



HAL
open science

Fixpoints of Types in Linear Logic from a Curry-Howard-Lambek Perspective

Farzad Jafarrahmani

► **To cite this version:**

Farzad Jafarrahmani. Fixpoints of Types in Linear Logic from a Curry-Howard-Lambek Perspective. Logic in Computer Science [cs.LO]. Université Paris Cité, 2023. English. NNT : . tel-04523738v1

HAL Id: tel-04523738

<https://hal.science/tel-04523738v1>

Submitted on 20 Nov 2023 (v1), last revised 27 Mar 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Université Paris Cité
École Doctorale 386 - Sciences Mathématiques de Paris Centre
Institut de recherche en informatique fondamentale (IRIF)

Fixpoints of Types in Linear Logic from a Curry-Howard-Lambek Perspective

par Farzad Jafarrahmani

Thèse de doctorat de mathématiques, logique et fondements de l'informatique dirigée par

Thomas EHRHARD et Alexis SAURIN

Jury

Thomas EHRHARD	DR CNRS & Université Paris Cité	Directeur de thèse
Alexis SAURIN	CR CNRS & Université Paris Cité	Directeur de thèse
Nicola GAMBINO	PR, University of Manchester	Rapporteur
Olivier LAURENT	DR CNRS & École Normale Supérieure de Lyon	Rapporteur
Marcelo FIORE	PR, University of Cambridge	Examinateur
Daniela PETRISAN	MC, Université Paris Cité	Examinatrice
Ralph MATTHES	CR CNRS & Université de Toulouse	Examinateur
Christine TASSON	PR, Sorbonne Université	Examinatrice

Présentée et soutenue publiquement le 25 Janvier 2023.

Resume

Cette thèse porte sur l'étude d'une extension de la logique linéaire propositionnelle avec des points fixes de type dans une perspective Curry-Howard-Lambek. La logique linéaire à points fixes de types, appelée μLL , nous permet d'avoir des preuves inductives et coinductives. Nous développons une sémantique catégorielle de μLL basée sur les catégories de Seely et sur des foncteurs "strong" agissant sur elles. Ensuite, nous introduisons et étudions μLLP comme une extension de la logique linéaire polarisée, avec plus petit et plus grand points fixes. Profitant des règles structurelles implicites de μLLP , nous introduisons une syntaxe de terme pour ce langage, dans l'esprit du λ -calcul classique et du système L. Nous équipons ce système logique d'une sémantique de réduction déterministe ainsi que d'une sémantique catégorielle. Nous examinons toujours notre sémantique catégorielle avec des cas concrets tels que la catégorie des ensembles et des relations, la catégorie des ensembles munis d'une notion de totalité (espaces de totalité non uniformes) et des relations qui préservent a totalité, et les espaces de cohérence avec totalité. Dans le cas de μLLP , nous prouvons un résultat d'adéquation pour μLLP entre sa sémantique opérationnelle et dénotationnelle, dont nous dérivons une propriété de normalisation grâce aux propriétés de l'interprétation de la totalité. Nous étudierons également les preuves non bien fondées en logique linéaire, que l'on peut voir comme une extension des preuves inductives, d'un point de vue sémantique dénotationnelle en faisant une relation entre condition de validité des preuves non bien fondées et interprétation de la totalité. Enfin, nous fournirons un modèle catégoriel pour les exponentielles codées à l'aide de l'opérateur de point fixe.

Keywords— logique linéaire, plus petit et plus grand points fixes, lambda-calcul classique, sémantique catégorielle, Sémantique dénotationnelle, preuve non bien fondés, logique linéaire polarisée

Abstract

This thesis is concerned with the studying of an extension of the propositional linear logic with fixpoints of type from a Curry-Howard-Lambek perspective. Linear logic with fixpoints of types, called μLL , allows us to have inductive and coinductive proofs. We develop a categorical semantics of μLL based on Seely categories and on strong functors acting on them. Then we introduce and study μLLP as an extension of Polarized Linear Logic, with least and greatest fixpoints. Taking advantage of the implicit structural rules of μLLP , we introduce a term syntax for this language, in the spirit of the classical λ -calculus and of system L. We equip this logical system with a deterministic reduction semantics as well as a categorical semantic. We always examine our categorical semantics with concrete cases such as the category of sets and relations, category of sets equipped with a notion of totality (non-uniform totality spaces) and relations preserving, and coherence spaces with totality. In the case of μLLP , we prove an adequacy result for μLLP between its operational and denotational semantics, from which we derive a normalization property thanks to the properties of the totality interpretation. We will also study non-wellfounded proofs in linear logic, which one can see as an extension of inductive proofs, from a denotational semantics point of view by making a relation between the validity condition for non-wellfounded proofs and totality interpretation. Finally, we will provide a categorical setting for the exponentials that are encoded using the fixpoints operator.

Keywords— linear logic, least and greatest fixed points, classical lambda-calculus, categorical semantics, denotational semantics, non-wellfounded proof, polarized linear logic

تقدیم به همسر

Acknowledgments

Foremost I would like to thank my supervisors Thomas Ehrhard and Alexis Saurin. I got in touch with Thomas Ehrhard in 2018 for doing my first master internship. When I started to work under his supervision, he taught me a great deal of denotational semantics. Afterward in 2019, I decided to work on the semantics of linear logic with fixpoints, so I asked Thomas Ehrhard and Alexis Saurin to work under their supervision, and they graciously accepted me as a PhD student. Discussions with them have been a tremendous help and influence, and they always had enormous patience with my questions and ideas. It was, and still is, very delightful to hear them once they talk passionately together in our meetings. So, it was a great opportunity for me to have both of them so that I could really see many deep and insightful discussions on both semantics and syntax at the same time ¹. Apart from science, they also taught me a lot about academia and its environment, and they were always also so kind to hear about my personal issue. Of course, I can not express in words how much I am so thankful to them. However, I will always remain in intellectual debt to both of them.

I am appreciative to Nicola Gambino and Olivier Laurent for having examined this manuscript, and a special thanks to Ralph Matthes for providing me with his comments about this manuscript.

In addition, I would like to thank you IRIF members to have interesting discussions on different topics related to my work and their field, and a special thanks to Pierre-Louis Curien, Hugo Herbelin, and Paul-André Melliès to talk about type theory and category theory. I also have been fortunate to have Abhishek De and Félix Castro as both my co-bureau and my friends. In addition to our friendship, I learned a lot from them about linear logic, realizability, set theory, etc.

I am also most grateful to Marcelo Fiore for hosting my visit at the university of Cambridge which was during my PhD. Apart from learning very interesting topics in category theory from him, working with him also helped me a lot to understand more my work. It was also a pleasure to talk with Zeinab Galal during that visit, and she introduced me 2-categorical model of linear logic.

A very special thanks to Noam Zeilberger: I did my second master internship in 2019 under his supervision. He taught me the fascinating topic of Fibred category, and also his work with Paul-André Melliès on Type Refinement Systems. During my PhD, I had a great opportunity to continue working with him on categorical models of circular definitions and proofs.

I am also so thankful to Mohammad Ardeshir and Mohammad Gholamzadeh Mahmoudi

¹of course, I just learned part of their insights

from whom I got familiar and learned mathematical logic and algebra. I did my bachelor thesis under supervision of Mohammad Gholamzadeh Mahmoudi on Ordered Field.

For friendship, I am grateful to Erfan Khaniki, Amirreza Laleh, Ahmadreza Rahimi, and Amir Azizi JirAbadi. Aside from having so many pleasant and unforgettable memories with them, discussion with them was always so instructive, and I learned a lot from them during my undergraduate study.

Last but not least, I want to thank my family for everything. Thanks to my parent for helping and supporting me to pursue my studies from the very beginning. It was always hard for them as I did almost half my studies far from home, so thanks for their patience. Thanks to Darya (my wife) for supporting me during my PhD. It is not always easy to live with a PhD student, so I am so thankful to her for her patience and moreover giving me energy and motivation during the hard times of my PhD.

Contents

Introduction	10
0 Background on denotational and categorical semantics of linear logic	25
0.1 Short history on categorical and denotational semantics	25
0.1.1 On categorical semantics	25
0.1.2 On denotational semantics	26
0.2 Linear logic with fixpoints	26
0.2.1 Syntax of LL	27
0.2.2 Syntax of μ LL	33
0.3 Categorical semantics of LL	37
0.3.1 Seely categories	38
0.3.2 Oplax monoidal comonads	39
0.3.3 Eilenberg-Moore category	39
0.3.4 Interpretation of LL proofs and formulas	40
0.4 Concrete models of LL	42
0.4.1 Sets and relations	42
0.4.2 Coherence spaces	43
0.4.3 Non-uniform coherence spaces	44
1 Categorical and denotational semantics of finitary linear logic with fixpoints (μLL)	48
1.1 Categorical semantics of finitary linear logic with fixpoints	48
1.1.1 The LL model of free comodules on a given coalgebra.	48
1.1.2 Strong functors on \mathcal{L}	49
1.1.3 A categorical axiomatization of models of μ LL	54
1.2 Denotational semantics of finitary linear logic with fixpoints	56
1.2.1 Rel as model of μ LL	56
1.2.2 Non-uniform totality spaces	60

CONTENTS

1.2.3	Coherence spaces with totality	69
1.3	Sum up of Chapter 1	77
2	Polarized LL with fixpoints and its semantics	78
2.1	A classical calculus for polarized linear logic with fixpoints ($\kappa\mu\text{LLP}$)	78
2.1.1	Term assignment for $\mu\text{LLP}_{\text{imp}}$	81
2.1.2	Substitution of terms in terms	83
2.1.3	Substitution of terms in formulas	85
2.1.4	Reduction relation	89
2.2	Normalization of $\kappa\mu\text{LLP}$	95
2.2.1	Failed attempts	96
2.2.2	An intersection typing system	102
2.2.3	The point typing system	104
2.2.4	Example: the integers	113
2.3	Categorical semantics of $\kappa\mu\text{LLP}$	116
2.3.1	Two concrete models of $\kappa\mu\text{LLP}$	128
2.4	Application: a normalization property	131
2.5	Sum up of Chapter 2	133
3	Non-well-founded LL with fixpoints and its denotational semantics	134
3.1	Non-well-founded LL with fixpoints (μLL_{∞})	135
3.2	Denotational semantics of μLL_{∞}	139
3.2.1	Soundness	140
3.2.2	On relation between totality semantics and syntactic validity	143
3.3	Inductive vs circular linear logic proofs	149
3.4	Sum up of Chapter 3	151
4	The magmatic quasi-exponential	152
4.1	A categorical setting for Tree exponentials	153
4.1.1	Functoriality of a FMQE	153
4.1.2	Comonadicity of an FMQE	154
4.1.3	Monoidality of an FMQE	155
4.1.4	Seely morphisms	159
4.2	Concrete models of LL with a tree-based comonad	160
4.2.1	Relational model of LL with tree exponentials	161
4.2.2	Coherence spaces with tree exponentials	165
4.3	Local Totality Spaces	167

4.3.1	COHLT is a *-autonomous category with finite product	168
4.3.2	Exponentials in COHLT	170
4.4	Sum up of Chapter 4	173
5	Conclusion and future work	174
5.1	A general categorical framework for computing recursive types in Nuts . . .	174
5.2	Full normalization of $\kappa\mu\text{LLP}$	175
5.3	Adding the general fixpoint operator to $\kappa\mu\text{LLP}$	175
5.4	Categorical model for non-well-founded linear (μLL_∞)	175
5.5	Investigation on Kleisli category of tree exponential comonad	176
5.6	Sequentiality and tree exponential semantic	176
	Bibliography	177
	Introduction en français	192
	Takeaway of this manuscript	200

Introduction

In logic we must reject all distinctions that are made from a purely psychological point of view. What is referred to as a deepening of logic by psychology is nothing but a falsification of it by psychology.

Gottlob Frege, Posthumous Writing

Logicism

The main project of Gottlob Frege in the foundation of mathematics, which nowadays is called “logicism”, is indeed an important starting point of contemporary logic. His first fundamental distinction in *Begriffsschrift* is between “content” and “judgement”, and this distinction separates Frege from many traditional discussions of logic ². In [Fre84], Frege presented a symbolic language as a foundation of arithmetic. Later on, Giuseppe Peano provided a formalization of logical inference in order to express formally proofs in arithmetic [vH67]. Russell also gave an axiomatic system in [Rus06], and he used the notation and formal rules of proof of Peano. David Hilbert also tried to provide a foundation of mathematics by developing a new system for mathematical proofs [Toe05]. We can find in [HA28] what is called nowadays the “Hilbert style” axiomatic system for predicate logic. The “Hilbert style” axiomatic system for predicate logic includes some axioms and two inference rules. A proof of a proposition A in the Hilbert system is a finite sequence A_1, \dots, A_n of propositions such that $A_n = A$ and for $i \leq n$, either A_i is an axiom or A_i is derived from A_j 's using the inference rules for $j < i$. The two inference rules of predicate logic in his system are the following where (MP) and (G) stand for Modus Ponens and Generality respectively:

$$\frac{A \quad A \rightarrow B}{B} \text{ (MP)} \quad \frac{A}{\forall x A(x)} \text{ (G)}$$

In the (G) rule, one should be careful that the variable x should not be free in all the formulas that are used to prove A . Hilbert wanted to have a sound and complete system. A system is called sound if all derivable formulas of that system are “true”. A system is called complete whenever a formula is true, then it should be also derivable in the logical system. Apart from soundness and completeness, Hilbert was also interested in questions of consistency and

²based on a lecture note by Jeff Speaks about Frege's *Begriffsschrift*. We refer interested readers to [FB97, KK95] to see some commentaries on Frege's work

decidability of the formal systems. Afterward, Kurt Gödel proved that it is impossible to have a complete formal system of elementary arithmetic [Göd31]. However, the question of consistency and completeness of predicate logic was still open, and Gerhard Gentzen started to work on the question of consistency of arithmetic. He first started working on the analysis of mathematical proofs as they are in practice. And this study led him to arrive at a calculus of natural deduction [Gen35]. One of his observations was to divide the actual proofs of theorems into two parts of assumptions and conclusions. Then analysing the conclusion was done based on what Gentzen called introduction rules, and analysing the assumptions was done based on what Gentzen called elimination rules [Gen35]. Introduction rules provide sufficient conditions for deriving a conclusion of a given form. As an example, it is enough to derive A and B separately to derive $A \wedge B$. Gentzen wrote this inference rule formally as follows and he called it the introduction of conjunction connective \wedge :

$$\frac{A \quad B}{A \wedge B}$$

The elimination rules will tell us what an immediate use of an assumption is. For example, if we assume that we have $A \wedge B$, then an immediate conclusion of this assumption can be A and B . So, we have the two following rules in the natural deduction that Gentzen called elimination of conjunction connective \wedge :

$$\frac{A \wedge B}{A} \quad \frac{A \wedge B}{B}$$

As another example of rules in the natural deduction, let us look at the implication. To conclude an implication $A \rightarrow B$, we first assume A and then try to derive B . However, we should notice that the assumption A is not a permanent assumption, so, this assumption will be “discharged”. We can use the assumption $A \rightarrow B$ if a deviation A has been found. So, the introduction and elimination rules of implication are as follows:

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} \quad \frac{A \quad A \rightarrow B}{B}$$

Gentzen noticed that if an introduction rule is followed by the corresponding elimination rule to prove a proposition A , then these two consecutive introduction and elimination rules seem unnecessary to prove A , and he called such introduction-elimination pattern “detours”. As two examples consider the following proofs of $A \wedge B$ and $A \rightarrow B$:

$$\frac{\frac{A \wedge B}{A} \quad B}{A \wedge B} \quad \frac{\frac{A \rightarrow B \quad [A]}{B}}{A \rightarrow B}$$

And the so-called detour conversion will remove such elimination-introduction pattern. This procedure is also called normalization, as it tries to convert non-normal proofs (those has detour) to the normal ones (no detour pattern in the proof). The normalization theorem is as follows: If a formula A has a proof, it also has a normal proof. Gentzen proved the normalization theorem for the intuitionistic natural deduction in [Gen35] and he wrote that he was not able to prove normalization for the classical natural deduction [PG08]³. If we

³According to [PG08], there are apparently two versions of Gentzen’s PhD thesis; the published one and the handwritten version. The handwritten version contains the detailed proof of normalization for the standard system of natural deduction that we are nowadays familiar with, and it seems this was not mentioned by people even a priori by Paul Bernays who has the handwritten version [PG08]

look at the introduction rules of conjunction and implication, we will notice that premisses (the formulas above the inference line) are subformulas of the conclusion, and the situation is converse for the elimination rules. Gentzen noticed that in normal proofs, this property, which is called sub-formula property, holds for the whole proof. That is to say, all formulas are subformulas of the conclusion [Gen35]. One can derive the consistency of a logical system using the normalization theorem and the subformula property as follows: Assume that the proof system is not consistent. Then by definition, it proves any formula and especially atomic formulas. Then the atomic formulas have also normal proofs. But there is neither an elimination rule nor an introduction rule on an atomic formula. So, in this way, one can obtain a syntactic proof of consistency for a logical system. However, we have not yet talked about the consistency of arithmetic. Gentzen tried to extend the natural deduction with a rule that corresponds to the principle of induction. And he wanted to derive consistency from the normalization and the subformula properties. But, he realized that this proof will not go through, as he saw it is impossible to have subformula property in the arithmetic [PG08]. Afterward, Gentzen developed another logical calculus, and he called it “Sequenzenkalkul” (sequent calculus). A sequent consists of a list Γ of formulas, a turnstile \vdash (original an arrow by Gentzen), and another list Δ of formulas, i.e., $\Gamma \vdash \Delta$. Take $\Gamma = A_1, \dots, A_k$ and $\Delta = B_1, \dots, B_l$, then one can read a sequent $\Gamma \vdash \Delta$ as follows: “The assumption $A_1 \wedge \dots \wedge A_k$ implies the conclusion $B_1 \vee \dots \vee B_l$ ”. In sequent calculus, we have two introduction rules (called left and right introduction rules) for any connective in contrast with the natural deduction we have introduction and elimination rules. Let us consider the case of conjunction and implication. The right introduction rule for conjunction is as follows:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

And the left introduction rules for conjunction are

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \quad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

One can try to interpret these rules with the reading of a sequent $\Gamma \vdash \Delta$ which is provided above. The left and right introduction rules for implication are as follows:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \quad \frac{\Gamma \vdash \Gamma', A \quad B, \Delta \vdash \Gamma''}{A \rightarrow B, \Gamma, \Delta \vdash \Gamma', \Gamma''}$$

Gentzen denoted by LK (respectively LJ) the classical (respectively intuitionistic) system in the sequent calculus style⁴. Similar to the detour in the natural deduction which was somehow implicit, there is a notion of the detour in the sequent calculus. However, this notion of detour explicitly appears in the sequent calculus as an inference rule which is called (cut) rule:

$$\frac{\Gamma \vdash A, \Delta \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Similar to the normalization theorem in the natural deduction, there is a theorem in sequent calculus that is called “cut elimination” theorem nowadays. This theorem says that if a sequent $\Gamma \vdash \Delta$ has a proof, then it also has a proof without using of (cut) rule. Those proofs are called cut-free proofs. Gentzen showed the cut elimination theorem for LK (and

⁴The rules that are presented above are actually the rules of a classical system. In the sequent calculus, there is a very nice characterization of intuitionistic system saying that a system is intuitionistic if there exists at most one formula on the right-hand side of \vdash .

consequently for LJ). Gentzen continued his plan on proving the consistency of arithmetic after working on the natural deduction and the sequent calculus in his thesis. And he proved the consistency of arithmetic [Gen39] using the now famous principle of transfinite induction up to the first epsilon-number ⁵. To see more details and other discussions on proof theory, we refer a reader to excellent and exciting references on proof theory such as [Bus98, Gir87b, GLT89, NvPR01, Pra91, vP13, TS00].

Type theory

Back in the 19th century, Georg Cantor in [CC74] launched a branch of mathematics called set theory. In that paper, he first proved that the collection of all real algebraic numbers can be placed into a one-to-one correspondence with the collection of all natural numbers. Then he proved that the real numbers cannot be put into a one-to-one correspondence with the natural numbers ⁶. After this discovery of Cantor, which is also called linear continuum, the concept of set theory started to evolve. Early on, some paradoxes arose in naive set theory; in particular the paradox by Bertrand Russell (and also by Ernst Zermelo) which is known as the paradox of self-reference. This issue was basically solved in two ways: Developing an axiomatic set theory by Ernst Zermelo and Abraham Fraenkel [Zer08, Fra22], and developing of type theory by Bertrand Russell and Alfred North Whitehead [WR13] which was simplified by Leon Chwistek [Chw22] and Frank Plumpton Ramsey [Ram31]. One can look at types as ranges of predicates meaning that for a given predicate $P(X)$ there is a class of objects, called type of X , such that the predicate $P(X)$ can only apply to members of type of X . One can also look at types as sets. Although one of the motivations was trying to solve the paradoxes of naive set theory, one can see a more conceptual purpose in type theory which was looking for a secure foundation of mathematics, and we can maybe say that this project is really started from Gottlob Frege.

Simple Theory of Types and λ -calculus

Along these lines of looking at foundations of mathematics, Alonzo Church in [Chu32], as he mentioned at the very beginning of his paper, provided a set of postulates for the foundation of formal logic. His system was based on function rather than set. Church first defined what he called “undefined terms”. Among the undefined terms, one can find these symbols: $\{\}()$, $\lambda[]$, Π , Σ , and $\&$, and he also allows to have variables. Then he defined the notion *well-formed* formula inductively: (1) variables are well-formed (2) Π , Σ , and $\&$ are well-formed (3) if M and N are well-formed then $\{M\}(N)$ is well-formed (4) if x is a variable and M is well-formed then $\lambda x[M]$ is well-formed. Although those are called undefined terms, Church had (as it is mentioned by himself in page 8 of [Chu32]) an intended meaning for them which was coming from a functional view. He says that if F is a function and A is a value, then $\{F\}(A)$ represents the application of the function F on the value A . He also uses the idea of Schonfinkel [Sch] to deal with multi variables functions which is now called Currying function. Then we will

⁵Gentzen actually presented different proofs for the consistency of arithmetic in different years, as there was some discussions by Gödel and Bernay on each of them [vP18]

⁶Actually, he first proved that there are infinitely many numbers in any interval $[a, b]$ (Cantor’s notation for interval was $(a \cdots b)$). Then, as two corollaries, first he proved there are infinitely many transcendental numbers in any interval. And secondly, real numbers cannot be written as an infinite sequence.

also see his postulates in order to do reasoning in the system. There we can find basically the first place he provides his conversion rules such as converting the formula $\{\lambda x[M]\}(N)$ into formula $S_N^x M$ which denoted the result of substituting N in M for x . This particular conversion is now called β reduction. Although Stephen Cole Kleene and John Barkley Rosser proved that Church's system was inconsistent [KR35], his system lead to introduce λ -calculus and simple theory of type in [Chu40, Chu41]. What we see in [Chu40] seems closer to what we nowadays called Higher-order sorted logic. Church first explained his hierarchy of types in [Chu40]: The class of type symbols is the least class of symbols which contains the symbols ι and o and is closed under the operation of $(\alpha\beta)$ from symbols α and β ⁷. In a similar way as he did in [Chu32], he defined his notion of well-defined formulas that were based on the notion of primitive symbols which were the same as undefined terms with the difference that now they have type as their subscript. Among the primitive symbols, one can see $\lambda, N_{oo}, A_{ooo}, \Pi_{o(o\alpha)}$. He called λ the improper symbols, and the others ones the proper symbols. Well-formed formulas is now defined inductively as follows: (1) proper symbols are well-formed (2) If x_β is a variable and M_α is a well-formed formula, then $(\lambda x_\beta M_\alpha)_{(\alpha\beta)}$ is a well-formed formula (3) If $F_{\alpha\beta}$ and B_β are well-formed formulas, then $(F_{\alpha\beta} B_\beta)_\alpha$ is a well-formed formula. As an example, the formula $B_o \vee C_o$ can be written as $A_{ooo} B_o C_o$, and the formula $\neg B_o$ can be written as $N_{oo} B_o$. This work is basically the first place that we can see a typed system based on a λ language which is based on the notion of functions and variables. However, the simple type theory is not the only outcome of Church's work in [Chu32]. The underlying pure λ -calculus was also another outcome. The very early work on the pure λ -calculus was begun to study its conversion theory (now called reduction theory) in [CR36] which Church and Rosser proved confluence theorem using the method of residuals. Further studies on λ -calculus were made in [Chu36a, Chu36b, Pe37] that introduce the notion of λ -definable function and prove that this notion coincides with the Herbrand-Gödel recursive functions. This also answered negatively the question that we mentioned above by Hilbert about the decidability of predicate logic. Church first showed that the convertibility problem for pure λ -calculus was undecidable, and then deduced the undecidability of predicate logic from this result [Chu36b, Chu36a].

Propositions-as-Types

Church gave his last version of the simple theory of type in [Chu40]. Afterward, there were many studies on Church's system and its extensions. For an instance, Leon Henkin in [Hen50] proved the completeness of simple type theory for his wider class of models. The question of completeness of predicate logic was answered positively by Gödel in [Göd29]. But, by Gödel's first incompleteness theorem, we can see that completeness of second order logic will be failed as a finite set of axioms for natural numbers can be formulated in a second order calculus with an added functional constant. However, Henkin provide a wider notion of models (he called them general models) in which he proved the completeness of simple theory of type (and calculi of higher order). About this wider notion, he says that "Roughly, these models consist of an arbitrary domain of individuals, as before, but now an arbitrary class of sets of ordered n -tuples of individuals as the range for functional variables of degree n ". There was also another study on Church's system by Peter Bruce Andrews that he extended Church's system to the theory of transfinite types [Gan68]. However, if we go back again before work of Church in [Chu40], we will see that Haskell Curry has also developed a type theory in

⁷This operation is now denoted by $\beta \rightarrow \alpha$

[Cur34, Cur35] with the aim of investigating the foundations of mathematics. He called his theory combinatory logic⁸. Combinatory logic is a formal theory based on a set of primitive axioms and rules of procedure and does not suppose the notion of variables. One of the main ideas of Curry was to decompose a substitution process into elemental steps, and to do so, he introduced a concept of combinator [Cur30]⁹. Getting back to Curry’s type theory in [Cur34, Cur35], one can see that his approach is different from the one by Church. Curry added a functionality constant \mathcal{F} in a way that an expression such as $\mathcal{F}abf$ means intuitively $\forall x(x \in a \Rightarrow f(x) \in b)$. In his approach, the range of f includes a and not exactly a . In other words, a term in Curry’s type theory can have an infinite number of types, whereas as we saw above in Church’s type theory, any well-formed formula has a unique type which is shown as its subscript¹⁰. In addition to the functional symbol \mathcal{F} , Curry in [Cur42] also considered another combinator \mathcal{P} to represent implication. He used the notation $\mathcal{P}ab$ for the implication that we now usually write it as $a \rightarrow b$, and then he provided some Postulates for \mathcal{P} and \mathcal{F} . In footnote 28 of [Cur42], Curry says the following: “Note the similarity of the postulates for \mathcal{F} and those for \mathcal{P} . If in any of the former postulates we change \mathcal{F} to \mathcal{P} and drop the combinator we have the corresponding postulate for \mathcal{P} .”. As noted in [CH06], we can say that here is the very first place that one can see more explicitly some sparks of the appearance of the *proposition-as-types*. Later on, Curry and Robert Feys in [Goo60], proved a formal statement for propositions-as-types that they called them “The P-to-F transformation” (Section 9.E of [Goo60]). There were two following axioms in their basic (type) theory where two combinators K and S are defined respectively as $Kxy = x$ and $Sxyz = xz(yz)$:

$$\vdash \mathcal{F}a(\mathcal{F}ba)K \qquad \vdash \mathcal{F}(\mathcal{F}a(\mathcal{F}bc))(\mathcal{F}(\mathcal{F}ab)(ac))S$$

If we replace the combinator \mathcal{F} with \mathcal{P} , and remove their subjects, then these two axioms will be as follows (remember that $\mathcal{P}ab$ is a notation for $a \rightarrow b$):

$$a \rightarrow (b \rightarrow a) \qquad (a \rightarrow (b \rightarrow c)) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c)$$

Moreover, there was also a rule for \mathcal{F} saying that $\mathcal{F}abf, ax \vdash b(fx)$. Under the same transformation, this rule for \mathcal{F} will be modus ponens. As we know, modus ponens plus the two axioms above will generate a theory of implication in the intuitionistic propositional logic. Hence, under this transformation, one can show every theorem of the Curry’s basic (type) theory is related to a theorem in the pure theory of implication in the intuitionistic propositional logic [Goo60]. By looking at the proof of this theorem and a partial converse of it in [Goo60], it can be observed that the transformation is not only restricted between propositions and types, and moreover there was a correspondence between type-assignment deductions and the proofs in the propositional logic. This fact and its extension to the λ -calculus (Church’s type theory) were mentioned by William Alvin Howard in [How80]¹¹. Howard called this correspondence formulae-as-types which is also known today as the Curry-Howard correspondence. This correspondence was also pointed out independently and with different motivations by Dana Scott in [Sco70] and Nicolaas Govert de Bruijn in [dB80] as part of the automath project. This

⁸The idea of having a syntax such as combinatory logic of Curry appeared also before him in a work by Moses Ilyich Schönfinkel [Sch]. Schönfinkel’s idea was to remove bound variables, which he thought are just auxiliary syntactic concepts

⁹As it is mentioned in [CH06], when Curry discovered [Sch], he admits “This paper anticipates much of what I have done”.

¹⁰This different typing is now called intrinsic and extrinsic typing due to John Reynolds [Rey98]

¹¹According to [CH06], this fact were pointed out by William Howard in a manuscript in 1969, and then published in [How80] in 1980

correspondence between proposition and types was also used by Per Martin-Löf, and then he introduced the intuitionistic type theory [Mar84]¹².

Normalization and Higher order type theories

We talked above about the normalization theorem in the natural deduction with the aim of eliminating detours in the proofs. By the Curry-Howard correspondence, we saw that proofs of a proposition will be transformed into terms of the corresponding type. Hence, one should have a similar notion to detours in terms which is now called redex (or β -redex). Hence the property of normalization in the natural deduction will be transformed to the following property in type theory: any typed term has a normal form which can be obtained via the reduction system of the given type theory. This property is sometimes called weak normalization which says that there exists at least one finite reduction of the terms. Similarly, a type theory has strong normalization, if all reductions of typed terms are finite. Alan Turing proved the weak normalization theorem for λ -terms with pure arrow types and β -reduction [CHS80]¹³ ¹⁴. Curry also proved the weak normalization using the cut elimination of the sequent calculus [Goo60]. As it is mentioned above, one of the questions of Hilbert was about the consistency of logical systems, and Gentzen proved the consistency of arithmetic. However, there is another proof of the consistency of arithmetic due to Gödel in his Dialectica paper [Göd58]. Roughly speaking, Gödel interpreted first order arithmetic in an impredicative primitive recursive functionals of finite type (we call it theory T), and he proved that all closed terms T of the integer type will be computed to a natural number. From this result, he deduced the consistency of intuitionistic first order arithmetic (Heyting arithmetic). Later on, William Walker Tait formalize the theory T as an extension of typed λ -calculus with a primitive-recursion operator and proved weak normalization of his typed system [Tai65]. Tait again provided another proof of normalization of his system in [Tai67]. In his second proof, he defined the so-called *computability predicate* by induction on types in a way that this predicate implies normalization. Then he proved by induction on terms that all terms satisfy this predicate. Although one can notice that one can adapt Tait's proof to have a strong normalization, Howard proved directly a strong normalization result [How70].

In the meantime, Gaisi Takeuti provided a sequent calculus for higher-order logic and he left the question of cut elimination for his system in [Tak53]. Afterward, Tait in [Tai68] and Dag Prawitz in [Pra67] proved that statement for second order predicate logic. Later on, Moto-o Takahashi in [Tit73] and Prawitz in [Pra68] provide the proof for the full Takeuti's system. However, if we look at all these proofs, we can not see an explicit procedure to construct the cut-free proof¹⁵. Jean-Yves Girard gave a constructive (algorithmic) proof of cut elimination for second and higher order logic in [Gir71, Gir72]. He first developed a typed system corresponding to the second order predicate logic and called it system F, and then proved the strong normalization result for his system. His idea was to develop Tait's computability predicate, and it is called *candidats de réductibilité* which is widely used in

¹²The very first work of Martin-Löf on the theory of type, which had an impredicative character, was published in 1971, and it turned out that it was inconsistent. Then he remove this impredicative character in his subsequent versions in 1972, 1975, 1982, and 1984 [CH06]

¹³This system is now called simply typed λ -calculus

¹⁴As it is noted in [CH06], this was the first proof of normalization which was before 1942. However, it took around 40 years to be published.

¹⁵That is why we nowadays distinguish between cut elimination and cut admissibility

proof theory and type theory. In system F, there is a type construction of the universal quantifier which is denoted by $\forall\alpha.\rho$ where α is a type variable. Apart from abstraction and application for terms, we also have abstraction and application for types. Let r be a term of type ρ , and the abstraction for type is the term $\Lambda\alpha r$ of type $\forall\alpha.\rho$. Let r be a term of type $\forall\alpha.\rho$, the application for type is the term $(r\sigma)$ of type $\rho[\alpha/\sigma]$. In $\forall\alpha.\rho$, the type variable α ranges over all types, including $\forall\alpha.\rho$. This impredicativity of the language made the proof of normalization of system F delicately complex. However, this universal quantifier construction, which is called polymorphism, was also seen useful in programming languages [Str00], and that was one of the motivations of John Charles Reynolds to develop, independently, a similar language to system F in [Rey93]. An example taken from Section 3.6.4 of [Str00]: Let f be a function of type $\alpha \rightarrow \beta$, and L be a list of elements of type α which will be denoted by $\text{list}(\alpha)$. Consider a function `map` that applies f respectively to each element of L so that it produces a $\text{list}(\beta)$ (`map(f, L)` will be of type $\text{list}(\beta)$). Now, using the universal quantifier of system F, one can assign the type $\forall\alpha\forall\beta(\alpha \rightarrow \beta) \rightarrow (\text{list}(\alpha) \rightarrow \text{list}(\beta))$ to term `map`. From another perspective, one can derive many other data types using polymorphism. For instance, type of integers can be written as $\forall\alpha(\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha)$ [Gir72], the boolean type can be written as $\forall\alpha.\alpha \rightarrow (\alpha \rightarrow \alpha)$, etc.

Birth of Linear Logic

One of the questions that arose after presenting system F and the other similar systems was asking about how we should interpret polymorphic types. Reynold conjectured in [Rey83] that there is a set-theoretic interpretation of polymorphic typed λ -calculus, and he disproved himself his conjecture ¹⁶. Anne Sjerp Troelstra in [Tro73] provided a semantic for system F based on partial equivalence relations (PER), i.e, binary relation satisfying the symmetry and transitivity conditions, over Kleene structure (\mathbb{N}, \cdot) where $n \cdot m$ is the result of applying the n -th partial recursive function to m as its argument. This model is called “hereditarily recursive operations of order 2” by Troelstra. As it is mentioned in [Gir86], there were many other attempts to interpret system F using some sort of Scott domains, but those were essentially the same as Troelstra’s; Girard says in [Gir86] “simply partial recursive functions are replaced by λ -terms, and these λ -terms are in turn interpreted in a Scott domain.”. However, new semantic ideas were presented in [Gir86] using *coherence spaces* which was called qualitative domains by Girard in [Gir88]. Now, we are in a position where linear logic was invented by Girard in [Gir87a]. Although there is almost no doubt that linear logic is coming from coherence spaces, we can also look at this sentence by Girard in [Gir87a] saying that “Let us explain what is the starting disturbance that led to linear logic: the semantics of coherent spaces,...”. In classical and intuitionistic logic, the multiplicative and additive rules are equivalent thanks to the free structural rules. However, in linear logic we do not have that freedom on structural rules, and those rules are only accessible on so-called modal formulas. Hence the equivalence of multiplicative and additive rules will be broken. Based on this fact, one has two different conjunctions, two different disjunctions, and the same on units of logic. In fact, as an example, one has both intuitionistic and classical disjunctions at the same time in linear logic. More precisely, there is a formula $A \oplus B$ in linear logic for any formula A and

¹⁶Reynold said if there is such an interpretation, then the endofunctor $(2^X)^X$ over category of sets and function has initial algebra, i.e, a set A such that $A \cong (2^A)^A$. But this is not possible by a cardinality argument

B together with the following rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}$$

Moreover, one also has a formula $A \wp B$ with the following rule:

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \wp B, \Delta}$$

And one can see that the formula $A \oplus A^\perp$ is not provable in general in linear logic where A^\perp is the linear version of negation. Whereas the formula $A \wp A^\perp$ is provable in linear logic. Hence linear logic provides a fine-grain analysis of proofs in intuitionistic and classical logic, and also of their cut-elimination. Based on this idea of having explicitly a restricted version of structural rules, linear logic provides us with a logical understanding of the evaluation strategies. Although the structure of the cut-elimination procedure is essential, one can see that this structure is quite problematic in sequent calculus as we have problems of the permutation of rules. That is to say one can have a single proof of a formula, whereas it can be represented in two different ways in sequent calculus. To avoid this problem, Girard gave a new syntax of proofs [Gir87a] which is called *proof-nets*. Cut elimination theorem was also obtained in [Gir87a] using normalization for proof-nets. However, as it is also mentioned in [Gir95], one can of course prove cut elimination for propositional linear logic directly in the sequent calculus using essentially the argument of Gentzen, and moreover, the proof can be simpler due to the restrictions on structural rules.

Fixpoint theory

If we split the title of the manuscript into three parts of “Fixed Points”, “Linear Logic”, and “Curry-Howard-Lambek”, till now we have almost talked about the last two, i.e. “Linear Logic”, and “Curry-Howard-Lambek”. It is said “almost”, since we did not talk about the role of Lambek. But a short history of categorical logic is provided in Section 0.1.1 which will talk about Lambek’s role in this story. But we really have not talked about the “Fixed Points” part yet, and this is what we are going to do now. The goal is to give a very short background on order-theoretic fixpoints. Once we are done with this, we will talk about the connection of these three parts which is the main goal of this manuscript.

Fixpoint theorems in mathematics are the results that says that an operation (or a function) f on a particular space has fixpoints, meaning that there is an element x of the space such that $f(x) = x$. Bronisław Knaster and Alfred Tarski in [Kna28] proved a set-theoretical fixpoint theorem saying that any function on the family of all subsets of a set, which is increasing under set-theoretical inclusion, has at least one fixpoint. Later on, Tarski gave a lattice-theoretic generalisation of the set-theoretical one ¹⁷. He proved that the set P of all fixpoints of a given increasing function f on a complete lattice (A, \leq) ¹⁸ is a non empty

¹⁷Although this result was published in 1955, Tarski says in [Tar55] that most of his results contained in that paper were obtained in 1939. He also tells us that in the meantime he was working on a research project in the foundations of mathematics which was founded by Office of Ordnance Research, U.S. Army

¹⁸A lattice is a partial order set (A, \leq) such that for any two elements $a, b \in A$ there is a least upper bound (join) and a greatest lower bound (meet). A lattice is said to be complete if every subset B of A has a least upper bound and a greatest lower bound

complete lattice (Theorem 1 of [Tar55]). In particular, we will have the least and greatest fixpoints of f , since P is a complete lattice. This theorem is an existential statement and it is not completely obvious how one can build the fixpoints. There is another fixpoint theorem that is more constructive in the sense that one can see a procedure to build the least fixpoint under an assumption on the function. This theorem is called *Kleene fixpoint theorem*¹⁹, and it says that if a function $f : X \rightarrow X$ on inductive poset X ²⁰ preserves directed joins then there it has a least fixpoint. Patrick Cousot and Radhia Cousot relaxed the assumption of preserving directed joins, and proved that any monotone function on a complete lattice has both the least and greatest fixpoints [CC79]. Their proof also provides a transfinite procedure to build fixpoints. They actually showed that the set of fixed points of f is the image of preclosure operations on the complete lattice which is defined as limits of stationary transfinite iteration sequences. Although their proof is constructive (in the sense of mentioned above) and relaxes the assumption of preserving directed joins, it works on a more restricted domain (complete lattice) compare to Kleene’s theorem which was on inductive poset. The good news is that there is a theorem by Dito Patariaia, called *Patariaia’s theorem*, which indeed proves that any monotone operation $f : X \rightarrow X$ on inductive poset X has a fixed point²¹. Paul Taylor in [Tay22] proved Patariaia’s theorem under the “special condition” which simplifies the original proof of Patariaia. This special condition is saying $\forall x, y \in X((x = f(x) \leq y = f(y)) \Rightarrow x = y)$.

The fixpoint theorems have many applications in mathematics. For example, one can see in [Kna28] that the set-theoretical fixpoint theorem gave us a generalization of the Cantor-Bernstein theorem, and also some applications topology are mentioned there. And of course, many other applications that are way much more than the author’s knowledge. To see the application that this manuscript is based on that, we should get back again to 1895 where Cantor introduced well-orderings or ordinals in [Can95, Can97], and that led Zermelo to generalize it to the notion of a well founded relation [Zer35]. Let A be a set and \prec be a well founded relation on A , then it obeys the well founded induction scheme:

$$\frac{\forall a \in A (\forall b \in A b \prec a \Rightarrow \Phi(b)) \Rightarrow \Phi(a)}{\forall a \in A \Phi(a)}$$

for any predicate Φ on A . This well founded induction makes us able to prove a predicate by induction. From this ability, John von Neumann proved in [vN] a recursion theorem that will make us able to define a function by recursion. The interesting point is that one can also prove the same recursion theorem using Patariaia’s theorem [Tay22]. So, here is a place where we can see, as an example, an application of fixpoint theory in the field of inductive definitions and proofs. In this spirit of using fixpoint theory, one can develop an induction principle in a sequent calculus style. For instance, the following rules can be seen as a kind of Patariaia’s theorem where $\mu X.F(X)$ is a notation for the least fixpoint of the operation F :

$$\frac{\Delta \vdash F(\mu X.F(X)), \Gamma}{\Delta \vdash \mu X.F(X), \Gamma} \quad \frac{F(S) \vdash S}{\mu X.F(X) \vdash S}$$

¹⁹The author is not able to find easily the first place that this theorem has appeared, so not fully sure why this is called Kleene fixpoint theorem. But most probably it is because of the similarity between this theorem and Kleene’s recursion theorems in [Kle38, Kle71].

²⁰An inductive poset is a poset with a least element, and admitting joins of all directed subsets.

²¹According to [Tay22], Patariaia never wrote it up formally himself, and his proof was simplified by Alex Simpson.

Along this approach, one can try to develop an extension of the propositional logical system by inference rules such as those above. We said propositional because, as it is mentioned above, one can encode fixpoints types in second order logical systems such as system F. However, along this connection of proof theory and fixpoint theory, Ralph Matthes developed an extension of system F with iteration and Primitive recursion on inductive Types [Mat99a], as he mentioned in [Mat99a] that he considered a “magnifying glasses of proof theory” on the theory of complete lattices. And, finally, we are relatively in a good position to talk about the topic of this manuscript which will be done in the following sections.

Fixpoints theory, Linear Logic, Curry-Howard-Lambek

Girard in [Gir92] suggested considering of extension of linear logic with fixpoints of formulas in an unpublished note, though the first comprehensive proof-theoretic investigation of such an extension of linear logic is by Baelde [Bae12]. He considered an extension μ MALL of Multiplicative Additive linear logic with induction and coinduction principles. It seems, apparently, his motivation was more coming from proof-search and system verification point of view and therefore his μ MALL logical system is a predicate calculus. We will look at that system from the fixpoint theory approach mentioned above together with a Curry-Howard-Lambek perspective. So, in the first place, we do not need to deal with a predicate calculus, and we stick to the proposition calculus setting. Unlike [Bae12] we include the exponentials in our system from the beginning, so we call it μ LL rather than propositional μ MALL. Exponentials are not considered in μ MALL because some form of exponential can be encoded using inductive/coinductive types, however, these exponentials are not fully satisfactory from our point of view because their denotational interpretation does not satisfy all required isomorphisms; specifically, the *Seely isos* are lacking (This is explained in Chapter 4). The μ LL system is described in Chapter 0, and there we can also see a brief background on linear logic, and its model. Now, we explain below what we have done in each chapter along with our main motivation which is making relations between fixpoints theory, linear logic, and Curry-Howard-Lambek correspondence.

Chapter 1

In Chapter 1, we first decided to investigate the denotational semantics of μ LL, whose definition does not rely on the rewrite system μ LL is equipped with, since the proof-theory (and hence the “operational semantics”) of μ LL is still under development. We develop there a categorical semantics of μ LL extending the standard notion of Seely category²² of classical linear logic. Such a model of μ LL consists of a Seely category \mathcal{L} and of a class of functors $\mathcal{L}^n \rightarrow \mathcal{L}$ for all possible arities n which will be used for interpreting μ LL formulas with free variables. These functors have to be equipped with a strength to deal properly with contexts in the ν rule. Then we develop a simple instance of this setting which consists in taking for \mathcal{L} the category of sets and relations, a well-known Seely model of linear logic. The strong functors we consider on this category, that we call *variable sets*, are the pairs $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ where $\widehat{\mathbb{F}}$ is the strength and $\overline{\mathbb{F}} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is a functor which is Scott-continuous in the sense

²²Sometimes called new-Seely category: it is a cartesian SMCC with a *-autonomous structure and a comonad $!_-$ with a strong symmetric monoidal structure from the cartesian product to the tensor product.

that it commutes with directed unions of morphisms which implies categorical cocontinuity on the category of sets and injections and maps inclusions to inclusions (this light additional requirement simplifies the presentation). There is no special requirement about the strength \mathbb{F} beyond naturality, monoidality and compatibility with the comultiplication of the comonad $!_.$ Variable sets form a Seely model of μLL where linear negation is the identity on objects, the formulas $\mu\zeta F$ and $\nu\zeta F$ are interpreted as the same variable set, exactly as \otimes and \wp are interpreted in the same way (and similarly for additives and exponentials): this denotational “degeneracy” at the level of types is a well known feature of **Rel** which doesn’t mean at all that the model is trivial; for instance normal multiplicative exponential LL proofs which have distinct relational interpretations have distinct associated proof-nets [dCdF12, dC16]. Then we “enrich” this model **Rel** by considering sets equipped with an additional structure of *totality*: a *non-uniform totality space* (NUTS) is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X)$ is a set of subsets which intuitively represent the total, that is, terminating computations of type X . This set $\mathcal{T}(X)$ is required to coincide with its bidual for a duality expressed in terms of non-empty intersections. Given two NUTS X and Y there is a natural notion of *total relation* $t \subseteq |X| \times |Y|$ giving rise to a category **Nuts** which is easily seen to be a Seely model of linear logic. To turn it into a categorical model of μLL , we need a notion of strong functors $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$. Rather than considering them directly as functors, we define *variable non-uniform totality spaces* (VNUTS) as pairs $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ where $|\mathbb{X}| : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is a variable set and, for each tuple $\vec{X} = (X_1, \dots, X_n)$ of VNUTS’s, $\mathcal{T}(\mathbb{X})(\vec{X})$ is a totality structure on the set $|\mathbb{X}|(|\vec{X}|)$. It is also required that the action of the functor $|\mathbb{X}|$ on **Nuts** morphisms and the strength $\hat{\mathbb{X}}$ respect this totality structures. Then it is easy to derive from such a VNUTS \mathbb{X} a strong functor $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ and we prove that, equipped with these strong functors, **Nuts** is a model of μLL . And finally, we will apply the same notion of totality on coherence spaces, and we will provide another concrete model of μLL based on coherence spaces with totality.

Chapter 2

In Chapter 2, along our path to make a relation between fixpoints, linear logic, and Curry-Howard-Lambek approach, we decided to develop a λ -calculus for this extension μLL . However, the explicit structural rules in linear logic make such a language very heavy, hard to use and analyze. In particular, explicit substitutions would be necessary in such a μLL -based syntax. However, Olivier Laurent in [Lau02] introduced a “polarized” linear logic LLP which relaxes the use of structural rules on so-called negative formulas. So, we take this advantage of the relaxed structural rules in LLP, and we call LLP_{imp} for the polarized linear logic with implicit structural rules.

We study $\mu\text{LLP}_{\text{imp}}$, an extension of LLP_{imp} with least and greatest fixpoints, by introducing and studying a system L calculus [CH00, CM10, Mun13, Mun09] for a polarized version of μLL . The search for a maximal syntactic simplicity guided our design of this calculus. We use the Greek letter κ for the name binder instead of the more traditional $\mu/\tilde{\mu}$ [Par92] which would lead to confusions with the standard notation associated with least and greatest fixpoints (μ/ν). Related to this syntactical simplicity is that a negative term or a command can be typed by a negative or a positive sequent so that there are actually five kinds of typing judgments and this partitioning is taken into account by the semantics. The polarization of fixpoints means that least fixpoints allow to define data-types (integers, lists, trees *etc.*)

while greatest fixpoints allow to define co-data-types, that is types of data-consumers ²³. We refer to [APTS13] for a detailed discussion of the computational duality between data and co-data-types. The typed calculus introduced in that way is called $\kappa\mu\text{LLP}$. It features natural *construction* rules associated with the positive connectives and constants (pairing for \otimes , injections for \oplus etc). It has a positive promotion construct $s^!$ for putting a negative term s in a box (or *thunk*) which can be used as a piece of data and a negative dereliction $\text{der } p$ which allows to open such a box. Just as the $\lambda\mu$ -calculus, $\kappa\mu\text{LLP}$ has names $\alpha, \beta \dots$ associated with the negative formulas of a sequent ²⁴. Since at most one positive formula can occur in a single-sided LLP sequent, we need only one variable that we denote as \bullet . There are several binders for names: one general binder $\kappa\alpha.c$ which allows to select a negative formula in the context, and the other ones are associated with \perp , \wp and Park's rule. All these binders produce a negative term whose type is a negative formula made active for further uses. There is also a binder $\tilde{\kappa}.c$ associated with the unique variable \bullet which produces a positive term. One crucial feature of \bullet is that it can occur only *linearly* in a command or negative term. Again, this is due to the fact that all the formulas in the context of a promotion must be negative. Notice also that \bullet cannot occur free in a positive term due to the fact that a positive sequent has exactly one positive formula. All these binders apply to *commands* which are cuts $s \star p$ between a negative term s and a positive term p . Our operational semantics provides only reduction relations for commands and can be seen as describing the interactions between positive constructors and negative destructors. One specific critical command is $\kappa\alpha.c \star \tilde{\kappa}.d$ which could *a priori* lead to $c[\tilde{\kappa}.d/\alpha]$ or $d[\kappa\alpha.c/\bullet]$; we choose the second option making our reduction semantics deterministic; we are actually defining a kind of abstract machine whose states are commands $s \star p$ where s is the program and p is the stack (there are no environments because substitutions are executed immediately).

Our goal in this chapter is twofold. On the one hand, we provide a categorical semantics of $\kappa\mu\text{LLP}$ building on the one provided in Chapter 1. Given a model $(\mathcal{L}, \vec{\mathcal{L}})$ (simply denoted as \mathcal{L}) of μLL the main idea is standard: interpret a closed positive formula P as an object of the Eilenberg-Moore category $\mathcal{L}^!$. This requires however to deal also with open positive formulas: we take them into account introducing the notion of *positive functors* which are strong functors (as specified by \mathcal{L}) equipped with a distributive law wrt. the comonad $!$, they are a functorial generalization of the notion of $!$ -coalgebra. We illustrate this semantics in the concrete models **Rel** and **Nuts**.

On the other hand, we also prove some form of normalization (cut-elimination) for $\kappa\mu\text{LLP}$. This turned out to be surprisingly difficult. The solution came from understanding that proving a termination property with respect to the relational semantics — saying very roughly that if a command has a non-empty relational semantics its reduction terminates — would be possible because the points of the relational model are *finite* trees on which induction is possible. Concretely this means that we associate sets of terms to points of the relational models and these sets are easily defined by induction. To make this proof of normalization more natural we also provide a presentation of the relational semantics of $\kappa\mu\text{LLP}$ as an intersection typing system. To derive from this relational normalization a standard normalization property, it is enough to prove that, in sufficiently many meaningful situations, when $c \vdash \alpha : N$,

²³This strongly suggests that lists and streams are not of the same nature, streams are not data but data consumers.

²⁴Indeed in the $\lambda\mu$ -calculus all formulas are negative and the names are associated with the formulas occurring on the right side of a sequent whereas variables are associated with formulas on the left.

the command c has a non-empty relational semantics. We do that for $N = ?\text{nat}$ where nat is a type of integers defined as a least fixpoint formula. The model **Nuts** of non-uniform *totality* spaces gives us precisely this information: any total subset of the relational interpretation of $?\text{nat}$ is non-empty and hence, in particular, the interpretation of c is non-empty.

Chapter 3

Till now, we have only considered finitary logical system with inductive and co-inductive types. However, in those systems, cut elimination theorem will not imply the sub-formula property as one basically need to guess an invariant in the inductive proofs, and this can not be avoided in these finitary systems. There are morally two ways to obtain a cut elimination result and sub-formula property in the logical system with inductive and co-inductive types: either considering infinitary logic in the sense of [ST58, Tar58], or considering non-well founded proofs meaning that one has infinite proofs in depth but finite branching in the sense of [Dou17, BDS16]. In this chapter, we actually consider non-well founded proofs from a denotational semantics point of view. In [Dou17, BDS16], the μMALL_∞ system, which is non-well-founded multiplicative and additive LL with two rules for unfolding fixed-points, is studied. They have defined a syntactic notion of validity on proofs in order to distinguish sound from unsound proofs. We consider an extension of μMALL_∞ , called μLL_∞ , with exponentials. We first recap the language and the inference rules of μLL_∞ . Then we will provide an interpretation for μMALL_∞ proofs in **Rel**. However, the proofs are possibly non-well-founded. So, we consider, as it is standard, all finite approximations of a proof, and then take the supremum of the interpretation of them. To define the precise definition of this approximation, we consider an extension of μLL_∞ with this rule: $\frac{}{\vdash \Gamma}^{(\Omega)}$ for any sequence Γ . We interpret this rule as the empty set, and this helps us to consider any finite sub-tree of a given μLL_∞ as a proof in this extension of μLL_∞ . As the cut elimination procedure in μLL_∞ is an infinite process, we will consider a notion of limit by defining a metric on the set of all μLL_∞ finite proofs. Based on this notion of metric, we define an equivalence relation on the collection of all Cauchy sequences. Then we will show the metric completion of the collection of all Cauchy sequences is isomorphic to set of all (potentially infinite) μLL_∞ proof (Although, what it is provided till now is standard in the literature, we will provide them in this chapter for the sake of self-containdness of the manuscript). We then proved the preservation by cut elimination procedure, which can be possibly an infinite reduction path, using a result showing that the interpretation of any Cauchy sequence is obtained by limsup of the interpretation of each proof in the sequence. Afterwards, we relate the validity condition and totality of **Nuts** by proving that each μLL_∞ valid proof will be interpreted as a total element in **Nuts**. Our proof method is similar to the proof of soundness of LKID^ω in [Bro06]. However the system of [Bro06] is classical logic with inductive definitions, and this proof is for a Tarskian semantics. So, we need to adapt that proof in two aspects: considering μLL_∞ instead of LKID^ω , and trying to deal with a denotational semantics instead of a Tarskian semantics. The adaptation for μLL_∞ is somehow done in [Dou17], since there is soundness theorem for μMALL_∞ with respect to the truncated truth semantics (a Tarskian semantics). So, basically, the main point of our proof is turning a Tarskian soundness theorem into a denotational soundness theorem. Finally, we end this chapter by proving that the semantic of proofs is preserved via the translation operation which sends any μLL proof to a μLL_∞ proof.

Chapter 4

In this chapter, we get back to the question that arose in the Chapter 0 about considering exponentials in μLL . In [Bae12], he did not consider the exponentials in his system of μMALL , and one the reason is that using an encoding $!A = 1 \& A \& (!A \otimes !A)$ of $!A$, one can derive all structural rules of exponentials in μMALL . However, this is not satisfactory from a denotational point of view. Although this encoded exponential $!^t$ is functorial and also has a comonad structure, it does not fulfill all the categorical requirements as stipulated in the definition of a Seely category, as it is also mentioned in [Laf88b, Laf88a]. More precisely, we will not have the Seely iso with this $!$ exponential (Proposition 6 of [Laf88b] is not iso but equivalence). In other words, the associated Kleisli category $\mathcal{L}_!$ is not a CCC. In this chapter, we will study a general categorical construction for this encoded exponential. Based on the notion of comagma in a category \mathcal{L} with a binary functor, we turn the class of comagmas over a given object into a category. Then we define the notion of *free magmatic quasi-exponential* (FMQE) as an operation which associates, with each object X of \mathcal{L} , a terminal object of the category comagmas over X . We will afterward show the functoriality, comonadicity, and monoidality of a FMQE. And we will end the categorical construction of $!^t$ by providing Seely morphisms.

We will also consider some concrete models as instances of this categorical structure. Interpreting this encoded $!^t$ as a binary tree, we will show that **Rel** and coherence spaces are indeed two concrete cases. We end this chapter by proving that the interpretation of a linear logic proof in coherence spaces with this tree-based exponential is related to its interpretation in non-uniform coherence spaces with Boudes's exponential. To do so, we use the logical relation method, and so, we define another instance of our categorical setting provide, and we called it *local totality spaces*. A local totality space (LTS) is a tuple generated by a coherence space, a non-uniform coherence space, an operation sending tree-based exponential to multiset exponential denoted as ρ , and a local totality gadget to define composition properly. And we will prove that if we apply the operation ρ on interpretation of linear logic proofs in coherence spaces with the tree-based exponential, we will get its interpretation in non-uniform coherence spaces.

We end up this manuscript by a concluding Chapter 5, in which we expose the perspectives of our work.

$\mathcal{L} \ ? \ \otimes \ \wp \ \oplus \ \& \ \mu \ \nu \ !$

We can summarize this manuscript by saying that it is about studying the connection between fixpoint theory, linear logic, and Curry-Howard-Lambek correspondence.

Chapter 0

Background on denotational and categorical semantics of linear logic

In this chapter, we will recall some background on proof theory of linear logic, denoted by LL, and its denotational and categorical semantic. As linear logic is a logic that is discovered from a denotational semantic (coherence spaces) [Gir87a], we start this chapter with a very short history on denotational and categorical semantics in general. In this manuscript, by “denotational semantics” we mean the construction of concrete models, and by “categorical semantics” we mean a categorical axiomatization of what the concrete models of a given logic/computational system are. And all the proof systems presented in this manuscript are classical (not intuitionistic).

0.1 Short history on categorical and denotational semantics

0.1.1 On categorical semantics

The connection between intuitionistic propositional logic and cartesian closed categories was introduced by Lawvere and Lambek [Lam68, Lam69, Law63] which is known by proposition-as-object and proof-as-morphism, and in fact it was the heart of categorical logic. Whereas we require some structure on the category (e.g., cartesian closed category), one needs also some other theories and structures in order to study predicate logic, dependent and polymorphic type theory, etc in a categorical setting. The idea started with Lawvere who introduced the notion of hyperdoctrine, a kind of indexed category or fibered category, to study logic [Law69, Law]. Then Seely based his logic on “a natural deduction formulation of intuitionistic, multisorted, first order predicate calculus with equality” and showed that hyperdoctrines are equivalent to that logic in more details [See83, See77]. Makkai and E. Reyes also wrote a book [MR77] on the connections of first order model theory and categories, and Makkai then published two papers [Mak93a, Mak93b] about “give an algebraic framework for the proof theory of intuitionistic predicate calculus” based on Lawvere’s notion of hyperdoctrine. In fact, as it is in title of the papers, those were focused on two completeness theorems: “Gödel completeness theorem” and “Kripke completeness theorem”. In the meantime there was a work by Benabou [Ben] (its source is not available to me). Of course we cannot refer to all the work that have been done due to the lack of space (and of course lack of knowledge of

the writer); among those we refer the reader to some particular ones [Cur89, Ču97, HP87].

0.1.2 On denotational semantics

In 1969, at the almost same date as the very early work on categorical logic by Lambek, Christopher Strachey and Dana Scott invented denotational semantics [Sco93, Sco72, DS71]. The goal was to provide a mathematical interpretation of functional programs that Strachey was promoting since 1960's. Strachey's idea was to look at functional programs as functions. However, one of the main problem was to see what sort of functions and spaces we should consider as the interpretation of programs. Scott was able to find an answer to this question, and he gave a meaning to the λ -calculus using ordered topological spaces and continuous functions. And so, he found a way to the solution of recursive type definitions which involve positive and negative occurrences of variables such as $D \simeq (D \Rightarrow D)$. It is worth mentioning that later on Freyd gave a categorical framework to the canonical solution of recursive types namely coinciding of initial algebra and final coalgebra of endofunctors. And then using that he developed solutions of fixpoint equations of mixed variant functors [Fre91, Fre92].

Later on, Berry developed the notion of stability in order to capture sequentiality in mathematical models [Ber78, Ber79]. And one can see in works of Lamarche and Kegelman [Keg99, Lam] that those two notions Scott-continuity and stability are indeed coming from a same notion, i.e, factorized domain. Girard also rediscovered this notion of stable function once he provided a model of system F (second-order typed λ -calculus) [Gir86] based on coherence spaces which is the origin of LL [Gir87a].

Scott mentioned in [Sco93] the *parallel or function* which is not definable in PCF (a term language for higher-order computable functions). Plotkin proved that Scott semantics (lattices and continuous functions) is fully abstract for PCF extended with a constant implementing the parallel or function [Plo77]. Notice that, as it is shown by Berry, *parallel or function* is not stable. Even by replacing Scott-semantics by stable semantics, we will not have full-abstraction of PCF. Actually, full abstraction fails because of so-called *Gustave function*¹. This function is indeed stable but not sequential in the sense of Sequentiality introduced by Jean Vuillemin. We refer to [BCL83, Mul85, BE91, Ehr95, Cur94, AJ94, JFM⁺96] to see more details on sequentiality problem and full abstraction. We just mentioned that sequentiality has been introduced before stability and not at all for the λ -calculus, but for analyzing the computation flow in the execution of (first order) recursive programs. Then stability has been discovered by Berry as a simplification of sequentiality because he realized that there was a real problem for extending sequentiality to higher order functions.

0.2 Linear logic with fixpoints

Linear logic (LL) was introduced by Jean-Yves Girard in his seminal work [Gir87a] that is a refinement of classical and intuitionistic logic. The origin of the discovery of this new logic comes from the semantical analysis of the coherence space denotational interpretation of System F [Gir86]. The relation between LL and linear algebra will be more clear in Section 0.4.2 where we recall the category of coherence spaces and linear morphisms. One can also look at

¹Based on the MPRI course by Pierre-Louis Curien: the name of the function is a joke, it was the nickname of Gérard Béry, because there was already two other Gérard in the lab where he worked.

the work by Lamarche [Lam92] to see a clearer relation between LL and linear algebra, and also some more recent works [Ong17, LMMP13] to see this relation. Tsukada and Asada generalized this linear algebraic perspective [TA22] based on categories of modules over special semirings (sigma-semirings [TA22] Definition 12). However, in this section we focus more on proof-theoretical (syntax) aspects of LL. For instance, one feature of LL, contrary to classical logic LK, is that we do not have a free access to structural rules, that is weakening and contraction rules. More precisely, we can only do weakening and contraction rules on so-called exponential formulas. In other words LL provides more control on structural rules.

In this section, we first recall briefly LL [Gir87a] in Section 0.2.1 and we will review some basic materials of proof theory of LL. And then we will describe the syntax of formulas and proofs of μ LL [Bae12] in Section 0.2.2.

0.2.1 Syntax of LL

The LL formulas are defined inductively as follows:

$$A, B, \dots := 1 \mid \perp \mid A \otimes B \mid A \wp B \mid 0 \mid \top \mid A \oplus B \mid A \& B \mid !A \mid ?A$$

We do not assume a negation connective explicitly in the syntax, whereas a negation operator is defined using De-Morgan duality by induction on formulas: $1^\perp = \perp$, $\perp^\perp = 1$, $(A \wp B)^\perp = A^\perp \otimes B^\perp$, $(A \otimes B)^\perp = A^\perp \wp B^\perp$, $0^\perp = \top$, $\top^\perp = 0$, $(A \& B)^\perp = A^\perp \oplus B^\perp$, $(A \oplus B)^\perp = A^\perp \& B^\perp$, $(!A)^\perp = ?A^\perp$, $(?A)^\perp = !A^\perp$. Obviously $A^{\perp\perp} = A$ for any formula A .

There is also no connective for implication in our presentation of LL. Instead, a linear implication is defined similarly to the decomposition $A \Rightarrow B = \neg A \vee B$ in classical logic, as $A \multimap B := A^\perp \wp B$.

The logical system of LL presented here is the usual single-sided sequent calculus of classical propositional LL [Gir87a], see also [Mel09] Section 3.1 and 3.13. In this setting we deal with sequents $\vdash A_1, \dots, A_n$ where the A_i 's are formulas. It is important to notice that the order of formulas in this list is not relevant, which means that we keep the exchange rule implicit as it is usual in sequent calculus. The inference rules of LL are given in Figure 1.

Remark 1 *If weakening and contraction were allowed for any arbitrary formulas, then \otimes and $\&$ would be identified. That is to say if we replace the \otimes with the $\&$ or vice versa, then the provability would be preserved. In this case (free weakening and contraction), we also have this identification between \oplus and \wp , as well as 1 and \top , 0 and \perp . One can summarise this identification by saying that if we have free weakening and contraction on any formula, then the multiplicatives (different context in the premises) and additives (same context in the premises) rules would be identified. And actually, based on this, there is a separation of linear logic connectives in multiplicatives and additives ones. The multiplicatives connectives are \otimes and \wp , and the additives connectives are \oplus and $\&$. The same is done for the unit; the multiplicatives units are 1 and \perp , and the additives units are 0 and \top .*

We say two LL formulas A and B are *equivalent* if both implications $A \multimap B$ and $B \multimap A$ are provable. Related to this notion of equivalence, there are also two other notions; namely *isomorphism* (stronger than equivalence) and *equiprovability* (weaker than equivalence). We say two LL formulas A and B are *equiprovable*, when $\vdash A$ is provable iff $\vdash B$ is provable.

The identity fragment:

$$\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)}$$

The multiplicative fragment:

$$\frac{}{\vdash 1} \text{ (1)} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \text{ (\otimes)} \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \text{ (\perp)} \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \text{ (\wp)}$$

The additive fragment:

$$\frac{}{\vdash \Gamma, \top} \text{ (\top)} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \text{ (\oplus_1)} \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \text{ (\oplus_2)} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \text{ (\&)}$$

The exponential fragment:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \text{ (?w)} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \text{ (?c)} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \text{ (?d)} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \text{ (?p)}$$

Figure 1: Inference rules of LL

For instance, $A \otimes B$ and $A \& B$ are equiprovable for any LL formula A and B . To define isomorphism, we first need to talk about cut-reduction rules of LL. We refer to [Gir87a] to see all cut-reduction rules, and we just bring some examples of those reductions here.

$$\frac{\frac{\frac{\pi_1}{\vdash \Gamma_1, A} \quad \frac{\pi_2}{\vdash \Gamma_2, B}}{\vdash \Gamma_1, \Gamma_2, A \otimes B} \text{ (\otimes)} \quad \frac{\frac{\pi_3}{\vdash A^\perp, B^\perp, \Delta}}{\vdash A^\perp \wp B^\perp, \Delta} \text{ (\wp)}}{\vdash \Gamma_1, \Gamma_2, \Delta} \text{ (cut)}}{\vdash \Gamma_1, \Gamma_2, \Delta} \text{ (cut)} \quad \xrightarrow{\otimes, \wp} \frac{\frac{\pi_2}{\vdash \Gamma_2, B} \quad \frac{\frac{\pi_1}{\vdash \Gamma_1, A} \quad \frac{\pi_3}{\vdash A^\perp, B^\perp, \Delta}}{\vdash B^\perp, \Gamma_1, \Delta} \text{ (cut)}}{\vdash \Gamma_1, \Gamma_2, \Delta} \text{ (cut)}$$

$$\frac{\frac{\frac{\pi_1}{\vdash A, B, \Gamma, C}}{\vdash A \wp B, \Gamma, C} \text{ (\wp)} \quad \frac{\pi_2}{\vdash C^\perp, \Delta} \text{ (cut)}}{\vdash A \wp B, \Gamma, \Delta} \text{ (cut)}}{\vdash A, B, \Gamma, \Delta} \text{ (cut)} \quad \xrightarrow{c\wp} \frac{\frac{\pi_1}{\vdash A, B, \Gamma, C} \quad \frac{\pi_2}{\vdash C^\perp, \Delta}}{\vdash A, B, \Gamma, \Delta} \text{ (cut)}}{\vdash A \wp B, \Gamma, \Delta} \text{ (\wp)}$$

$$\frac{\frac{\frac{\pi_1}{\vdash ?\Gamma, A} \quad \frac{\pi_2}{\vdash ?A^\perp, ?A^\perp, \Delta}}{\vdash ?\Gamma, !A} \text{ (?p)} \quad \frac{\pi_2}{\vdash ?A^\perp, \Delta} \text{ (c)}}{\vdash ?\Gamma, \Delta} \text{ (cut)}}{\vdash ?\Gamma, \Delta} \text{ (cut)} \quad \xrightarrow{!, c} \frac{\frac{\pi_1}{\vdash ?\Gamma, A} \quad \frac{\frac{\pi_2}{\vdash ?\Gamma, !A} \quad \frac{\pi_2}{\vdash ?A^\perp, ?A^\perp, \Delta}}{\vdash ?A^\perp, ?\Gamma, \Delta} \text{ (cut)}}{\vdash ?\Gamma, ?\Gamma, \Delta} \text{ (c)}}{\vdash ?\Gamma, \Delta} \text{ (c)}$$

And then LL enjoys cut-elimination theorem [Gir87a]:

Theorem 1 *A sequent $\vdash \Gamma$ has a LL proof iff it has a LL proof that does not use the (cut) rule.*

This theorem has many consequences such as the *subformula* property:

Theorem 2 *A sequent $\vdash \Gamma$ is provable iff there is a LL proof of $\vdash \Gamma$ such that each intermediate conclusion is made of subformulas of the formulas of Γ .*

And as one of the main purposes of cut-elimination theorem, one also has consistency of LL:

Theorem 3 *It is impossible in the LL proof system to prove that $\vdash A$ is provable for any LL formula A .*

Remark 2 *There is an extension of LL with the the mix rules:*

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} \text{ (mix}_2\text{)} \qquad \frac{}{\vdash} \text{ (mix}_0\text{)}$$

One can prove 1 and \perp are equivalent using these two rules, and also show that $\vdash \perp$ is provable. However, this doesn't entail that LL is not consistent, because $\perp \multimap A$ for any formula A is not still provable.

As another proof-theoretical property of LL, one can also prove the η -expansion theorem:

Theorem 4 *Let $\vdash \Gamma$ be a LL sequent and π be an LL proof. Then there is a proof π' of $\vdash \Gamma$ in which the (ax) rule is not used. Moreover, there is a cut-free π' .*

Proof: It is enough to prove that the sequent $\vdash A, A^\perp$ has a cut-free proof in which the (ax) rule is not used. And this can be done by a straightforward induction on A . \blacksquare

Now we can define the isomorphism of two LL formulas. Two LL formulas A and B are isomorphic, denoted by $A \cong B$, if there are two proofs π of $\vdash A^\perp, B$ and π' of $\vdash A, B^\perp$ such that eliminating the (cut) rule of

$$\frac{\frac{\pi}{\vdash A^\perp, B} \quad \frac{\pi'}{\vdash A, B^\perp}}{\vdash A, A} \text{ (cut)}$$

ends with an η -equivalent proof to $\frac{}{\vdash A, A} \text{ (ax)}$ ², and moreover eliminating the (cut) rule of

$$\frac{\frac{\pi}{\vdash A^\perp, B} \quad \frac{\pi'}{\vdash A, B^\perp}}{\vdash B, B} \text{ (cut)}$$

ends with an η -equivalent proof to $\frac{}{\vdash B, B} \text{ (ax)}$. As an example of two isomorphic formulas, we will investigate the example $!(A \& B) \cong !A \otimes !B$ which is known as *Seely iso*. Proof π of $\vdash !(A \& B) \multimap !A \otimes !B$ is as follows:

$$\frac{\frac{\frac{\frac{\frac{}{\vdash A^\perp, A} \text{ (ax)}}{\vdash A^\perp \oplus B^\perp, A} \text{ (}\oplus_1\text{)}}{\vdash ?(A^\perp \oplus B^\perp), A} \text{ (d)}}{\vdash ?(A^\perp \oplus B^\perp), !A} \text{ (p)}}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !A \otimes !B} \text{ (}\otimes\text{)}} \text{ (c)}$$

²By η -equivalent, we mean proof equivalence modulo commutation of rules and η -expansion, and η -expansion can be defined systematically by induction on A .

And proof π' of $\vdash !A \otimes !B \multimap !(A \& B)$:

$$\begin{array}{c}
 \frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{}{\vdash B^\perp, B} \text{ (ax)} \\
 \frac{}{\vdash ?A^\perp, A} \text{ (d)} \quad \frac{}{\vdash ?B^\perp, B} \text{ (d)} \\
 \frac{}{\vdash ?A^\perp, ?B^\perp, A} \text{ (w)} \quad \frac{}{\vdash ?A^\perp, ?B^\perp, B} \text{ (w)} \\
 \hline
 \frac{}{\vdash ?A^\perp, ?B^\perp, A \& B} \text{ (&)} \\
 \frac{}{\vdash ?A^\perp, ?B^\perp, !(A \& B)} \text{ (p)} \\
 \hline
 \frac{}{\vdash ?A^\perp \wp ?B^\perp, !(A \& B)} \text{ (\wp)}
 \end{array}$$

Now, if we try to eliminate the (cut) rule of

$$\frac{\frac{}{\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B} \pi \quad \frac{}{\vdash ?A^\perp \wp ?B^\perp, !(A \& B)} \pi'}{\vdash ?(A^\perp \oplus B^\perp), !(A \& B)} \text{ (cut)}$$

we will end with the following proof ρ :

$$\begin{array}{c}
 \frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{}{\vdash B^\perp, B} \text{ (ax)} \\
 \frac{}{\vdash A^\perp \oplus B^\perp, A} \text{ (\oplus}_1\text{)} \quad \frac{}{\vdash A^\perp \oplus B^\perp, B} \text{ (\oplus}_2\text{)} \\
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), A} \text{ (d)} \quad \frac{}{\vdash ?(A^\perp \oplus B^\perp), B} \text{ (d)} \\
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A} \text{ (w)} \quad \frac{}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), B} \text{ (w)} \\
 \hline
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), A \& B} \text{ (&)} \\
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), ?(A^\perp \oplus B^\perp), !(A \& B)} \text{ (p)} \\
 \hline
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), !(A \& B)} \text{ (c)}
 \end{array}$$

However, an η -equivalent proof to $\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$ is the following proof:

$$\begin{array}{c}
 \frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{}{\vdash B^\perp, B} \text{ (ax)} \\
 \frac{}{\vdash A^\perp \oplus B^\perp, A} \text{ (\oplus}_1\text{)} \quad \frac{}{\vdash A^\perp \oplus B^\perp, B} \text{ (\oplus}_2\text{)} \\
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), A} \text{ (d)} \quad \frac{}{\vdash ?(A^\perp \oplus B^\perp), B} \text{ (d)} \\
 \hline
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), A \& B} \text{ (&)} \\
 \frac{}{\vdash ?(A^\perp \oplus B^\perp), !(A \& B)} \text{ (p)}
 \end{array}$$

And the proof ρ is not exactly an η -equivalent proof to $\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$. In the literature, there are some equivalence relations on LL proofs making the proof ρ and the η -expansion of $\vdash ?(A^\perp \oplus B^\perp), !(A \& B)$ in a same class. Those usually called R etor e equivalences. For instance, one can identify the two following proofs:

$$\frac{\frac{}{\vdash ?A, \Gamma} \pi}{\vdash ?A, ?A, \Gamma} \text{ (w)} \quad \sim \quad \frac{}{\vdash ?A, \Gamma} \pi \\
 \frac{}{\vdash ?A, \Gamma} \text{ (c)}$$

We have a similar situation if we try to eliminate the (cut) rule of

$$\frac{\frac{\pi}{\vdash ?(A^\perp \oplus B^\perp), !A \otimes !B} \quad \frac{\pi'}{\vdash ?A^\perp \wp ?B^\perp, !(A \& B)}}{\vdash !A \otimes !B, ?A^\perp \wp ?B^\perp} \text{ (cut)}$$

The proof above will be normalized to the following proof ρ' :

$$\frac{\frac{\frac{\frac{\frac{}{\vdash A^\perp, A} \text{ (ax)}}{\vdash ?A^\perp, A} \text{ (d)}}{\vdash ?A^\perp, ?B^\perp, A} \text{ (w)}}{\vdash ?A^\perp, ?B^\perp, !A} \text{ (p)}}{\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp} \text{ (\otimes)}} \quad \frac{\frac{\frac{\frac{\frac{}{\vdash B^\perp, B} \text{ (ax)}}{\vdash ?B^\perp, B} \text{ (d)}}{\vdash ?A^\perp, ?B^\perp, B} \text{ (w)}}{\vdash ?A^\perp, ?B^\perp, !B} \text{ (p)}}{\vdash !A \otimes !B, ?A^\perp, ?B^\perp, ?A^\perp, ?B^\perp} \text{ (\otimes)}}}{\vdash !A \otimes !B, ?A^\perp, ?B^\perp} \text{ (c), (c)}} \text{ (\wp)}$$

whereas the η -expansion of $\vdash !A \otimes !B, ?A^\perp \wp ?B^\perp$ is as follows:

$$\frac{\frac{\frac{\frac{\frac{}{\vdash A, A^\perp} \text{ (ax)}}{\vdash A, ?A^\perp} \text{ (d)}}{\vdash !A, ?A^\perp} \text{ (p)}}{\vdash !A \otimes !B, ?A^\perp, ?B^\perp} \text{ (\otimes)}} \quad \frac{\frac{\frac{\frac{\frac{}{\vdash B, B^\perp} \text{ (ax)}}{\vdash B, ?B^\perp} \text{ (d)}}{\vdash !B, ?B^\perp} \text{ (p)}}{\vdash !A \otimes !B, ?A^\perp, ?B^\perp} \text{ (\otimes)}}}{\vdash !A \otimes !B, ?A^\perp \wp ?B^\perp} \text{ (\wp)}$$

However, one can ask this question that why we really need to have Rétoré equivalences. There are different ways to think about this question. For instance, there is Call-by-name Girard's translation of intuitionistic logic to LL that translates $A \Rightarrow B$ into $!A \multimap B$. If one wants to prove that cut-reductions of LL simulate the β -reduction of intuitionistic logic, we will see that Rétoré equivalences are essential.

Remark 3 *One can also think about this question more semantically (categorically) once we provide the categorical model LL in Section 0.3: All Rétoré equivalences indeed hold in any model of LL, and those are crucial to prove the soundness theorem (Theorem 6). And moreover, the Kleisli category $\mathcal{L}_!$ for any model \mathcal{L} of LL is a CCC category and therefore model of simple typed λ -calculus, and this is true due to fact that we have the Seely iso in \mathcal{L} .*

We end our presentation of LL with a very short discussion on *reversibility*.

Theorem 5 *The connectives $\wp, \perp, \&$ are reversible. That is to say:*

- *A sequent $\vdash \Gamma, A \wp B$ is provable iff $\vdash \Gamma, A, B$ is provable.*
- *A sequent $\vdash \Gamma, \perp$ is provable iff $\vdash \Gamma$ is provable.*
- *A sequent $\vdash \Gamma, A \& B$ is provable iff $\vdash \Gamma, A$ and $\vdash \Gamma, B$ are provable.*

Proof: The direction from right to left is trivial for all cases using the inference rules of LL (Figure 1). For the other direction of all cases, one can proceed by induction on the proof. ■

The reversible connectives are called the *negative* connectives of LL, and their dual are called *positive* connectives. Corresponding to this reversibility property of negative connectives, there is a property for positive connectives, which is called *focalization* [And92]. In Chapter 2, we will discuss about these positive and negative connectives [Lau02] in more details. However, our view on the polarity is more from a Curry-Howard perspective rather than reversibility/focalization.

Term calculus for LL

One of the engrossing and active research areas of proof theory is try to develop a term calculus for a given logic in the sens of Curry-Howard correspondence. And this correspondence is sometimes done via the sequent calculus style and sometimes via the natural deduction. Moreover, from the categorical point of view, it is quite useful to develop an internal language for certain categories, in the same sense that simply-typed λ -calculus (augmented by appropriate structures for products and the terminal object) is the internal language for cartesian closed categories. The same question is studied for different fragments of LL in the literature, and indeed there are many different languages that each of them has some advantages and disadvantages; we will discuss about them briefly and refer the reader to more suitable references. Perhaps one can refer to [Jay89, Lam89] as one of the earliest attempt to have such a language for monoidal categories. In [Abr93] we see a term calculus for intuitionistic and classical LL based on the sequent calculus in which one can find a concrete computational interpretation of LL, based on the Curry-Howard correspondence. Their calculus for classical LL is based on a syntax of the concurrency theory, and one of their motivation to use such a syntax is because of the (cut) rule in classical LL, since the (cut) rule in classical LL is fully symmetric whereas this is not the case in intuitionistic LL. In intuitionistic LL the left premise is distinguished from the right as that the cut formula appears in the output position (i.e. right side of turnstyle \vdash) in one, and in input position (i.e. left side of turnstyle \vdash) in the other. In term of programming languages, we can say that the (cut) rule in intuitionistic LL is noncommutative operation of function composition while the (cut) rule of classical LL is commutative which can be seen as *parallel composition* in the concurrency theory [Mil80, Mil92]. However, their systems lack two important properties; the *substitution* property and (consequently) *subject reduction*. This important issue is addressed in [BBdPH93, BBdPH92] for the intuitionistic LL without the additive connectives. Their approach is basically in two ways; the sequent calculus style (using the underlying categorical structure) and the natural deduction one, and it is shown that those two approaches produce equivalent systems. However, the relation between their process of cut-elimination and their proof normalisation is not clear. So, one needs to consider [Zuc74, Pot77] in their linear type system. In the same time, and independently to [BBdPH92], a similar work as [BBdPH92] have been done in [MRA93]. However, in [MRA93], they have moreover proved that their term calculus is indeed an internal language to autonomous (symmetric monoidal closed categories) categories. And they have also shown the coherence theorem of Kelly and Mac Lane as an application of their theorem [KM71, Mac63]. Afterwards, in [KhLO98], one can find an extension of [MRA93] to a classical version using a Parigot-style μ -abstraction. More precisely, they provide a term calculus for classical multiplicative linear logic, and they

show that their calculus is an internal language for $*$ -autonomous categories. One can also find an extension of the previous work to multiplicative and additives linear logic in [CP05] using a π -calculus syntax [BS94].

While talking about a term calculus for LL, we should of course talk about an important innovation of [Gir87a] which is the introduction of *proofs net*. One can think about proof-nets as a kind of natural deduction for LL. Although proof-nets provide a kind of term calculus for LL and moreover those give rise to an elegant proof for the coherence theorems of Kelly and Mac Lane [Blu93, BCST96], proof-nets have two major drawbacks. One is handling of the units of LL, and the other one is that dealing with additives connectives is not obvious. But we must admit that proofs net allow us to discard a lot of peripheral syntax that one can have in a term calculus for the sequent calculus; in particular the problematic commuting conversions are removed in the case of proofs net.

This question of having a term calculus for the sequent calculus LL has a difficulty that is probably not only related to LL, but more to the sequent calculus itself. One of the benefits of having a term calculus for the sequent calculus is that this kind of language has a symmetry, and one can also see implicit symmetries such as input/output in programming languages. So, having a term calculus for Gentzen's LJ and LK [Gen64] is studied in the literature. We mention some of them such as [UB99, Her95, Her94, UB99, Gal93, BTKP93]. Another work that led to a term calculus for linear logic is [CH00] in which a syntax for LK (and LJ) is provided. It is called $\bar{\lambda}\mu\bar{\mu}$ -calculus which is basically a λ -calculus plus the control operators [Par92]. One of the important issue in the classical languages is the Lafont's critical pair. This issue have been solved in [CH00] using a symmetry of call-by-name and call-by-value reduction strategy. Based on the $\bar{\lambda}\mu\bar{\mu}$ -calculus, we see another solution to this problem of critical pair in [CH00, Mum09]. There, they tackle this problem from the focalisation point of view. They provide a language called system L, as traditionally sequent calculus system names begin with the letter L. In this style, we have developed a term calculus for an extension of linear logic with fixpoints in Chapter 2.

0.2.2 Syntax of μ LL

We assume to be given an infinite set of propositional variables \mathcal{V} (ranged over by Greek letters $\zeta, \xi \dots$). We introduce a language of propositional LL formulas with least and greatest fixed points.

$$\begin{aligned}
 A, B, \dots := & 1 \mid \perp \mid A \otimes B \mid A \wp B \\
 & \mid 0 \mid \top \mid A \oplus B \mid A \& B \\
 & \mid !A \mid ?A \\
 & \mid \zeta \mid \mu\zeta A \mid \nu\zeta A.
 \end{aligned}$$

The notion of closed types is defined as usual, the two last constructions being the only binders.

We can define two basic operations on formulas.

- *Substitution*: $A[B/\zeta]$, taking care of avoiding the capture of free variables (uses α -conversion).

The identity fragment:

$$\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)}$$

The multiplicative fragment:

$$\frac{}{\vdash 1} \text{ (1)} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \text{ (\otimes)} \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp} \text{ (\perp)} \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \text{ (\wp)}$$

The additive fragment:

$$\frac{}{\vdash \Gamma, \top} \text{ (\top)} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \text{ (\oplus_1)} \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \text{ (\oplus_2)} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} \text{ (\&)}$$

The exponential fragment:

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \text{ (?w)} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \text{ (?c)} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \text{ (?d)} \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \text{ (?p)}$$

The fixed point fragment:

$$\frac{\vdash \Gamma, F [\mu\zeta F/\zeta]}{\vdash \Gamma, \mu\zeta F} \text{ (\mu-fold)} \quad \frac{\vdash \Delta, A \quad \vdash ?\Gamma, A^\perp, F [A/\zeta]}{\vdash \Delta, ?\Gamma, \nu\zeta F} \text{ (\nu-rec)}$$

Figure 2: Inference rules of μLL

- *Negation* or *dualization*: extended LL negation with $\zeta^\perp = \zeta$, $(\mu\zeta A)^\perp = \nu\zeta A^\perp$ and $(\nu\zeta A)^\perp = \mu\zeta A^\perp$. Obviously $A^{\perp\perp} = A$ for any formula A .

Remark 4 *The only subtle point of this definition is negation of propositional variables: $\zeta^\perp = \zeta$. This entails $(B [A/\zeta])^\perp = B^\perp [A^\perp/\zeta]$ by an easy induction on B . If we consider B as a compound logical connective with placeholders labeled by variables then B^\perp is its De Morgan dual. This definition of ζ^\perp is also a natural way of preventing the introduction of fixed points wrt. variables with negative occurrences. As an illustration, if we define as usual $A \multimap B$ as $A^\perp \wp B$ then we can define $E = \mu\zeta (1 \& (!\zeta \multimap \zeta))$ which looks like the definition of a model of the pure λ -calculus as a recursive type. But this is only an illusion since we actually have $E = \mu\zeta (1 \& (? \zeta \wp \zeta))$ so that $!E \multimap E$ is not a retract of E . And indeed if it were possible to define a type D such that $!D \multimap D$ is isomorphic to (or is a retract of) D then we would be able to type all pure λ -terms in our system and this would contradict the fact that μLL enjoys strong normalization and has a denotational semantics based on totality as shown below.*

Our logical system extends the usual single-sided sequent calculus of classical propositional LL [Gir87a], see also [Mel09] Section 3.1 and 3.13.

We give now the deduction rules, in a standard single-sided LL sequent calculus as in [Bae12] (Figure 2), and we use only closed formulas in the sequents. Basically, the deduction rules of μLL are the ones for LL plus $(\mu\text{-fold})$ and $(\nu\text{-rec})$ rules.

By taking, in the last rule, $\Delta = A^\perp$ and proving the left premise by an axiom, we obtain

the following derived rule:

$$\frac{\vdash ?\Gamma, A^\perp, F[A/\zeta]}{\vdash ?\Gamma, A^\perp, \nu\zeta F} (\nu - \text{rec}')$$

In Chapter 3 we will study another extension of LL which is non-well-founded LL [Dou17, BDS16].

The only cut-elimination rule of μLL that we give is $(\mu\text{-fold})/(\nu\text{-rec})$, in Section 0.2.2; for the other ones, see for instance [Gir87a] or any other presentation of the classical LL Sequent Calculus. We refer to [Bae12] for a proof that μMALL (multiplicative and additive linear logic with fixpoints of types) admits cut-elimination. Observe that a cut-free proof has not the sub-formula property in general because of rule $(\nu - \text{rec})$. But Baelde's theorem makes sure that a proof of a sequent *which does not contain any ν -formula* has a cut-free proof with the sub-formula property.

Remark 5 *If we consider the $(\nu - \text{rec}')$ rule instead of the $(\nu - \text{rec})$ rule, then the cut-elimination does not hold, as one can find the following counter-example in [Dou17]:*

$$\frac{\frac{\overline{\vdash 0, \top}}{\vdash 0, \nu\zeta\zeta} (\top) \quad \frac{\overline{\vdash 0, 0, \top}}{\vdash 0, 0, \nu\zeta\zeta} (\text{cut})}{\vdash 0, 0, \nu\zeta\zeta} (\nu - \text{rec}')$$

However, without the (cut) rule, the sequent $\vdash 0, 0, \nu\zeta F$ is not provable in μLL simply because there is no rule on 0 and the context are not ? formulas, so we cannot apply the $(\nu - \text{rec}')$ rule. But this is not surprising, since there is some sort of integrated cut formula in the $(\nu - \text{rec})$ rule, i.e, the formula A . And the price to pay to have the cut-elimination theorem for μLL is to loose the sub-formula property. The situation is general for finitary inductive logical systems, as it is well explained in [ML71] as follows:

“The opinion seems to have been generally accepted that there be no real cut elimination theorem for first order arithmetic and that such a theorem could only be obtained by eliminating the induction schema in favour of the ω -rule. However, when arithmetic is formulated as a theory of inductive definitions, it becomes possible to formulate and prove a cut elimination theorem which is just as natural and basic as the one for pure first order logic, although, like in second order logic, the subformula principle is necessarily lost.”

Syntactic functoriality of formulas

The reduction rule for the $(\mu - \text{fold})/(\nu - \text{rec})$ cut requires the possibility of substituting a proof for a propositional variable in a formula. More precisely, let $(\zeta, \xi_1, \dots, \xi_k)$ be a list of pairwise distinct propositional variables containing all the free variables of a formula F and let $\vec{C} = (C_1, \dots, C_k)$ be a sequence of closed formulas. Let π be a proof³ of $\vdash ?\Gamma, A^\perp, B$, then one defines a proof $F[\pi/\zeta, \vec{C}/\vec{\xi}]$ of

$$\vdash ?\Gamma, (F[A/\zeta, \vec{C}/\vec{\xi}])^\perp, F[B/\zeta, \vec{C}/\vec{\xi}]$$

³Again the fact that the formulas of the context bear a ?_ is absolutely necessary to make this definition possible.

by induction on F , adapting the corresponding definition in [Bae12]. We illustrate this definition by two inductive steps.

Assume first that $F = \mu\xi G$ (so that $(\zeta, \xi, \xi_1, \dots, \xi_k)$ is a list of pairwise distinct variables containing all free variables of G). Let $G' = G[\vec{C}/\vec{\xi}]$ whose only possible free variables are ζ and ξ . The proof $F[\pi/\zeta, \vec{C}/\vec{\xi}]$ is defined by

$$\frac{\begin{array}{c} \vdots \\ G[\pi/\zeta, (\mu\xi G') [B/\zeta] / \xi, \vec{C}/\vec{\xi}] \\ \vdots \end{array} \quad \frac{\vdash ?\Gamma, (G'[A/\zeta, (\mu\xi G') [B/\zeta] / \xi)]^\perp, G'[B/\zeta, (\mu\xi G') [B/\zeta] / \xi]}{\vdash ?\Gamma, (G'[A/\zeta, (\mu\xi G') [B/\zeta] / \xi)]^\perp, (\mu\xi G') [B/\zeta]} \text{ (\mu - fold)}}{\vdash ?\Gamma, ((\mu\xi G') [A/\zeta])^\perp, (\mu\xi G') [B/\zeta]} \text{ (\nu - rec')}$$

Notice that this case uses the additional parameters \vec{C} in the definition of this substitution with $k+1$ parameters in the inductive hypothesis. To see that the last inference in this deduction is an instance of $(\nu\text{-rec}'$), set $H = (G')^\perp [A^\perp/\zeta]$ and notice that $(G'[A/\zeta, (\mu\xi G') [B/\zeta] / \xi])^\perp = H[((\mu\xi G') [A/\zeta])^\perp / \xi]$ and $((\mu\xi G') [A/\zeta])^\perp = \nu\xi H$. Another example is $F = G_1 \otimes G_2$: $F[\pi/\zeta, \vec{C}/\vec{\xi}]$ is defined as

$$\frac{\begin{array}{c} \vdots \\ G_1[\pi/\zeta, \vec{C}/\vec{\xi}] \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ G_2[\pi/\zeta, \vec{C}/\vec{\xi}] \\ \vdots \end{array} \quad \frac{\vdash ?\Gamma, (G'_1 [A/\zeta])^\perp, G'_1 [B/\zeta] \quad \vdash ?\Gamma, (G'_2 [A/\zeta])^\perp, G'_2 [B/\zeta]}{\vdash ?\Gamma, ?\Gamma, (G'_1 [A/\zeta])^\perp, (G'_2 [A/\zeta])^\perp, G'_1 [B/\zeta] \otimes G'_2 [B/\zeta]} \text{ (\otimes)}}{\frac{\vdash ?\Gamma, (G'_1 [A/\zeta])^\perp, (G'_2 [A/\zeta])^\perp, G'_1 [B/\zeta] \otimes G'_2 [B/\zeta]}{\vdash ?\Gamma, (G'_1 [A/\zeta])^\perp, (G'_2 [A/\zeta])^\perp, G'_1 [B/\zeta] \otimes G'_2 [B/\zeta]} \text{ (c)}}{\vdash ?\Gamma, (G'_1 [A/\zeta])^\perp \wp (G'_2 [A/\zeta])^\perp, G'_1 [B/\zeta] \otimes G'_2 [B/\zeta]} \text{ (\wp)}} \text{ (\wp)}$$

Observe that we use in an essential way the fact that all formulas of the context are of shape $?H$ (even if F is exponential-free) when we apply contraction rules on this context. Notice again that the exchange rule is kept implicit.

