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INSTITUT  
POLYTECHNIQUE  
DE PARIS

Habilitation à Diriger les Recherches

# Monotonicity in Mean Field Games and Dynamics of the Spectrum of Large Random Matrices

**CHARLES BERTUCCI**

Chargé de recherche CNRS, CMAP UMR7641

Le 13/06/2022,

Sur avis des rapports de :

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Professeur, Princeton University

Benoit Perthame  
Professeur, Sorbonne Université

Nizar Touzi  
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# Introduction

## General Introduction

This manuscript is a summary of part of the work I have been conducting for the last three years. The writing of this document is in general submitted to few rules, I decided here to juxtapose themes which are not related and omit some aspects of my work that may have seemed, at first, more relevant. I hope the result gives an honest (and favorable) opinion on my work.

Most of my research is concerned with the analysis of mathematical models of large systems of interacting agents or particles. Many of these models originate from socio-economic sciences and fall under the scope of the theory of Mean Field Games (MFG for short), which is to be presented in details later on. My work on this topic has focused on both using this theory to model several situations, and on developing a mathematical framework to study those games. A condensate of this research is to be the first part of this *habilitation* while the second part is to be devoted to results on the evolution of the spectrum of large random matrices.

## Organisation of the manuscript

This manuscript is organized as follows.

In the first chapter, I am almost exclusively concerned with MFG. In a first time I recall some basic facts about this theory. Then, I explain how we can characterize the value of such a MFG using a notion of solution of the master equation that I call monotone. I present results of existence, uniqueness and stability for this notion. Later on, I show how such a formulation can help to characterize the value of MFG which seem too singular to allow a proper partial differential equation (PDE for short) to be written for the master equation. I finish this chapter with the presentation of a work concerned with the study of the master equation for the mean field planning problem.

In the second chapter, I present a new approach on the dynamics of the spectrum of large random matrices. After some reminders on the known results in this theory, I

explain how we can characterize the limit evolution of the spectrum using a uniqueness result for a non linear PDE. This uniqueness result relies on a comparison result that can be interpreted as a spectral dominance propagation result. This method allows to generalize and unify the literature on this topic in a single framework.

Both the two chapters ends with bibliographical comments and future perspectives.

A list of articles I have written or co-signed is at the end of the manuscript.

## Notation

- Given a metric space  $(E, d)$ ,  $\mathcal{M}(E)$  and  $\mathcal{P}(E)$  denote respectively the set of measures and probability measures on  $E$ . Unless otherwise stated,  $\mathcal{P}(E)$  is equipped with the Monge-Kantorovich distance

$$\mathbf{d}_1(\mu, \nu) := \sup \left\{ \int_E f d(\mu - \nu) \right\}, \quad (1)$$

where the supremum is taken over all Lipschitz continuous functions with a Lipschitz constant at most one.

- More generally,  $\mathbf{d}_p$  stands for the  $p$  Wasserstein distance.
- Given a topological space  $E$ , a space  $E'$  such that there is a duality product  $\langle \cdot, \cdot \rangle$  between  $E$  and  $E'$ , a function  $f : E \rightarrow E'$  is said to be monotone if for any  $x, y \in E$

$$\langle f(x) - f(y), x - y \rangle \geq 0. \quad (2)$$

The function  $f$  is said to be strictly monotone if an equality in the previous inequality implies that  $f(x) = f(y)$ .

- In the following,  $\langle \cdot, \cdot \rangle$  shall denote indistinguishably the duality product between two elements of appropriate spaces. That is, it can be the euclidean scalar product, the scalar product for functions in  $L^2$  or the duality product between measures and continuous functions for instance.
- The  $d$  dimensional positive orthant  $\mathbb{O}_d$  is defined by  $\mathbb{O}_d := \{x \in \mathbb{R}^d, \forall 1 \leq i \leq d, x_i \geq 0\}$ .
- The adjoint of a linear map  $T$  is denoted by  $T^*$ .
- For  $x, y \in \mathbb{R}^d$ , I note  $x \leq y$  when, for  $1 \leq i \leq d, x_i \leq y_i$ .
- For  $R \geq 0$ ,  $B_R^1 := \{q \in \mathbb{R}^d, q \geq 0, q_1 + \dots + q_d \leq R\}$ .
- The set of, real, symmetric  $n \times n$  matrices is denoted  $S_n(\mathbb{R})$ .
- The norm  $\|\cdot\|_{n+\alpha}$  denotes the usual Hölder norm  $\mathcal{C}^{n,\alpha}$ .

- For a measurable function  $f : E \rightarrow F$  and a measure  $\mu \in \mathcal{M}(E)$ ,  $f_{\#}\mu$  denotes the image measure of  $\mu$  by  $f$ .
- For  $\mathcal{O} \subset \mathbb{R}^d$  and a function  $U : \mathcal{P}(\mathcal{O}) \rightarrow \mathbb{R}$ , we note for  $y \in \mathcal{O}$ , when it is defined

$$\frac{\delta U}{\delta m}(m, y) = \lim_{\theta \rightarrow 0} \frac{U((1 - \theta)m + \theta\delta_y) - U(m)}{\theta}. \quad (3)$$

If the function  $\frac{\delta U}{\delta m}(m) \in \mathcal{C}(\mathcal{O})$ , it is sometimes called the flat derivative of  $U$  with respect to  $m$ .



# Chapter 1

## Monotonicity in Mean Field Games

### 1.1 Background on Mean Field Games

#### 1.1.1 A brief history

The theory of MFG is concerned with the study of dynamic games involving an infinite number of non-atomic agents. To trace back the beginning of a mathematical theory is often a complicated task and the MFG one is no exception. If it is clear that the development of the mathematical toolbox, used in this theory, dates back to the work of Lasry and Lions [LL07; Lio11], it is also clear that lots of MFG models have been studied before. Let me insist in particular on the models introduced in [KS98; SW86] and on the models arising from engineering [HCM03; HMC06; HCM07]. A posteriori, the most powerful tools developed by Lasry and Lions are i) the master equation and ii) the fact that there is uniqueness of Nash equilibria of the game in a monotone regime. I shall explain this concepts after having introduced two based-case models.

#### 1.1.2 A continuous framework

Let me consider a setting which shall be in force for a major part of this manuscript. The duration of the game is  $t_f > 0$ , the state space of the players is  $\mathcal{O} := \mathbb{T}^d$ , the  $d$ -dimensional torus, where  $d \geq 1$  is an integer. Each player controls its state  $X$  which evolves according to the stochastic differential equation (SDE)

$$dX_t = \alpha_t dt + \sqrt{2\sigma} dW_t, \quad (1.1)$$

where  $(W_t)_{t \geq 0}$  is a standard  $d$ -dimensional Brownian motion on a standard filtered probability space  $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$  and  $(\alpha_t)_{t \in [0, t_f]}$  is the control of the players which has to satisfy some measurability assumptions with regards to the state of this player. We interpret here  $(W_t)_{t \geq 0}$  as an individual noise term. By this, we imply that two different players are facing independent realizations of  $(W_t)_{t \geq 0}$ . The cost paid by the player whose

trajectory and control are respectively  $(X_s)_{s \in [0, t_f]}$  and  $(\alpha_s)_{s \in [0, t_f]}$  is

$$\int_0^{t_f} L(X_s, \alpha_s, m_s) ds + \mathcal{G}(X_{t_f}, m_{t_f}), \quad (1.2)$$

where  $L$  and  $\mathcal{G}$  are functions on which assumptions are to be made later on, and  $(m_s)_{s \in [0, t_f]}$  is the measure valued process which describe the repartition of the players in the state space. That is, at any time  $t$ , for any measurable set  $A \subset \mathcal{O}$ ,  $m_t(A)$  is the number of players whose state is in  $A$  at the time  $t$ . The costs  $L$  and  $\mathcal{G}$  are common to all players.

The Fenchel conjugate of  $L$  is the Hamiltonian  $H(x, p, m) := \sup_{\alpha} \{-\alpha \cdot p - L(x, \alpha, m)\}$ . If a player anticipates the evolution  $(\tilde{m}_s)_{s \in [0, t_f]}$  for the repartition of players, then its optimization problem is described by the Hamilton-Jacobi-Bellman (HJB for short) equation

$$\begin{cases} -\partial_t u - \sigma \Delta u + H(x, \nabla u, \tilde{m}) = 0 & \text{in } (0, t_f) \times \mathcal{O}, \\ u|_{t=t_f} = \mathcal{G}(\tilde{m}_{t_f}) & \text{in } \mathcal{O}. \end{cases} \quad (1.3)$$

The unknown of this HJB equation is the value  $u$  of a player, given its anticipation  $(\tilde{m}_s)_{s \in [0, t_f]}$ . This dynamic programming approach yields the optimal control of a player (once again given its anticipation) in closed loop form with

$$\alpha^*(s, x) := -D_p H(x, \nabla u(s, x), \tilde{m}_s). \quad (1.4)$$

Hence, if a strategic equilibrium is reached, and the initial repartition of players is  $m_0 \in \mathcal{M}(\mathcal{O})$ , then the repartition of players has to satisfy, in the sense of distributions, the Kolmogorov equation

$$\begin{cases} \partial_t m - \sigma \Delta m - \operatorname{div}(D_p H(x, \nabla u, m)m) = 0 & \text{in } (0, t_f) \times \mathcal{O}, \\ m|_{t=0} = m_0, \end{cases} \quad (1.5)$$

where  $u$  is the solution of the HJB equation (1.3) with anticipation  $\tilde{m} = m$ . Remark that the fact that the noises affecting any two players are independent is crucial to allow the previous equation to be true.

### 1.1.3 A finite state framework

In this setting, the state space of the players is the finite set  $\{1, \dots, d\}$  and the duration of the game is still denoted by  $t_f > 0$ . I do not describe precisely either the trajectories of the players or the cost functions they are facing. In this setting I adopt a more abstract approach. Let me only state that, the analogue of (1.3) is the following backward differential equation

$$\begin{cases} \dot{V}^i(t) = G^i(V(t), \tilde{Y}(t)), t \in (0, t_f), i \in \{1, \dots, d\}, \\ V(t_f) = U_0(Y(t_f)), \end{cases} \quad (1.6)$$

where  $V^i(t)$  represents the value of a player in the state  $i$  at the time  $t$ , given that its anticipation on the repartition of players is given by  $(\tilde{Y}_s)_{s \in [0, t_f]}$ . In the previous  $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a function on which assumptions shall be made later on and  $U_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is the terminal cost of the players.

On the other hand, the analogue of the Kolmogorov equation is here given by

$$\begin{cases} \dot{Y}^i(t) = F^i(V(t), Y(t)), t \in (0, t_f), i \in \{1, \dots, d\}, \\ Y(0) = q_0, \end{cases} \quad (1.7)$$

where  $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a function on which assumptions shall be made later on,  $q_0 \in \mathbb{R}^d$  represents a sort of initial distribution of players and  $Y(t)$  the state of this distribution at time  $t$ .

#### 1.1.4 Nash equilibria of the MFG

I focus for the moment on the continuous state space case. Even though I do not want to enter into the precise formulation of the underlying game (strategies of the players, information and so on), it is clear that for each Nash equilibrium of the game, one expect to have a solution  $(u, m)$  of the coupled equations (1.3)-(1.5) (where  $\tilde{m} = m$  of course). I precisely choose this as a definition of a Nash equilibrium in the previous game, when all the information concerning the state space of the players is public, that is an equilibrium is a solution  $(u, m)$  of (1.3)-(1.5).

