho(dx) \right), & \rho \in \mathcal{P}(\mathbb{R}^2), \end{cases} \quad (3.1.5)$$

where $\text{Lip}(\mathbb{R}^2; \mathbb{R})$ denotes the set of Lipschitz function from \mathbb{R}^2 into \mathbb{R} with

$$\tilde{H}(t, \rho, \partial_\rho v, \partial_x \partial_\rho v, \tilde{z}) := \int_{\mathbb{R}^2} \tilde{\mathbb{L}}^{\tilde{z}} v(t, \rho) \rho(dx)$$

and where for any $\varphi : [0, T] \times \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R}$, which is continuously differentiable in t and twice continuously differentiable in ρ , for any $(t, \rho) \in [0, T] \times \mathcal{P}(\mathbb{R}^2)$, $\tilde{z} \in \text{Lip}(\mathbb{R}^2; \mathbb{R})$, the map $\tilde{\mathbb{L}}^{\tilde{z}} \varphi(t, \rho)$ is defined from \mathbb{R}^2 into \mathbb{R} by

$$\tilde{\mathbb{L}}^{\tilde{z}} \varphi(t, \rho)(x) := \partial_\rho \varphi(t, \rho) \cdot C(t, x, \rho, q_t, \tilde{z}(x)) + \frac{1}{2} \text{Tr} [\partial_x \partial_\rho \varphi(t, \rho)(x) S S^\top(t, x, \tilde{z}(x))], \quad x \in \mathbb{R}^2.$$

Similarly to Theorem 2.1, a verification argument leads to the following result (see [Élie et al., 2018, Theorem 4.2])

Theorem 3.1. *Assume that there exists a smooth solution $v : [0, T] \times \mathcal{P}(\mathbb{R}^2)$ such that $v(t, \cdot)$ is twice continuously differentiable on $\mathcal{P}(\mathbb{R}^2)$ and such that $v(\cdot, \rho)$ is continuously differentiable on $[0, T]$ solving (3.1.5) such that the supremum is attained for some optimizer of \tilde{H} denoted by $\tilde{Z}^*(t, \cdot, \rho) \in \text{Lip}(\mathbb{R}^2; \mathbb{R})$ in \mathcal{Z} such that (**MV-SDE**) admits a solution $(M^{\tilde{Z}^*}, \mu, q)$. Then $U_0^P = \sup_{Y_0 \geq R_0} v(0, \mu_0)$, and $Z^* := \tilde{Z}^*(\cdot, M^{\tilde{Z}^*}, \mu)$ is optimal in the problem of the Principal (3.1.4).*

3.1.3 Solvable examples

3.1.3.1 Example 1: Quadratic dynamics and power cost

We define for any $(s, x, \mu, q, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{C}) \times \mathfrak{P}(\mathbb{R}) \times \mathbb{R}_+$, and any $\alpha, \beta_1, \beta_2, \gamma \in \mathbb{R}$

$$b(s, x, \mu, q, a) := a + \alpha x + \beta_1 \int_{\mathbb{R}} z d\mu_s(z) + \beta_2 \int_{\mathbb{R}} z dq_s(z) - \gamma V_\mu(s),$$

$$V_\mu(s) := \int_{\mathbb{R}} |z|^2 d\mu_s(z) - \left| \int_{\mathbb{R}} z d\mu_s(z) \right|^2, \text{ and for fixed } n > 1, c(s, x, \mu, q, a) := c \frac{|a|^n}{n}, c > 0.$$

For the sake of simplicity, we assume in this applicative section that $k = 0$ (no discount factor), $U_A(x) = x$, $x \in \mathbb{R}$ (the Agent is risk neutral). To alleviate notations, we omit the dependence with respect to χ in every objects previously defined and we denote $\kappa := \alpha + \beta_1$. We also assume that the Principal is risk neutral. In our particular case, we can solve explicitly HJB equation (3.1.5) and using a verification result we can solve completely the problem of the Principal (3.1.4).

Theorem 3.2. *The optimal contract for the problem of the Principal is*

$$\xi^* := \delta + \beta_1(1 + \beta_2) \int_0^T e^{(\alpha+\beta_1)(T-t)} X_t dt + (1 + \beta_2) \left(X_T - e^{(\alpha+\beta_1)T} X_0 \right)$$

for some constant δ that can be explicitly computed. Besides, the contract ξ^* is a Normal random variable and

$$\xi^* \sim \mathcal{N} \left(R_0 + \frac{(1 + \beta_2)^{\frac{n}{n-1}}}{nc} \left(e^{\frac{n}{n-1}(\alpha+\beta_1)(T)} - 1 \right); \sigma^2(1 + \beta_2)^2 \left(e^{2(\alpha+\beta_1)T} - 1 \right) \right).$$

The associated optimal effort of the Agent is deterministic and given by

$$a_u^* := (1 + \beta_2)^{\frac{1}{n-1}} \left(\frac{e^{(\alpha+\beta_1)(T-u)}}{c} \right)^{\frac{1}{n-1}}, u \in [0, T].$$

We sum up the sensitivities with respect to the parameters in the next table and corresponding economical interpretations are given in [Élie et al., 2018, Section 5.1.3]

	c	α	β_1	β_2	γ
Expectation of ξ^*	\searrow	\nearrow	\nearrow	\nearrow	$=$
Variance of ξ^*	$=$	\nearrow	\nearrow	\nearrow	$=$
Fixed salary part δ	\searrow	\searrow	\searrow	\searrow	\nearrow
Optimal effort of the Agent	\searrow	\nearrow	\nearrow	\nearrow	$=$
Expected gain of the Principal	\searrow	\nearrow	\nearrow	\nearrow	\searrow
Variance of the terminal gain of the Principal	$=$	$=$	\nearrow	\nearrow	$=$

3.1.3.2 Convergence of the N -players' problem

We now come back to Example 1 in Section 3.1.3.1 and we study the link with the corresponding N -players' game investigated in Section 2.3, see Theorem 3.3. Let $\beta_2 = \gamma = 0$.

Theorem 3.3. *Assume that for any $i = 1, \dots, N$, $(\lambda_0^N)^i = \lambda_0$. We have the following two properties.*

(i) *For any $i \in \{1, \dots, N\}$ we have $(a_t^{N,*})^i = a_t^*$, i.e. the optimal effort of the i th Agent in the N players model coincides with the optimal effort of the Agent in the mean-field model.*

(ii) *For any $i \in \{1, \dots, N\}$ we have*

$$\mathbb{P}_N^{a_N,*} \circ ((\xi^{N,*})^i)^{-1} \xrightarrow[N \rightarrow \infty]{\text{weakly}} \mathbb{P}^{a^*} \circ (\xi^*)^{-1}.$$

Remark 3.1. *The convergence of the value functions associated with the N -players' model and the mean field model is direct, since we can compute explicitly all the relevant quantities. In a more general case, one can reasonably expect to get the convergence of the value functions of both the Principal and the Agents, and therefore the convergence of the optimal contracts (seen as the terminal values of the*

continuation utilities of the Agents) of the N -players' model to the mean field model, when N goes to $+\infty$. We refer to [Cardaliaguet et al., 2019, Theorem 2.13]. However, the convergence of the optimal efforts in the Nash equilibria of the N -players' model, to the equilibria in the mean field model, remains a much harder problem, since it would necessarily involve studying the convergence of the derivatives of the value functions. We leave these very interesting questions for future research and we also refer to [Djete, 2020] for recent results on the convergence of Nash equilibria.

3.1.3.3 Example 2: extension to mean-variance type problems

We consider the same model than Example 1 above with $n = 2$ but the payoff of the Principal is penalised by the covariance between the project and the salary given to any Agent. More precisely, the Principal has to solve for some positive constants $\lambda_X, \lambda_\xi, \lambda_{X\xi}$

$$U_0^P(Y_0) = \sup_{(X,Z) \in \mathcal{X} \times \mathcal{Z}(X)} \mathbb{E}^{\mathbb{P}^*} [X_T - \xi] - \lambda_X \text{Var}_{\mathbb{P}^*}(X_T) - \lambda_\xi \text{Var}_{\mathbb{P}^*}(\xi) - \lambda_{X\xi} \text{Cov}_{\mathbb{P}^*}(X_T - \xi), \quad (3.1.6)$$

where $\text{Var}_{\mathbb{P}^*}$ denotes the variance under \mathbb{P}^* . We thus have a slight modification of HJB equation (3.1.5).

In this case, one gets

$$a_t^* = \frac{1 + \beta_2}{c(1 + 2(\lambda_\xi + \lambda_{X\xi})c\sigma^2)} e^{(\alpha + \beta_1)(T-t)} + \frac{2\lambda_{X\xi}\sigma^2}{1 + 2(\lambda_\xi + \lambda_{X\xi})c\sigma^2} e^{\alpha(T-t)}$$

with optimal contract

$$\xi^* = C + \beta_1 \frac{1 + \beta_2}{1 + 2(\lambda_\xi + \lambda_{X\xi})c\sigma^2} \int_0^T e^{\kappa(T-t)} X_t dt + \frac{1 + \beta_2 + 2c\lambda_{X\xi}\sigma^2}{1 + 2(\lambda_\xi + \lambda_{X\xi})c\sigma^2} X_T,$$

where C is an explicit constant. Besides,

$$\xi^* \sim \mathcal{N} \left(R_0 + \frac{c}{2} \int_0^T |a_t^*|^2 dt ; \int_0^T c^2 \sigma^2 |a_t^*|^2 dt \right).$$

We sum up the sensitivities to the parameters $\lambda_X, \lambda_\xi, \lambda_{X\xi}$ of the model in the next tabular.

	λ_X	λ_ξ	$\lambda_{X\xi}$
Optimal effort of the Agent	=	\searrow	\searrow
Expectation of ξ^*	=	\searrow	\searrow
Variance of ξ^*	=	\searrow	\searrow

- The optimal effort of the agent is decreasing with the penalisation of both the variance of ξ and the one of $X_T - \xi$. Hence, the optimal contract provides incentives for Agents to provide less efforts.
- Let $\beta_1 = \beta_2 = 0$, corresponding to the situation where there are no interactions between the project dynamics. If $\lambda_\xi = \lambda_X = 0$, we retrieve a similar solution as the one obtained in the previous study. Indeed, when both Principal and Agents are risk neutral, the optimal solution without penalisation already exhibits no variance, since the Agent keeps all the risk. Hence, it is also optimal for a Principal with mean-variance criterion.
- The penalisation with respect to the dispersion of the project values has absolutely no effect on the optimal effort or the optimal contract. This is due to the fact that the variance of the output is not impacted by a deterministic effort and hence by the optimal one.

3.2 Application to electricity demand response management

We now apply this study to optimal energy demand response management extending the previous study to volatility controlled problems with common noise. We aim at designing demand response contracts between an electricity producer (the Principal) and a continuum of consumers (the crowd of Agents). This is a direct extension [Aïd et al., 2018] to a mean field of agents.

In this problem, the electricity consumer has a consumption composed by a deterministic seasonal component and a deviation, denoted by X and called the baseline, of this deterministic component. He benefits from deviating to his baseline with a utility function denoted by f and he has d different usage of electricity. On the other hand, the electricity producer is penalized by the generation cost of the produced energy, denoted by g , and the cost induced by the variation of production denoted by θ .

An effort ν induces a separable cost $c(\nu) := \frac{1}{2}c_\alpha(\alpha) + \frac{1}{2}c_\beta(\beta)$. We use the same specification of the cost function c for the representative consumer as in [Aïd et al., 2018]:

$$c_\alpha(a) := \sum_{k=1}^d \frac{(a^k)^2}{\rho^k} \quad \text{and} \quad c_\beta(b) := \sum_{k=1}^d \frac{(\sigma^k)^2}{\lambda^k \eta^k} \left((b^k)^{-\eta^k} - 1 \right),$$

for fixed $(\rho, \lambda, \eta) \in (0, +\infty)^d \times (0, +\infty)^d \times (1, +\infty)^d$ and with $a \in A := ([0, \rho^k A_{\max}])_{k \leq 1}$ and $b \in B := [B_{\min}, 1]^d$. The cost of the effort in the drift term of X , denoted by c_α , is a classical quadratic cost function, meaning that no effort for the Agent induces no intrinsic cost, and such that he has no interest to provide negative efforts. The cost associated with the effort in the volatility of X prohibits the Agent from removing the volatility ($b^k > 0$) and is equal to zero if the Agent makes no effort (case $b^k = 1$).

The producer proposes a compensation ξ at a maturity T for the consumer to reduce his nominal level of consumption and his consumption deviation variability for each usage of electricity. In a moral hazard situation, the producer observes X but not the effort $\nu := (\alpha, \beta)$ made to achieve this reduction.

Extended [Aïd et al., 2018] to a continuum of identical and correlated consumers, we have to find a way for the Principal to benefit from this MF of consumers given that the producer knows the law of the consumption of the pool of (identical) consumers. The producer designs a new contract in order to penalise/reward a consumer who makes less/more effort than the rest of the pool given a reservation utility R_0 of the consumer.

Merely speaking, the contract should consists in two parts: a classical part indexed on the deviation consumption of the Agent and its variability similar to [Aïd et al., 2018], an additional part indexed on the law of the deviation consumption of others.

The correlation is modelled by adding a common noise W° referring to [Carmona and Delarue, 2018]. In this case, for a typical small consumer, the deviation of a typical small consumer³ from his baseline consumption is given by

$$X_t = x_0 - \int_0^t \alpha_s \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s) \cdot dW_s + \int_0^t \sigma^\circ dW_s^\circ, \quad \mathbb{P}^\nu - a.s., \quad \text{with } \sigma(b) := (\sigma^1 \sqrt{b^1}, \dots, \sigma^d \sqrt{b^d})^\top,$$

where W is a d -dimensional idiosyncratic noise, under the effort \mathbb{P}^ν (see [Élie et al., 2019] for more technical details) of a consumer and W° is a one dimensional common noise. We consider a CARA utility function U_A for the consumer with risk-aversion parameter $R_A > 0$, so that $U_A(x) = -e^{-R_A x}$. The problem of the representative consumer for ξ fixed is thus to solve

$$V_0^A(\xi, \hat{\mu}) := \mathbb{E}^{\mathbb{P}^\nu} \left[U_A(\xi + \int_0^T f(X_s) ds - \int_0^T c(\nu_s) ds) \right], \quad (3.2.1)$$

³Small means that we consider a representative agent who does not impact the global consumption of the continuum.

where $\hat{\mu}$ is the conditional law of the deviation consumption of other consumers w.r.t the common noise. Following [Carmona et al., 2016], we can provide the following informal definition of a mean-field equilibrium for the pool of consumer, extending Definition 3.1 to volatility controlled problems and by adding a common noise

Definition 3.2. *A mean-field equilibrium is a pair (\mathbb{P}^*, μ^*) such that*

$$V_0^A(\xi, \mu^*) = \mathbb{E}^{\mathbb{P}^*} \left[U_A \left(\xi + \int_0^T f(X_s) ds - \int_0^T c(\nu^{\mathbb{P}^*}) ds \right) \right],$$

μ_t^* is the conditional law of X^* with respect to the common noise, where X^* is the optimal deviation consumption of each Agent. We denote by $\mathcal{M}^*(\xi)$ the set of mean-field equilibria associated with the contract ξ .

Hence, the problem of the producer under moral hazard is

$$J_0^P = \sup_{\xi} \sup_{(\mathbb{P}, \mu) \in \mathcal{M}^*(\xi)} \mathbb{E}^{\mathbb{P}^*} \left[U_P \left(-\xi - \int_0^T g(X_s) ds - \frac{\theta}{2} \langle X \rangle_T \right) \right],$$

where the producer is either risk-averse with a CARA utility function $U_P(x) = -e^{-R_P x}$ with $R_P > 0$ or risk-neutral with $U_P(x) = x$.

Following the previous *modus operandi*, we first focus on (3.2.1) as a mean-field game played by the consumers. Intuitively, we expect that the continuation utility V_t^A of the consumer, given a contract ξ and efforts of other consumers subsumed by the distribution $\hat{\mu}$, may be written as

$$V_t^A = v^A(t, X_{t^\wedge}, \hat{\mu}_t),$$

that is, the process V^A at time t depends on t , on the path history of X and on the conditional law $\hat{\mu}$ of deviation consumption of others denoted by \hat{X} given by

$$\hat{X}_t = \hat{X}_0 - \int_0^t \alpha_s^{\hat{\mathbb{P}}} \cdot \mathbf{1}_d ds + \int_0^t \sigma(\beta_s^{\hat{\mathbb{P}}}) \cdot d\hat{W}_s + \int_0^t \sigma^\circ dW_s^\circ, \quad t \in [0, T].$$

In the Markovian framework, if the value function is smooth enough in the sense of [Carmona and Delarue, 2018, Section 4.3.4], the chain rule with common noise for function of both the state and the measure (see [Carmona and Delarue, 2018, Theorem 4.17]) motivate us to introduce the following set of simple contracts

Definition 3.3. *We denote by Ξ_S the space of simple contract $\xi_T^{\xi_0, \zeta}$ the contract parametrised by predictable processes $\zeta := (Z, Z^\mu, \Gamma)$ in a space \mathcal{V} with integrability conditions, and $\xi_0 \in \mathbb{R}$, such that⁴*

$$\begin{aligned} \xi_t^{\xi_0, \zeta} &= \xi_0 - \int_0^t \mathcal{H}(X_s, \hat{\mu}_s, \zeta_s, \hat{\alpha}_s) ds + \int_0^t Z_s dX_s + \int_0^t \hat{\mathbb{E}}^{\hat{\mu}_s} [Z_s^\mu(\hat{X}_{s^\wedge}) d\hat{X}_s] + \frac{1}{2} \int_0^t (\Gamma_s + R_A Z_s^2) d\langle X \rangle_s \\ &+ \frac{1}{2} R_A \int_0^t \hat{\mathbb{E}}^{\hat{\mu}_s} \check{\mathbb{E}}^{\hat{\mu}_s} [Z_s^\mu(\hat{X}_{s^\wedge}) Z_s^\mu(\check{X}_{s^\wedge}) d\langle \hat{X}, \check{X} \rangle_s] + R_A \int_0^t Z_s \hat{\mathbb{E}}^{\hat{\mu}_s} [Z_s^\mu(\hat{X}_{s^\wedge}) d\langle X, \hat{X} \rangle_s], \end{aligned}$$

where

$$\mathcal{H}(x, \hat{\mu}, \zeta, \hat{\alpha}) := \frac{1}{2} H_d(z) + \frac{1}{2} H_v(\gamma) + H_c(x, \gamma) - \hat{\mathbb{E}}^{\hat{\mu}} [z^\mu \hat{\alpha} \cdot \mathbf{1}_d],$$

with

$$\begin{aligned} H_d(z) &:= - \inf_{a \in A} \{2za \cdot \mathbf{1}_d + c_\alpha(a)\}, \quad H_v(\gamma) := - \inf_{b \in B} \{c_\beta(b) - \gamma \Sigma(b)\}, \quad H_c(x, \gamma) := \frac{1}{2} \gamma (\sigma^\circ)^2 + f(x), \\ H_o(\hat{\mu}, \zeta, \hat{\nu}) &:= - \hat{\mathbb{E}}^{\hat{\mu}_t} [z^\mu(\hat{X}_{t^\wedge}) \hat{\alpha} \cdot \mathbf{1}_d] + (\sigma^\circ)^2 \hat{\mathbb{E}}^{\hat{\mu}_t} [\gamma^\mu(\hat{X}_{t^\wedge})] + \frac{1}{2} \hat{\mathbb{E}}^{\hat{\mu}_t} [\gamma^{\mu,1}(\hat{X}_{t^\wedge}) (\Sigma(\hat{\beta}) + (\sigma^\circ)^2)] \\ &+ \frac{1}{2} \hat{\mathbb{E}}^{\hat{\mu}_t} \check{\mathbb{E}}^{\hat{\mu}_t} [\gamma^{\mu,2}(\hat{X}_{t^\wedge}, \check{X}_{t^\wedge}) (\sigma^\circ)^2]. \end{aligned}$$

⁴ \check{X} denotes a copy of \hat{X} . We refer to [Élie et al., 2019, Section 3] for a rigorous statement.

A contract in Ξ_S contract is mainly composed of two parts: one is an indexation on the process controlled by the consumer, that is to say his deviation consumption, the other one is an indexation on other consumers through the law $\hat{\mu}$. In particular, similarly to [Aïd et al., 2018], the contract has a linear part in the level of consumption deviation X and the corresponding quadratic variation $\langle X \rangle$, with linearity coefficients Z and Γ . This part of the contract is the classic contract for drift and volatility control. The constant part with the function \mathcal{H} is composed by the certainty equivalent of the utility gain of the consumer that can be achieved by an optimal response to the contract and the other terms corresponds to compensations for the infinitesimal payment given to the others and compensation for the covariation induced by the two infinitesimal payments.

Theorem 3.4. *Given a contract $\xi_T^{\xi_0, \zeta} \in \Xi_S$. indexed by the triple of parameters $\zeta := (Z, Z^\mu, \Gamma) \in \mathcal{V}$, there exists a unique mean-field equilibrium in the sense of Definition 3.2 denoted by (\mathbb{P}^*, μ^*) where*

1. the optimal drift effort of the consumer is given by the process $\alpha^* := a^*(Z)$ where

$$a^{k, \star}(z) := \rho^k(z^- \wedge A_{\max}), \quad z \in \mathbb{R}, \quad k = 1, \dots, d;$$

2. the optimal volatility effort of the consumer is given by the process $\beta^* := b^*(\Gamma)$ where

$$b^{k, \star}(\gamma) := 1 \wedge (\lambda^k \gamma^-)^{\frac{-1}{\eta^k + 1}} \vee B_{\min}, \quad \gamma \in \mathbb{R}, \quad k = 1, \dots, d;$$

3. \mathbb{P}^* is the law of X driven by optimal controls

$$dX_t = -\bar{\rho}(Z_t^- \wedge A_{\max})dt + \sigma^*(\Gamma_t) \cdot dW_t + \sigma^\circ dW_t^\circ;$$

4. μ^* is the conditional law of X given \mathbb{F}° .

We in particular see that $\alpha^* \sim Z^-$, $\beta^* \sim \Gamma^-$ meaning that the more Z , resp. Γ , is negative, the more the Agent reduces his consumption deviation in average, resp. deviation volatility. Moreover, Z^μ has no influence so that a consumer optimizes independently of what his neighbours do even if the price depends on the global demand.