$(\mu\text{-fold})/(\nu\text{-rec})$ Cut elimination rule

We can now provide the $(\mu\text{-fold})/(\nu\text{-rec})$ reduction rule. Let θ be the following proof:

$$\frac{\frac{\begin{array}{c} \vdots \pi \\ \vdash \Lambda, F[\mu\zeta F/\zeta] \end{array} \text{ (\mu - fold)}}{\vdash \Lambda, \mu\zeta F} \quad \frac{\begin{array}{c} \vdots \lambda \\ \vdash \Delta, A^\perp \end{array} \quad \frac{\begin{array}{c} \vdots \rho \\ \vdash ?\Gamma, A, (F[A/\zeta])^\perp \end{array} \text{ (\nu - rec)}}{\vdash \Delta, ?\Gamma, (\mu\zeta F)^\perp}}{\vdash \Lambda, \Delta, ?\Gamma}$$

and let ρ' be the proof

$$\frac{\begin{array}{c} \vdots \rho \\ \vdash ?\Gamma, A, (F[A/\zeta])^\perp \end{array} \text{ (\nu - rec')}}{\vdash ?\Gamma, A, (\mu\zeta F)^\perp}$$

Then θ reduces to the following proof

$$\frac{\frac{\frac{\vdots F[\rho'/\zeta]}{\vdots \pi} \quad \vdots \rho}{\vdash \Lambda, F[\mu\zeta F/\zeta]} \quad \vdash \Lambda, F[\mu\zeta F/\zeta]}{\vdash \Lambda, ?\Gamma, F[A/\zeta]} \quad \vdash ?\Gamma, A, (F[A/\zeta])^\perp}{\frac{\frac{\vdash \Lambda, ?\Gamma, ?\Gamma, A}{\vdash \Lambda, ?\Gamma, A} \quad (c) \quad \vdots \lambda}{\vdash \Delta, \Lambda, ?\Gamma}}$$

We end this section by some remarks about the differences between our system μLL and Baelde's μMALL :

- Remark 6** • *Baelde's logical system is a higher-order predicate calculus whereas our system is a propositional calculus. Indeed, Baelde is mainly interested in applying μMALL to program verification where the predicate calculus is essential for expressing properties of programs. We have a Curry-Howard perspective where formulas are seen as types and proofs as programs and where a propositional logical system is sufficient.*
- *Our system has exponentials whereas Baelde's system has not because they can be encoded in μMALL to some extent. However the exponentials encoded in that way do not satisfy all required isos (in particular the "Seely morphisms" are not isos with Baelde's exponentials) and this is a serious issue if we want to encode some form of λ -calculus in the system and consider it as a programming language.*
 - *Our $(\nu - \text{rec})$ rule differs from Baelde's by the fact that we admit a context in the right premise. Notice that all formulas of this context must bear a $?_$ modality: this restriction is absolutely crucial for allowing to express the cut-elimination rule in Section 0.2.2 which uses an operation of substitution of proofs in formulas and this operation uses structural rules on the context. The semantic counterpart of this operation is described in Section 1.1.2 where it appears clearly that it uses the fact that P is an object of \mathcal{L}^\perp . So such a version of $(\nu - \text{rec})$ with a context would be problematic in Baelde's system, due to the absence of built-in exponentials.*

0.3 Categorical semantics of LL

Two early formulations of categorical model of intuitionistic LL are the one by Lafont [Laf88b] and by Seely [See89]. The formulation by Lafont, which is based on a free construction of the exponentials, will not capture some concrete models of LL. For example, coherence spaces with the Girard exponential [Gir87a] is not a free one. For that reason, a more relaxed definition of Lafont categories is presented in Section 5 of [Mel03]. Regarding the Seely formulation in [See89], Benton, Bierman, de Paiva and Hyland in [Hy197, BBdPH92] pointed out that Seely's formulation is not complete. And more rigorously, Bierman showed in [Bie94, Bie95] that Seely's category is not a sound model of intuitionistic LL. Hence Bierman in [Bie94] add a new assumption in Seely's formulation, and then proved its soundness using another formulation, so-called linear categories introduced in [BBdPH92]. Afterwards, Benton in [Ben95] provides *linear-non-linear* category definition which seems a simpler definition at first sight rather than Seely and linear categories.

In this thesis, we consider the notion of *Seely category* as presented in [Mel09]. Although, we do not recall all technical details, we ensure that a reader can follow the thesis with the provided definitions. And we refer to [Mel09] for all the other technical materials. We also assume that a reader is familiar with the following notion in the category theory: Categories, Functors, Initial and Final Objects, Natural Transformations, Products and Coproducts, CCC categories, Monoidal and *-autonomous categories, Monads and Comonads, Strong and Lax monoidal functors, Algebra and Coalgebra of endofunctors. And we refer the reader to [Mac71, Bar79] to see all those notions in the category theory.

0.3.1 Seely categories

We define the basic notion of categorical model of LL (our main reference is the notion of a *Seely category* as presented in [Mel09]. We refer to that survey for all the technical material that we do not recall here).

A Seely category is a symmetric monoidal closed category (SMCC) $(\mathcal{L}, \otimes, 1, \lambda, \rho, \alpha, \gamma)$ where $\lambda_X \in \mathcal{L}(1 \otimes X, X)$, $\rho_X \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X,Y,Z} \in \mathcal{L}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$ and $\gamma_{X,Y} \in \mathcal{L}(X \otimes Y, Y \otimes X)$ are natural isomorphisms satisfying coherence diagrams that we do not record here. We use $X \multimap Y$ for the object of linear morphisms from X to Y , $\text{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$ for the evaluation morphism and cur for the linear curryfication map $\mathcal{L}(Z \otimes X, Y) \rightarrow \mathcal{L}(Z, X \multimap Y)$. We assume \mathcal{L} to be *-autonomous with dualizing object \perp (this object is part of the structure of a Seely category). We use X^\perp for the object $X \multimap \perp$ of \mathcal{L} (the dual, or linear negation, of X). It is also assumed that \mathcal{L} is cartesian with final object \top , product $X_1 \& X_2$ with projections pr_1, pr_2 . We will denote by t_X the unique morphism from an object X to \top . So, if there are morphisms $f_1 : X \rightarrow X_1$ and $f_2 : X \rightarrow X_2$, then there is a unique morphism $\langle f_1, f_2 \rangle : X \rightarrow X_1 \& X_2$ such that $\text{pr}_i \circ \langle f_1, f_2 \rangle = f_i$ for $i = 1, 2$. By *-autonomy \mathcal{L} is cocartesian with initial object 0 , coproduct \oplus and injections $\bar{\pi}_i$.

We also assume to be given a comonad $!_ : \mathcal{L} \rightarrow \mathcal{L}$ with counit $\text{der}_X \in \mathcal{L}(!X, X)$ (*derelection*) and comultiplication $\text{dig}_X \in \mathcal{L}(!X, !!X)$ (*digging*) together with a symmetric monoidal structure (Seely natural isos $\text{m}^0 : 1 \rightarrow !\top$ and m^2 with $\text{m}_{X_1, X_2}^2 : !X_1 \otimes !X_2 \rightarrow !(X_1 \& X_2)$ for the functor $!_$, from the symmetric monoidal category $(\mathcal{L}, \&)$ to the symmetric monoidal category (\mathcal{L}, \otimes) satisfying an additional coherence condition wrt. dig . This strong monoidal structure allows to define a lax monoidal structure (μ^0, μ^2) of $!_$ from (\mathcal{L}, \otimes) to itself. More precisely $\mu^0 \in \mathcal{L}(1, !1)$ and $\mu_{X_1, X_2}^2 \in \mathcal{L}(!X_1 \otimes !X_2, !(X_1 \otimes X_2))$ are defined using m^0 and m^2 (and are not isos in most cases). Then, by induction on n , one has $\mu^n \in \mathcal{L}(!X_1 \otimes \dots \otimes !X_n, !(X_1 \otimes \dots \otimes X_n))$. Also, for each object $X \in \text{Obj}(\mathcal{L})$, there is a canonical structure of commutative \otimes -comonoid on $!X$ given by $\text{w}_X \in \mathcal{L}(!X, 1)$ and $\text{contr}_X \in \mathcal{L}(!X, !X \otimes !X)$. The definition of these morphisms involves all the structure of $!_$ explained above, and in particular the Seely isos.

In Chapter 4, we will use the fact that the following equation holds

$$\text{n}_{X_1, X_2}^2 = (!\text{pr}_1 \otimes !\text{pr}_2) \text{contr}_{X_1 \& X_2} \quad (1)$$

and also, as a consequence:

$$\begin{aligned} \text{w}_{X_1 \& X_2} \text{m}_{X_1, X_2}^2 &= \text{w}_{X_1} \otimes \text{w}_{X_2} \\ \text{der}_{X_1 \& X_2} \text{m}_{X_1, X_2}^2 &= \langle \text{der}_{X_1} \otimes \text{w}_{X_2}, \text{w}_{X_1} \otimes \text{der}_{X_2} \rangle \\ \text{contr}_{X_1 \& X_2} \text{m}_{X_1, X_2}^2 &= \left(\text{m}_{X_1, X_2}^2 \otimes \text{m}_{X_1, X_2}^2 \right) \gamma_{2,3} (\text{contr}_{X_1} \otimes \text{contr}_{X_2}) \end{aligned} \quad (2)$$

We use $?_$ for the “De Morgan dual” of $!_$: $?X = (!((X)^\perp))^\perp$ and similarly for morphisms. It is a monad on \mathcal{L} .

0.3.2 Oplax monoidal comonads

Let \mathcal{M} be a symmetric monoidal category (with the same notations as above for the tensor product) and $(T, \epsilon, \mu) : \mathcal{M} \rightarrow \mathcal{M}$ be a comonad (ϵ is the counit and μ the comultiplication). An *oplax monoidal* structure on T consists of a morphism $\theta^0 \in \mathcal{M}(T1, 1)$ and a natural transformation $\theta_{X_1, X_2}^2 \in \mathcal{M}(T(X_1 \otimes X_2), T(X_1) \otimes T(X_2))$ subject to standard symmetric monoidality and compatibility with ϵ and μ , this latter reading $(\epsilon_{X_1} \otimes \epsilon_{X_2}) \theta_{X_1, X_2} = \epsilon_{X_1 \otimes X_2}$ and:

$$\begin{array}{ccc} T(X_1 \otimes X_2) & \xrightarrow{\theta_{X_1, X_2}^2} & TX_1 \otimes TX_2 \xrightarrow{\mu_{X_1} \otimes \mu_{X_2}} T^2 X_1 \otimes T^2 X_2 \\ \mu_{X_1 \otimes X_2} \downarrow & & \uparrow \theta_{TX_1, TX_2} \\ T^2(X_1 \otimes X_2) & \xrightarrow{T(\theta_{X_1, X_2}^2)} & T(TX_1 \otimes TX_2) \end{array}$$

Then the Kleisli category \mathcal{M}_T has a canonical symmetric monoidal structure, with unit 1 and tensor product $X_1 \otimes X_2$ defined as in \mathcal{M} for objects. Given $f_i \in \mathcal{M}_T(X_i, Y_i)$, $f_1 \otimes_T f_2 \in \mathcal{M}_T(X_1 \otimes X_2, Y_1 \otimes Y_2)$ is defined as

$$T(X_1 \otimes X_2) \xrightarrow{\theta_{X_1, X_2}^2} TX_1 \otimes TX_2 \xrightarrow{f_1 \otimes f_2} Y_1 \otimes Y_2 .$$

Let $F_T : \mathcal{M} \rightarrow \mathcal{M}_T$ be the canonical functor which acts as the identity on objects and maps $f \in \mathcal{M}(X, Y)$ to $f \epsilon_X \in \mathcal{M}_T(X, Y)$.

0.3.3 Eilenberg-Moore category

Let \mathcal{L} be a Seelye category. Since $!_$ is a comonad we can define the category $\mathcal{L}^!$ of $!$ -coalgebras (Eilenberg-Moore category of $!_$). An object of this category is a pair $P = (\underline{P}, h_P)$ where $\underline{P} \in \text{Obj}(\mathcal{L})$ and $h_P \in \mathcal{L}(\underline{P}, !\underline{P})$ is such that the following diagrams commute:

$$\begin{array}{ccc} \underline{P} & \xrightarrow{h_P} & !\underline{P} \\ \text{id} \searrow & & \downarrow \text{der}_P \\ & & \underline{P} \end{array} \quad \begin{array}{ccc} \underline{P} & \xrightarrow{h_P} & !\underline{P} \\ h_P \downarrow & & \downarrow \text{dig}_P \\ !\underline{P} & \xrightarrow{!h_P} & !!\underline{P} \end{array}$$

Then $f \in \mathcal{L}^!(P, Q)$ if $f \in \mathcal{L}(\underline{P}, \underline{Q})$ and the following diagram commutes:

$$\begin{array}{ccc} \underline{P} & \xrightarrow{f} & \underline{Q} \\ h_P \downarrow & & \downarrow h_Q \\ !\underline{P} & \xrightarrow{!f} & !\underline{Q} \end{array}$$

The functor $!_$ can be seen as a functor from \mathcal{L} to $\mathcal{L}^!$ mapping X to $(!X, \text{dig}_X)$ and $f \in \mathcal{L}(X, Y)$ to $!f$. It is right adjoint to the forgetful functor $\mathcal{L}^! \rightarrow \mathcal{L}$. Given $f \in \mathcal{L}(\underline{P}, X)$, we

use $f^! \in \mathcal{L}^!(P, !X)$ for the morphism associated with f by this adjunction, one has $f^! = !f h_P$. If $g \in \mathcal{L}^!(Q, P)$, we have $f^! g = (f g)^!$. Then $\mathcal{L}^!$ is cartesian with final object $(1, h_1 = \mu^0)$ still denoted as 1 and product $P_1 \otimes P_2 = (\underline{P}_1 \otimes \underline{P}_2, h_{P_1 \otimes P_2})$ with $h_{P_1 \otimes P_2}$ defined as follows:

$$\underline{P}_1 \otimes \underline{P}_2 \xrightarrow{h_{P_1 \otimes P_2}} !\underline{P}_1 \otimes !\underline{P}_2 \xrightarrow{\mu_{\underline{P}_1, \underline{P}_2}^2} !(\underline{P}_1 \otimes \underline{P}_2) .$$

This category is also cocartesian with initial object $(0, h_0)$ (h_0 comes from the initiality of 0) still denoted as 0 and coproduct $P_1 \oplus P_2 = (\underline{P}_1 \oplus \underline{P}_2, h_{P_1 \oplus P_2})$ with $h_{P_1 \oplus P_2}$ defined as follows.

For $i = 1, 2$ one defines $h^i : P_i \rightarrow !(P_1 \oplus P_2)$ as $\underline{P}_1 \xrightarrow{h_{P_1}} !\underline{P}_1 \xrightarrow{! \bar{\pi}_i} !(P_1 \oplus P_2)$, and then $h_{P_1 \oplus P_2}$ is the unique morphism $\underline{P}_1 \oplus \underline{P}_2 \rightarrow !(P_1 \oplus P_2)$ such that $h_{P_1 \oplus P_2} \bar{\pi}_i = h_i$ for $i = 1, 2$. More details can be found in [Mel09]. We use $\text{contr}_P \in \mathcal{L}^!(P, P \otimes P)$ (*contraction*) for the diagonal and $\text{w}_P \in \mathcal{L}^!(P, 1)$ (*weakening*) for the unique morphism to the final object. These morphisms turn \underline{P} into a commutative \otimes -comonoid, and are defined as the two following morphisms:

$$\begin{aligned} \underline{P} &\xrightarrow{h_P} !\underline{P} \xrightarrow{!(\text{Id}, \text{Id})} !(\underline{P} \& \underline{P}) \xrightarrow{(\text{m}^2)^{-1}} !\underline{P} \otimes !\underline{P} \xrightarrow{\text{der}_{\underline{P}} \otimes \text{der}_{\underline{P}}} \underline{P} \otimes \underline{P} \\ \underline{P} &\xrightarrow{h_P} !\underline{P} \xrightarrow{! \tau_{\underline{P}}} !(\top) \xrightarrow{(\text{m}^0)^{-1}} 1 \end{aligned}$$

We summarize the Seely's formulation of categorical models of linear logic that we will use several times in this thesis as follows:

$$\underbrace{\left(\underbrace{\underbrace{\langle L, \otimes, 1, \lambda, \rho, \alpha, \gamma \rangle}_{\text{symmetric monoidal}}, \text{ev}, \text{cur}, \perp, \&, \top}_{\text{symmetric monoidal closed}}, \underbrace{!, \text{der}_X, \text{dig}_X}_{\text{comonad}}, \underbrace{\text{m}^0, \text{m}^2}_{\text{strong symmetric monoidal functor}} \right)}_{\text{Model of LL}} \text{*-autonomous cartesian}$$

0.3.4 Interpretation of LL proofs and formulas

Interpretation of LL formulas are basically using the structure of Seely category \mathcal{L} . Interpretation of LL proofs is defined by induction on the last inference rule. So, one only needs to provide interpretation of each inference rule in \mathcal{L} which is done in Figure 3. The general idea is to interpret a proof π of $\vdash \Gamma, \Delta$ as a morphism $\llbracket \pi \rrbracket$ in $\mathcal{L}(\llbracket \Gamma \rrbracket^\perp, \llbracket \Delta \rrbracket)$ where $\Gamma = A_1, \dots, A_n$ (respectively $\Delta = B_1, \dots, B_m$) and $\llbracket \Gamma \rrbracket^\perp = (\llbracket A_1 \rrbracket)^\perp \otimes (\llbracket A_2 \rrbracket)^\perp \otimes \dots$ (respectively $\llbracket \Delta \rrbracket = \llbracket B_1 \rrbracket \wp \llbracket B_2 \rrbracket \wp \dots$). In this thesis, we keep symmetric monoidality isomorphisms of \mathcal{L} and of $!_-$ implicit (see for instance [Ehr18] how *monoidal trees* allow to take them into account).

And then one can prove the following [Mel09]:

Theorem 6 *If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of LL, then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$.*

$$\begin{aligned}
 \left[\frac{}{\vdash A, A^\perp} \text{(ax)} \right] &= \text{Id}_A \quad \left[\frac{\begin{array}{c} \vdots \pi_1 \quad \vdots \pi_2 \\ \vdash \Gamma, A \quad \vdash A^\perp, \Delta \end{array}}{\vdash \Gamma, \Delta} \text{(cut)} \right] = ([\Gamma])^\perp \xrightarrow{[\pi_1]} [A] \xrightarrow{[\pi_2]} [\Delta] \\
 \left[\frac{}{\vdash 1} \text{(1)} \right] &= \text{Id}_1 \quad \left[\frac{\begin{array}{c} \vdots \pi_1 \quad \vdots \pi_2 \\ \vdash \Gamma, A \quad \vdash \Delta, B \end{array}}{\vdash \Gamma, \Delta, A \otimes B} \text{(\otimes)} \right] = ([\Gamma])^\perp \otimes ([\Delta])^\perp \xrightarrow{[\pi_1] \otimes [\pi_2]} [A] \otimes [B] \\
 \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash \Gamma \end{array}}{\vdash \Gamma, \perp} \text{(\perp)} \right] &= \text{cur}([\pi]) \quad \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash \Gamma, A, B \end{array}}{\vdash \Gamma, A \wp B} \text{(\wp)} \right] = [\pi] \quad \left[\frac{}{\vdash \Gamma, \top} \text{(\top)} \right] = \mathbf{t}_{([\Gamma])^\perp} \\
 \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash \Gamma, A_i \end{array}}{\vdash \Gamma, A_1 \oplus A_2} \text{(\oplus}_i\text{)} \right] &= ([\Gamma])^\perp \xrightarrow{[\pi]} [A_i] \xrightarrow{\bar{\pi}_i} [A_1] \oplus [A_2] \\
 \left[\frac{\begin{array}{c} \vdots \pi_1 \quad \vdots \pi_2 \\ \vdash \Gamma, A \quad \vdash \Gamma, B \end{array}}{\vdash \Gamma, A \& B} \text{(\&)} \right] &= ([\Gamma])^\perp \xrightarrow{\langle [\pi_1], [\pi_2] \rangle} [A] \& [B] \\
 \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash \Gamma \end{array}}{\vdash \Gamma, ?A} \text{(w)} \right] &= \text{cur}(f) \text{ where } f \text{ is } \quad \begin{array}{c} ([\Gamma])^\perp \otimes [!A^\perp] \xrightarrow{\text{Id} \otimes \mathbf{w}_{A^\perp}} ([\Gamma])^\perp \otimes 1 \xrightarrow{\simeq} ([\Gamma])^\perp \\ \downarrow [\pi] \\ \perp \end{array} \\
 \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash \Gamma, ?A, ?A \end{array}}{\vdash \Gamma, ?A} \text{(c)} \right] &= \text{cur}(f) \text{ where } f \text{ is } \quad \begin{array}{c} ([\Gamma])^\perp \otimes (!A^\perp \otimes !A^\perp) \xrightarrow{\text{Id} \otimes \text{contr}_{A^\perp}} ([\Gamma])^\perp \otimes !A^\perp \xrightarrow{[\pi]} \perp \end{array} \\
 \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash \Gamma, A \end{array}}{\vdash \Gamma, ?A} \text{(d)} \right] &= \text{cur}(f) \text{ where } f = ([\Gamma])^\perp \otimes [!A^\perp] \xrightarrow{\text{Id} \otimes \text{der}_{A^\perp}} ([\Gamma])^\perp \otimes A^\perp \xrightarrow{[\pi]} \perp \\
 \left[\frac{\begin{array}{c} \vdots \pi \\ \vdash ?\Gamma, A \end{array}}{\vdash ?\Gamma, !A} \text{(p)} \right] &= [\otimes_{B_i^\perp \in \Gamma} (!B_i)] \xrightarrow{\otimes \text{dig}_{B_i}} [\otimes_{B_i^\perp \in \Gamma} !!B_i] \xrightarrow{\mu^n} !([\otimes_{B_i^\perp \in \Gamma} !B_i]) \xrightarrow{![\pi]} ![A]
 \end{aligned}$$

Figure 3: Interpretation of LL rules

0.4 Concrete models of LL

In this section we recall three concrete models of LL that we will use later. We start by recalling the relational model of LL.

0.4.1 Sets and relations

The category **Rel** has sets as objects, and relations as morphisms: given sets E and F , $\mathbf{Rel}(E, F) = \mathcal{P}(E \times F)$. Identity is the diagonal relation and composition is the usual composition of relations, denoted by simple juxtaposition; Let $t \in \mathbf{Rel}(E, F)$ and $s \in \mathbf{Rel}(F, G)$, then $s \circ t = \{(a, c) \in E \times G \mid \exists b \in F ((a, b) \in t \text{ and } (b, c) \in s)\}$. If $t \in \mathbf{Rel}(E, F)$ and $u \subseteq E$ then $t \cdot u = \{b \in F \mid \exists a \in u (a, b) \in t\}$.

This category is a well-known model of LL in which $1 = \perp = \{*\}$, $E \otimes F = (E \multimap F) = E \wp F = E \times F$ so that $E^\perp = E$. As to the additives, $0 = \top = \emptyset$ and $\&_{i \in I} E_i = \oplus_{i \in I} E_i = \bigcup_{i \in I} \{i\} \times E_i$. The symmetric monoidal structure of **Rel** is closed, and this closedness manifests itself by the existence of the canonical ev and $\text{cur}()$ operations:

$$\begin{aligned} \text{cur}(f) &= \{(a, (b, c)) \mid ((a, b), c) \in f\} \\ \text{ev}_{E,F} &= \{(((a, b), a), b) \mid a \in E \wedge b \in F\} \end{aligned}$$

The exponentials are given by $!E = ?E = \mathcal{M}_{\text{fin}}(E)$ (finite multisets of elements of E). For the additives and multiplicatives, the operations on morphisms are defined in the obvious way. Let us be more specific about the exponentials. Given $s \in \mathbf{Rel}(E, F)$, $!s \in \mathbf{Rel}(!E, !F)$ is $!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid \forall i (a_i, b_i) \in s\}$, $\text{der } E \in \mathbf{Rel}(!E, E)$ is given by $\text{der } E = \{([a], a) \mid a \in E\}$ and $\text{dig}_E \in \mathbf{Rel}(!E, !!E)$ is given by $\text{dig}_E = \{(m_1 + \dots + m_n, [m_1, \dots, m_n]) \mid \forall i m_i \in \mathcal{M}_{\text{fin}}(E)\}$. Last $m^0 \in \mathbf{Rel}(1, !\top)$ is $m^0 = \{(*, [])\}$ and $m_{E,F}^2 \in \mathbf{Rel}(!E \otimes !F, !(E \& F))$ is given by

$$\begin{aligned} m_{E,F}^2 &= \{([a_1, \dots, a_k], [b_1, \dots, b_l]), [(1, a_1), \dots, (1, a_k), (2, b_1), \dots, (2, b_l)]\} \\ &\quad \mid a_1, \dots, a_k \in E \text{ and } b_1, \dots, b_l \in F. \end{aligned}$$

Weakening $w_E \in \mathbf{Rel}(!E, 1)$ and $\text{contr}_E \in \mathbf{Rel}(!E, !E \otimes !E)$ are given by $w_E = \{([], *)\}$ and $\text{contr}_E = \{(m_1 + m_2, (m_1, m_2)) \mid m_i \in \mathcal{M}_{\text{fin}}(E) \text{ for } i = 1, 2\}$.

We end this section with an auxiliary lemma about the Eilenberg-Moore category of **Rel** that will be useful in Chapter 2.

Lemma 7 *If P is an object of $\mathbf{Rel}^!$, then any morphism $f \in \mathbf{Rel}^!(1, P)$ is non-empty.*

Proof: Since $f \in \mathbf{Rel}^!(1, P)$, one has the commutations of
$$\begin{array}{ccc} \underline{1} & \xrightarrow{f} & P \\ h_1 \downarrow & & \downarrow h_P \\ \underline{!1} & \xrightarrow{!f} & !P \end{array}$$
 Let assume f

is the empty morphism. So, one has $(*, []) \in (!f) \circ (h_1)$. But $(*, []) \notin (h_P) \circ (f) = \emptyset$ which contradicts commutation of the diagram. ■

0.4.2 Coherence spaces

We consider now the case where \mathcal{L} is the category **Coh** of coherence spaces and linear maps, a well-known model of LL introduced in [Gir86, Gir87a].

Definition 8 A coherence space is a structure $E = (|E|, \supseteq_E)$ where $|E|$ is a set called the web of E and \supseteq_E is a binary, reflexive and symmetric relation on $|E|$.

A clique of E is a subset u of $|E|$ such that $\forall a_1, a_2 \in u \ a_1 \supseteq_E a_2$. We use $\text{Cl}(E)$ for the set of all cliques of E that we consider as a domain, the order relation on $\text{Cl}(E)$ being always inclusion. Observe indeed that $\emptyset \in \text{Cl}(E)$ (that is $\text{Cl}(E)$ has a least element), if $u \subseteq v$ and $v \in \text{Cl}(E)$ then $u \in \text{Cl}(E)$ and last if $D \subseteq \text{Cl}(E)$ is directed then $\bigcup D \in \text{Cl}(E)$.

Coherence spaces as a model of LL

Given coherence spaces E and F we define a coherence space $E \multimap F$ whose web is $|E| \times |F|$ and coherence is: $(a_1, b_1) \supseteq_{E \multimap F} (a_2, b_2)$ if $a_1 \supseteq_E a_2 \Rightarrow (b_1 \supseteq_F b_2 \text{ and } b_1 = b_2 \Rightarrow a_1 = a_2)$.

We now define category **Coh** that has coherence spaces as objects, and homsets $\mathbf{Coh}(E, F) = \text{Cl}(E \multimap F)$. In this category the identities are the diagonal relations and composition is the ordinary composition of relations.

Remark 7 It can be useful to keep in mind that these morphisms can be considered as linear functions: a function $f : \text{Cl}(E) \rightarrow \text{Cl}(F)$ is linear if it is stable (that is $\forall u_1, u_2 \in \text{Cl}(E) \ u_1 \cup u_2 \in \text{Cl}(E) \Rightarrow f(u_1 \cap u_2) = f(u_1) \cap f(u_2)$) and commutes with arbitrary well-defined unions of cliques. Such a function f has a trace $\text{tr}f = \{(a, b) \in |E| \times |F| \mid b \in f(\{a\})\}$ and this trace operation defines a bijection between $\text{Cl}(E \multimap F)$ and the set of all linear functions from $\text{Cl}(E)$ to $\text{Cl}(F)$. The converse of this operation maps $t \in \text{Cl}(E \multimap F)$ to the function $\text{fun}(t) : \text{Cl}(E) \rightarrow \text{Cl}(F)$ defined by $\text{fun}(t)(u) = \{b \in |F| \mid \exists a \in u \ (a, b) \in t\}$. We will always write $t.u$ instead of $\text{fun}(t)(u)$. In this manuscript we stick to the relational point of view on morphisms.

This category is monoidal, with tensor product $E_1 \otimes E_2$ having $|E_1| \times |E_2|$ as web and $(a_1, a_2) \supseteq_{E_1 \otimes E_2} (a'_1, a'_2)$ if $a_i \supseteq_{E_i} a'_i$ for $i = 1, 2$. Given $t_i \in \mathcal{L}(E_i, F_i)$ for $i = 1, 2$, one defines $t_1 \otimes t_2$ as $\{((a_1, a_2), (b_1, b_2)) \mid (a_i, b_i) \in t_i \text{ for } i = 1, 2\} \in \mathcal{L}(E_1 \otimes E_2, F_1 \otimes F_2)$ as easily checked. So \otimes is a functor $\mathcal{L}^2 \rightarrow \mathcal{L}$, which equips \mathcal{L} with an obvious symmetric monoidal structure that we will not make explicit here, for a unit object $1 = (\{*\}, =)$. This category is monoidal closed with $E \multimap F$ object of morphisms from E to F (and evaluation and curryfication morphism $\text{ev} \in \mathcal{L}((E \multimap F) \otimes E, F)$ defined by $\text{ev} = \{(((a, b), a), b) \mid a \in |E| \text{ and } b \in |F|\}$ and $\text{cur}(f) = \{(a, (b, c)) \mid ((a, b), c) \in f\}$). Taking $\perp = 1$ as dualizing object, \mathcal{L} is easily seen to be $*$ -autonomous and the corresponding orthogonality is a functor $(_)^\perp : \mathcal{L}^{\text{op}} \rightarrow \mathcal{L}$ where $(E)^\perp = (|E|, \succsim_E) \simeq (E \multimap \perp)$ (by a trivial iso), the *incoherence* binary relation \succsim_E being defined by $a_1 \succsim_E a_2$ if $a_1 \supseteq_E a_2 \Rightarrow a_1 = a_2$. The transpose $(t)^\perp$ of $t \in \mathcal{L}(E, F)$ is simply $\{(b, a) \mid (a, b) \in t\}$. Under this linear negation, the De Morgan dual (par or cotensor) of the tensor product is $E_1 \wp E_2 = (E_1^\perp \otimes E_2^\perp)^\perp$ whose web is $|E_1| \times |E_2|$ and whose coherence relation is characterized by: $(a_1, a_2) \supseteq_{E_1 \wp E_2} (a'_1, a'_2)$ iff $a_i \supseteq_{E_i} a'_i$ for $i = 1$ or $i = 2$ (where $a \supseteq_E b$ means $a \supseteq_E b$ and $a \neq b$ and is called *strict coherence*; *strict incoherence* \smile_E is defined similarly). Remember that, with these notations, $E \multimap F = E^\perp \wp F$.

The category **Coh** has a final object $\top = (\emptyset, \emptyset)$ and a cartesian product $E_1 \& E_2 = (\{1\} \times |E_1| \cup \{2\} \times |E_2|, \supset_{E_1 \& E_2})$ where the coherence relation is defined by: $(i, a) \supset_{E_1 \& E_2} (j, a')$ if $i = j \Rightarrow a \supset_{E_i} a'$, the associated projections $\text{pr}_i \in \mathcal{L}(E_1 \& E_2, E_i)$ being $\text{pr}_i = \{((i, a), a) \mid a \in |E_i|\}$. Dually the initial object is $0 = \top^\perp = \top$ and the coproduct is $E_1 \oplus E_2 = (E_1^\perp \& E_2^\perp)^\perp$ whose web is $\{1\} \times |E_1| \cup \{2\} \times |E_2|$ and whose coherence is characterized by $(i, a) \supset_{E_1 \oplus E_2} (j, a')$ if $i = j$ and $a \supset_{E_i} a'$. There are canonical injections $E_i \rightarrow E_1 \oplus E_2$ which are the transposes of the projections defined above.

We define $!E$ as the coherence space whose web is the set of all finite elements of $\text{Cl}(E)$ and the coherence is: $u_1 \supset_{!E} u_2$ if $u_1 \cup u_2 \in \text{Cl}(E)$ (that is $\forall a_1 \in u_1 \forall a_2 \in u_2 \ a_1 \supset_E a_2$). This operation is a functor: given $t \in \mathcal{L}(E, F)$ one sets $!t = \{(\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}) \in |!E| \times |!F| \mid \forall i \ (a_i, b_i) \in t\}$. The comonad structure of this functor is given by the natural transformations $\text{der } E = \{(\{a\}, a) \mid a \in |E|\} \in \mathcal{L}(!E, E)$ (dereliction) and $\text{dig}_E = \{(u_1 \cup \dots \cup u_n, \{u_1, \dots, u_n\}) \mid u_1, \dots, u_n \in |!E| \text{ with } u_1 \cup \dots \cup u_n \in \text{Cl}(E)\}$ (digging). Last, there is an obvious isomorphism $\mathbf{m}^0 \in \mathcal{L}(1, !\top)$ and a natural isomorphism $\mathbf{m}_{E_1, E_2}^2 \in \mathcal{L}(!E_1 \otimes !E_2, !(E_1 \& E_2))$ (these isos defining a strong monoidal structure), satisfying an additional technical condition explained in [Mel09] for instance.

0.4.3 Non-uniform coherence spaces

Before going to details of Non-uniform coherence spaces, we first try to provide some motivation on why this notion is introduced.

As we saw in Section 0.4.1, we have $A^\perp = A$ in **Rel**. So, in this sense, one can say that **Rel** is a degenerate model to interpret types. However, this does not mean necessarily that it is a degenerate model for interpreting proofs. And one can also say that it is quite the contrary; as we see in [dCdF12], two cut-free proof-nets which have the same interpretation in **Rel** are “essentially” equal up to the equivalence on proofs induced by Rétoré’s reduction relation, including the fact that (w) is neutral for (c) and that (c) is associative and commutative [Ret93]. Nevertheless, the interpretation of a proof in **Rel** provides almost no information about it, i.e, given a proof π of $\vdash \Gamma$ we only know that $\llbracket \pi \rrbracket_{\mathbf{Rel}} \subseteq \llbracket \Gamma \rrbracket_{\mathbf{Rel}}$ but we do not know if this subset has a specific property or not. This issue is solved in coherence spaces, since we know that the interpretation of a proof would be an element of cliques, i.e., $\llbracket \pi \rrbracket_{\mathbf{Coh}} \in \text{Cl}(\llbracket \Gamma \rrbracket_{\mathbf{Coh}})$. For instance, in coherence spaces, the cliques of $1 \oplus 1$ are $\{(1, *)\}, \{(2, *)\}$ and \emptyset which are the basically two boolean values and the undefined one. So, by the information that $\llbracket \pi \rrbracket_{\mathbf{Coh}} \in \text{Cl}(\llbracket 1 \oplus 1 \rrbracket_{\mathbf{Coh}})$, we know that this proof π is either true or false or a non-terminating proof. Moreover, the proof cannot take both values true and false, and hence that the model **Coh** says something non-trivial about the determinism of the syntax. Notice that if we moreover consider interpretation of $1 \oplus 1$ in **Nuts** (Section 1.2.2) we know that $\llbracket \pi \rrbracket_{\mathbf{Nuts}}$ cannot be empty, so π is certainly not non-terminating proofs. And the reason for it is that $\mathcal{T}(\llbracket 1 \oplus 1 \rrbracket_{\mathbf{Nuts}})$ is $\{\{(1, *)\}, \{(2, *)\}\}$ ($\emptyset \notin \mathcal{T}(\llbracket 1 \oplus 1 \rrbracket_{\mathbf{Nuts}})$) and $\llbracket \pi \rrbracket_{\mathbf{Nuts}} \in \mathcal{T}(\llbracket 1 \oplus 1 \rrbracket_{\mathbf{Nuts}})$.

Although we know that the interpretation of any LL proof is the same in both model **Rel** and **Nuts** ($\llbracket \pi \rrbracket_{\mathbf{Rel}} = \llbracket \pi \rrbracket_{\mathbf{Nuts}}$), this is not the case between the models **Coh** and **Rel**. As an example, consider the following proof π of $\vdash ?(\perp \& \perp), 1 \oplus 1$ which is basically a program of

type $(?(\perp \& \perp)) \wp (1 \oplus 1) = !(1 \oplus 1) \multimap (1 \oplus 1)$ ⁴.

$$\begin{array}{c}
 \frac{\frac{\overline{\vdash 1} \text{ (1)}}{\vdash 1 \oplus 1} \text{ (}\oplus_1\text{)} \quad \frac{\overline{\vdash 1} \text{ (1)}}{\vdash 1 \oplus 1} \text{ (}\oplus_2\text{)}}{\vdash \perp, \perp, 1 \oplus 1} \text{ (}\perp\text{), (}\perp\text{)} \quad \frac{\overline{\vdash 1} \text{ (1)}}{\vdash 1 \oplus 1} \text{ (}\oplus_1\text{)} \quad \frac{\overline{\vdash 1} \text{ (1)}}{\vdash 1 \oplus 1} \text{ (}\oplus_2\text{)}}{\vdash \perp, \perp, 1 \oplus 1} \text{ (}\perp\text{), (}\perp\text{)} \\
 \frac{\vdash \perp, \perp, 1 \oplus 1}{\vdash \perp, \perp \& \perp, 1 \oplus 1} \text{ (}\&\text{)} \quad \frac{\vdash \perp, \perp, 1 \oplus 1}{\vdash \perp, \perp \& \perp, 1 \oplus 1} \text{ (}\&\text{)} \\
 \frac{\vdash \perp \& \perp, \perp \& \perp, 1 \oplus 1}{\vdash ?(\perp \& \perp), ?(\perp \& \perp), 1 \oplus 1} \text{ (d), (d)} \\
 \frac{\vdash ?(\perp \& \perp), ?(\perp \& \perp), 1 \oplus 1}{\vdash ?(\perp \& \perp), 1 \oplus 1} \text{ (c)}
 \end{array}$$

First, for the interpretation of $?(\perp \& \perp)$ in **Rel** and **Coh** we have:

$$\llbracket ?(\perp \& \perp) \rrbracket_{\mathbf{Rel}} = \mathcal{M}_{\text{fin}}(\{(1, *), (2, *)\})$$

$\llbracket ?(\perp \& \perp) \rrbracket_{\mathbf{Coh}} = \llbracket !(1 \oplus 1) \rrbracket_{\mathbf{Coh}} = \text{Cl}(\llbracket (1 \oplus 1) \rrbracket_{\mathbf{Coh}}) = \{\emptyset, \{(1, *)\}, \{(2, *)\}\}$, since all cliques of $\llbracket (1 \oplus 1) \rrbracket_{\mathbf{Coh}}$ are finite. Although we have $[(1, *), (2, *)] \in \llbracket ?(\perp \& \perp) \rrbracket_{\mathbf{Rel}}$, as you see $\{(1, *), (2, *)\} \notin \llbracket ?(\perp \& \perp) \rrbracket_{\mathbf{Coh}}$ ⁵.

If we compute the interpretation of π in **Rel** and **Coh** according to Figure 3, Section 0.4.1, and Section 0.4.2, then we have

$$\llbracket \pi \rrbracket_{\mathbf{Rel}} = \{([(1, *), (1, *)], (1, *)), [(1, *), (2, *)], (2, *)), [(2, *), (1, *)], (1, *)), [(2, *), (2, *)], (2, *)\}$$

$$\llbracket \pi \rrbracket_{\mathbf{Coh}} = \{(\{(1, *), (1, *)\}, (1, *)), (\{(2, *), (2, *)\}, (2, *))\}.$$

So, we do not have these two points $([(1, *), (2, *)], (2, *)), [(2, *), (1, *)], (1, *)$ in the interpretation of π in coherence spaces⁶.

Remark 8 *In order to have these two points $([(1, *), (2, *)], (2, *)), [(2, *), (1, *)], (1, *)$ in coherence spaces, one might think that we can redefine $!|E|$ as $\mathcal{M}_{\text{fin}}(|E|)$. If we do this, then of course, we have $[(1, *), (2, *)]$ in $!|E|$. But we have $([(1, *), (2, *)], (2, *)) \sim_{!(1 \oplus 1) \multimap (1 \oplus 1)} ([[(2, *), (1, *)], (1, *)))$, so, what we obtain is not a model of LL, since the proof is interpreted by something which is not a clique.*

So, we can say that one of the main goals of Non-uniform coherence spaces is to define a model of LL that lies over **Rel**, i.e., For any formula A of LL we would have $\llbracket [A] \rrbracket_{\mathbf{NCoh}} = \llbracket [A] \rrbracket_{\mathbf{Rel}}$ where $\llbracket \cdot \rrbracket_{\mathbf{NCoh}}$ is the interpretation of A in non-uniform coherence spaces.

Definition 9 *A non-uniform coherence space is a triple $E = (|E|, \frown_E, \smile_E)$ where $|E|$ is a set and \frown_E and \smile_E are disjoint binary and symmetric relations on $|E|$, called strict coherence and strict incoherence respectively.*

There is also a binary relation on $|E|$, called *neutrality* and denoted by ν_E , which is defined as complementary set of $\smile_E \cup \frown_E$.

⁴This example and the following explanations are basically coming from two lectures by Thomas Ehrhard and Christine Tasson in CIRM Linear Logic Winter School.

⁵One can also take finite multi-cliques which are multisets whose supports are cliques as web of $!E$, as observed first by Van de Wiel [vdW87] indeed one obtains in that way a nice example of Lafont category.

⁶One might consider this as a negative feature of coherence spaces. However, if we cut the proof π with any proof ρ of $\vdash !(1 \oplus 1)$, we will see that indeed the two middle branches of proof π will not appear in the cut-elimination process. So, one can consider this as positive feature of coherence spaces that would guess what are the parts of the proof that will not be considered in cut-elimination.

We also use the following notations: $\supset_E = \frown_E \cup \nu_E$ (called large coherence) and $\succsim_E = \smile_E \cup \nu_E$ (called large incoherence) which are symmetric relations on $|E|$.

Remark 9 *This neutrality ν is clearly symmetric, but neither reflexive nor anti-reflexive in general.*

Definition 10 *Given a non-uniform coherence space E , a clique of E is a subset u of $|E|$ such that $\forall a, a' \in u (a \supset_E a')$. Then we define $\text{NCl}(E)$ as the set of all cliques of E .*

Apart from Definition 9, a non-uniform coherence space can be defined equivalently by providing any pair of relations among the following ones, satisfying some conditions as follows:

- Two symmetric relations \supset_E and \frown_E such that $\frown_E \subseteq \supset_E$.
- \supset_E and ν_E such that $\nu_E \subseteq \supset_E$.
- \frown_E and ν_E such that $\frown_E \cap \nu_E = \emptyset$.
- and the duals of the cases above, i.e, considering \smile_E instead of \frown_E .

The non-uniform coherence space E^\perp is defined as $E^\perp = (|E|, \smile_E, \frown_E)$. So, we have $\frown_{E^\perp} = \smile_E$ and $\smile_{E^\perp} = \frown_E$, and we also obviously have $E^{\perp\perp} = E$.

Given two non-uniform coherence spaces E_1 and E_2 , then $|E_1 \otimes E_2|$ is defined as $|E_1| \times |E_2|$, and $(a_1, a_2) \supset_E (a'_1, a'_2)$ if $a_i \supset_{E_i} a'_i$ for $i = 1, 2$, and $(a_1, a_2) \nu_E (a'_1, a'_2)$ if $a_i \nu_{E_i} a'_i$ for $i = 1, 2$.

Then one sets $E \multimap F$ as $(E \otimes F^\perp)^\perp$. If we unfold the definition, we have $|E \multimap F| = |E| \times |F|$ and the following for $\supset_{E \multimap F}$ and $\nu_{E \multimap F}$:

- $(a, b) \supset_{E \multimap F} (a', b')$ if $a \supset_E a' \Rightarrow b \supset_F b'$ and $a \frown_E a' \Rightarrow b \frown_F b'$.
- $(a, b) \nu_{E \multimap F} (a', b')$ if $a \nu_E a'$ and $b \nu_F b'$.

Now, one can define the category **NCoh**; it has non-uniform coherence spaces as objects, and $\text{NCoh}(E, F)$ is defined as $\text{NCl}(E \multimap F)$. The identity relation Id_E obviously is in $\text{NCl}(E \multimap E)$, and the composition in this category is the relational composition defined in Section 0.4.1 (one should check that the composition of two morphisms is a morphism and that this is an easy verification). This category is easily seen to be symmetric monoidal using the \otimes defined on objects above, and one can extend it to morphisms in the same way as in **Rel**. The structural isos of being symmetric monoidal also are defined as in **Rel**. The neutral element of \otimes is $1 = (\{*\}, \emptyset, \emptyset)$. This category **NCoh** is also monoidal closed using $E \multimap F$ defined above on objects, and evaluation and curryfication is defined as in **Rel**. It is also $*$ -autonomous with dualizing object $\perp = 1$. **NCoh** is also a cartesian category. The terminal object is $\top = (\emptyset, \emptyset, \emptyset)$. Given two non-uniform coherence spaces E_1 and E_2 , the cartesian product is defined as $|E_1 \& E_2| = (\{1\} \times |E_1|) \cup (\{2\} \times |E_2|)$ and

- $(i, a) \supset_{E_1 \& E_2} (j, b)$ if $i = j \Rightarrow a \supset_{E_i} b$
- $(i, a) \nu_{E_1 \& E_2} (j, b)$ if $i = j$ and $a \supset_{E_i} b$

The projection morphisms are defined as in **Rel**.

For the exponentials, there are many ways to define it in non-uniform coherence spaces. In this thesis, we will provide the definition of Boudes's exponential [Bou02, Bou11], as we are going to use this definition later in Chapter 4.

Boudes' the exponential

Given a non-uniform coherence space E , we denote by $!_b E$ the Boudes' the exponential of **NCoh**. Its web $|!_b E|$ is defined as $\mathcal{M}_{\text{fin}}(|E|)$ and

- Given two multiset $m = [a_1, \dots, a_n]$ and $m' = [a'_1, \dots, a'_{n'}]$, we have $m \supset_{!_b E} m'$ if $\forall i, j (a_i \supset_E a'_j)$
- $m \nu_{!_b E} m'$ if $m \supset_{!_b E} m'$ and $m = [a_1, \dots, a_n]$ and $m' = [a'_1, \dots, a'_n]$ (same length) such that $\forall i (a_i \nu_E a'_i)$.

One can show that $!$ is a functor from **NCoh** to **NCoh**; its action on objects has already been defined, and on morphisms it is defined as in **Rel**. And if we take the same definitions for **der** and **dig** as in **Rel**, one can prove that these relations are morphisms in **NCoh** and turn $!_b$ into a comonad.

And for the Seely iso, if we take again the one mentioned for **Rel** in Section 0.4.1, we have the fact that $m^0 \in \mathbf{NCoh}(1 \multimap !_b \top)$ and $m^2_{E,F} \in \mathbf{NCI}(!_b E \otimes !_b F \multimap !_b (E \& F))$. And this ends the proof that **NCoh** is a model of **LL**.

Now if we get back to the example that we started with it at the beginning of this section, one can see that $([(1, *), (2, *)], (1, *)) \supset_{!_b (1 \oplus 1) \multimap (1 \oplus 1)} ([(1, *), (2, *)], (2, *))$ whereas this is not the case in usual coherence spaces. This is due to the fact that $[(1, *), (2, *)]$ is strictly incoherent with itself, which was impossible in **Coh**.

We end this section by the following remark.

Remark 10 *One can see the usual coherence spaces as a special case of non-uniform coherence spaces. So, the coherence spaces can be defined equivalently by saying that E is a coherence spaces if it is a non-uniform coherence spaces and $a \nu_E b \Leftrightarrow a = b$.*

*There is also a definition in the literature by Boudes saying that a non-uniform coherence space E is a Boudes space E if $a \nu_E b \Rightarrow a = b$. One can define category **NCohB** of Boudes space and linear morphisms, and prove that indeed **NCohB**, as a full subcategory of **NCoh**, is a model of **LL**.*

Chapter 1

Categorical and denotational semantics of finitary linear logic with fixpoints (μLL)

In this chapter, we develop a denotational semantics of LL with least and greatest fixed points in a general categorical setting based on the notion of Seely category and on strong functors acting on them. We exhibit a simple instance of this setting in the category of sets and relations, where least and greatest fixed points are interpreted in the same way, and in a category of sets equipped with a notion of totality (non-uniform totality spaces) and relations preserving totality, where least and greatest fixed points have distinct interpretations.

1.1 Categorical semantics of finitary linear logic with fixpoints

To define our notion of model of μLL , we first need to remind the notion of free comodules on a given coalgebra, and strong functors.

1.1.1 The LL model of free comodules on a given coalgebra.

Given an object¹ P of $\mathcal{L}^!$, we can define a functor $\text{fc}_P : \mathcal{L} \rightarrow \mathcal{L}$ which maps an object X to $\underline{P} \otimes X$ and a morphism f to $\underline{P} \otimes f$. This functor is clearly an oplax monoidal comonad (with structure maps defined using w_P , contr_P and the monoidal structure of \mathcal{L})². A coalgebra for this comonad is a P -comodule³. By Section 0.3.2 the Kleisli category $\mathcal{L}[P] = \mathcal{L}_{\text{fc}_P}$ of this comonad (that is, the category of free P -comodules) has a canonical structure of SMC. We set $\mathbf{F}_P = \mathbf{F}_{\text{fc}_P} : \mathcal{L} \rightarrow \mathcal{L}[P]$. Girard shows [Gir99] that $\mathcal{L}[P]$ is a Seely model of LL with operations on objects defined in the same way as in \mathcal{L} , and using the coalgebra structure of

¹In this thesis we could restrict to the case where P is a tensor of “free coalgebras” ($!X_i, \text{dig}_{X_i}$) but it is more natural to deal with the general case, which will be quite useful in further work.

²The definition of this comonad uses only the comonoid structure of \underline{P} . The $!_-$ -coalgebra structure will be used later.

³This is just the dual notion of the standard algebraic notion of an M -module which can be defined as soon as a commutative \otimes -monoid M is given.

P for the operations on morphisms. Intuitively, P should be considered as a given context and $\mathcal{L}[P]$ as a model in this context. This idea appears at various places in the literature [CFM16, UV08]. Let us summarize this construction. If $f_i \in \mathcal{L}[P](X_i, Y_i)$ for $i = 1, 2$ then $f_1 \otimes_P f_2 = f_1 \otimes_{\text{fc}_P} f_2 \in \mathcal{L}[P](X_1 \otimes X_2, Y_1 \otimes Y_2)$ is given by

$$\underline{P} \otimes X_1 \otimes X_2 \xrightarrow{\text{contr}_{\underline{P}} \otimes \text{ld}} \underline{P} \otimes \underline{P} \otimes X_1 \otimes X_2 \xrightarrow{\cong} \underline{P} \otimes X_1 \otimes \underline{P} \otimes X_2 \xrightarrow{f_1 \otimes f_2} Y_1 \otimes Y_2$$

The object of linear morphisms from X to Y in $\mathcal{L}[P]$ is $X \multimap Y$, and the evaluation morphism $\text{ev}_P \in \mathcal{L}[P]((X \multimap Y) \otimes X, Y)$ is simply $F_P(\text{ev})$. Then it is easy to check that if $f \in \mathcal{L}[P](Z \otimes X, Y)$, that is $f \in \mathcal{L}(\underline{P} \otimes Z \otimes X, Y)$, the morphism $\text{cur} f \in \mathcal{L}[P](Z, X \multimap Y)$ satisfies the required monoidal closeness equations. With these definitions, the category $\mathcal{L}[P]$ is $*$ -autonomous, with \perp as dualizing object. Specifically, given $f \in \mathcal{L}[P](X, Y)$, then $f^{\perp[P]}$ is the following composition of morphisms: $\underline{P} \otimes Y^{\perp} \xrightarrow{P \otimes (f)^{\perp}} \underline{P} \otimes (\underline{P} \multimap X^{\perp}) \xrightarrow{\text{ev}} X^{\perp}$. Using implicitly the iso between $(Z \otimes X)^{\perp}$ and $Z \multimap X^{\perp}$, and the $*$ -autonomy of \mathcal{L} allows to prove that indeed $f^{\perp[P]\perp[P]} = f$.

The category $\mathcal{L}[P]$ is easily seen to be cartesian with \top as final object, $X_1 \& X_2$ as cartesian product (and projections defined in the obvious way, applying F_P to the projections of \mathcal{L}). Last we define a functor $!_{P-} : \mathcal{L}[P] \rightarrow \mathcal{L}[P]$ by $!_{P}X = !X$ and, given $f \in \mathcal{L}[P](X, Y)$, we define $!_{P}f \in \mathcal{L}[P](!X, !Y)$ as $\underline{P} \otimes !X \xrightarrow{h_P \otimes !X} !\underline{P} \otimes !X \xrightarrow{\mu^2} !(\underline{P} \otimes X) \xrightarrow{!f} !Y$, and this functor has a comonad structure $(\text{der}[P], \text{dig}[P])$ defined by $\text{der}[P] = F_P(\text{der})$ and $\text{dig}[P] = F_P(\text{dig})$ ⁴.

Remark 11 Any $p \in \mathcal{L}^!(P, Q)$ induces a functor $\mathcal{L}[p] : \mathcal{L}[Q] \rightarrow \mathcal{L}[P]$ which acts as the identity on objects and maps $f \in \mathcal{L}[Q](X, Y)$ to $\mathcal{L}[p](f) = f(p \otimes X) \in \mathcal{L}[P](X, Y)$. This functor is strict monoidal symmetric and preserves all the constructions of LL, for instance $\mathcal{L}[p](\text{dig}[Q]) = \text{dig}[P]$ (simply because $\mathcal{L}[p] \circ F_Q = F_P$) and also $\mathcal{L}[p](!_Q f) = !_P(\mathcal{L}[p](f))$. We can actually consider $\mathcal{L}[_]$ as a functor from $\mathcal{L}^{\text{!op}}$ to the category of Seely categories and functors which preserve their structure on the nose. This functor could probably more suitably be considered as a fibration in the line of [PR97], Section 7.

1.1.2 Strong functors on \mathcal{L}

Given $n \in \mathbb{N}$, an n -ary strong functor on \mathcal{L} is a pair $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ where $\overline{\mathbb{F}} : \mathcal{L}^n \rightarrow \mathcal{L}$ is a functor and $\widehat{\mathbb{F}}_{X, \vec{Y}} \in \mathcal{L}(!X \otimes \overline{\mathbb{F}}(\vec{Y}), \overline{\mathbb{F}}(!X \otimes \vec{Y}))$ is a natural transformation, called the *strength* of \mathbb{F} . We use the notation $Z \otimes (Y_1, \dots, Y_n) = (Z \otimes Y_1, \dots, Z \otimes Y_n)$. It is assumed moreover that the diagrams of Figure 1.1 commute, expressing the monoidality of this strength as well as its compatibility with the comultiplication of $!_{-}$.

The main purpose of this definition is that for any object P of $\mathcal{L}^!$ one can lift \mathbb{F} to a functor $\mathbb{F}[P] : \mathcal{L}[P]^n \rightarrow \mathcal{L}[P]$ as follows. First one sets $\mathbb{F}[P](\vec{X}) = \overline{\mathbb{F}}(\vec{X})$. Then, given $\vec{f} \in \mathcal{L}[P]^n(\vec{X}, \vec{Y})$ we define $\mathbb{F}[P](\vec{f}) \in \mathcal{L}[P](\mathbb{F}(\vec{X}), \mathbb{F}(\vec{Y}))$ as the following morphism:

$$\underline{P} \otimes \mathbb{F}(\vec{X}) \xrightarrow{h_P \otimes \text{ld}} !\underline{P} \otimes \mathbb{F}(\vec{X}) \xrightarrow{\widehat{\mathbb{F}}} \mathbb{F}(!\underline{P} \otimes \vec{X}) \xrightarrow{\overline{\mathbb{F}}(\text{der } P \otimes \vec{X})} \mathbb{F}(\underline{P} \otimes \vec{X}) \xrightarrow{\mathbb{F}(\vec{f})} \mathbb{F}(\vec{Y})$$

The fact that we have defined a functor results from the three diagrams of Figure 1.1 and from the definition of w_P and contr_P based on the Seely isomorphisms.

⁴The definition of $!_{P}f$ requires P to be a $!$ -coalgebra and not simply a commutative \otimes -comonoid. Of course if $!$ is the free exponential as in [Gir99] the latter condition implies the former.

$$\begin{array}{ccc}
(!X_1 \otimes !X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) & \xrightarrow{m^2 \otimes \overline{\mathbb{F}}(\vec{Y})} & !(X_1 \& X_2) \otimes \overline{\mathbb{F}}(\vec{Y}) \\
!X_1 \otimes \widehat{\mathbb{F}}_{X_2, \vec{Y}} \downarrow & & \downarrow \widehat{\mathbb{F}}_{X_1 \& X_2, \vec{Y}} \\
!X_1 \otimes \overline{\mathbb{F}}(!X_2 \otimes \vec{Y}) & & \\
\widehat{\mathbb{F}}_{X_1, !X_2 \otimes \vec{Y}} \downarrow & & \\
\overline{\mathbb{F}}(!X_1 \otimes !X_2 \otimes \overline{\mathbb{F}}(\vec{Y})) & \xrightarrow{\overline{\mathbb{F}}(m^2 \otimes \vec{Y})} & \overline{\mathbb{F}}(!(X_1 \& X_2) \otimes \vec{Y}) \\
\end{array}
\qquad
\begin{array}{ccc}
1 \otimes \overline{\mathbb{F}}(\vec{Y}) & \xrightarrow{m^0 \otimes \overline{\mathbb{F}}(\vec{Y})} & !\top \otimes \overline{\mathbb{F}}(\vec{Y}) \\
\downarrow \cong & & \downarrow \widehat{\mathbb{F}}_{\top, \vec{Y}} \\
\overline{\mathbb{F}}(1 \otimes \vec{Y}) & \xrightarrow{\overline{\mathbb{F}}(m^0 \otimes \vec{Y})} & \overline{\mathbb{F}}(!\top \otimes \vec{Y}) \\
\end{array}$$

$$\begin{array}{ccc}
!X \otimes \overline{\mathbb{F}}(\vec{Y}) & \xrightarrow{\text{dig}_X \otimes \overline{\mathbb{F}}(\vec{Y})} & !!X \otimes \overline{\mathbb{F}}(\vec{Y}) \\
\widehat{\mathbb{F}}_{X, \vec{Y}} \downarrow & & \downarrow \widehat{\mathbb{F}}_{!X, \vec{Y}} \\
\overline{\mathbb{F}}(!X \otimes \vec{Y}) & \xrightarrow{\overline{\mathbb{F}}(\text{dig}_X \otimes \vec{Y})} & \overline{\mathbb{F}}(!!X \otimes \vec{Y}) \\
\end{array}$$

Figure 1.1: Monoidality and dig diagrams for strong functors

Operations on strong functors

There is an obvious unary identity strong functor \mathcal{I} and for each object Y of \mathcal{L} there is an n -ary Y -valued constant strong functor \mathcal{K}^Y ; in the first case the strength natural transformation is the identity morphism and in the second case, it is defined using $w_{!X}$. Let \mathbb{F} be an n -ary strong functor and $\mathbb{G}_1, \dots, \mathbb{G}_n$ be k -ary strong functors. Then one defines a k -ary strong functor $\mathbb{H} = \mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n)$: the functorial component $\overline{\mathbb{H}}$ is defined in the obvious compositional way.

The strength is $!X \otimes \overline{\mathbb{H}}(\vec{Y}) \xrightarrow{\widehat{\mathbb{F}}} \overline{\mathbb{F}}(!X \otimes \overline{\mathbb{G}_i}(\vec{Y}))_{i=1}^n \xrightarrow{\overline{\mathbb{F}}((\widehat{\mathbb{G}_i})_{i=1}^k)} \overline{\mathbb{F}}((\overline{\mathbb{G}_i}(!X \otimes \vec{Y}))_{i=1}^n)$, and is easily seen to satisfy the commutations of Figure 1.1. Given an n -ary strong functor, we can define its *De Morgan dual* $(\mathbb{F})^\perp$ which is also an n -ary strong functor. On objects, we set $(\overline{\mathbb{F}})^\perp(\vec{Y}) = (\overline{\mathbb{F}}(\vec{Y}^\perp))^\perp$ and similarly for morphisms. The strength of \mathbb{F}^\perp is defined as the Curry transpose of the following morphism (remember that $!X \multimap \vec{Y}^\perp = (!X \otimes \vec{Y})^\perp$ up to canonical iso):

$$\begin{array}{ccc}
!X \otimes (\overline{\mathbb{F}}((\vec{Y}^\perp)^\perp))^\perp \otimes \overline{\mathbb{F}}(!X \multimap \vec{Y}^\perp) & \xrightarrow{\cong} & !X \otimes \overline{\mathbb{F}}(!X \multimap (\vec{Y}^\perp)^\perp) \otimes (\overline{\mathbb{F}}(\vec{Y}^\perp))^\perp \\
& & \downarrow \widehat{\mathbb{F}} \otimes \text{Id} \\
& & \overline{\mathbb{F}}(!X \otimes (!X \multimap \vec{Y}^\perp)) \otimes (\overline{\mathbb{F}}(\vec{Y}^\perp))^\perp \\
& & \downarrow \overline{\mathbb{F}}(\text{ev}) \otimes \text{Id} \\
\perp & \xleftarrow{\text{ev } \gamma} & \overline{\mathbb{F}}(\vec{Y}^\perp) \otimes (\overline{\mathbb{F}}(\vec{Y}^\perp))^\perp
\end{array}$$

Then it is possible to prove, using the $*$ -autonomy of \mathcal{L} , that $\mathbb{F}^{\perp\perp}$ and \mathbb{F} are canonically isomorphic (as strong functors)⁵. As a direct consequence of the definition of $(\mathbb{F})^\perp$ and of the canonical iso between $\mathbb{F}^{\perp\perp}$ and \mathbb{F} we get:

Lemma 11 $(\mathbb{F} \circ (\mathbb{G}_1, \dots, \mathbb{G}_n))^\perp = (\mathbb{F})^\perp \circ ((\mathbb{G}_1)^\perp, \dots, (\mathbb{G}_n)^\perp)$ up to canonical iso.

⁵In the concrete settings considered in this thesis, these canonical isos are actually identity maps.

The bifunctor \otimes can be turned into a strong functor: one defines the strength as⁶

$$!X \otimes Y_1 \otimes Y_2 \xrightarrow{\text{contr}_{!X} \otimes \text{Id}} !X \otimes !X \otimes Y_1 \otimes Y_2 \xrightarrow{\cong} !X \otimes Y_1 \otimes !X \otimes Y_2$$

By De Morgan duality, this endows \mathfrak{A} with a strength as well. The bifunctor \oplus is also endowed with a strength, simply using the distributivity of \otimes over \oplus (which results from the monoidal closedness of \mathcal{L}). By duality again, $\&$ inherits a strength. The functor $!_-$ is equipped with the strength $!X \otimes !Y \xrightarrow{\text{dig}_X \otimes !Y} !!X \otimes !Y \xrightarrow{\mu^2} !(X \otimes Y)$.

Fixed Points of strong functors.

The following facts are standard in the literature on fixed points of functors [Poh73, Ada74].

Definition 12 *Let \mathcal{A} be a category and $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a functor. A coalgebra⁷ of \mathcal{F} is a pair (A, f) where A is an object of \mathcal{A} and $f \in \mathcal{A}(A, \mathcal{F}(A))$. Given two coalgebras (A, f) and (A', f') of \mathcal{F} , a coalgebra morphism from (A, f) to (A', f') is an $h \in \mathcal{A}(A, A')$ such that $f' h = \mathcal{F}(h) f$. The category of coalgebras of the functor \mathcal{F} will be denoted as $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$. The notion of algebra of an endofunctor is defined dually (reverse the directions of the arrows f and f') and the corresponding category is denoted as $\mathbf{Alg}_{\mathcal{A}}(\mathcal{F})$.*

By Lambek's Lemma, if (A, f) with $f \in \mathcal{A}(A, \mathcal{F}(A))$ is a final object in $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})$ then f is an iso. We assume that this iso is always the identity⁸ as this holds in our concrete models so that this final object $(\nu\mathcal{F}, \text{Id})$ satisfies $\mathcal{F}(\nu\mathcal{F}) = \nu\mathcal{F}$. We focus on coalgebras rather than algebras for reasons which will become clear when we deal with fixed points of strong functors. This universal property of $\nu\mathcal{F}$ gives us a powerful tool for proving equalities of morphisms.

Lemma 13 *Let $A \in \text{Obj}(\mathcal{A})$ and let $f_1, f_2 \in \mathcal{A}(A, \nu\mathcal{F})$. If there exists $l \in \mathcal{A}(A, \mathcal{F}(A))$ such that $\mathcal{F}(f_i)l = f_i$ for $i = 1, 2$, then $f_1 = f_2$.*

Proof: Since $\mathcal{F}(f_i)l = f_i$ for $i = 1, 2$, we have $f_i \in \mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})((A, l), (\nu\mathcal{F}, \text{Id}))$ for $i = 1, 2$. $(\nu\mathcal{F}, \text{Id})$ is the final object in $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F})((A, l), (\nu\mathcal{F}, \text{Id}))$, so there is a unique morphism from (A, l) to $(\nu\mathcal{F}, \text{Id})$. Hence $f_1 = f_2$. \blacksquare

Let $\mathcal{F} : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{A}$ be a functor, then we denote by \mathcal{F}_B the partial application of \mathcal{F} , i.e., $\mathcal{F}_B : \mathcal{A} \rightarrow \mathcal{A}$ where $\mathcal{F}_B(A) = \mathcal{F}(B, A)$.

Lemma 14 *Let $\mathcal{F} : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{A}$ be a functor such that, for all $B \in \text{Obj}(\mathcal{B})$, the category $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F}_B)$ has a final object. Then there is a functor $\nu\mathcal{F}$ such that $(\nu\mathcal{F}(B), \text{Id})$ is the final object of $\mathbf{Coalg}_{\mathcal{A}}(\mathcal{F}_B)$ (so that $\mathcal{F}(B, \nu\mathcal{F}(B)) = \nu\mathcal{F}(B)$) for each $B \in \text{Obj}(\mathcal{B})$, and, for each $g \in \mathcal{B}(B, B')$, $\nu\mathcal{F}(g)$ is uniquely characterized by $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$.*

⁶This definition, as well as the following one, shows that our assumption that the strength is available for "context object" of shape $!X$ only cannot be disposed of.

⁷Not to be confused with the coalgebras of Section 0.3.3 which must satisfy additional properties of compatibility with the comonad structure of $!_-$.

⁸This assumption is highly debatable from the view point of category theory where the notion of equality of objects is not really meaningful.

Proof: We have $\mathcal{F}(g, \nu\mathcal{F}(B)) \in \mathcal{A}(\nu\mathcal{F}(B), \mathcal{F}(B', \nu\mathcal{F}(B)))$ thus defining a $\mathcal{F}_{B'}$ -coalgebra structure on $\nu\mathcal{F}(B)$ and hence there exists a unique morphism $\nu\mathcal{F}(g) : \nu\mathcal{F}(B) \rightarrow \nu\mathcal{F}(B')$ such that

$$\mathcal{F}(B', \nu\mathcal{F}(g)) \mathcal{F}(g, \nu\mathcal{F}(B)) = \nu\mathcal{F}(g),$$

that is $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$.

Functoriality follows: consider also $g' \in \mathcal{B}(B', B'')$, then we know that $h = \nu\mathcal{F}(g'g)$ satisfies $\mathcal{F}(g'g, h) = h$ by the definition above. Now $h' = \nu\mathcal{F}(g')\nu\mathcal{F}(g)$ satisfies the same equation by functoriality of \mathcal{F} and because $\mathcal{F}(g, \nu\mathcal{F}(g)) = \nu\mathcal{F}(g)$ and $\mathcal{F}(g', \nu\mathcal{F}(g')) = \nu\mathcal{F}(g')$, and hence $h' = h$ by Lemma 13, taking $l = \mathcal{F}(g'g, \nu\mathcal{F}(B))$. In the same way one proves that $\nu\mathcal{F}(\text{Id}) = \text{Id}$. \blacksquare

We consider now the same $\nu\mathcal{F}$ operation applied to strong functors on a model \mathcal{L} of LL. Let \mathbb{F} be an $n+1$ -ary strong functor on \mathcal{L} (so that \mathbb{F} is a functor $\mathcal{L}^{n+1} \rightarrow \mathcal{L}$). Assume that for each $\vec{X} \in \text{Obj}(\mathcal{L}^n)$ the category $\mathbf{Coalg}_{\mathcal{L}}(\mathbb{F}_{\vec{X}})$ has a final object. We have defined $\nu\mathbb{F} : \mathcal{L}^n \rightarrow \mathcal{L}$ characterized by $\mathbb{F}(\vec{X}, \nu\mathbb{F}(\vec{X})) = \nu\mathbb{F}(\vec{X})$ and $\mathbb{F}(\vec{f}, \nu\mathbb{F}(\vec{f})) = \nu\mathbb{F}(\vec{f})$ for all $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ (Lemma 14). For each $Y, \vec{X} \in \mathcal{L}$, we define $\widehat{\nu\mathbb{F}}_{Y, \vec{X}} \in \mathcal{L}(!Y \otimes \nu\mathbb{F}(\vec{X}), \nu\mathbb{F}(!Y \otimes \vec{X}))$. We have

$$!Y \otimes \nu\mathbb{F}(\vec{X}) = !Y \otimes \mathbb{F}(\vec{X}, \nu\mathbb{F}(\vec{X})) \xrightarrow{\widehat{\mathbb{F}}_{Y, (\vec{X}, \nu\mathbb{F}(\vec{X}))}} \mathbb{F}(!Y \otimes \vec{X}, !Y \otimes \nu\mathbb{F}(\vec{X}))$$

exhibiting a $\mathbb{F}_{!Y \otimes \vec{X}}$ -coalgebra structure on $!Y \otimes \nu\mathbb{F}(\vec{X})$. Since $\nu\mathbb{F}(!Y \otimes \vec{X})$ is the final coalgebra of the functor $\mathbb{F}_{!Y \otimes \vec{X}}$, we define $\widehat{\nu\mathbb{F}}_{Y, \vec{X}}$ as the unique morphism $!Y \otimes \nu\mathbb{F}(\vec{X}) \rightarrow \nu\mathbb{F}(!Y \otimes \vec{X})$ such that the following diagram commutes

$$\begin{array}{ccc} !Y \otimes \nu\mathbb{F}(\vec{X}) & \xrightarrow{\widehat{\mathbb{F}}_{Y, (\vec{X}, \nu\mathbb{F}(\vec{X}))}} & \mathbb{F}(!Y \otimes \vec{X}, !Y \otimes \nu\mathbb{F}(\vec{X})) \\ & \searrow \widehat{\nu\mathbb{F}}_{Y, \vec{X}} & \downarrow \mathbb{F}(!Y \otimes \vec{X}, \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \\ & & \mathbb{F}(!Y \otimes \vec{X}, \nu\mathbb{F}(!Y \otimes \vec{X})) = \nu\mathbb{F}(!Y \otimes \vec{X}) \end{array}$$

Lemma 15 *Let \mathbb{F} be an $n+1$ -ary strong functor on \mathcal{L} such that for each $\vec{X} \in \text{Obj}(\mathcal{L}^n)$, the category $\mathbf{Coalg}_{\mathcal{L}}(\mathbb{F}_{\vec{X}})$ has a final object $\nu\mathbb{F}_{\vec{X}}$. Then there is a unique n -ary strong functor $\nu\mathbb{F}$ such that $\nu\mathbb{F}(\vec{X}) = \nu\mathbb{F}_{\vec{X}}$ (and hence $\mathbb{F}(\vec{X}, \nu\mathbb{F}(\vec{X})) = \nu\mathbb{F}(\vec{X})$), $\mathbb{F}(\vec{f}, \nu\mathbb{F}(\vec{f})) = \nu\mathbb{F}(\vec{f})$ for all $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ and $\mathbb{F}(!Y \otimes \vec{X}, \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \widehat{\mathbb{F}}_{Y, (\vec{X}, \nu\mathbb{F}(\vec{X}))} = \widehat{\nu\mathbb{F}}_{Y, \vec{X}}$.*

Proof: The part of the statement which concerns the functor $\nu\mathbb{F}$ is a direct application of Lemma 14 so we only have to deal with the strength. Let us prove naturality so let $\vec{f} \in \mathcal{L}^n(\vec{X}, \vec{X}')$ and $g \in \mathcal{L}(Y, Y')$, we must prove that the following diagram commutes

$$\begin{array}{ccc} !Y \otimes \nu\mathbb{F}(\vec{X}) & \xrightarrow{\widehat{\nu\mathbb{F}}_{Y, \vec{X}}} & \nu\mathbb{F}(!Y \otimes \vec{X}) \\ !g \otimes \nu\mathbb{F}(\vec{f}) \downarrow & & \downarrow \nu\mathbb{F}(!g \otimes \vec{f}) \\ !Y' \otimes \nu\mathbb{F}(\vec{X}') & \xrightarrow{\widehat{\nu\mathbb{F}}_{Y', \vec{X}'}} & \nu\mathbb{F}(!Y' \otimes \vec{X}') \end{array}$$

Let $h_1 = \widehat{\nu\mathbb{F}}_{Y', \vec{X}'} (!g \otimes \overline{\nu\mathbb{F}}(\vec{f}))$ and $h_2 = \overline{\nu\mathbb{F}}(!g \otimes \vec{f}) \widehat{\nu\mathbb{F}}_{Y, \vec{X}}$ be the two morphisms we must prove equal. We use Lemma 13, taking the following morphism l .

$$\begin{aligned} !Y \otimes \overline{\nu\mathbb{F}}(\vec{X}) &= !Y \otimes \mathbb{F}(\vec{X}, \overline{\nu\mathbb{F}}(\vec{X})) \\ &\quad \downarrow \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ \mathbb{F}(!Y \otimes \vec{X}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) & \\ &\quad \downarrow \mathbb{F}(!g \otimes \vec{f}, \text{Id}) \\ \mathbb{F}(!Y' \otimes \vec{X}', !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) & \end{aligned}$$

With these notations we have

$$\begin{aligned} \mathbb{F}(!Y' \otimes \vec{X}', h_1) \circ l &= \mathbb{F}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}) \circ \mathbb{F}(!Y' \otimes \vec{X}', !g \otimes \overline{\nu\mathbb{F}}(\vec{f})) \\ &\quad \circ \mathbb{F}(!g \otimes \vec{f}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \circ \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ &= \mathbb{F}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}) \circ \mathbb{F}(!g \otimes \vec{f}, !g \otimes \overline{\nu\mathbb{F}}(\vec{f})) \circ \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ &= \mathbb{F}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y', \vec{X}'}) \circ \widehat{\mathbb{F}}_{Y', (\vec{X}', \overline{\nu\mathbb{F}}(\vec{X}'))} \circ (!g \otimes \mathbb{F}(\vec{f}, \overline{\nu\mathbb{F}}(\vec{f}))) \\ &\quad \text{by naturality of } \widehat{\mathbb{F}} \\ &= \widehat{\nu\mathbb{F}}_{Y', \vec{X}'} \circ (!g \otimes \mathbb{F}(\vec{f}, \overline{\nu\mathbb{F}}(\vec{f}))) \text{ by (1.1.2)} \\ &= \widehat{\nu\mathbb{F}}_{Y', \vec{X}'} \circ (!g \otimes \overline{\nu\mathbb{F}}(\vec{f})) \text{ by Lemma 14} \end{aligned}$$

so that $\mathbb{F}(!Y' \otimes \vec{X}', h_1) l = h_1$ as required. On the other hand we have

$$\begin{aligned} \mathbb{F}(!Y' \otimes \vec{X}', h_2) \circ l &= \mathbb{F}(!Y' \otimes \vec{X}', \overline{\nu\mathbb{F}}(!g \otimes \vec{f})) \circ \mathbb{F}(!Y' \otimes \vec{X}', \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \\ &\quad \circ \mathbb{F}(!g \otimes \vec{f}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \circ \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ &= \mathbb{F}(!Y' \otimes \vec{X}', \overline{\nu\mathbb{F}}(!g \otimes \vec{f})) \circ \mathbb{F}(!g \otimes \vec{f}, !Y \otimes \overline{\nu\mathbb{F}}(\vec{X})) \\ &\quad \circ \mathbb{F}(!Y \otimes \vec{X}, \widehat{\nu\mathbb{F}}_{Y, \vec{X}}) \circ \widehat{\mathbb{F}}_{Y, (\vec{X}, \overline{\nu\mathbb{F}}(\vec{X}))} \\ &= \mathbb{F}(!g \otimes \vec{f}, \overline{\nu\mathbb{F}}(!g \otimes \vec{f})) \circ \widehat{\nu\mathbb{F}}_{Y, \vec{X}} \text{ by (1.1.2)} \\ &= \overline{\nu\mathbb{F}}(!g \otimes \vec{f}) \circ \widehat{\nu\mathbb{F}}_{Y, \vec{X}} \text{ by Lemma 14} \end{aligned}$$

so that $\mathbb{F}(!Y' \otimes \vec{X}', h_2) l = h_2$ which proves our contention. The commutation of the diagrams of Figure 1.1 for $\widehat{\nu\mathbb{F}}$ is proven similarly, and we only deal with the last one. We must prove the commutation of the following diagram:

$$\begin{array}{ccc} !X \otimes \overline{\nu\mathbb{F}}(\vec{Y}) & \xrightarrow{\text{dig}_X \otimes \overline{\nu\mathbb{F}}(\vec{Y})} & !!X \otimes \overline{\nu\mathbb{F}}(\vec{Y}) \\ \widehat{\nu\mathbb{F}}_{X, \vec{Y}} \downarrow & & \downarrow \widehat{\nu\mathbb{F}}_{!X, \vec{Y}} \\ \overline{\nu\mathbb{F}}(!X \otimes \vec{Y}) & \xrightarrow{\overline{\nu\mathbb{F}}(\text{dig}_X \otimes \vec{Y})} & \overline{\nu\mathbb{F}}(!!X \otimes \vec{Y}) \end{array}$$

Using Lemma 14, one can reduce this diagram to the following one where $h = \overline{\mathbb{F}}(\text{dig}_X \otimes \overrightarrow{Y}, \nu\overline{\mathbb{F}}(\text{dig}_X \otimes \overrightarrow{Y}))$ and $g = \overline{\mathbb{F}}(\text{dig}_X \otimes \overrightarrow{Y}, \text{dig}_X \otimes \nu\overline{\mathbb{F}}(\overrightarrow{Y}))$:

$$\begin{array}{ccc}
 !X \otimes \nu\overline{\mathbb{F}}(\overrightarrow{Y}) & \xrightarrow{\text{dig}_X \otimes \nu\overline{\mathbb{F}}(\overrightarrow{Y})} & !!X \otimes \nu\overline{\mathbb{F}}(\overrightarrow{Y}) \\
 \downarrow \widehat{\nu\overline{\mathbb{F}}}_{X, \overrightarrow{Y}} & \searrow & \downarrow \widehat{\nu\overline{\mathbb{F}}}_{!!X, \overrightarrow{Y}} \\
 & !X \otimes \overline{\mathbb{F}}(\overrightarrow{Y}, \nu\overline{\mathbb{F}}(\overrightarrow{Y})) \xrightarrow{\text{dig}_X \otimes \overline{\mathbb{F}}(\overrightarrow{Y}, \nu\overline{\mathbb{F}}(\overrightarrow{Y}))} & !!X \otimes \overline{\mathbb{F}}(\overrightarrow{Y}, \nu\overline{\mathbb{F}}(\overrightarrow{Y})) \\
 & \downarrow \widehat{\mathbb{F}} & \downarrow \widehat{\mathbb{F}} \\
 & \overline{\mathbb{F}}(!X \otimes \overrightarrow{Y}, !X \otimes \nu\overline{\mathbb{F}}(\overrightarrow{Y})) \xrightarrow{g} & \overline{\mathbb{F}}(!!X \otimes \overrightarrow{Y}, !!X \otimes \nu\overline{\mathbb{F}}(\overrightarrow{Y})) \\
 & \downarrow \overline{\mathbb{F}}(\text{Id}, \widehat{\nu\overline{\mathbb{F}}}_{X, \overrightarrow{Y}}) & \downarrow \overline{\mathbb{F}}(\text{Id}, \widehat{\nu\overline{\mathbb{F}}}_{!!X, \overrightarrow{Y}}) \\
 & \overline{\mathbb{F}}(!X \otimes \overrightarrow{Y}, \nu\overline{\mathbb{F}}(!X \otimes \overrightarrow{Y})) \xrightarrow{h} & \overline{\mathbb{F}}(!!X \otimes \overrightarrow{Y}, \nu\overline{\mathbb{F}}(!!X \otimes \overrightarrow{Y})) \\
 \downarrow \widehat{\nu\overline{\mathbb{F}}}_{X, \overrightarrow{Y}} & \searrow & \downarrow \widehat{\nu\overline{\mathbb{F}}}_{!!X, \overrightarrow{Y}} \\
 \nu\overline{\mathbb{F}}(!X \otimes \overrightarrow{Y}) & \xrightarrow{\nu\overline{\mathbb{F}}(\text{dig}_X \otimes \overrightarrow{Y})} & \nu\overline{\mathbb{F}}(!!X \otimes \overrightarrow{Y})
 \end{array}$$

Commutation of left and right trapezoids is coming from Diagram 1.1.2. Commutation of the rectangle above is coming from the fact \mathbb{F} is a strong functor (last diagram of Figure 1.1). And finally the commutation of rectangle below will be deduced from functoriality of \mathbb{F} and last diagram of Figure 1.1. \blacksquare

Lemma 16 *Let \mathbb{F} be an $n + 1$ -ary strong functor on \mathcal{L} such that for each $\overrightarrow{X} \in \text{Obj}(\mathcal{L}^n)$, the category $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\overrightarrow{X}})$ has an initial object $\mu\overline{\mathbb{F}}_{\overrightarrow{X}}$. Then there is a unique n -ary strong functor $\mu\overline{\mathbb{F}}$ such that $\mu\overline{\mathbb{F}}(\overrightarrow{X}) = \mu\overline{\mathbb{F}}_{\overrightarrow{X}}$ (and hence $\overline{\mathbb{F}}(\overrightarrow{X}, \mu\overline{\mathbb{F}}(\overrightarrow{X})) = \mu\overline{\mathbb{F}}(\overrightarrow{X})$), $\overline{\mathbb{F}}(\overrightarrow{f}, \mu\overline{\mathbb{F}}(\overrightarrow{f})) = \mu\overline{\mathbb{F}}(\overrightarrow{f})$ for all $\overrightarrow{f} \in \mathcal{L}^n(\overrightarrow{X}, \overrightarrow{X}')$ and $\overline{\mathbb{F}}(!Y \otimes \overrightarrow{X}, \widehat{\mu\overline{\mathbb{F}}}_{Y, \overrightarrow{X}}) \widehat{\mathbb{F}}_{Y, (\overrightarrow{X}, \mu\overline{\mathbb{F}}(\overrightarrow{X}))} = \widehat{\mu\overline{\mathbb{F}}}_{Y, \overrightarrow{X}}$. Moreover $((\mu\overline{\mathbb{F}})^\perp)^\perp = \nu((\mathbb{F})^\perp)$*

Proof: Apply Lemma 15 to the strong functor $(\mathbb{F})^\perp$. \blacksquare

1.1.3 A categorical axiomatization of models of μLL

We are now ready to axiomatize our notion of models of μLL . Our general definition of Seely categorical model of μLL is based on the notions and results above. We refer in particular to Section 1.1.2 for the basic definitions of operations on strong functors in our LL categorical setting.

Definition 17 *A categorical model or Seely model of μLL is a pair $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ where*

1. \mathcal{L} is a Seely category
2. $\overrightarrow{\mathcal{L}} = (\mathcal{L}_n)_{n \in \mathbb{N}}$ where \mathcal{L}_n is a class of strong functors $\mathcal{L}^n \rightarrow \mathcal{L}$, and $\mathcal{L}_0 = \text{Obj}(\mathcal{L})$
3. if $\mathbb{X} \in \mathcal{L}_n$ and $\mathbb{X}_i \in \mathcal{L}_k$ (for $i = 1, \dots, n$) then $\mathbb{X} \circ \overrightarrow{\mathbb{X}} \in \mathcal{L}_k$ and all k projection strong functors $\mathcal{L}^k \rightarrow \mathcal{L}$ belong to \mathcal{L}_k
4. the strong functors \otimes and $\&$ belong to \mathcal{L}_2 , the strong functor $!_-$ belongs to \mathcal{L}_1 and, if $\mathbb{X} \in \mathcal{L}_n$, then $(\mathbb{X})^\perp \in \mathcal{L}_n$

5. and last, for all $\mathbb{X} \in \mathcal{L}_1$ the category $\mathbf{Coalg}_{\mathcal{L}}(\overline{\mathbb{X}})$ (see Section 1.1.2) has a final object. Moreover, for any $\mathbb{X} \in \mathcal{L}_{k+1}$, the associated strong functor $\nu_{\mathbb{X}} : \mathcal{L}^k \rightarrow \mathcal{L}$ (see Lemma 15) belongs to \mathcal{L}_k .

Our goal is now to provide the interpretation of μLL formulas and proofs in such a model.

Remark 12 One can certainly also define a notion of categorical model of μLL in a linear-non-linear adjunction setting as presented in [Mel09]. This is postponed to further work. And one can also probably adapt this definition to the case of intuitionistic LL, without $*$ -autonomy.

Interpreting formulas and proofs and Soundness

We just outline the interpretation of μLL formulas and proofs, as we will see all the details in Chapter 2.

We assume to be given a μLL Seely model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$, see Section 1.1.3. With any formula A and any repetition-free sequence $\overrightarrow{\zeta} = (\zeta_1, \dots, \zeta_k)$ of type variables containing all the free variables of A , we associate $\llbracket A \rrbracket_{\overrightarrow{\zeta}} \in \mathcal{L}_k$ in the obvious way, for instance $\llbracket A \otimes B \rrbracket_{\overrightarrow{\zeta}} = \otimes \circ (\llbracket A \rrbracket_{\overrightarrow{\zeta}}, \llbracket B \rrbracket_{\overrightarrow{\zeta}}) \in \mathcal{L}_k$ by conditions (4) and (3) in Definition 17 and $\llbracket \nu \zeta A \rrbracket_{\overrightarrow{\zeta}} = \nu(\llbracket A \rrbracket_{\overrightarrow{\zeta}, \zeta})$ using condition (5). Then $\llbracket A^\perp \rrbracket_{\overrightarrow{\zeta}} = (\llbracket A \rrbracket_{\overrightarrow{\zeta}})^\perp$ up to a natural isomorphism. With any $\Gamma = (A_1, \dots, A_n)$ we associate an object $\llbracket \Gamma \rrbracket$ of \mathcal{L} and with any proof π of $\vdash \Gamma$ we associate a morphism $\llbracket \pi \rrbracket \in \mathcal{L}(1, \llbracket \Gamma \rrbracket)$ using the categorical constructs of \mathcal{L} in a straightforward way, see [Mel09]. Then one proves that if π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules, then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$. This is done by an inspection of the various cut-elimination rules. In the case of $(\mu - \text{fold})/(\nu - \text{rec})$ cut-elimination, we need the following lemma that we state in a rough way (again, isos are kept implicit).

Lemma 18 Let $\Gamma = (D_1, \dots, D_n)$ be a sequence of closed formulas, F be a formula and $\zeta, \xi_1, \dots, \xi_k$ be a repetition-free list of variables containing all the free variables of F . Let π be a proof of $\vdash ?\Gamma, A^\perp, B$ (so that, setting $X = \llbracket D_1^\perp \rrbracket \& \dots \& \llbracket D_n^\perp \rrbracket$, we can consider that $\llbracket \pi \rrbracket \in \mathcal{L}(!X \otimes \llbracket A \rrbracket, \llbracket B \rrbracket)$) and let $\overrightarrow{C} = (C_1, \dots, C_k)$ be a list of closed formulas. Then

$$\llbracket F[\pi/\zeta, \overrightarrow{C}/\overrightarrow{\xi}] \rrbracket = \overline{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}}(\llbracket \pi \rrbracket, \mathbf{w}_X \otimes \overline{\llbracket C \rrbracket}) (\widehat{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}})_{X, (\llbracket A \rrbracket, \overline{\llbracket C \rrbracket})}.$$

The proof of Lemma 18 is completely similar to the last case of Lemma 98 (Figure 2.10), and to avoid repetition, we only provide the details for Lemma 98. Notice that $\overline{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}}(\llbracket \pi \rrbracket, \mathbf{w}_X \otimes \overline{\llbracket C \rrbracket}) \in \mathcal{L}(\overline{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}}(!X \otimes \llbracket A \rrbracket, !X \otimes \overline{\llbracket C \rrbracket}), \overline{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}}(\llbracket B \rrbracket, \overline{\llbracket C \rrbracket}))$ and that $(\widehat{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}})_{X, (\llbracket A \rrbracket, \overline{\llbracket C \rrbracket})}$ belongs to $\mathcal{L}(!X \otimes \overline{\llbracket F \rrbracket}_{\zeta, \overrightarrow{\xi}}(\llbracket A \rrbracket, \overline{\llbracket C \rrbracket}), \llbracket F \rrbracket_{\zeta, \overrightarrow{\xi}}(!X \otimes \llbracket A \rrbracket, !X \otimes \overline{\llbracket C \rrbracket}))$.

Theorem 19 If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of μLL , then $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$.

The proof of Theorem 19 is completely similar to Theorem 99, and to avoid repetition, we only do the details for Theorem 99.

1.2 Denotational semantics of finitary linear logic with fixpoints

In this section, we provide some concrete models for μLL . The first model is **Rel**, and the second of is non-uniform totality spaces (**Nuts**). One of our motivations to provide **Nuts** is that the interpretation of least and greatest fixpoints collapse in the case **Rel** whereas they are distinct in **Nuts**. We also examine the same idea on coherence spaces, and similarly coherence spaces with totality.

1.2.1 Rel as model of μLL

Computing recursive types in **Rel** is quite well-studied in the literature. And one can find different vocabulary in different references. For example, we can see in [Fio96] the notion of cpo-functors and algebraic complete categories which **Rel** is an instance. In [Wan79, SP82] we can see the notion of locally-continuous functor. So, to have a coherent notation with the other parts, we provide all details of computing least and greatest fixpoint of linear logic in **Rel** in this section. There is no novelty here. We refer an interested reader to [AMM18] for a survey on fixed points of endofunctors, and to [Fio96] to see more details of axiomatic categorical domain theory.

Locally continuous functors on Rel

Definition 20 A functor $\mathbb{F} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is locally continuous⁹ if, for all $\vec{E}, \vec{F} \in \mathbf{Rel}^n$ and all directed set $D \subseteq \mathbf{Rel}^n(\vec{E}, \vec{F})$, one has $\mathbb{F}(\bigcup D) = \bigcup \{\mathbb{F}(\vec{s}) \mid \vec{s} \in D\}$.

Definition 20 implies in particular that if $\vec{s} \subseteq \vec{t}$, one has $\mathbb{F}(\vec{s}) \subseteq \mathbb{F}(\vec{t})$ (taking $D = \{\vec{s}, \vec{t}\}$). To simplify notations assume that $n = 1$ (but what follows holds for all values of n).

Lemma 21 Let E and F be sets and let $s \in \mathbf{Rel}(E, F)$ and $t \in \mathbf{Rel}(F, E)$. Assume that $ts = \text{Id}_E$ and that $st \subseteq \text{Id}_F$. Then s is an injective function and $t = \{(b, a) \in F \times E \mid (a, b) \in s\}$.

Proof: Let $a \in E$, since $(a, a) \in \text{Id}_E = ts$, there must exist $b \in F$ such that $(a, b) \in s$ and $(b, a) \in t$. If $(a, b') \in s$ then $(b, b') \in st \subseteq \text{Id}_F$ and hence $b' = b$. It follows that s is a total function $E \rightarrow F$. Let $(a, b) \in s$ (that is $a \in E$ and $b = s(a)$). Since $ts = \text{Id}_E$, we must have $(b, a) \in t$. Conversely let $(b, a) \in t$, we have $(b, s(a)) \in st$ and hence $b = s(a)$. We have proven that $t = \{(s(a), a) \mid a \in E\}$. If $a, a' \in E$ satisfy $s(a) = s(a')$ we have therefore $(a, a') \in ts = \text{Id}_E$ and hence $a = a'$; this shows that s is injective. ■

Lemma 22 Let $\mathbb{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ be a locally-continuous functor. Assume that $E \subseteq F$ and let $\eta_{E,F}^+ = \{(a, a) \mid a \in E\} \in \mathbf{Rel}(E, F)$ and $\eta_{E,F}^- = \{(a, a) \mid a \in E\} \in \mathbf{Rel}(F, E)$. Then $\mathbb{F}(\eta_{E,F}^+) \in \mathbf{Rel}(\mathbb{F}(E), \mathbb{F}(F))$ is an injective function.

⁹The right setting to express this property would be that of *cpo-enriched categories* but we do not really need this general concept here.

Proof: We have $\eta_{E,F}^- \eta_{E,F}^+ = \text{Id}_E$ and $\eta_{E,F}^+ \eta_{E,F}^- \subseteq \text{Id}_F$ and hence $\mathbb{F}(\eta_{E,F}^-) \mathbb{F}(\eta_{E,F}^+) = \text{Id}$ by functoriality and $\mathbb{F}(\eta_{E,F}^+) \mathbb{F}(\eta_{E,F}^-) \subseteq \text{Id}$ by local-continuity. The announced property results from Lemma 21. ■

Let \mathbf{Rel}^\subseteq be the category whose objects are sets and morphisms are set inclusions (so that $\mathbf{Rel}^\subseteq(E, F)$ has $\eta_{E,F}^+$ as unique element if $E \subseteq F$ and is empty otherwise). Then η^+ can be thought of as the “inclusion functor” $\mathbf{Rel}^\subseteq \rightarrow \mathbf{Rel}$, acting as the identity on objects. Obviously, \mathbf{Rel}^\subseteq is cocomplete¹⁰.

Proposition 23 *If $\mathbb{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ is locally-continuous then $\mathbb{F}\eta^+ : \mathbf{Rel}^\subseteq \rightarrow \mathbf{Rel}$ is directed-cocontinuous (that is, preserves the colimits of directed sets of sets).*

Proof: Let \mathcal{D} be a directed set of sets and let H be a set. For each $E \in \mathcal{D}$ let $s_E \in \mathbf{Rel}(\mathbb{F}(E), H)$ so that $(s_E)_{E \in \mathcal{D}}$ defines a cocone, that is, for each $E, F \in \mathcal{D}$ such that $E \subseteq F$, one has $s_E = s_F \mathbb{F}(\eta_{E,F}^+)$. Let $L = \bigcup \mathcal{D}$. Let $s \in \mathbf{Rel}(\mathbb{F}(L), H)$ be given by $s = \bigcup_{E \in \mathcal{D}} s_E \mathbb{F}(\eta_{E,L}^-)$. Let $E \in \mathcal{D}$, we have $s \mathbb{F}(\eta_{E,L}^+) = \bigcup_{F \in \mathcal{D}} s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+)$ so that $s_E \subseteq s \mathbb{F}(\eta_{E,L}^+)$ (since $s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+) = s_E$ when $F = E$).

We prove the converse inclusion. Let $F \in \mathcal{D}$ and let $G \in \mathcal{D}$ be such that $E, F \subseteq G$. We have

$$\begin{aligned} s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+) &= s_F \mathbb{F}(\eta_{F,G}^- \eta_{G,L}^- \eta_{G,L}^+ \eta_{E,G}^+) = s_F \mathbb{F}(\eta_{F,G}^- \eta_{E,G}^+) \\ &= s_G \mathbb{F}(\eta_{F,G}^+) \mathbb{F}(\eta_{F,G}^- \eta_{E,G}^+) \\ &\subseteq s_G \mathbb{F}(\eta_{E,G}^+) = s_E \end{aligned}$$

where we have used the fact that $\eta_{F,G}^+ \eta_{F,G}^- \subseteq \text{Id}_G$ and hence $\mathbb{F}(\eta_{F,G}^+ \eta_{F,G}^-) \subseteq \text{Id}_{\mathbb{F}(G)}$ by local-continuity of \mathbb{F} . So $s_F \mathbb{F}(\eta_{F,L}^- \eta_{E,L}^+) \subseteq s_E$ for all $F \in \mathcal{D}$ and hence $s \mathbb{F}(\eta_{E,L}^+) \subseteq s_E$ as contended.

Let now $s' \in \mathbf{Rel}(\mathbb{F}(L), H)$ be such that $s' \mathbb{F}(\eta_{E,L}^+) = s_E$ for each $E \in \mathcal{D}$, we show that $s' = s$ thus proving the uniqueness part of the universal property. For $E \in \mathcal{D}$, let $\theta_E = \eta_{E,L}^+ \eta_{E,L}^- \in \mathbf{Rel}(L, L)$. Then $(\theta_E)_{E \in \mathcal{D}}$ is a directed family (for \subseteq) and $\bigcup_{E \in \mathcal{D}} \theta_E = \text{Id}_L$. By local-continuity of \mathbb{F} , we have

$$s' = s' \text{Id}_{\mathbb{F}(L)} = s' \bigcup_{E \in \mathcal{D}} \mathbb{F}(\theta_E) = \bigcup_{E \in \mathcal{D}} s' \mathbb{F}(\eta_{E,L}^+) \mathbb{F}(\eta_{E,L}^-) = \bigcup_{E \in \mathcal{D}} s_E \mathbb{F}(\eta_{E,L}^-) = s$$

by our assumption on s' and by definition of s . This shows that the cocone $(\mathbb{F}(\eta_{E,L}^+))_{E \in \mathcal{D}}$ on $\mathbb{F}\eta^+$ is colimiting, thus proving that $\mathbb{F}\eta^+$ is directed cocontinuous. ■

We know that a locally-continuous functor \mathbb{F} maps inclusions to injections, we shall say that \mathbb{F} is *strict* if it maps inclusions to inclusions, that is, if $E \subseteq F$ then $\mathbb{F}(E) \subseteq \mathbb{F}(F)$ and $\mathbb{F}(\eta_{E,F}^+) = \eta_{\mathbb{F}(E), \mathbb{F}(F)}^+$ (which implies $\mathbb{F}(\eta_{E,F}^-) = \eta_{\mathbb{F}(E), \mathbb{F}(F)}^-$). As a direct consequence of Proposition 23, we get:

Lemma 24 *If \mathbb{F} is strict locally-continuous then, for any directed set of sets \mathcal{D} , one has $\mathbb{F}(\bigcup \mathcal{D}) = \bigcup_{E \in \mathcal{D}} \mathbb{F}(E)$.*

¹⁰Notice that it is not complete, for instance it has no final object.