The existence of Nash equilibria is pretty standard under some continuity assumptions on  $H$  and  $\mathcal{G}$ . I indicate to the interested reader the following references [LL07; Car+15; CD18a; Por15] for various results of existence. A surprising feature of MFG is that there is a quite general regime in which uniqueness of an equilibrium holds. It is called the monotone regime. To simplify a bit the notation (and the assumptions under which the monotone regime is in force), I assume from now on that  $H$  is of the form

$$H(x, p, m) = H(x, p) - f(m)(x), \quad (1.8)$$

for some function  $f : \mathcal{M}(\mathcal{O}) \rightarrow \mathcal{C}(\mathcal{O})$ . The monotone regime holds as soon as

**Hypothesis 1.1.1.** *The Hamiltonian  $H$  is convex in  $p$ . The functions  $f$  and  $\mathcal{G}$  are monotone (seen as functions from  $\mathcal{M}(\mathcal{O})$  into  $\mathcal{C}(\mathcal{O})$ ).*

The monotone regime is almost sufficient to provide a uniqueness result for MFG Nash equilibria.

**Theorem 1.1.1** (Lasry & Lions). *Under Hypothesis 1.1.1, if  $f$  and  $\mathcal{G}$  are strictly monotone, there exists at most one Nash equilibrium  $(u, m)$  such that  $u$  is a classical solution of the HJB equation and  $m$  is a solution of the Kolmogorov equation in the sense of distribution.*

This result can in fact be established under more general assumptions, namely concerning the additional strict monotonicity that is assumed on  $f$  and the regularity of  $u$

and  $m$ . Indeed it is sufficient to have some duality between the equation satisfied by  $u$  on one side, and  $m$  on the other side, and vice versa so that if  $m$  is more regular, then  $u$  can be asked to be less regular.

In the finite state space setting, the monotone regime holds when

**Hypothesis 1.1.2.** *The operator  $(G, F) : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$  is monotone.  $U_0$  is monotone.*

A similar result of uniqueness can be formulated in this context.

In general, the monotone regime is thought as an adversarial one for the underlying game. Indeed, in this context, the monotonicity of the costs creates incentives for the players to spread in the state space. Moreover it is natural from a game theory perspective that the uniqueness of equilibria holds under this type of adversarial assumptions.

### 1.1.5 The value of a mean field game and the master equation

It is a general feature of game theory that the set of equilibria of a game is an unstable object. The MFG theory does not seem to be an exception to this phenomenon [CV17; Cir19]. On the other hand, it is also a general feature that when a game has an adversarial structure (zero-sum game, dissipative games), a concept of value can be defined. In such an adversarial context, the value of a game is in general a stable object. Formally this is a consequence of the fact that players can correctly anticipate the behavior of the players as they are competing with themselves. Games of coordination proved to be much more involved for instance, and yield in general to quite unstable situations.

Armed with the result of uniqueness of Lasry and Lions, one can define a concept of value in a MFG in the monotone regime. Indeed, from the uniqueness of Nash equilibria, given a remaining time in the game, and the knowledge of the distribution of players, then the value of the MFG is the value of the problem of a single player, given that it anticipates the unique equilibrium. More precisely, if we denote by  $(u_{s,\mu}, m_{s,\mu})$  the unique equilibrium of the game with  $s = t_f$  and  $m_0 = \mu$ , then the value  $U$  of the MFG can be defined as

$$U(s, x, \mu) = u_{s,\mu}(0, x), \text{ for } (s, x, \mu) \in (0, \infty) \times \mathcal{O} \times \mathcal{P}(\mathcal{O}). \quad (1.9)$$

Let me note that in some sense I have reversed time between  $u$  and  $U$ . Formally, the value function  $U$  is then naturally a solution of the infinite dimensional PDE for which (1.3)-(1.5) is a system of characteristics :

$$\begin{aligned} \partial_t U + H(x, \nabla U) - \sigma \Delta U + \left\langle \frac{\delta U}{\delta m}, -\Delta m - \operatorname{div}(D_p H(\nabla U) m) \right\rangle &= f(x, m) \\ &\text{in } (0, \infty) \times \mathcal{O} \times \mathcal{P}(\mathcal{O}), \end{aligned} \quad (1.10)$$

together with the initial condition

$$U|_{t=0} = \mathcal{G}. \quad (1.11)$$

This equation is called the master equation. One of the advantages of the master equation approach, in the monotone regime, is that it allows us to model quite easily MFG in which the structure of randomness is more involved than the one we presented. Indeed, when the randomness affecting the players is not distributed in an i.i.d. fashion among them, working with an analogue of the system (1.3)-(1.5) is, in general, more difficult than to work with the corresponding master equation. In a situation in which the noise is not i.i.d., I shall say that a common noise is present in the MFG.

Equation (1.10) has been mostly studied using the methods of characteristics, that is by an extensive study of the system (1.3)-(1.5) and a systematic use of (1.9). This lead to results concerning (1.10) which demand quite strong assumptions. I shall present in the next section a new approach on (1.10).

In the finite state space case, the master equation takes the form

$$\partial_t U + (F(q, U) \cdot \nabla_q)U = G(q, U) \text{ in } (0, \infty) \times \mathbb{R}^d, \quad (1.12)$$

with the initial condition

$$U|_{t=0} = U_0, \quad (1.13)$$

where the unknown  $U : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is the value function of the MFG. Let me insist on the fact that in this situation, the variable  $q$  stands for the repartition of players, i.e. it is the analogue of the variable  $m$  in the continuous state space.

In both the previous cases, the fact that the master equation is set on either  $\mathcal{M}(\mathcal{O})$  or  $\mathcal{P}(\mathcal{O})$  (or  $\mathbb{R}^d$  or  $\mathbb{O}_d$  for the finite state case) is very model dependent and has to do with an eventual entry or exit of players that I shall not discuss here.

Let me insist on the fact that the concept of value for a MFG does not make much sense outside the monotone regime (or any other regime that yields uniqueness of the Nash equilibria of the MFG). Hence seeking for a solution of either (1.10) or (1.12) in the most general setting is a queer question from a game theoretic point of view.

### 1.1.6 Common noise in mean field games

As already mentioned, the master equation approach is useful to analyze MFG involving a common noise. Without entering into too much details about the underlying models, I present here several master equations involving different common noises.

In the finite state space case, if at random Poisson times, of intensity  $\lambda > 0$ , the players are all affected by the linear map  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , then the master equation becomes

$$\partial_t U + (F(q, U) \cdot \nabla_q)U + \lambda(U - T^*(U(t, Tq))) = G(q, U) \text{ in } (0, \infty) \times \mathbb{R}^d. \quad (1.14)$$

A nice example to illustrate such a noise is the case in which, at certain random times, all the players of states 1 arrive in state 2 and vice versa. The operator  $T$  associated

to this transformation is then simply  $T(q) = (q_2, q_1, q_3, \dots, q_d)$ . The analogue of this situation in the continuous state space case yields the master equation

$$\begin{aligned} \partial_t U + H(x, \nabla U) - \sigma \Delta U + \left\langle \frac{\delta U}{\delta m}, -\Delta m - \operatorname{div}(D_p H(\nabla U)m) \right\rangle \\ + \lambda(U - \mathcal{T}^* U(x, \mathcal{T}m)) = f(x, m) \text{ in } (0, \infty) \times \mathcal{O} \times \mathcal{P}(\mathcal{O}), \end{aligned} \quad (1.15)$$

where  $\mathcal{T} : \mathcal{P}(\mathcal{O}) \rightarrow \mathcal{P}(\mathcal{O})$  denotes the linear map which acts on the players at the random times.

Another situation MFG can model is a one in which the costs of the players, i.e.  $f$  and  $\mathcal{G}$ , depend on an additional parameter  $p \in \mathbb{T}^n$  which evolves, independently of the players, according to

$$dp_t = b(p_t)dt + \sqrt{2}dW'_t, \quad (1.16)$$

where  $(W'_t)_{t \geq 0}$  is a  $n$ -dimensional Brownian motion on the aforementioned probability space. In this case, the master equation is given by

$$\begin{aligned} \partial_t U + H(x, \nabla_x U) - \sigma \Delta U + \left\langle \frac{\delta U}{\delta m}, -\Delta m - \operatorname{div}(D_p H(\nabla_x U)m) \right\rangle \\ - \Delta_p U - b(p)\nabla_p U = f(x, m, p) \text{ in } (0, \infty) \times \mathcal{O} \times \mathcal{P}(\mathcal{O}) \times \mathbb{R}^n, \\ U|_{t=0} = \mathcal{G} \text{ in } \mathcal{O} \times \mathcal{M}(\mathcal{O}) \times \mathbb{R}^n, \end{aligned} \quad (1.17)$$

where the unknown is the value function  $U : (0, \infty) \times \mathcal{O} \times \mathcal{P}(\mathcal{O}) \times \mathbb{R}^n \rightarrow \mathbb{R}$ .

Another type of common noise is a one in which all the trajectories of the players are perturbed by the same Brownian motion. The master equation associated to this noise contains second order terms with respect to the measure argument. For the sake of clarity, I shall not study such models here.

Obviously, all the previous situations can happen simultaneously and richer and richer models can always be imagined.

### 1.1.7 Bibliographical comments

In this section I present a, non-exhaustive, list of works, related or not to this manuscript, which present an interest for the MFG theory. As already mentioned above, MFG dates back to the work of Lasry and Lions [LL06a; LL06b; LL07; Lio11]. Quite rapidly, classical questions have been treated, such as numerical methods [ACD10; ACCD12], long time behavior of equilibria [Car+12; Car+13; CP19a] or several regularity issues [Lio11; Por15]. A probabilistic approach to MFG have also been introduced in [CD13] and developed in [CD18a; CD18b; CDL16]. The master equation was introduced in [Lio11]. After several papers on this topic [CCD14; CD14], the work [Car+19] established a robust framework to study such master equations. Around the same time as when the seminal papers of Lasry and Lions were published, the works [HCM03; HMC06; HCM07] proposed a derivation of the system (1.3)-(1.5) by taking limits of  $N$  players stochastic

games as  $N \rightarrow \infty$ . The problem of the convergence of such games toward a MFG limit has been the subject of several works since, like [Car+19; Cec+19; DLR19; DLR20; Lac20; LF21] to mention a few. Most of the previous works are concerned with continuous state spaces. MFG in finite state spaces have been the subject of several works such as [GMS13; CF20; CP19b].

## 1.2 Monotonicity to the rescue of the master equation : a first look

In this section, I present a notion of solution of the master equation I introduced in [Ber21b; Ber21a]. This notion allows us to work with value functions which are merely continuous in their measure argument ( $x \in \mathbb{R}^d$  for finite states and  $m \in \mathcal{M}(\mathcal{O})$  for a continuous state space). Moreover, as we shall see, this notion requires weaker assumptions than the existing literature to obtain results of existence and stability. This notion clearly relies on the monotone structure of the games in question.

### 1.2.1 A formal proof of uniqueness and the definitions of solutions

I focus here on the finite state space case. Let me recall that, under reasonable assumptions on  $F$  and  $G$ , the uniqueness of a smooth solution of (1.12) is standard in the PDE theory. However, finding a formal proof of uniqueness of smooth solutions, which does not rely too much on the smoothness of the solutions is often helpful. The following (formal) proof, borrowed from [Lio11], is of this type.

