Holography, hydrodynamics and strongly-correlated condensed matter

Blaise Gouteraux

INSTITUT POLYTECHNIQUE DE PARIS

Habilitation à diriger les recherches

Thèse de doctorat

Holography, hydrodynamics and strongly-correlated condensed matter

Thèse d'habilitation à diriger les recherches de l'Institut Polytechnique de Paris
préparée à l'École polytechnique

École doctorale n°626 École doctorale de l'Institut Polytechnique de Paris (EDIPP)
Spécialité: Physique

Thèse présentée et soutenue à Palaiseau, le 09 juin 2022, par

BLAISE GOUTÉRAUX

Composition du Jury :

Bachas Costas Directeur de Recherches, CNRS, Laboratoire de Physique de l'Ecole Normale Supérieure, Paris, France	President
Christopher Herzog Professor in Theoretical Physics, King's College, London, UK	Reviewer
Pavel Kovtun Professor of Physics and Astronomy, Dept of Physics and Astronomy, University of Victoria, Victoria, Canada	Reviewer
Karl Landsteiner Cientifico Titular, Consejo Superior de Investigaciones Cientifias, Instituto de Física Teórica, Universidad Autónoma de Madrid, Madrid, Spain	Reviewer
Antoine Georges Professeur, CPHT, École Polytechnique, Palaiseau, France	Examiner
Kristan Jensen Assistant Professor, Dept of Physics and Astronomy, University of Victoria, Victoria, Canada	Examiner
Jan Zaanen Professor of Theoretical Physics, Instituut-Lorentz, Leiden University, Leiden, The Netherlands	Examiner

ACKNOWLEDGMENTS

I sincerely thank all my collaborators, and everyone with whom I have discussed Physics over the years. I would especially like to thank: my PhD advisor, Christos Charmousis, who guided my first steps as a researcher, taught me to be tenacious in my calculations, and gave me a taste for black holes in Einstein-Maxwell-Dilaton theories; to Elias Kiritsis, with whom I learned about the AdS/CFT correspondence and that it could be applied to Condensed Matter; to Sean Hartnoll and Jan Zaanen, who helped me to develop my intuition and showed me all that Condensed Matter had to offer; to Richard Davison, for all things related to hydrodynamics and black hole quasinormal modes. Thanks again to Elias and Sean for their warm hospitality in Crete and in California.

I also thank everyone who has placed their trust in me: the Physics Department and the Center for Theoretical Physics at Ecole Polytechnique for offering me to join them and welcoming me among them, and the members in my research group, with whom I've greatly enjoyed doing research these past years.

Thanks as well to the reviewers and the rest of the jury, for the time and effort they have put into reading through this thesis and accompanying me through this rite of passage into a full-fledged researcher.

Finally, I would like to wholeheartedly thank my family for their continued and unwavering support throughout my career.

My research is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No758759).

CONTENTS

I. INTRODUCTION

Strongly-correlated states of matter present a serious theoretical challenge, as perturbation theory typically fails to describe them. An archetypal example are high critical temperature (T_c) superconductors [1], which have resisted theoretical efforts to account for their phenomenology since their discovery, [2, 3]. The absence of long-lived quasiparticles, as reported by photo-emission experiments, and their unconventional transport properties are two signatures of their incompatibility with the Fermi liquid paradigm, [4]. The Hubbard model [5], marginal Fermi liquid theory [6] and various field theories with a large (infinite) number of degrees of freedom [7] aided by random interactions [8] all provide some degree of insight into this problem.

Progress in understanding the physics at play in these systems has been complicated by the variety of phases (most famously superconductivity and the pseudogap, but also antiferromagnetism, spin and charge and pair density waves, spin glasses...) that appear to be competing (or working in concert) in different regions of the phase diagram characterized by temperature, doping, magnetic field, pressure, etc.

Hydrodynamics and effective field theory methods [9–11] offer a complementary avenue to more microscopic approaches by eschewing the microscopic details of strongly-correlated systems, as proved to be particularly relevant for graphene [12–14], and more qualitatively for bad metals [15]. These approaches rely on identifying the relevant infrared degrees of freedom, based on the symmetries of the system, and writing the most general action/equations of motion compatible with these symmetries in a derivative expansion. The control parameter in this expansion is the cutoff of the infrared theory, where microscopic degrees of freedom cannot be neglected anymore. It is closely related to the time and length scales where local equilibrium is established. The price to pay is that the analysis is limited to low energies, late times and long distances, and breaks down near a phase transition or as temperature goes to zero (in particular in the vicinity of any quantum critical point where fluctuation

effects cannot be neglected). Besides the equation of state, effective approaches also need to be supplemented with transport coefficients which parametrize the various terms in the derivative expansion, the values of which can only be computed within a microscopically complete theory or measured experimentally.

Gauge/gravity duality (also referred to colloquially as holography) maps a strongly-coupled, large N matrix model (where N is the rank of the gauge group) to classical Einstein gravity coupled to a set of matter fields, [16]. The application of this set of techniques to strongly-correlated condensed matter systems has been intensively pursued in the past fifteen years, [17, 18]. The original duality [16] relates a specific gauge theory to a specific string theory, so that in principle microscopic degrees of freedom on both sides of the duality can be matched (in practice, this can be technically involved). A more common approach in applied gauge/gravity duality is the bottom up one, where the dual field theory is not known precisely, nor is it clear that the classical gravity dual can be promoted to a full quantum gravity. A charged black hole is dual to a nonzero density, thermal state in the dual field theory. The horizon of the black hole provides a thermal bath of degrees of freedom in which excitations can relax. At long distances and late times, the hydrodynamic equations emerge at the boundary from bulk dynamics, [19]. Within the range of validity of the low energy classical gravity theory, the equation of state and transport coefficients of the dual effective field theory can be computed. There has been a rich dialogue between gauge/gravity duality and hydrodynamics, the most famous example being perhaps the KSS lower bound on the shear viscosity to entropy ratio, [20], which originated from the calculation of the shear viscosity in the $\mathcal{N} = 4$ super Yang-Mills plasma [21].

In applying these tools, identifying the right set of symmetries is paramount, as this will dictate the starting point of the effective approach. This does so by determining the set of hydrodynamic conservation equations governing the low energy dynamics of the system in one case, or by acting as a guiding principle to write down the appropriate bulk action/field Ansatz in the second.

In the absence of any microscopic model to compute transport coefficients, fundamental bounds allow to constrain the parameters of the effective field theory – in some circumstances, these bounds can be saturated, and then offer more insight into the low-energy dynamics.

Positivity of the divergence of a suitable entropy current has historically played an essen-

tial role in constructing consistent hydrodynamic theories (together with Onsager reciprocity relations), either by trimming down the number of transport coefficients or bounding the remaining ones. However, these bounds are typically not tight, and transport coefficients are far from saturating them (though we will see circumstances later on in this manuscript where precisely this happens, leading to interesting phenomenological consequences).

Quantum mechanical bounds have proven to be a more fruitful avenue. The Mott-Ioffe-Regel bound famously bounds from above the resistivity of metals, under the assumption of long-lived quasiparticles – or turning this around, its violation is a hallmark of non-quasiparticle transport and strong electronic correlations. It can be stated starting from the Drude model of electronic transport as

$$\rho_{dc} \lesssim \frac{\hbar}{k_F^{d-2} e^2} \tag{1}$$

after using $k_F \ell \gtrsim 1$: the quasiparticle mean free path ℓ should be longer than their Compton wavelength. This bound is commonly violated at room temperature by so-called bad metals, [322, 323]. To get (1), we used a lower bound on the mean free path. If instead we bound the quasiparticle lifetime through $\tau k_B T \gtrsim \hbar$, we find [15]

$$\rho_{dc} \lesssim \frac{\hbar k_B T}{E_F k_F^{d-2} e^2} \tag{2}$$

so that the resistivity of quasiparticle metals is bounded from above by T-linearity. Many strongly-correlated metals appear to saturate this bound – which one may summarize succintly by the slogan: 'strong correlations tend to push quantum many-body systems as close to bounds as allowed by quantum mechanics'.

For a strongly-correlated, diffusive metal where quasiparticles are short-lived and momentum is relaxed, the resistivity is more usefully restated as an Einstein relation between the conductivity, the diffusivity and the static susceptibility. The bound (2) then becomes a lower bound on the diffusivity [15]

$$D \gtrsim v^2 \frac{\hbar}{k_B T} \tag{3}$$

where v is some characteristic infrared velocity, which has been argued to be the Fermi velocity in a metal with a Fermi surface, the speed of light in relativistic systems, or the butterfly velocity (which controls the ballistic spread of quantum chaos) in quantum chaotic systems, [146]. A closely related bound has been rigorously established on the rate of propagation of quantum chaos, $\Lambda_L \leq 2\pi k_B T/\hbar$, where Λ_L is the Lyapunov exponent, [26].

As we have noted, the bound (2) follows from putting together the Drude formula for electronic transport and the quantum mechanical bound

$$\tau \gtrsim \tau_P \equiv \frac{\hbar}{k_B T} \tag{4}$$

This is tantamount to saying that quantum dissipative processes cannot occur on timescales shorter than the 'Planckian' timescale τ_P, [7, 22, 23].

While these bounds are not proven in general, they have received much theoretical support, especially (3), from holographic calculations [146, 149, 150] or the Sachdev-Ye-Kitaev (SYK) model, [170, 178]. The link between hydrodynamics and quantum chaos has been strengthened by noting that energy diffusion is intimately related to the Lyapunov exponent and the butterfly velocity $D_T \sim v_B^2 \lambda_L$ through the phenomenon of pole-skipping, [85, 154, 182–184]. Several experiments also appear consistent with these expectations, [30–33, 325**?**].[1]

In this thesis, we will describe recent progress in applying effective hydrodynamic and holographic approaches on two classes of problems:

1. **Transport in the vicinity of quantum critical points or phases.** In many-body quantum systems with strong interactions, the characteristic timescales relevant for a variety of dynamical processes are short, and are set by the inverse temperature $\tau_P = \hbar/(k_B T)$. For example, the 'Planckian' timescale τ_P [7, 22, 23] has been shown to control the onset of hydrodynamics in holographic plasmas, the post-quench equilibration of the SYK model, as well as the Lyapunov exponent characterising the growth rate of chaos in both of the aforementioned kinds of theories [24–29]. Transport measurements in the strange metallic phase of high-T_c superconductors [30–33, 325] further support the conjecture that τ_P fundamentally bounds the dynamics of strongly correlated phases [15, 22, 23, 34].

   Indeed, in the vicinity of a quantum critical point (QCP), T is the only energy scale and so the importance of τ_P is manifest [7]. However, there are circumstances in which non-universal effects are important and lead to dynamics that survive on timescales much longer than τ_P. The most familiar example is near a QCP where translational

[1] The more recent conjectured violation of the Planckian bound reported in [**?**] is subject to caution. What they are reporting really is a violation of (2), which only signals that their system is not described by quasiparticles. It only turns into a violation of (4) if one is willing to accept that the resistivity is simply given by the Drude formula, which is unlikely.

symmetry is broken by an irrelevant coupling g [35–39], leading to the slow relaxation of momentum and a parametrically small resistivity. More generally, whenever the dynamics near a QCP is sensitive to a dangerously irrelevant coupling, τ_P is no longer privileged since the irrelevant coupling provides an additional energy scale [7]. In such situations, it is not obvious what the relevant timescales for dynamical processes are.

2. **Phases with spontaneously broken translations.** This is motivated by the ubiquity of such phases in the phase diagram of strongly-correlated electron materials, in particular cuprate or iron-based high T_c superconductors, kagome materials, organic conductors, transition metals dichalcogenides, etc. While translational 'spin-charge stripe' order was long anticipated on theoretical grounds to play an important role in underdoped cuprates and other doped Mott insulators [40–46], and was experimentally confirmed subsequently in most families of underdoped cuprate materials [47] as well as in numerical studies of the Hubbard model [48, 49], recent experiments suggest that charge density fluctuations [50] and short-range charge density wave order are actually found across the phase diagram all the way to the edge of the superconducting dome, [51–60].[2] These observations prompted a number of investigations of the impact of such fluctuating charge order on transport and spectroscopic experiments, [62–67].

   In seeking to apply effective field theory methods to this problem, one is inevitably confronted with the impact of disorder and other sources of explicit translation symmetry breaking on the dynamics of the charge density wave, leading to the phenomenon of pinning, [68]. When the explicit breaking is weak, the symmetry rules which usually tightly constrain effective field theories are relaxed and it becomes more arduous to develop a consistent double expansion, in powers of the strength of the explicit breaking and of the effective field theory cut-off. On the other hand, gauge/gravity duality allows to model such phases from first principles, and is particularly appealing testing arena for effective field theories with approximate symmetries.

   Thus, by considering holographic phases breaking translations pseudo-spontaneously and how their low energy dynamics matched to field theory expectations, great progress was made in the last few years: the incorporation of the physics of explicit symmetry breaking in the hydrodynamics of phases breaking translations spontaneously

[2] A recent numerical study of the Hubbard model also reports fluctuating stripes across the phase diagram, [61].

[63, 64, 67, 69]; the construction of simpler homogeneous holographic models for (pseudo-)spontaneously breaking translations [65, 70–78], which provided a far more tractable platform to compare with hydrodynamic predictions; a thorough analysis of how background strain and external sources enter in the hydrodynamic theory, [79, 80], and subsequent comparison with holographic constructions [81].

In section II, we review the hydrodynamics of charged fluids and their linear excitations. In section III, we give an overview of holographic methods and the holographic modeling of quantum critical points. In section IV, we review transport near quantum critical points using gauge/gravity duality. In section V, we discuss recent progress in incorporating background strain in the hydrodynamics of spontaneously broken translation phases, without assuming any particular boost symmetry, and expand on checks of this theory by various holographic models. Next, in section VI, we turn to the physics of pseudo-spontaneous translation symmetry breaking in hydrodynamics and in holography, and discuss phenomenological implications. In section VII, we discuss Wigner solid electronic phases and their quantitative description in terms of the hydrodynamic theory of a pinned magnetophonon. Finally, we close in section VIII with a discussion on the relevance of pseudo-spontaneous breaking of translations to strange metallic transport in strongly-correlated materials and cuprate high T_c superconductors in particular.

Some of the content of this manuscript has been borrowed from some of my previous work, specifically [63–65, 67, 82–86]

II. HYDRODYNAMICS OF CHARGED FLUIDS

A. Generalities

Symmetries and their spontaneous breaking provide a natural route to classifying states of matter, formalized by Landau's theory of second-order phase transitions [87]. This is an example of effective field theory, valid around the critical temperature at which the phase transition occurs, where the relevant degrees of freedom are only the order parameter and its fluctuations.

Hydrodynamics and its extensions to non-liquid states of matter (e.g. elasticity theory) [88] constitute another class of effective field theories, which describe the long-distance,

late-time dynamics of the system. Microscopic degrees of freedom are integrated out in this limit, and are reorganized into fast and slow degrees of freedom. Fast degrees of freedom equilibrate on time and length scales short compared to the local equilibration scales, which are typically set by the temperature of the system. Slow degrees of freedom are protected by symmetries and need to be retained in the effective field theory – they are the conserved densities of the system, such as energy, charge or momentum. Their evolution is described by conservation laws descending from the symmetries previously mentioned. They cannot decay locally and are transported away on scales much larger than the local equilibration scales to other regions of the system by hydrodynamic modes, such as sound or diffusion.

More concretely, the equations of motion for the conserved densities n^a take the form

$$\dot{n}_a(t,x) + \nabla \cdot j_a(t,x) = 0\,. \tag{5}$$

Upper dots stand for time derivatives, $\dot{} \equiv \partial/\partial_t$. For a fluid with a conserved $U(1)$ charge, the n^a's are the set of energy ε, momentum π^i and charge n densities. The spatial currents j^a are generally not slow operators.[3] They decay locally in the thermal bath of conserved densities, so that their vacuum expectations values obey so-called constitutive relations, i.e. local expansions in terms of the densities and external sources:[4]

$$\langle j_a \rangle = \alpha^{(0)}_{ab} \langle n^b \rangle + \alpha^{(1)}_{ab} \nabla \langle n^b \rangle + \cdots \tag{6}$$

The angular brackets denote a thermal average. The $\alpha^{(0),(1)}_{ab}$ are transport coefficient matrices, order by order in the gradient expansion, with dots denoting higher-order terms. Which of these coefficients are nonzero depends on the details of the system and the symmetry breaking pattern. The underlying reason why such expansions are possible is related to the central assumption to hydrodynamics: all microscopic, high energy modes relax on short scales of the order the thermalization time/length, and can be integrated out. At longer scales, only hydrodynamic fields are retained and are the sole source of non-analyticities in the retarded Green's functions. In other words, in the hydrodynamic regime, the retarded Green's functions only contain the gapless hydrodynamic poles.

[3] In special cases they can be. For example, in a system invariant under Galilean boosts, the corresponding Ward identity gives an operator equation between the charge current and momentum operators, so that the charge current is a slow operator in this case. In a relativistic system, the Lorentz boosts Ward identity equates the momentum and energy current operators, so the energy current becomes a slow operator as well.

[4] Throughout this manuscript, we work in a hydrodynamic frame where the time components of the conserved currents match the microscopic conserved densities, and dissipative corrections only enter with spatial gradients. Any time derivative correction can be traded for spatial derivatives by using the equations of motion and constitutive relations at lower order in derivatives.

In this manuscript, we will limit ourselves to expansions to first order in gradients. We will also ignore the effects of fluctuations, [89, 90], which generally spoil the analyticity of retarded Green's functions and of the dispersion relations of the hydrodynamic modes beyond first order in gradient terms. In gauge/gravity duality, these fluctuations are suppressed by the $N \to +\infty$ limit, [91].

Inserting (6) in (5), these become evolution equations for the vevs of the conserved densities, which can now be solved. Taking a spatial Fourier transform and dropping angular brackets for convenience, we obtain a set of dynamical equations given by

$$\dot{n}_a(t,q) + M_{ab}(q) \cdot n^b(t,q) = 0\,. \tag{7}$$

By construction, the matrix $M_{ab}(q) = M_0 + M_1 q + M_2 q^2 + \cdots$ has a local expansion in powers of the wave-vector q, with each term suppressed by the cutoff length of hydrodynamics ℓ_{th} (ie $q\ell_{th} \ll 1$ and $M_1 \sim \mathcal{O}(\ell_{th})$, $M_2 \sim \mathcal{O}(\ell_{th}^2)$, etc.).

We would now like to compute the retarded Green's functions of the system. As usual, this implies turning on a time-dependent deformation of the Hamiltonian

$$H_o \mapsto H(t) = H_o - \int d^d x\, n^a(t,x) \delta\mu_{e,a}(t,x) \tag{8}$$

(with d the number of spatial dimensions), upon which the equations of motion become

$$\dot{n}^a(t,q) + M^a_b(q) \cdot \left(n^b(t,q) - \chi^b_c\, \delta\mu^c_e \right) = 0\,. \tag{9}$$

Here χ is the matrix of static susceptibilities, obtained by functional differentiation of the equilibrium free energy

$$\chi_{ab}(x-x') = -\frac{\delta^2 W[\mu_e]}{\delta\mu^a_e(x)\delta\mu^b_e(x')}\,, \tag{10}$$

where $W = -T \log \mathrm{Tr} e^{-\beta H}$. This matrix encodes the linear response of the system to static perturbations $\delta\mu_e(x)$. It should be positive definite in order for the system to be locally thermodynamically stable.

Taking a Laplace transform of (9) (see [9] for more details) leads to the retarded Green's functions

$$G^R_{ab}(\omega,q) \equiv \frac{\delta n^a(\omega,q)}{\delta\mu^b_e(\omega,q)} = -\left(i\omega - M\right)^{-1} \cdot M \cdot \chi \tag{11}$$

where ω is the frequency. The hydrodynamic poles of the system are found by solving the equation $\det(-i\omega + M) = 0$. As a point of reference, in the case of a single conserved $U(1)$,

the constitutive relation for the spatial current compatible with invariance under parity and time reversal and with external sources turned on is

$$j^i = -D_n \left(\nabla^i n - \chi_{nn} \nabla^i \delta\mu_e \right) + \cdots , \quad i = 1 \ldots d , \tag{12}$$

leading to a quadratically dispersing, diffusive mode $\omega = -iD_n q^2 + \cdots$. The diffusivity can be measured by the following Kubo formula

$$D_n = \frac{1}{\chi_{nn}} \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} \mathrm{Im} G^R_{nn}(\omega, q) . \tag{13}$$

Instead, in the longitudinal sector, a neutral fluid would have two linearly dispersing sound modes $\omega = \pm c_s q - i\frac{\Gamma}{2} q^2$, where the longitudinal sound velocity is determined by the various static susceptibilities and the sound attenuation Γ by first-order in gradients dissipative corrections to the constitutive relation of the energy current and stress tensor, see e.g. [88].

Hydrodynamics only gives access to gapless poles with a vanishing dispersion relation at zero wavevector $\omega(q = 0) = 0$, the hydrodynamic modes. Non-hydrodynamic, gapped modes of the system cannot reliably be included in the hydrodynamics in general, except in certain special circumstances, for instance when the gap is generated by breaking weakly one of the symmetries of the system [92, 93]. Later in this manuscript, we will explain how to incorporate such weakly-gapped degrees of freedom in the low energy effective field theory. Generic gapped modes which do not fall in the previous category typically signal the breakdown of the effective field theory description [94, 95] and can only be accounted for by supplementing hydrodynamics with a microscopic completion.[5]

B. Linearized excitations in charged fluids

Linearised hydrodynamics is the theory that describes the transport of small perturbations of charge and heat over long distance and timescales in a system which is locally in thermal equilibrium. We will assume that our system has both translational and rotational symmetry, but no particular form of boost symmetry.[6]

[5] Remarkably, when the gradient expansion can be systematically computed in a microscopically complete framework, the dispersion relation of gapped modes can be obtained by resumming the hydrodynamic series, [94].

[6] See [96–101] for more complete treatments of fluid hydrodynamics without boosts.

The hydrodynamic variables are the long wavelength perturbations of the entropy density δs, charge density δn and momentum density π^i. In the absence of external sources, the linearized conservation equations are

$$\begin{aligned}
&\partial_t \delta s + \partial \cdot (\delta j_q/T) = 0, \\
&\partial_t \delta n + \partial \cdot \delta j = 0, \\
&\partial_t \delta \pi^i + \partial_j \delta \tau^{ji} = 0.
\end{aligned} \tag{14}$$

The perturbations $(\delta s, \delta n, \delta \pi^i)$ are sourced by temperature, chemical potential, and velocity perturbations $(\delta T, \delta \mu, \delta v^i)$. The static susceptibility matrix χ relating these quantities has the form

$$\begin{pmatrix} \delta n \\ \delta s \\ \delta \pi^i \end{pmatrix} = \begin{pmatrix} \chi_{nn} & \chi_{ns} & 0 \\ \chi_{ns} & \chi_{ss} & 0 \\ 0 & 0 & \chi_{\pi\pi} \end{pmatrix} \begin{pmatrix} \delta \mu \\ \delta T \\ \delta v^i \end{pmatrix}. \tag{15}$$

In d spatial dimensions, and in the absence of external sources, the constitutive relations for the charge, heat, and momentum currents are (neglecting terms of order ∂^2 and higher)

$$\delta j^i = n \delta v^i - \sigma_o \partial^i \delta \mu - \alpha_o \partial^i \delta T, \tag{16}$$

$$\delta j_q^i = sT \delta v^i - T \alpha_o \partial^i \delta \mu - \bar{\kappa}_o \partial^i \delta T, \tag{17}$$

$$\delta \tau^{ij} = \delta^{ij} \delta p - \eta \left(\partial^i \delta v^j + \partial^j \delta v^i - \frac{2}{d} \delta^{ij} \left(\partial \cdot \delta v \right) \right) - \zeta \left(\partial \cdot \delta v \right) \delta^{ij}, \tag{18}$$

where $\delta p = s \delta T + n \delta \mu$ is the pressure fluctuation, η and ζ are the shear and bulk viscosities and σ_o, α_o and $\bar{\kappa}_0$ are further dissipative transport coefficients related to the thermoelectric conductivities $\sigma, \alpha, \bar{\kappa}$ by

$$\sigma(\omega) = \frac{n^2}{\chi_{\pi\pi}} \frac{i}{\omega} + \sigma_o \,, \quad \alpha(\omega) = \frac{sn}{\chi_{\pi\pi}} \frac{i}{\omega} + \alpha_o \,, \quad \bar{\kappa}(\omega) = \frac{Ts^2}{\chi_{\pi\pi}} \frac{i}{\omega} + \bar{\kappa}_o. \tag{19}$$

The divergent low frequency parts of the thermoelectric conductivities are a consequence of the non-zero static susceptibilities between the thermoelectric currents and the conserved momentum [37]

$$\chi_{\pi j} = n \,, \quad \chi_{\pi j_q} = sT. \tag{20}$$

The longitudinal excitations of this theory consist of a diffusive excitation with dispersion relation

$$\omega_D = -iDk^2, \qquad D = \frac{s^2 T^2 \sigma_o + \bar{\kappa}_0 T n^2 - 2nsT^2 \alpha_o}{T^2 \left(s^2 \chi_{nn} + n^2 \chi_{ss} - 2sn \chi_{ns} \right)}, \tag{21}$$

as well as two sound modes with dispersion relations

$$\begin{aligned}\omega^{\pm} &= \pm\sqrt{\frac{n^2\chi_{ss}+s^2\chi_{nn}-2sn\chi_{sn}}{\chi_{\pi\pi}\left(\chi_{ss}\chi_{nn}-\chi_{sn}^2\right)}}k-\frac{i\Gamma}{2}k^2,\\ \Gamma &= \frac{2\eta\left(1-\frac{1}{d}\right)+\zeta}{\chi_{\pi\pi}}-D+\frac{\sigma_o\chi_{ss}-2\alpha_o\chi_{ns}+\frac{\bar{\kappa}_0}{T}\chi_{nn}}{\chi_{ss}\chi_{nn}-\chi_{sn}^2}.\end{aligned} \tag{22}$$

The sound waves are 'coherent' [15] excitations that transport perturbations of both long-lived momentum density $\delta\pi^i$ and of pressure δp through the system. The importance of pressure fluctuations in the transport of momentum density is obvious from the form of the stress tensor (18).

In contrast to this, the diffusive mode is an 'incoherent' excitation of the system, in that it does not transport long-lived momentum density. We can define perturbations of an 'incoherent' density $\delta n_{\rm inc}$

$$\delta n_{\rm inc} \equiv s^2 T\delta\left(n/s\right) = T\left(s\delta n - n\delta s\right), \tag{23}$$

which obeys the conservation equation

$$\partial_t\delta n_{\rm inc} + \partial\cdot\delta j_{\rm inc} = 0, \qquad \delta j_{\rm inc} \equiv sT\delta j - n\delta j_q. \tag{24}$$

The incoherent perturbations $\delta n_{\rm inc}$ and $\delta j_{\rm inc}$ are special because they do not overlap with fluctuations in the pressure and momentum density i.e. their static susceptibilities vanish

$$\chi_{n_{\rm inc}p} = 0, \qquad \chi_{j_{\rm inc}\pi} = 0. \tag{25}$$

The consequence of this is, that to leading order in the hydrodynamic limit[7] the retarded Green's function of $\delta n_{\rm inc}$ has the simple diffusive form

$$G^R_{n_{\rm inc}n_{\rm inc}}(\omega,k) = \frac{-k^2\left(T^2s^2\sigma_o - 2nsT^2\alpha_o + n^2T\bar{\kappa}_o\right)}{-i\omega+Dk^2}. \tag{26}$$

It does not have poles corresponding to the propagation of the sound waves (22).[8] The diffusivity D in equation (21) obeys the Einstein relation

$$D = \frac{\sigma_{\rm inc}}{\chi_{\rm inc}}, \tag{27}$$

[7] Explicitly, we let $\omega\to\lambda^2\omega$, $k\to\lambda k$ and expand the Green's function at leading order in λ.

[8] i.e. the full Green's function does not have sound poles when expanded in the limit $\lambda\to 0$ with $\omega\to\lambda\omega$, $k\to\lambda k$.

where $\sigma_{\rm inc}$ and $\chi_{\rm inc}$ are the dc conductivity and static susceptibility of $\delta n_{\rm inc}$

$$\begin{aligned}\sigma_{\rm inc} &\equiv \lim_{\omega\to 0}\lim_{k\to 0}\frac{i}{\omega}G^R_{j_{\rm inc}j_{\rm inc}}(\omega,k) = \lim_{\omega\to 0}\lim_{k\to 0}\frac{i}{\omega}\frac{\omega^2}{k^2}G^R_{n_{\rm inc}n_{\rm inc}}(\omega,k) = T^2s^2\sigma_o - 2nsT^2\alpha_o + n^2T\bar{\kappa}_o,\\ \chi_{\rm inc} &\equiv -\lim_{k\to 0}\lim_{\omega\to 0}G^R_{n_{\rm inc}n_{\rm inc}}(\omega,k) = T^2\left(n^2\chi_{ss} - 2sn\chi_{ns} + s^2\chi_{nn}\right).\end{aligned} \tag{28}$$

Note that in contrast to the dc conductivities of charge and heat individually, the dc conductivity of $\delta n_{\rm inc}$ is finite. This is because $\delta j_{\rm inc}$ has no overlap with the conserved momentum [102]. It is this independence from momentum conservation that makes it possible for $\sigma_{\rm inc}$ to be controlled by the underlying infrared QCP (as we will discuss in section IV A), unlike the electrical conductivity. Also note that while the values of the incoherent conductivity and susceptibility depend on the overall normalization of $\delta n_{\rm inc}$, the value of the diffusivity D does not.

Another useful expression for $\chi_{\rm inc}$ can be found by using thermodynamic identities on the expression in equation (28) and taking a low temperature limit

$$\chi_{\rm inc}\left(T\to 0\right) \longrightarrow Tn^2c_n, \tag{29}$$

where $c_n = T\left(\partial s/\partial T\right)_n$ is the specific heat at fixed charge density. We have assumed that, in this low temperature limit, the charge susceptibility is finite while the entropy density is a power law $s\sim T^\alpha$ with $\alpha\geq 0$. These conditions will be valid in all of the holographic theories we examine later in section III B and section IV.

We can also evaluate the susceptibility of pressure at low temperatures to find

$$\chi_{pp}(T\to 0)\to\frac{n^2}{\chi_{nn}}. \tag{30}$$

In holographic theories at non-zero density, the chemical potential μ of the theory is not a near-horizon property, but depends on knowledge of the entire spacetime. As a consequence, the low temperature static charge susceptibility $\chi_{nn} = \left(\partial n/\partial\mu\right)_T$ at low temperature is not an IR property of these theories.

By diagonalizing the susceptibility matrix in the basis $(\delta n_{\rm inc}, \delta p)$, we have separated its components into IR and UV-dominated pieces.

The incoherent conductivity also controls the dissipative dynamics of another set of physical processes in this system.

The hydrodynamic equations (14) to (18) are valid in the absence of external sources. In the presence of a small electric field δE_i, the momentum conservation equation should be

modified to

$$\partial_t \delta\pi^i + \partial_j \delta\tau^{ji} = n\delta E_i, \tag{31}$$

while $\partial_i \delta\mu$ should be replaced with $\partial_i \delta\mu - \delta E_i$ in the constitutive relations (16) to (18). The presence of an external electric field affects the measured conductivities and diffusivities. One experimentally relevant configuration is when there is an electric field such that no current flows (open-circuit boundary conditions). The open circuit dc thermal conductivity is simply related to the incoherent conductivity by

$$\kappa \equiv -\left.\frac{\delta j_s}{\partial \delta T}\right|_{\delta j=0} = \frac{\sigma_{\text{inc}}}{Tn^2}. \tag{32}$$

Similarly, under the condition $\partial \cdot \delta j = 0$ (i.e. charge perturbations are static $\partial_t \delta n = 0$), there is a hydrodynamic diffusion equation

$$\partial_t \left(\delta T + \frac{\chi_{\pi\pi} \left(n\alpha_o - s\sigma_o \right)}{n^2 \left(\partial s / \partial T \right)_n} \partial \cdot \delta v \right) = D_T \partial^2 \left(\delta T + \frac{\chi_{\pi\pi} \left(n\alpha_o - s\sigma_o \right)}{n^2 \left(\partial s / \partial T \right)_n} \partial \cdot \delta v \right) + O(\partial^3) \tag{33}$$

with a 'thermal' diffusivity

$$D_T \equiv \frac{\kappa}{c_n}, \tag{34}$$

which is simply related to σ_{inc} by equation (32).