Remark 3.2. *Until now, we supposed that the Principal is not allowed to directly index compensations on the common noise. We notice that the contract in Ξ_S can be written in the following way*

$$\begin{aligned} \xi_t &= \xi_0 - \int_0^t \mathcal{H}^\circ(X_s, \zeta_s) ds + \int_0^t Z_s dX_s + \sigma^\circ \int_0^t \hat{\mathbb{E}}^{\hat{\mu}_s} [Z_s^\mu (\hat{X}_{s \wedge \cdot})] dW_s^\circ + \frac{1}{2} \int_0^t (\Gamma_s + R_A Z_s^2) d\langle X \rangle_s \\ &\quad + \frac{1}{2} R_A \int_0^t \hat{\mathbb{E}}^{\hat{\mu}_s} \tilde{\mathbb{E}}^{\tilde{\mu}_s} [Z_s^\mu (\hat{X}_{s \wedge \cdot}) Z_s^\mu (\tilde{X}_{s \wedge \cdot})] d\langle \hat{X}, \tilde{X} \rangle_s + R_A \int_0^t Z_s \hat{\mathbb{E}}^{\hat{\mu}_s} [Z_s^\mu (\hat{X}_{s \wedge \cdot})] d\langle X, \hat{X} \rangle_s, \end{aligned}$$

$$\mathcal{H}^\circ(x, \zeta) = \frac{1}{2} H_d(z) + \frac{1}{2} H_v(\gamma) + H_c(x, \gamma),$$

and thus does not depend on the others' effort $\hat{\alpha}$ anymore. We can even go further in the simplifications noting that, given the common noise, \hat{X} and \tilde{X} are independent. Hence, by setting $\bar{Z}_s^\mu = \hat{\mathbb{E}}^{\hat{\mu}_s} [Z_s^\mu (\hat{X}_{s \wedge \cdot})]$, the actual compensation parameter of the contract in this case is the triple $\bar{\zeta} := (Z, \bar{Z}^\mu, \Gamma) \in \bar{\mathcal{V}}$ and the form of contracts is

$$\xi_0 - \int_0^t \mathcal{H}^\circ(X_s, \bar{\zeta}_s) ds + \int_0^t Z_s dX_s + \sigma^\circ \int_0^t \bar{Z}_s^\mu dW_s^\circ + \frac{1}{2} \int_0^t (\Gamma_s + R_A \bar{Z}_s^2) d\langle X \rangle_s + \frac{1}{2} R_A (\sigma^\circ)^2 \int_0^t \bar{Z}_s^\mu (\bar{Z}_s^\mu + 2Z_s) ds.$$

This shows that indexing on the conditional law is actually a hidden indexing on the common noise.

Following the general approach of [Cvitanic et al., 2019] we expect that there is no loss of generality for the Principal to restrict to contracts in Ξ_S , similarly to the previous chapter. We show here that this general result also extends to cases where volatility can be controlled as well, using 2BSDEs theory. Moreover, any contract $\xi \in \Xi_S$ leads to a unique mean–field equilibrium by Theorem 3.4. For notational simplicity, we define for any $(\xi_0, \zeta) \in \mathbb{R} \times \mathcal{V}$ the following process

$$L_t^{\xi_0, \zeta} := \xi_t^{\xi_0, \zeta} + \int_0^t g(X_s) ds + \frac{\theta}{2} \int_0^t d\langle X \rangle_s, \text{ for } t \in [0, T].$$

Theorem 3.5. *Let \mathcal{F}_T° be the natural filtration of W_T° . We have*

$$V_0^P = \sup_{(\xi_0, \zeta) \in [U_A^{-1}(R_0), +\infty) \times \mathcal{V}} \mathbb{E}^{\mathbb{P}^*} [U^P(-\mathbb{E}^{\mathbb{P}^*}[L_T^{\xi_0, \zeta} | \mathcal{F}_T^\circ])] = \sup_{\zeta \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^*} \left[U^P \left(-\mathbb{E}^{\mathbb{P}^*} \left[L_T^{U_A^{-1}(R_0), \zeta} | \mathcal{F}_T^\circ \right] \right) \right].$$

Similarly to (3.1.4), we have reduced the problem to a stochastic control problem with two state variables: the conditional law of X and conditional law of L . This enables us to use a verification procedure to solve the problem, see [Élie et al., 2019, Section 4.1].

From now, following the application proposed in [Aïd et al., 2018], we will focus on linear energy value discrepancy by assuming that

$$\text{(EVD)} \quad (f - g)(x) = \delta x, \quad \delta \in \mathbb{R}.$$

In this case, comparing the Agents' preferences for their consumption (the function f) with the Principal's preference for production (the function g) can be summarised by studying the sign of δ . More precisely, a positive δ means that an increase in consumption provides more utility to the consumers than it is costly for the producer. Therefore, a reduction of the consumption has a negative effect more important on the consumers' utilities than the positive effect on the producer's utility. Similarly, δ negative means that an increase of consumption induces more cost for the producer than the benefit generated for the consumer. We thus obtain, by recalling the notations in Remark 3.2

Proposition 3.2. *Under (EVD), the optimal payment rate process $\bar{\zeta}^* = (Z^*, \bar{Z}^{\mu, *}, \Gamma^*)$ is a deterministic function of time, independent of σ° and is defined by*

$$Z_t^* = \text{Arg min}_{z \in \mathbb{R}} \bar{h}(t, z), \quad \bar{Z}_t^{\mu, *} = -Z_t^* + \frac{R_P}{R_A + R_P} \delta(T - t) \quad \text{and} \quad \Gamma_t^* = -\max \left\{ \theta + R_A(Z_t^*)^2, \frac{1}{\lambda} \right\}.$$

for all $t \in [0, T]$, with

$$\bar{h}(t, z) := F_0(\theta + R_A z^2) + \bar{\rho}((z^- \wedge A_{\max}) + \delta(T - t))^2, \quad (t, z) \in [0, T] \times \mathbb{R}.$$

The above proposition underlines the fact that the optimal payment rates (Z^*, Γ^*) are the same, in both cases of a risk–averse or risk–neutral Principal. Hence, the efforts of the consumers on their deviation consumptions will be the same, whatever the risk aversion of the Principal. The Principal controls the risk she wants to bear thanks to the control \bar{Z}^μ . Indeed, in the risk–neutral case, the Principal does not care about the risk, hence \bar{Z}^μ is such that the contract does not depend on the common noise. On the other hand, in the risk–averse case, the Agent is remunerated for a part of the common noise: the risk induced by the common noise is shared between the Agent and the Principal.

On the benefit of the mean–field indexation. We study the benefits of indexing the contracts on the distribution of the deviation of other consumers. We thus consider, as a benchmark case, the producer's problem when incentives are limited to payments for efforts of the considered consumer. This corresponds to the case $\zeta^0 := (Z, 0, \Gamma)$ instead of $\zeta := (Z, Z^\mu, \Gamma)$. We assume that $\delta \leq 0$ and we use the same parameters than [Aïd et al., 2018]. Comparing the two models we obtain Figure 3.1 and

we see that the Principal can incentivise the consumers to make more effort on the drift and on the volatility for the entire duration of the contract by indexing on the others' consumptions. Figure 3.2 show that, in most cases, our new contracts lead to a significant decrease in consumption on average and on volatility compared to classical contracts. However, when the risk-aversion of the Principal increases, this gain becomes negative. Nevertheless, it can be stressed that, on the one hand, for the parameters calibrated in [Aïd et al., 2018], there is a significant gain regardless of the correlation with the common noise. On the other hand, even if the Principal has a relatively high risk-aversion, our new contracts allow a reduction on average and on volatility of the consumption if it is strongly impacted by weather conditions. Therefore, our contracts can help to better manage consumption cut-off during peak demand due to climate hazard. Figure 3.2, right side, gives the relative gain of value function for the Principal by indexing on the consumption of others. We see that this gain is more important depending on σ° .

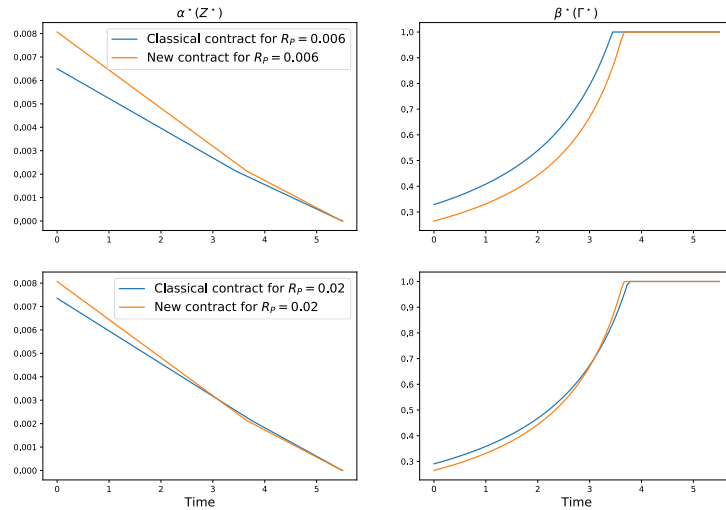


Figure 3.1: Comparison of efforts in the linear EVD case. $R_P = 0.006$ (upper graphs) or $R_P = 0.02$ (lower graphs).

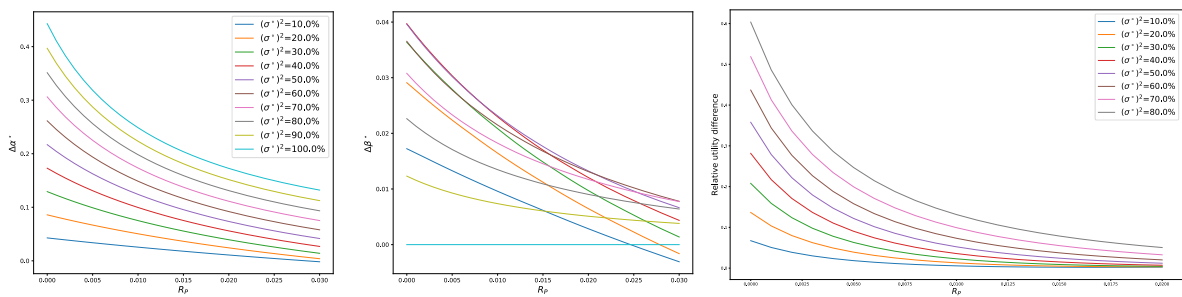


Figure 3.2: Relative gain in mean (left), in volatility (center) of the consumption and relative utility difference with respect to the risk aversion parameter R_P and the correlation with the common noise σ° (center) in the linear EVD case.

One Agent and N –Principals: Common Agency Problems

We consider a problem of contract theory in which several Principals hire a common Agent, extending [Bernheim and Whinston, 1985] to continuous-time models. We show that optimal contracts should satisfy some equilibrium conditions and we reduce the optimisation problem of the Principals to a system of coupled Hamilton-Jacobi-Bellman equations. For the sake of simplicity, we focus in the report on a linear-quadratic example. More generally, we provide conditions ensuring that for risk-neutral Principals, the system of coupled HJB equations admits a solution, see [Mastrolia and Ren, 2018] for more details.

4.1 The model

In this chapter, we focus on a multi-Principals/single Agent problem in the continuous setting. More precisely, we assume in this chapter that several N –Principals aim at hiring one common agent who works simultaneously for all the Principals to manage N risky projects. We work under the same framework than Section 2.1.1.1 above by denoting X instead of X^N to simplify the notations. The vector X is composed by different projects given by its components X^i .

The volatility Σ satisfying Assumption (Σ) in Section 2.1.1.1, taking values in $\mathcal{M}_N(\mathbb{R})$, characterizes the correlation of noises among the different projects, satisfying ASSUMPTION (Σ) in Section 2.1.1.1. A control process¹ is an \mathbb{F} –progressive process α such that

$$X_t = x + \int_0^t b(s, X_s, \alpha_s) ds + \int_0^t \Sigma(s, X_s) dW_s^{\mathbb{P}^\alpha}, \quad \mathbb{P}^\alpha\text{-a.s.},$$

where

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} := \mathcal{E} \left(\int_0^t b(s, X_s, \alpha_s) \cdot \Sigma^{-1}(s, X_s) dW_s^{\mathbb{P}^\alpha} \right)_{t \in [0, T]}, \quad W^{\mathbb{P}^\alpha} \text{ is a } \mathbb{P}^\alpha\text{-Brownian motion},$$

and $b : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a drift function with components denoted by b^i . The process α is regarded as the effort of the Agent, and influences the output X through the drift function b . We assume that the Agent plays an effort $\alpha \in \mathbb{R}^N$ on the panel of projects X but that only α^i impacts the i th project with $1 \leq i \leq N$, so that $b(t, x, a) = (b^i(t, x, a^i))_{1 \leq i \leq N}^\top$.

The Agent derives a utility given the salaries from the N Principals at the terminal time T , and his effort reduces his general payoff through the cost function $c : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}_+$. More precisely, each Principal, *e.g.* Principal i , proposes to the Agent a contract with the salary denoted by ξ^i , a \mathbb{R} -valued \mathcal{F}_T -measurable random variable. The total salary for the Agent is thus $\xi \cdot \mathbf{1}_N$, where $\xi = (\xi^1, \dots, \xi^N)^\top$, with integrability conditions. We consider the exponential utility for the Agent with risk aversion parameter $R_A > 0$, so that given a panel of contracts ξ proposed by the N Principals, the Agent's problem is

$$U_0^A(\xi) := \sup_{\alpha} u_0^A(\xi, \alpha), \quad \text{with } u_0^A(\xi, \alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[-\exp \left(-R_A \left(\xi \cdot \mathbf{1}_N - \int_0^T c(t, X_t, \alpha_t) dt \right) \right) \right]. \quad (4.1.1)$$

¹Adding integrability assumptions, see [Mastrolia and Ren, 2018, Definition 2.2].

Denote by $\mathcal{A}^*(\xi)$ the set of best responses of the Agent, given a vector of salaries ξ . Given the best response, we will study Nash equilibriums among the N -Principals when they try to maximize their own utilities. Let $R_0 \in \mathbb{R}$, be the Agent reservation utility, that is, the N -Principals have to solve their utility maximization problem under the constraint that the set of contracts ξ proposed to the Agent satisfies $U_0^A(\xi) \geq R_0$.

We now introduce the problem of the i th Principal given that the others have proposed ξ^j with $j \neq i$ to the Agent

$$U_0^{P_i}((\xi^j)_{j \neq i}) := \sup_{\xi^i, U_0^A(\xi^i) \geq R_0} \sup_{\alpha^* \in \mathcal{A}^*(\xi)} \mathbb{E}^{\nu^*} [U_{P_i}(\ell_i(X_T) - \xi^i)], \quad (4.1.2)$$

where U_P^i is a utility function, assumed to be non-decreasing and concave, $\ell_i : \mathbb{R}^N \rightarrow \mathbb{R}$ is a liquidation function of linear growth. Without communication considered, the Principals will simultaneously solve their problem leading to reach a Nash equilibrium in the following sense

Definition 4.1 (Nash equilibrium). *A contract ξ^* is a Nash equilibrium for the N Principals if, for any $\alpha^* \in \mathcal{A}(\xi^*)$, it satisfies*

$$\sup_{\xi^i} \mathbb{E}^{\alpha^*(\xi^*)} [U_{P_i}(\ell_i(X_T) - \xi^i)] = \mathbb{E}^{\alpha^*(\xi^*)} [U_{P_i}(\ell_i(X_T) - \xi^{i,*})], \quad 1 \leq i \leq N \text{ and } U_0^A(\xi^*) \geq R_0.$$

From now², the following assumption is enforced

ASSUMPTION (LQ). *We assume that $N = 2$ and*

- (i) *the drift b is a linear function of the effort such that $b(t, x, \alpha) := K\alpha$, where K is a diagonal matrix and the coefficients on the diagonal represents the efficiency of the Agent with project 1 and 2; the map c is the classical quadratic cost function defined by $c(t, x, \alpha) = \frac{\|\alpha_t\|^2}{2}$; $\Sigma \in \mathcal{M}_2(\mathbb{R})$ is invertible;*
- (ii) *$\ell_i(x) := (1 + \ell_i)x_i - \ell_i x_j$, $1 \leq i \neq j \leq 2$ with appetite parameters $\ell_1, \ell_2 \in [0, 1]$. ℓ^i is typically the appetite parameter of Principal i toward project i . This situation fits exactly with noncooperative Principals' behaviors investigated in [Bernheim and Whinston, 1985];*
- (ii) *the Principals are risk-neutral, i.e. $U_{P_i}(x) = x$.*

4.2 Solving the bi-level optimization with common agency

4.2.1 The Agent problem

We thus introduce the following set of admissible contract

$$\Xi_2 := \left\{ \xi : U_0^A(\xi) \geq R_0, \exists Y_0 \in \mathbb{R}, \exists Z \in \mathcal{Z} : \xi \cdot \mathbf{1}_2 = Y_0 + \int_0^T G(Z_t)dt + \int_0^T Z_t \cdot dX_t, \mathbb{P}\text{-a.s.} \right\},$$

with integrability properties for the space \mathcal{Z} of \mathbb{R}^2 -valued process Z , where $G(z) := \frac{R_A}{2} \|\Sigma^\top z\|^2 - \sup_{a \in \mathbb{R}} (Ka \cdot z - \frac{\|a\|^2}{2})$, $(s, x, z) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$. The following proposition reveals the optimal effort of the Agent for any contract in Ξ_2

Proposition 4.1. *Let $\xi \in \Xi_2$. $U_0^A(\xi) = -e^{-R_A Y_0}$ and $\alpha_t^* := KZ_t \cdot \mathbf{1}_2$ is the optimal effort of the Agent.*

Note that, in this case for any $\xi \in \Xi_2$ the optimal control α^* is unique. As explained in the first chapter and Proposition 2.1, as soon as ξ is enough integrable one expect that the set Ξ_2 is not restrictive,

²see [Mastrolia and Ren, 2018] for a more general model

by using BSDE theory³ or martingale representation theorem for the problem of the Agent. This will provide the simple decomposition of $\xi \cdot \mathbf{1}_2$ only. However and as mentioned in the first chapter, the main difficulty is to reduced the problem of the Principals (4.1.2) to a stochastic control problem providing a smooth decomposition of ξ^i . Here, the situation is a more complex since the problem of the Agent will lead to a representation of the aggregated compensation $\xi \cdot \mathbf{1}_2$ only and not on the component ξ^i .

4.2.2 Equilibrium necessary conditions and optimal contract

Our aim is to obtain the existence of a Nash equilibrium in the sense of Definition 4.1. We thus force the contract to have a simple form from the beginning so that in this chapter, we are only interested in Nash equilibriums in the following sets of contracts:

$$\tilde{\Xi}_2 := \left\{ \xi \in \Xi_2 : \exists (y, \gamma, \zeta), \xi^i = y^i + \int_0^T \gamma_s^i ds + \int_0^T \zeta_s^{:,i} \cdot dX_s, \forall i \in \{1, 2\} \right\},$$

with $y \in \mathbb{R}^2$ and where γ and ζ are \mathbb{R}^2 -valued and $\mathcal{M}_2(\mathbb{R})$ -valued processes respectively with integrability conditions. In this case, we obtain the following equilibrium conditions

Proposition 4.2 (Necessary condition of Nash equilibrium). *Let $\xi^* = (\xi^{i,*})_{1 \leq i \leq N}$ be a Nash equilibrium in $\tilde{\Xi}_2$ such that $\xi^{i,*}$ is characterized by the triplet $(y^i, \gamma^{i,*}, \zeta^{i,*})$. Then, the following equilibrium condition holds for $(y^i, \gamma^{i,*}, \zeta^{i,*})_{i \in \{1, 2\}}$:*

$$y^1 + y^2 = Y_0, \quad \gamma_s^{1,*} + \gamma_s^{2,*} = G(\zeta_s^{1,*} + \zeta_s^{2,*}), \quad \zeta_s^{1,*} + \zeta_s^{2,*} = Z_s. \quad (4.2.1)$$

We are going to fully characterize a Nash equilibrium. We start from the problem of Principal 1 recalling (4.1.2) with $i = 1$. Denote $r_0 := -\frac{\ln(-R_0)}{R_A}$, so that we have $U_0^A(\xi) \geq R_0 \iff Y_0 \geq r_0$, thanks to Proposition 4.1. It follows from Proposition 4.2 that given ξ^2 with characteristic triplet (y^2, γ^2, ζ^2)

$$\begin{aligned} U_0^{P1}(\xi^2) &= \sup_{\xi^1, U_0^A(\xi) \geq R_0} \mathbb{E}^{\alpha^*} [\ell_1(X_T) - \xi \cdot \mathbf{1}_2 + \xi^2] = \sup_{y^1 \geq r_0 - y^2} \sup_{\substack{(\gamma^1, \zeta^1) \\ \gamma_s^1 + \gamma_s^2 = G(s, X_s, \zeta_s^1 + \zeta_s^2)}} \mathbb{E}^{\alpha^*} [\ell_1(X_T) - Y_T^{y^1, \gamma^1, \zeta^1}] \\ &= u^1(0, x, r_0 - y^2), \end{aligned}$$

where

$$u^1(0, x, y^1) := \sup_{\zeta^1} \mathbb{E}^{\alpha^*} [\ell(X_T) - Y_T^{y^1, \gamma^1, \zeta^1}] \quad \text{and} \quad Y_T^{y^1, \gamma^1, \zeta^1} = y^1 + \int_0^T (G(\zeta_s^2 + \zeta_s^1) - \gamma_s^2) ds + \int_0^T \zeta_s^1 \cdot dX_s.$$

Therefore, the problem of the Principal (4.1.2) in $\tilde{\Xi}_2$ coincides with a stochastic control problem with two state variables: the output X and the value process $Y^{y^1, \gamma^1, \zeta^1}$ and controlled by the coefficient ζ^1 only (from (4.2.1)). The associated HJB equation is

$$\begin{cases} -\partial_t u^1(t, x, y) - \sup_{\beta^1 \in \mathbb{R}^N} H(\nabla_x u^1, \partial_y u^1, \nabla_x^2 u^1, \partial_{yy}^2 u^1, \partial_{x,y}^2 u^1, \gamma_t^2, \zeta_t^2, \zeta^1) = 0, \\ u^1(T, x, y) = \ell_1(x) - y, \quad (t, x, y) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}. \end{cases} \quad (4.2.2)$$

with H defined for any $(t, x, p, \tilde{p}, q, \tilde{q}, r, \gamma, z, \zeta) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathcal{M}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ by

$$\begin{aligned} H(t, x, p, \tilde{p}, q, \tilde{q}, r, \gamma, z, \zeta) &:= p \cdot K^2(z + \zeta) + \tilde{p} (G(z + \zeta) - \gamma + \zeta \cdot K^2(z + \zeta)) \\ &\quad + \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top q) + \frac{1}{2} \beta^\top \Sigma \Sigma^\top \zeta \tilde{q} + \zeta^\top \Sigma \Sigma^\top r. \end{aligned}$$

³We can for instance use [Briand and Hu, 2008] since dealing with exponential utility necessarily leads to BSDE with a generator having a quadratic growth with respect to the z component.