Consider  $U$  and  $V$  two smooth solutions of (1.12). Let me define  $W$  by

$$W(t, q, \tilde{q}) = \langle U(t, q) - V(t, \tilde{q}), q - \tilde{q} \rangle. \quad (1.18)$$

As a consequence of the rule of derivation of products, the function  $W$  satisfies on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$\begin{aligned} \partial_t W + F(q, U) \cdot \nabla_q W + F(\tilde{q}, V) \cdot \nabla_{\tilde{q}} W = & \langle G(q, U) - G(\tilde{q}, V), q - \tilde{q} \rangle \\ & + \langle F(q, U) - F(\tilde{q}, V), U - V \rangle, \end{aligned} \quad (1.19)$$

together with

$$W|_{t=0} = \langle U_0(q) - U_0(\tilde{q}), q - \tilde{q} \rangle. \quad (1.20)$$

In the monotone regime, the right hand side of the two previous equations are non-negative. Hence, a comparison principle yields that  $W$  is non-negative everywhere. Writing  $\tilde{q} = q - \epsilon z$  and passing to the limit  $\epsilon \rightarrow 0$ , we deduce from the non-negativity of  $W$  that

$$\forall t > 0, q, z \in \mathbb{R}^d, \langle U(t, q) - V(t, q), z \rangle \geq 0. \quad (1.21)$$

Hence the equality between  $U$  and  $V$ .

It is now possible to remark that in the previous proof, the only regularity really needed on  $U$  and  $V$  was one which allows us to use a comparison principle like result. From, at least, the theory of viscosity solutions [CL83; CIL92], we know that  $U$  and  $V$  do not need to be differentiable to "maintain" this proof. Moreover, let me observe that in the previous proof, because the main argument is on  $W$ , we do not need exactly the fact that  $U$  solves the master equation in the whole space, but only the fact that we have information on the equation satisfied by  $t, q \rightarrow \langle U(t, q) - V, q - \tilde{q} \rangle$  for some  $V, \tilde{q} \in \mathbb{R}^d$ , thanks to the doubling of variables. Those remarks lead to the following definition of solution of (1.12), which is set on  $B_R^1 := \{q \in \mathbb{R}^d, q \geq 0, q_1 + \dots + q_d \leq R\}$  for some  $R > 0$  to simplify the discussion.

**Definition 1.2.1.** *A continuous function  $U : (0, \infty) \times B_R^1 \rightarrow \mathbb{R}^d$  is a monotone solution of (1.12) on  $(0, \infty) \times B_R^1$  if*

- for any  $V \in \mathbb{R}^d, \tilde{q} \in B_R^1$ , and  $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$  a smooth function, for any  $(t_0, q_0) \in (0, \infty) \times B_R^1$  point of strict minimum of  $(t, q) \rightarrow \langle U(t, q) - V, q - \tilde{q} \rangle - \vartheta(t)$  on  $[0, t_0] \times B_R^1$ , the following holds :

$$\frac{d}{dt}\vartheta(t_0) \geq \langle F(q_0, U(t_0, q_0)), U(t_0, q_0) - V \rangle + \langle G(q_0, U(t_0, q_0)), q_0 - \tilde{q} \rangle. \quad (1.22)$$

- The initial condition holds.

$$U|_{t=0}(q) = U_0(q) \text{ on } B_R^1. \quad (1.23)$$

This definition captures the fact that, when one wants to apply a comparison principle as in the previous proof of uniqueness, it should be possible.

**Remark 1.2.1.** *The fact that the information is only required at point of strict minimum instead of points of minimum is not innocuous and shall be useful later on.*

The same ideas can be developed in the continuous state space case, when the equation is set on  $\mathcal{P}(\mathcal{O})$  to fix ideas, and leads to

**Definition 1.2.2.** *A continuous function  $U \in \mathcal{C}([0, \infty) \times \mathcal{O} \times \mathcal{P}(\mathcal{O}))$ , smooth in its second argument, is a monotone solution of (1.10) if*

- for any  $\mathcal{C}^2$  function  $\phi : \mathbb{T}^d \rightarrow \mathbb{R}$ , for any measure  $\nu \in \mathcal{M}(\mathbb{T}^d)$ , for any smooth function  $\vartheta : [0, \infty) \rightarrow \mathbb{R}$  and any point  $(t_0, m_0) \in (0, \infty) \times \mathcal{P}(\mathbb{T}^d)$  of strict minimum of  $(t, m) \rightarrow \langle U(t, \cdot, m) - \phi, m - \nu \rangle - \vartheta(t)$  on  $[0, t_0] \times \mathcal{P}(\mathcal{O})$ , the following holds

$$\begin{aligned} \frac{d\vartheta}{dt}(t_0) + \langle -\sigma\Delta U + H(\cdot, \nabla_x U), m_0 - \nu \rangle &\geq \langle f(\cdot, m_0), m_0 - \nu \rangle \\ &- \langle U - \phi, \text{div}(D_p H(\nabla_x U)m_0) \rangle - \sigma\langle \Delta(U - \phi), m_0 \rangle. \end{aligned} \quad (1.24)$$

- the initial condition holds

$$U(0, \cdot, \cdot) = \mathcal{G}(\cdot, \cdot). \quad (1.25)$$



Maybe the main difference here lies in the fact that some regularity is required on the value function  $U$  with respect to the  $x$  variable. Some assumptions on this regularity seems to be necessary, although it seems clear that it can be reduced to merely  $C^{1,1}$  continuity. However, because we are interested in the case  $\sigma > 0$ , the value function is sufficiently regular and we do not have to worry about such a difficulty here.

## 1.2.2 Uniqueness and stability of monotone solutions

Being nearly built on a proof of uniqueness, the following results should be no surprise to the reader.

**Theorem 1.2.1.** *In the monotone regime, under a stability assumption on  $F$  at the boundary of  $B_R^1$ , there exists at most one monotone solution of (1.12).*

**Theorem 1.2.2.** *In the monotone regime, two monotone solutions  $U$  and  $V$  of (1.10) are such that  $\nabla_x U = \nabla_x V$ . If  $f$  is strictly monotone, there exists at most one monotone solution of (1.10).*

The additional assumption on  $f$  has to do with the fact that because we are only considering the master equation on  $\mathcal{P}(\mathbb{T}^d)$ , some information lacks, in some sense, compare to what we would have obtained if the equation was set on  $\mathcal{M}(\mathbb{T}^d)$ . More precisely, this has to do with the fact that  $(\mathcal{P}(\mathcal{O}) - \mathcal{P}(\mathcal{O}))^\perp$  is the set of constant functions.<sup>1</sup>

Due to the fact that we are only looking at strict minima in Definitions 1.2.1 and 1.2.2, stability results are quite easy to obtain. The following two results hold.

**Theorem 1.2.3.** *Assume that  $((F_n, G_n))_{n \geq 0}$  converges toward  $(F, G) : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$  in a suitable sense, and that a sequence of monotone solutions  $(U_n)_{n \geq 0}$  of (1.12) (associated with  $F_n$  and  $G_n$ ) converges locally uniformly toward some function  $U$ . Then  $U$  is a monotone solution of (1.12) associated with  $F$  and  $G$ .*

**Theorem 1.2.4.** *Assume that  $((H_n, f_n))_{n \geq 0}$  converges toward  $(H, f)$ ,  $H : \mathcal{O} \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $f : \mathcal{M}(\mathbb{T}^d) \rightarrow \mathcal{C}(\mathcal{O})$  in a suitable sense, and that a sequence of monotone solutions  $(U_n)_{n \geq 0}$  of (1.10) (associated with  $H_n$  and  $f_n$ ) converges, in a suitable sense, toward some function  $U$ . Then  $U$  is a monotone solution of (1.12) associated with  $H$  and  $f$ .*

These stability results are particularly convenient to establish new results of existence, by considering approximating sequence for general classes of couplings  $F, G, H$  and  $f$ . Moreover, such results could also have been stated concerning the stability of the solutions with respect to other parameters of the model, like the initial condition or the common noise for instance.

**Remark 1.2.2.** *Analogous results could have been established for master equations involving first order common noise terms such as (1.14), (1.15) or (1.17), namely by treating the additional terms exactly as the Hamiltonian for instance.*

<sup>1</sup>The formal orthogonality is understood through the  $L^2$  scalar product.

### 1.2.3 Another stability result : from finite state space toward a continuous one

Another type of stability that holds for monotone solutions is a one about the nature of the state space. With Alekos Cecchin, we proved in [BC22] a result of convergence of MFG with finite states toward a MFG with a continuous state space. The framework we placed ourselves in is the following.

At the limit : The continuous state space is the unit circle, hence  $\mathbb{T}$  of dimension 1. Both the Hamiltonian  $H$  and the costs  $f$  and  $\mathcal{G}$  are smooth and the monotone regime holds. There is a common noise of the form of the one modeled in the equation (1.15), with a smooth linear operator  $\mathcal{T}$  which is given as a convolution with a smooth kernel  $K$ .

At the discrete level : The state space is  $E_d := \{0, d^{-1}, \dots, d^{-1}(d-1)\}$  whose elements can also be denoted as elements of  $\mathbb{Z}/(d\mathbb{Z})$ . The discretization of the master equation is similar to what one has for numerical methods using finite difference schemes. That is, the discretized Hamiltonian  $g_d$  is given as a function  $E_d \times \mathbb{R}^2 \rightarrow \mathbb{R}, x, p_1, p_2 \rightarrow g_d(x, p_1, p_2)$ , which is non-increasing in its first argument, non-decreasing with respect to the second and satisfies the consistency condition  $g_d(x, p, p) = H(x, p)$ . Hence the discretization  $G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  of the Hamiltonian (with the running cost and the i.i.d. noise) reads

$$G^i(q, U) := -g\left(d^{-1}i, \frac{U^{i+1} - U^i}{d^{-1}}, \frac{U^i - U^{i-1}}{d^{-1}}\right) + \sigma \frac{U^{i+1} + U^{i-1} - 2U^i}{d^{-2}} + f(m_q)(d^{-1}i), \quad (1.26)$$

where  $m_q := \sum_{i=1}^d q_i \delta_{d^{-1}i}$ . Thus, the discretized master equation is given by

$$\begin{aligned} \partial_t U(t, x, q) - (D_U G(q, U) \cdot \nabla_q) U(t, x, q) + \lambda(U - T_d^* U(t, x, T_d q)) \\ = G^x(q, U) \text{ on } (0, \infty) \times E_d \times P_d, \end{aligned} \quad (1.27)$$

where  $P_d := \{q \in \mathbb{O}_d \mid \sum_i q_i = 1\}$  which is isomorphic to  $B_1^1 \cap \mathbb{O}_{d-1}$ , and  $T_d$  is a standard discretization of the mapping  $\mathcal{T}$  that we do not detail here. The initial condition is simply given by

$$U|_{t=0}(x, q) = \mathcal{G}(m_q)(x), \text{ on } E_d \times P_d. \quad (1.28)$$

Since the monotone regime holds in the continuous setting, it also holds in this discretized one because of the discretization we chose. It is a classical application of the existing literature that there exists a unique monotone solution of (1.27). The results we were able to prove with Alekos Cecchin is

**Theorem 1.2.5.** *Let  $(U_d)_{d \geq 1}$  be the sequence of monotone solutions of (1.27). This sequence converges toward  $U$ , the unique solution of (1.10) in the following sense*

$$U_d(t, d^{-1}i, m_q) \xrightarrow{|d^{-1}i-x| \rightarrow 0, d_1(m_q, m) \rightarrow 0} U(t, x, m), \quad (1.29)$$

*uniformly in  $t \geq 0, x \in \mathbb{T}, m \in \mathcal{P}(\mathbb{T})$ . Moreover, there is a uniform rate of convergence of order  $d^{-\frac{1}{3}}$  if  $U$  is a classical solution of (1.10).*

#### 1.2.4 Remark on a fundamental tool used with monotone solutions

Even though the proofs are mainly omitted in this *manuscript d'HDR*, I now make some comments on them. The proofs of uniqueness are established following a doubling of variables technique. Namely we define  $W$  as in (1.18) and use the definition of monotone solutions at the point of minimum of  $W$ . Since the definition of monotone solution requires a minimum to be strict, a problem might arise if the minimum of  $W$  is not strict. The key argument I used here, is the so-called Stegall's Lemma. This result gives conditions under which one can perturb a function by a small linear map, so that the perturbed function has a strict minimum. Such a result turns out to be deeply entangled with geometric properties of the set on which the function takes its arguments. I recommend the reader the following for more details on this question [Phe09; Ste78; Ste86; Bou81].