In the low temperature limit described above, D therefore coincides with D_T

$$D_T = \frac{\kappa}{c_n} = \frac{\sigma_{\text{inc}}}{Tn^2 c_n} = \lim_{T\to 0} \frac{\sigma_{\text{inc}}}{\chi_{\text{inc}}} = \lim_{T\to 0} D, \tag{35}$$

due to the relation (29).

III. HOLOGRAPHIC METHODS

A. Generalities

The hydrodynamic construction outlined in the previous section can be systematically carried out order by order in the gradient expansion. The procedure quickly becomes intractable analytically due to the proliferation of terms to be considered, [103]. Moreover, the equation of state and each transport coefficient needs to be measured experimentally or computed in a microscopic model.

Most microscopic models, nevertheless, face serious difficulties whenever the system under investigation is either strongly interacting, made of a large number of constituents, placed

at finite chemical potential, finite temperature or when its real time dynamics is considered. Under these circumstances, the AdS-CFT correspondence[9] provides a self-consistent framework to attack these problems and guide new interdisciplinary explorations. Holography posits a duality between a large class of quantum field theories with gauge group of dimension N and higher-dimensional gravitational theories (for details we refer to a number of reviews and textbooks now available in the literature [17, 18, 106]). The duality was originally discovered in the context of string theory [16, 107, 108], which provides a precise formulation of the conjecture, between a supersymmetric gauge theory ($\mathcal{N} = 4$ super Yang-Mills with gauge group $SU(N)$) and a string theory (type IIB string theory on $\mathrm{AdS}_5 \times S^5$), now widely accepted as proven. In the bottom-up approach, the simplifying limit of classical gravity coupled to matter fields without extended objects (D-branes) is taken and corresponds to considering a dual field theory in the regime of strong 't Hooft coupling and in the large N limit[10] and is known as the bottom-up approach. Bottom-up holographic methods have been applied in several directions such as quantum chromodynamics (QCD) and heavy ion collisions [110, 111], condensed matter many-body systems and quantum information [17, 18, 112–114].

From a formal point of view, the duality is built on the identification of the field theory partition function with sources on with the gravitational on-shell path integral, [107, 108]:

$$Z_{QFT}[s_a(x)] = Z_{gravity}[s_a(x)] \tag{36}$$

with

$$Z_{QFT}[\{s_a(x)\}] = \langle e^{i\sum_a \int dx\, s_a(x) O_a(x)} \rangle\,, \quad Z_{gravity}[\{s_a(x)\}] = \int^{\phi_a \to s_a} (\Pi_a \mathcal{D}\phi_a)\, e^{iS[\{\phi_a\}]} \tag{37}$$

The field theory operators O_a and sources s_a are given by specific coefficients of the asymptotic expansion of dynamical fields ϕ_a living in the curved, higher-dimensional, bulk spacetime through a by now well-established holographic dictionary, [128]. Local symmetries in the bulk are mapped into global symmetries of the boundary field theory. Thermal, finite density states in the dual field theory are captured by gravitational charged black hole solutions in the bulk, with the field theory temperature given by the Hawking temperature at

[9] Anti de Sitter spacetime [107] is a maximally symmetric spacetime which is a solution of Einstein's equations with a negative cosmological constant. It is endowed with a conformal timelike boundary. A conformal field theory is invariant under the conformal group, which contains spacetime translations, spatial rotations, Lorentz boosts and special conformal transformations. It is also scale invariant.

[10] Here, N is the rank of the dual gauge field theory [109]. In the absence of a precisely identified dual field theory, this limit has to be understood as a large number of degrees of freedom.

the event horizon and the chemical potential by the boundary value of the bulk gauge field. From this gravitational background, all thermodynamic quantities can be computed, as well as the static susceptibilities. Linear perturbations of the gravitational solution together with appropriate boundary conditions for the bulk fields [115] yield the real-time, space-dependent retarded Green's functions, the poles of which are given by the quasi-normal modes of the black hole solution. This linear analysis also gives access to all linear transport coefficients through the appropriate Kubo formulas. This way, one can obtain the dispersion relations of both hydrodynamic and gapped excitations in the dual field theory, [116, 117]. The low energy excitations of charged black holes have been successfully matched to the predictions of charged, relativistic linearized hydrodynamics [118–122].[11] Besides providing a concrete test bed for hydrodynamics, holography is a microscopically complete framework which allows to compute all transport coefficients. This has led to important insights on strongly-coupled dynamics, such as the celebrated lower bound on the shear viscosity to entropy density ratio [20] or the fast hydrodynamization of the Quark-Gluon-Plasma [28].

The interplay between quantum criticality and holography was immediately the subject of much attention, and contributed to kickstart the program of applying holographic methods to strongly-correlated condensed matter systems, see eg the review [18] for an exhaustive review of this literature. In the next subsection, we introduce holographic quantum critical points and their scaling properties.

B. Holographic quantum critical points

In this section, we will describe the properties of the holographic QCPs of [129–135]. A reader who is very familiar with these QCPs can safely skip this section.

The holographic QCPs that we study arise at the IR endpoint of the RG flows generated by relevant deformations of UV conformal fixed points. Using gauge/gravity duality, RG flows of this kind can be captured by the gravitational Einstein-Maxwell-scalar action

$$S_{UV} = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{Z(\phi)}{4}F^2 - V(\phi)\right). \tag{38}$$

The relevant deformations that can be captured by this action include those that generate a non-zero density. We are interested in solutions that near the boundary ($u \to \infty$) have

[11] The full nonlinear structure of the hydrodynamic theory can be also derived from the gravitational equations using the fluid-gravity correspondence [123].

an asymptotic metric that is AdS_{d+2} together with the near-boundary behaviour for the potentials

$$V(\phi \to 0) \to -d(d+1) + m^2\phi^2/2\,, \qquad Z(\phi \to 0) \to 1\,. \tag{39}$$

By choosing the scalar couplings $V(\phi)$ and $Z(\phi)$ appropriately, we can find gravitational solutions which are dual to field theories that are governed by IR quantum critical points. The metric of these solutions depart from AdS away from the boundary, and in the deep interior become scale-covariant (at zero temperature). Solutions of this kind arise when the scalar field ϕ has a runaway behaviour in the deep interior 'IR region', where the scalar couplings can be approximated by exponentials

$$V(\phi \to \infty) \longrightarrow V_0 e^{-\delta\phi}\,, \qquad Z(\phi \to \infty) \longrightarrow Z_0 e^{\gamma\phi}\,, \tag{40}$$

where δ, γ are real numbers that will be constrained shortly. Interpreting the radial coordinate as the energy scale of the dual field theory in the usual way, we expect the IR properties of the dual quantum field theory to be governed by this IR region of the spacetime. The effective action governing the IR region is

$$S_{IR} = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - V_0 e^{-\delta\phi} - \frac{Z_0 e^{\gamma\phi}}{4}F^2\right). \tag{41}$$

As is now well-understood [132, 133, 135], the various terms in the IR effective action are on different footings. The first three terms proportional to the Ricci curvature, the scalar kinetic term and the scalar potential directly support the IR solution dual to the QCP and their effects cannot be neglected [129] in order to obtain consistent solutions. The last term, proportional to F^2, parameterises a deformation of the QCP. Depending on the value of the exponent γ, this deformation will be either marginal or irrelevant. It captures the effects of nonzero density (particle-hole symmetry breaking) near the QCP. When it is marginal, it directly sources the IR solution. When it is irrelevant, it sources corrections to the IR solution that grow towards the boundary of the IR region of the spacetime.

We will now describe the IR solutions in greater detail. The action (41) admits zero temperature scaling solutions [135], which are naturally parametrized by two scaling exponents: $\{z, \theta\}$ [130, 131, 135]. These solutions (valid deep in the interior of the bulk) are dual

to an IR quantum critical point, and have the form

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{r}{L}\right)^{2\frac{\theta}{d}} \left(-\frac{L^{2z}}{r^{2z}} L_t^2 \mathrm{d}t^2 + \frac{\tilde{L}^2 \mathrm{d}r^2}{r^2} + \frac{L^2}{r^2} L_x^2 \mathrm{d}\boldsymbol{x}^2 \right), \qquad \phi = \kappa \ln\left(\frac{r}{L}\right), \\ \kappa^2 &= \frac{2}{d}(d-\theta)(dz-d-\theta), \qquad \kappa\delta = 2\frac{\theta}{d}. \end{aligned} \tag{42}$$

The regime of the full geometry where this IR solution is valid is controlled by the scale L: for $\theta < d$ ($\theta > d$), it is valid for $r \gg L$ ($r \ll L$) assuming that that the asymptotically AdS boundary is at $r \to 0$ ($r \to \infty$). The r coordinate does not extend all the way to the AdS boundary. The values of the scales L_t and L_x will be determined in a non-trivial way by the flow to the asymptotically AdS solution, and so we keep them as free parameters. Their dependence on the UV sources depends on the specific RG flow considered.

Our choice of coordinates makes it manifest that the zero temperature metric transforms covariantly under the scaling

$$t \to \Lambda^z t, \qquad (r, \boldsymbol{x}) \to \Lambda(r, \boldsymbol{x}), \tag{43}$$

and therefore that z is the dynamical critical exponent of the critical point. The zero temperature metric is only covariant (rather than invariant) under this transformation when there is a non-zero hyperscaling violation exponent θ. Hyperscaling violation is directly related to the IR running of the scalar ϕ, and to the fact that the scale L has not decoupled from the IR theory. θ determines the effective spatial dimensionality $d-\theta$ of the IR quantum critical state. This statement can be made precise by embedding such solutions in higher or lower-dimensional spacetimes [130, 136, 137].

We have not yet given an expression for the value of the dynamical exponent z. To do so, we need to consider the deformations of the QCP due to the gauge field F term in the IR effective action (41).

1. *Marginal deformation* ($z \neq 1$)

The first possibility is that the deformation of interest is marginal. That is, it does not give rise to terms with different powers of r in the solution (42). This implies

$$\begin{aligned} A &= A_0 \left(\frac{r}{L}\right)^{\theta-d-z} L_t \mathrm{d}t, \qquad A_0^2 = \frac{2(z-1)}{Z_0(d+z-\theta)}, \\ \kappa\gamma &= 2d - 2(d-1)\frac{\theta}{d}, \qquad \tilde{L}^2 = \frac{(d+z-\theta)(d-1+z-\theta)}{-V_0}. \end{aligned} \tag{44}$$

The IR metric enjoys nonrelativistic scaling $z \neq 1$. The relations between (δ, γ) and (z, θ) can be inverted to express z and θ in terms of parameters in the effective action. Thus the scaling exponents of the IR QCP are completely determined by specifying the gravitational action.

When a small temperature T is turned on, the solutions with marginal deformations change: there is a horizon at $r = r_h$ and

$$\mathrm{d}s^2 = \left(\frac{r}{L}\right)^{2\frac{\theta}{d}} \left(-\frac{L^{2z} f}{r^{2z}} L_t^2 \mathrm{d}t^2 + \frac{\tilde{L}^2 \mathrm{d}r^2}{r^2 f} + \frac{L^2}{r^2} L_x^2 \mathrm{d}\boldsymbol{x}^2 \right), \quad f = 1 - \left(\frac{r}{r_h}\right)^{d+z-\theta} . \tag{45}$$

These IR solutions are dual to the thermal state of the quantum critical theory, with T related to r_h by (49) below.

Even at non-zero temperature, there is a region $L \ll r \ll r_h$ where the metric looks like the zero temperature form (42). One simple example of this is the AdS-Reissner-Nordström solution, which is characterized by an $\mathrm{AdS}_2 \times \mathrm{R}^d$ zero temperature IR geometry. The AdS_2 can be placed at nonzero temperature, which describes small departures from the zero temperature state.

Imposing the null energy condition and positivity of the low temperature heat capacity results in the following restrictions on the allowed parameter space of IR solutions

$$\frac{d-\theta}{z} \geq 0\,, \quad (d-\theta)(dz - d - \theta) \geq 0\,, \quad (z-1)(d+z-\theta) \geq 0\,. \tag{46}$$

2. Irrelevant deformation $(z = 1)$

The second possibility we would like to consider is that the deformation of interest is irrelevant. For these cases the $T = 0$ IR solution has $z = 1$, and the deformation sources power law (in r) corrections to this solution that grow towards the edge of the IR region. This means that the IR solution is like a 'CFT' in $d-\theta$ spatial dimensions, in the presence of an irrelevant deformation parameterised by a coupling $g = A_0$. As dilatations are broken by $\theta \neq 0$, it is not an actual CFT, but it can be endowed with a generalized conformal structure in the sense of [136, 137]. The full solution is obtained by solving for the backreaction of the irrelevant field A order-by-order in g. The leading corrections take the form (where the subscript $g = 0$ means (42) with $z = 1$)

$$\mathrm{d}s^2 = \mathrm{d}s^2_{g=0} \left(1 + \#g^2 \left(\frac{r}{L}\right)^{2\Delta_g} + O(r^{4\Delta_g})\right), \quad \phi = \phi_{g=0} + \#g^2 \left(\frac{r}{L}\right)^{2\Delta_g} + O(r^{4\Delta_g})\,. \tag{47}$$

The corrections are quadratic in the irrelevant coupling as the corresponding field appears quadratically in the scalar and Einstein equations.

(47) is supplemented by

$$A_t = L_t A_0 \left(\frac{r}{L}\right)^{\theta-d-1-2\Delta_g} \left(1 + \#A_0^2 \left(\frac{r}{L}\right)^{2\Delta_{A_0}} + O(r^{4\Delta_{A_0}})\right) , \quad 2\Delta_{A_0} = 2(d-\theta) - \kappa\gamma + \frac{2}{d}\theta \, . \tag{48}$$

The critical states (42) with $z = 1$ can also be placed at a small, nonzero temperature by introducing an event horizon at $r = r_h$. Close to the horizon, the metric will have the form (45) with $z = 1$. The zero temperature form (47) will be recovered in the range $L \ll r \ll r_h$.

Since $z = 1$, the inequalities (46) enforce that $\theta < 0$. This in turn implies that the expansions in (47) can only make sense if $\Delta_g < 0$ (the IR is located at $r \to +\infty$ in these coordinates). This is precisely what we expect for the dimension of an irrelevant coupling. The irrelevant coupling g breaks the $z = 1$ symmetry of the QCP, in tandem with breaking the particle-hole symmetry of the QCP. g also breaks Lorentz boost symmetry. The dimension of the irrelevant coupling is primarily determined by the value of γ.

3. More on irrelevant deformations of holographic QCPs

We will now characterise more precisely the scaling properties of the $z = 1$ IR solutions by relating Δ_{A_0} (defined above) to the dimension of an irrelevant coupling. A_t has two independent modes: in addition to the A_0 mode that scales as $r^{\theta-d-1+2\Delta_{A_0}}$, there is a constant r^0 mode that is allowed by U(1) gauge invariance. As usual in holographic theories, we would like to interpret one of these modes as the source of an operator that is irrelevant near the IR critical point, and the other as the corresponding expectation value [17, 18, 106]. From their radial dependence, we see that the difference between the IR scaling dimension of the source and the expectation value is $\theta - d - 1 + 2\Delta_{A_0}$. Combining this with the fact that their dimensions should sum to $d + 1 - \theta$, due to the effective dimensionality of the IR critical point, we are left with two possible values for the dimensionality of the operator. We choose the positive value $\Delta_{irr} = d + 1 - \theta - \Delta_{A_0} > 0$, and so the source has dimension $\Delta_{A_0} < 0$. Note that since $\Delta_{irr} > d+1-\theta$, this is an irrelevant deformation of the IR critical point.

We interpret the mode which grows fastest near the boundary of the IR solution as the source, and hence A_0 is the source for an irrelevant deformation of the IR critical point with

dimension Δ_{A_0} given by (48). This irrelevant source produces a deformation of the metric which vanishes in the deep IR, as expected. This is in contrast to the constant mode (the expectation value), which would backreact on the metric in a way that grows in the IR and destroys the critical point, consistent with the discussion in [138].

From (44), we see that for the $z \neq 1$ QCPs, $g = A_0$ is a marginal coupling.

4. IR scaling of thermodynamic observables

Physically, we expect that the scaling dimension of the irrelevant coupling will control the IR behaviour of certain observables. In this section, we specifically comment on thermodynamic observables. We will address the impact of these irrelevant deformations on transport in section IV A.

The scaling properties of the solutions we have just described result in many field theory observables exhibiting scaling behaviour at low temperatures and frequencies. This scaling behaviour can be understood in terms of the anomalous IR scaling dimensions of entropy and charge density in these solutions [130, 131, 133, 135, 139].

Once the values of V_0 and Z_0 in the IR action have been fixed, the zero temperature IR solution is characterized by the parameters L_t, L_x and L. Changing the values of these parameters does not induce any RG flow (i.e. any new radial dependence in the IR solution) and so they are marginal. There are two important deformations that do change the radial dependence of the IR solution. The first is turning on r_h i.e. turning on a non-zero temperature. The second is turning on the coupling $g = A_0$, which for $z = 1$ solutions is an irrelevant deformation that induces an RG flow.[12]

We will now assign IR scaling dimensions by determining how quantities depend on these two IR scales – the temperature T and the irrelevant coupling g. This is straightforward for quantities which can be expressed explicitly in terms of the near-horizon gravitational solution, as their T and g dependence is then manifest. The Hawking temperature is related to the horizon radius by

$$T = \frac{(d + z - \theta)}{4\pi \tilde{L}} L_t \left(\frac{r_h}{L}\right)^{-z} \quad \Rightarrow \quad [T] = z, \tag{49}$$

[12] There are additional irrelevant deformations, but since they do not affect the low temperature and low frequency behaviour of the observables of interest in this work, we will not discuss them.

in our conventions where $[r] = -1$. This result is consistent with interpreting T as an inverse time, where $[t] = -z$ in line with the scaling transformation (43). The entropy density s and charge density n can be calculated from the area of the horizon and the electric flux emitted by the horizon, and are given by

$$s = 4\pi L_x^d \left(\frac{r_h}{L}\right)^{\theta-d}, \qquad n = L_x^d \frac{(\theta - d - z - 2\Delta_g)Z_0}{\tilde{L}} A_0. \tag{50}$$

We can therefore assign them the dimensions[13]

$$[s] = d - \theta, \qquad [n] = \Delta_{A_0}, \tag{51}$$

using equation (49).

It is also convenient to define an anomalous scaling dimension Φ for the charge density [139, 140] via[14]

$$[n] = \Delta_{A_0} = d - \theta + \Phi. \tag{52}$$

We observe that even for the marginal case, $\Phi = \theta - d$ is non trivial and implies $[n] = 0$. We will see later that this scaling assignment correctly reproduces the explicit low T scaling of transport observables.

The authors of [140] used (51) as a starting point and derived a complete scaling theory of transport observables in quantum critical states of this kind. They showed that a variety of magnetothermoelectric transport observables (but not thermodynamic observables) experimentally measured in cuprate high T_c superconductors could be understood from this phenomenological theory, assuming that these observables are insensitive to irrelevant deformations. The inclusion of non-zero anomalous dimensions is an improvement on previous applications of scaling theories to the cuprates [141] (see [2, 142] for more discussion of the role of quantum criticality in the cuprates).

We emphasise that these scaling dimensions do not tell us anything about how n and s depend on the sources which deform the UV CFT, such as the chemical potential μ. At zero temperature, the UV sources generate an RG flow to the IR solution, where the values of all of the parameters in the IR solution A_0, L_t, L_x etc. will depend non-trivially on the UV sources. Access to the entire RG flow is needed to reconstruct this dependence. But the dependence of n and s on r_h does tell us their T dependence at low T, as the other parameters

[13] In the absence of hyperscaling violation, [s]=d as expected. Hyperscaling violation acts as an effective spatial dimensionality $d - \theta$.

[14] This is anomalous since in $d - \theta$ effective spatial dimensions, we would expect $[n] = d - \theta = [s]$.

in the IR solution A_0, L_t, L_x etc. are T-independent in the limit $T \to 0$ (otherwise the zero temperature IR solution would not exist).

For the $z \neq 1$ solutions, $[A_0] = 0$ (i.e. this coupling is marginal) and so the scaling dimensions of s and n indicate their dependence on the only dimensionful scale T,[15] $s \sim T^{(d-\theta)/z}$ and $n \sim T^0$. For $z = 1$ solutions, the total scaling dimensions of s and n tell us the combined dependence on T and the dimensionful irrelevant coupling A_0. The information about how they separately depend on A_0 and on T is not captured by the total dimension. More information – the explicit expressions (50) – are needed to determine this separate dependence, $s \sim (A_0)^0\, T^{d-\theta}$ and $n \sim A_0\, T^0$.

As we have emphasised, in general the relation between the UV sources and the IR sources of the theory is not simple. However, the fact that the charge density n (50) is directly proportional to A_0 at zero temperature means that it can sometimes be helpful to think of A_0 as an 'IR charge density'.

IV. TRANSPORT NEAR HOLOGRAPHIC QUANTUM CRITICAL POINTS

A. Holographic quantum critical points with $1 \leq z < +\infty$

In [82, 143], we have computed the low-frequency response to a small oscillating electric field – the ac conductivity – near the holographic QCPs described in section III B. We only outline the main results here and refer to [82, 143] for details of the calculation.

We find that the conductivity takes the general form

$$\sigma(\omega) = \frac{n^2}{\chi_{\pi\pi}} \frac{i}{\omega} + (\chi_{\pi\pi})^2 \sigma_{\rm inc}(\omega)\,. \tag{53}$$

The first term arises from momentum conservation and captures the contribution of momentum-dragging processes to the current. The second term is the incoherent contribution described in the section II B, which does not drag momentum.[16] It reads

$$\sigma_{\rm inc}(\omega) = \frac{\sigma^{dc}_{\rm inc}}{1 - i\omega\tau_{eq}}\,, \quad \sigma^{dc}_{\rm inc} = \sigma_o(\chi_{\pi\pi})^2. \tag{54}$$

This calculation is valid in the limit where $\omega \ll T$. This means that it only makes sense to keep the effects of the timescale τ_{eq} in the conductivity (ie extend the validity of (54) to

[15] In fact, as A_0 can be replaced with Z_0 by equation (44) for these solutions, it is not very meaningful to ask how IR observables depend on A_0.

[16] This can be seen more precisely by putting together (19), (28) and recalling that in a Lorentz-invariant theory, $\alpha_o = -\mu\sigma_o/T$, $\bar\kappa_o = \mu^2\sigma_o/T^2$ and $\chi_{\pi\pi} = \varepsilon + p = sT + n\mu$ after imposing the Lorentz boost ward identity $\pi^i = j^i_\varepsilon = j^i_q - \mu j^i$.

frequencies of order $\omega\tau_{eq} \sim 1$ instead of $\omega \ll T$) if this timescale is parametrically longer than the timescale set by temperature, $\tau_{eq}T \gg 1$. A precise expression for τ_{eq} in terms of bulk fields can be found in [82, 143].

For holographic QCPs with $z \neq 1$, we find that in fact $T\tau_{eq} \sim 1$, so that the expression for the conductivity near the QCP (53) is actually of the same form as (19) for charged fluids. The effects of the QCP manifest themselves in the temperature dependence of $\sigma^{dc}_{\rm inc}$. In holographic theories of the kind (38), we showed in [102] that $\sigma^{dc}_{\rm inc} = (sT)^2(s/4\pi)^{(d-2)/2}Z(\phi(r_h))$. All the quantities appearing in $\sigma_{\rm inc}$ are determined by the black hole horizon, or in other words, are IR quantities that are not directly sensitive to details of the UV. So $\sigma_{\rm inc}$ is a contribution to the conductivity which is determined by the IR, and is amenable to an IR scaling analysis. In contrast, σ_o involves $\chi_{\pi\pi} = sT + n\mu \simeq \mu n$ at low temperatures, and μ is not an IR quantity (it comes from the boundary condition at the UV AdS boundary).

Instead, QCPs with $z = 1$ feature a dangerously irrelevant deformation, which manifests itself by sourcing a long-lived mode (a quasinormal mode of the dual black hole) at low temperatures in the vicinity of the QCP. This mode has a lifetime τ_{eq}, which schematically takes the form

$$\tau_{eq} \sim \frac{1}{T}\left(\frac{T^{\Delta_g}}{g}\right)^2 , \tag{55}$$

where g is the irrelevant coupling and Δ_g its dimension. As by definition of an irrelevant deformation $\Delta_g < 0$, the timescale $\tau_{eq} \gg 1/T$ as $T \to 0$. This implies that relativistic hydrodynamics has a significantly restricted range of validity near the QCP (it is valid for times $t \gtrsim \tau_{eq} \gg 1/T$ rather than $t \gtrsim 1/T$), as the dangerously irrelevant coupling slows down the return to equilibrium. This affects the low frequency dependence of the incoherent conductivity (54), which can display a coherent, Drude-like peak centered at $\omega = 0$.

Furthermore, not only the incoherent dc conductivity, but also its associated susceptibility and diffusivity (which obeys an Einstein relation), are consistent with a scaling theory based on the scaling hypothesis:

$$[s] = d - \theta\,, \qquad [n] = d - \theta + \Phi\,. \tag{56}$$

At low temperatures, all of these quantities are directly determined by the near-horizon solution, and so we can explicitly determine their dependence on temperature and on irrelevant couplings. While the incoherent dc conductivity always scales with temperature as predicted by the scaling theory, the associated susceptibility and diffusivity do not. However,

the temperature scaling of these latter observables is consistent with the scaling theory once their dependence on a dangerously irrelevant coupling is accounted for. Such a dangerously irrelevant deformation is present for IR QCPs states with $z = 1$ and $\theta \neq 0$. Further, the low frequency scaling of the ac (time-dependent) incoherent conductivity at zero temperature is also consistent with this scaling theory, once its dependence on the irrelevant coupling is accounted for.

Our results are further evidence that IR scaling theories are a helpful way to understand the properties of near-horizon observables in holographic theories, but that not all such observables obey naive temperature scaling. In other words, some care must be taken when characterizing the degree to which near-horizon observables are universal.

Irrelevant deformations are also known to play an important role in the thermal diffusivity D_T near the $z = 1$ quantum critical points that we study [144]. More precisely, for a generic holographic quantum critical state D_T, written in units of the Planckian time $\tau_P = \hbar/k_B T$ and the butterfly velocity v_B (an IR velocity that quantifies the spread of quantum chaos [145, 146]), is a simple universal constant. However, for $z = 1$ critical states this universal relation breaks down as D_T becomes sensitive to irrelevant deformations of the critical point. We showed that both D_T and τ_{eq} are in fact controlled by the same dangerously irrelevant coupling and therefore that τ_{eq} (rather than τ_P) is the timescale controlling thermal diffusion in these systems

$$D_T = \frac{2}{d+1-\theta} v_B^2 \tau_{eq} \,. \tag{57}$$

This is consistent with the conjectures of [147, 148]. Near translational invariant quantum critical states, we furthermore showed in (35) that the usual 'incoherent' diffusion constant D of relativistic hydrodynamics is equal to D_T and thus $D \sim v_B^2 \tau_{eq}$ in these cases also.

One of the key results of this section is the identification of the physical timescale τ_{eq} which controls the thermal diffusivity near $z = 1$ IR QCPs through equation (57). As anticipated in [147, 148], this timescale is the equilibration timescale of the system, which in our case is much longer than that set by temperature $\tau \sim 1/T$, see (55). In the case we have studied (translation-invariant QCPs), this timescale corresponds to the lifetime of the longest-lived non-hydrodynamic excitation near the QCP. This differs from previous holographic results, where the timescale appearing in (57) was found to be $\tau_{eq} \sim 1/T$ [146, 149–155].

It is natural to ask whether the velocity in the relation (57) should also be interpreted not as the butterfly velocity but as the velocity of a long-lived collective excitation of the

system. Indeed, we note that for $z = 1$ holographic systems, the speed of sound in the infrared spacetime[17] $v_s^2 = 2v_B^2/(d+1-\theta) = 1/(d-\theta)$ is precisely equal to the speed in (57). We anticipate that the systems we study here, in which the $z = 1$ spacetime arises at the IR endpoint of the RG flow, will also support a collective mode with this speed and that it is the speed of this mode that sets the diffusivity.

In ongoing work [156], we are showing that the linearized equations of motion reveal that the low-energy EFT at low temperatures features an emergent $(d-1)$-form symmetry with a mixed 't Hooft anomaly when an external gauge field coupling to the global U(1) symmetry is turned on. This is similar to the EFT for superfluids where the Josephson relation is traded for the conservation of the winding number, [157] (see also the discussion in section V E). This EFT features a gapless, propagating mode (the superfluid Goldstone in the EFT for superfluids), which has the speed $v_s^2 = 1/(d-\theta)$ mentioned in the previous paragraph. At nonzero temperature, this emergent symmetry is broken by the irrelevant operator A_o that puts the system at nonzero density and gives rise to the long timescale τ_{eq}.

Thus the relation (57) is seen to be on the same footing as another case which has been studied in great detail: that of a neutral fluid with slowly relaxing momentum [92]. There it was shown that while transport is diffusive at late times, at earlier times the diffusive mode undergoes a collision and turns into a pair of 'sound' modes with a long lifetime and speed v_s. The thermal diffusivity is $D_T = v_s^2 \tau_{eq}$, see equation (2.17) of [92].

Similar relations arise in other hydrodynamic theories where a gapless mode becomes long-lived, for instance in the hydrodynamic theories of fluctuating superconductivity [158] and fluctuating, pinned charge density waves [64]. The velocities are respectively the superfluid sound velocity and the shear transverse sound velocity, respectively. The same relations can also be derived in probe brane scenarios [159–161], states with generalised global symmetries [162], magnetohydrodynamics and Müller-Israel-Stewart theory [93].

In the more generic cases in which irrelevant deformations are unimportant and therefore τ_{eq} is not parametrically long, the thermal diffusivity near the QCP is [144]

$$D_T = \frac{z}{4\pi(z-1)} \times v_B^2 T^{-1} \,. \tag{58}$$

In light of the previous discussion, it would be interesting to understand if this relation can also be refined by quantitatively identifying a lifetime $\tau_{eq} = \# T^{-1}$ and velocity $v_s^2 = \# v_B^2$ of

[17] By this we mean the speed of the sound mode obtained by promoting the IR spacetime to the full spacetime as in [136].

a collective mode in these systems such that $D_T = v_s^2 \tau_{eq}$. Such a relation would indicate that it is not the butterfly velocity v_B that fundamentally sets the thermal diffusivity, but instead the velocity of a collective mode that transports energy through the system. It would also hint at a relation between the velocities of collective modes and the butterfly velocity near quantum critical points. While the jury is still out for finite values of z, we investigated this question for $z = +\infty$ (that is, with an $\mathrm{AdS}_2 \times \mathrm{R}^2$ geometry) in [85], to which we now turn.

B. Hydrodynamic diffusion and its breakdown near AdS_2 quantum critical points

Even in the absence of a weakly broken symmetry, the breakdown of hydrodynamics can be characterized by an energy scale ω_{eq} and wavenumber k_{eq}, which are sensitive to the system's microscopic details. ω_{eq} and k_{eq} are defined as the absolute values of the complex frequency ω and complex wavenumber k at which the hydrodynamic pole of the retarded Green's function first collides with a non-hydrodynamic pole or branch point [94, 95, 163, 164].[18] The convergence properties of the real-space hydrodynamic gradient expansion are governed by k_{eq} [165, 166], which also coincides with the radius of convergence of the small-k expansion of the hydrodynamic dispersion relation $\omega_{\mathrm{hydro}}(k)$. See [167–170] for other recent applications of this.

In [85], we have studied the breakdown of hydrodynamics in certain low temperature (T) states dual to black holes with nearly-extremal $\mathrm{AdS}_2 \times \mathrm{R}^2$ near-horizon metrics. Such states are closely related to the Sachdev-Ye-Kitaev (SYK)-like models of electrons in strange metals, which are governed by the same type of infra-red fixed point in the limit of large number of fermions and strong interactions [25, 27, 171–180]. Specifically, we have studied the AdS_4-Reissner-Nordström (AdS_4-RN) black brane, and the breakdown of the hydrodynamics governing the diffusive transport of energy, charge and momentum in their dual states.[19] The states we are interested in do not include any slow modes in the sense described in the previous section. Instead, local equilibration is controlled by the incoherent dynamics that follow from the presence of an $\mathrm{AdS}_2 \times \mathrm{R}^2$ fixed point. We identified simple, general results for the local equilibration scales ω_{eq} and k_{eq} and confirmed that these also apply to the SYK chain model studied in [170] in the limit of strong interactions.