In order to find a Nash equilibrium, we assume that both Principals are reduced to solve (4.2.2) simultaneously. In the particular example studied we have the following lemma

Lemma 4.1. *The following system of PDEs admits a solution (v^1, v^2) in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^2)$*

$$\begin{cases} -\partial_t v^1(t, x) - \left\{ \nabla_x v^1 \cdot K^2(\zeta_t^{1,\star} + \zeta_t^{2,\star}) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top \nabla_x^2 v^1) - \frac{R_A}{2} \|\Sigma^\top(\zeta_t^{1,\star} + \zeta_t^{2,\star})\|^2 \right. \\ \quad \left. + \frac{\|K(\zeta_t^{1,\star} + \zeta_t^{2,\star})\|^2}{2} + \gamma_t^{2,\star} - K^2(\zeta_t^{1,\star} + \zeta_t^{2,\star}) \cdot \zeta_t^{1,\star} \right\} = 0 \\ v^1(T, x) = (1 + \ell_1)x_1 - \ell_1 x_2, \end{cases} \quad (4.2.3)$$

and

$$\begin{cases} -\partial_t v^2(t, x) - \left\{ \nabla_x v^2 \cdot K^2(\zeta_t^{1,\star} + \zeta_t^{2,\star}) + \frac{1}{2} \text{Tr}(\Sigma \Sigma^\top \nabla_x^2 v^2) - \frac{R_A}{2} \|\Sigma^\top(\zeta_t^{1,\star} + \zeta_t^{2,\star})\|^2 \right. \\ \quad \left. + \frac{\|K(\zeta_t^{1,\star} + \zeta_t^{2,\star})\|^2}{2} + \gamma_t^{1,\star} - K^2(\zeta_t^{1,\star} + \zeta_t^{2,\star}) \cdot \zeta_t^{2,\star} \right\} = 0 \\ v^2(T, x) = (1 + \ell_2)x_2 - \ell_2 x_1, \end{cases} \quad (4.2.4)$$

with

$$\begin{cases} \zeta^{1,\star} = \nabla_x v^1 - K^{-2} R_A \Sigma \Sigma^\top M \Lambda \\ \zeta^{2,\star} = \nabla_x v^2 - K^{-2} R_A \Sigma \Sigma^\top M \Lambda \\ \gamma^{1,\star} + \gamma^{2,\star} = \frac{R_A}{2} \|\Sigma^\top(\zeta_t^{1,\star} + \zeta_t^{2,\star})\|^2 - \frac{K^2 \|\zeta_t^{1,\star} + \zeta_t^{2,\star}\|^2}{2}, \end{cases} \quad (4.2.5)$$

where $M := (I + 2K^{-2} R_A \Sigma \Sigma^\top)^{-1}$ given by

$$v^1(t, x) := \tilde{\lambda}(T - t) + (1 + \ell_1, -\ell_1) \cdot x \text{ and } v^2(t, x) := \tilde{\lambda}(T - t) + (-\ell_2, 1 + \ell_2) \cdot x,$$

with

$$\tilde{\lambda} := \Lambda \cdot K^2 M \Lambda - \frac{5R_A}{4} \|\Sigma^\top M \Lambda\|^2 - \frac{3}{4} \|K M \Lambda\|^2, \quad \Lambda := (1 + \ell_1 - \ell_2, 1 + \ell_2 - \ell_1)^\top.$$

Remark 4.1 (Idea of the proof). *The main idea of proving the result above is to study the "aggregated" equation. Let us do the following intuitive analysis. We have*

$$\bar{\zeta} = \nabla_x V - 2K^{-2} R_A \Sigma \Sigma^\top \bar{\zeta}, \quad \bar{\zeta} := \zeta^{1,\star} + \zeta^{2,\star},$$

where $V = v^1 + v^2$, and thus $\bar{\zeta} = M \nabla_x V$. Note now that there exists a unique solution to the PDE associated with V given by $V(t, x) = x \cdot \Lambda + \lambda(T - t)$ with $\lambda = \Lambda \cdot K^2 M \Lambda - \frac{R_A}{2} \|\Sigma^\top M \Lambda\|^2 - \frac{\|K M \Lambda\|^2}{2}$. Hence, we have thus able to decoupled the system and we can prove the lemma.

Theorem 4.1 (Verification – Optimal contracts). *Define for $i \in \{1, 2\}$*

$$\xi^{i,\star} = y_i + \int_0^T \gamma_s^{i,\star} ds + \int_0^T \zeta_s^{i,\star} \cdot dX_s,$$

with $\gamma^{i,\star}, \zeta^{i,\star}$ given by Lemma 4.1 above and $y^1 + y^2 = r_0$. Then, the set of contract $(\xi^{i,\star})_{i \in \{1, 2\}}$ is an admissible Nash equilibrium and $\alpha^\star := K M \Lambda$ is the optimal effort of the Agent.

Remark 4.2. *Note in particular that there is a degrees of liberty in the choices of y^i and $\gamma^{i,\star}$. Hence, there is no uniqueness of Nash equilibria.*

4.2.3 Main theoretical results in [Mastrolia and Ren, 2018]

For $N \in \mathbb{N}^*$, we can weaken Assumption **(LQ)** – (i) – (ii) – (iii) to obtain again the same kind of equilibrium conditions than those in Proposition 4.2, see [Mastrolia and Ren, 2018, Section 4.1] together with a verification theorem to obtain Nash equilibria under existence of solutions to fully coupled system of HJB equations, see [Mastrolia and Ren, 2018, Section 4.2]. As soon as the Principals are risk neutral, we are able to solve the system of HJB equations fully coupled by using similar procedure than Remark 4.1. We refer to [Mastrolia and Ren, 2018, Section 4.3] for the details.

4.2.4 Common agency *versus* Aggregated offer

In this section, we compare the competitive common agency example studied in the previous section with the model in which the Principals can be aggregated into a parent firm. The problem can be thus reduced to one single classical Principal-Agent model, in the spirit of Chapter 2 with $N = 1$ by aggregating the two payoffs of the Principals. In particular, the aggregated Principal manages to do the following optimization under the optimal effort of the Agent:

$$U_0 = \sup_Z \mathbb{E}^* [(1 + \ell_1 - \ell_2)X_T^1 + (1 + \ell_2 - \ell_1)X_T^2 - \xi]$$

As before, we can calculate the optimal control $Z^* = M_a \Lambda$ with $M_a := (R_A \Sigma \Sigma^\top + K^2)^{-1} K^2$. Further, the optimal effort of the Agent should be $\nu_{P_f}^* = K M_a \Lambda$. By comparing $\alpha_{P_f}^*$ and α^* in (4.1), we immediately have the following conclusion extending the result of [Bernheim and Whinston, 1985] to continuous-time models

Proposition 4.3. *The effort of the common Agent in the competitive model coincides with the effort of the common Agent in the aggregated model if and only if the Agent is risk-neutral, that is, $R_A = 0$. Moreover, the non-risk part of the remuneration is higher for the Agent if he is employed by a parent firm compared to the non-risk part of the remuneration of the Agent hired by two different firms.*

We refer to [Mastrolia and Ren, 2018, Section 5] for additional interpretations.

Perspectives

We have seen in [Élie et al., 2019] that an indexation on the other consumers' deviation baseline benefits to the energy producer. One possible extension could be to design market mechanisms for optimal allocation of local flexibility (consumption, storage) to reduce the strain on distribution networks. In fact, 80% of renewable production is connected to the distribution networks which were not designed for this usage, and if one wants to avoid the high costs of expanding these networks, a possibility is to use new "smart" technologies and innovative market mechanisms (*e.g.*, peer to peer markets). This investigation is one research task of the ANR project EcoRees, PI: Peter Tankov, and is currently investigated.

The article [Mastrolia and Ren, 2018] was a first step in the investigation of a multiple Principals/Agent problems in contract theory. At the opposite of common agency problem, other works have investigated a competition problem between two principals who want the exclusive service of an agent. For instance, the recent work [Hu et al., 2019] considers infinitely many Principals hiring an Agent, switching from a Principal to another. In [Rothschild and Stiglitz, 1976] the authors have studied insurance markets in which principals are identical and compete for a single agent, then generalized in [Biglaiser and Mezzetti, 1993]. This problem is not considered here, because we allows the agents to be remunerated by all the Principals, and it will be left for future researches. The continuous-time Agent/Multi-Principals model, outside the common agency framework or switching strategies in a mean-field case, is still an open question. What happens if the Principals play a Nash equilibrium in a general class of contract, without restricting the study to contracts with smooth decomposition, see Section 4.2.2 above, and for risk-averse Principal? As we will see in Chapter 6 Section 6.2, by assuming symmetry between the players, we are able to get the existence of a Nash equilibrium in a particular application. What about more general cases? This challenging problem is currently studied with Nizar Touzi. It would lead to a lot of interesting applications both in finance, see Chapter 8, or in population dynamics, see Chapter 11.

We now briefly recall the study made in [Hernández-Santibáñez and Mastrolia, 2019]. We have investigated a Principal-Agent problem with moral hazard under Knightian uncertainty. More exactly, both the Principal and the Agent solve their utility maximization problems under volatility uncertainty controlled by a third exogenous player called the Nature. Hence, both the Agent and the Principal solve a zero-sum game against the Nature. As an extension of [Cvitanić et al., 2019], we characterize the best reaction effort of the agent through the solution to a second order BSDE (see [Hernández-Santibáñez and Mastrolia, 2019, Theorem 3.1]). We have showed that the value of the problem of the Principal is the viscosity solution of an Hamilton-Jacobi-Bellman-Isaacs equation, without needing a dynamic programming principle, weakening the conditions in [Mastrolia and Possamaï, 2018], by using stochastic Perron's method, see [Hernández-Santibáñez and Mastrolia, 2019, Theorem 4.1]. Although we allow the set of volatility values to be time-dependent, we do not specify how it could evolve with time. A reinforcement learning procedure similar to [Bielecki et al., 2019] would lead to sharper estimates of the volatility to be updated into the values of both the Agent and the Principal.

Finally I am currently working with Emma Hubert on characterizations of Nash equilibria for stochastic control problems with volatility controlled. This is strongly linked to the existence of solutions to multi-dimensional second order BSDE, as an extension of [Soner et al., 2012] or [Possamaï et al., 2018] to the multi-dimensional case, which has not been studied yet.

Part II

Financial regulation and market microstructure

Make-take fees for market making regulation

In this chapter, we consider an exchange who wishes to set suitable make-take fees to attract liquidity on its platform. Using a principal-agent approach, in the spirit of Part I above, we are able to describe in quasi-explicit form the optimal contract to propose to a market maker. This contract depends essentially on the market maker inventory trajectory and on the volatility of the asset. We also provide the optimal quotes that should be displayed by the market maker. We extend the study to oligopoly of symmetrical exchanges. These two sections are extracted from [El Euch et al., 2018].

6.1 One market maker regulated by one exchange

Our starting point is the seminal work [Avellaneda and Stoikov, 2008]. Our objective is to derive optimal make-take fees in order to monitor the behavior of a market maker on a platform acting according to the optimal market making model of [Avellaneda and Stoikov, 2008] or [Guéant et al., 2013].

6.1.1 The model

Let $T > 0$ be a final horizon time, Ω_c be the set of continuous functions from $[0, T]$ into \mathbb{R} , Ω_d the set of piecewise constant càdlàg functions from $[0, T]$ into \mathbb{N} , and $\Omega := \Omega_c \times (\Omega_d)^2$ with corresponding Borel algebra \mathcal{F} . The observable state is the canonical process $(\chi_t)_{t \in [0, T]} = (S_t, N_t^a, N_t^b)_{t \in [0, T]}$ of the measurable space (Ω, \mathcal{F}) :

$$S_t(\omega) := s(t), \quad N_t^a(\omega) := n^a(t), \quad N_t^b(\omega) := n^b(t), \quad \text{for all } t \in [0, T], \quad \omega = (s, n^a, n^b) \in \Omega,$$

with canonical completed filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]} = (\mathcal{F}_t^c \otimes (\mathcal{F}_t^d)^{\otimes 2})_{t \in [0, T]}$.

The trading activity is reduced to a single risky asset S with observable efficient price S , that can be seen in practice as the mid-price of the asset, defined by

$$S_t := S_0 + \sigma W_t, \quad t \in [0, T], \tag{6.1.1}$$

for some Brownian motion W , initial price $S_0 > 0$, and constant volatility $\sigma > 0$. The market maker chooses processes denoted by δ^b and δ^a respectively so as to fix publicly available bid and ask offer prices:

$$P_t^b := S_t - \delta_t^b \quad \text{and} \quad P_t^a := S_t + \delta_t^a, \quad t \in [0, T].$$

The arrival of bid and ask market orders is modeled by a counting process (N^b, N^a) with unit jumps, so that no more than one market order can occur at each time. We introduce the inventory process of the market maker Q :

$$Q_t := N_t^b - N_t^a \in \mathbb{N} \cap [-\bar{q}, \bar{q}], \quad t \in [0, T],$$

where $N_0^b = N_0^a = 0$ and, as in [Guéant et al., 2013], we impose a critical absolute inventory $\bar{q} \in \mathbb{N}$ above which the market maker stops quoting on the ask or bid side. Let $c > 0$ be the take-fee collected by the exchange. To illustrate the impact of the posted prices on the transactions arrival process (N^b, N^a) ,

the corresponding intensity process depends on the departure of the transaction price from the efficient price, i.e. $c + \delta_t^i$, $i \in \{b, a\}$, as follows:

$$\lambda_t^{i,\delta} := \lambda(\delta_t^i) \mathbf{1}_{\{\varepsilon_i Q_t > -\bar{q}\}}, \quad i \in \{b, a\}, \quad (\varepsilon_b, \varepsilon_a) = (-1, 1), \quad \text{with } \lambda(x) := A e^{-k \frac{(x+c)}{\sigma}}, \quad (6.1.2)$$

for some fixed positive constants A and k . The dependence on the volatility parameter σ reproduces the stylized fact that the average number of trades per unit of time is a decreasing function of the ratio between spread and volatility. The arrival of orders is obviously decreasing with the fee c .

Recall the the market-maker proposes the bid and ask spread δ^b and δ^a . The set of admissible controls \mathcal{A} is the collection of all predictable processes $\delta = (\delta^a, \delta^b)$ uniformly bounded by δ_∞ , some sufficiently large positive constant. Each control process $\delta \in \mathcal{A}$ induces

– the market maker profit and loss process:

$$\text{PL}_t^\delta := X_t^\delta + Q_t S_t, \quad \text{where } X_t^\delta := \int_0^t P_r^a dN_r^a - \int_0^t P_r^b dN_r^b, \quad t \in [0, T], \quad (6.1.3)$$

as the sum of the cash flow X^δ and the inventory risk QS ,

– and a probability measure \mathbb{P}^δ under which S is driven by (6.1.1), and

$$\tilde{N}_t^{i,\delta} := N_t^i - \int_0^t \lambda_r^{i,\delta} dr, \quad t \in [0, T], \quad i \in \{b, a\}, \quad \text{are martingales.} \quad (6.1.4)$$

equivalent to \mathbb{P}^0 and explicitly given by a Doleans-Dade exponential martingale associated with the change of measure leading to (6.1.4). In particular, all probability measures \mathbb{P}^δ are equivalent.

Recall that the exchange aims at encouraging the market maker to reduce his spread so as to enhance market liquidity on the platform. This is achieved by setting the terms of an incentive contract defined by an \mathcal{F}_T -measurable random variable ξ in a class of admissible contracts Ξ having specific integrability conditions (see the definition below). In other words, the compensation ξ may depend on the whole paths of the contractible variables N^a , N^b and S . Given this additional revenue, the market maker's objective is to solve

$$\begin{aligned} V_{\text{MM}}(\xi) &:= \sup_{\delta \in \mathcal{A}} J_{\text{MM}}(\delta, \xi), \quad \text{where } J_{\text{MM}}(\delta, \xi) := \mathbb{E}^\delta \left[-e^{-\gamma(\xi + \text{PL}_T^\delta)} \right] \\ &= \mathbb{E}^\delta \left[-e^{-\gamma(\xi + \int_0^T \delta_t^a dN_t^a + \delta_t^b dN_t^b + Q_t dS_t)} \right]. \end{aligned} \quad (6.1.5)$$

Here, $\gamma > 0$ is the absolute risk aversion parameter of the CARA market maker. Remember that The exchange receives a fixed fee $c > 0$ for each market order that occurs in the market and then collects at time T the total revenue $c(N_T^a + N_T^b) - \xi$. The choice of the contract ξ given the best reaction spread $\hat{\delta}(\xi)$ solving (6.1.5) is dictated by the utility maximization problem

$$V_0^E := \sup_{\xi \in \Xi} \mathbb{E}^{\hat{\delta}(\xi)} \left[-e^{-\eta(c(N_T^a + N_T^b) - \xi)} \right], \quad (6.1.6)$$

where $\eta > 0$ is the exchange's absolute risk aversion parameter, and the set of admissible contracts Ξ is the collection of all contracts satisfying

- the participation constraint $V_{\text{MM}}(\xi) \geq R$, with $R < 0$,
- the integrability conditions¹

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta \left[e^{\eta' \xi} \right] < \infty \quad \text{and} \quad \sup_{\delta \in \mathcal{A}} \mathbb{E}^\delta \left[e^{-\gamma' \xi} \right] < \infty, \quad \text{for some } \eta' > \eta, \quad \gamma' > \gamma.$$

¹This is Condition (a) of the first chapter see Section 2.1.2.2. Echoing Remark 2.2, these integrability conditions ensure that there exists an optimizer, so that Condition (b) is automatically satisfied. See [El Euch et al., 2018, Theorem 3.1].

6.1.2 The optimal contract

We thus follow the *modus operandi* which consists in solving the problem of the market maker given ξ . For $(\delta, z, q) \in [-\delta_\infty, \delta_\infty]^2 \times \mathbb{R}^3 \times \mathbb{Z}$, with $\delta = (\delta^a, \delta^b)$ and $z = (z^S, z^a, z^b)$, we define

$$h(\delta, z, q) := \sum_{i=b,a} \frac{1 - e^{-\gamma(z^i + \delta^i)}}{\gamma} \lambda(\delta^i) \mathbf{1}_{\{\varepsilon_i q > -\bar{q}\}} \quad \text{and} \quad H(z, q) := \sup_{|\delta^a| \vee |\delta^b| \leq \delta_\infty} h(\delta, z, q).$$

with $(\varepsilon_b, \varepsilon_a) = (-1, 1)$. For an arbitrary constant $Y_0 \in \mathbb{R}$ and predictable processes $Z = (Z^S, Z^a, Z^b)$, with $\int_0^T (|Z_t^S|^2 + |H(Z_t, Q_t)|) dt < \infty$, we introduce the process

$$Y_t^{Y_0, Z} = Y_0 + \int_0^t Z_r^a dN_r^a + Z_r^b dN_r^b + Z_r^S dS_r + \left(\frac{1}{2} \gamma \sigma^2 (Z_r^S + Q_r)^2 - H(Z_r, Q_r) \right) dr,$$

and we denote by \mathcal{Z} the collection of all such processes Z with integrability conditions².

Theorem 6.1. (i) Any contract $\xi \in \Xi$ has a unique representation as $\xi = Y_T^{Y_0, Z}$, for some $(Y_0, Z) \in [\hat{Y}_0, +\infty) \times \mathcal{Z}$ with $\hat{Y}_0 := -\frac{1}{\gamma} \log(-R)$.

(ii) Under this representation, the market maker utility value is $V_{\text{MM}}(\xi) = -e^{-\gamma Y_0}$, with the following optimal bid-ask policy

$$\hat{\delta}_t^i(\xi) = \Delta(Z_t^i), \quad i \in \{b, a\}, \quad \text{where} \quad \Delta(z) := (-\delta_\infty) \vee \left\{ -z + \frac{1}{\gamma} \log \left(1 + \frac{\sigma \gamma}{k} \right) \right\} \wedge \delta_\infty.$$

Hence, the problem of the exchange (6.1.6) becomes

$$V_0^E = e^{\eta \hat{Y}_0} v_0^E, \quad \text{where} \quad v_0^E := \sup_{Z \in \mathcal{Z}} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0, Z})} \left[-e^{-\eta(c(N_T^a + N_T^b) - Y_T^{0, Z})} \right]. \quad (6.1.7)$$

The problem of the exchange is thus reduced to a standard stochastic Markovian control problem. Since all the coefficients in (6.1.7) do not depend on S , we guess that the HJB equation associated to (6.1.7) reduces to

$$\partial_t v(t, q) + H_E(q, v(t, q), v(t, q+1), v(t, q-1)) = 0, \quad q \in \{-\bar{q}, \dots, \bar{q}\}, \quad t \in [0, T], \quad (6.1.8)$$

with boundary condition $v|_{t=T} = -1$, with Hamiltonian $H_E : [-\bar{q}, \bar{q}] \times (-\infty, 0]^3 \rightarrow \mathbb{R}$:

$$H_E(q, y, y_+, y_-) = H_E^1(q, y) + \mathbf{1}_{\{q > -\bar{q}\}} H_E^0(y, y_-) + \mathbf{1}_{\{q < \bar{q}\}} H_E^0(y, y_+),$$

and

$$H_E^1(q, y) = \sup_{z_s \in \mathbb{R}} h_E^1(q, y, z_s), \quad \text{and} \quad h_E^1(q, y, z_s) = \frac{\eta \sigma^2}{2} y (\gamma(z_s + q)^2 + \eta z_s^2),$$

$$H_E^0(y, y') = \sup_{\zeta \in \mathbb{R}} h_E^0(y, y', \zeta) \quad \text{and} \quad h_E^0(y, y', \zeta) = \lambda(\Delta(\zeta)) \left[y' e^{\eta(\zeta - c)} - y \left(1 + \eta \frac{1 - e^{-\gamma(\zeta + \Delta(\zeta))}}{\gamma} \right) \right],$$

with maximizers $\hat{z} = (\hat{z}^s, \hat{z}^a, \hat{z}^b)$ given by:

$$\hat{z}^s(t, q) = -\frac{\gamma}{\gamma + \eta} q, \quad \hat{z}^a(t, q) = \hat{\zeta}(v(t, q), v(t, q-1)), \quad \hat{z}^b(t, q) = \hat{\zeta}(v(t, q), v(t, q+1)), \quad (6.1.9)$$

$$\text{with} \quad \hat{\zeta}(y, y') = \zeta_0 + \frac{1}{\eta} \log \left(\frac{y}{y'} \right), \quad \zeta_0 = c + \frac{1}{\eta} \log \left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)} \right).$$

Inspired by [Guéant et al., 2013], by introducing $u := (-v)^{-\frac{k}{\sigma \eta}}$, this equation can be reduced to a linear equation that can be explicitly solved and with solution u given by

$$u(t, q) = \mathbb{E} \left[e^{\int_t^T (-C_1(Q_s^{t,q})^2 + \bar{\lambda}_s + \underline{\lambda}_s) ds} \right], \quad (6.1.10)$$

where $Q_s^{t,q} = q + \int_t^s d(\bar{N}_u - \underline{N}_u)$, and (\bar{N}, \underline{N}) is a two-dimensional point process with intensity $(\bar{\lambda}_s, \underline{\lambda}_s) = C_1'(\mathbf{1}_{\{Q_{s-} < -\bar{q}\}}, \mathbf{1}_{\{Q_{s-} > -\bar{q}\}})$, for some explicit constants C_1, C_1' .