In the case of a finite state space, such a result is quite easy to obtain. I have been able to give, what I think is, a new proof of this result using Alexandrov's Theorem for convex functions, and then extend this proof to the case of the Wasserstein space, which did not clearly, at least for me, fall under the scope of the existing literature. However when reading [Bou81], it seems clear that such an extension was already well understood by several persons.

Seeing that defining monotone solutions at point of strict minima only raises some difficulties, one may ask why not consider all the point of minima. Clearly this approach yields a stronger notion of solutions (in the sense that it is more restrictive), which, in view of the uniqueness result I stated, is not necessary. Moreover, the aforementioned stability results are then more difficult to prove since points of minimum are not always limits of points of minimum.

#### 1.2.5 Existence of monotone solutions

Clearly, the notion of monotone solution is consistent with the usual notion of classical solutions of the master equation. However the converse has no reason to be true in general and I have established the existence of monotone solutions for (1.10) under assumptions for which no existence of classical solution is known. I focus here on the continuous state space, as the finite case is somehow similar.

The main idea to prove such results is to find an a priori estimate on solutions of master equations. We then introduce a sequence of sufficiently smooth couplings  $(f_n)_{n \geq 0}$  such that

- For all  $n \geq 0$ , there exists a classical solution of (1.10) associated to  $f_n$ .
- The sequence  $(f_n)_{n \geq 0}$  converges locally uniformly toward  $f_\infty$ .
- The a priori estimate is satisfied for all  $n \in \mathbb{N} \cup \{\infty\}$ .

Such an a priori estimate is for instance

**Proposition 1.2.1.** *Assume that  $U$  is a classical solution of (1.10) and that there exists  $C > 0, \alpha, \beta \in (0, 1)$  such that*

- $$\sup_{\mu, \nu \in \mathcal{P}(\mathbb{T}^d)} \frac{\|f(\mu) - f(\nu)\|_{1+\alpha}}{\mathbf{d}_1(\mu, \nu)^\beta} + \frac{\|U_0(\mu) - U_0(\nu)\|_{2+\alpha}}{\mathbf{d}_1(\mu, \nu)^\beta} \leq C. \quad (1.30)$$

- $H$  is uniformly convex and smooth.

Then there exists  $C' > 0$  depending only on  $C, \alpha$  and  $\beta$  such that

$$|U(t, x, m) - U(t', x', m')| \leq C'(|t - t'|^{\frac{\gamma}{2}} + |x - x'| + \mathbf{d}_1(m, m')^\gamma), \quad (1.31)$$

where  $\gamma = (2(\beta^{-1} - \frac{1}{2}))^{-1} \in (0, 1)$ .

This a priori estimate is helpful to prove the two results

**Theorem 1.2.6.** *Under the assumptions of the previous Proposition, if  $f$  can be approximated by smooth functions, then there exists a monotone solution of (1.10).*

**Theorem 1.2.7.** *Under the assumptions of the previous Theorem, if  $\mathcal{T}$  is a convolution operator with a smooth kernel, then there exists a monotone solution of (1.15).*

## 1.2.6 Bibliographical comments

The work presented in this section originates from [Ber21b; Ber21a; BC22]. Several other works have been devoted to the study of the master equation. In finite state spaces, let me mention [Cec+19] in dimension  $d = 1$  without monotonicity, [Bay+21] with regularizing noises, [CD21] in the potential case and [BLL19]. In the continuous state space case, additionally to [Car+19], several works have been conducted such as [MZ20; GM20; Gan+21]. In [MZ20], the authors gave a characterization of the value function which relies on the stability of the value, due to the monotone regime. Such an idea is also present in this section, even if we are here able to characterize the value function without using the system of characteristics. In [GM20] the authors are concerned with the potential case, in which the master equation is reduced to a Hamilton-Jacobi equation in infinite dimension. In [Gan+21], the authors studied the master equation under other geometric assumptions than the monotone one (which they called displacement monotonicity by reference to displacement convexity). Let us also mention the paper [IZ21] which is concerned with the study of the master equation in a non-monotone regime. Up to this point, no general framework has been proposed and we think that this paper can start one, in the monotone regime.

More recently, [CS21] used the ideas of this part to treat the case of a common noise which yields second order terms in the master equation, in the case  $\sigma = 0$  (i.e. no individual noise).

Concerning the convergence of MFG with finite states toward MFG with a continuum of states, to my knowledge, only the work [HS19] addresses this question directly, although most of the literature on numerical methods for MFG is obviously concerned with this question, see for instance the seminal works [ACD10; ACCD12].

## 1.3 Monotone solutions : a tool to write down singular mean field games master equations

In this section, I explain how monotone solutions enable us to characterize value functions for MFG which seem too singular to actually write down a master equation as a PDE.

### 1.3.1 MFG of optimal stopping with finite states

Let me introduce a slight precision of the model in finite state space that I have been looking at until now. Since the aim of this section is to derive a master equation for a MFG in which the players can exit the game, the variable  $q \in \mathbb{R}^d$  which describes the repartition of players has to be closer to a measure. That is why I shall interpret here exactly  $q^i$  as the mass of players in the state  $i \in \{1, \dots, d\}$ . Moreover, for technical reasons, we only look at the stationary discounted master equation, i.e. the horizon of the underlying game is  $+\infty$  and the players discount their cost at rate  $r > 0$ . We also consider a common noise modeled by  $\lambda > 0$  and a linear map  $T$  such that  $T(B_1^1) \subset B_1^1$ . This leads to the following master equation (without exiting players for the moment)

$$rU + (F(q, U) \cdot \nabla_q)U + \lambda(U - T^*U(Tq)) = G(q, U) \text{ in } \mathbb{O}_d, \quad (1.32)$$

where the unknown is  $U : \mathbb{O}_d \rightarrow \mathbb{R}^d$ . The monotone regime is in force and  $F$  satisfies the stability condition

$$F^i(q, p) \leq 0 \text{ if } q^i = 0, \forall p \in \mathbb{R}^d, \quad (1.33)$$

as well as a growth condition for  $x$  large. Such conditions ensure that i) the evolution of the repartition of players induced by the master equation can indeed be interpreted as a repartition of players, i.e. it is valued in  $\mathbb{O}_d$  at anytime, ii) starting from a given total mass of players, this total mass shall be bounded uniformly in time.

Let me now turn to the understanding of the effect of allowing the players to leave the game at their will. Because, if suddenly a lot of players decide to leave the game, the underlying evolution of the density of players is singular, writing the master equation is not an obvious task. I proceed as in [Ber18] and introduce first a penalized game in which players cannot exactly leave as soon as they want, but rather control the bounded intensity of a Poisson clock whose ringing implies their exit. Each player is supposed to be assigned to an independent clock. This means that the players can try to leave the game but that they shall in general wait for some (random) time before being able to leave it. This feature makes the game more regular and allows us to write a master equation. If I bound the intensity of the clocks by  $\epsilon^{-1}$  for  $\epsilon > 0$  and if the cost to leave the game is 0, then the master equation is

$$rU + \left( \left( (F(q, U) + \frac{1}{\epsilon} \beta'(U) * q) \right) \cdot \nabla_q \right) U + \frac{1}{\epsilon} \beta(U) = G(q, U) \text{ in } \mathbb{O}_d, \quad (1.34)$$

where  $\beta(s)$  stands for the positive part of  $s \in \mathbb{R}$ ,  $*$  is the term by term product and  $\beta(U)$  and  $\beta'(U)$  are understood component wise.

The first remark I make is that whatever  $\epsilon > 0$ , (1.34) satisfies the assumptions of the monotone regime, and thus, is well-posed (at least formally since  $\beta'$  is singular).

The second remark I make is that in the limit  $\epsilon \rightarrow 0$ , we expect the value  $U$  of the MFG to be non-positive component wise. Indeed because the cost to leave the game is 0, the value of the game cannot exceed it.

Following the ideas of section 1.2.1, to define a notion of monotone solution, we only need information on the point  $q_0$  of (strict) minimum of

$$W : q \rightarrow \langle U(q) - V, x - \tilde{q} \rangle, \quad (1.35)$$

for  $\tilde{q} \in \mathbb{O}_d$  and  $V \in \mathbb{R}^d$  such that  $V \leq 0$ . Indeed we only need to take  $V$  which is non-positive component wise since, to establish a result of uniqueness,  $V$  shall be another monotone solution of the master equation, thus non-positive component wise.

Let me now remark that for such  $\tilde{q} \in \mathbb{O}_d$  and  $V \in \mathbb{R}^d, V \leq 0$ , if  $U$  is a (classical) monotone solution of (1.34), then if  $q_0$  is a strict minimum of  $W$  defined in (1.35), the following holds

$$r \langle U(q_0), q_0 - \tilde{q} \rangle \geq \langle G(q_0, U(q_0)), q_0 - \tilde{q} \rangle + \langle F(q_0, U(q_0)), U(q_0) - V \rangle. \quad (1.36)$$

This is the result of a simple computation involving  $\beta$ . It is quite remarkable that, in the previous equation, no term involves neither  $\beta$  nor  $\epsilon$ , and thus should hold in the limit  $\epsilon \rightarrow 0$ . This naturally leads us to the following definition

**Definition 1.3.1.** *A continuous function  $U : \mathbb{O}_d \rightarrow \mathbb{R}^d$  is a monotone solution of the master equation of the MFG with optimal stopping with exit cost 0 if*

- For all  $q \in \mathbb{O}_d$ ,  $U(q) \leq 0$ .
- For all  $V \in \mathbb{R}^d, V \leq 0, \tilde{q} \in \mathbb{O}_d, q_0 \in \mathbb{O}_d$  point of strict minimum of  $q \rightarrow \langle U(q) - V, q - \tilde{q} \rangle$ , the following holds

$$r \langle U(q_0), q_0 - \tilde{q} \rangle \geq \langle G(q_0, U(q_0)), q_0 - \tilde{q} \rangle + \langle F(q_0, U(q_0)), U(q_0) - V \rangle. \quad (1.37)$$

As usual, the same properties of uniqueness and stability hold for this notion of solution. The existence of such a monotone solution can be established through an a priori estimate on (1.34), independent of  $\epsilon > 0$  and the use of the stability of the solutions when  $\epsilon \rightarrow 0$ . In this situation, due to the singular nature of the underlying evolution of the repartition of players, the fact that information is only asked at points of strict minima for monotone solutions is very helpful. Let me also remark that in this situation, deriving the strategies of the players at the equilibrium is not necessary easy. I refer to [Ber21b] for more details on these questions.

### 1.3.2 A comment on the method we just presented

Before presenting the same method on another example, let me highlight some key features of the previous section.

First of all, the master equation has not been properly written. However a characterization of the value of the MFG was still possible. The main difficulty to effectively derive the master equation as a PDE lies in the fact that I am not able to identify the limit as  $\epsilon \rightarrow 0$  of the term

$$q \rightarrow \frac{1}{\epsilon} \beta'(U(q)) * q, \quad (1.38)$$

even for a known function  $U$ . This term, which is already quite ill defined for  $\epsilon > 0$  since  $\beta$  is not smooth at 0, has no clear limit when  $\epsilon \rightarrow 0$ . This is mainly because we expect  $U$  to converge toward 0 on non negligible sets. Let me recall that this term models the dynamics of the repartition of the players in the state space. The term  $\epsilon^{-1} \beta(U)$  is well understood in the limit thanks to the theory of obstacle problems [LS67; Caf98].