Variable sets and basic constructions on them

Definition 25 An n -ary variable set (\mathbf{Vst}) is a strong functor $\mathbb{V} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ such that $\overline{\mathbb{V}}$ is locally-continuous and strict.

By the general considerations of Section 1.1.2, we know that there is a constant strong functor $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$ with value E for each set E , that there are projection strong functors $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$, that \times (that is \otimes) and $+$ (that is \oplus) define strong functors $\mathbf{Rel}^2 \rightarrow \mathbf{Rel}$, that $\mathcal{M}_{\text{fin}}(_)$ (that is $!_$) defines a strong functor $\mathbf{Rel} \rightarrow \mathbf{Rel}$, that strong functors on \mathbf{Rel} are stable under composition, and that if \mathbb{V} is a strong functor $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$ then there is a “dual” strong functor $(\mathbb{V})^\perp$ (which is actually identical to \mathbb{V} in this very simple model). We have only to check that for each of the strong functors \mathbb{V} defined in that way, the underlying functor $\overline{\mathbb{V}}$ is a strict locally-continuous functor.

We deal with $!_$ and composition, the other cases are similar. The underlying functor of $!_$ is $\mathcal{M} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ defined by $\mathcal{M}(E) = \mathcal{M}_{\text{fin}}(E)$, $\mathcal{M}(s) = \{([a_1, \dots, a_k], [b_1, \dots, b_k]) \mid (a_i, b_i) \in s \text{ for } i = 1, \dots, k\}$ if $s \in \mathbf{Rel}(E, F)$. First if $s \subseteq t \in \mathbf{Rel}(E, F)$, it follows from the definition that $\mathcal{M}(s) \subseteq \mathcal{M}(t)$. Let $D \subseteq \mathbf{Rel}(E, F)$ be directed, we prove $\mathcal{M}(\bigcup D) \subseteq \bigcup_{s \in D} \mathcal{M}(s)$: an element of $\mathcal{M}(\bigcup D)$ is a pair $([a_1, \dots, a_k], [b_1, \dots, b_k])$ with $(a_i, b_i) \in \bigcup D$ for $i = 1, \dots, k$. Since D is directed, there is an $s \in D$ such that $(a_i, b_i) \in s$ for $i = 1, \dots, k$ and the inclusion follows. Strictness is obvious.

Composing variable sets Let $\mathbb{V}_i : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ be variable sets for $i = 1, \dots, k$ and let $\mathbb{W} : \mathbf{Rel}^k \rightarrow \mathbf{Rel}$ be a variable set. Then the functor $\overline{\mathbb{W}} \circ \overrightarrow{\mathbb{V}} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is clearly strict locally-continuous (since these conditions are preservation properties) from which it follows that the strong functor $\mathbb{U} = \overline{\mathbb{W}} \circ \overrightarrow{\mathbb{V}}$ is a variable type.

Fixed point of a variable set

Let $\mathbb{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ be a strict locally-continuous functor. Since $\emptyset \subseteq \mathbb{F}(\emptyset)$ we have $\mathbb{F}^n(\emptyset) \subseteq \mathbb{F}^{n+1}(\emptyset)$ for all $n \in \mathbb{N}$, by induction on n and hence $\mathbb{F}(\bigcup_{n=0}^{\infty} \mathbb{F}^n(\emptyset)) = \bigcup_{n=0}^{\infty} \mathbb{F}^n(\emptyset)$ by Lemma 24 since $\{\mathbb{F}^n(\emptyset) \mid n \in \mathbb{N}\}$ is directed. Let $\sigma\mathbb{F} = \bigcup_{n=0}^{\infty} \mathbb{F}^n(\emptyset)$, so that $(\sigma\mathbb{F}, \text{Id}_{\sigma\mathbb{F}})$ is an \mathbb{F} -coalgebra.

Lemma 26 The coalgebra $(\sigma\mathbb{F}, \text{Id})$ is final in $\mathbf{Coalg}_{\mathbf{Rel}}(\mathbb{F})$.

Proof: Let (E, t) be an \mathbb{F} -coalgebra. Let $e = \emptyset \in \mathbf{Rel}(E, \emptyset)$ (this is the unique morphism to the final object of \mathbf{Rel}). We define a sequence $e_n \in \mathbf{Rel}(E, \sigma\mathbb{F})$ as follows: $e_0 = \emptyset$ and $e_{n+1} = \mathbb{F}(e_n)t$. Then $e_n \subseteq e_{n+1}$ for all n by an easy induction, using the fact that \mathbb{F} is locally-continuous. Let $e = \bigcup_{n=0}^{\infty} e_n \in \mathbf{Rel}(E, \sigma\mathbb{F})$, by locally-continuity of \mathbb{F} we have $\mathbb{F}(e)t = (\bigcup_{n=0}^{\infty} \mathbb{F}(e_n))t = \bigcup_{n=0}^{\infty} (\mathbb{F}(e_n)t) = \bigcup_{n=0}^{\infty} e_{n+1} = e$ which means that $e \in \mathbf{Coalg}_{\mathbf{Rel}}(\mathbb{F})((E, t), (\sigma\mathbb{F}, \text{Id}))$. We end the proof by showing that e is the unique such morphism, so let

$$e' \in \mathbf{Coalg}_{\mathbf{Rel}}(\mathbb{F})((E, t), (\sigma\mathbb{F}, \text{Id})),$$

which means that $e' \in \mathbf{Rel}(E, \sigma\mathbb{F})$ and $\mathbb{F}(e')t = e'$.

Let $i_n \in \mathbf{Rel}(\sigma\mathbb{F}, \sigma\mathbb{F})$ be defined by induction by $i_0 = \emptyset$ and $i_{n+1} = \mathbb{F}(i_n)$. Then $(i_n)_{n \in \mathbb{N}}$ is monotone and $\bigcup_{n=0}^{\infty} i_n = \mathbf{Id}$ by definition of $\sigma\mathbb{F}$. We prove by induction on n that $\forall n \in \mathbb{N} \ i_n e' = i_n e$. Clearly $i_0 e' = i_0 e = \emptyset$. Next

$$\begin{aligned} i_{n+1} e' &= \mathbb{F}(i_n) \mathbb{F}(e') t = \mathbb{F}(i_n e') t = \mathbb{F}(i_n e) t \quad \text{by induction hypothesis} \\ &= i_{n+1} e. \end{aligned}$$

Therefore $e' = (\bigcup_{n \in \mathbb{N}} i_n) e' = \bigcup_{n \in \mathbb{N}} (i_n e') = \bigcup_{n \in \mathbb{N}} (i_n e) = e$. \blacksquare

Notice that $(\sigma\mathbb{F}, \mathbf{Id})$ is also an initial object in $\mathbf{Alg}_{\mathbf{Rel}}(\mathbb{F})$. When we insist on considering $\sigma\mathbb{F}$ as a final coalgebra, we denote it as $\nu\mathbb{F}$.

Lemma 27 *Let $\mathbb{F} : \mathbf{Rel}^{n+1} \rightarrow \mathbf{Rel}$ be a strict locally-continuous functor. The functor $\nu\mathbb{F} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is strict locally-continuous.*

Proof: As usual we assume that $n = 1$ to increase readability. We need to prove first that $\nu\mathbb{F}$ is monotone on morphisms, so let $s, t \in \mathbf{Rel}(E, F)$ with $s \subseteq t$. We have $\nu\mathbb{F}(s) = \bigcup_{n \in \mathbb{N}} s_n$ and $\nu\mathbb{F}(t) = \bigcup_{n \in \mathbb{N}} t_n$ with $s_0 = t_0 = \emptyset$, $s_{n+1} = \mathbb{F}(s, s_n)$ and $t_{n+1} = \mathbb{F}(t, t_n)$. By induction and monotonicity of \mathbb{F} we have $\forall n \in \mathbb{N} \ s_n \subseteq t_n$ and hence $\nu\mathbb{F}(s) \subseteq \nu\mathbb{F}(t)$. Let us prove now local-continuity so let $D \subseteq \mathbf{Rel}(E, F)$ be directed and let $t = \bigcup D$, we prove that $\nu\mathbb{F}(t) = \bigcup_{s \in D} \nu\mathbb{F}(s) \in \mathbf{Rel}(\nu\mathbb{F}(E), \nu\mathbb{F}(F))$ using Lemma 13 (with the notations of that lemma, we take $l = \mathbb{F}(t, \nu\mathbb{F}(E))$). We have $\mathbb{F}_F(\nu\mathbb{F}(t)) \mathbb{F}(t, \nu\mathbb{F}(E)) = \nu\mathbb{F}(t)$ by definition of the functor $\nu\mathbb{F}$ and

$$\begin{aligned} \mathbb{F}_F\left(\bigcup_{s \in D} \nu\mathbb{F}(s)\right) \mathbb{F}(t, \nu\mathbb{F}(E)) &= \bigcup_{s \in D} \mathbb{F}(F, \nu\mathbb{F}(s)) \bigcup_{s \in D} \mathbb{F}(s, \nu\mathbb{F}(E)) \quad \text{by locally-cont.} \\ &= \bigcup_{s \in D} \mathbb{F}(s, \nu\mathbb{F}(s)) = \bigcup_{s \in D} \nu\mathbb{F}(s). \end{aligned}$$

In the second equation, we used the facts that D is directed and the monotonicity of \mathbb{F} and $\nu\mathbb{F}$ on morphisms.

Let $E \subseteq F$, we prove that $\nu\mathbb{F}(E) \subseteq \nu\mathbb{F}(F)$. This results from the observation that if $E' \subseteq F'$, then $\mathbb{F}_E(E') \subseteq \mathbb{F}_F(F')$ and hence $\forall n \in \mathbb{N} \ \mathbb{F}_E^n(\emptyset) \subseteq \mathbb{F}_F^n(\emptyset)$. Let us check that $\nu\mathbb{F}(\eta_{E,F}^+) = \eta_{\nu\mathbb{F}(E), \nu\mathbb{F}(F)}^+ \in \mathbf{Rel}(\nu\mathbb{F}(E), \nu\mathbb{F}(F))$. We have $\mathbb{F}(F, \nu\mathbb{F}(\eta_{E,F}^+)) \mathbb{F}(\eta_{E,F}^+, \nu\mathbb{F}(E)) = \mathbb{F}(\eta_{E,F}^+, \nu\mathbb{F}(\eta_{E,F}^+)) = \nu\mathbb{F}(\eta_{E,F}^+)$ by definition of the functor $\nu\mathbb{F}$ and

$$\mathbb{F}(F, \eta_{\nu\mathbb{F}(E), \nu\mathbb{F}(F)}^+) \mathbb{F}(\eta_{E,F}^+, \nu\mathbb{F}(E)) = \eta_{\mathbb{F}(E, \nu\mathbb{F}(E)), \mathbb{F}(F, \nu\mathbb{F}(F))}^+ = \eta_{\nu\mathbb{F}(E), \nu\mathbb{F}(F)}^+$$

by strictness of \mathbb{F} . The equation follows by Lemma 13, so that the functor $\nu\mathbb{F}$ is strict. \blacksquare

Let $\mathbb{V} : \mathbf{Rel}^{n+1} \rightarrow \mathbf{Rel}$ be a variable set, by Lemma 15, there is a unique strong functor $\nu\mathbb{V} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ which is characterized by: $\overline{\nu\mathbb{V}}(\vec{E}) = \nu\overline{\mathbb{V}}_{\vec{E}}$, for each $\vec{s} \in \mathbf{Rel}^n(\vec{E}, \vec{F})$, $\overline{\nu\mathbb{V}}(\vec{s}) = \overline{\mathbb{V}}(\vec{s}, \nu\overline{\mathbb{V}}(\vec{s}))$ and last $\overline{\mathbb{V}}(!E \otimes \vec{F}, \widehat{\mathbb{V}}_{E, \vec{F}}) = \widehat{\mathbb{V}}_{E, \vec{F}}$.

Lemma 28 *The functor $\nu\mathbb{V}$ is a variable set.*

Proof: By the conditions above satisfied by $\nu\mathbb{V}$ we have that $\overline{\nu\mathbb{V}} = \nu\overline{\mathbb{V}}$ and hence $\overline{\nu\mathbb{V}}$ is strict locally-continuous by Lemma 27. \blacksquare

A model of μLL based on variable sets.

Let \mathbf{Rel}_n be the class of all n -ary \mathbf{Vst} , so that $\mathbf{Rel}_0 = \text{Obj}(\mathbf{Rel})$. The fact that $(\mathbf{Rel}, (\mathbf{Rel}_n)_{n \in \mathbb{N}})$ is a Seely model of μLL in the sense of Section 1.1.3 results mainly from the fact that we take *all* variable sets in the \mathbf{Rel}_n 's so that there is essentially nothing to check. More explicitly:

Theorem 29 $(\mathbf{Rel}, (\mathbf{Rel}_n)_{n \in \mathbb{N}})$ is a Seely model of μLL .

Proof: (1) holds by Section 0.4.1, (2) holds by construction, (3) holds by the fact that variable sets compose as explained in Section 1.2.1 (notice that this condition refers to the general composition of strong functors defined in Section 1.1.2), (4) holds by Section 1.2.1 and by the fact that the De Morgan dual of a strong functor is strong, see Section 1.1.2 and (5) holds by Section 1.2.1. \blacksquare

1.2.2 Non-uniform totality spaces

As we saw in Section 1.2.1, the interpretation of least and greatest fixpoints are not distinguished in \mathbf{Rel} . We now move to our second concrete model of μLL in order to distinguish least and greatest fixpoints. This second model is based on the notion of totality on top of \mathbf{Rel} , and we denoted it by \mathbf{Nuts} . As our categorical model of μLL lies over a categorical model of LL , we first need to prove that \mathbf{Nuts} indeed is a model of linear logic. To do so, we show that \mathbf{Nuts} is actually an instance of Seely category mentioned in Chapter 0.

Let us first define this category.

Definitions of \mathbf{Nuts}

Let E be a set and $\mathcal{T} \subseteq \mathcal{P}(E)$. We define

$$(\mathcal{T})^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}.$$

If $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{P}(E)$ then $(\mathcal{T})^\perp \subseteq (\mathcal{S})^\perp$. We also have $\mathcal{T} \subseteq \mathcal{T}^{\perp\perp}$ and therefore $\mathcal{T}^{\perp\perp\perp} = (\mathcal{T})^\perp$.

Lemma 30 Let $\mathcal{T} \subseteq \mathcal{P}(E)$, then $\mathcal{T}^{\perp\perp} = \uparrow\mathcal{T} = \{v \subseteq E \mid \exists u \in \mathcal{T} \ u \subseteq v\}$.

Proof: The \supseteq direction is obvious, let us prove the converse so let $u \subseteq E$ and assume that $u \notin \uparrow\mathcal{T}$. This means that for each $v \in \mathcal{T}$ there exists $a(v) \in v$ such that $a(v) \notin u$. Let $u' = \{a(v) \mid v \in \mathcal{T}\} \subseteq E$. By construction we have $u' \in (\mathcal{T})^\perp$ and $u \cap u' = \emptyset$. This shows that $u \notin \mathcal{T}^{\perp\perp}$. \blacksquare

Definition 31 A non-uniform totality space (NUTS) is a pair $X = (|X|, \mathcal{T}(X))$ where $|X|$ is a set and $\mathcal{T}(X) \subseteq \mathcal{P}(|X|)$ satisfies $\mathcal{T}(X) = \mathcal{T}(X)^{\perp\perp}$, that is $\mathcal{T}(X) = \uparrow\mathcal{T}(X)$, in other words, $\mathcal{T}(X)$ is upwards closed wrt. inclusion.

And we set $X^\perp = (|X|, (\mathcal{T}(X))^\perp)$.

Here is an example: take $X = (\mathbb{N}, \mathcal{T}(X))$ where $\mathcal{T}(X)$ is the set of all infinite subsets of \mathbb{N} . It is a NUTS because a superset of an infinite set is infinite. Then $|X^\perp| = \mathbb{N}$ and $\mathcal{T}(X^\perp)$

is the set of all cofinite subsets of \mathbb{N} (the subsets u of \mathbb{N} such that $\mathbb{N} \setminus u$ is finite). If, with the same web \mathbb{N} , we take $\mathcal{T}(X) = \{u \subseteq \mathbb{N} \mid u \neq \emptyset\}$ (again $\mathcal{T}(X) = \uparrow \mathcal{T}(X)$ obviously), then $\mathcal{T}(X^\perp) = \{\mathbb{N}\}$.

Lemma 32 *For a NUTS X we have $\emptyset \in \mathcal{T}(X) \Leftrightarrow \mathcal{T}(X^\perp) = \emptyset$.*

We define four basic NUTS:

- $0 = (\emptyset, \emptyset)$
- $\top = (\emptyset, \{\emptyset\})$
- $1 = (\{*\}, \{\{*\}\})$
- $\perp = 1 = (\{*\}, \{\{*\}\})$

Given NUTS X_1 and X_2 we define a NUTS $X_1 \otimes X_2$ by $|X_1 \otimes X_2| = |X_1| \times |X_2|$ and

$$\begin{aligned} \mathcal{T}(X_1 \otimes X_2) &= \{u_1 \times u_2 \mid u_i \in \mathcal{T}(X_i) \text{ for } i = 1, 2\}^{\perp\perp} \\ &= \uparrow \{u_1 \times u_2 \mid u_i \in \mathcal{T}(X_i) \text{ for } i = 1, 2\}. \end{aligned}$$

And then we define $X \multimap Y = (X \otimes Y^\perp)^\perp$.

Lemma 33 $t \in \mathcal{T}(X \multimap Y) \Leftrightarrow \forall u \in \mathcal{T}(X) \ t \cdot u \in \mathcal{T}(Y)$.

Proof: Let $t \in \mathcal{T}(X \multimap Y)$ and let $u \in \mathcal{T}(X)$. Let $v' \in (\mathcal{T}(Y))^\perp$, since $u \times v' \in \mathcal{T}(X \otimes Y^\perp)$ we have $t \cap (u \times v') \neq \emptyset$ and hence $(t \cdot u) \cap v' \neq \emptyset$. Therefore $t \cdot u \in \mathcal{T}(Y)^{\perp\perp} = \mathcal{T}(Y)$. Conversely assume that $\forall u \in \mathcal{T}(X) \ t \cdot u \in \mathcal{T}(Y)$. Let $u \in \mathcal{T}(X)$ and $v' \in \mathcal{T}(Y^\perp) = (\mathcal{T}(Y))^\perp$. Since $t \cdot u \in \mathcal{T}(Y)$ we have $(t \cdot u) \cap v' \neq \emptyset$ and hence $t \cap (u \times v') \neq \emptyset$ and this shows that $t \in \mathcal{T}(X \multimap Y)$. ■

We define the category **Nuts** whose objects are the NUTS and $\mathbf{Nuts}(U, V) = \mathcal{T}(U \multimap V)$, composition being defined as the usual composition in **Rel** (relational composition) and identities as the diagonal relations. Lemma 33 shows that we have indeed defined a category.

Multiplicative structure

Lemma 34 *Let X and Y be NUTS and $t \in \mathbf{Nuts}(X, Y)$. Then t is an iso in **Nuts** iff t is (the graph of) a bijection $|X| \rightarrow |Y|$ such that $\forall u \subseteq |X| \ (u \in \mathcal{T}(X) \Leftrightarrow t(u) \in \mathcal{T}(Y))$.*

Proof: Assume that t is an iso in **Nuts** so that there is $t' \in \mathbf{Nuts}(Y, X)$ such that $t' t = \text{Id}_{|X|}$ and $t t' = \text{Id}_{|Y|}$ and since we know that the isos in **Rel** are the bijections we know that t is a bijection. The fact that $\forall u \subseteq |X| \ u \in \mathcal{T}(X) \Leftrightarrow t(u) \in \mathcal{T}(Y)$ results from the fact that both t and $t' = t^{-1}$ are morphisms in **Nuts**.

The converse implication is obvious. ■

Lemma 35 *Let $t \subseteq |X| \times |Y|$. One has $t \in \mathbf{Nuts}(X, Y)$ iff $(t)^\perp = \{(b, a) \mid (a, b) \in t\} \in \mathbf{Nuts}(Y^\perp, X^\perp)$.*

Proof: This is an obvious consequence of Lemma 33 and of the fact that $(X \multimap Y) = (X \otimes Y^\perp)^\perp$ and $(Y^\perp \multimap X^\perp) = (Y^\perp \otimes X)^\perp$. ■

Lemma 36 *Let $t \subseteq |X_1 \otimes X_2 \multimap Y|$. One has $t \in \mathbf{Nuts}(X_1 \otimes X_2, Y)$ iff for all $u_1 \in \mathcal{T}(X_1)$ and $u_2 \in \mathcal{T}(X_2)$ one has $t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$.*

Proof: The condition is obviously necessary, let us prove that it is sufficient so assume that t fulfills it and let us prove that $t \in \mathcal{T}(X_1 \otimes X_2 \multimap Y)$. To this end it suffices to prove that $(t)^\perp \in \mathcal{T}(Y^\perp \multimap (X_1 \otimes X_2)^\perp)$. So let $v' \in \mathcal{T}(Y^\perp)$ and let us prove that $(t)^\perp \cdot v' \in \mathcal{T}((X_1 \otimes X_2)^\perp) = (\{u_1 \otimes u_2 \mid u_1 \in \mathcal{T}(X_1) \text{ and } u_2 \in \mathcal{T}(X_2)\})^\perp$. So let $u_i \in \mathcal{T}(X_i)$ for $i = 1, 2$. We know that $t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$ and hence $(t \cdot (u_1 \otimes u_2)) \cap v' \neq \emptyset$, that is $(u_1 \otimes u_2) \cap ((t)^\perp \cdot v') \neq \emptyset$, proving our contention. ■

Lemma 37 *The bijection $\alpha_{|X_1|, |X_2|, |Y|}$ is an isomorphism from $(X_1 \otimes X_2) \multimap Y$ to $X_1 \multimap (X_2 \multimap Y)$.*

Proof: Let $t \in \mathcal{T}((X_1 \otimes X_2) \multimap Y)$ and let us prove that $s = \alpha \cdot t \in \mathcal{T}(X_1 \multimap (X_2 \multimap Y))$. Given $u_i \in \mathcal{T}(X_i)$ it suffices to prove that $(t \cdot u_1) \cdot u_2 \in \mathcal{T}(Y)$ which results from the fact that $(s \cdot u_1) \cdot u_2 = t \cdot (u_1 \otimes u_2)$. Conversely let $s \in \mathcal{T}(X_1 \multimap (X_2 \multimap Y))$ and let us prove that $t = \alpha^{-1} \cdot s \in \mathcal{T}((X_1 \otimes X_2) \multimap Y)$. This results from lemma 36 and from the equation $(s \cdot u_1) \cdot u_2 = t \cdot (u_1 \otimes u_2)$. ■

We turn now that \otimes into a functor, its action on morphisms being defined as in **Rel**. Let $t_i \in \mathbf{Nuts}(X_i, Y_i)$ for $i = 1, 2$, we have $t_1 \otimes t_2 \in \mathbf{Nuts}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ by Lemma 36 and by the equation

$$(t_1 \otimes t_2) \cdot (u_1 \otimes u_2) = (t_1 \cdot u_1) \otimes (t_2 \cdot u_2).$$

This functor is monoidal, with unit 1 and symmetric monoidality isomorphisms λ, ρ, γ and α defined as in **Rel**. The only non-trivial thing to check is that α is indeed a morphism, namely

$$\alpha_{|X_1|, |X_2|, |X_3|} \in \mathbf{Nuts}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3)).$$

This results from Lemma 37 and from the observation that

$$\begin{aligned} ((X_1 \otimes X_2) \otimes X_3)^\perp &= ((X_1 \otimes X_2) \multimap X_3^\perp) \\ (X_1 \otimes (X_2 \otimes X_3))^\perp &= (X_1 \multimap (X_2 \multimap X_3^\perp)). \end{aligned}$$

The SMC category **Nuts** is closed, with $X \multimap Y$ as internal hom object from X to Y , and evaluation morphism

$$\text{ev} = \{(((a, b), a), b) \mid a \in |X| \text{ and } b \in |Y|\}$$

which indeed belongs to $\mathbf{Nuts}((X \multimap Y) \otimes X, Y)$ by Lemma 36 since, for all $t \in \mathcal{T}(X \multimap Y)$ and $u \in \mathcal{T}(X)$ we have

$$\text{ev}(t \otimes u) = t u \in \mathcal{T}(Y).$$

This category **Nuts** is also *-autonomous with dualizing object $\perp = 1$.

Additive structure

Let $(X_i)_{i \in I}$ be a family of objects of **Nuts**. We define $X = \&_{i \in I} X_i$ as follows: $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$ and

$$\mathcal{T}(X) = \{u \subseteq |X| \mid \forall i \in I \text{ pr}_i \cdot u \in \mathcal{T}(X_i)\}.$$

It is clear that $\mathcal{T}(X) = \uparrow \mathcal{T}(X)$ and hence X is an object of **Nuts**. By definition of X and by Lemma 33 we have $\forall i \in I \text{ pr}_i \in \mathbf{Nuts}(X, X_i)$. Given $\vec{t} = (t_i)_{i \in I}$ with $\forall i \in I t_i \in \mathbf{Nuts}(Y, X_i)$, we have $\langle \vec{t} \rangle \in \mathbf{Nuts}(Y, X)$ as easily checked (using Lemma 33 again). It follows that $(\&_{i \in I} X_i, (\text{pr}_i)_{i \in I})$ is the cartesian product of the X_i 's in **Nuts**.

This shows that the category **Nuts** has all countable products and hence is cartesian.

Since it is *-autonomous, the category **Nuts** is also cocartesian, coproduct being given by $\oplus_{i \in I} X_i = (\&_{i \in I} (X_i)^\perp)^\perp$. It follows that we have

$$\oplus_{i \in I} X_i = \left(\bigcup_{i \in I} \{i\} \times |X_i|, \left\{ v \subseteq \bigcup_{i \in I} \{i\} \times |X_i| \mid \exists i \in I \exists u \in \mathcal{T}(X_i) \{i\} \times u \subseteq v \right\} \right).$$

Indeed, let $v \in (\mathcal{T}(\&_{i \in I} (X_i)^\perp))^\perp$. This means that for any family $(u'_i)_{i \in I}$ such that $\forall i \in I u'_i \in (\mathcal{T}(X_i))^\perp$, one has $v \cap (\bigcup_{i \in I} \{i\} \times u'_i) \neq \emptyset$. If, for some $i \in I$ and $u \in \mathcal{T}(X_i)$, one has $\{i\} \times u \subseteq v$, this condition obviously holds. Otherwise, this means that $\forall i \in I \text{ pr}_i(v) \notin \mathcal{T}(X_i) = \mathcal{T}(X_i)^\perp$, and hence, for all $i \in I$, there exists $u'_i \in \mathcal{T}(X_i)^\perp$ such that $\text{pr}_i(v) \cap u'_i = \emptyset$. Then we have $v \cap (\bigcup_{i \in I} \{i\} \times u'_i) = \emptyset$ contradicting our assumption about v since $\bigcup_{i \in I} \{i\} \times u'_i \in \mathcal{T}(\&_{i \in I} (X_i)^\perp)$. The injections $\bar{\pi}_j \in \mathbf{Nuts}(X_j, \oplus_{i \in I} X_i)$ are those of **Rel**. Also, given $t_i \in \mathbf{Nuts}(X_i, X)$, the co-tupling $[\vec{t}] \in \mathbf{Nuts}(\oplus_{i \in I} X_i, X)$ is defined as in **Rel**.

Notice that the final object is $\top = (\emptyset, \{\emptyset\})$ and that $0 = \top^\perp = (\emptyset, \emptyset)$.

Exponential

This exponential is an extension of the multiset exponential of **Rel** with totality. Remember that $u^{(1)} = \mathcal{M}_{\text{fin}}(u)$. We set $!X = \mathcal{M}_{\text{fin}}(|X|)$ and

$$\mathcal{T}(!X) = \left\{ u^{(1)} \mid u \in \mathcal{T}(X) \right\}^{\perp\perp} = \uparrow \left\{ u^{(1)} \mid u \in \mathcal{T}(X) \right\}.$$

Lemma 38 *Let $t \subseteq \mathcal{M}_{\text{fin}}(|X|) \times |Y|$. One has $t \in \mathbf{Nuts}(!X, Y)$ iff for all $u \in \mathcal{T}(X)$ one has $t \cdot u^{(1)} \in \mathcal{T}(Y)$.*

Proof: The condition is obviously necessary, so let us assume that it holds. By Lemma 35, it suffices to prove that $(t)^\perp \in \mathbf{Nuts}(Y^\perp, (!X)^\perp)$. Let $v' \in \mathcal{T}(Y^\perp)$, we prove that $(t)^\perp \cdot v' \in (\mathcal{T}(!Y))^\perp$. So let $u \in \mathcal{T}(X)$, since $t \cdot u^{(1)} \in \mathcal{T}(Y)$ and hence $(t \cdot u^{(1)}) \cap v' \neq \emptyset$, that is $((t)^\perp \cdot v') \cap u^{(1)} \neq \emptyset$. ■

Lemma 39 *Let $t \subseteq \mathcal{M}_{\text{fin}}(|X_1|) \times \mathcal{M}_{\text{fin}}(|X_2|) \times |Y|$. One has $t \in \mathbf{Nuts}(!X_1 \otimes !X_2, Y)$ iff for all $u_1 \in \mathcal{T}(X_1)$ and $u_2 \in \mathcal{T}(X_2)$, one has $t \cdot (u_1^{(1)} \otimes u_2^{(1)}) \in \mathcal{T}(Y)$.*

Proof: The condition is necessary since, if $u_1 \in \mathcal{T}(X_1)$ and $u_2 \in \mathcal{T}(X_2)$, then $u_1^{(1)} \otimes u_2^{(1)} \in \mathcal{T}(!X_1 \otimes !X_2)$. So assume that it holds. Let $t' = \text{cur}(t) \in \mathbf{Rel}(|X_1| \multimap (|X_2| \multimap |Y|))$. Let $u_1 \in \mathcal{T}(X_1)$, we have $t' \cdot u_1^{(1)} \in \mathcal{P}(!X_2 \multimap Y)$. Let $u_2 \in \mathcal{T}(X_2)$, we have $(t' \cdot u_1^{(1)}) \cdot u_2^{(1)} = t \cdot (u_1^{(1)} \otimes u_2^{(1)}) \in \mathcal{T}(Y)$ by our assumption. It follows by Lemma 38 that $t' \cdot u_1^{(1)} \in \mathcal{T}(!X_2 \multimap Y)$ and since this holds for any $u_1 \in \mathcal{T}(X_1)$ we actually have $t' \in \mathbf{Nuts}(!X_1, !X_2 \multimap Y)$. It follows that $t = \text{cur}^{-1}(t') \in \mathbf{Nuts}(!X_1 \otimes !X_2, Y)$ as contended. ■

Lemma 39 generalizes easily to n-ary tensors \otimes .

Lemma 40 *For any $t \in \mathbf{Nuts}(X, Y)$, one has $!t \in \mathbf{Nuts}(!X, !Y)$.*

Proof: By Lemma 38 and the fact that $!t \cdot u^{(1)} = (t \cdot u)^{(1)}$. ■

Nuts as a model of LL: To prove that **Nuts** is a categorical model of LL, it suffices to show that the various relational morphisms defining the strong symmetric monoidal monadic structure of $!_-$ in **Rel** (see Section 0.4.1) are actually morphisms in **Nuts**. This is essentially straightforward and based on Lemma 38.

Lemma 41 *Equipped with der , dig , \mathfrak{m}^0 and \mathfrak{m}^2 defined as in **Rel**, $!_-$ is a symmetric monoidal comonad which turns **Nuts** into a Seely model of LL.*

Proof: Given an object X of **Nuts**, we set $\text{der}_X = \text{der}_{|X|} \in \mathbf{Rel}(|!X|, |X|)$ and $\text{dig}_X = \text{dig}_{|X|} \in \mathbf{Rel}(|!X|, |!X|)$. Given $u \in \mathcal{T}(X)$, we have $\text{der}_X \cdot u^{(1)} = u \in \mathcal{T}(X)$ and $\text{dig}_X \cdot u^{(1)} = u^{(1)(1)} \in \mathcal{T}(!X)$. It follows by Lemma 38 that $\text{der}_X \in \mathbf{Nuts}(!X, X)$ and $\text{dig}_X \in \mathbf{Nuts}(!X, !X)$.

Naturality and monadicity trivially hold because they hold in **Rel**: we have an obvious faithful forgetful functor **Nuts** \rightarrow **Rel** which commutes with all LL categorical constructs.

We are left with defining the strong monoidal structure of $!_-$ (Seely isomorphisms), for $\mathfrak{m}^0 \in \mathbf{Nuts}(1, !\top)$ we take the same morphism as in **Rel**. And we set $\mathfrak{m}_{X_1, X_2}^2 = \mathfrak{m}_{|X_1|, |X_2|}^2 \in \mathbf{Rel}(|!X_1 \otimes !X_2|, |(X_1 \& X_2)|)$. Let $u_i \in \mathcal{T}(X_i)$ for $i = 1, 2$. We have $\mathfrak{m}_{X_1, X_2}^2 \cdot (u_1^{(1)} \otimes u_2^{(1)}) = (u_1 \& u_2)^{(1)} \in \mathcal{T}(!(X_1 \& X_2))$ since $u_1 \& u_2 \in \mathcal{T}(X_1 \& X_2)$, and hence by Lemma 39 we have $\mathfrak{m}_{X_1, X_2}^2 \in \mathbf{Nuts}(!X_1 \otimes !X_2, !(X_1 \& X_2))$. Any element w of $\mathcal{T}(X_1 \& X_2)$ is of shape $w = u_1 \& u_2$ with $u_i \in \mathcal{T}(X_i)$, namely $u_i = \text{pr}_i \cdot w$. We have $(\mathfrak{m}_{X_1, X_2}^2)^{-1} \cdot w^{(1)} = u_1^{(1)} \otimes u_2^{(1)} \in \mathcal{T}(!X_1 \otimes !X_2)$ and hence by Lemma 38 we have $(\mathfrak{m}_{X_1, X_2}^2)^{-1} \in \mathbf{Nuts}(!(X_1 \& X_2), !X_1 \otimes !X_2)$. This ends the proof that **Nuts** is a model of classical LL since the required commutations obviously hold because they hold in **Rel**. ■

Variable non-uniform totality spaces (VNUTS)

Let E be a set, we use $\text{Tot}(E)$ for the set of all *totality candidates* on E , that is, of all subsets \mathcal{T} of $\mathcal{P}(E)$ such that $\mathcal{T} = \mathcal{T}^{\perp\perp}$ (remember that $(\mathcal{T})^\perp = \{u' \subseteq E \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}$). In other words $\mathcal{T} \in \text{Tot}(E)$ means that $\mathcal{T} = \uparrow\mathcal{T}$ by Lemma 30.

Lemma 42 *Ordered by \subseteq , this set $\text{Tot}(E)$ is a complete lattice.*

Proof: Indeed, let $\Theta \subseteq \text{Tot}(E)$, we have

$$\cap\Theta = \left\{ u \subseteq E \mid \forall \mathcal{T} \in \Theta \forall u' \in (\mathcal{T})^\perp \ u \cap u' \neq \emptyset \right\} = \left(\bigcup \{ (\mathcal{T})^\perp \mid \mathcal{T} \in \Theta \} \right)^\perp$$

from which it follows that $\cap\Theta \in \text{Tot}(E)$. ■

Definition 43 Let $n \in \mathbb{N}$, an n -ary VNUTS is a pair $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ where $|\mathbb{X}| : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ is a variable set $|\mathbb{X}| = (|\overline{\mathbb{X}}|, |\widehat{\mathbb{X}}|)$ (see Section 1.2.1) and $\mathcal{T}(\mathbb{X})$ is an operation which associates with each n -tuple \vec{X} of objects of **Nuts** an element $\mathcal{T}(\mathbb{X})(\vec{X})$ of $\text{Tot}(|\overline{\mathbb{X}}|(|\vec{X}|))$ in such a way that

1. for any $\vec{t} \in \mathbf{Nuts}^n(\vec{X}, \vec{Y})$, the element $|\overline{\mathbb{X}}|(\vec{t})$ of $\mathbf{Rel}(|\overline{\mathbb{X}}|(|\vec{X}|), |\overline{\mathbb{X}}|(|\vec{Y}|))$ belongs to $\mathbf{Nuts}(|\overline{\mathbb{X}}|(\vec{X}), |\overline{\mathbb{X}}|(\vec{Y}))$ (where $|\overline{\mathbb{X}}|(\vec{X})$ denotes the NUTS $(|\overline{\mathbb{X}}|(|\vec{X}|), \mathcal{T}(\mathbb{X})(\vec{X}))$)
2. and for any $\vec{Y} \in \text{Obj}(\mathbf{Nuts}^n)$ and any $X \in \text{Obj}(\mathbf{Nuts})$ one has $|\widehat{\mathbb{X}}|_{|X|, |\vec{Y}|} \in \mathbf{Nuts}(!X \otimes |\overline{\mathbb{X}}|(\vec{Y}), |\overline{\mathbb{X}}|(!X \otimes \vec{Y}))$. In other words, for $u \in \mathcal{T}(X)$ and $v \in \text{Tot}(\mathbb{X})(\vec{Y})$, one has $|\widehat{\mathbb{X}}|_{|X|, |\vec{Y}|} \cdot (u^{(!)} \otimes v) \in \text{Tot}(\mathbb{X})(!X \otimes \vec{Y})$.

Remark 13 The most obvious approach to define VNUTS would have been to define them directly as functors $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$. We prefer to define a VNUTS as a gadget from which such a functor can be derived (Lemma 44). Because we know that **Rel** underlies VNUTS and that in **Rel** we have a very simple fixpoint for all **Vst** (and these **Vst** are really defined as functors) of which we take advantage for defining VNUTS least and greatest fixpoints by a simple application of Knaster–Tarski theorem on top of the set obtained by this **Vst** fixpoint which is the same for both fixpoints.

Remark 14 It is essential to keep in mind that, if $\vec{f} \in \mathbf{Nuts}(\vec{X}, \vec{Y})$ then actually $\vec{f} \in \mathbf{Rel}(|\vec{X}|, |\vec{Y}|)$ so that the morphism $|\overline{\mathbb{X}}|(\vec{f}) \in \mathbf{Rel}(|\overline{\mathbb{X}}|(|\vec{X}|), |\overline{\mathbb{X}}|(|\vec{Y}|))$ is defined, independently of the notions of totality on \vec{X} and \vec{Y} and similarly for $|\widehat{\mathbb{X}}|_{\vec{Y}, \vec{X}}$. This decoupling of the totality-free part of the notions involved from the totality dependent ones makes life much simpler. This definition also defines VNUTS as a generalization of **Nuts**: a web (which is now a **Vst** and not a set) plus an additional totality structure (which is parameterized). We will use exactly the same definition for variable coherence spaces with totality (Definition 58). It would be nice to find the general categorical framework for this kind of 2-level definition (See Section 5.1 for more details on this).

Lemma 44 Any VNUTS $\mathbb{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ induces a strong functor $\mathcal{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ which satisfies

- $|\mathcal{X}(\vec{X})| = |\overline{\mathbb{X}}|(|\vec{X}|)$,
- $\mathcal{T}(\mathcal{X}(\vec{X})) = \mathcal{T}(\mathbb{X})(\vec{X})$,
- $\mathcal{X}(\vec{t}) = |\overline{\mathbb{X}}|(\vec{t}) \in \mathbf{Nuts}(|\overline{\mathbb{X}}|(\vec{X}), |\overline{\mathbb{X}}|(\vec{Y}))$ for $\vec{t} \in \mathbf{Nuts}(\vec{X}, \vec{Y})$,
- and $\widehat{\mathcal{X}}_{X, \vec{Y}} = |\widehat{\mathbb{X}}|_{|X|, |\vec{Y}|}$

and \mathbb{X} can be retrieved from \mathcal{X} .

Proof: It is clear that \mathcal{X} so defined is a strong functor. Let us check that \mathbb{X} can be retrieved from \mathcal{X} . Given a set E , $(E, \mathcal{P}(E))$ is a NUTS that we denote as $\mathfrak{p}(E)$. Notice that \mathfrak{p} can be extended into a functor $\mathbf{Rel} \rightarrow \mathbf{Nuts}$ which acts as the identity on morphisms. There is also a forgetful functor $\mathfrak{u} : \mathbf{Nuts} \rightarrow \mathbf{Rel}$ which maps X to $|X|$ and acts as the identity on morphisms (btw. \mathfrak{p} is right adjoint to \mathfrak{u}). Let \mathbb{X} be a unary VNUTS and let $\mathcal{X} : \mathbf{Nuts} \rightarrow \mathbf{Nuts}$ be the associated strong functor. Then we have $|\mathbb{X}| = \mathfrak{u} \circ \overline{\mathcal{X}} \circ \mathfrak{p}$ and $|\widehat{\mathbb{X}}|_{E,F} = \widehat{\mathcal{X}}_{\mathfrak{p}(E),\mathfrak{p}(F)}$ for any sets E and F . Last, given a NUTS X , we have that $\mathcal{T}(\mathbb{X})(X)$ is just the totality component of the NUTS $\mathcal{X}(X)$. This shows that \mathbb{X} is determined by \mathcal{X} as contended. \blacksquare

For this reason we use \mathbb{X} to denote the functor \mathcal{X} .

Given $n \in \mathbb{N}$ let \mathbf{Vnuts}_n be the class of strong n -ary VNUTS. We identify \mathbf{Vnuts}_0 with the class of objects of the Seely category \mathbf{Nuts} . The following refers to Definition 17.

Theorem 45 ($\mathbf{Nuts}, (\mathbf{Vnuts}_n)_{n \in \mathbb{N}}$) *is a Seely model of μ LL.*

Proof: Concerning Condition (3), let $(\mathbb{X}_i)_{i=1}^k$ be elements of \mathbf{Vnuts}_n and let $\mathbb{X} \in \mathbf{Vnuts}_k$. Considering \mathbb{X} and the \mathbb{X}_i 's as strong functors, we know that $\mathbb{X} \circ \overline{\mathbb{X}}$ is a strong functor $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$. We simply have to exhibit a VNUTS whose associated strong functor is $\mathbb{X} \circ \overline{\mathbb{X}}$. Let $\mathbb{F} = |\mathbb{X}| \circ |\overline{\mathbb{X}}|$ (composition of variable sets, Section 1.2.1). Let $\overline{\mathbb{X}} \in \mathbf{Nuts}^n$, each $\overline{\mathbb{X}_i}(\overline{\mathbb{X}})$ is an object of \mathbf{Nuts} and hence $(|\mathbb{F}|(|\overline{\mathbb{X}}|), \mathcal{T}(\mathbb{X})(\overline{\mathbb{X}_1}(\overline{\mathbb{X}}), \dots, \overline{\mathbb{X}_k}(\overline{\mathbb{X}})))$ is a NUTS. Moreover given $\overrightarrow{t} \in \mathbf{Nuts}^n(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$, we know that for each $i = 1, \dots, k$, one has $\overline{\mathbb{X}_i}(\overrightarrow{t}) \in \mathbf{Nuts}(\overline{\mathbb{X}_i}(\overline{\mathbb{X}}), \overline{\mathbb{X}_i}(\overline{\mathbb{Y}}))$ since \mathbb{X}_i is a VNUTS. Since \mathbb{X} is a VNUTS we have $|\mathbb{F}|(\overrightarrow{t}) \in \mathbf{Nuts}(\mathbb{X}(\overline{\mathbb{X}_1}(\overline{\mathbb{X}}), \dots, \overline{\mathbb{X}_k}(\overline{\mathbb{X}})), \mathbb{X}(\overline{\mathbb{X}_1}(\overline{\mathbb{Y}}), \dots, \overline{\mathbb{X}_k}(\overline{\mathbb{Y}})))$.

Let $X \in \mathbf{Obj}(\mathbf{Nuts})$ and $\overline{\mathbb{Y}} \in \mathbf{Obj}(\mathbf{Nuts}^k)$. For $i = 1, \dots, k$ we know that $\widehat{\mathbb{X}}_{X, \overline{\mathbb{Y}}} \in \mathbf{Nuts}(!X \otimes \overline{\mathbb{X}_i}(\overline{\mathbb{Y}}), \overline{\mathbb{X}_i}(!X \otimes \overline{\mathbb{Y}}))$. Therefore

$$\overline{\mathbb{X}}(\widehat{\mathbb{X}}_{X, \overline{\mathbb{Y}}})_{i=1}^k \in \mathbf{Nuts}(\overline{\mathbb{X}}(!X \otimes \overline{\mathbb{X}_i}(\overline{\mathbb{Y}}))_{i=1}^k, \overline{\mathbb{X}}(\overline{\mathbb{X}_i}(!X \otimes \overline{\mathbb{Y}}))_{i=1}^k)$$

and hence

$$\overline{\mathbb{X}}(\widehat{\mathbb{X}}_{X, \overline{\mathbb{Y}}})_{i=1}^k \widehat{\mathbb{X}}_{X, (\overline{\mathbb{X}_i}(\overline{\mathbb{Y}}))_{i=1}^k} \in \mathbf{Nuts}(!X \otimes \overline{\mathbb{X}}(\overline{\mathbb{X}_i}(\overline{\mathbb{Y}}))_{i=1}^k, \overline{\mathbb{X}}(\overline{\mathbb{X}_i}(!X \otimes \overline{\mathbb{Y}}))_{i=1}^k).$$

Moreover we have

$$\begin{aligned} \widehat{\mathbb{F}}_{|X|, |\overline{\mathbb{Y}}|} &= |\overline{\mathbb{X}}|(|\widehat{\mathbb{X}}_{X, \overline{\mathbb{Y}}}|_{|X|, |\overline{\mathbb{Y}}|})_{i=1}^k |\widehat{\mathbb{X}}|_{|X|, (|\overline{\mathbb{X}_i}(\overline{\mathbb{Y}})|)_{i=1}^k} \quad \text{by definition of } \mathbb{F} \\ &= |\overline{\mathbb{X}}|(|\widehat{\mathbb{X}}_{X, \overline{\mathbb{Y}}}|_{|X|, |\overline{\mathbb{Y}}|})_{i=1}^k |\widehat{\mathbb{X}}|_{|X|, (|\overline{\mathbb{X}_i}(\overline{\mathbb{Y}})|)_{i=1}^k} \\ &= \overline{\mathbb{X}}(\widehat{\mathbb{X}}_{X, \overline{\mathbb{Y}}})_{i=1}^k \widehat{\mathbb{X}}_{X, (\overline{\mathbb{X}_i}(\overline{\mathbb{Y}}))_{i=1}^k} \end{aligned}$$

using again the fact that \mathbb{X} and the \mathbb{X}_i 's are VNUTS. This shows that the pair $\mathbb{Y} = (|\mathbb{Y}|, \mathcal{T}(\mathbb{Y}))$ given by $|\mathbb{Y}| = \mathbb{F}$ and $\mathcal{T}(\mathbb{Y})(\overline{\mathbb{X}}) = \mathcal{T}(\mathbb{X})(\overline{\mathbb{X}_1}(\overline{\mathbb{X}}), \dots, \overline{\mathbb{X}_k}(\overline{\mathbb{X}}))$ is a VNUTS whose associated strong functor is $\mathbb{X} \circ \overline{\mathbb{X}}$ thus proving our contention.

Concerning Condition (4), let us deal only with the case of $!_-$, the others being similar. We have to exhibit a unary VNUTS \mathbb{X} whose associated strong functor $\mathbf{Nuts} \rightarrow \mathbf{Nuts}$ coincides with $!_-$ (which is known to be a strong functor $\mathbf{Nuts} \rightarrow \mathbf{Nuts}$ by Section 1.2.2 and by the general considerations of Section 1.1.2). For $|\mathbb{X}|$, which has to be a variable set

$\mathbf{Rel} \rightarrow \mathbf{Rel}$, we take the interpretation \mathbb{E} of $!_-$ in the model \mathbf{Rel} (Section 1.2.1) which is an element of \mathbf{Rel}_1 , that is, a unary variable set. Next, given $X \in \mathbf{Obj}(\mathbf{Nuts})$, we take $\mathcal{T}(\mathbb{X})(X) = \mathcal{T}(!X)$. Condition (1) in the definition of VNUTS holds by functoriality of $!_-$ on \mathbf{Nuts} . Condition (2) holds by the definition of $\widehat{\mathbb{F}}_{|X|,|Y|}$ as described in Section 1.1.2 which coincides with $\mu^2(\text{dig}_X \otimes !Y) \in \mathbf{Nuts}(!X \otimes !Y, !(X \otimes Y))$.

Let us now turn to Condition (5) which is a bit more challenging.

Let first $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ be a unary VNUTS. Let $E = \sigma|\mathbb{X}|$ which is the least set such that $|\mathbb{X}|(E) = E$, that is $E = \bigcup_{n=0}^{\infty} |\mathbb{X}|^n(\emptyset)$. Let $\Phi : \mathbf{Tot}(E) \rightarrow \mathbf{Tot}(E)$ be defined as follows: given $\mathcal{T} \in \mathbf{Tot}(E)$, then (E, \mathcal{T}) is a NUTS, and we set $\Phi(\mathcal{T}) = \mathcal{T}(\mathbb{X})(E, \mathcal{T}) \in \mathbf{Tot}(|\mathbb{X}|(E)) = \mathbf{Tot}(E)$. This function Φ is monotone. Let indeed $\mathcal{S}, \mathcal{T} \in \mathbf{Tot}(E)$ with $\mathcal{S} \subseteq \mathcal{T}$. Then we have $\text{ld} \in \mathbf{Nuts}((E, \mathcal{S}), (E, \mathcal{T}))$ and therefore, by Condition (1) satisfied by \mathbb{X} , we have $\text{ld} = |\mathbb{X}|(\text{ld}) \in \mathbf{Nuts}(|\mathbb{X}|(E, \mathcal{S}), |\mathbb{X}|(E, \mathcal{T})) = \mathbf{Nuts}((E, \Phi(\mathcal{S})), (E, \Phi(\mathcal{T})))$ which means that $\Phi(\mathcal{S}) \subseteq \Phi(\mathcal{T})$. By the Knaster-Tarski Theorem [Tar55, Kna28] (remember that $\mathbf{Tot}(E)$ is a complete lattice), Φ has a greatest fixpoint \mathcal{T} that we can describe as follows. Let $(\mathcal{T}_\alpha)_{\alpha \in \mathbb{O}}$, where \mathbb{O} is the class of ordinals, be defined by: $\mathcal{T}_0 = \mathcal{P}(E)$ (the largest possible notion of totality on E), $\mathcal{T}_{\alpha+1} = \Phi(\mathcal{T}_\alpha)$ and $\mathcal{T}_\lambda = \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha$ when λ is a limit ordinal. This sequence is decreasing (easy induction on ordinals using the monotonicity of Φ) and there is an ordinal θ such that $\mathcal{T}_{\theta+1} = \mathcal{T}_\theta$ (by a cardinality argument; we can assume that θ is the least such ordinal). The greatest fixpoint of Φ is then \mathcal{T}_θ as easily checked.

By construction $((E, \mathcal{T}_\theta), \text{ld})$ is an object of $\mathbf{Coalg}_{\mathbf{Nuts}}(|\mathbb{X}|)$, we prove that it is the final object. So let (Y, t) be another object of the same category. Since $(|Y|, t)$ is an object of $\mathbf{Coalg}_{\mathbf{Rel}}(|\mathbb{X}|)$ and since (E, ld) is the final object in that category, we know by Lemma 26

that there is exactly one $e \in \mathbf{Rel}(|Y|, E)$ such that
$$\begin{array}{ccc} (|Y|) & \xrightarrow{e} & E \\ & \downarrow t & \nearrow \text{ld} \\ & & |\mathbb{X}|(e) \\ \widehat{|\mathbb{X}|}(|Y|) & & \end{array} .$$
 We prove that

actually $e \in \mathbf{Nuts}(Y, (E, \mathcal{T}_\theta))$ so let $v \in \mathcal{T}(Y)$. We prove by induction on the ordinal α that $e \cdot v \in \mathcal{T}_\alpha$. For $\alpha = 0$ it is obvious since $\mathcal{T}_0 = \mathcal{P}(E)$. Assume that the property holds for α and let us prove it for $\alpha + 1$. We have $t \cdot v \in \mathcal{T}(\mathbb{X})(Y) = \mathcal{T}(|\mathbb{X}|(Y))$ since $t \in \mathbf{Nuts}(Y, |\mathbb{X}|(Y))$. Since $|\mathbb{X}|(e) \in \mathbf{Nuts}(|\mathbb{X}|(Y), |\mathbb{X}|(E, \mathcal{T}_\alpha))$ and since $|\mathbb{X}|(E, \mathcal{T}_\alpha) = (E, \mathcal{T}_{\alpha+1})$ we have $(|\mathbb{X}|(e) t) \cdot v \in \mathcal{T}_{\alpha+1}$, that is $e \cdot v \in \mathcal{T}_{\alpha+1}$. Last if λ is a limit ordinal and if we assume $\forall \alpha < \lambda e \cdot v \in \mathcal{T}_\alpha$ we have $e \cdot v \in \bigcap_{\alpha < \lambda} \mathcal{T}_\alpha = \mathcal{T}_\lambda$. Therefore $e \cdot v \in \mathcal{T}_\theta$. We use $\nu|\mathbb{X}|$ to denote this final coalgebra (E, \mathcal{T}_θ) (its definition depends only on $|\mathbb{X}|$ and does not involve the strength $\widehat{|\mathbb{X}|}$).

So we have proven the first part of Condition (5) in the definition of a Seely model of μLL (see Section 17). As to the second part, let \mathbb{X} be an $n + 1$ -ary VNUTS. We know by the general Lemma 15 that there is a uniquely defined strong functor $\nu\mathbb{X} : \mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ such that

- $\overline{\nu\mathbb{X}}(\vec{X}) = \nu(\overline{\mathbb{X}}_{\vec{X}})$, so that $\overline{\mathbb{X}}(\vec{X}, \overline{\nu\mathbb{X}}(\vec{X})) = \overline{\nu\mathbb{X}}(\vec{X})$, for all $\vec{X} \in \mathbf{Obj}(\mathbf{Nuts}^n)$,
- $\overline{\mathbb{X}}(\vec{t}, \overline{\nu\mathbb{X}}(\vec{t})) = \overline{\nu\mathbb{X}}(\vec{t})$ for all $\vec{t} \in \mathbf{Nuts}(\vec{X}, \vec{Y})$
- and $\overline{\mathbb{X}}(Y \otimes \vec{X}, \widehat{\nu\mathbb{X}}_{Y, \vec{X}}) \widehat{\mathbb{X}}_{Y, (\vec{X}, \overline{\nu\mathbb{X}}(\vec{X}))} = \widehat{\nu\mathbb{X}}_{Y, \vec{X}}$ for all $Y \in \mathbf{Obj}(\mathbf{Nuts})$ and $\vec{X} \in \mathbf{Obj}(\mathbf{Nuts}^n)$.

To end the proof, it will be enough to exhibit an n -ary VNUTS $\mathbb{Y} = (|\mathbb{Y}|, \mathcal{T}(\mathbb{Y}))$ whose associated strong functor coincides with $\nu\mathbb{X}$. We know that $|\mathbb{X}|$ is a variable set $\mathbf{Rel}^{n+1} \rightarrow \mathbf{Rel}$ so let

$\mathbb{F} = \nu|\mathbb{X}| = \sigma|\mathbb{X}|$ which is a variable set $\mathbf{Rel}^n \rightarrow \mathbf{Rel}$ (see Section 1.2.1). Let $\vec{X} \in \text{Obj}(\mathbf{Nuts}^n)$, we have $|\overline{\nu\mathbb{X}}(\vec{X})| = |\nu(\overline{\mathbb{X}}_{\vec{X}})| = \bigcup_{n=0}^{\infty} |\overline{\mathbb{X}}_{\vec{X}}|^n(\emptyset) = \overline{\mathbb{F}}(|\vec{X}|)$. Let $\vec{t} \in \mathbf{Nuts}^n(\vec{X}, \vec{Y})$, then $\overline{\nu\mathbb{X}}(\vec{t})$ is the unique element s of $\mathbf{Nuts}(\overline{\nu\mathbb{X}}(\vec{X}), \overline{\nu\mathbb{X}}(\vec{Y})) \subseteq \mathbf{Rel}(\mathbb{F}(|\vec{X}|), \mathbb{F}(|\vec{Y}|))$ which satisfies $\overline{\mathbb{X}}(\vec{t}, s) = s$, that is $|\overline{\mathbb{X}}|(\vec{t}, s) = s$, which means that $\overline{\nu\mathbb{X}}(\vec{t}) = s = \overline{\mathbb{F}}(\vec{t})$. By a completely similar uniqueness argument we have $\widehat{\nu\mathbb{X}}_{X, \vec{Y}} = \widehat{\mathbb{F}}_{|X|, |\vec{Y}|}$ for all $X \in \text{Obj}(\mathbf{Nuts})$ and $\vec{Y} \in \text{Obj}(\mathbf{Nuts}^n)$. So we set $|\mathbb{Y}| = \mathbb{F}$.

Next, given $\vec{X} \in \text{Obj}(\mathbf{Nuts}^n)$ we set $\mathcal{T}(\mathbb{Y})(\vec{X}) = \mathcal{T}(\overline{\nu\mathbb{X}}(\vec{X})) \in \text{Tot}(|\overline{\nu\mathbb{X}}(\vec{X})|) = \text{Tot}(\overline{\mathbb{F}}(|\vec{X}|))$. Given $\vec{t} \in \mathbf{Nuts}(\vec{X}, \vec{Y})$ we have

$$\overline{\mathbb{F}}(\vec{t}) = \overline{\nu\mathbb{X}}(\vec{t}) \in \mathbf{Nuts}((\overline{\mathbb{F}}(|\vec{X}|), \mathcal{T}(\mathbb{Y})(\vec{X})), (\overline{\mathbb{F}}(|\vec{Y}|), \mathcal{T}(\mathbb{Y})(\vec{Y})))$$

since $(\overline{\mathbb{F}}(|\vec{X}|), \mathcal{T}(\mathbb{Y})(\vec{X})) = \overline{\nu\mathbb{X}}(\vec{X})$ and similarly for \vec{Y} . Last, since $\widehat{\mathbb{F}}_{|X|, |\vec{Y}|} = \widehat{\nu\mathbb{X}}_{X, \vec{Y}} \in \mathbf{Nuts}(!X \otimes \overline{\nu\mathbb{X}}(\vec{Y}), \overline{\nu\mathbb{X}}(X \otimes \vec{Y}))$ we know that $\mathbb{Y} = (|\mathbb{Y}|, \mathcal{T}(\mathbb{Y}))$ is a VNUTS whose associated strong functor is $\nu\mathbb{X}$. This ends the proof that $(\mathbf{Nuts}, (\mathbf{Vnuts}_n)_{n \in \mathbb{N}})$ is a Seely model of μLL . \blacksquare

Remark 15 For any closed formula A , the web of its interpretation $\llbracket A \rrbracket^{\mathbf{Nuts}}$ in \mathbf{Nuts} coincides with its interpretation $\llbracket A \rrbracket^{\mathbf{Rel}}$ in \mathbf{Rel} . It is also easy to check that for any proof π of $\vdash A$, one has $\llbracket \pi \rrbracket^{\mathbf{Nuts}} = \llbracket \pi \rrbracket^{\mathbf{Rel}}$ (this can be formalized by a structure preserving functor $\mathbf{Nuts} \rightarrow \mathbf{Rel}$ which acts trivially on morphisms).

We end this section by providing interpretation of some of data-types as examples.

Integers: The type of “flat integers” is defined by $\iota = \mu\zeta(1 \oplus \zeta)$. In \mathbf{Rel} , $1 \oplus \zeta$ is interpreted as the unary variable set $\llbracket 1 \oplus \zeta \rrbracket_{\zeta}^{\mathbf{Rel}} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ which maps a set E to $1 \oplus E = \{(1, *)\} \cup (\{2\} \times E)$. Hence $\llbracket \iota \rrbracket^{\mathbf{Rel}}$ is the least set such that $\llbracket \iota \rrbracket = \{(1, *)\} \cup (\{2\} \times \llbracket \iota \rrbracket)$ that is, the set of all tuples $\bar{n} = (2, (2, (\dots (1, *) \dots)))$ where n is the number of occurrences of 2, that is $\llbracket \iota \rrbracket^{\mathbf{Rel}} = \mathbb{N}$ up to renaming¹¹. We have $|\llbracket \iota \rrbracket^{\mathbf{Nuts}}| = \llbracket \iota \rrbracket^{\mathbf{Rel}} = \mathbb{N}$ and we compute $\mathcal{T}(\llbracket \iota \rrbracket^{\mathbf{Nuts}})$ dually wrt. the proof of Theorem 45: it is the least fixed point of the operator $\Phi : \text{Tot}(\mathbb{N}) \rightarrow \text{Tot}(\mathbb{N})$ (remember that $\text{Tot}(\mathbb{N})$ is just the set of all \subseteq -upwards-closed subsets of $\mathcal{P}(\mathbb{N})$) such that, if $\mathcal{T} \in \text{Tot}(\mathbb{N})$ then $\Phi(\mathcal{T}) = \{u \subseteq \mathbb{N} \mid 0 \in u \text{ or } \{n \in \mathbb{N} \mid n+1 \in u\} \in \mathcal{T}\}$. Therefore $\text{Tot}(\llbracket \iota \rrbracket^{\mathbf{Nuts}}) = \{u \subseteq \mathbb{N} \mid u \neq \emptyset\}$. So if π is a proof of $\vdash \iota$, we know that $\llbracket \pi \rrbracket^{\mathbf{Rel}} = \llbracket \pi \rrbracket^{\mathbf{Nuts}} \in \mathcal{T}(\llbracket \iota \rrbracket^{\mathbf{Nuts}})$, and hence is a non-empty set. Using an additional notion of coherence (which can be fully compatible with \mathbf{Rel} as in the non-uniform coherence space model of [BE01, Bou11]) we can also prove that $\llbracket \pi \rrbracket^{\mathbf{Rel}}$ has at most one element, and hence is a singleton $\{n\}$ (See section 1.2.3). This is a denotational version of normalization expressing that indeed π “has a value” (and actually exactly one, which expresses a weak form of confluence). We will use crucially this observation in the proof of Theorem 106.

Binary trees with integer leaves: This type can be defined as $\tau = \mu\zeta(\iota \oplus (\zeta \otimes \zeta))$. Then an element of $\llbracket \tau \rrbracket^{\mathbf{Rel}} = |\llbracket \tau \rrbracket^{\mathbf{Nuts}}|$ is an element of the set described by the following syntax: $\alpha, \beta, \dots := \langle n \rangle \mid \langle \alpha, \beta \rangle$. A computation similar to the previous one shows that $\text{Tot}(\llbracket \tau \rrbracket^{\mathbf{Nuts}}) = \{u \subseteq \llbracket \tau \rrbracket^{\mathbf{Rel}} \mid u \neq \emptyset\}$.

¹¹One can take alternatively this set $\llbracket \iota \rrbracket^{\mathbf{Rel}}$ as a definition of integers in order to not deal with the problem of equality

An empty type of streams of integers: After reading [BDS16], one could be tempted to define the type of streams of integers as $\sigma = \nu\zeta(\iota \otimes \zeta)$. The variable set $\llbracket \iota \otimes \zeta \rrbracket_{\zeta}^{\mathbf{Rel}} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ maps a set E to $\mathbb{N} \times E$. The least fixed point of this operation on sets is \emptyset and hence $\llbracket \sigma \rrbracket^{\mathbf{Nuts}} = \emptyset$ and notice that $\mathbf{Tot}(\emptyset) = \{\emptyset, \{\emptyset\}\}$. In that case, the operation $\Phi : \mathbf{Tot}(\emptyset) \rightarrow \mathbf{Tot}(\emptyset)$ maps \mathcal{T} to $\{u \times v \mid v \in \mathcal{T} \text{ and } u \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}\}$ and hence $\{\emptyset\}$ to itself. It follows that $\mathcal{T}(\llbracket \sigma \rrbracket^{\mathbf{Nuts}}) = \{\emptyset\}$ that is $\llbracket \sigma \rrbracket^{\mathbf{Nuts}} = \top$, the final object of \mathbf{Nuts} . What is the meaning of this trivial interpretation? It simply reflects that, though σ has a lot of non trivial proofs in $\mu\mathbf{LL}$, it is impossible to extract any finite information from these proofs within $\mu\mathbf{LL}$, and accordingly all these proofs are interpreted as \emptyset .

Theorem 46 *In $\mu\mathbf{LL}$, there is no proof of $\vdash (\sigma)^\perp, \iota$.*

In other words there is no proof of $\vdash \sigma \multimap \iota$ in $\mu\mathbf{LL}$; typically a function extracting the first element of a stream would be a proof of this type... if it would exist! Here is the proof: if π were a proof of $\vdash (\sigma)^\perp, \iota$, we would have $\llbracket \pi \rrbracket \in \mathbf{Nuts}(\llbracket \sigma \rrbracket^{\mathbf{Nuts}}, \llbracket \iota \rrbracket^{\mathbf{Nuts}})$ and hence $\llbracket \pi \rrbracket \cdot \emptyset \in \mathcal{T}(\llbracket \iota \rrbracket^{\mathbf{Nuts}})$ which is not the case since $\llbracket \pi \rrbracket \cdot \emptyset = \emptyset$. If such infinite types are meaningful in a proof-search perspective, their significance as data-types in a Curry-Howard approach to $\mu\mathbf{LL}$ is dubious.

A non-empty type of streams of integers: We set now $\sigma = \nu\zeta(1 \& (\iota \otimes \zeta))$. This type looks like the previous one, but the type 1 leaves space for *partial* empty streams. Warning: it is *not* a type of finite or infinite streams; the $\&$ means that this empty stream will not be a total element: it will have to be complemented by some total element from the right argument of the $\&$. More precisely $\llbracket 1 \& (\mathbb{N} \otimes \zeta) \rrbracket_{\zeta}^{\mathbf{Rel}} : \mathbf{Rel} \rightarrow \mathbf{Rel}$ is the variable set which maps a set E to $\{(1, *)\} \cup \{2\} \times \mathbb{N} \times E$ so that up to renaming $\llbracket \sigma \rrbracket^{\mathbf{Nuts}} = \mathbb{N}^{<\omega}$ (all *finite sequences* of integers). In this case, the operator $\Phi : \mathbf{Tot}(\mathbb{N}^{<\omega}) \rightarrow \mathbf{Tot}(\mathbb{N}^{<\omega})$ maps \mathcal{T} to

$$\{v \subseteq \mathbb{N}^{<\omega} \mid () \in v \text{ and } \exists n \in \mathbb{N}, u \in \mathcal{T} \{n\} \times u \subseteq v\}$$

where we use $()$ for the empty sequence. So for instance

$$\begin{aligned} \Phi^0(\mathcal{P}(\mathbb{N}^{<\omega})) &= \mathcal{P}(\mathbb{N}^{<\omega}). \\ \Phi^1(\mathcal{P}(\mathbb{N}^{<\omega})) &= \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid () \in u\}. \\ \Phi^3(\mathcal{P}(\mathbb{N}^{<\omega})) &= \{u \in \mathcal{P}(\mathbb{N}^{<\omega}) \mid \exists n_1, n_2 \text{ } (), (n_1), (n_1, n_2) \in u\}. \end{aligned}$$

The greatest fixed point is reached in ω steps:

$$\mathbf{Tot}(\llbracket \sigma \rrbracket^{\mathbf{Nuts}}) = \bigcap_{n < \omega} \Phi^n(\mathcal{P}(\mathbb{N}^{<\omega})) = \{u \subseteq \mathbb{N}^{<\omega} \mid \exists f \in \mathbb{N}^\omega \forall k < \omega (f(1), \dots, f(k)) \in u\}.$$

So a total subset of $\llbracket \sigma \rrbracket^{\mathbf{Nuts}}$ must contain (at least) an infinite stream of integer. For this type of streams σ it is easy to build a proof of $\vdash (\sigma)^\perp, \iota$ extracting the first element of a stream, interpreted as $\{((n), n) \mid n \in \mathbb{N}\}$.

1.2.3 Coherence spaces with totality

We showed that \mathbf{Rel} is model of $\mu\mathbf{LL}$, and on top of \mathbf{Rel} , we provided another model of $\mu\mathbf{LL}$, called \mathbf{Nuts} . We believe that one can apply the same idea to almost all models of linear logic,

that is to say, take a model of linear logic and try to build another model on top of it via an appropriate notion of totality. In this section, we examine this idea on coherence spaces.

We first show that the category of coherence spaces (**Coh**) is indeed a model of μLL . The methods developed in this section are by no way specific to coherence spaces and could be used in many other models of **LL** (relational semantics, Scott semantics, hypercoherence spaces, probabilistic coherence spaces, game models, up to some adaptations since these are not models of classical **LL**, etc).

Coherence spaces as a model of μLL

Let E and F be coherence spaces, we write $E \subseteq F$ if $|E| \subseteq |F|$ and $\forall a, a' \in |E| \quad a \circ_E a' \Leftrightarrow a \circ_F a'$.

Observe that when $E \subseteq F$, one has two linear morphisms $\eta_{E,F}^+ \in \mathbf{Coh}(E, F)$ and $\eta_{E,F}^- \in \mathbf{Coh}(F, E)$ given by $\eta_{E,F}^+ = \eta_{E,F}^- = \{(a, a) \mid a \in |E|\}$.

They satisfy $\eta_{E,F}^- \cdot \eta_{E,F}^+ = \text{Id}_E$ and $\eta_{E,F}^+ \cdot \eta_{E,F}^- \subseteq \text{Id}_F$, defining an embedding-retraction pair of coherence spaces as considered for instance in [Gir86] (all embedding-retraction pairs are of that shape, up to isomorphism of coherence spaces). One major feature of this order relation is that it makes linear negation monotonic, making life quite easy when one needs to compute fixed points of arbitrary **LL** formulas in this model.

Lemma 47 *The relation \subseteq is a partial order relation on coherence spaces, and we have $E \subseteq F \Leftrightarrow E^\perp \subseteq F^\perp$.*

We use **Coh** for the class of coherence spaces ordered under the \subseteq partial order relation. This partially ordered class has a least element denoted as \emptyset (the coherence space which has an empty web).

Any countable directed subset \mathcal{E} of **Coh** has a lub $\cup \mathcal{E}$, which is the coherence space defined by $|\cup \mathcal{E}| = \cup_{E \in \mathcal{E}} |E|$ and, for all $a, a' \in |\cup \mathcal{E}|$, one has $a \circ_{\cup \mathcal{E}} a'$ iff $a \circ_E a'$ for some $E \in \mathcal{E}$.

The family \mathcal{E} gives rise to two diagrams in the category **Coh**:

- the *inductive diagram* \mathcal{E}^+ with morphisms $\eta_{E,F}^+ \in \mathbf{Coh}(E, F)$ when $E, F \in \mathcal{E}$ with $E \subseteq F$
- and the *projective diagram* \mathcal{E}^- with morphisms $\eta_{E,F}^- \in \mathbf{Coh}(F, E)$ when $E, F \in \mathcal{E}$ with $E \subseteq F$.

Lemma 48 *Let $G = \cup \mathcal{E}$. Then, in **Coh**, the cocone $(E, \eta_{E,G}^+)_{E \in \mathcal{E}}$ is the colimit of the inductive diagram \mathcal{E}^+ and the cone $(E, \eta_{E,G}^-)_{E \in \mathcal{E}}$ is the limit of the projective diagram \mathcal{E}^- .*

Proof: We prove the first statement, the second one following by duality. Let F be a coherence space and, for each $E \in \mathcal{E}$ let $t_E \in \mathbf{Coh}(E, F)$ defining a cocone based on \mathcal{E}^+ , which means $\forall E, E' \in \mathcal{E} \quad E \subseteq E' \Rightarrow t_{E'} \eta_{E,E'}^+ = t_E$, that is $\forall E, E' \in \mathcal{E} \quad E \subseteq E' \Rightarrow t_E = t_{E'} \cap |E| \times |F|$. Then the unique morphism $t \in \mathbf{Coh}(G, F)$ such that $\forall E \in \mathcal{E} \quad t \eta_{E,G}^+ = t_E$ is given by $t = \cup_{E \in \mathcal{E}} t_E$ as easily checked. ■

Definition 49 A functor $\mathcal{F} : \mathbf{Coh}^n \rightarrow \mathbf{Coh}$ is continuous on objects if whenever $\vec{E}, \vec{F} \in \mathbf{Coh}^n$ satisfy $E_i \subseteq F_i$ for $i = 1, \dots, n$, one has $\mathcal{F}(\vec{E}) \subseteq \mathcal{F}(\vec{F})$ and $\mathcal{F}(\eta_{E_1, F_1}^+, \dots, \eta_{E_n, F_n}^+) = \eta_{\mathcal{F}(\vec{E}), \mathcal{F}(\vec{F})}^+$ and $\mathcal{F}(\eta_{E_1, F_1}^-, \dots, \eta_{E_n, F_n}^-) = \eta_{\mathcal{F}(\vec{E}), \mathcal{F}(\vec{F})}^-$. Moreover, \mathcal{F} commutes with the lubs of countable directed families of coherence spaces. In other words, for any countable directed families of coherence spaces $\mathcal{E}_1, \dots, \mathcal{E}_n$, one has $\mathcal{F}(\cup \mathcal{E}_1, \dots, \cup \mathcal{E}_n) = \cup \{\mathcal{F}(E_1, \dots, E_n) \mid E_1 \in \mathcal{E}_1, \dots, E_n \in \mathcal{E}_n\}$.

One says that \mathcal{F} is continuous on morphisms if when $\vec{f}, \vec{g} \in \mathbf{Coh}^n(\vec{E}, \vec{F})$ satisfy $\vec{f} \subseteq \vec{g}$ (that is $\forall i f_i \subseteq g_i$) one has $\mathcal{F}(\vec{f}) \subseteq \mathcal{F}(\vec{g})$ and, if D is a directed subset of $\mathbf{Coh}^n(\vec{E}, \vec{F})$, one has $\mathcal{F}(\cup D) = \cup_{\vec{f} \in D} \mathcal{F}(\vec{f})$ (equivalently $\mathcal{F}(\cup D) \supseteq \cup_{\vec{f} \in D} \mathcal{F}(\vec{f})$).

Last one says that \mathcal{F} is continuous if it is both continuous on objects and on morphisms.

Notice that this property is preserved by composition and duality (setting, consistently with Section 1.1.2, $((\mathcal{F})^\perp)(\vec{E}) = (\mathcal{F}((\vec{E})^\perp))^\perp$ and similarly for morphisms).

Remark 16 One can alternatively deal with Definition 49 in a similar way as what we saw for **Rel** where everything was reduced to local continuity. If we do so, then essentially using similar facts as Lemmas 22, 24 and Proposition 23, the second part of Definition 49 implies its first part. However, we preferred not to do so to simplify the presentation for the case of coherence spaces.

Definition 50 A (n -ary) variable coherence space (VCS) is a strong functor $\mathbb{E} : \mathbf{Coh}^n \rightarrow \mathbf{Coh}$ such that $\overline{\mathbb{E}}$ is continuous.

Proposition 51 The operations $\otimes, \wp, \oplus, \&, !$ and $?$ are VCSs and VCSs are closed under De Morgan duality and composition.

Proof: This results immediately from the properties of strong functors stated in Section 1.1.2 and from straightforward computations (for the continuity statement). ■

Least fixed point of a VCS, universal properties wrt. algebras and coalgebras

Let $\mathcal{F} : \mathbf{Coh} \rightarrow \mathbf{Coh}$ be continuous on objects (in the sense of Definition 49). Then we have $\emptyset \subseteq \mathcal{F}(\emptyset) \subseteq \dots \subseteq \mathcal{F}^n(\emptyset) \subseteq \mathcal{F}^{n+1}(\emptyset) \subseteq \dots$ as shown by an easy induction on n . We set $\sigma \mathcal{F} = \cup_{n=0}^\infty \mathcal{F}^n(\emptyset)$. By Scott continuity of \mathcal{F} , we have $\mathcal{F}(\sigma \mathcal{F}) = \sigma \mathcal{F}$.

Lemma 52 One has $\sigma((\mathcal{F})^\perp) = (\sigma \mathcal{F})^\perp$. If moreover \mathcal{F} is continuous on morphisms then $\sigma \mathcal{F}$ is at the same time the initial object of $\mathbf{Alg}_{\mathbf{Coh}}(\mathcal{F})$ and the final object of $\mathbf{Coalg}_{\mathbf{Coh}}(\mathcal{F})$.

Proof: The first statement results from the observation that $((\mathcal{F})^\perp)^n = (\mathcal{F}^n)^\perp$. For the second statement, since $\sigma((\mathcal{F})^\perp) = (\sigma \mathcal{F})^\perp$, it suffices to prove that $(\sigma \mathcal{F}, \text{Id})$ is initial in $\mathbf{Alg}_{\mathbf{Coh}}(\mathcal{F})$. This results easily from Lemma 48 and from continuity on morphisms. ■

Let \mathbb{E} be an $n + 1$ -ary VCS. Applying Lemma 52 to the functors $\overline{\mathbb{E}}_{\vec{E}}$ for all $\vec{E} \in \mathbf{Coh}^n$, Lemma 16 shows that there is an n -ary strong functor $\mu \mathbb{E}$ uniquely determined by the following equations

- $\overline{\mu\mathbb{E}}(\vec{E}) = \sigma \overline{\mathbb{E}}_{\vec{E}}$
- $\mathbb{E}(\vec{f}, \overline{\mu\mathbb{E}}(\vec{f})) = \overline{\mu\mathbb{E}}(\vec{f})$ for $\vec{f} \in \mathbf{Coh}(\vec{E}, \vec{E}')$
- and $\mathbb{E}(!F \otimes \vec{E}, \widehat{\mu\mathbb{E}}_{F, \vec{E}}) \widehat{\mathbb{E}}_{F, (\vec{E}, \overline{\mu\mathbb{E}}(\vec{E}))} = \widehat{\mu\mathbb{E}}_{F, \vec{E}}$.

Proposition 53 *The functor $\overline{\mu\mathbb{E}}$ is a variable coherence space. Defining the dual operation as $\nu\mathbb{E} = (\mu((\mathbb{E})^\perp))^\perp$, one has $\nu\mathbb{E} = \overline{\mu\mathbb{E}}$. We use $\sigma \mathbb{E}$ for this unique (final and initial) fixed point VCS.*

Proof: The proof that $\overline{\mu\mathbb{E}}$ is monotonic and Scott continuous on $\mathbf{Coh}_{\subseteq}^n$ is a standard domain-theoretic verification. We are left with proving that given \vec{E}, \vec{F} in \mathbf{Coh}^n such that $\vec{E} \subseteq \vec{F}$, one has

$$\overline{\mu\mathbb{E}}(\eta_{\vec{E}, \vec{F}}^+) = \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^+ \quad \overline{\mu\mathbb{E}}(\eta_{\vec{E}, \vec{F}}^-) = \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^-.$$

Let us prove the first equation, the proof of the second one being completely similar. By Lemma 13, it suffices to prove $\mathbb{E}(\eta_{\vec{E}, \vec{F}}^+, \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^+) = \eta_{\overline{\mu\mathbb{E}}(\vec{E}), \overline{\mu\mathbb{E}}(\vec{F})}^+$ which in turn results from the assumption that \mathbb{E} is a VCS.

The identity $\mu\mathbb{E} = \nu\mathbb{E}$ results from the uniqueness statements of Lemmas 15 and 16 and from the fact that $\overline{\nu\mathbb{E}}(\vec{E}) = \sigma \overline{\mathbb{E}}_{\vec{E}} = \overline{\mu\mathbb{E}}(\vec{E})$. \blacksquare

So, we have seen that \mathbf{Coh} is a model of μLL , and just as in \mathbf{Rel} , the least and greatest fixpoints are interpreted similarly. Now, we build a model on top of \mathbf{Coh} with a notion of totality.

Let \mathbf{Vcs}_n be the class of all n -ary VCS, so that $\mathbf{Vcs}_0 = \text{Obj}(\mathbf{Coh})$.

Theorem 54 $(\mathbf{Coh}, (\mathbf{Vcs}_n)_{n \in \mathbb{N}})$ is a Seely model of μLL

Proof: By a direct application of Lemma 52, Propositions 53 and 51. \blacksquare

Coherence spaces with totality: Let E be a coherence space and let $\mathcal{A} \subseteq \text{Cl}(E)$. We set

$$(\mathcal{A})^\perp = \{x' \in \text{Cl}(E^\perp) \mid \forall x \in \mathcal{A} \ x \cap x' \neq \emptyset\}.$$

Observe that if $x \cap x' \neq \emptyset$ then this intersection has exactly one element, due to the fact that x and x' are cliques in E and E^\perp respectively.

If $\mathcal{A}, \mathcal{B} \subseteq \text{Cl}(E)$ and $\mathcal{A} \subseteq \mathcal{B}$, we have $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$, and also $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$. Therefore $\mathcal{A}^\perp = \mathcal{A}^{\perp\perp\perp}$.

A *totality candidate* on E is a set $\mathcal{T} \subseteq \text{Cl}(E)$ such that $\mathcal{T}^{\perp\perp} = \mathcal{T}$, or equivalently $\mathcal{T}^{\perp\perp} \subseteq \mathcal{T}$. This property is equivalent to the existence of a “predual” of \mathcal{T} , that is, of a set $\mathcal{A} \subseteq \text{Cl}(E^\perp)$ such that $\mathcal{T} = \mathcal{A}^\perp$. We use $\text{Tot}(E)$ for the set of all totality candidates of the coherence space E , and we consider this set as a poset, equipped with inclusion.

Remark 17 *We do not have such a simple characterization of totality candidates here as in \mathbf{Nuts} where they are upward closed sets (Lemma 30), this is also a good reason to move from Girard’s coherence spaces to non-uniform totality spaces where one can develop the theory of totality independently from the theory of coherence.*

Lemma 55 *The poset $\text{Tot}(E)$ is a complete lattice.*

Proof: Let $\Theta \subseteq \text{Tot}(E)$ and let $\Theta' = \{(\mathcal{T})^\perp \mid \mathcal{T} \in \Theta\}$, which is a subset of $\text{Tot}(E^\perp)$. Given $x \in \text{Cl}(E)$, we have $x \in \cap\Theta$ iff for all $\mathcal{T}' \in \Theta'$ and all $x' \in \mathcal{T}'$, $x \cap x' \neq \emptyset$. In other words $\cap\Theta = (\cup_{\mathcal{T} \in \Theta} (\mathcal{T})^\perp)^\perp \in \text{Tot}(E)$. ■

The greatest element of $\text{Tot}(E)$ is $\text{Cl}(E)$ and its least element is $(\text{Cl}(E^\perp))^\perp = \emptyset$ as easily checked. Any subset Θ of $\text{Tot}(E)$ has a least upper bound $\vee\Theta$ which is given by $\vee\Theta = (\cup\Theta)^{\perp\perp}$ and this biorthogonal closure cannot be disposed of in general ($\cup\Theta$ is not necessarily a totality candidate). It is useful to observe that the map $\mathcal{T} \mapsto (\mathcal{T})^\perp$ is an isomorphism between the complete lattices $\text{Tot}(E^\perp)$ and $\text{Tot}(E)^{\text{op}}$.

Definition 56 *A coherence space with totality is a pair $X = (\underline{X}, \top X)$ where \underline{X} is a coherence space (the carrier) and $\top X \in \text{Tot}(\underline{X})$.*

Coherence spaces with totality as a model of LL

Let X and Y be coherence spaces with totality, we define a coherence space with totality $X \multimap Y$ by $\underline{X \multimap Y} = \underline{X} \multimap \underline{Y}$ and $\top(X \multimap Y) = \{t \in \text{Cl}(\underline{X} \multimap \underline{Y}) \mid \forall x \in \top X \ t \cdot x \in \top Y\} = (\{x \otimes y' \mid x \in \top X \text{ and } y' \in \top Y^\perp\})^\perp$, this latter equation resulting from the equivalence $(t \cdot x) \cap y' \neq \emptyset \Leftrightarrow t \cap (x \otimes y') \neq \emptyset$. It is clear that if $s \in \top(X \multimap Y)$ and $t \in \top(Y \multimap Z)$ then $t \cdot s \in \top(X \multimap Z)$, and also that $\text{Id} \in \top(X \multimap X)$, hence we have defined a category that we denote as **CohT**.

We equip 1 and \perp with the same totality, namely $\{\{*\}\}$. We define $X_1 \otimes X_2$ by $\underline{X_1 \otimes X_2} = \underline{X_1} \otimes \underline{X_2}$ and $\top(X_1 \otimes X_2) = \{x_1 \otimes x_2 \mid x_i \in \top X_i \text{ for } i = 1, 2\}^{\perp\perp}$, so that $X \otimes Y = (X \multimap Y^\perp)^\perp$. Then it is easy to check that **CohT** is $*$ -autonomous, *with the same operations on morphisms as in Coh* (for instance one checks that if $t_i \in \text{CohT}(X_i, Y_i)$ then $t_1 \otimes t_2 \in \text{CohT}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ which is easy using the following lemma.

Lemma 57 *Let $t \in \text{Coh}(\underline{X}, \underline{Y})$ and let $\mathcal{A} \subseteq \text{Cl}(\underline{X})$ be such that $\top X = \mathcal{A}^{\perp\perp}$. If $\forall x \in \mathcal{A} \ t \cdot x \in \top Y$ then $t \in \text{CohT}(X, Y)$.*

Proof: Let $x \in \top X$, we have to prove that $t \cdot x \in \top Y$ so let $y' \in \top Y^\perp$, we must prove that $(t \cdot x) \cap y' \neq \emptyset$. This statement is equivalent to $t \cap (x \otimes y') \neq \emptyset \Leftrightarrow (t^\perp \cdot y') \cap x \neq \emptyset$. So we must prove $\forall x \in \top X \ \forall y' \in \top Y^\perp \ (t^\perp \cdot y') \cap x \neq \emptyset$, that is $t^\perp \in \text{CohT}(Y^\perp, X^\perp)$. So let $y' \in \top Y^\perp$, we must prove that $t^\perp \cdot y' \in \top X^\perp = (\mathcal{A})^\perp$ which results from our assumption by the same reasoning. ■

Lemma 57 is a useful tool for proving that a linear morphism (a morphism in **Coh** between the carriers of two coherence spaces with totality) is total.

Using Lemma 57, one proves easily that $\text{ev} \in \text{CohT}((X \multimap Y) \otimes X, Y)$ etc). Similarly one shows that the cartesian structure on **Coh** gives rise to a cartesian structure on **CohT**: $\underline{X_1} \& \underline{X_2} = \underline{X_1} \& \underline{X_2}$ and $\{1\} \times x_1 \cup \{2\} \times x_2 \in \top(X_1 \& X_2)$ if $x_i \in \top X_i$ for $i = 1, 2$. The total cliques of $X_1 \oplus X_2$ are the $\{i\} \times z$ for $i = 1, 2$ and $z \in \top X_i$. Notice that \top and 0 are different coherence spaces with totality: $\top\top = \{\emptyset\}$ and $\top 0 = \emptyset$.

Last $!X$ is given by $\underline{!X} = \underline{!X}$ and $\top(!X) = \{x^! \mid x \in \top X\}^{\perp\perp}$ (where $x^! = \mathcal{P}_{\text{fin}}(x)$). Then one proves easily that $t \in \text{CohT}(X, Y) \Rightarrow !t \in \text{CohT}(!X, !Y)$ again using Lemma 57.

It is also easy to check that $\text{der}_{\underline{X}} \in \mathbf{CohT}(!X, X)$ and that $\text{dig}_{\underline{X}} \in \mathbf{CohT}(!X, !!X)$ so we denote these morphisms as der_X and dig_X turning “!” into a comonad on \mathbf{CohT} . The same holds for the monoidal structure (Seely isomorphisms): $\mathbf{m}^0 \in \mathbf{CohT}(1, !\top)$ and $\mathbf{m}_{X_1, X_2}^2 \in \mathbf{CohT}(!X_1 \otimes !X_2, !(X_1 \& X_2))$.

Coherence spaces with totality as a model of μ LL

We first need to define a notion of variable coherence space with totality.

To make the notations more readable, when \mathbb{E} is a VCS (see Definition 50), we use $\hat{\mathbb{E}}$ (instead of \mathbb{E}) to denote its functorial part. We keep denoting as $\hat{\mathbb{E}}$ the associated strength natural transformation.

Definition 58 *An n -ary variable coherence space with totality (VCST) is a pair $\mathbb{X} = (\underline{\mathbb{X}}, \top\mathbb{X})$ where*

- $\underline{\mathbb{X}} : \mathbf{Coh}^n \rightarrow \mathbf{Coh}$ is an n -ary VCS called the carrier of \mathbb{X}
- and $\top\mathbb{X}$ is an operation, called the totality of \mathbb{X} , which, with each n -tuple \vec{X} of coherence spaces with totality, associates $\top\mathbb{X}(\vec{X}) \in \mathbf{Tot}(\underline{\mathbb{X}}(\vec{X}))$ — and we use the notation $\mathbb{X}(\vec{X})$ for the coherence space with totality $(\underline{\mathbb{X}}(\vec{X}), \top\mathbb{X}(\vec{X}))$.

Moreover the two following properties must hold.

- If \vec{X} and \vec{Y} are objects of \mathbf{CohT}^n and $\vec{f} \in \mathbf{CohT}^n(\vec{X}, \vec{Y})$, then the \mathbf{Coh} morphism $\underline{\mathbb{X}}(\vec{f})$ belongs actually to $\mathbf{CohT}(\mathbb{X}(\vec{X}), \mathbb{X}(\vec{Y}))$, so that \mathbb{X} defines a functor $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$ (denoted simply as \mathbb{X}).
- If \vec{X} is an object of \mathbf{CohT}^n and Y is an object of \mathbf{CohT} then the \mathbf{Coh} morphism $\hat{\underline{\mathbb{X}}}_{Y, \vec{X}}$ belongs actually to $\mathbf{CohT}(!Y \otimes \mathbb{X}(\vec{X}), \mathbb{X}(!Y \otimes \vec{X}))$. We denote this total morphism as $\hat{\mathbb{X}}_{Y, \vec{X}}$.

So we can consider \mathbb{X} as a strong functor $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$ (the monoidality diagram commutations of Figure 1.1 hold because the LL operations on morphisms are interpreted in the same way in \mathbf{CohT} and in \mathbf{Coh}).

The following remark is essentially the same as Remark 14.

Remark 18 *It is essential to keep in mind that, if $\vec{f} \in \mathbf{CohT}(\vec{X}, \vec{Y})$ then actually $\vec{f} \in \mathbf{Coh}(\vec{X}, \vec{Y})$ so that the morphism $\underline{\mathbb{X}}(\vec{f}) \in \mathbf{Coh}(\underline{\mathbb{X}}(\vec{X}), \underline{\mathbb{X}}(\vec{Y}))$ is defined, independently of the notions of totality on \vec{X} and \vec{Y} and similarly for $\hat{\underline{\mathbb{X}}}_{Y, \vec{X}}$.*

The following remark is similar to Lemma 44, we do not provide the proof because it is similar to the proof of that lemma (Remember that \mathbb{X}^\perp is defined in Section 1.1.2).

Remark 19 *Strictly speaking, an n -ary VCST \mathbb{X} is not a strong functor $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$ but a structure which induces — as explained above — such a strong functor \mathcal{F} , that we have denoted simply as \mathbb{X} . This choice of notation is motivated by the fact that \mathbb{X} can very simply*

be recovered from \mathcal{F} . We have indeed a forgetful functor $\mathcal{U} : \mathbf{CohT} \rightarrow \mathbf{Coh}$ which maps X to \underline{X} and acts as the identity on morphisms. This functor has a left adjoint $\mathcal{Z} : \mathbf{Coh} \rightarrow \mathbf{CohT}$ which maps a coherence space E to (E, \emptyset) (no cliques of E are total) and acts as the identity on morphisms. Then we have $\underline{\mathbb{X}} = \mathcal{U} \circ \mathcal{F} \circ \mathcal{Z} = \mathcal{U} \circ \mathcal{F} \circ (\mathcal{Z})^\perp$ (for the functorial part of $\underline{\mathbb{X}}$) and for the strength $\widehat{\underline{\mathbb{X}}}_{F, \vec{E}} = \widehat{\mathcal{F}}_{\mathcal{Z}(F), \mathcal{Z}(\vec{E})}$, and $\mathcal{T} = \mathbb{T}\mathbb{X}(\vec{X})$ is defined by the fact that the coherence space with totality $\mathcal{F}(\vec{X})$ is of shape (F, \mathcal{T}) . In these definitions, the choice of \mathcal{Z} as “inverse” of \mathcal{U} is arbitrary. By the definition of VCSTs we could have used the right adjoint $(\mathcal{Z})^\perp$ (it maps E to $(E, \mathcal{C}(E))$ where all cliques are total) or any other functor $\mathcal{Y} : \mathbf{Coh} \rightarrow \mathbf{CohT}$ such that $\mathcal{U} \circ \mathcal{Y} = \text{Id}$ instead: the resulting \mathbb{X} would have been the same. For these reasons, it is meaningful to consider VCSTs as strong functors $\mathbf{CohT}^n \rightarrow \mathbf{CohT}$, what we do now.

This observation also motivates our general notion of model presented in Definition 17.

As a consequence of Proposition 51, we have the following Proposition:

Proposition 59 *The operations $\otimes, \wp, \oplus, \&, !$ and $?$ are VCSTs and VCSTs are closed under De Morgan duality and composition.*

Fixed Points of VCST's

We deal first with least fixed points of unary VCST's, so let \mathbb{X} be a unary VCST (whose strength is not used in this first step). We define a coherence space with totality $\mu \mathbb{X}$. First, we set $\underline{\mu \mathbb{X}} = \sigma \underline{\mathbb{X}}$.

We define a map $\Theta(\mathbb{X}) : \text{Tot}(\underline{\mu \mathbb{X}}) \rightarrow \text{Tot}(\underline{\mu \mathbb{X}})$ as follows: if $\mathcal{T} \in \text{Tot}(\underline{\mu \mathbb{X}})$, then $\mathbb{T}\mathbb{X}(\underline{\mu \mathbb{X}}, \mathcal{T}) \in \text{Tot}(\underline{\mathbb{X}}(\underline{\mu \mathbb{X}})) = \text{Tot}(\underline{\mu \mathbb{X}})$ and we set $\Theta(\mathbb{X})(\mathcal{T}) = \mathbb{T}\mathbb{X}(\underline{\mu \mathbb{X}}, \mathcal{T})$. We contend that this mapping is monotonic on the lattice $\text{Tot}(\underline{\mu \mathbb{X}})$. Assume that $\mathcal{T}, \mathcal{T}' \in \text{Tot}(\underline{\mu \mathbb{X}})$ with $\mathcal{T} \subseteq \mathcal{T}'$. Then $\text{Id} \in \mathbf{CohT}((\underline{\mu \mathbb{X}}, \mathcal{T}), (\underline{\mu \mathbb{X}}, \mathcal{T}'))$ (see Section 1.2.3) and hence $\text{Id} = \underline{\mathbb{X}}(\text{Id}) \in \mathbf{CohT}((\underline{\mu \mathbb{X}}, \Theta(\mathbb{X})(\mathcal{T})), (\underline{\mu \mathbb{X}}, \Theta(\mathbb{X})(\mathcal{T}')))$ by Definition 58, from which it follows that $\Theta(\mathbb{X})(\mathcal{T}) \subseteq \Theta(\mathbb{X})(\mathcal{T}')$.

Let \mathcal{U} be the least fixed point of $\Theta(\mathbb{X})$ (applying Knaster-Tarski's Theorem), we set $\mathcal{T}(\underline{\mu \mathbb{X}}) = \mathcal{U}$ and this ends the definition of the coherence space with totality $\mu \mathbb{X}$, which satisfies $\mathbb{X}(\underline{\mu \mathbb{X}}) = \underline{\mu \mathbb{X}}$. Now we prove that it is initial in $\mathbf{Alg}_{\mathbf{CohT}}(\mathbb{X})$.

For this we shall use the following sequence of candidates of totality for $\underline{\mu \mathbb{X}}$, indexed by ordinals: $\mathcal{U}_{\alpha+1} = \Theta(\mathbb{X})(\mathcal{U}_\alpha)$ and $\mathcal{U}_\lambda = (\bigcup_{\alpha < \lambda} \mathcal{U}_\alpha)^{\perp\perp}$ when λ is a limit ordinal. By Knaster-Tarski's Theorem, we know that there is an ordinal θ such that $\mathcal{U}_{\theta+1} = \mathcal{U}_\theta$, and that we have $\mathcal{U} = \mathcal{U}_\theta$.