²This integrability conditions implies that $\xi = Y_T^{Y_0, Z}$ is in Ξ .

Theorem 6.2 (Verification – Optimal contract). *Assume that $\delta_\infty \geq \Delta_\infty$, with explicit constant Δ_∞ . Then the optimal contract for the problem of the exchange (6.1.6) is given by*

$$\hat{\xi} = \hat{Y}_0 + \int_0^T \hat{Z}_r^a dN_r^a + \hat{Z}_r^b dN_r^b + \hat{Z}_r^S dS_r + \left(\frac{1}{2} \gamma \sigma^2 (\hat{Z}_r^S + Q_r)^2 - H(\hat{Z}_r, Q_r) \right) dr, \quad (6.1.11)$$

with $\hat{Z}_r^S = \hat{z}^s(r, Q_{r-})$, $\hat{Z}_r^a = \hat{z}^a(r, Q_{r-})$, and $\hat{Z}_r^b = \hat{z}^b(r, Q_{r-})$ as defined in (6.1.9). The market maker's optimal effort is given by

$$\hat{\delta}_t^a = \hat{\delta}_t^a(\hat{\xi}) = -\hat{Z}_t^a + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right), \quad \hat{\delta}_t^b = \hat{\delta}_t^b(\hat{\xi}) = -\hat{Z}_t^b + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right). \quad (6.1.12)$$

Moreover, the value function of the exchange is $V_0^E = v(0, Q_0)$ and does not depend on c .

Remark 6.1. *Remember from Section 2.2.2 that we have to check the admissibility of the contract $\hat{\xi}$. From (6.1.10), we deduce that the optimizers $\hat{z}^S, \hat{z}^a, \hat{z}^b$ are bounded so that $\hat{\xi} \in \Xi$ and thus admissible.*

6.1.3 Interpretations and economical insights

6.1.3.1 Policy implications

The processes \hat{Z}^a , \hat{Z}^b and \hat{Z}^S defining the optimal contract have natural interpretations. At least for large inventories and using the same kind of expansions than [Guéant et al., 2013, Section 4] and [Avellaneda and Stoikov, 2008, Section 3.2] where same type of PDE as ours is considered, we have

$$\hat{Z}^i \underset{|q| \rightarrow +\infty}{\sim} \zeta_0 + \varepsilon_i \frac{\sigma}{2k} \sqrt{\frac{C_1}{C_1'}} (2Q_{t-} - \varepsilon_i), \quad i \in \{a, b\}, \quad (6.1.13)$$

Thus, when the inventory is highly positive, the exchange provides incentives to the market-maker so that it attracts buy market orders and discourage him from more sell market orders, and vice versa for a negative inventory. The integral $\int_0^T \hat{Z}_r^S dS_r$ can be understood as a risk sharing term. More precisely, $\int_0^t Q_r dS_r$ corresponds to the price driven component of the inventory risk $Q_t S_t$. Hence, the exchange supports the proportion $\frac{\gamma}{\gamma+\eta}$ of this risk so that the market maker maintains reasonable quotes despite some inventory.

Notice that for a highly risk averse exchange, *i.e.* $\eta \nearrow \infty$, $\int_0^T \hat{Z}_r^a dN_r^a + \hat{Z}_r^b dN_r^b \approx c(N_T^a + N_T^b)$, $\hat{Z}_r^S \approx 0$, meaning that the exchange transfers to the market maker the total fee. This is the so-called *selling the firm* effect, as the exchange delegates all benefit to the market maker.

Until now, we have focused on the maker part of the make-take fees problem since we have considered that the taker cost c is fixed. Nevertheless, our approach also enables us to suggest the exchange a relevant value for c and so to also calibrate optimally the taker cost. Hence if for example the exchange targets a spread close to one tick, he can put for $\sigma\gamma/k$ small enough $c \approx \frac{\sigma}{k} - \frac{1}{2}\text{Tick}$.

6.1.3.2 On the benefit of incentive policies

We compare our setting with the situation without incentive policy from an exchange towards market making activities which corresponds to the problem of optimal market making considered in [Avellaneda and Stoikov, 2008, Guéant et al., 2013]. First note in Figure 6.1 that the spread is reduced thanks to the optimal contract between the market maker and the exchange. This is not surprising since in our case the exchange aims at increasing the market order flow by proposing an incentive contract to the market maker inducing a spread reduction. Since the spread is tighter during the trading period under an incentive policy from the exchange, the arrival intensity of market orders is more important

and hence the market is more liquid as shown in Figure 6.1. On the bottom of Figure 6.1, we see that when the inventory is positive and very large, the ask spreads are negative, meaning that the market maker is ready to sell at prices lower than the efficient price in order to attract market orders and reduce his inventory risk. On the contrary, if the inventory is negative and very large, in both situations, its ask quotes are well above the efficient price in order to repulse the arrival of buy market orders. However, since in our case the exchange remunerates the market maker for each arrival of market order, we get that the ask spread with contract $\hat{\delta}^a$ is smaller than the ask spread without contract. A symmetric conclusion holds for the bid part of the spread.

In Figure 6.2, we plot on the left side the cumulated P&L of the market maker under the optimal contract given by $PL_t^* = PL_t^{\hat{\delta}} + Y_t^{\hat{Y}_0, \hat{Z}}$, together with his P&L without intervention of the exchange. To make the two quantities comparable, we choose \hat{Y}_0 in (6.1.11) so that the market maker gets the same utility in both situations, we thus see that he is indifferent with or without exchange intervention. On the right side, we compare the profit and loss of the exchange in the two considered cases. We see that the initial P&L of the contracting exchange is negative because of the initial payment \hat{Y}_0 . However it finally exceeds, with a smaller standard deviation, the P&L in the situation without incentive policy from the exchange. Hence the incentive policy of the exchange proves to be successful.

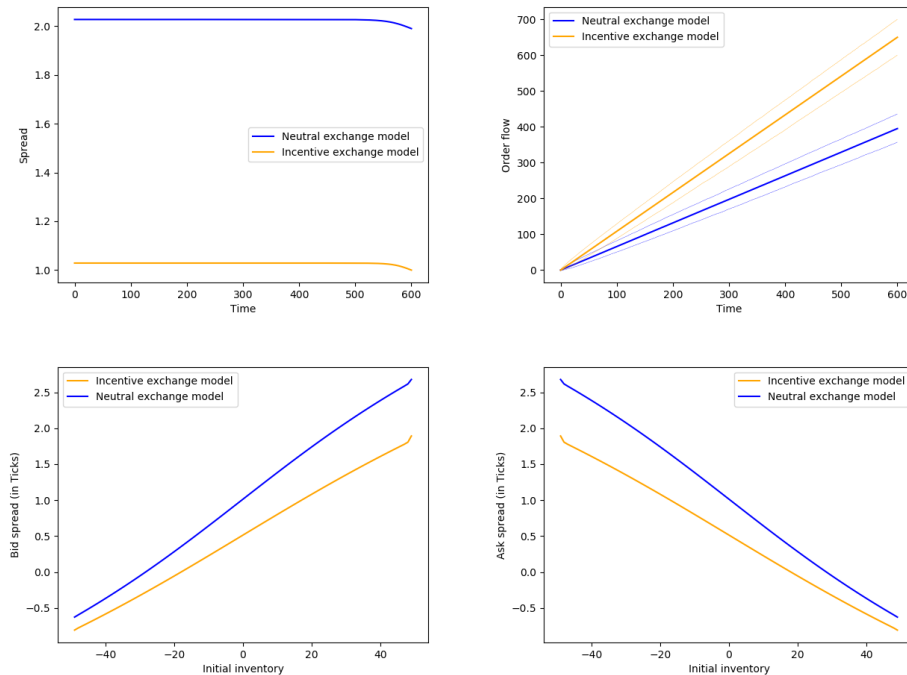


Figure 6.1: Average spread (upper left side) and average order flow (upper right side) on $[0, T]$ and bid spread (bottom left) and ask spread (bottom right) at time 0 with/without incentive.

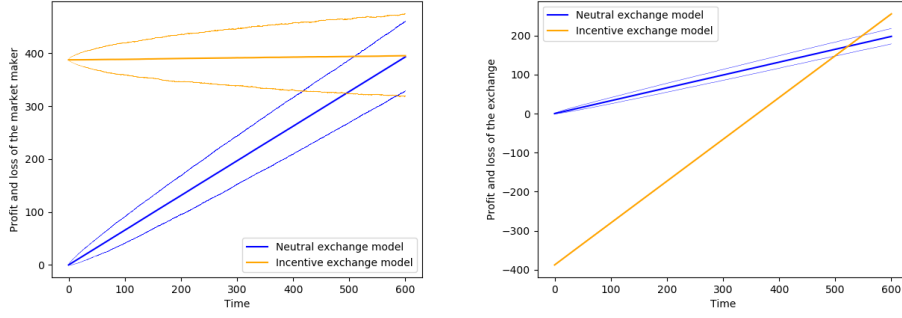


Figure 6.2: Average (virtual) P&L accumulated during the trading period by the market-maker (left side) and average P&L of the exchange (right side) with/without incentive, with 95% confidence interval .

6.2 Extension to an oligopoly of symmetrical exchanges

This section refers to [El Euch et al., 2018, Section 6]. We extend the previous model to N -exchanges hiring one market maker, in the spirit of Chapter 4 and [Mastrolia and Ren, 2018].

6.2.1 The market-maker/ N -exchanges problem and optimal contract

We assume that the N exchanges are identical and each one receives $\frac{c}{N}$ per unit of trade. The market maker receives the aggregation of the N salary. The market maker receives the aggregation of the compensation given by the N exchanges denoted by $\bar{\xi} = \xi + \tilde{\xi}$, where ξ and $\tilde{\xi}$ are respectively the remuneration given by a representative exchange and the aggregation of the $N - 1$ others. Hence, $\bar{\xi}$ inherits all the technical assumptions made previously on ξ (for only one exchange), since the problem of the market maker is similar by considering $\bar{\xi}$ for his compensation. Consequently, the market maker's problem returns an optimal spread $\hat{\delta}(\bar{\xi})$ so that Theorem 6.1 holds by considering $\bar{\xi}$. In view of the symmetry assumption made on the exchanges, any exchange aims at solving

$$V_0^E(\tilde{\xi}) = \sup_{\xi \in \Xi} \mathbb{E}^{\hat{\delta}(\xi + \tilde{\xi})} \left[-e^{-\eta(\frac{c}{N}(N_T^a + N_T^b) - \xi)} \right], \quad (6.2.1)$$

where $\tilde{\xi}$ is fixed, and $\eta > 0$ is the common risk aversion parameter of the N exchanges. Without communicating, the problem of the oligopoly of exchange is reduced to the existence of a Nash equilibrium.

Definition 6.1 (Nash equilibrium and symmetric Nash equilibrium). *A N -tuple $(\xi^e)_{1 \leq e \leq N}$ is a Nash equilibrium if for any $e \in \{1, \dots, N\}$ we have*

$$V_0^E(\xi^e) = \mathbb{E}^{\hat{\delta}(\sum_{j=1}^N \xi^j)} \left[-e^{-\eta(\frac{c}{N}(N_T^a + N_T^b) - \xi^e)} \right].$$

A N -tuple of contracts $(\xi^e)_{1 \leq e \leq N}$ is a symmetric Nash equilibrium if $(\xi^e)_{1 \leq e \leq N}$ is a Nash equilibrium such that $\xi^1 = \dots = \xi^N$. We denote by $\mathcal{S}^N := \{\xi^0 : (\xi^0, \dots, \xi^0) \in \Xi^N\}$ the collection of all such symmetric Nash equilibria.

From Theorem 6.1, it follows that any symmetric Nash equilibrium $\xi^0 \in \mathcal{S}^N$ is induced by a pair $(\tilde{y}_0, \tilde{Z}) \in [\hat{Y}_0, +\infty) \times \mathcal{Z}$ such that

$$\begin{aligned} \xi^0 &= \frac{1}{N} Y_T^{\tilde{y}_0, \tilde{Z}} = \frac{\tilde{y}_0}{N} + \int_0^T \frac{1}{N} \tilde{Z}_r d\chi_r + \frac{\gamma \sigma^2}{2N} (\tilde{Z}_r^S + Q_r)^2 dr - \frac{1}{N} \sum_{i=a,b} H^i(\tilde{Z}_r^i, Q_r) dr, \\ &= \frac{\tilde{y}_0}{N} + \int_0^T \zeta_r^0 d\chi_r + \frac{\gamma \sigma^2}{2N} (N \zeta_r^{S,0} + Q_r)^2 dr - \frac{1}{N} \sum_{i=a,b} H^i(N \zeta_r^{i,0}, Q_r) dr, \end{aligned} \quad (6.2.2)$$

with $\zeta^0 = \frac{\tilde{z}}{N}$, and

$$H^i(z, q) = \lambda(\hat{\delta}(z)) \frac{\sigma}{k + \sigma\gamma} \mathbf{1}_{\varepsilon_i q < \bar{Q}}, \quad \text{with } (\varepsilon_b, \varepsilon_a) = (-1, 1).$$

Definition 6.2 (Markovian Nash equilibrium). *A symmetric Nash equilibrium $\xi^0 \in \mathcal{S}^N$ is Markovian if the coefficients ζ^0 appearing in (6.2.2) is given by $\zeta_t^0 = \zeta^0(t, Q_t)$ for some deterministic function ζ^0 .*

Similarly to the one exchange problem studied previously, we introduce the HJB equation

$$v_N|_{t=T} = -1 \text{ and } \partial_t v_N(t, q) - \eta v_N(t, q) \hat{F}(t, q, v_N(t, q), v_N(t, q+1), v_N(t, q-1)) = 0, \quad (6.2.3)$$

with

$$\hat{F}(t, q, y, y^+, y^-) = \sup_{\zeta^S} F^S(t, q, \zeta^S) + \sup_{\zeta} F^0(t, q, y, y^+, \zeta) \mathbf{1}_{q < \bar{q}} + \sup_{\zeta} F^0(t, q, y, y^-, \zeta) \mathbf{1}_{q > -\bar{q}},$$

and

$$\begin{aligned} F^S(t, q, z) &= -\hat{H}^S(q, \zeta^{S,0}(q), z) - \frac{\eta}{2} \sigma^2 |z|^2, \\ F^0(t, q, y, y', z) &= -\lambda(\hat{\delta}(z + (N-1)\tilde{\zeta}^0(y, y'))) \left(\frac{y'}{y} \frac{e^{\eta(z - \frac{c}{N})}}{\eta} - \frac{1}{\eta} - \frac{\sigma}{k + \sigma\gamma} \right) \\ &\quad - \lambda(\hat{\delta}(N\tilde{\zeta}^0(y, y'))) \frac{(N-1)\sigma}{N(k + \sigma\gamma)}, \\ \hat{H}^S(q, \tilde{z}, z) &= \frac{1}{2} \sigma^2 \gamma \left[(z + (N-1)\tilde{z} + q)^2 - \frac{N-1}{N} (N\tilde{z} + q)^2 \right], \\ \zeta^{S,0}(q) &= -\frac{\gamma}{\eta + N\gamma} q, \quad \tilde{\zeta}^0(y, y') = \hat{\zeta}(y, y') + \frac{1-N}{N} c. \end{aligned}$$

Hence, by denoting $\hat{\zeta}^S, \hat{\zeta}^N$ the optimizers of F^S and F^0 respectively, we get

$$\hat{\zeta}^S(q) = \zeta^{S,0}(q), \quad \hat{\zeta}^N(y, y') = \tilde{\zeta}^0(y, y').$$

By setting $u_N = (-v_N)^{-\frac{kN}{\sigma\eta}}$ and using direct substitution, we obtain again a linear ODE that can be explicitly solved. We thus have the following result

Theorem 6.3 (Verification – Optimal contract). *There is a unique symmetric Nash equilibrium $\xi^0 \in \mathcal{S}^N$ defined by*

$$\xi^0 = \frac{\widehat{Y}_0}{N} + \int_0^T \zeta_r^0 d\chi_r + \frac{\gamma\sigma^2}{2N} (N\zeta_r^{S,0} + Q_r)^2 dr - \frac{1}{N} \sum_{i=a,b} H^i(N\zeta_r^{i,0}, Q_r) dr, \quad (6.2.4)$$

where, for $i \in \{a, b\}$ and $(\varepsilon_b, \varepsilon_a) = (-1, 1)$,

$$\zeta_r^{S,0} = -\frac{\gamma}{\eta + N\gamma} Q_r, \quad \zeta_r^{i,0} = \zeta_N + \frac{1}{\eta} \log \left(\frac{v_N(r, Q_r)}{v_N(r, Q_r - \varepsilon_i)} \right),$$

with

$$\zeta_N = \frac{c}{N} + \frac{1}{\eta} \log \left(1 - \frac{\sigma^2 \gamma \eta}{(\sigma\eta + k)(\sigma\gamma + k)} \right).$$

In particular, this unique symmetric Nash equilibrium is Markovian.

6.2.2 Economical insights

Notice that the total compensation $N\xi^0$ obtained by the market maker in the N -symmetric exchanges situation differs from the optimal contract $\hat{\xi}$ of the one-exchange situation in Theorem 6.2. By using the same arguments than those in Section 6.1.3.1 we note that

$$\zeta_t^{i,0} \sim \zeta_N + \varepsilon_i \frac{\sigma}{2kN} \sqrt{\frac{C_N}{C'_N}} (2Q_{t-} - \varepsilon_i), \quad i \in \{a, b\}$$

for $i \in \{a, b\}$. Again, when the inventory is highly positive, the exchanges provide incentives to the market-maker to attract buy market orders and discourage additional sell market orders, and *vice versa* for a negative inventory. As ζ_N and $\frac{C_N}{C'_N}$ are decreasing with respect to N , this effect is reduced when the number of platforms increases. Moreover the optimal spread is not a monotonic function of N for a given c , thus suggesting that an optimal number of exchanges may exist. Finally, note that when N becomes large, $N\zeta_r^{S,0} \sim -Q_r$. In other words, for a large number of platforms, the inventory risk is transferred to the oligopoly of exchanges.

Auction market design and extensions to hybrid markets

We first consider an auction market in which market makers fill the order book during a given time period while some other investors send market orders. We define the clearing price of the auction as the price maximizing the exchanged volume at the clearing time according to the supply and demand of each market participants. Then we derive in a semi-explicit form the error made between this clearing price and the efficient price as a function of the auction duration. We study the impact of the behavior of market takers on this error. We compute the optimal duration of the auctions for 77 stocks traded on Euronext and compare the quality of price formation process under this optimal value to the case of a continuous limit order book. This part is extracted from [Jusselin et al., 2019], see Section 7.1. Then, we model a hybrid market in which the traders (brokers) can trade with a market maker at a fixed price in a CLOB and can trigger a double auction when needed (namely ad hoc auctions market). We describe the optimal strategies and show existence results for Nash equilibria. This part is extracted from [Derchu et al., 2020], see Section 7.2.

7.1 Auction markets and optimal auction durations

7.1.1 The auction mechanism

We consider an auction market organized in independent sequential auctions triggered by market orders. We work here under regenerative properties of the market so that we restrict¹ the study for the sake of simplicity in this report to one auction. After the opening of the market or after the clearing of an auction, a new auction starts when a first market order is sent. We write τ_1^{op} for opening time of the auction and τ_1^{cl} for the clearing time of the auction. An auction has a duration of h seconds and allows market takers to meet. Note that when $h = 0$, our model fits with a CLOB market.

At time τ_1^{op} , market participants can send market or limit orders. At the auction clearing time $\tau_1^{cl} = \tau_1^{op} + h$, a clearing price, denoted by $P_{\tau_1^{cl}}^{cl}$, is set to maximize the exchanged volume. More precisely, sellers who are willing to sell below the price $P_{\tau_1^{cl}}^{cl}$ sell their shares to buyers who are willing to buy above $P_{\tau_1^{cl}}^{cl}$. Each cleared share is sold at the clearing price, independently of the posted price of the associated limit order if any (to the benefit of participants sending limit orders).

7.1.2 Market makers and market takers

Along the day, market makers arrive randomly in the market and send limit orders to fill the LOB. During the market phase market makers arrival times are given by $(\tau_k^{1,mm})_{k \geq 0}$ where $\tau_k^{1,mm}$ is the k -th event time of a counting process $(N_s^{1,mm})_{s \geq 0}$. We describe the liquidity provided by the k -th market maker by its supply function ς_k . This supply function depends on the market maker's view on the efficient price at the clearing time when he sends his limit orders. The efficient price process is $(P_s^*)_{s \geq 0}$ and can be seen as the average of market makers' opinions at a given time on the "fair" value of the

¹In particular, to obtain a regenerative market we consider that after the auction clearing time market makers cancel their unmatched limit orders. See [Jusselin et al., 2019] for more details in the general sequential auctions model.

underlying asset. It satisfies $P_s^* = P_0 + \sigma_f W_s$ with W a Brownian motion and σ_f a positive constant. When positive, the quantity $\varsigma_k(p)$ represents the number of shares the k -th market maker is willing to sell at price p or above. Negative values correspond to shares he is willing to buy at price p or below. We assume that

$$\varsigma_k(p) = K(p - \tilde{P}_k), \text{ with } \tilde{P}_k = \mathbb{E}[P_{\tau_1^{cl}}^* | \mathcal{F}_{\tau_k^{1,mm}}] + g_k,$$

where \tilde{P}_k is the view on the price of the asset by the k -th market maker when he sends his orders and K a positive constant, $(g_k)_{k>0}$ a sequence of i.i.d random variables with variance σ^2 representing the estimation noise in the inference of the efficient price by the market maker independent of all other processes. In this case we get

$$\tilde{P}_k = P_{\tau_k^{1,mm}}^* + g_k.$$

In this model we see that at the clearing there is $N_{\tau_1^{cl}}^{mm}$ market makers in the LOB.

During the market phase the arrival time of the k -th buy (resp. sell) market order is given by $\tau_k^{1,a}$ (resp. $\tau_k^{1,b}$) where $\tau_k^{1,a}$ (resp. $\tau_k^{1,b}$) is the k -th event time of the counting process $(N_s^{1,a})_{s \geq 0}$ (resp. $(N_s^{1,b})_{s \geq 0}$). Consequently the opening time of auction is $\tau^{op} = \tau_1^a \wedge \tau_1^b$. We suppose that each market taker sends market orders with constant volume v . Moreover we assume that $(N^{1,mm}, N^{1,a}, N^{1,b})$ is independent of the efficient price P^* . We define I as the cumulated imbalance of the market takers: $I_t = vN_t^a - vN_t^b$. The aggregated demand of the market takers at the clearing of the auction is thus given by $I_{\tau_1^{cl}}$.