The main advantages of monotone solutions here, is that we do not need the precise limit of this operator to characterize the value function  $U$ . We only need to understand how it behaves on certain functions. Indeed, formally, if we denote by  $A[U, q]$  this limit operator, the associated term appearing in the master equation is

$$(A[U, q] \cdot \nabla_q)U. \quad (1.39)$$

On the other hand, one can view monotone solutions as the following. We look at the PDE satisfied by  $q \rightarrow W(q) := \langle U(q) - V, q - \tilde{q} \rangle$ . By doing so, the terms resulting of (1.39) in the PDE satisfied by  $W$  can be written as

$$A[U, q] \cdot \nabla_q W - \langle A[U, q], U - V \rangle. \quad (1.40)$$

Hence, because we are only interested in points of minimum of  $W$ , we can attribute a sign to the first term. Most of the time we would like this term to be 0 but because, for instance, minima can be located at the boundary of the domain, only a sign is in general available. This yields that formally, only the effect of  $A$  on the difference  $U - V$  is needed to characterize a value function.

This is, in my opinion, a remarkable feature, namely in terms of duality, as  $A$  was the operator through which the density of players evolve, and I am here stating that, in the monotone regime, to characterize a value for the MFG, we only need to understand it on value functions.

Let me conclude this section by stating that I do not claim that this method is sufficient (even though it should be at least helpful) to characterize the equilibria of the game, i.e. the precise evolution of the population of players. At the moment it only brings a characterization of the value of the MFG.

### 1.3.3 Mean field games with incomplete information

In this section, I present a MFG in which the players do not observe the repartition of the other players, but only the payments of all the players. Such games are meaningful, namely for their applications in Economics or Telecommunications. This section is taken from [Ber22].

The framework of this game is the same as the one we presented in section 1.1.2, except for the fact that the initial (and thus possibly for all time) repartition of players is unknown. However, at an any time  $t$ , the players observe the whole function  $f(m_t) : \mathcal{O} \rightarrow \mathbb{R}$ , where  $m_t$  is the actual, unknown, repartition of players at time  $t$ . Clearly, if  $f$  is injective seen as  $\mathcal{P}(\mathbb{T}^d) \rightarrow \mathcal{C}(\mathcal{O})$ , then the setting is the same as the usual, since the players can infer the repartition of players from the payments. Obviously, this is not always the case, in particular in Economics where the payments are quite often only a function of a few moments of the repartition of players.

In such a context, the value of the MFG cannot be a function of the repartition, as it is unknown. It is tempting to see it as a function of the observed payments, however this would not take into account the fact that the players can learn in the game. That is they accumulate knowledge through the evolution of the payments. Hence, they build a belief on the unknown repartition of players. It is somehow classical from game theory that this belief, which is common since all the information is public, becomes part of the state of the game and thus that the value function depends on it.

As it was the case for optimal stopping in MFG, the precise evolution of the belief of the players is difficult to grasp. Since the leitmotiv of this section is that only a weak understanding of this evolution is necessary to define a notion of value function through monotone solutions, this does not seem to be an insurmountable problem. In any case I now present a formal description of this evolution to introduce the way I am going to define a value function.

The belief of the players  $\mu$  is an element of  $\mathcal{P}(\mathcal{P}(\mathbb{T}^d))$  which has to be coherent with the fact that the payments are known. Hence, it can only be supported on a set of the form  $f^{-1}(\{g\})$  for some  $g \in \mathcal{C}(\mathcal{O})$ . Let me define

$$\mathcal{A} := \{\nu \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d)), \exists g \in f(\mathcal{P}(\mathbb{T}^d)), \nu \text{ almost everywhere } f(m) = g\}. \quad (1.41)$$

If  $f : \mathcal{M}(\mathcal{O}) \rightarrow \mathcal{C}(\mathcal{O})$  is continuous then  $\mathcal{A}$  is compact.

### A formal derivation of the evolution of the belief

Let me fix for the moment the strategies of the players to be given in feedback by the function  $b$ . Denote by  $K_s$  the semi-group generated by the Kolmogorov equation with drift  $b$ , that is  $K_s m_0$  is the solution of

$$\begin{aligned} \partial_t m - \sigma \Delta m + \operatorname{div}(bm) &= 0 \text{ in } (0, \infty) \times \mathbb{T}^d, \\ m|_{t=0} &= m_0, \end{aligned} \quad (1.42)$$

taken at time  $s \geq 0$ . Define the total information received by the players up to time  $t$  when the initial repartition of players is  $m$  by the function

$$\mathcal{F}(t, m) = (f(K_s m))_{s \in [0, t]}. \quad (1.43)$$

The common belief of the players evolves, formally, as a mix of the two following rules.



1. The measure  $\mu \in \mathcal{A}$  is weighting elements of  $\mathcal{P}(\mathbb{T}^d)$  which are transported along the **same** Kolmogorov equation. Indeed, the strategies of the players, hence the drift in the Kolmogorov equation, cannot depend on the different elements of  $\mathcal{P}(\mathbb{T}^d)$  which are "weighted" by  $\mu$  as I supposed the value is a function only of the belief.
2. At any time, the belief  $\mu$  is disintegrated (or conditioned) along the payment function  $f$  into  $(\mu_\theta)_{\theta \in f(\mathcal{P}(\mathbb{T}^d))}$  and the belief  $\mu_\theta$  which corresponds to the observed payments  $\theta$  becomes the new belief. Let me refer to chapter 45 of [Fre00] for more information on the disintegration (or conditioning) of measures.

Given a drift  $b$  for the Kolmogorov equation, an initial belief  $\mu_0 \in \mathcal{A}$ , we can indeed compute the evolution of the belief for  $\mu_0$  almost every  $m_0 \in \mathcal{P}(\mathbb{T}^d)$ . The belief  $\mu_t$  at time  $t$  is simply given by the "free" evolution (i.e. like if only the first rule above applied) of the conditioned initial belief to the whole information. This can be summarized in the relation

$$\mu_t = (K_t)_\# \nu_0, \quad (1.44)$$

where  $\nu_0$  is the measure obtained by conditioning (or disintegrating)  $\mu_0$  with the whole information  $\mathcal{F}(t, m_0)$ , along  $\mathcal{F}(t, \cdot)$ .

Now that we have defined an evolution for the belief, depending on the actual repartition of players, we can define an operator  $A$  associated to this evolution. If we assume that the players are risk neutral, according to their belief, it is natural to define

$$A[b, \mu, \phi] := \lim_{t \rightarrow 0} \mathbb{E}_\mu \left[ \frac{\phi(\mu_t^m) - \phi(\mu)}{t} \right], \quad (1.45)$$

where  $\phi : \mathcal{A} \rightarrow \mathbb{R}$  and  $\mu_t^m$  denotes the belief one obtains at time  $t$ , if the initial belief is  $\mu$  and the initial repartition of players is  $m$ . The expectation means that  $m$  is seen as a random variable of law  $\mu$ . Of course  $b$  is the drift in the associated Kolmogorov PDE. I do not claim that  $A$  is well defined here.

### The value function for such MFG

Assuming that  $A$  is indeed well defined, which is not the case, the "formal" master equation one obtains is

$$\begin{aligned} \partial_t U - \sigma U + H(x, \nabla_x U) - A[-D_p H(\nabla_x U), \mu, U] &= \tilde{f}(x, \mu), \text{ in } (0, \infty) \times \mathcal{O} \times \mathcal{A}, \\ U|_{t=0}(x, \mu) &= \tilde{\mathcal{G}}(x, \mu) \text{ in } \mathcal{O} \times \mathcal{A}, \end{aligned} \quad (1.46)$$

where  $\tilde{f}(x, \mu) = \int_{\mathcal{P}(\mathcal{O})} f(x, m) \mu(dm)$  and  $\tilde{\mathcal{G}}(x, \mu) = \int_{\mathcal{P}(\mathcal{O})} \mathcal{G}(x, m) \mu(dm)$ . Remark that those cost functions are written as integrals against  $\mu$ , but since the value of the MFG is only defined on  $\mathcal{A}$ , the first one could have been replaced by  $f(m)$  for any  $m$  in the support of  $\mu$ .

As already mentioned, we only need to understand this operator  $A$  on a small class of functions to define a notion of value of the MFG using the techniques of monotone solutions.

Here a comment has to be made on the fact that we have changed the underlying dual structure of the MFG. Indeed, previously, we used everywhere the implicit duality between  $\mathcal{C}(\mathcal{O})$  and  $\mathcal{P}(\mathcal{O})$ . Here we have to understand the duality between  $\mathcal{C}(\mathcal{O})$  and  $\mathcal{P}(\mathcal{P}(\mathcal{O}))$ . I shall here choose the natural duality product

$$\langle \phi, \mu \rangle := \int_{\mathcal{P}(\mathcal{O})} \int_{\mathcal{O}} \phi(x) m(dx) \mu(dm), \quad (1.47)$$

where  $\phi \in \mathcal{C}(\mathcal{O})$  and  $\mu \in \mathcal{P}(\mathcal{P}(\mathcal{O}))$ . Having this in mind, we only have to understand how  $A$  behaves on linear functions on  $\mathcal{P}(\mathcal{P}(\mathcal{O}))$  of the form  $\Psi : \mu \rightarrow \langle \phi, \mu \rangle$  for  $\phi \in \mathcal{C}^2(\mathcal{O})$ . It is quite immediate to check that  $A$  is indeed well defined on such functions and that, whatever  $f$  or the drift  $b$ ,

$$A[b, \mu, \Psi] := \int_{\mathcal{P}(\mathcal{O})} \int_{\mathbb{T}^d} \sigma \Delta \phi(x) + b \cdot \nabla_x \phi(x) m(dx) \mu(dm). \quad (1.48)$$

Recalling section 1.3.2, this naturally leads us to the following definition of monotone solution.

**Definition 1.3.2.** *We say that a continuous function  $U : [0, T] \times \mathbb{T}^d \times \mathcal{A} \rightarrow \mathbb{R}$ , smooth in its second argument, is a value of the MFG with observed payments and unknown repartition of players if :*

- for any  $\mathcal{C}^2$  function  $\phi \in \mathcal{C}(\mathcal{O})$ , for any measure  $\nu \in \mathcal{M}(\mathcal{P}(\mathcal{O}))$ , for any smooth function  $\vartheta : [0, \infty) \rightarrow \mathbb{R}$  and any point  $(t_0, \mu_0) \in (0, \infty) \times \mathcal{A}$  of minimum of  $(t, \mu) \rightarrow \langle U(t, \cdot, \mu) - \phi, \mu - \nu \rangle - \vartheta(t)$  on  $[0, t_0] \times \mathcal{A}$ , the following holds

$$\begin{aligned} \frac{d\vartheta}{dt}(t_0) + \langle -\sigma \Delta U + H(\cdot, \nabla_x U), \mu_0 - \nu \rangle &\geq \langle \tilde{f}(\cdot, \mu_0), \mu_0 - \nu \rangle \\ &- \int_{\mathcal{P}(\mathcal{O})} \langle U - \phi, \sigma \Delta m + \operatorname{div}(D_p H(\nabla_x U) m) \rangle \mu(dm). \end{aligned} \quad (1.49)$$

- the initial condition holds

$$U|_{t=0}(x, \mu) = \tilde{\mathcal{G}}(x, \mu). \quad (1.50)$$

It remains to check that this MFG has indeed a monotone structure, so that the previous definition can be helpful to characterize a unique value for the MFG. The following proposition gives a favorable answer to this question.