Proposition 60 *$\underline{\mu \mathbb{X}}$ is initial in the category $\mathbf{Alg}_{\mathbf{CohT}}(\mathbb{X})$.*

Proof: Let (X, g) be an object in $\mathbf{Alg}_{\mathbf{CohT}}(\mathbb{X})$, that is $g \in \mathbf{CohT}(\mathbb{X}(X), X)$. This means in particular that $g \in \mathbf{Coh}(\underline{\mathbb{X}}(X), \underline{X})$ so that, by Proposition 53, we know that there is exactly one morphism $\hat{g} \in \mathbf{Coh}(\underline{\mu \mathbb{X}}, \underline{X})$ such that

$$g \cdot \underline{\mathbb{X}}(\hat{g}) = \hat{g}.$$

We have to prove that $\hat{g} \in \mathbf{CohT}(\mu\mathbb{X}, X)$. By induction on the ordinal α , we prove that

$$\hat{g} \in \mathbf{CohT}((\underline{\mu}\mathbb{X}, \mathcal{U}_\alpha), X)$$

for all ordinal α . Assume first that the property holds for α and let us prove it for $\alpha + 1$. By Definition 58 we get $\underline{\mathbb{X}}(\hat{g}) \in \mathbf{CohT}((\underline{\mu}\mathbb{X}, \mathcal{U}_{\alpha+1}), \mathbb{X}(X))$ and hence $\hat{g} = g \cdot \underline{\mathbb{X}}(\hat{g}) \in \mathbf{CohT}((\underline{\mu}\mathbb{X}, \mathcal{U}_{\alpha+1}), X)$. Let now λ be a limit ordinal and assume that $\hat{g} \in \mathbf{CohT}((\underline{\mu}\mathbb{X}, \mathcal{U}_\alpha), X)$ for all $\alpha < \lambda$. It will be sufficient to prove that $(\hat{g})^\perp \in \mathbf{CohT}(X^\perp, (\bigcup_{\alpha < \lambda} \mathcal{U}_\alpha)^\perp)$ so let $x' \in (\mathbb{T}X)^\perp$, we must prove that $(\hat{g})^\perp \cdot x' \in (\bigcup_{\alpha < \lambda} \mathcal{U}_\alpha)^\perp$ so let $y \in \mathcal{U}_\alpha$ for some $\alpha < \lambda$, we must prove that $((\hat{g})^\perp \cdot x') \cap y \neq \emptyset$, that is $x' \cap g \cdot y \neq \emptyset$ which results from our inductive hypothesis applied to ordinal α .

So we have proven the existence of $\hat{g} \in \mathbf{CohT}(\mu\mathbb{X}, X)$ such that $g \cdot \underline{\mathbb{X}}(\hat{g}) = \hat{g}$. Uniqueness follows from the uniqueness property for $\underline{\mu}\mathbb{X}$. ■

We consider now the case of several variables, so let \mathbb{X} be an $n + 1$ -ary VCST. Given $\vec{X} \in \mathbf{CohT}^n$ consider the unary VCST $\mathbb{X}_{\vec{X}}$ defined as follows: $\underline{\mathbb{X}}_{\vec{X}} = \underline{\mathbb{X}}_{\vec{X}}$ and $\mathbb{T}(\mathbb{X}_{\vec{X}})(X) = \mathbb{T}\mathbb{X}(\vec{X}, X)$ (the strength can be defined in a similar way though this is not needed actually because the proof of Proposition 60 does not involve the strength). Then by Proposition 60 applied to $\mathbb{X}_{\vec{X}}$ and Lemma 16 we have an n -ary strong functor $\Phi = (\overline{\Phi}, \widehat{\Phi})$ on \mathbf{CohT} such that $\overline{\Phi}(\vec{X}) = \mu(\underline{\mathbb{X}}_{\vec{X}})$ and whose action on morphisms and strength are uniquely characterized by

- $\underline{\mathbb{X}}(\vec{f}, \overline{\Phi}(\vec{f})) = \overline{\Phi}(\vec{f})$ for all $\vec{f} \in \mathbf{CohT}^n(\vec{X}, \vec{X}')$
- and $\underline{\mathbb{X}}(!Y \otimes \vec{X}, \widehat{\Phi}_{Y, \vec{X}}) \widehat{\mathbb{X}}_{Y, (\vec{X}, \mu\overline{\Phi}(\vec{X}))} = \widehat{\Phi}_{Y, \vec{X}}$.

By Proposition 53, the first equation implies that $\overline{\Phi}(\vec{f}) = \sigma \underline{\mathbb{X}}(\vec{f})$ (remember that actually $\vec{f} \in \mathbf{Coh}^n(\vec{X}, \vec{X}')$ and that $\sigma \underline{\mathbb{X}}$ is an n -ary VCS characterized by that proposition) and the second equation shows that $\widehat{\Phi}_{Y, \vec{X}} = \sigma \widehat{\underline{\mathbb{X}}}_{Y, \vec{X}}$. This proves that $\sigma \underline{\mathbb{X}}(\vec{f}) \in \mathbf{CohT}(\mu(\underline{\mathbb{X}}_{\vec{X}}), \mu(\underline{\mathbb{X}}_{\vec{X}}'))$ and that $\sigma \widehat{\underline{\mathbb{X}}}_{Y, \vec{X}} \in \mathbf{CohT}(Y \otimes \mu(\underline{\mathbb{X}}_{\vec{X}}), \mu(\underline{\mathbb{X}}_{Y \otimes \vec{X}}))$. Therefore we have defined a VCST $\mu\mathbb{X}$ whose carrier $\underline{\mu}\mathbb{X}$ is the VCS $\sigma \underline{\mathbb{X}}$ and whose totality $\mathbb{T}(\mu\mathbb{X})$ is such that $(\sigma \underline{\mathbb{X}}(\vec{X}), \mathbb{T}(\mu\mathbb{X})(\vec{X})) = \mu(\underline{\mathbb{X}}_{\vec{X}})$ for all $\vec{X} \in \mathbf{CohT}^n$. We can summarize our constructions as follows.

Theorem 61 *Let \mathbb{X} be an $n + 1$ -ary VCST. There is a unique VCST $\mu\mathbb{X}$ whose carrier is $\sigma \underline{\mathbb{X}}$ and whose totality is such that $\underline{\mathbb{X}}(\vec{X}, \mu\mathbb{X}(\vec{X})) = \mu\mathbb{X}(\vec{X})$ and $(\mu\mathbb{X}(\vec{X}), \text{Id})$ is initial in the category $\mathbf{Alg}_{\mathbf{CohT}}(\mathbb{X}_{\vec{X}})$.*

Moreover, we have provided a “concrete” way for defining this operation (which involves an ordinal iteration).

Now we can define “greatest fixed points” by De Morgan duality. So let \mathbb{X} be an $n + 1$ -ary VCST. Given an n -tuple of coherence spaces with totality \vec{X} , we set $\nu \mathbb{X}(\vec{X}) = (\mu((\mathbb{X})^\perp)((\vec{X})^\perp))^\perp$. More precisely, this means that the carrier of $\nu \mathbb{X}$ is the VCS $\sigma \underline{\mathbb{X}}$ (the *very same* as for $\mu \mathbb{X}$), and that $\mathbb{T}(\nu \mathbb{X})(\vec{X}) \in \mathbf{Tot}(\sigma \underline{\mathbb{X}}(\vec{X}))$ is given by $\mathbb{T}(\nu \mathbb{X})(\vec{X}) = (\mathbb{T}(\mu((\mathbb{X})^\perp))((\vec{X})^\perp))^\perp$ which indeed

makes sense because $\top\mu((\mathbb{X})^\perp)((\vec{X})^\perp) \in \text{Tot}(\sigma(\mathbb{X})^\perp((\vec{X})^\perp))$ and $\sigma(\mathbb{X})^\perp((\vec{X})^\perp) = (\sigma\mathbb{X}(\vec{X}))^\perp$ by definition of the De Morgan dual of a VCS.

More concretely, this means that $\top(\nu\mathbb{X})(\vec{X}) = (\sigma\mathbb{X}(\vec{X}), \mathcal{V})$ where \mathcal{V} is the greatest totality candidate of $\sigma\mathbb{X}(\vec{X})$ such that $F(\mathcal{V}) = \mathcal{V}$ where $F(\mathcal{T}) = \top\mathbb{X}(\vec{X}, (\sigma\mathbb{X}(\vec{X}), \mathcal{T}))$. In other words, $\mathcal{V} = \bigcap_\alpha F^\alpha(\text{Cl}(\sigma\mathbb{X}(\vec{X})))$ where $F^\lambda = \bigcap_{\alpha < \lambda} F^\alpha$ for limit ordinal λ , and $F^{\alpha+1} = F \circ F^\alpha$.


Given $n \in \mathbb{N}$ let \mathbf{Vcst}_n be the class of strong n -ary VCST. We identify \mathbf{Vcst}_0 with the class of objects of the Seely category \mathbf{CohT} .

Theorem 62 $(\mathbf{CohT}, (\mathbf{Vcst}_n)_{n \in \mathbb{N}})$ is a Seely model of μLL

Proof: By a direct application of Proposition 59 Proposition 60, and Theorem 61 ■

1.3 Sum up of Chapter 1

Sum up of Chapter 1

- Axiomatize a notion of categorical model of μLL , and prove its soundness: Definition 17 and Theorem 19.
- Provide some concrete models of μLL : 
 - **Rel**: Theorem 29.
 - **Nuts**: Theorem 45.
 - **Coh**: Theorem 54.
 - **CohT**: Theorem 62.
- One can have similar constructions as what has been done in this chapter for the other concrete models such as Scott model (which are based on the category of preorders and downward-closed relations) and hypercoherences. So, a general categorical setting taking into account all these situations would be quite useful (see Section 5.1 for some discussion about this).

Chapter 2

Polarized LL with fixpoints and its semantics

In this chapter, we introduce and study μ LLP, which can be viewed both as an extension of Laurent’s Polarized LL, LLP, with least and greatest fixpoints, and as a polarized version of Baelde’s LL with fixpoints (μ MALL and μ LL). We take advantage of the implicit structural rules of μ LLP to introduce a term syntax for this language, in the spirit of the classical lambda-calculus and of system L in the style of Curien, Herbelin and Munch-Maccagnoni [CH00, CM10, Mun13]. Our motivation to deal with a polarized language is as follows: We want to see μ LL as a programming language. However, the explicit structural rules in LL make such a language very heavy, hard to use and analyze. In particular, explicit substitutions would be necessary in such a μ LL-based syntax. So there is a real benefit in using μ LLP instead of μ LL, because in μ LLP we will be able to use variables exactly as we use variables in the λ -calculus, that is, keeping implicit the rules of weakening, contraction and promotion. The distinction linear vs. non-linear variables will not completely disappear but will be almost undetectable (see Section 2.1.1). The fact that the resulting μ LLP calculus contains also classical principles (call/cc) is a cherry on top of the cake but was not our initial motivation.

We equip this language with a deterministic reduction semantics as well as a denotational semantics based on the notion of non-uniform totality spaces and the notion of categorical model for LL with fixpoint introduced in Chapter 1. We prove an adequacy result for μ LLP between these operational and denotational semantics, from which we derive a normalization property for μ LLP thanks to the properties of the totality interpretation.

2.1 A classical calculus for polarized linear logic with fixpoints ($\kappa\mu$ LLP)

Olivier Laurent in [Lau02] introduced a polarized linear logic LLP which relaxes the use of structural rules on negative formulas. This system is better suited than LL for encoding classical logic, as one can encode μ LK into LLP using a straightforward extension of so-called Girard’s translation. In the sense of Curry-Howard paradigm, we will develop an extension of Laurent’s Polarized Linear Logic with least and greatest fixed points, μ LLP, which one can

$$\begin{array}{c}
 \frac{}{\vdash N, N^\perp} \text{ (ax)} \quad \frac{}{\vdash 1} \text{ (1)} \quad \frac{\vdash \mathcal{N}_1, P_1 \quad \vdash \mathcal{N}_2, P_2}{\vdash \mathcal{N}_1, \mathcal{N}_2, P_1 \otimes P_2} \text{ (\otimes)} \quad \frac{\vdash \mathcal{N}, P_i \quad i \in \{1, 2\}}{\vdash \mathcal{N}, P_1 \oplus P_2} \text{ (\oplus)}_i \\
 \\
 \frac{}{\vdash \mathcal{N}[, P], \top} \text{ (\top)} \quad \frac{\vdash \mathcal{N}[, P], N_1 \quad \vdash \mathcal{N}[, P], N_2}{\vdash \mathcal{N}[, P], N_1 \& N_2} \text{ (\&)} \quad \frac{\vdash \mathcal{N}[, P], N_1, N_2}{\vdash \mathcal{N}[, P], N_1 \wp N_2} \text{ (\wp)} \\
 \\
 \frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N} \text{ (!)} \quad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P} \text{ (d)} \quad \frac{\vdash \mathcal{N}[, P]}{\vdash \mathcal{N}[, P], N} \text{ (w)} \quad \frac{\vdash \mathcal{N}[, P], N, N}{\vdash \mathcal{N}[, P], N} \text{ (c)} \\
 \\
 \frac{\vdash \mathcal{N}[, P]}{\vdash \mathcal{N}[, P], \perp} \text{ (\perp)} \quad \frac{\vdash \mathcal{N}_1[, P], N \quad \vdash \mathcal{N}_2, N^\perp}{\vdash \mathcal{N}_1[, P], \mathcal{N}_2} \text{ (cut)}
 \end{array}$$

Figure 2.1: LLP inference rules

$$\begin{array}{c}
 \frac{}{\vdash \mathcal{N}, P, P^\perp} \text{ (ax)} \quad \frac{}{\vdash \mathcal{N}, 1} \text{ (1)} \quad \frac{\vdash \mathcal{N}, P_1 \quad \vdash \mathcal{N}, P_2}{\vdash \mathcal{N}, P_1 \otimes P_2} \text{ (\otimes)} \quad \frac{\vdash \mathcal{N}, P_i \quad i \in \{1, 2\}}{\vdash \mathcal{N}, P_1 \oplus P_2} \text{ (\oplus)}_i \\
 \\
 \frac{}{\vdash \mathcal{N}[, P], \top} \text{ (\top)} \quad \frac{\vdash \mathcal{N}[, P], N_1 \quad \vdash \mathcal{N}[, P], N_2}{\vdash \mathcal{N}[, P], N_1 \& N_2} \text{ (\&)} \quad \frac{\vdash \mathcal{N}[, P], N_1, N_2}{\vdash \mathcal{N}[, P], N_1 \wp N_2} \text{ (\wp)} \\
 \\
 \frac{\vdash \mathcal{N}[, P]}{\vdash \mathcal{N}[, P], \perp} \text{ (\perp)} \quad \frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N} \text{ (!)} \quad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P} \text{ (d)} \quad \frac{\vdash \mathcal{N}[, P], N \quad \vdash \mathcal{N}, N^\perp}{\vdash \mathcal{N}[, P]} \text{ (cut)}
 \end{array}$$

 Figure 2.2: LLP_{imp} inference rules

also see this extension as a polarized version of Baelde's Linear Logic with fixed points and exponentials.

Definition 63 *The formula of LLP is described as follows:*

$$N := \perp \mid N_1 \wp N_2 \mid \top \mid N_1 \& N_2 \mid ?P \quad P := 1 \mid P_1 \otimes P_2 \mid 0 \mid P_1 \oplus P_2 \mid !N$$

Notice that we are not considering the so-called weakly negative (respectively weakly positive) formula, i.e. the formula of the shape $?N$ (respectively $!P$). The inferences rules of LLP are the restriction of LL rules to polarized formula plus structural rules for any negative formula. As a side-effect of this restriction, one can check that all the rules preserve the fact that at most one positive formula may occur in a sequent. The inferences rules are provided in Figure 2.1 where $\mathcal{N} = N_1, \dots, N_n$ is a negative context, and $\mathcal{N}[, P]$ is a context which includes at most one positive formula P . Notice that whenever we use $\mathcal{N}[, P]$ in the inference rules such as the ($\&$) rule, the contexts of the premises must be the same.

In the presentation of Figure 2.1, the structural rules are explicit. One can equivalently present LLP system so that structural rules are implicit. In order to that, we add a negative context in the (ax) rule of Figure 2.1, and we also assume that the context \mathcal{N}_1 and \mathcal{N}_2 of the (\otimes) of Figure 2.1 are the same. We call the system LLP_{imp}, and the rules of LLP_{imp} are provided in Figure 2.2.

Proposition 64 *The sequent $\vdash \mathcal{N}[, P]$ is provable in LLP if and only if is provable in LLP_{imp}.*

From now on, we mostly work with the LLP_{imp} system unless it is explicitly mentioned that we are considering the LLP system.

Now, we would like to add the fixpoint formulas and rules to the LLP_{imp} . First we extend the syntax of formulas of LLP as the following definition, and we call them μLLP formulas.

Definition 65 *Given an infinite set of literals denoted ζ, ξ, \dots , we define the μLLP formulas as follows:*

$$\begin{aligned} N &:= \zeta^- \mid \perp \mid N_1 \wp N_2 \mid \top \mid N_1 \& N_2 \mid ?P \mid \nu\zeta.N \\ P &:= \zeta^+ \mid 1 \mid P_1 \otimes P_2 \mid 0 \mid P_1 \oplus P_2 \mid !N \mid \mu\zeta.P \end{aligned}$$

The reason why we have considered the $\mu\zeta.P$ (respectively $\nu\zeta.N$) as a positive (respectively negative) formula will be clear once we introduce the semantic of the system in Section 2.3.

Linear negation is defined as usual by induction on formulas, turns positive formulas into negative ones and conversely. The main cases are $(\zeta^+)^{\perp} = \zeta^-$, $(\zeta^-)^{\perp} = \zeta^+$, $(\mu\zeta.P)^{\perp} = \nu\zeta.P^{\perp}$ and $(\nu\zeta.N)^{\perp} = \mu\zeta.N^{\perp}$.

Remark 20 *With each literal ζ are associated two distinct variables ζ^+ and ζ^- which are formulas (a literal is not a formula). In the formula $\mu\zeta.P$, only the occurrences of ζ^+ are bound and dually for $\nu\zeta.N$. For example, if we allow to have a formula such as $N = \nu\zeta.(?\zeta^+ \wp ?1)$, then formula N will satisfy the equation $N = !N \multimap ?1$ which typically leads to non terminating computations.*

Although not strictly necessary, we adopt the convention that for a given literal ζ it is never the case that both ζ^+ and ζ^- occur in a given formula. This property can be obtained by α -renaming for closed formulas. For instance, instead of writing $\mu\zeta!(\nu\zeta.(\zeta^- \& ?\zeta^+))$, we write $\mu\zeta!(\nu\xi.(\xi^- \& ?\zeta^+))$.

Before moving to the inference rules for fixpoint formulas, we need to define an important operation which is substitution of formulas in formulas, it is defined in the obvious way (performing as usual α -renaming of type variables when needed). More precisely we define by induction on a negative or positive formula A :

- substitution of a positive formula P for a positive variable ζ^+ in A , written $A[P/\zeta^+]$
- and substitution of a negative formula N for a negative variable ζ^- in A , written $A[N/\zeta^-]$,

both formulas having the same polarity as A . We use $\text{fv}(A)$ for the set of free variables of the type A .

Lemma 66 *One has $A[P/\zeta^+]^{\perp} = A^{\perp}[P^{\perp}/\zeta^-]$ and $A[N/\zeta^-]^{\perp} = A^{\perp}[N^{\perp}/\zeta^+]$.*

The proof is a simple induction on A .

We shall quite often deal with formulas A where pairwise distinct positive variables $\zeta_1^+, \dots, \zeta_k^+$ are substituted by P_1, \dots, P_k and pairwise distinct negative variables ξ_1^-, \dots, ξ_n^- are substituted by N_1, \dots, N_n in parallel. Again the definition is a straightforward induction

on A but the use of such parallel substitutions can lead to quite heavy notations. We use letters such as π, ρ to denote such *type valuations* $\pi = (N_1/\xi_1^-, \dots, N_n/\xi_n^-, P_1/\zeta_1^+, \dots, P_k/\zeta_k^+)$ and write $A[\pi]$ rather than $A[N_1/\xi_1^-, \dots, N_n/\xi_n^-, P_1/\zeta_1^+, \dots, P_k/\zeta_k^+]$. We set $\pi^\perp = (N_1^\perp/\xi_1^+, \dots, N_n^\perp/\xi_n^+, P_1^\perp/\zeta_1^-, \dots, P_k^\perp/\zeta_k^-)$. We use $\text{dom}(\pi)$ for the domain of π which is the set $\{\zeta_1^+, \dots, \zeta_k^+, \xi_1^-, \dots, \xi_n^-\}$. We use $\pi \cdot (P/\zeta^+)$ and $\pi \cdot (N/\zeta^-)$ to denote extensions of such valuations (assuming of course that $\zeta^+, \zeta^- \notin \text{dom}(\pi)$). We say that π is closed if all formulas P_i, N_j 's are closed.

The inference rule for the introduction of (μ) is as follows:

$$\frac{\vdash \mathcal{N}, P [\mu\zeta.P/\zeta^+]}{\vdash \mathcal{N}, \mu\zeta.P} (\mu)$$

For introduction of (ν) , we have two following possibilities:

$$\frac{\vdash \mathcal{N}, Q, N [Q^\perp/\xi^-]}{\vdash \mathcal{N}, Q, \nu\xi.N} (\nu) \quad \frac{\vdash \mathcal{N}[P], Q^\perp \quad \vdash \mathcal{N}, Q, N [Q^\perp/\xi^-]}{\vdash \mathcal{N}[P], \nu\xi.N} (\nu')$$

Notice that, the existence of a positive formula in the context of (ν) (respectively the context of the second premises of (ν') rule) is necessary as we should substitute a negative formula in N for ξ^- .

For a technical reason, which is discussed in Remark 21, we will consider the (ν') rule as the introduction rule of (ν) .

We call $\mu\text{LLP}_{\text{imp}}$ for the system which adds the (μ) and (ν') rules to LLP_{imp} , and it has μLLP formulas (Definition 65) as its set of formulas. Similarly, we call μLLP for the system which adds the (μ) rule and the following one to LLP :

$$\frac{\vdash \mathcal{N}_1[P], Q^\perp \quad \vdash \mathcal{N}_2, Q, N [Q^\perp/\xi^-]}{\vdash \mathcal{N}_1, \mathcal{N}_2[P], \nu\xi.N}$$

Using Proposition 64, we have:

Proposition 67 *The sequent $\vdash \mathcal{N}[P]$ is provable in $\mu\text{LLP}_{\text{imp}}$ if and only if is provable in μLLP .*

2.1.1 Term assignment for $\mu\text{LLP}_{\text{imp}}$

We now assign terms for the $\mu\text{LLP}_{\text{imp}}$ logic to develop a kind of λ -calculus based on the sequent calculus. We do it step by step as follows.

Positive terms and negative terms: Based on the distinction between positive and negative formulas in LLP_{imp} , it is natural to have the same distinction on terms. We make this distinction apparent in the typing system by the use of three different kinds of judgments. From now on, by \mathcal{N} we mean $(\alpha_1 : N_1, \dots, \alpha_n : N_n)$.

- Postive term in a negative context: $\vdash \mathcal{N} \mid p : P$
- Negative term in a positive context: $\vdash \mathcal{N}, P \mid t : N$

- Negative term in a negative context: $\vdash \mathcal{N} \mid t : N$

Hence, we have two axiom rules, called **(t-n)** and **(t-•)**:

$$\frac{}{\vdash \mathcal{N}, \alpha : P^\perp \mid \alpha : P} \text{ (t-n)} \quad \frac{}{\vdash \mathcal{N}, P \mid \bullet : P^\perp} \text{ (t-•)}$$

The expression \bullet should be considered as a variable, the unique variable of negative type. Because of this uniqueness we do not need to mention it in the context and \mathcal{N}, P should be read as $\mathcal{N}, \bullet : P$.

Commands: We introduce commands c to represent cuts, and they can be typed by two different kinds of judgments:

- command in a positive context: $c \vdash \mathcal{N}, P$
- command in a negative context: $c \vdash \mathcal{N}$

Such a command $c = t \star p$ is then typed as follows:

$$\frac{\vdash \mathcal{N}[, P] \mid t : N \quad \vdash \mathcal{N} \mid p : N^\perp}{t \star p \vdash \mathcal{N}[, P]} \text{ (t-cut)}$$

Using these five kinds of judgment it is now easy to associate a term or command construction with each of the rules of $\mu\text{LLP}_{\text{imp}}$ listed in Fig 2.2. However this term assignment is not satisfactory because it does not satisfy for instance the following expected property:

A sequent $\vdash N_1, \dots, N_n[, P]$ is provable in $\mu\text{LLP}_{\text{imp}}$ iff there is a command c such that $c \vdash \mathcal{N}[, P]$.

The main reason for inability of proving the statement above is explained as follows: one basically needs to construct a negative term t from a command $c \vdash \mathcal{N}[, P], \alpha : N$ such that $\vdash \mathcal{N}[, P], \mid t : N$. Similarly, one needs to construct a positive term p from a command $c \vdash \mathcal{N}, P$ such that $\vdash \mathcal{N} \mid p : P$. But there are no such constructions in our system so far. So, we just simply add to our system the rules to do so. One can see a similar constructions in [Par92, CH00]; those construction are denoted by μ and $\tilde{\mu}$. However, as we already have used the notation μ for the least fixpoint formulas, we use κ and $\tilde{\kappa}$. instead as follows:

$$\frac{c \vdash \mathcal{N}, \alpha : N[, P]}{\vdash \mathcal{N}[, P] \mid \kappa \alpha . c : N} \text{ (t-}\kappa) \quad \frac{c \vdash \mathcal{N}, P}{\vdash \mathcal{N} \mid \tilde{\kappa} . c : P} \text{ (t-}\tilde{\kappa})$$

Extended with these two rules ((t- κ) and (t- $\tilde{\kappa}$)), our system is called $\kappa\mu\text{LLP}$, and its syntax and typing rules are given in Figure 2.3. The positive variable α is bounded in the negative command $\kappa \alpha . c$, and \bullet is bounded in the positive term $\tilde{\kappa} . d$ and in the command c of $\bar{\kappa}_{R^\perp, \zeta} \alpha . (c; s)$.

Lemma 68 *If α does not occur in the negative context \mathcal{N} , the following holds.*

- If $c \vdash \mathcal{N}[, P]$ then $c \vdash \mathcal{N}, \alpha : N[, P]$.
- If $\vdash \mathcal{N}[, P] \mid t : M$ then $\vdash \mathcal{N}, \alpha : N[, P] \mid t : M$.

- If $\vdash \mathcal{N} \mid p : P$ then $\vdash \mathcal{N}, \alpha : N \mid p : P$

In other words, we can use freely weakening on negative formulas in the context. Assume that $\vdash \mathcal{N}, P \mid s : N$. We have $\vdash \mathcal{N}, \alpha : N \mid \alpha : N^\perp$ and hence $s \star \alpha \vdash \mathcal{N}, \alpha : N, P$ so that $\vdash \mathcal{N}, \alpha : N \mid \tilde{\kappa}.(s \star \alpha) : P$, we shall often use this kind of *change of active formula*.

Proposition 69 *A sequent $\vdash N_1, \dots, N_n, P$ is provable in $\mu\text{LLP}_{\text{imp}}$ iff there is a command c such that $c \vdash \mathcal{N}[, P]$ in $\kappa\mu\text{LLP}$.*

Proof: Straightforward induction on derivation trees and terms. For left to right one always builds a command c . For the other direction, for each of the five kinds of judgments one associates with each typing derivation a $\mu\text{LLP}_{\text{imp}}$ proof of the underlying sequent. ■

We denote by π^* the command c which corresponds to μLLP proof π . And conversely, given a command c , we denote by $c^\#$ the proof π which corresponds to proof of the underlying sequent of c .

And using Proposition 67, a direct conclusion of Proposition 69 is the following:

Corollary 70 *A sequent $\vdash N_1, \dots, N_n, P$ is provable in μLLP iff there is a command c such that $c \vdash \mathcal{N}[, P]$ in $\kappa\mu\text{LLP}$.*

Before describing the reduction system of $\kappa\mu\text{LLP}$, we first need to speak about two important notions of substitution which are discussed in Section 2.1.2 and 2.1.3.

2.1.2 Substitution of terms in terms

Substitution of positive terms for a variable α and negative terms for \bullet are defined in the obvious way for all terms and commands. Notice that we have $(\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s)) [t/\bullet] = \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; (s [t/\bullet]))$ as \bullet is bounded in c (this is essential in our proof of Lemma 72). We have two substitution lemmas as follows that one can prove straightforwardly by mutual induction on terms and commands:

Lemma 71 *Assume that $\vdash \mathcal{N} \mid p : P$.*

- If $c \vdash \mathcal{N}, \alpha : P^\perp[, Q]$ then $c [p/\alpha] \vdash \mathcal{N}[, Q]$,
- if $\vdash \mathcal{N}, \alpha : P^\perp[, Q] \mid t : N$ then $\vdash \mathcal{N}[, Q] \mid t [p/\alpha] : N$,
- and if $\vdash \mathcal{N}, \alpha : P^\perp \mid q : Q$ then $\vdash \mathcal{N} \mid q [p/\alpha] : Q$.

Proof: The proof is by mutual induction on terms and commands. We just provide the details of three cases below, and the other cases are essentially similar.

▷ If we have $(t \star q) \vdash \mathcal{N}, \alpha : P^\perp[, Q]$ so that $\vdash \mathcal{N}, \alpha : P^\perp[, Q] \mid t : N$ and $\vdash \mathcal{N}, \alpha : P^\perp \mid q : Q$: By induction hypothesis, we have $\vdash \mathcal{N}[, Q] \mid t [p/\alpha] : N$ and $\vdash \mathcal{N} \mid q [p/\alpha] : Q$. Hence using the (**t-cut**) rule, we have $(t \star q) [p/\alpha] \vdash \mathcal{N}[, Q]$, since $(t \star q) [p/\alpha] = ((t [p/\alpha]) \star (q [p/\alpha]))$ by definition.

$p, q, \dots := \alpha \mid () \mid (p_1, p_2) \mid \text{in}_i(p) \mid \tilde{\kappa}.c \mid \text{fd}(p) \mid t^!$
 $s, t, \dots := \bullet \mid \langle \rangle \mid \langle t_1, t_2 \rangle \mid \kappa\alpha.c \mid \kappa_{\perp}.c \mid \kappa(\alpha_1, \alpha_2).c \mid \bar{\kappa}_{N, \zeta}\alpha.(c; s) \mid \text{der } p$
 $c, d, \dots := t \star p$

$$\begin{array}{c}
 \frac{}{\vdash \mathcal{N}, \alpha : P^{\perp} \mid \alpha : P} \text{ (t-n)} \quad \frac{}{\vdash \mathcal{N} \mid () : 1} \text{ (t-1)} \quad \frac{\vdash \mathcal{N}[P] \mid t : N \quad \vdash \mathcal{N} \mid p : N^{\perp}}{t \star p \vdash \mathcal{N}[P]} \text{ (t-cut)} \\
 \\
 \frac{\vdash \mathcal{N} \mid p_1 : P_1 \quad \vdash \mathcal{N} \mid p_2 : P_2}{\vdash \mathcal{N} \mid (p_1, p_2) : P_1 \otimes P_2} \text{ (t-}\otimes\text{)} \quad \frac{\vdash \mathcal{N} \mid p : P_i \quad i \in \{1, 2\}}{\vdash \mathcal{N} \mid \text{in}_i(p) : P_1 \oplus P_2} \text{ (t-}\oplus\text{)} \\
 \\
 \frac{c \vdash \mathcal{N}, P}{\vdash \mathcal{N} \mid \tilde{\kappa}.c : P} \text{ (t-}\tilde{\kappa}\text{)} \quad \frac{\vdash \mathcal{N} \mid p : P[\mu\zeta.P/\zeta^+]}{\vdash \mathcal{N} \mid \text{fd}(p) : \mu\zeta.P} \text{ (t-}\mu\text{)} \quad \frac{\vdash \mathcal{N} \mid t : N}{\vdash \mathcal{N} \mid t^! : !N} \text{ (t-!)} \\
 \\
 \frac{}{\vdash \mathcal{N}, P \mid \bullet : P^{\perp}} \text{ (t-}\bullet\text{)} \quad \frac{}{\vdash \mathcal{N}[P] \mid \langle \rangle : \top} \text{ (t-}\top\text{)} \\
 \\
 \frac{\vdash \mathcal{N}[P] \mid t_1 : N_1 \quad \vdash \mathcal{N}[P] \mid t_2 : N_2}{\vdash \mathcal{N}[P] \mid \langle t_1, t_2 \rangle : N_1 \& N_2} \text{ (t-}\&\text{)} \quad \frac{c \vdash \mathcal{N}, \alpha : N[P]}{\vdash \mathcal{N}[P] \mid \kappa\alpha.c : N} \text{ (t-}\kappa\text{)} \\
 \\
 \frac{c \vdash \mathcal{N}[P]}{\vdash \mathcal{N}[P] \mid \kappa_{\perp}.c : \perp} \text{ (t-}\perp\text{)} \quad \frac{c \vdash \mathcal{N}, \alpha_1 : N_1, \alpha_2 : N_2[P]}{\vdash \mathcal{N}[P] \mid \kappa(\alpha_1, \alpha_2).c : N_1 \wp N_2} \text{ (t-}\wp\text{)} \\
 \\
 \frac{c \vdash \mathcal{N}, P, \alpha : N[P^{\perp}/\zeta^-] \quad \vdash \mathcal{N}[Q] \mid s : P^{\perp}}{\vdash \mathcal{N}[Q] \mid \bar{\kappa}_{N, \zeta}\alpha.(c; s) : \nu\zeta.N} \text{ (t-}\nu\text{)} \quad \frac{\vdash \mathcal{N} \mid p : P}{\vdash \mathcal{N} \mid \text{der } p : ?P} \text{ (t-?)}
 \end{array}$$

 Figure 2.3: Syntax of terms and typing rules of $\kappa\mu\text{LLP}$

▷ If we have $\vdash \mathcal{N}, \beta : S^\perp[Q] \mid \bar{\kappa}_{R^\perp, \zeta} \beta.(c; s) : \nu \zeta.R^\perp$ so that $c \vdash \mathcal{N}, \alpha : P^\perp, S, \beta : R^\perp [S^\perp/\zeta^-]$ and $\vdash \mathcal{N}, \alpha : P^\perp[Q] \mid s : S^\perp$: By induction hypothesis, we have $c[p/\alpha] \vdash \mathcal{N}, S, \beta : R^\perp [S^\perp/\zeta^-]$ and $\vdash \mathcal{N}[Q] \mid s[p/\alpha] : S^\perp$. Hence using the $(\mathbf{t}\nu)$ rule, we have $\vdash \mathcal{N}, \beta : P^\perp[Q] \mid \bar{\kappa}_{R^\perp, \zeta} \beta.((c[p/\alpha]); (s[p/\alpha])) : \nu \zeta.R^\perp$.

▷ If we have $\vdash \mathcal{N}, \alpha : P^\perp \mid \text{fd}(q) : \mu \zeta.Q$ so that $\vdash \mathcal{N}, \alpha : P^\perp \mid q : Q [\mu \zeta.Q/\zeta^+]$. By induction hypothesis, we have $\vdash \mathcal{N} \mid q[p/\alpha] : Q [\mu \zeta.Q/\zeta^+]$. Hence using the $(\mathbf{t}\mu)$ rule, we have $\vdash \mathcal{N} \mid (\text{fd}(q)) [p/\alpha] : \mu \zeta.Q$. ■

Lemma 72 *Assume that $\vdash \mathcal{N}[Q] \mid t : N$.*

- *If $c \vdash \mathcal{N}, N^\perp$ then $c[t/\bullet] \vdash \mathcal{N}[Q]$ and*
- *if $\vdash \mathcal{N}, N^\perp \mid s : M$ then $\vdash \mathcal{N}[Q] \mid s[t/\bullet] : M$.*

Proof: The proof is by mutual induction on terms and commands. We just provide the details of two cases below, and the other cases are essentially similar.

▷ If we have $\vdash \mathcal{N}, N^\perp \mid \kappa \alpha.c : M$ so that $c \vdash \mathcal{N}, \alpha : M, N^\perp$. By induction hypothesis, we have $c[t/\bullet] \vdash \mathcal{N}, \alpha : M[Q]$. Hence using the $(\mathbf{t}\kappa)$ rule, we have $\vdash \mathcal{N} \mid \kappa \alpha.(c[t/\bullet]) : M$.

▷ If we have $\vdash \mathcal{N}, N^\perp \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s) : \nu \zeta.R^\perp$ so that $c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-]$ and $\vdash \mathcal{N}, N^\perp \mid s : P^\perp$ for some P . By induction hypothesis, we have $\vdash \mathcal{N}[Q] \mid s[t/\bullet] : P^\perp$. Hence using the $(\mathbf{t}\nu)$ rule, we have $\vdash \mathcal{N}, N^\perp \mid (\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s)) [t/\bullet] : \nu \zeta.R^\perp$, since $(\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s)) [t/\bullet] = \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; (s[t/\bullet]))$ by definition. ■

Remark 21 *If we had used the apparently simpler rule and term construction $(\mathbf{t}\nu^1)$ below instead of our $(\mathbf{t}\nu)$, we would not have been able to prove Lemma 72.*

$$\frac{c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-]}{\vdash \mathcal{N}, P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : \nu \zeta.R^\perp} (\mathbf{t}\nu^1)$$

Let us explain why. Suppose that we are proving the second item of Lemma 72, and s is $\bar{\kappa}_{R^\perp, \zeta} \alpha.c$ and $N = P^\perp$ such that $\vdash \mathcal{N}, P \mid s : \nu \zeta.R^\perp$, and we also have $\vdash \mathcal{N} \mid t : N$. By induction hypothesis, we have $c[t/\bullet] \vdash \mathcal{N}, \alpha : R^\perp [P^\perp/\zeta^-]$. But we cannot apply the rule $(\mathbf{t}\nu^1)$ on $c[t/\bullet]$, since there is no positive formula in the context. So, that is why we chose to work with the (ν') rule rather than the (ν) rule. However, the $(\mathbf{t}\nu)$ rule is derivable in $\kappa\mu\text{LLP}$ as follows:

$$\frac{c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-] \quad \frac{}{\vdash \mathcal{N}, P \mid \bullet : P^\perp} (\mathbf{t}\bullet)}{\vdash \mathcal{N}, P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet) : \nu \zeta.R^\perp} (\mathbf{t}\nu)$$

We use $(\mathbf{t}\nu^1)$ for this most important derived rule and use the notation $\bar{\kappa}_{R^\perp, \zeta} \alpha.c$ for $\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet)$.

2.1.3 Substitution of terms in formulas

In this section, we will explain the notion of *functoriality* in $\kappa\mu\text{LLP}$ which is in some sense the generalization of η -expansion. This notion will be essential to deal with the normalization proof as we will see in Section 2.1.4.

Assume that $\vdash \mathcal{N}, P \mid s : N$, let Q be a positive formula and let ζ be a variable. Let also π be a type valuation such that $\zeta^+, \zeta^- \notin \text{dom}(\pi)$. Then we can define a negative term $Q[\pi, s/\zeta^+]$ (called *positive substitution*) in such a way that

$$\vdash \mathcal{N}, Q[\pi, P/\zeta^+] \mid Q[\pi, s/\zeta^+] : Q^\perp[\pi^\perp, N/\zeta^-]$$

In order to make this more readable, let us remove the context \mathcal{N} and the term annotations. So, assume that we have a proof of $\vdash P, N$ and let Q be a positive formula that has ζ as its only free variable. Then we have a proof of

$$\vdash Q[P/\zeta^+], Q^\perp[N/\zeta^-]$$

One can try to construct the negative term $Q[\pi, s/\zeta^+]$ by induction on formula Q . If we do so, then we will see that we need also the following negative term (called *negative substitution*) once Q is $!N$ for some negative formula N :

$$\vdash \mathcal{N}, Q[\pi, N/\zeta^-] \mid Q[\pi, s/\zeta^-] : Q^\perp[\pi^\perp, P/\zeta^+]$$

If we again remove context \mathcal{N} and term annotation, we will need a proof of under the same assumption for s :

$$\vdash Q[N/\zeta^-], Q^\perp[P/\zeta^+]$$

Remark 22 We will see later in Section 2.3 that a proof of $\vdash P, N$ will be interpreted as a morphism $\llbracket N^\perp \rrbracket \rightarrow \llbracket P \rrbracket$. And a proof of $\vdash Q[P/\zeta^+], Q^\perp[N/\zeta^-]$ will be interpreted as a morphism $\llbracket Q \rrbracket(\llbracket N^\perp \rrbracket) \rightarrow \llbracket Q \rrbracket(\llbracket P \rrbracket)$, and this just says $\llbracket Q \rrbracket$ is a covariant functor. However, a proof of $\vdash Q[N/\zeta^-], Q^\perp[P/\zeta^+]$ will be interpreted as a morphism $\llbracket Q \rrbracket(\llbracket P^\perp \rrbracket) \rightarrow \llbracket Q \rrbracket(\llbracket N \rrbracket)$, whereas if $\llbracket Q \rrbracket$ were a contravariant functor we would have a morphism $\llbracket Q \rrbracket(\llbracket P \rrbracket) \rightarrow \llbracket Q \rrbracket(\llbracket N^\perp \rrbracket)$. Hence, although positive substitution corresponds to covariant functoriality, negative substitution will not correspond to contravariant functoriality.

We now state and prove our statement about substitution of terms in formulas:

Proposition 73 Assume that $\vdash \mathcal{N}, P \mid s : N$, let Q be a positive formula and let ζ be a variable. Let also π be a type valuation such that $\zeta^+, \zeta^- \notin \text{dom}(\pi)$. Then we can define two negative terms $Q[\pi, s/\zeta^+]$ and $Q[\pi, s/\zeta^-]$ in such a way that

$$\begin{aligned} \vdash \mathcal{N}, Q[\pi, P/\zeta^+] \mid Q[\pi, s/\zeta^+] &: Q^\perp[\pi^\perp, N/\zeta^-] \\ \vdash \mathcal{N}, Q[\pi, N/\zeta^-] \mid Q[\pi, s/\zeta^-] &: Q^\perp[\pi^\perp, P/\zeta^+] . \end{aligned}$$

In the first case we assume that $\text{fv}(Q) \subseteq \text{dom}(\pi) \cup \{\zeta^+\}$ and in the second case, that $\text{fv}(Q) \subseteq \text{dom}(\pi) \cup \{\zeta^-\}$.

Proof: We will define the two terms $Q[\pi, s/\zeta^+]$ and $Q[\pi, s/\zeta^-]$ in a simultaneous induction on Q .

- ▷ If $Q = \zeta^+$ then $Q[\pi, s/\zeta^+] = s$.
- ▷ If $Q = \zeta_i^+ \in \text{dom}(\pi)$ then $Q[\pi, s/\zeta^+] = \bullet$ with $\vdash \mathcal{N}, P_i \mid \bullet : P_i^\perp$.
- ▷ If $Q = 1$ then we set $Q[\pi, s/\zeta^+] = \bullet$ with $\vdash \mathcal{N}, 1 \mid \bullet : \perp$.

▷ If $Q = Q_1 \otimes Q_2$. Let $s_i = Q_i [\pi, s/\zeta^+]$. By inductive hypothesis we have $\vdash \mathcal{N}, Q_i [\pi, P/\zeta^+] \mid s_i : Q_i^\perp [\pi^\perp, N/\zeta^-]$.

$$\frac{\frac{\frac{\vdash \mathcal{N}, \alpha_i : Q_i^\perp [\pi^\perp, N/\zeta^-] \mid \tilde{\kappa}.(s_i \star \alpha_i) : Q_i [\pi, P/\zeta^+] \quad \text{for } i = 1, 2}{\vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, N/\zeta^-], \alpha_2 : Q_2^\perp [\pi^\perp, N/\zeta^-] \mid (\tilde{\kappa}.s_1 \star \alpha_1, \tilde{\kappa}.s_2 \star \alpha_2) : (Q_1 \otimes Q_2) [\pi, P/\zeta^+]} \quad (\mathbf{t}\text{-}\otimes)}{\vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, N/\zeta^-], \alpha_2 : Q_2^\perp [\pi^\perp, N/\zeta^-], (Q_1 \otimes Q_2) [\pi, P/\zeta^+]} \quad \pi \quad (\mathbf{t}\text{-cut})}{\frac{\bullet \star (\tilde{\kappa}.s_1 \star \alpha_1, \tilde{\kappa}.s_2 \star \alpha_2) \vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, N/\zeta^-], \alpha_2 : Q_2^\perp [\pi^\perp, N/\zeta^-], (Q_1 \otimes Q_2) [\pi, P/\zeta^+]}{\vdash \mathcal{N}, Q [\pi, P/\zeta^+] \mid \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.s_1 \star \alpha_1, \tilde{\kappa}.s_2 \star \alpha_2)) : Q^\perp [\pi^\perp, N/\zeta^-]} \quad (\mathbf{t}\text{-}\wp)} \quad (\mathbf{t}\text{-n})$$

where π is $\vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, N/\zeta^-], \alpha_2 : Q_2^\perp [\pi^\perp, N/\zeta^-], (Q_1 \otimes Q_2) [\pi, P/\zeta^+] \mid \bullet : Q_1^\perp \wp Q_2^\perp$.

Hence we set $(Q_1 \otimes Q_2) [\pi, s/\zeta^+] = \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.s_1 \star \alpha_1, \tilde{\kappa}.s_2 \star \alpha_2))$.

▷ Assume $Q = 0$. We have $\vdash \mathcal{N}, 0 \mid \langle \rangle : \top$ and we set $0 [\pi, s/\zeta^+] = \langle \rangle$.

▷ If $Q = Q_1 \oplus Q_2$, we use the same notations as for \otimes . We have

$$\frac{\frac{\frac{\vdash \mathcal{N}, \alpha_i : Q_i^\perp [\pi^\perp, N/\zeta^-] \mid \tilde{\kappa}.(s_i \star \alpha_i) : Q_i [\pi, P/\zeta^+]}{\vdash \mathcal{N}, \alpha_i : Q_i^\perp [\pi^\perp, N/\zeta^-] \mid \text{in}_i(\tilde{\kappa}.(s_i \star \alpha_i)) : (Q_1 \oplus Q_2) [\pi, P/\zeta^+]} \quad (\mathbf{t}\text{-}\oplus)}{\vdash \mathcal{N}, \alpha_i : Q_i^\perp [\pi^\perp, N/\zeta^-] \mid \text{in}_i(\tilde{\kappa}.(s_i \star \alpha_i)) : (Q_1 \oplus Q_2) [\pi, P/\zeta^+]} \quad \pi \quad (\mathbf{t}\text{-cut})}{\frac{(\bullet \star \text{in}_i(\tilde{\kappa}.(s_i \star \alpha_i))) \vdash \mathcal{N}, (Q_1 \oplus Q_2) [\pi, P/\zeta^+], \alpha_i : Q_i^\perp [\pi^\perp, N/\zeta^-]}{\vdash \mathcal{N}, (Q_1 \oplus Q_2) [\pi, P/\zeta^+] \mid \kappa\alpha_i.(\bullet \star \text{in}_i(\tilde{\kappa}.(s_i \star \alpha_i))) : Q_i^\perp [\pi^\perp, N/\zeta^-]} \quad (\mathbf{t}\text{-}\kappa)} \quad \text{for } i = 1, 2}{\frac{\vdash \mathcal{N}, (Q_1 \oplus Q_2) [\pi, P/\zeta^+] \mid \kappa\alpha_1.(\bullet \star \text{in}_1(\tilde{\kappa}.(s_1 \star \alpha_1))), \kappa\alpha_2.(\bullet \star \text{in}_2(\tilde{\kappa}.(s_2 \star \alpha_2)))}{\vdash \mathcal{N}, Q [\pi, P/\zeta^+] \mid \langle \kappa\alpha_1.(\bullet \star \text{in}_1(\tilde{\kappa}.(s_1 \star \alpha_1))), \kappa\alpha_2.(\bullet \star \text{in}_2(\tilde{\kappa}.(s_2 \star \alpha_2))) \rangle : Q^\perp [\pi^\perp, N/\zeta^-]} \quad (\mathbf{t}\text{-}\&)} \quad (\mathbf{t}\text{-}\bullet)$$

where $\pi \vdash \mathcal{N}, \alpha_i : Q_i^\perp [\pi^\perp, N/\zeta^-], (Q_1 \oplus Q_2) [\pi, P/\zeta^+] \mid \bullet : ((Q_1 \oplus Q_2) [\pi, P/\zeta^+])^\perp$. Hence we set $(Q_1 \oplus Q_2) [\pi, s/\zeta^+] = \langle \kappa\alpha_1.(\bullet \star \text{in}_1(\tilde{\kappa}.(s_1 \star \alpha_1))), \kappa\alpha_2.(\bullet \star \text{in}_2(\tilde{\kappa}.(s_2 \star \alpha_2))) \rangle$.

▷ Assume that $Q = \mu\xi.R$ and let us set $R_P = R [P/\zeta^+]$ and $R_N = R [N^\perp/\zeta^+]$. Let $\rho = \pi \cdot (\mu\xi.R_P [\pi] / \xi^+)$ (it is here that one sees why the type valuation π is required). By inductive hypothesis we have $t = R [\rho, s/\zeta^+]$ such that $\vdash \mathcal{N}, R_P [\rho] \mid t : R_N^\perp [\rho^\perp]$.

Notice that, due to the definition of ρ , we have $R_S [\rho] = R_S [\pi] [\mu\xi.R_P [\pi] / \xi^+]$ for $S = P, N$.

$$\frac{\frac{\frac{\frac{\vdash \mathcal{N}, \alpha : R_N^\perp [\rho^\perp], R_P [\rho] \mid t : R_N^\perp [\rho^\perp] \quad \vdash \mathcal{N}, \alpha : R_N^\perp [\rho^\perp] \mid \alpha : (R_N^\perp [\rho^\perp])^\perp}{(t \star \alpha) \vdash \mathcal{N}, \alpha : R_N^\perp [\rho^\perp], R_P [\rho]} \quad (\mathbf{t}\text{-cut})}{\vdash \mathcal{N}, \alpha : R_N^\perp [\rho^\perp] \mid \tilde{\kappa}.(t \star \alpha) : R_P [\rho]} \quad (\mathbf{t}\text{-}\mu)}{\vdash \mathcal{N}, \alpha : R_N^\perp [\rho^\perp] \mid \text{fd}(\tilde{\kappa}.(t \star \alpha)) : \mu\xi.R_P [\pi]} \quad (\mathbf{t}\text{-cut})}{\frac{(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha))) \vdash \mathcal{N}, \mu\xi.R_P [\pi], \alpha : R_N^\perp [\rho^\perp] = (R_N [\pi] [\mu\xi.R_P [\pi] / \xi^+])^\perp}{\vdash \mathcal{N}, Q [\pi, P/\zeta^+] \mid \bar{\kappa}_{R_N^\perp[\pi^\perp], \zeta} \alpha.(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha))) : Q^\perp [\pi^\perp, N/\zeta^-]} \quad (\mathbf{t}\text{-}\nu^1)}$$

where π

$$\frac{}{\vdash \mathcal{N}, \alpha : R_N^\perp [\rho^\perp], \mu\xi.R_P [\pi] \mid \bullet : (\mu\xi.R_P [\pi])^\perp} \quad (\mathbf{t}\text{-}\bullet)$$

So we define $(\mu\xi.R) [\pi, s/\zeta^+]$ as $\bar{\kappa}_{R_N^\perp[\pi^\perp], \zeta} \alpha.(\bullet \star \text{fd}(\tilde{\kappa}.(t \star \alpha)))$.

▷ Assume last that $Q = !R^\perp$.

By inductive hypothesis we have defined $t = R [\pi^\perp, s/\zeta^-]$ which satisfies $\vdash \mathcal{N}, R [\pi^\perp, N/\zeta^-] \mid t : R^\perp [\pi, P/\zeta^+]$

$$\begin{array}{c}
 \vdash \mathcal{N}, R [\pi^\perp, N/\zeta^-] \mid t : R^\perp [\pi, P/\zeta^+] \\
 \hline
 \vdash \mathcal{N}, R [\pi^\perp, N/\zeta^-], \alpha : R^\perp [\pi, P/\zeta^+] \mid t : R^\perp [\pi, P/\zeta^+] \quad \pi_3 \\
 \hline
 (t \star \alpha) \vdash \mathcal{N}, \alpha : R^\perp [\pi, P/\zeta^+], R [\pi^\perp, N/\zeta^-] \quad (\mathbf{t}\text{-}\tilde{\kappa}) \\
 \hline
 \vdash \mathcal{N}, \alpha : R^\perp [\pi, P/\zeta^+] \mid \tilde{\kappa}.(t \star \alpha) : R [\pi^\perp, N/\zeta^-] \quad (\mathbf{t}\text{-}?) \\
 \hline
 \vdash \mathcal{N}, \alpha : R^\perp [\pi, P/\zeta^+] \mid \text{der } \tilde{\kappa}.(t \star \alpha) : ?R [\pi^\perp, N/\zeta^-] \\
 \hline
 \vdash \mathcal{N}, \alpha : R^\perp [\pi, P/\zeta^+], \beta : ?R [\pi^\perp, N/\zeta^-] \mid \text{der } \tilde{\kappa}.(t \star \alpha) : ?R [\pi^\perp, N/\zeta^-] \quad \pi_1 \\
 \hline
 (\text{der } \tilde{\kappa}.(t \star \alpha) \star \beta) \vdash \mathcal{N}, \beta : ?R [\pi^\perp, N/\zeta^-], \alpha : R^\perp [\pi, P/\zeta^+] \quad (\mathbf{t}\text{-}\kappa) \\
 \hline
 \vdash \mathcal{N}, \beta : ?R [\pi^\perp, N/\zeta^-] \mid \kappa\alpha.(\text{der } \tilde{\kappa}.(t \star \alpha) \star \beta) : R^\perp [\pi, P/\zeta^+] \quad (\mathbf{t}\text{-}!) \\
 \hline
 \vdash \mathcal{N}, \beta : ?R [\pi^\perp, N/\zeta^-] \mid \kappa\alpha.(\text{der } \tilde{\kappa}.(t \star \alpha) \star \beta)^\dagger : !R^\perp [\pi, P/\zeta^+] \quad \pi_2 \\
 \hline
 (\bullet \star \kappa\alpha.(\text{der } \tilde{\kappa}.(t \star \alpha) \star \beta)^\dagger) \vdash \mathcal{N}, !R^\perp [\pi, P/\zeta^+], \beta : ?R [\pi^\perp, N/\zeta^-] \quad (\mathbf{t}\text{-}\kappa) \\
 \hline
 \vdash \mathcal{N}, !R^\perp [\pi, P/\zeta^+] \mid \kappa\beta.(\bullet \star \kappa\alpha.(\text{der } \tilde{\kappa}.(t \star \alpha) \star \beta)^\dagger) : ?R [\pi^\perp, N/\zeta^-] \quad (\mathbf{t}\text{-cut})
 \end{array}$$

where π_1 is a proof of $\vdash \mathcal{N}, \alpha : R^\perp [\pi, P/\zeta^+], \beta : ?R [\pi^\perp, N/\zeta^-] \mid \beta : (?R [\pi^\perp, N/\zeta^-])^\perp$ using $(\mathbf{t}\text{-}\mathbf{n})$ rule. And π_2 is a proof of $\vdash \mathcal{N}, \beta : ?R [\pi^\perp, N/\zeta^-], !R^\perp [\pi, P/\zeta^+] \mid \bullet : (!R^\perp [\pi, P/\zeta^+])^\perp$ using $(\mathbf{t}\text{-}\bullet)$ rule. And π_3 is a proof $\vdash \mathcal{N}, \alpha : R^\perp [\pi, P/\zeta^+] \mid \alpha : (R^\perp [\pi, P/\zeta^+])^\perp$ using $(\mathbf{t}\text{-}\mathbf{n})$ rule.

So we define $!R^\perp [\pi, s/\zeta^+]$ as the term $\kappa\beta.(\bullet \star \kappa\alpha.(\text{der } \tilde{\kappa}.(t \star \alpha) \star \beta)^\dagger)$

In the same induction we define completely similarly the negative substitution $Q [\pi, s/\zeta^-]$. Let us do only the case where $Q = Q_1 \otimes Q_2$.

Let $s_i = Q_i [\pi, s/\zeta^-]$. By inductive hypothesis we have $\vdash \mathcal{N}, Q_i [\pi, N/\zeta^-] \mid s_i : Q_i^\perp [\pi^\perp, P/\zeta^+]$.

$$\begin{array}{c}
 \vdash \mathcal{N}, \alpha_i : Q_i^\perp [\pi^\perp, P/\zeta^+] \mid \tilde{\kappa}.(s_i \star \alpha_i) : Q_i [\pi, N/\zeta^-] \quad \text{for } i = 1, 2 \\
 \hline
 \vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, P/\zeta^+], \alpha_2 : Q_2^\perp [\pi^\perp, P/\zeta^+] \mid (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)) : (Q_1 \otimes Q_2) [\pi, N/\zeta^-] \quad \pi \\
 \hline
 \bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)) \vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, P/\zeta^+], \alpha_2 : Q_2^\perp [\pi^\perp, P/\zeta^+], (Q_1 \otimes Q_2) [\pi, N/\zeta^-] \quad (\mathbf{t}\text{-}\mathfrak{Y}) \\
 \hline
 \vdash \mathcal{N}, Q [\pi, N/\zeta^-] \mid \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2))) : Q^\perp [\pi^\perp, P/\zeta^+] \\
 \hline
 (\mathbf{t}\text{-}\mathbf{n})
 \end{array}$$

where π is $\vdash \mathcal{N}, \alpha_1 : Q_1^\perp [\pi^\perp, P/\zeta^+], \alpha_2 : Q_2^\perp [\pi^\perp, P/\zeta^+], (Q_1 \otimes Q_2) [\pi, N/\zeta^-] \mid \bullet : Q_1^\perp \mathfrak{Y} Q_2^\perp$.

Hence we set $(Q_1 \otimes Q_2) [\pi, s/\zeta^-] = \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \alpha_1), \tilde{\kappa}.(s_2 \star \alpha_2)))$.

■

2.1.4 Reduction relation

The idea of our reduction relation for $\kappa\mu\text{LLP}$ system is coming from the cut-elimination rules on sequent calculus proofs of $\mu\text{LLP}_{\text{imp}}$. As we see in Proposition 69, the sequent calculus proofs of $\mu\text{LLP}_{\text{imp}}$ will correspond to commands of $\kappa\mu\text{LLP}$. So, we only equip commands (and not terms) with a rewriting relation \rightarrow . There are two categories of cut-elimination rules; the principal cases and the commutative rules. Hence, we will also consider these two cases here.

Let us first look at principal cases, and assume that we are in the following situation:

$$\frac{\frac{\frac{\vdash N_1, \dots, N_n, P_1 \quad \vdash N_1, \dots, N_n, P_2}{\vdash N_1, \dots, N_n, P_1 \otimes P_2} (\otimes) \quad \frac{\vdash N_1, \dots, N_n[Q], P_1^\perp, P_2^\perp}{\vdash N_1, \dots, N_n[Q], P_1^\perp \wp P_2^\perp} (\wp)}{\vdash N_1, \dots, N_n[Q]} (\text{cut})$$

This proof reduces to

$$\frac{\frac{\frac{\vdash N_1, \dots, N_n, P_1 \quad \vdash N_1, \dots, N_n[Q], P_1^\perp, P_2^\perp}{\vdash N_1, \dots, N_n[Q], P_2^\perp} (\text{cut})}{\vdash N_1, \dots, N_n[Q], P_2^\perp} (\text{cut}) \quad \vdash N_1, \dots, N_n, P_2}{\vdash N_1, \dots, N_n[Q]} (\text{cut})$$

This situation will be translated into $\kappa\mu\text{LLP}$ as follows.

$$\frac{\frac{\frac{d_1 \vdash \mathcal{N}, P_1}{\vdash \mathcal{N} \mid \tilde{\kappa}.d_1 : P_1} (\mathbf{t}\text{-}\tilde{\kappa}) \quad \frac{d_2 \vdash \mathcal{N}, P_2}{\vdash \mathcal{N} \mid \tilde{\kappa}.d_2 : P_2} (\mathbf{t}\text{-}\tilde{\kappa})}{\vdash \mathcal{N} \mid (\tilde{\kappa}.d_1, \tilde{\kappa}.d_2) : P_1 \otimes P_2} (\mathbf{t}\text{-}\otimes) \quad \frac{c \vdash \mathcal{N}[Q], \alpha_1 : P_1^\perp, \alpha_2 : P_2^\perp}{\vdash \mathcal{N}[Q] \mid \kappa(\alpha_1, \alpha_2).c : P_1^\perp \wp P_2^\perp} (\mathbf{t}\text{-}\wp)}{\kappa(\alpha_1, \alpha_2).c \star (\tilde{\kappa}.d_1, \tilde{\kappa}.d_2) \vdash \mathcal{N}[Q]} (\mathbf{t}\text{-}\text{cut})$$

And this proof reduces to

$$\frac{\frac{\frac{d_1 \vdash \mathcal{N}, P_1}{\vdash \mathcal{N} \mid \tilde{\kappa}.d_1 : P_1} (\mathbf{t}\text{-}\tilde{\kappa}) \quad c \vdash \mathcal{N}[Q], \alpha_1 : P_1^\perp, \alpha_2 : P_2^\perp}{c[\tilde{\kappa}.d_1/\alpha_1] \vdash \mathcal{N}[Q], \alpha_2 : P_2} \text{ Lemma 71} \quad \frac{d_2 \vdash \mathcal{N}, P_2}{\vdash \mathcal{N} \mid \tilde{\kappa}.d_2 : P_2} (\mathbf{t}\text{-}\tilde{\kappa})}{c[\tilde{\kappa}.d_1/\alpha_1][\tilde{\kappa}.d_2/\alpha_2] \vdash \mathcal{N}[Q]} \text{ Lemma 71}$$

So, we will consider the following rule in our reduction system:

$$\kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rightarrow c[p_1/\alpha_1, p_2/\alpha_2].$$

Similarly, one can obtain the rules corresponding to the principal cases of $(\oplus) - (\&)$, $(!) - (\mathbf{d})$, $(\mu) - (\nu')$, and $(\perp) - (1)$. We only explain the case of $(\mu) - (\nu')$ as it shows the motivation of Section 2.1.3, and the other cases are similar to what we showed above for $(\otimes) - (\wp)$.

Let assume that we are in the following situation:

$$\frac{\frac{c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-] \quad \vdash \mathcal{N}[Q] \mid s : P^\perp}{\vdash \mathcal{N}[Q] \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s) : \nu \zeta.R^\perp} (\mathbf{t}\text{-}\nu) \quad \frac{\vdash \mathcal{N} \mid p : R [\mu \zeta.R/\zeta^+]}{\vdash \mathcal{N} \mid \text{fd}(p) : \mu \zeta.R} (\mathbf{t}\text{-}\mu)}{\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s) \star \text{fd}(p) \vdash \mathcal{N}[Q]} (\mathbf{t}\text{-}\text{cut})$$

The proof above will reduce to the following:

$$\begin{array}{c}
 \frac{\frac{c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp / \zeta^-]}{\vdash \mathcal{N}, \alpha : R^\perp [P^\perp / \zeta^-] [Q] \mid s : P^\perp} \text{Lemma 71}}{\frac{c[s/\bullet] \vdash \mathcal{N}, \alpha : R^\perp [P^\perp / \zeta^-] [Q]}{\vdash \mathcal{N}, \alpha : R^\perp [P^\perp / \zeta^-] [Q]} \text{(t-}\kappa\text{)}}} \\
 \frac{\pi}{\frac{\vdash \mathcal{N}, \alpha : R^\perp [P^\perp / \zeta^-] [Q]}{\vdash \mathcal{N}, \alpha : R^\perp [P^\perp / \zeta^-] [Q]} \text{Lemma 71}} \\
 \frac{\vdash \mathcal{N}, \alpha : R^\perp [P^\perp / \zeta^-] [Q]}{(R [\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet) / \zeta^+] \star p) [\kappa \alpha.(c[s/\bullet]) / \bullet] \vdash}
 \end{array}$$

where π is the following proof:

$$\frac{\frac{\frac{c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp / \zeta^-]}{\vdash \mathcal{N}, P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet) : \nu \zeta . R^\perp} \text{(t-}\nu^1\text{)}}{R} \text{Proposition 73}}{\frac{\vdash \mathcal{N}, R [P / \zeta^+] \mid R [\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet) / \zeta^+] : R^\perp [\nu \zeta . R^\perp / \zeta^-]}{\vdash \mathcal{N} \mid p : R [\mu \zeta . R / \zeta^+]} \text{(t-cut)}}} \\
 R [\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet) / \zeta^+] \star p \vdash \mathcal{N}, R [P / \zeta^+]$$

So, we will also consider the following rule in our reduction system:

$$\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; s) \star \text{fd}(p) \rightarrow (R [\bar{\kappa}_{R^\perp, \zeta} \alpha.(c; \bullet) / \zeta^+] \star p) [\kappa \alpha.(c[s/\bullet]) / \bullet].$$

We now turn to the commutative cases. Let us consider as an example the case of (\mathfrak{A}) rule in sequent calculus of $\mu\text{LLP}_{\text{imp}}$. So, we are in the following situation:

$$\frac{\frac{\vdash N_1, \dots, N_n, M_1, M_2, P^\perp}{\vdash N_1, \dots, N_n, M_1 \mathfrak{A} M_2, P^\perp} \text{(}\mathfrak{A}\text{)}}{\frac{\vdash N_1, \dots, N_n, M_1 \mathfrak{A} M_2, P^\perp}{\vdash N_1, \dots, N_n, M_1 \mathfrak{A} M_2} \text{(cut)}}$$

The proof above reduces to the following proof:

$$\frac{\frac{\vdash N_1, \dots, N_n, M_1, M_2, P^\perp}{\vdash N_1, \dots, N_n, M_1, M_2} \text{(cut)}}{\frac{\vdash N_1, \dots, N_n, M_1, M_2}{\vdash N_1, \dots, N_n, M_1 \mathfrak{A} M_2} \text{(}\mathfrak{A}\text{)}}$$

This situation will be translated to $\kappa\mu\text{LLP}$ as follows. Assume that we have the proof below:

$$\frac{\frac{\frac{c \vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2, \beta : P^\perp}{\vdash \mathcal{N}, \beta : P^\perp \mid \kappa(\alpha_1, \alpha_2).c : M_1 \mathfrak{A} M_2} \text{(t-}\mathfrak{A}\text{)}}{\frac{\vdash \mathcal{N}, \beta : P^\perp \mid \kappa(\alpha_1, \alpha_2).c : M_1 \mathfrak{A} M_2}{\vdash \mathcal{N}, \beta : P^\perp \mid \kappa(\alpha_1, \alpha_2).c : M_1 \mathfrak{A} M_2} \text{(t-cut)}}} \\
 \frac{\frac{\kappa(\alpha_1, \alpha_2).c \star \gamma \vdash \mathcal{N}, \beta : P^\perp, \gamma : M_1 \mathfrak{A} M_2}{\vdash \mathcal{N}, \gamma : M_1 \mathfrak{A} M_2 \mid \kappa \beta.(\kappa(\alpha_1, \alpha_2).c \star \gamma) : P^\perp} \text{(t-}\kappa\text{)}}{\frac{\vdash \mathcal{N}, \gamma : M_1 \mathfrak{A} M_2 \mid \kappa \beta.(\kappa(\alpha_1, \alpha_2).c \star \gamma) : P^\perp}{\kappa \beta.(\kappa(\alpha_1, \alpha_2).c \star \gamma) \star \tilde{\kappa}.d \vdash \mathcal{N}, \gamma : M_1 \mathfrak{A} M_2} \text{(t-cut)}}} \pi_1$$

where π_1 and π_2 are the following proofs (from left to right respectively)

$$\frac{d \vdash \mathcal{N}, P}{\vdash \mathcal{N} \mid \tilde{\kappa}.d : P} \text{(t-}\tilde{\kappa}\text{)} \quad \frac{}{\vdash \mathcal{N}, \beta : P^\perp, \gamma : M_1 \mathfrak{A} M_2 \mid \gamma : (M_1 \mathfrak{A} M_2)^\perp} \text{(t-n)}$$

This proof will reduce to:

$$\frac{c \vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2, \beta : P^\perp \quad \frac{d \vdash \mathcal{N}, P}{\vdash \mathcal{N} \mid \tilde{\kappa}.d : P} \text{ (t-}\tilde{\kappa}\text{)}}{\vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2 \mid \tilde{\kappa}.d : P} \text{ Lemma 71}$$

$$\frac{\frac{c[\tilde{\kappa}.d/\beta] \vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2}{\vdash \mathcal{N} \mid \kappa(\alpha_1, \alpha_2).(c[\tilde{\kappa}.d/\beta]) : M_1 \wp M_2} \text{ (t-}\wp\text{)}}{(\kappa(\alpha_1, \alpha_2).(c[\tilde{\kappa}.d/\beta])) \star \gamma \vdash \mathcal{N}, \gamma : M_1 \wp M_2} \pi \text{ (t-cut)}$$

where π is proof of $\vdash \mathcal{N}, \gamma : M_1 \wp M_2 \mid \gamma : (M_1 \wp M_2)^\perp$ using the **(t-n)** rule.

So, we see that the command $\kappa\beta.(\kappa(\alpha_1, \alpha_2).c\star\gamma)\star\tilde{\kappa}.d$ is reduced to $(\kappa(\alpha_1, \alpha_2).(c[\tilde{\kappa}.d/\beta]))\star\gamma$. Basically, what happened is that the command $\kappa\beta.(\kappa(\alpha_1, \alpha_2).c\star\gamma)\star\tilde{\kappa}.d$ is reducing to $(\kappa(\alpha_1, \alpha_2).c\star\gamma)[\tilde{\kappa}.d/\beta]$ which is equal to $(\kappa(\alpha_1, \alpha_2).(c[\tilde{\kappa}.d/\beta]))\star\gamma$.

If we look at the other commutative cases, we see that in general, we have this pattern: a command $(\kappa\beta.c)\star\tilde{\kappa}.d$ reduces to $c[\tilde{\kappa}.d/\beta]$.

We only considered the case of a negative cut-formula. Now, we do the same analysis for the case where the cut formula is positive. So, let us say, as an example, we have the following proof:

$$\frac{\frac{\vdash N_1, \dots, N_n, M_1, M_2, P}{\vdash N_1, \dots, N_n, M_1 \wp M_2, P} \text{ (}\wp\text{)} \quad \vdash N_1, \dots, N_n, P^\perp}{\vdash N_1, \dots, N_n, M_1 \wp M_2} \text{ (cut)}$$

And this proof reduces to

$$\frac{\frac{\vdash N_1, \dots, N_n, M_1, M_2, P \quad \vdash N_1, \dots, N_n, P^\perp}{\vdash N_1, \dots, N_n, M_1 \wp M_2} \text{ (cut)}}{\vdash N_1, \dots, N_n, M_1 \wp M_2} \text{ (}\wp\text{)}$$

This situation is translated to $\kappa\mu\text{LLP}$ as follows. We first have the proof below:

$$\frac{\frac{c \vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2, P}{\vdash \mathcal{N}, P \mid \kappa(\alpha_1, \alpha_2).c : M_1 \wp M_2} \text{ (t-}\wp\text{)} \quad \frac{}{\vdash \mathcal{N}, \gamma : M_1 \wp M_2 \mid \gamma : (M_1 \wp M_2)^\perp} \text{ (t-n)}}{\vdash \mathcal{N}, \gamma : M_1 \wp M_2 \mid \kappa(\alpha_1, \alpha_2).c : M_1 \wp M_2} \text{ (t-cut)}$$

$$\frac{\frac{\kappa(\alpha_1, \alpha_2).c\star\gamma \vdash \mathcal{N}, P, \gamma : M_1 \wp M_2}{\vdash \mathcal{N}, \gamma : M_1 \wp M_2 \mid \tilde{\kappa}.(\kappa(\alpha_1, \alpha_2).c\star\gamma) : P} \text{ (t-}\tilde{\kappa}\text{)}}{(\kappa\gamma.d)\star\tilde{\kappa}.(\kappa(\alpha_1, \alpha_2).c\star\gamma) \vdash \mathcal{N}, \gamma : M_1 \wp M_2} \pi \text{ (t-cut)}$$

where π is $\frac{d \vdash \mathcal{N}, \gamma : P^\perp}{\vdash \mathcal{N} \mid \kappa\gamma.d : P^\perp} \text{ (t-}\tilde{\kappa}\text{)}$. And the proof above will be reduced to:

$$\frac{c \vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2, \beta : P \quad \frac{d \vdash \mathcal{N}, \gamma : P^\perp}{\vdash \mathcal{N} \mid \kappa\gamma.d : P^\perp} \text{ (t-}\tilde{\kappa}\text{)}}{\vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2 \mid \kappa\gamma.d : P^\perp} \text{ Lemma 72}$$

$$\frac{\frac{c[\kappa\gamma.d/\bullet] \vdash \mathcal{N}, \alpha_1 : M_1, \alpha_2 : M_2}{\vdash \mathcal{N} \mid \kappa(\alpha_1, \alpha_2).(c[\kappa\gamma.d/\bullet]) : M_1 \wp M_2} \text{ (t-}\wp\text{)}}{(\kappa(\alpha_1, \alpha_2).(c[\kappa\gamma.d/\bullet])) \star \gamma \vdash \mathcal{N}, \gamma : M_1 \wp M_2} \pi \text{ (t-cut)}$$

where π is proof $\vdash \mathcal{N}, \gamma : M_1 \wp M_2 \mid \gamma : (M_1 \wp M_2)^\perp$ using **(t-n)** rule.

So, we see that one can have this reduction: $(\kappa\gamma.d)\star\tilde{\kappa}.(\kappa(\alpha_1, \alpha_2).c\star\gamma) \rightarrow (\kappa(\alpha_1, \alpha_2).(c[\kappa\gamma.d/\bullet]))\star\gamma$. And indeed the reduced command is $(\kappa(\alpha_1, \alpha_2).c\star\gamma)[\kappa\gamma.d/\bullet]$. If we look at the other commutative cases, we see that in general, we have this pattern: $\kappa\gamma.d\star\tilde{\kappa}.c \rightarrow c[\kappa\gamma.d/\bullet]$.

Therefore, we ended up with a critical pair, as we have this situation:

$$\kappa\gamma.d\star\tilde{\kappa}.c \rightarrow d[\tilde{\kappa}.c/\gamma] \text{ and } \kappa\gamma.d\star\tilde{\kappa}.c \rightarrow c[\kappa\gamma.d/\bullet].$$

Both reductions are compatible with the denotational semantics, and this semantics is non-trivial in the sense that the booleans are not identified by the semantics, and so we are not at all in the situation of the Lafont critical pair of LK which identifies, by conversion, any two proofs of the same formula. However, due the following motivations, we decided to have a deterministic reduction system, i.e, we only allow $\kappa\gamma.d\star\tilde{\kappa}.c \rightarrow c[\kappa\gamma.d/\bullet]$ in our system.

Our motivations is as follows: We are looking for a minimal reduction system so that one can prove normalization. Hence, we equip commands with a deterministic rewriting relation \rightarrow specified in Figure 2.4. By $p \notin \tilde{\kappa}$, we mean p is not $\tilde{\kappa}.d$ for any command d .

As a direct conclusion of Lemma 71 and Lemma 72, we have the *subject reduction* property:

Proposition 74 *If $c \rightarrow^* d$ and $c \vdash \mathcal{N}[P]$, then $d \vdash \mathcal{N}[P]$.*

Remark 23 *Notice that although the reduction system given in Figure 2.4 is quite close to the cut-elimination rules of $\mu\text{LLP}_{\text{imp}}$ as explained above, it does not completely simulate the cut-elimination of $\mu\text{LLP}_{\text{imp}}$ (and μLLP), simply because of the axiom/cut case. Assume that we have the following proof π :*

$$\frac{\frac{}{\vdash N_1, \dots, N_n, P, P^\perp} \text{ (ax)} \quad \frac{\pi_1}{\vdash N_1, \dots, N_n, P} \text{ (cut)}}{\vdash N_1, \dots, N_n, P} \text{ (cut)}$$

This proof reduces to the following proof π_1 :

$$\frac{\pi_1}{\vdash N_1, \dots, N_n, P}$$

However, $\pi^ = \bullet \star \pi_1^*$ which does not certainly reduce to π_1^* . The converse simulation also does not hold in general. That is to say there are some reductions in the $\kappa\mu\text{LLP}$ that are not available in $\mu\text{LLP}_{\text{imp}}$. For instance, consider the command $c = t \star p$ of type $c \vdash \beta : P^\perp \& P^\perp, P \otimes P$ where $t = \kappa\alpha.(\bullet \star (\alpha, \alpha))$ and $p = \tilde{\kappa}.(\langle \bullet, \bullet \rangle \star \beta)$. The negative term t will be translated to the following $\mu\text{LLP}_{\text{imp}}$ proof π_1 :*

$$\frac{\frac{}{\vdash P^\perp \& P^\perp, P^\perp, P \otimes P, P^\perp \wp P^\perp} \text{ (ax)} \quad \frac{\frac{}{\vdash P^\perp \& P^\perp, P^\perp, P} \text{ (ax)} \quad \frac{}{\vdash P^\perp \& P^\perp, P^\perp, P} \text{ (ax)}}{\vdash P^\perp \& P^\perp, P^\perp, P \otimes P} \text{ (cut)}}{\vdash P^\perp \& P^\perp, P^\perp, P \otimes P} \text{ (cut)}$$

The positive term p will be translated to the following $\mu\text{LLP}_{\text{imp}}$ proof π_2

$$\frac{\frac{}{\vdash P^\perp \& P^\perp, P \oplus P} \text{ (ax)} \quad \frac{\frac{}{\vdash P^\perp \& P^\perp, P^\perp, P} \text{ (ax)} \quad \frac{}{\vdash P^\perp \& P^\perp, P^\perp, P} \text{ (ax)}}{\vdash P^\perp \& P^\perp, P, P^\perp \& P^\perp} \text{ (cut)}}{\vdash P^\perp \& P^\perp, P} \text{ (cut)}$$

$$\begin{array}{l}
 s \star \tilde{\kappa}.c \rightarrow c[s/\bullet] \\
 \langle s_1, s_2 \rangle \star \text{in}_i(p) \rightarrow s_i \star p \\
 \kappa_{\perp}.c \star () \rightarrow c \\
 \bar{\kappa}_{R^{\perp}, \zeta} \alpha.(c; s) \star \text{fd}(p) \rightarrow (R[\bar{\kappa}_{R^{\perp}, \zeta} \alpha.c/\zeta^+] \star p)[\kappa \alpha.(c[s/\bullet])/\bullet]
 \end{array}
 \qquad
 \begin{array}{l}
 \kappa \alpha.c \star p \rightarrow c[p/\alpha] \quad \text{if } p \notin \tilde{\kappa} \\
 \kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rightarrow c[p_1/\alpha_1, p_2/\alpha_2] \\
 \text{der } p \star s' \rightarrow s \star p
 \end{array}$$

Figure 2.4: Reduction of commands

Hence, we have the following proof π as the translation of the command c :

$$\frac{\frac{\pi_1}{\vdash P^{\perp} \& P^{\perp}, P^{\perp}, P \otimes P} \quad \frac{\pi_2}{\vdash P^{\perp} \& P^{\perp}, P}}{\vdash P^{\perp} \& P^{\perp}, P \otimes P} \text{ (cut)}$$

And there is no reduction rule after on the proof above in $\mu\text{LLP}_{\text{imp}}$ unless one want to use the contraction rule as an admissible rule in $\mu\text{LLP}_{\text{imp}}$. However, by the reduction system given in Figure 2.4, we have $c = t \star p \rightarrow \langle \bullet, \bullet \rangle \star \beta[t/\bullet] = \langle t, t \rangle \star \beta = d$. The command d will be translated to the following $\mu\text{LLP}_{\text{imp}}$ proof ρ :

$$\frac{\frac{\frac{\pi_1}{\vdash P^{\perp} \& P^{\perp}, P^{\perp}, P \otimes P} \quad \frac{\pi_1}{\vdash P^{\perp} \& P^{\perp}, P^{\perp}, P \otimes P}}{\vdash P^{\perp} \& P^{\perp}, P \otimes P, P^{\perp} \& P^{\perp}} \text{ (\&)} \quad \frac{}{\vdash P^{\perp} \& P^{\perp}, P \oplus P} \text{ (ax)}}{\vdash P^{\perp} \& P^{\perp}, P \otimes P} \text{ (cut)}$$

Therefore, we saw that the reduction $\kappa \alpha.c \star \tilde{\kappa}.d \rightarrow d[\kappa \alpha.c/\bullet]$ is not available in $\mu\text{LLP}_{\text{imp}}$ in general.

Lemma 75 *If $c \vdash \mathcal{N}, P$, $c \rightarrow c'$ and $\vdash \mathcal{N} \mid s : P^{\perp}$, then $c[s/\bullet] \rightarrow c'[s/\bullet]$. If $c \vdash \mathcal{N}, \alpha : P^{\perp}[Q]$, $c \rightarrow c'$ and $\vdash \mathcal{N} \mid p : P$ with $p \notin \tilde{\kappa}$ then $c[p/\alpha] \rightarrow c'[p/\alpha]$.*

Proof: The proof is a simple inspection of the reduction rules, and we just provide details of two cases.

Assume that $c = t \star \tilde{\kappa}.d$ so that $c' = d[t/\bullet]$. Then we have $c[s/\bullet] = (t[s/\bullet]) \star \tilde{\kappa}.d \rightarrow d[t[s/\bullet]/\bullet]$, and $d[t/\bullet][s/\bullet] = d[t[s/\bullet]/\bullet]$. Hence $c[s/\bullet] \rightarrow c'[s/\bullet]$.

One also has $c[p/\alpha] = (t[p/\alpha]) \star ((\tilde{\kappa}.d)[p/\alpha]) \rightarrow (d[p/\alpha])[t[p/\alpha]/\bullet]$, and $(d[t/\bullet])[p/\alpha] = (d[p/\alpha])[t[p/\alpha]/\bullet]$. Hence $c[p/\alpha] \rightarrow c'[p/\alpha]$.

Assume that $c = \kappa \beta.c \star q$ and $c' = c[q/\beta]$ where $q \notin \tilde{\kappa}$. Then we have $(\kappa \beta.c \star q)[s/\bullet] = \kappa \beta.(c[s/\bullet]) \star (q[s/\bullet]) \rightarrow (c[s/\bullet])[q[s/\bullet]/\beta]$, and $(c[q/\beta])[s/\bullet] = (c[s/\bullet])[q[s/\bullet]/\beta]$. Hence $c[s/\bullet] \rightarrow c'[s/\bullet]$. One also has $(\kappa \beta.c \star q)[p/\alpha] = \kappa \beta.(c[p/\alpha]) \star (q[p/\alpha]) \rightarrow c[p/\alpha][(q[p/\alpha])/\beta]$, and $c[q/\beta][p/\alpha] = c[p/\alpha][(q[p/\alpha])/\beta]$. Hence $c[p/\alpha] \rightarrow c'[p/\alpha]$. Notice that $q[p/\alpha] \notin \tilde{\kappa}$, since $p \notin \tilde{\kappa}$. \blacksquare

Example

Before proving a normalization theorem, we try to run some examples here in order to see some programs using our $\kappa\mu\text{LLP}$ calculus.

We define a type of integers as $\mathbf{nat} = \mu\zeta.(1 \oplus \zeta^+)$. We set $\underline{0} = \text{fd}(\text{in}_1())$ so that $\vdash \mathcal{N} \mid \underline{0} : \mathbf{nat}$, and given a positive term p such that $\vdash \mathcal{N} \mid p : \mathbf{nat}$ we set $\text{suc } p = \text{fd}(\text{in}_2(p))$ so that $\vdash \mathcal{N} \mid \text{suc } p : \mathbf{nat}$. Now, let us look at the instance of the $(\mathbf{t}\nu)$ rule for $\mathbf{nat}^\perp = \nu\zeta.(\perp \& \zeta^-)$. Let p, t and s be terms such that $\vdash \mathcal{N}, P \mid t : P^\perp$, $\vdash \mathcal{N} \mid p : P$, and $\vdash \mathcal{N}[Q] \mid s : P^\perp$. Then we build a term as follows:

$$\frac{\frac{\frac{\frac{\frac{\vdash \mathcal{N} \mid p : P}{\vdash \mathcal{N}, \alpha : (\perp \& P^\perp) \mid p : P} \pi_4 \text{ (t-cut)}}{\bullet \star p \vdash \mathcal{N}, \alpha : (\perp \& P^\perp), P} \text{ (t-}\perp\text{)}}{\vdash \mathcal{N}, \alpha : (\perp \& P^\perp), P \mid \kappa_\perp.(\bullet \star p) : \perp} \pi_3 \text{ (t-}\mathfrak{A}\text{)}}{\vdash \mathcal{N}, \alpha : (\perp \& P^\perp), P \mid \langle \kappa_\perp.(\bullet \star p), t \rangle : \perp \& P^\perp} \pi_2 \text{ (t-cut)}}{\frac{(\langle \kappa_\perp.(\bullet \star p), t \rangle) \star \alpha \vdash \mathcal{N}, P, \alpha : (\perp \& P^\perp)}{\vdash \mathcal{N}[Q] \mid \bar{\kappa}_{\perp \& \zeta^-, \zeta} \alpha.(((\langle \kappa_\perp.(\bullet \star p), t \rangle) \star \alpha); s) : \mathbf{nat}^\perp} \pi_1 \text{ (t-}\nu\text{)}}$$

where π_1 is the proof of $\vdash \mathcal{N}[Q] \mid s : P^\perp$, π_2 is the proof of $\vdash \mathcal{N}, \alpha : (\perp \& P^\perp) \mid \alpha : (\perp \& P^\perp)^\perp$ using $(\mathbf{t-n})$ rule, π_3 is $\frac{\vdash \mathcal{N}, P \mid t : P^\perp}{\vdash \mathcal{N}, \alpha : (\perp \& P^\perp), P \mid t : P^\perp}$, and π_4 is the proof of $\vdash \mathcal{N}, \alpha : (\perp \& P^\perp), P \mid \bullet : P^\perp$ using $(\mathbf{t}\bullet)$ rule.

As a notation, we set $\text{it}(p, t; s) = \bar{\kappa}_{\perp \& \zeta^-, \zeta} \alpha.(((\langle \kappa_\perp.(\bullet \star p), t \rangle) \star \alpha); s)$.

In summary, the following results of an instance of the $(\mathbf{t}\nu)$ rule at type \mathbf{nat}^\perp :

$$\frac{\vdash \mathcal{N} \mid p : P \quad \vdash \mathcal{N}, P \mid t : P^\perp \quad \vdash \mathcal{N}[Q] \mid s : P^\perp}{\vdash \mathcal{N}[Q] \mid \text{it}(p, t; s) : \mathbf{nat}^\perp} \mathbf{itr}_{\mathbf{nat}}$$

If we consider the $(\mathbf{t}\nu^1)$ rule instead of $(\mathbf{t}\nu)$, then we would have the following rule where $\text{it}^1(p, t) = \bar{\kappa}_{\perp \& \zeta^-, \zeta} \alpha.(((\langle \kappa_\perp.(\bullet \star p), t \rangle) \star \alpha) = \bar{\kappa}_{R^\perp, \zeta} \alpha.(((\langle \kappa_\perp.(\bullet \star p), t \rangle) \star \alpha); \bullet)$:

$$\frac{\vdash \mathcal{N} \mid p : P \quad \vdash \mathcal{N}, P \mid t : P^\perp}{\vdash \mathcal{N}, P \mid \text{it}^1(p, t) : \mathbf{nat}^\perp} \mathbf{itr}_{\mathbf{nat}}^1$$

So, one can see that $\text{it}^1(p, t) = \text{it}(p, t; \bullet)$.

Now, suppose that we have $\vdash \mathcal{N} \mid \text{suc } q : \mathbf{nat}$ such that q is not a $\tilde{\kappa}$. term. Then, we have the following computation of $\text{it}(p, t; s) \star \text{suc } q$ where $s_1 = \kappa\alpha_1.(\bullet \star \text{in}_1(\tilde{\kappa}.((\bullet \star [\text{it}^1(p, t)/\zeta^+]) \star \alpha_1))) = \kappa\alpha_1.(\bullet \star \text{in}_1(\tilde{\kappa}.(\bullet \star \alpha_1)))$ and $s_2 = \kappa\alpha_2.(\bullet \star \text{in}_2(\tilde{\kappa}.((\zeta^+ [\text{it}^1(p, t)/\zeta^+]) \star \alpha_2))) = \kappa\alpha_2.(\bullet \star \text{in}_2(\tilde{\kappa}.(\text{it}^1(p, t) \star \alpha_2)))$.

$$\begin{aligned}
 & \text{it}(p, t; s) \star \text{succ } q \\
 & \rightarrow ((1 \oplus \zeta^+) \left[\text{it}^1(p, t) / \zeta^+ \right] \star \text{in}_2 q) [\kappa\alpha.(((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) / \bullet] \\
 & = (\langle s_1, s_2 \rangle \star \text{in}_2 q) [\kappa\alpha.(((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) / \bullet] \\
 & \rightarrow s_2 \star q [\kappa\alpha.(((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) / \bullet] \\
 & \rightarrow ((\bullet \star \text{in}_2(\tilde{\kappa}.(\text{it}^1(p, t) \star \alpha_2))) [q/\alpha_2]) [\kappa\alpha.(((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) / \bullet] \\
 & = ((\bullet \star \text{in}_2(\tilde{\kappa}.(\text{it}^1(p, t) \star q)))) [\kappa\alpha.(((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) / \bullet] \\
 & = (\kappa\alpha.(((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) \star \text{in}_2(\tilde{\kappa}.(\text{it}^1(p, t) \star q))) \\
 & \rightarrow (((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star \alpha) [s/\bullet]) \left[(\text{in}_2(\tilde{\kappa}.(\text{it}^1(p, t) \star q))) / \alpha \right] \\
 & = (((\langle \kappa_{\perp}.(\bullet \star p), t \rangle) \star ((\text{in}_2(\tilde{\kappa}.(\text{it}^1(p, t) \star q)))) [s/\bullet] \\
 & \rightarrow t \star (\tilde{\kappa}.(\text{it}^1(p, t) \star q)) [s/\bullet] \rightarrow ((\text{it}^1(p, t) \star q)) [t/\bullet] [s/\bullet] = ((\text{it}(p, t; \bullet) \star q)) [t/\bullet] [s/\bullet] \\
 & = (\text{it}(p, t; t [s/\bullet]) \star q)
 \end{aligned}$$

And one checks similarly that $\text{it}(p, t; s) \star \underline{0}$ reduces to $s \star p$.

As a very first example of function on natural numbers, one can define the successor function Suc as $\text{it}(\text{succ } \underline{0}, \kappa\alpha.(\bullet \star \text{fd}(\text{in}_2(\alpha)))) ; \bullet$. Then, one can show by induction on \underline{n} that $\text{Suc} \star \underline{n} \rightarrow^* \bullet \star \text{succ } \underline{n}$.

As another example of function on natural numbers, one can define the addition function as $\text{add} = \kappa(\alpha_1, \alpha_2).(\bullet \star (\tilde{\kappa}.(\text{it}(\alpha_1, \kappa\gamma.(\bullet \star \text{fd}(\text{in}_2(\gamma)))) ; \bullet) \star \alpha_2))$. Then one can show, by induction on \underline{n} , that $\text{add} \star (\underline{m}, \underline{n}) \rightarrow^* \bullet \star (\underline{m} + \underline{n})$.

One can try translate the Gödel's system \mathbb{T} into $\kappa\mu\text{LLP}$. However, notice that we should translate the type of integers of system \mathbb{T} into $?\text{nat}$ and not nat . Let us explain why. Let us consider a presentation of system \mathbb{T} which has two kinds of types: *data-types* which are associated with positive formulas of $\kappa\mu\text{LLP}$ and *function types* $\sigma \Rightarrow \tau$. In this system \mathbb{T} there will be at least a data-type of integers ι associated with nat and possibly other ones, for instance a type of binary trees with leaves labeled by natural numbers associated with $\mu\zeta.(\text{nat} \oplus (\zeta \otimes \zeta))$. We would like to use the Girard encoding $(\sigma \Rightarrow \tau)^- = !\sigma^- \multimap \tau^-$ where as usual $A \multimap B = A^\perp \wp B$. In other words $(\sigma \Rightarrow \tau)^- = ?(\sigma^-)^\perp \wp \tau^-$. The polarity constraints of $\kappa\mu\text{LLP}$ require σ^- to be negative, this prevents us from setting simply $\iota^- = \text{nat}$ since nat is positive.

For benefiting from the structural rules available for free on all negative formulas in $\kappa\mu\text{LLP}$ one can introduce also a positive translation defined by $\iota^+ = \text{nat}$ (and similarly for all data-types of \mathbb{T}) and $\sigma^+ = !\sigma^-$ if σ is not a data-type and then the negative translation can be defined by $\iota^- = ?\text{nat}$ (and similarly for all data-types) and $(\sigma \Rightarrow \tau)^- = (\sigma^+)^\perp \wp \tau^-$. For instance $((\iota \Rightarrow \iota) \Rightarrow \iota)^- = ?(\text{nat} \otimes !\text{nat}^\perp) \wp ?\text{nat}$. A closed term of type ι of \mathbb{T} will therefore be translated into a $\kappa\mu\text{LLP}$ negative term t such that $\vdash | t : ?\text{nat}$ and we can form $c = t \star \alpha$.

2.2 Normalization of $\kappa\mu\text{LLP}$

In this section, we are trying to prove a normalization theorem for our system. Ideally, we would like to prove that if we have a command $c \vdash \mathcal{N}[P]$, then c is strongly normalizing.

Notice that as we have a deterministic reduction system, there would be no difference between normalization and strong normalization. Before going to the details of our proof, we first provide some intuitions on what are the main difficulties of this theorem for $\kappa\mu\text{LLP}$.

One of the usual techniques to prove such a statement is via logical relations. Vaguely, one needs to define an interpretation for each type as a set of terms of this type, denoted as $|P|$ and $|N|$, in order to prove the following statement, so-called interpretation lemma (imagine for the moment that we are considering an hypothetical system which would also have reductions on terms).

If $\mathcal{N} = (\alpha_1 : N_1, \dots, \alpha_k : N_k)$ and $p_i \in |(N_i)^\perp|$ for $i = 1, \dots, k$, then

1. if $c \vdash \mathcal{N}, [P]$ then $c[p_i/\alpha_i]_{i=1}^n$ is strongly normalizing,
2. if $\vdash \mathcal{N}, [P] \mid t : N$, then $t[p_i/\alpha_i]_{i=1}^n$ is strongly normalizing,
3. if $\vdash \mathcal{N} \mid p : P$, then $p[p_i/\alpha_i]_{i=1}^n$ is strongly normalizing.

Usually, one proves such a statement by induction on proofs. A direct consequence of this lemma, if we have $c \vdash [P]$ then c is normalizing.

First of all, the definition of this interpretation can be neither by induction on types nor by induction on terms: not on types, since we have fixpoint types, and not on term, since we have exponential types. However, one can try to define an indexed version of this interpretation, namely $|P|_\alpha$ for α 's being in a certain class. If we consider ordinal numbers, and define $|P|_\alpha$ by induction on ordinal α , we see that at some point we need an auxiliary lemma whose proof is as difficult as the normalization theorem we are trying to prove. So, we need to define $|P|_\alpha$ by induction on something else. What we do here is that we define $|P|_\alpha$ by induction on points α of the interpretation of P in the relational model of LL. These points are always finite objects, whatever P is, which makes this approach possible. The main goal of Section 2.2.2 is to make this idea precise.

If a reader would like to skip the details of the proofs, he/she can just look at the statement of Theorem 84, and then its application in Section 2.4. To understand fully the statement of Theorem 84, one needs to look at Figure 2.5 and 2.6.

The next section (Section 2.2.1) provides more details of some failed attempts to prove normalization of $\kappa\mu\text{LLP}$, and it will not be related to the other parts of the manuscript. So, a reader can completely skip this section.

2.2.1 Failed attempts

In this section, we consider the system $\kappa\mu\text{LLP}$ extended with a constant \circ which is a new command, typed by $\circ \vdash$. We introduce this \circ only to simplify definitions (because otherwise there are no closed commands) and in particular to allow the simple definition of \perp given below.