In practice it is very unlikely that a market taker sends a market order if there is no liquidity in the LOB and a situation with empty LOB is very unrealistic. A way to adapt the non empty LOB assumption setting is to consider that the first market maker always arrives before the auction clearing occurs. It means that $\tau_1^{mm} < (\tau_1^a \wedge \tau_1^b) + h$. We refer to Assumption 2 in [Jusselin et al., 2019] for a mathematical characterization of this assumption.

We sum up the auction mechanism in Figure 7.1 for two consecutive auctions that can be reduced under regenerative property to one auction with $i = 1$ and $\tau_0^{cl} = 0$ by convention.

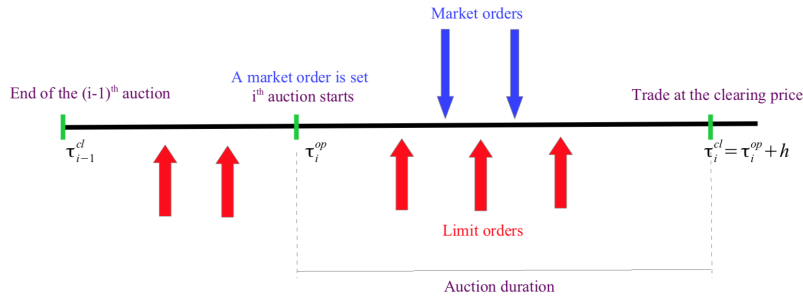


Figure 7.1: Auction mechanism.

7.1.3 Clearing rule and metric for the market quality

We now explain how the clearing price is settled at the end of an auction. At time $\tau_1^{cl} = \tau_1^{op} + h$, a clearing price $P_{\tau_1^{cl}}^{cl}$ is set in order to maximize the exchanged volume. This clearing rule is used in most of electronic markets for the opening and clearing auctions. In order to maximize volume at the clearing time, the optimal clearing price has to satisfied an equilibrium between supply-demand functions of all market participants together (market makers and market takers), corresponding to a zero of the aggregated supply-demand of the agents. Consequently, the clearing price $P_{\tau_1^{cl}}^{cl}$ of the auction can be

found solving the following equation:

$$\sum_{k=1}^{N_{\tau_1^{cl}}^{1,mm}} s_k(p) - I_{\tau_1^{cl}} = 0 \iff P_{\tau_1^{cl}}^{cl} = P_{\tau_1^{cl}}^{mid} + \frac{1}{K} \frac{I_{\tau_1^{cl}}}{N_{\tau_1^{cl}}^{1,mm}}. \quad (7.1.1)$$

where $P_{\tau_1^{cl}}^{mid}$ is the mid-price of the LOB as the obtained clearing price without taking into account market defined by:

$$P_{\tau_1^{cl}}^{mid} = \frac{1}{N_{\tau_1^{cl}}^{1,mm}} \sum_{k=1}^{N_{\tau_1^{cl}}^{1,mm}} \tilde{P}_k. \quad (7.1.2)$$

One of the main roles of a financial market is to reveal with accuracy the price of the underlying asset, guaranteeing fair transaction prices to market participants. In our framework, this is equivalent to have a clearing price close to the efficient price. Therefore a natural criterion to compare different auction durations is to assess, with respect to the auctions duration, the accumulated error between the efficient price and the clearing prices over the day. Thus, for each auction, we consider the quadratic deviation between the clearing price and the efficient price and weight this deviation by the time to wait until a new price is set, that is

$$Z_t^h = \int_0^t (\bar{P}_s^{cl} - \bar{P}_s)^2 ds,$$

where the processes \bar{P}_s^{cl} and \bar{P}_s are respectively the clearing and efficient price at the last auction clearing time before time s . By using regenerative properties, see Lemma 2.1 in [Jusselin et al., 2019], the asymptotic error is thus given by $E(h) := \mathbb{E}[(P_{\tau_1^{cl}}^{cl} - P_{\tau_1^{cl}}^*)^2]$. A duration h^* is optimal if it is a minimizer of the function E . We also consider the efficiency of the mid-price defined in (7.1.2), denoted by E^{mid} and given by $E^{mid}(h) := \mathbb{E}[(P_{\tau_1^{op}+h}^{mid} - P_{\tau_1^{op}+h}^*)^2]$. Therefore, we get the following theorem

Theorem 7.1. *The quality of price formation process metric satisfies:*

$$E(h) = E^{mid}(h) + \frac{\mathbb{E}[I_{\tau_1^{cl}}^2]}{K^2} (1 - e^{-\mu h} \frac{\nu}{\nu + \mu})^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t} e^{-\mu t} \int_0^{\mu t} \frac{1}{s} \int_0^s \frac{e^u - 1}{u} du ds dt,$$

with $E^{mid}(h)$ given by

$$(1 - e^{-\mu h} \frac{\nu}{\nu + \mu})^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t} \left((\sigma_f^2 \frac{t}{6} + \sigma^2) e^{-\mu t} \int_0^{\mu t} \frac{e^s - 1}{s} ds + \sigma_f^2 \frac{t}{3} (1 - e^{-\mu t}) \right) dt.$$

Naive market takers case. The only quantity depending on market takers behavior is $\mathbb{E}[I_{\tau_1^{op}+h}^2]$. Theorem 7.1 enables us to compute the function E and therefore to find the optimal auction duration by minimizing E . We can consider the standard assumption that N^a and N^b are independent Poisson processes with intensity $\nu/2$ along the auction. In this case $\mathbb{E}[I_{\tau_1^{op}+h}^2] = \nu^2(\nu h + 1)$. Therefore the function $h \rightarrow E(h)$ of Theorem 7.1 becomes fully explicit. In the next section, we will assume that market makers are strategical in their intensity arrival.

7.1.4 Strategic market takers

In this section, we consider that market takers aim at minimizing their trading costs by adapting their trading intensities to the market state. We formalize this into a competitive game between buying and selling market takers. We show that this game admits a Nash equilibrium. Moreover, when market takers follow the strategies corresponding to this Nash equilibrium, we compute the key quantity $\mathbb{E}[I_{\tau_1^{op}+h}^2]$ appearing in $E(h)$.

Trading costs of market takers. We model the aggregated group of buying (resp. selling) market takers as a single player called Player a (resp. b). During the auctions, Player a (resp. b) controls the intensity λ^a (resp. λ^b) of the arrival process N^a (resp. N^b), wishing to get minimal costs. To reproduce the fact that market takers intensities can neither be too high nor too low, we assume that λ^a and λ^b are bounded from above and below by two positive constants λ_+ and λ_- .

In the long run, the average trading cost of buying market takers is given² as a multiple of $\mathbb{E}[N_h^a(N_h^a - N_h^b)]$, when $(N_0^a, N_0^b) = (1, 0)$ and when $(N_0^a, N_0^b) = (0, 1)$. Consequently, for any $(\alpha, \beta) \in \mathbb{N}^2$, we consider the more general problem for Player a minimizing $\mathbb{E}[N_h^a(N_h^a - N_h^b)|(N_0^a, N_0^b) = (\alpha, \beta)]$. In the same way and simultaneously, Player b minimizes $\mathbb{E}[N_h^b(N_h^b - N_h^a)|(N_0^a, N_0^b) = (\alpha, \beta)]$. Each player aims at deriving its own trading intensity which will lead to the smallest possible trading costs for him.

Nash equilibrium. We now give our result on the existence of a Nash equilibrium in this game of competing market takers. We consider that market takers control their trading intensities. The set of admissible controls is denoted by \mathcal{U} and defined as the set of \mathbb{F} -predictable processes with values in $[\lambda_-, \lambda_+]$ for fixed $0 < \lambda_- \leq \lambda_+$. Any strategies $(\lambda_a, \lambda_b) \in \mathcal{U}^2$ of Player a and b induces a probability measure $\mathbb{P}^{\lambda_a, \lambda_b}$ such that $N^a - \int_0^\cdot \lambda_s^a ds$ and $N^b - \int_0^\cdot \lambda_s^b ds$ are martingales under $\mathbb{P}^{\lambda_a, \lambda_b}$. In order to minimize its costs, Player a solves

$$\inf_{\lambda^a \in \mathcal{U}} V_h^{a, \alpha, \beta}(\lambda_a, \lambda_b), \quad (7.1.3)$$

with $V_h^{a, \alpha, \beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a, \lambda_b}}[N_h^a(N_h^a - N_h^b)|(N_0^a, N_0^b) = (\alpha, \beta)]$, for fixed λ_b chosen by the selling market taker, Player b . Symmetrically, Player b solves

$$\inf_{\lambda^b \in \mathcal{U}} V_h^{b, \alpha, \beta}(\lambda_a, \lambda_b), \quad (7.1.4)$$

with $V_h^{b, \alpha, \beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a, \lambda_b}}[N_h^b(N_h^b - N_h^a)|(N_0^a, N_0^b) = (\alpha, \beta)]$ for fixed λ_a chosen by the buying market taker, Player a . A Nash equilibrium is obtained if the two optimization problems (7.1.3) and (7.1.4) can be addressed simultaneously.

Note that this framework is realistic regarding the information observable by market takers. Indeed we only assume that market takers observe market orders imbalance. This information is for example available on the Euronext platform for the opening and closing auctions and on the auctions market of BATS-Cboe.

We prove that this game indeed admits a (non-necessarily unique) Nash equilibrium with corresponding optimal controls $(\lambda_a^*, \lambda_b^*)$. More precisely using these notations we have the following result.

Theorem 7.2. *There exists a Nash equilibrium to the simultaneous optimization problem (7.1.3)-(7.1.4) given by some Markovian controls³ $(\lambda_a^*, \lambda_b^*)$ satisfying*

$$\inf_{\lambda^a \in \mathcal{U}} V_h^{a, \alpha, \beta}(\lambda_a, \lambda_b^*) = \mathbb{E}^{\mathbb{P}^{\lambda_a^*, \lambda_b^*}}[N_h^a(N_h^a - N_h^b)|(N_0^a, N_0^b) = (\alpha, \beta)]$$

and

$$\inf_{\lambda^b \in \mathcal{U}} V_h^{b, \alpha, \beta}(\lambda_a^*, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a^*, \lambda_b^*}}[N_h^b(N_h^b - N_h^a)|(N_0^a, N_0^b) = (\alpha, \beta)].$$

Under the Nash equilibrium $(\lambda_a^*, \lambda_b^*)$, we have $\mathbb{E}[I_{\tau_1^{op} + h}^2] = V_h^{a, 1, 0}(\lambda_a^*, \lambda_b^*) + V_h^{b, 1, 0}(\lambda_a^*, \lambda_b^*)$.

The proof of Theorem 7.2 also provides a numerical method to approximate $V_h^{a, 1, 0}(\lambda_a^*, \lambda_b^*)$ and $V_h^{b, 1, 0}(\lambda_a^*, \lambda_b^*)$ using solutions of some integro-differential equations. It is particularly important since it enables us to compute optimal auction durations when market takers are playing the Nash equilibrium. This is because the function E of Theorem 7.1 explicitly depends on $V_h^{a, 1, 0}(\lambda_a^*, \lambda_b^*)$ and $V_h^{b, 1, 0}(\lambda_a^*, \lambda_b^*)$.

²See [Jusselin et al., 2019, Section 3.1]

³The notion of Markovian control has to be understood in the sense of [Carmona and Delarue, 2018, Definition 2.10]

	DurationPoisson	DiffrelPoisson	DurationNash	DiffrelNash
Bouygues	228 [226;230]	1%	152 [150;153]	20%
Alstom	0 [0;0]	0%	180 [178;181]	14%
Air France - KLM	295 [292;297]	17%	218 [216;220]	35%
Veolia Environ.	350 [346;353]	3%	253 [251;256]	21%
Schneider Electric	246 [245;248]	39%	171 [170;172]	65%
Peugeot	386 [383;389]	10%	282 [280;285]	29%
Vinci	350 [348;353]	39%	252 [250;253]	64%
Suez Environnement	0 [0;0]	0%	311 [308;315]	14%
Danone	204 [203;206]	15%	146 [145;147]	35%

Table 7.1: Optimal auction durations (in seconds) with a 90% confidence interval w.r.t. v .

7.1.5 Optimal auction duration

We have access to intra-day market data for 77 of the most liquid stocks traded on Euronext exchange, for all trading days of September 2018. For each stock, every trade is reported with the following information: Timestamp of the trade; traded volume; execution price; best bid and ask prices just before the transaction; volumes at best bid and best ask just before the transaction. We can estimate ν by the average number of market orders per day divided by the duration of a trading day; v by the average volume of a market order; μ and K , see [Jusselin et al., 2019, Section 4.2.2]; the volatility σ_f of the efficient price from the five minutes sampling based realized volatility of the traded price and σ as the implicit spread of the asset. Finally we set $\lambda^+ = 2\nu$ and $\lambda^- = \nu/4$.

We provide in Table 7.1 the optimal auction durations for some stocks traded on Euronext. The first column is the stock name. In the Poisson (resp. Nash) case, the second (resp. fourth) column is the optimal duration in seconds. The third (resp. fourth) one is the relative difference of quality of the price formation process between the optimal duration case and the CLOB situation: $(E(0) - E(h^*))/E(h^*)$. In the optimal durations columns we provide estimated optimal durations together with 90% confidence interval. Note that CLOBs are sometimes optimal. When they are not, the difference in the values of the metric for $h = 0$ and $h = h^*$ is typically not very large. On BATS-Cboe the auction duration is approximately 100ms which is very small compared to the typical optimal auction durations we find. We also see that the more the market takers are strategic, the more the optimal duration reduces (except when $h^* = 0$).

Policy implications and economical insights. The main take away of our analysis is that *one size does not fit all*: first in the spirit of [Budish et al., 2015] we confirm that the nowadays almost universal CLOB mechanism (for liquid assets) may be suboptimal. We indeed show that auctions are quite often preferable in term of market quality but CLOB can be optimal sometimes. Second, the auction duration has to depend on some fundamental parameters of the considered asset (such as liquidity and volatility). Our work also underlines the crucial need of thorough quantitative analysis as a preliminary task to market structure modification. Here such an analysis enables us to compute optimal frequencies for auctions. We find that they are diverse but that reasonable order of magnitude is of a few minutes, which is probably fast enough for large investors. This is in contrast with the intuitive idea that in fast electronic markets auction frequency should be necessarily very high. Such a result is due to the fact that our criterion aims at finding the best price discovery mechanism and therefore we somehow take the investor point of view.

7.2 AHEAD: *Ad Hoc* Electronic Auction Design

On the one hand, in frequent batch auction the market participants have to wait for the end of the auction to actually trade the asset and so liquidate their volumes. On the other hand, and beyond the possible arbitrage opportunities discussed in the introduction of the previous section, CLOBs cannot allow market taker to modify the configuration of the market if the price is really outside their preferences and may lead to huge trading costs. In this section, we study a hybrid market between the CLOB and the frequent auctions market which we call a market with *ad hoc* auctions. The mechanism of such market is the following: at the beginning of the trading day, a price denoted by P is fixed. Market makers continuously quote limit orders but those limit orders cannot be crossed with one another. Trades can only be triggered by sending a special type of order, which we call a market order by analogy with market orders in the CLOB. However, contrary to the CLOB, the only trade price possible is P , and, as a consequence, a market order triggers a trade only if there is a waiting limit order which can be crossed with it. Also, when the market takers are no longer content with the fixed price P , they can send a signal which triggers an auction of fixed duration h . The auction unfolds as a classical double auction. All the limit orders sent by the market-makers and the market orders sent by the market makers are taken into account in order to determine an auction clearing price. Once the auction is over, a new price is fixed and a new market phase begins.

The *ad hoc* auction is thus decomposed in two phases: the first one is a continuous phase during which market takers trade at a fixed price P , the second consecutive one is the auction triggered by a market taker similar to the previous section in which a clearing price has to be set in order to ensure large volume traded on the exchange. Let $\mathcal{T}_{0,T}$ the set of stopping time in $[0, T]$. We denote by τ^a and τ^b in $\mathcal{T}_{0,T}$ the stopping times chosen Player a and Player b respectively to trigger the auction. An auction starts at time $\tau = \tau^a \wedge \tau^b \wedge T$, considering that if the players do not trigger one themselves before time T , an auction is automatically triggered at time T . As T stands for the duration of a trading day, this situation is similar to real-world CLOB markets, which usually have a closing auction.

7.2.1 The continuous trading phase

As in the previous section, we assume that there is an efficient price for the asset $P_t^* := P_0 + \sigma W_t$. During the continuous auction phase, the market makers accept all the orders from Player a (buy orders) if $P > P_t^*$. In this case, a quantity of 1 is traded at price P . Symmetrically, the market makers accept all the orders from Player b (sell orders) if $P < P_t^*$. Hence, a quantity of 1 is traded at price P . We introduce the processes N^a and N^b describing the number of orders sent by Player a and Player b which are not rejected by the market makers and defined by $N_t^a = \int_0^t (\mathbf{1}_{s \leq \tau} \mathbf{1}_{P > P_s^*} + \mathbf{1}_{s > \tau}) d\tilde{N}_s^a$ and $N_t^b = \int_0^t (\mathbf{1}_{s \leq \tau} \mathbf{1}_{P < P_s^*} + \mathbf{1}_{s > \tau}) d\tilde{N}_s^b$, where \tilde{N}^a, \tilde{N}^b are counting process with intensity λ^a and λ^b controlled by Player a and Player b respectively in a set of admissible control \mathcal{U} . We assume that both players a and b are penalized during continuous market phase if they do not trade the right volumes. We model this penalization by paying a fix positive constant q proportionally to the square of the quadratic error between the optimal volume and the effective number of trades along the continuous trading period. In other words, during the continuous market phase the costs of players a and b are respectively given for any $t \in [0, T + h]$ by

$$\begin{aligned} L_t^a &= q \int_0^{t \wedge \tau} (v^a s - N_s^a)^2 ds + \int_0^{t \wedge \tau} (P - P_t^*) \mathbf{1}_{P > P_t^*} dN_t^a \\ L_t^b &= q \int_0^{t \wedge \tau} (v^b s - N_s^b)^2 ds - \int_0^{t \wedge \tau} (P - P_t^*) \mathbf{1}_{P < P_t^*} dN_t^b. \end{aligned} \tag{7.2.1}$$

The first term of the following equations represents the penalty if the number of trades does not reach

the fixed amounts and the second term is the cost of the market takers resulting from their trading activities compared to the benchmark price P^* .

7.2.2 The auction

We now introduce a mechanism which forces a market taker to trade a minimum amount in an auction he initiated. Indeed, we would like to have only meaningful auctions, that is to say auctions should take place when the price P is no longer satisfactory. Requiring a minimum traded volume tends to make the auction price go against the market participant who has triggered the auction. Thus, a trader triggers an auction when he really needs to. This can also be seen as a constraint or a cost associated with triggering an auction. This mechanism is modelled by a fixed number of orders automatically records by the exchange for a player triggering an auction. Let $\hat{n} \in \mathbb{N}$ be the number of orders automatically recorded by the exchange for a player when he triggers an auction alone before T , and let $\hat{n}_{ab} \in \mathbb{N}$ be the number of orders automatically recorded by the market for both players when they trigger an auction simultaneously before T . We define two \mathcal{F}_τ -measurable random variables N_+^a , N_+^b , representing the number of orders automatically recorded by the market for Player a and Player b when the auction starts, by

$$N_+^a = \hat{n}\mathbf{1}_{\tau^a < \tau^b, \tau^a < T} + \hat{n}_{ab}\mathbf{1}_{\tau^a = \tau^b < T} \quad \text{and} \quad N_+^b = \hat{n}\mathbf{1}_{\tau^b < \tau^a, \tau^b < T} + \hat{n}_{ab}\mathbf{1}_{\tau^a = \tau^b < T}.$$

Similarly to [Jusselin et al., 2019], see Section 7.1, during the auction, the market makers are willing to buy or sell at a certain price. We model it by saying that they quote a mid-price $\tilde{P}_{\tau+h}$ and a slope $K \in \mathbb{R}$, meaning that they quote a volume $K(p - \tilde{P}_{\tau+h})$ at time $\tau + h$ when the auction price is $p \in \mathbb{R}$. Recall that the auction price P^{auc} fixed at the clearing time $\tau + h$ of the auction will be solution to the equation which equals supply and demand so that

$$P^{auc} = \tilde{P}_{\tau+h} + \frac{(N_{\tau+h}^a - N_\tau^a + N_+^a) - (N_{\tau+h}^b - N_\tau^b + N_+^b)}{K} \quad (7.2.2)$$

In other words, at the end of the auction, Player a buys $N_{\tau+h}^a - N_\tau^a + N_+^a$ units at a price P^{auc} and Player b sells $N_{\tau+h}^b - N_\tau^b + N_+^b$ units at a price P^{auc} . We set $\tilde{P}_{\tau+h} = P_{\tau+h}^*$. According to the above remark it is true up to a change of K .

We now turn to the costs paid by Player a and b during the auction. We again assume that both players a and b are penalized during the auction if they do not trade at the rate v^a and v^b respectively. The penalization is again assumed to be quadratic with parameter $q > 0$. Hence, the penalizations of Player a and Player b during the auction are respectively given by

$$\mathcal{C}_{auc}^a = qh(v^a(\tau + h) - N_{\tau+h}^a - N_+^a)^2 \quad \text{and} \quad \mathcal{C}_{auc}^b = qh(v^b(\tau + h) - N_{\tau+h}^b - N_+^b)^2.$$

As in [Jusselin et al., 2019], the cost resulting from trading activities of market taker a is given by $N_{\tau,\tau+h}^a(P^{auc} - P_{\tau+h}^*)$ while the gain of b resulting from his trades is $N_{\tau,\tau+h}^b(P^{auc} - P_{\tau+h}^*)$, where $N_{\tau,\tau+h}^a = N_{\tau+h}^a - N_\tau^a + N_+^a$ and $N_{\tau,\tau+h}^b = N_{\tau+h}^b - N_\tau^b + N_+^b$ denote respectively the number of ask and bid orders sent during the auction.