**Proposition 1.3.1.** *Assume that  $f : \mathcal{P}(\mathcal{O}) \rightarrow \mathcal{C}(\mathcal{O})$  is a monotone operator and define  $\tilde{f} : \mu \rightarrow \int_{\mathcal{P}(\mathcal{O})} f(m) \mu(dm)$ . Then  $\tilde{f}$  is a monotone operator on  $\mathcal{A}$ .*

The previous statement is no longer true in general if one is interested in the monotonicity of  $\tilde{f}$  on the whole  $\mathcal{P}(\mathcal{O})$ .

For the moment, I only know partial results of uniqueness which are summarize in the next result.

**Theorem 1.3.1.** *Assume that we are in the monotone regime and that  $f$  is strictly monotone. Then two value functions  $U$  and  $V$  of the MFG in the sense of Definition 1.3.2 satisfy*

- $U(t, \cdot, \mu) = V(t, \cdot, \mu)$  for any  $t \geq 0$  and  $\mu = \delta_m$  for any  $m \in \mathcal{P}(\mathcal{O})$ .
- For any  $t \geq 0, \mu \in \mathcal{A}$ ,  $(U - V) \in (f^{-1}(\tilde{f}(\mu)) - f^{-1}(\tilde{f}(\mu)))^\perp$ .<sup>2</sup>
- If  $f$  only depends on  $m$  through its first moment, then  $D_x^2 U = D_x^2 V$  everywhere.

Let me comment the second point of this result. By trying the usual proof of uniqueness on this new problem, one can indeed derive as usually the fact that for all  $t \geq 0, \mu, \nu \in \mathcal{A}$ ,

$$\langle U(t, \mu) - V(t, \nu), \mu - \nu \rangle \geq 0. \quad (1.51)$$

However, because  $\mathcal{A}$  is not convex, this only yields the second point in the Theorem and not a full equality between  $U$  and  $V$ . Despite being a complete uniqueness result, it may still be helpful to analyze some problems. Moreover this difficulty is overcome partially if  $f$  only depends on  $m$  through its first moment.

Another open question to this section is the question of existence of such a monotone solution.

### 1.3.4 Bibliographical comments

MFG in which agents have "singular" controls, such as leaving the game, or impulse controls, have been the subject of several works, although none of them is concerned with the master equation. The reader can refer to [GP15; Nut18; CDL17; NSMT20; Ber18; BDT20] for optimal stopping for instance, to [Ber20b; GP16] for impulse controls, and to [Cam+20] for singular controls. Section 1.3.1 is taken from [Ber21b].

MFG with incomplete information on the repartition of players have only been studied in an explicitly solvable case for the moment to my knowledge [CJ18]. A model in which all the players are learning an unknown parameter has been derived by P.-L. Lions. Other framework in which agents do not know their individual state, but use some filtering for example, have been proposed, such as [CK16], but they are not related to the aforementioned studied. Section 1.3.3 originates from [Ber22].

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<sup>2</sup>Once again the orthogonality is understood by extending the  $L^2$  scalar product.

## 1.4 Another example of the usefulness of monotonicity in mean field games

### 1.4.1 The mean field planning problem

In [BLL21] we studied with J.-M. Lasry and P.-L. Lions the so called mean field planning problem, in the finite state space setting, in the presence of a common noise. The mean field planning problem is a variant of MFG which aims at modeling situations in which, whatever the initial repartition of players, they will end up in the same repartition at the end of the game. If the situations at interest were not games but optimization problems, this behavior shall be modeled by a constraint at the final time. However this is not the case and the "target" final repartition is not a constraint for the players. Indeed they cannot control this repartition as they are, by nature, infinitesimal. Hence, in order for them to behave so that they will indeed be distributed along the required distribution at the final time, they shall be given strong incentives to do so.

### 1.4.2 Mathematical analysis

More precisely, we were interested with taking the limit  $\epsilon \rightarrow 0$  of the master equation

$$\begin{aligned} (i) \quad & \partial_t U + (F(q, U) \cdot \nabla_q)U + \lambda(U - T^*U(t, Tq)) = G(q, U) \text{ on } (0, \infty) \times \mathbb{R}^d, \\ (ii) \quad & U|_{t=0} = \frac{1}{\epsilon}(q - q_0) \text{ on } \mathbb{R}^d. \end{aligned} \tag{1.52}$$

In this master equation,  $q_0$  is the final target repartition and  $\epsilon^{-1} > 0$  represents the strength of the incentives given to the players. Indeed since  $\epsilon^{-1}(q - q_0)$  represents the terminal cost faced by the players, when the repartition is  $q$ , they should tend to be finally distributed along  $q_0$ .

Let me recall that (1.52) is a classical master equation (for  $\epsilon > 0$ ) and that in a monotone regime, it does not raise any new mathematical difficulty. I shall thus only focus on the passage to the limit  $\epsilon \rightarrow 0$ , and on the singularity that it creates at  $t = 0$ .

The monotone structure provides two key arguments in the analysis of this limit. The first one is a, uniform in  $\epsilon$ , a priori estimate on solutions of (1.52). The second one is a characterization of the limit  $t \rightarrow 0$  in the limit equation.

The a priori estimate that we were able to prove requires more restrictive assumptions than the monotone regime and is stated as follows.

**Proposition 1.4.1.** *Assume that there exists  $\alpha > 0$  such that for any  $q, \tilde{q}, U, V \in \mathbb{R}^d$*

$$\langle G(q, U) - G(\tilde{q}, V), q - \tilde{q} \rangle + \langle F(q, U) - F(\tilde{q}, V), U - V \rangle \geq \alpha \|U - V\|^2. \tag{1.53}$$

*Then, there exists  $C > 0$  such that any smooth solution  $U$  of (1.52)-(i), which is monotone at  $t = 0$ , satisfies*

$$\forall t \in (0, 1], q \in \mathbb{R}^d, \|D_q U(t, q)\| \leq \frac{C}{t}. \tag{1.54}$$

In particular,  $C$  does not depend on  $U|_{t=0}$ .

This regularity estimate is somehow sufficient to study the master equation away from  $t = 0$ .

It remains to understand the behavior of the solution near the singularity. To analyse this phenomenon, we used once again the monotonicity of the master equation. Let me remark first that

$$\frac{1}{\epsilon}(\cdot - q_0) \xrightarrow{\mathcal{G}} A_{q_0}, \quad (1.55)$$

where  $A_{q_0}$  is the maximal monotone operator from  $\mathbb{R}^d$  to itself whose domain is reduced to  $\{q_0\}$  and where the previous convergence is in the sense of graphs. Hence, it is natural to expect the value function  $U$  of the mean field planning problem to converge toward  $A_{q_0}$  in the sense of graphs as  $t \rightarrow 0$ . We were able both to establish that this is the case, and that it is sufficient to characterize the value of the mean field planning problem. To sum up, we proved in [BLL21] the

**Theorem 1.4.1.** *Under the assumptions of the previous Proposition, there is a unique function  $U$  satisfying (1.54), (1.52)-(i) almost everywhere and*

$$U(t) \xrightarrow[t \rightarrow 0]{\mathcal{G}} A_{q_0}. \quad (1.56)$$

This Theorem is proved namely by using the Yosida's approximation, which is possible here, since for all  $t \geq 0$ ,  $U(t)$  is a monotone operator.

From a modeling perspective, we have in fact established that the incentives taken as the limit of  $\epsilon^{-1}(q - q_0)$  are indeed suitable to force the population of players to be distributed according to  $q_0$  at the final time, whatever the state of the game. In particular despite the presence of a common noise, which tends to perturb the final distribution of players.

### 1.4.3 Bibliographical comments

The mean field planning problem has been the subject of several works, none of which were interested in either the finite state case, the master equation, nor the presence of a common noise. The problem was at first stated in [Lio11] and then studied in [Por14; ACCD12]. More recently it has been investigated in more general framework in [OPS19; Gra+19]. As mentioned, all these works are concerned with the well-posedness of the forward-backward systems which model Nash equilibria of this mean field planning problem.

## 1.5 A conclusive remark on Mean Field Games

Before passing to some future perspectives concerning MFG, let me conclude this part on MFG. In my opinion, the main interest of the previous development is to present a point of view, which has been particularly helpful for my study of MFG. This point of view

consists in seeing the value of a MFG, when it can be defined, as the central mathematical object of interest. This is thought by opposition to studying Nash equilibria and thus the precise evolution of the repartition of players. This leads, in section 1.2, to the development of a theory of the value function which is based on an intrinsic proof of uniqueness, rather than by using the underlying Nash equilibria. This proved to be an efficient way of establishing new results of stability and existence. Furthermore I explained in section 1.3 how such a point of view allowed to characterize value functions of MFG, for which the precise characterization of Nash equilibria does not seem to allow to write a master equation.

Finally let me mention that in several models such as [Ber+20; Ach+20], the concepts of value and master equation are meaningful, whereas no precise game is defined between the players. Hence in those models, which have been numerically confronted to actual data and gave satisfying results, no precise equilibria are modeled but a mathematical analysis of the master equation proved to be helpful. In my opinion, this further justifies the approach adopted here.

## 1.6 Future perspectives

A lot of questions remain open in the MFG theory. Here are a few that shall be subjects of my research in the near future and on which I am currently working.

### Numerical methods and constructive proofs of existence for the value function

If the master equation is getting better understood from a theoretical standpoint, the development of practical methods to actually construct the value function remains for the most part an open question. Several methods are of course possible in a finite state space when the number of states (hence the dimension) is low. In the general case, we are facing the problem of the curse of dimensionality as the set of probability measures gets bigger and bigger (possibly to an infinite dimensional limit). Different methods involving neural networks have been tried for such equations [CL21; CL19], but a general understanding is still missing.

A breakthrough would be a new simple constructive proof of existence of a value function, which does not use implicitly either the characteristics method or a compactness argument. I am currently trying to find such an algorithm, namely by making an extensive use of the monotonicity, in particular to use a (constructive) fixed point result for monotone maps. I am currently discussing these ideas with Sylvain Sorin.

In the same time, we plan, with Yves Achdou, Louis Bertucci, Jean-Michel Lasry and Pierre-Louis Lions, to use neural networks to solve master equations in infinite dimension. We hope that by using neural networks only *à bon escient*, we can actually develop an algorithm to approach the value function, at least on a meaningful subset of  $\mathcal{P}(\mathcal{O})$ . This plan is related to the previous one in the sense that here again, we are looking to use neural networks only to implement numerically a theoretic constructive

proof of existence of the value function. Such a proof remains to be found.

### The mean field planning problem in a continuous state space

The extension of the result of section 1.4 to a continuous state space is not trivial. It requires in particular new regularity estimates on the master equation and a better understanding of the behavior of the solution near the singularity, since the use of the Yosida's approximation does not seem to be possible anymore here. It seems that a weak characterization of the singularity, using monotone solutions, could overcome the latter of the two difficulties mentioned here.

Let me insist on the fact that such a study seems to be of importance from the applications it could have. Indeed, in the so-called potential case, the mean field planning problem reduces to a sort of optimal transport problem. Being able to treat the presence of a common noise in such problems should allow us to model a great variety of problems arising from Economics.

### Extension of the notion of monotone solution to other geometric structure

As mentioned in [Ber21b] (from a remark of P.-L. Lions), all the machinery of monotone solutions can be immediately adapted to more general settings than the monotone one. Indeed, in the finite state space case for instance, if there is no common noise, and  $G$  and  $F$  satisfy for all  $q, \tilde{q}, U, V \in \mathbb{R}^d$

$$\langle G(q, U) - G(\tilde{q}, V), \phi(q) - \phi(\tilde{q}) \rangle + \langle F(q, U) - F(\tilde{q}, V), UD_q\phi(q) - VD_q\phi(\tilde{q}) \rangle \geq 0 \quad (1.57)$$

for some  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , then we can establish the uniqueness of what I call  $\phi$ -monotone solutions of (1.12) (a notion which I do not detail here) which satisfy

$$\langle U(q) - U(\tilde{q}), \phi(q) - \phi(\tilde{q}) \rangle \geq 0. \quad (1.58)$$

Of course some assumptions on  $\phi$  are needed to develop such a theory.