[18] Non-analyticities due to interactions between hydrodynamic modes provide an independent mechanism for the breakdown of hydrodynamics (see e.g. [9] for a review), but these are expected to be suppressed in the large N limit [91].

[19] We also studied the diffusion of energy in a state with momentum relaxation. We obtained similar results as in translation-invariant states.

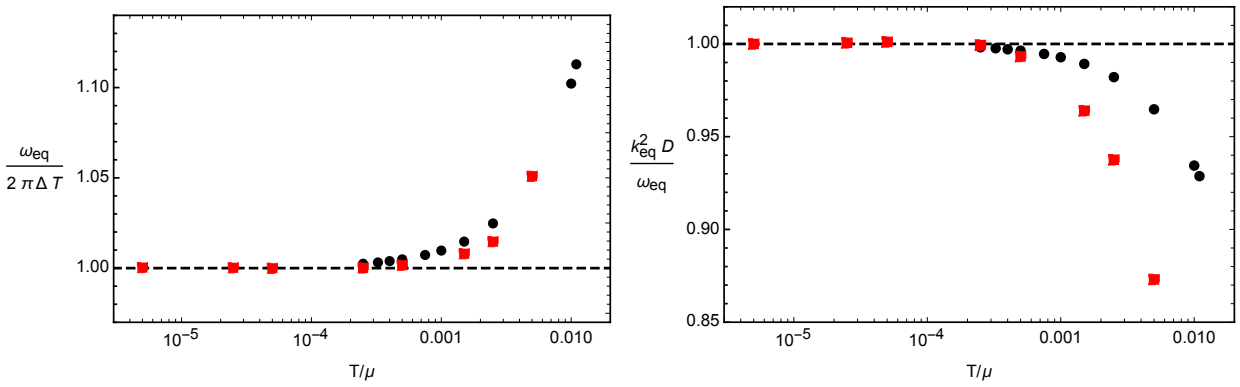

Figure 1. Numerically obtained local equilibration data for diffusive hydrodynamics in $G_{\varepsilon\varepsilon}$ (black circles) and $G_{\pi\pi}$ (red squares) of the charged state.

Our first result (see figure 1) was that the breakdown is caused by modes associated to the AdS_2 region of the geometry, and as a consequence ω_{eq} is set by universal (i.e. infra-red) data via

$$\omega_{eq} \to 2\pi\Delta T \quad \text{as} \quad T \to 0, \tag{59}$$

where Δ is the infra-red scaling dimension of the least irrelevant operator that couples to the diffusion mode. This is in contrast to systems with a weakly broken symmetry, for which $\omega_{eq} \ll T$. More precisely, we find that at small k and T the Fourier space locations of the longest-lived non-hydrodynamic poles are inherited from infra-red Green's functions, and are located at $\omega_n = -i(n+\Delta)2\pi T$ for non-negative integers n. The breakdown is characterized by a collision, parametrically close to the imaginary ω axis, between the $n = 0$ mode (which has a weak k-dependence) and the hydrodynamic mode. This collision manifests itself as a branch-point singularity in the dispersion relation of the mode.

Secondly, we found that at low temperatures the corrections to the quadratic approximation $-iDk^2$ to the exact hydrodynamic dispersion relation are parametrically small such that the collision occurs when k is almost real and

$$k_{eq}^2 \to \frac{\omega_{eq}}{D} \quad \text{as} \quad T \to 0. \tag{60}$$

In other words, the scales k_{eq} and ω_{eq} governing the regime of validity of hydrodynamics are set simply by the diffusivity D and the scaling dimension Δ. In some of the examples we studied (those involving diffusion of energy), the relevant diffusivity is controlled by an irrelevant deformation of the AdS_2 fixed point and in these cases the result (60) indicates that

k_{eq} is controlled by the same irrelevant deformation. A priori, the result (60) is quite surprising: it relates the radius of convergence to just the leading order term in the hydrodynamic expansion. This is a consequence of the AdS_2 fixed point.

By rearranging equation (60) we obtain an answer to the question raised in [15] (and in the previous section) of what the underlying velocity and time scales are that govern the diffusivity in non-quasiparticle systems. In all our examples they are set by the local equilibration scales

$$D \to v_{eq}^2 \tau_{eq}, \quad \text{as} \quad T \to 0, \tag{61}$$

where $v_{eq} \equiv \omega_{eq}/k_{eq}$ and $\tau_{eq} \equiv \omega_{eq}^{-1}$ are the velocity and timescale associated to local equilibration.

In the cases where diffusive hydrodynamics breaks down due to a parametrically slow mode protected by a weakly broken symmetry, D is typically set by τ_{eq} and the speed of the propagating mode that dominates following the breakdown.[20] We emphasize that the breakdown of hydrodynamics is qualitatively different in the cases we study: there is not a single slow mode but a tower of AdS_2 modes with parametrically similar lifetimes set by ω_n, and the breakdown does not produce a propagating mode with velocity $v \simeq v_{eq}$. More generally, the local equilibration time has been argued to set an upper bound on the diffusivity in [147, 148]. All examples that we study are consistent with a bound of the form $D \lesssim v_{eq}^2 \tau_{eq}$ for the range of parameters we have investigated.

In the absence of a slow mode, it was proposed that low temperature diffusivities are set by the butterfly velocity v_B and Lyapunov time τ_L that characterize the onset of scrambling following thermalization of the system [146]. This was shown to robustly apply to the diffusivity of energy density D_ε in holographic theories and SYK-like models [144, 149, 150, 178, 179, 181]. For the examples we study, $\tau_L^{-1} = 2\pi T$ and

$$D_\varepsilon \to v_B^2 \tau_L \quad \text{as} \quad T \to 0, \tag{62}$$

which is furthermore true in general for states governed by an infra-red AdS_2 with the universal deformation [150].

As for our result (61), equation (62) can be viewed as a consequence of the excellent applicability of the quadratic approximation to the exact hydrodynamic dispersion relation

[20] Unlike here, τ_{eq} is often defined by the lifetime of a $k = 0$ mode. In the convention of this section, these examples have $D \to v_{eq}^2 \tau_{eq}/2$ in the limit of slow relaxation, as the $k = 0$ mode has a gap $\omega = -i2/(\tau_{eq})$.

up to the relevant scale. Specifically, pole-skipping analysis suggests that the energy diffusion mode satisfies $\omega_{\text{hydro}}(k = iv_B^{-1}\tau_L^{-1}) = i\tau_L^{-1}$ [154, 182, 183], from which (62) follows assuming corrections to the quadratic, diffusive form $-iD_\varepsilon k^2$ at $k = iv_B^{-1}\tau_L^{-1}$ are parametrically small as $T \to 0$.

Our result (61) is more general than (62) in that it is true for all diffusivities in the examples we study, not just the diffusion of energy. Fundamentally this is because, by definition, all diffusive modes pass through the location set by (ω_{eq}, k_{eq}), while only the energy diffusion mode satisfies the pole-skipping constraint above [184]. As a consequence, this provides a new perspective on, and generalization of, the relations between equilibration, transport and scrambling and their applications in AdS_2/SYK-like models of electrons in strange metals.

There is good reason to expect that at least some of our results will generalise beyond the specific examples studied here to other states governed by AdS_2 infra-red fixed points. While our key observation that the quadratic approximation to the hydrodynamic dispersion relation works parametrically well even for wavenumbers $k^2 \sim T$ seems unusual, it is non-trivially consistent with the result (62) that is indeed true for holographic $AdS_2 \times R^2$ fixed points with a universal deformation [150] as well as in related SYK chain models [178, 179].

There are more general holographic and SYK-like systems governed by AdS_2 fixed points that exhibit additional diffusive modes beyond the two types we have studied. Of particular interest are non-translationally invariant systems with a $U(1)$ symmetry, for which an Einstein relation relates the electrical resistivity to a diffusivity [15, 150, 179]. If our results (59) and (60) extend to such modes, they will therefore also provide a simple relation between the phenomenologically important electrical resistivity and the local equilibration scales of such strongly correlated systems.

Confirmation of the broader applicability of our result (61) for $AdS_2 \times R^2$ solutions would be an important step for quantifying diffusivities near general infra-red fixed points. One way to do this would be to identify a speed u and timescale τ such that in general $D \sim u^2\tau$ with the coefficient being T-independent. This is difficult even for the relatively simple case of holographic energy diffusion, primarily because there are two exceptional types of fixed point where dangerously irrelevant deformations take over the properties of the mode: $AdS_2 \times R^2$ fixed points (i.e. dynamical critical exponent $z = \infty$) [150] and relativistic fixed points (i.e. $z = 1$) [144]. If our result does generalize to $AdS_2 \times R^2$ solutions (including

those with non-universal deformations), both of these exceptional cases will be consistent with the identification $u = v_{eq}$ and $\tau = \tau_{eq}$.[21,22] Provided that naive T-scaling holds for the equilibration scales in the other cases ($\tau_{eq} \sim 1/T$ and $v_{eq} \sim T^{1-1/z}$), which seems likely, this identification will then work for all fixed points.

V. HYDRODYNAMICS OF PHASES WITH BROKEN TRANSLATIONS

Continuous, global symmetries can be spontaneously broken in the ground state (see [125] for a recent introduction). Formally, this means that the ground state is invariant under a smaller set of symmetries than the Hamiltonian of the system. A corresponding number of gapless modes, the Goldstone bosons, appear in the spectrum. In the simplest case of an internal symmetry and when the broken generators commute, the number of Goldstones is given by the number of broken generators. In other cases, such as that of spacetime symmetries [187], the counting rule is more complicated, see [188] for a review.

Hydrodynamics can be advantageously extended to systems with spontaneously broken symmetries, like superfluid Helium or crystalline solids, [88]. The set of slow degrees of freedom is enlarged to include the Goldstone modes, the dynamical evolution of which is described by so-called Josephson equations (historically, the Josephson equation describes the phase difference in a superconductor in the presence of an external voltage).

In this section, we focus on the case of broken translations, such as crystalline solids, charge density waves and Wigner crystals. We incorporate the effects of background strain, which provd important to match to existing holographic studies. Moreover, strain/pressure is also a common experimental tool in the investigation of broken translation phases in strongly-correlated materials and has a strong effect on the onset of the charge density wave and superconducting phases, [189, 190].

A. Thermodynamics

For simplicity, we assume the system is two-dimensional ($d = 2$), isotropic and that translations are spontaneously broken in all spatial directions – extensions to anisotropic or

[21] We expect the result of footnote 20 to apply to the $z = 1$ cases due to the existence of a parametrically slow mode [82, 143].

[22] Another family of holographic $z = 1$ fixed points was studied in [185, 186]. These geometries do not have a slow mode, but in a certain limit also display an emergent SL(2,R) symmetry and a spectrum of infra-red mode similar to the one studied in this work.

higher-dimensional crystals are conceptually straightforward but technically tedious, due to more complicated tensor structures and a larger number of transport coefficients. We will not consider the coupling to background sources, which can be realized along the lines of [79, 80]. We also do not assume any particular boost symmetry.[23]

Since spatial translations are spontaneously broken in all directions, we expect as many Goldstone modes as there are broken translations.[24] The Goldstones of broken translations are often called phonons and are related to the displacements of the underlying crystal structure, which we denote by u^i. They transform non-linearly under spatial translations $x^i \to x^i + a^i$ as

$$u^i \to u^i + a^i \,. \tag{63}$$

The free energy of the system must be invariant under these shifts, so that it only depends on the non-linear, Lagrange strain tensor, [88]

$$u_{ij} = \nabla_{(i} u_{j)} + \frac{1}{2} \nabla_i u_k \nabla_j u^k \,. \tag{64}$$

Lower-case latin indices $i, j, \ldots$ are raised an lowered with the Kronecker delta δ_{ij} and run over spatial dimensions.

The elastic part of the equilibrium free energy density of the system is then, by isotropy,

$$f_{el} = \frac{B_o(X,Y)}{2} X^2 + G_o(X,Y)\, Y \tag{65}$$

where we have defined $X \equiv u^i_i$, $Y \equiv u_{ij}u^{ij} - \frac{1}{2}(u^i_i)$ and suppressed the dependence on temperature and chemical potential for now.[25] The first parameter X corresponds to a purely volumetric deformation, $\Delta V/V = X$, while the second, Y, to a deviatoric deformation which modifies the shape of the material but not its total volume. Accordingly, the coefficients B_o, G_o are the bare nonlinear bulk and shear moduli, respectively. If translations are broken in one dimension only, there is only a bulk modulus. Both quantities are nonlinear functions of the deformation parameters X, Y and, in what follows, of temperature and chemical potential as well.

Since we wish to compute the linear response of the system to external perturbations, the first step is to determine the static susceptibilities. To this end, we expand the static

[23] See [96–101] for more complete treatments of fluid hydrodynamics without boosts.

[24] Rotations are also spontaneously broken, but do not have independent Goldstones, per the Goldstone counting theorems for broken spacetime symmetries [187]. The underlying reason is that translations and rotations are not independent local transformations.

[25] One could equivalently define a generic function $f_{el}(X,Y)$. Our parametrization makes the linear limit $X, Y \to 0$ and the limit of zero background strain clearer. In $d > 2$, this function would depend on d independent scalars, [191, 192].

free energy to quadratic order in fluctuations using[26]

$$u_i = m\, x^i + (1+m)\delta\phi_i(x)\,, \quad i = \{x, y\}\,, \tag{66}$$

where m is a real parameter, and we have assumed that the linear perturbations $\delta\phi_i$ depend only on one spatial dimension. As we are about to see, the $\delta\phi_i$ are the Goldstones modes of the system.

A nonzero m represents a nonzero, isotropic background strain on the state:

$$X = u^l_l = u_o + \mathcal{O}(\nabla)\,, \quad u_o \equiv m(m+2)\,. \tag{67}$$

For this state, $Y = \mathcal{O}(\nabla^2)$. In this configuration, the equilibrium free energy density f_{el}, and the bare elastic moduli B_o, G_o are functions of u_o.

Anisotropic strains can easily be considered by allowing for additional off-diagonal terms of the type $u_i \sim x_j$. The first term mx^i in Eq.(66) can be thought as the additional displacement from the would-be equilibrium configuration $m = 0$ to the actual configuration with background strain $m \neq 0$. In this sense, for $m \neq 0$, the Goldstone fields $\delta\phi_i(x)$ parametrize the low-energy dynamics around the configuration with background strain u_o.

Plugging in the expansion (66) in the elastic free energy (65) and expanding to quadratic order in fluctuations, we obtain

$$f_{el} = \frac{u_o^2}{2} B_o - p_{el}\lambda_\parallel + \frac{G}{2}\left(\lambda_\perp\right)^2 + \frac{1}{2}\left(B+G\right)\left(\lambda_\parallel\right)^2 \tag{68}$$

where it is convenient to define the longitudinal and transverse Goldstones $\lambda_\parallel = \nabla\cdot\delta\phi$, $\lambda_\perp = \nabla\times\delta\phi$. We have also defined a background elastic pressure $p_{el} = -(1+u_o)\partial_{u_o}(u_o^2 B_o)/2$ due to the nonzero background strain u_o, as well as the renormalized, effective nonlinear bulk and shear moduli

$$\begin{aligned} B \,&\equiv \frac{1}{2}(1+u_o)^2\partial^2_{u_o}\left(u_o^2 B_o\right)\,, \\ G \,&\equiv (1+u_o)^2 G_o + \frac{u_o^2}{2}(1+u_o)^2\partial_Y B_o - p_{el}\,. \end{aligned} \tag{69}$$

The bare bulk and shear moduli are renormalized due to the nonzero background strain. In the limit of zero strain, we recover $B(u_o = 0) = B_o$ and $G(u_o = 0) = G_o$.

The free energy in Eq.(68) now displays a linear term in $\delta\phi_x$ with coefficient p_{el}.[27] This linear term implies that the system is not in mechanical equilibrium whenever m is nonzero:

[26] This maps to the formulation of [79] as follows: $u^i \leftrightarrow \Phi^I - x^I$, $u^{ij} \leftrightarrow h^{IJ} - \delta^{IJ}$, $m = \frac{1}{\alpha} - 1$.

[27] This term also appears in the relativistic treatment of [79]. Here we generate it by expanding around the state with background strain in Eq.(66). In [79], p_{el} is introduced directly as a force contribution to the free energy. There, the reference contribution is not assumed to minimize the free energy. After matching conventions, both approaches give the same results.

a background strain is applied to the system, through non-trivial boundary conditions and the resulting sum of external forces does not vanish. Instead, in mechanical equilibrium, there is no background displacement, $m = 0$, and $p_{el} = 0$. The standard treatment of elasticity theory (see e.g. [88]) assumes the reference configuration, the choice of which is arbitrary, to correspond with the state of mechanical equilibrium.

Turning on background external sources $f \mapsto f - s_\parallel \lambda_\parallel - s_\perp \lambda_\perp$, we obtain the Goldstone static susceptibilities after integrating out the Goldstones:[28]

$$\begin{aligned} \chi_{\lambda_\parallel \lambda_\parallel} &\equiv -\frac{\partial^{(2)} f_{el}}{\partial {s_\parallel}^2} = \frac{1}{B+G}\,, \\ \chi_{\lambda_\perp \lambda_\perp} &\equiv -\frac{\partial^{(2)} f_{el}}{\partial {s_\perp}^2} = \frac{1}{G}\,. \end{aligned} \tag{70}$$

$B + G$ and G should both be positive definite in order for the phase to be locally thermodynamically stable, which follows from the usual requirement that the determinant of the Hessian of the free energy be positive definite. We will see later on that this ensures that sound modes have a positive velocity squared. Through (69), we observe that both the bulk and shear moduli have a non-trivial dependence on background strain. Varying the background strain may lead to thermodynamic instabilities, signaled by divergences in the static susceptibilities (70) when G or $B + G$ change sign. Determining whether these instabilities are actually present requires to know their functional dependence on u_o and is beyond the effective field theory approach.

By their properties under parity transformations $x \mapsto -x$, we also expect the longitudinal phonon $\lambda_\parallel$ to couple to entropy and charge. To this end, we include temperature and chemical potential dependence in the bare moduli $B_o(u_i^i, T, \mu)$ and $G_o(u_i^i, T, \mu)$ in (65). Linearizing around (66) together with $\{T, \mu\} = \{T_o, \mu_o\} + \{\delta T, \delta\mu\}$ allows to identify the off-diagonal susceptibilities

$$\begin{aligned} \chi_{n\lambda_\parallel} &\equiv -\frac{\partial^{(2)} f_{el}}{\partial s_\parallel \partial \mu} = \frac{\partial_\mu p_{el}}{B+G}\,, \\ \chi_{s\lambda_\parallel} &\equiv -\frac{\partial^{(2)} f_{el}}{\partial s_\parallel \partial T} = \frac{\partial_T p_{el}}{B+G}\,. \end{aligned} \tag{71}$$

They are nonzero even in the absence of background strain and correspond physically to the chemical and thermal expansion of the system under strain.

[28] This leads to $s_\perp = G\lambda_\perp$, $s_\parallel = (G+B)\lambda_\parallel$.

The full longitudinal static susceptibility matrix reads:

$$\chi_{o,\parallel} = \begin{pmatrix} \chi_{nn} & \chi_{n\varepsilon} & 0 & \chi_{n\lambda_\parallel} \\ \chi_{n\varepsilon} & \chi_{\varepsilon\varepsilon} & 0 & \chi_{\varepsilon\lambda_\parallel} \\ 0 & 0 & \chi_{\pi\pi} & 0 \\ \chi_{n\lambda_\parallel} & \chi_{\varepsilon\lambda_\parallel} & 0 & \chi_{\lambda_\parallel\lambda_\parallel} \end{pmatrix} . \tag{72}$$

The equality of off-diagonal components follows from invariance under PT symmetry.

In the transverse sector, the susceptibility matrix $\chi_{o,\perp}$ is diagonal with the two nonzero elements $\chi_{\lambda_\perp\lambda_\perp}$, given by (70), and $\chi_{\pi_\perp\pi_\perp} = \chi_{\pi\pi}$, by isotropy.

B. Dynamics

We are now ready to state the equations that govern the dynamics of the system in the hydrodynamic regime. Assuming rotation, translation and $U(1)$ symmetry, these are the conservation of energy, charge and momentum density

$$\dot{\varepsilon} + \nabla \cdot j^i_\varepsilon = 0\,, \quad \dot{n} + \nabla \cdot j = 0\,, \quad \dot{\pi}^i + \nabla_j \tau^{ji} = 0\,, \tag{73}$$

together with the Josephson equation for the dynamic evolution of the Goldstones:

$$\frac{\mathrm{d}}{\mathrm{dt}} u^i = -v^i + \dots\,. \tag{74}$$

$\mathrm{d}/\mathrm{dt} \equiv \partial_t + v^i \nabla_i$ stands for the material derivative and the dots for dissipative corrections to this relation.

We can arrive at (74) in the following way. The Goldstone fields are canonically conjugate to the momentum density, i.e. the conserved charge that generates the broken symmetry:

$$i\,[\pi^i(x), u^j(x')] = -\delta^{(2)}(x - x')\left(\delta^{ij} + \nabla^i u^j\right)\,. \tag{75}$$

Then, we deform the Hamiltonian by an external velocity source $H_o \mapsto H = H_o - \int d^2x\, \pi_i v^i_e$ and use the Schrödinger equation to compute the time evolution of the displacement:

$$\dot{u}^i = i\,[H, u^i] = v^i_e + v^j_e \nabla_j u^i\,. \tag{76}$$

Since u^i must be time-independent in thermodynamic equilibrium ($v^i = v^i_e$), this means that the Josephson relation must take the form

$$\dot{u}^i = (v^j_e - v^j)\left(\delta^i_j + \nabla_j u^i\right) + \tilde{u}^i\,, \tag{77}$$

in agreement with (9). Taking a divergence or a curl of (77) with sources off leads to (74). We have allowed for a possible dissipative correction $\tilde{u}^i$.

In our thermodynamic ensemble, the first law of thermodynamics is

$$df = -s\, dT - n\, d\mu + h^{ij} d\left(\nabla_i u_j\right), \tag{78}$$

where $h^{ij} \equiv \partial f / \partial(\nabla_i u_j)$ is

$$\begin{aligned} h^{ij} = &\left(X B_o + \frac{X^2}{2} \partial_X B_o + Y \partial_X G_o \right) X^{ij} \\ &+ \left(2 G_o + Y \partial_Y G_o + \frac{X^2}{2} \partial_Y B_o \right) Y^{ij} , \end{aligned} \tag{79}$$

with

$$\begin{aligned} X^{ij} &= \frac{\partial X}{\partial \nabla_i u_j} = \delta^{ij} + \nabla^i u^j , \\ Y^{ij} &= \frac{\partial Y}{\partial \nabla_i u_j} = 2\left(u^{ij} + u^{ik} \nabla_k u^j\right) - X X^{ij} . \end{aligned} \tag{80}$$

Using that the entropy density must be conserved $\dot{s} + \nabla_i (s v^i + \tilde{j}_q^i / T) = 0$ in the absence of dissipative (gradient) corrections, the ideal constitutive relations are found to be

$$j_\varepsilon^i = (\varepsilon + p)\, v^i + h^{ij} v_j + h^{il} v^j \nabla_j u_l + \tilde{j}_\varepsilon^i , \tag{81}$$

$$\tau^{ij} = p\, \delta^{ij} + h^{ij} + h^{il} \nabla^j u^l + v^i \pi^j + \tilde{\tau}^{ij} , \tag{82}$$

$$j^i = n\, v^i + \tilde{j}^i . \tag{83}$$

Here p is the thermodynamic pressure, which verifies $p = -f = -\varepsilon + sT + n\mu + v_k \pi^k$. It is straightforward to verify that the stress tensor τ^{ji} is symmetric by substituting the expression for h^{ij} in terms of u^{ij} in (81). $\tilde{j}_q^i$, $\tilde{j}_\varepsilon^i$, $\tilde{\tau}^{ji}$ and $\tilde{j}^i$ all stand for dissipative corrections which are at least first order in gradients.

The form of dissipative corrections are determined by a well-known algorithm. We start by allowing all possible terms that are spatial derivatives of the fields (the conserved densities and the Goldstones) consistent with the symmetries – for instance, we do not allow terms that violate parity. Then, we require that these terms do not lead to non-localities in the equations of motion. Finally, we check that the entropy current is positive definite, also imposing Onsager relations. The outcome of this procedure, which we detail in appendix A,

leads to the following constitutive relations:

$$\begin{aligned}
\tilde{j}^i &= -\sigma_o^{ij}\nabla_j\mu - \alpha_o^{ij}\nabla_j T - \frac{1}{2}\xi_\mu^{ij}\nabla^k h_{kj}\,,\\
\frac{\tilde{j}_q^i}{T} &= -\alpha_o^{ij}\nabla_j\mu - \frac{\bar{\kappa}_o^{ij}}{T}\nabla_j T - \frac{1}{2}\xi_T^{ij}\nabla^k h_{kj}\,,\\
\tilde{\tau}^{ij} &= -\eta^{ijkl}\nabla_{(k}v_{l)}\,,\\
\tilde{u}^i &= \xi_\mu^{ij}\nabla_j\mu + \xi_T^{ij}\nabla_j T + \xi_h^{ij}\nabla^k h_{kj}\,,\\
\tilde{j}_\varepsilon^i &= \tilde{j}_q^i + \mu\tilde{j}^i - h_{ij}\tilde{u}^j + v_j\tilde{\tau}^{ij}\,.
\end{aligned} \tag{84}$$

In the absence of background strain, all the transport matrices would have a trivial index structure and depend on temperature and chemical potential only,[29] e.g. $\sigma_o^{ij} = \sigma_{(o)}(T,\mu)\delta^{ij}$ or $\eta^{ijkl}\nabla_{(k}v_{l)} = -\eta\sigma^{ij} - \frac{2}{d}\zeta\partial_k v^k\delta^{ij}$, where we have defined the shear rate tensor $\sigma_{ij} = \nabla_{(i}v_{j)} - \frac{2}{d}\nabla_k v^k\delta_{ij}$.

In the presence of background strain, the strain tensor u_{ij} provides an independent rank-2 tensor. This gives rise to new terms in the transport matrices, which all take the form $\sigma_o^{ij} = \sigma_{(o)}(T,\mu,u_o)\delta^{ij} + \sigma_{(u)}(T,\mu,u_o)u^{ij}$ in $d = 2$[30] with some arbitrary dependence on u_o (since $X = u_o$ and $Y = 0$ when evaluated on the background (66)). There is more freedom in the viscosity rank-4 tensor, which takes the general form

$$\begin{aligned}
\eta_{ijkl} =& 2\,\eta^{(0)}\left(\delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl}\right) + \zeta^{(0)}\delta_{ij}\delta_{kl}\\
&+ 2\,\eta^{(u)}\left(\delta_{ik}u_{jl} - \frac{1}{2}\delta_{ij}u_{kl} - \frac{1}{2}u_{ij}\delta_{kl} + \frac{1}{4}u_m^m\delta_{ij}\delta_{kl}\right)\\
&+ 2\left(\zeta^{(u)} + \bar{\zeta}^{(u)}\right)\delta_{ij}u_{\langle kl\rangle} + 2\left(\zeta^{(u)} - \bar{\zeta}^{(u)}\right)u_{\langle ij\rangle}\delta_{kl}
\end{aligned} \tag{85}$$

where angular brackets stand for the transverse, traceless part of the tensor.

[29] Throughout this manuscript, we do not consider states with a background fluid velocity. See [193], [96–101].

[30] In $d > 2$, additional tensor structures such as $u_k^i u^{kj}$ would enter. To map to the formulation of [79], higher-order terms in their strain u^{IJ} need to be considered.

C. Linear response

With the constitutive relations in hand, we can now investigate the linear response of the system about the equilibrium state

$$\begin{pmatrix} T \\ s \\ \mu \\ n \\ v^i \\ \pi^i \\ u^i \end{pmatrix} = \begin{pmatrix} T_o \\ s_o \\ \mu_o \\ n_o \\ 0 \\ 0 \\ mx^i \end{pmatrix} + \begin{pmatrix} \delta T \\ \delta s \\ \delta \mu \\ \delta n \\ \delta v^i \\ \delta \pi^i \\ \delta \phi^i \end{pmatrix} e^{-i\omega t + iqx} \,. \tag{86}$$

We have made use of the underlying translation invariance of the system to decompose the linear perturbation in plane waves.

We start with the transverse sector. In contrast to the fluid case discussed at the end of Section II (see also [9]), the transverse Goldstone field mixes with transverse momentum to form a pair of sound modes propagating in opposite directions:

$$\omega = \pm\, q \sqrt{\frac{G}{\chi_{\pi\pi}}} - \frac{i}{2}\left(\frac{\eta}{\chi_{\pi\pi}} + G\,\xi\right) q^2 + O(q^3)\,. \tag{87}$$

This is the celebrated shear sound mode of crystalline solids. Its velocity is real provided that the matrix of static susceptibility is positive definite, which implies $G > 0$ and $\chi_{\pi\pi} > 0$. The sound attenuation receives two contributions

$$\eta \equiv \eta^{(0)} + \frac{u_o}{2}\eta^{(u)}\,, \quad \xi \equiv \frac{1}{1+u_0}\left(\xi_h^{(0)} + \frac{u_0}{2}\xi_h^{(u)}\right)\,. \tag{88}$$

Here, as in the longitudinal sector, we find that the effect of the extra terms in the constitutive relations due to background strain can be hidden away in a redefinition of the transport coefficients contributing to linear response. This is advantageous as this means there is no proliferation of transport coefficients. For instance, the shear Kubo formula that usually measures the shear viscosity for a fluid becomes

$$\eta^{(0)} + \frac{u_o}{2}\eta^{(u)} \equiv \eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{\tau^{xy}\tau^{xy}}(\omega, q=0) \tag{89}$$

and it is the linear combination (88) which appears, not the individual transport coefficients $\eta^{(0)}$ and $\eta^{(u)}$.

There is a similar Kubo formula for ξ_μ:

$$\xi_\mu = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{j\dot{u}}(\omega, q=0) \tag{90}$$

and for ξ:

$$\xi = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{\dot{u}\dot{u}}(\omega, q=0) \tag{91}$$

which as we will see defines the Goldstone diffusivity.

In the longitudinal sector, there are four modes: two sound modes propagating in opposite directions and two diffusive modes. Their expressions are in general quite complicated, and so we report them only for a neutral, relativistic system (in which one of the diffusive mode disappears)[31]

$$\begin{aligned}
&\omega = \pm c_\parallel q - \frac{i}{2}\Gamma_\parallel q^2\,, \qquad \omega = -iD_\parallel q^2\,, \qquad D_\parallel = \frac{(B+G)\,\chi_{\pi\pi}\,\xi}{{c_\parallel}^2\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon}}\,,\\
&{c_\parallel}^2 = \frac{B+G}{\chi_{\pi\pi}} + \frac{T^2\left(s_o - \chi_{sh_\parallel}\right)^2}{\chi_{\pi\pi}\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon}}\,, \quad \Gamma_\parallel = \frac{\eta+\zeta}{\chi_{\pi\pi}} + \frac{\xi\,\chi_{\pi\pi}}{{c_\parallel}^2}\left({c_\parallel}^2 - \frac{T\left(s_o - \chi_{sh_\parallel}\right)}{\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon}}\right)^2\,,
\end{aligned} \tag{92}$$

Here $\chi_{sh_\parallel} = \partial s/\partial\lambda_\parallel = -\partial p_{el}/\partial T_0$, and similarly $\chi^{(\lambda_\parallel)}_{\varepsilon\varepsilon} = c_v T_0$ (with c_v the heat capacity), computed fixing $\lambda_\parallel$. Similarly to the transverse sector, only certain linear combinations of transport coefficients appear (e.g. ξ instead of both $\xi^{(0)}_h$ and $\xi^{(u)}_h$). After matching conventions, these expressions agree with [79, 80]. The modes can be worked out in full generality (absence of boost symmetry, finite density, nonzero background strain), but become rather complicated. The appearance of instabilities related to a change of sign of $B+G$ is manifest in the expression for $D_\parallel$, as the corresponding purely imaginary mode would then cross to the upper half complex frequency plane.