7.2.3 Nash equilibrium

Since P^{auc} is given by (7.2.2) the total cost of Players a and the gain of Player b are respectively given by

$$\frac{L_\tau^a + \mathcal{C}_{auc}^a + \frac{N_{\tau,\tau+h}^a(N_{\tau,\tau+h}^a - N_{\tau,\tau+h}^b)}{K}}{\tau + h} \quad \text{and} \quad \frac{-L_\tau^b - \mathcal{C}_{auc}^b + \frac{N_{\tau,\tau+h}^b(N_{\tau,\tau+h}^a - N_{\tau,\tau+h}^b)}{K}}{\tau + h}.$$

For $x_0 \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N} \times \mathbb{R} \times \mathbb{R}$ and a pair of controls $((\tau^a, \lambda^a), (\tau^b, \lambda^b))$, let

$$J^a(x, (\tau^a, \lambda^a), (\tau^b, \lambda^b)) = \mathbb{E}^{\lambda^a, \lambda^b} \left[\frac{L_\tau^a + C_{auc}^a + \frac{N_{\tau, \tau+h}^a (N_{\tau, \tau+h}^a - N_{\tau, \tau+h}^b)}{K}}{\tau + h} \mid (P_0^*, N_0^a, N_0^b, L_0^a, L_0^b) = x_0 \right] \quad (7.2.3)$$

and

$$J^b(x, (\tau^a, \lambda^a), (\tau^b, \lambda^b)) = \mathbb{E}^{\lambda^a, \lambda^b} \left[\frac{-L_\tau^b - C_{auc}^b + \frac{N_{\tau, \tau+h}^b (N_{\tau, \tau+h}^a - N_{\tau, \tau+h}^b)}{K}}{\tau + h} \mid (P_0^*, N_0^a, N_0^b, L_0^a, L_0^b) = x_0 \right], \quad (7.2.4)$$

with $\tau = \tau^a \wedge \tau^b \wedge T$. Since Player a aims at minimizing his cost along the day and Player b aims at maximizing his gain, we define the notion of open-loop Nash equilibrium by

Definition 7.1 (Open-Loop Nash Equilibrium (OLNE)). *Given $x_0 \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N} \times \mathbb{R} \times \mathbb{R}$, we say that the pair of controls $((\tau^{a,*}, \lambda^{a,*}), (\tau^{b,*}, \lambda^{b,*}))$ is an OLNE of the game if*

$$\begin{cases} J^a(x_0, (\tau^{a,*}, \lambda^{a,*}), (\tau^{b,*}, \lambda^{b,*})) \leq J^a(x_0, (\tau^a, \lambda^a), (\tau^{b,*}, \lambda^{b,*})) & \forall (\tau^a, \lambda^a) \in \mathcal{T}_{0,T} \times \mathcal{U} \\ J^b(x_0, (\tau^{a,*}, \lambda^{a,*}), (\tau^{b,*}, \lambda^{b,*})) \geq J^b(x_0, (\tau^{a,*}, \lambda^{a,*}), (\tau^b, \lambda^b)) & \forall (\tau^b, \lambda^b) \in \mathcal{T}_{0,T} \times \mathcal{U} \end{cases}$$

In general, the existence of an OLNE is reduced to find a solution to a fully coupled HJB equation. Here the situation is a little bit more intricate than Section 7.1. We thus relax a little bit the study and we look for an OLNE in the case where the stopping times can only take discrete values. Set $\delta \in \mathbb{R}^+$ such that $\frac{T}{\delta} \in \mathbb{N}$. For simplicity, we consider $\mathcal{T}_{k\delta, T}^d$, for $k = 0, \dots, \frac{T}{\delta}$ the set of stopping times which values $\{k\delta, (k+1)\delta, \dots, T\}$ -valued.

Definition 7.2 (Open-Loop Nash Equilibrium for the discrete game (OLNED)). *Let $k \in \{0, \dots, T/\delta\}$, $x_0 \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N} \times \mathbb{R} \times \mathbb{R}$. We say that $((\tau^{a,*}, \lambda^{a,*}), (\tau^{b,*}, \lambda^{b,*})) \in (\mathcal{T}_{k\delta, T}^d \times \mathcal{U})^2$ is an open-loop Nash equilibrium of the discretized game (OLNED for short) starting at time $k\delta$ if it is a solution to the game*

$$\begin{cases} \mathbb{E}_{k\delta}^{\lambda^{a,*}, \lambda^{b,*}} \left[\frac{L_{\tilde{\tau}^a}^a + C_{auc}^a + \frac{N_{\tilde{\tau}^a, \tilde{\tau}^a+h}^a (N_{\tilde{\tau}^a, \tilde{\tau}^a+h}^a - N_{\tilde{\tau}^a, \tilde{\tau}^a+h}^b)}{K}}{\tilde{\tau}^a + h} \right] = \operatorname{ess\,inf}_{\substack{\tau^a \in \mathcal{T}_{k\delta, T}^d, \\ \lambda^a \in \mathcal{U}}} \mathbb{E}_{k\delta}^{\lambda^a, \lambda^{b,*}} \left[\frac{L_{\tilde{\tau}^a}^a + C_{auc}^a + \frac{N_{\tilde{\tau}^a, \tilde{\tau}^a+h}^a (N_{\tilde{\tau}^a, \tilde{\tau}^a+h}^a - N_{\tilde{\tau}^a, \tilde{\tau}^a+h}^b)}{K}}{\tilde{\tau}^a + h} \right] \\ \mathbb{E}_{k\delta}^{\lambda^{a,*}, \lambda^{b,*}} \left[\frac{-L_{\tilde{\tau}^b}^b - C_{auc}^b + \frac{N_{\tilde{\tau}^b, \tilde{\tau}^b+h}^b (N_{\tilde{\tau}^b, \tilde{\tau}^b+h}^a - N_{\tilde{\tau}^b, \tilde{\tau}^b+h}^b)}{K}}{\tilde{\tau}^b + h} \right] = \operatorname{ess\,sup}_{\substack{\tau^b \in \mathcal{T}_{k\delta, T}^d, \\ \lambda^b \in \mathcal{U}}} \mathbb{E}_{k\delta}^{\lambda^{a,*}, \lambda^b} \left[\frac{-L_{\tilde{\tau}^b}^b - C_{auc}^b + \frac{N_{\tilde{\tau}^b, \tilde{\tau}^b+h}^b (N_{\tilde{\tau}^b, \tilde{\tau}^b+h}^a - N_{\tilde{\tau}^b, \tilde{\tau}^b+h}^b)}{K}}{\tilde{\tau}^b + h} \right] \end{cases}$$

a.s., with $\tilde{\tau}^a = \tau^a \wedge \tau^{b,*} \wedge T$ and $\tilde{\tau}^b = \tau^{a,*} \wedge \tau^b \wedge T$.

7.2.4 Main theoretical results and economical insights

We are able to prove that there exists an OLNED as soon as there is no cost for triggering an auction, i.e. $\hat{n} = \hat{n}_{ab} = 0$. The proof is based on a backward induction to compute at each step the optimizers of the game. In the general case, by looking for generalized stopping times instead of stopping times, we prove that there exists a mixed open-loop Nash equilibrium for the discrete game considered. Finally, by slightly modifying the definitions of C_{auc}^a and C_{auc}^b replacing⁴ $v^a \tau$ and $v^b \tau$ by the nearest integer, we get the following theorem making the link with the continuous-time setting

Theorem 7.3 (OLNED and ε -OLNE). *Let $((\tau^a, \lambda^a), (\tau^b, \lambda^b)) \in (\mathcal{T}_{0,T}^d \times \mathcal{U}) \times (\mathcal{T}_{0,T}^d \times \mathcal{U})$ be the strategies associated to an OLNED starting at 0. Let $\varepsilon > 0$. Then, for δ small enough:*

$$\begin{aligned} \mathbb{E}^{\lambda^a, \lambda^b} \left[\frac{L_{\tau^a \wedge \tau^b}^a + \xi_{\tau^a \wedge \tau^b}^a}{\tau^a \wedge \tau^b + h} \right] &\leq \inf_{\tau \in \mathcal{T}_{0,T}, \lambda \in \mathcal{U}} \mathbb{E}^{\lambda^a, \lambda^b} \left[\frac{L_{\tau \wedge \tau^b}^a + \xi_{\tau \wedge \tau^b}^a}{\tau \wedge \tau^b + h} \right] + \varepsilon \\ \mathbb{E}^{\lambda^a, \lambda^b} \left[\frac{-L_{\tau^a \wedge \tau^b}^b + \xi_{\tau^a \wedge \tau^b}^b}{\tau^a \wedge \tau^b + h} \right] &\geq \sup_{\tau \in \mathcal{T}_{0,T}, \lambda \in \mathcal{U}} \mathbb{E}^{\lambda^a, \lambda^b} \left[\frac{-L_{\tau^a \wedge \tau}^b + \xi_{\tau^a \wedge \tau}^b}{\tau^a \wedge \tau + h} \right] - \varepsilon \end{aligned}$$

⁴We have checked that all the previous results could be proved in this slightly modified setting.

Numerical analysis: CLOB vs. sequential auctions vs. *ad hoc* auctions. The average duration for the continuous phase is given by $E_0 = \mathbb{E}^{\lambda^{a,*}, \lambda^{b,*}}[\tau]$, where $(\lambda^{a,*}, \lambda^{b,*})$ and τ are Nash equilibrium computed numerically. We take v^a and v^b of order 0.1, so we expect roughly 2 trades every 10 seconds, which corresponds to the case of reasonably liquid assets. We fix $T = 100s$. We take $q = 0.01, h = 20$ and $\hat{n} = n_{ab} = 3$. We sum up in Table 7.2 the value functions and average durations for *ad hoc* auctions, expensive periodic auctions ($\hat{n} = 3$ and no trading allowed in the continuous phase), periodic auctions similar to Section 7.1 ($\hat{n} = 1$ and no trading allowed in the continuous phase) and CLOBs.

Market design	V^a (1e-6)				Average duration			
	$h = 20, \hat{n} = 3$		$h = 20, \hat{n} = 1$	CLOB	$h = 20, \hat{n} = 3$		$h = 20, \hat{n} = 1$	CLOB
continuous trading is allowed	Yes	No	No	No	Yes	No	No	No
$v^a = 0.1, v^b = 0.1$	3685.6	384.6	0.0	10000.0	21.3s	6.0s	0.0s	10.0s
$v^a = 0.05, v^b = 0.1$	392.8	-6666.7	-5000.0	5000.0	33.3s	10.0s	0.0s	20.0s
$v^a = 0.1, v^b = 0.05$	7800.0	11606.7	10000.0	10000.0	33.3s	10.0s	0.0s	10.0s
$v^a = 0.15, v^b = 0.1$	9397.9	8680.0	10000.0	15000.0	9.0s	5.0s	0.0s	6.7s
$v^a = 0.1, v^b = 0.15$	-2841.8	0.0	0.0	10000.0	9.0s	5.0s	0.0s	10s

Table 7.2: V^a and average duration of the continuous trading phase with $q = 0.01$.

If $v^a = v^b = 0.1$, we numerically observe that Player a has much more to lose when Player b can trade with the market maker than to earn when he can trade with the market maker. Hence, the case $\hat{n} = 3$ provides an intermediary between periodic auctions and the CLOB in terms of value functions.

If v^a and v^b are small and asymmetric (either $v^a = 0.05$ and $v^b = 0.1$ or $v^a = 0.1$ and $v^b = 0.05$), we observe that the player with the larger target benefits from the *ad hoc* auctions. We explain it as follows: if the larger player can trade with the market maker, he is able to liquidate his temporary surplus at a low cost with the market maker and he suffers less from price impact in the auction, which is more balanced than in the case without continuous trading. In this case, it is too costly for the smaller player to trigger an auction, since \hat{n} is too high compared to the target 0.05. The larger player is thus the first to trigger the auction if the price becomes too unfavourable. Otherwise, if the smaller player can trade with the market maker during the continuous trading phase, the larger player triggers the auction to prevent him from a too unfavourable price at the auction. In both case, the larger player triggers first the auction and benefit from the continuous trading phase.

We now turn to $v^a = 0.1$ and $v^b = 0.15$. As before, if Player a , the smaller player, can trade with the market maker, he liquidates a part of his volume but Player b , the larger player, triggers an auction to avoid Player a to trade with the market maker. The larger player triggers the auction quickly since $\hat{n} = 3$ coincides with his target. If now Player b can trade with the market maker, unlike the previous case, the auction triggering cost is reasonable for Player a . The continuous phase appears has an opportunity for Player a to arbitrate Player b . Reversely, when $v^a = 0.15$ and $v^b = 0.1$, we see that the value functions of Player a are similar by considering *ad hoc* auction or classical periodic auctions. For large investors, *ad hoc* auctions benefits from the smaller trader compared to periodic auctions and CLOB without impacting significantly the value of the larger player.

Policy implications. For large investors, *ad hoc* auctions are always a suitable design when the other investors are smaller. Moreover and compared to sequential auctions, the volume made with the market maker allows them to reduce their volume imbalance with respect to the smaller investors in the auctions phases. For small investors, strategic consideration plays an important role. Essentially, if a small investor is still large enough to trigger auctions without too much cost, the *ad hoc* mechanism will be beneficial for him. Otherwise, sequential auctions are more interesting from their viewpoint.

Perspectives

A first extension to [El Euch et al., 2018] that I am currently studying with Nizar Touzi is to consider an oligopoly of exchanges monitoring jointly a common market maker. Remember that if all the exchanges are symmetrical, we are able to solve explicitly the problem again, see Section 6.2. It is however not clear whether the competition of exchanges reduces the spread or not, suggesting an optimal number of platforms on the market. The case of fully different exchanges is still an open question and it echoes Section 5 above and [Mastrolia and Ren, 2018]. An other interesting investigation is to note that in practice it is more likely that the trader will stick to one exchange rather than splitting the order equally among many, which is what we assume in the paper. Even in this very simple case we have seen that the investigation requires very involved arguments. Note also that in practice in order to mitigate market impact, market takers may split their trades between exchanges although prices are similar across venues. This implies to randomize the payoff distribution of the exchanges and is left for future researches.

I have also extended [El Euch et al., 2018] to a model with a market maker monitored by one exchange and having the possibility to trade on a lit or a dark pool in [Baldacci et al., 2019]. Dark pools are Over-The-Counter markets allowing anonymous trades. This is particularly interesting to liquidate a high volume of orders. An interesting extension in the spirit again of [Mastrolia and Ren, 2018] could be to consider several type of platforms available for the market maker and to study the impact of the competition in terms of liquidity.

We now turn to Section 7.1. The main technical assumption is that the market is regenerative so that each auctions are somehow independent of the previous one. Deleting this regenerative property seems to be a big challenge but it will may emphasize some price manipulation from market makers, that we assumed to be non-strategic in [Jusselin et al., 2019] for instance. Also, in this model on the one hand, the participant cannot delete their orders¹ and on the other hand at the clearing time the auction is cleaned the that the no-matching order are cancelled. It will be interesting to relax these assumption to emphasize some flaws in auction markets.

Additionally, note that in the CLOB case, usual parameters that the exchange can adjust to improve market microstructure are tick sizes and make-take fees. Given the order of magnitude of the auctions durations we find, the tick size will probably play a minor role in the market dynamics. However, the effects of a make-take fees schedule are yet to be investigated, as well as the impact of auctions in a situation with multiple competing exchanges, specially in the spirit of Chapter 6.

Finally, in [Derchu et al., 2020] randomizing the target volumes and making them unobservable would be more realistic and it would probably lead to more trading in the continuous phase, although it also leads to considerable technical difficulties.

¹In [Jusselin et al., 2019, Appendix D] we give an extension in which market makers cancel their orders with a fixed and exogenous probability. A natural extension may be to consider market makers controlling the cancellation of their orders.

Part III

Population Dynamics, Monitoring and Control

Regulation of renewal natural resource exploitation

We investigate the impact of a monitoring policy imposed on an agent exploiting a possibly renewable natural resource. We adopt a principal-agent model in which the Principal looks for a contract, i.e. taxes/compensations, leading the Agent to a certain level of exploitation. For a given contract, we first describe the Agent's optimal harvest using the BSDE theory. Under regularity and boundedness assumptions on the coefficients, we express almost optimal contracts as solutions to HJB equations. We then extend the result to coefficients with less regularity and logistic dynamics for the natural resource. We end by numerical examples to illustrate the impact of the regulation in our model. This chapter sums up [Kharroubi et al., 2019].

9.1 The natural resource and the monitoring problem

We fix a deterministic time horizon $T > 0$ and we suppose that the natural resource abundance X_t^μ at time $t \geq 0$ is given by¹

$$X_t^\mu = X_0 + \int_0^t X_s^\mu (\lambda - \mu(X_s^\mu)) ds + \int_0^t \sigma X_s^\mu dW_s, \quad t \in [0, T], \quad (9.1.1)$$

where X_0 , λ and σ are positive constants. The quantities X_0 and λ correspond to the initial natural resource abundance and the growth rate respectively. The map μ represents the competition inside the species considered or more generally an auto-degradation parameter for a natural resource. We assume that the map μ from \mathbb{R}_+ to \mathbb{R}_+ is such that (9.1.1) admits a unique strong solution in $\mathbb{S}^2(\mathbb{R})$, for instance if the map $x \mapsto x\mu(x)$ is Lipschitz continuous or when $\mu(x) = x$ on \mathbb{R}_+ coinciding with the so-called logistic equation. The ecological interpretation of this model is the following. At time t , if the coefficient $\mu(X_t^\mu)$ is larger than λ then the drift of the diffusion is negative. Therefore the abundance of the natural resource X_t^μ decreases in mean. Conversely, if $\mu(X_t^\mu)$ is smaller than λ then the drift of the diffusion is positive. Hence, the abundance X_t^μ increases in mean. For more details we refer to [Méléard and Bansaye, 2015].

We consider an agent who tries to make profit from the natural resource seen, named natural resource manager. We suppose that this agent owns facilities to either harvest or renew this resource. We assume that his action happens continuously in time and we denote by α_t his intervention rate at time t , i.e. the abundance X_t^μ will decrease of an amount $\alpha_t X_t^\mu$ per unit of time. This means that if the intervention rate α_t is positive (resp. negative), the Agent harvests (resp. renews) the natural resource. We denote by \mathcal{A} the set of \mathbb{F} -adapted processes defined on $[0, T]$ and valued in $[-\underline{M}, \overline{M}]$ where \underline{M} and \overline{M} are two nonnegative constants. If the Agent is prohibited to renew the resource then $\underline{M} = 0$. This set \mathcal{A} is called the set of admissible actions. Hence, for $\alpha \in \mathcal{A}$, the dynamics of X can be rewritten under the probability \mathbb{P}^α induced by α and equivalent to \mathbb{P} as

$$X_t^\mu = x + \int_0^t (X_s^\mu (\lambda - \mu(X_s^\mu)) - \alpha_s X_s^\mu) ds + \int_0^t \sigma X_s^\mu dW_s^\alpha, \quad t \in [0, T],$$

¹We also refer to [Saphores, 2003, Evans et al., 2015, Kharroubi et al., 2019] for natural resource dynamics given by as solutions of the logistic stochastic differential equation applied to stochastic control problems.

where $W^\alpha := W + \int_0^\cdot \alpha_s/\sigma ds$ is a \mathbb{P}^α -Brownian motion, by using Girsanov Theorem. This new dynamics reflects the evolution of the population with a rate α_t per unit of time. Hence, $\alpha_t X_t^\mu$ has to be seen as the speed of the exploitation of the natural resource at time t .

Let $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a price function per unit of the natural resource on the market and given by $p(X_t^\mu)$ at time $t \geq 0$ with² $p(x) = Px^{-1}$, $x > 0$. This price function p allows to take into account the dependence w.r.t. the abundance (the more abundant the resource is, the cheaper it will be and conversely). It reflects the inability to buy the natural resource once it is extinct. We assume that the manager sells the harvested resource on the market at price $p(X_t^\mu)$ per unit at time t if α_t is positive, and pays the price $p(X_t^\mu)$ per unit of natural resource at time t if α_t is negative to renew this one. This provides the global amount $\int_0^T p(X_t^\mu) X_t^\mu \alpha_t dt$ over the time horizon $[0, T]$.

We also suppose that giving an effort is costly for the manager and we consider the classical quadratic cost function $k : \mathbb{R} \rightarrow \mathbb{R}_+$ given by $k(\alpha) = \frac{|\alpha|^2}{2}$, $\alpha \in \mathbb{R}$. Thus, the Agent is penalized by the instantaneous amount $k(\alpha_t)$ per unit of time for a given effort $\alpha \in \mathcal{A}$. This leads to the global payment $\int_0^T k(\alpha_t) dt$ over the considered time horizon $[0, T]$.

In our investigation, we recall that the activity of the natural resource manager is regulated by an institution (usually an environment administration) who is taking care about the size of the remaining natural resource. To avoid an over-exploitation, the regulator imposes a tax on the Agent depending on the remaining resource. This tax amount is represented by an \mathcal{F}_T -measurable random variable $\xi \in \mathcal{C}$ and is paid at time T . Note that ξ can be either positive or negative. In this last case, it means that the regulator gives a compensation to the manager.

Throughout the chapter we assume that the Agent's preferences are given by the exponential utility function u_A defined by $u_A(x) := -\exp(-\gamma x)$, $x \in \mathbb{R}$, where γ is a positive constant corresponding to the risk aversion of the Agent. We define the value function $V_A(\xi)$ of the Agent associated to the taxation policy ξ by

$$V_A(\xi) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[-\exp \left(-\gamma \left(\int_0^T p(X_s^\mu) X_s^\mu \alpha_s ds - \int_0^T \frac{|\alpha_s|^2}{2} ds - \xi \right) \right) \right]. \quad (9.1.2)$$

For a fixed tax ξ , we denote by $\mathcal{A}^*(\xi)$ the set of optimal efforts for the fixed tax ξ in (9.1.2). The aim of the regulator is to stabilize the resource population at a fixed target size at the maturity T . For that, a tax ξ is chosen to incentivize the Agent to manage the natural resource so that the remaining population is close to the targeted size. Hence, the regulator benefits from the tax paid by the Agent and is penalized through a cost function f depending on the size of the resource at maturity T . We have in mind $f(x) = c(\beta - x)^+$ meaning that the regulator targets a population size $\beta > 0$ at time T for the sustainability of the resource and pays the cost c per unit if the natural resource is over-consumed. This function f can be seen as the amount that the regulator must pay to reintroduce the missing resource. We also introduce a reserve utility R which is a negative constant. This reserve means that the regulator cannot penalize too strongly the Agent for economical reasons so that the utility $V_A(\xi)$ expected by the Agent has to be greater than R . We denote by Ξ_R^μ the set of such admissible taxes with exponential integrability conditions. Hence, as usual in moral hazard problems the regulator solves

$$V_R^P = \sup_{\xi \in \Xi_R^\mu} V^P(\xi), \text{ with } V^P(\xi) = \sup_{\alpha \in \mathcal{A}^*(\xi)} \mathbb{E}^{\mathbb{P}^\alpha} \left[\xi - f(X_T^\mu) \right]. \quad (9.1.3)$$

9.2 Optimal contract

Applying the *modus operandi* introduced in the first chapter, we first solve the optimal problem of the Agent (9.1.2) under taxation policy $\xi \in \Xi_R^\mu$. We set $\tilde{R} := \frac{\log(-R)}{\gamma}$.

²We can generalize the choice of p , see [Kharroubi et al., 2019].