Definition 76 *Given a closed positive formula P (respectively closed negative formula N), we use $\text{PR}(P)$ (respectively $\text{PR}(N)$) for the set of all positive terms $\vdash \mid p : P$ (respectively all negative terms $\vdash \mid t : N$). We use \perp for the set of all normalizing command $c \vdash$. Let $\mathcal{R} \subseteq \text{PR}(P)$, we define $\mathcal{R}^\perp \subseteq \text{PR}(P^\perp)$ by*

$$\mathcal{R}^\perp = \{t \in \text{PR}(P^\perp) \mid \forall p \in \mathcal{R}(t \star p \in \perp)\}$$

As usual the following properties hold:

- $\mathcal{R} \subseteq \mathcal{S} \Rightarrow \mathcal{S}^\perp \subseteq \mathcal{R}^\perp$
- $\mathcal{R} \subseteq \mathcal{R}^{\perp\perp}$
- $(\bigcup_{i \in I} \mathcal{R}_i)^\perp = \bigcap_{i \in I} \mathcal{R}_i^\perp$

A reducibility candidate for P (respectively N) is a $\mathcal{R} \subseteq \text{PR}(P)$ (respectively $\mathcal{R} \subseteq \text{PR}(N)$) such that $\mathcal{R} = \mathcal{R}^{\perp\perp}$. We use $\text{RC}(P)$ (respectively $\text{RC}(N)$) for the set of all reducibility candidates of P (respectively N).

Given a formula P , two sequence $\vec{\zeta}^+ = \zeta_1^+, \dots, \zeta_k^+$, $\vec{\zeta}^- = \zeta_1^-, \dots, \zeta_l^-$ of pairwise distinct respectively positive and negative variables which contain all the free variables of P , two sequences S_1, \dots, S_k and N_1, \dots, N_l of respectively positive and negative closed formulae, and two sequences $\mathcal{S}_1, \dots, \mathcal{S}_k$, $\mathcal{N}_1, \dots, \mathcal{N}_l$ of reducibility candidates such that $\mathcal{S}_i \in \text{RC}(S_i)$ for $i = 1, \dots, k$ and $\mathcal{N}_j \in \text{RC}(N_j)$ for $j = 1, \dots, l$, we define $|P|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) \in \text{RC}(P[\vec{S}/\vec{\zeta}^+, \vec{N}/\vec{\zeta}^-])$ by induction on P . For the case of negative formulas we use De Morgan duality along

$$|N|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = (|N^\perp|(\vec{\mathcal{S}}^\perp : \vec{S}^\perp/\vec{\zeta}^-, \vec{\mathcal{N}}^\perp : \vec{N}^\perp/\vec{\zeta}^-))^\perp$$

- $|1|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = \{()\}^{\perp\perp}$.
- $|P_1 \otimes P_2|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = \{(p_1, p_2) \mid p_i \in |P_i|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-)\}^{\perp\perp}$.
- $|0|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = \text{PR}(0)$.
- $|P_1 \oplus P_2|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = \{\text{in}_i(p_i) \mid p_i \in |P_i|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-)\}^{\perp\perp}$.
- $!|N|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = \{t^! \mid t \in |N|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-)\}^{\perp\perp}$.
- $|\mu\xi.P|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-) = \text{lfp}(\Phi)$ where

$$\Phi : \text{RC}(\mu\xi.P[\vec{S}/\vec{\zeta}^+, \vec{N}/\vec{\zeta}^-]) \rightarrow \text{RC}(\mu\xi.P[\vec{S}/\vec{\zeta}^+, \vec{N}/\vec{\zeta}^-])$$

is defined as follows. Let $\mathcal{R} \in \text{RC}(\mu\xi.P[\vec{S}/\vec{\zeta}^+, \vec{N}/\vec{\zeta}^-])$, we set

$$\Phi(\mathcal{R}) = \{\text{fd}(p) \mid p \in |P|(\vec{\mathcal{S}} : \vec{S}/\vec{\zeta}^+, \mathcal{R} : \mu\xi.P[\vec{S}/\vec{\zeta}^+, \vec{N}/\vec{\zeta}^-]/\xi^+, \vec{\mathcal{N}} : \vec{N}/\vec{\zeta}^-)\}^{\perp\perp}$$

Now, we would like to prove the following statement.

Proposition Attempt If $\mathcal{N} = (\alpha_1 : N_1, \dots, \alpha_k : N_k)$ and $p_i \in |N_i^\perp|$ for $i = 1, \dots, k$, then

1. If $c \vdash \mathcal{N}$ then $c[p_i/\alpha_i]_{i=1}^n \in \perp$.
2. If $c \vdash \mathcal{N}, P$ then for all $t \in |P^\perp|$ we have $c[\vec{p}/\vec{\alpha}, t/\bullet] \in \perp$
3. if $\vdash \mathcal{N} \mid t : N$, then $t[p_i/\alpha_i]_{i=1}^n \in |N|$.

4. if $\vdash \mathcal{N}, P \mid t : N$, then $t[\vec{p}/\vec{\alpha}] \in |N|$.
5. if $\vdash \mathcal{N} \mid p : P$, then $p[p_i/\alpha_i]_{i=1}^n \in |P|$.

Proof Attempt If we try to prove this statement by mutual induction on typing derivations, one can see a difficulty in the third case. For instance, assume that $N = N_1 \& N_2$ so that we have $\vdash \mathcal{N} \mid t_i : N_i$ for $i = 1, 2$. By induction hypothesis, one has $t'_i = t_i[p_j/\alpha_j]_{j=1}^n \in |N_i|$ for $i = 1, 2$. So, we have $\forall q \in |N_i^\perp| (t_i \star q \in \perp)$ for $i = 1, 2$. In particular, we have $t_i \star \tilde{\kappa}.c \rightarrow c[t_i/\bullet] \in \perp$ for $\tilde{\kappa}.c \in |N_i^\perp|$. We need to prove $\langle t_1, t_2 \rangle \in |N_1 \& N_2| = |N_1^\perp \oplus N_2^\perp|^\perp$. Take a positive term $\tilde{\kappa}.d \in |N_1^\perp \oplus N_2^\perp|$. By our reduction system (Figure 2.4) one has $\langle t_1, t_2 \rangle \star \tilde{\kappa}.d \rightarrow d[\langle t_1, t_2 \rangle/\bullet]$. However we do not see how to use the induction hypothesis to prove $d[\langle t_1, t_2 \rangle/\bullet] \in \perp$. \otimes

The issue is that the relation between d and c can be complicated. For instance, one can imagine $d = \bullet \star \text{in}_1(\tilde{\kappa}.c)$. In this case, we have $d[\langle t_1, t_2 \rangle/\bullet] = \langle t_1, t_2 \rangle \star \text{in}_1(\tilde{\kappa}.c) \rightarrow t_1 \star \tilde{\kappa}.c \rightarrow c[t_1/\bullet]$, and hence we can use the induction hypothesis to deduce that $d[\langle t_1, t_2 \rangle/\bullet] \in \perp$. But d can also be $\bullet \star \tilde{\kappa}.(\bullet \star \text{in}_i(\tilde{\kappa}.c))$ or $\bullet \star \tilde{\kappa}.(\bullet \star \tilde{\kappa}.(\bullet \star \text{in}_i(\tilde{\kappa}.c)))$ and so on and so forth.

As we see we need some sort of induction on d in order to be able to use the inductive hypothesis on $\langle t_1, t_2 \rangle$. Evidently, this difficulty is in the case that we have a positive term $p \in \tilde{\kappa}$. So, we tried to change a bit our definition in order to implement this required further induction on d , and we describe now this idea. We first provide the following definition for any ordinal α :

- $\perp_\alpha(P) := \{c \mid c \vdash P \text{ and } \exists p \in (|P|_\alpha \setminus \tilde{\kappa}) c \rightarrow^* \bullet \star p\}$

where $|P|_\alpha$ is defined as follows for the successor ordinal:

- $|1|_{\alpha+1} = \{()\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(1)\}$.
- $|P_1 \otimes P_2|_{\alpha+1} = \{(p_1, p_2) \mid p_i \in |P_i|_\alpha\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(P_1 \otimes P_2)\}$.
- $|\top|_{\alpha+1} = \{\langle \rangle\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(\top)\}$.
- $|P_1 \oplus P_2|_{\alpha+1} = \{\text{in}_i(p_i) \mid p_i \in |P_i|_\alpha\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(P_1 \oplus P_2)\}$
- $!N|_{\alpha+1} = \{t^! \mid \forall q \in |N^\perp|_\alpha (t \star q \in \perp)\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(!N)\}$.
- $|\mu\xi.P|_{\alpha+1} = \{\text{fd}(p) \mid p \in |P[\mu\xi.P/\xi^+]|_\alpha\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(\mu\xi.P)\}$.

And for the limit ordinal λ , we take $|P|_\lambda = \bigcup_{\alpha < \lambda} |P|_\alpha$. And we also define $|P| := \bigcup |P|_\alpha$ and $\perp(P) := \bigcup \perp_\alpha(P)$. The interpretation of negative type is defined by duality, that is to say $|N|_{\alpha+1} = (|N^\perp|_{\alpha+1})^\perp$, $|N|_\lambda = \bigcap_{\alpha < \lambda} |N|_\alpha$, and $|N| = \bigcap |N|_\alpha$.

Notice that one cannot define $|P|$ by induction on formula because of basically two reasons. First $\perp(P)$ depends on $|P|$ and $|P|$ depends on $\perp(P)$. Secondly, in the last case, the formula $P[\mu\xi.P/\xi^+]$ might be bigger than $\mu\xi.P$. Moreover, one cannot also define $|P|$ by induction on terms, since in the case of $!N$ the positive term q can be bigger than the negative term t .

Let us say why the problem that we had above for the case of $N = N_1 \& N_2$ has now vanished with these new definitions. Assume that we have $\vdash \mathcal{N} \mid t_i : N_i$ for $i = 1, 2$. By induction hypothesis, we have $t'_i = t_i[p_j/\alpha_j]_{j=1}^n \in |N_i| = \bigcap |N_i|_\alpha$. So, we have $\forall \alpha \forall q \in |N_i^\perp|_\alpha (t_i \star q \in \perp)$ for $i = 1, 2$. We need to show $\langle t_1, t_2 \rangle \in |N_1 \& N_2| = |N_1^\perp \oplus N_2^\perp|^\perp$. This is equivalent to show the following:

$$\forall \alpha \forall p \in |N_1^\perp \oplus N_2^\perp|_\alpha (\langle t_1, t_2 \rangle \star p \in \perp)$$

We prove this statement by induction on α .

Let first assume that α is a successor ordinal $\gamma + 1$. Then either $p = \text{in}_i(p_i)$ for $p_i \in |N_i^\perp|_\gamma$ or $p = \tilde{\kappa}.d$ where $d \in \perp_{N_1^\perp \oplus N_2^\perp}(\gamma)$. If $p = \text{in}_i(p_i)$, then we have $\langle t_1, t_2 \rangle \star \text{in}_i(p_i) \rightarrow t_i \star p_i \in \perp$ by induction hypothesis on $\langle t_1, t_2 \rangle$. If $p = \tilde{\kappa}.d$, then we have $\langle t_1, t_2 \rangle \star \tilde{\kappa}.d \rightarrow d[\langle t_1, t_2 \rangle / \bullet]$. By the assumption, we have $d \in \perp_{N_1^\perp \oplus N_2^\perp}(\gamma)$, so, by definition $d \rightarrow^* \bullet \star p$ for $p \in |N_1^\perp \oplus N_2^\perp|_\gamma$. By Lemma 75, we have $d[\langle t_1, t_2 \rangle / \bullet] \rightarrow^* \bullet \star p[\langle t_1, t_2 \rangle / \bullet] = \langle t_1, t_2 \rangle \star p$. Notice that $p \notin \tilde{\kappa}$, so $p = \text{in}_i(r)$ where $r \in |N_i^\perp|_{\gamma'}$ for $\gamma' \leq \gamma < \gamma + 1$. Therefore we can use the induction hypothesis on γ' , and hence we have $\langle t_1, t_2 \rangle \star \text{in}_i(r) \rightarrow t_i \star r \in \perp$.

Now assume that α is a limit ordinal. The property holds obviously for any $p \in |N_1^\perp \oplus N_2^\perp|_\gamma$ for $\gamma < \alpha$ by the induction hypothesis. Hence, the property holds for any $p \in \bigcap_{\gamma < \alpha} |N_1^\perp \oplus N_2^\perp|_\gamma$.

However, we still have an issue even with this idea. To see the problem, let us try to compute $|P|$ where $P = \mu\zeta.!\zeta$.

- $|P|_{\alpha+1} = \{\text{fd}(p) \mid p \in |!P|_\alpha\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(P)\}$
- $|!P|_{\alpha+1} = \{t^! \mid \forall q \in |!P^\perp|_\alpha (t \star q \in \perp)\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(!P)\}$
- $|!P^\perp|_{\alpha+1} = \{s^! \mid \forall r \in |P|_\alpha (s \star r \in \perp)\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha(!P^\perp)\}$

We said earlier that for the limit ordinal λ , we take $|P|_\lambda = \bigcup_{\alpha < \lambda} |P|_\alpha$. However, to take the union for the limit ordinal, one needs to provide an increasing sequence of $|P|_\alpha$'s. And due to the definition above, one consequently needs to have an increasing sequence $|!P|_\alpha$'s. And to do so, the sequence $|!P^\perp|_\alpha$'s must be decreasing. So, although $!P^\perp$ is a positive formula, we should take $|!P^\perp|_\lambda = \bigcap_{\alpha < \lambda} |!P^\perp|_\alpha$ for the limit ordinal λ . Therefore, what we learnt from this example is that we need to consider both increasing and decreasing sequences $|P|_\alpha$'s for a positive type P (and similarly for negative type N).

First we define $\perp_\alpha^\sigma(P)$ for a positive type P and an ordinal α where $\sigma \in \{\uparrow, \downarrow\}$:

- $\perp_\alpha^\sigma(P) := \{c \mid c \vdash P \text{ and } \exists p \in (|P|_\alpha^\sigma \setminus \tilde{\kappa}) c \rightarrow^* \bullet \star p\}$

And $|P|_\alpha^\sigma$ is described as follows. Given a positive type P and an ordinal α , we define two sequences $|P|_\alpha^\uparrow$ and $|P|_\alpha^\downarrow$ as follows by induction on α . For the successor ordinal, we take the following definition where $-\sigma$ changes the direction of the arrow::

- $|1|_{\alpha+1}^\sigma = \{()\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha^\sigma(1)\}$.
- $|P_1 \otimes P_2|_{\alpha+1}^\sigma = \{(p_1, p_2) \mid p_i \in |P_i|_\alpha^\sigma\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha^\sigma(P_1 \otimes P_2)\}$.
- $|\top|_{\alpha+1}^\sigma = \{\langle \rangle\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha^\sigma(\top)\}$.
- $|P_1 \oplus P_2|_{\alpha+1}^\sigma = \{\text{in}_i(p_i) \mid p_i \in |P_i|_\alpha^\sigma\} \cup \{\tilde{\kappa}.d \mid d \in \perp_\alpha^\sigma(P_1 \oplus P_2)\}$

- $!N|_{\alpha+1}^{\sigma} = \{t^! \mid \forall q \in |N^{\perp}|_{\alpha}^{-\sigma}(t \star q \in \perp)\} \cup \{\tilde{\kappa}.d \mid d \in \perp_{\alpha}^{\sigma}(!N)\}$.
- $|\mu\xi.P|_{\alpha+1}^{\sigma} = \{\text{fd}(p) \mid p \in |P|_{\alpha}^{\sigma}[\mu\zeta.P/\zeta^+]\} \cup \{\tilde{\kappa}.d \mid d \in \perp_{\alpha}^{\sigma}(\mu\xi.P)\}$.

For the limit ordinal α we will take the following:

$$\begin{cases} |P|_{\alpha}^{\uparrow} = \bigcup_{\beta < \alpha} |P|_{\beta}^{\uparrow} \\ |P|_{\alpha}^{\downarrow} = \bigcap_{\beta < \alpha} |P|_{\beta}^{\downarrow} \end{cases}$$

And finally, we take $|P|^{\uparrow} = \bigcup |P|_{\alpha}^{\uparrow}$ and $|P|^{\downarrow} = \bigcap |P|_{\alpha}^{\downarrow}$. We also need to change the statement of the interpretation lemma as follows:

Proposition Attempt If $\mathcal{N} = (\alpha_1 : N_1, \dots, \alpha_k : N_k)$ and $p_i \in |N_i^{\perp}|^{\uparrow}$ for $i = 1, \dots, k$, then

1. If $c \vdash \mathcal{N}$ then $c[p_i/\alpha_i]_{i=1}^n \in \perp$.
2. If $c \vdash \mathcal{N}, P$ then for all $t \in |P^{\perp}|^{\downarrow}$ we have $c[\vec{p}/\vec{\alpha}, t/\bullet] \in \perp$.
3. If $\vdash \mathcal{N} \mid t : N$, then $t[p_i/\alpha_i]_{i=1}^n \in |N|^{\downarrow}$.
4. If $\vdash \mathcal{N}, P \mid t : N$, then $t[\vec{p}/\vec{\alpha}] \in |N|^{\downarrow}$.
5. If $\vdash \mathcal{N} \mid p : P$, then $p[p_i/\alpha_i]_{i=1}^n \in |P|^{\uparrow}$.

Now, if we try to prove our main statement (the interpretation lemma), we will see that one needs to verify that $|P|^{\uparrow} = |P|^{\downarrow}$. To see this, let us try to prove the fifth case of the interpretation lemma when $P = !N$ so that we have $\vdash \mathcal{N} \mid t^! : !N$. By the induction hypothesis, we have $t' = t[p_i/\alpha_i]_{i=1}^n \in |N|^{\downarrow} = (|N^{\perp}|^{\uparrow})^{\downarrow} = (\bigcup |N^{\perp}|_{\alpha}^{\uparrow})^{\downarrow}$. Hence we have $t' \star q \in \perp$ for any $q \in \bigcup |N^{\perp}|_{\alpha}^{\uparrow}$. We need to show $(t')^! \in |!N|^{\uparrow} = \bigcup |!N|_{\alpha}^{\uparrow}$. That is to say, one needs to show that there is an ordinal α such that $(t')^! \in |!N|_{\alpha+1}^{\uparrow}$. By definition, this is equivalent to prove that $t' \star q \in \perp$ for any $q \in |N^{\perp}|_{\alpha}^{\downarrow}$. Hence, as we see, to use the induction hypothesis, it is enough to prove that $|N^{\perp}|_{\alpha}^{\downarrow} \subseteq \bigcup_{\beta < \alpha} |N^{\perp}|_{\beta}^{\uparrow}$ for any α . That is to say, we need to prove $|N^{\perp}|^{\downarrow} = |N^{\perp}|^{\uparrow}$. Similarly, we need to show $|P|^{\uparrow} \subseteq |P|^{\downarrow}$ for the third case of the interpretation lemma when $N = ?P$.

One can prove $|P|^{\uparrow} \subseteq |P|^{\downarrow}$ by induction on P . As an instance, let us suppose that $P = !N$, and take a positive term $p \in |!N|^{\uparrow}$. By definition, there is an ordinal β such that $p \in |!N|_{\beta+1}^{\uparrow}$. One needs to prove that $p \in |!N|_{\alpha}^{\downarrow}$ for all ordinal α . We prove this by induction on α . Assume that $p = t^!$, so we have $t \star q \in \perp$ for all $q \in |N^{\perp}|_{\beta}^{\downarrow}$. Now suppose that we have $p \in |!N|_{\gamma}^{\downarrow}$, and we want to show $p \in |!N|_{\gamma+1}^{\downarrow}$. To do so, it is enough to prove that $t \star q \in \perp$ for all $q \in |N^{\perp}|_{\gamma}^{\uparrow}$. Take an arbitrary $q \in |N^{\perp}|_{\gamma}^{\uparrow}$. Then $q \in |N^{\perp}|^{\uparrow}$. Moreover $q \in |N^{\perp}|^{\downarrow}$, since since we have $|N^{\perp}|^{\uparrow} \subseteq |N^{\perp}|^{\downarrow}$ by the induction hypothesis. Therefore $q \in |N^{\perp}|_{\delta}^{\downarrow}$ for any ordinal γ , and in particular $q \in |N^{\perp}|_{\beta}^{\downarrow}$. Hence $t \star q \in \perp$. If $p = \tilde{\kappa}.d$, then by definition $d \in \perp_{\beta}^{\uparrow}(!N)$. So, $d \rightarrow^* \bullet \star q$ such that $q \in |!N|_{\beta}^{\uparrow}$. To show $\tilde{\kappa}.d \in |!N|_{\gamma+1}^{\downarrow}$, we need to prove $d \in \perp_{\gamma}^{\downarrow}(!N)$. That is to say $d \rightarrow^* \bullet \star q$ such that $q \in |!N|_{\gamma}^{\downarrow}$. And this is true, since $q \in |!N|_{\beta}^{\uparrow}$ implies that $q \in |!N|_{\gamma}^{\downarrow}$ by induction hypothesis on the ordinal. The case of the limit ordinal can be treated in a similar manner. Notice that in order to deal with the case that $P = \mu\xi.P$, we cannot

simply prove it by induction on P . One needs to make the statement stronger in order to use an inductive proof. However, this is not the main issue, and this can be solved by considering the following statement: let P be a positive formula and π be a type valuation, then one has $|P[\pi]|^\uparrow \subseteq |P[\pi]|^\downarrow$.

However, the other direction ($|P|^\downarrow \subseteq |P|^\uparrow$), seems more difficult, let us show why. As usual, we want to prove $|P|^\downarrow \subseteq |P|^\uparrow$ by induction on P (to have simpler notations, we just drop the substitution with the type valuation π). Assume that $P = \mu\xi.Q$. Take a positive term $p = \tilde{\kappa}.d \in |\mu\xi.Q|^\downarrow$. So, we have $p \in |\mu\xi.Q|_{\alpha+1}^\downarrow$ for all ordinal α . That is to say $d \rightarrow^* \bullet \star \text{fd}(q)$ such that $\text{fd}(q) \in |\mu\xi.Q|_\alpha^\downarrow$. We now need to show $p \in |\mu\xi.Q|^\uparrow$. So, one needs to prove that there is an ordinal γ such that $p \in |\mu\xi.Q|_{\gamma+1}^\uparrow$. This is equivalent to prove that $d \rightarrow^* \bullet \star \text{fd}(q)$ such that $\text{fd}(q) \in |\mu\xi.Q|_\gamma^\uparrow$. However, we do not see any evidence on how to prove $\text{fd}(q) \in |\mu\xi.Q|_\gamma^\uparrow$. Especially, imagine that $q = \tilde{\kappa}.c$ such that $c \rightarrow^* \bullet \star p$. In this case, proving $p \in |\mu\xi.Q|_{\gamma+1}^\uparrow$ is reduced to proving $p \in |\mu\xi.Q|_\beta^\uparrow$ for some ordinal $\beta \leq \gamma$. However, the sequence $|\mu\xi.Q|_\alpha^\uparrow$ is an increasing sequence, so $|\mu\xi.Q|_\beta^\uparrow$ can be strictly smaller than $|\mu\xi.Q|_{\gamma+1}^\uparrow$. Of course, if one can prove that there is no such a positive term $\tilde{\kappa}.d$ reducing to $\bullet \star \text{fd}(q)$ in many steps such that $q = \tilde{\kappa}.c$ and $c \rightarrow^* \bullet \star p$, then we can manage to prove this direction $|P|^\downarrow \subseteq |P|^\uparrow$. But the bad news is that apparently disproving the existence of this positive term $\tilde{\kappa}.d$ is the same as proving normalization. So we got stuck here!



Let us summarize what the main parts of our failure attempts are in order to understand the system $\kappa\mu\text{LLP}$ better:

1. We first tried the reducibility candidates method in a similar way as proving normalization of second order linear logic [Gir87a]: our failure was due to the existence of $\tilde{\kappa}$ terms. The normalization of μLL is indeed proved by this reducibility candidates method in [Bae12]. However, we do not see such $\tilde{\kappa}$ terms (proofs) in μLL , and the existence of such positive terms in our calculus is due to polarization.
2. To solve the first issue, we provided another possible definition of interpretation of formulas, and also a new set $\perp_\alpha(P)$ for any positive formula P and ordinal α . We defined these notions by induction on α due to the existence of fixpoints of types, exponentials, and again $\tilde{\kappa}$ terms.
3. Afterward, we saw that indeed one needs to consider both increasing and decreasing sequences of $|P|_\alpha$'s to deal with the limit ordinals. The main reason for this is actually the changes of polarity in formulas as we saw the issue in the example $\mu\zeta.!?\zeta$.

In one sentence, we can say that apparently the difficulty is due to the coexistence of polarization and fixpoints of types.

Remark 24 *It is worth mentioning that maybe the difficulty of the normalization proof depends only on the presentation of the syntax. For instance, as suggested by one of the reviewer (Olivier Laurent), one can consider an intuitionistic style of our system $\kappa\mu\text{LLP}$ and then try to prove the normalization theorem using the very first failed attempt or the methods mentioned in [Mat98, Mat99b, Mat99a] (Indeed, if we do not have fixpoint, then our second failed attempt proved the normalization of the resulting system). Based on this intuitionistic style, one can also try to develop a natural deduction system corresponding to $\kappa\mu\text{LLP}$, however it is not clear what rule we should consider for the Park's rule in the case of natural deduction. Nevertheless, the method that we consider in this manuscript is based on two denotational models (**Rel** and **Nuts**), in which we first show that, by induction on the derivations in the intersection type system given in Section 2.2.2, if the interpretation of a term is non empty then it terminates. Then using a denotational totality argument, we show that the interpretation of a term is not empty.*

2.2.2 An intersection typing system

The syntax of points

Let I be an infinite and countable set of indices (we can take $I = \mathbb{N}$). The syntax of *relational types* or *points* is given in Figure 2.5, as well as the size $\text{sz}(a)$ of a point, which is an integer ≥ 1 . Given a point a and a literal ξ , we define a finite subset $\text{rg}_\xi(a)$ of I as follows:

- $\text{rg}_\xi^* = \emptyset$, $\text{rg}_\xi(a, b) = \text{rg}_\xi a \cup \text{rg}_\xi b$, $\text{rg}_\xi(j, \xi) = \{j\}$, $\text{rg}_\xi(j, \zeta) = \emptyset$ if $\zeta \neq \xi$, $\text{rg}_\xi(i, a) = \text{rg}_\xi a$, $\text{rg}_\xi[a_1, \dots, a_n] = \bigcup_{i=1}^n \text{rg}_\xi a_i$, and $\text{rg}_\xi \sigma(a) = \text{rg}_\xi a$.

We give a typing system for these points in Figure 2.5. Its main purpose is to enforce that, when $a : P$, given a literal ζ , the indices of I associated with ζ in a are pairwise distinct.

Given $a^0 : R$, a literal ζ and a family of points $\vec{b} = (b_j)_{j \in \text{rg}_\zeta a^0}$ such that $b_j : P$ for all j and such that for any literal ξ the sets $\text{rg}_\xi b_j$ are pairwise disjoint and disjoint from $\text{rg}_\xi a^0$ (when these disjointness conditions hold we say that the pair (a^0, \vec{b}) is *adapted*) then we define in the obvious way the point $a^0 \{b_j / (j, \zeta)\}_{j \in J}$ for $J = \text{rg}_\zeta a^0$ such that $a^0 \{b_j / (j, \zeta)\}_{j \in J} : R [P/\zeta^+]$. One proves easily that, for any literal $\xi \neq \zeta$

$$\text{rg}_\xi a^0 \{b_j / (j, \zeta)\}_{j \in J} = \text{rg}_\xi a^0 \uplus \bigsqcup_{j \in \text{rg}_\zeta a^0} \text{rg}_\xi b_j. \quad (2.1)$$

Crucially, this point substitution is in some sense reversible.

Lemma 77 *Let R and P be positive formulas and let ζ be a literal. Let a be a point such that $a : R [P/\zeta^+]$. Let $J \subseteq I$ be an infinite set. There is a point a^0 such that $a^0 : R$ and $\text{rg}_\zeta a^0 \subseteq J$ and there is a family of points $\vec{b} = (b_j)_{j \in \text{rg}_\zeta a^0}$ such that $b_j : P$ for all $j \in \text{rg}_\zeta a^0$, (a^0, \vec{b}) is adapted and $a = a^0 \{b_j / (j, \zeta)\}_{j \in \text{rg}_\zeta a^0}$.*

Proof: By induction on a assuming that we have a derivation of $a : R [P/\zeta^+]$. We consider several cases, according to the shape of R .

$$a, b, \dots := * \mid (j, \zeta) \mid (a, b) \mid (i, a) \mid [a_1, \dots, a_n] \mid \sigma(a) \quad \text{with } j \in I \text{ and } i \in \{1, 2\}$$

$$\text{sz}(*) = \text{sz}(j, \zeta) = 1 \quad \text{sz}(a, b) = \text{sz}(a) + \text{sz}(b)$$

$$\text{sz}(i, a) = \text{sz}(\sigma(a)) = 1 + \text{sz}(a) \quad \text{sz}([a_1, \dots, a_n]) = 1 + \sum_{i=1}^n \text{sz}(a_i)$$

$$\frac{}{(j, \zeta) : \zeta} \text{ (p-var)} \quad \frac{}{* : 1} \text{ (p-1)} \quad \frac{a : P \quad b : Q \quad \forall \xi (\text{rg}_\xi a \cap \text{rg}_\xi b = \emptyset)}{(a, b) : P \otimes Q} \text{ (p-}\otimes\text{)}$$

$$\frac{a : P [\mu\zeta.P/\zeta^+]}{\sigma(a) : \mu\zeta.P} \text{ (p-}\mu\text{)} \quad \frac{a : P_i}{(i, a) : P_1 \oplus P_2} \text{ (p-}\oplus\text{)}$$

$$\frac{(a_i : N^\perp)_{i=1}^n \quad \forall \xi \forall i \neq i' (\text{rg}_\xi a_i \cap \text{rg}_\xi a_{i'} = \emptyset)}{[a_1, \dots, a_n] : !N} \text{ (p-!)}$$

$$\frac{\frac{k \in \mathbb{N}}{* \tilde{1} k [*]} \text{ (s-1)} \quad \frac{a^i \tilde{P}_i [a_1^i, \dots, a_n^i] \text{ for } i = 1, 2}{(a^1, a^2) \widetilde{P_1 \otimes P_2} [(a_1^1, a_1^2), \dots, (a_n^1, a_n^2)]} \text{ (s-}\otimes\text{)}}{} \text{ (s-!)}$$

$$\frac{\underline{k} = \uplus_{i=1}^n J_i \quad (a_j : N^\perp)_{j=1}^k}{[a_1, \dots, a_k] \widetilde{!N} [[a_j \mid j \in J_i] \mid i = 1, \dots, n]} \text{ (s-!)}$$

$$\frac{i \in \{1, 2\} \quad a \tilde{P}_i [a_1, \dots, a_n]}{(i, a) \widetilde{P_1 \oplus P_2} [(i, a_1), \dots, (i, a_n)]} \text{ (s-}\oplus\text{)} \quad \frac{a R [\mu\zeta.\widetilde{R}/\zeta^+] [a_j \mid j \in J]}{\sigma(a) \widetilde{\mu\zeta.R} [\sigma(a_j) \mid j \in J]} \text{ (s-}\mu\text{)}$$

Figure 2.5: Syntax, size, typing rules and structural relation for points

▷ If $R = \zeta^+$ we choose $j \in J$ and set $a^0 = (j, \zeta)$, $b_j = a$. This is possible because J is infinite and hence non-empty.

▷ If $R = \xi^+ \neq \zeta^+$ we must have $a = (j, \xi)$ and we set $a^0 = a$. In that case we have $\text{rg}_\zeta a^0 = \emptyset$ and so we have no b_k 's to define.

▷ If $R = 1$ we must have $a = *$ and we take $a^0 = a$. As before $\text{rg}_\zeta a^0 = \emptyset$ and so we have no b_k 's to define.

▷ If R is $R_1 \otimes R_2$ so that $a = (a_1, a_2)$ with $a_i : R_i [P/\zeta^+]$ for $i = 1, 2$. Let $J_1, J_2 \subseteq J$ be disjoint and infinite. By inductive hypothesis, for $i = 1, 2$, we can find $a_i^0 : R$ with $\text{rg}_\zeta a_i^0 \subseteq J_i$ as well as families $\vec{b}(i) = (b(i)_j)_{j \in \text{rg}_\zeta a_i^0}$ such that $\forall j b(i)_j : P$, $(a_i^0, \vec{b}(i))$ is adapted and $a_i = a_i^0 \{b(i)_j / (j, \zeta)\}_{j \in \text{rg}_\zeta a_i^0}$. For $j \in \text{rg}_\zeta a^0 = \text{rg}_\zeta a_1^0 \uplus \text{rg}_\zeta a_2^0$ we set $b_j = b(i)_j$ where $i \in \{1, 2\}$ is uniquely determined by $j \in \text{rg}_\zeta a_i^0$, defining $\vec{b} = (b_j)_{j \in \text{rg}_\zeta a^0}$. Let ξ be a literal, $j, j' \in \text{rg}_\xi a^0$ with $j \neq j'$ and let $K = \text{rg}_\xi b_j \cap \text{rg}_\xi b_{j'}$. If $j, j' \in \text{rg}_\xi a_i^0$ for $i = 1$ or $i = 2$ then $K = \emptyset$ since $(a_i^0, \vec{b}(i))$ is adapted. If $j \in \text{rg}_\xi a_i^0$ and $j' \in \text{rg}_\xi a_{3-i}^0$ then $K = \emptyset$ because we know that $\text{rg}_\xi b_j \subseteq \text{rg}_\zeta a_i$: this is due to the fact that, by inductive hypothesis $a_i = a_i^0 \{b_k / (k, \zeta)\}_{k \in \text{rg}_\zeta a_i^0}$. Similarly $\text{rg}_\xi b_{j'} \subseteq \text{rg}_\zeta a_{3-i}$ and moreover $\text{rg}_\zeta a_i \cap \text{rg}_\zeta a_{3-i} = \emptyset$ because $a : R_1 [P/\zeta^+] \otimes R_2 [P/\zeta^+]$. For the same reason, if $j \in \text{rg}_\zeta a_i^0$ we have $\text{rg}_\xi b(i)_j \cap \text{rg}_\zeta a_{3-i}^0 = \emptyset$. It follows that the pair (a^0, \vec{b}) is adapted. The fact that $a = a^0 \{b_j / (j, \zeta)\}_{j \in \text{rg}_\zeta a^0}$ is an immediate consequence of the inductive hypothesis.

▷ The case $R = !R_0$ (and hence $a = [a_1, \dots, a_n]$) is dealt with similarly (applying the inductive hypothesis to the a_i 's).

▷ The case $R = R_1 \oplus R_2$ is straightforward: we have $a = (i, a_0)$ with $a_0 : R_i [P/\zeta^+]$ for $i = 1$ or $i = 2$ and the inductive hypothesis directly applies to a_0 .

▷ Last if $R = \mu\xi.Q$ with $\xi \neq \zeta$ and ξ does not occur in P , so that $a = \sigma(a_0)$ and that we have a derivation of $a_0 : R'$ where $R' = Q [P/\zeta^+] [\mu\xi.Q [P/\zeta^+]/\xi^+] = Q [R/\xi^+] [P/\zeta^+]$. By inductive hypothesis applied to a_0 there is $a_0^0 : Q [R/\xi^+]$ and a family $(c_j)_{j \in \text{rg}_\xi(a_0^0)}$ satisfying the required properties wrt. a^0 . We take $a^0 = \sigma(a_0^0)$ so that $\text{rg}_\zeta a^0 = \text{rg}_\zeta a_0^0$ and we set $b_j = c_j$ for each j in that set. ■

Given a closed positive type P , we define a binary relation \tilde{P} between points a and multisets of points $[a_1, \dots, a_n]$ where $a, a_1, \dots, a_n : P$. The definition is provided as a deduction system in Figure 2.5. The role of this relation is to deal with the structural rules in the point deduction system of Fig 2.6. Notice that in each of the deduction rules the sum of the sizes of the points occurring on the left in the premises is strictly smaller than the size of the point occurring on the left in the conclusion. So the size of such a deduction tree is upper-bounded by the size of the point occurring on the left in its conclusion.

2.2.3 The point typing system

A *negative point typing context* is a sequence $\Phi = (\alpha_1 : a_1 : N_1, \dots, \alpha_k : a_k : N_k)$ where the α_i 's are pairwise distinct and $a_i : N_i^\perp$ for each i . A *positive point typing context* is a sequence $\Phi, a : P$ with $a : P$. In these rules we use Φ to denote the context $(\alpha_i : a_i : P_i^\perp)_{i=1}^n$ and \mathcal{N} for the ordinary typing context $(\alpha_i : P_i^\perp)_{i=1}^n$. All rules but $(\mathbf{i}\nu)$ are given in Figure 2.6. Notice

$$\begin{array}{c}
 \frac{(a_i \widetilde{P}_i [])_{i \in \underline{n} \setminus \{j\}}}{\vdash \Phi \mid \alpha_j : a_j : P_j} \text{ (i-n)} \quad \frac{(a_i \widetilde{P}_i [])_{i=1}^n}{\vdash \Phi \mid () : * : 1} \text{ (i-1)} \\
 \\
 \frac{(\vdash (\alpha_i : a_i^j : P_i^\perp)_{i=1}^n \mid p_j : b_j : Q_j)_{j=1,2} \quad (a_i \widetilde{P}_i [a_i^1, a_i^2])_{i=1}^n}{\vdash \Phi \mid (p_1, p_2) : (b_1, b_2) : Q_1 \otimes Q_2} \text{ (i-}\otimes\text{)} \\
 \\
 \frac{\vdash \Phi \mid p : a : Q_i}{\vdash \Phi \mid \text{in}_i(p) : (i, a) : Q_1 \oplus Q_2} \text{ (i-}\oplus\text{)} \quad \frac{c \vdash \Phi, a : P}{\vdash \Phi \mid \tilde{\kappa}.c : a : P} \text{ (i-}\tilde{\kappa}\text{)} \quad \frac{\vdash \Phi \mid p : a : P [\mu\zeta.P/\zeta^+]}{\vdash \Phi \mid \text{fd}(p) : \sigma(a) : \mu\zeta.P} \text{ (i-}\mu\text{)} \\
 \\
 \frac{(\vdash (\alpha_i : a_i^j : P_i^\perp)_{i=1}^n \mid s : b_j : N)_{j \in J} \quad (a_i \widetilde{P}_i [a_i^j \mid j \in J])_{i=1}^n}{\vdash \Phi \mid s^! : [b_j \mid j \in J] : !N} \text{ (i-!)} \\
 \\
 \frac{(a_i \widetilde{P}_i [])_{i=1}^n}{\vdash \Phi, a : P \mid \bullet : a : P^\perp} \text{ (i-}\bullet\text{)} \quad \frac{c \vdash \Phi, a : P}{\vdash \Phi, a : P \mid \kappa_\perp.c : * : \perp} \text{ (i-}\perp\text{)} \\
 \\
 \frac{c \vdash \Phi, \alpha : a : N[b, P]}{\vdash \Phi, b : P \mid \kappa\alpha.c : a : N} \text{ (i-}\kappa\text{)} \quad \frac{c \vdash \Phi, \alpha_1 : a_1 : N_1, \alpha_2 : a_2 : N_1[b, P]}{\vdash \Phi, b : P \mid \kappa(\alpha_1, \alpha_2).c : (a_1, a_2) : N_1 \wp N_2} \text{ (i-}\wp\text{)} \\
 \\
 \frac{\vdash \Phi, a : P \mid s_i : b : N_i \quad \vdash \mathcal{N}[P] \mid s_{3-i} : N_{3-i}}{\vdash \Phi, a : P \mid \langle s_1, s_2 \rangle : (i, b) : N_1 \& N_2} \text{ (i-}\&\text{)} \quad \frac{\vdash \Phi \mid p : a : P}{\vdash \Phi \mid \text{der } p : [a] : ?P} \text{ (i-?)} \\
 \\
 \frac{\vdash (\alpha_i : a_i^1 : P_i^\perp)[, a : P] \mid s : b : P^\perp \quad \vdash (\alpha_i : a_i^2 : P_i^\perp) \mid p : b : P \quad (a_i \widetilde{P}_i [a_i^1, a_i^2])_{i=1}^n}{(s \star p) \vdash \Phi, a : P} \text{ (i-cut)}
 \end{array}$$

 Figure 2.6: Point deduction system — the rule (i- ν) is given in the body of Section 2.2.3.

that there are two instances of the rules $(\mathbf{i}\oplus)$ and $(\mathbf{i}\&)$, one for $i = 1$ and one for $i = 2$. We give now the lacking $(\mathbf{i}\nu)$ inference rule: if $h : Q$, $d : R$, $(b_l : P)_{l \in L}$ where $L = \text{rg}_\zeta d$, and

- $\vdash (\alpha_i : a_i'' : P_i^\perp)_{i=1}^n [h : Q] \mid s : b : P^\perp$
- $c \vdash (\alpha_i : a_i' : P_i^\perp)_{i=1}^n, b : P, \alpha : d\{b_l/(l, \zeta)\}_{l \in L} : R^\perp \left[P^\perp / \zeta^- \right]$
- $(\vdash (\alpha_i : a_i^l : P_i^\perp)_{i=1}^n, b_l : P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : f_l : \nu \zeta.R^\perp)_{l \in L}$
- $a_i \widetilde{P}_i [a_i', a_i''] + [a_i^l \mid l \in L]$ for $i = 1, \dots, n$

then $\vdash \Phi [h : Q] \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : \sigma(d)\{f_l/(l, \zeta)\}_{l \in L} : \nu \zeta.R^\perp$.

Upon taking $s = \bullet$ we obtain the following derived rule $(\mathbf{i}\nu^1)$ (with the same notations as above). If

- $c \vdash (\alpha_i : a_i' : P_i^\perp)_{i=1}^n, b : P, \alpha : d\{b_l/(l, \zeta)\}_{l \in L} : R^\perp \left[P^\perp / \zeta^- \right]$
- $(\vdash (\alpha_i : a_i^l : P_i^\perp)_{i=1}^n, b_l : P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : f_l : \nu \zeta.R^\perp)_{l \in L}$
- $a_i \widetilde{P}_i [a_i'] + [a_i^l \mid l \in L]$ for $i = 1, \dots, n$

then $\vdash \Phi, b : P \mid \bar{\kappa}_{R^\perp, \zeta} \alpha.c : \sigma(d)\{f_l/(l, \zeta)\}_{l \in L} : \nu \zeta.R^\perp$.

We will provide an example of our point deduction system in Section 2.2.4. The typing system can seem complicated but it is completely imposed by the relational semantics, we had actually no choice.

Interpretation of points

Given a set \mathcal{P} of positive terms p such that $\vdash p : P$, we set

- $\mathcal{P}^\bullet = \{c \mid c \vdash P \text{ and } \exists p \in (\mathcal{P} \setminus \tilde{\kappa}) c \rightarrow^* \bullet \star p\}$
- $\mathcal{P}^{\tilde{\kappa}} = \mathcal{P} \cup \{\tilde{\kappa}.c \mid c \in \mathcal{P}^\bullet\}$

so that any $p \in \mathcal{P}^{\tilde{\kappa}}$ satisfies $\vdash p : P$. We set $\perp = \text{SN}$, that is all commands c such that there is no command d that $c \rightarrow d$. Given $a : P$ we define $\perp(a : P) = |a|_{P^\bullet}$ and we set $|a|_P = \|\!|a|\!\|_P^{\tilde{\kappa}}$ where

- $\|\!|\ast\!\|_1 = \{()\}$
- $\|\!(a_1, a_2)\!\|_{P_1 \otimes P_2} = \{(p_1, p_2) \mid p_i \in |a_i|_{P_i} \text{ for } i = 1, 2\}$
- $\|\!(i, a)\!\|_{P_1 \oplus P_2} = \{\text{in}_i(p) \mid p \in |a|_{P_i}\}$ for $i = 1, 2$
- $\|\![a_j \mid j \in J]\!\|_{!N} = \{t^! \mid \forall j \in J \forall p \in |a_j|_{N^\perp} t \star p \in \perp\}$
- $\|\!\sigma(a)\!\|_{\mu\zeta.P} = \{\text{fd}(p) \mid p \in |a|_{P[\mu\zeta.P/\zeta^+]}\}$.

This definition is a simple definition by induction on points, even if recursive calls involve larger μLLP formulas as parameters as in the last case. Notice that $\|a\|_P = |a|_P \setminus \tilde{\kappa}$ since $\|a\|_P \cap \tilde{\kappa} = \emptyset$. As an auxiliary notion, given $a : P$ and $b : N^\perp$ we set $|b|_N(a : P) = \{s \mid \vdash P \mid s : N \text{ and } \forall p \in |b|_{N^\perp} s \star p \in \perp(a : P)\}$. We will also use $|b|_N$ for the set of s such that $\vdash s : N$ and $\forall p \in |b|_{N^\perp} s \star p \in \perp$. The following lemma is a tree-typed version of the functoriality of formulas with respect to the proofs.

Lemma 78 *Let $a : P$, $b : Q$ and s be such that $\vdash P \mid s : Q^\perp$. If $\forall p \in \|b\|_Q s \star p \in \perp(a : P)$ then $s \in |b|_{Q^\perp}(a : P)$.*

Proof: Let $p = \tilde{\kappa}.c \in |b|_Q$ (because of the assumption that we have in the statement of the lemma, we only have to consider the case where $p \in \tilde{\kappa}$). This means that $c \rightarrow^* \bullet \star q$ with $q \in |b|_Q \setminus \tilde{\kappa} = \|b\|_Q$. It follows that $s \star p \rightarrow c[s/\bullet] \rightarrow^* (\bullet \star q)[s/\bullet] = s \star q \in \perp(a : P)$ (notice that \bullet cannot occur free in q since q is a typed positive term). It follows that $s \star p \in \perp(a : P)$. ■

Lemma 79 *If a $\tilde{P} [a_1, \dots, a_n]$ then $\|a\|_P \subseteq \|a_i\|_P$ and $|a|_P \subseteq |a_i|_P$ for $1 \leq i \leq n$.*

Proof: By induction on the derivation of a $\tilde{P} [a_1, \dots, a_n]$ we prove that $\|a\|_P \subseteq \|a_i\|_P$ which implies the announced inclusions.

- If the derivation consists of (s-1) then we have $P = 1$ and $\forall i a = a_i = *$ so that the statement obviously holds.
- If the derivation ends with (s- \otimes) then $P = P_1 \otimes P_2$, $a = (a^1, a^2)$, $a_i = (a_i^1, a_i^2)$ and $a^j \tilde{P}_j [a_1^j, \dots, a_n^j]$ for $j = 1, 2$. The inductive hypothesis gives $\|a^j\|_{P_j} \subseteq \|a_i^j\|_{P_j}$ for $j = 1, 2$ and for each i whence the announced inclusion by definition of $\|(a^1, a^2)\|_{P_1 \otimes P_2}$.
- The case where the last rule is (s- \oplus) is similar.
- If the last rule is (s-!) then $P = !N$, $a = [b_1, \dots, b_k]$, $a_i = [b_j \mid j \in J_i]$ with $\underline{k} = \uplus_{i=1}^n J_i$. Let $p \in \|a\|_P$ so that $p = s^!$ with $\forall j \in \underline{k} \forall q \in |b_j|_{N^\perp} s \star q \in \perp$. So *a fortiori* for each $i \in \underline{n}$ one has $\forall j \in J_i \forall q \in |b_j|_{N^\perp} s \star q \in \perp$, that is $p \in \|a_i\|_P$.
- If the last rule is (s- μ) so that $P = \mu\zeta.Q$, $a = \sigma(b)$, $a_i = \sigma(b_i)$ for $i \in \underline{n}$ and we have $b \widetilde{Q[P/\zeta^+]} [b_1, \dots, b_n]$. Let $p \in |a|_P$, which means that $p = \text{fd}(q)$ with $q \in |b|_{Q[P/\zeta^+]}$. For $i \in \underline{n}$ we have $q \in \|b_i\|_{Q[P/\zeta^+]}$ by inductive hypothesis and hence $p \in \|a_i\|_P$. ■

Lemma 80 *Let P be a closed positive formula and let $a : P$. If $b : P$, one has a $\tilde{P} [b]$ iff $a = b$. Let $(b_j : P)_{j \in J}$, $(J_k)_{k \in K}$ be such that $\uplus_{k \in K} J_k = J$. Then one has a $\tilde{P} [b_j \mid j \in J]$ iff there is $(c_k : P)_{k \in K}$ such that a $\tilde{P} [c_k \mid k \in K]$ and $c_k \tilde{P} [b_j \mid j \in J_k]$ for each $k \in K$.*

Proof: Straightforward induction on the size of a . ■

Definition 81 We know that $a \tilde{P} [a]$, and since $[a] = [a] + []$, there are $a', e \in P$, by Lemma 80, such that $a \tilde{P} [a', e]$, $a \tilde{P} P[a]$, and $e \tilde{P} []$. Therefore we have $a = a'$. This implies in particular $\exists e : P \tilde{P} []$. Such an e is called a *conutral point* of P .

By Lemma 80, if $a \tilde{P} [e, b]$ and e is conutral then $a = b$ (by taking $J_1 = \emptyset$, $J_2 = \{1\}$, $K = \{1, 2\}$, $c_1 = e$, $c_2 = b$, and $b_2 = b$), and if $a : P$ there is a conutral e such that $a \tilde{P} [e, a]$ (which depends generally on a). Conutral points are generally not unique: for instance in $1 \oplus 1$, both $(1, *)$ and $(2, *)$ are conutral.

The following lemma is functoriality of our point deduction system. We saw functoriality of $\kappa\mu\text{LLP}$ in Section 2.1.3, and we will see the semantical functoriality in Lemma 87.

Lemma 82 Let Q be a positive formula, η a literal, π a closed type valuation such that all free type variables of Q , but possibly η^+, η^- , are in $\text{dom}(\pi)$. Let $d : Q[\pi]$, assume that $\vdash (\alpha_i : a_i^j : N_i)_{i=1}^n, b_j : P \mid s : c_j : N$ for each $j \in \text{rg}_\eta d$. If $(a_i : N_i^\perp)_{i=1}^n$ are such that for all $i \in \underline{n}$ one has $a_i \widetilde{N_i^\perp} [a_i^j \mid j \in \text{rg}_\eta d]$. Then

$$\vdash (\alpha_i : a_i : N_i), d\{b_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q[\pi, P/\eta^+] \mid Q[\pi, s/\eta^+] : d\{c_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q^\perp[\pi^\perp, N/\eta^-] \quad (2.2)$$

and

$$\vdash (\alpha_i : a_i : N_i), d\{c_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q[\pi, N/\eta^-] \mid Q[\pi, s/\eta^-] : d\{b_j/(j, \eta)\}_{j \in \text{rg}_\eta d} : Q^\perp[\pi^\perp, P/\eta^+] \quad (2.3)$$

Proof: By induction on the pairs (Q, d) , ordered lexicographically, following the definition of substitution of terms in formulas in Section 2.1.3. It is important to notice that the universal quantification on π is part of the statement we prove by induction. We set $J = \text{rg}_\eta d$.

▷ If $Q = 1$ then $d = *$, $J = \emptyset$, $Q[\pi, s/\eta^+] = \bullet$, $d\{b_j/(j, \eta)\}_{j \in J} = d\{c_j/(j, \eta)\}_{j \in J} = *$ and $(a_i \widetilde{P_i} [])_{i=1}^n$. We have $\vdash (\alpha_i : a_i : N_i)_{i=1}^n, * : 1 \mid \bullet : * : \perp$ as required.

▷ The case $Q = \zeta^+ \neq \eta^+$ is similar.

▷ If $Q = \eta^+$, $d = (j, \eta)$ for some $j \in I$ then $J = \{j\}$ and hence $d\{b_k/(k, \eta)\}_{k \in J} = b_j$ and $d\{c_k/(k, \eta)\}_{k \in J} = c_j$. Moreover we have $(a_i^j \widetilde{P_i} [a_i])_{i=1}^n$ so that $a_i^j = a_i$ for each $i \in \underline{n}$ by Lemma 80. We have $Q[\pi, s/\eta^+] = s$ and the first conclusion is identical to the typing assumption on s . Since $Q[\pi, s/\eta^-] = \bullet$ the second conclusion is obtained as in the previous cases.

▷ If $Q = Q_1 \otimes Q_2$ then $d = (d_1, d_2)$ with $(d_k : Q_k[\pi])_{k=1,2}$. Let $K = \{1, 2\}$ and $J_k = \text{rg}_\eta d_k$ for $k \in K$ so that $J = J_1 \uplus J_2$. By Lemma 80, for each $i \in \underline{n}$ and $k \in K$ there is $a(k)_i : N_i^\perp$ such that $(a(k)_i \widetilde{N_i^\perp} [a_i^j \mid j \in J_k])_{i=1}^n$ for $k \in K$ and $(a_i \widetilde{N_i^\perp} [a(1)_i, a(2)_i])_{i=1}^n$. By inductive

hypothesis we have

$$\begin{aligned} \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, d_k\{b_j/(j, \eta)\}_{j \in J_k} : Q \left[\pi, P/\eta^+ \right] \mid \\ : Q_k \left[\pi, s/\eta^+ \right] : d_k\{c_j/(j, \eta)\}_{j \in J_k} Q^\perp \left[\pi^\perp, N/\eta^- \right] \end{aligned} \quad (2.4)$$

for $k \in K$. Pick some coneutral $(e_i : N_i)_{i=1}^n$ and $f_k : Q_k \left[\pi, N^\perp/\eta^+ \right]$, we have

$$\begin{aligned} \vdash (\alpha_i : e_i : N_i)_{i=1}^n, \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k^\perp \left[\pi^\perp, N/\eta^- \right], \beta_{3-k} : f_{3-k} : Q_{3-k}^\perp \left[\pi^\perp, N/\eta^- \right] \mid \\ : \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} Q_k \left[\pi, N^\perp/\eta^+ \right] \end{aligned} \quad (2.5)$$

Since $(a(k)_i \widetilde{N}_i^\perp [e_i, a(k)_i])_{i=1}^n$ we get the following by the rules **(i-cut)** and **(i-n)**

$$\begin{aligned} s_k \star \beta_k \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, d_k\{b_j/(j, \eta)\}_{j \in J_k} : Q_k \left[\pi, P/\eta^+ \right], \\ \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k^\perp \left[\pi^\perp, N/\eta^- \right], \beta_{3-k} : f_{3-k} : Q_{3-k}^\perp \left[\pi^\perp, N/\eta^- \right] \end{aligned} \quad (2.6)$$

where $s_k = Q_k \left[\pi, s/\eta^+ \right]$ and consequently

$$\begin{aligned} \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \beta_k : d_k\{c_j/(j, \eta)\}_{j \in J_k} : Q_k^\perp \left[\pi^\perp, N/\eta^- \right], \beta_{3-k} : f_{3-k} : Q_{3-k}^\perp \left[\pi^\perp, N/\eta^- \right] \mid \\ : \tilde{\kappa}.(s_k \star \beta_k) : d_k\{b_j/(j, \eta)\}_{j \in J_k} Q \left[\pi, P/\eta^+ \right] \end{aligned} \quad (2.7)$$

Using the fact that $(a_i \widetilde{N}_i^\perp [a(1)_i, a(2)_i])_{i=1}^n$ and the coneutrality of f_1, f_2 we get

$$\begin{aligned} \vdash (\alpha_i : a_i : N_i)_{i=1}^n, \beta_1 : d_1\{c_j/(j, \eta)\}_{j \in J_1} : Q_1^\perp \left[\pi^\perp, N/\eta^- \right], \beta_2 : d_2\{c_j/(j, \eta)\}_{j \in J_2} : Q_2^\perp \left[\pi^\perp, N/\eta^- \right] \mid \\ : (\tilde{\kappa}.(s_1 \star \beta_1), \tilde{\kappa}.(s_2 \star \beta_2)) : (d_1\{b_j/(j, \eta)\}_{j \in J_1}, d_2\{b_j/(j, \eta)\}_{j \in J_2}) Q \left[\pi, P/\eta^+ \right] \end{aligned} \quad (2.8)$$

Since we have $d\{b_j/(j, \eta)\}_{j \in J} = (d_1\{b_j/(j, \eta)\}_{j \in J_1}, d_2\{b_j/(j, \eta)\}_{j \in J_2})$ and similarly for $d\{c_j/(j, \eta)\}_{j \in J}$, we finally get

$$\begin{aligned} \vdash (\alpha_i : a_i : N_i)_{i=1}^n, d\{b_j/(j, \eta)\}_{j \in J} : Q \left[\pi, P/\eta^+ \right] \mid \kappa(\beta_1, \beta_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \beta_1), \\)\tilde{\kappa}.(s_2 \star \beta_2)) : d\{c_j/(j, \eta)\}_{j \in J} : Q^\perp \left[\pi^\perp, N/\eta^- \right] \end{aligned} \quad (2.9)$$

as contended.

▷ The case $Q = Q_1 \oplus Q_2$ is similar. The second conclusion is obtained in the same way.

▷ Assume that $Q = !R^\perp$, so that $d = [d_k \mid k \in K]$ with $d_k : R \left[\pi^\perp \right]$. For $k \in K$ let $J_k = \text{rg}_\eta d_k$ so that $J = \biguplus J_k$. By Lemma 80, for each $i \in \underline{n}$ and $k \in K$ there is $a(k)_i : N_i^\perp$ such that $(a(k)_i \widetilde{N}_i^\perp [a_i^j \mid j \in J_k])_{i=1}^n$ for $k \in K$ and $(a_i \widetilde{N}_i^\perp [a(k)_i \mid k \in K])_{i=1}^n$. By inductive hypothesis, for each $k \in K$ we have

$$\vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, d_k\{c_j/(j, \eta)\}_{j \in J_k} : R \left[\pi, N/\eta^- \right] \mid t : d_k\{b_j/(j, \eta)\}_{j \in J_k} : R^\perp \left[\pi^\perp, P/\eta^+ \right] \quad (2.10)$$

where $t = R[\pi, s/\eta^-]$, so

$$\begin{aligned} \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \beta : d_k\{b_j/(j, \eta)\}_{j \in J_k} : R^\perp[\pi^\perp, P/\eta^+] \mid \\ : \tilde{\kappa}.(t \star \beta) : d_k\{c_j/(j, \eta)\}_{j \in J_k} R[\pi, N/\eta^-] \end{aligned} \quad (2.11)$$

So

$$\begin{aligned} \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \beta : d_k\{b_j/(j, \eta)\}_{j \in J_k} : R^\perp[\pi^\perp, P/\eta^+] \mid \\ : \text{der } \tilde{\kappa}.(t \star \beta) : [d_k\{c_j/(j, \eta)\}_{j \in J_k}]?R[\pi, N/\eta^-] \end{aligned} \quad (2.12)$$

and hence

$$\begin{aligned} \vdash (\alpha_i : a(k)_i : N_i)_{i=1}^n, \gamma : [d_k\{b_j/(j, \eta)\}_{j \in J_k}] : ?R[\pi, N/\eta^-] \mid \\ : \kappa\beta.(\text{der } \tilde{\kappa}.(t \star \beta) \star \gamma) : d_k\{c_j/(j, \eta)\}_{j \in J_k} R^\perp[\pi^\perp, P/\eta^+] \end{aligned} \quad (2.13)$$

Since $(a_i \widetilde{N_i^\perp} [a(k)_i \mid k \in K])_{i=1}^n$ and $[d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] !R^\perp[\widetilde{\pi^\perp}, \widetilde{N^\perp}/\eta^+] [[d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K]$ we have

$$\begin{aligned} \vdash (\alpha_i : a_i : N_i)_{i=1}^n, \gamma : [d_k\{b_j/(j, \eta)\}_{j \in J_k} \mid k \in K] : ?R[\pi, N/\eta^-] \mid \\ : (\kappa\beta.(\text{der } \tilde{\kappa}.(t \star \beta) \star \gamma))^\dagger : [d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] !R^\perp[\pi^\perp, P/\eta^+] \end{aligned} \quad (2.14)$$

and therefore we have

$$\begin{aligned} \vdash (\alpha_i : a_i : N_i)_{i=1}^n, [d_k\{c_j/(j, \eta)\}_{j \in J_k} \mid k \in K] : !R^\perp[\pi^\perp, P/\eta^+] \mid \\ : \kappa\gamma.(\bullet \star \kappa\beta.(\text{der } \tilde{\kappa}.(t \star \beta) \star \gamma)^\dagger) : [d_k\{b_j/(j, \eta)\}_{j \in J_k} \mid k \in K]?R[\pi, N/\eta^-] \end{aligned} \quad (2.15)$$

as required. The second conclusion is dealt with similarly.

▷ Assume that $Q = \mu\zeta.R$. We set $R_P = R[P/\eta^+]$ and $R_N = R[N^\perp/\eta^+]$. We have $d = \sigma(d^0)$ with $d^0 : R[\pi][Q[\pi]/\zeta^+]$. By Lemma 77 we can find $f : R[\pi]$ as well as a family $(d_l : Q[\pi])_{l \in L}$ where $L = \text{rg}_\zeta f$ such that $d^0 = f\{d_l/(l, \zeta)\}_{l \in L}$. Let $J = \text{rg}_\eta d$, we have $J = J' \uplus \uplus_{l \in L} J_l$ where $J' = \text{rg}_\eta f$ and $J_l = \text{rg}_\eta d_l$ for each $l \in L$. By Lemma 80 we can find $a'_i, a_i^l : N_i^\perp$ such that $a_i \widetilde{N_i^\perp} [a'_i] + [a_i^l \mid l \in L]$ and $a'_i \widetilde{N_i^\perp} [a_i^j \mid j \in J']$ and $a_i^l \widetilde{N_i^\perp} [a_i^j \mid j \in J_l]$, for all $i = 1, \dots, n$. Let $l \in L$. We set $f_l = d_l\{b_j/(j, \eta)\}_{j \in J_l}$ and $g_l = d_l\{c_j/(j, \eta)\}_{j \in J_l}$. By inductive hypothesis (since $\text{sz } d_l \leq \text{sz } d^0 < \text{sz } d$) we have, for all $l \in L$,

$$\vdash (\alpha_i : a_i^l : N_i)_{i=1}^n, f_l : \mu\zeta.R_P[\pi] \mid Q[\pi, s/\eta^+] : g_l : \nu\zeta.R_N^\perp[\pi^\perp] \quad (2.16)$$

Let $\rho = \pi \cdot (\mu\zeta.R_P[\pi]/\zeta^+)$. Notice that all the free variables of R , but possibly η^+, η^- are in $\text{dom}(\rho)$. Let $d^1 = f\{f_l/(l, \zeta)\}_{l \in L}$ so that $d^1 : R[\rho]$ since $f : R[\pi]$ and $f_l : \mu\zeta.R_P[\pi]$. Notice that $\text{rg}_\eta d^1 = J'$. We apply the inductive hypothesis to (R, d^1) and get

$$\begin{aligned} \vdash (\alpha_i : a_i^l : N_i)_{i=1}^n, d^1\{b_j/(j, \eta)\}_{j \in J'} : R[\rho, P/\eta^+] \mid \\ : R[\rho, s/\eta^+] : d^1\{c_j/(j, \eta)\}_{j \in J'} R^\perp[\rho^\perp, N/\eta^-] \end{aligned} \quad (2.17)$$

Notice that $R[\rho, P/\zeta^+] = R_P[\pi][\mu\zeta.R_P[\pi]/\zeta^+]$ and $R[\rho, N^\perp/\zeta^+] = R_{N^\perp}[\pi][\mu\zeta.R_P[\pi]/\zeta^+]$ so that

$$\begin{aligned} \bullet \star \text{fd}(\tilde{\kappa}.(R[\rho, s/\eta^+] \star \alpha)) \vdash (\alpha_i : a'_i : N_i)_{i=1}^n, \sigma(d^1\{b_j/(j, \eta)\}_{j \in J'}) : \mu\zeta.R_P[\pi], \\ \alpha : d^1\{c_j/(j, \eta)\}_{j \in J'} : R_N[\pi]^\perp [(\mu\zeta.R_P[\pi])^\perp/\zeta^-] \end{aligned} \quad (2.18)$$

We have

$$\begin{aligned} \sigma(d^1\{b_j/(j, \eta)\}_{j \in J'}) &= d\{b_j/(j, \eta)\}_{j \in J} \\ d^1\{c_j/(j, \eta)\}_{j \in J'} &= f\{c_j/(j, \eta)\}_{j \in J'}\{f_l/(l, \zeta)\}_{l \in L} \end{aligned}$$

hence by (2.16) and (2.18) applying rule $(\mathbf{i}\nu^1)$ and using also the fact that $a_i \widetilde{N}_i^\perp [a'_i] + [a'_i \mid l \in L]$, we get

$$\begin{aligned} \vdash (\alpha_i : a_i : N_i)_{i=1}^n, d\{b_j/(j, \eta)\}_{j \in J} : \mu\zeta.R_P[\pi] \mid \\ : \mu\zeta.R[\pi, s/\eta^+] : \sigma(f)\{c_j/(j, \eta)\}_{j \in J'}\{g_l/(l, \zeta)\}_{l \in L} \nu\zeta.(R_N[\pi])^\perp \end{aligned} \quad (2.19)$$

and notice that $\sigma(f)\{c_j/(j, \eta)\}_{j \in J'}\{g_l/(l, \zeta)\}_{l \in L} = d\{c_j/(j, J)\}_{j \in \text{rg}_\eta d}$ since $g_l = d_l\{c_j/(j, \eta)\}_{j \in J}$ and $d = \sigma(f)\{d_l/(l, \zeta)\}_{l \in L}$; the announced statement is proven. For the second conclusion we proceed similarly. \blacksquare

Notice that we do not need this current format of Lemma 82. However we at least need to have Lemma 82 without the points just to say that $Q[\pi, s/\eta^+]$ and $Q[\pi, s/\eta^-]$ are well-typed, and moreover it shows some sort of completeness of this realizability semantics for $Q[\pi, s/\eta^+]$ and $Q[\pi, s/\eta^-]$.

Lemma 83 *Let Q be a positive formula, η a literal, π a closed type valuation such that all free type variables of Q , but possibly η^+, η^- , are in $\text{dom}(\pi)$ and let $d : Q[\pi]$. If $s \in |c_j|_N(b_j : P)$ for all $j \in J = \text{rg}_\eta d$ then $Q[\pi, s/\eta^+] \in |d(\vec{c})|_{Q^\perp[\pi^\perp, N/\eta^-]}(d(\vec{b}) : Q[\pi, P/\eta^+])$ where $d(\vec{b}) = d\{b_j/(j, \eta)\}_{j \in \text{rg}_\eta d}$ and similarly for $d(\vec{c})$. And we have $Q[\pi, s/\eta^-] \in |d(\vec{b})|_{Q^\perp[\pi^\perp, P/\eta^+]}(d(\vec{c}) : Q[\pi, N/\eta^-])$.*

Proof: By induction on the pairs (Q, d) , ordered lexicographically, following the definition of substitution of terms in formulas in Section 2.1.3.

▷ The cases $Q = 1$ and $Q = \eta^+$ are trivial.

▷ If $Q = Q_1 \otimes Q_2$ then $d = (d_1, d_2)$ with $(d_k : Q_k[\pi])_{k=1,2}$ and $J = J_1 \uplus J_2$ where $J_k = \text{rg}_\eta d_k$ and by inductive hypothesis we have $s_k = Q_k[\pi, s/\eta^+] \in |g_k|_{N_k}(f_k : P_k)$ where $f_k = d_k(\vec{b})$, $N_k = Q_k^\perp[\pi^\perp, N/\eta^-]$, $g_k = d_k(\vec{c})$ and $P_k = Q_k[\pi, P/\eta^+]$ for $k = 1, 2$. It suffices to prove that $t \in |(g_1, g_2)|_{N_1 \wp N_2}((f_1, f_2) : P_1 \otimes P_2)$ where $t = \kappa(\beta_1, \beta_2).(\bullet \star (\tilde{\kappa}.(s_1 \star \beta_1), \tilde{\kappa}.(s_2 \star \beta_2)))$. We use Lemma 78 so let $q_k \in |g_k|_{Q_k^\perp}$ for $k = 1, 2$, we have $t \star (q_1, q_2) \rightarrow (\bullet \star (\tilde{\kappa}.(s_1 \star q_1), \tilde{\kappa}.(s_2 \star q_2)))$. We have $s_k \star q_k \in \perp(f_k : P_k)$ and hence $(\tilde{\kappa}.(s_1 \star q_1), \tilde{\kappa}.(s_2 \star q_2)) \in \|(f_1, f_2)\|_{P_1 \otimes P_2}$ so that $\bullet \star (\tilde{\kappa}.(s_1 \star q_1), \tilde{\kappa}.(s_2 \star q_2)) \in \perp((f_1, f_2) : P_1 \otimes P_2)$ and hence $t \star (q_1, q_2) \in \perp((f_1, f_2) : P_1 \otimes P_2)$. The case $Q = Q_1 \oplus Q_2$ is similar.

\triangleright If $Q = !R^\perp$ then $d = [d_1, \dots, d_k]$ with $J = \uplus_{l=1}^k J_l$ where $J_l = \text{rg}_\eta d_l$ for each $l \in \underline{k}$. By inductive hypothesis $s' = R[\pi^\perp, s/\eta^-] \in |d_l(\vec{b})|_{R^\perp[\pi, P/\eta^+]}$ ($d_l(\vec{c}) : R[\pi^\perp, N/\eta^-]$) for each $l \in \underline{k}$ and we must prove that $t \in |[d_l(\vec{c}) \mid l \in L]|_{?R[\pi^\perp, N/\eta^-]}$ ($[d_l(\vec{b}) \mid l \in L] : !R^\perp[\pi, P/\eta^+]$) where $t = !R^\perp[\pi, s/\eta^+] = \kappa\beta.(\bullet \star (\kappa\alpha.(\text{der } \tilde{\kappa}.(s' \star \alpha) \star \beta)))$ where $s' = R[\pi^\perp, s/\eta^-]$ and for this we apply Lemma 78. Let $s_1 \in \bigcap_{l \in L} |d_l(\vec{c})|_{R^\perp[\pi, N^\perp/\eta^+]}$, it suffices to prove that $t \star s_1^\perp \in \perp\!\!\!\perp([d_l(\vec{b}) \mid l \in L] : !R^\perp[\pi, P/\eta^+])$. We have $t \star s_1^\perp \rightarrow \bullet \star (\kappa\alpha.(\text{der } \tilde{\kappa}.(s' \star \alpha) \star s_1^\perp))^\perp$ so it is sufficient to prove $(\kappa\alpha.(\text{der } \tilde{\kappa}.(s' \star \alpha) \star s_1^\perp))^\perp \in |[d_l(\vec{b}) \mid l \in L]|_{!R^\perp[\pi, P/\eta^+]}$, and hence it is enough to prove that $\kappa\alpha.(\text{der } \tilde{\kappa}.(s' \star \alpha) \star s_1^\perp) \in \bigcap_{l \in L} |d_l(\vec{b})|_{R^\perp[\pi, P/\eta^+]}$. So let $q \in \bigcup_{l \in L} |d_l(\vec{b})|_{R[\pi^\perp, P^\perp/\eta^-]}$, it suffices to prove that $(\text{der } \tilde{\kappa}.(s' \star q) \star s_1^\perp) \in \perp\!\!\!\perp$. Let $l \in L$ such that $q \in |d_l(\vec{b})|_{R[\pi^\perp, P^\perp/\eta^-]}$, since $s' \in |d_l(\vec{b})|_{R^\perp[\pi, P/\eta^+]}$ ($d_l(\vec{c}) : R[\pi^\perp, N/\eta^-]$) we have $s' \star q \in \perp\!\!\!\perp(d_l(\vec{c}) : R[\pi^\perp, N/\eta^-])$ and hence $\tilde{\kappa}.(s' \star q) \in |d_l(\vec{c})|_{R[\pi^\perp, N/\eta^-]}$ and this holds for this particular $l \in L$, that is $\tilde{\kappa}.(s' \star q) \in \bigcup_{l \in L} |d_l(\vec{c})|_{R[\pi^\perp, N/\eta^-]}$. Since $s_1 \in \bigcap_{l \in L} |d_l(\vec{c})|_{R^\perp[\pi, N^\perp/\eta^+]}$ it follows that $s_1 \star \tilde{\kappa}.(s' \star q) \in \perp\!\!\!\perp$, hence $\text{der } \tilde{\kappa}.(s' \star q) \star s_1^\perp \in \perp\!\!\!\perp$ as expected since $\text{der } \tilde{\kappa}.(s' \star q) \star s_1^\perp \rightarrow s_1 \star \tilde{\kappa}.(s' \star q)$.

\triangleright Assume that $Q = \mu\zeta.R$. We set $R_P = R[P/\eta^+]$ and $R_N = R[N^\perp/\eta^+]$, $Q_P = Q[P/\eta^+]$ and $Q_N = Q[N^\perp/\eta^+]$, and also $t = Q[\pi, s/\eta^+]$. We also use $\rho = \pi \cdot (Q_P[\pi]/\zeta^+)$. We have $d = \sigma(d^0)$ with $d^0 : R[\pi][Q[\pi]/\zeta^+]$. By Lemma 77 we can find $f : R[\pi]$ as well as a family $(d_l : Q[\pi])_{l \in L}$ where $L = \text{rg}_\zeta f$ such that $d^0 = f\{d_l/(l, \zeta)\}_{l \in L}$. Let $J = \text{rg}_\eta d$, we have $J = J' \uplus \uplus_{l \in L} J_l$ where $J' = \text{rg}_\eta f$ and $J_l = \text{rg}_\eta d_l$ for each $l \in L$. By inductive hypothesis applied to (Q, d_l) (since $\text{sz } d_l \leq \text{sz } d^0 < \text{sz } d$) we have $t \in |g_l|_{Q_N^\perp[\pi^\perp]}$ ($f_l : Q_P[\pi]$) where $f_l = d_l(\vec{b})$ and $g_l = d_l(\vec{c})$ for each $l \in L$, since $\text{sz } d_l \leq \text{sz } d^0 < \text{sz } d$. Notice that $f(\vec{c})\{g_l/(l, \zeta)\}_{l \in L} = d^0(\vec{c})$ by definition of the g_l 's. We must prove that $t \in |d(\vec{c})|_{Q_N^\perp[\pi^\perp]}$ ($d(\vec{b}) : Q_P[\pi]$) so let $p \in \|d(\vec{c})\|_{Q_N[\pi]}$, it suffices to prove that $t \star p \in \perp\!\!\!\perp(d(\vec{b}) : Q_P[\pi])$. We have $p = \text{fd}(q)$ with $q \in |d^0(\vec{c})|_{R_N[\pi, Q_N[\pi]/\zeta^+]}$. We have (see Section 2.1.3) $t = Q[\pi, s/\eta^+] = \bar{\kappa}_{Q_N^\perp[\pi^\perp], \zeta} \alpha.c$ where $c = \bullet \star \text{fd}(\tilde{\kappa}.(s' \star \alpha))$ where $s' = R[\rho, s/\eta^+]$. So $t \star p \rightarrow (R_N[\pi, t/\zeta^+] \star q) [\kappa\alpha.c/\bullet]$.

Notice that $f(\vec{c}) = f\{c_j/(j, \eta)\}_{j \in J'}$ satisfies $f(\vec{c}) : R_N[\pi]$ and hence by inductive hypothesis applied to $(R, f(\vec{c}))$ we have that $R_N[\pi, t/\zeta^+]$ belongs to the set

$$|f(\vec{c})\{g_l/(l, \zeta)\}_{l \in L}|_{R_N^\perp[\pi^\perp, Q_N^\perp[\pi^\perp]/\zeta^-]}(f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L} : R_N[\pi, Q_P[\pi]/\zeta^+])$$

since we have seen that $t \in |g_l|_{Q_N^\perp[\pi^\perp]}$ ($f_l : Q_P[\pi]$) for each $l \in L$. Since $q \in |d^0(\vec{c})|_{R_N[\pi, Q_N[\pi]/\zeta^+]}$ and $d^0(\vec{c}) = f(\vec{c})\{g_l/(l, \zeta)\}_{l \in L}$ it follows that $c' = R_N[\pi, t/\zeta^+] \star q \in \perp\!\!\!\perp(f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L} : R_N[\pi, Q_P[\pi]/\zeta^+])$ which means $c' \rightarrow^* \bullet \star r$ for some $r \in \|f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L}\|_{R_N[\pi, Q_P[\pi]/\zeta^+]}$ and therefore $c' [\kappa\alpha.c/\bullet] \rightarrow^* \kappa\alpha.c \star r = \kappa\alpha.(\bullet \star \text{fd}(\tilde{\kappa}.(s' \star \alpha))) \star r \rightarrow \bullet \star \text{fd}(\tilde{\kappa}.(s' \star r))$. Notice next that $f' = f\{f_l/(l, \zeta)\}_{l \in L} : R[\pi, Q_P[\pi]/\zeta^+] = R[\rho]$. Hence by induction hypothesis applied to (R, f') and since $f'(\vec{c}) = f(\vec{c})\{f_l/(l, \zeta)\}_{l \in L}$ we have $s' \in |f'(\vec{c})|_{R^\perp[\rho^\perp, N/\eta^-]}$ ($f'(\vec{b}) : R[\rho, P/\eta^+]$) and hence $s' \star r \in \perp\!\!\!\perp(f'(\vec{b}) : R[\rho, P/\eta^+])$. Notice that $f'(\vec{b}) = d^0(\vec{b})$ and hence we have $\tilde{\kappa}.(s' \star r) \in |d^0(\vec{b})|_{R[\rho, P/\eta^+]}$ and therefore, by definition of ρ , $\text{fd}(\tilde{\kappa}.(s' \star r)) \in \|d(\vec{b})\|_{Q_P[\pi]}$. So we have $\bullet \star \text{fd}(\tilde{\kappa}.(s' \star r)) \in \perp\!\!\!\perp(d(\vec{b}) : Q_P[\pi])$ and hence $t \star p \in \perp\!\!\!\perp(d(\vec{b}) :$

$Q_P [\pi]$) as contended.

▷ The second statement of the lemma is proven similarly in the same induction of course since the case $Q = !R^\perp$ (change of polarity) for a given statement uses the other one as an inductive hypothesis. ■

2.2.4 Example: the integers

We continue the example initiated in Section 2.1.4. There is a bijection from \mathbb{N} to the points $a : \text{nat}$ which maps 0 to $\bar{0} = \sigma(1, *)$ and $n + 1$ to $\overline{n+1} = \sigma(2, \bar{n})$. With these notations the following rules are derivable in the point typing system.

- If $(a_i \widetilde{N}_i [])_{i=1}^n$ then $\vdash (\alpha_i : a_i : P_i)_{i=1}^n \mid \bar{0} : \text{nat}$.
- If $\vdash \Phi \mid p : \bar{k} : \text{nat}$ then $\vdash \Phi \mid \text{succ } p : \overline{k+1} : \text{nat}$.
- If $\vdash (\alpha_i : a_i^1 : N_i)_{i=1}^n \mid p : a : P, \vdash (\alpha_i : a_i^2 : N_i)_{i=1}^n [b : Q] \mid a : s : P^\perp$ and $(a_i \widetilde{N}_i^\perp [a_i^1, a_i^2])_{i=1}^n$ then $\vdash (\alpha_i : a_i : N_i)_{i=1}^n [b : Q] \mid \text{it}(p, t; s) : \bar{0} : \text{nat}^\perp$.
- If $\vdash (\alpha_i : a_i^1 : N_i)_{i=1}^n [b : Q] \mid s : a : P^\perp, \vdash (\alpha_i : a_i^2 : N_i)_{i=1}^n, a : P \mid t : a' : P^\perp, \vdash (\alpha_i : a_i^3 : N_i)_{i=1}^n, a' : P \mid \text{it}(p, t; \bullet) : \bar{n} : \text{nat}^\perp$ and $(a_i \widetilde{N}_i^\perp [a_i^1, a_i^2, a_i^3])_{i=1}^n$ then $\vdash (\alpha_i : a_i : N_i)_{i=1}^n [b : Q] \mid \text{it}(p, t; s) : \overline{n+1} : \text{nat}^\perp$.

This shows semantically that $\text{it}(p, t; s)$ is an iterator, with p corresponding to the base case, t to the inductive step and s to a continuation using the result of the iteration.

Interpretation Theorem

Using the previous lemmas it is not very hard to relate the point typing system with normalization.

Theorem 84 *Let $\Phi = (\alpha_i : a_i : P_i^\perp)_{i=1}^n$ be a point context and let $a : P$, let $p_i \in |a_i|_{P_i}$ for $i = 1, \dots, n$.*

1. *If $c \vdash \Phi$ then $c[p_i/\alpha_i]_{i=1}^n \in \perp$.*
2. *If $c \vdash \Phi, a : P$ then $c[p_i/\alpha_i]_{i=1}^n \in \perp(a : P)$.*
3. *If $\vdash \Phi \mid s : b : N$ then $s[p_i/\alpha_i]_{i=1}^n \in |b|_N$.*
4. *If $\vdash \Phi, a : P \mid t : b : N$ then $t[p_i/\alpha_i]_{i=1}^n \in |b|_N(a : P)$.*
5. *If $\vdash \Phi \mid p : a : P$ then $p[p_i/\alpha_i]_{i=1}^n \in |a|_P$.*

Proof: By induction on the point derivation δ for c, t and p . To increase readability we use c' for $c[p_i/\alpha_i]_{i=1}^n$ and similarly for s and p . In the induction, we use the notations introduced in the statement of the theorem to avoid boring sentences introducing new symbols. But one has to keep in mind that the statement proven by induction contains the universal quantification on the p_i 's.

▷ δ consists of **(i-n)** so that we are in case **(5)** with $p = \alpha_j$ for some $j \in \underline{n}$, and $a = a_j$. In that case we have $p' = p_j$ and the expected conclusion follows from $\|a_j\|_{P_j} \subseteq |a_j|_{P_j}$.

▷ δ consists of **(i-1)** so that we are in case **(5)** with $p = ()$ and $a = *$. We have $p' = ()$ so that $p' \in \|*\|_1 \subseteq |*|_1$.

▷ δ ends with a **(i- \otimes)** so that we are in case **(5)** with $p = (q_1, q_2)$, $a = (b_1, b_2)$ and we have subderivations δ_j of $\vdash (\alpha_i : a_i^j : P_i^\perp)_{i=1}^n \mid q_j : b_j : Q_j$ for $j \in \underline{2}$ and moreover $a_i \widetilde{P}_i [a_i^1, a_i^2]$ for all $i \in \underline{n}$. For each $i \in \underline{n}$ we know that $p_i \in \|a_i\|_{P_i}$ and hence by Lemma 79 we have $p_i \in \|a_i^j\|_{P_i}$ for each $i \in \underline{n}$ and $j \in \underline{2}$. Hence by inductive hypothesis $q_j' \in |b_j|_{Q_j}$ so that $p' = (q_1', q_2') \in \|(b_1, b_2)\|_{Q_1 \otimes Q_2} \subseteq |(b_1, b_2)|_{Q_1 \otimes Q_2}$.

▷ δ ends with a left **(i- \oplus)** the case of a right **(i- \oplus)** being of course completely similar. We are in case **(5)** and $P = Q_1 \oplus Q_2$, $p = \text{in}_1(q)$ and $a = (1, b)$ and we have a subderivation δ_1 of $\vdash \Phi \mid q : b : Q_1$ so that by inductive hypothesis $q' \in |b|_{Q_1}$ and hence $p' = \text{in}_1(q') \in \|(1, b)\|_{Q_1 \oplus Q_2} \subseteq |(1, b)|_{Q_1 \oplus Q_2}$.

▷ δ ends with a **(i- $\tilde{\kappa}$)**. We are in case **(5)** with $p = \tilde{\kappa}.c$ and we have a subderivation δ_1 of δ which has $c \vdash \Phi, a : P$ as conclusion. By inductive hypothesis we have $c' \in \perp(a : P) = |a|_{P^\bullet} = \|a\|_{P^\bullet}$ and hence $p' = \tilde{\kappa}.c' \in |a|_P$ as required.

▷ δ ends with **(i- μ)** so that we are in case **(5)** with $P = \mu\zeta.Q$, $p = \text{fd}(q)$ and $a = \sigma(b)$ and δ has a subderivation δ_1 whose conclusion is $\vdash \Phi \mid q : b : Q [P/\zeta^+]$. By inductive hypothesis we have $q' \in |b|_{Q[P/\zeta^+]}$ and hence $p' = \text{fd}(q') \in \|\sigma(b)\|_{\mu\zeta.Q} \subseteq |a|_P$.

▷ δ ends with **(i- $!$)** so that we are in case **(5)** with $P = !N$, $p = s^!$ and $a = [b_j \mid j \in J]$ and δ has a subderivation δ_j of $\vdash (\alpha_i : a_i^j : P_i^\perp) \mid s : b_j : N$ for each $j \in J$ and moreover we have $a_i \widetilde{P}_i [a_i^j \mid j \in J]$ for each $i \in \underline{n}$. Since $\|a_i\|_{P_i} \subseteq \|a_i^j\|_{P_i}$ for each $i \in \underline{n}$ (Lemma 79) and $j \in J$ we have $s' \in |b_j|_N$ by inductive hypothesis applied to δ_j for each $j \in J$ and hence $p' = (s')^! \in \|[b_j \mid j \in J]\|_{!N} \subseteq |a|_P$.

▷ δ is **(i- \bullet)** so that we are in case **(4)** with $N = P^\perp$, $s = \bullet$, $b = a$ and $a_i \widetilde{P}_i []$ for each $i \in \underline{n}$. We have $s' = \bullet$ so that for all $p \in \|a\|_P$ we have $s' \star p \in \perp(a : P)$ which means that $s' \in |a|_{P^\perp(a : P)}$ as required.

▷ δ ends with a left **(i- $\&$)** (the case of a right **(i- $\&$)** is of course completely similar) and we are in case **(3)** with $N = N_1 \& N_2$, $a = (1, a_1)$, $s = \langle s_1, s_2 \rangle$ and we have a subderivation δ_1 whose conclusion is $\vdash \Phi \mid s_1 : a_1 : N_1$. The other subderivation δ_2 makes sure that s_2 is well typed in the typing system of Figure 2.3. Let $p \in \|(1, a_1)\|_{N_1^\perp \oplus N_2^\perp}$ that is $p = \text{in}_1(p_1)$ with $p_1 \in |a_1|_{N_1^\perp}$. We have $s' \star p = \langle s_1', s_2' \rangle \star \text{in}_1(p_1) \rightarrow s_1' \star p_1 \in \perp$ by inductive hypothesis and hence $s' \in |a|_N$.

▷ δ ends with a left **(i- $\&$)** (the case of a right **(i- $\&$)** is of course completely similar) and we are in case **(4)** with $N = N_1 \& N_2$, $b = (1, b_1)$, $s = \langle s_1, s_2 \rangle$ and we have a subderivation δ_1 whose conclusion is $\vdash \Phi, a : P \mid s_1 : b_1 : N_1$. The other subderivation δ_2 makes sure that s_2 is well typed in the typing system of Figure 2.3. Let $p \in \|(1, b_1)\|_{N_1^\perp \oplus N_2^\perp}$ that is $p = \text{in}_1(p_1)$ with $p_1 \in |b_1|_{N_1^\perp}$. We have $s' \star p = \langle s_1', s_2' \rangle \star \text{in}_1(p_1) \rightarrow s_1' \star p_1 \in \perp(a : P)$ by inductive hypothesis and hence $s' \in |b|_N(a : P)$.

▷ δ ends with **(i- κ)** and we are in case **(3)** with $s = \kappa\alpha.c$ and we have a subderivation δ_1 whose conclusion is $c \vdash \Phi, \alpha : a : N$. Let $p \in \|a\|_{N^\perp}$ we have $s' \star p \rightarrow c' [p/\alpha] \in \perp$ by inductive

hypothesis. It follows that $s' \in |a|_N$.

▷ δ ends with **(i- κ)** and we are in case (4) with $s = \kappa\alpha.c$ and we have a subderivation δ_1 whose conclusion is $c \vdash \Phi, a : P, \alpha : b : N$. Let $p \in \|b\|_{N^\perp}$ we have $s' \star p \rightarrow c' [p/\alpha] \in \perp\!\!\!\perp(a : P)$ by inductive hypothesis. It follows that $s' \in |b|_N(a : P)$ as required.

▷ δ ends with **(i- \perp)** and we are in case (3) with $N = \perp$, $a = *$, $s = \kappa_\perp.c$ and δ has a subderivation δ_1 whose conclusion is $c \vdash \Phi$. We have $s' \star () = \kappa_\perp.c' \star () \rightarrow c'$ and since, by inductive hypothesis, $c' \in \perp\!\!\!\perp$, it follows that $s' \in |*|_\perp$.

▷ δ ends with **(i- \perp)** and we are in case (4) with $N = \perp$, $b = *$, $s = \kappa_\perp.c$ and δ has a subderivation δ_1 whose conclusion is $c \vdash \Phi$. We have $s' \star () = \kappa_\perp.c' \star () \rightarrow c'$ and since, by inductive hypothesis, $c' \in \perp\!\!\!\perp(a : P)$, it follows that $s' \in |*|_\perp(a : P)$.

▷ δ ends with **(i- \wp)** and we are in case (3) with $N = N_1 \wp N_2$, $a = (a_1, a_2)$, $s = \kappa(\alpha_1, \alpha_2).c$ and we have a subderivation δ_1 whose conclusion is $c \vdash \Phi, \alpha_1 : a_1 : N_1, \alpha_2 : a_2 : N_2$. Let $p \in \|(a_1, a_2)\|_{N_1^\perp \otimes N_2^\perp}$, that is $p = (q_1, q_2)$ with $q_j \in |a_j|_{N_j^\perp}$ for $j \in \underline{2}$. We have $s' \star p \rightarrow c' [q_1/\alpha_1, q_2/\alpha_2] \in \perp\!\!\!\perp$ by inductive hypothesis. It follows that $s' \in |a|_N$.

▷ δ ends with **(i- \wp)** and we are in case (4) with $N = N_1 \wp N_2$, $b = (b_1, b_2)$, $s = \kappa(\alpha_1, \alpha_2).c$ and we have a subderivation δ_1 whose conclusion is $c \vdash \Phi, \alpha_1 : b_1 : N_1, \alpha_2 : b_2 : N_2, a : P$. Let $p \in \|(b_1, b_2)\|_{N_1^\perp \otimes N_2^\perp}$, that is $p = (q_1, q_2)$ with $q_j \in |b_j|_{N_j^\perp}$ for $j \in \underline{2}$. We have $s' \star p \rightarrow c' [q_1/\alpha_1, q_2/\alpha_2] \in \perp\!\!\!\perp(a : P)$ by inductive hypothesis. It follows that $s' \in |b|_N(a : P)$.

▷ δ ends with **(i-?)** so that we are in case (3) (notice that there no case (4) for this rule) with $N = ?Q$, $a = [b]$, $s = \text{der } p$ and δ has a subderivation δ_1 whose conclusion is $\vdash \Phi \mid p : b : Q$. Let $p \in \|[b]\|_{Q^\perp}$, that is $p = t'$ where $t \in |b|_{Q^\perp}$. We have $s' \star p \rightarrow t \star p' \in \perp\!\!\!\perp$ since $p' \in |b|_Q$ by inductive hypothesis.

▷ δ ends with **(i-cut)** and we are in case (1) with $c = s \star p$ and δ has two subderivations δ_1 and δ_2 with conclusions $\vdash (\alpha_i : a_i^1 : P_i^\perp)_{i=1}^n \mid s : b : Q^\perp$ and $\vdash (\alpha_i : a_i^2 : P_i^\perp)_{i=1}^n \mid p : b : Q$ and $a_i \widetilde{P}_i [a_i^1, a_i^2]$ for all $i \in \underline{n}$ so that by Lemma 79 we have $p_i \in |a_i^j|_{P_i}$ for all $i \in \underline{n}$ and $j \in \underline{2}$. By inductive hypothesis we have $s' \in |b|_{Q^\perp}$ and $p' \in |b|_Q$ so that $c' = s' \star p' \in \perp\!\!\!\perp$.

▷ δ ends with **(i-cut)** and we are in case (2) with $c = s \star p$ and δ has two subderivations δ_1 and δ_2 with conclusions $\vdash (\alpha_i : a_i^1 : P_i^\perp)_{i=1}^n, a : P \mid s : b : Q^\perp$ and $\vdash (\alpha_i : a_i^2 : P_i^\perp)_{i=1}^n \mid p : b : Q$ and $a_i \widetilde{P}_i [a_i^1, a_i^2]$ for all $i \in \underline{n}$ so that by Lemma 79 we have $p_i \in |a_i^j|_{P_i}$ for all $i \in \underline{n}$ and $j \in \underline{2}$. By inductive hypothesis we have $s' \in |b|_{Q^\perp}(a : P)$ and $p' \in |b|_Q$ so that $c' = s' \star p' \in \perp\!\!\!\perp(a : P)$.

▷ δ ends with **(i- ν)**, remember that this rule is given at the beginning of Section 2.2.3. Assume that we are in case (4) with $N = \nu\zeta.R^\perp$, $s = \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; t)$, we have a point $d : R$, a subderivation δ'' of δ whose conclusion is $\vdash (\alpha_i : a_i'' : P_i^\perp), a : P \mid t : h : Q^\perp$, a subderivation δ' of δ whose conclusion is $c \vdash (\alpha_i : a_i' : P_i^\perp), h : Q, \alpha : d\{h_l/(l, \eta)\}_{l \in L} : R^\perp [Q^\perp/\zeta^-]$, and for each $l \in L = \text{rg}_\zeta d$ we have a subderivation δ_l of δ whose conclusion is $\vdash (\alpha_i : a_i^l : P_i^\perp)_{i=1}^n, h_l : Q \mid u : f_l : \nu\zeta.R^\perp$ where $u = \bar{\kappa}_{R^\perp, \zeta}\alpha.c = \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; \bullet)$. With these notations we have $b = \sigma(d)\{f_l/(l, \zeta)\}_{l \in L}$. Moreover $a_i \widetilde{P}_i [a_i', a_i''] + [a_i^l \mid l \in L]$ so that by Lemma 79 we have $p_i \in |a_i^l|_{P_i}$ for all $i \in \underline{n}$ and $l \in L$, and $p_i \in |a_i^j|_{P_i}$, $p_i \in |a_i''|_{P_i}$ for all $i \in \underline{n}$. Let $p \in \|b\|_{N^\perp}$ so that $p = \text{fd}(q)$ with $q \in |d\{f_l/(l, \zeta)\}_{l \in L}|_{R[N^\perp/\zeta^+]}$. Then we have $s' \star p \rightarrow (R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet]/\bullet]$. By inductive hypothesis applied to the δ_l 's we have $u' \in |f_l|_N(h_l : Q)$ for each $l \in L$ and hence by Lemma 83, $R[u'/\zeta^+] \in |d\{f_l/(l, \zeta)\}_{l \in L}|_{R[N/\zeta^-]}(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$, therefore:

$R[u'/\zeta^+] \star q \in \perp(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$ which means that $R[u'/\zeta^+] \star q \rightarrow^* \bullet \star r$ with $r \in \|\!|d\{h_l/(l, \zeta)\}_{l \in L}\|\!|_{R[Q/\zeta^+]}$. It follows that $(R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet] \rightarrow^* \kappa\alpha.c' [t'/\bullet] \star r \rightarrow c' [r/\alpha] [t'/\bullet]$ since $r \notin \tilde{\kappa}$, r is closed and t' has no free names. By inductive hypothesis applied to δ' (with substituting positive terms the p_i 's and r which satisfies $r \in \|\!|d\{h_l/(l, \zeta)\}_{l \in L}\|\!|_{R[Q/\zeta^+]}$ as required) we have $c' [r/\alpha] \in \perp(h : Q)$ and hence $\tilde{\kappa}.c' [r/\alpha] \in |h|_Q$. We also have $t' \in |h|_{Q^\perp}(a : P)$ by inductive hypothesis and hence $t' \star \tilde{\kappa}.(c' [r/\alpha]) \in \perp(a : P)$ so that $c' [r/\alpha] [t'/\bullet] \in \perp(a : P)$ and hence $s' \star p \in \perp(a : P)$. Since this holds for all $p \in \|\!|b\|\!|_{N^\perp}$ we have proven that $s' \in |b|_N(a : P)$ as required.