At very low temperatures, we expect the sound modes to be mostly carried by the Goldstone field and the longitudinal momentum density, while the diffusive mode corresponds to thermal diffusion. At temperatures close to the critical temperature, the sound modes are carried by thermal and momentum fluctuations, while the diffusive mode is predominantly carried by the Goldstone mode.

[31] See the appendix A for details on how to take this limit. See also [80] for the complete expressions at finite density.

D. Holography

The holographic description of broken-symmetry, strongly-coupled phases of matter with an eye towards condensed matter was initiated in [194–196] by considering the spontaneous breaking of a global $U(1)$ symmetry – a superfluid state. Superfluid hydrodynamics correctly predicts the low energy dynamics of holographic superfluids, [197–202].

Closely following the global $U(1)$ case, instabilities towards holographic spatially modulated phases breaking translations spontaneously were investigated as well, [203–207]. In these studies, one looks for a spatially modulated, normalizable bulk mode (i.e., without a source at the boundary) in the translation-invariant, homogeneous bulk geometry. The outcome of this analysis is an instability curve displaying the maximum temperature at which the mode can be found vs the wavevector of the modulation, i.e., the onset of the instability. The apex of this curve gives the thermodynamically preferred wavevector with the highest critical temperature, below which a fully backreacted, spatially modulated phase breaking translations spontaneously can be expected to be found. As such, when the preferred trajectory within the instability curve is followed, these phases are true global minima of the thermodynamic free energy, [208]. The breaking of parity and time reversal through Chern-Simons couplings in the bulk and/or an external magnetic field played an important role originally to generate the instabilities, but is not always necessary, [209]. The original works focused on inhomogeneous instabilities, but helical phases proved easier to construct at first, [210–212]. Backreacted inhomogeneous phases spatially modulated along one direction were constructed in [213–215], bearing on the expertise developed to construct explicit holographic lattices. Generalizations to two-dimensional, checkerboard or triangular patterns are found in [216, 217], with the triangular lattice providing the minimum free energy state. Remarkably, these phases all include circulating current loops together with spontaneous parity breaking, which is reminiscent of the loop current order proposed to underlie the pseudogap phase of underdoped cuprate high T_c superconductors, [218]. This phenomenology is a direct consequence of the bulk Chern-Simons term.

Probe branes constructions can also display spatially modulated instabilities [219–221]. Being top-down models descending from specific string theory realizations, they have the advantage of offering a more precise field theory interpretation. On the other hand, it is not clear how one should interpret the spontaneous spatial modulation of charge and current

densities, since in these setups momentum and temperature fluctuations are frozen.

Phases in which the breaking of translations and of a global $U(1)$ are intertwined are of interest to model pair density wave phases [222], which are thought to play an important role in the phase diagram of underdoped cuprate high T_c superconductors. They have been argued to be the mother phase from which daughter charge density wave and superconducting phases emerge. Holographic realization of these phases are found in [223–225]. These constructions rely on a combination of bulk Chern-Simons terms and the introduction of Stückelberg scalars, which naturally give rise to pair density wave phases where the condensate is spatially modulated with a zero average and periodicity which is twice that of the charge density wave.

While hydrodynamics for Galilean-invariant phases with broken translations is an old subject [88, 226–228], it was revisited in [64], which incorporated phase relaxation by defects and pinning by disorder in the hydrodynamic framework without assuming Galilean invariance. Many holographic studies were subsequently carried out, intent on verifying the match between the holographic and hydrodynamic approaches, [73, 76, 79–81, 84, 229–231]. Importantly, a consistent hydrodynamic construction with nonzero background strain and with coupling to external sources, [79, 80].

The presence of an isotropic background strain (equivalently, a background elastic pressure p_{el}) is a common feature of many of these homogeneous holographic models, which at an operational level can be directly observed by identifying an extra contribution p_{el} to the (relativistic) momentum susceptibility, $\chi_{\pi\pi} = \varepsilon + p - p_{el}$. More importantly, this implies that the states considered are not global (or even local) minima of the holographic thermodynamic free energy, [208, 217] (when the free energy is minimized, $p_{el} = 0$). In spite of this, they are locally thermodynamically stable, with a positive definite static susceptibility matrix. Accordingly, they do not have poles in the upper half complex frequency plane.

Increasing the background strain gives additional contributions to the effective elastic moduli through (69). Depending on the specific functional dependence on strain, this may lead to thermodynamic instabilities if the effective elastic moduli vanish (this leads in turn to a divergence of the corresponding susceptibilities (70)). These thermodynamic instabilities have dynamical counterparts, as e.g. the transverse sound velocity (87) becomes complex or the longitudinal diffusive mode $D_{\parallel}$ (92) crosses to the upper half plane. The conjectured endpoint of this instability is the nucleation of topological defects, which relax the back-

ground strain, and probably leads to a plastic behaviour and the failure of the rigidity of the system.[32]

In holographic systems, the black hole horizon provides a large bath of $\mathcal{O}(N^2)$ degrees of freedom. It is natural to expect that the Goldstone can relax into this bath. This is embodied by a modernized version of the "membrane paradigm", [236, 237], whereby transport coefficients characterizing linear response are expressed in terms of the background solution evaluated on the black hole horizon through the construction of radially conserved bulk fluxes, [238],[239, 240].

This was used to great effect to compute the linear, relativistic transport coefficients (84) in holographic models of spontaneously broken translations, either homogeneous [84, 229] or inhomogeneous, [241, 242]. As an illustration, we can consider the model of [84]:

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{2}\partial\phi^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi) - \frac{1}{2}Y(\phi)\sum_{i=1}^{2}\delta^{IJ}\partial\psi_I\partial\psi_J \right], \tag{93}$$

with $I, J = 1, 2$ run over the spatial coordinates of the boundary. The model has a global shift symmetry $\psi_I \to \psi_I + c_I$ where c_I is a real constant. Picking the bulk Ansatz $\psi_I = k\delta_{Ii}x^i$ breaks both spatial translations $x^i \to x^i + a^i$ and the shift symmetry, but preserves a diagonal subgroup. The scalar couplings $V(\phi)$, $Z(\phi)$ and $Y(\phi)$ are chosen so as to allow an asymptotically anti de Sitter UV spacetime. Moreover, picking $Y(\phi) \sim \phi^2$ for small ϕ, together with spontaneous Dirichlet conditions for ϕ, results in spontaneous breaking of translations (see [84] for details of the model).

The dissipative coefficients then take the relatively simple form:

$$\sigma_o = \frac{\left(sT + k^2 I_Y\right)^2}{\left(sT + \mu n + k^2 I_Y\right)^2} Z_h + \frac{4\pi k^2 (I_Y)^2 n^2}{sY_h \left(sT + \mu n + k^2 I_Y\right)^2}, \tag{94}$$

$$\xi_\mu = -\frac{4\pi I_Y n \left(sT + \mu n\right)}{sY_h \left(sT + \mu n + k^2 I_Y\right)^2} - \mu \frac{\left(sT + k^2 I_Y\right)}{\left(sT + \mu n + k^2 I_Y\right)^2} Z_h, \tag{95}$$

$$\xi = \frac{4\pi \left(sT + \mu n\right)^2}{k^2 sY_h \left(sT + \mu n + k^2 I_Y\right)^2} + \frac{\mu^2 Z_h}{\left(sT + \mu n + k^2 I_Y\right)^2}. \tag{96}$$

Here $I_Y \equiv \int_{r_h}^{0} \sqrt{g_{rr} g_{tt}} Y(\phi) dr$, with r_h the horizon location, g_{rr} and g_{tt} the dr^2 and dt^2 metric elements. Moreover $Z_h \equiv Z(\phi(r_h))$, $Y_h \equiv Y(\phi(r_h))$.

Together with knowledge of the static susceptibilities and of the value of the shear viscosity η, this allows to check that the dispersion relation of transverse hydrodynamics modes

[32] The Landau instability for superfluids at large background superfluid velocities [232] is also driven by the change of sign of a static susceptibility. In other words, a thermodynamic instability leads to a dynamical instability, which is manifested by a pole crossing to the upper half complex frequency plane, [233]. This explains results reported in bottom-up holographic models [234, 235].

(87) matches the gapless quasinormal modes in the holographic model of [84], see figure 2. While this was not clear at the time when [84] appeared, but as we explained in the previous sections, the effects of background strain that are inherent to these holographic models can be hidden away in a redefinition of the momentum static susceptibility $\chi_{\pi\pi}$. Also, what (94)-(96) are really computing are the linear combinations of the sort (88). The match between the longitudinal dispersion relations and the holographic quasinormal modes in a slightly different holographic (massive gravity) model has been demonstrated in [76, 81].

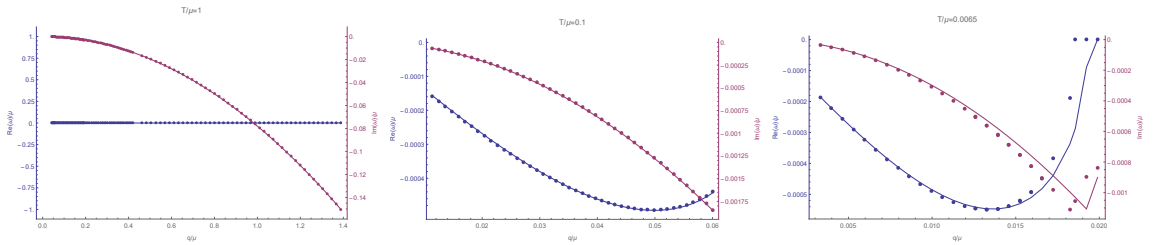

Figure 2. Match between the dispersion relation of transverse hydrodynamics modes (87) (solid lines) and the gapless quasinormal modes in the holographic model of [84] (dots) at various temperatures.

In homogeneous holographic models, it is very well understood how to encode for quantum critical infrared fixed points with broken translations, [135, 243]. Near such critical phases, it was observed [84] that some of the transport coefficients are not independent

$$\xi_\mu = -\left(\frac{\mu}{\chi_{\pi j_q}}\right)\sigma_o\,, \quad \xi = \left(\frac{\mu}{\chi_{\pi j_q}}\right)^2\sigma_o\,, \tag{97}$$

where for relativistic phases $\chi_{\pi j_q} = s_o T_o - p_{el}$. Effectively, the Goldstone relaxation processes are governed by the incoherent (i.e. without momentum drag) diffusivity σ_o, which also controls the thermal diffusivity with open circuit boundary conditions, [82, 102]. This saturates the entropy bound $\xi_\mu^2 \leq \sigma_o \xi_\mu$ obeyed by these coefficients.

This can be understood as arising from the dominance of the following effective interaction between the momentum and the heat current j_q^i in the infrared Hamiltonian:

$$\Delta H = \frac{1}{\chi_{\pi j_q}}\int d^d x\, \pi_i j_q^i\,. \tag{98}$$

This in turn implies that

$$\dot{u}^i = i\,[H, u^i] = \frac{j_q^i}{\chi_{\pi j_q}}\,. \tag{99}$$

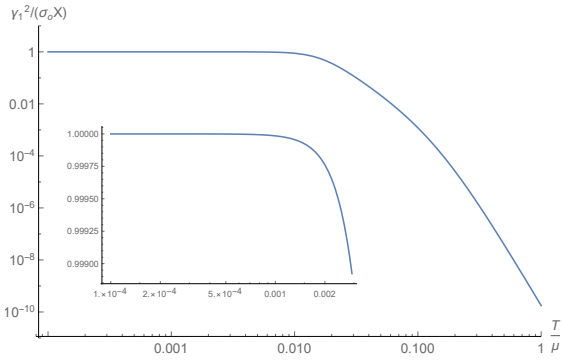

Figure 3. Low-temperature saturation of the entropy bound by the diffusive transport coefficients in (97), from [84] (notwithstanding some small notational changes compared to the present manuscript).

Plugging this in the Kubo formulæ for ξ_μ and ξ, (90), (91), and evaluating them, leads to (97).

The reader may wonder why the specific coupling (98) appears rather than some arbitrary linear combination of the electric and heat currents. It is plausible that this is an artifact of the homogeneous holographic Q-lattice/massive gravity models, where the heat current plays a distinguished role in relaxation processes, [78, 244]. Whether this remains true in homogeneous helical models [72, 212] or in inhomogeneous models [213–215] is an open question. An obvious avenue for future research would be to work out what the analytical 'horizon' formulae for the diffusive transport coefficients are, in analogy to (94)-(96) for the homogeneous model of [84]. Another interesting question is how the temperature dependence of these transport coefficients changes when there is no background strain – this may already be seen for σ_o in the models of [74, 229] and it would be interesting to determine whether this is an artifact of these models.[33]

E. Emergent higher-form symmetries and topological defects

Ordinary, 0-form symmetries (such as a global U(1)) give rise to conserved, one-form currents (e.g. $\nabla_\mu J^\mu = 0$ in relativistic notation). The associated conserved charges are point-like objects. [246] pointed out the existence of more general symmetries associated to

[33] Probe brane models can also display spontaneous breaking of translations, [220, 221]. Importantly, because of the presence of an additional long-lived mode [159–161, 245] (see the discussion in section IV A), the hydrodynamics presented in this section does not apply directly to these holographic models.

differential forms of a higher rank. A prototypical example is the U(1) of electromagnetism in four spacetime dimensions. There, the Bianchi identity can be reformulated as the conservation equation of a magnetic U(1) symmetry by Hodge dualizing the Maxwell field strength $J^{\mu\nu} = 1/2\,\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The charge $Q = \int_\Sigma \star J$ counts the number of magnetic lines across a codimension-2 surface Σ, and its associated conserved current is now a two-form $\nabla_\mu J^{\mu\nu} = 0$. Among various applications, this provides a starting point for a consistent formulation of magnetohydrodynamics [247].

A similar treatment can be applied to phases with a spontaneously broken global U(1) symmetry (superfluids). Keeping to relativistic notation, the absence of topological defects (vortices) implies that derivatives commute, $\nabla_{[\mu}\nabla_{\nu]}\phi = 0$, where ϕ here is the superfluid phase. Defining $J^{\mu\nu} = \epsilon^{\mu\nu\rho}\nabla_\rho\phi$ leads to an emergent conservation equation for the higher-form symmetry $\mathrm{U}(1)_w$ associated to the conservation of winding planes, $\nabla_\mu J^{\mu\nu} = 0$, [157]. Indeed, the condensate vanishes at the location of a defect and its phase is not defined there. The phase picks up a nonzero winding number around a defect, $n = 1/(2\pi)\oint d\phi \neq 0$.

For superfluids, [157] showed that this emergent symmetry is in fact anomalous: upon coupling to a background gauge field, its conservation equation is broken by a mixed anomaly given by the electric field. This is a mixed anomaly as it is the source of the microscopic U(1) which breaks the conservation of the emergent $\mathrm{U}(1)_w$.

Emergent symmetries are often anomalous and their higher-form generalizations are no exception, [246, 248]. Such anomalies give rise to anomaly matching conditions, which put strong constraints on the hydrodynamic gradient expansion, [157], and give rise to dissipationless transport, [249]. It would be interesting to derive analogous constraints on the hydrodynamics of broken translation phases. Preliminary investigations of the higher-form symmetry formulation[34] of these phases can be found in [79, 93].

In the condensed phase, the winding operators $W_{ij} = \int d^2x \nabla_i u_j$ are conserved and measure elastic deformations of the crystal/density wave. They lead to undamped propagation of uniform bulk and shear strains, e.g.[35]

$$\eta(\omega) \equiv \frac{i}{\omega} G^R_{\tau^{xy}\tau^{xy}}(\omega, q = 0) = \eta + G\,\frac{i}{\omega}\,. \tag{100}$$

This infinite dc 'shear conductivity' is the analogue of dissipationless charge transport in superfluids.

[34] This differs from the dual formulations of [46, 250].

[35] Notice the difference with the Kubo formula in Eq.(89) in which the divergent $1/\omega$ term would not appear.

At finite temperatures, bound pairs of defects/anti-defects (dislocations or disclinations) nucleate. Above the Berezinsky-Kosterlitz-Thouless temperature, thermal fluctuations lead to their unbinding and they become mobile – the BKT phase transition, [251–254].[36] Mobile defects relax the windings, and the corresponding emergent symmetry is explicitly broken. This leads to relaxation of the longitudinal and transverse phonons,

$$\dot{\lambda}_{\parallel,\perp} = -\Omega_{\parallel,\perp}\lambda_{\parallel,\perp} + \dots \tag{101}$$

This equation is valid when the anisotropic rates $\Omega_\parallel$ and $\Omega_\perp$ are small, close to the BKT phase transition. The phase relaxation rates are set by the viscosities of the normal phase and the density of free defects n_f, e.g. $\Omega_\perp \sim n_f/\eta_{normal}$, [254].[37] 'Climb' motion of dislocations is usually suppressed compared to 'glide', i.e. $\Omega_\parallel \ll \Omega_\perp$. In the language of higher-form symmetries, the emergent higher-form symmetry counting winding planes is broken by irrelevant operators (the defects), [157].

Evaluating (100) again, the viscosities of the condensed phase are finite but large, $\eta(\omega = 0) = \eta + G/\Omega_\perp$.

VI. PSEUDO-SPONTANEOUS BREAKING OF TRANSLATIONS

The total momentum of the system is always conserved, due to the translation invariance of the ambient spacetime in which the crystal lives. Thus, the emergent continuous translation symmetry at long distances in crystalline solids cannot be explicitly broken. In systems at finite density such as metals, the conduction electrons (or more generally the charge carriers at strong coupling) can be considered in some regimes (typically, low enough temperatures) to be weakly-coupled to lattice degrees of freedom and other sources of inelastic scattering. The electron momentum then becomes approximately conserved, with an emergent electronic translation symmetry in the infrared broken by irrelevant operators (such umklapp, impurities). In an electronic charge density wave or Wigner crystal phase, electronic translations are spontaneously broken and give rise to a spatially modulated electronic density of states (see [68] for a review). New Goldstone degrees of freedom emerge, called phasons or sometimes phonons by abuse of terminology (not to be confused with the phonons of the underlying lattice).

[36] See [43–46, 250] for the quantum case.

[37] See [64] for a memory matrix calculation of these rates.

It then becomes interesting to study how the weak explicit breaking mentioned above affects the dynamics of the Goldstones. These acquire both a small mass q_o[38] and a damping Ω, leading to a nonzero real and imaginary part in their $q = 0$ dispersion relation, respectively. Phenomenologically, the spontaneous, spatially modulated phase is no longer free to slide and is pinned at a frequency $\omega_o \sim c_s q_o$ proportional to the mass of the Goldstone – which now has a finite correlation length. Correspondingly, there is a gap in the real part of the the frequency-dependent conductivity with a peak at a frequency $\omega \sim \omega_o$, representing the energy cost to de-pin the density wave.[39]

Pinning of charge density waves is an old subject, [256, 257], and was confirmed in many experiments on quasi one-dimensional materials, [68]. It was revived in recent years, spurred on by a combination of mounting experimental evidence on the role of charge density wave phases or fluctuations across the phase diagram of cuprate high T_c superconductors, [51–56, 58–60] (see [57] for a review); theoretical developments on the application of hydrodynamics and related effective field theoretic descriptions of transport to strongly-correlated electronic materials, [15, 34, 258, 259]; and the development of holographic methods for phases with broken translations.

Following the initial work of [64], which incorporated pinning by explicit breaking of translations and damping by defects into hydrodynamics, a number of groups set out to investigate these phases using holographic methods. The original expectation was that these systems would display a pinning frequency ω_o and a momentum relaxation rate Γ, but no phase relaxation rate Ω, as none of these holographic models included mobile elastic defects.[40] It then initially came as a surprise when it was recognized that they exhibited a finite phase relaxation rate governed by the pseudo-Goldstone mass and diffusivity $\Omega = G q_o^2 \xi$, [65, 262] with further confirmations in [77, 78, 230, 263–265].

The main theoretical achievement of this collective effort is the construction of a hydrodynamic theory of pseudo-spontaneously broken translations, [67, 69], which explains the observations above and which we now describe. For simplicity we will consider states without background strain throughout this section, but this can be incorporated straightforwardly, [69].

[38] By a similar mechanism that leads to the Gell-Mann Oakes Renner (GMOR) relation [255] for pion masses in QCD.

[39] If disorder or lattice effects are strong, the density wave is strongly-pinned and locked at impurity sites.

[40] Though see [260, 261] for a holographic construction of phases with static discommensurations.

A. Hydrodynamics

When translations are weakly broken explicitly, the free energy at quadratic order in fluctuations now includes a mass term for the Goldstone modes[41]

$$\delta f^{(2)} = \frac{B+G}{2}\left(\nabla^i \delta\phi_i\right)^2 + \frac{G}{2}\left(\nabla \times \delta\phi\right)^2 + \frac{G\, q_o^2}{2}\,\delta\phi_i \delta\phi^i \tag{102}$$

which shifts the unpinned static susceptibility matrices $\chi_{o,\parallel}$ and $\chi_{o,\perp}$ as

$$\chi_o^{-1} \mapsto \chi^{-1} = \chi_o^{-1} + \Delta\chi^{-1}\,, \tag{103}$$

where $\Delta\chi^{-1}$ is a matrix whose only nonzero elements are $(\Delta\chi^{-1})_{\lambda_\parallel\lambda_\parallel} = (\Delta\chi^{-1})_{\lambda_\perp\lambda_\perp} = G\, q_o^2/q^2$. As a result, the static susceptibility matrix χ becomes nonlocal.

The charge and energy conservation equations in (73) remain unchanged. On the other hand, since translations are broken explicitly, momentum is no longer conserved

$$\dot\pi^i + \nabla_j \tau^{ji} = -\Gamma\pi^i - G q_o^2 \delta\phi^i\,. \tag{104}$$

The Γ term is allowed on general grounds and captures momentum relaxation, while the second term encodes the effects of the mass of the Goldstone, and can be derived by computing $\dot\pi^i = i[H,\pi^i]$ including a mass deformation (102) in the Hamiltonian H and using the commutator (75).

The constitutive relations and the Josephson equation can all contain terms linear in ϕ^i without any spatial gradient, since the shift symmetry is broken. These terms are constrained by locality and Onsager relations. After imposing these constraints, the constitutive relations and the Josephson equation read

$$\begin{aligned} j^i =& -\sigma_o \nabla^i\mu - \alpha_o \nabla^i T + \xi_\mu h^i\,,\\ \frac{\tilde{j}_q^i}{T} =& -\alpha_o \nabla^i\mu - \frac{\bar\kappa_o}{T}\nabla^i T + \xi_T h^i\,,\\ \tau^{ij} =& -\eta\,\sigma^{ij} - \zeta \nabla\cdot v\,\delta^{ij}\,,\\ \dot\phi^i =& v^i + \xi_\mu \nabla^i\mu + \xi_T \nabla^i T - \xi\, h^i \end{aligned} \tag{105}$$

where $h^i = \partial f/\partial\phi^i = G q_o^2 \delta\phi^i - \nabla_j h^{ji}$ and in the absence of background strain the transport coefficients are no longer matrices. These dissipative corrections ensure that the equations

[41] The mass term can be thought to originate from expanding a $\cos u_i$ deformation of the Hamiltonian of the system to quadratic order in fluctuations, so the u_i are still compact scalars.

of motion remain local [67] and that the divergence of the entropy current is positive, [69]. Translating the h^i terms to fields ϕ^i generates new relaxation terms in the constitutive relations and Josephson equations, proportional to q_o^2 and various dissipative transport coefficients; ξ_μ, ξ_T and ξ. For instance, the Josephson equations take the form

$$\dot{\delta}\phi^i = -\Omega\,\delta\phi^i + O(\nabla^i) \tag{106}$$

where the damping term

$$\Omega = G q_o^2 \xi \tag{107}$$

is universally determined by the Goldstone mass and ξ. The parameter ξ is a diffusive transport coefficient of the translation invariant theory which enters in the attenuation of sound and diffusive modes of section V and encodes dissipation of the Goldstone mode in the thermal bath over long distances.

In the framework of effective theories, (107) is a direct consequence of locality [67] or the second law of thermodynamics [69] with external sources on.[42]

In the presence of explicit breaking, Ω captures the relaxation of the pseudo-Goldstone mode in the surrounding bath of thermal excitations. In [265], it was shown that in the absence of a gap the time-dependent Ginzburg-Landau equation gives a good account of the dynamics of these systems near T_c. For one-dimensional systems with quasi-perfect nesting of the modulation wavevector and gapping of the Fermi surface, the charge density wave formation is described by the Peierls instability, [68]. The gap equation is BCS-like and the density of uncondensed electrons is exponentially suppressed at low temperatures. In this case, there are very few thermal excitations that the pseudo-Goldstone can relax into and we expect the damping Ω to be suppressed, which explains why it has not been discussed in previous literature, [68]. In other words, in the absence of a thermal bath, the Goldstone mode is gapped and cannot 'leak' to arbitrarily low energies.

Pinning also introduces new relaxation parameters in the constitutive relations for the currents

$$j^i = n v^i + \Omega_n \delta\phi^i + O(\nabla)\,, \quad \frac{j_q^i}{T} = s v^i + \Omega_s \delta\phi^i + O(\nabla) \tag{108}$$

with

$$\Omega_n = G q_o^2 \xi_\mu\,, \quad \Omega_s = G q_o^2 \xi_T\,. \tag{109}$$

[42] Analogous relations apply for other symmetry-broken phases, such as superfluids [67, 69, 266], QCD in the chiral limit [267, 268], nematic phases, (anti-)ferromagnets [67].

With translations broken explicitly weakly, the quasi-normal modes of the system have both an imaginary and a real gap

$$\omega_{\pm} = \pm\sqrt{\frac{G}{\chi_{\pi\pi}}}q_o - \frac{i}{2}\left(\Gamma + Gq_o^2\xi\right) + O\left(q^2, g^3\right) . \tag{110}$$

In the equation above, we have assumed the scaling $q_o \sim g$, $\Gamma \sim g^2$, where g is the source of the microscopic operator breaking translations explicitly. This assumption can be lifted, and then the dispersion relation takes a more complicated form. The expression (110) makes manifest the damped oscillator behavior of the system, with a pinning frequency $\omega_o \equiv q_o\sqrt{G/\chi_{\pi\pi}}$, and two contributions to the damping rate: $Gq_o^2\xi$ takes a universal form in terms of parameters of the effective field theory, while Γ does not. The only gapless modes left are two diffusive modes transporting charge and thermal fluctuations. Their expressions, as well as the leading q-dependence of the gapped modes can easily be computed with (105) in hand, but their expressions are not particularly illuminating and we leave it to the interested reader to write them down.

In [69], extra coefficients have been reported when coupling to external sources. Since these terms originate from extra freedom in how currents are coupled to external sources when symmetries are explicitly broken, they only appear in the numerator of retarded Green's functions and do not affect the poles. In particular, they do not affect the relations (107) and (109). It is also not known at the time of writing this manuscript how they affect the electric conductivity, which will be our primary focus in the next section. For simplicity, we will then omit these terms and refer to [69] for details. This is justified to some extent by the fact that these terms are either absent from or can be redefined away in the holographic models with pseudo-spontaneous breaking investigated so far, [65, 75, 77, 78, 262, 266] (see discussion below (113)).

B. Charge transport in pinned crystals

In a translation-invariant system at nonzero density, the electric conductivity is infinite in the dc limit $\sigma_{dc} \equiv \sigma(\omega = 0)$. This is because at nonzero density the electric current, which is a fast mode, overlaps with the (electronic) momentum density, which is conserved. This is manifested in a nonzero cross-susceptibility χ_{JP} between the charge and momentum operators. Hence the electric current cannot relax, which manifests itself as a divergence of

the zero frequency conductivity. This can be proven rigorously on general grounds using the memory matrix formalism, see e.g. [18]. This continues to be true when translations are spontaneously broken (e.g. for an electronic charge density wave in a clean system). Using the hydrodynamics equations of the previous sections, the conductivity can be obtained from the Ward identity for charge conservation:[43]

$$\sigma(\omega) \equiv \frac{i}{\omega} G^R_{jj}(\omega, q=0) = \frac{i}{\omega} \lim_{q\to 0} \frac{\omega^2}{q^2} G^R_{nn}(\omega, q) \tag{111}$$

and is found to be

$$\sigma(\omega) = \sigma_o + \frac{n_o^2}{\chi_{\pi\pi}} \frac{i}{\omega} \,. \tag{112}$$

The $\omega = 0$ pole in the imaginary part is physical and cannot be removed by contact terms. As announced, its residue is directly proportional to the off-diagonal susceptibility $\chi_{JP} = n_o$, which is identified as the charge density of the system. It gives rise to a delta function in the real part through Kramers-Krönig relations. There is also a finite contribution to the real part, captured by the transport coefficient σ_o. It is always nonzero except in a Galilean-invariant system, where it vanishes as a consequence of the Ward identity for Galilean boosts, $j^i = \pi^i$ (where for simplicity we set the electric charge and particle mass to unity in this formula). Intuitively, it is the contribution to electric transport of 'incoherent' processes (meaning which do not give rise to dissipationless current) [102]. It has no equivalent in a simple quasi-particle picture. It would also be present in a translation-invariant fluid without Galilean boosts, and there it transports fluctuations of entropy per unit charge $\delta(n/s)$ diffusively, [18] (when translations are spontaneously broken, the eigenmode is more complicated due to the coupling to the longitudinal Goldstone).

When translations are explicitly broken, the electronic momentum is no longer conserved. In the regime where it relaxes slowly enough to be kept in the effective field theory as a light mode, the conductivity is strongly modified. It is helpful to first consider the case without spontaneous breaking, [269]. The only relaxation parameter is the momentum relaxation rate Γ, and the electric conductivity becomes

$$\sigma(\omega) = \sigma_o + \frac{\tilde{n}^2}{\widetilde{\chi}_{\pi\pi}} \frac{1}{\Gamma - i\omega} + \mathcal{O}(\Gamma^0) \,. \tag{113}$$

The $\omega = 0$ pole is now located at $\omega = -i\Gamma$ and is identified with slowly-relaxing momentum. In real space, we expect $\langle \pi^i(t) \rangle \sim \pi_0^i e^{-\Gamma t}$. The real part of the conductivity shows a sharp

[43] While in hydrodynamics the continuity equation is a dynamical equation for the time evolution of vevs of operators, the Ward identity is a consequence of the U(1) symmetry and is more fundamental. It is an operator equation which can be used inside Green's functions.

peak centered at zero frequency (the Drude peak), of width Γ and weight $\tilde{n}^2/(\widetilde{\chi}_{\pi\pi}\Gamma)$. Here we have defined tilded quantities (eg $\tilde{n} = n_o + \Gamma n_1 + \ldots$) to signal that the Drude weight is expected to receive $\mathcal{O}(\Gamma)$ corrections when translations are explicitly broken. Such corrections were computed in [270] in the 'axion' holographic model of [271] and can be rewritten in the form (113) by redefining the Drude weight appropriately. A first comparison with the results of [69] specialized to the pure explicit case shows that these match to some of the new transport coefficients discussed in this work, [272].

In the weakly-relaxing regime, $\Gamma \ll \Lambda$ (Λ being the thermalization scale), the dc conductivity $\sigma_{dc} = \sigma_o + \tilde{n}^2/(\widetilde{\chi}_{\pi\pi}\Gamma) \simeq n_o^2/(\chi_{\pi\pi}\Gamma)$ is large and completely dominated by this 'Drude' contribution. The system is a hydrodynamic metal where the electronic momentum relaxes by inelastic scattering off impurities or by Umklapp processes.

By contrast, when translations are pseudo-spontaneously broken, the frequency-dependent conductivity becomes

$$\sigma(\omega) = \sigma_o + \frac{\frac{n_o^2}{\chi_{\pi\pi}}(i\omega - \Omega) + 2n_o\Omega_n + \frac{\Omega_n^2}{\omega_o^2}(\Gamma - i\omega)}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_o^2}\,. \tag{114}$$

Compared to the case without spontaneous breaking of translations, we observe new contributions to inelastic scattering, proportional to q_o^2 and contained in the ω_o, Ω and Ω_n terms. The lineshape interpolates between a Lorentzian centered at ω_o when Ω and Ω_n can be neglected (matching previous hydrodynamic treatments of the collective zero mode, [68]), and a Drude-like peak centered at $\omega = 0$ when the damping rates become more important, as is illustrated in figure 4.