9.2.1 The best-reaction effort of the natural resource manager

Theorem 9.1. *Let $\xi \in \Xi_R^\mu$. There exists a unique pair $(Y_0, Z) \in (-\infty, \tilde{R}] \times \mathbb{H}^2(\mathbb{R})$ such that*

- *the tax has the following decomposition*

$$\xi = Y_0 - \int_0^T (g(X_t^\mu, Z_t) + \frac{\sigma^2}{2} \gamma |Z_t|^2) dt + \int_0^T \sigma Z_t dW_t, \quad (9.2.1)$$

where g is defined for any $(x, z) \in \mathbb{R}_+ \times \mathbb{R}$ by $g(x, z) = \frac{|a^*(x, z)|^2}{2} - p(x)xa^*(x, z) - a^*(x, z)z$ with

$$a^*(x, z) = ((p(x)x + z) \vee (-M)) \wedge \overline{M}, \quad (9.2.2)$$

- *the value of the Agent is given by $V_A(\xi) = -\exp(\gamma Y_0)$, and the process $\alpha^*(\xi)$ defined by $\alpha_t^*(\xi) = \alpha^*(X_t^\mu, Z_t)$ is the unique optimal effort associated with the tax ξ given by (9.2.1).*

9.2.2 Regulator problem and main technical issues

Given the previous theorem, $V_R^P = \sup_{Y_0 \leq \tilde{R}, Z \in \mathcal{Z}} \mathbb{E}^{\mathbb{P}^{a^*(X^\mu, Z)}} [Y_T^{Y_0, Z, \mu} - f(X_T^\mu)]$, with

$$Y_t^{Y_0, Z, \mu} = Y_0 - \int_0^t (k(\alpha_s^*) - p(X_s^\mu)X_s^\mu \alpha_s^*(X_s^\mu, Z_s) + \frac{\sigma^2}{2} \gamma |Z_s|^2) dt + \int_0^t \sigma Z_s dW_s^*, \quad t \in [0, T],$$

where W^* stands for $W^{a^*(X^\mu, Z)}$ and $\mathcal{Z} \subset \mathbb{H}^2(\mathbb{R})$ so that $Y_T^{Y_0, Z} \in \Xi_R^\mu$ for any $Z \in \mathcal{Z}$. We notice that the function to maximize in V_R^P is non-decreasing w.r.t. the variable Y_0 . Therefore the constraint $Y_0 \leq \tilde{R}$ is saturated so that

$$V_R^P = \sup_{Z \in \mathcal{Z}} \mathbb{E}^{a^*(X^\mu, Z)} [Y_T^{\tilde{R}, Z, \mu} - f(X_T^\mu)]. \quad (9.2.3)$$

We introduce the related HJB PDE given by

$$\begin{cases} -\partial_t v - H(x, \partial_x v(t, x), \partial_{xx} v(t, x)) = 0, & (t, x) \in [0, T) \times \mathbb{R}_+^*, \\ v(T, x) = -f(x), & x \in \mathbb{R}_+^*, \end{cases} \quad (9.2.4)$$

where the Hamiltonian H is given by

$$\begin{aligned} H(x, \delta_1, \delta_2) = \sup_{z \in \mathbb{R}} & \left\{ xp(x)a^*(x, z) - k(a^*(x, z)) - \frac{\sigma^2}{2} \gamma z^2 + x(\lambda - \mu(x) - a^*(x, z))\delta_1 \right\} \\ & + \frac{\sigma^2}{2} x^2 \delta_2, \quad (x, \delta_1, \delta_2) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Main issues:

- In the logistic case, *i.e.* $\mu(x) = \mu x$, we obtain a term of the form $x^2 \partial_x v$. It prevents us to use (as far as we know) regular solution to PDEs.
- a^* involved irregular coefficients.
- the first order condition gives an optimal Z as a function of $X \partial_x v$ that has to satisfy the technical conditions to be sure that the final contract is indeed admissible, to echo Section 2.2.2.

We thus have to regularize a little bit the PDE. We extend PDE (9.2.4) to the whole domain $[0, T] \times \mathbb{R}$ by considering the change of variable $w(t, y) := v(t, e^y)$ for any $(t, y) \in [0, T] \times \mathbb{R}$. We get the following PDE

$$\begin{cases} -\partial_t w - \mathcal{H}(y, \partial_y w(t, y), \partial_{yy} w(t, y)) = 0, & (t, y) \in [0, T) \times \mathbb{R}, \\ w(T, y) = -f(e^y), & y \in \mathbb{R}, \end{cases} \quad (9.2.5)$$

where

$$\mathcal{H}(y, \delta_1, \delta_2) := \sup_{z \in \mathbb{R}} \left\{ e^y p(e^y) a^*(e^y, z) - \frac{a^*(e^y, z)^2}{2} - \frac{\sigma^2}{2} \gamma z^2 + \left(\lambda - \frac{\sigma^2}{2} - \mu(e^y) - a^*(e^y, z) \right) \delta_1 \right\} + \frac{\sigma^2}{2} \delta_2, \quad (y, \delta_1, \delta_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

Bounded μ and smooth cost function. To overcome the previous issues, we provide a smooth approximation \mathcal{H}_ε of \mathcal{H} for which we get regular solutions.

Lemma 9.1. *Assume that μ is bounded and $\mu, f, y \mapsto e^y p(e^y)$ are smooth enough, then there exists a family $\{\mathcal{H}_\varepsilon, \varepsilon > 0\}$ of functions from \mathbb{R}^3 to \mathbb{R} such that the PDE*

$$\begin{cases} -\partial_t w_\varepsilon - \mathcal{H}_\varepsilon(y, \partial_y w_\varepsilon(t, y), \partial_{yy} w_\varepsilon(t, y)) = 0, & (t, y) \in [0, T] \times \mathbb{R}, \\ w_\varepsilon(T, y) = -f(e^y), & y \in \mathbb{R}, \end{cases}$$

admits a unique solution w_ε in $C^{2+\nu}([0, T] \times \mathbb{R})$ and for any $\varepsilon > 0$

$$\sup_{\mathbb{R}^3} |\mathcal{H} - \mathcal{H}_\varepsilon| \leq \varepsilon.$$

In this lemma, the family \mathcal{H}_ε can be built explicitly and PDE results are deduced from [Oleinik and Kruzhkov, 1961].

Theorem 9.2 (ε -optimal contract). *Under the assumptions of Lemma 9.1, for any $\varepsilon > 0$, the tax policy ξ_ε given by*

$$\xi_\varepsilon = \tilde{R} - \int_0^T (g(X_t^\mu, Z_t^\varepsilon) + \frac{1}{2} \sigma^2 \gamma |Z_t^\varepsilon|^2 + Z_t^\varepsilon (\lambda - \mu(X_t^\mu))) dt + \int_0^T \frac{Z_t^\varepsilon}{X_t^\mu} dX_t^\mu,$$

where

$$Z_t^\varepsilon = -\frac{\partial_x w_\varepsilon(t, \log(X_t^\mu))}{1 + \gamma \sigma^2}, \quad t \in [0, T], \quad (9.2.6)$$

is $2T\varepsilon$ -optimal for the regulator problem:

$$V_R^P \leq \mathbb{E}^{a^*(X^\mu, Z^\varepsilon)} [\xi_\varepsilon - f(X_T^\mu)] + 2T\varepsilon.$$

In particular, Z^ε is bounded, so ξ_ε satisfies the integrability conditions required to be admissible in Ξ_R^μ .

Extension to logistic equation and continuous cost function. We now relax the previous assumptions by assuming that $\mu(x) = x$ and f is bounded and only continuous on \mathbb{R} . Let f_n be a mollifier of f (explicit formula) and set $\mu_n(x) := x \mathbf{1}_{x \in [-e^n, e^n]} + e^n \mathbf{1}_{x > e^n} - e^n \mathbf{1}_{x < -e^n}$. We also set $\tau_n := \inf\{t \in [0, T], X_t \geq e^n\}$. Finally, we consider $Z^{\varepsilon, n}$ an almost-optimal control for the regulator problem associated with μ^n and f^n defined in Theorem 9.2 and we set

$$\tilde{Z}_t^{\varepsilon, n} := Z_t^{\varepsilon, n} \mathbf{1}_{[0, \tau_n]}(t).$$

Theorem 9.3 (ε -optimal contract for the logistic case). *For the logistic model with continuous and bounded cost f (possibly irregular), we have $V_R^P < +\infty$ and*

$$\limsup_{n \rightarrow +\infty} \left(V_R^P - \mathbb{E}^{\mathbb{P}^{a^*(X, \tilde{Z}^{\varepsilon, n})}} [\tilde{\xi}_{\varepsilon, n} - f(X_T)] \right) \leq 2T\varepsilon,$$

with $\tilde{\xi}_{\varepsilon, n} := Y_T^{R, \tilde{Z}^{\varepsilon, n}}$.

9.3 Numerical illustration and policy implications

Consider the problem of the regulator without regulation policy

$$\bar{V}_A := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^\alpha \left[- \exp \left(- \gamma \left(\int_0^T p(X_s^\mu) \alpha_s X_s^\mu ds - \int_0^T \frac{|\alpha_s|^2}{2} ds \right) \right) \right] = - \exp(-\gamma P^2 T).$$

If the regulator chooses $R = \bar{V}_A$ then any admissible tax ξ will satisfy $V_A(\xi) \geq \bar{V}_A$. In other words, the choice of R ensures a non-punitive regulation policy. We now give some numerical results to illustrate our theoretical results. For that we consider $f(x) = (c - \frac{c}{\beta}x)\mathbf{1}_{x < \beta}$ where c is the cost of the resource for the regulator and β is the target size of the population. We use the following parameters: $\gamma = 0.1$, $\lambda = 1.2$, $\sigma = 0.1$, $p = 1$, $T = 1$, $\beta = 0.9$, $c = 3$, $\underline{M} = \bar{M} = 10$, $n = 100$, $\varepsilon = 0.01$ with $X_0 = 1.2$.

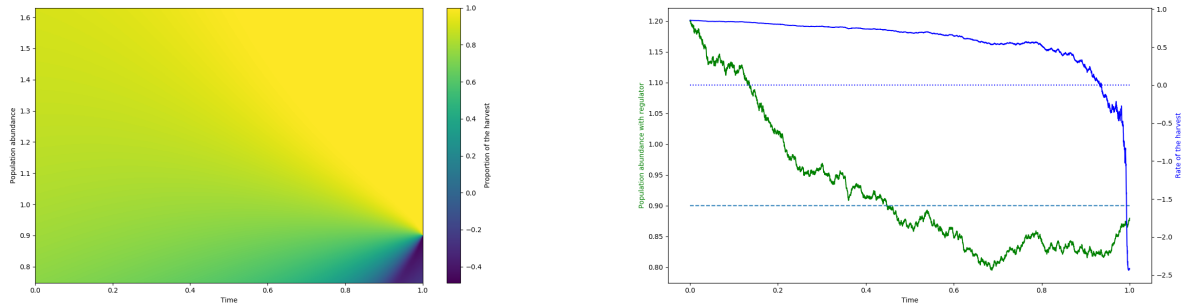


Figure 9.1: (left side) The optimal harvest rate w.r.t. the time t and the population abundance X_t . (right side) A trajectory of the optimally controlled population abundance (green curve, y -axis on the left) and the associated optimal harvest rate (blue curve, y -axis on the right) w.r.t. the time (x -axis). The dotted line corresponds to $\alpha = 0$, and the dashed line corresponds to $X_t^{\alpha^*} = \beta$

Figure 9.1 shows the Agent harvests moderately at the beginning and the rate is increasing w.r.t. the abundance population. On the contrary, at times close to the maturity the strategy depends on the abundance of the resource. Indeed, for an abundance below the target β , *i.e.* $X_t^\alpha < 0.9$, the Agent renews the population, and for an abundance higher than β , *i.e.* $X_t^\alpha \geq 0.9$, the Agent harvests. On the right side, Figure 9.1 shows that the Agent harvests with an important rate at the beginning : $\alpha_t^*(\xi)$ is around 0.6. As he get closer to the maturity, the Agent slows down the harvest and then renew the resource. This can be interpreted as follows. The Agent harvests with a high rate and does not care about the tax at maturity since the population has time to regenerate itself. Getting closer to the maturity, the Agent take into account the tax and slows down the harvest. When very close to maturity, $t \approx 0.93$, the Agent renews the population to ensure an abundance close to the target $\beta = 0.9$ to limit the tax. This shows that the incentive policy is efficient.

We now study the sensitivity of the incentive policy w.r.t. the target β and the renewal cost c (see Figure 9.3).

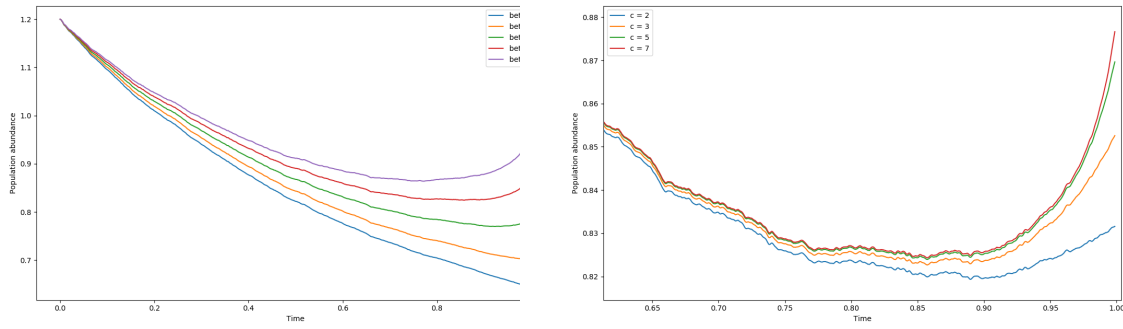


Figure 9.2: (left side) Evolution in mean of the population abundance w.r.t. the time for different values of β ; (right side) evolution in mean of the population abundance w.r.t. the time for different values of c .

Note that at each time the mean of the population abundance is more important as β is larger. This shows that the choice of β influences the behavior of the resource manager: the most important β is, the least the Agent harvests. We also notice that for each value of β , the mean terminal value reaches the target, which also shows the incentive effect of the parameter β . Moreover, the highest the cost c is, the most the Agent is concerned, through the incentive policy, by the size of the population at the end.

We now compare the situation for which the Agent can renew the population abundance (that is $\underline{M} > 0$) with the situation for which the Agent can only harvest (that is $\underline{M} = 0$) in Figure 9.3.

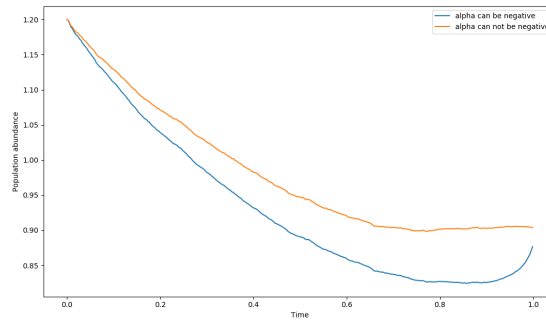


Figure 9.3: Evolution in mean of the population abundance w.r.t. the time when the Agent can (blue curve) and cannot (orange curve) renew the population abundance.

We remark that at each time the population abundance is more important in mean if the Agent cannot renew the resource. This is explained by the fact that if the resource is not renewable, the Agent reduces his harvesting rate in prevision of the terminal tax. On the contrary, if the resource can be renewed, the Agent harvests more to generate a higher profit since he can reduce the terminal tax by renewing the resource at the end.

Optimal lockdown and testing policy in a SIR model

We consider the control of a pandemic via lockdown and testing policies, through a stochastic SIR compartmental model. When the epidemic is ongoing, the population can reduce interactions between individuals in order to decrease the rate of transmission of the disease, and thus limit the epidemic. However, this effort comes at a social cost for the population. Therefore, the government can put in place a policy of incentives to encourage the lockdown of the population. On the other hand, the government may also choose a testing policy in order to know more precisely the spread of the epidemic. We provide numerical simulations and interpretations of the results. This Chapter sums up some results in [Hubert et al., 2020].

10.1 The SIR stochastic model with testing policy

10.1.1 Birth/death/recovery and interaction rates

We will focus on is the classical SIR (Susceptible–Infected–Recovered) compartment model. The SIR model is used to describe epidemics in which infected individuals develop immunity to the virus. The class S represents the *Susceptible*, the class I represents the *Infected* and the class R , represents the *Recovered*, *i.e.*, individuals who have contracted the disease, are now cured, and therefore immune to the considered virus. The birth and (natural) death rates among the population are given by the parameters λ and μ , reflecting the demographic dynamics unrelated to the epidemic. The death rate associated to the disease is denoted by γ . We denote by ρ the recovery rate, which is assumed to be a fixed non–negative constant. All these parameters are assumed to be constant and exogenous.

Let β be the transmission rate of the disease. In our framework, β is endogenous and time–dependent, in order to model the influence that the population can have on this transmission rate depending on two factors: the disease characteristics and the contact rate within the population. Although the population cannot modify the disease characteristics, each individual can choose (or be incentivised) to reduce his/her contact rate with other individuals in the population. From now on, β will thus denote the time–dependent transmission rate of the disease, controlled by the population. More precisely, we fix some constant β^{\max} as the maximum rate of interaction that can be considered, with values in $B := [0, \beta^{\max}]$. We denote by \mathcal{B} the corresponding set of processes.

Therefore, during the epidemic, each individual can be either 'Susceptible' or 'Infected' or in 'Recovery', and (S_t, I_t, R_t) denotes the proportion of each category at time $t \geq 0$.

10.1.2 Stochastic model and testing policy

Without a large testing campaign, it seems complicated to know precisely the proportion of infected in the population. This is particularly true in the case of the COVID–19 epidemic: the absence of symptoms for a significant proportion of infected individuals leads to uncertainty about the actual number of susceptible and infected. Consequently, it seems more realistic in our study to turn to stochastic controlled models. The volatility is partly represented by a fixed and deterministic parameter

$\sigma > 0$, and by a time-dependent process α , representing the actions of the government in terms of testing policy. In our model, an increase of the number of tests in the population, leads to a decrease in the volatility of the processes S and I and so to decrease α . Hence, both the population and the government have a clearer view of the number of susceptible and infected, and thus on the epidemic. However, this strategy comes at an economic cost for the government. We then assume that, without any specific effort of the government, α is equal to 1. The control α of the government is assumed to be A -valued, with $A := [\varepsilon, 1]$ ¹ for $\varepsilon \in (0, 1)$. We denote by \mathcal{A} the corresponding set of processes.

In addition, the testing policy allows the government to isolate individuals with positive test results. Therefore, the control α also has an impact on the effective transmission rate of the disease. More precisely, without any testing policy, *i.e.* $\alpha = 1$, the government cannot isolate contaminated individuals efficiently. In this case, all infected people spread the disease, and the transmission rate of the virus is given by β . Conversely, if a testing policy is put into place by the government, *i.e.* when $\alpha < 1$, we consider that individuals with positive test results can be isolated, and as a consequence less infected people spread the disease. In this case, the effective transmission rate is lower. We however do not assume that the impact of the testing policy on the volatility of S and I , and on the transmission rate has the same magnitude. Indeed, we expect a lower reduction of the effective transmission rate, compared to the volatility reduction for a given policy α . This should be understood as a manifestation of the fact that it is easier to reduce the uncertainty on the number of infected people, compared to actually isolate individuals who have been identified as infected. We thus assume a linear dependency with respect to α for the volatility of both S and I , while the effective transmission rate is chosen equal to $\beta\sqrt{\alpha}$, so that the number of infected people spreading the disease at time t is given by $\sqrt{\alpha}I_t$.

Inspired by the stochastic SIR model in [Tornatore et al. 2005], the dynamic of the epidemic is

$$\begin{cases} S_t = s_0 + \int_0^t (\lambda - \mu S_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds + \int_0^t \sigma \alpha_s S_s I_s dW_s, & t \in [0, T], \\ I_t = i_0 - \int_0^t ((\mu + \rho + \gamma) I_s - \beta_s \sqrt{\alpha_s} S_s I_s) ds - \int_0^t \sigma \alpha_s S_s I_s dW_s, & t \in [0, T], \\ R_t = r_0 + \int_0^t (\rho I_s - \mu R_s) ds, & t \in [0, T]. \end{cases} \quad (10.1.1)$$

10.2 The optimal lockdown/testing policy

The COVID-19 pandemic has emphasised that a control policy has to be established with penalties, if lockdown measures are not respected by the population. We study here how a lockdown policy can limit the number of infected people during an epidemic, with uncertainties on the actual number of affected individuals, and on their level of adherence to such a policy. This problem perfectly fits with a classical principal-agent problem with moral hazard, between the Principal (the government) proposing a policy to an Agent (the population) to interact optimally in order to reduce the spread of the disease. More precisely, at time 0, the government informs the population about its testing policy $\alpha \in \mathcal{A}$, as well as its fine policy $\chi \in \hat{\mathfrak{E}}$ where $\hat{\mathfrak{E}} \subset \mathfrak{C}$ with integrability conditions for the lockdown period $[0, T]$. Knowing this, the population will choose an interacting behaviour according to the following rules

1. an increase in the tax lowers its utility. We model it by a mixed CARA-risk-neutral utility function

$$U(x) := \frac{1 - e^{-\theta_p x}}{\theta_p} + \phi_p x, \quad x \in \mathbb{R},$$

¹The lower bound ε shows that it is not possible, or prohibitively expensive, to cancel completely the uncertainty in the disease's dynamics, by taking α to be 0.

where $\theta_p > 0$ is the risk-aversion of the population, and $\phi_p > 0$, so that $U(0) = 0$, and U is an increasing and strictly concave bijection from \mathbb{R} to \mathbb{R} . For later use, we record that the inverse of U , denoted by $U^{(-1)}$, has an explicit form

$$U^{(-1)}(y) := \frac{1}{\theta_p} \text{LambertW}\left(\phi_p^{-1} e^{\frac{1-\theta_p y}{\phi_p}}\right) + \frac{\theta_p y - 1}{\theta_p \phi_p}, \quad y \in \mathbb{R}.$$

2. an increase in the level of interaction (up to a specific threshold, namely $\bar{\beta}$) improves its well-being although the population is scared of having a large number of infected people. We model it by $u : [0, T] \times B \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of the form $u(t, b, i) := -u_I(i) - u_\beta(t, b)$ where the function $u_I : \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the fear of the infection for the population. In order to choose this function, we would like model the fact that when the proportion of infected is close to 0, the population underestimates the epidemic, while when this proportion becomes large, the population becomes irrationally afraid. Therefore, we choose $u_I(t, i) = c_p i^3$, $(t, i) \in [0, T] \times \mathbb{R}_+$, for some $c_p \geq 0$. Next, the function u_β represents the sensitivity of the population with respect to the initial transmission rate $\bar{\beta} \in B$ of the disease, *i.e.*, without any lockdown measure. During the lockdown period, the social cost of distancing measures becomes more and more important for the population, and we thus expect the cost u_β to also reflect this sensitivity with respect to time. More precisely, $u_\beta(t, b) := \eta_p \psi(t) (\bar{\beta} - b)^2 / 2$, for some $\eta_p > 0$, to insist on the fact that it is costly for the population to deviate from its usual contact rate, where $\psi(t) := e^{\tau_p t}$, for some $\tau_p > 0$ to represent the increasing aversion to the lockdown for the population as time passes.