$\triangleright \delta$ ends with **(i- ν)** and we are in case **(3)** with $N = \nu\zeta.R^\perp$, $s = \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; t)$, we have a point $d : R$, a subderivation δ'' of δ whose conclusion is $\vdash (\alpha_i : a_i'' : P_i^\perp) \mid t : h : Q^\perp$, a subderivation δ' of δ whose conclusion is

$$c \vdash (\alpha_i : a_i' : P_i^\perp), h : Q, \alpha : d\{h_l/(l, \eta)\}_{l \in L} : R^\perp [Q^\perp/\zeta^-]$$

and for each $l \in L = \text{rg}_\zeta d$ we have a subderivation δ_l of δ whose conclusion is

$$\vdash (\alpha_i : a_i^l : P_i^\perp)_{i=1}^n, h_l : Q \mid u : f_l : \nu\zeta.R^\perp$$

where $u = \bar{\kappa}_{R^\perp, \zeta}\alpha.c = \bar{\kappa}_{R^\perp, \zeta}\alpha.(c; \bullet)$. With these notations we have $b = \sigma(d)\{f_l/(l, \zeta)\}_{l \in L}$. Moreover $a_i \tilde{P}_i [a_i', a_i''] + [a_i^l \mid l \in L]$ so that by Lemma 79 we have $p_i \in |a_i^l|_{P_i}$ for all $i \in \underline{n}$ and $l \in L$, and $p_i \in |a_i'|_{P_i}$, $p_i \in |a_i''|_{P_i}$ for all $i \in \underline{n}$. Let $p \in \|\!|b\|\!|_{N^\perp}$ so that $p = \text{fd}(q)$ with $q \in \|\!|d\{f_l/(l, \zeta)\}_{l \in L}\|\!|_{R[N^\perp/\zeta^+]}$. Then we have $s' \star p \rightarrow (R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet]$. By inductive hypothesis applied to the δ_l 's we have $u' \in |f_l|_N(h_l : Q)$ for each $l \in L$ and hence, by Lemma 83,

$$R[u'/\zeta^+] \in \|\!|d\{f_l/(l, \zeta)\}_{l \in L}\|\!|_{R[N/\zeta^-]}(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$$

therefore $R[u'/\zeta^+] \star q \in \perp(d\{h_l/(l, \zeta)\}_{l \in L} : R[Q/\zeta^+])$ which means that $R[u'/\zeta^+] \star q \rightarrow^* \bullet \star r$ with $r \in \|\!|d\{h_l/(l, \zeta)\}_{l \in L}\|\!|_{R[Q/\zeta^+]}$. It follows that

$$(R[u'/\zeta^+] \star q) [\kappa\alpha.c' [t'/\bullet] / \bullet] \rightarrow^* \kappa\alpha.c' [t'/\bullet] \star r \rightarrow c' [r/\alpha] [t'/\bullet]$$

since $r \notin \tilde{\kappa}$, r is closed and t' is closed. By inductive hypothesis applied to δ' (with substituting positive terms the p_i 's and r which satisfies $r \in \|\!|d\{h_l/(l, \zeta)\}_{l \in L}\|\!|_{R[Q/\zeta^+]}$ as required) we have $c' [r/\alpha] \in \perp(h : Q)$ and hence $\tilde{\kappa}.(c' [r/\alpha]) \in |h|_Q$. We also have $t' \in |h|_{Q^\perp}$ by inductive hypothesis and hence $t' \star \tilde{\kappa}.(c' [r/\alpha]) \in \perp$ so that $c' [r/\alpha] [t'/\bullet] \in \perp$ and hence $s' \star p \in \perp$. Since this holds for all $p \in \|\!|b\|\!|_{N^\perp}$ we have proven that $s' \in |b|_N$ as required. \blacksquare

2.3 Categorical semantics of $\kappa\mu\text{LLP}$

We build a categorical model on top of the categorical model of μLL provided in Chapter 1. We start from the following definition which is one of the central definitions of our notion of models of $\kappa\mu\text{LLP}$. We assume to be given a model \mathcal{L} of μLL .

We write X, Y, \dots for objects of the category \mathcal{L}^\dagger , and A, B, \dots for those of \mathcal{L} .

$$\begin{array}{ccc}
 & & \mathbb{P}(\vec{A}, \vec{!B}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} \mathbb{!}\mathbb{P}(\vec{A}, \vec{B}) \\
 \mathbb{P}(\vec{A}, \vec{!B}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} \mathbb{!}\mathbb{P}(\vec{A}, \vec{B}) & \mathbb{P}(\vec{A}, \text{dig}_{\vec{B}}) \downarrow & \mathbb{P}(\vec{A}, \vec{!B}) \downarrow \\
 \mathbb{P}(\vec{A}, \text{der}_{\vec{B}}) \searrow & \mathbb{P}(\vec{A}, \mathbb{!}\vec{B}) & \mathbb{!}\mathbb{P}(\vec{A}, \vec{B}) \\
 & \tilde{\mathbb{P}}_{\vec{A}, \vec{!B}} \downarrow & \downarrow \text{dig}_{\mathbb{P}(\vec{A}, \vec{B})} \\
 & \mathbb{!}\mathbb{P}(\vec{A}, \vec{!B}) \xrightarrow{\mathbb{!}\tilde{\mathbb{P}}_{\vec{A}, \vec{B}}} \mathbb{!}\mathbb{!}\mathbb{P}(\vec{A}, \vec{B}) & \\
 & & \mathbb{!}C \otimes \mathbb{P}(\vec{A}, \vec{!B}) \xrightarrow{\widehat{\mathbb{P}}_{\mathbb{!}C, (\vec{A}, \vec{B})}} \mathbb{P}(\mathbb{!}C \otimes \vec{A}, \mathbb{!}C \otimes \vec{!B}) \\
 & & \mathbb{!}C \otimes \tilde{\mathbb{P}}_{\vec{A}, \vec{B}} \downarrow \qquad \qquad \qquad \downarrow \mathbb{P}(\text{der}_{\mathbb{!}C \otimes \vec{A}}, \mu_{\mathbb{!}C, \vec{B}}^2) \\
 & & \mathbb{!}C \otimes \mathbb{!}\mathbb{P}(\vec{A}, \vec{B}) \qquad \qquad \mathbb{P}(\mathbb{!}C \otimes \vec{A}, \mathbb{!}(\mathbb{!}C \otimes \vec{B})) \\
 & & \mu_{\mathbb{!}C, \mathbb{P}(\vec{A}, \vec{B})}^2 \downarrow \qquad \qquad \qquad \downarrow \tilde{\mathbb{P}}_{\mathbb{!}C \otimes \vec{A}, \mathbb{!}C \otimes \vec{B}} \\
 & & \mathbb{!}(\mathbb{!}C \otimes \mathbb{P}(\vec{A}, \vec{B})) \xrightarrow{\widehat{\mathbb{P}}_{\mathbb{!}C, (\vec{A}, \vec{B})}} \mathbb{!}\mathbb{P}(\mathbb{!}C \otimes \vec{A}, \mathbb{!}C \otimes \vec{B})
 \end{array}$$

 Figure 2.7: Compatibility of $\tilde{\mathbb{P}}$ with der , dig , and $\widehat{\mathbb{P}}$

Definition 85 Let n and p be two integers. A (n, p) -positive functor \mathbb{P} is a pair $(\mathbb{P}, \tilde{\mathbb{P}})$ where $\mathbb{P} \in \mathcal{L}_{n+p}$ and $\tilde{\mathbb{P}}_{\vec{A}, \vec{B}} \in \mathcal{L}(\mathbb{P}(\vec{A}, \vec{!B}), \mathbb{!}\mathbb{P}(\vec{A}, \vec{B}))$ is a natural transformation. It is assumed moreover that the diagrams of Figure 2.7 commute, expressing the compatibility of $\tilde{\mathbb{P}}$ with der , dig , and $\widehat{\mathbb{P}}$, and therefore $\tilde{\mathbb{P}}$ is called the distributive law of \mathbb{P} .

It is assumed that the length for \vec{A} is n and for \vec{B} is p . We will always use this kind of convention tacitly in the sequel.

Lemma 86 Any (n, p) -positive functor \mathbb{P} induces a functor $\mathbb{P}^+ : \mathcal{L}^n \times (\mathcal{L}^!)^p \rightarrow \mathcal{L}^!$.

Proof: Given $(\vec{A}, \vec{Y}) \in \mathcal{L}^n \times (\mathcal{L}^!)^p$ where $\vec{Y} = (\vec{Y}, h_{\vec{Y}})$, we define $\mathbb{P}^+(\vec{A}, \vec{Y})$ as $\mathbb{P}(\vec{A}, \vec{Y})$ and $h_{\mathbb{P}^+(\vec{A}, \vec{Y})}$ as the following composition of morphisms in \mathcal{L} :

$$\mathbb{P}(\vec{A}, \vec{Y}) \xrightarrow{\mathbb{P}(\vec{A}, h_{\vec{Y}})} \mathbb{P}(\vec{A}, \vec{!Y}) \xrightarrow{\tilde{\mathbb{P}}_{\vec{A}, \vec{Y}}} \mathbb{!}\mathbb{P}(\vec{A}, \vec{Y})$$

Let $(f_1, f_2) \in (\mathcal{L}^n \times (\mathcal{L}^!)^p)((\vec{A}_1, \vec{Y}_1), (\vec{A}_2, \vec{Y}_2))$, we define $\mathbb{P}^+(f_1, f_2) \in \mathcal{L}^!(\mathbb{P}^+(\vec{A}_1, \vec{Y}_1), \mathbb{P}^+(\vec{A}_2, \vec{Y}_2))$ as $\mathbb{P}(f_1, f_2)$. And the following diagram commutes which shows that $\mathbb{P}(f_1, f_2)$ is indeed a mor-

phism in $\mathcal{L}^!$.

$$\begin{array}{ccc}
 \mathbb{P}(\vec{A}_1, \vec{Y}_1) & \xrightarrow{\mathbb{P}(f_1, f_2)} & \mathbb{P}(\vec{A}_2, \vec{Y}_2) \\
 \mathbb{P}(\vec{A}_1, h_{\vec{Y}_1}) \downarrow & & \downarrow \mathbb{P}(\vec{A}_2, h_{\vec{Y}_2}) \\
 \mathbb{P}(\vec{A}_1, !\vec{Y}_1) & & \mathbb{P}(\vec{A}_2, !\vec{Y}_2) \\
 \tilde{\mathbb{P}}_{\vec{A}_1, \vec{Y}_1} \downarrow & & \downarrow \tilde{\mathbb{P}}_{\vec{A}_2, \vec{Y}_2} \\
 !\mathbb{P}(\vec{A}_1, \vec{Y}_1) & \xrightarrow{!\mathbb{P}(f_1, f_2)} & !\mathbb{P}(\vec{A}_2, \vec{Y}_2)
 \end{array}$$

■

As a consequence of Lemma 86, a $(0, 0)$ -positive functor \mathbb{P} induces an object of $\mathcal{L}^!$, and in that case $\tilde{\mathbb{P}} = h_{\mathbb{P}}$.

We recall that $\mathcal{L}[Z]$ is the Kleisli category of the comonad (fc_Z, W_Z, C_Z) where $\text{fc}_Z : \mathcal{L} \rightarrow \mathcal{L}$ is the functor which maps an object A to $Z \otimes A$ and a morphism f to $Z \otimes f$, and W_Z, C_Z are weakening and contraction morphism respectively.

Recall from Chapter 1 (Section 1.1.2) that one can extend a strong functor \mathbb{F} to a functor $\mathbb{F}[Z] : \mathcal{L}[Z]^k \rightarrow \mathcal{L}[Z]$ for a given object $Z = (Z, h_Z) \in \mathcal{L}^!$. On objects, one sets $\mathbb{F}[Z](\vec{A}) = \overline{\mathbb{F}}(\vec{A})$. And given a morphism $\vec{f} \in \mathcal{L}[Z]^k(\vec{A}_1, \vec{A}_2)$, we define $\mathbb{F}[Z](\vec{f})$ as

$$Z \otimes \overline{\mathbb{F}}(A_1) \xrightarrow{(h_Z \otimes \overline{\mathbb{F}}(A_1))} !Z \otimes \overline{\mathbb{F}}(A_1) \xrightarrow{(\widehat{\mathbb{F}}_{Z, \vec{A}_1})} \overline{\mathbb{F}}(!Z \otimes A_1) \xrightarrow{(\overline{\mathbb{F}}(\text{der}_Z \otimes \vec{A}_1))} \overline{\mathbb{F}}(Z \otimes A_1) \xrightarrow{(\overline{\mathbb{F}}(\vec{f}))} \overline{\mathbb{F}}(A_2)$$

Lemma 87 *Let $\mathbb{P} = (\mathbb{P}, \tilde{\mathbb{P}})$ be a positive functor and $f \in \mathcal{L}^!(Z \otimes X, Y)$. If \mathbb{P} is a $(n, p+1)$ -positive functor, then*

$\mathbb{P}[Z](\vec{\text{id}}, \vec{\text{id}}, f) \in \mathcal{L}^!(Z \otimes \mathbb{P}^+(\vec{A}, \vec{Y}_1, X), \mathbb{P}^+(\vec{A}, \vec{Y}_1, Y))$. And if \mathbb{P} is a $(n+1, p)$ -positive functor, then

$\mathbb{P}[Z](f^\perp, \vec{\text{id}}, \vec{\text{id}}) \in \mathcal{L}^!(Z \otimes \mathbb{P}^+((Y)^\perp, \vec{A}, \vec{Y}_1), \mathbb{P}^+((X)^\perp, \vec{A}, \vec{Y}_1))$.

Proof: The first item holds because of commutation of the following diagram :

$$\begin{array}{ccc}
 Z \otimes \mathbb{P}(\vec{A}, \vec{Y}_1, X) & \xrightarrow{\mathbb{P}[Z](\vec{\text{id}}, \vec{\text{id}}, f)} & \mathbb{P}(\vec{A}, \vec{Y}_1, Y) \\
 h_{Z \otimes \mathbb{P}^+(\vec{A}, \vec{Y}_1, X)} \downarrow & & \downarrow h_{\mathbb{P}^+(\vec{A}, \vec{Y}_1, Y)} \\
 ! (Z \otimes \mathbb{P}(\vec{A}, \vec{Y}_1, X)) & \xrightarrow{!(\mathbb{P}[Z](\vec{\text{id}}, \vec{\text{id}}, f))} & !\mathbb{P}(\vec{A}, \vec{Y}_1, Y)
 \end{array}$$

And the second item is similar to the first one.

■

Operations on positive functors

LL operations on positive functors: Given a (n, p) -positive functor \mathbb{P} , strong functors $\mathbb{F}_1, \dots, \mathbb{F}_n$ in \mathcal{L}_k , and (n', p') -positive functors $\mathbb{Q}_1, \dots, \mathbb{Q}_p$ such that $n' + p' = k$, one can define a (n', p') -positive functor

$\mathbb{R} = \mathbb{P} \circ (\mathbb{F}_1, \dots, \mathbb{F}_n, \mathbb{Q}_1, \dots, \mathbb{Q}_p)$: the strong functor \mathbb{R} is just $\mathbb{P} \circ (\mathbb{F}_1, \dots, \mathbb{F}_n, \underline{\mathbb{Q}}_1, \dots, \underline{\mathbb{Q}}_p)$ as

in Chapter 1. The distributive law is defined as follows, and satisfies the commutations of Fig. 2.7.

$$\begin{aligned}
 \mathbb{R}(\vec{A}, \vec{B}) &= \mathbb{P}((\mathbb{F}_i(\vec{A}, \vec{B}))_{i=1}^n, (\mathbb{Q}_i(\vec{A}, \vec{B}))_{i=1}^p) \\
 &\quad \downarrow \mathbb{P}((\mathbb{F}_i(\vec{A}, \text{der}_{\vec{B}}))_{i=1}^n, (\mathbb{Q}_i)_{i=n+1}^p) \\
 &= \mathbb{P}((\mathbb{F}_i(\vec{A}, \vec{B}))_{i=1}^n, (\mathbb{Q}_i(\vec{A}, \vec{B}))_{i=1}^p) \\
 &\quad \downarrow \mathbb{P}(\mathbb{F}_i(\vec{A}, \vec{B})_{i=1}^n, \mathbb{Q}_i(\vec{A}, \vec{B})) \\
 &= \mathbb{P}((\mathbb{F}_i(\vec{A}, \vec{B}))_{i=1}^n, (\mathbb{Q}_i(\vec{A}, \vec{B}))_{i=1}^p) = \mathbb{R}(\vec{A}, \vec{B})
 \end{aligned}$$

The bifunctor \otimes can be turned into a $(0, 2)$ -positive functor: the distributive law is $\mu_{A,B}^2 \in \mathcal{L}(!A \otimes !B, !(A \otimes B))$, and it satisfies commutations of Fig. 2.7.

The bifunctor \oplus can be turned into a $(0, 2)$ -positive functor: the distributive law is $[\pi_1, \pi_2] \in \mathcal{L}(!A \oplus !B, !(A \oplus B))$, and it satisfies commutations of Fig. 2.7.

The functor $!$ is a $(1, 0)$ -positive functor: the distributive law is $\text{dig}_A \in \mathcal{L}(!A, !!A)$, and satisfies commutations of Fig. 2.7.

Let \mathbb{P} be a (n, p) -positive functor. One can define its De Morgan dual, denoted as \mathbb{P}^\perp , as a strong functor: $\mathbb{P}^\perp = \mathbb{P}^\perp$. Notice that \mathbb{P}^\perp is just a strong functor, so we do not have a distributive law for it.

Fixpoint of positive functors: Let \mathbb{P} be a $(n, p+1)$ -positive functor. We must define a (n, p) -positive functor $\mu\mathbb{P}$. We set $\mu\mathbb{P} = \mu\mathbb{P}$. The distributive law $\mu\mathbb{P}$ must be a natural transformation. $\widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}} \in \mathcal{L}(\mu\mathbb{P}(\vec{A}, \vec{B}), \mathbb{P}(\vec{A}, \vec{B}))$. To define it, we first notice that $(\mathbb{P}(\vec{A}, \vec{B}), \widetilde{\mathbb{P}}_{\vec{A}, (\vec{B}, \mu\mathbb{P}(\vec{A}, \vec{B}))})$ is an object of $\mathbf{Coalg}_{\mathcal{L}}(\mathbb{P}_{\vec{A}, \vec{B}})$. So, by the universal property of $\mu\mathbb{P}(\vec{A}, \vec{B})$, there is a unique morphism $\widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{P}(\vec{A}, \vec{B}, \mu\mathbb{P}(\vec{A}, \vec{B})) & \xrightarrow{\mathbb{P}(\vec{A}, \vec{B}, \widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}})} & \mathbb{P}(\vec{A}, \vec{B}, \mathbb{P}(\vec{A}, \vec{B})) \\
 \simeq \downarrow & & \downarrow \widetilde{\mathbb{P}}_{\vec{A}, (\vec{B}, \mu\mathbb{P}(\vec{A}, \vec{B}))} \\
 \mu\mathbb{P}(\vec{A}, \vec{B}) & \xrightarrow{\widetilde{\mu\mathbb{P}}_{\vec{A}, \vec{B}}} & \mathbb{P}(\vec{A}, \vec{B}) \xleftarrow{\simeq} \mathbb{P}(\vec{A}, \vec{B}, \mu\mathbb{P}(\vec{A}, \vec{B}))
 \end{array}$$

Lemma 88 Given a strong functor $\mathbb{F} \in \mathcal{L}_{k+1}$ and an object $Z = (Z, h_Z)$ in $\mathcal{L}^!$, there is a unique functor $\mu(\mathbb{F}[Z]) : \mathcal{L}[Z]^k \rightarrow \mathcal{L}[Z]$ such that $\mu(\mathbb{F}[Z])(\vec{A})$ is the initial object of the category $\mathbf{Alg}_{\mathcal{L}[Z]}(\mathbb{F}_{\vec{A}})$ for any object $\vec{A} \in \mathcal{L}^k$, and $\mu(\mathbb{F}[Z])(\vec{f})$ is the unique morphism making the following diagram commute for any $\vec{f} \in \mathcal{L}[Z]^k(\vec{B}_1, \vec{B}_2)$:

$$\begin{array}{ccccc}
 \mu(\mathbb{F}[Z])(\vec{B}_1) & \xrightarrow{\mu(\mathbb{F}[Z])(\vec{f})} & \mu(\mathbb{F}[Z])(\vec{B}_2) & & \\
 \simeq \downarrow & & & \swarrow \simeq & \\
 \mathbb{F}_{\vec{B}_1}(\mu(\mathbb{F}[Z])(\vec{B}_1)) & \xrightarrow{\mathbb{F}_{\vec{B}_1}(\mu(\mathbb{F}[Z])(\vec{f}))} & \mathbb{F}_{\vec{B}_1}(\mu(\mathbb{F}[Z])(\vec{B}_2)) & \xrightarrow{\mathbb{F}(\vec{f}, \text{id})} & \mathbb{F}_{\vec{B}_2}(\mu(\mathbb{F}[Z])(\vec{B}_2))
 \end{array}$$

where the composition is considered in the category $\mathcal{L}[Z]$.

Proof: Since $\mathbb{F} \in \mathcal{L}^{k+1}$, there is a functor $\overline{\mu\mathbb{F}} : \mathcal{L}^k \rightarrow \mathcal{L}$ such that $\overline{\mu\mathbb{F}}(\vec{A})$ is the initial object of the category $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{A}})$ for any object $A \in \mathcal{L}$.

Now, by the construction above (see Section 1.1.2 for details), one can extend $\mu\mathbb{F}$ to a functor $(\mu\mathbb{F})[Z] : \mathcal{L}[Z]^k \rightarrow \mathcal{L}[Z]$. We take $(\mu\mathbb{F})[Z]$ as the unique functor $\mu(\mathbb{F}[Z])$. Let us take an object $A \in \mathcal{L}$. We have $(\mu\mathbb{F})[Z](\vec{A}) = \overline{\mu\mathbb{F}}(\vec{A})$. We need to show that $\mu\mathbb{F}(\vec{A})$ is the initial object of the category $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})$. First observe that $(\overline{\mu\mathbb{F}}(\vec{A}), h)$ is an object of $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})$ where h is the following:

$$\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) \xrightarrow{W_Z \otimes \text{Id}} \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) \xrightarrow{\simeq} \overline{\mu\mathbb{F}}(\vec{A})$$

So, let us take an object (B, g) of $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})$ where $g \in \mathcal{L}[Z](\overline{\mathbb{F}}_{\vec{A}}(B), B) = \mathcal{L}(\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B), B)$. We need to provide a morphism $\tilde{g} \in \mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{F}}_{\vec{A}})(\overline{\mu\mathbb{F}}(\vec{A}), B)$. Having a morphism $\tilde{g} \in \mathcal{L}(\underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A}), B)$ is equivalent to having a morphism $\text{cur}'(\tilde{g}) \in \mathcal{L}(\overline{\mu\mathbb{F}}(\vec{A}), \underline{Z} \multimap B)$. Since $\overline{\mu\mathbb{F}}(\vec{A})$ is the initial object of $\mathbf{Alg}_{\mathcal{L}}(\overline{\mathbb{F}}_{\vec{A}})$, it is enough to have a morphism $g' \in \mathcal{L}(\overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B), \underline{Z} \multimap B)$ in order to have $\text{cur}'(\tilde{g})$. And this is equivalent to provide a morphism $\text{cur}'^{-1}(g') \in \mathcal{L}(\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B), B)$ which is as follows:

$$\begin{array}{ccc} \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B) & \xrightarrow{C_Z \otimes \text{Id}} & \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B) \\ & & \downarrow \text{Id} \otimes h_Z \otimes \text{Id} \\ \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(!\underline{Z} \otimes (\underline{Z} \multimap B)) & \xleftarrow{\text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}} & \underline{Z} \otimes !\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \multimap B) \\ \text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}(\text{der}_{\underline{Z}} \otimes \text{Id}) \downarrow & & \\ \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \otimes (\underline{Z} \multimap B)) & \xrightarrow{\text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}(\text{ev})} & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B) \\ & & \downarrow g \\ & & B \end{array}$$

So, \tilde{g} is $\text{cur}'^{-1}(g')$, and it satisfies the following diagram

$$\begin{array}{ccc} \underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A}) & \xrightarrow{\tilde{g}} & B \\ \simeq \uparrow & & \uparrow g \\ \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & & \\ \downarrow C_Z \otimes \text{Id} & & \\ \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & \xrightarrow{\text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}[Z](\tilde{g})} & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B) \end{array} \quad (2.20)$$

Because one can see that the following diagram commutes which is just unfolding (considering

the composition in \mathcal{L} rather than $\mathcal{L}[Z]$ of Diagram 2.20:

$$\begin{array}{ccc}
 \underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A}) & \xrightarrow{\tilde{g}} & B \\
 \uparrow \text{Id} \otimes (W_Z \otimes \simeq) & & \uparrow g \\
 \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & & \\
 \uparrow C_{\underline{Z}} \otimes \text{Id} & & \\
 \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(B) \\
 \downarrow C_Z \otimes \text{Id} & & \uparrow \text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}(\tilde{g}) \\
 \underline{Z} \otimes \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A})) \\
 \downarrow \text{Id} \otimes h_Z \text{Id} & & \uparrow \text{Id} \otimes \overline{\mathbb{F}}_{\vec{A}}(\text{der}_{\underline{Z}} \otimes \text{Id}) \\
 \underline{Z} \otimes !\underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(\overline{\mu\mathbb{F}}(\vec{A})) & \xrightarrow{\text{Id} \otimes \widehat{\overline{\mathbb{F}}_{\vec{A}}}} & \underline{Z} \otimes \overline{\mathbb{F}}_{\vec{A}}(!\underline{Z} \otimes \overline{\mu\mathbb{F}}(\vec{A}))
 \end{array}$$

■

Lemma 89 *Given a morphism $g \in \mathcal{L}[Z](\overline{\mathbb{Q}}(\underline{Y}), \underline{Y})$ where \mathbb{Q} is a $(0, 1)$ -positive functor, there is a unique morphism $\tilde{g} \in \mathcal{L}[Z](\underline{\mu\mathbb{Q}}, \underline{Y})$ such that $\tilde{g} \in \mathbf{Alg}_{\mathcal{L}[Z]}(\mathbb{Q}[Z])(\underline{\mu\mathbb{Q}}, \underline{Y})$.*

Proof: By Lemma 88, we know that there is a unique object $\mu(\mathbb{Q}[Z]) \in \mathcal{L}[Z]$ which is the initial object of the category $\mathbf{Alg}_{\mathcal{L}[Z]}(\overline{\mathbb{Q}})$. Since $g \in \mathcal{L}[Z](\overline{\mathbb{Q}}(\underline{Y}), \underline{Y})$, there is a unique morphism $\tilde{g} \in \mathcal{L}[Z](\mu(\mathbb{Q}[Z]), \underline{Y})$ by the universal property of the initial algebra of the endofunctor $\overline{\mathbb{Q}}$ on the category $\mathcal{L}[Z]$. Hence, to conclude the statement of the lemmas, it is enough to show that $\mu(\mathbb{Q}[Z]) = \underline{\mu\mathbb{Q}}$. As we saw in the proof of Lemma 88, one has $\mu(\mathbb{Q}[Z]) = (\mu\mathbb{Q})[Z]$, and $(\mu\mathbb{Q})[Z]$ is the same as $\underline{\mu\mathbb{Q}}$ (considering $\underline{\mu\mathbb{Q}}$, equivalently, as $\underline{\mu\mathbb{Q}} : 1 \rightarrow \mathcal{L}$). And finally, by definition (Section 2.3), we have $\underline{\mu\mathbb{Q}} = \underline{\mu\mathbb{Q}}$. ■

Lemma 90 *If \mathbb{Q} is a $(0, 1)$ -positive functor, and g is a morphism $g \in \mathcal{L}^!(Z \otimes \mathbb{Q}^+(Y), Y)$, then there is a unique morphism $\tilde{g} \in \mathcal{L}^!(Z \otimes (\mu\mathbb{Q})^+, Y)$ such that $\tilde{g} \in \mathbf{Alg}_{\mathcal{L}[Z]}(\mathbb{Q}[Z])(\underline{\mu\mathbb{Q}}, \underline{Y})$.*

Proof: By definition, we have $\underline{\mu\mathbb{Q}^+} = \underline{\mu\mathbb{Q}} = \underline{\mu\mathbb{Q}} = \underline{\mu\mathbb{Q}}$. The following diagram commutes which shows that \tilde{g} is indeed a co-algebra morphism. The proof of this commutation is shown in Figure 2.8.

$$\begin{array}{ccc}
 !\underline{Z} \otimes \underline{\mu\mathbb{Q}} & \xrightarrow{\tilde{g}} & \underline{Y} \\
 \downarrow h_{!\underline{Z} \otimes \underline{\mu\mathbb{Q}}} & & \downarrow h_Y \\
 !(!\underline{Z} \otimes \underline{\mu\mathbb{Q}}) & \xrightarrow{!\tilde{g}} & !\underline{Y}
 \end{array}$$

■

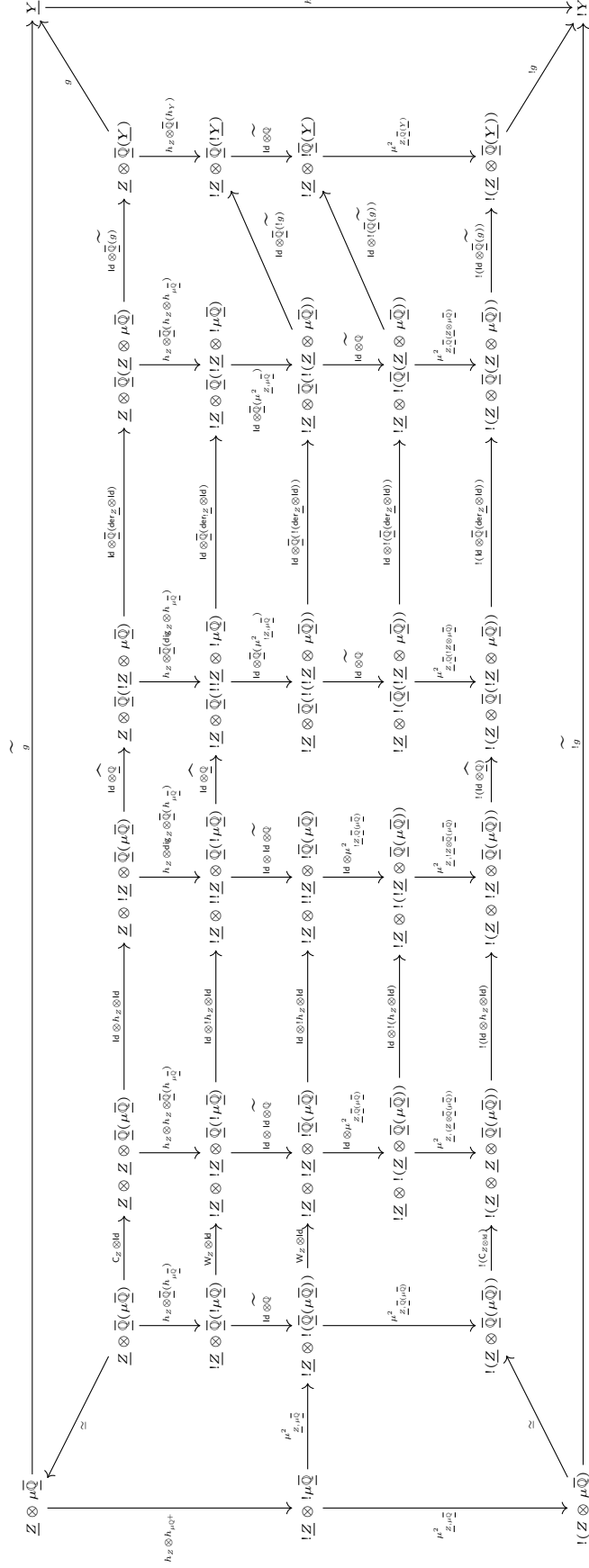


Figure 2.8: Proof of Lemma 90

Interpretation of proofs and formulas

Definition 91 $(\vec{\zeta}^-, \vec{\xi}^+) = (\zeta_1^-, \dots, \zeta_n^-, \xi_1^+, \dots, \xi_p^+)$ is adapted to P (respectively N) if $(\vec{\zeta}^-, \vec{\xi}^+)$ is repetition-free (no literal appears twice in the whole list) and all the free variables of P (respectively N) appear in that list.

Let P be a positive formula and assume that $(\vec{\zeta}^-, \vec{\xi}^+) = (\zeta_1^-, \dots, \zeta_n^-, \xi_1^+, \dots, \xi_p^+)$ is adapted to P . We define its interpretation as a (n, p) -positive functor $\llbracket P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}$. If $n = p = 0$ we simply write $\llbracket P \rrbracket$.

If N is a negative formula and $(\vec{\zeta}^-, \vec{\xi}^+) = (\zeta_1^-, \dots, \zeta_n^-, \xi_1^+, \dots, \xi_p^+)$ is adapted to N we define its interpretation as $(\llbracket N^\perp \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})^\perp$.

The definition of $\llbracket P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}$ is by induction on P and uses the constructions of Section 2.3 in the obvious way: $\llbracket P_1 \otimes P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = \otimes \circ (\llbracket P_1 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}, \llbracket P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})$, $\llbracket P_1 \oplus P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = \oplus \circ (\llbracket P_1 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+}, \llbracket P_2 \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})$, $\llbracket !N \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = ! \circ (\llbracket N^\perp \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+})^\perp$, and $\llbracket \mu Z.P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} = \mu(\llbracket P \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+, Z})$.

Lemma 92 Let A be a formula and π be a type valuation $N_1/\xi_1^-, \dots, N_n/\xi_n^-, P_1/\zeta_1^+, \dots, P_k/\zeta_k^+$. Then

$$\llbracket A[\pi] \rrbracket = \llbracket A \rrbracket_{\vec{\zeta}^-, \vec{\xi}^+} \circ (\llbracket N_1 \rrbracket, \dots, \llbracket N_n \rrbracket, \llbracket P_1 \rrbracket, \dots, \llbracket P_k \rrbracket).$$

The proof of Lemma 92 is by a straightforward induction on A . Notice that the substituting formulas are closed, so the composition in Lemma 92 is just an application.

We define the interpretation of terms, as usual, by induction on proofs based on Figure 2.3. As we have different judgments, we must have different ways of interpreting terms which is explained as follows:

- if $\vdash \mathcal{N} \mid p : P$, then $\llbracket p \rrbracket_{\mathcal{N}}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket, \llbracket P \rrbracket)$;
- if $\vdash \mathcal{N}, P \mid t : N$, then $\llbracket t \rrbracket_{\mathcal{N}, P}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket, \llbracket P \rrbracket)$;
- if $c \vdash \mathcal{N}, P$, then $\llbracket c \rrbracket_{\mathcal{N}, P}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket, \llbracket P \rrbracket)$;
- if $\vdash \mathcal{N} \mid t : N$, then $\llbracket t \rrbracket_{\mathcal{N}} \in \mathcal{L}(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket, \perp)$;
- if $c \vdash \mathcal{N}$, then $\llbracket c \rrbracket_{\mathcal{N}} \in \mathcal{L}(\llbracket \mathcal{N}^\perp \rrbracket, \perp)$.

We only give the full details of the interpretation of $\bar{\kappa}_{N, \zeta} \alpha.(c; s)$: the other cases are defined diagrammatically in Figure 2.9.

Remember that $c \vdash \mathcal{N}, P, \alpha : R^\perp [P^\perp/\zeta^-]$ and $\vdash \mathcal{N}[Q] \mid s : P^\perp$. By induction hypothesis, we know that $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp [P/\zeta^+] \rrbracket, \llbracket P \rrbracket)$, and $\llbracket s \rrbracket_{\mathcal{N}}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket, \llbracket Q \rrbracket)$.

By Lemma 92, $\llbracket N^\perp [P/\zeta^+] \rrbracket = (\llbracket N^\perp \rrbracket_{\zeta^+}) \circ (\llbracket P \rrbracket)$. So, we have $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+ \in \mathcal{L}^!(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket_{\zeta^+}(\llbracket P \rrbracket), \llbracket P \rrbracket)$, and we denote $\llbracket c \rrbracket_{\mathcal{N}, P, N[P^\perp/\zeta^-]}^+$ by f . Hence, by Lemma 90, we have a

morphism $\tilde{f} \in \mathcal{L}'(\llbracket \mathcal{N}^\perp \rrbracket \otimes \mu \llbracket N^\perp \rrbracket_{\zeta^+}, \llbracket P \rrbracket)$. And by the interpretation of formula, we know that $\llbracket \mu \zeta^+ . N^\perp \rrbracket = \mu \llbracket N^\perp \rrbracket_{\zeta^+}$. So, we have $\tilde{f} \in \mathcal{L}'(\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mu \zeta^+ . N^\perp \rrbracket, \llbracket P \rrbracket)$. Notice that \tilde{f} is unique satisfying the condition of Lemma 90. So we define $\llbracket \bar{\kappa}_{N, \zeta \alpha} . (c; s) \rrbracket_{\mathcal{N}, Q}^+$ as the following morphism in the category \mathcal{L}'

$$\begin{array}{ccc} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mu \zeta^+ . N^\perp \rrbracket & \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mu \zeta^+ . N^\perp \rrbracket \\ & & \downarrow \text{Id} \otimes \tilde{f} \\ \llbracket Q \rrbracket & \xleftarrow{\llbracket s \rrbracket_{\mathcal{N}, Q}^+} & \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \end{array}$$

Soundness

We first state the substitution lemmas (Lemmas 93, 94 & 95).

Lemma 93 Assume that $\vdash \mathcal{N} \mid p : P$.

$$c \vdash \mathcal{N}, \alpha : P^\perp, \text{ then } \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, P^\perp}^+} \perp$$

$$\llbracket c/p/\alpha \rrbracket_{\mathcal{N}}$$

$$c \vdash \mathcal{N}, \alpha : P^\perp, Q, \text{ then } \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \xrightarrow{\llbracket c \rrbracket_{\mathcal{N}, P^\perp, Q}^+} \llbracket Q \rrbracket$$

$$\llbracket c/p/\alpha \rrbracket_{\mathcal{N}}^+$$

If $\vdash \mathcal{N}, \alpha : P^\perp \mid t : N$, then

$$\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+ \otimes \text{Id}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{cur}^{-1}(\llbracket t \rrbracket_{\mathcal{N}, P} \rrbracket)} \perp$$

$$\llbracket t/p/\alpha \rrbracket_{\mathcal{N}}$$

If $\vdash \mathcal{N}, \alpha : P^\perp, Q \mid t : N$, then

$$\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+ \otimes \text{Id}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\llbracket t \rrbracket_{\mathcal{N}, P^\perp, Q}^+} \llbracket Q \rrbracket$$

$$\llbracket t/p/\alpha \rrbracket_{\mathcal{N}}^+$$

$$\text{If } \vdash \mathcal{N}, \alpha : P^\perp \mid q : Q, \text{ then } \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket P \rrbracket \xrightarrow{\llbracket q \rrbracket_{\mathcal{N}, P^\perp}^+} \llbracket Q \rrbracket$$

$$\llbracket q/p/\alpha \rrbracket_{\mathcal{N}}^+$$

Lemma 94 Assume that $\vdash \mathcal{N}, Q \mid t : N$.

$$\text{If } c \vdash \mathcal{N}, N^\perp, \text{ then } \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket c \rrbracket_{\mathcal{N}, N^\perp}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \xrightarrow{\llbracket t \rrbracket_{\mathcal{N}, Q}^+} \llbracket Q \rrbracket$$

$$\llbracket c/t/\bullet \rrbracket_{\mathcal{N}, Q}^+$$

If $\vdash \mathcal{N}, N^\perp \mid s : M$, then

$$\begin{array}{c} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket s \rrbracket_{\mathcal{N}, N^\perp}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \xrightarrow{\llbracket t \rrbracket_{\mathcal{N}, Q}^+} \llbracket Q \rrbracket \\ \searrow \hspace{15em} \nearrow \\ \llbracket s[t/\bullet] \rrbracket_{\mathcal{N}, Q}^+ \end{array}$$

Lemma 95 Assume that $\vdash \mathcal{N} \mid t : N$.

$$\text{If } c \vdash \mathcal{N}, N^\perp, \text{ then } \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket c \rrbracket_{\mathcal{N}, N^\perp}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \xrightarrow{\llbracket t \rrbracket_{\mathcal{N}} \perp} \perp \\ \searrow \hspace{15em} \nearrow \\ \llbracket c[t/\bullet] \rrbracket_{\mathcal{N}}$$

If $\vdash \mathcal{N}, N^\perp \mid s : M$, then

$$\begin{array}{c} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket \xrightarrow{C_{\llbracket \mathcal{N}^\perp \rrbracket} \otimes \text{Id}} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket M^\perp \rrbracket \xrightarrow{\text{Id} \otimes \llbracket s \rrbracket_{\mathcal{N}, N^\perp}^+} \llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket N^\perp \rrbracket \xrightarrow{\llbracket t \rrbracket_{\mathcal{N}} \perp} \perp \\ \searrow \hspace{15em} \nearrow \\ \llbracket s[t/\bullet] \rrbracket_{\mathcal{N}} \end{array}$$

The following lemma relates syntactic functoriality (section 2.1.3) and the semantical one (Lemma 87).

Lemma 96 Assume that $\vdash \mathcal{N}, P \mid s : N$, let Q be a positive formula and let ζ be a variable. Let also π be a type valuation such that $\zeta^+, \zeta^- \notin \text{dom}(\pi)$. Then

$$\begin{aligned} \llbracket Q[\pi, s/\zeta^+] \rrbracket_{\mathcal{N}, Q[\pi, P/\zeta^+]}^+ &= \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+} \llbracket \llbracket \mathcal{N}^\perp \rrbracket \rrbracket (\overrightarrow{\text{Id}}, \overrightarrow{\text{Id}}, \llbracket s \rrbracket_{\mathcal{N}, P}^+) \\ \llbracket Q[\pi, s/\zeta^-] \rrbracket_{\mathcal{N}, Q[\pi, N/\zeta^-]}^+ &= \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^-} \llbracket \llbracket \mathcal{N}^\perp \rrbracket \rrbracket ((\llbracket s \rrbracket_{\mathcal{N}, P}^+)^{\perp}, \overrightarrow{\text{Id}}, \overrightarrow{\text{Id}}) \end{aligned}$$

Proof of this lemma is a tedious verification of interpretations of $Q[\pi, s/\zeta^+]$ and $Q[\pi, s/\zeta^-]$ based on the proof of Proposition 73.

As a direct conclusion of Lemma 87 and Lemma 96, we have the following corollary:

Corollary 97 Assume that $\vdash \mathcal{N}, P \mid s : N$, let Q be a positive formula and let ζ be a variable. Let also π be a type valuation such that $\zeta^+, \zeta^- \notin \text{dom}(\pi)$. Then

$$\begin{aligned} \llbracket Q[\pi, s/\zeta^+] \rrbracket_{\mathcal{N}, Q[\pi, P/\zeta^+]}^+ &\in \mathcal{L}^! (\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}^+ (\overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}, \llbracket N^\perp \rrbracket), \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}^+ (\overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}, \llbracket P \rrbracket})) \\ \llbracket Q[\pi, s/\zeta^-] \rrbracket_{\mathcal{N}, Q[\pi, N/\zeta^-]}^+ &\in \mathcal{L}^! (\llbracket \mathcal{N}^\perp \rrbracket \otimes \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}^+ (\overrightarrow{\llbracket P \rrbracket}^{\perp}, \overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket}), \llbracket Q \rrbracket_{\text{dom}(\pi), \zeta^+}^+ (\llbracket N^\perp \rrbracket^{\perp}, \overrightarrow{\llbracket N \rrbracket}, \overrightarrow{\llbracket P \rrbracket})) \end{aligned}$$

Lemma 98 If $c \rightarrow d$, then either $\llbracket c \rrbracket_{\mathcal{N}} = \llbracket d \rrbracket_{\mathcal{N}}$ or $\llbracket c \rrbracket_{\mathcal{N}, P}^+ = \llbracket d \rrbracket_{\mathcal{N}, P}^+$ depending on the typing derivation of c and d .

Proof: We prove it by case analysis of c and d based on the reduction system in Figure 2.4.

▷ If we have $s \star \tilde{\kappa}.c \rightarrow c[s/\bullet]$:

$$\begin{aligned} \llbracket s \star \tilde{\kappa}.c \rrbracket_{\mathcal{N}} &= ((\llbracket s \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket \tilde{\kappa}.c \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= ((\llbracket s \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket c \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= \llbracket c[s/\bullet] \rrbracket_{\mathcal{N}} \text{ by Lemma 95} \end{aligned}$$

▷ If we have $\kappa\alpha.c \star p \rightarrow c[p/\alpha]$:

$$\begin{aligned} \llbracket \kappa\alpha.c \star p \rrbracket_{\mathcal{N}} &= ((\llbracket \kappa\alpha.c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= ((\llbracket c \rrbracket_{\mathcal{N}, P^{\perp}}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= \llbracket c[p/\alpha] \rrbracket_{\mathcal{N}} \text{ by Lemma 93} \end{aligned}$$

▷ If we have $\langle s_1, s_2 \rangle \star \text{in}_i(p) \rightarrow s_i \star p$:

$$\begin{aligned} \llbracket \langle s_1, s_2 \rangle \star \text{in}_i(p) \rrbracket_{\mathcal{N}} &= ((\llbracket \langle s_1, s_2 \rangle \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket \text{in}_i(p) \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= ((\llbracket s_1 \rrbracket_{\mathcal{N}}, \llbracket s_2 \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes ((\tilde{\pi}_i) \circ (\llbracket p \rrbracket_{\mathcal{N}}^{\dagger})))) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= ((\llbracket s_i \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \quad \text{since } \mathcal{L} \text{ is co-cartesian} \\ &= \llbracket s_i \star p \rrbracket_{\mathcal{N}} \end{aligned}$$

▷ If we have $\kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rightarrow c[p_1/\alpha_1, p_2/\alpha_2]$:

$$\begin{aligned} \llbracket \kappa(\alpha_1, \alpha_2).c \star (p_1, p_2) \rrbracket_{\mathcal{N}} &= ((\llbracket \kappa(\alpha_1, \alpha_2).c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket (p_1, p_2) \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= ((\llbracket c \rrbracket_{\mathcal{N}, P_1^{\perp}, P_2^{\perp}}) \circ (\text{Id} \otimes ((\llbracket p_1 \rrbracket_{\mathcal{N}}^{\dagger} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^{\dagger}) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket})))) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \text{ by Diagram (2.21)} \\ &= ((\llbracket c \rrbracket_{\mathcal{N}, P_1^{\perp}, P_2^{\perp}}) \circ ((\text{Id} \otimes \text{Id} \otimes \llbracket p_1 \rrbracket_{\mathcal{N}}^{\dagger} \otimes \mathbf{W}_{\llbracket P_2 \rrbracket}) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \otimes P_2 \rrbracket}))) \circ (\text{Id} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^{\dagger}) \circ \mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= ((\llbracket c[p_1/\alpha_1] \rrbracket_{\mathcal{N}_2}) \circ (\text{Id} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^{\dagger})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\ &= \llbracket (c[p_1/\alpha_1])[p_2/\alpha_2] \rrbracket_{\mathcal{N}} \\ &= \llbracket c[p_1/\alpha_1, p_2/\alpha_2] \rrbracket_{\mathcal{N}} \end{aligned}$$

$$\begin{array}{ccc} \llbracket \mathcal{N}^{\perp} \rrbracket & \xrightarrow{\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}} & \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket \\ \downarrow \mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket} & & \downarrow \text{Id}_{\llbracket \mathcal{N}^{\perp} \rrbracket} \otimes \mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket} \\ \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket & & \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket \\ \downarrow \text{Id}_{\llbracket \mathcal{N}^{\perp} \rrbracket} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^{\dagger} & & \downarrow \text{Id}_{\llbracket \mathcal{N}^{\perp} \rrbracket} \otimes \llbracket p_1 \rrbracket_{\mathcal{N}}^{\dagger} \otimes \llbracket p_2 \rrbracket_{\mathcal{N}}^{\dagger} \\ \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P_2 \rrbracket & & \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P_1 \rrbracket \otimes \llbracket P_2 \rrbracket \\ \downarrow \mathbf{C}_{\llbracket \mathcal{N} \otimes P_2 \rrbracket} & & \downarrow \llbracket c \rrbracket_{\mathcal{N}, P_1^{\perp}, P_2^{\perp}} \\ \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P_2 \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P_2 \rrbracket & & \perp \\ \downarrow \text{Id} \otimes \text{Id} \otimes \llbracket p_1 \rrbracket_{\mathcal{N}}^{\dagger} \otimes \mathbf{W}_{\llbracket P_2 \rrbracket} & \nearrow \llbracket c \rrbracket_{\mathcal{N}, P_1^{\perp}, P_2^{\perp}} & \\ \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P_2 \rrbracket \otimes \llbracket P_1 \rrbracket & & \end{array} \quad (2.21)$$

▷ If we have $\kappa_{\perp}.c \star () \rightarrow c$:

$$\begin{aligned}
 \llbracket \kappa_{\perp}.c \star () \rrbracket_{\mathcal{N}} &= ((\llbracket \kappa_{\perp}.c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket () \rrbracket)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\
 &= ((\llbracket c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket () \rrbracket)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\
 &= ((\llbracket c \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \mathbf{W}_{\llbracket \mathcal{N}^{\perp} \rrbracket})) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\
 &= (\llbracket c \rrbracket_{\mathcal{N}}) \circ (\text{Id}) = \llbracket c \rrbracket_{\mathcal{N}}
 \end{aligned}$$

If we have $\text{der } p \star s^! \rightarrow s \star p$:

$$\begin{aligned}
 \llbracket \text{der } p \star s^! \rrbracket_{\mathcal{N}} &= ((\llbracket \text{der } p \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket s^! \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\
 &= ((\text{cur}^{-1}((\text{der}_{\llbracket P^{\perp} \rrbracket})^{\perp} \circ \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ ((\text{Id} \otimes !(\llbracket \text{cur}(s)_{\llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P, \perp \rrbracket} \rrbracket_{\mathcal{N}})) \circ (h_{\llbracket \mathcal{N}^{\perp} \rrbracket})))) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \\
 &= ((\llbracket s \rrbracket_{\mathcal{N}}) \circ (\text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+)) \circ (\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}) \text{ by Diagram 2.22}
 \end{aligned}$$

$$\begin{array}{ccc}
 \llbracket \mathcal{N}^{\perp} \rrbracket & \xrightarrow{\mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket}} & \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket \\
 \downarrow \mathbf{C}_{\llbracket \mathcal{N}^{\perp} \rrbracket} & & \downarrow \text{Id} \otimes h_{\llbracket \mathcal{N}^{\perp} \rrbracket} \\
 \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket \mathcal{N}^{\perp} \rrbracket & & \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket !\mathcal{N}^{\perp} \rrbracket \\
 \downarrow \text{Id} \otimes \llbracket p \rrbracket_{\mathcal{N}}^+ & & \downarrow \text{Id} \otimes !(\llbracket \text{cur}(s)_{\llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P, \perp \rrbracket} \rrbracket_{\mathcal{N}}) \\
 \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket P \rrbracket & & \llbracket \mathcal{N}^{\perp} \rrbracket \otimes \llbracket !P \rrbracket \\
 \downarrow \llbracket s \rrbracket_{\mathcal{N}} & \swarrow \text{cur}^{-1}((\text{der}_{\llbracket P^{\perp} \rrbracket})^{\perp} \circ \llbracket p \rrbracket_{\mathcal{N}}^+) & \\
 \perp & &
 \end{array} \tag{2.22}$$

▷ If we have $\bar{\kappa}_{Q^{\perp}, \zeta} \alpha.c \star \text{fd}(p) \rightarrow (Q \left[\bar{\kappa}_{Q^{\perp}, \zeta} \alpha.c / \zeta^+ \right] \star p) [\kappa \alpha.c / \bullet]$: See Figure 2.10 ■

Notice that the computations in proof of Lemma 98 can be done diagrammatically to see the typing, as what we did in Diagram 2.21 and Diagram 2.22, but this would take a lot of time and space.

Theorem 99 *If $c \rightarrow^* d$, then either $\llbracket c \rrbracket_{\mathcal{N}} = \llbracket d \rrbracket_{\mathcal{N}}$ or $\llbracket c \rrbracket_{\mathcal{N}, P}^+ = \llbracket d \rrbracket_{\mathcal{N}, P}^+$ depending on the typing derivation of c and d .*

Proof: By induction on the length of reductions from c to d , and using Lemma 98. ■

2.3.1 Two concrete models of $\kappa\mu\text{LLP}$

In this section, we will show how **Rel** and **Nuts** are indeed two instances of our categorical model of $\kappa\mu\text{LLP}$. To do so, we need to extend the notion of variable set, definition 25, to deal the notion of positive functor. For **Rel**, it is quite easy and as follows.

Let \mathbf{Rel}_n be the class of all n -ary variable sets, so that $\mathbf{Rel}_0 = \text{Obj}(\mathbf{Rel})$. As we saw in Chapter 1, $(\mathbf{Rel}, (\mathbf{Rel}_n)_{n \in \mathbb{N}})$ is indeed a model of μLL . So, as we saw in Section 2.3, we can define positive functor on top of \mathbf{Rel}_n in order to consider **Rel** a model of $\kappa\mu\text{LLP}$. And this is just an instance of the general definition of a positive functor in the general case. Hence a

positive $n + p$ -ary variable set is a (n, p) - positive functor $\mathbb{P} : \mathbf{Rel}^{n+p} \rightarrow \mathbf{Rel}$ such that $\underline{\mathbb{P}}$ is an $n + p$ -ary variable set.

Before going to deal with **Nuts**, we first relate **Rel** and the point typing system.

Let P be a closed formula. In Section 2.2.2 we have introduced a point typing system. The points of this typing system are essentially the same thing as points of the relational model, the only difference being that a point $a : P$ contains the σb construct at places corresponding to occurrences of μ or ν type constructs in P . It is easy to see that there is a bijective correspondence between the $a : P$ and the $a' \in \llbracket P \rrbracket$ in **Rel**. To simplify notations we consider this correspondence as the identity function.

Lemma 100 *Let P be a closed positive formula. Then $a \tilde{P} [a_1, \dots, a_n]$ iff $(a, [a_1, \dots, a_n]) \in \llbracket P \rrbracket$.*

The proof is a straightforward case analysis.

Theorem 101 *Let $\Phi = (\alpha_1 : a_1 : N_1, \dots, \alpha_k : a_k : N_k)$ be a negative point typing context and let $\mathcal{N} = (\alpha_1 : N_1, \dots, \alpha_n : N_n)$. Then*

- $(a_1, \dots, a_n, b) \in \llbracket t \rrbracket_{\mathcal{N}}^{\mathbf{Rel}}$ iff $t \vdash \Phi \mid b : N$
- $(a_1, \dots, a_n, c, b) \in \llbracket t \rrbracket_{\mathcal{N}, P}^{\mathbf{Rel}}$ iff $t \vdash \Phi, c : P \mid b : N$
- $(a_1, \dots, a_n, b) \in \llbracket q \rrbracket_{\mathcal{N}}^{\mathbf{Rel}}$ iff $q \vdash \Phi \mid b : P$
- $(a_1, \dots, a_n) \in \llbracket c \rrbracket_{\mathcal{N}}^{\mathbf{Rel}}$ iff $c \vdash \Phi$
- $(a_1, \dots, a_n, b) \in \llbracket c \rrbracket_{\mathcal{N}, P}^{\mathbf{Rel}}$ iff $c \vdash \Phi, b : P$

The proof of Theorem 101 is a simple verification, and it uses Lemma 100.

Non-uniform totality spaces as a model of $\kappa\mu\text{LLP}$

Just as what we did for VNUTS, we introduce a combinatorial gadget that we will use to describe more explicitly positive functors in the **Nuts** model. This can be done because composition in **Rel** and in **Nuts**, as well as all the operations of μLL , are defined exactly in the same way.

Definition 102 *Let $n, p \in \mathbb{N}$, an $n + p$ -ary positive VNUTS is a pair $\mathbb{P} = (|\mathbb{P}|, \mathcal{T}(\mathbb{P}))$ such that*

1. $|\mathbb{P}|$ is an $n + p$ -ary positive variable set,
2. $\underline{\mathbb{P}} = (|\mathbb{P}|, \mathcal{T}(\mathbb{P}))$ is an $n + p$ -ary VNUTS, and
3. for any $\vec{A}, \vec{B} \in \text{Obj}(\mathbf{Nuts}^n)$ one has $|\tilde{\mathbb{P}}|_{|\vec{A}|, |\vec{B}|} \in \mathbf{Nuts}(|\mathbb{P}|(\vec{A}, !\vec{B}), !(|\mathbb{P}|(\vec{A}, \vec{B})))$.

Lemma 103 *Any $n + p$ -ary positive VNUTS $\mathbb{P} : \mathbf{Nuts}^{n+p} \rightarrow \mathbf{Nuts}$ induces a (n, p) - positive functor $\mathcal{P} : \mathbf{Nuts}^{n+p} \rightarrow \mathbf{Nuts}$ which satisfies*

- $|\overline{\mathcal{P}}(\vec{A}, \vec{B})| = |\overline{\mathbb{P}}(|\vec{A}|, |\vec{B}|)|$,
- $\mathcal{T}(\overline{\mathcal{P}}(\vec{A}, \vec{B})) = \mathcal{T}(\mathbb{P})(\vec{A}, \vec{B})$,
- $\overline{\mathcal{P}}(\vec{t}) = |\overline{\mathbb{P}}(\vec{t})| \in \mathbf{Nuts}(\overline{\mathbb{P}}(\vec{A}_1, \vec{B}_1), \overline{\mathbb{P}}(A_2, \vec{B}_2))$. for $\vec{t} \in \mathbf{Nuts}^{n+p}((\vec{A}_1, \vec{B}_1), (A_2, \vec{B}_2))$,
- $\widehat{\mathcal{P}}_{A, \vec{B}} = |\widehat{\mathbb{P}}_{|\vec{A}|, |\vec{B}|}|$ and $\widetilde{\mathcal{P}}_{\vec{A}, \vec{B}} = |\widetilde{\mathbb{P}}_{|\vec{A}|, |\vec{B}|}|$

and \mathbb{P} can be retrieved from \mathcal{P} .

Examples of nat

As we can see in Chapter 1, the interpretation of \mathbf{nat} in \mathbf{Nuts} is a totality space $(\mathbb{N}, \{u \subseteq \mathbb{N} \mid u \neq \emptyset\})$. So, $[\mathbf{nat}^\perp] = (\mathbb{N}, \{\mathbb{N}\})$. Hence $[!\mathbf{nat}^\perp] = (\mathcal{M}_{\text{fin}}(\mathbb{N}), \{\mathcal{M}_{\text{fin}}(\mathbb{N})\})$, since $\{\mathcal{M}_{\text{fin}}(\mathbb{N})\} = \uparrow \{u^{(1)} \mid u \in \{\mathbb{N}\}\}$. The inductive definition of h_{nat} means this set is the least one satisfying

- $(\overline{0}, k[\overline{0}]) \in h_{\text{nat}}$ for any $k \in \mathbb{N}$, and
- If $(\overline{n}, [\overline{n}_1, \dots, \overline{n}_k]) \in h_{\text{nat}}$, then $(\overline{n+1}, [\overline{n_1+1}, \dots, \overline{n_k+1}]) \in h_{\text{nat}}$.

Hence we have $h_{\text{nat}} = \{(\overline{n}, k[\overline{n}]) \mid k, n \in \mathbb{N}\}$.

2.4 Application: a normalization property

To follow this section, one might need to look at some observations in the previous section and chapter, namely Theorem 101 and Lemma 100, and Lemma 7 in Chapter 0.

As the first application of Theorem 84, one can prove the following:

Theorem 104 *If $c \vdash P$, then c is strongly normalizing.*

Proof: Since $[P]_{\mathbf{Rel}} \neq \emptyset$ and $[c]_{\mathbf{Rel}}$ is a morphism in \mathbf{Rel}^\dagger , by Lemma 7, we know that $[c]_{\mathbf{Rel}} \neq \emptyset$. So, there exists $a \in [c]$. By Theorem 101, we have $c \vdash a : P$. By Theorem 84, we have $c \in \perp(a : P)$. By definition of $\perp(a : P)$, there is a $p \in (|a|_P \setminus \tilde{\kappa})$ such that $c \rightarrow^* \bullet \star p$. ■

As an example of theorem above, if we have $c \vdash \mathbf{nat}$, then c is strongly normalizing, meaning that $c \rightarrow^* \bullet \star p$ such that $p \in |n|_{\text{nat}}$ for some n where $n : \mathbf{nat}$. And using Lemma 105, one can extract algorithmically the value of n from p .

In order to state and prove the next lemma, first we define two mutually recursive partial functions val (on terms p such that $\vdash p : \mathbf{nat}$) and val^\oplus (on terms q such that $\vdash q : 1 \oplus \mathbf{nat}$) with values in \mathbb{N} . If $\vdash p : \mathbf{nat}$ then

- if $p = \text{fd}(q)$ then we have $\vdash q : 1 \oplus \mathbf{nat}$ and we take $\text{val } p = \text{val}^\oplus q$;
- else if $p = \tilde{\kappa}.c$ then if $c \rightarrow^* \bullet \star p_0 \in \perp$ (that is $p_0 \notin \tilde{\kappa}$) then $\vdash p_0 : \mathbf{nat}$ and we take $\text{val } p = \text{val } p_0$.

And if $\vdash | q : 1 \oplus \text{nat}$ then

- if $q = \text{in}_1 r$ then if $r = ()$ or if $r = \tilde{\kappa}.c$ and $c \rightarrow^* \bullet \star ()$, then $\text{val}^\oplus q = 0$;
- if $q = \text{in}_2 p'$ then we have $\vdash | p' : \text{nat}$ and we take $\text{val}^\oplus q = 1 + \text{val} p'$;
- if $q = \tilde{\kappa}.c$ and $c \rightarrow^* \bullet \star q_0 \in \perp\!\!\!\perp$ (that is $q_0 \notin \tilde{\kappa}$) then we have $\vdash | q_0 : 1 \oplus \text{nat}$ and we take $\text{val}^\oplus q = \text{val}^\oplus q_0$.

Lemma 105 *If $\vdash | p : \text{nat}$ and $p \in |n|_{\text{nat}}$ for some $n \in \mathbb{N}$ then $\text{val} p$ is defined and has n as value.*

The proof is a straightforward application of the definition of $|n|_{\text{nat}}$.

As it is mentioned in Section 2.1.4, we need to consider $?\text{nat}$ to encode the type of natural numbers from Gödel's system \mathbb{T} into $\kappa\mu\text{LLP}$. So, it is worth trying to prove that if we have $c \vdash ?\text{nat}$, then c is normalizing, and one can indeed extract an integer from c . However, to prove such a result, one needs to look at the interpretation of c in non-uniform totality spaces, mentioned in Section 2.3.1.

Theorem 106 *If $c \vdash \alpha : ?\text{nat}$, then c is normalizing.*

Proof: Extend $\kappa\mu\text{LLP}$ with a constant \circ which is a new command, typed by $\circ \vdash \mathcal{N}$ where \mathcal{N} is an arbitrary *negative* context¹. We also extend the point typing system with the rule $\circ \vdash (\alpha_i : a_i : N_i)_{i=1}^n$ under the proviso that $a_i \in \widetilde{N_i^\perp}$ for all $i \in \underline{n}$. The benefit of this extension is that now $\perp\!\!\!\perp$ contains closed commands.

We have $\vdash | () : 1, \vdash | 1 | \bullet : \perp$ and $\vdash | \kappa_\perp.\circ : \perp$. Therefore $\vdash | \text{rd} = \text{it}(\bullet, \kappa_\perp.\circ) : \text{nat}^\perp$. This negative term is a “reader of integer” which behaves as follows: $\text{rd} \star 0 \rightarrow^* \circ$ and $\text{rd} \star \text{succ } p \rightarrow^* \text{rd} \star p$. By induction on $n \in \mathbb{N}$ it is not hard to check that $\forall n \in \mathbb{N} \vdash | \text{rd} : \bar{n} : \text{nat}^\perp$. Let $m = [\bar{n}_1, \dots, \bar{n}_k] : ?\text{nat}$, we have therefore $\vdash | \text{rd}^\dagger : m : !\text{nat}^\perp$ and hence $\text{rd}^\dagger \in \|m\|_{!\text{nat}^\perp}$ by Theorem 84.

If $c \vdash \alpha : m = [\bar{n}_1, \dots, \bar{n}_k] : ?\text{nat}$ then by Theorem 84 we have $c \left[\text{rd}^\dagger / \alpha \right] \in \perp\!\!\!\perp$. By Lemma 75 this implies that the reduction of c terminates, so that $c \rightarrow^* t \star \alpha$ where $\vdash | \alpha : m_1 : ?\text{nat} | t : m_2 : ?\text{nat}$, t is not of shape $\kappa\beta.d$ and $m = m_1 + m_2$ (because we have $m \in \widetilde{!\text{nat}^\perp} [m_1, m_2]$ by the (**i-cut**) rule). So we must have $t = \text{der } p$ with $\vdash | \alpha : m_1 : ?\text{nat} | p : \bar{n} : \text{nat}$ for some $n \in \mathbb{N}$ and $m_2 = [\bar{n}]$. So by Theorem 84 we have $p \left[\text{rd}^\dagger / \alpha \right] \in |\bar{n}|_{\text{nat}}$ and hence by Lemma 105 $\text{val } p = n$. As a whole we have described an algorithm which, under the assumption that $c \vdash \alpha : m : ?\text{nat}$ for some m , produces an integer n that we denote as $\text{val}_\alpha c$, and we have $m = m_1 + [\bar{n}]$.

By the results of Section 2.3 and 2.3.1 we know that the interpretation of $\llbracket c \rrbracket_{?\text{nat}}$ in **Rel** belongs to $\mathcal{T}(\llbracket ?\text{nat} \rrbracket)$ where $\llbracket ?\text{nat} \rrbracket$ is the interpretation of $?\text{nat}$ in **Nuts** (that is $\llbracket c \rrbracket_{?\text{nat}}$ is *total* in the interpretation of that type). As explained in Section 2.3.1 $\llbracket \text{nat}^\perp \rrbracket = (\mathbb{N}, \{\mathbb{N}\})$ and hence $\mathcal{T}(\llbracket !\text{nat}^\perp \rrbracket) = \{\mathcal{M}_{\text{fin}}(\mathbb{N})\}$. Therefore $\llbracket c \rrbracket_{?\text{nat}} \cap \mathcal{M}_{\text{fin}}(\mathbb{N}) \neq \emptyset$ that is $\llbracket c \rrbracket_{?\text{nat}} \neq \emptyset$ so by Theorem 101 there is $m : ?\text{nat}$ such that $c \vdash \alpha : m : ?\text{nat}$.

¹This may seem surprising at first sight but remember that weakening is freely available for all negative formulas. So the real meaning of this rule is $\circ \vdash$ which is the familiar 0-ary mix rule of LL. It is easy to check that all properties of $\kappa\mu\text{LLP}$ proven so far are still valid for this extension.

■

Moreover, assume $c \vdash \alpha : ?\text{nat}$ and c does not contain \circ . By the considerations above the integer $n = \text{val}_\alpha c$ is well defined (and we have given an algorithm to compute it consisting in executing $\kappa\mu\text{LLP}$ commands). Moreover $c \vdash \alpha : m_1 + [\bar{n}] : ?\text{nat}$ for some $m_1 : ?\text{nat}$. Using the model **RelW** introduced in [AE15] (it is a variation on the relational model where each object is a set E equipped with a function $E \rightarrow \mathbb{Z}$) it is possible to prove that, because c does not contain \circ , one has $m_1 = []$. So we actually have $c \vdash \alpha : [\bar{n}] : ?\text{nat}$, that is, the value obtained by execution coincides with the value provided by the semantics.

2.5 Sum up of Chapter 2

Sum up of Chapter 1

- Developing a polarized λ -calculus for LL with fixpoints: $\kappa\mu\text{LLP}$.
- Prove interpretation lemma and a normalization theorem for $\kappa\mu\text{LLP}$: Theorem 84, 104, and 106.
- Providing a categorical model for $\kappa\mu\text{LLP}$ and prove its soundness: Theorem 99.
- Providing two concrete models of $\kappa\mu\text{LLP}$: **Rel** and **Nuts**.

Chapter 3

Non-well-founded LL with fixpoints and its denotational semantics

In Chapter 0, we saw the system μLL which is a version of propositional LL with least and greatest fixed points extending propositional μMALL with exponentials [Bae12]. In [Dou17, BDS16], the μMALL_∞ system, which is non-well-founded multiplicative and additive LL with two rules for unfolding fixed-points, is studied. They have defined a syntactic notion of validity on proofs in order to distinguish sound from unsound proofs. The same idea is done for the additive linear logic in [FS13, San02], and they moreover provided a categorical semantics for their system. However, this notion of validity condition goes back to μ -calculus [Wal00, NW96]. A similar notion of validity of infinite proof can be found also in [Bro06, BS07].

We consider an extension of μMALL_∞ , called μLL_∞ , with exponentials. One of our purposes in this chapter is to develop a more Curry-Howard oriented point of view on μLL_∞ through the denotational semantics. At least this Curry-Howard perspective will help us to formalize syntactical statement about the meaning of proofs as we discussed an example of it in Section 3.3. Moreover as there are different validity conditions on the μLL_∞ proofs (such as straight-thread, bouncing, etc. [BDKS20]), we hope that the denotational semantic shed light on the comparison of those different validity conditions.

As we saw in Chapter 1, there is a categorical semantics of μLL , and two instances of this categorical setting are **Rel** (the category of sets and relations), and **Nuts** (the category of sets equipped with a notion of totality and relations preserving it).

We first recap the language and the inference rules of μLL_∞ in Section 3.1. In Section 3.2, the interpretation of formula and proofs in **Rel** and **Nuts** are provided together with a soundness result in 3.2.1. Finally, the main contribution is established in Section 3.2.2 which relates validity condition and totality of **Nuts**. More precisely, we show that each μLL_∞ valid proof will be interpreted as a total element in **Nuts**.

3.1 Non-well-founded LL with fixpoints (μLL_∞)

The syntax of μLL_∞ formulas is exactly the same as the one for μLL in 0.2.2. We just recall it here again to have a self-contained text for this section.

$A, B, \dots := 1 \mid \perp \mid A \otimes B \mid A \wp B \mid 0 \mid \top \mid A \oplus B \mid A \& B \mid !A \mid ?A \mid \zeta \mid \mu\zeta A \mid \nu\zeta A$

The inference rules of μLL_∞ is the extension of rules of [Dou17, BDS16] with exponential rules of LL. In other words, the inference rules of μLL_∞ are the rules of LL (presented in Chapter 0, and [Gir87a]) plus the two following ones:

$$\frac{\vdash \Gamma, F [\mu\zeta F / \zeta]}{\vdash \Gamma, \mu\zeta F} \quad (\mu - \text{fold}) \quad \frac{\vdash \Gamma, F [\nu\zeta F / \zeta]}{\vdash \Gamma, \nu\zeta F} \quad (\nu - \text{fold})$$

A μLL_∞ pre-proof is a possibly infinite tree, generated by the inference rules of μLL_∞ . Among all μLL_∞ pre-proofs, the regular/circular proofs are the ones that have finitely many sub-trees. Those circular proofs can be represented with finite proof-trees having back-edges or labels.

As an example, consider the type of natural numbers $\text{nat} = \mu\zeta.(1 \oplus \zeta)$ and its dual $\text{nat}^\perp = \nu\zeta.(\perp \& \zeta)$. The two following proofs, that correspond to the constant zero and identity functions on natural numbers, are two examples of circular μLL_∞ proofs. The $*$ on the left side of the turnstyle symbol in $\vdash \text{nat}, \text{nat}^\perp$ means that the proof of this sequent is the same as the proof we are defining, that is the proof of the conclusion sequent. So this proof is a representation of a really infinite proof tree.

$$\frac{\frac{\frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu - \text{fold})}{\vdash \text{nat}} (\perp)}{\vdash \text{nat}, \perp} \quad \frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu - \text{fold}) \quad \frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1) \quad \frac{\frac{\frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu - \text{fold})}{\vdash \text{nat}} (\perp)}{\vdash \text{nat}, \perp} \quad \frac{\frac{\frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu - \text{fold})}{\vdash \text{nat}} (\perp)}{\vdash \text{nat}, \perp} \quad \frac{\frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu - \text{fold}) \quad \frac{\frac{\overline{\vdash 1} (1)}{\vdash 1} (\oplus_1)}{\vdash 1 \oplus \text{nat}} (\mu - \text{fold})}{\vdash \text{nat}, \text{nat}^\perp} (\oplus_2)}{\vdash \text{nat}, \text{nat}^\perp} (\&)}{\vdash \text{nat}, \perp \& \text{nat}^\perp} (\&)}{\vdash \text{nat}, \perp \& \text{nat}^\perp} (\nu - \text{fold}) \quad \frac{\vdash \text{nat}, \perp \& \text{nat}^\perp}{\vdash \text{nat}, \text{nat}^\perp} (\&)}{\vdash \text{nat}, \text{nat}^\perp} (\nu - \text{fold})$$

However, in general the pre-proofs can be unsound. For instance one can provide a pre-proof for any sequent $\vdash \Gamma$ (and in particular a pre-proof of the empty sequent \vdash) as follows:

$$\frac{\frac{\frac{\vdots}{\vdash \nu\zeta.\zeta} (\nu - \text{fold})}{\vdash \nu\zeta.\zeta} (\nu - \text{fold})}{\vdash \Gamma} \quad \frac{\frac{\frac{\vdots}{\vdash \Gamma, \mu\zeta.\zeta} (\mu - \text{fold})}{\vdash \Gamma, \mu\zeta.\zeta} (\mu - \text{fold})}{\vdash \Gamma} (\text{cut})$$

In [Dou17, BDS16], a criterion, called *validity condition*, is provided in order to distinguish proper proofs from pre-proofs. We only sum up this criterion here and provide some examples, and we refer to [Dou17, BDS16] for more details.

Definition 107 We define the relation \rightarrow_{FL} on formulas as follows:

- $A * B \rightarrow_{FL} A$ and $A * B \rightarrow_{FL} B$ where $*$ is a binary LL connective.

- $@A \rightarrow_{FL} A$ where $@$ is either $!$ or $?$.
- $\sigma\zeta F \rightarrow_{FL} F[\sigma\zeta F/\zeta]$ where σ is either ν or μ .

The Fischer-Ladner sub-formula of a formula F is the formula G such that $F \rightarrow_{FL}^* G$.

Lemma 108 (Corollary 2.1 of [Dou17]) *The Fischer-Ladner closure of a formula (the set of its Fischer-Ladner sub-formulas) is finite.*

Definition 109 *The usual sub-formula relation on μLL_∞ is defined as follows:*

- $A * B \rightarrow_{sub} A$ and $A * B \rightarrow_{sub} B$ where $*$ is a binary LL connective.
- $@A \rightarrow_{sub} A$ where $@$ is either $!$ or $?$.
- $\sigma\zeta F \rightarrow_{sub} F$ where σ is either ν or μ .

The usual sub-formula of a formula F is the formula G such that $F \rightarrow_{sub}^* G$.

Notice that the usual sub-formula relation is an ordering, so, we write $A \leq_{sub} B$ if A is sub-formula of B , i.e, we have $A \rightarrow_{sub}^* B$.

Remark 25 *The variable ζ is a subformula of $A = (\nu\zeta.\zeta)\otimes\zeta$. However, there are two different ζ in the formula A , one is the bound ζ and the other is the free one. To distinguish them, one can use the notion of occurrence [Dou17]. As we also need this notion for Definition 118, we provide it here.*

Definition 110 (Definition 1.21 of [Dou17]) *Let Σ be the alphabet $\{l, r, i\}$. An address is a word over the alphabet Σ . The empty word is denoted by ϵ . An occurrence, is a formula A together with an address α , denoted as A_α . If α is sub-word of β , we denote it by $\alpha \preceq_{sw} \beta$.*

Definition 111 *We say than an occurrence B_β is sub-occurrence of A_α if $B \leq_{sub} A$ and $\beta \preceq_{sw} \alpha$.*

The notion of Fischer-Ladner-suboccurrence is similar to the previous one:

Definition 112 *We say that B_β is a FL-sub-occurrence of A_α if $A \rightarrow_{FL}^* B$ and $\beta \preceq_{sw} \alpha$.*

To have simpler notations, we will drop the addresses of an occurrence and simply write it as a formula when the address can be recovered from context easily.

Definition 113 *A thread is a sequence $t = (A_i)_{i \in \omega}$ of occurrences such that for all i either A_{i+1} is a FL-sub-occurrence of A_i or $A_i = A_{i+1}$.*

Now to define the criterion, one also needs to annotate all formulas in a proof by an address. One can do this automatically by defining it on inferences rules of μLL_∞ . This is done in Section 1.5.2 of [Dou17]. We just provide these annotations for a few rules, and one can see in an obvious way how to do the same for the others. For instance the (\mathfrak{A}) and $(\&)$ rule will be as follows:

$$\frac{\vdash \Gamma, A_{\alpha l}, B_{\alpha r}}{\vdash \Gamma, (A \wp B)_{\alpha}} (\wp) \quad \frac{\vdash \Gamma, A_{\alpha l} \quad \vdash \Gamma, B_{\alpha r}}{\vdash \Gamma, (A \& B)_{\alpha}} (\&)$$

The pattern for fixpoint rules and exponentials rules is as follows:

$$\frac{\vdash \Gamma, A_{\alpha i}}{\vdash ?\Gamma, (!A)_{\alpha}} (\text{p}) \quad \frac{\vdash \Gamma, (F[\nu\zeta F/\zeta])_{\alpha i}}{\vdash \Gamma, (\nu\zeta F)_{\alpha}} (\nu - \text{fold})$$

Definition 114 A branch of a proof is a sequence of sequent $(\Gamma_i^{j_i})_{i \in \omega}$ for $j_i \in \{1, 2\}$ such that $\Gamma_{i+1}^{j_{i+1}}$ is the j_i th premises of $\Gamma_i^{j_i}$ in the proof tree for each i .

So with each branch is associated a set of threads. Notice that this thread is not unique in general. For instance, we have two threads in the following proof; one is $t_1 = (\mu\zeta \zeta)_{\alpha}, (\mu\zeta \zeta)_{\alpha}, (\mu\zeta \zeta)_{\alpha}, \dots$ and the other one is $t_2 = (\nu\zeta \zeta)_{\beta}, (\nu\zeta \zeta)_{\beta i}, (\nu\zeta \zeta)_{\beta i i}, \dots$. Since the only rule applying in the following proof is the $(\nu - \text{fold})$ rule, the thread corresponding to the $\mu\zeta \zeta$ is stationary.

$$\frac{\vdots}{\vdash \mu\zeta \cdot \zeta, \nu\zeta \cdot \zeta} (\nu - \text{fold})$$

$$\frac{\vdash \mu\zeta \cdot \zeta, \nu\zeta \cdot \zeta}{\vdash \mu\zeta \cdot \zeta, \nu\zeta \cdot \zeta} (\nu - \text{fold})$$

Definition 115 If $t = (A_i)_{i \in \omega}$ is a thread we use \bar{t} for the sequence obtained by forgetting the addresses of the occurrences of t . And we denote by $\text{Inf}(t)$ the set of formulas that occurs infinitely often in \bar{t} .

For example, \bar{t}_1 and \bar{t}_2 of two threads t_1 and t_2 of the example above are $\mu\zeta \zeta, \mu\zeta \zeta, \dots$ and $\nu\zeta \zeta, \nu\zeta \zeta, \dots$ respectively.

Proposition 116 If a thread t is coming from a branch of μLL_{∞} pre-proof, then $\text{Inf}(t)$ admits a minimum with respect to the usual sub-formula ordering \leq_{sub} defined in 109.

Proof: Proposition 2.7 of [Dou17]. The idea of the proof is based on the observation that $\text{Inf}(t)$ forms a cycle, and roughly speaking, the minimum of $\text{Inf}(t)$ is the nearest to the root in that cycle. \blacksquare

Now, we have all the required material to define the notion of valid threads and then valid proofs.

Definition 117 A valid thread t is a non-stationary thread such that $\min(\text{Inf}(t))$ is a ν -formula with respect to the usual sub-formula ordering \leq_{sub} .

Definition 118 A valid μLL_{∞} proof π is a pre-proof π such that for any infinite branch $\gamma = (\vdash \Gamma_i)_{i \in \omega}$, there is a non stationary valid thread $t = (A_i)_{i > j}$ where $j \in \omega$ and $\forall i > j (A_i \in \Gamma_i)$ and A_{i+1} is a suboccurrence of A_i .

Syntactic functoriality of formulas in μLL_∞

Let $(\zeta, \xi_1, \dots, \xi_k)$ be a list of pairwise distinct propositional variables containing all the free variables of a formula F and let $\vec{C} = (C_1, \dots, C_k)$ be a sequence of closed formulas. Then the following rule is admissible in μLL_∞ :

$$\frac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (F[A/\zeta, \vec{C}/\vec{\xi}])^\perp, F[B/\zeta, \vec{C}/\vec{\xi}]} (\mathfrak{F}_F)$$

The proof is done by induction on the formula F . We will provide two cases below and refer to [Dou17] for the other cases (Definition 2.38 of [Dou17]). Let $F = G_1 \otimes G_2$, then we have the following pre-proof where $G'_i = G \left[\frac{\vec{C}}{\xi} \right]$ for $i = 1, 2$:

$$\frac{\frac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (G'_1[A/\zeta])^\perp, G'_1[B/\zeta]} (\mathfrak{F}_{G_1}) \quad \frac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (G'_2[A/\zeta])^\perp, G'_2[B/\zeta]} (\mathfrak{F}_{G_2})}{\vdash ?\Gamma, ?\Gamma, (G'_1[A/\zeta])^\perp, (G'_2[A/\zeta])^\perp, G'_1[B/\zeta] \otimes G'_2[B/\zeta]} (\otimes)} (\text{c})$$

$$\frac{\vdash ?\Gamma, (G'_1[A/\zeta])^\perp, (G'_2[A/\zeta])^\perp, G'_1[B/\zeta] \otimes G'_2[B/\zeta]}{\vdash ?\Gamma, (G'_1[A/\zeta])^\perp \wp (G'_2[A/\zeta])^\perp, G'_1[B/\zeta] \otimes G'_2[B/\zeta]} (\wp)$$

By induction hypothesis, the pre-proofs of $?\Gamma, (G'_1[A/\zeta])^\perp, G'_1[B/\zeta]$ and $?\Gamma, (G'_2[A/\zeta])^\perp, G'_2[B/\zeta]$ that are obtained respectively by the (\mathfrak{F}_{G_1}) and (\mathfrak{F}_{G_2}) rule are valid proofs. There is neither $(\nu - \text{fold})$ rule nor $(\mu - \text{fold})$ rule in the rest of the pre-proof, so the whole pre-proof is a valid proof. Now assume that $F = \nu\xi G$ (so that $(\zeta, \xi, \xi_1, \dots, \xi_k)$ is a list of pairwise distinct variables containing all free variables of G). Let $G' = G \left[\frac{\vec{C}}{\xi} \right]$ whose only possible free variables are ζ and ξ . Then we have:

$$\frac{\frac{\vdash ?\Gamma, A^\perp, B}{\vdash ?\Gamma, (G'[A/\zeta, (\nu\xi G') [A/\zeta] / \xi)]^\perp, G'[B/\zeta, (\nu\xi G') [B/\zeta] / \xi]} (\mathfrak{F}_{G[(\nu\xi G')[B/\zeta] / \xi]})}{\frac{\vdash ?\Gamma, ((\nu\xi G') [A/\zeta])^\perp, G'[B/\zeta, (\nu\xi G') [B/\zeta] / \xi]}{\vdash ?\Gamma, ((\nu\xi G') [A/\zeta])^\perp, (\nu\xi G') [B/\zeta]} (\nu - \text{fold})} (\mu - \text{fold})$$

The validity of this proof relies on the fact that the thread crossing $(\nu\xi G') [B/\zeta]$ is the validating thread (See Proposition 2.13 of [Dou17]). The case where $F = \nu\xi G$ is symmetrically the same as $F = \nu\xi G$.

3.2 Denotational semantics of μLL_∞

In this section, we will provide an interpretation for μLL_∞ proofs in **Rel**. As the proof trees are possibly non-well-founded, we cannot define their interpretation inductively. The idea is to consider all finite approximations of the proof, and then take the union of the interpretation of them. The precise definition of this approximation is defined in Section 3.2.1. Notice that there is another way of interpreting infinite proofs in [KPP21], based on a notion of *computation* and a well-founded relation on them.

3.2.1 Soundness

Our goal in this section is to prove that if a μLL_∞ proof π reduces to π' via the cut-elimination process, then $\llbracket \pi \rrbracket_{\mathbf{Rel}} = \llbracket \pi' \rrbracket_{\mathbf{Rel}}$. Notice that this reduction can be potentially an infinite reduction, so, one needs to define a notion of limit of an infinite reduction sequence. And to do so, we need to define precisely the notion infinite sequences of proofs. This is done in Section 3.2.1.

The cut-elimination theorem on μMALL_∞ is provided in [Dou17], and it is extended to μLL_∞ in []. The set of primitive (single step) reduction rules of μLL_∞ are the ones for LL plus the following ones (Figure 3.2 of [Dou17]).

The proof

$$\frac{\frac{\frac{\pi}{\vdash \Gamma, F[\mu\zeta F/\zeta]} \quad (\mu\text{-fold}) \quad \frac{\frac{\pi'}{\vdash \Delta, F^\perp[\nu\zeta F^\perp/\zeta]} \quad (\nu\text{-fold})}{\vdash \Delta, \nu\zeta F^\perp} \quad (\text{cut})}}{\vdash \Gamma, \Delta}}$$

will be reduced to

$$\frac{\frac{\pi}{\vdash \Gamma, F[\mu\zeta F/\zeta]} \quad \frac{\pi'}{\vdash \Delta, F^\perp[\nu\zeta F^\perp/\zeta]}}{\vdash \Gamma, \Delta} \quad (\text{cut})$$

Metric completion of finite proofs

What we provide in this section is standard in the literature, and is provided here for the sake of self-containdness. In this section (and only here), we assume that in the inference rules of μLL_∞ , we also have this rule: $\frac{}{\vdash \Gamma} \quad (\Omega)$ for any sequence Γ . And we interpret this rule as the empty set. The reason that we consider this assumption will be clear later, for instance in Definition 121.

Definition 119 *Given a μLL_∞ pre-proof π , we associate a set $\text{Pos}(\pi)$ of positions corresponding to each sequent of π as follows:*

- $\langle 0 \rangle \in \text{Pos}(\pi)$
- Let r be an occurrence of an inference rule in π and that $\langle x \rangle$, which belongs to $\text{Pos}(\pi)$, is the location of this occurrence in π
 - If $r \in \{(\otimes), (\&), (\text{cut})\}$, then both $\langle x0 \rangle$ and $\langle x1 \rangle$ are in $\text{Pos}(\pi)$;
 - Otherwise $\langle x0 \rangle \in \text{Pos}(\pi)$.

The elements of $\text{Pos}(\pi)$ are finite sequences of 0 and 1.

Definition 120 *Given a pre-proof π and $p \in \text{Pos}(\pi)$, we denote by $\text{Proof}(\pi, p)$ the last sequent of the sub-pre-proof of π rooted at position p .*

As an example, consider the following proof π :

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{}{\vdash A^\perp, A} \text{(ax)}}{\vdash A^\perp, A} \text{(d)}}{\vdash ?A^\perp, A} \text{(w)}}{\vdash ?A^\perp, ?B^\perp, A} \text{(w)} \quad \frac{\frac{\frac{\frac{}{\vdash B^\perp, B} \text{(ax)}}{\vdash B^\perp, B} \text{(d)}}{\vdash ?B^\perp, B} \text{(w)}}{\vdash ?A^\perp, ?B^\perp, B} \text{(w)}}{\vdash ?A^\perp, ?B^\perp, A \& B} \text{(\&)} \\
 \frac{\vdash ?A^\perp, ?B^\perp, A \& B}{\vdash ?A^\perp, ?B^\perp, !(A \& B)} \text{(p)}
 \end{array}$$

Then one can represent it by the $\text{Pos}(\pi)$ as follows which is also annotated by the sequents. One can also label the edges by the inference rules.

$$\begin{array}{c}
 \vdash ?A^\perp, ?B^\perp, !(A \& B) \langle 0 \rangle \\
 | \\
 \vdash ?A^\perp, ?B^\perp, A \& B \langle 00 \rangle \\
 \swarrow \quad \searrow \\
 \vdash ?A^\perp, ?B^\perp, A \langle 000 \rangle \quad \vdash ?A^\perp, ?B^\perp, B \langle 001 \rangle \\
 | \quad \quad | \\
 \vdash ?A^\perp, A \langle 0000 \rangle \quad \vdash ?B^\perp, B \langle 0010 \rangle \\
 | \quad \quad | \\
 \vdash A^\perp, A \langle 00000 \rangle \quad \vdash B^\perp, B \langle 00100 \rangle
 \end{array}$$

Definition 121 Let π be a pre-proof and P be a prefix-closed subset of $\text{Pos}(\pi)$. We denote by $\pi(P)$ the sub-pre-proof of π whose set of positions is P , i.e., $\text{Pos}(\pi(P)) = P$.

Notice that if we do not assume having the (Ω) rule, then $\pi(P)$ might not exist.

Definition 122 If π is a pre-proof we denote by $\text{Pos}_i(\pi)$ the subset of $\text{Pos}(\pi)$ that contains only all position of length i , i.e., $\text{Pos}_i(\pi) = \pi(\text{Pos}(\pi) \cap \{0, 1\}^i)$.

Let \mathcal{X} be the set of all μLL_∞ finite proofs. One can define a distance $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, 1]$: $d(\pi, \pi') = 0$ if two proofs π and π' are identical, otherwise $d(\pi, \pi') = \frac{1}{2^k}$, where k is the length of the shortest position at which π and π' differ.

Denote by $C[\mathcal{X}]$ the collection of all Cauchy sequences in \mathcal{X} . Define a relation \sim on $C[\mathcal{X}]$ by

$$(\pi_n) \sim (\pi'_n) \Leftrightarrow \lim_{n \rightarrow \infty} d(\pi_n, \pi'_n) = 0$$

It is easy to see that this is an equivalence relation on $C[\mathcal{X}]$. This definition does not depend on the choice of representatives in the two equivalence classes. Let \mathcal{X}^* be the set of all equivalence classes for \sim . One can define the metric d^* on \mathcal{X}^* as follows where $[(\pi_n)]$ is an equivalence class:

$$d^*([(\pi_n)], [(\pi'_n)]) = \lim_{n \rightarrow \infty} d(\pi_n, \pi'_n)$$

The metric space (\mathcal{X}^*, d^*) is called *metric completion* of \mathcal{X} , and there is standard result showing that this is a complete space.

Proposition 123 *Let \mathcal{X}_∞ be set of all (potentially infinite) μLL_∞ proofs. Then the metric space (\mathcal{X}^*, d^*) is isomorphic to \mathcal{X}_∞ .*

Proof: Since the completion of a metric space is unique up to isometry, it is enough to show that (\mathcal{X}_∞, d') is the completion of \mathcal{X} for a metric d' . That is to show \mathcal{X} is dense in \mathcal{X}_∞ for taking d' same as d .

Take $\pi \in \mathcal{X}_\infty$. Consider the sequence (π_n) where $\pi_n = \pi(\bigcup_{i < n} \text{Pos}_i(\pi))$. We have now $d(\pi, \pi_n) = \frac{1}{2^n}$, so, π is the limit of the sequence (π_n) of finite proofs. ■

As the direct conclusion of 123, the metric space (\mathcal{X}_∞, d) is complete, that is to say every Cauchy sequence of proofs in \mathcal{X}_∞ has a limit inside of \mathcal{X}_∞ .

Remark 26 *In the cut-elimination process of μLL_∞ , for any natural number n , the number of steps of the sequence which reduces a (cut) rule at depth less than n is finite []. So, the cut-elimination reduction has countable length.*

We saw that the metric space (\mathcal{X}_∞, d) is a complete space, but this was a result of the proposition 123. Here we show the completeness of this metric space directly.

Proposition 124 *The metric space (\mathcal{X}_∞, d) is complete.*

Proof: Take a Cauchy sequence (π_n) . First, we define the set P as $\bigcup_i \bigcap_{j > i} \text{Pos}(\pi_i)$. And we also provide a function f that sends a $p \in P$ to a sequent as follows: Since $p \in P$, $\exists i \forall j > i (p \in \text{Pos}(\pi_j) \wedge (\text{Proof}(\pi_i, p) = \text{Proof}(\pi_j, p)))$. So, we define $f(p)$ as $\text{Proof}(\pi_i, p)$ (this does not depend on the choice of i). Now since the sequence (π_n) is a Cauchy sequence, we have $\forall k, \exists N \forall i, j > N (d(\pi_i, \pi_j) < \frac{1}{2^k})$, and therefore $d(\Pi(P, f), \pi_i) < \frac{1}{2^k}$ where $\Pi(P, f)$ is the pre-proof tree that has P as set of its positions and it is labeled by element of $f(P)$ (one can deduce it by the contradiction). Hence the proof $\Pi(P, f)$ is the limit of the (π_n) . ■

We will use this direct proof later in proof of Lemma 126.

We can now provide our definition for interpretation of μLL_∞ pre-proof in **Rel**, and prove that this interpretation preserves the semantics through a Cauchy sequence of proofs.

First, one can define interpretation of a finite μLL_∞ proofs by induction on last inference rule. The interpretation of LL rules is provided in Figure 3, and we just need to take an instance of it for **Rel**. So, we only need to say how we interpret the $(\nu - \text{fold})$ and $(\mu - \text{fold})$ rules in **Rel**, and this is done, in a obvious way, as follows:

$$\left[\left[\frac{\vdots \pi}{\vdash \Gamma, F [\mu\zeta F / \zeta]} (\mu - \text{fold}) \right] \right] = \llbracket \pi \rrbracket \quad \left[\left[\frac{\vdots \pi}{\vdash \Gamma, F [\nu\zeta F / \zeta]} (\nu - \text{fold}) \right] \right] = \llbracket \pi \rrbracket$$

We also take the empty set as the interpretation of the (Ω) rule. Finally, give a μLL_∞ proof π , we define $\llbracket \pi \rrbracket_{\mathbf{Rel}} = \bigcup_{\rho \in \text{fin}(\pi)} \llbracket \rho \rrbracket_{\mathbf{Rel}}$ where $\text{fin}(\pi)$ is the set of all finite sub-pre-proof of π (we are allowed to do this, since we added the (Ω) rule).

Combing Theorem 6 and computing the interpretation of two proofs given by cut-elimination rule of $(\mu - \text{fold}) - (\nu - \text{fold})$ case, one can show the following:

Theorem 125 *Given two finite μLL_∞ proofs π and π' such that π' is obtained from π via one cut-elimination rule, then $\llbracket \pi \rrbracket_{\mathbf{Rel}} = \llbracket \pi' \rrbracket_{\mathbf{Rel}}$.*

Lemma 126 *Let (π_i) be a Cauchy sequence. Then $\llbracket \lim_{n \rightarrow \infty} \pi_i \rrbracket_{\mathbf{Rel}} = \bigcup_i \bigcap_{j > i} \llbracket \pi_j \rrbracket_{\mathbf{Rel}}$.*

Proof: By Proposition 124, $\lim_{n \rightarrow \infty} \pi_i = \Pi(P, f)$ (we are using a notation introduced in the proof of Proposition 124). By definition, $\llbracket \Pi(P, f) \rrbracket_{\mathbf{Rel}} = \bigcup_{\pi \in \text{fin}(\Pi(P, f))} \llbracket \pi \rrbracket_{\mathbf{Rel}}$. Take a $\pi' \in \text{fin}(\Pi(P, f))$. For each $p \in \text{fin}(\Pi(P, f))$, we have $\exists i_p \forall j > i_p (p \in \text{Pos}(\pi_j) \wedge (\text{Proof}(\pi_j, p) = \text{Proof}(\pi', p)))$, by definition. Let i be the maximum among all i_p 's (The set $\text{Pos}(\pi')$ is finite). Then for all $j > i$ we have $\pi' \in \pi_j$. Hence we have the following:

$$\forall \pi' \in \text{fin}(\Pi(P, f)) \quad \forall p \in \pi' \quad \exists i \quad \forall j > i \quad (p \in \pi_j \wedge (\text{Proof}(\pi_j, p) = \text{Proof}(\pi', p)))$$

And that is to say for each $\pi' \in \text{fin}(\Pi(P, f))$, there exists an i such that for all $j > i$, π' is a finite sub-pre-proof of all π_j . Hence $\llbracket \pi' \rrbracket_{\mathbf{Rel}}$ is a subset of $\llbracket \pi_j \rrbracket_{\mathbf{Rel}}$ for all $j > i$, so, $\llbracket \pi' \rrbracket_{\mathbf{Rel}} \subseteq \bigcap_{j > i} \llbracket \pi_j \rrbracket_{\mathbf{Rel}}$. ■

Theorem 127 *Let (π_i) be a Cauchy sequence such that $\forall i, j$ we have $\llbracket \pi_i \rrbracket_{\mathbf{Rel}} = \llbracket \pi_j \rrbracket_{\mathbf{Rel}}$ (We denote by $\llbracket \pi \rrbracket_{\mathbf{Rel}}$ the $\llbracket \pi_i \rrbracket_{\mathbf{Rel}}$ for any i). Then $\llbracket \lim_{n \rightarrow \infty} \pi_i \rrbracket_{\mathbf{Rel}} = \llbracket \pi \rrbracket_{\mathbf{Rel}}$.*

Proof:

$$\begin{aligned} \llbracket \lim_{n \rightarrow \infty} \pi_i \rrbracket_{\mathbf{Rel}} &= \bigcup_i \bigcap_{j > i} \llbracket \pi_j \rrbracket_{\mathbf{Rel}} && \text{By Lemma 126} \\ &= \bigcup_i \llbracket \pi_i \rrbracket_{\mathbf{Rel}} \\ &= \llbracket \pi \rrbracket_{\mathbf{Rel}} \end{aligned}$$

■

And, we can now prove the soundness theorem for μLL_∞ as a direct conclusion of Theorem 125 and Theorem 127:

Corollary 128 *If π and π' are proofs of $\vdash \Gamma$ and π reduces to π' by the cut-elimination rules of μLL_∞ , then $\llbracket \pi \rrbracket_{\mathbf{Rel}} = \llbracket \pi' \rrbracket_{\mathbf{Rel}}$.*

3.2.2 On relation between totality semantics and syntactic validity

What we have seen till now in Section 3.2 is the interpretation of μLL_∞ proofs in \mathbf{Rel} and a soundness theorem for μLL_∞ with respect to \mathbf{Rel} . However, as one might notice, we did not talk about valid proofs. Indeed, Lemma 126 is true in general for any Cauchy sequence of μLL_∞ pre-proofs (not necessary the valid ones). In this section, we provide a denotational account of the validity criterion using the model introduced in Chapter 2, i.e, **Nuts**.