The dc conductivity

$$\sigma_{dc} = \sigma_o + \frac{\frac{n_0^2}{\chi_{\pi\pi}}\Omega - 2n_o\Omega_n - \frac{\Omega_n^2}{\omega_o^2}\Gamma}{\Gamma\Omega + \omega_o^2} \tag{115}$$

is non-vanishing due to the nonzero symmetry-breaking terms Ω, Ω_n and to the 'non-Galilean' transport coefficient σ_o. Previous hydrodynamic treatments (see e.g. [68]) usually assume the Galilean limit, where the coefficients σ_o, ξ_μ and consequently Ω_n would be zero, but did not account for Ω in the dynamics of the collective mode. Here, in the Galilean limit (setting $n_o = ne$ and $\chi_{\pi\pi} = mn$, where e is the electron unit charge, n the density and m the mass), the resistivity $\rho \equiv 1/\sigma(\omega = 0)$ is

$$\rho_{Galilean} = \frac{m}{ne^2}\Gamma + \frac{1}{(ne)^2\xi}\,. \tag{116}$$

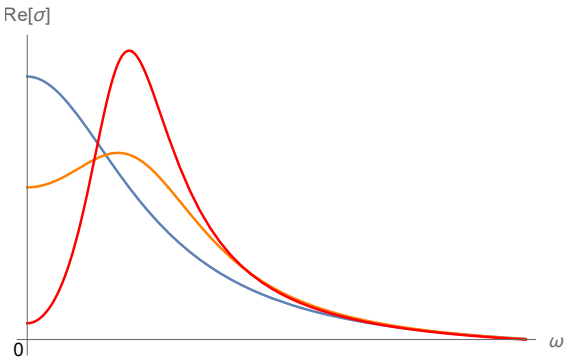

Figure 4. Schematic representation of the ac conductivity (114) in the Galilean limit when Ω is dialed from large (blue solid line) to small (red solid line), keeping all other parameters fixed. The transfer of spectral weight from the zero frequency Drude-like peak to the off-axis peak is evident.

In other words, we do not expect a translation-broken phase such as a charge density wave to be necessarily insulating: the inelastic scattering of the Goldstone into the bath of thermal excitations provides a conduction channel. In electronic charge density wave materials, such as those reviewed in [68], the Fermi surface may only be partially gapped in the charge density wave phase. The second term in (116) captures the inelastic scattering of the Goldstone mode into these uncondensed electrons. Instead, if the Fermi surface is fully gapped, $\Omega = 0$ and the collective mode does not contribute to dc transport, as in [68]. Said otherwise, in the absence of gapless thermal excitations, there is a finite energy cost to make the density wave slide.

The resistivity (116) takes a Drude-like form with a 'transport scattering rate'

$$\frac{1}{\tau_{tr}} = \Gamma + \frac{1}{(mn)\xi} \,. \tag{117}$$

In general, this picture is misleading, as there is no single pole located at $\omega = -i/\tau_{tr}$ as in the Drude model (113). Rather, both poles (110) give important contributions to the lineshape. When the Goldstone damping rate is large compared to Γ and ω_o, the poles are located at

$$\omega_- = -i\left(\Omega - \frac{\omega_o^2}{\Omega} + \dots\right) , \quad \omega_+ = -i\left(\Gamma + \frac{\omega_o^2}{\Omega} + \dots\right) , \tag{118}$$

where the dots denote subleading $1/\Omega$ corrections. The ω_- pole recedes deep in the lower-half plane and drops out of the effective theory, while the ω_+ one remains long-lived. Accordingly, the ac conductivity becomes Drude-like, as in the blue solid line of figure 4. In [65], this process takes place at very low temperatures.

Above T_c, we also expect to recover a Drude-like conductivity as in (113), dominated by a single pole $\omega \simeq -i\Gamma$. In [265], it was shown that this occurs through a vanishing of the residue of the ω_- pole, while Ω remains finite through the phase transition. Above T_c, there is no condensate but Ω captures the fluctuations of the order parameter.

Moving away once again from the Galilean limit, the dc conductivity (115) no longer depends on the explicit symmetry-breaking parameter q_o after inserting the relations (107) and (109). At low temperatures, inelastic scattering off impurities is expected to dominate the momentum relaxation rate, and does not contribute any temperature dependence $\Gamma \sim T^0$. Then, the primary temperature dependence of the resistivity originates from incoherent scattering processes encapsulated in the diffusive transport coefficients σ_o, ξ_μ and ξ, in sharp contrast to metallic phases. In a metal, extrinsic processes dominate the resistivity through the scattering rate, $\rho_{dc} \sim \Gamma$, while when translations are pseudo-spontaneously broken, intrinsic ones do.

C. Holography

The explicit breaking of translations in holographic models has been the subject of sustained attention over the past fifteen years. Early papers mostly focused on incorporating the effects of weak breaking of translations in hydrodynamics and computing the rate of momentum relaxation using memory matrix methods, [37, 269, 273–275]. Non-perturbative results became available a little later thanks to two types of developments.

Holographic lattices breaking translations explicitly were constructed numerically in [276, 277],[44] using advanced numerical methods to solve Einstein's equations allowing for inhomogeneous spatial dependence (see [281–283] for reviews of these methods). Spacetimes with a randomly disordered source at the boundary have been constructed analytically in a perturbative expansion in the strength of (marginal) disorder in [284–286], as well as non-perturbatively using numerical methods, and revealed an emergent Lifshitz scaling geometry at low temperatures. This behavior may only be transient at intermediate energy scales and not the true ground state, [287, 288].

Homogeneous models provide another particularly appealing avenue. In these holographic setups, translations are broken but the background metric and the dual stress tensor remain

[44] See [240, 278–280] for further numerical constructions of holographic lattices.

homogeneous (independent of the spatial coordinates). This property is due to existence of specific global structures which mix with spacetime symmetries, leading to a tremendous simplification in the computations of physical observables. The homogeneous models fall into different classes: (I) de Rham-Gabadadze-Tolley (dRGT) massive gravity theory [289],, (II) 'axion' models [134, 135, 271, 290], (III) Q-lattices [291], (IV) higher-forms models [162][45], (V) helical lattices [203, 243, 293, 294]. Irrespective of the specific holographic model employed, in the regime of weak explicit breaking, the low energy dynamics match well with the field theory expectations for a metallic phase with slowly-relaxing momentum, [92, 258, 269, 273, 275, 295].

Pseudo-spontaneous breaking of translation symmetry has been implemented in several holographic models in the past years [65, 70–72, 75, 77, 78, 230, 260, 263–265, 296–298]. Independently of the concrete model at hand, this limit is always achieved by introducing on top of the purely spontaneous state a small space-dependent source for a boundary operator which is therefore responsible for the explicit translation symmetry breaking.[46]

This body of work firmly established the validity of the hydrodynamic theory of pseudo-phonons presented above [47] and more specifically of the relation (107). Recent works [67, 69] further confirmed that this relation is not an artifact either of the homogeneity of the holographic models used or of the large N limit inherent to the holographic approach. In the left panel of figure 5, we show the excellent match between the ac conductivity computed in the holographic model of [65][48] and the hydrodynamic prediction (114). In the right panel, we show the match between the quasinormal modes computed numerically and the hydrodynamic prediction for the poles including the relation (107).

Holographic models can easily account for phases which are either insulating, where the resistivity diverges towards low temperatures, [72, 260], or metallic, [65], with a vanishing resistivity at low temperatures. The former case is in some respects more similar to conventional charge density wave systems, in the sense that a gap forms and the damping rate Ω does not make a large contribution to the dc conductivity, as evidenced from the negligible value of the dc conductivity compared to the height of the off-axis peak in the ac conductivity, [263]. An important difference is that the gap is algebraic, and the resistivity diverges

[45] Static black hole solutions including matter in the form of free scalar and p-form fields first appeared in the gravity communityt in [292].

[46] In some of the examples, this boundary operator is the same which breaks translations spontaneously.

[47] For the reasons mentioned in footnote 33, the hydrodynamics of probe brane setups are of a different nature.

[48] This is the same model as in (93), but now the scalar ϕ has a source ϕ_s such that the vev of the operator dual to ϕ is much larger than its source, corresponding to the pseudo-spontaneous regime.

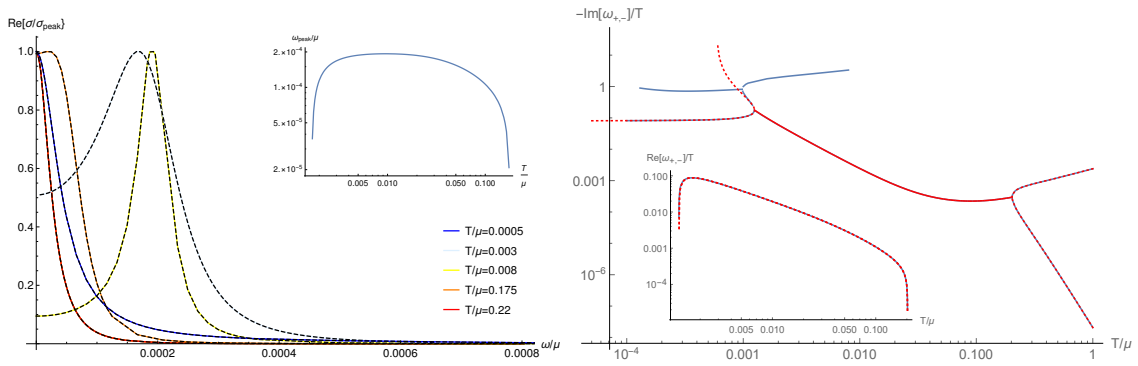

Figure 5. Left: comparison between the holographic (solid lines) and hydrodynamic (dashed lines) ac conductivity at various temperatures. Right: comparison between the holographic quasinormal modes (solid lines) and the hydrodynamic prediction (dashed lines) together with (107).

like a power-law. In the helical, homogeneous setup of [263], this scaling is rooted in the critical behavior of the infra-red geometry, in the near-horizon, near-extremal limit. Indeed, as is well-known in holographic models, such critical geometries leave a strong imprint on the scaling of transport observables at low temperatures, [135, 243]. It is then surprising that the resistivity continues to scale in the inhomogeneous construction of [260], even though there is no evidence so far of scaling behavior in the geometry. A better understanding of this result remains an open question.

In the metallic case, an inverse transfer of spectral weight is observed [65] as the off-axis peak in the ac conductivity smoothly interpolates back to a Drude-like peak at zero frequency upon lowering the temperature, as depicted in figure 5. This is accompanied by a non-trivial motion of the poles in the lower half complex frequency plane. At low enough temperature, the poles are once again purely imaginary, and the width of the Drude-like peak is controlled by the pole closest to the real axis. Its partner quickly recedes down the axis and becomes incoherent. Whether this behavior can be reproduced in a more realistic, inhomogeneous state is not known. Nonetheless, it bears intriguing resemblance to what is observed experimentally in cuprate high T_c superconductors and many other strongly-correlated materials, as we will describe in section VIII.

Given that $\Omega = Gq_o^2\xi$ and $\Omega_n = Gq_o^2\xi_\mu$, the same effective interaction we described in section V D around (98) operates near homogeneous holographic quantum critical phases with pseudo-spontaneously broken translations (keeping in mind further checks are needed

in more complicated holographic models).[49] This further implies that the low-temperature resistivity is controlled by a single, diffusive transport coefficient σ_o of the clean state, with subleading contributions from explicit symmetry breaking (assuming disorder and umklapp processes to be irrelevant and/or contribute no significant temperature dependence to the momentum relaxation rate), $\rho_{dc} \simeq (sT/\mu n_o)^2/\sigma_o + \mu\Gamma/n_o$. As the transport coefficient σ_o can be computed in terms of data at the black hole horizon, it is sensitive to the scaling properties of the low temperature critical phase, and hence so is the resistivity. This does not suffice to explain the results of [260] but resonates with the scaling form of the low temperature resistivity uncovered there.

When the spontaneous spatially modulated structure is coupled to an explicit lattice, one expects their periodicities to become commensurate for sufficiently large lattice strength. This phenomenon is beyond homogeneous constructions, [299]. Instead, more realistic inhomogeneous constructions display commensurability effects [300]. The black hole horizon is strongly spatially modulated by the spontaneous structure, which is weak in the ultraviolet near the boundary (since it is not sourced) but important in the infrared. The explicit lattice is strong in the ultraviolet but irrelevant (weak) in the infrared. The commensurability that develops between these two structures is a reflection of a strong UV-IR mixing upon increasing the UV lattice strength and turns the system into a Mott insulator, [260], albeit with an algebraic rather than exponential gap and reminiscent of underdoped cuprates.

VII. PINNED WIGNER CRYSTALS

The interplay between magnetic fields and broken translations gives rise to a rich variety of physical phenomena, of great interest both experimentally and theoretically. In sufficiently large magnetic fields, the ground state of two-dimensional electrons systems such those arising in GaAs/GaAlAs heterostructures is expected to be a Wigner solid [302, 303]. Re-entrant insulating phases between quantum Hall plateaux [304, 305] are also naturally interpreted as Wigner solids, as evidenced by threshold behavior in their nonlinear conductivity [305–307].

In the presence of an external magnetic field, phases with spontaneously broken trans-

[49] Recently, [**?**] examined this issue in Bianchi VII and inhomogeneous models. However their conclusions rely on fitting ac thermoelectric conductivities on the functional form predicted by hydrodynamics rather than determining all coefficients appearing in hydrodynamics and comparing this to the exact numerics. Moreover in these models the consequences for thermoelectric of the breaking PT symmetry have not been determined. Further work seems needed to draw more definitive conclusions.

lations feature a complex structure of low-energy excitations. As the magnetic field breaks time reversal and parity symmetry, the longitudinal and transverse phonons are no longer independent degrees of freedom, $[\varphi_i, \varphi_j] \neq 0$, and hybridize into a gapless magnetophonon and a gapped magnetoplasmon. The reduction of the number of independent Goldstones follows from general arguments, [309–311]. The magnetophonon is a type-II Goldstone boson with quadratic dispersion $\mathrm{Re}(\omega) \sim k^2$, [308]. In the following, we will assume that we are at sufficiently large magnetic field and focus on frequencies and wavevectors small enough that the magnetoplasmon gap, set by the cyclotron frequency, is large, and only the magnetophonon is retained in the effective field theory.

Early accounts of the dynamics of two-dimensional pinned charge density waves in presence of an external magnetic field are given in [312, 313]. Their hydrodynamics were revisited recently in [67, 83, 314]. In the presence of pinning and a magnetic field, new relations of the type (107) arise.

More precisely, in a magnetic field, the 'magnetic momenta' $\mathcal{P}_i = P_i + \int d^2x n A_i$ do not commute: $[P_i, P_j] = -i\epsilon_{ij} BN$, where N is the charge operator. As we now describe, this is a consequence of the following non-dissipative effective Lagrangian for the magnetophonon:

$$\mathcal{L} = \epsilon^{ij}\varphi_i\dot{\varphi}_j - \frac{1}{2}\varphi_i\left[\omega_{\mathrm{pk}}\delta^{ij} - \mu^{ijab}\nabla_a\nabla_b + \ldots\right]\varphi_j + s_i\varphi_i\,. \tag{119}$$

where the leading time derivative term is allowed as parity and time reversal symmetries are broken by the nonzero magnetic field, and where $\omega_c \equiv nB/\chi_{\pi\pi}$ is the cyclotron frequency. The pinning term $\omega_o \neq 0$ captures the leading effects of translation symmetry breaking. The stiffness tensor μ_{ijab} encodes the elastic response. By isotropy and PT symmetry, it takes the form

$$\mu_{ijab}k^a k^b = \left[\kappa k_i k_j + \mu k^2 \delta_{ij}\right]\,, \tag{120}$$

where the stiffnesses must satisfy κ, $\kappa + \mu > 0$ for the matrix of static susceptibilities to be positive definite.[50]

Let us concentrate on the first term, which is the leading term in the absence of pinning $\omega_o = 0$. Upon quantization, this gives the canonical commutator $[\varphi^i(x), \varphi^j(y)] = -i\epsilon_{ij}\delta(x-y)$. The φ^i transform under translations $x^i \mapsto x^i - c^i$ the usual way $\varphi^i \mapsto \varphi^i + c^i$, which leads through Noether argument to the conserved densities $\pi^i \sim \epsilon^{ij}\varphi_j$. Then $\mathcal{P}^i = \int d^2x \pi^i = \sqrt{nB}\int d^2x \epsilon^{ij}\varphi_j$ reproduces the magnetic algebra above.

[50] We are changing our notation for the elastic moduli in order to avoid confusion with the magnetic field B.

Defining the 'source'

$$h_i \equiv \frac{\partial f}{\partial \varphi_i} = \left[\delta_{ij}\omega_{\rm pk} - \mu^{ijab}\nabla_a\nabla_b + \ldots\right]\varphi_j\,, \tag{121}$$

the equation of motion from (119) provides the non-dissipative 'Josephson relation' for the magnetophonon

$$\dot{\varphi}_i = \epsilon_{ij}\left(s_j - h_j\right)\,, \tag{122}$$

so that in the static limit $s_j = h_j$. This can be solved to find the dispersion relation

$$\omega(k) = \pm\sqrt{\left(\omega_{\rm pk} + k^2\mu\right)\left(\omega_{\rm pk} + k^2(\mu+\kappa)\right)}\,. \tag{123}$$

In the absence of pinning we find a pair of gapless propagating modes with dispersion relation $\omega \sim \pm k^2$ characteristic of magnetophonons and type II Goldstones. In this limit, (123) gives the pinned magnetophonon gap $\omega = \pm\omega_{pk}$, which remains small for small disorder strength and large magnetic field.

We are now ready to formulate the general dissipative hydrodynamic theory of magnetophonons coupled to charge fluctuations. We will ignore heat fluctuations for simplicity, these can be incorporated by a straightforward extension of what follows. The slow hydrodynamic modes are therefore just the charge density n, which satisfies a continuity equation $\dot{n} + \nabla\cdot j = 0$, and the magnetophonon φ_i. The general Josephson relation consistent with isotropy is then

$$\dot{\varphi}_i = \epsilon_{ij}\left(s_j - h_j\right) - (\Omega\delta^{ij} - \widetilde{\mu}^{ijab}\nabla_a\nabla_b)\varphi_j + (\gamma)_{ij}\left(E_j - \nabla_j\mu_e\right) + D_\varphi s_i + \ldots\,, \tag{124}$$

where $\widetilde{\mu}^{ijab}$ contains dissipative transport coefficients and has the same structure as the stiffness tensor in (120), and $(\gamma)_{ij} \equiv \gamma\delta_{ij} + \sqrt{n/B}\epsilon_{ij}$. The last term is fixed by observing that it is equal to $\chi_{\varphi_y j_x}$ and using $\mathcal{P}_i = \sqrt{nB}\epsilon_{ij}\varphi^j$. A hydrodynamic coefficient is dissipative if it needs to change sign for the above expression to be invariant under PT. These terms are (Ω, $\widetilde{\mu}$, γ, D_φ), which contribute to entropy production.

The current constitutive relation takes the form

$$j_i = (\sigma_o)_{ij}\left(E_j - \nabla_j\mu_e\right) + \gamma_{ij}\left(s_j - h_j\right)\,, \tag{125}$$

where the second term is fixed by Onsager relations (PT symmetry).

Locality of the equations of motion with sources turned on fixes

$$\Omega = D_\varphi\omega_{\rm pk}\,, \quad \widetilde{\mu}^{ijab} = D_\varphi\mu^{ijab} \tag{126}$$

These are analogous relations to those found at zero magnetic field. The Josephson equation now takes the simple form:

$$\dot{\varphi}_i = \epsilon_{ij}\,(s_j - h_j) + (\gamma)_{ij}\,(E_j - \nabla_j \mu_e) + D_\varphi (s_i - h_i) + \dots\,. \tag{127}$$

The frequency-dependent electric conductivities are

$$\sigma_{xx}(\omega) = \sigma_o + \sqrt{\frac{n}{B}}\,\omega_{\rm pk}\frac{(1-a^2)(-i\omega+\Omega) - 2a\omega_{\rm pk}}{(-i\omega+\Omega)^2 + \omega_{\rm pk}^2}\,, \tag{128a}$$

$$\sigma_{xy}(\omega) = \sigma_o^H + \sqrt{\frac{n}{B}}\,\omega_{\rm pk}\frac{-2a(-i\omega+\Omega) + (a^2-1)\omega_{\rm pk}}{(-i\omega+\Omega)^2 + \omega_{\rm pk}^2}\,. \tag{128b}$$

The physical meaning of the various terms is apparent: σ_o and σ_o^H describe current dissipation into modes other than the magnetophonon, Ω is the phase relaxation rate, setting the width of the peak, $\omega_{\rm pk}$ is the pinning frequency, $a \equiv \gamma\sqrt{B/n}$ determines the deviation from a strict Lorentzian form.

In [83], we considered the match between the hydrodynamic ac conductivity and experimental measurement in GaAs heterojunctions, [301, 317, 318], in which Wigner crystallization occurs at large enough magnetic fields in between Quantum Hall plateaux. The ac conductivity data can be fitted to (128), and gives a very good match, see figure 6. This implies that the hydrodynamic theory we have just written down gives quantitatively correct account of the dynamics in the Wigner solid phase and of the approach to its melting point.

In the presence of gapless charged degrees of freedom, whose contribution to the current is given by σ_o, we expect a new relaxation channel for the magnetophonon to open. An especially universal coupling is to the current operator:

$$H_{\rm dis} = \sqrt{B/n}\int d^2x\,\epsilon_{ij}\varphi_i(x) j_j(x)\,. \tag{129}$$

The factor of ϵ is necessary for the coupling to respect PT symmetry. This interaction leads to $\dot{\varphi}_i = j_i\sqrt{B/n}$, which is exactly the relationship obtained by using (127) together with (125) to solve for $\dot{\phi}$ in terms of j in the absence of sources and dissipation.

Recalling that $\dot{\varphi}_i^{k=0} = i[H_{\rm dis}, \varphi_i^{k=0}]$, together with the Kubo formulae (in the clean system without disorder):

$$D_\varphi = \lim_{\omega\to 0}\frac{1}{\omega}\Im G^R_{\dot{\varphi}_x\dot{\varphi}_x}(\omega)\,, \tag{130a}$$

$$\gamma = \lim_{\omega\to 0}\frac{1}{\omega}\Im G^R_{j_x\dot{\varphi}_x}(\omega)\,, \tag{130b}$$

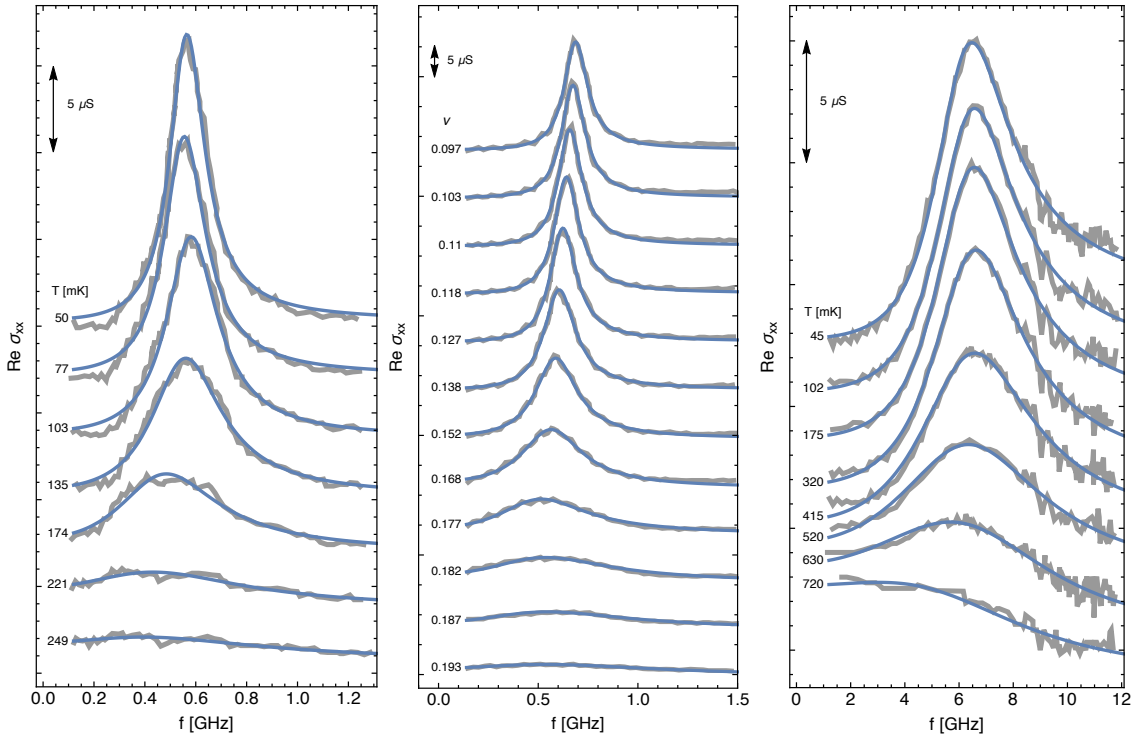

Figure 6. **Fits of the magnetophonon resonance.** Blue curves are fits to (128). Gray is data. Curves are offset relative to each other, absolute offset has not been fit and is not shown (see main text). Left: Temperature dependence of the resonance in a sample with $\nu = 0.128$ and $B = 18\,\mathrm{T}$, data from [315]. Centre: Filling fraction dependence of the resonance in a sample ('sample P') with $n = 7.7 \times 10^{10}\,\mathrm{cm}^{-2}$ and $T = 80\,\mathrm{mK}$, data from [301]. Right: Temperature dependence of the resonance in a sample with $\nu = 0.16$ and $B = 10.3\,\mathrm{T}$, data from [316].

the coupling (129) implies that

$$D_\varphi = \frac{B}{n}\sigma_o\,, \quad \gamma_{\mathrm{dis}} = \frac{\sigma_o}{\sqrt{n/B}}\,. \tag{131}$$

In turn the contribution to the damping rate Ω from pinning (126) becomes

$$\Omega_{\mathrm{dis}} = \frac{B\omega_{\mathrm{pk}}\sigma_o}{n}\,. \tag{132}$$

This relaxation mechanism into a hydrodynamic current is reminiscent of the analogous mechanism in holographic systems discussed in sections V D and VI C.

Topological defects are expected to give an independent contribution to Ω. These rates can be computed using the memory matrix formalism along the lines of [64, 158, 319], giving

the Bardeen-Stephen-like expressions

$$\Omega_{\rm vor} = \frac{2x}{\sigma_{\rm n}} \frac{n}{B} \omega_{\rm pk} \,, \qquad \gamma_{\rm vor} = x \sqrt{\frac{n}{B}} \frac{\sigma_{\rm n}^H}{\sigma_{\rm n}} \,. \tag{133}$$

Here x is the fraction of the total area covered by mobile vortex cores, while $\sigma_{\rm n}$ and $\sigma_{\rm n}^H$ are the longitudinal and Hall conductivities of the normal state in the core. At low temperatures one can expect that $\sigma_{\rm n}^H = n/B$, leading to the relation $\Omega_{\rm vor}/\omega_{\rm pk} = 2\gamma_{\rm vor}/\sqrt{n/B} = 2a_{\rm vor}$. More generally, even when the cores are not large, one still expects $\Omega \sim \gamma \sim x$.

The discussion of dissipation mechanisms above led to the expressions (132) and (133) for the dissipative parameters. These in turn lead to the relations

$$\frac{\Omega_{\rm dis}}{a_{\rm dis}\, \omega_{\rm pk}} = 1 \,, \qquad \frac{\Omega_{\rm vor}}{a_{\rm vor}\, \omega_{\rm pk}} = 2 \,. \tag{134}$$

Interestingly, these values seem to account well for fits to the experimental results at low temperatures or strong magnetic fields, see figure 7.

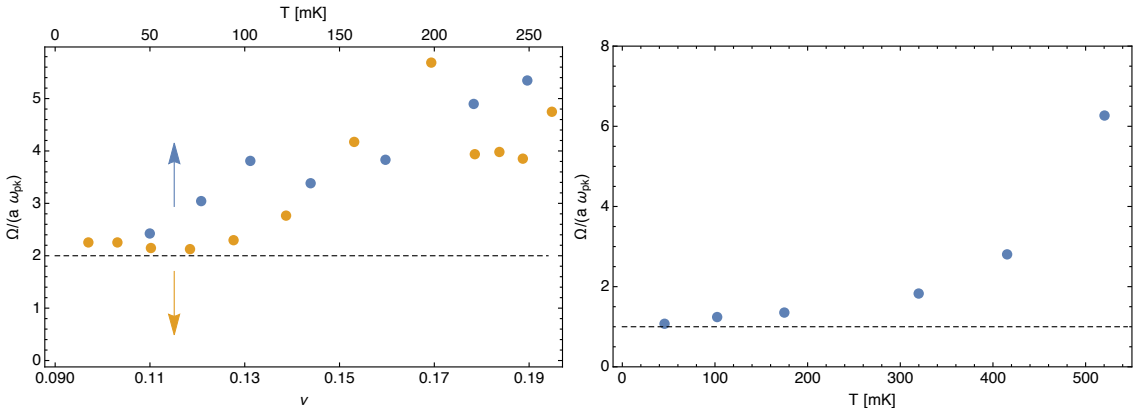

Figure 7. **Ratio of coefficients suggests different dissipative mechanisms** are at work in the less disordered and more disordered samples. Left: The low temperature/low filling behavior of the cleaner samples (leftmost and center plots in Fig. 6) is consistent with dissipation into mobile dislocations, shown as a dashed black line. Right: The low temperature behavior of the more disordered sample (rightmost plot in Fig. 6) is consistent with the universal phase dissipation into currents, shown as a dashed black line.

There are several avenues for future research. The results presented above are mostly based on [83]. The more recent works [67, 69]. [67] used locality of the equations of motion in the presence of external sources to constrain the dissipative coefficients of the theory with pinning. This led to the relation (126) for the damping rate Ω in the presence of pinning.

On the other hand, the contribution of vortices is simply modeled through an $\Omega\varphi$ term. It is not obvious how this term should be modified in order to consistently account for pinning and locality.

[69] pointed out new transport coefficients in the presence of pinning, arising from the ambiguity of coupling to external sources when the symmetries are only approximate. These terms should also be present with a nonzero magnetic field. Homogeneous holographic models with broken translations and a magnetic field have been studied in [231, 320, 321]. It remains to be seen if these coefficients can identified in these constructions.

VIII. TRANSPORT IN STRANGE METALS AND PSEUDO-SPONTANEOUS BREAKING OF TRANSLATIONS

Can the physics of pseudo-spontaneous breaking of translations shed light on the phenomenology of high T_c superconductors, in particular on their strange metallic phase? Transport experiments famously measure a resistivity linear in temperature [30] which extends for optimally doped samples from above room temperature to the lowest temperatures experimentally available when superconductivity is suppressed by an external magnetic field. This observation brings two important puzzles. The absence of resistivity saturation at high temperatures violates the Mott-Ioffe-Regel bound [322, 323] and precludes any notion of quasiparticle-based transport, calling for other descriptions of transport in systems with short-lived excitations [15]. Charge transport in conventional metals with long-lived quasiparticles is often analyzed with the Drude model. Applying this framework to the resistivity of strange metals identifies a 'Planckian' scattering rate, [32], which on theoretical grounds can be argued to be the shortest relaxation timescale consistent with Heisenberg's uncertainty principle, [7, 22] – see [23] for a recent review on Planckian dissipation in metals and bounds on transport.

At low temperatures, the persistence of a T-linear component in the resistivity over a range of doping [324, 325] clashes both with the Fermi liquid prediction of a T^2 resistivity, which is only fully recovered beyond the superconducting dome for very overdoped samples, and with conventional expectations of transport in the vicinity of a quantum critical point, [7], where quantum critical behavior is expected in a quantum critical cone at nonzero temperatures, but which should shrink to a point at zero temperature.

The slope of the T-linear resistivity appears to be of the same order of magnitude across different materials, [325], which hints at a universal mechanism underpinning this phenomenon. Extrapolations of the resistivity to zero temperature show that the disorder of the sample does not play an important role, with values of the residual resistivity varying sometimes over an order of magnitude or more across materials. Additional support for this comes from ion-irradiation transport experiments [326–329], which show that resistivity curves simply shift upwards without any change in the slope of the T-linear component when disorder is increased.