Therefore, for a given test/tax policy (α, χ) the population aims at solving

$$V_0^A(\alpha, \chi) := \sup_{\beta \in B} \mathbb{E} \left[\int_0^T u(t, \beta_t, I_t) dt + U(-\chi) \right], \quad (10.2.1)$$

A benchmark case: optimal interaction without taxation and testing policies. Before turning to the principal-agent problem itself, we aim at solving (10.2.1) for $\alpha = 1$ fixed, and $\chi = 0$ (no tax). Hence, without tax, the population's problem boils down to a standard control problem, with two state variables S and I that can be solved numerically.

Representation of χ and solution to the population problem (10.2.1). Again, given an admissible contract, namely a testing policy $\alpha \in \mathcal{A}$ and a tax $\chi \in \hat{\Xi}$, we can prove by using martingale representation that there exist a unique Y_0 and Z such that

$$U(-\chi) = Y_0 - \int_0^T \left(Z_t(\mu + \nu + \gamma + \rho) I_t + u(t, \beta_t^*, I_t) - \beta_t^* \sqrt{\alpha_t} S_t I_t Z_t \right) dt - \int_0^T Z_t dI_t, \quad (10.2.2)$$

where $\beta_t^* := b^*(t, S_t, I_t, Z_t \sqrt{\alpha_t})$ is the optimal contact rate with

$$b^*(t, s, i, z) := \begin{cases} \beta^{\max}, & \text{if } z < -\frac{\eta_p \psi(t)}{si} (\beta^{\max} - \bar{\beta}), \\ \bar{\beta} - \frac{siz}{\eta_p \psi(t)}, & \text{if } z \in \left[-\frac{\eta_p \psi(t) (\beta^{\max} - \bar{\beta})}{si}, \frac{\bar{\beta} \eta_p \psi(t)}{si} \right], \\ 0, & \text{if } z > \frac{\bar{\beta} \eta_p \psi(t)}{si}. \end{cases}$$

Since U is invertible (10.2.2) gives a representation of χ . Referring for instance to [Cvitanić et al., 2019] the contract is usually the sum of three components: a constant similar to Y_0 , chosen by the Principal in order to satisfy the participation constraint of the Agent; an integral with respect to time $t \in [0, T]$ of the Agent's Hamiltonian; a stochastic integral with respect to the controlled process I .

Remark 10.1. From [Cvitanić et al., 2019, Equation (2.1)], the drift of (S, I) should be proportional to its volatility which is obviously impossible here since there is only one Brownian motion for two state variables. However, and somewhat surprisingly, the form we exhibit for the tax is actually strongly related to the usual representation. The reason for this is twofold. First, up to the sign, the volatilities in the dynamics of both S and I are exactly the same. Second, both the processes S and I are driven by the same Brownian motion W . Therefore, intuitively, in order to provide incentives to the population, the government can afford to index the tax on only one of the two processes. Mathematically, it is also straightforward to show that given an arbitrary decomposition of the process Z in (10.2.2) of the form $Z =: Z^s - Z^i$, we have

$$U(-\chi) = Y_0 - \int_0^T H(t, S_t, I_t, Z_t^s, Z_t^i) dt + \int_0^T Z_t^s dS_t + \int_0^T Z_t^i dI_t,$$

where H is the Hamiltonian of the population, fitting exactly with the form in [Cvitanić et al., 2019].

Government optimization problem The government aims at minimising the number of infected people until the end of the quarantine period. We assume that it has to pay the instantaneous cost c implied by the proportion of infected people during the quarantine period, and given by $c(i) := c_g(i + i^2)$, $i \in \mathbb{R}_+$ for some $c_g > 0$. The quadratic growth highlights the cost induced by the saturation of the health-care system when the number of infected is too high. In addition to that, the government has a cost k associated to the testing policy, defined for some $\eta_g > 0$ by $k(a) := \kappa_g(a^{-\eta_g} - 1)$, $a \in A$. This function highlights the fact that it is very costly, if not impossible, to eliminate the uncertainty associated with the epidemic. Thus the government problem is

$$V_0^P := \sup_{(\alpha, \chi) \in \Xi} \mathbb{E}^{\beta^*} \left[\chi - \int_0^T (c(I_t) + k(\alpha_t)) dt \right], \quad (10.2.3)$$

where the set Ξ takes into account the so-called participation constraint for the population. This means that the government is benevolent, which translates into the fact that it has committed to ensure that the living conditions of the population do not fall below a minimal level. Mathematically, Ξ is the set of $(\alpha, \chi) \in \mathcal{A} \times \hat{\Xi}$ such that $V_0^A(\alpha, \chi) \geq \underline{v}$, where the minimal utility $\underline{v} \in \mathbb{R}$ is given. It thus remains to solve the government's problem in order to determine the optimal choice of Y_0 and Z .

Optimal contract. The optimal process Z as well as the optimal texting policy α are determined so as to maximise the government's Hamiltonian, given by

$$H^P(t, s, i, p, M) = \sup_{z \in \mathbb{R}, a \in A} \left\{ b^*(t, s, i, z\sqrt{a})\sqrt{a}si(p_2 - p_1) + \frac{1}{2}\sigma^2 a^2 (si)^2 f(z, M) - k(a) - u^*(t, s, i, z, a)p_3 \right\} \\ + (\lambda - \mu s + \nu i)p_1 - (\mu + \nu + \gamma + \rho)ip_2 - c(i), \quad (t, s, i, p, M) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R}^3 \times \mathbb{S}^3,$$

and where for all $(t, s, i, z, M) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{S}^3$

$$f(z, M) := M_{11} - 2M_{12} + M_{22} - 2z(M_{23} - M_{13}) + z^2 M_{33}, \quad \text{and } u^*(t, s, i, z, a) := u(t, b^*(t, s, i, z\sqrt{a}), i, z).$$

Finally, the problem is reduced to solve numerically the following HJB equation, for all $t \in [0, T]$ and $x := (s, i, y) \in \mathbb{R}^3$

$$-\partial_t v(t, x) - H^P(t, x, \nabla_x v, D_x^2 v) = 0, \quad (t, x) \in \mathcal{O}, \quad v(T, x) = -U^{(-1)}(y), \quad x \in \mathcal{O}_T, \quad (10.2.4)$$

where the natural domain over which the above PDE must be solved is

$$\mathcal{O} := \{(t, s, i, y) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R} : 0 < s + i < F(t, s_0, i_0)\}, \quad \mathcal{O}_T := \{(s, i, y) \in \mathbb{R}_+^2 \times \mathbb{R} : 0 < s + i < F(T, s_0, i_0)\},$$

for an explicit function F given with respect to $\lambda, \mu, \gamma, \rho$. Unfortunately, we are not able to check the admissibility of the optimal χ^* since we have no information on the optimizer Z^* . This thus leads to a numerical analysis only to approach the optimal contract by choosing $Z \in [-z_{max}, z_{max}]$, $z_{max} > 0$.

10.3 Numerical results and policy implications

Parameters	c_p	η_p	θ_p	τ_p	ϕ_p	β^{\max}	κ_g	c_g	η_g	ε
Values	0.5	1	4	0	0.5	0.2	0.5	1	0.01	0.01

	T	(s_0, i_0, r_0)	(λ, μ)	γ	ν	ρ	σ	$\bar{\beta}$
SIR model (10.1.1)	200	$(0.99984, 1.07 \times 10^{-4}, 5.3 \times 10^{-5})$	$(0, 0)$	0.01	0	0.1	0.1	0.2

Table 10.1: Set of parameters for cost and utility functions of the population and the government and set of parameters for the simulations of SIR model inspired by [Élie et al., 2020]

We assume (as a worse-case scenario) that the government ensures at the same living conditions the population would have had in the event of an uncontrolled epidemic. Mathematically, $\underline{v} = -c_p \mathbb{E}^{\mathbb{P}^{1, \bar{\beta}}} \left[\int_0^T I_t^3 dt \right]$.

Lockdown policy, without testing:

The epidemic is delayed. In Figure 10.1 we give one (representative²) trajectory of the proportion of I (left) and optimal β for the benchmark and under contract policy without testing (right) for different Z_{\max} . Compared to the benchmark case, we observe that the optimal lockdown policy slows the spread of infection and thus delays the epidemic peak, and even prevents it in some cases. This lockdown incentive strategy therefore responds to the main motivation for managing the COVID-19 pandemic, namely to slow the spread of the epidemic.

Policy implications. In Figure 10.2 we compute the corresponding trajectory of the optimal compensation Z . First remark that the shape of the optimal indexation parameter rate Z remains the same, regardless of the simulation and the value of Z_{\max} . The control takes the most negative value possible ($-Z_{\max}$) for about 20 days, then increases almost instantaneously to reach the maximum value Z_{\max} , before slowly decreasing to 0. First, at the beginning of the epidemic, it seems optimal to give to the population a compensation (corresponding to a negative tax) as maximal as possible, by setting $Z = -Z_{\max}$. The fact that the same phenomenon appeared in virtually all our simulations tends to show that it is actually significant. We interpret this as a the government anticipating the negative consequences of the lockdown policy by immediately providing monetary relief to the population. This is exactly what happened in several countries, for instance in the USA with the stimulus checks sent to every citizens, and our model endogenously reproduces this aspect. Policy-wise, it also shows that maximum efficiency for such stimulus packages is attained when they are provided to the population as early as possible. After this initial phase, when the epidemic spreads among the population, the government suddenly increases Z , so that the tax becomes positive and is in fact maximum, in order to deter people from interacting. Approaching the maturity, the government keeps on reducing the tax, and thus eases little by little the lockdown. Note that near the maturity T , the government gives again the maximal compensation. However, this is most likely an artefact due to boundary effects in the simulations at T , and would disappear for long time horizons.

²The general shape is unchanged for other trajectories.

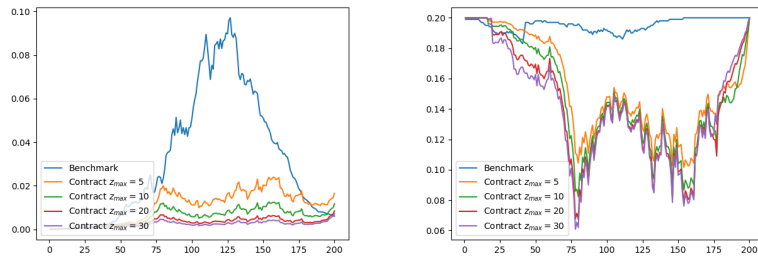


Figure 10.1: One trajectory of the proportion of I (left) and optimal β for the benchmark and under contract policy without testing (right) for different Z_{\max} .

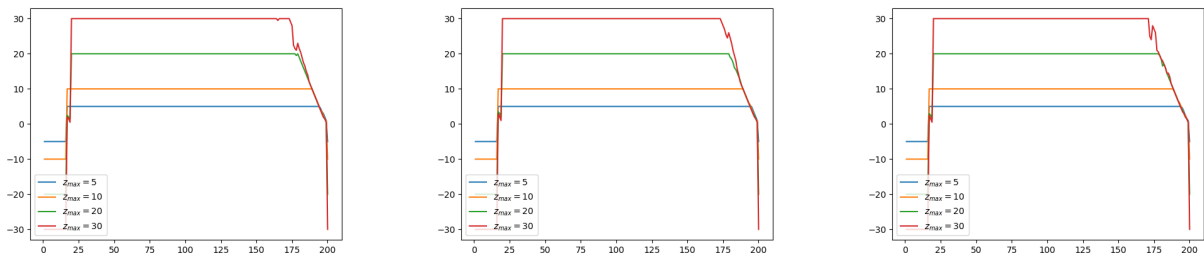


Figure 10.2: Optimal control Z for $A = \{1\}$ and different values for Z_{\max} on three trajectories.

Tax policy with testing: Relaxed lockdown but lower effective transmission rate We now turn to the effect of the testing policy. Figure 10.3 left part shows that the optimal contact rate is closer to the initial level $\bar{\beta}$, which should induce a more violent spreading of the disease. However, the control α , representing the testing policy see Figure 10.3 center, balances this effect. Indeed, the testing allows an isolation of targeted infected individual, and therefore contribute to the decrease of the effective transmission rate of the disease, represented in Figure 10.3 right.

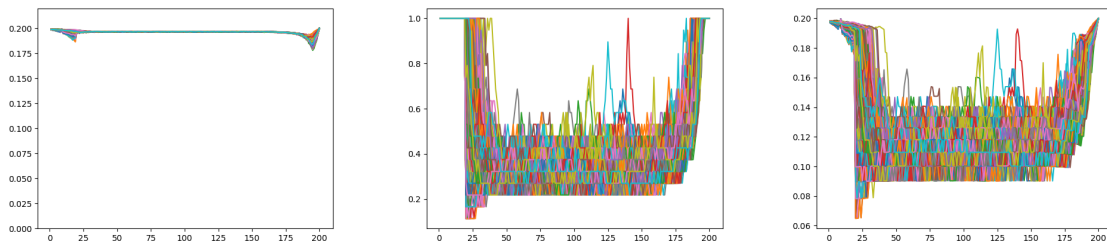


Figure 10.3: 500 simulations of optimal β (left), α (center) and effective rate $\sqrt{\alpha}\beta$ (right) with testing

Conclusion: the benefits of anticipative testing policies with isolation to relax the lockdown. Figure 10.4 compares three trajectories of the optimal effective transmission rate for the benchmark, the case with tax policy/without testing and the case tax and testing policies. Note that the effective rate is reduced when we add a testing policy with isolation more quickly at the beginning of the disease. Consequently, we see that the proportion of infected is reduced in Figure 10.5 when the government establishes both tax policy and testing with isolation. In addition to that, the testing and isolation policy is a greatly beneficial for the population since the interaction rate β is very close to the initial

level $\bar{\beta}$. However, we observe that the monetary relief for the optimal tax Z disappears at the beginning in more simulations, see Figure 10.6. In this case the government asks less effort to the population and so the monetary relief underlined in the no-testing policy case is not necessary.

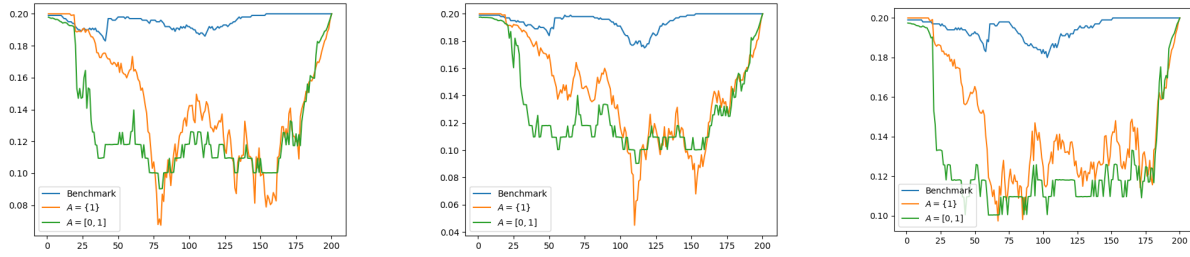


Figure 10.4: Optimal effective rate trajectories $\sqrt{\alpha\beta}$.

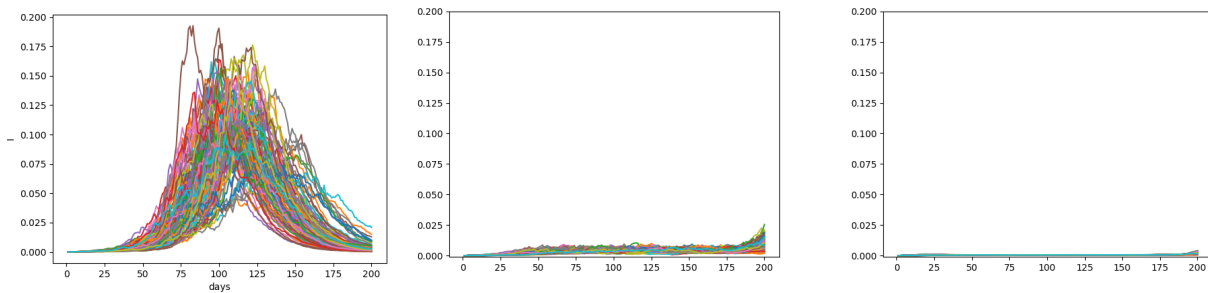


Figure 10.5: Proportion of I without contract and without testing policy (left); with tax policy and without testing policy (center); with tax policy and with testing policy (right).

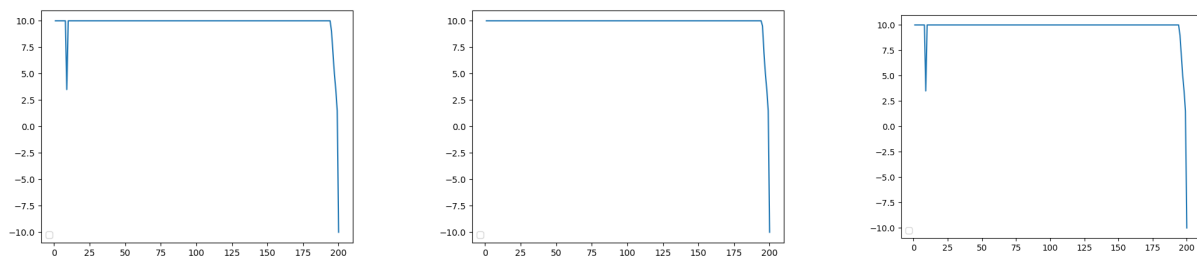


Figure 10.6: Three optimal trajectories of Z , for $A = [\varepsilon, 1]$.

Perspectives

As an application of contract theory, I have studied in [Kharroubi et al., 2019] the monitoring of natural renewal resource management. This kind of problem appears in the so-called *tragedy of commons* in which several resource managers harvest a common resource in a specific land. It is then optimal for each individual to harvest it until extinction of the resource. Several remedies has been proposed to fix this problem. In particular, Elinor Ostrom has proposed monitoring policies between managers to balance the resource exploitation. A first extension of [Kharroubi et al., 2019] could be to consider many resource managers monitored by a regulator to model mathematically this tragedy of commons and to find an optimal incentive policy to improve the resource management. Several situations may be compared (non-cooperative managers, central planner monitoring the resource managers...) leading to different mechanisms.

The initial idea of Ostrom to fix tragedy of commons problem was to proposes a common taxation policy in which many resource managers monitor themselves directly by fixing taxation/compensation policy in their own group. The second extension of this work could be to find equilibria in a non-Markovian stochastic differential game, in the spirit of [Mastrolia and Ren, 2018]. These questions will be part of future researches.

We are currently trying to extend [Kharroubi et al., 2019] by considering singular controls in the dynamic of the resource. This requires to define a stochastic control problem with singular controls in a weak formulation, that is currently a work in progress with Idris Kharroubi and Thomas Lim.

In [Kharroubi et al., 2019], we have decided to model the resource with the stochastic (continuous) logistic equation. However, from a population dynamic point of view, we wondered if this kind of equation is robust in terms of birth/death point processes. Consider as a first step a resource manager aiming at stabilizing the population by harvesting the resource (at a level $\tilde{x} > 0$ fixed) with cost induced by his/her efforts. In a discrete framework, let N^b and N^d be respectively birth and death counting processes with birth/death rates ν and μ in a population and α the corresponding effort to manage the resource. The resource manager has thus to solve

$$V_0^K = \sup_{\alpha \in \mathcal{A}^K} \mathbb{E}^{K,\alpha} [-\gamma(X_T^K - \tilde{x})^2 - \int_0^T \frac{(\alpha_s X_s^K)^2}{2} ds],$$

where $X^{K,n} = n + \frac{N^b - N^d}{K}$ is the rescaled population with parameter $K > 0$. From [Bansaye and Méléard, 2015] we know that the corresponding model in continuous time is given by the unique solution X of

$$dX_t = (\nu - \mu - \alpha_t)X_t dt + \sigma\sqrt{X_t}dW_t.$$

The same control problem in the continuous framework becomes

$$V_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^\alpha [-\gamma(X_T - \tilde{x})^2 - \int_0^T \frac{(\alpha_s X_s)^2}{2} ds].$$

We have seen in [Jusselin and Mastrolia, 2019] that stochastic control problems are robust, meaning that both the value functions and the optimal controls of stochastic control problems in a discrete model

converge to the corresponding optimal values for continuous model (potentially weakly), see Figures 11.1 and 11.2. The next step could be to check if this stability is still ensured for Stackelberg games in the spirit of [Kharroubi et al., 2019] and so for the monitoring of resource management.

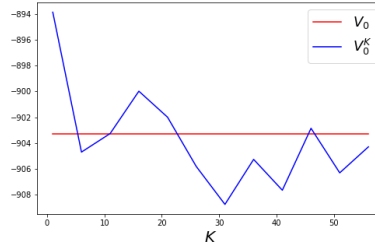


Figure 11.1: Convergence of $(V_0^K)_{K \geq 0}$ towards V_0 .

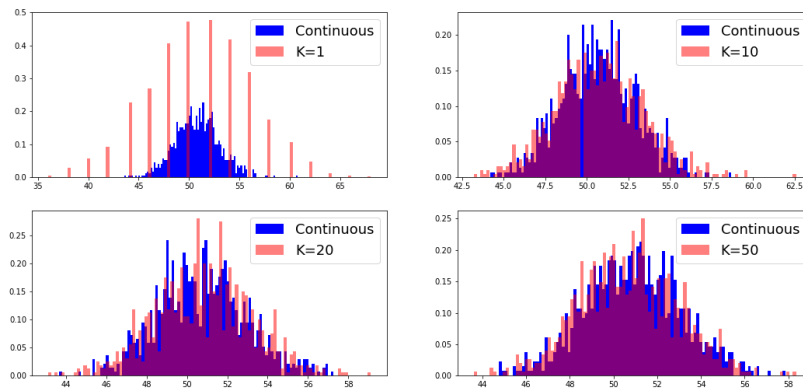


Figure 11.2: Empirical distribution of the discrete optimal control for different values of K (in red) compared to the distribution of the continuous optimal control.

Finally, I am currently studying with Caroline Hillairet and Wissal Sabbagh a model in cyber-criminality, in which the owner of a computer cluster aims at preventing cyberattacks with an antivirus protection. In this case, the cluster can be identified with an SIR-type compartment model. He/she thus has to find the optimal strategy by solving a Nash game with the hacker than can be extended to mean-field interactions in the spirit of [Kolokoltso and Bensoussan, 2016].

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