We prove the main result of this chapter which says that the interpretation of any valid proof is a total element, i.e. theorem 134. The proof method is similar to the proof of

soundness of LKID^ω in [Bro06]. However the system of [Bro06] is classical logic with inductive definitions, and this proof is for a Tarskian semantics. We need to adapt that proof in two aspects: considering μLL_∞ instead of LKID^ω , and trying to deal with a denotational semantics instead of a Tarskian semantics. The adaptation for μLL_∞ is somehow done in [Dou17], since there is soundness theorem for μMALL_∞ with respect to the truncated truth semantics (a Tarskian semantics). So, basically, the main point of our proof is turning a Tarskian soundness theorem into a denotational soundness theorem.

From now on, when we write the interpretation of formula as $\llbracket F \rrbracket$, we mean its interpretation in **Nuts**. And when we write the interpretation of proof as $\llbracket \pi \rrbracket$, we mean its interpretation in **Rel**, i.e., $\llbracket \pi \rrbracket_{\mathbf{Rel}}$. However, at the end of this chapter, we will see that indeed this $\llbracket \pi \rrbracket_{\mathbf{Rel}}$ is a total element, so, it is indeed in **Nuts** (but this needs to be proven).

As we saw in the proof of Theorem 45, given a closed formula $\nu\zeta F$, we can define its interpretation in **Nuts** by a transfinite induction (using Knaster–Tarski theorem) considering sequences of totality candidate as follows:

- $U_0^A = \mathcal{P}(\llbracket \nu\zeta F \rrbracket_{\mathbf{Rel}})$ where $\mathcal{P}(X)$ is the power set of X .
- $U_{\alpha+1}^A = \mathcal{T}(\llbracket F \rrbracket)(\llbracket \nu\zeta F \rrbracket_{\mathbf{Rel}}, U_\alpha^A)$.
- $U_\delta^A = \bigcap_{\alpha < \delta} U_\alpha^A$
- and finally, there is an ordinal λ such that $U_\lambda = U_{\lambda+1}$, and we use λ_A for the least such ordinal.

To have simpler notation, we use the notation U_α (and U_λ) freely without mentioning the formula. One can find what the corresponding formula is from the context.

The following definition is borrowed from [Dou17].

Definition 129 *The marked formulas of μLL_∞ are defined as follows where α is an ordinal:*

$$A, B, \dots := 1 \mid 0 \mid \perp \mid \top \mid A \oplus B \mid A \otimes B \mid A \& B \mid A \wp B \mid ?A \mid !B \mid \zeta \mid \mu\zeta F \mid \nu^\alpha\zeta.F \quad (3.1)$$

We denote by A° the label-stripped formula A .

The interpretation of $\nu^\alpha\zeta.F$ in **Nuts** is $\llbracket \nu^\alpha\zeta.F \rrbracket = (\llbracket \nu\zeta F \rrbracket_{\mathbf{Rel}}, U_\alpha)$, and the other marked formulas are interpreted as usual.

Proposition 130 *Let A be a μLL_∞ formula. Then we have $\llbracket \bar{A} \rrbracket = \llbracket A \rrbracket$ where \bar{A} is the marked formula, obtained from A by marking every ν binder of A by the ordinal λ_A .*

The proof of this proposition is obvious.

Lemma 131 *If A is a μLL_∞ formula and $t \notin \mathcal{T}(\llbracket \nu^\alpha\zeta.F \rrbracket)$ ($t \subseteq \llbracket \nu^\alpha\zeta.F \rrbracket$), then there exists an ordinal $\gamma < \alpha$ such that $t \notin \mathcal{T}(\llbracket F [\nu^\gamma\zeta.F/\zeta] \rrbracket)$.*

Proof: If α is a successor ordinal $\delta + 1$ then $U_\alpha = \mathcal{T}(\llbracket F \rrbracket)(\llbracket \nu \zeta F \rrbracket_{\mathbf{Rel}}, U_\delta)$ by definition, and obviously $t \notin \mathcal{T}(\llbracket F \rrbracket)(\llbracket \nu \zeta F \rrbracket_{\mathbf{Rel}}, U_\delta)$. And so $t \notin \mathcal{T}(\llbracket F [\nu \zeta F / \zeta] \rrbracket)$ for $\gamma = \delta$.

If α is a limit ordinal, then: $U_\alpha = \bigcap_{\gamma < \alpha} U_\gamma$, and $t \notin \bigcap_{\gamma < \alpha} U_\gamma = \bigcap_{\delta+1 < \alpha} U_{\delta+1}$. So, there exists an ordinal $\delta + 1 < \alpha$ such that $t \notin U_{\delta+1}$ and we continue as before. ■

Lemma 132 $\mathcal{T}(\llbracket F [\mu \zeta F / \zeta] \rrbracket) = \mathcal{T}(\llbracket \mu \zeta F \rrbracket)$.

Proof: The interpretation of $\mu \zeta F$ is the least fixed-point of θ_F where θ_F is $\mathcal{T}(\llbracket F \rrbracket)$. So, we have:

$$\begin{aligned} \mathcal{T}(\llbracket \mu \zeta F \rrbracket) &= \theta_F(\mathcal{T}(\llbracket \mu \zeta F \rrbracket)) \\ &= \mathcal{T}(\llbracket F \rrbracket)(\llbracket \mu \zeta F \rrbracket_{\mathbf{Rel}}, \mathcal{T}(\llbracket \mu \zeta F \rrbracket)) && \text{by definition of } \theta_F \\ &= \mathcal{T}(\llbracket F [\mu \zeta F / \zeta] \rrbracket) && \text{by Lemma 92} \end{aligned}$$

■

Lemma 133 *If π is a proof of $\vdash \Gamma$ and $\llbracket \pi \rrbracket \notin \mathcal{T}(\llbracket \Gamma \rrbracket)$, then*

1. π has an infinite branch $\gamma = (\vdash \Gamma_i)_{i \in \omega}$ such that $\llbracket \pi_i \rrbracket \notin \mathcal{T}(\llbracket \Gamma_i \rrbracket)$ where π_i is the sub-proof of π rooted in $\vdash \Gamma_i$;
2. and there exists a sequence of functions $(f_i)_{i \in \omega}$ where f_i maps all formulas D of Γ_i to a marked formula $f_i(D)$ such that
 - $(f_i(D))^\circ = D$,
 - one can write $\Gamma_i = \Gamma'_i, C$,
 - and there exists $x \in \mathcal{T}(\llbracket (f_i(\Gamma'_i))^\perp \rrbracket)$ such that $\llbracket \pi_i \rrbracket . x \notin \mathcal{T}(\llbracket f_i(C) \rrbracket)$ where $\Gamma'_i = A_1^i, \dots, A_{n_i}^i$ and $\llbracket (f_i(\Gamma'_i))^\perp \rrbracket = (\llbracket f_i(A_1^i) \rrbracket)^\perp \otimes \dots \otimes (\llbracket f_i(A_{n_i}^i) \rrbracket)^\perp$.

Proof: We set $\Gamma_0 = \Gamma$, and $f_0(D) = \overline{D}$ for all $D \in \Gamma_0$:

- Since $\pi_0 = \pi$, $\llbracket \pi_0 \rrbracket \notin \mathcal{T}(\llbracket \Gamma_0 \rrbracket)$.
- Let C be the principal formula in Γ_0 . The sequent $\vdash f_0(\Gamma_0)$ is denotationally the same as $\vdash (f_0(\Gamma'_0))^\perp \multimap f_0(C)$. By the proposition 130, $\llbracket f_0(D) \rrbracket = \llbracket D \rrbracket$ for all $D \in \Gamma_0$. So, $\llbracket \pi_0 \rrbracket \notin \mathcal{T}(f_0(\Gamma_0))$. That is to say $\llbracket \pi_0 \rrbracket \notin \mathcal{T}(\llbracket (f_0(\Gamma'_0))^\perp \multimap f_0(C) \rrbracket)$. Therefore, by the lemma 33, there exists $x \in \mathcal{T}(\llbracket (f_0(\Gamma'_0))^\perp \rrbracket)$ such that $\llbracket \pi_0 \rrbracket . x \notin \mathcal{T}(\llbracket f_0(C) \rrbracket)$.

Suppose that we have provided Γ_i and f_i for $i \in \omega$. We then define Γ_{i+1} and f_{i+1} depending on the rule applied on $\vdash \Gamma_i$ in π . Let us assume that the formula C is the principal in Γ_i :

- If $C = C_1 \wp C_2$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. $f_i(C) = B_C^1 \wp B_C^2$ where B_C^1 and B_C^2 are two marked formulas, so, we set $f_{i+1}(C_1) = B_C^1$, $f_{i+1}(C_2) = B_C^2$, and $f_{i+1}(F) = f_i(F)$ for the other $F \in \Gamma_{i+1}$:

- Since Γ_i is obtained by applying the \wp rule on Γ_{i+1} , we have $\llbracket \pi_{i+1} \rrbracket = \llbracket \pi_i \rrbracket$, and $\llbracket \Gamma_{i+1} \rrbracket = \llbracket \Gamma_i \rrbracket$. By induction hypothesis, $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket \Gamma_{i+1} \rrbracket)$.
- By induction hypothesis, there exists $x \in \mathcal{T}(\llbracket (f_i(\Gamma'_i))^{\perp} \rrbracket)$ such that $\llbracket \pi_i \rrbracket.x \notin \mathcal{T}(\llbracket f_i(C) \rrbracket)$. So, $\llbracket \pi_{i+1} \rrbracket.x = \llbracket \pi_i \rrbracket.x \notin \mathcal{T}(\llbracket B_C^1 \wp B_C^2 \rrbracket) = ((\mathcal{T}(\llbracket ((B_C^1))^{\perp} \rrbracket) \otimes \llbracket ((B_C^2))^{\perp} \rrbracket))$. So, there is a $y \in \mathcal{T}(\llbracket ((B_C^1))^{\perp} \rrbracket \otimes \llbracket ((B_C^2))^{\perp} \rrbracket)$ such that $\llbracket \pi_{i+1} \rrbracket.x \cap y \neq \emptyset$. Since $y \in \mathcal{T}(\llbracket (B_C^1)^{\perp} \rrbracket \otimes \llbracket (B_C^2)^{\perp} \rrbracket)$, there is $u' \in \mathcal{T}(\llbracket (B_C^1)^{\perp} \rrbracket)$ and $v' \in \mathcal{T}(\llbracket (B_C^2)^{\perp} \rrbracket)$ such that $u' \times v' \subseteq y$. So, $\llbracket \pi_{i+1} \rrbracket.x \cap (u' \times v') = \emptyset$. This statement is equivalent to $(\llbracket \pi_{i+1} \rrbracket.x).u' \cap v' \neq \emptyset$. $\llbracket \pi_{i+1} \rrbracket.x \in$, and this is equivalent to $\llbracket \pi_{i+1} \rrbracket.(x \times u') \cap v' \neq \emptyset$. We have shown till now that there exists $v' \in \mathcal{T}(\llbracket (B_C^1)^{\perp} \rrbracket)$ such that $\llbracket \pi_{i+1} \rrbracket.x' \cap v' \neq \emptyset$ where $x' = x \times u'$. So, by definition, $\llbracket \pi_{i+1} \rrbracket.x' \notin \mathcal{T}(\llbracket B_C^1 \rrbracket)$.
- If $C = C_1 \oplus C_2$, then we proceed as above.
- If $C = C_1 \otimes C_2$. Let us call Γ_{i+1}^1 and Γ_{i+1}^2 for the two premises of $\vdash \Gamma_i$. $f_i(C) = B_C^1 \otimes B_C^2$ where B_C^1 and B_C^2 are two marked formulas. Since $\llbracket \pi_i \rrbracket \notin \mathcal{T}(\llbracket \Gamma_i \rrbracket)$, we have $\llbracket \pi_{i+1}^j \rrbracket \notin \mathcal{T}(\llbracket \Gamma_{i+1}^j \rrbracket)$ for either $j = 1$ or $j = 2$ where π_{i+1}^1 (respectively π_{i+1}^2) is the left (respectively the right) subproof of π_i . Let us assume that it is true for $j = 1$ (the proof of the case $j = 2$ is identical to the case $j = 1$). So we set $\Gamma_{i+1} = \Gamma_{i+1}^1$, $f_{i+1}(C_1) = B_C^1$, and $f_{i+1}(D) = f_i(D)$ for the other $D \in \Gamma_{i+1}^1$.
 - By induction hypothesis, there exists $x' \in \mathcal{T}(\llbracket (f_i(\Gamma_{i+1}^1) \wp \Gamma_{i+1}^2))^{\perp} \rrbracket)$ such that $\llbracket \pi_i \rrbracket.x' \notin \mathcal{T}(\llbracket B_C^1 \otimes B_C^2 \rrbracket)$. Hence $\llbracket \pi_i \rrbracket \notin \mathcal{T}(\llbracket f_i(\Gamma_i) \rrbracket)$ by Lemma 33. So, we have $\llbracket \pi_{i+1}^j \rrbracket \notin \mathcal{T}(\llbracket f_{i+1}(\Gamma_{i+1}^j) \wp B_C^j \rrbracket)$ for either $j = 1$ or $j = 2$. Let us assume that is true for $j = 1$ (the proof of the case $j = 2$ is identical to the case $j = 1$). So, $\llbracket \pi_{i+1}^1 \rrbracket \notin \mathcal{T}(\llbracket (f_{i+1}(\Gamma_{i+1}^1))^{\perp} \multimap B_C^1 \rrbracket)$. And therefore, using lemma 33, there is a $y \in \llbracket (f_{i+1}(\Gamma_{i+1}^1))^{\perp} \rrbracket$ such that $\llbracket \pi_{i+1}^1 \rrbracket.y \notin \mathcal{T}(\llbracket B_C^1 \rrbracket)$.
- If $C = C_1 \& C_2$, then we proceed as above.
- If $C = \mu\zeta F$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. Wlog let us say $\Gamma_i = A_1^i, \dots, A_{n_i}^i, \mu\zeta F$. $f_i(C) = \mu\zeta B_C$ where B_C is a marked formula. By induction hypothesis, there exists $x \in \mathcal{T}(\llbracket (f_i(\Gamma'_i))^{\perp} \rrbracket)$ such that $\llbracket \pi_i \rrbracket.x \notin \mathcal{T}(\llbracket \mu\zeta B_C \rrbracket)$ where $\Gamma'_i = A_1^i, \dots, A_{n_i}^i$. So, $\llbracket \pi_{i+1} \rrbracket.x \notin \mathcal{T}(\llbracket B_C [\mu\zeta B_C / \zeta] \rrbracket)$, since $\llbracket \pi_{i+1} \rrbracket = \llbracket \pi_i \rrbracket$ and lemma 132. Then we set $f_{i+1}(F[C/\zeta]) = B_C [\mu\zeta B_C / \zeta]$ and $f_{i+1}(D) = f_i(D)$ for all the other formula $D \in \Gamma_{i+1}$ in order to have the second property of the lemma 133.
- If $C = \nu\zeta F$, then Γ_{i+1} is the unique premise of $\vdash \Gamma_i$. Wlog, let us say $\Gamma_i = A_1^i, \dots, A_{n_i}^i, \nu\zeta F$. $f_i(C) = \nu^{\theta}\zeta.B_C$ where B_C is a marked formula. By induction hypothesis, there exists $x \in \mathcal{T}(\llbracket (f_i(\Gamma'_i))^{\perp} \rrbracket)$ such that $\llbracket \pi_i \rrbracket.x \notin \mathcal{T}(\llbracket \nu^{\theta}\zeta.B_C \rrbracket)$ where $\Gamma'_i = A_1^i, \dots, A_{n_i}^i$. By Lemma 131, there is an ordinal $\delta < \theta$ such that $\llbracket \pi_{i+1} \rrbracket.x \notin \mathcal{T}(\llbracket B_C [\nu^{\delta}\zeta.B_C / \zeta] \rrbracket)$, since $\llbracket \pi_{i+1} \rrbracket = \llbracket \pi_i \rrbracket$. So, we set $f_{i+1}(F[C/\zeta]) = f_i(F) [\nu^{\delta}\zeta.B_C / \zeta]$ and $f_{i+1}(D) = f_i(D)$ for all the other formula $D \in \Gamma_{i+1}$ in order to have the second property of the lemma.
- If the rule applied to $\vdash \Gamma_i$ is a (cut) rule on the C . Let us say Γ_i is Γ_i^1, Γ_i^2 . By induction hypothesis, $\llbracket \pi_i \rrbracket \notin \mathcal{T}(\llbracket \Gamma_i \rrbracket)$. So, we have either $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket \Gamma_i^1 \wp \Gamma_i^2 \rrbracket)$ or $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket \Gamma_i^2 \wp \Gamma_i^1 \rrbracket)$. Wlog let us say $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket \Gamma_i^1 \wp \Gamma_i^2 \rrbracket)$. Then we take $\Gamma_{i+1} = \Gamma_i^1, C$. And for the f_{i+1} , we define $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma_i^1$, and $f_i(C) = \overline{C}$.

- By induction hypothesis, $\llbracket \pi_i \rrbracket \notin \mathcal{T}(\llbracket f_i(\Gamma_i) \rrbracket)$. So, we have either $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket f_i(\Gamma_i^1) \wp \overline{C} \rrbracket)$ or $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket f_i(\Gamma_i^1) \wp C^\perp \rrbracket)$. So, we can use lemma 33 to deduce the second property as we proceed as the case $C = C_1 \otimes C_2$.
- If the rule applied to $\vdash \Gamma_i$ is a (w) rule, then Γ_{i+1} is the unique premise of the (w) rule. And $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma_{i+1}$. We have $\llbracket \pi_{i+1} \rrbracket \notin \mathcal{T}(\llbracket f_i(\Gamma_{i+1}) \rrbracket) = \mathcal{T}(\llbracket f_{i+1}(\Gamma_{i+1}) \rrbracket)$, since $\llbracket \pi_i \rrbracket \notin \mathcal{T}(\llbracket f_i(\Gamma_i) \rrbracket)$ (here we are also using Theorem 19 of μLL).
- If the rule applied to $\vdash \Gamma_i$ is (c) rule on the formula $?C$, then we proceed as above.
- If the rule applied to $\vdash \Gamma_i$ is (d) rule on the formula $?C$. Let us say $\Gamma_i = \Gamma'_i, ?C$. Then $\Gamma_{i+1} = \Gamma'_i, C$. $f_{i+1}(D) = f_i(D)$ for all $D \in \Gamma'_i$. $f_i(?C) = ?B_C$ where B_C is a marked formula. Then we take $f_{i+1}(C) = B_C$. To show the second property, we can again use Theorem 19 of μLL .
- If the rule applied to $\vdash \Gamma_i$ is (p) rule on the formula $!C$, then we proceed as above.

■

Now, we can state and prove our the main result of this chapter.

Theorem 134 *If π is a valid proof of the sequent $\vdash \Gamma$, then $\llbracket \pi \rrbracket \in \mathcal{T}(\llbracket \Gamma \rrbracket)$.*

Proof: Let us assume $\llbracket \pi \rrbracket \notin \mathcal{T}(\llbracket \Gamma \rrbracket)$. We can then apply Lemma 133 to obtain an infinite branch $(\vdash \Gamma_i)_{i \in \omega}$ and a sequence $(f_i)_{i \in \omega}$ satisfying properties 1 and 2 of Lemma 133. By the definition of valid proof (Definition 118), there exists a valid thread $t = (F_i)_{i \in \omega}$ for the infinite branch $(\vdash \Gamma_i)_{i \in \omega}$. Let $\nu \zeta F$ be the minimal formula formula of t . So, there are infinitely many times in t that we use a ν rule to unfold $\nu \zeta F$. Let $(i_k)_{k \in \omega}$ be the sequence of indices where $\nu \zeta F$ gets unfolded. Then $(\nu \zeta F)_{\alpha_{i_k}}$ is sub-occurrence (Definition 112) of $(\nu \zeta F)_{\alpha_{i'_k}}$ for $k' \geq k$ where α_{i_k} (respectively $\alpha_{i'_k}$) is the address of $\nu \zeta F$ in sequent i_k (respectively i'_k). By the property 2 of Lemma 133, $f_{i_k}(\nu \zeta F) = \nu^{\alpha_k} \zeta.f_{i_k}(F)$. Therefore, by the property 2 of Lemma 133 and by the construction of the f_i in the proof of Lemma 133, the sequence $(\alpha_k)_{k \in \omega}$ is strictly decreasing. As this contradicts the well-foundedness property of the ordinals we obtain the required contradiction and conclude that $\llbracket \pi \rrbracket \in \mathcal{T}(\llbracket \Gamma \rrbracket)$. ■

We end this chapter by the following two remarks.

Remark 27 *One might think of the following statement as the converse of Theorem 134. If π is a pre-proof of the sequent $\vdash \Gamma$ such that $\llbracket \pi \rrbracket \in \mathcal{T}(\llbracket \Gamma \rrbracket)$, then π is a valid proof. This statement is not necessarily true, and there are many counterexamples indeed. For instance, take $F = \mu \zeta (\perp \& (\zeta \wp \zeta))$ and $G = \nu \xi (1 \oplus (\xi \wp \xi))$ and the following pre-proof π where $\sigma = ((\nu - \text{fold}), (\oplus_2), (\wp))^\omega$:*

$$\frac{\frac{\frac{\overline{\vdash 1}}{\vdash 1} \text{ (1)}}{\vdash 1 \oplus (G \wp G)} \text{ } (\oplus_1)}{\frac{\vdash G}{\vdash \perp, G} \text{ } (\perp)} \text{ } (\nu\text{-fold}) \quad \frac{\frac{\frac{\overline{\vdash \perp, F, G, G}^\sigma}{\vdash \perp, F, G \wp G} \text{ } (\wp)}{\vdash \perp, F, 1 \oplus (G \wp G)} \text{ } (\oplus_2)}{\vdash \perp, F, G} \text{ } (\nu\text{-fold}) \quad \frac{\frac{\frac{\overline{\vdash \perp, F, F, G}^\sigma}{\vdash \perp \& (F \wp F), F, F, G} \text{ } (\&)}{\vdash F, F, F, G} \text{ } (\wp)}{\vdash F \wp F, F, G} \text{ } (\&)}{\vdash \perp \& (F \wp F), F, G} \text{ } (\mu\text{-fold})}$$

This pre-proof is not valid, since there is no valid thread in the rightmost branch. The interpretation of π in \mathbf{Rel} is $\llbracket \pi \rrbracket_{\mathbf{Rel}} = \{((1, *), (1, *))\}$. However, $\llbracket \pi \rrbracket_{\mathbf{Rel}} \in \mathcal{T}(\llbracket F \wp G \rrbracket)$.

Notice that there are two ways to see that $\llbracket \pi \rrbracket_{\mathbf{Rel}} \in \mathcal{T}(\llbracket F \wp G \rrbracket)$. One can compute the interpretation of the formula $F \wp G$ in \mathbf{Nuts} . And one can also provide a valid proof π' of $\vdash F, G$ such that $\llbracket \pi \rrbracket_{\mathbf{Rel}} = \llbracket \pi' \rrbracket_{\mathbf{Rel}}$. Consider indeed the following pre-proof π' :

$$\frac{\frac{\frac{\overline{\vdash 1}}{\vdash 1} \text{ (1)}}{\vdash 1 \oplus (G \wp G)} \text{ } (\oplus_1)}{\frac{\vdash G}{\vdash \perp, G} \text{ } (\perp)} \text{ } (\nu\text{-fold}) \quad \frac{\frac{\frac{\overline{\vdash F \wp F, G, G}^\omega}{\vdash F \wp F, G \wp G} \text{ } (\wp)}{\vdash F \wp F, 1 \oplus (G \wp G)} \text{ } (\oplus_2)}{\vdash F \wp F, G} \text{ } (\nu\text{-fold})}$$

This proof π' is a valid proof, since the thread $t = G_\alpha, (1 \oplus (G \wp G))_{\alpha i}, (G \wp G)_{\alpha i r}, G_{\alpha i r i}, \dots$ is a valid thread ($\min(\text{Inf}(t)) = G$). We also have $\llbracket \pi' \rrbracket_{\mathbf{Rel}} = \{((1, *), (1, *))\}$, and hence using Theorem 134, we know that $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket \in \mathcal{T}(\llbracket F \wp G \rrbracket)$.

Remark 28 The following example is another example of non-valid proof whose interpretation is total.

$$\frac{\frac{\overline{\vdash \nu\zeta.\zeta, \mu\zeta.\zeta} \text{ (ax)}}{\vdash \nu\zeta.\zeta, \mu\zeta.\zeta} \text{ } (\nu\text{-fold})}{\vdash \nu\zeta.\zeta} \text{ } (\text{cut}) \quad * \vdash \nu\zeta.\zeta \text{ } (\text{cut})$$

This example differs however from the one of Remark 27. It is true that this pre-proof does not respect the validity criterion of, but it is valid with respect to the more recent criterion of [Dou17, BDS16], but this proof is considered as a valid proof in a more recent work [BDKS20]. That is why we hope that denotational semantic helps us to understand which validity conditions is more appropriate. However, this is just a hope till now. The only thing that we can say for the moment is that the notion of totality provides a sort of maximal notion for validity as valid proofs should be interpreted as total elements. To see this, let us assume that there is another notion of validity which will not be interpreted as total elements. In particular, take a proof π of $\vdash \text{nat}^\perp, \text{nat}$ such that $\llbracket \pi \rrbracket \notin \mathcal{T}(\llbracket \text{nat} \rrbracket \multimap \llbracket \text{nat} \rrbracket)$.

As $\mathcal{T}([\text{nat}] \multimap [\text{nat}]) = \{f \subseteq \mathbb{N} \times \mathbb{N} \mid \forall n \exists m \text{ s.t. } (n, m) \in f\}$, then this says that there is a finite, hence valid, proof σ of a natural number such that the cut-elimination procedure of π and σ will not terminate. Hence we can learn that that notion of validity is not a good one, as it does not enjoy cut-elimination. Hence one can also ask this question: can we extend such a reasoning to all μLL_∞ types? The same question is asked by Girard for second order type, and it is still an open question [Gir86].

3.3 Inductive vs circular linear logic proofs

In this section we will talk about the comparison between μLL proofs and μLL_∞ circular proofs. As it is mentioned in [Dou17], if a sequent $\vdash \Gamma$ is provable in μLL , then it is provable in μLL_∞ . This can be done by translating a μLL proof π of $\vdash \Gamma$ into a circular μLL_∞ proof π' of $\vdash \Gamma$ that we will denote it by $\text{Trans}(\pi)$. This translation can be done by induction on π :

- We have the following for $r \in \{(1), (\text{ax}), (\perp), (\wp), (\top), (\oplus_1), (\oplus_2), (\text{w}), (\text{c}), (\text{d}), (\text{p}), (\mu\text{-fold})\}$:

$$\text{Trans} \left(\frac{\pi}{\frac{\vdash \Delta}{\vdash \Gamma}} r \right) = \frac{\text{Trans}(\pi)}{\frac{\vdash \Delta}{\vdash \Gamma}} r$$

- We have the following for $r \in \{(\text{cut}), (\otimes), (\&)\}$:

$$\text{Trans} \left(\frac{\frac{\pi_1}{\vdash \Delta_1} \quad \frac{\pi_2}{\vdash \Delta_2}}{\vdash \Gamma} r \right) = \frac{\text{Trans}(\pi_1) \quad \text{Trans}(\pi_2)}{\vdash \Gamma} r$$

- And finally $\text{Trans} \left(\frac{\frac{\pi}{\vdash ?\Gamma, A^\perp, F[A/\zeta]} (\nu\text{-rec}')}{\vdash ?\Gamma, A^\perp, \nu\zeta F} \right)$ is the following circular proof using the functoriality of formulas given in Section 3.1:

$$\frac{\frac{\frac{\pi}{\vdash ?\Gamma, A^\perp, F[A/\zeta]} \quad \frac{\frac{* \vdash ?\Gamma, A^\perp, \nu\zeta F}{\vdash ?\Gamma, (F[A/\zeta])^\perp, F[\nu\zeta F/\zeta]} (\wp_F)}{\vdash ?\Gamma, (F[A/\zeta])^\perp, \nu\zeta F} (\nu\text{-fold})}{\vdash ?\Gamma, ?\Gamma, A^\perp, \nu\zeta F} (\text{cut})}{* \vdash ?\Gamma, A^\perp, \nu\zeta F} (\text{c})$$

And finally, Proposition 2.14 of [Dou17] ensures that $\text{Trans}(\pi)$ is a valid μLL_∞ proof. Our main goal in the section is to prove that the semantic is preserved via this operation $\text{Trans}()$. To do so, first of all, we need to say what the interpretation of a μLL_∞ circular proof is in any categorical model of μLL . The interpretation of each inference rule of μLL_∞ is exactly identical to the inference rule of μLL given in Section 1.1.3. To interpret the μLL_∞ circular proofs, the general idea is to associate a system of equation on the morphisms of the given category to the proof, and then proving that it has a solution which we take it as the interpretation of the circular proof. This is done in the case of additive linear logic in [FS13, San02]. However, in this manuscript we only do this on the circular proofs that are coming from the translation of an inductive proof, i.e, image of the operation $\text{Trans}()$, and we leave this question for all μLL_∞ circular proofs to a future work.

Theorem 135 *Let π be a μLL proof. Then we have $\llbracket \pi \rrbracket = \llbracket \text{Trans}(\pi) \rrbracket$ where the interpretation is given in a model $(\mathcal{L}, \overrightarrow{\mathcal{L}})$ of μLL .*

Proof: The proof is by induction on π . Let us assume that the last inference rule is a $(\nu - \text{rec}')$ rule so that π is the following proof:

$$\frac{\pi' \quad \vdash ?\Gamma, A^\perp, F[A/\zeta]}{\vdash ?\Gamma, A^\perp, \nu\zeta F} (\nu - \text{rec}')$$

Let $f = \llbracket \text{Trans}(\pi) \rrbracket$. By definition of $\text{Trans}(\pi)$ given above, f should satisfy the following diagram:

$$\begin{array}{ccc} \llbracket \Gamma^\perp \rrbracket \otimes A & \xrightarrow{f} & \llbracket \nu\zeta F \rrbracket \\ \downarrow C_{\llbracket \Gamma^\perp \rrbracket} \otimes \text{Id} & & \uparrow \simeq \\ \llbracket \Gamma^\perp \rrbracket \otimes \llbracket \Gamma^\perp \rrbracket \otimes A & & \overline{\llbracket F \rrbracket}(\llbracket \nu\zeta F \rrbracket) \\ \downarrow \text{Id} \otimes \llbracket \pi' \rrbracket & & \uparrow \overline{\llbracket F \rrbracket}(f) \\ \llbracket \Gamma^\perp \rrbracket \otimes \overline{\llbracket F \rrbracket}(\llbracket A \rrbracket) & \xrightarrow{\widehat{\llbracket F \rrbracket}} & \overline{\llbracket F \rrbracket}(\llbracket \Gamma^\perp \rrbracket \otimes \llbracket A \rrbracket) \end{array}$$

By the construction given in Section 1.1.3, the interpretation of π is the unique morphism $\llbracket \pi \rrbracket \in \mathcal{L}(\llbracket \Gamma^\perp \rrbracket \otimes A, \llbracket \nu\zeta F \rrbracket)$ satisfying the following diagram:

$$\begin{array}{ccc} \llbracket \Gamma^\perp \rrbracket \otimes A & \xrightarrow{\llbracket \pi \rrbracket} & \llbracket \nu\zeta F \rrbracket \\ \downarrow C_{\llbracket \Gamma^\perp \rrbracket} \otimes \text{Id} & & \uparrow \simeq \\ \llbracket \Gamma^\perp \rrbracket \otimes \llbracket \Gamma^\perp \rrbracket \otimes A & & \overline{\llbracket F \rrbracket}(\llbracket \nu\zeta F \rrbracket) \\ \downarrow \text{Id} \otimes \llbracket \pi' \rrbracket & & \uparrow \overline{\llbracket F \rrbracket}(\llbracket \pi \rrbracket) \\ \llbracket \Gamma^\perp \rrbracket \otimes \overline{\llbracket F \rrbracket}(\llbracket A \rrbracket) & \xrightarrow{\widehat{\llbracket F \rrbracket}} & \overline{\llbracket F \rrbracket}(\llbracket \Gamma^\perp \rrbracket \otimes \llbracket A \rrbracket) \end{array}$$

Hence, by Lemma 13, we have $\llbracket \pi \rrbracket = \llbracket \text{Trans}(\pi) \rrbracket$. ■

We end this chapter by the following remark:

Remark 29 *Using the operator $\text{Trans}()$, we saw that if a sequent $\vdash \Gamma$ is provable in μLL , then it is also provable in μLL_∞ . However, the converse of this statement is an open question. By the converse, we mean the following question. If a sequent $\vdash \Gamma$ has a circular μLL_∞ proof, then is it true that it also has a μLL proof? The same question is also asked in the literature for other logical systems such as the inductive definitions in classical logic for the first-order*

language [BS10] and it is shown that it has a negative answer for that system [BT19]. It is also shown that this question has a positive answer for intuitionistic logic when both systems (inductive and circular) contain Heyting arithmetic [BT17]. For LL, what we know is that the provability of μLL_∞ circular proofs is strictly included in the provability of arbitrary μLL_∞ proofs based on the recent result of [DDS22] (Theorem 18 of [DDS22]).

3.4 Sum up of Chapter 3

Sum up of Chapter 3

- Provide two denotational models for μLL_∞ in **Rel** and **Nuts**, and prove its soundness: Theorem 128.
- Relate the syntactic validity conditions on μLL_∞ pre-proofs and totality notion of **Nuts**: Theorem 134.
- The semantic is preserved via the translation of the μLL proofs into μLL_∞ proofs: Theorem 135.

4.1 A categorical setting for Tree exponentials

Let \mathcal{L} be a category, \otimes be a binary functor $\mathcal{L}^2 \rightarrow \mathcal{L}$ and 1 be an object of \mathcal{L} .

A *comagma* in \mathcal{L} is a triple $M = (\underline{M}, w_M, c_M)$ where \underline{M} is an object of \mathcal{L} , $w_M \in \mathcal{L}(\underline{M}, 1)$ and $c_M \in \mathcal{L}(\underline{M}, \underline{M} \otimes \underline{M})$, without further conditions. Given comagmas M and N , a comagma morphism $M \rightarrow N$ is an element f of $\mathcal{L}(\underline{M}, \underline{N})$ such that the diagrams

$$\begin{array}{ccc} \underline{M} & \xrightarrow{f} & \underline{N} \\ w_M \searrow & & \swarrow w_N \\ & 1 & \end{array} \qquad \begin{array}{ccc} \underline{M} & \xrightarrow{f} & \underline{N} \\ c_M \downarrow & & \downarrow c_N \\ \underline{M} \otimes \underline{M} & \xrightarrow{f \otimes f} & \underline{N} \otimes \underline{N} \end{array}$$

commute. We use $\mathbf{Comag}(\mathcal{L})$ for this category.

Given an object X of \mathcal{L} , a comagma over X is a tuple $M = (\underline{M}, w_M, c_M, d_M)$ where $(\underline{M}, w_M, c_M)$ is a comagma (simply denoted as M) and $d_M \in \mathcal{L}(\underline{M}, X)$.

Given an object X of \mathcal{L} , we can organize the class of comagmas over X into a category $\mathbf{Comag}_X(\mathcal{L})$: an element of $\mathbf{Comag}_X(\mathcal{L})(M, N)$ is an $f \in \mathbf{Comag}(\mathcal{L})(M, N)$ such that

$$\begin{array}{ccc} \underline{M} & \xrightarrow{f} & \underline{N} \\ d_M \searrow & & \swarrow d_N \\ & X & \end{array}$$

A *free magmatic quasi-exponential* (FMQE) is an operation which, with each object X of \mathcal{L} , associates a terminal object $E^t(X) = (!^t X, w_X^t, c_X^t, der_X^t)$ of $\mathbf{Comag}_X(\mathcal{L})$.

4.1.1 Functoriality of a FMQE

Assume we are given such an FMQE. Let $f \in \mathcal{L}(X, Y)$, we have a functor $f_* : \mathbf{Comag}_X(\mathcal{L}) \rightarrow \mathbf{Comag}_Y(\mathcal{L})$ defined on objects by $f_*(M) = (\underline{M}, w_M, c_M, f d_M)$ and acting as the identity on morphisms. Since $E^t(Y)$ is terminal in $\mathbf{Comag}_Y(\mathcal{L})$, the set $\mathbf{Comag}_Y(\mathcal{L})(f_*(E^t(X)), E^t(Y))$ has exactly one element that we denote as $!^t f$. This morphism is therefore an element of $\mathcal{L}(!^t X, !^t Y)$ which is completely characterized by the following commutations

$$\begin{array}{ccc} !^t X & \xrightarrow{!^t f} & !^t Y \\ der_X^t \downarrow & & \downarrow der_Y^t \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} !^t X & \xrightarrow{!^t f} & !^t Y \\ w_X^t \searrow & & \swarrow w_Y^t \\ & 1 & \end{array} \qquad \begin{array}{ccc} !^t X & \xrightarrow{!^t f} & !^t Y \\ c_X^t \downarrow & & \downarrow c_Y^t \\ !^t X \otimes !^t X & \xrightarrow{!^t f \otimes !^t f} & !^t Y \otimes !^t Y \end{array}$$

Functoriality of the operation $!^t _$ follows easily from this uniqueness. It is also clear that $(d_X)_{X \in \text{Obj}(\mathcal{L})}$ is a natural transformation.

Remark 30 *One could have done something slightly simpler to deal with functoriality of a FMQE, as one can see a FMQE boils down to a left adjoint to the forgetful functor $\mathbf{Comag}(\mathcal{L}) \rightarrow \mathcal{L}$, rather than using our definition in terms of terminal object. However we preferred this more concrete, equivalent, presentation which has also the advantage of defining the FMQE as an operation on objects and not on morphisms a priori.*

4.1.2 Comonadicity of an FMQE

In the category $\mathbf{Comag}_{!^t X}(\mathcal{L})$, we have the object $!X = (!^t X, w_X^t, c_X^t, \text{Id}_{!^t X})$. Let dig_X^t be the unique element of $\mathbf{Comag}_{!^t X}(\mathcal{L})(!X, \mathbf{E}^t(!^t X))$. So $\text{dig}_X^t \in \mathcal{L}(!^t X, !^t !^t X)$ is completely characterized by the following commutations:

$$\begin{array}{ccc}
 !^t X & \xrightarrow{\text{dig}_X^t} & !^t !^t X \\
 \text{Id}_{!^t X} \searrow & & \swarrow \text{der}_{!^t X}^t \\
 & !^t X &
 \end{array}
 \quad
 \begin{array}{ccc}
 !^t X & \xrightarrow{\text{dig}_X^t} & !^t !^t X \\
 w_X^t \searrow & & \swarrow w_{!^t X}^t \\
 & 1 &
 \end{array}
 \quad
 \begin{array}{ccc}
 !^t X & \xrightarrow{\text{dig}_X^t} & !^t !^t X \\
 c_X^t \downarrow & & \downarrow c_X^t \\
 !^t X \otimes !^t X & \xrightarrow{\text{dig}_X^t \otimes \text{dig}_X^t} & !^t !^t X \otimes !^t !^t X
 \end{array}$$

Let us check that dig_X^t is natural in X so let $f \in \mathcal{L}(X, Y)$. We must prove that

$$\begin{array}{ccc}
 !^t X & \xrightarrow{!^t f} & !^t Y \\
 \text{dig}_X^t \downarrow & & \downarrow \text{dig}_Y^t \\
 !^t !^t X & \xrightarrow{!^t !^t f} & !^t !^t Y
 \end{array}$$

Let $M = (!^t f)_*(!X) = (!^t X, w_X^t, c_X^t, !^t f) \in \text{Obj}(\mathbf{Comag}_{!^t Y}(\mathcal{L}))$ (see Section 4.1.1 to recall definition of f_*). We clearly have $\text{dig}_Y^t !^t f \in \mathbf{Comag}_{!^t Y}(\mathcal{L})(M, \mathbf{E}^t(!^t Y))$, so it suffices to prove that $!^t !^t f \text{dig}_X^t \in \mathbf{Comag}_{!^t Y}(\mathcal{L})(M, \mathbf{E}^t(!^t Y))$ since $\mathbf{E}^t(!^t Y)$ is the terminal object of $\mathbf{Comag}_{!^t Y}(\mathcal{L})$. We have

$$\text{dig}_X^t \in \mathbf{Comag}_{!^t X}(\mathcal{L})(!X, \mathbf{E}^t(!^t X))$$

and hence, applying the $(!^t f)_*$ functor we get

$$\text{dig}_X^t \in \mathbf{Comag}_{!^t Y}(\mathcal{L})(M, (!^t f)_*(\mathbf{E}^t(!^t X)))$$

and since, by definition, $!^t !^t f \in \mathbf{Comag}_{!^t Y}(\mathcal{L})((!^t f)_*(\mathbf{E}^t(!^t X)), \mathbf{E}^t(!^t Y))$, we get the expected property.

Next we must check that

$$\begin{array}{ccc}
 !^t X & \xrightarrow{\text{dig}_X^t} & !^t !^t X \\
 \text{Id}_X \searrow & & \swarrow !^t(\text{der}_X^t) \\
 & !^t X &
 \end{array}$$

and for this it suffices to prove that $!^t(\text{der}_X^t) \text{dig}_X^t \in \mathbf{Comag}_X(\mathcal{L})(\mathbf{E}^t(X), \mathbf{E}^t(X))$. We have $\text{der}_X^t \in \mathcal{L}(!^t X, X)$ and hence $!^t(\text{der}_X^t) \in \mathbf{Comag}_X(\mathcal{L})((\text{der}_X^t)_*(\mathbf{E}^t(!^t X)), \mathbf{E}^t(X))$. On the other hand $\text{dig}_X^t \in \mathbf{Comag}_{!^t X}(\mathcal{L})(!^t X, \mathbf{E}^t(!^t X))$ and we end the proof by applying the $(\text{der}_X^t)_*$ functor to that morphism and observing that $(\text{der}_X^t)_*(!^t X) = \mathbf{E}^t(X)$.

Last we have to prove that

$$\begin{array}{ccc}
 !^t X & \xrightarrow{\text{dig}_X^t} & !^t !^t X \\
 \text{dig}_X^t \downarrow & & \downarrow \text{dig}_{!^t X}^t \\
 !^t !^t X & \xrightarrow{!^t(\text{dig}_X^t)} & !^t !^t !^t X
 \end{array}$$

Let $M = (\text{dig}_X^\dagger)_*(!X) \in \text{Obj}(\mathbf{Comag}_{!^\dagger X}(\mathcal{L}))$; applying the $(\text{dig}_X^\dagger)_*$ functor we get

$$\text{dig}_X^\dagger \in \mathbf{Comag}_{!^\dagger X}(\mathcal{L})(M, (\text{dig}_X^\dagger)_*(\mathbf{E}^\dagger(!^\dagger X)))$$

and $!(\text{dig}_X^\dagger) \in \mathbf{Comag}_{!^\dagger X}(\mathcal{L})((\text{dig}_X^\dagger)_*(\mathbf{E}^\dagger(!^\dagger X)), \mathbf{E}^\dagger(!^\dagger X))$ and hence it suffices to prove that

$$\text{dig}_{!^\dagger X}^\dagger \text{dig}_X^\dagger \in \mathbf{Comag}_{!^\dagger X}(\mathcal{L})(M, \mathbf{E}^\dagger(!^\dagger X)).$$

We have $\text{dig}_{!^\dagger X}^\dagger \in \mathbf{Comag}_{!^\dagger X}(\mathcal{L})(!^\dagger X, \mathbf{E}^\dagger(!^\dagger X))$ by definition and, trivially we also have $\text{dig}_X^\dagger \in \mathbf{Comag}_{!^\dagger X}(\mathcal{L})(M, !^\dagger X)$ which proves our contention.

Lemma 136 *Equipped with the natural transformations $(\text{der}_X^\dagger)_{X \in \text{Obj}(\mathcal{L})}$ and $(\text{dig}_X^\dagger)_{X \in \text{Obj}(\mathcal{L})}$, the functor $!^\dagger_-$ is a comonad $\mathcal{L} \rightarrow \mathcal{L}$. It will be called the free comagma comonad.*

4.1.3 Monoidality of an FMQE

Notice that so far all this can be done without any monoidality structure on top of $(\mathcal{L}, \otimes, 1)$. Such extremely weak assumptions are not sufficient for the sequel.

We assume from now on that a symmetric monoidal structure is given on \mathcal{L} , so that we are given isomorphisms $\lambda_X \in \mathcal{L}(1 \otimes X, X)$, $\rho \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X_1, X_2, X_3} \in \mathcal{L}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$ and $\gamma_{X_1, X_2} \in \mathcal{L}(X_1 \otimes X_2, X_2 \otimes X_1)$, subject to the usual coherence commutation diagrams [Mac71].

This allows to equip 1 with a structure of comagma over itself, we denote by 1 the corresponding object $(1, \mathbf{w}_1, \mathbf{c}_1, \mathbf{d}_1)$ where $\mathbf{w}_1 = \mathbf{d}_1 = \text{id}_1$, $\mathbf{c}_1 = \lambda_1^{-1} = \rho_1^{-1}$. Since $\mathbf{E}^\dagger(1)$ is terminal in $\mathbf{Comag}_1(\mathcal{L})$, there is exactly one morphism $\mu^{t,0} \in \mathbf{Comag}_1(\mathcal{L})(1, \mathbf{E}^\dagger(1))$. In other words, it is the unique element of $\mathcal{L}(1, !^\dagger 1)$ such that

$$\begin{array}{ccc} 1 \xrightarrow{\mu^{t,0}} !^\dagger 1 & 1 \xrightarrow{\mu^{t,0}} !^\dagger 1 & 1 \xrightarrow{\mu^{t,0}} !^\dagger 1 \\ \searrow \text{id}_1 \downarrow \text{der}_1^\dagger & \searrow \text{id}_1 \downarrow \mathbf{w}_1^\dagger & \downarrow \mathbf{c}_1 \downarrow \mathbf{c}_1^\dagger \\ 1 & 1 & 1 \otimes 1 \xrightarrow{\mu^{t,0} \otimes \mu^{t,0}} !^\dagger 1 \otimes !^\dagger 1 \end{array}$$

Let now X_1 and X_2 be objects of \mathcal{L} . For $i = 1, 2$ let M_i be a comagma over X_i . We equip $\underline{M}_1 \otimes \underline{M}_2$ with a structure of comagma over $X_1 \otimes X_2$. We use $M_1 \otimes M_2$ for the corresponding comagma $(\underline{M}_1 \otimes \underline{M}_2, \mathbf{w}_{M_1 \otimes M_2}, \mathbf{c}_{M_1 \otimes M_2}, \mathbf{d}_{M_1 \otimes M_2})$. The structure maps are defined as the following compositions of morphisms

$$\begin{array}{ccc} \underline{M}_1 \otimes \underline{M}_2 & \underline{M}_1 \otimes \underline{M}_2 & \underline{M}_1 \otimes \underline{M}_2 \\ \downarrow \mathbf{w}_{M_1} \otimes \mathbf{w}_{M_2} & \downarrow \mathbf{c}_{M_1} \otimes \mathbf{c}_{M_2} & \downarrow \mathbf{d}_{M_1} \otimes \mathbf{d}_{M_2} \\ 1 \otimes 1 & (\underline{M}_1 \otimes \underline{M}_1) \otimes (\underline{M}_2 \otimes \underline{M}_2) & X_1 \otimes X_2 \\ \downarrow \lambda_1 = \rho_1 & \downarrow \phi & \\ 1 & (\underline{M}_1 \otimes \underline{M}_2) \otimes (\underline{M}_1 \otimes \underline{M}_2) & \end{array}$$

where ϕ is an iso obtained by combining instances of α and γ (any well-typed combination will do by the coherence theorem). Let moreover $f_i \in \mathbf{Comag}_{X_i}(\mathcal{L})(M_i, N_i)$ for $i = 1, 2$. Then it is easily checked that $f_1 \otimes f_2 \in \mathbf{Comag}_{X_1 \otimes X_2}(\mathcal{L})(M_1 \otimes M_2, N_1 \otimes N_2)$.

Let $\mu_{X_1, X_2}^{t,2}$ be the unique element of

$$\mathbf{Comag}_{X_1 \otimes X_2}(\mathcal{L})(\mathbf{E}^t(X_1) \otimes \mathbf{E}^t(X_2), \mathbf{E}^t(X_1 \otimes X_2)).$$

Hence $\mu_{X_1, X_2}^{t,2} \in \mathcal{L}(!^t X_1 \otimes !^t X_2, !^t(X_1 \otimes X_2))$ is characterized by the following commutations

$$\begin{array}{ccc} !^t X_1 \otimes !^t X_2 & \xrightarrow{\mu_{X_1, X_2}^{t,2}} & !^t(X_1 \otimes X_2) \\ \text{der}_{X_1}^t \otimes \text{der}_{X_2}^t \searrow & & \downarrow \text{der}_{X_1 \otimes X_2}^t \\ & & X_1 \otimes X_2 \end{array} \quad \begin{array}{ccc} !^t X_1 \otimes !^t X_2 & \xrightarrow{\mu_{X_1, X_2}^{t,2}} & !^t(X_1 \otimes X_2) \\ w_{X_1}^t \otimes w_{X_2}^t \downarrow & & \downarrow w_{X_1 \otimes X_2}^t \\ 1 \otimes 1 & \xrightarrow{\lambda_1 = \rho_1} & 1 \end{array}$$

$$\begin{array}{ccc} !^t X_1 \otimes !^t X_2 & \xrightarrow{\mu_{X_1, X_2}^{t,2}} & !^t(X_1 \otimes X_2) \\ \phi(c_{X_1}^t \otimes c_{X_2}^t) \downarrow & & \downarrow c_{X_1 \otimes X_2}^t \\ (!^t X_1 \otimes !^t X_2) \otimes (!^t X_1 \otimes !^t X_2) & \xrightarrow{\mu_{X_1, X_2}^{t,2} \otimes \mu_{X_1, X_2}^{t,2}} & !^t(X_1 \otimes X_2) \otimes !^t(X_1 \otimes X_2) \end{array}$$

Now we prove that $(!^t _, \mu^{t,0}, \mu^{t,2})$ is a lax symmetric monoidal functor from the SMC $(\mathcal{L}, 1, \otimes)$ to itself. We first prove the following commutation (the symmetric one where we swap the role of 1 and X is proven similarly).

$$\begin{array}{ccc} 1 \otimes !^t X & \xrightarrow{\mu^{t,0} \otimes !^t X} & !^t 1 \otimes !^t X \xrightarrow{\mu_{1, X}^{t,2}} & !^t(1 \otimes X) \\ & \searrow \lambda_{!^t X} & & \downarrow !^t \lambda_X \\ & & & !^t X \end{array}$$

Let $M = (\lambda_X)_*(1 \otimes \mathbf{E}^t(X)) \in \text{Obj}(\mathbf{Comag}_X(\mathcal{L}))$, so that $\underline{M} = 1 \otimes !^t X$, $d_M = \lambda_X(1 \otimes \text{der}_X^t)$, $w_M = w_{1 \otimes !^t X}$ and $c_M = c_{1 \otimes !^t X}$. The following diagrams commute

$$\begin{array}{ccc} 1 \otimes !^t X & \xrightarrow{\lambda_{!^t X}} & !^t X \\ 1 \otimes \text{der}_X^t \downarrow & & \downarrow \text{der}_X^t \\ 1 \otimes X & \xrightarrow{\lambda_X} & X \end{array} \quad \begin{array}{ccc} 1 \otimes !^t X & \xrightarrow{\lambda_{!^t X}} & !^t X \\ 1 \otimes w_X^t \downarrow & & \downarrow w_X^t \\ 1 \otimes 1 & \xrightarrow{\lambda_1} & 1 \end{array}$$

by naturality of λ . The diagram

$$\begin{array}{ccc} 1 \otimes !^t X & \xrightarrow{\lambda_{!^t X}} & !^t X \\ \lambda_1^{-1} \otimes c_X^t \downarrow & & \downarrow c_X^t \\ (1 \otimes 1) \otimes (!^t X \otimes !^t X) & & \\ \phi \downarrow & & \\ (1 \otimes !^t X) \otimes (1 \otimes !^t X) & \xrightarrow{\lambda_{!^t X} \otimes \lambda_{!^t X}} & !^t X \otimes !^t X \end{array}$$

commutes for the same reason (writing $\lambda_1^{-1} \otimes c_X^t = (\lambda_1^{-1} \otimes (!^t X \otimes !^t X))(1 \otimes c_X^t)$), plus the coherence diagrams in \mathcal{L} . This means that $\lambda_{!^t X} \in \mathbf{Comag}_X(\mathcal{L})(M, \mathbf{E}^t(X))$. To prove the

required commutation it will be sufficient to prove that

$$!^t \lambda_X \mu_{1,X}^{t,2} \left(\mu^{t,0} \otimes !^t X \right) \in \mathbf{Comag}_X(\mathcal{L})(M, \mathbf{E}^t(X))$$

Since $\mu^{t,0} \in \mathbf{Comag}_1(\mathcal{L})(1, \mathbf{E}^t(1))$ we have $\mu^{t,0} \otimes !^t X \in \mathbf{Comag}_{1 \otimes X}(\mathcal{L})(1 \otimes \mathbf{E}^t(X), \mathbf{E}^t(1) \otimes \mathbf{E}^t(X))$. By definition we have $\mu_{1,X}^{t,2} \in \mathbf{Comag}_{1 \otimes X}(\mathcal{L})(\mathbf{E}^t(1) \otimes \mathbf{E}^t(X), \mathbf{E}^t(1 \otimes X))$. Hence $\mu_{1,X}^{t,2} (\mu^{t,0} \otimes !^t X) \in \mathbf{Comag}_{1 \otimes X}(\mathcal{L})(1 \otimes \mathbf{E}^t(X), \mathbf{E}^t(1 \otimes X))$. Applying the $(\lambda_X)_*$ functor we get $\mu_{1,X}^{t,2} (\mu^{t,0} \otimes !^t X) \in \mathbf{Comag}_X(\mathcal{L})(M, (\lambda_X)_*(\mathbf{E}^t(1 \otimes X)))$. We get the required property by the fact that $!^t \lambda_X \in \mathbf{Comag}_X(\mathcal{L})((\lambda_X)_*(\mathbf{E}^t(1 \otimes X)), \mathbf{E}^t(X))$.

We prove similarly the commutation

$$\begin{array}{ccc} (!^t X_1 \otimes !^t X_2) \otimes !^t X_3 & \xrightarrow{\mu_{X_1, X_2}^{t,2} \otimes !^t X_3} & !^t (X_1 \otimes X_2) \otimes !^t X_3 \xrightarrow{\mu_{X_1 \otimes X_2, X_3}^{t,2}} !^t ((X_1 \otimes X_2) \otimes X_3) \\ \alpha_{!^t X_1, !^t X_2, !^t X_3} \downarrow & & \downarrow !^t \alpha_{X_1, X_2, X_3} \\ !^t X_1 \otimes (!^t X_2 \otimes !^t X_3) & \xrightarrow{!^t X_1 \otimes \mu_{X_2, X_3}^{t,2}} & !^t X_1 \otimes !^t (X_2 \otimes X_3) \xrightarrow{\mu_{X_1, X_2 \otimes X_3}^{t,2}} !^t (X_1 \otimes (X_2 \otimes X_3)) \end{array}$$

By similar arguments we have

$$\begin{aligned} & \mu_{X_1 \otimes X_2, X_3}^{t,2} \left(\mu_{X_1, X_2}^{t,2} \otimes !^t X_3 \right) \\ & \in \mathbf{Comag}_{(X_1 \otimes X_2) \otimes X_3}(\mathcal{L})((\mathbf{E}^t(X_1) \otimes \mathbf{E}^t(X_2)) \otimes \mathbf{E}^t(X_3), \mathbf{E}^t((X_1 \otimes X_2) \otimes X_3)). \end{aligned}$$

Let $M = (\alpha_{X_1, X_2, X_3})_*((\mathbf{E}^t(X_1) \otimes \mathbf{E}^t(X_2)) \otimes \mathbf{E}^t(X_3)) \in \mathbf{Obj}(\mathbf{Comag}_{X_1 \otimes (X_2 \otimes X_3)}(\mathcal{L}))$. Applying the $(\alpha_{X_1, X_2, X_3})_*$ functor we get

$$\begin{aligned} & \mu_{X_1 \otimes X_2, X_3}^{t,2} \left(\mu_{X_1, X_2}^{t,2} \otimes !^t X_3 \right) \\ & \in \mathbf{Comag}_{X_1 \otimes (X_2 \otimes X_3)}(\mathcal{L})(M, (\alpha_{X_1, X_2, X_3})_*(\mathbf{E}^t((X_1 \otimes X_2) \otimes X_3))) \end{aligned}$$

and hence

$$\begin{aligned} & !^t \alpha_{X_1, X_2, X_3} \mu_{X_1 \otimes X_2, X_3}^{t,2} \left(\mu_{X_1, X_2}^{t,2} \otimes !^t X_3 \right) \\ & \in \mathbf{Comag}_{X_1 \otimes (X_2 \otimes X_3)}(\mathcal{L})(M, \mathbf{E}^t(X_1 \otimes (X_2 \otimes X_3))). \end{aligned}$$

The required commutation follows from $\alpha_{!^t X_1, !^t X_2, !^t X_3} \in \mathbf{Comag}_{X_1 \otimes (X_2 \otimes X_3)}(\mathcal{L})(M, \mathbf{E}^t(X_1) \otimes (\mathbf{E}^t(X_2) \otimes \mathbf{E}^t(X_3)))$ (by naturality of α plus monoidality of \mathcal{L}) and by the usual argument using the fact that $\mathbf{E}^t(X_1 \otimes (X_2 \otimes X_3))$ is terminal in $\mathbf{Comag}_{X_1 \otimes (X_2 \otimes X_3)}(\mathcal{L})$.

Symmetry, which is the commutation

$$\begin{array}{ccc} !^t X_1 \otimes !^t X_2 & \xrightarrow{\mu_{X_1, X_2}^{t,2}} & !^t (X_1 \otimes X_2) \\ \gamma_{!^t X_1, !^t X_2} \downarrow & & \downarrow !^t \gamma_{X_1, X_2} \\ !^t X_2 \otimes !^t X_1 & \xrightarrow{\mu_{X_2, X_1}^{t,2}} & !^t (X_2 \otimes X_1) \end{array}$$

is proven similarly (using now the functor $(\gamma_{X_1, X_2})_*$).

The following commutation also holds

$$\begin{array}{ccc}
 !^t X_1 \otimes !^t X_2 & \xrightarrow{\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t} & !^t !^t X_1 \otimes !^t !^t X_2 \\
 \downarrow \mu_{X_1, X_2}^{t,2} & & \downarrow \mu_{!^t X_1, !^t X_2}^{t,2} \\
 & & !(^t X_1 \otimes !^t X_2) \\
 & & \downarrow !^t (\mu_{X_1, X_2}^{t,2}) \\
 !^t (X_1 \otimes X_2) & \xrightarrow{\text{dig}_{X_1 \otimes X_2}^t} & !^t !^t (X_1 \otimes X_2)
 \end{array}$$

Since $\mu_{X_1, X_2}^{t,2} \in \mathbf{Comag}_{X_1 \otimes X_2}(\mathcal{L})(\mathbf{E}^t(X_1) \otimes \mathbf{E}^t(X_2), \mathbf{E}^t(X_1 \otimes X_2))$, we have trivially $\mu_{X_1, X_2}^{t,2} \in \mathbf{Comag}_{!^t(X_1 \otimes X_2)}(\mathcal{L})(M, !^t(X_1 \otimes X_2))$ where M is given by

$$\begin{aligned}
 \underline{M} &= !^t X_1 \otimes !^t X_2 \\
 \mathbf{w}_M &= \mathbf{w}_{\mathbf{E}^t(X_1) \otimes \mathbf{E}^t(X_2)} \\
 \mathbf{c}_M &= \mathbf{c}_{\mathbf{E}^t(X_1) \otimes \mathbf{E}^t(X_2)} \\
 \mathbf{d}_M &= \mu_{X_1, X_2}^{t,2}
 \end{aligned}$$

and by definition $\text{dig}_{X_1 \otimes X_2}^t \in \mathbf{Comag}_{!^t(X_1 \otimes X_2)}(\mathcal{L})(!^t(X_1 \otimes X_2), \mathbf{E}^t(!^t(X_1 \otimes X_2)))$ so, by the universal property of $\mathbf{E}^t(!^t(X_1 \otimes X_2))$ it will be sufficient to prove that

$$!^t (\mu_{X_1, X_2}^{t,2}) \mu_{!^t X_1, !^t X_2}^{t,2} (\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t) \in \mathbf{Comag}_{!^t(X_1 \otimes X_2)}(\mathcal{L})(M, \mathbf{E}^t(!^t(X_1 \otimes X_2))).$$

We have $\text{dig}_{X_i}^t \in \mathbf{Comag}_{!^t X_i}(\mathcal{L})(!^t X_i, \mathbf{E}^t(!^t X_i))$ for $i = 1, 2$, hence

$$\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t \in \mathbf{Comag}_{!^t X_1 \otimes !^t X_2}(\mathcal{L})(!^t X_1 \otimes !^t X_2, \mathbf{E}^t(!^t X_1) \otimes \mathbf{E}^t(!^t X_2)).$$

Hence

$$\mu_{!^t X_1, !^t X_2}^{t,2} (\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t) \in \mathbf{Comag}_{!^t X_1 \otimes !^t X_2}(\mathcal{L})(!^t X_1 \otimes !^t X_2, \mathbf{E}^t(!^t X_1 \otimes !^t X_2)).$$

Next since $\mu_{X_1, X_2}^{t,2} \in \mathcal{L}(!^t X_1 \otimes !^t X_2, !^t(X_1 \otimes X_2))$ we have

$$!^t \mu_{X_1, X_2}^{t,2} \in \mathbf{Comag}_{!^t(X_1 \otimes X_2)}(\mathcal{L})\left(\left(\mu_{X_1, X_2}^{t,2}\right)_* (!^t X_1 \otimes !^t X_2), \mathbf{E}^t(!^t(X_1 \otimes X_2))\right)$$

so applying the functor $\left(\mu_{X_1, X_2}^{t,2}\right)_*$ to $\mu_{!^t X_1, !^t X_2}^{t,2} (\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t)$ we get the required property by observing also that

$$\left(\mu_{X_1, X_2}^{t,2}\right)_* (!^t X_1 \otimes !^t X_2) = M.$$

So we have proven the following result.

Lemma 137 *The structure $(!^t _, \text{der}^t, \text{dig}^t, \mu^{t,0}, \mu^{t,2})$ is a symmetric monoidal comonad on the SMC $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \gamma)$.*

We end this section by analysing the Seely morphisms in this categorical setting.

4.1.4 Seely morphisms

We assume now that \mathcal{L} is also cartesian, with terminal object \top and cartesian product $\&$ (projections pr_i , tupling $\langle _, _ \rangle$).

We define $\mathfrak{m}^{t,0} = !^t t \mu^{t,0} \in \mathcal{L}(1, !\top)$ where t is the unique element of $\mathcal{L}(1, \top)$ and $\mathfrak{m}_{X_1, X_2}^{t,2} \in \mathcal{L}(!^t X_1 \otimes !^t X_2, !^t (X_1 \& X_2))$ is defined as the following composition of morphisms

$$\begin{array}{ccc} !^t X_1 \otimes !^t X_2 & & \\ \downarrow \text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t & & \\ !^t !^t X_1 \otimes !^t !^t X_2 & & \\ \downarrow \mu_{!^t X_1, !^t X_2}^{t,2} & \text{!} \langle \text{der}_{X_1}^t ({}^! X_1 \otimes w_{X_2}^t), \text{der}_{X_1}^t (w_{X_1}^t \otimes {}^! X_2) \rangle & \\ !^t (!^t X_1 \otimes !^t X_2) & \xrightarrow{\quad} & !^t (X_1 \& X_2) \end{array}$$

where we leave the isos λ and ρ implicit. It results from this definition that $\mathfrak{m}^{t,2}$ is a natural transformation.

The following diagrams

$$\begin{array}{ccccc} 1 \otimes !^t X & \xrightarrow{\mathfrak{m}^{t,0} \otimes !^t X} & !^t \top \otimes !^t X & \xrightarrow{\mathfrak{m}_{\top, X}^{t,2}} & !^t (\top \& X) \\ & \searrow \lambda_{!^t X} & & & \downarrow !^t \text{pr}_2 \\ & & & & !^t X \end{array}$$

and the symmetric one, as well as

$$\begin{array}{ccccc} (!^t X_1 \otimes !^t X_2) \otimes !^t X_3 & \xrightarrow{\mathfrak{m}_{X_1, X_2}^{t,2} \otimes !^t X_3} & !^t (X_1 \& X_2) \otimes !^t X_3 & \xrightarrow{\mathfrak{m}_{X_1 \& X_2, X_3}^{t,2}} & !^t ((X_1 \& X_2) \& X_3) \\ \downarrow \alpha_{!^t X_1, !^t X_2, !^t X_3} & & \downarrow \text{!} \langle \text{pr}_1 \text{pr}_1, \langle \text{pr}_2 \text{pr}_1, \text{pr}_2 \rangle \rangle & & \downarrow \\ !^t X_1 \otimes (!^t X_2 \otimes !^t X_3) & \xrightarrow{\text{!} X_1 \otimes \mathfrak{m}_{X_2, X_3}^{t,2}} & !^t X_1 \otimes !^t (X_2 \& X_3) & \xrightarrow{\mathfrak{m}_{X_1, X_2 \& X_3}^{t,2}} & !^t (X_1 \& (X_2 \& X_3)) \end{array}$$

and

$$\begin{array}{ccc} !^t X_1 \otimes !^t X_2 & \xrightarrow{\mathfrak{m}_{X_1, X_2}^{t,2}} & !^t (X_1 \& X_2) \\ \downarrow \text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t & & \downarrow \text{dig}_{X_1 \& X_2}^t \\ !^t !^t X_1 \otimes !^t !^t X_2 & \xrightarrow{\mathfrak{m}_{!^t X_1, !^t X_2}^{t,2}} & !^t (!^t X_1 \& !^t X_2) \\ & & \downarrow \text{!} \langle !^t \text{pr}_1, !^t \text{pr}_2 \rangle \end{array}$$

commute, as proven by simple diagram chasing.

With ordinary exponentials, the Seely monoidality morphisms are isos. This is not the case here. However, there are “Seely morphisms” in the converse direction $\mathfrak{n}^{t,0} = w_{\top}^t \in \mathcal{L}(!\top, 1)$ and $\mathfrak{n}^{t,2} \in \mathcal{L}(!^t (X_1 \& X_2), !^t X_1 \otimes !^t X_2)$ defined as the following composition of morphisms

$$\begin{array}{c}
 !^t(X_1 \& X_2) \\
 \downarrow c_{X_1 \& X_2}^t \\
 !^t(X_1 \& X_2) \otimes !^t(X_1 \& X_2) \\
 \downarrow !^t \text{pr}_1 \otimes !^t \text{pr}_2 \\
 !^t X_1 \otimes !^t X_2
 \end{array}$$

which is natural in X_1 and X_2 by construction.

If we try to compute two compositions $(m_{X_1, X_2}^{t,2}) \circ (n_{X_1, X_2}^{t,2})$ and $(n_{X_1, X_2}^{t,2}) \circ (m_{X_1, X_2}^{t,2})$ we will see that none of them is the identity, and they look like what follows. For $(n_{X_1, X_2}^{t,2}) \circ (m_{X_1, X_2}^{t,2})$ we have

$$\begin{array}{ccc}
 !^t X_1 \otimes !^t X_2 & & !^t X_1 \otimes !^t X_2 \\
 \downarrow c_{X_1}^t \otimes c_{X_2}^t & & \uparrow (!^t \text{der}_{X_1}^t) \otimes (!^t \text{der}_{X_2}^t) \\
 (!^t X_1 \otimes !^t X_1) \otimes !^t X_2 \otimes !^t X_2 & & !^t !^t X_1 \otimes !^t !^t X_2 \\
 \downarrow \phi & & \uparrow !^t (\text{Id}_{X_1} \otimes w_{X_2}^t) \otimes !^t (w_{X_1}^t \otimes \text{Id}_{X_2}) \\
 (!^t X_1 \otimes !^t X_2) \otimes (!^t X_1 \otimes !^t X_2) & & \\
 \downarrow (\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t) \otimes (\text{dig}_{X_1}^t \otimes \text{dig}_{X_2}^t) & & \\
 (!^t !^t X_1 \otimes !^t !^t X_1) \otimes (!^t !^t X_2 \otimes !^t !^t X_2) & \xrightarrow{\mu_{!^t X_1, !^t X_2}^{t,2} \otimes \mu_{!^t X_1, !^t X_2}^{t,2}} & !^t (!^t X_1 \otimes !^t X_2) \otimes !^t (!^t X_1 \otimes !^t X_2)
 \end{array}$$

And for $(m_{X_1, X_2}^{t,2}) \circ (n_{X_1, X_2}^{t,2})$ we have

$$\begin{array}{ccc}
 !^t(X_1 \& X_2) & & !^t(X \& Y) \\
 \downarrow \text{dig}_{X_1 \& X_2}^t & & \uparrow !^t(\langle f_1, f_2 \rangle) \\
 !^t !^t(X_1 \& X_2) & \xrightarrow{!^t c_{!^t(X_1 \& X_2)}^t} & !^t((!^t(X_1 \& X_2)) \otimes (!^t(X_1 \& X_2)))
 \end{array}$$

where f_i for $i = 1, 2$ is as follows:

$$!^t(X_1 \& X_2) \otimes !^t(X_1 \& X_2) \xrightarrow{\text{Id} \otimes w_{!^t(X_1 \& X_2)}^t} !^t(X_1 \& X_2) \xrightarrow{\text{der}_{!^t(X_1 \& X_2)}^t} !^t(X_1 \& X_2) \xrightarrow{\text{pr}_i} X_i$$

4.2 Concrete models of LL with a tree-based comonad

In this section, we will provide concrete models of LL based on the tree exponentials mentioned in the previous section and at the beginning of this chapter. We will examine this tree exponential in two models of LL, the relational model **Rel** and the coherence space model **Coh**.

4.2.1 Relational model of LL with tree exponentials

Let $\mathcal{L} = \mathbf{Rel}$ with its usual tensor product \otimes (cartesian product of sets) and tensor unit (singleton 1). Given a set E we use $\mathbf{tree}(E)$ for the set of binary trees defined by the following syntax:

$$\alpha, \beta, \dots := \langle \rangle \mid \langle a \rangle \mid \langle \alpha, \beta \rangle \quad a \in E$$

Then we set $!^t E = \mathbf{tree}(E)$ and

$$\begin{aligned} \mathbf{w}_E^t &= \{(\langle \rangle, *)\} \in \mathbf{Rel}(!^t E, 1) \\ \mathbf{c}_E^t &= \{(\langle \alpha_1, \alpha_2 \rangle, (\alpha_1, \alpha_2)) \mid \alpha_i \in !^t E \text{ for } i = 1, 2\} \in \mathbf{Rel}(!^t E, !^t E \otimes !^t E) \\ \mathbf{d}_E^t &= \{(\langle a \rangle, a) \mid a \in E\} \in \mathbf{Rel}(!^t E, E) \end{aligned} \quad (4.1)$$

So, we denote the object $\mathbf{E}^t(E)$ as $\mathbf{E}^t(E) = (\mathbf{tree}(E), \mathbf{w}_E^t, \mathbf{c}_E^t, \mathbf{d}_E^t)$ where the components are defined as above. We also use the notation $\mathbf{TreeSize}(\sigma)$ for the obvious definition of size of the binary tree σ .

Proposition 138 *The object $\mathbf{E}^t(E)$ of $\mathbf{Comag}_E(\mathbf{Rel})$ is terminal object.*

Proof: Let M be another comagma on E , remember this means that we have as set \underline{M} together with relations $\mathbf{w}_M \in \mathbf{Rel}(\underline{M}, 1)$, $\mathbf{c}_M \in \mathbf{Rel}(\underline{M}, \underline{M} \otimes \underline{M})$ and $\mathbf{d}_M \in \mathbf{Rel}(\underline{M}, E)$. Let $t \in \mathbf{Comag}_E(\mathbf{Rel})(M, \mathbf{E}^t(E))$, which means that the following diagrams commute in \mathbf{Rel}

$$\begin{array}{ccc} \underline{M} \xrightarrow{t} !^t E & \underline{M} \xrightarrow{t} !^t E & \underline{M} \xrightarrow{t} !^t E \\ \mathbf{w}_M \searrow & \mathbf{c}_M \downarrow & \mathbf{d}_M \searrow \\ & \underline{M} \otimes \underline{M} \xrightarrow{t \otimes t} !^t E \otimes !^t E & E \\ & \mathbf{w}_E^t \swarrow & \mathbf{d}_E^t \swarrow \\ & 1 & \end{array}$$

Let $m \in \underline{M}$. If $(m, \langle \rangle) \in t$ then by the first diagram we have $(m, *) \in \mathbf{w}_M$. And conversely by the same diagram if $(m, *) \in \mathbf{w}_M$ we must have $(m, \langle \rangle) \in t$. Let now also $a \in E$. By the same reasoning using the third diagram, $(m, \langle a \rangle) \in t \Leftrightarrow (m, a) \in \mathbf{d}_M$. Now assume that $(m, \langle \alpha_1, \alpha_2 \rangle) \in t$. By the second diagram there must exist $m_1, m_2 \in \underline{M}$ such that $(m, (m_1, m_2)) \in \mathbf{c}_M$ and $(m_i, \alpha_i) \in t$ for $i = 1, 2$. The converse is also true: if $(m_i, \alpha_i) \in t$ for $i = 1, 2$ with $(m, (m_1, m_2)) \in \mathbf{c}_M$, then we must have $(m, \langle \alpha_1, \alpha_2 \rangle) \in t$. This means that there is only one $t \in \mathcal{L}(\underline{M}, !^t E)$ satisfying these commutations, which is given by the following inductive definition:

$$\begin{aligned} t &= \{(m, \langle \rangle) \mid (m, *) \in \mathbf{w}_M\} \cup \{(m, \langle a \rangle) \mid (m, a) \in \mathbf{d}_M\} \\ &\quad \cup \{(m, \langle \alpha_1, \alpha_2 \rangle) \mid \exists m_1, m_2 \in \underline{M} (m, (m_1, m_2)) \in \mathbf{c}_M \text{ and } (m_i, \alpha_i) \in t \text{ for } i = 1, 2\}. \end{aligned}$$

■

Functoriality of $\mathbf{tree}(_)$

Given $t \in \mathbf{Rel}(E, F)$, we compute $!^t t$. Since $\mathbf{w}_F^t !t = \mathbf{w}_E^t$ we must have $(\alpha, \langle \rangle) \in !^t t \Leftrightarrow \alpha = \langle \rangle$. Since $\mathbf{d}_F^t !t = t \mathbf{d}_E^t$, for all $\alpha \in !^t E$ and $b \in F$ we must have $(\alpha, \langle b \rangle) \in !^t t$ iff there is

$a \in E$ such that $\alpha = \langle a \rangle$ and $(a, b) \in t$. Last since $\mathbf{c}_F^t \text{!}^t = (\text{!}^t t \otimes \text{!}^t t) \mathbf{c}_E^t$ we must have that, for any $\alpha \in \text{!}^t E$ and $\beta_1, \beta_2 \in \text{!}^t$, one has $(\alpha, \langle \beta_1, \beta_2 \rangle) \in \text{!}^t t$ iff there are $\alpha_1, \alpha_2 \in \text{!}^t E$ such that $\alpha = \langle \alpha_1, \alpha_2 \rangle$ and $(\alpha_i, \beta_i) \in \text{!}^t t$. This means that $\text{!}^t t$ is given by the following inductive definition:

$$\begin{aligned} \text{!}^t t = & \{(\langle \rangle, \langle \rangle)\} \cup \{(\langle a \rangle, \langle b \rangle) \mid (a, b) \in t\} \\ & \cup \{(\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle) \mid (\alpha_i, \beta_i) \in \text{!}^t t \text{ for } i = 1, 2\} \end{aligned} \quad (4.2)$$

Hence, one can easily check the following proposition, as the computation above satisfies the general characterization given in Section 4.1.1:

Proposition 139 *The operation $\text{tree}(_)$ defines a functor on \mathbf{Rel} .*

Monadicity of $\text{tree}(_)$

We now compute $\text{dig}_E^t \in \mathcal{L}(\text{!}^t E, \text{!}^t \text{!}^t E)$. The condition $\mathbf{w}_{\text{!}^t E}^t \text{dig}_E^t = \mathbf{w}_E^t$ means that $(\langle \rangle, \langle \rangle) \in \text{dig}_E^t$. The condition $\mathbf{der}_{\text{!}^t E}^t \text{dig}_E^t = \text{Id}_{\text{!}^t E}$ means that for any $\alpha, \beta \in \text{!}^t E$, one has $(\alpha, \langle \beta \rangle) \in \text{dig}_E^t$ iff $\alpha = \beta$. The condition $\mathbf{c}_{\text{!}^t E}^t \text{dig}_E^t = (\text{dig}_E^t \otimes \text{dig}_E^t) \mathbf{c}_E^t$ means that for all $A_1, A_2 \in \text{!}^t \text{!}^t E$ and $\alpha \in \text{!}^t E$, one has $(\alpha, \langle A_1, A_2 \rangle) \in \text{dig}_E^t$ iff there are $\alpha_1, \alpha_2 \in \text{!}^t E$ such that $\alpha = \langle \alpha_1, \alpha_2 \rangle$ and $(\alpha_i, A_i) \in \text{dig}_E^t$ for $i = 1, 2$. So dig_E^t is given by the following inductive definition:

$$\begin{aligned} \text{dig}_E^t = & \{(\langle \rangle, \langle \rangle)\} \cup \{(\alpha, \langle \alpha \rangle) \mid \alpha \in \text{!}^t E\} \\ & \cup \{(\langle \alpha_1, \alpha_2 \rangle, \langle A_1, A_2 \rangle) \mid (\alpha_i, A_i) \in \text{dig}_E^t \text{ for } i = 1, 2\} . \end{aligned} \quad (4.3)$$

Hence we have easily the following proposition, as the computation dig^t above satisfies the general pattern given in Section 4.1.2.

Proposition 140 *The triple $(\text{tree}(_), \text{der}^t \text{dig}^t)$ is a comonad over \mathbf{Rel} .*

To have simpler notation we introduce a function $\text{flat}_E : \text{tree}(\text{tree}(E)) \rightarrow \text{tree}(E)$ defined inductively by

$$\begin{aligned} \text{flat}_E(\langle \rangle) &= \langle \rangle \\ \text{flat}_E(\langle \alpha \rangle) &= \alpha \\ \text{flat}_E(\langle A_1, A_2 \rangle) &= \langle \text{flat}_E(A_1), \text{flat}_E(A_2) \rangle . \end{aligned}$$

Then it is easy to check that

$$\text{dig}_E^t = \{(\text{flat}_E(A), A) \mid A \in \text{!}^t \text{!}^t E\} \quad (4.4)$$

Symmetric monoidality of tree ($_$)

Now we compute $\mu^{t,0} \in \mathcal{L}(1, !^t 1)$. By the equation $\text{der}_1^t \mu^{t,0} = \text{Id}_1$ we have $(*, \langle * \rangle) \in \mu^{t,0}$. By the equation $w_1^t \mu^{t,0} = \text{Id}_1$ we have $(*, \langle \rangle) \in \mu^{t,0}$. And by the equation $c_1^t \mu^{t,0} = (\mu^{t,0} \otimes \mu^{t,0}) c_1$ we have that, for any $\alpha_1, \alpha_2 \in !^t 1$, $(*, \langle \alpha_1, \alpha_2 \rangle) \in \mu^{t,0}$ iff $(*, \alpha_1) \in \mu^{t,0}$ and $(*, \alpha_2) \in \mu^{t,0}$. So $\mu^{t,0}$ is inductively defined by

$$\mu^{t,0} = \{(*, \langle \rangle), (*, \langle * \rangle)\} \cup \left\{ (*, \langle \alpha_1, \alpha_2 \rangle) \mid (*, \alpha_i) \in \mu^{t,0} \text{ for } i = 1, 2 \right\} \quad (4.5)$$

so that actually $\mu^{t,0} = \{(*, \alpha) \mid \alpha \in !^t 1\}$.

Next we compute $\mu_{E_1, E_2}^{t,2} \in \mathcal{L}(!^t E_1 \otimes !^t E_2, !^t (E_1 \otimes E_2))$ for two sets E_1 and E_2 . By equation $\text{der}_{E_1 \otimes E_2}^t \mu_{E_1, E_2}^{t,2} = \text{der}_{E_1}^t \otimes \text{der}_{E_2}^t$, given $\alpha^i \in !^t E_i$ and $a^i \in E_i$ (for $i = 1, 2$), we have $((\alpha^1, \alpha^2), \langle (a^1, a^2) \rangle) \in \mu_{E_1, E_2}^{t,2}$ iff $\alpha^i = \langle a^i \rangle$ for $i = 1, 2$. By equation $w_{E_1 \otimes E_2}^t \mu_{E_1, E_2}^{t,2} = \lambda_1 (w_{E_1}^t \otimes w_{E_2}^t)$ we have $((\alpha^1, \alpha^2), \langle \rangle) \in \mu_{E_1, E_2}^{t,2}$ iff $\alpha^i = \langle \rangle$ for $i = 1, 2$. By equation $c_{E_1 \otimes E_2}^t \mu_{E_1, E_2}^{t,2} = (\mu_{E_1, E_2}^{t,2} \otimes \mu_{E_1, E_2}^{t,2}) \phi (c_{E_1}^t \otimes c_{E_2}^t)$, for any $\alpha^1, \alpha^2 \in !^t E_i$ and $\beta_1, \beta_2 \in !^t (E_1 \otimes E_2)$ we have $((\alpha^1, \alpha^2), \langle \beta_1, \beta_2 \rangle) \in \mu_{E_1, E_2}^{t,2}$ iff there are $\alpha_j^i, \alpha_j^i \in !^t E_i$ such that $\alpha^i = \langle \alpha_j^i, \alpha_j^i \rangle$ and $((\alpha_j^1, \alpha_j^2), \beta_j) \in \mu_{E_1, E_2}^{t,2}$ for $j = 1, 2$. So $\mu_{E_1, E_2}^{t,2}$ is given by the following inductive definition

$$\begin{aligned} \mu_{E_1, E_2}^{t,2} = & \{(\langle \rangle, \langle \rangle), \langle \rangle\} \cup \left\{ ((\langle a^1 \rangle, \langle a^2 \rangle), \langle (a^1, a^2) \rangle) \mid a^i \in E_i \text{ for } i = 1, 2 \right\} \\ & \cup \left\{ ((\langle \alpha_j^1, \alpha_j^2 \rangle, \langle \alpha_j^1, \alpha_j^2 \rangle), \langle \beta_1, \beta_2 \rangle) \mid ((\alpha_j^1, \alpha_j^2), \beta_j) \in \mu_{E_1, E_2}^{t,2} \text{ for } j = 1, 2 \right\} \end{aligned} \quad (4.6)$$

As this computation above is coming from a general characterization given in Section 4.1.3, we easily have

Proposition 141 *The structure $(!^t _ , \text{der}^t, \text{dig}^t, \mu^{t,0}, \mu^{t,2})$ is a symmetric monoidal comonad on \mathbf{Rel} .*

For $j = 1, 2$ let $\text{split}_j : \text{tree}(E_1 \times E_2) \rightarrow \text{tree}(E_j)$ be the function defined inductively by

$$\begin{aligned} \text{split}_j(\langle \rangle) &= \langle \rangle \\ \text{split}_j(\langle (a^1, a^2) \rangle) &= \langle a^j \rangle \\ \text{split}_j(\langle \beta_1, \beta_2 \rangle) &= \langle \text{split}_j(\beta_1), \text{split}_j(\beta_2) \rangle. \end{aligned}$$

then an easy induction on β shows that

$$\mu_{E_1, E_2}^{t,2} = \{(\langle \text{split}_1(\beta), \text{split}_2(\beta) \rangle, \beta) \mid \beta \in !^t (E_1 \otimes E_2)\}$$

And finally, one can compute the Seely morphisms in \mathbf{Rel} with this tree ($_$) exponential, according to Section 4.1.4.

Seely morphisms

The unique $t \in \mathbf{Rel}(1, \top)$ is $t = \emptyset$. Hence $!^t t \in \mathbf{Rel}(!^t 1, !^t \top)$ is $!^t t = \{(\alpha, \alpha) \mid \alpha \in \text{tree}(\emptyset)\}$. It follows from the general definition of $\mathbf{m}^{t,0}$ that

$$\mathbf{m}^{t,0} = \{(*, \alpha) \mid \alpha \in \text{tree}(\emptyset)\}.$$

Notice that $\text{tree}(\emptyset)$ is the set of all binary trees whose all leaves are $\langle \rangle$. Unlike $\mathcal{M}_{\text{fin}}(\emptyset) = \{\emptyset\}$, it is an infinite set.

We have

$$\begin{aligned} \text{der}_{E_1}^t (!^t E_1 \otimes \mathbf{w}_{E_2}^t) &= \left\{ (\langle \langle a^1 \rangle, \langle \rangle \rangle, a^1) \mid a^1 \in E_1 \right\} \in \mathbf{Rel}(!^t E_1 \otimes !^t E_2, E_1) \\ \text{der}_{E_2}^t (\mathbf{w}_{E_1}^t \otimes !^t E_2) &= \left\{ (\langle \langle \rangle, \langle a^2 \rangle \rangle, a^2) \mid a^2 \in E_2 \right\} \in \mathbf{Rel}(!^t E_1 \otimes !^t E_2, E_2) \end{aligned}$$

Let $\text{ptr}_i : \text{tree}(E_1 \& E_2) \rightarrow \text{tree}(E_1 \otimes E_2)$ be defined, for $i = 1, 2$, by induction

$$\begin{aligned} \text{ptr}_i(\langle \rangle) &= \langle \rangle \\ \text{ptr}_i(\langle \langle i, a \rangle \rangle) &= \langle a \rangle \\ \text{ptr}_i(\langle \langle 3-i, b \rangle \rangle) &= \langle \rangle \\ \text{ptr}_i(\langle \beta_1, \beta_2 \rangle) &= \langle \text{ptr}_i(\beta_1), \text{ptr}_i(\beta_2) \rangle \end{aligned}$$

Then using the following relation s

$$s = !^t \langle \text{der}_{E_1}^t (!^t E_1 \otimes \mathbf{w}_{E_2}^t), \text{der}_{E_2}^t (\mathbf{w}_{E_1}^t \otimes !^t E_2) \rangle \in \mathbf{Rel}(!^t (!^t E_1 \otimes !^t E_2), !^t (E_1 \& E_2)).$$

we have

$$\begin{aligned} \mathbf{m}_{X_1, X_2}^{\mathbf{t}, 2} &= \{(\langle \rangle, \langle \rangle)\} \\ &\cup \left\{ (\langle \langle a^1 \rangle, \langle \rangle \rangle, \langle \langle 1, a^1 \rangle \rangle) \mid a^1 \in E_1 \right\} \cup \left\{ (\langle \langle \rangle, \langle a^2 \rangle \rangle, \langle \langle 2, a^2 \rangle \rangle) \mid a^2 \in E_2 \right\} \\ &\cup \left\{ (\langle \langle \beta_{1,1}, \beta_{2,1} \rangle, \langle \beta_{1,2}, \beta_{2,2} \rangle \rangle, \langle \gamma_1, \gamma_2 \rangle) \mid ((\beta_{1,1}, \beta_{1,2}), \gamma_1), ((\beta_{2,1}, \beta_{2,2}), \gamma_1) \in s \right\} \end{aligned}$$

By an easy induction on elements of $\text{tree}(E_1 \& E_2)$, we have

$$\mathbf{m}_{X_1, X_2}^{\mathbf{t}, 2} = \{(\langle \text{ptr}_1(\beta), \text{ptr}_2(\beta) \rangle, \beta) \mid \beta \in !^t (E_1 \& E_2)\}$$

Similarly, using the categorical construction of $\mathbf{n}_{X_1, X_2}^{\mathbf{t}, 2}$ given in Section 4.1.4, we have

$$\mathbf{n}_{X_1, X_2}^{\mathbf{t}, 2} = \{(\langle \langle \text{Int}_1(\alpha_1), \text{Int}_2(\alpha_2) \rangle \rangle, (\alpha_1, \alpha_2)) \mid (\alpha_1, \alpha_2) \in !^t (E_1) \otimes !^t (E_2)\}$$

where $\text{Int}_i : \text{tree}(E_i) \rightarrow \text{tree}(E_1 \& E_2)$ is defined as follows for $i = 1, 2$ by induction on $\text{tree}(E_i)$:

$$\begin{aligned} \text{Int}_i(\langle \rangle) &= \langle \rangle \\ \text{Int}_i(\langle a \rangle) &= \langle \langle i, a \rangle \rangle \\ \text{Int}_i(\langle \beta_1, \beta_2 \rangle) &= \langle \text{Int}_i(\beta_1), \text{Int}_i(\beta_2) \rangle \end{aligned}$$

So, if we compute the two compositions $(\mathbf{m}_{X_1, X_2}^{\mathbf{t}, 2}) \circ (\mathbf{n}_{X_1, X_2}^{\mathbf{t}, 2})$ and $(\mathbf{n}_{X_1, X_2}^{\mathbf{t}, 2}) \circ (\mathbf{m}_{X_1, X_2}^{\mathbf{t}, 2})$, we will see that none of the is the identity in this concrete case of \mathbf{Rel} , as it is mentioned generally in Section 4.1.4.

4.2.2 Coherence spaces with tree exponentials

Given a coherence space $E = (|E|, \circlearrowright_E)$, we define the coherence space $!^t E = (\text{tree}(|E|), \circlearrowright_{!^t E})$ where $\circlearrowright_{!^t E}$ is defined inductively as follows:

1. $\langle \rangle \circlearrowright_{!^t E} \sigma$ for any $\sigma \in \text{tree}(|E|)$,
2. $\langle \alpha \rangle \circlearrowright_{!^t E} \langle \beta \rangle$ if $\alpha \circlearrowright_E \beta$ for all $\alpha, \beta \in |E|$,
3. $\langle \alpha \rangle \circlearrowright_{!^t E} \langle \sigma_1, \sigma_2 \rangle$ for all $\alpha \in |E|$ and for all $\sigma_i \in \text{tree}(|E|)$ for $i = 1, 2$,
4. $\langle \sigma_1, \sigma_2 \rangle \circlearrowright_{!^t E} \langle \tau_1, \tau_2 \rangle$ if $\sigma_i \circlearrowright_{!^t E} \tau_i$ for all $\sigma_i, \tau_i \in \text{tree}(|E|)$ and for $i = 1, 2$.

To prove that category **Coh** is an instance of the categorical setting given in Section 4.1, we have almost nothing to do. Basically, we just check that all definitions are the same as for **Rel**, given in Section 4.2.1, and prove that those definitions are indeed a clique in the appropriate coherence space. We will do this in the following Propositions 142, 143 and 144.

Proposition 142 *The operation $\text{tree}(_)$ defines a functor on **Coh**.*

Proof: Let us assume that $t \in \text{Cl}(E \multimap F)$. We need to show $!^t t \in \text{Cl}(!^t E \multimap !^t F)$ where $!^t t$ is defined in equation 4.2. Let $(\sigma, \tau), (\sigma', \tau') \in !^t t$. We prove that $(\sigma, \tau) \circlearrowright_{!^t E \multimap !^t F} (\sigma', \tau')$ by induction on $\text{TreeSize}(\sigma) + \text{TreeSize}(\sigma')$. So assume that $\sigma \circlearrowright_{!^t E} \sigma'$. We consider the following cases and in each case we prove that $\tau \circlearrowright_{!^t F} \tau'$ and that, if $\tau = \tau'$ then $\sigma = \sigma'$:

- $(\langle \rangle, \langle \rangle) \circlearrowright_{!^t E \multimap !^t F} (\langle a \rangle, \langle b \rangle)$: This is true, since we have $\langle \rangle \circlearrowright_{!^t E} \langle a \rangle$ and $\langle \rangle \circlearrowright_{!^t F} \langle b \rangle$ by definition of $\circlearrowright_{!^t _}$ (Item 1).
- $(\langle \rangle, \langle \rangle) \circlearrowright_{!^t E \multimap !^t F} (\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle)$: Same as the previous case.
- $(\langle a \rangle, \langle b \rangle) \circlearrowright_{!^t E \multimap !^t F} (\langle a' \rangle, \langle b' \rangle)$: Assume $\langle a \rangle \circlearrowright_{!^t E} \langle a' \rangle$, so, by Item 2, we have $a \circlearrowright_E a'$. Since $t \in \text{Cl}(E \multimap F)$, we have $b \circlearrowright_F b'$, and then $\langle b \rangle \circlearrowright_{!^t F} \langle b' \rangle$. Now Assume $\langle a \rangle \circlearrowright_{!^t E} \langle a' \rangle$ and $\langle b \rangle = \langle b' \rangle$. So, we have $a = a'$, and therefore $\langle a \rangle = \langle a' \rangle$.
- $(\langle a \rangle, \langle b \rangle) \circlearrowright_{!^t E \multimap !^t F} (\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle)$: This case is vacuously true, since we do not have $\langle a \rangle \circlearrowright_{!^t E} \langle \alpha_1, \alpha_2 \rangle$.
- $(\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle) \circlearrowright_{!^t E \multimap !^t F} (\langle \alpha'_1, \alpha'_2 \rangle, \langle \beta'_1, \beta'_2 \rangle)$: Assume that $\langle \alpha_1, \alpha_2 \rangle \circlearrowright_{!^t E} \langle \alpha'_1, \alpha'_2 \rangle$. By Item 4, we have $\alpha_i \circlearrowright_{!^t E} \alpha'_i$ for $i = 1, 2$. By Definition 4.2, we have $(\alpha_i, \beta_i) \in !^t t$ and $(\alpha'_i, \beta'_i) \in !^t t$. So, by induction hypothesis, we have $(\alpha_i, \beta_i) \circlearrowright_{!^t E \multimap !^t F} (\alpha'_i, \beta'_i)$ for $i = 1, 2$. Hence we have $\beta_i \circlearrowright_{!^t F} \beta'_i$. And therefore $\langle \beta_1, \beta_2 \rangle \circlearrowright_{!^t F} \langle \beta'_1, \beta'_2 \rangle$. The reasoning is the same for the assumption that $\langle \alpha_1, \alpha_2 \rangle \circlearrowright_{!^t E} \langle \alpha'_1, \alpha'_2 \rangle$ and $\langle \beta_1, \beta_2 \rangle = \langle \beta'_1, \beta'_2 \rangle$.

■

Definition of natural transformations der_E^t and dig_E^t is respectively $\text{der}_{|E|}^t$ and $\text{dig}_{|E|}^t$.

Proposition 143 *The triple $(\text{tree}(_), \text{der}_E^t, \text{dig}_E^t)$ is a comonad over **Coh**.*

Proof: We only need to show that $\text{der}_E^t \in \text{Cl}(!^t E \multimap E)$ and $\text{dig}_E^t : \text{Cl}(!^t E \multimap !^t !^t E)$ for der_E^t and dig_E^t defined respectively in 4.1 and 4.3. Assume $\langle a \rangle \supset_{!^t E} \langle b \rangle$. So, we have $a \supset_E b$. If we moreover assume that $a = b$, then we certainly have $\langle a \rangle = \langle b \rangle$. Hence $(\langle a \rangle, a) \supset_{!^t E \multimap E} (\langle b \rangle, b)$.