The simplest example being of course to choose  $\phi = \pm Id$ . In the same spirit, I plan to investigate a systematic study of master equations for which a geometric structure allows us to build a sort of monotone like notion of solution.

### Master equation for optimal stopping in a continuous state space

If results of uniqueness and stability for a monotone solution of MFG with optimal stopping, in a continuous state space, seem quite easy to obtain, the question of the existence of such a solution remains an open problem. The main difficulty here consists in obtaining a proper regularity estimate on the value function (in the measure argument) to gain some compactness on any approximating sequence. Formal computations on this question suggest strongly that such an estimate is true.

## Mean field games with incomplete information

The question of MFG with incomplete information is a brand new one in the theory of MFG. I hope to continue the work I have presented here. Namely by investigating the question of existence and by refining the result of uniqueness I gave above. I am also trying to find setups in which such developments are of interest, namely for their applications in the management of decentralized systems. In other words, more and more decision makers who have to manage (or control) systems made of a huge number of agents are trying to decentralize the decision making to the agents themselves. However, in a realistic framework, these agents do not have access to all the information in the game but only to certain quantities, such as the cost they are facing for instance, hence the usefulness of such models.

## Applications of mean field games

Even though I omit them in this manuscript, modeling questions, especially using the theory of MFG, play a central role in my work. Let me give two such modeling problems which I am investigating at the moment.

The first one is concerned with the understanding of the interconnection between the different agents of the financial world, namely on the problem called systemic risk. Roughly speaking, this domain is concerned with how the default of one or several actors in the system can lead all the system to collapse. A huge literature exists on this topic, but most of it is purely empirical or based on agents which do not make anticipations. The rationality of the agents is an important factor which should be taken into account, such as in the bank runs models for instance. Recently, [DT21] addressed this question and I plan to pursue such modeling efforts.

The second one is concerned with the modeling of several aspects of the cryptocurrencies industry. Without entering into too much details, let me mention that cryptocurrencies, especially bitcoin, are protocols which offer a decentralized solution to the problem of consensus, and thus enable to avoid the use of a regulator. This decentralized solution takes the form of a game which takes place between a huge number of players. It is thus a perfect actual situation to study, by means of the theory of MFG. Let me mention that since such protocols have been well thought of (see for instance [Nak08]), quite often, they possess a monotone-like structure which yields some stability. In [Ber+20], we proposed a model to study the proof of work protocol with L. Bertucci, J.-M. Lasry and P.-L. Lions. We plan to extend this type of modeling to study the proof of stake protocol in the near future.



## Chapter 2

# Dynamics of the spectrum of large random matrices

This part is a summary of the study [Ber+21a] which is a collaboration with M. Debbah, J.-M. Lasry and P.-L. Lions.

### 2.1 A brief, thus incomplete, history of large random matrices

The origin of the study of large random matrices seems to be in the work of Wishart [Wis28], who was then mostly interested with statistics. The field gained drastically in importance when physicists, and in particular Wigner [Wig67], used large random matrices to describe the nuclei of large atoms. The most striking results obtained at that moment were the convergence of the spectrum of several random matrices, when properly renormalized, to deterministic objects, in the limit of matrices of infinite size. In the case of symmetric matrices, the spectrum converges toward measures supported on the real line such as the semi-circle law [Wig67] or the Marchenko-Pastur distribution [MP67]. In the case of non-symmetric matrices, the spectrum converges toward a measure on the complex plane such as in [Gin65]. Several other properties have been studied such as the law of the top eigenvalue [TW02] or the presence of outliers in the spectrum [BAP05].

Independently to these "static" results, Dyson introduced a dynamical model of random matrices to analyze complex systems [Dys62]. The so-called Dyson Brownian motion allowed the derivation of a system of interacting SDE which is solved by the eigenvalues of a matrix valued random process. Since then, such systems have attracted a tremendous attention from the mathematical, in particular the probabilistic, community. Concerning dynamical models, results of convergence, as the size of the matrix increases, were first established in [Cha92; RS93]. More refined and general results were obtained in [DG01; Fon04]. Finally, let me indicate the book [AGZ10] for a better, and incommensurably more complete, presentation of random matrices.

## 2.2 Motivation for this study

The starting point of the work M. Debbah, J.-M. Lasry, P.-L. Lions and myself did was the realization that all the proofs of the convergence results mentioned above rely on exact analytical computations, rather than on quantitative estimates. Whereas, the number and generality of these results suggest that key structural aspects should be at the origin of the stability of such systems. We believe that our work identify such a structural stability, and that this new point of view can be helpful, in particular to generalize those results. That is why, in the same spirit as [Ber+21a], the following starts by the presentation of the simplest model, before passing to generalizations.

## 2.3 The Dyson model

I present in this section what I shall refer to as the Dyson setting (or case) in the following.

Consider  $N \geq 1$  and the  $S_N(\mathbb{R})$  valued process  $(M_t^N)_{t \geq 0}$  defined by

$$\forall 1 \leq i, j \leq N, t \geq 0, d(M_t^N)_{ij} = \frac{1}{\sqrt{N}}(dW_t^{ij} + dW_t^{ji}), \quad (2.1)$$

where  $(W^{ij})_{1 \leq i, j}$  is a collection of independent Brownian motions on a standard filtered probability space  $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ . The ordered spectrum of  $M_t^N$  is denoted by  $\{\lambda_{1,t}^N, \dots, \lambda_{N,t}^N\}$  and the associated empirical measure is

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_{i,t}^N}. \quad (2.2)$$

As Dyson observed, the eigenvalues satisfy, in law, the system

$$d\lambda_{i,t}^N = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_{i,t}^N - \lambda_{j,t}^N} dt + \frac{\sqrt{2}}{\sqrt{N}} dB_t^i, \quad (2.3)$$

where  $(B^i)_{1 \leq i}$  is a collection of independent Brownian motion on  $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ . Moreover, it is known that, under the assumption that  $\mu_0^N$  converges, as  $N \rightarrow \infty$ , in law toward some measure  $\mu_0 \in \mathcal{P}(\mathbb{R})$ , then for any  $t \geq 0$ ,  $(\mu_t^N)_{N \geq 1}$  converges in law toward  $\mu_t \in \mathcal{P}(\mathbb{R})$ , characterized by the fact that the deterministic process  $(\mu_t)_{t \geq 0}$  is the only weak solution, in a sense I do not make precise here, of

$$\begin{aligned} \partial_t \mu + \partial_x(\mu H[\mu]) &= 0 \text{ in } (0, \infty) \times \mathbb{R}, \\ \mu|_{t=0} &= \mu_0, \end{aligned} \quad (2.4)$$

where  $H$  is the interaction operator, defined on smooth <sup>1</sup> functions  $\phi$  by

$$H[\phi](x) := \int_{\mathbb{R}} \frac{\phi(y) - \phi(x)}{x - y} dy = \int_{\mathbb{R}} \frac{\phi(y)}{x - y} dy. \quad (2.5)$$

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<sup>1</sup> $C^{0,1}$  with decay at infinity for instance

The second inequality above is understood in the sense of principal value. That is, I stipulate that

$$\int_{\mathbb{R}} \frac{1}{x} dx = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon^{-1}}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{\epsilon^{-1}} \frac{1}{x} dx = 0. \quad (2.6)$$

## 2.4 Spectral dominance

Here I first recall a dominance property for systems such as (2.3), which has been known for quite some time in the literature [Šni02; AGZ10], and which we used systematically to establish our results. I call this notion spectral dominance. It can be stated as

**Proposition 2.4.1** (Sniady, Anderson-Guionnet-Zeitouni). *Consider  $(\lambda_{i,t}^N)_{1 \leq i \leq N}$  and  $(\tilde{\lambda}_{i,t}^N)_{1 \leq i \leq N}$  two strong solutions of (2.3) such that  $\lambda_{i,0} \leq \tilde{\lambda}_{i,0}$  for all  $1 \leq i \leq N$ . Then for all  $t \geq 0, 1 \leq i \leq N, \lambda_{i,t} \leq \tilde{\lambda}_{i,t}$ .*

In the continuous limit, such a property is obviously still true and I state it on the spatial primitive  $F$  of  $\mu$ , that is on  $F(t, x) = \mu(t, ] - \infty, x])$ . The continuous comparison result can be given as

**Proposition 2.4.2.** *Let  $\mu_1$  and  $\mu_2$  be two smooth solutions of (2.4) such that  $F_1|_{t=0} \leq F_2|_{t=0}$ . For all time  $t \geq 0, F_1(t) \leq F_2(t)$ .*

In the previous,  $F_1$  and  $F_2$  are obviously the spatial primitives of  $\mu_1$  and  $\mu_2$ . To better understand this result, let me remark that if  $\mu$  is a solution of (2.4), then  $F$  is a solution of

$$\partial_t F + \partial_x F \tilde{H}[F] = 0 \text{ in } (0, \infty) \times \mathbb{R}, \quad (2.7)$$

where  $\tilde{H}$  is the half Laplacian operator, defined on smooth <sup>2</sup> functions  $\phi$  by

$$\tilde{H}[\phi](x) = \int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{(x - y)^2} dy. \quad (2.8)$$

The Proposition 2.4.2 can be established easily by using a comparison principle on (2.7), which holds for non-decreasing solutions. This is mainly due to the fact that  $\tilde{H}$  is an elliptic operator, more than to its explicit form. This remark allowed us to build a theory of well-posedness of the equation (2.7) (and thus of (2.4)) which is robust to quite a lot of generalizations. Such a theory, i.e. one based on a comparison principle, relies of course on the notion of viscosity solutions, which I shall not detail here for the sake of clarity. Let me refer to [CL83; CIL92] for a general presentation of viscosity solutions and to [Awa91; Ari08; BI08] for a presentation of viscosity solutions of PDE involving integral operators such as (2.8).

The rest of this part is mostly devoted to the presentation of such a theory on equations of a generality which contains (2.4).

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<sup>2</sup> $C^{1,1}$  with decay at infinity for instance

## 2.5 A general equation for the dynamics of spectrum of large random matrices

A quick derivation of the analogous of the system (2.3) for more general models than the Dyson Brownian motion suggests that a general form for such systems is

$$d\lambda_{i,t} = \frac{1}{N} \sum_{j \neq i} \frac{f(\lambda_{i,t}, \lambda_{j,t})}{\lambda_{i,t} - \lambda_{j,t}} dt + \epsilon(\lambda_{i,t}, N) dB_t^i, \quad (2.9)$$

where  $(B^i)_{1 \leq i}$  is a collection of independent Brownian motions on  $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$  and  $\epsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function such that  $\epsilon(\lambda, N) \rightarrow 0$  sufficiently fast as  $N \rightarrow \infty$ , for any  $\lambda^3$ .

Obviously the Dyson case is of this form, but so is the Wishart case [Bru91]. The Wishart case is concerned with the dynamics of the eigenvalues of the matrix  $X_t X_t^T$ , where  $(X_t)_{t \geq 0}$  is a  $n \times m$  matrix valued random process whose coefficients are all independent Brownian motions. For such a case, up to the addition of a constant term  $f(x, y) = x + y$  and  $\epsilon(\lambda, N) = \frac{2}{N} \sqrt{\lambda}$ .