Transport experiments also report a T^2 scaling of the cotangent of the Hall angle, [330], and a magnetoresistance linear in the magnetic field over a range of dopings at large magnetic fields, [331–333], prompting suggestions that the scattering rate should also be linear in field at high fields. This is once again at odds with quasiparticle-based transport and the Boltzmann equation, which predicts that the resistivity and the Hall angle are controlled by the same transport timescale, and that the magnetoresistance is quadratic in field.[51] Instead, the different temperature dependencies of the resistivity and Hall angle are often interpreted in a two-timescale scenario, [334–336]. More generally, there is some experimental support for two sectors contributing to transport, one coherent and the other incoherent, [333, 337, 338].

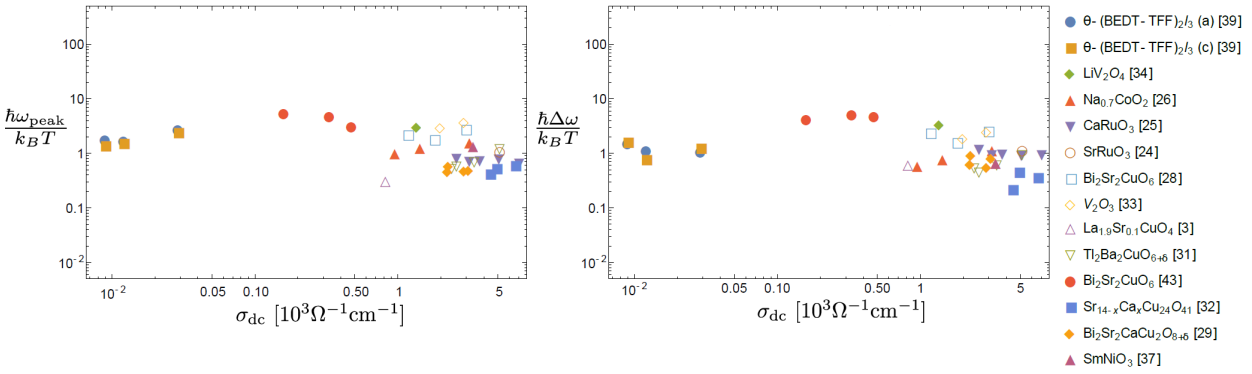

Figure 8. Location of the off-axis peak (left) and peak width (center) times the Planckian timescale $\tau_P = \hbar/(k_BT)$ at high temperatures in bad metals, demonstrating Planckian dissipation. References on the right can be found in [63].

[51] In overdoped Nd-LSCO and LSCO, the Fermi surface is strongly anisotropic due to the close proximity of a van Hove singularity. The resistivity data is well-fitted by an isotropic, linear in T inelastic scattering rate (whose origin remains unexplained) and an anisotropic scattering rate arising from the anisotropic Fermi surface, [**?**]. It has been argued that in these materials, Boltzmann transports predicts that the anisotropic scattering rate also gives rise to the linear in field behavior of the magnetoresistance, and so that the isotropic scattering rate does not depend on field, [**?**]. This is specific to the La-based cuprate family and likely does not extend to other cuprates where the Fermi surface does not have the same level of anisotropy, and also does not apply to the strange metallic regime, [**?**].

Turning to optics, the ac in-plane conductivity in the strange metal regime above the temperature at which superconductivity sets in is Drude-like, with a peak centered at zero frequency and a width of order T. At higher temperatures, a number of compounds reveal a transfer of spectral weight and the zero frequency peak moves off-axis to a nonzero frequency, [63, 323]. This happens as the resistivity crosses the Mott-Ioffe-Regel limit. Moreover, the peak location and width both scale approximately linearly with temperature, as shown on figure 8, suggesting another manifestation of Planckian dissipation.

The ac conductivity also features an infrared contribution [339–341] extending beyond the peak, which scales as $|\sigma(\omega)| \sim \omega^{-2/3}$, [342, 343]. This is weaker than the expected Drude scaling $|\sigma(\omega)| \sim \omega^{-1}$. This resonates with the two-component analysis of transport, however fits to optics data typically assume that the dc conductivity solely originates from the Drude component, ascribing a frequency dependence to the infrared component which vanishes as $\omega \to 0$. It would be interesting to investigate to what extent this constraint in fitting optics can be relaxed and cross-referenced to dc transport data.

These experimental facts pose an immediate conundrum when attempting to interpret them in the framework of a metal with slowly-relaxing momentum. The ac conductivity at not too high temperatures suggests a Drude analysis may work, but fails to account for the appearance of an off-axis peak at higher temperatures or for the infrared non-Drude contribution. The ac conductivity of a slowly-relaxing metal is given in (113). If momentum relaxes weakly, Γ must be small compared to some parameter determining the scale at which other degrees of freedom start to be important, usually temperature. But this theoretical assumption contradicts the experimental observation that $\Gamma \sim \mathcal{O}(T)$, raising the question of which scale plays the role of cut-off of the low-energy effective theory. Moreover, for slowly-relaxing momentum, the dc conductivity is dominated by the 'coherent' contribution from the Drude peak, $\sigma_{coh} \sim n_o^2/(\chi_{\pi\pi}\Gamma) + \mathcal{O}(\Gamma^0)$. However in this picture Γ strongly depends on disorder strength, [18], and so this is not supported by experimentally, since the slope of the resistivity does not change under irradiation by ions, [326–329].

While experimentally difficult to establish, the notion of coherent and incoherent charge transport in a slowly-relaxing metal is easy to understand from a theoretical standpoint. All that is required is to give up Galilean invariance, which imposes that the electric current is equal to the momentum density – thereby killing any incoherent contribution to transport. Doing so, new processes are allowed that conduct charge but do not drag momentum, and

neatly encapsulated in appearance of the transport coefficient σ_o in the dc conductivity (113). These processes naturally appear in hydrodynamics [102], memory matrix approaches [18, 258] and in holographic models [270].

Relaxing Galilean invariance is not enough though, as in a metal with slowly relaxing momentum such incoherent processes inevitably give contributions to transport (of order Γ^0) subleading compared to the coherent contribution (of order $1/\Gamma$). There are several avenues one can think of to suppress the coherent contribution to transport:

i) suppress the Drude weight through some emergent particle-hole symmetry that would effectively set $n_o = 0$;

ii) assume strong explicit breaking of translations;

iii) more radically, require that $\chi_{\pi\pi} \to +\infty$, [249];

iv) short-circuit the large contribution from slowly-relaxing momentum by assuming translations are spontaneously broken, [63].

Strange metals arise in doped Mott insulators, which leads to disregard i) (in contrast to the example of graphene near the charge neutrality point). The ability to synthesize very clean samples with a low residual resistivity [332] also works against ii). iii) was recently considered [249]. There, the authors argue that strange metals arise in the vicinity of an ordered phase where the order parameter has the same symmetries as loop currents [218] and that this would lead to the divergence of all susceptibilities in the same symmetry sector. It is interesting to note that holographic checkerboards, [216, 217, 225], naturally feature such current loops intertwined with translation symmetry breaking thanks to the bulk Chern-Simons terms.

Finally, let us consider (iv) how pseudo-spontaneous translation symmetry breaking may shed light on transport in strange metals. Further motivation for this is found in recent reports of charge density fluctuations across the phase diagram, [51–60], rather than restricted to the underdoped regime as previous experiments suggested, [2]. This is backed up by numerical (Determinant Quantum Monte Carlo) studies of the Hubbard model which also report intertwined charge and spin stripes at optimal doping and in the overdoped regime, [61]. [344] made a direct link between the T-linear resistivity and fluctuating charge density waves by showing that the suppression of static charge order through strain (leaving only

the fluctuations) restores the T-linearity of the resistivity. Theoretical arguments on the impact of fluctuating charge density wave order on strange metal transport have been given in [62–67] (see as well the earlier references [43–45, 345] where the emphasis is more on the underdoped range).

Let us first discuss the ac conductivity in a pinned crystal, (114). It is straightforward to see that the frequency dependence deriving from this formula interpolates between a Drude-like peak centered at $\omega = 0$ if pinning q_o is sufficiently weak compared to the typical frequency scales set by Γ and Ω, and an off-axis peak once pinning becomes stronger. The precise inequality can be derived from (114), asking when all maxima in Re $\sigma(\omega)$ are for $\omega = 0$ or complex frequencies:

$$\omega_o^2 > \frac{\Omega^3}{\Gamma + 2\Omega} \tag{135}$$

If we assume the same scaling as under equation (110), $\omega_o \sim g$, $\Gamma \sim g^2$, $\Omega \sim g^2$, then to leading order in small g, the location of the peak $\omega_{\text{peak}} \simeq \omega_o + O(g^3)$. If charge density wave fluctuations are (at least partially) responsible for the strange metallic behavior, it would be natural to assume that $\omega_o \sim T$ and this would lead to an approximately linear temperature dependence of the off-axis peak location. As the temperature is decreased below the energy scale set by disorder, we might expect the inequality (135) to be violated, upon which the peak returns on-axis.

In spectroscopic experiments, whether an off-axis peak develops at high temperatures seems very material-dependent – materials where this behavior is seen are compiled in [63], see figure 8. For some of these materials, several data points as a function of temperature are available, but not quite enough to resolve the temperature dependence of the peak location as it is returning on-axis, which would be very helpful to determine more precisely.

In $YBa_2Cu_4O_8$, the ac conductivity interpolates from Drude-like to an off-axis peak upon Zn-disordering, [346]. This is in qualitative agreement with charge transport in the pseudo-spontaneous regime, since stronger disorder will lead to an increase in the pseudo-Goldstone mass q_o and in the pinning frequency ω_o. It would be interesting to better understand the effects of Zn-doping on pinning charge density wave fluctuations in scattering experiments, [347–349], especially in light of the results in [52]. In [348], it was shown that Zn-doping gradually kills static spin-stripe order, leaving only charge density fluctuations. As we have already commented on, destroying static spin-stripe order through strain [344] restores T-linearity in the resistivity.

This is as far as effective approaches can take us, since to determine how the frequency dependence of the conductivity varies in any given system requires a microscopic calculation or an experimental measurement. In gauge/gravity duality models, the peak can remain off-axis at all temperatures in the ordered phase, [72, 260, 263, 265], or interpolate between being on-axis and off-axis, [65, 75].

Turning now to dc transport, it is clear that by looking at the dc conductivity of a pinned crystal (115) alone, it will be hard to disentangle the individual contributions of various scattering processes.[52] This said, we can distinguish two types of processes:

- First, extrinsic processes, encapsulated in the momentum relaxation rate Γ. This is through this relaxation coefficient that e.g. disorder or umklapp processes feed in the dc conductivity. Their scaling is expected to be sensitive to irrelevant deformations and to the details of the disorder distribution, leading to scattering rates $\Gamma_{ext} \sim T(g/T^{\Delta_g})^2 \ll T$, [37, 143, 275]. For this reason, it is unlikely they are the origin of the T-linear resistivity.

- Second, intrinsic processes, coming from dissipation into the bath of thermal, critical excitations, encapsulated in transport coefficients such as σ_o, ξ_μ and ξ.

  These are much stronger candidates as the source of T-linear resistivity. Gauge/gravity duality allows to easily calculate these transport coefficients and verify that indeed they their temperature dependence reflects the scaling properties of the underlying critical phase, [82, 102, 143]. These results have inspired scaling theories to explain transport data in cuprates, such as [140, 351]. A crucial extra ingredient compared to previous attempts at a scaling theory (e.g. [141]) is the introduction of anomalous scaling dimensions for the charge density at the critical point, [133, 135, 139, 352].[53]

As we have already emphasized in section VI B, introducing pseudo-spontaneous breaking of translations short-circuits the extrinsic contribution to the resistivity, which is now $\rho_{dc} \sim \mathcal{O}(\Gamma^0)$ rather than $\mathcal{O}(1/\Gamma)$ in a metal. The order $O(\Gamma^0)$ terms are determined by σ_o, ξ_μ and ξ, are intrinsic and are dominant against the extrinsic $O(\Gamma)$ terms. From (114) and (115),

[52] See [350] for an attempt at fitting the hydrodynamic theory of pinned charge density waves in a magnetic field to magnetotransport data in a cuprate. While this analysis has the merit of fitting a consistent set of data on a single material, the set of data used does not allow to unambiguously determine all the parameters in the effective theory.

[53] Holographic models combining explicit breaking of translations and these new scaling laws met with difficulties, [353–356], including matching all scaling laws and or suppressing the coherent, extrinsic contribution to the conductivity from momentum without resorting to strong explicit breaking. This task is made harder by the experimental hurdle of producing thermoelectric transport data displaying clean scaling laws over sufficiently large ranges of temperature.

it is clear that they contribute both to the coherent (the peak) and to the incoherent (the infrared band) parts of the conductivity.This gives further motivation to revisit the two-component analysis of ac conductivity data, which customarily assumes that the infrared band does not contribute to the dc conductivity.

It is reasonable to assume that the temperature dependence of $\Gamma \sim \gamma_0 + \gamma_2 T^2 + \ldots$, capturing the expected temperature dependence of disorder (the zero temperature residual resistivity) and umklapp (the Fermi liquid-like behavior recovered outside the superconducting dome). What might the temperature dependence of intrinsic processes be? This is a difficult question, barring a concrete microscopic model of cuprates. The scaling theories alluded to above give one possible answer, but have not been extended to the pseudo-spontaneous case yet.

An alternative relies on theoretical arguments by which diffusivities D in strongly-correlated systems tend to saturate a Planckian bound, [15],

$$D \gtrsim v^2 \tau_{Pl} \,, \quad \tau_{Pl} = \frac{\hbar}{k_B T} \,. \tag{136}$$

Here v is some characteristic velocity, which is sometimes argued to the Fermi velocity, the Lieb-Robinson velocity or the butterfly velocity (see e.g. [146]). Through this general mechanism, applied to the diffusive transport coefficients σ_o, ξ_μ and ξ, we may expect various disorder-independent, T-linear contributions to the resistivity, split between the coherent and incoherent terms. This resonates with the analysis of the magnetoresistance data of [333], which found necessary to include a T-linear component in both coherent and incoherent contributions.

The diffusivity σ_o is directly related to the thermal diffusivity, [143]. Energy diffusion is likely to be universal in a critical phase. Indeed, measurements of this observable in hole-, [357], and electron-doped cuprates, [358], as well as in crystalline insulators, [359, 360], all suggest that the thermal diffusivity in these materials is close to a Planckian bound.

The reader may legitimately wonder why the same ought to hold for the Goldstone diffusive coefficient ξ. The Goldstones are weakly-coupled in the low energy effective field theory, [10, 191, 361], and so it does not naturally follow that they relax on Planckian scales (the attenuation of superfluid phonons being a case in point). On the other hand, in sections V D, VI C and VII, we have highlighted a dissipation mechanism into hydrodynamic currents at play both in holographic systems and in 2d electron gases hosting Wigner crystal phases.

This mechanism links the Goldstone diffusivity ξ to the thermal diffusivity, which itself is likely to be close to a Planckian bound in a strongly-correlated system.

Pseudo-spontaneous breaking of translations thus appears to be a promising avenue to understand various features of strange and bad metals. While it is difficult to be more conclusive at this stage, further analyses of experimental data, revolving around the influence of disorder on charge density fluctuations, a systematic analysis of charge, heat and magneto-transport data on the same compound, and a refinement of the two-component analysis of optics data, may give further support to this hypothesis or disprove it.

Appendix A: Positivity of entropy production

In this Appendix, we give more details on the steps leading to the Lorentz invariant constitutive relations (84). Using (81) together with the first law of thermodynamics

$$Tds = d\varepsilon - \mu dn - v_i d\pi^i - h^{ij} d\left(\nabla_i u_j\right) \tag{A1}$$

as well as the equations of motion, the divergence of the entropy current is found to be

$$T\dot{s} + T\nabla_i\left(\frac{j_q^i}{T}\right) = \tilde{u}^j\left(K_j + \nabla^i h_{ij}\right) - \tilde{j}_q^i\frac{\nabla_i T}{T} - \tilde{j}^i\nabla_i\mu - \nabla_i v_j\tilde{\tau}^{ij}\,, \tag{A2}$$

with

$$j_q^i = Tsv^i + \tilde{j}_q^i\,, \quad \tilde{j}_q^i = \tilde{j}_\varepsilon^i - \mu\tilde{j}^i + h_{ij}\tilde{u}^j - v_j\tilde{\tau}^{ij}\,. \tag{A3}$$

Here we have turned on an external source for the u^i, $f_{el} \mapsto f_{el} - K_i u^i$, which we take to be first order in gradients $K_i \sim \mathcal{O}(\nabla)$.

The right-hand side must be positive so that entropy is not destroyed by dissipative processes. This constrains the constitutive relations to take the following form:[54]

$$\begin{aligned}
\tilde{j}^i &= -\sigma_o^{ij}\nabla_j\mu - \alpha_o^{ij}\nabla_j T - \gamma_\mu^{ij}\left(K_j + \nabla^k h_{kj}\right)\,,\\
\frac{\tilde{j}_q^i}{T} &= -\bar{\alpha}_o^{ij}\nabla_j\mu - \frac{\bar{\kappa}_o^{ij}}{T}\nabla_j T - \gamma_T^{ij}\left(K_j + \nabla^k h_{kj}\right)\,,\\
\tilde{\tau}^{ij} &= -\eta^{ijkl}\nabla_{(k}v_{l)}\,,\\
\tilde{u}^i &= \xi_\mu^{ij}\nabla_j\mu + \xi_T^{ij}\nabla_j T + \xi_h^{ij}\left(K_j + \nabla^k h_{kj}\right)\,.
\end{aligned} \tag{A4}$$

[54] The ideal equations of motion are used to remove all time derivatives in the constitutive relations, and we choose a frame such that the conserved densities are not corrected at first order in gradients. See [9, 193] for a discussion on the role of frames in relativistic hydrodynamics, and [96–101] for hydrodynamics without boosts.

Turning on the external source K_j is necessary to remove terms like $\nabla_j h^k_k$, which otherwise would appear to be allowed. In the main text and in the remainder of this Appendix, we now turn off the external sources.

The Onsager relations can be imposed either on the matrix of retarded Green's function

$$S \cdot \big(G^R(\omega,-q)\big)^T = G^R(\omega,q)\cdot S\,, \tag{A5}$$

or, as is often simpler, directly on the $M\cdot\chi$ matrix

$$S\cdot\big(M(-q)\cdot\chi\big)^T = M(q)\cdot\chi\cdot S\,, \tag{A6}$$

where M is defined from the equations of motion and with S being the matrix of time-reversal eigenvalues of the corresponding fields $(n,\varepsilon,\pi_\parallel,\lambda_\parallel,\pi_\perp,\lambda_\perp)$. Here $S=\mathrm{diag}(1,1,-1,1,-1,1)$.

The $M\cdot\chi$ matrix reads

$$M\cdot\chi = \begin{pmatrix} \sigma_o q^2 & \alpha_o q^2 & iqn & \gamma_\mu q^2 & 0 & 0\\ \bar\alpha_o q^2 & \frac{\bar\kappa_o}{T}q^2 & iqs & \gamma_T q^2 & 0 & 0\\ iqn & iqs & (\zeta+\eta)q^2 & -iq & 0 & 0\\ \xi_\mu q^2 & \xi_T q^2 & -iq & \xi q^2 & 0 & 0\\ 0 & 0 & 0 & 0 & \eta q^2 & -iq\\ 0 & 0 & 0 & 0 & -iq & \xi q^2 \end{pmatrix}. \tag{A7}$$

The Onsager relations further fix

$$\gamma_\mu = \xi_\mu\,,\quad \gamma_T = \xi_T \tag{A8}$$

Recall that all the transport coefficient matrices and tensors are decomposed as e.g. $\sigma_o^{ij} = \sigma_{(o)}\delta^{ij} + \sigma_{(u)}u^{ij}$, and the final coefficient appearing in (A7) is a linear combination of $\sigma_{(o)}$ and $\sigma_{(u)}$, for instance $\sigma_o = \sigma_{(o)} + (u_o/2)\sigma_{(u)}$.

At linearized level, it is enough for us to impose positivity of (A7), but in general, one should instead require the quadratic form on the right-hand side of (A2) to be positive definite. Positivity of (A7) follows if all eigenvalues are positive, which in turn is equivalent to all principal minors of this matrix being positive. The following constraints are sufficient to that effect

$$\begin{aligned} &\sigma_o\,,\bar\kappa_o\,,\eta\,,\zeta+\eta\geq 0\,,\\ &\sigma_o\bar\kappa_o \geq T\alpha_o^2\,,\quad \sigma_o\xi\geq\xi_\mu^2\,,\quad \bar\kappa_o\xi\geq T\xi_T^2\,. \end{aligned} \tag{A9}$$

The Lorentz boost Ward identity implies that $j^i_\varepsilon = \pi^i$. At ideal level, using (81) this fixes

$$\chi_{\pi\pi} = \varepsilon + p - p_{el}\,, \tag{A10}$$

while at first order in gradients, from (A4) and (A8), the following relations between the longitudinal transport coefficients

$$\begin{aligned} &T\xi_T + \mu\xi_\mu - p_{el}\xi = 0 \\ &T\alpha_o + \mu\sigma_o - p_{el}\xi_\mu = 0 \\ &\mu\alpha_o + \bar{\kappa}_o - p_{el}\xi_T = 0\,, \end{aligned} \tag{A11}$$

or in matrix form:

$$\begin{aligned} &\alpha_o^{ij} + \frac{\mu}{T}\sigma_o^{ij} + \frac{1}{T}h_{ik}\xi_\mu^{kj} = 0 \\ &\kappa_o^{ij} + \frac{\mu}{T}\alpha_o^{ij} + \frac{1}{T}h_{ik}\xi_T^{kj} = 0 \\ &\xi_T^{ij} + \frac{\mu}{T}\xi_\mu^{ij} + \frac{1}{T}h_{ik}\xi_h^{kj} = 0\,. \end{aligned} \tag{A12}$$

The constitutive relations then become

$$\begin{aligned} &\tilde{j}^i = -T\sigma_o^{ij}\nabla_j\frac{\mu}{T} - \gamma_\mu^{ij}\nabla^k h_{kj}\,, \\ &\frac{\tilde{j}_q^i}{T} = \left(\mu\sigma_o^{ij} + h^{il}\xi_\mu^{lj}\right)\nabla_j\frac{\mu}{T} - \left(\mu\xi_\mu^{ij} + h^i{}_l\xi_h^{lj}\right)\nabla^k\frac{h_{kj}}{T}\,, \\ &\tilde{\tau}^{ij} = -\eta^{ijkl}\nabla_{(k}v_{l)}\,, \\ &\tilde{u}^i = T\xi_\mu^{ij}\nabla_j\frac{\mu}{T} + \xi_h^{ij}\nabla^k\frac{h_{kj}}{T}\,. \end{aligned} \tag{A13}$$

In the Galilean limit, the Galilean boost Ward identity enforces $j^i \propto \pi^i$ and instead:

$$\sigma_o^{ij} = 0\,, \quad \alpha_o^{ij} = 0\,, \quad \xi_\mu^{ij} = 0\,. \tag{A14}$$

---

[1] J.G. Bednorz and K.A. Müller, *Possible high T_c superconductivity in the Ba-La-Cu-O system*, *Zeitschrift für Physik B Condensed Matter* **64** (1986) 189.

[2] B. Keimer, S.A. Kivelson, M.R. Norman, S. Uchida and J. Zaanen, *From quantum matter to high-temperature superconductivity in copper oxides*, *Nature* **518** (2015) 179.

[3] A. Alexandradinata, N. Armitage, A. Baydin, W. Bi, Y. Cao, H.J. Changlani et al., *The future of the correlated electron problem*, 2010.00584.

[4] L. Landau, E. Lifshitz and L. Pitaevskii, *Course of Theoretical Physics: Statistical Physics, Part 2 : by E.M. Lifshitz and L.P. Pitaevskii*, no. vol. 9, Oxford, Pergamon Press (1980).

[5] D.P. Arovas, E. Berg, S.A. Kivelson and S. Raghu, *The Hubbard model*, *Annual Review of Condensed Matter Physics* **13** (2021) .

[6] C.M. Varma, P.B. Littlewood, S. Schmitt-Rink, E. Abrahams and A.E. Ruckenstein, *Phenomenology of the normal state of Cu-O high-temperature superconductors*, *Phys. Rev. Lett.* **63** (1989) 1996.

[7] S. Sachdev, *Quantum phase transitions*, Cambridge university press (2011).

[8] D. Chowdhury, A. Georges, O. Parcollet and S. Sachdev, *Sachdev-Ye-Kitaev Models and Beyond: A Window into Non-Fermi Liquids*, 2109.05037.

[9] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, *J. Phys.* **A45** (2012) 473001 [1205.5040].

[10] A. Nicolis, R. Penco, F. Piazza and R. Rattazzi, *Zoology of condensed matter: Framids, ordinary stuff, extra-ordinary stuff*, *JHEP* **06** (2015) 155 [1501.03845].

[11] H. Liu and P. Glorioso, *Lectures on non-equilibrium effective field theories and fluctuating hydrodynamics*, *PoS* **TASI2017** (2018) 008 [1805.09331].

[12] J. Crossno, J.K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas et al., *Observation of the Dirac fluid and the breakdown of the Wiedemann-Franz law in graphene*, *Science* **351** (2016) 1058 [1509.04713].

[13] D.A. Bandurin, I. Torre, R.K. Kumar, M. Ben Shalom, A. Tomadin, A. Principi et al., *Negative local resistance caused by viscous electron backflow in graphene*, *Science* **351** (2016) 1055 [http://science.sciencemag.org/content/351/6277/1055.full.pdf].

[14] A. Lucas and K.C. Fong, *Hydrodynamics of electrons in graphene*, *J. Phys. Condens. Matter* **30** (2018) 053001 [1710.08425].

[15] S.A. Hartnoll, *Theory of universal incoherent metallic transport*, *Nature Phys.* **11** (2015) 54 [1405.3651].

[16] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv.*

Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[17] J. Zaanen, Y.-W. Sun, Y. Liu and K. Schalm, *Holographic Duality in Condensed Matter Physics*, Cambridge Univ. Press (2015).

[18] S.A. Hartnoll, A. Lucas and S. Sachdev, *Holographic quantum matter*, 1612.07324.

[19] V.E. Hubeny, S. Minwalla and M. Rangamani, *The fluid/gravity correspondence*, in *Black holes in higher dimensions*, pp. 348–383, 2012 [1107.5780].

[20] P. Kovtun, D.T. Son and A.O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601 [hep-th/0405231].

[21] G. Policastro, D.T. Son and A.O. Starinets, *The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [hep-th/0104066].

[22] J. Zaanen, *Why the temperature is high*, *Nature* **430** (2004) 512.

[23] S.A. Hartnoll and A.P. Mackenzie, *Planckian Dissipation in Metals*, 2107.07802.

[24] S.H. Shenker and D. Stanford, *Black holes and the butterfly effect*, *JHEP* **03** (2014) 067 [1306.0622].

[25] A. Kitaev, *Talks at KITP, Talks at 'Entanglement in Strongly-Correlated Quantum Matter', KITP* (2015) .

[26] J. Maldacena, S.H. Shenker and D. Stanford, *A bound on chaos*, *JHEP* **08** (2016) 106 [1503.01409].

[27] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev.* **D94** (2016) 106002 [1604.07818].

[28] M.P. Heller, *Holography, Hydrodynamization and Heavy-Ion Collisions*, *Acta Phys. Polon. B* **47** (2016) 2581 [1610.02023].

[29] A. Eberlein, V. Kasper, S. Sachdev and J. Steinberg, *Quantum quench of the Sachdev-Ye-Kitaev Model*, *Phys. Rev.* **B96** (2017) 205123 [1706.07803].

[30] M. Gurvitch and A.T. Fiory, *Resistivity of* $\mathrm{la_{1.825}sr_{0.175}cuo_4}$ *and* $\mathrm{yba_2cu_3o_7}$ *to 1100 k: Absence of saturation and its implications*, *Phys. Rev. Lett.* **59** (1987) 1337.

[31] J. Zhang, E.M. Levenson-Falk, B. Ramshaw, D. Bonn, R. Liang, W. Hardy et al., *Anomalous Thermal Diffusivity in Underdoped* $YBa_2Cu_3O_{6+x}$, *Proceedings of the National Academy of Sciences* **114** (2017) 5378 [1610.05845].

[32] J.A.N. Bruin, H. Sakai, R.S. Perry and A.P. Mackenzie, *Similarity of scattering rates in metals showing T-linear resistivity*, *Science* **339** (2013) 804.

[33] J. Gooth, F. Menges, N. Kumar, V. Süb, C. Shekhar, Y. Sun et al., *Thermal and electrical signatures of a hydrodynamic electron fluid in tungsten diphosphide*, *Nature Communications* **9** (2018) 4093 [1706.05925].

[34] J. Zaanen, *Planckian dissipation, minimal viscosity and the transport in cuprate strange metals*, *SciPost Phys.* **6** (2019) 061 [1807.10951].

[35] W. Götze and P. Wölfle, *Homogeneous dynamical conductivity of simple metals*, *Phys. Rev. B* **6** (1972) 1226.

[36] A. Rosch and N. Andrei, *Conductivity of a clean one-dimensional wire*, *Phys. Rev. Lett.* **85** (2000) 1092.

[37] S.A. Hartnoll and D.M. Hofman, *Locally Critical Resistivities from Umklapp Scattering*, *Phys. Rev. Lett.* **108** (2012) 241601 [1201.3917].

[38] S.A. Hartnoll, R. Mahajan, M. Punk and S. Sachdev, *Transport near the Ising-nematic quantum critical point of metals in two dimensions*, *Phys. Rev.* **B89** (2014) 155130 [1401.7012].

[39] A.A. Patel and S. Sachdev, *DC resistivity at the onset of spin density wave order in two-dimensional metals*, *Phys. Rev.* **B90** (2014) 165146 [1408.6549].

[40] J. Zaanen and O. Gunnarsson, *Charged magnetic domain lines and the magnetism of high-T_c oxides*, *Phys. Rev. B* **40** (1989) 7391.

[41] K. Machida, *Magnetism in la2cuo4 based compounds*, *Physica C: Superconductivity* **158** (1989) 192 .

[42] D. Poilblanc and T.M. Rice, *Charged solitons in the Hartree-Fock approximation to the large-U Hubbard model*, *Phys. Rev. B* **39** (1989) 9749.

[43] S.A. Kivelson, E. Fradkin and V.J. Emery, *Electronic liquid-crystal phases of a doped Mott insulator*, *Nature* **393** (1998) 550 EP .

[44] D.F. Mross and T. Senthil, *Theory of a continuous stripe melting transition in a two-dimensional metal: A possible application to cuprate superconductors*, *Phys. Rev. Lett.* **108** (2012) 267001.

[45] D.F. Mross and T. Senthil, *Stripe melting and quantum criticality in correlated metals*, *Phys. Rev. B* **86** (2012) 115138.

[46] A.J. Beekman, J. Nissinen, K. Wu, K. Liu, R.-J. Slager, Z. Nussinov et al., *Dual gauge field theory of quantum liquid crystals in two dimensions*, *Phys. Rept.* **683** (2017) 1

[1603.04254].

[47] A. Frano, S. Blanco-Canosa, B. Keimer and R.J. Birgeneau, *Charge ordering in superconducting copper oxides*, *Journal of Physics: Condensed Matter* **32** (2020) 374005.

[48] E.W. Huang, C.B. Mendl, S. Liu, S. Johnston, H.-C. Jiang, B. Moritz et al., *Numerical evidence of fluctuating stripes in the normal state of high-Tc cuprate superconductors*, *Science* **358** (2017) 1161–1164.

[49] B.-X. Zheng, C.-M. Chung, P. Corboz, G. Ehlers, M.-P. Qin, R.M. Noack et al., *Stripe order in the underdoped region of the two-dimensional Hubbard model*, *Science* **358** (2017) 1155–1160.

[50] S. Kivelson, I. Bindloss, E. Fradkin, V. Oganesyan, J. Tranquada, A. Kapitulnik et al., *How to detect fluctuating stripes in the high-temperature superconductors*, *Rev. Mod. Phys.* **75** (2003) 1201 [cond-mat/0210683].

[51] Y.Y. Peng, R. Fumagalli, Y. Ding, M. Minola, S. Caprara, D. Betto et al., *Re-entrant charge order in overdoped (bi,pb)2.12sr1.88cuo6+δ outside the pseudogap regime*, *Nature Materials* **17** (2018) 697–702 [1705.06165].

[52] R. Arpaia, S. Caprara, R. Fumagalli, G.D. Vecchi, Y.Y. Peng, E. Andersson et al., *Dynamical charge density fluctuations pervading the phase diagram of a Cu-based high-Tc superconductor*, *Science* **365** (2019) 906–910.