In general, such systems as (2.9) are only valid for  $S_N(\mathbb{R})$ -valued process, i.e. when the spectrum is real. The complex case is much more involved. The equivalent of (2.7) for the system (2.9), is a PDE which is of the form

$$\partial_t F + (\partial_x F)L[F] + B(x)\partial_x F = 0 \text{ in } (0, \infty) \times \mathbb{R}, \quad (2.10)$$

where  $B$  is a drift term and  $L$  an integral operator which results from the mean field interaction between the eigenvalues. Typically  $L$  is defined on smooth functions by

$$L[\phi](x) := \int_{\mathbb{R}} \frac{\sigma(x, y)(\phi(x) - \phi(y))}{(x - y)^2} dy = \int_{\mathbb{R}} \frac{g(x, z)(\phi(x) - \phi(x + z))}{z^2} dz, \quad (2.11)$$

for  $\sigma, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  given functions. For instance, in the Wishart case,  $B$  is constant, and  $\sigma(x, y) = 2x\mathbb{1}_{y \geq 0}$ .

We were able to prove the following.

**Theorem 2.5.1.** *Assume that there exists  $C, \alpha_0 > 0$  such that*

- (i)  $g \geq 0$ ,
  - (ii)  $|g(x, z) - g(y, z)| \leq C|x - y|$ , for  $x, y, z \in \mathbb{R}$ ,
  - (iii)  $C^{-1} \leq g \leq C$  on  $\mathbb{R} \times (-\alpha_0, \alpha_0)$ ,
  - (iv)  $\left| \frac{\partial_x g(x, z)}{g(x, z)} - \frac{\partial_x g(x, 0)}{g(x, 0)} \right| \leq C|z|$  on  $\mathbb{R} \times (-\alpha_0, \alpha_0)$ ,
  - (v)  $B(x) - B(y) \geq -C(x - y)$  for  $x, y \in \mathbb{R}$ .
- (2.12)

*Then a comparison principle holds for (2.10) and there exists a unique viscosity solution of (2.10) given that the initial condition is smooth enough.*

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<sup>3</sup>In general, if the decay is not fast enough, one cannot guarantee the existence of a strong solution of the system of SDE.

I indicate to the interested reader [Ber+21a] in which a more precise statement is made, in particular in terms of the definition of a viscosity solution.

Let me comment on the assumptions of the previous result. The non-negativity of  $g$  is only necessary for the comparison principle part. By relaxing this assumption we can still prove the existence and uniqueness of a viscosity solution. The estimate (2.12)-(ii) and the upper bound in (2.12)-(iii) are quite standard and the lower bound in (2.12)-(iii) is helpful, together with (2.12)-(iv) to establish the propagation of the Lipschitz regularity of the solution. The assumption (2.12)-(v) is somehow the minimal known assumption for which a comparison result can be obtained for (2.10), even when  $L = 0$ . The existence and uniqueness are proved by using a Lipschitz regularization of the initial condition and by proving results of well-posedness for Lipschitz solutions.

Finally, let me remark that the Wishart case does not satisfy the assumptions of the Theorem, although a study of the singularity at  $y = 0$  yields the same type of results.

## 2.6 The convergence of the spectrum and other properties of the Dyson Brownian motion

In order to completely justify the use of viscosity solutions to treat such problems, we proved in [Ber+21a] a convergence result for the empirical measure of the solution of (2.9). Even though we only prove it in the Dyson case, it is quite easy to treat the general case, at least once some compactness is established, by using a similar approach. The method we used consists in making a systematic use of the spectral dominance to prove the

**Theorem 2.6.1.** *Assume that the empirical measure  $\mu_0^N$  of initial conditions of (2.3) converges, almost surely, weakly toward a measure  $\mu_0$ . Then, almost surely, the sequence of spatial primitives  $(F_N)_{N \geq 1}$  converges almost everywhere toward the unique viscosity solution  $F$  of (2.7) which satisfies  $F(0, x) = \mu_0((-\infty, x])$  almost everywhere.*

Another striking property of the Dyson Brownian motion is that the semi-group generated by the equation (2.4) is a contraction for any  $p$  Wasserstein distance for  $1 \leq p \leq \infty$ . It can also be formulated as

**Proposition 2.6.1.** *Consider  $(\mu_t)_{t \geq 0}$  and  $(\tilde{\mu}_t)_{t \geq 0}$  the solutions<sup>4</sup> of (2.4) associated to the initial conditions  $\mu_0$  and  $\tilde{\mu}_0$  respectively. Then, for any  $t \geq 0$ ,*

$$\mathbf{d}_p(\mu_t, \tilde{\mu}_t) \leq \mathbf{d}_p(\mu_0, \tilde{\mu}_0). \quad (2.13)$$

This result is rather immediate application of a famous Lemma by Crandall and Tartar [CT80].

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<sup>4</sup>The solution selected by the fact that their spatial primitive is the unique viscosity solution of (2.7).

## 2.7 Bibliographical comments

In this section I make some comment on existing results and on how they compare to the results above. The derivation of (2.3) is due to Dyson and its generalization to the so called Wishart case can be found in [Bru91]. The convergence, in law, of the empirical measure toward a deterministic process was first established in [Cha92; RS93] in the Dyson case with  $B = 0$ . This results were later on generalized in [DG01] to the Wishart and unitary cases, and in which more precise results of convergence were given. The characterization of the limit deterministic process as the solution of a PDE relies in the previous on either : explicit computation of the moments, or taking the Stieljes or Fourier transform of the equation and reducing the problem to a well known equation. The generalization of such methods to general drifts  $B$  or operator  $L$  is a difficult question. To our knowledge, such results exist mainly in [DG01; Fon04] for quite restrictive classes of drifts  $B$ . A more general method has been proposed in [LLX20] which relies on gradient flow like techniques and allows for general drifts  $B$  in the Dyson case. From the point of view of viscosity solutions, Theorem 2.5.1 is an improvement of a result which can be found in [BI08].

## 2.8 Perspectives

In this section, I elaborate on some perspectives concerning this spectral dominance approach.

### 2.8.1 General dynamics and the control of such systems

The aim of this section is somehow the main motivation for the previous analysis. Indeed, because such systems arise in Physics (in particular in complex systems), in Telecommunications or in Finance, it is natural to consider i) refinements of such systems to broaden the range of applications of such models and ii) the problem of controlling such systems. In general, the latter requires to have a good understanding of the former, as optimal controls often possess only few regularity.

If I believe that the previous study provides interesting results to study such systems in generality, several topics seem to remain mostly unstudied at the moment and such a topic is obviously still full of questions. Let me mention two such topics here. The first one is the case of several systems described by (2.4) interacting between each others.

The second one is the derivation of equations for systems in which the size of the matrix can depend on the time. Such models are helpful to address situations in which the "physical" system at interest is not closed, i.e. the number of particles in it can evolve with time. Obviously such a variant is only meaningful in the limit if a significant proportion of particles (or columns and lines of the matrix) are leaving or entering. If, in simple cases, such models may simply lead to the addition of constants in the drift  $B$  in (2.10), it could lead toward stochastic PDE or different non-linearities for more general models.

The question of the control, or optimal control, of such systems should lead to new mathematical equations. Let me give a few examples. First, if one looks at the optimal control problem in which the controlled state evolves according to an equation similar to (2.4), then the value function of such a problem is a solution of a Hamilton-Jacobi-Bellman equation set on  $\mathcal{P}(\mathbb{R})$  of a new type. Moreover, depending on the nature of the control, along the optimal trajectories, a new system of PDE can characterize the optimal control for such a problem. Such a phenomenon is now classical, it appears for instance in optimal transport problem [BB00] or potential MFG [Car+19], but its consequence on such a model is mostly unknown.

Another interesting development is the optimal control of a random process in a free probability space. Even though I do not take the time to enter in details in this analogy here, there exists a strong link between the theory of random matrices and free probabilities [VDN92], which is a theory concerned with non-commutative random variables. In this theory, (2.4) naturally arises as the PDE satisfied by the law of the free Brownian motion, see [BS98] for stochastic calculus with free Brownian motion. It should be interesting to study the link between the aforementioned problem and the optimal control of a free random process, in particular to derive the associated Hamilton-Jacobi-Bellman equation.

## 2.8.2 Derivation of equations in the case of complex eigenvalues

Characterizing the spectrum of large random matrices, without the symmetry condition that I imposed earlier, is a notoriously more difficult question. For static cases, the question has been solved. When all the entries of the matrix are i.i.d. centered random variables with sufficient integrability conditions, the limit spectral measure is a uniform distribution on a complex circle of centre 0 [Gin65]. This result has been extended, namely to more general structure of independence between the coefficients. However, when the coefficients of the matrix are stochastic processes, no analogous to (2.3) has been found. It is sometimes conjectured in the literature that such a system should contain the same interaction as in the symmetric case. That is a pairwise repulsive interaction which is inversely proportional to the euclidean distance between two eigenvalues, but in the complex plan this time. This should lead to the following system for the dynamics of the complex eigenvalues  $(\lambda_t^i)_{1 \leq i \leq N, t \geq 0}$  of a certain matrix

$$d\lambda_t^i = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_t^i - \lambda_t^j} dt + \epsilon(N, \lambda_t^i) dB_t^i, \quad (2.14)$$

where the  $(B_t^i)_{t \geq 0}$  are independent Brownian motion on the complex plane and  $\epsilon$  a given function. For the moment such a system has not been derived. The point of view that one could use to address such a question could be to analyse precisely, as it is already done in the static case, the covariance matrix associated to the matrix in question. Hence, maybe, analyzing in details the covariance matrix (which is a Wishart process) using the techniques mentioned in this part of the manuscript could lead to the derivation of the analogue of (2.3).

### 2.8.3 Proving a regularizing effects for such equations

As already mentioned, several techniques, including the one in question in this manuscript, allow to select or characterize the good solution of equation (2.4) given an initial condition. Starting with  $\mu_0 = \delta_0$ , the solution of (2.4) is given by the semi-circle law

$$\mu(t, x) = \frac{2}{\pi t} \sqrt{t - x^2} \mathbb{1}_{x \in [-\sqrt{t}, \sqrt{t}]}.$$
 (2.15)

It is a remarkable property of this solution that it instantly (i.e. for any  $t > 0$ ) becomes  $\mathcal{C}^{\frac{1}{2}}$  regular, starting from a Dirac mass. Moreover, it is  $\mathcal{C}^\infty$  on the interior of its support. It seems that this regularizing property is a general feature of this equation and that such a property holds for any initial condition (which is a measure). We can also conjecture a similar result for (2.10) under assumptions on the operator  $L$ .

### 2.8.4 The question of outliers in the spectrum

An important feature of random matrix theory is that a sort of independence between the coefficients of the matrix yields, after proper renormalization, that the spectral measure of a random matrix tends to be a "smooth" object, such as the semi-circle law or the Marshenko-Pastur distribution. In general, the top eigenvalue converges nicely toward the supremum of the support of the limit distribution, see for instance [TW02] in the, static, Dyson case. However, in several practical situations, a few top eigenvalues seem to be outliers to this kind of distribution. More than being empirical or statistical defaults, these outliers often carry most of the information on the model whereas the rest of the spectral measure (or bulk) is mostly the noise in the model. Understanding the behavior of the outliers, in particular how they interact with the bulk is a fundamental question. The first work in this direction is the so-called BBP transition introduced in [BAP05]. This transition characterizes conditions under which an outlier can "escape" or "enter" the bulk. It seems to be interesting to develop and analyze more involved models, in particular ones in which the outliers and the bulk interact with each other. The question of knowing wether or not such interactions perturb or preserve the stability introduced in this part of the manuscript seems to be of interest.



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