[53] H. Miao, G. Fabbris, R.J. Koch, D.G. Mazzone, C.S. Nelson, R. Acevedo-Esteves et al., *Charge density waves in cuprate superconductors beyond the critical doping*, *npj Quantum Materials* **6** (2021) [2001.10294].

[54] J.Q. Lin, H. Miao, D.G. Mazzone, G.D. Gu, A. Nag, A.C. Walters et al., *Strongly Correlated Charge Density Wave in* $La_{2-x}Sr_xCuO_4$ *Evidenced by Doping-Dependent Phonon Anomaly*, *Phys. Rev. Lett.* **124** (2020) 207005 [2001.10312].

[55] W.S. Lee, K.-J. Zhou, M. Hepting, J. Li, A. Nag, A.C. Walters et al., *Spectroscopic fingerprint of charge order melting driven by quantum fluctuations in a cuprate*, *Nature Physics* **17** (2020) 53–57 [2007.02464].

[56] Q. Ma, K.C. Rule, Z.W. Cronkwright, M. Dragomir, G. Mitchell, E.M. Smith et al., *Parallel spin stripes and their coexistence with superconducting ground states at optimal and high doping in* $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, *Phys. Rev. Research* **3** (2021) 023151 [2009.04627].

[57] R. Arpaia and G. Ghiringhelli, *Charge order at high temperature in cuprate*

superconductors, *Journal of the Physical Society of Japan* **90** (2021) 111005.

[58] C.C. Tam, M. Zhu, J. Ayres, K. Kummer, F. Yakhou-Harris, J.R. Cooper et al., *Charge density waves and Fermi surface reconstruction in the clean overdoped cuprate superconductor Tl2Ba2CuO6+δ*, *Nature Communications* **13** (2022) [2109.04279].

[59] S. Kawasaki, M. Ito, D. Kamijima, C. Lin and G.-q. Zheng, *Charge order and fluctuations in $Bi_2Sr_{2-x}La_xCuO_6+\delta$ revealed by 63,65cu-nuclear magnetic resonance*, *Journal of the Physical Society of Japan* **90** (2021) 111008 [2108.08150].

[60] S. Lee, E.W. Huang, T.A. Johnson, X. Guo, A.A. Husain, M. Mitrano et al., *Generic character of charge and spin density waves in superconducting cuprates*, *arXiv preprint arXiv:2110.13991* (2021) [:2110.13991].

[61] E.W. Huang, T. Liu, W.O. Wang, H.-C. Jiang, P. Mai, T.A. Maier et al., *Fluctuating intertwined stripes in the strange metal regime of the Hubbard model*, 2202.08845.

[62] S. Caprara, C. Di Castro, G. Seibold and M. Grilli, *Dynamical charge density waves rule the phase diagram of cuprates*, *Phys. Rev. B* **95** (2017) 224511.

[63] L.V. Delacrétaz, B. Goutéraux, S.A. Hartnoll and A. Karlsson, *Bad Metals from Fluctuating Density Waves*, *SciPost Phys.* **3** (2017) 025 [1612.04381].

[64] L.V. Delacrétaz, B. Goutéraux, S.A. Hartnoll and A. Karlsson, *Theory of hydrodynamic transport in fluctuating electronic charge density wave states*, *Phys. Rev. B* **96** (2017) 195128 [1702.05104].

[65] A. Amoretti, D. Areán, B. Goutéraux and D. Musso, *Universal relaxation in a holographic metallic density wave phase*, *Phys. Rev. Lett.* **123** (2019) 211602 [1812.08118].

[66] G. Seibold, R. Arpaia, Y.Y. Peng, R. Fumagalli, L. Braicovich, C. Di Castro et al., *Strange metal behaviour from charge density fluctuations in cuprates*, *Communications Physics* **4** (2021) .

[67] L.V. Delacrétaz, B. Goutéraux and V. Ziogas, *Damping of Pseudo-Goldstone Fields*, 2111.13459.

[68] G. Grüner, *The dynamics of charge-density waves*, *Rev. Mod. Phys.* **60** (1988) 1129.

[69] J. Armas, A. Jain and R. Lier, *Approximate symmetries, pseudo-Goldstones, and the second law of thermodynamics*, 2112.14373.

[70] A. Amoretti, D. Areán, R. Argurio, D. Musso and L.A. Pando Zayas, *A holographic perspective on phonons and pseudo-phonons*, *JHEP* **05** (2017) 051 [1611.09344].

[71] L. Alberte, M. Ammon, M. Baggioli, A. Jiménez and O. Pujolàs, *Black hole elasticity and gapped transverse phonons in holography*, *JHEP* **01** (2018) 129 [1708.08477].

[72] T. Andrade, M. Baggioli, A. Krikun and N. Poovuttikul, *Pinning of longitudinal phonons in holographic spontaneous helices*, *JHEP* **02** (2018) 085 [1708.08306].

[73] L. Alberte, M. Ammon, A. Jiménez-Alba, M. Baggioli and O. Pujolàs, *Holographic Phonons*, *Phys. Rev. Lett.* **120** (2018) 171602 [1711.03100].

[74] A. Amoretti, D. Areán, B. Goutéraux and D. Musso, *Effective holographic theory of charge density waves*, *Phys. Rev. D* **97** (2018) 086017 [1711.06610].

[75] A. Donos and C. Pantelidou, *Holographic transport and density waves*, *JHEP* **05** (2019) 079 [1903.05114].

[76] M. Ammon, M. Baggioli, S. Gray and S. Grieninger, *Longitudinal Sound and Diffusion in Holographic Massive Gravity*, *JHEP* **10** (2019) 064 [1905.09164].

[77] M. Ammon, M. Baggioli and A. Jiménez-Alba, *A Unified Description of Translational Symmetry Breaking in Holography*, *JHEP* **09** (2019) 124 [1904.05785].

[78] A. Donos, D. Martin, C. Pantelidou and V. Ziogas, *Incoherent hydrodynamics and density waves*, *Class. Quant. Grav.* **37** (2020) 045005 [1906.03132].

[79] J. Armas and A. Jain, *Viscoelastic hydrodynamics and holography*, *JHEP* **01** (2020) 126 [1908.01175].

[80] J. Armas and A. Jain, *Hydrodynamics for charge density waves and their holographic duals*, *Phys. Rev. D* **101** (2020) 121901 [2001.07357].

[81] M. Ammon, M. Baggioli, S. Gray, S. Grieninger and A. Jain, *On the Hydrodynamic Description of Holographic Viscoelastic Models*, 2001.05737.

[82] R.A. Davison, S.A. Gentle and B. Goutéraux, *Impact of irrelevant deformations on thermodynamics and transport in holographic quantum critical states*, *Phys. Rev. D* **100** (2019) 086020 [1812.11060].

[83] L.V. Delacrétaz, B. Goutéraux, S.A. Hartnoll and A. Karlsson, *Theory of collective magnetophonon resonance and melting of a field-induced Wigner solid*, *Phys. Rev. B* **100** (2019) 085140 [1904.04872].

[84] A. Amoretti, D. Areán, B. Goutéraux and D. Musso, *Diffusion and universal relaxation of holographic phonons*, *JHEP* **10** (2019) 068 [1904.11445].

[85] D. Arean, R.A. Davison, B. Goutéraux and K. Suzuki, *Hydrodynamic Diffusion and Its*

Breakdown near AdS2 Quantum Critical Points, *Phys. Rev. X* **11** (2021) 031024 [2011.12301].

[86] M. Baggioli and B. Goutéraux, *Effective and holographic theories of strongly-correlated phases of matter with broken translations*, 2203.03298.

[87] L.D. Landau and E.M. Lifshitz, *Statistical Physics: Volume 5*, vol. 5, Elsevier (2013).

[88] P. Chaikin and T. Lubensky, *Principles of Condensed Matter Physics*, Cambridge University Press (2000).

[89] I. De Schepper, H. Van Beyeren and M. Ernst, *The nonexistence of the linear diffusion equation beyond fick's law*, *Physica* **75** (1974) 1.

[90] D. Forster, D.R. Nelson and M.J. Stephen, *Large-distance and long-time properties of a randomly stirred fluid*, *Phys. Rev. A* **16** (1977) 732.

[91] P. Kovtun and L.G. Yaffe, *Hydrodynamic fluctuations, long time tails, and supersymmetry*, *Phys. Rev. D* **68** (2003) 025007 [hep-th/0303010].

[92] R.A. Davison and B. Goutéraux, *Momentum dissipation and effective theories of coherent and incoherent transport*, *JHEP* **01** (2015) 039 [1411.1062].

[93] S. Grozdanov, A. Lucas and N. Poovuttikul, *Holography and hydrodynamics with weakly broken symmetries*, *Phys. Rev. D* **99** (2019) 086012 [1810.10016].

[94] B. Withers, *Short-lived modes from hydrodynamic dispersion relations*, *JHEP* **06** (2018) 059 [1803.08058].

[95] S. Grozdanov, P.K. Kovtun, A.O. Starinets and P. Tadić, *Convergence of the Gradient Expansion in Hydrodynamics*, *Phys. Rev. Lett.* **122** (2019) 251601 [1904.01018].

[96] J. de Boer, J. Hartong, N.A. Obers, W. Sybesma and S. Vandoren, *Perfect Fluids*, *SciPost Phys.* **5** (2018) 003 [1710.04708].

[97] J. de Boer, J. Hartong, N.A. Obers, W. Sybesma and S. Vandoren, *Hydrodynamic Modes of Homogeneous and Isotropic Fluids*, *SciPost Phys.* **5** (2018) 014 [1710.06885].

[98] I. Novak, J. Sonner and B. Withers, *Hydrodynamics without boosts*, *JHEP* **07** (2020) 165 [1911.02578].

[99] N. Poovuttikul and W. Sybesma, *First order non-Lorentzian fluids, entropy production and linear instabilities*, *Phys. Rev. D* **102** (2020) 065007 [1911.00010].

[100] J. de Boer, J. Hartong, E. Have, N.A. Obers and W. Sybesma, *Non-Boost Invariant Fluid Dynamics*, *SciPost Phys.* **9** (2020) 018 [2004.10759].

[101] J. Armas and A. Jain, *Effective field theory for hydrodynamics without boosts*, *SciPost Phys.* **11** (2021) 054 [2010.15782].

[102] R.A. Davison, B. Goutéraux and S.A. Hartnoll, *Incoherent transport in clean quantum critical metals*, *JHEP* **10** (2015) 112 [1507.07137].

[103] S. Grozdanov and N. Kaplis, *Constructing higher-order hydrodynamics: The third order*, *Phys. Rev. D* **93** (2016) 066012 [1507.02461].

[104] G.W. Gibbons, *Anti-de-sitter spacetime and its uses*, in *Mathematical and quantum aspects of relativity and cosmology*, pp. 102–142, Springer (2000).

[105] P. Ginsparg, *Applied conformal field theory*, *arXiv preprint hep-th/9108028* (1988) .

[106] M. Ammon and J. Erdmenger, *Gauge/gravity duality*, Cambridge University Press (2015).

[107] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[108] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109].

[109] G. 't Hooft, *Large N*, in *The Phenomenology of Large N(c) QCD*, pp. 3–18, 4, 2002, DOI [hep-th/0204069].

[110] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U.A. Wiedemann, *Gauge/String Duality, Hot QCD and Heavy Ion Collisions*, Cambridge University Press (2014), 10.1017/CBO9781139136747, [1101.0618].

[111] J. Berges, M.P. Heller, A. Mazeliauskas and R. Venugopalan, *QCD thermalization: Ab initio approaches and interdisciplinary connections*, *Rev. Mod. Phys.* **93** (2021) 035003 [2005.12299].

[112] M. Rangamani and T. Takayanagi, *Holographic entanglement entropy*, in *Holographic Entanglement Entropy*, pp. 35–47, Springer (2017).

[113] H. Liu and J. Sonner, *Holographic systems far from equilibrium: a review*, 1810.02367.

[114] H. Liu and J. Sonner, *Quantum many-body physics from a gravitational lens*, *Nature Rev. Phys.* **2** (2020) 615 [2004.06159].

[115] D.T. Son and A.O. Starinets, *Minkowski space correlators in AdS / CFT correspondence: Recipe and applications*, *JHEP* **09** (2002) 042 [hep-th/0205051].

[116] P.K. Kovtun and A.O. Starinets, *Quasinormal modes and holography*, *Phys. Rev.* **D72** (2005) 086009 [hep-th/0506184].

[117] E. Berti, V. Cardoso and A.O. Starinets, *Quasinormal modes of black holes and black branes*, *Class. Quant. Grav.* **26** (2009) 163001 [0905.2975].

[118] G. Policastro, D.T. Son and A.O. Starinets, *From AdS / CFT correspondence to hydrodynamics*, *JHEP* **09** (2002) 043 [hep-th/0205052].

[119] G. Policastro, D.T. Son and A.O. Starinets, *From AdS / CFT correspondence to hydrodynamics. 2. Sound waves*, *JHEP* **12** (2002) 054 [hep-th/0210220].

[120] R. Baier, P. Romatschke, D.T. Son, A.O. Starinets and M.A. Stephanov, *Relativistic viscous hydrodynamics, conformal invariance, and holography*, *JHEP* **04** (2008) 100 [0712.2451].

[121] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **01** (2009) 055 [0809.2488].

[122] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, *Hydrodynamics from charged black branes*, *JHEP* **01** (2011) 094 [0809.2596].

[123] M. Rangamani, *Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence*, *Class. Quant. Grav.* **26** (2009) 224003 [0905.4352].

[124] S. Cremonini, *The Shear Viscosity to Entropy Ratio: A Status Report*, *Mod. Phys. Lett.* **B25** (2011) 1867 [1108.0677].

[125] A.J. Beekman, L. Rademaker and J. van Wezel, *An Introduction to Spontaneous Symmetry Breaking*, 1909.01820.

[126] S. Weinberg, *Approximate symmetries and pseudo-goldstone bosons*, *Phys. Rev. Lett.* **29** (1972) 1698.

[127] C. Burgess, *Goldstone and pseudoGoldstone bosons in nuclear, particle and condensed matter physics*, *Phys. Rept.* **330** (2000) 193 [hep-th/9808176].

[128] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [hep-th/0209067].

[129] C. Charmousis, B. Gouteraux, B.S. Kim, E. Kiritsis and R. Meyer, *Effective Holographic Theories for low-temperature condensed matter systems*, *JHEP* **11** (2010) 151 [1005.4690].

[130] B. Goutéraux and E. Kiritsis, *Generalized Holographic Quantum Criticality at Finite Density*, *JHEP* **1112** (2011) 036 [1107.2116].

[131] L. Huijse, S. Sachdev and B. Swingle, *Hidden Fermi surfaces in compressible states of gauge-gravity duality*, *Phys.Rev.* **B85** (2012) 035121 [1112.0573].

[132] B. Goutéraux and E. Kiritsis, *Quantum critical lines in holographic phases with (un)broken*

symmetry, *JHEP* **04** (2013) 053 [1212.2625].

[133] B. Goutéraux, *Universal scaling properties of extremal cohesive holographic phases*, *JHEP* **01** (2014) 080 [1308.2084].

[134] A. Donos and J.P. Gauntlett, *Novel metals and insulators from holography*, *JHEP* **06** (2014) 007 [1401.5077].

[135] B. Goutéraux, *Charge transport in holography with momentum dissipation*, *JHEP* **04** (2014) 181 [1401.5436].

[136] I. Kanitscheider and K. Skenderis, *Universal hydrodynamics of non-conformal branes*, *JHEP* **04** (2009) 062 [0901.1487].

[137] B. Goutéraux, J. Smolic, M. Smolic, K. Skenderis and M. Taylor, *Holography for Einstein-Maxwell-dilaton theories from generalized dimensional reduction*, *JHEP* **01** (2012) 089 [1110.2320].

[138] E. Kiritsis, F. Nitti and L.S. Pimenta, *Exotic RG Flows from Holography*, *Fortsch. Phys.* **65** (2017) 1600120 [1611.05493].

[139] A. Karch, *Conductivities for Hyperscaling Violating Geometries*, *JHEP* **06** (2014) 140 [1405.2926].

[140] S.A. Hartnoll and A. Karch, *Scaling theory of the cuprate strange metals*, *Phys. Rev. B* **91** (2015) 155126 [1501.03165].

[141] P. Phillips and C. Chamon, *Breakdown of One-Paramater Scaling in Quantum Critical Scenarios for the High-Temperature Copper-oxide Superconductors*, *Phys. Rev. Lett.* **95** (2005) 107002 [cond-mat/0412179].

[142] S. Sachdev and B. Keimer, *Quantum Criticality*, *Phys. Today* **64N2** (2011) 29 [1102.4628].

[143] R.A. Davison, S.A. Gentle and B. Gouteraux, *Slow relaxation and diffusion in holographic quantum critical phases*, *Phys. Rev. Lett.* **123** (2019) 141601 [1808.05659].

[144] M. Blake, R.A. Davison and S. Sachdev, *Thermal diffusivity and chaos in metals without quasiparticles*, *Phys. Rev.* **D96** (2017) 106008 [1705.07896].

[145] D.A. Roberts and B. Swingle, *Lieb-Robinson Bound and the Butterfly Effect in Quantum Field Theories*, *Phys. Rev. Lett.* **117** (2016) 091602 [1603.09298].

[146] M. Blake, *Universal Charge Diffusion and the Butterfly Effect in Holographic Theories*, *Phys. Rev. Lett.* **117** (2016) 091601 [1603.08510].

[147] T. Hartman, S.A. Hartnoll and R. Mahajan, *Upper Bound on Diffusivity*, *Phys. Rev. Lett.*

119 (2017) 141601 [1706.00019].

[148] A. Lucas, *Constraints on hydrodynamics from many-body quantum chaos*, 1710.01005.

[149] M. Blake, *Universal Diffusion in Incoherent Black Holes*, *Phys. Rev.* **D94** (2016) 086014 [1604.01754].

[150] M. Blake and A. Donos, *Diffusion and Chaos from near AdS_2 horizons*, *JHEP* **02** (2017) 013 [1611.09380].

[151] R.A. Davison, S.s. Grozdanov, S. Janiszewski and M. Kaminski, *Momentum and charge transport in non-relativistic holographic fluids from Hořava gravity*, *JHEP* **11** (2016) 170 [1606.06747].

[152] M. Baggioli, B. Goutéraux, E. Kiritsis and W.-J. Li, *Higher derivative corrections to incoherent metallic transport in holography*, *JHEP* **03** (2017) 170 [1612.05500].

[153] K.-Y. Kim and C. Niu, *Diffusion and Butterfly Velocity at Finite Density*, *JHEP* **06** (2017) 030 [1704.00947].

[154] M. Blake, H. Lee and H. Liu, *A quantum hydrodynamical description for scrambling and many-body chaos*, *JHEP* **10** (2018) 127 [1801.00010].

[155] H.-S. Jeong, Y. Ahn, D. Ahn, C. Niu, W.-J. Li and K.-Y. Kim, *Thermal diffusivity and butterfly velocity in anisotropic Q-Lattice models*, *JHEP* **01** (2018) 140 [1708.08822].

[156] R.A. Davison, B. Goutéraux and E. Mefford, *Emergent anomalous generalized symmetries in holgraphic quantum matter*, *to appear* (2022) .

[157] L.V. Delacrétaz, D.M. Hofman and G. Mathys, *Superfluids as Higher-form Anomalies*, *SciPost Phys.* **8** (2020) 047 [1908.06977].

[158] R.A. Davison, L.V. Delacrétaz, B. Goutéraux and S.A. Hartnoll, *Hydrodynamic theory of quantum fluctuating superconductivity*, *Phys. Rev.* **B94** (2016) 054502 [1602.08171].

[159] A. Karch, D.T. Son and A.O. Starinets, *Holographic quantum liquid*, *Physical Review Letters* **102** (2009) [0806.3796].

[160] R.A. Davison and A.O. Starinets, *Holographic zero sound at finite temperature*, *Phys. Rev. D* **85** (2012) 026004 [1109.6343].

[161] C.-F. Chen and A. Lucas, *Origin of the Drude peak and of zero sound in probe brane holography*, *Phys. Lett. B* **774** (2017) 569 [1709.01520].

[162] S. Grozdanov and N. Poovuttikul, *Generalized global symmetries in states with dynamical defects: The case of the transverse sound in field theory and holography*, *Phys. Rev.* **D97**

(2018) 106005 [1801.03199].

[163] S. Grozdanov, P.K. Kovtun, A.O. Starinets and P. Tadić, *The complex life of hydrodynamic modes*, *JHEP* **11** (2019) 097 [1904.12862].

[164] M.P. Heller, A. Serantes, M. Spaliński, V. Svensson and B. Withers, *Convergence of hydrodynamic modes: insights from kinetic theory and holography*, 2012.15393.

[165] M.P. Heller, A. Serantes, M. Spaliński, V. Svensson and B. Withers, *The hydrodynamic gradient expansion in linear response theory*, 2007.05524.

[166] M.P. Heller, R.A. Janik and P. Witaszczyk, *Hydrodynamic Gradient Expansion in Gauge Theory Plasmas*, *Phys. Rev. Lett.* **110** (2013) 211602 [1302.0697].

[167] N. Abbasi and S. Tahery, *Complexified quasinormal modes and the pole-skipping in a holographic system at finite chemical potential*, *JHEP* **10** (2020) 076 [2007.10024].

[168] A. Jansen and C. Pantelidou, *Quasinormal modes in charged fluids at complex momentum*, *JHEP* **10** (2020) 121 [2007.14418].

[169] S. Grozdanov, *Bounds on transport from univalence and pole-skipping*, 2008.00888.

[170] C. Choi, M. Mezei and G. Sárosi, *Pole skipping away from maximal chaos*, 2010.08558.

[171] S. Sachdev and J. Ye, *Gapless spin-fluid ground state in a random quantum Heisenberg magnet*, *Phys. Rev. Lett.* **70** (1993) 3339 [cond-mat/9212030].

[172] S. Sachdev, *Holographic metals and the fractionalized Fermi liquid*, *Phys. Rev. Lett.* **105** (2010) 151602 [1006.3794].

[173] A. Almheiri and J. Polchinski, *Models of AdS_2 backreaction and holography*, *JHEP* **11** (2015) 014 [1402.6334].

[174] S. Sachdev, *Bekenstein-Hawking Entropy and Strange Metals*, *Phys. Rev. X* **5** (2015) 041025 [1506.05111].

[175] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [1606.01857].

[176] K. Jensen, *Chaos in AdS_2 Holography*, *Phys. Rev. Lett.* **117** (2016) 111601 [1605.06098].

[177] J. Engelsöy, T.G. Mertens and H. Verlinde, *An investigation of AdS_2 backreaction and holography*, *JHEP* **07** (2016) 139 [1606.03438].

[178] Y. Gu, X.-L. Qi and D. Stanford, *Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models*, *JHEP* **05** (2017) 125 [1609.07832].

[179] R.A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen and S. Sachdev, *Thermoelectric*

transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography, *Phys. Rev. B* **95** (2017) 155131 [1612.00849].

[180] S. Sachdev, *Universal low temperature theory of charged black holes with AdS_2 horizons*, *J. Math. Phys.* **60** (2019) 052303 [1902.04078].

[181] H. Guo, Y. Gu and S. Sachdev, *Transport and chaos in lattice Sachdev-Ye-Kitaev models*, *Phys. Rev. B* **100** (2019) 045140 [1904.02174].

[182] S. Grozdanov, K. Schalm and V. Scopelliti, *Black hole scrambling from hydrodynamics*, *Phys. Rev. Lett.* **120** (2018) 231601 [1710.00921].

[183] M. Blake, R.A. Davison, S. Grozdanov and H. Liu, *Many-body chaos and energy dynamics in holography*, *JHEP* **10** (2018) 035 [1809.01169].

[184] M. Blake, R.A. Davison and D. Vegh, *Horizon constraints on holographic Green's functions*, *JHEP* **01** (2020) 077 [1904.12883].

[185] P. Betzios, U. Gürsoy, M. Järvinen and G. Policastro, *Quasinormal modes of a strongly coupled nonconformal plasma and approach to criticality*, *Phys. Rev. D* **97** (2018) 081901 [1708.02252].

[186] P. Betzios, U. Gürsoy, M. Järvinen and G. Policastro, *Fluctuations in a nonconformal holographic plasma at criticality*, *Phys. Rev. D* **101** (2020) 086026 [1807.01718].

[187] I. Low and A.V. Manohar, *Spontaneously broken spacetime symmetries and goldstone's theorem*, *Phys. Rev. Lett.* **88** (2002) 101602.

[188] H. Watanabe, *Counting Rules of Nambu–Goldstone Modes*, *Ann. Rev. Condensed Matter Phys.* **11** (2020) 169 [1904.00569].

[189] C.W. Hicks, D.O. Brodsky, E.A. Yelland, A.S. Gibbs, J.A.N. Bruin, M.E. Barber et al., *Strong Increase of T_c of Sr_2RuO_4 Under Both Tensile and Compressive Strain*, *Science* **344** (2014) 283.

[190] H.-H. Kim, S.M. Souliou, M.E. Barber, E. Lefrançois, M. Minola, M. Tortora et al., *Uniaxial pressure control of competing orders in a high-temperature superconductor*, *Science* **362** (2018) 1040.

[191] A. Nicolis, R. Penco and R.A. Rosen, *Relativistic Fluids, Superfluids, Solids and Supersolids from a Coset Construction*, *Phys. Rev.* **D89** (2014) 045002 [1307.0517].

[192] A. Esposito, S. Garcia-Saenz, A. Nicolis and R. Penco, *Conformal solids and holography*, *JHEP* **12** (2017) 113 [1708.09391].

[193] P. Kovtun, *First-order relativistic hydrodynamics is stable*, *JHEP* **10** (2019) 034 [1907.08191].

[194] S.S. Gubser, *Breaking an Abelian gauge symmetry near a black hole horizon*, *Phys. Rev. D* **78** (2008) 065034 [0801.2977].

[195] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Building a Holographic Superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601 [0803.3295].

[196] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Holographic Superconductors*, *JHEP* **12** (2008) 015 [0810.1563].

[197] C.P. Herzog, P.K. Kovtun and D.T. Son, *Holographic model of superfluidity*, *Phys. Rev. D* **79** (2009) 066002 [0809.4870].

[198] J. Sonner and B. Withers, *A gravity derivation of the Tisza-Landau Model in AdS/CFT*, *Phys. Rev. D* **82** (2010) 026001 [1004.2707].

[199] C.P. Herzog, N. Lisker, P. Surowka and A. Yarom, *Transport in holographic superfluids*, *JHEP* **08** (2011) 052 [1101.3330].

[200] J. Bhattacharya, S. Bhattacharyya and S. Minwalla, *Dissipative Superfluid dynamics from gravity*, *JHEP* **04** (2011) 125 [1101.3332].

[201] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and A. Yarom, *A Theory of first order dissipative superfluid dynamics*, *JHEP* **05** (2014) 147 [1105.3733].

[202] D. Arean, M. Baggioli, S. Grieninger and K. Landsteiner, *A Holographic Superfluid Symphony*, *JHEP* **21** (2020) 206 [2107.08802].

[203] S. Nakamura, H. Ooguri and C.-S. Park, *Gravity Dual of Spatially Modulated Phase*, *Phys. Rev. D* **81** (2010) 044018 [0911.0679].

[204] A. Donos and J.P. Gauntlett, *Holographic striped phases*, *JHEP* **08** (2011) 140 [1106.2004].

[205] A. Donos, J.P. Gauntlett and C. Pantelidou, *Spatially modulated instabilities of magnetic black branes*, *JHEP* **01** (2012) 061 [1109.0471].

[206] S. Cremonini and A. Sinkovics, *Spatially Modulated Instabilities of Geometries with Hyperscaling Violation*, *JHEP* **01** (2014) 099 [1212.4172].

[207] J. Alsup, E. Papantonopoulos, G. Siopsis and K. Yeter, *Spontaneously Generated Inhomogeneous Phases via Holography*, *Phys. Rev. D* **88** (2013) 105028 [1305.2507].

[208] A. Donos and J.P. Gauntlett, *On the thermodynamics of periodic AdS black branes*, *JHEP* **10** (2013) 038 [1306.4937].

[209] A. Donos and J.P. Gauntlett, *Holographic charge density waves*, *Phys. Rev. D* **87** (2013) 126008 [1303.4398].

[210] A. Donos and J.P. Gauntlett, *Holographic helical superconductors*, *JHEP* **12** (2011) 091 [1109.3866].

[211] A. Donos and J.P. Gauntlett, *Helical superconducting black holes*, *Phys. Rev. Lett.* **108** (2012) 211601 [1203.0533].

[212] A. Donos and J.P. Gauntlett, *Black holes dual to helical current phases*, *Phys. Rev. D* **86** (2012) 064010 [1204.1734].

[213] M. Rozali, D. Smyth, E. Sorkin and J.B. Stang, *Holographic Stripes*, *Phys. Rev. Lett.* **110** (2013) 201603 [1211.5600].

[214] A. Donos, *Striped phases from holography*, *JHEP* **05** (2013) 059 [1303.7211].

[215] B. Withers, *Black branes dual to striped phases*, *Class. Quant. Grav.* **30** (2013) 155025 [1304.0129].

[216] B. Withers, *Holographic Checkerboards*, *JHEP* **09** (2014) 102 [1407.1085].

[217] A. Donos and J.P. Gauntlett, *Minimally packed phases in holography*, *JHEP* **03** (2016) 148 [1512.06861].

[218] C.M. Varma, *Pseudogap phase and the quantum-critical point in copper-oxide metals*, *Phys. Rev. Lett.* **83** (1999) 3538.

[219] N. Jokela, M. Jarvinen and M. Lippert, *Fluctuations and instabilities of a holographic metal*, *JHEP* **02** (2013) 007 [1211.1381].

[220] N. Jokela, M. Jarvinen and M. Lippert, *Gravity dual of spin and charge density waves*, *JHEP* **12** (2014) 083 [1408.1397].

[221] N. Jokela, M. Jarvinen and M. Lippert, *Holographic sliding stripes*, *Phys. Rev. D* **95** (2017) 086006 [1612.07323].

[222] E. Fradkin, S.A. Kivelson and J.M. Tranquada, *Colloquium: Theory of intertwined orders in high temperature superconductors*, *Reviews of Modern Physics* **87** (2015) 457–482 [1407.4480].

[223] S. Cremonini, L. Li and J. Ren, *Holographic Pair and Charge Density Waves*, *Phys. Rev. D* **95** (2017) 041901 [1612.04385].

[224] S. Cremonini, L. Li and J. Ren, *Intertwined Orders in Holography: Pair and Charge Density Waves*, *JHEP* **08** (2017) 081 [1705.05390].

[225] R.-G. Cai, L. Li, Y.-Q. Wang and J. Zaanen, *Intertwined Order and Holography: The Case of Parity Breaking Pair Density Waves*, *Phys. Rev. Lett.* **119** (2017) 181601 [1706.01470].

[226] P.C. Martin, O. Parodi and P.S. Pershan, *Unified hydrodynamic theory for crystals, liquid crystals, and normal fluids*, *Phys. Rev. A* **6** (1972) 2401.

[227] P.D. Fleming III and C. Cohen, *Hydrodynamics of solids*, *Physical Review B* **13** (1976) 500.

[228] C.P. Enz, *Two-fluid hydrodynamic description of ordered systems*, *Rev. Mod. Phys.* **46** (1974) 705.

[229] A. Amoretti, D. Areán, B. Goutéraux and D. Musso, *DC resistivity of quantum critical, charge density wave states from gauge-gravity duality*, *Phys. Rev. Lett.* **120** (2018) 171603 [1712.07994].

[230] M. Baggioli and S. Grieninger, *Zoology of solid & fluid holography — Goldstone modes and phase relaxation*, *JHEP* **10** (2019) 235 [1905.09488].

[231] M. Baggioli, S. Grieninger and L. Li, *Magnetophonons & type-B Goldstones from Hydrodynamics to Holography*, 2005.01725.

[232] A. Schmitt, *Introduction to Superfluidity: Field-theoretical approach and applications*, vol. 888 (2015), 10.1007/978-3-319-07947-9, [1404.1284].

[233] B. Goutéraux, E. Mefford and F. Sottovia, *Thermodynamic origin of the Landau instability in superfluids*, *to appear* (2022) .

[234] I. Amado, D. Areán, A. Jiménez-Alba, K. Landsteiner, L. Melgar and I. Salazar Landea, *Holographic Superfluids and the Landau Criterion*, *JHEP* **02** (2014) 063 [1307.8100].

[235] S. Lan, H. Liu, Y. Tian and H. Zhang, *Landau Instability and soliton formations*, 2010.06232.

[236] T. Damour and M. Lilley, *String theory, gravity and experiment*, *Les Houches* **87** (2008) 371 [0802.4169].

[237] K.S. Thorne, R.H. Price and D.A. MacDonald, *Black holes: The membrane paradigm* (1986).

[238] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, *Phys. Rev.* **D79** (2009) 025023 [0809.3808].

[239] A. Donos and J.P. Gauntlett, *Thermoelectric DC conductivities from black hole horizons*, *JHEP* **11** (2014) 081 [1406.4742].

[240] A. Donos and J.P. Gauntlett, *The thermoelectric properties of inhomogeneous holographic*

lattices, *JHEP* **01** (2015) 035 [1409.6875].

[241] A. Donos, J.P. Gauntlett, T. Griffin and V. Ziogas, *Incoherent transport for phases that spontaneously break translations*, *JHEP* **04** (2018) 053 [1801.09084].

[242] B. Goutéraux, N. Jokela and A. Pönni, *Incoherent conductivity of holographic charge density waves*, *JHEP* **07** (2018) 004 [1803.03089].

[243] A. Donos, B. Goutéraux and E. Kiritsis, *Holographic Metals and Insulators with Helical Symmetry*, *JHEP* **09** (2014) 038 [1406.6351].

[244] M. Blake, *Momentum relaxation from the fluid/gravity correspondence*, *JHEP* **09** (2015) 010 [1505.06992].

[245] D. Nickel and D.T. Son, *Deconstructing holographic liquids*, *New J. Phys.* **13** (2011) 075010 [1009.3094].

[246] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172 [1412.5148].

[247] S. Grozdanov, D.M. Hofman and N. Iqbal, *Generalized global symmetries and dissipative magnetohydrodynamics*, *Phys. Rev. D* **95** (2017) 096003 [1610.07392].

[248] M.J. Landry, *Higher-form symmetries and 't Hooft anomalies in non-equilibrium systems*, 2101.02210.

[249] D.V. Else and T. Senthil, *Critical drag as a mechanism for resistivity*, *Physical Review B* **104** (2021) .

[250] A.J. Beekman, J. Nissinen, K. Wu and J. Zaanen, *Dual gauge field theory of quantum liquid crystals in three dimensions*, *Phys. Rev.* **B96** (2017) 165115 [1703.03157].

[251] J.M. Kosterlitz and D.J. Thouless, *Ordering, metastability and phase transitions in two-dimensional systems*, *Journal of Physics C: Solid State Physics* **6** (1973) 1181.

[252] B.I. Halperin and D.R. Nelson, *Theory of two-dimensional melting*, *Phys. Rev. Lett.* **41** (1978) 121.

[253] D.R. Nelson and B.I. Halperin, *Dislocation-mediated melting in two dimensions*, *Phys. Rev. B* **19** (1979) 2457.

[254] A. Zippelius, B.I. Halperin and D.R. Nelson, *Dynamics of two-dimensional melting*, *Phys. Rev. B* **22** (1980) 2514.

[255] M. Gell-Mann, R.J. Oakes and B. Renner, *Behavior of current divergences under* $\mathrm{su}_3 \times \mathrm{su}_3$, *Phys. Rev.* **175** (1968) 2195.

[256] P. Lee, T. Rice and P. Anderson, *Conductivity from charge or spin density waves*, *Solid State Communications* **88** (1993) 1001.

[257] H. Fukuyama and P.A. Lee, *Dynamics of the charge-density wave. i. impurity pinning in a single chain*, *Phys. Rev. B* **17** (1978) 535.

[258] A. Lucas and S. Sachdev, *Memory matrix theory of magnetotransport in strange metals*, *Phys. Rev. B* **91** (2015) 195122 [1502.04704].

[259] L. Levitov and G. Falkovich, *Electron viscosity, current vortices and negative nonlocal resistance in graphene*, *Nature Physics* **12** (2016) 672–676.

[260] T. Andrade, A. Krikun, K. Schalm and J. Zaanen, *Doping the holographic Mott insulator*, *Nature Phys.* **14** (2018) 1049 [1710.05791].

[261] A. Krikun, *Holographic discommensurations*, *JHEP* **12** (2018) 030 [1710.05801].

[262] A. Donos, D. Martin, C. Pantelidou and V. Ziogas, *Hydrodynamics of broken global symmetries in the bulk*, *JHEP* **10** (2019) 218 [1905.00398].

[263] T. Andrade and A. Krikun, *Coherent vs incoherent transport in holographic strange insulators*, *JHEP* **05** (2019) 119 [1812.08132].

[264] A. Amoretti, D. Areán, B. Goutéraux and D. Musso, *Gapless and gapped holographic phonons*, *JHEP* **01** (2020) 058 [1910.11330].

[265] T. Andrade, M. Baggioli and A. Krikun, *Phase relaxation and pattern formation in holographic gapless charge density waves*, 2009.05551.

[266] M. Ammon, D. Arean, M. Baggioli, S. Gray and S. Grieninger, *Pseudo-spontaneous $U(1)$ Symmetry Breaking in Hydrodynamics and Holography*, 2111.10305.

[267] E. Grossi, A. Soloviev, D. Teaney and F. Yan, *Transport and hydrodynamics in the chiral limit*, *Phys. Rev. D* **102** (2020) 014042 [2005.02885].

[268] E. Grossi, A. Soloviev, D. Teaney and F. Yan, *Soft pions and transport near the chiral critical point*, *Phys. Rev. D* **104** (2021) 034025 [2101.10847].

[269] S.A. Hartnoll, P.K. Kovtun, M. Muller and S. Sachdev, *Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes*, *Phys. Rev.* **B76** (2007) 144502 [0706.3215].

[270] R.A. Davison and B. Goutéraux, *Dissecting holographic conductivities*, *JHEP* **09** (2015) 090 [1505.05092].

[271] T. Andrade and B. Withers, *A simple holographic model of momentum relaxation*, *JHEP*

05 (2014) 101 [1311.5157].

[272] B. Goutéraux and A. Shukla, *work in progress*, .

[273] S.A. Hartnoll and C.P. Herzog, *Impure AdS/CFT correspondence*, *Phys. Rev. D* **77** (2008) 106009 [0801.1693].

[274] S.A. Hartnoll, J. Polchinski, E. Silverstein and D. Tong, *Towards strange metallic holography*, *JHEP* **04** (2010) 120 [0912.1061].

[275] A. Lucas, S. Sachdev and K. Schalm, *Scale-invariant hyperscaling-violating holographic theories and the resistivity of strange metals with random-field disorder*, *Phys. Rev. D* **89** (2014) 066018 [1401.7993].

[276] G.T. Horowitz, J.E. Santos and D. Tong, *Optical Conductivity with Holographic Lattices*, *JHEP* **07** (2012) 168 [1204.0519].

[277] G.T. Horowitz, J.E. Santos and D. Tong, *Further Evidence for Lattice-Induced Scaling*, *JHEP* **11** (2012) 102 [1209.1098].

[278] P. Chesler, A. Lucas and S. Sachdev, *Conformal field theories in a periodic potential: results from holography and field theory*, *Phys. Rev. D* **89** (2014) 026005 [1308.0329].

[279] M. Rangamani, M. Rozali and D. Smyth, *Spatial Modulation and Conductivities in Effective Holographic Theories*, *JHEP* **07** (2015) 024 [1505.05171].

[280] B.W. Langley, G. Vanacore and P.W. Phillips, *Absence of Power-Law Mid-Infrared Conductivity in Gravitational Crystals*, *JHEP* **10** (2015) 163 [1506.06769].

[281] O.J. Dias, J.E. Santos and B. Way, *Numerical Methods for Finding Stationary Gravitational Solutions*, *Class. Quant. Grav.* **33** (2016) 133001 [1510.02804].

[282] T. Andrade, *Holographic Lattices and Numerical Techniques*, 12, 2017 [1712.00548].

[283] A. Krikun, *Numerical Solution of the Boundary Value Problems for Partial Differential Equations. Crash course for holographer*, 1, 2018 [1801.01483].

[284] S.A. Hartnoll and J.E. Santos, *Disordered horizons: Holography of randomly disordered fixed points*, *Phys. Rev. Lett.* **112** (2014) 231601 [1402.0872].

[285] S.A. Hartnoll, D.M. Ramirez and J.E. Santos, *Emergent scale invariance of disordered horizons*, *JHEP* **09** (2015) 160 [1504.03324].

[286] S.A. Hartnoll, D.M. Ramirez and J.E. Santos, *Thermal conductivity at a disordered quantum critical point*, *JHEP* **04** (2016) 022 [1508.04435].

[287] K. Ganesan and A. Lucas, *Breakdown of emergent Lifshitz symmetry in holographic matter*

with Harris-marginal disorder, *JHEP* **06** (2020) 023 [2004.06543].

[288] K. Ganesan, A. Lucas and L. Radzihovsky, *Renormalization group in quantum critical theories with Harris-marginal disorder*, *Phys. Rev. D* **105** (2022) 066016 [2110.11978].

[289] D. Vegh, *Holography without translational symmetry*, 1301.0537.

[290] M. Baggioli and O. Pujolas, *Electron-Phonon Interactions, Metal-Insulator Transitions, and Holographic Massive Gravity*, *Phys. Rev. Lett.* **114** (2015) 251602 [1411.1003].

[291] A. Donos and J.P. Gauntlett, *Holographic Q-lattices*, *JHEP* **04** (2014) 040 [1311.3292].

[292] Y. Bardoux, M.M. Caldarelli and C. Charmousis, *Shaping black holes with free fields*, *JHEP* **05** (2012) 054 [1202.4458].

[293] N. Iizuka, S. Kachru, N. Kundu, P. Narayan, N. Sircar and S.P. Trivedi, *Bianchi Attractors: A Classification of Extremal Black Brane Geometries*, *JHEP* **07** (2012) 193 [1201.4861].

[294] A. Donos and S.A. Hartnoll, *Interaction-driven localization in holography*, *Nature Phys.* **9** (2013) 649 [1212.2998].

[295] R.A. Davison, *Momentum relaxation in holographic massive gravity*, *Phys. Rev. D* **88** (2013) 086003 [1306.5792].

[296] Y. Ling, C. Niu, J. Wu, Z. Xian and H.-b. Zhang, *Metal-insulator Transition by Holographic Charge Density Waves*, *Phys. Rev. Lett.* **113** (2014) 091602 [1404.0777].

[297] N. Jokela, M. Jarvinen and M. Lippert, *Pinning of holographic sliding stripes*, *Phys. Rev. D* **96** (2017) 106017 [1708.07837].

[298] W.-J. Li and J.-P. Wu, *A simple holographic model for spontaneous breaking of translational symmetry*, 1808.03142.

[299] T. Andrade and A. Krikun, *Commensurability effects in holographic homogeneous lattices*, *JHEP* **05** (2016) 039 [1512.02465].

[300] T. Andrade and A. Krikun, *Commensurate lock-in in holographic non-homogeneous lattices*, *JHEP* **03** (2017) 168 [1701.04625].

[301] Y.P. Chen, *Quantum solids of two dimensional electrons in magnetic fields*, Ph.D. thesis, Princeton U, Dept. of Electrical Engineering, 2005. https://search.proquest.com/docview/305420029.

[302] M. Shayegan, *Case for the magnetic-field-induced two-dimensional wigner crystal*, in *Perspectives in Quantum Hall Effects*, pp. 343–384, John Wiley & Sons, Ltd (2007), https://onlinelibrary.wiley.com/doi/abs/10.1002/9783527617258.ch9.

[303] Schulz, H.J., *Domain walls in a doped antiferromagnet*, *J. Phys. France* **50** (1989) 2833.

[304] H.W. Jiang, R.L. Willett, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer and K.W. West, *Quantum liquid versus electron solid around ν=1/5 Landau-level filling*, *Phys. Rev. Lett.* **65** (1990) 633.

[305] H.W. Jiang, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer and K.W. West, *Magnetotransport studies of the insulating phase around ν=1/5 Landau-level filling*, *Phys. Rev. B* **44** (1991) 8107.

[306] V.J. Goldman, M. Santos, M. Shayegan and J.E. Cunningham, *Evidence for two-dimentional quantum Wigner crystal*, *Phys. Rev. Lett.* **65** (1990) 2189.

[307] F.I.B. Williams, P.A. Wright, R.G. Clark, E.Y. Andrei, G. Deville, D.C. Glattli et al., *Conduction threshold and pinning frequency of magnetically induced Wigner solid*, *Phys. Rev. Lett.* **66** (1991) 3285.

[308] H. Watanabe and H. Murayama, *Effective Lagrangian for Nonrelativistic Systems*, *Phys. Rev. X* **4** (2014) 031057 [1402.7066].

[309] H. Watanabe and T. Brauner, *On the number of Nambu-Goldstone bosons and its relation to charge densities*, *Phys. Rev. D* **84** (2011) 125013 [1109.6327].

[310] H. Watanabe and H. Murayama, *Unified description of nambu-goldstone bosons without lorentz invariance*, *Phys. Rev. Lett.* **108** (2012) 251602.

[311] Y. Hidaka, *Counting rule for nambu-goldstone modes in nonrelativistic systems*, *Phys. Rev. Lett.* **110** (2013) 091601.

[312] H. Fukuyama and P.A. Lee, *Pinning and conductivity of two-dimensional charge-density waves in magnetic fields*, *Phys. Rev. B* **18** (1978) 6245.

[313] B.G.A. Normand, P.B. Littlewood and A.J. Millis, *Pinning and conductivity of a two-dimensional charge-density wave in a strong magnetic field*, *Phys. Rev. B* **46** (1992) 3920.

[314] A. Amoretti, D. Arean, D.K. Brattan and N. Magnoli, *Hydrodynamic magneto-transport in charge density wave states*, *JHEP* **05** (2021) 027 [2101.05343].

[315] Y.P. Chen, G. Sambandamurthy, Z.H. Wang, R.M. Lewis, L.W. Engel, D.C. Tsui et al., *Melting of a 2d quantum electron solid in high magnetic field*, *Nature Physics* **2** (2006) 452.

[316] Y.P. Chen, G. Sambandamurthy, L.W. Engel, D.C. Tsui, L.N. Pfeiffer and K.W. West, *Microwave resonance study of melting in high magnetic field Wigner solid*, *International*

Journal of Modern Physics B **21** (2007) 1379.

[317] Y.P. Chen, G. Sambandamurthy, Z.H. Wang, R.M. Lewis, L.W. Engel, D.C. Tsui et al., *Melting of a 2d quantum electron solid in high magnetic field*, *Nature Physics* **2** (2006) 452–455.

[318] Y.P. Chen, G. Sambandamurthy, L.W. Engel, D.C. Tsui, L.N. Pfeiffer and K.W. West, *Microwave Resonance Study of Melting in High Magnetic Field Wigner Solid*, *International Journal of Modern Physics B* **21** (2007) 1379.

[319] L.V. Delacrétaz and S.A. Hartnoll, *Theory of the supercyclotron resonance and Hall response in anomalous two-dimensional metals*, *Phys. Rev. B* **97** (2018) 220506.

[320] A. Amoretti, D. Arean, D.K. Brattan and L. Martinoia, *Hydrodynamic magneto-transport in holographic charge density wave states*, *JHEP* **11** (2021) 011 [2107.00519].

[321] A. Donos, C. Pantelidou and V. Ziogas, *Incoherent hydrodynamics of density waves in magnetic fields*, *JHEP* **05** (2021) 270 [2101.06230].

[322] O. Gunnarsson, M. Calandra and J.E. Han, *Colloquium: Saturation of electrical resistivity*, *Reviews of Modern Physics* **75** (2003) 1085.

[323] N.E. Hussey, K. Takenaka and H. Takagi, *Universality of the Mott–Ioffe–Regel limit in metals*, *Philosophical Magazine* **84** (2004) 2847–2864.

[324] R.A. Cooper, Y. Wang, B. Vignolle, O.J. Lipscombe, S.M. Hayden, Y. Tanabe et al., *Anomalous criticality in the electrical resistivity of La2–xSrxCuO4*, *Science* **323** (2009) 603–607.

[325] A. Legros, S. Benhabib, W. Tabis, F. Laliberté, M. Dion, M. Lizaire et al., *Universal T-linear resistivity and planckian dissipation in overdoped cuprates*, *Nature Physics* **15** (2019) 142.

[326] F. Rullier-Albenque, A. Legris, H. Berger and L. Forro, *Effect of electron irradiation in Bi2Sr2CaCu2O8 and Bi2Sr2CuO6 superconductors*, *Physica C: Superconductivity* **254** (1995) 88–92.

[327] F. Rullier-Albenque, P.A. Vieillefond, J.F. Marucco, D. Colson, V. Viallet and P. Lejay, *Electron irradiation effects in YBa2Cu3O6+x and HgBa2Ca2Cu3O8+δ crystals*, *Physica C: Superconductivity* **282–287** (1997) 1199–1200.

[328] F. Rullier-Albenque, P.A. Vieillefond, H. Alloul, A.W. Tyler, P. Lejay and J.F. Marucco, *Universal Tc depression by irradiation defects in underdoped and overdoped cuprates*,

Europhysics Letters (EPL) **50** (2000) 81–87.

[329] F. Rullier-Albenque, H. Alloul and R. Tourbot, *Influence of pair breaking and phase fluctuations on disordered high T_c cuprate superconductors*, *Physical Review Letters* **91** (2003) 047001.

[330] T.R. Chien, Z.Z. Wang and N.P. Ong, *Effect of Zn impurities on the normal-state Hall angle in single-crystal $YBa_2Cu_3 - x\ Zn_xO_{7-\delta}$*, *Physical Review Letters* **67** (1991) 2088–2091.

[331] I.M. Hayes, N.P. Breznay, T. Helm, P. Moll, M. Wartenbe, R.D. McDonald et al., *Magnetoresistance near a quantum critical point*, *arXiv:1412.6484 [cond-mat]* (2014) .

[332] P. Giraldo-Gallo, J.A. Galvis, Z. Stegen, K.A. Modic, F.F. Balakirev, J.B. Betts et al., *Scale-invariant magnetoresistance in a cuprate superconductor*, *Science* **361** (2018) 479–481.

[333] J. Ayres, M. Berben, M. Culo, Y.-T. Hsu, E. van Heumen, Y. Huang et al., *Incoherent transport across the strange metal regime of highly overdoped cuprates*, *Nature* **595** (2021) 661–666.

[334] P.W. Anderson, *Hall effect in the two-dimensional Luttinger liquid*, *Physical Review Letters* **67** (1991) 2092–2094.

[335] P. Coleman, A.J. Schofield and A.M. Tsvelik, *Phenomenological Transport Equation for the Cuprate Metals*, *Physical Review Letters* **76** (1996) 1324–1327.

[336] P. Coleman, A.J. Schofield and A.M. Tsvelik, *How should we interpret the two transport relaxation times in the cuprates?*, *Journal of Physics: Condensed Matter* **8** (1996) 9985–10015.

[337] S. Licciardello, N. Maksimovic, J. Ayres, J. Buhot, M. Culo, B. Bryant et al., *Coexistence of orbital and quantum critical magnetoresistance in $FeSe_{1-x}S_x$*, *Physical Review Research* **1** (2019) 023011.

[338] M. Culo, C. Duffy, J. Ayres, M. Berben, Y.-T. Hsu, R.D.H. Hinlopen et al., *Possible superconductivity from incoherent carriers in overdoped cuprates*, *SciPost Phys.* **11** (2021) 12.

[339] S. Uchida, T. Ido, H. Takagi, T. Arima, Y. Tokura and S. Tajima, *Optical spectra of $La_{2-x}Sr_xCuO_4$: Effect of carrier doping on the electronic structure of the CuO_2 plane*, *Phys. Rev. B* **43** (1991) 7942.

[340] M.A. Quijada, D.B. Tanner, F.C. Chou, D.C. Johnston and S.W. Cheong, *Optical properties of single-crystal $la_2Cuo_{4+\delta}$*, *Phys. Rev. B* **52** (1995) 15485.

[341] M.A. Quijada, D.B. Tanner, R.J. Kelley, M. Onellion, H. Berger and G. Margaritondo, *Anisotropy in the ab-plane optical properties of* $Bi_2Sr_2CaCu_2O_8$ *single-domain crystals*, *Phys. Rev. B* **60** (1999) 14917.

[342] D.v.d. Marel, H.J.A. Molegraaf, J. Zaanen, Z. Nussinov, F. Carbone, A. Damascelli et al., *Powerlaw optical conductivity with a constant phase angle in high Tc superconductors*, *Nature* **425** (2003) 271–274 [cond-mat/0309172].

[343] J. Hwang, T. Timusk and G.D. Gu, *Doping dependent optical properties of Bi2Sr2CaCu2O8+d*, *Journal of Physics: Condensed Matter* **19** (2007) 125208.

[344] E. Wahlberg, R. Arpaia, G. Seibold, M. Rossi, R. Fumagalli, E. Trabaldo et al., *Restored strange metal phase through suppression of charge density waves in underdoped* $YBa_2Cu_3O_7 - \delta$, *Science* **373** (2021) 1506 [2009.08398].

[345] L. Taillefer, *Scattering and pairing in cuprate superconductors*, *Annual Review of Condensed Matter Physics* **1** (2010) 51 [https://doi.org/10.1146/annurev-conmatphys-070909-104117].

[346] D.N. Basov, B. Dabrowski and T. Timusk, *Infrared Probe of Transition from Superconductor to Nonmetal in* $YBa_2(Cu_{1-x}Zn_x)_4O_8$, *Phys. Rev. Lett.* **81** (1998) 2132.

[347] A. Suchaneck, V. Hinkov, D. Haug, L. Schulz, C. Bernhard, A. Ivanov et al., *Incommensurate Magnetic Order and Dynamics Induced by Spinless Impurities in* $YBa_2Cu_3\mathbf{O}_{6.6}$, *Phys. Rev. Lett.* **105** (2010) 037207.

[348] Z. Guguchia, B. Roessli, R. Khasanov, A. Amato, E. Pomjakushina, K. Conder et al., *Complementary Response of Static Spin-Stripe Order and Superconductivity to Nonmagnetic Impurities in Cuprates*, *Phys. Rev. Lett.* **119** (2017) 087002.

[349] P.M. Lozano, G.D. Gu, J.M. Tranquada and Q. Li, *Experimental evidence that zinc impurities pin pair-density-wave order in* $La_{2-x}Ba_xCuO_4$, *Phys. Rev. B* **103** (2021) L020502.

[350] A. Amoretti, M. Meinero, D.K. Brattan, F. Caglieris, E. Giannini, M. Affronte et al., *Hydrodynamical description for magneto-transport in the strange metal phase of Bi-2201*, *Phys. Rev. Res.* **2** (2020) 023387 [1909.07991].

[351] A. Karch, K. Limtragool and P.W. Phillips, *Unparticles and Anomalous Dimensions in the Cuprates*, *JHEP* **03** (2016) 175 [1511.02868].

[352] G. La Nave, K. Limtragool and P.W. Phillips, *Fractional Electromagnetism in Quantum*

Matter and High-Energy Physics, *Rev. Mod. Phys.* **91** (2019) 021003 [1904.01023].

[353] R.A. Davison, K. Schalm and J. Zaanen, *Holographic duality and the resistivity of strange metals*, *Phys. Rev. B* **89** (2014) 245116 [1311.2451].

[354] M. Blake and A. Donos, *Quantum Critical Transport and the Hall Angle*, *Phys. Rev. Lett.* **114** (2015) 021601 [1406.1659].

[355] A. Amoretti, M. Baggioli, N. Magnoli and D. Musso, *Chasing the cuprates with dilatonic dyons*, *JHEP* **06** (2016) 113 [1603.03029].

[356] E. Blauvelt, S. Cremonini, A. Hoover, L. Li and S. Waskie, *Holographic model for the anomalous scalings of the cuprates*, *Phys. Rev. D* **97** (2018) 061901 [1710.01326].

[357] J. Zhang, E.M. Levenson-Falk, B.J. Ramshaw, D.A. Bonn, R. Liang, W.N. Hardy et al., *Anomalous thermal diffusivity in underdoped yba2cu3o6+x*, *Proceedings of the National Academy of Sciences* **114** (2017) 5378 [https://www.pnas.org/content/114/21/5378.full.pdf].

[358] J. Zhang, E.D. Kountz, E.M. Levenson-Falk, D. Song, R.L. Greene and A. Kapitulnik, *Thermal diffusivity above the mott-ioffe-regel limit*, *Phys. Rev. B* **100** (2019) 241114.

[359] K. Behnia and A. Kapitulnik, *A lower bound to the thermal diffusivity of insulators*, *Journal of Physics: Condensed Matter* **31** (2019) 405702.

[360] C.H. Mousatov and S.A. Hartnoll, *On the Planckian bound for heat diffusion in insulators*, *Nature Physics* **16** (2020) 579–584 [1908.04792].

[361] D.T. Son, *Low-energy quantum effective action for relativistic superfluids*, hep-ph/0204199.

ECOLE DOCTORALE

Titre : Holographie, hydrodynamique et systèmes de matière condensée fortement corrélés

Mots clés : Théories des champs effectives; dualité entre théories de jauge et de gravitation; brisure spontanée de symétrie; métaux étranges.

Résumé : Dans les systèmes de matière condensée conventionnels, les interactions sont faibles et les excitations à basses énergies formées de quasiparticules, qui sont simplement des électrons habillés par les interactions. Les métaux ordinaires sont donc bien décrits par la théorie du liquide de Fermi, de même que les supraconducteurs ordinaires par la théorie de Bardeen-Cooper-Schrieffer. Toutefois, lorsque les interactions sont fortes, les quasiparticules ont une durée de vie très courte et ne constituent pas une base appropriée pour la description de la dynamique aux basses énergies. L'hydrodynamique et ses extensions offrent une autre perspective, dont l'hypothèse de départ est que l'équilibre local s'établit rapidement. On peut alors caractériser le système par la dynamique d'un petit nombre de variables collectives avec une longue durée de vie, qui découlent des symétries présentes.
La dualité entre gravitation et théorie de jauge suppose qu'un état thermal à densité non-nulle dans une théorie des champs fortement couplée peut être décrit de manière équivalente par un trou noir chargé. Cela permet donc de reformuler la dynamique d'un système fortement couplé comme un problème gravitationnel de résolution des équations d'Einstein en présence de champs de matière dans les espaces-temps dits d'Anti de Sitter.
Dans cette thèse, nous passons en revue la manière dont ces approches permettent de mieux comprendre la dynamique au voisinage de points critiques quantiques, ainsi que le mariage des brisures spontanée et explicite des translations spatiales dans les phases électroniques d'ordre de charge et de solides de Wigner, tout en soulignant la manière dont ces résultats ont émergé d'un dialogue fructueux entre la théorie des champs et la gravité. Pour conclure, nous examinons de manière critique l'hypothèse que la brisure pseudo-spontanée des translations spatiales soit à l'origine des propriétés de transport de chaleur et de charge dans les phases de métaux étranges, dont les fondements théoriques restent largement mystérieux plus de trente-cinq ans après leur découverte dans les supraconducteurs à haute température critique.

Title : Holography, hydrodynamics and strongly-correlated condensed matter

Keywords : Effective field theories; gauge/gravity duality; spontaneous symmetry breaking; strange metals

Abstract : In conventional condensed matter systems, interactions are weak and the low-energy excitations are quasiparticles, essentially electrons dressed by weak interactions. Ordinary metals are well-described by Fermi liquid theory and ordinary superconductors by Bardeen-Cooper-Schrieffer theory. On the other hand, when interactions are strong, quasiparticles are short-lived and are not a good starting point for the low-energy dynamics. Hydrodynamics and its extensions provides an alternative perspective that starts from the assumption of fast local equilibration. The system is then characterized by the dynamics of a few long-lived, collective variables following from the symmetries.
Gauge/gravity duality posits that a thermal state at nonzero density in a strongly-coupled quantum field theory is dual to a charged black hole. The strongly-coupled dynamics of a quantum system are then mapped to a gravitational problem where the goal is to solve Einstein's equations coupled to matter in Anti de Sitter spacetimes.
In this thesis, we review how these methods may be used to gain insight on the dynamics in the vicinity of quantum critical points, as well as on the interplay between spontaneous and explicit breaking of spatial translation symmetry in electronic charge density waves and Wigner solids, emphasising how progress was made through a fruitful dialogue between field theory and gravity. We close by a critical discussion on how pseudo-spontaneous breaking of translations may inform us on transport phenomenology in strange metallic phases, whose theoretical underpinnings remain unexplained over thirty-five years after their discovery in high T_c superconductors.

Institut Polytechnique de Paris
91120 Palaiseau, France

ECOLE DOCTORALE

Titre : Holographie, hydrodynamique et systèmes de matière condensée fortement corrélés

Mots clés : Théories des champs effectives; dualité entre théories de jauge et de gravitation; brisure spontanée de symétrie; métaux étranges.

Résumé : Dans les systèmes de matière condensée conventionnels, les interactions sont faibles et les excitations à basses énergies formées de quasiparticules, qui sont simplement des électrons habillés par les interactions. Les métaux ordinaires sont donc bien décrits par la théorie du liquide de Fermi, de même que les supraconducteurs ordinaires par la théorie de Bardeen-Cooper-Schrieffer. Toutefois, lorsque les interactions sont fortes, les quasiparticules ont une durée de vie très courte et ne constituent pas une base appropriée pour la description de la dynamique aux basses énergies. L'hydrodynamique et ses extensions offrent une autre perspective, dont l'hypothèse de départ est que l'équilibre local s'établit rapidement. On peut alors caractériser le système par la dynamique d'un petit nombre de variables collectives avec une longue durée de vie, qui découlent des symétries présentes.
La dualité entre gravitation et théorie de jauge suppose qu'un état thermal à densité non-nulle dans une théorie des champs fortement couplée peut être décrit de manière équivalente par un trou noir chargé. Cela permet donc de reformuler la dynamique d'un système fortement couplé comme un problème gravitationnel de résolution des équations d'Einstein en présence de champs de matière dans les espaces-temps dits d'Anti de Sitter.
Dans cette thèse, nous passons en revue la manière dont ces approches permettent de mieux comprendre la dynamique au voisinage de points critiques quantiques, ainsi que le mariage des brisures spontanée et explicite des translations spatiales dans les phases électroniques d'ordre de charge et de solides de Wigner, tout en soulignant la manière dont ces résultats ont émergé d'un dialogue fructueux entre la théorie des champs et la gravité. Pour conclure, nous examinons de manière critique l'hypothèse que la brisure pseudo-spontanée des translations spatiales soit à l'origine des propriétés de transport de chaleur et de charge dans les phases de métaux étranges, dont les fondements théoriques restent largement mystérieux plus de trente-cinq ans après leur découverte dans les supraconducteurs à haute température critique.

Title : Holography, hydrodynamics and strongly-correlated condensed matter

Keywords : Effective field theories; gauge/gravity duality; spontaneous symmetry breaking; strange metals

Abstract : In conventional condensed matter systems, interactions are weak and the low-energy excitations are quasiparticles, essentially electrons dressed by weak interactions. Ordinary metals are well-described by Fermi liquid theory and ordinary superconductors by Bardeen-Cooper-Schrieffer theory. On the other hand, when interactions are strong, quasiparticles are short-lived and are not a good starting point for the low-energy dynamics. Hydrodynamics and its extensions provides an alternative perspective that starts from the assumption of fast local equilibration. The system is then characterized by the dynamics of a few long-lived, collective variables following from the symmetries.
Gauge/gravity duality posits that a thermal state at nonzero density in a strongly-coupled quantum field theory is dual to a charged black hole. The strongly-coupled dynamics of a quantum system are then mapped to a gravitational problem where the goal is to solve Einstein's equations coupled to matter in Anti de Sitter spacetimes.
In this thesis, we review how these methods may be used to gain insight on the dynamics in the vicinity of quantum critical points, as well as on the interplay between spontaneous and explicit breaking of spatial translation symmetry in electronic charge density waves and Wigner solids, emphasising how progress was made through a fruitful dialogue between field theory and gravity. We close by a critical discussion on how pseudo-spontaneous breaking of translations may inform us on transport phenomenology in strange metallic phases, whose theoretical underpinnings remain unexplained over thirty-five years after their discovery in high T_c superconductors.

Institut Polytechnique de Paris
91120 Palaiseau, France