For dig_E^t , let $(\sigma, \tau), (\sigma', \tau') \in \text{dig}_E^t$. We prove that $(\sigma, \tau) \supset_{!^t E \multimap !^t !^t E} (\sigma', \tau')$ by induction on $\text{TreeSize}(\sigma) + \text{TreeSize}(\sigma')$. So assume that $\sigma \supset_{!^t E} \sigma'$. We consider the following cases and in each case we prove that $\tau \supset_{!^t !^t E} \tau'$ and that, if $\tau = \tau'$ then $\sigma = \sigma'$:

- $(\langle \rangle, \langle \rangle) \supset_{!^t E \multimap !^t !^t E} (\alpha, \langle \alpha \rangle)$: This is true, since we have $\langle \rangle \supset_{!^t !^t E} \langle \alpha \rangle$ by Item 1.
- $(\langle \rangle, \langle \rangle) \supset_{!^t E \multimap !^t !^t E} (\langle \alpha_1, \alpha_2 \rangle, \langle A_1, A_2 \rangle)$ where $(\alpha_i, A_i) \in \text{dig}_E^t$: This case is the same as the previous one.
- $(\alpha, \langle \alpha \rangle) \supset_{!^t E \multimap !^t !^t E} (\beta, \langle \beta \rangle)$: Assume $\alpha \supset_{!^t E} \beta$, then by Item 2 we have $\langle \alpha \rangle \supset_{!^t !^t E} \langle \beta \rangle$. And if $\alpha = \beta$, we obviously have $\langle \langle \alpha \rangle \rangle = \langle \langle \beta \rangle \rangle$.
- $(\alpha, \langle \alpha \rangle) \supset_{!^t E \multimap !^t !^t E} (\langle \alpha_1, \alpha_2 \rangle, \langle A_1, A_2 \rangle)$ where $(\alpha_i, A_i) \in \text{dig}_E^t$ $i = 1, 2$: This case is vacuously true, since we do not have $\alpha \supset_{!^t !^t E} \langle \alpha_1, \alpha_2 \rangle$.
- $(\langle \alpha_1, \alpha_2 \rangle, \langle A_1, A_2 \rangle) \supset_{!^t E \multimap !^t !^t E} (\langle \alpha'_1, \alpha'_2 \rangle, \langle A'_1, A'_2 \rangle)$ where $(\alpha_i, A_i) \in \text{dig}_E^t$ and $(\alpha'_i, A'_i) \in \text{dig}_E^t$ for $i = 1, 2$: Assume that $(\alpha_1, \alpha_2) \supset_{!^t !^t E} (\alpha'_1, \alpha'_2)$. By Item 4, we have $\alpha_i \supset_{!^t !^t E} \alpha'_i$ for $i = 1, 2$. Since $(\alpha_i, A_i) \in \text{dig}_E^t$, we therefore have $A_i \supset_{!^t E} A'_i$ for $i = 1, 2$, and hence $\langle A_1, A_2 \rangle \supset_{!^t E} \langle A'_1, A'_2 \rangle$. The reasoning is the same for the assumption that $(\langle \alpha_1, \alpha_2 \rangle, \langle A_1, A_2 \rangle) \supset_{!^t E \multimap !^t !^t E} (\langle \alpha'_1, \alpha'_2 \rangle, \langle A'_1, A'_2 \rangle)$ and $\langle A_1, A_2 \rangle = \langle A'_1, A'_2 \rangle$.

■

Proposition 144 *The structure $(!^t _, \text{der}^t, \text{dig}^t, \mu^{t,0}, \mu^{t,2})$ is a symmetric monoidal comonad on **Coh**.*

Proof: We only need to show that $\mu^{t,0} \in \text{Cl}(1 \multimap !^t 1)$ and $\mu_{E,F}^{t,2} \in \text{Cl}(!^t E_1 \otimes !^t E_2 \multimap !^t (E_1 \otimes E_2))$ for $\mu^{t,0}$ and $\mu^{t,2}$ defined respectively in 4.5 and 4.6. For $\mu^{t,0}$, let $(\sigma, \tau), (\sigma', \tau') \in \mu^{t,0}$. We prove that $(\sigma, \tau) \supset_{1 \multimap !^t 1} (\sigma', \tau')$ by induction on $\text{TreeSize}(\sigma) + \text{TreeSize}(\sigma')$. So assume that $\sigma \supset_1 \sigma'$. We consider the following cases and in each case we prove that $\tau \supset_{!^t 1} \tau'$ and that, if $\tau = \tau'$ then $\sigma = \sigma'$:

- $(*, \langle \rangle) \supset_{1 \multimap !^t 1} (*, \langle * \rangle)$: This is true, since $\langle \rangle \supset_{!^t 1} \langle * \rangle$ by Item 1.
- $(*, \langle \rangle) \supset_{1 \multimap !^t 1} (*, \langle \alpha_1, \alpha_2 \rangle)$: This case is the same as above.
- $(*, \langle * \rangle) \supset_{1 \multimap !^t 1} (*, \langle \alpha_1, \alpha_2 \rangle)$: We have $\langle * \rangle \supset_{!^t 1} \langle \alpha_1, \alpha_2 \rangle$ by Item 3.
- $(*, \langle \alpha_1, \alpha_2 \rangle) \supset_{1 \multimap !^t 1} (*, \langle \alpha'_1, \alpha'_2 \rangle)$: Since we have $(*, \alpha_i) \in \mu^{t,0}$ and $(*, \alpha'_i) \in \mu^{t,0}$, then $\alpha_i \supset_{!^t 1} \alpha'_i$ for $i = 1, 2$. Hence, by Item 4, we have $\langle \alpha_1, \alpha_2 \rangle \supset_{!^t 1} \langle \alpha'_1, \alpha'_2 \rangle$.

To show $\mu_{E,F}^{t,2} \in \text{Cl}(!^t E_1 \otimes !^t E_2 \multimap !^t (E_1 \otimes E_2))$, let $(\sigma, \tau), (\sigma', \tau') \in \mu_{E,F}^{t,2}$. We prove that $(\sigma, \tau) \supset_{!^t E_1 \otimes !^t E_2 \multimap !^t (E_1 \otimes E_2)} (\sigma', \tau')$ by induction on $\text{TreeSize}(\sigma) + \text{TreeSize}(\sigma')$. So assume that $\sigma \supset_{!^t E_1 \otimes !^t E_2} \sigma'$. We consider the following cases and in each case we prove that $\tau \supset_{!^t (E_1 \otimes E_2)} \tau'$ and that, if $\tau = \tau'$ then $\sigma = \sigma'$:

- $((\langle \rangle, \langle \rangle), \langle \rangle) \circ_{(\text{!}E_1 \otimes \text{!}E_2) \rightarrow \text{!}(E_1 \otimes E_2)} ((\langle a^1 \rangle, \langle a^2 \rangle), \langle (a^1, a^2) \rangle)$: This case is true, since $\langle \rangle \circ_{\text{!}(E_1 \otimes E_2)}$ $\langle (a^1, a^2) \rangle$ by Item 1.
- $((\langle \rangle, \langle \rangle), \langle \rangle) \circ_{(\text{!}E_1 \otimes \text{!}E_2) \rightarrow \text{!}(E_1 \otimes E_2)} ((\langle \alpha_1^1, \alpha_2^1 \rangle, \langle \alpha_1^2, \alpha_2^2 \rangle), \langle \beta_1, \beta_2 \rangle)$: This case is the same as the previous one.
- $((\langle a^1 \rangle, \langle a^2 \rangle), \langle (a^1, a^2) \rangle) \circ_{(\text{!}E_1 \otimes \text{!}E_2) \rightarrow \text{!}(E_1 \otimes E_2)} ((\langle \alpha_1^1, \alpha_2^1 \rangle, \langle \alpha_1^2, \alpha_2^2 \rangle), \langle \beta_1, \beta_2 \rangle)$ where $((\alpha_j^1, \alpha_j^2), \beta_j) \in \mu_{E_1, E_2}^{\text{t}, 2}$ for $j = 1, 2$: This case is vacuously true, since we do not have $\langle (a^1, a^2) \rangle \circ_{\text{!}(E_1 \otimes E_2)}$ $\langle \alpha_1^1, \alpha_2^1 \rangle, \langle \alpha_1^2, \alpha_2^2 \rangle$.
- $((\langle a^1 \rangle, \langle a^2 \rangle), \langle (a^1, a^2) \rangle) \circ_{(\text{!}E_1 \otimes \text{!}E_2) \rightarrow \text{!}(E_1 \otimes E_2)} (((\langle a^1 \rangle)', \langle (a^2) \rangle'), \langle (a^1)', (a^2) \rangle')$: Assume that $\langle (a^1, a^2) \rangle \circ_{\text{!}(E_1 \otimes \text{!}E_2)}$ $((\langle a^1 \rangle)', \langle (a^2) \rangle')$. Then we have $\langle a^i \rangle \circ_{\text{!}E_i}$ $\langle (a^i) \rangle'$ for $i = 1, 2$. Then $a^i \circ_{E_i}$ $(a^i)'$. And so we have $(a^1, a^2) \circ_{E_1 \otimes E_2}$ $((a^1)', (a^2)')$. Therefore $\langle (a^1, a^2) \rangle \circ_{\text{!}(E_1 \otimes E_2)}$ $\langle ((a^1)', (a^2)') \rangle$. We have a similar reasoning if we assume moreover that $\langle (a^1, a^2) \rangle = \langle ((a^1)', (a^2)') \rangle$.
- $((\langle \alpha_1^1, \alpha_2^1 \rangle, \langle \alpha_1^2, \alpha_2^2 \rangle), \langle \beta_1, \beta_2 \rangle) \circ_{(\text{!}E_1 \otimes \text{!}E_2) \rightarrow \text{!}(E_1 \otimes E_2)} (((\langle \alpha_1^1 \rangle', \langle \alpha_2^1 \rangle'), \langle \alpha_1^2 \rangle', \langle \alpha_2^2 \rangle'), \langle \beta_1', \beta_2' \rangle)$ where $((\alpha_j^1, \alpha_j^2), \beta_j) \in \mu_{E_1, E_2}^{\text{t}, 2}$ and $((\langle \alpha_j^1 \rangle', \langle \alpha_j^2 \rangle'), \beta_j') \in \mu_{E_1, E_2}^{\text{t}, 2}$ for $j = 1, 2$:
 Assume that $\langle \alpha_1^1, \alpha_2^1 \rangle, \langle \alpha_1^2, \alpha_2^2 \rangle \circ_{\text{!}(E_1 \otimes \text{!}E_2)}$ $((\langle \alpha_1^1 \rangle', \langle \alpha_2^1 \rangle'), \langle \alpha_1^2 \rangle', \langle \alpha_2^2 \rangle')$. So, we have $\langle \alpha_j^i \rangle \circ_{\text{!}E_i}$ $\langle (\alpha_j^i) \rangle'$ for $i = 1, 2$. Therefore $\alpha_j^i \circ_{E_i}$ $(\alpha_j^i)'$ for $j = 1, 2$. Then we have $(\alpha_j^1, \alpha_j^2) \circ_{\text{!}(E_1 \otimes \text{!}E_2)}$ $((\alpha_j^1)', (\alpha_j^2)')$ for $j = 1, 2$. Hence we have $\beta_j \circ_{\text{!}(E_1 \otimes E_2)}$ β_j' for $j = 1, 2$. And finally, we have $\langle \beta_1, \beta_2 \rangle \circ_{\text{!}(E_1 \otimes E_2)}$ $\langle \beta_1', \beta_2' \rangle$. We have a similar reasoning if we moreover assume that $\langle \beta_1, \beta_2 \rangle = \langle \beta_1', \beta_2' \rangle$.

■

4.3 Local Totality Spaces

In this section, we will relate the coherence spaces with tree exponentials (presented in Section 4.2.2) and non-uniform coherence spaces with the Boudes exponential (presented in Section 0.4.3). To do so, we use the logical relation method, and so, we define another instance of our categorical setting provide in Section 4.1, and we call it local totality spaces.

Definition 145 A local totality space (LTS) E is a tuple $E = (E^G, E^N, \rho_E, \mathbb{T}_E)$ where E^G is a coherence space, E^N is a non-uniform coherence space, $\rho_E : |E^G| \rightarrow |E^N|$ is a function, and $\mathbb{T}_E = (\mathbb{T}_E(a))_{a \in |E^N|}$ such that $\forall a \in |E^N|$ we have $\mathbb{T}_E(a) \in \text{Tot}(\rho_E^{-1}(a))$ and $\emptyset \notin \mathbb{T}_E(a) \neq \emptyset$.

Notice that that $\rho_E^{-1}(a)$ is the subcoherence space of E^G whose web is $\rho_E^{-1}(a)$. And remember that $\text{Tot}(X) = \{\mathcal{T} \subseteq \mathcal{P}(X) \mid \mathcal{T} = \mathcal{T}^{\perp\perp}\}$ where $\mathcal{T}^\perp = \{u' \subseteq X \mid \forall u \in \mathcal{T} \ u \cap u' \neq \emptyset\}$.

The letters G and N in Definition 145 stand for Girard and non-uniform coherence spaces, respectively.

Definition 146 Given a local totality space E , we define the set of cliques of E , denoted as $\text{Cl}_{\text{LT}}(E)$, as follows:

$$\text{Cl}_{\text{LT}}(E) = \{u \in \text{Cl}(E^G) \mid \rho_E(u) \in \text{Cl}(E^N) \wedge \forall a \in \rho_E(u) \ (u \cap \rho_E^{-1}(a)) \in \mathbb{T}_E(a)\}.$$

4.3.1 COHLT is a *-autonomous category with finite product

Given two local totality spaces E and F , we define local totality space $E \otimes F$ as $(E^{\mathbb{G}} \otimes F^{\mathbb{G}}, E^{\mathbb{N}} \otimes F^{\mathbb{N}}, \rho_{E \otimes F}, \mathbb{T}_{E \otimes F})$ where $E^{\mathbb{G}} \otimes F^{\mathbb{G}}$ and $E^{\mathbb{N}} \otimes F^{\mathbb{N}}$ are defined in Section 0.4.2 and Section 0.4.3 respectively. And $\rho_{E \otimes F}(a, b) = (\rho_E(a), \rho_F(b))$. And we also define $\mathbb{T}_{E \otimes F}$ as

$$\mathbb{T}_{E \otimes F}(a, b) = \{u \times v \mid u \in \mathbb{T}_E(a) \wedge v \in \mathbb{T}_F(b)\}^{\perp\perp}.$$

This $\perp\perp$ is computed in coherence space $\rho_E^{-1}(a) \otimes \rho_F^{-1}(b)$ as subcoherence space $E^{\mathbb{G}} \otimes F^{\mathbb{G}}$. And notice that $\mathbb{T}_{E \otimes F}((a, b)) \in \text{Tot}(\rho_{E \otimes F}^{-1}(a, b))$ by definition, and $\emptyset \notin \mathbb{T}_{E \otimes F}((a, b)) \neq \emptyset$, sine that is the case for $\emptyset \notin \mathbb{T}_E(a) \neq \emptyset$ and $\emptyset \notin \mathbb{T}_F(b) \neq \emptyset$. Hence, indeed $E \otimes F$ is a LTS.

Given a LTS E , we define E^\perp as $((E^\perp)^{\mathbb{G}}, (E^\perp)^{\mathbb{N}}, \rho_{E^\perp}, \mathbb{T}_{E^\perp}^\perp)$ where $\rho_{E^\perp}(a) = \rho_E(a)$. E^\perp is obviously a LTS, and it relies on the fact that $\emptyset \notin \mathbb{T}_E(a) \neq \emptyset$. And then we define $E \multimap F$ as $(E \otimes F^\perp)^\perp$.

We will use the two following lemmas (Lemma 147, and Lemma 148) many times in this chapter, and we sometimes omit to mention them explicitly.

Lemma 147 *Let $t \in \text{Cl}_{\text{LT}}(E \multimap F)$. Then $t \in \mathbb{T}_{E \multimap F}((a, b)) \Leftrightarrow \forall u \in \mathbb{T}_E(a) t \cdot u \in \mathbb{T}_F(b)$.*

Proof: Let $t \in \mathbb{T}_{E \multimap F}((a, b))$ and let $u \in \mathbb{T}_E(a)$. Let $v' \in (\mathbb{T}_F(b))^\perp$, since $u \times v' \in \mathbb{T}_{E \otimes F}((a, b))$ we have $t \cap (u \times v') \neq \emptyset$ and hence $(t \cdot u) \cap v' \neq \emptyset$. Therefore $t \cdot u \in \mathbb{T}_F(b)^{\perp\perp} = \mathbb{T}_F(b)$. Conversely assume that $\forall u \in \mathbb{T}_E(a) t \cdot u \in \mathbb{T}_F(b)$. Let $u \in \mathbb{T}_E(a)$ and $v' \in \mathbb{T}_F(b)^\perp = (\mathbb{T}_F(b))^\perp$. Since $t \cdot u \in \mathbb{T}_F(b)$ we have $(t \cdot u) \cap v' \neq \emptyset$ and hence $t \cap (u \times v') \neq \emptyset$ and this shows that $t \in \mathbb{T}_{E \multimap F}((a, b))$. ■

The proof of the following is the same as the proof of Lemma 57.

Lemma 148 *Let $t \subseteq |E^{\mathbb{N}}| \times |F^{\mathbb{N}}|$, $a \in |E^{\mathbb{N}}|$, $b \in |F^{\mathbb{N}}|$, and $\mathcal{A} \subseteq \mathcal{P}(\rho_E^{-1}(a))$ be such that $\mathbb{T}_E(a) = \mathcal{A}^{\perp\perp}$. If $\forall x \in \mathcal{A} t x \in \mathbb{T}_F(b)$ then $t \in \text{Cl}_{\text{LT}}(E \multimap F)$.*

We now define the category COHLT of local totality spaces that includes local totality spaces as objects, and as morphisms $\text{COHLT}(E, F) = \text{Cl}_{\text{LT}}(E \multimap F)$. We take the diagonal morphism $\text{Id}_{|E^{\mathbb{G}}|}$ as the identity morphism on E . For the composition, we take the relational composition of **Rel**. Proposition 151 ensures that what we have defined is indeed a category.

The following lemma is the main justification of the our definition of LTS.

Lemma 149 *Given $f \in \text{Cl}_{\text{LT}}(E \multimap F)$ and $g \in \text{Cl}_{\text{LT}}(F \multimap G)$, we have $\rho_{E \multimap G}(g \circ f) = (\rho_{F \multimap G}(g)) \circ (\rho_{E \multimap F}(f))$.*

Proof:

▷ **Proof of $\rho_{E \multimap G}(g \circ f) \subseteq (\rho_{F \multimap G}(g)) \circ (\rho_{E \multimap F}(f))$:** Take $(a, c) \in \rho_{E \multimap G}(g \circ f)$, then there is a $(\alpha, \gamma) \in (g) \circ (f)$ such that $\rho_{E \multimap G}((\alpha, \gamma)) = (a, c)$. So, $\rho_E(\alpha) = a$ and $\rho_G(\gamma) = c$. Since $(\alpha, \gamma) \in (g) \circ (f)$, there is a $\beta \in$ such that $(\alpha, \beta) \in f$ and $(\beta, \gamma) \in g$. Let $b = \rho_F(\beta)$ so that there is a $b \in |F^{\mathbb{N}}|$ such that $(a, b) \in \rho_{E \multimap F}(f)$ and $(b, c) \in \rho_{F \multimap G}(g)$. Hence $(a, c) \in (\rho_{F \multimap G}(g)) \circ (\rho_{E \multimap F}(f))$.

▷ **Proof of $\rho_{E \multimap G}(g \circ f) \supseteq (\rho_{F \multimap G}(g)) \circ (\rho_{E \multimap F}(f))$:** Take $(a, c) \in (\rho_{F \multimap G}(g)) \circ (\rho_{E \multimap F}(f))$, so, there is a $b \in |F^{\mathbb{N}}|$ such that $(a, b) \in \rho_{E \multimap F}(f)$ and $(b, c) \in \rho_{F \multimap G}(g)$. Since $f \in \text{Cl}_{\text{LT}}(E \multimap F)$ (respectively $g \in \text{Cl}_{\text{LT}}(F \multimap G)$), we have $f \cap \rho_{E \multimap F}^{-1}(a, b) \in \mathbb{T}_{E \multimap F}(a, b)$ (respectively $g \cap \rho_{F \multimap G}^{-1}(b, c) \in \mathbb{T}_{F \multimap G}(b, c)$). We know that for any $u \in \mathbb{T}_E(a)$ we have $(f \cap \rho_{E \multimap F}^{-1}(a, b)) \cdot u \in \mathbb{T}_F(b)$. Similarly, we have $(g \cap \rho_{F \multimap G}^{-1}(b, c)) \cdot ((f \cap \rho_{E \multimap F}^{-1}(a, b)) \cdot u) \in \mathbb{T}_G(c)$. Since $\emptyset \notin \mathbb{T}_G(c) \neq \emptyset$ and $\emptyset \notin \mathbb{T}_F(b) \neq \emptyset$, one can find a $\gamma \in \mathbb{T}_G(c)$ and $\beta \in (f \cap \rho_{E \multimap F}^{-1}(a, b)) \cdot u$ such that $(\beta, \gamma) \in g \cap \rho_{F \multimap G}^{-1}(b, c)$, and so, $(\beta, \gamma) \in g$. Similarly, since $\emptyset \notin \mathbb{T}_E(a) \neq \emptyset$, there is a $\alpha \in u$ such that $(\alpha, \beta) \in f \cap \rho_{E \multimap F}^{-1}(a, b)$, and so, $(\alpha, \beta) \in f$. Therefore, there is a $(\alpha, \gamma) \in (g) \circ (f)$ such that $\rho_{E \multimap G}((\alpha, \gamma)) = (a, c)$. Hence $(a, c) \in \rho_{E \multimap G}(g \circ f)$. ■

As a direct conclusion of Lemma 149, we also have

Lemma 150 *Given $f \in \text{Cl}_{\text{LT}}(E \multimap F)$ $g \in \text{Cl}_{\text{LT}}(F \multimap G)$, $(a, c) \in (g \circ f)$, $(a, b) \in f$ and $(b, c) \in g$, we have $\rho_{E \multimap G}^{-1}(a, c) = (\rho_{F \multimap G}^{-1}(b, c)) \circ (\rho_{E \multimap F}^{-1}(a, b))$*

Proposition 151

1. $\text{Id}_{|E^{\mathbb{G}}|} \in \text{Cl}_{\text{LT}}(E \multimap E)$
2. *Given two morphisms $f \in \text{Cl}_{\text{LT}}(E \multimap F)$ and $g \in \text{Cl}_{\text{LT}}(F \multimap G)$, we have $g \circ f \in \text{Cl}_{\text{LT}}(E \multimap G)$.*

Proof: First of all we have $\rho_{E \multimap E}(\text{Id}_{|E^{\mathbb{G}}|}) \in \text{Cl}(E^{\mathbb{N}} \multimap E^{\mathbb{N}})$, since $\rho_{E \multimap E}(\text{Id}_{|E^{\mathbb{G}}|}) \subseteq \text{Id}_{|E^{\mathbb{G}}|}$. Given $(a, a) \in \rho_{E \multimap E}(\text{Id}_{|E^{\mathbb{G}}|})$, let us assume that $u \in \mathbb{T}_E(a)$. Then we have $(\text{Id}_{|E^{\mathbb{G}}|} \cap \rho_{E \multimap E}^{-1}((a, a)))(u) = u$, since $u \subseteq \rho_E^{-1}(a)$. Therefore we have the first item.

For the second item, $\rho_{E \multimap G}(g \circ f) = (\rho_{F \multimap G}(g)) \circ (\rho_{E \multimap F}(f))$ by Lemma 149. And we also have $\rho_{F \multimap G}(g) \in \text{Cl}((F \multimap G)^{\mathbb{N}})$ and $\rho_{E \multimap F}(f) \in \text{Cl}((E \multimap F)^{\mathbb{N}})$. Hence, $\rho_{E \multimap G}((g) \circ (f)) \in \text{Cl}(E \multimap G^{\mathbb{N}})$. Given $(a, c) \in \rho_{E \multimap G}(g \circ f)$, suppose that $u \in \mathbb{T}_E(a)$. By Lemma 149, there exists a b such that $(a, b) \in \rho_{E \multimap F}(f)$ and $(b, c) \in \rho_{F \multimap G}(g)$. Since $f \in \mathbb{T}_{E \multimap F}$, we have $f \cap \rho_{E \multimap F}^{-1}(a, b) \in \mathbb{T}_{E \multimap F}((a, b))$. Therefore $x = (f \cap \rho_{E \multimap F}^{-1}(a, b))(u) \in \mathbb{T}_F(b)$ by Lemma 147. And by a similar reasoning, we have $(g \cap \rho_{F \multimap G}^{-1}(b, c))(x) \in \mathbb{T}_G(c)$. And we also have $((g \circ f) \cap \rho_{E \multimap G}^{-1}(a, c))(u) = (g \cap \rho_{F \multimap G}^{-1}(b, c))(x)$ by Lemma 150. Hence the second item is true, by Lemma 147. ■

Now, we are going to prove that COHLT is a monoidal closed category. We just take all definitions provided in Section 0.4.1, and show that they define cliques in the appropriate LTS.

Lemma 152 *Let $f \subseteq |E^{\mathbb{N}}| \times |F^{\mathbb{N}}|$. One has $f \in \text{Cl}_{\text{LT}}(E \multimap F)$ iff $f^\perp = \{(b, a) \mid (a, b) \in f\} \in \text{Cl}_{\text{LT}}(F^\perp \multimap E^\perp)$.*

Proof: First we have $\rho_{E \multimap F}(f) \text{Cl}(E^{\mathbb{N}} \multimap F^{\mathbb{N}})$ iff $\rho_{F^\perp \multimap E^\perp}(f^\perp) \text{Cl}((F^\perp)^{\mathbb{N}} \multimap (E^\perp)^{\mathbb{N}})$, since $\rho_{E^\perp}(a) = \rho_E(a)$ and by results of Section 0.4.3.

To show $f \cap \rho_{E \multimap F}^{-1}(a, b) \in \mathbb{T}_{E \multimap F}((a, b))$ iff $f^\perp \cap \rho_{F^\perp \multimap E^\perp}^{-1}(b, a) \in \mathbb{T}_{F^\perp \multimap E^\perp}((b, a))$ we can use Lemma 147 and the fact that $(E \multimap F) = ((E \otimes F^\perp)^\perp)^\perp$ and $(F^\perp \multimap E^\perp) = (F^\perp \otimes E)^\perp$. ■

By the same proof as that of Lemma 36, one has the following:

Lemma 153 $f \in \mathbb{T}_{(E_1 \otimes E_2) \multimap F}$ iff for all $u_1 \in \mathbb{T}_{E_1}$ and $u_2 \in \mathbb{T}_{E_2}$ one has $f \cdot (u_1 \otimes u_2) \in \mathbb{T}_F$.

As we saw in two previous lemma, one can prove easily Lemma 154 and Lemma 155 using results of Sections 1.2.2 and 0.4.3.

Lemma 154 The bijection $\alpha_{|X_1|, |X_2|, |Y|}$ is an isomorphism from $(E_1 \otimes E_2) \multimap F$ to $E_1 \multimap (E_2 \multimap F)$.

Lemma 155 Given $f \in \mathbb{T}_{E \otimes F \multimap G}$, then $\text{cur}(f) \in \mathbb{T}_{E \multimap (F \multimap G)}$. We also have $\text{ev}_{E, F} \in \mathbb{T}_{(E \multimap F) \otimes E \multimap F}$.

And we finally define two objects $1 = (1^{\mathbb{N}}, 1^{\mathbb{G}}, \text{Id}, \{\{*\}\})$. What we have done so far was proof of the following theorem:

Theorem 156 The category COHLT is a *-autonomous category.

One can also equip category COHLT with a product. Given two LTS E_1 and E_2 , one can define $E_1 \& E_2$ as $((E_1 \& E_2)^{\mathbb{G}}, (E_1 \& E_2)^{\mathbb{N}}, \rho_{E_1 \& E_2}, \mathbb{T}_{E_1 \& E_2})$ where $\rho_{E_1 \& E_2}((i, a)) = (i, \rho_{E_i}(a))$ for $i = 1, 2$, and

$$\mathbb{T}_{E_1 \& E_2}((i, a)) = \left\{ u \subseteq \rho_{E_1 \& E_2}^{-1}(i, a) \mid \text{pr}_i \cdot u \in \mathbb{T}_{E_i}(a) \right\}.$$

Then we have $\mathbb{T}_{E_1 \& E_2}((i, a)) \in \text{Tot}(\rho_{E_1 \& E_2}^{-1}(i, a))$ by definition, and we also have $\emptyset \notin \mathbb{T}_{E_1 \& E_2}((i, a)) \neq \emptyset$, since $\emptyset \notin \mathbb{T}_{E_i} a \neq \emptyset$. So, indeed $E_1 \& E_2$ is a LTS.

Since COHLT is a *-autonomous category, one can define co-product $E_1 \oplus E_2 = (E_1^\perp \& E_2^\perp)^\perp$. And finally, the terminal object of COHLT is $\top = (\top^{\mathbb{G}}, \top^{\mathbb{N}}, \emptyset, \{\emptyset\})$ and that $0 = \top^\perp = (\top^{\mathbb{G}}, \top^{\mathbb{N}}, \emptyset, \emptyset)$.

4.3.2 Exponentials in COHLT

Given a LTS E . We define $!E = (!^t E^{\mathbb{G}}, !^b E^{\mathbb{N}}, \rho_{!E}, \mathbb{T}_{!E})$ where $\rho_{!E}$ is defined inductively as follows ($!^b _$ and $!^t _$ are defined respectively in Sections 0.4.3 and 4.2.2):

- $\rho_{!E}(\langle \rangle) = \square$,
- $\rho_{!E}(\langle a \rangle) = [\rho_E(a)]$,
- $\rho_{!E}(\langle \alpha, \beta \rangle) = \rho_{!E}(\alpha) + \rho_{!E}(\beta)$.

$\mathbb{T}_{!E}(m)$ is defined as follows for a given $m = [a_1, \dots, a_k] \in !^b E^{\mathbb{N}}$:

$$\mathbb{T}_{!E}(m) = \{ U \in \text{Cl}(!^t E^{\mathbb{G}}) \mid U \subset \rho_{!E}^{-1}(m) \wedge \exists (u_i \in \mathbb{T}_E(a_i))_{i=1}^k (\text{TreeLeaf}(\vec{u}) \subset U) \}^{\perp\perp}$$

where $\text{TreeLeaf}(u_1, \dots, u_k)$ is the set $\{\tau \in \text{tree}(\bigcup u_i) \mid \exists (\alpha_i \in u_i)_{i=1}^k (\text{L}(\tau) = [\alpha_1, \dots, \alpha_k])\}$, and $\text{L}(_)$ is the function that return the leaves of a tree.

Lemma 157 *Let E be a local totality space, then $!E$ is a local totality space.*

Proof: We obviously have $\mathbb{T}_{!E}(m) \in \text{Tot}(\rho_{!E}^{-1}(m))$ since we took bi-orthogonal in the definition. First assume that $m = []$ then one can take $U = \text{tree}(\emptyset) \in \mathbb{T}_{!E}([])$, so, $\emptyset \notin \mathbb{T}_{!E}([]) \neq \emptyset$. And obviously, $\emptyset \notin \mathbb{T}_{!E}(m) \neq \emptyset$ if $m \neq []$, since $\emptyset \notin \mathbb{T}_{E_i}(a_i) \neq \emptyset$ for $i = 1, \dots, k$. ■

Lemma 158 *Let E and F be two local totality spaces, then $\rho_{!E \multimap !F}(!^t f) = !(\rho_{E \multimap F}(f))$.*

Proof:

▷ **Proof of $\rho_{!E \multimap !F}(!^t f) \subset !(\rho_{E \multimap F}(f))$:** Take $(m, n) \in \rho_{!E \multimap !F}(!^t f)$ where $m = [a_1, \dots, a_k]$ and $n = [b_1, \dots, b_k]$. Then there is $(\tau, \sigma) \in !^t f$ such that $(m, n) = \rho_{!E \multimap !F}((\tau, \sigma)) = \rho_{!E}(\tau) \times \rho_{!F}(\sigma)$. So, we have $m = [\rho_E(\alpha_1)] + \dots + [\rho_E(\alpha_k)]$ and $n = [\rho_F(\beta_1)] + \dots + [\rho_F(\beta_k)]$ where $\alpha_i \in \mathbb{L}(\tau)$ and $\beta_i \in \mathbb{L}(\sigma)$ for $i \in \{1, \dots, k\}$. Therefore, $a_i = \rho_E(\alpha_i)$ and $b_i = \rho_F(\beta_i)$. So, there are $(\alpha_i, \beta_i) \in f$ such that $(a_i, b_i) = \rho_{E \multimap F}(\alpha_i, \beta_i)$ for $i \in \{1, \dots, k\}$. Therefore we have $(a_i, b_i) \in \rho_{E \multimap F}(f)$ for $i \in \{1, \dots, k\}$. Hence $(m, n) \in !(\rho_{E \multimap F}(f))$.

▷ **Proof of $\rho_{!E \multimap !F}(!^t f) \supset !(\rho_{E \multimap F}(f))$:** All steps of above also hold in the other direction. ■

Lemma 159 *Let E and F be two local totality spaces, and $f \in \text{Cl}(E^{\mathbb{G}} \multimap F^{\mathbb{G}})$ such that $\rho_{E \multimap F}(f) \in \text{Cl}(E^{\mathbb{N}} \multimap F^{\mathbb{N}})$. Then $\rho_{!E \multimap !F}(!^t f) \in \text{Cl}(!^b E^{\mathbb{N}} \multimap !^b F^{\mathbb{N}})$.*

Proof: By Lemma 158, $\rho_{!E \multimap !F}(!^t f) = !(\rho_{E \multimap F}(f))$. And since $\rho_{E \multimap F}(f) \in \text{Cl}(E^{\mathbb{N}} \multimap F^{\mathbb{N}})$, we have $!(\rho_{E \multimap F}(f)) \in \text{Cl}(!^b E^{\mathbb{N}} \multimap !^b F^{\mathbb{N}})$. ■

Lemma 160 *Let E and F be two local totality spaces, and $f \in \text{Cl}_{\text{LT}}(E \multimap F)$. Then $\forall (m, n) \in \rho_{!E \multimap !F}(!^t f)$ one has $!^t f \cap \rho_{!E \multimap !F}^{-1}(m, n) \in \mathbb{T}_{!E \multimap !F}(m, n)$.*

Proof: Take $(m, n) \in \rho_{!E \multimap !F}(!^t f)$ where $m = [a_1, \dots, a_k]$ and $n = [b_1, \dots, b_k]$ with $(a_i, b_i) \in \rho_{E \multimap F}(f)$. Let $U \in \mathbb{T}_{!E}(m)$. One also has $U \subset \rho_{!E}^{-1}(m)$ and assume that there is a collection $(u_i \in \mathbb{T}_E(a_i))_{i=1}^k$ such that $\text{TreeLeaf}(\vec{u}) \subset U$. Notice that one can also remove this assumption of existence of u_i 's using Lemma 148.

Since $f \in \text{Cl}_{\text{LT}}(E \multimap F)$, we have $f \cap \rho_{E \multimap F}^{-1}(a_i, b_i) \in \mathbb{T}_{E \multimap F}$. By Lemmas 147, $(f \cap \rho_{E \multimap F}^{-1}(a_i, b_i)) \cdot u_i \in \mathbb{T}_F(b_i)$ for the given $u_i \in \mathbb{T}_E(a_i)$. So, let $v_i = (f \cap \rho_{E \multimap F}^{-1}(a_i, b_i)) \cdot u_i$ for $i = 1, \dots, k$. Hence $\text{TreeLeaf}(\vec{v}) \subseteq (!^t f \cap \rho_{!E \multimap !F}(m, n)) \cdot U$.

Moreover, since $U \subseteq \rho_{!E}^{-1}(m)$, one has $(!^t f \cap \rho_{!E \multimap !F}(m, n)) \cdot U \in \rho_{!F}^{-1}(n)$. And this ends the proof that $!^t f \cap \rho_{!E \multimap !F}^{-1}(m, n) \in \mathbb{T}_{!E \multimap !F}(m, n)$. ■

As an immediate application of Lemmas 157, 159 and 160, we have the following fact:

Proposition 161 *The operation $!^t$ defines an endofunctor on the category COHLT.*

Lemma 162 *One has $\text{der}_E^t \in \text{Cl}_{\text{LT}}(!E \multimap E)$, and $\text{dig}_E^t \in \text{Cl}_{\text{LT}}(!E \multimap !!E)$.*

Proof: Let $(m, a) \in \rho_{!E \multimap E}(\text{der}_E^\dagger)$ and $U \in \mathbb{T}_{!E}(m)$. By definition of der_E (defined in Section 0.4.1), m should be $[a]$. Let $u \in \mathbb{T}_E(a)$ such that $\text{TreeLeaf}(u) \subseteq U$. u cannot be empty, so, because $U \subseteq \rho_{!E}^{-1}([a])$, we have $u = \{a \mid \langle a \rangle \in U\}$. One also has $\text{der}_E^\dagger \cdot U = \{a \mid \langle a \rangle \in U\}$. Hence $\text{der}_E^\dagger \cdot U \in \mathbb{T}_E(a)$. And as $\rho_{!E \multimap E}(\text{der}_E^\dagger) = \text{der}_E$, we also have $\rho_{!E \multimap E}(\text{der}_E^\dagger) \in \text{Cl}(!_b E^{\mathbb{N}} \multimap E^{\mathbb{N}})$ by Section 0.4.3. So, we end with the proof of $\text{der}_E^\dagger \in \text{Cl}_{\text{LT}}(!E \multimap E)$.

One has $\rho_{!E \multimap !E}(\text{dig}_E^\dagger) = \text{dig}_E$. Now, let $(m_1 + \dots + m_n, [m_1, \dots, m_n]) \in \rho_{!E \multimap !E}(\text{dig}_E^\dagger)$ and $U \in \mathbb{T}_{!E}((m_1 + \dots + m_n))$ where $m_j = [a_i \mid i \in I_j]$ (I_j 's are disjoint). And so, $[m_1, \dots, m_n] = [a_i \mid i \in I]$ where $I = \bigcup_{j=1}^n I_j$. Let the collection $(u_i \in \mathbb{T}_E(a_i))_{i \in I}$ be such that $\text{TreeLeaf}(\vec{u}) \subseteq U$. Now, we define $V_j = \text{TreeLeaf}((u_i)_{i \in I_j})$. Take $T \in \text{TreeLeaf}(\vec{V})$. Therefore one has $\text{flat}_E(T) \in U$. Hence, by Definition 4.4, we have $T \in \text{dig}_E^\dagger \cdot U$. And since $(\text{dig}_E^\dagger \cap \rho_{!E \multimap !E}^{-1}(m_1 + \dots + m_n, [m_1, \dots, m_n])) \cdot U = \text{dig}_E^\dagger \cdot U$, we have $(\text{dig}_E^\dagger \cap \rho_{!E \multimap !E}^{-1}(m_1 + \dots + m_n, [m_1, \dots, m_n])) \cdot U \in \mathbb{T}_{!E}([m_1, \dots, m_n])$. Hence, one has $\text{dig}_E^\dagger \in \mathbb{T}_{!E \multimap !E}((m_1 + \dots + m_n, [m_1, \dots, m_n]))$. And as $\rho_{!E \multimap !E}(\text{dig}_E^\dagger) = \text{dig}_E$, we also have $\rho_{!E \multimap !E}(\text{dig}_E^\dagger) \in \text{Cl}(!_b !_b E^{\mathbb{N}} \multimap !_b E^{\mathbb{N}})$ by Section 0.4.3. So, we are done with the proof of $\text{dig}_E^\dagger \in \text{Cl}_{\text{LT}}(!E \multimap !E)$. ■

As an immediate application of Proposition 143 and Lemma 162, we have the following fact:

Proposition 163 *The triple $(\text{tree}(_), \text{der}^\dagger, \text{dig}^\dagger)$ is a comonad over COHLT.*

Lemma 164 *One has $\mu^{t,0} \in \text{Cl}_{\text{LT}}(1 \multimap !1)$ and $\mu^{t,2} \in \text{EFCl}_{\text{LT}}((!E_1 \otimes !E_2) \multimap !(E_1 \otimes E_2))$*

Proof: Since COHLT is that a symmetric monoidal category (part of Theorem 156), and $(\text{tree}(_), \text{der}^\dagger, \text{dig}^\dagger)$ is a comonad over COHLT (Proposition 143), by the general construction given in Section 4.1.3, we have $\mu^{t,0} \in \text{COHLT}(1, !1)$ and $\mu^{t,2} \in \text{COHLT}(!E_1 \otimes !E_2, !(E_1 \otimes E_2))$. ■

And finally, as an immediate application of Proposition 144 and Lemma 164, we have the following fact:

Proposition 165 *The structure $(!^t _, \text{der}^\dagger, \text{dig}^\dagger, \mu^{t,0}, \mu^{t,2})$ is a symmetric monoidal comonad on the SMC COHLT.*

We denote by $\llbracket A \rrbracket_{\text{CohT}}$ (respectively $\llbracket A \rrbracket_{\text{COHLT}}$) the interpretation of formula A in the coherence spaces with the tree exponential (respectively the category COHLT).

Now, we have all material to prove one of our main goal of this chapter which is the following theorem.

Theorem 166 *Let π be a LL proof $\vdash \Gamma$. Then $\rho_{\llbracket \Gamma \rrbracket_{\text{COHLT}}}(\llbracket \pi \rrbracket_{\text{CohT}}) = \llbracket \pi \rrbracket_{\text{NCoh}}$.*

Proof: There is a forgetful functor $U : \text{COHLT} \rightarrow \mathbf{Coh}$ sending object $E = (E^{\mathbb{G}}, E^{\mathbb{N}}, \rho_E, \mathbb{T}_E)$ to $E^{\mathbb{G}}$ and it acts as identity on morphisms. By Theorem 156, Propositions 143, and Propositions 165, we can compute $\llbracket \pi \rrbracket_{\text{COHLT}}$ using the general construction given in Figure 3, and one can see that $U(\llbracket \pi \rrbracket_{\text{COHLT}}) = \llbracket \pi \rrbracket_{\text{COHLT}} = \llbracket \pi \rrbracket_{\text{CohT}}$ as U preserves all the linear logic structure, since it acts on morphism (cliques) as identity. There is also a functor $V : \text{COHLT} \rightarrow \mathbf{NCoh}$ sending object $E = (E^{\mathbb{G}}, E^{\mathbb{N}}, \rho_E, \mathbb{T}_E)$ to $E^{\mathbb{N}}$ and a morphism

$u \in \text{COHLT}(E, F) = \text{Cl}_{\text{LT}}(E \multimap F)$ to $\rho_{E \multimap F}(u) \in \text{Cl}(E^{\mathbf{N}} \multimap F^{\mathbf{N}}) = \mathbf{NCoh}(E, F)$ using Definition 146. The functor V preserves all the linear logic structure: For the symmetric monoidal closed structure, this is obvious, since it acts as identity on those structures. This is also the case for $!^{\dagger}$ by Lemma 158, and this holds also for der^{\dagger} and dig^{\dagger} by the proof of Lemma 162. Hence, we have $V(\llbracket \pi \rrbracket_{\text{COHLT}}) = \rho_{\llbracket \Gamma \rrbracket_{\text{COHLT}}}(\llbracket \pi \rrbracket_{\text{COHLT}}) = \rho_{\llbracket \Gamma \rrbracket_{\text{COHLT}}}(\llbracket \pi \rrbracket_{\text{CohT}}) = \llbracket \pi \rrbracket_{\mathbf{NCoh}}$. ■

4.4 Sum up of Chapter 4

Sum up of Chapter 4

- Provide a categorical setting for Tree exponentials: Section 4.1
- Examine the mentioned categorical setting on two well-known models of LL, i.e, **Rel** and **Coh**: Propositions 140 and 141 for **Rel**. Propositions 143 and 144 for **Coh**.
- Provide a new concrete instances of the mentioned categorical setting, i.e, Local Totality Spaces. Theorem 156, Propositions 163 and 165.
- Relate interpretations of LL proofs in coherence spaces with the tree exponential and non-uniform coherence spaces: Theorem 166.

Chapter 5

Conclusion and future work

In the last chapter we list some further research and open questions related to this thesis.

5.1 A general categorical framework for computing recursive types in **Nuts**

We have done in Chapter 1 that **Nuts** (non-uniform totality spaces) is model of μLL . This has two parts; first showing that **Nuts** is model of **LL**, and secondly that an endofunctor on **Nuts**, under some assumptions, has both initial algebra and final coalgebra. To deal with the second part, one can summarize the situation as follows.

Let F be an endofunctor on **Rel** which has an initial algebra and a final algebra and let \bar{F} be a lifting of F to **Nuts**. That is to say one has the commutation of the following square where U is the forgetful functor:

$$\begin{array}{ccc} \mathbf{Nuts} & \xrightarrow{\bar{F}} & \mathbf{Nuts} \\ U \downarrow & & U \downarrow \\ \mathbf{Rel} & \xrightarrow{F} & \mathbf{Rel} \end{array}$$

Then we showed that \bar{F} also has both initial algebra and final coalgebra. So, one can ask this natural question: what is the categorical setting for this construction that we have in **Nuts**.

We can make this question more precise by noticing that category **Nuts** is an example of slack orthogonality category (Definition 47 of [HS03]). So, the question is as follows:

Question 1 *Let \mathcal{L} be a model of **LL**. Let J be an object of \mathcal{L} , and consider the slack orthogonality category $\mathcal{S}_J(\mathcal{L})$. And let F be an endofunctor on \mathcal{L} which has both an initial algebra and a final algebra. Does the endofunctor $\bar{F} : \mathcal{S}_J(\mathcal{L}) \rightarrow \mathcal{S}_J(\mathcal{L})$ have initial algebra and final coalgebra where \bar{F} is the lifting of F ?*

5.2 Full normalization of $\kappa\mu\text{LLP}$

We proved an adequacy theorem in Chapter 2 for $\kappa\mu\text{LLP}$ (Theorem 84). As two applications of this theorem, we saw two normalization theorem: for command $c \vdash P$ (Theorem 104), and for command $c \vdash ?\text{nat}$. So, a very natural question is that can we extend this result to a command $c \vdash \mathcal{N}[, P]$.

Question 2 *Given a command $c \vdash \mathcal{N}[, P]$, is c strongly normalizing?*

We have some ideas to tackle this question: The idea is to extend $\kappa\mu\text{LLP}$ with a constant \circ which is a new command, typed by $\circ \vdash \mathcal{N}$ where \mathcal{N} is an arbitrary *negative* context. This may seem surprising at first sight but remember that weakening is freely available for all negative formulas. So the real meaning of this rule is $\circ \vdash$ which is the familiar 0-ary mix rule of LL. It is easy to check that all properties of $\kappa\mu\text{LLP}$ proven so far are still valid for this extension. We also need to extend the point typing system with the rule $\circ \vdash (\alpha_i : a_i : N_i)_{i=1}^n$ under the proviso that $a_i \widetilde{N_i^\perp} []$ for all $i \in \underline{n}$. The benefit of this extension is that now $\underline{\perp}$ contains closed commands.

5.3 Adding the general fixpoint operator to $\kappa\mu\text{LLP}$

One can imagine an extension of $\kappa\mu\text{LLP}$ with a general fixpoint construction. That is to say adding the following rule or maybe some other formulation (One can also look at [Ehr16, ET19] to see this kind of language). And moreover, we also have two unfolding rule for μ and ν .

$$\frac{\vdash \mathcal{N}, \alpha : ?N^\perp \mid t : N}{\vdash \mathcal{N} \mid \text{fix}_\alpha(t) : N} \text{ (fix)} \quad \frac{\vdash \mathcal{N} \mid p : P [\mu\zeta.P/\zeta^+]}{\vdash \mathcal{N} \mid \text{fd}(p) : \mu\zeta.P} \text{ (t-}\mu\text{)} \quad \frac{\vdash \mathcal{N}[, Q] \mid t : N [\nu\zeta.N/\zeta^-]}{\vdash \mathcal{N}[, Q] \mid \text{fd}(t) : \nu\zeta.N} \text{ (}\nu\text{-fold)}$$

One can also extend the reduction system of $\kappa\mu\text{LLP}$ by the following rules:

$$\begin{aligned} \text{fix}_\alpha(t) \star p &\rightarrow (t \star p) \left[\text{fix}_\alpha(t)^\dagger / \alpha \right] \\ \text{fd}(t) \star \text{fd}(p) &\rightarrow t \star p \end{aligned}$$

Notice that it is not clear that if one can derive the (t- ν) rule of $\kappa\mu\text{LLP}$ using the rules above and the LL ones. However, there is a strong feeling that one can prove using the categorical semantic that this cannot be true. So, perhaps we have to deal with the (t- ν) rule directly but this makes adequacy theorem more complicated. Or perhaps we need another formulation of the (fix) rule.

Question 3 *If we want to add a general fix operator to $\kappa\mu\text{LLP}$, what formulation should we consider? And what sort of normalization theorem can we prove?*

5.4 Categorical model for non-well-founded linear (μLL_∞)

We only provided concrete models of μLL_∞ in Chapter 3, and not a categorical model. For instance, in [FS13, San02], there is categorical model for the circular proof in the additive fragment of LL. So, one can wonder if that can be extended to full LL. Circular proofs are

the Non-well-founded proofs that have a finite representation. The very general idea would be to interpret circular proofs as a system of equations and try to find the solution of that system using final coalgebra. However, one the main difficulty is that it is not clear how to understand the syntactic validity condition (Definition 118) categorically.

If we do not restrict ourselves to circular proofs and consider all valid μLL_∞ proofs, we almost do not hope to provide a categorical model for them. Because it is not even clear how to interpret an arbitrary non-well founded proof. Of course, one can assume some structure on the category in order to interpret those proof. For instance, one can work with CPO-enrich categories, and interpret proofs using the same idea as we did in **Rel**. Nevertheless, it worth trying to find a class of categories as model of μLL_∞ , rather than finding a free category for μLL_∞ logic in the sense of what we have for CCC categories and simple typed λ -calculus.

Question 4 *What is the categorical model of circular μLL_∞ ?*

5.5 Investigation on Kleisli category of tree exponential comonad

As we saw in Chapter4, the Kleisli category of tree exponential comonad is not a CCC category. So, in particular, the semantic does not preserve the β -reduction of simply typed λ -calculus $((\lambda x.u)t \rightarrow u[t/x])$. However, one might gain by comparing the interpretation of $(\lambda x.u)t$ and $u[t/x]$ in those Kleisli category.

Question 5 *What sort of information one might obtain by computing the interpretation of $(\lambda x.u)t$ and $u[t/x]$ in those Kleisli category of tree exponential comonad in different structure such as **Rel**, **Coh**, and **COHLT**?*

5.6 Sequentiality and tree exponential semantic

Let us consider the following LL proof π_1 :

$$\begin{array}{c}
 \frac{\frac{\frac{\vdash 1}{\vdash 1 \oplus 1} (\oplus_1)}{\vdash \perp, 1 \oplus 1} (\perp)}{\vdash (\perp \& \perp), 1 \oplus 1} \quad \frac{\frac{\frac{\vdash 1}{\vdash 1 \oplus 1} (\oplus_2)}{\vdash \perp, 1 \oplus 1} (\perp)}{\vdash (\perp \& \perp), 1 \oplus 1} (\&)}{\vdash ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\oplus_2)}{\vdash (\perp, ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\perp)} \\
 \frac{\frac{\frac{\frac{\frac{\vdash 1}{\vdash 1 \oplus 1} (\oplus_2)}{\vdash \perp, 1 \oplus 1} (\perp)}{\vdash (\perp \& \perp), 1 \oplus 1} (\&)}{\vdash ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\oplus_2)}{\vdash (\perp, ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\perp)}{\vdash (\perp \& \perp, ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\&)} \\
 \frac{\vdash (\perp \& \perp, ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\oplus_1)}{\vdash ((\perp \& \perp) \oplus (\perp \& \perp)), ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\oplus_1)} \\
 \frac{\vdash ((\perp \& \perp) \oplus (\perp \& \perp)), ((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\text{d}), (\text{d})}{\vdash ((\perp \& \perp) \oplus (\perp \& \perp)), ?((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\text{c})} \\
 \frac{\vdash ((\perp \& \perp) \oplus (\perp \& \perp)), ?((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} {\vdash ?((\perp \& \perp) \oplus (\perp \& \perp)), 1 \oplus 1} (\text{c})}
 \end{array}$$

We denote by **t** (respectively **f**) point $(1, *)$ (respectively $(2, *)$), and denote by *it* (reps. *if*) point $(i, (1, *))$ (respectively $(i, (2, *))$) for $i = 1, 2$.

Interpretation of π_1 in **Rel** is $\{([1\mathbf{t}, 2\mathbf{t}], \mathbf{t}), ([1\mathbf{t}, 2\mathbf{f}], \mathbf{f}), ([1\mathbf{f}, 2\mathbf{t}], \mathbf{f}), ([1\mathbf{f}, 2\mathbf{f}], \mathbf{f})\}$. And if we consider the **Rel** with tree exponential, one has $\{(\langle\langle 1\mathbf{t}\rangle, \langle 2\mathbf{t}\rangle\rangle, \mathbf{t}), (\langle\langle 1\mathbf{t}\rangle, \langle 2\mathbf{f}\rangle\rangle, \mathbf{f}), (\langle\langle 1\mathbf{f}\rangle, \langle 2\mathbf{t}\rangle\rangle, \mathbf{f}), (\langle\langle 1\mathbf{f}\rangle, \langle 2\mathbf{f}\rangle\rangle, \mathbf{f})\}$ as the interpretation of π_1 .

Now, let proof π_2 be the proof which is the same as π_1 but we do (\oplus_1) on the yellow formula instead of doing (\oplus_1) on the green formula. Then interpretation of π_2 is the same as π_1 in **Rel** with the multiset exponential, as the order does not matter in a multiset. However, interpretation of π_2 in **Rel** with tree exponential will be $\{(\langle\langle 2\mathbf{t}\rangle, \langle 1\mathbf{t}\rangle\rangle, \mathbf{t}), (\langle\langle 2\mathbf{f}\rangle, \langle 1\mathbf{t}\rangle\rangle, \mathbf{f}), (\langle\langle 2\mathbf{t}\rangle, \langle 1\mathbf{f}\rangle\rangle, \mathbf{f}), (\langle\langle 2\mathbf{f}\rangle, \langle 1\mathbf{f}\rangle\rangle, \mathbf{f})\}$.

One can also obtain, symmetrically, proof π_3 and π_4 that do (\oplus_2) on the red formula and (\oplus_2) on the blue formula respectively instead of doing (\oplus_1) on the green formula in π_1 . Then one can see that whereas interpretation of π_3 and π_4 are same in **Rel**, those are different in **Rel** with tree exponential. So, one has four different interpretation in **Rel** with tree exponential for those four (not necessarily “different”) proofs. One can see such a phenomenon usually in sequential function and game semantics.

Question 6 *Is there any relation between the tree exponential semantic and game semantic of LL?*

Bibliography

- [Abr93] Samson Abramsky. Computational interpretations of linear logic. *Theoretical Computer Science*, 111:3–57, 1993.
- [Ada74] Jiri Adamek. Free algebras and automata realizations in the language of categories. *Commentationes Mathematicae Universitatis Carolinae*, 015(4):589–602, 1974.
- [AE15] Shahin Amini and Thomas Ehrhard. On classical pcf, linear logic and the MIX rule. In Stephan Kreutzer, editor, *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany*, volume 41 of *LIPICs*, pages 582–596. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2015.
- [AJ94] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *The Journal of Symbolic Logic*, 59(2):543–574, 1994.
- [AMM18] Jiri Adámek, Stefan Milius, and Lawrence S. Moss. Fixed points of functors. *J. Log. Algebraic Methods Program.*, 95:41–81, 2018.
- [And92] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
- [APTS13] Andreas Abel, Brigitte Pientka, David Thibodeau, and Anton Setzer. Copatterns: programming infinite structures by observations. In *POPL*, pages 27–38. ACM, 2013.
- [Bae12] David Baelde. Least and Greatest Fixed Points in Linear Logic. *ACM Trans. Comput. Log.*, 13(1):2:1–2:44, 2012.
- [Bar79] Michael Barr. **- Autonomous Categories*. Springer Berlin Heidelberg, 1979.
- [BBdPH92] Nick Benton, Gavin Bierman, Valeria de Paiva, and Martin Hyland. Term assignment for intuitionistic linear logic. Technical report, 1992.
- [BBdPH93] Nick Benton, Gavin Bierman, Valeria de Paiva, and Martin Hyland. A term calculus for intuitionistic linear logic. In Marc Bezem and Jan Friso Groote, editors, *Typed Lambda Calculi and Applications*, pages 75–90, Berlin, Heidelberg, 1993. Springer Berlin Heidelberg.
- [BCL83] Gérard Berry, Pierre-Louis Curien, and Jean-Jacques Levy. Full abstraction for sequential languages : The states of the art. Research Report RR-0197, INRIA, 1983.

- [BCST96] R.F. Blute, J.R.B. Cockett, R.A.G. Seely, and T.H. Trimble. Natural deduction and coherence for weakly distributive categories. *Journal of Pure and Applied Algebra*, 113(3):229–296, 1996.
- [BDKS20] David Baelde, Amina Doumane, Denis Kuperberg, and Alexis Saurin. Bouncing threads for infinitary and circular proofs, 2020.
- [BDS16] David Baelde, Amina Doumane, and Alexis Saurin. Infinitary Proof Theory: the Multiplicative Additive Case. In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 - September 1, 2016, Marseille, France*, volume 62 of *LIPICs*, pages 42:1–42:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.
- [BE91] A. Bucciarelli and T. Ehrhard. Sequentiality and strong stability. In *[1991] Proceedings Sixth Annual IEEE Symposium on Logic in Computer Science*, pages 138–145, 1991.
- [BE01] Antonio Bucciarelli and Thomas Ehrhard. On phase semantics and denotational semantics: the exponentials. 109(3):205–241, 2001.
- [Ben] Jean Benabou. Logique categorique lecture notes of a course by J. Benabou in montreal (1974).
- [Ben95] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models. In Leszek Pacholski and Jerzy Tiuryn, editors, *Computer Science Logic*, pages 121–135, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg.
- [Ber78] Gérard Berry. Stable models of typed λ -calculi. In Giorgio Ausiello and Corrado Böhm, editors, *Automata, Languages and Programming*, pages 72–89, Berlin, Heidelberg, 1978. Springer Berlin Heidelberg.
- [Ber79] Gérard Berry. *Modèles complètement adéquats et stables des lambda-calculs typés*. Thèse de doctorat, Université Paris VII, 1979.
- [Bie94] G.M. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, University of Cambridge, 1994.
- [Bie95] G. M. Bierman. What is a categorical model of intuitionistic linear logic? In Mariangiola Dezani-Ciancaglini and Gordon Plotkin, editors, *Typed Lambda Calculi and Applications*, pages 78–93, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg.
- [Blu93] Richard Blute. Linear logic, coherence and dinaturality. *Theoretical Computer Science*, 115(1):3–41, 1993.
- [Bou02] Pierre Boudes. Non-uniform hypercoherences. In Richard Blute and Peter Selinger, editors, *Category Theory and Computer Science, CTCS 2002, Ottawa, Canada, August 15-17, 2002*, volume 69 of *Electronic Notes in Theoretical Computer Science*, pages 62–82. Elsevier, 2002.
- [Bou11] Pierre Boudes. Non-uniform (hyper/multi)coherence spaces. *Math. Struct. Comput. Sci.*, 21(1):1–40, 2011.

BIBLIOGRAPHY

- [Bro06] James Brotherston. *Sequent Calculus Proof Systems for Inductive Definitions*. PhD thesis, University of Edinburgh, November 2006.
- [BS94] G. Bellin and P.J. Scott. On the π -calculus and linear logic. *Theoretical Computer Science*, 135(1):11–65, 1994.
- [BS07] James Brotherston and Alex Simpson. Complete sequent calculi for induction and infinite descent. In *Proceedings of LICS-22*, pages 51–60. IEEE Computer Society, July 2007.
- [BS10] James Brotherston and Alex Simpson. Sequent calculi for induction and infinite descent. *Journal of Logic and Computation*, 21(6):1177–1216, 10 2010.
- [BT17] Stefano Berardi and Makoto Tatsuta. Equivalence of intuitionistic inductive definitions and intuitionistic cyclic proofs under arithmetic. *CoRR*, abs/1712.03502, 2017.
- [BT19] Stefano Berardi and Makoto Tatsuta. Classical System of Martin-Lof’s Inductive Definitions is not Equivalent to Cyclic Proofs. *Logical Methods in Computer Science*, Volume 15, Issue 3, August 2019.
- [BTKP93] V. Breazu-Tannen, D. Kesner, and L. Puel. A typed pattern calculus. In *[1993] Proceedings Eighth Annual IEEE Symposium on Logic in Computer Science*, pages 262–274, 1993.
- [Bus98] Samuel R. Buss. *Handbook of Proof Theory*. Elsevier, 1998.
- [Can95] G. Cantor. *Beiträge zur Begründung der transfiniten Mengenlehre. 1. (Erster Artikel)*. Number v. 1. Teubner, 1895.
- [Can97] Georg Cantor. Beiträge zur begründung der transfiniten mengenlehre. (zweiter artikel.). *Mathematische Annalen*, 49:207–246, 1897.
- [CC74] Cantor and Cantor. Ueber eine eigenschaft des inbegriffs aller reellen algebraischen zahlen. 1874(77):258–262, 1874.
- [CC79] Patrick Cousot and Radhia Cousot. Constructive versions of Tarski’s fixed point theorems. *Pacific Journal of Mathematics*, 82(1):43 – 57, 1979.
- [CFM16] Pierre-Louis Curien, Marcelo P. Fiore, and Guillaume Munch-Maccagnoni. A theory of effects and resources: adjunction models and polarised calculi. In Rastislav Bodík and Rupak Majumdar, editors, *Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2016, St. Petersburg, FL, USA, January 20 - 22, 2016*, pages 44–56. ACM, 2016.
- [CH00] Pierre-Louis Curien and Hugo Herbelin. The duality of computation. In *ICFP*, pages 233–243. ACM, 2000.
- [CH06] Felice Cardone and Roger Hindley. History of lambda-calculus and combinatory logic. 2006.

- [CHS80] Haskell B. Curry, J. Roger Hindley, and Jonathan P. Seldin. To h.b. curry: Essays on combinatory logic, lambda calculus, and formalism. 1980.
- [Chu32] Alonzo Church. A set of postulates for the foundation of logic. *Annals of Mathematics*, 33(2):346–366, 1932.
- [Chu36a] Alonzo Church. A note on the entscheidungsproblem. *Journal of Symbolic Logic*, 1(1):40–41, 1936.
- [Chu36b] Alonzo Church. An unsolvable problem of elementary number theory. *American Journal of Mathematics*, 58:345, 1936.
- [Chu40] Alonzo Church. A formulation of the simple theory of types. *The Journal of Symbolic Logic*, 5(2):56–68, 1940.
- [Chu41] Alonzo Church. *The Calculi of Lambda Conversion. (AM-6)*. Princeton University Press, 1941.
- [Chw22] Leon Chwistek. über die antinomien der prinzipien der mathematik. *Mathematische Zeitschrift*, 14:236–243, 1922.
- [CM10] Pierre-Louis Curien and Guillaume Munch-Maccagnoni. The duality of computation under focus. In Cristian S. Calude and Vladimiro Sassone, editors, *Theoretical Computer Science - 6th IFIP TC 1/WG 2.2 International Conference, TCS 2010, Held as Part of WCC 2010, Brisbane, Australia, September 20-23, 2010. Proceedings*, volume 323 of *IFIP Advances in Information and Communication Technology*, pages 165–181. Springer, 2010.
- [CP05] J.R.B. Cockett and C.A. Pastro. A language for multiplicative-additive linear logic. *Electronic Notes in Theoretical Computer Science*, 122:23–65, 2005. Proceedings of the 10th Conference on Category Theory in Computer Science (CTCS 2004).
- [CR36] Alonzo Church and J. B. Rosser. Some properties of conversion. *Transactions of the American Mathematical Society*, 39(3):472–482, 1936.
- [Cur30] H. B. Curry. Grundlagen der kombinatorischen logik. *American Journal of Mathematics*, 52(3):509–536, 1930.
- [Cur34] H. B. Curry. Functionality in combinatory logic. *Proceedings of the National Academy of Sciences of the United States of America*, 20(11):584–590, 1934.
- [Cur35] H. B. Curry. First properties of functionality in combinatory logic. *Tohoku Mathematical Journal, First Series*, 41:371–401, 1935.
- [Cur42] Haskell B. Curry. The combinatory foundations of mathematical logic. *The Journal of Symbolic Logic*, 7(2):49–64, 1942.
- [Cur89] Pierre-Louis Curien. Alpha-conversion, conditions on variables and categorical logic. *Studia Logica*, 48(3):319–360, 1989.
- [Cur94] Pierre-Louis Curien. *Categorical Combinators, Sequential Algorithms, and Functional Programming (2nd Ed.)*. Birkhauser Boston Inc., USA, 1994.

BIBLIOGRAPHY

- [dB80] Nicolaas Govert de Bruijn. A survey of the project automath. *Studies in logic and the foundations of mathematics*, 133:141–161, 1980.
- [dC16] Daniel de Carvalho. The Relational Model Is Injective for Multiplicative Exponential Linear Logic. In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic, CSL 2016, August 29 - September 1, 2016, Marseille, France*, volume 62 of *LIPICs*, pages 41:1–41:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.
- [dCdF12] Daniel de Carvalho and Lorenzo Tortora de Falco. The relational model is injective for multiplicative exponential linear logic (without weakenings). *Ann. Pure Appl. Log.*, 163(9):1210–1236, 2012.
- [DDS22] Anupam Das, Abhishek De, and Alexis Saurin. Decision problems for linear logic with least and greatest fixed points. In Amy P. Felty, editor, *7th International Conference on Formal Structures for Computation and Deduction, FSCD 2022, August 2-5, 2022, Haifa, Israel*, volume 228 of *LIPICs*, pages 20:1–20:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.
- [Dou17] Amina Doumane. *On the infinitary proof theory of logics with fixed points*. PhD thesis, Paris Diderot University, 2017.
- [DS71] Christopher Strachey Dana Scott. Towards a mathematical semantics for computer languages. Technical report, 1971.
- [Ehr95] Thomas Ehrhard. Hypercoherences: A strongly stable model of linear logic. In *Proceedings of the Workshop on Advances in Linear Logic*, page 83–108, USA, 1995. Cambridge University Press.
- [Ehr16] Thomas Ehrhard. Call-by-push-value from a linear logic point of view. In Peter Thiemann, editor, *Programming Languages and Systems - 25th European Symposium on Programming, ESOP 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, volume 9632 of *Lecture Notes in Computer Science*, pages 202–228. Springer, 2016.
- [Ehr18] Thomas Ehrhard. An introduction to differential linear logic: proof-nets, models and antiderivatives. 28(7):995–1060, 2018.
- [ET19] Thomas Ehrhard and Christine Tasson. Probabilistic call by push value. *Log. Methods Comput. Sci.*, 15(1), 2019.
- [FB97] Gottlob Frege and Michael Beaney. *The Frege Reader*. Oxford, England: Blackwell, 1997.
- [Fio96] Marcelo P. Fiore. *Axiomatic Domain Theory in Categories of Partial Maps*. Distinguished Dissertations in Computer Science. Cambridge University Press, 1996.
- [Fra22] A. Fraenkel. Zu den grundlagen der cantor-zermeloschen mengenlehre. *Mathematische Annalen*, 86:230–237, 1922.

- [Fre84] Gottlob Frege. *The Foundations of Arithmetic: A Logico-Mathematical Enquiry Into the Concept of Number*. translated by J.L. Austin, Oxford: Blackwell, second revised edition, 1974, 1884.
- [Fre91] Peter Freyd. Algebraically complete categories. In Aurelio Carboni, Maria Cristina Pedicchio, and Giuseppe Rosolini, editors, *Category Theory*, pages 95–104, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.
- [Fre92] P. J. Freyd. *Remarks on algebraically compact categories*, page 95–106. London Mathematical Society Lecture Note Series. Cambridge University Press, 1992.
- [FS13] Jérôme Fortier and Luigi Santocanale. Cuts for circular proofs: semantics and cut-elimination. In Simona Ronchi Della Rocca, editor, *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy*, volume 23 of *LIPICs*, pages 248–262. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013.
- [Gal93] Jean Gallier. Constructive logics part i: A tutorial on proof systems and typed λ -calculi. *Theoretical Computer Science*, 110(2):249–339, 1993.
- [Gan68] R. O. Gandy. P. b. andrews. a transfinite type theory with type variables. studies in logic and the foundations of mathematics. north-holland publishing company, amsterdam1965, xv 143 pp. *Journal of Symbolic Logic*, 33(1):112–113, 1968.
- [Gen35] Gerhard Gentzen. *Untersuchungen über das logische Schliessen*. Ph.d. thesis, Universität Göttingen, March 1935.
- [Gen39] G. Gentzen. Die gegenwärtige lage in der mathematischen grundlagenforschung. — neue fassung des widerspruchsfreiheitsbeweises für die reine zahlentheorie. 1939.
- [Gen64] Gerhard Gentzen. Investigations into logical deduction. *American Philosophical Quarterly*, 1(4):288–306, 1964.
- [Gir71] J. Y. Girard. Une extension de l'interprétation de Godel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. 63:63–92, 1971.
- [Gir72] J.Y. Girard. *Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieur*. Éditeur inconnu, 1972.
- [Gir86] Jean-Yves Girard. The system F of variable types, fifteen years later. *Theoretical Computer Science*, 45:159–192, 1986.
- [Gir87a] Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987.
- [Gir87b] J.Y. Girard. *Proof Theory and Logical Complexity*. Number v. 1 in Collection Gratianus: Section Monographs. Bibliopolis, 1987.
- [Gir88] Jean-Yves Girard. Normal functors, power series and λ -calculus. *Annals of Pure and Applied Logic*, 37(2):129–177, 1988.
- [Gir92] Jean-Yves Girard. Fixpoint theorem in linear logic. An email posting to the mailing list linear@cs.stanford.edu, 1992.

BIBLIOGRAPHY

- [Gir95] Jean-Yves Girard. Linear logic: Its syntax and semantics. In Jean-Yves Girard, Yves Lafont, and Laurent Regnier, editors, *Advances in Linear Logic*. Cambridge University Press, 1995.
- [Gir99] Jean-Yves Girard. On denotational completeness. *Theoretical Computer Science*, 227:249–273, 1999.
- [GLT89] J.Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1989.
- [Göd29] Kurt Gödel. Über die vollständigkeit des logikkalküls, 1929.
- [Göd31] Kurt Gödel. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für Mathematik und Physik*, 38:173–198, 1931.
- [Göd58] Kurt Gödel. Über eine bisher noch nicht benützte erweiterung des finiten standpunktes. *Dialectica*, 12(3):280, 1958.
- [Goo60] R. L. Goodstein. Combinatory logic. by h. b. curry and r. feys. pp. 417. 42s. 1958. (north holland publishing co., amsterdam). *The Mathematical Gazette*, 44(347):77–77, 1960.
- [HA28] D. Hilbert and W. Ackermann. *Grundzüge der theoretischen Logik*. Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. J. Springer, 1928.
- [Hen50] Leon Henkin. Completeness in the theory of types. *The Journal of Symbolic Logic*, 15(2):81–91, 1950.
- [Her94] Hugo Herbelin. A lambda-calculus structure isomorphic to gentzen-style sequent calculus structure. In Leszek Pacholski and Jerzy Tiuryn, editors, *Computer Science Logic, 8th International Workshop, CSL '94, Kazimierz, Poland, September 25-30, 1994, Selected Papers*, volume 933 of *Lecture Notes in Computer Science*, pages 61–75. Springer, 1994.
- [Her95] Hugo Herbelin. *Séquents qu'on calcule: de l'interprétation du calcul des séquents comme calcul de lambda-termes et comme calcul de stratégies gagnantes. (Computing with sequents: on the interpretation of sequent calculus as a calculus of lambda-terms and as a calculus of winning strategies)*. PhD thesis, Paris Diderot University, France, 1995.
- [How70] W.A. Howard. Assignment of ordinals to terms for primitive recursive functionals of finite type. In A. Kino, J. Myhill, and R.E. Vesley, editors, *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968*, volume 60 of *Studies in Logic and the Foundations of Mathematics*, pages 443–458. Elsevier, 1970.
- [How80] William Alvin Howard. The formulae-as-types notion of construction. In Haskell Curry, Hindley B., Seldin J. Roger, and P. Jonathan, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Academic Press, 1980.

- [HP87] J. M. E. Hyland and Andrew M. Pitts. The theory of constructions: Categorical semantics and topos-theoretic models. 1987.
- [HS03] Martin Hyland and Andrea Schalk. Glueing and orthogonality for models of linear logic. *Theoretical Computer Science*, 294(1):183–231, 2003. Category Theory and Computer Science.
- [Hyl97] Martin Hyland. *Game Semantics*, page 131–184. Publications of the Newton Institute. Cambridge University Press, 1997.
- [Jay89] C. Barry Jay. Languages for monoidal categories. *Journal of Pure and Applied Algebra*, 59(1):61–85, 1989.
- [JFM⁺96] Achim Jung, Marcelo P. Fiore, Eugenio Moggi, Peter W. O’Hearn, Jon G. Riecke, Giuseppe Rosolini, and Ian David Bede Stark. Domains and denotational semantics history accomplishments and open problems. 1996.
- [Keg99] Mathias Kegelmann. Factorisation systems on domains. *Appl. Categorical Struct.*, 7(1-2):113–128, 1999.
- [KhLO98] T. W. Koh and C. h. L. Ong. Type theories for autonomous and *-autonomous categories: I. type theories and rewrite systems - ii. internal languages and coherence theorems, 1998.
- [KK95] A. Kenny and P.V.C.A. Kenny. *Frege: An Introduction to the Founder of Modern Analytic Philosophy*. Penguin philosophy. Penguin Books, 1995.
- [Kle38] S. C. Kleene. On notation for ordinal numbers. *Journal of Symbolic Logic*, 3(4):150–155, 1938.
- [Kle71] S.C. Kleene. *Introduction to Metamathematics*. Bibliotheca Mathematica, a Series of Monographs on Pure and. Wolters-Noordhoff, 1971.
- [KM71] G. M. Kelly and Saunders MacLane. Coherence in closed categories. *Journal of Pure and Applied Algebra*, 1:97–140, 1971.
- [Kna28] Bronisław Knaster. Un théorème sur les fonctions d’ensembles. *Ann. Soc. Polon. Math.* 6, pages 133–134, 1928.
- [KPP21] Denis Kuperberg, Laureline Pinault, and Damien Pous. Cyclic proofs, system t, and the power of contraction. *Proc. ACM Program. Lang.*, 5(POPL), January 2021.
- [KR35] S. C. Kleene and J. B. Rosser. The inconsistency of certain formal logics. *Annals of Mathematics*, 36(3):630–636, 1935.
- [Laf88a] Yves Lafont. The linear abstract machine. *Theor. Comput. Sci.*, 59:157–180, 1988.
- [Laf88b] Yves Lafont. *Logiques, catégories machines : implantation de langages de programmation guidée par la logique catégorique*. PhD thesis, 1988.

BIBLIOGRAPHY

- [Lam] François Lamarche. A large cartesian closed category of domains.
- [Lam68] Joachim Lambek. Deductive systems and categories i. syntactic calculus and residuated categories. *Math. Syst. Theory*, 2(4):287–318, 1968.
- [Lam69] Joachim Lambek. Deductive systems and categories ii. standard constructions and closed categories. In Peter J. Hilton, editor, *Category Theory, Homology Theory and their Applications I*, pages 76–122, Berlin, Heidelberg, 1969. Springer Berlin Heidelberg.
- [Lam89] J. Lambek. *Multicategories Revisited*. Rapport du Department of Mathematics and Statistics. McGill University, Department of Mathematics and Statistics, 1989.
- [Lam92] François Lamarche. Quantitative domains and infinitary algebras. *Theoretical Computer Science*, 94(1):37–62, 1992.
- [Lau02] Olivier Laurent. *Etude de la polarisation en logique*. Thèse de doctorat, Université Aix-Marseille II, March 2002.
- [Law] Francis William Lawvere. Equality in hyperdoctrines and comprehension scheme as an adjoint functor. In A. Heller, editor, *Applications of Categorical Algebra*, pages 1-14, Providence, 1970. AMS.
- [Law63] F. William Lawvere. *Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories*. Ph.d. thesis, Columbia University, 1963.
- [Law69] F. William Lawvere. Adjointness in foundations. *Dialectica*, 23(3/4):281–296, 1969.
- [LMMP13] Jim Laird, Giulio Manzonetto, Guy McCusker, and Michele Pagani. Weighted relational models of typed lambda-calculi. In *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 301–310, 2013.
- [Mac63] Saunders MacLane. Natural associativity and commutativity. *Rice Institute Pamphlet - Rice University Studies*, 49:28–46, 1963.
- [Mac71] Saunders MacLane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
- [Mak93a] Michael Makkai. The fibrational formulation of intuitionistic predicate logic i: completeness according to gödel, kripke, and läuchli, part 1. *Notre Dame J. Formal Log.*, 34:334–377, 1993.
- [Mak93b] Michael Makkai. The fibrational formulation of intuitionistic predicate logic i: completeness according to gödel, kripke, and läuchli, part 2. *Notre Dame J. Formal Log.*, 34:471–498, 1993.
- [Mar84] Per Martin-Löf. *Intuitionistic Type Theory*. Bibliopolis, 1984.

- [Mat98] Ralph Matthes. Monotone fixed-point types and strong normalization. In Georg Gottlob, Etienne Grandjean, and Katrin Seyr, editors, *Computer Science Logic, 12th International Workshop, CSL '98, Annual Conference of the EACSL, Brno, Czech Republic, August 24-28, 1998, Proceedings*, volume 1584 of *Lecture Notes in Computer Science*, pages 298–312. Springer, 1998.
- [Mat99a] Ralph Matthes. *Extensions of system F by iteration and primitive recursion on monotone inductive types*. PhD thesis, Ludwig Maximilian University of Munich, Germany, 1999.
- [Mat99b] Ralph Matthes. Monotone (co)inductive types and positive fixed-point types. *RAIRO Theor. Informatics Appl.*, 33(4/5):309–328, 1999.
- [Mel03] Paul-André Melliès. Categorical models of linear logic revisited. working paper or preprint, October 2003.
- [Mel09] Paul-André Melliès. Categorical semantics of linear logic. *Interactive models of computation and program behaviour, PES*, volume 27, 2009.
- [Mil80] Robin Milner. A calculus of communicating systems. In *Lecture Notes in Computer Science*, 1980.
- [Mil92] Robin Milner. Functions as processes. *Mathematical Structures in Computer Science*, 2:119 – 141, 1992.
- [ML71] Per Martin-Löf. Hauptsatz for the intuitionistic theory of iterated inductive definitions. In J.E. Fenstad, editor, *Proceedings of the Second Scandinavian Logic Symposium*, volume 63 of *Studies in Logic and the Foundations of Mathematics*, pages 179–216. Elsevier, 1971.
- [MR77] Michael Makkai and Gonzalo E. Reyes. *First order categorical logic*. Springer-Verlag, Berlin-Heidelberg-New York, 1977.
- [MRA93] Ian Mackie, Leopoldo Román, and Samson Abramsky. An internal language for autonomous categories. *Applied Categorical Structures*, 1:311–343, 1993.
- [Mul85] Ketan D. Mulmuley. *Full Abstraction and Semantic Equivalence*. PhD thesis, USA, 1985. AAI8601184.
- [Mun09] Guillaume Munch-Maccagnoni. Focalisation and classical realisability. In Erich Grädel and Reinhard Kahle, editors, *Computer Science Logic, 23rd international Workshop, CSL 2009, 18th Annual Conference of the EACSL, Coimbra, Portugal, September 7-11, 2009. Proceedings*, volume 5771 of *Lecture Notes in Computer Science*, pages 409–423. Springer, 2009.
- [Mun13] Guillaume Munch-Maccagnoni. *Syntax and Models of a non-Associative Composition of Programs and Proofs. (Syntaxe et modèles d'une composition non-associative des programmes et des preuves)*. PhD thesis, Paris Diderot University, France, 2013.
- [NvPR01] Sara Negri, Jan von Plato, and Aarne Ranta. *Structural Proof Theory*. Cambridge University Press, 2001.

BIBLIOGRAPHY

- [NW96] Damian Niwiński and Igor Walukiewicz. Games for the μ -calculus. *Theoretical Computer Science*, 163(1):99–116, 1996.
- [Ong17] C.-H. Luke Ong. Quantitative semantics of the lambda calculus: Some generalisations of the relational model. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12, 2017.
- [Par92] Michel Parigot. $\lambda\mu$ -calculus: An algorithmic interpretation of classical natural deduction. In Andrei Voronkov, editor, *Logic Programming and Automated Reasoning*, pages 190–201, Berlin, Heidelberg, 1992. Springer Berlin Heidelberg.
- [PG08] Jan Von Plato and G. Gentzen. Gentzen’s proof of normalization for natural deduction. *The Bulletin of Symbolic Logic*, 14(2):240–257, 2008.
- [Plo77] G.D. Plotkin. Lcf considered as a programming language. *Theoretical Computer Science*, 5(3):223–255, 1977.
- [Poh73] Veřa Pohlová. On sums in generalized algebraic categories. *Czechoslovak Mathematical Journal*, 23(2):235–251, 1973.
- [Pot77] Garrel Pottinger. Normalization as a homomorphic image of cut-elimination. *Annals of Mathematical Logic*, 12(3):323–357, 1977.
- [PR97] John Power and Edmund Robinson. Premonoidal Categories and Notions of Computation. 7(5):453–468, 1997.
- [Pra67] Dag Prawitz. Completeness and hauptsatz for second order logic. *Theoria*, 33(3):246–258, 1967.
- [Pra68] Dag Prawitz. Hauptsatz for higher order logic. *Journal of Symbolic Logic*, 33:452 – 457, 1968.
- [Pra91] Dag Prawitz. Gaisi takeuti. proof theory. studies in logic and the foundations of mathematics, vol. 81. north-holland publishing company, amsterdam and oxford, and american elsevier publishing company, new york, 1975. *Journal of Symbolic Logic*, 56(3):1094, 1991.
- [Pé37] Rózsa Péter. S. c. kleene. λ -definability and recursiveness. duke mathematical journal, bd. 2 (1936), s. 340–353. *Journal of Symbolic Logic*, 2(1):38–39, 1937.
- [Qua95] Myriam Quatrini. *Sémantique cohérente des exponentielles : de la logique linéaire à la logique classique*. PhD thesis, 1995.
- [Ram31] Frank Plumpton Ramsey. *The Foundations of Mathematics and Other Logical Essays*. Edited by R.B. Braithwaite, with a Pref. By G.E. Moore. –. London, England: Routledge and Kegan Paul, 1931.
- [Ret93] Christian Retoré. *Réseaux et séquents ordonnés. (Ordered sequents and proof nets)*. PhD thesis, Paris Diderot University, France, 1993.
- [Rey83] John C. Reynolds. Types, abstraction and parametric polymorphism. In *IFIP Congress*, 1983.

- [Rey93] John C. Reynolds. The discoveries of continuations. *LISP and Symbolic Computation*, 6:233–247, 1993.
- [Rey98] John C. Reynolds. *Theories of Programming Languages*. Cambridge University Press, 1998.
- [Rus06] Bertrand Russell. The theory of implication. *American Journal of Mathematics*, 28(2):159–202, 1906.
- [San02] Luigi Santocanale. A calculus of circular proofs and its categorical semantics. In Mogens Nielsen and Uffe Engberg, editors, *Foundations of Software Science and Computation Structures, 5th International Conference, FOSSACS 2002. Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2002 Grenoble, France, April 8-12, 2002, Proceedings*, volume 2303 of *Lecture Notes in Computer Science*, pages 357–371. Springer, 2002.
- [Sch] Moses Schönfinkel. Über die bausteine der mathematischen logik. *Mathematische Annalen*, 92:305–316.
- [Sco70] Dana Scott. Constructive validity. In M. Laudet, D. Lacombe, L. Nolin, and M. Schützenberger, editors, *Symposium on Automatic Demonstration*, pages 237–275, Berlin, Heidelberg, 1970. Springer Berlin Heidelberg.
- [Sco72] Dana Scott. Continuous lattices. In F. W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, pages 97–136, Berlin, Heidelberg, 1972. Springer Berlin Heidelberg.
- [Sco93] Dana S. Scott. A type-theoretical alternative to iswim, cuch, owhy. *Theor. Comput. Sci.*, 121:411–440, 1993.
- [See77] R.A.G Seely. *Hyperdoctrines and Natural Deduction: Some Connections between Proof Theory and Category Theory*. Thesis (ph.d.), University of Cambridge, 1977.
- [See83] Robert A. G. Seely. Hyperdoctrines, natural deduction and the beck condition. *Mathematical Logic Quarterly*, 29(10):505–542, 1983.
- [See89] R.A.G. Seely. Linear logic, *-autonomous categories and cofree coalgebras. In *IN CATEGORIES IN COMPUTER SCIENCE AND LOGIC*, pages 371–382. American Mathematical Society, 1989.
- [SP82] M. B. Smyth and G. D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal on Computing*, 11(4):761–783, 1982.
- [ST58] Dana S. Scott and Alfred Tarski. The sentential calculus with infinitely long expressions. *Colloquium Mathematicum*, 6:165–170, 1958.
- [Str00] Christopher S. Strachey. Fundamental concepts in programming languages. *Higher-Order and Symbolic Computation*, 13:11–49, 2000.
- [TA22] Takeshi Tsukada and Kazuyuki Asada. Linear-algebraic models of linear logic as categories of modules over sigma-semirings, 2022.

BIBLIOGRAPHY

- [Tai65] William W. Tait. Infinitely long terms of transfinite type. *Studies in logic and the foundations of mathematics*, 40:176–185, 1965.
- [Tai67] W. W. Tait. Intensional interpretations of functionals of finite type i. *The Journal of Symbolic Logic*, 32(2):198–212, 1967.
- [Tai68] W. W. Tait. A nonconstructive proof of gentzen’s hauptsatz for second order predicate logic. *Journal of Symbolic Logic*, 33(2):289–290, 1968.
- [Tak53] Gaisi Takeuti. On a generalized logic calculus. *Japanese journal of mathematics :transactions and abstracts*, 23:39–96, 1953.
- [Tar55] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.
- [Tar58] Alfred Tarski. Remarks on predicate logic with infinitely long expressions. *Colloquium Mathematicum*, 6:171–176, 1958.
- [Tay22] Paul Taylor. Well founded coalgebras and recursion. 2022.
- [Tit73] Satoko Titani. A proof of the cut-elimination theorem in simple type theory. *The Journal of Symbolic Logic*, 38(2):215–226, 1973.
- [Toe05] Michael Toepell. Chapter 55 - david hilbert, grundlagen der geometrie, first edition (1899). In I. Grattan-Guinness, Roger Cooke, Leo Corry, Pierre Crépel, and Niccolo Guicciardini, editors, *Landmark Writings in Western Mathematics 1640-1940*, pages 710–723. Elsevier Science, Amsterdam, 2005.
- [Tro73] Anne Sjerp Troelstra. Notes on intuitionistic second order arithmetic. 1973.
- [TS00] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2 edition, 2000.
- [UB99] C. Urban and G. M. Bierman. Strong normalisation of cut-elimination in classical logic. In Jean-Yves Girard, editor, *Typed Lambda Calculi and Applications*, pages 365–380, Berlin, Heidelberg, 1999. Springer Berlin Heidelberg.
- [UV08] Tarmo Uustalu and Varmo Vene. Comonadic Notions of Computation. In Jiri Adámek and Clemens Kupke, editors, *Proceedings of the Ninth Workshop on Coalgebraic Methods in Computer Science, CMCS 2008, Budapest, Hungary, April 4-6, 2008*, volume 203 of *Electronic Notes in Theoretical Computer Science*, pages 263–284. Elsevier, 2008.
- [vdW87] J. van de Wiele. PhD thesis, 1987.
- [vH67] J. van Heijenoort. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. Source books in the history of the sciences. Harvard University Press, 1967.
- [vN] John von Neumann. Über die definition durch transfinite induktion und verwandte fragen der allgemeinen mengenlehre. *Mathematische Annalen*, 99:373–391.

- [vP13] Jan von Plato. *Elements of Logical Reasoning*. Cambridge and New York: Cambridge University Press, 2013.
- [vP18] Jan von Plato. The Development of Proof Theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2018 edition, 2018.
- [Wal00] Igor Walukiewicz. Completeness of kozen’s axiomatisation of the propositional μ -calculus. *Information and Computation*, 157(1):142–182, 2000.
- [Wan79] Mitchell Wand. Fixed-point constructions in order-enriched categories. *Theoretical Computer Science*, 8(1):13–30, 1979.
- [WR13] A. N. Whitehead and B. Russell. Principia mathematica. *Revue de Métaphysique et de Morale*, 19(2):19–19, 1910-1913.
- [Zer08] E. Zermelo. Untersuchungen über die grundlagen der mengenlehre. i. *Mathematische Annalen*, 65:261–281, 1908.
- [Zer35] Ernest Zermelo. Grundlagen einer allgemeinen theorie der mathematischen satzsysteme. *Fundamenta Mathematicae*, 25(1):136–146, 1935.
- [Zuc74] J. Zucker. The correspondence between cut-elimination and normalization. *Annals of Mathematical Logic*, 7(1):1–112, 1974.
- [Ču97] Djordje Čubrić. On the semantics of the universal quantifier. *Annals of Pure and Applied Logic*, 87(3):209–239, 1997.

Introduction en français

Dans cette thèse, nous étudions le point fixe des types en logique linéaire à partir d’une correspondance de Curry-Howard-Lambek.

Girard dans [Gir92] a suggéré d’envisager l’extension de la logique linéaire avec des points fixes de formules dans une note non publiée, bien que la première étude complète de la théorie de la preuve d’une telle extension de la logique linéaire est de Baelde [Bae12]. Il a considéré une extension μMALL de la logique linéaire additive multiplicative avec les principes d’induction et de coinduction. Il semble, apparemment, que sa motivation vienne plus du point de vue de la “proof search” et de la vérification du système et donc son système logique μMALL est un calcul de prédicat. Nous examinerons ce système à partir de l’approche de la théorie du point fixe avec une perspective Curry-Howard-Lambek. Ainsi, nous n’avons pas besoin de traiter un calcul de prédicat, et nous nous en tenons au cadre du calcul de proposition. Contrairement à [Bae12], nous incluons les exponentielles dans notre système depuis le début, nous l’appelons donc μLL plutôt que propositionnel μMALL . Les exponentielles ne sont pas prises en compte dans μMALL parce qu’une certaine forme d’exponentielle peut être codée en utilisant des types inductifs/coinductifs, cependant ces exponentielles ne sont pas entièrement satisfaisantes de notre point de vue car leur interprétation dénotationnelle ne satisfait pas tous les isomorphismes requis ; spécifiquement, les *Seely isos* manquent (Ceci est expliqué dans le chapitre 4). Le système μLL est décrit dans le Chapitre 0, et là nous pouvons également voir un bref aperçu de la logique linéaire et de son modèle.

Maintenant, nous expliquons ci-dessous ce que nous avons fait dans chaque chapitre selon notre motivation principale qui est de faire la relation entre la théorie des points fixes, la logique linéaire et la correspondance de Curry-Howard-Lambek.

Chapitre 1

Dans le chapitre 1, nous avons d’abord décidé d’étudier la sémantique dénotationnelle de μLL , dont la définition ne repose pas sur la réécriture système μLL est équipé, puisque la théorie de la preuve (et donc la “sémantique opérationnelle”) de μLL est encore en cours de développement. Nous y développons une sémantique catégorielle de μLL étendant la notion standard de catégorie Seely ¹ de la logique linéaire classique. Un tel modèle de μLL est constitué d’une catégorie de Seely \mathcal{L} et d’une classe de foncteurs $\mathcal{L}^n \rightarrow \mathcal{L}$ pour toutes les arités possibles n qui seront utilisées pour interpréter μLL formules à variables libres. Ces

¹Parfois appelée catégorie new-Seely : c’est un SMCC cartésien avec une structure *-autonome et une comonade $!_-$ avec une structure monoïdale symétrique forte du produit cartésien au produit tensoriel.

foncteurs doivent être dotés d’une “strength” pour traiter correctement les contextes dans la règle ν . Puis nous développons une instance simple de ce cadre qui consiste à prendre pour \mathcal{L} la catégorie des ensembles et des relations, un modèle Seely bien connu de la logique linéaire. Les foncteurs “strong” que nous considérons sur cette catégorie, que nous appelons *ensembles de variables*, sont les couples $\mathbb{F} = (\overline{\mathbb{F}}, \widehat{\mathbb{F}})$ où $\widehat{\mathbb{F}}$ est la strength et $\overline{\mathbb{F}} : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ est un foncteur continu de Scott dans le sens où il commute avec les unions dirigées de morphismes ce qui implique une cocontinuité catégorielle sur la catégorie des ensembles et des injections et fait correspondre les inclusions aux inclusions (cette légère exigence supplémentaire simplifie la présentation). Il n’y a pas d’exigence particulière concernant la force $\widehat{\mathbb{F}}$ au-delà de la naturalité, de la monoïdalité et de la compatibilité avec la comultiplication de la comonade $!_-$. Les ensembles de variables forment un modèle Seely de $\mu\mathbf{LL}$ où la négation linéaire est l’identité sur les objets, les formules $\mu\zeta F$ et $\nu\zeta F$ sont interprétées comme le même ensemble de variables, exactement comme \otimes et \wp sont interprétés de la même manière (et de même pour les additifs et les exponentielles) : la “dégénérescence” dénotationnelle au niveau des types est une caractéristique bien connue de \mathbf{Rel} qui ne signifie pas du tout que le modèle est trivial ; par exemple, les preuves exponentielles multiplicatives normales \mathbf{LL} qui ont des interprétations relationnelles distinctes ont des réseaux de preuves associés distincts [dCdF12, dC16]. Alors on “enrichit” ce modèle \mathbf{Rel} en considérant des ensembles munis d’une structure supplémentaire de *totalité* : un *espace de totalité non uniforme* (NUTS) est un couple $X = (|X|, \mathcal{T}(X))$ où $|X|$ est un ensemble et $\mathcal{T}(X)$ est un ensemble de sous-ensembles qui intuitivement représentent le total, c’est-à-dire terminer les calculs de type X . Cet ensemble $\mathcal{T}(X)$ doit coïncider avec son bidual pour une dualité exprimée en termes d’intersections non vides. Étant donné deux NUTS X et Y , il existe une notion naturelle de *relation totale* $t \subseteq |X| \times |Y|$ donnant lieu à une catégorie \mathbf{Nuts} qui est facilement considérée comme un modèle Seely de logique linéaire. Pour en faire un modèle catégoriel de $\mu\mathbf{LL}$, nous avons besoin d’une notion de foncteurs “strong” $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$. Plutôt que de les considérer directement comme des foncteurs, nous définissons les “non-uniform variable totality spaces” (VNUTS) comme des paires $\mathbb{X} = (|\mathbb{X}|, \mathcal{T}(\mathbb{X}))$ où $|\mathbb{X}| : \mathbf{Rel}^n \rightarrow \mathbf{Rel}$ est un ensemble de variables et, pour chaque tuple $\vec{X} = (X_1, \dots, X_n)$ de VNUTS, $\mathcal{T}(\mathbb{X})(Vect X)$ est une structure de totalité sur l’ensemble $|\mathbb{X}|(|\vec{X}|)$. Il faut aussi que l’action du foncteur $|\mathbb{X}|$ sur les morphismes \mathbf{Nuts} et la “strength” $\widehat{\mathbb{X}}$ respectent cette structure de totalité. Alors il est facile de dériver d’un tel VNUTS \mathbb{X} un foncteur “strong” $\mathbf{Nuts}^n \rightarrow \mathbf{Nuts}$ et on montre que, muni de ces foncteurs “strong”, \mathbf{Nuts} est un modèle de $\mu\mathbf{LL}$. Et enfin, nous appliquerons la même notion de totalité sur des espaces de cohérence, et nous fournirons un autre modèle concret de $\mu\mathbf{LL}$ basé sur des espaces de cohérence avec totalité.

Chapitre 2

Dans le chapitre 2, le long de notre parcours pour établir une relation entre les points fixes, la logique linéaire et l’approche de Curry-Howard-Lambek, nous avons décidé de développer un λ -calcul pour cette extension $\mu\mathbf{LL}$. Cependant, les règles structurelles explicites de la logique linéaire rendent un tel langage très lourd, difficile à utiliser et à analyser. En particulier, des substitutions explicites seraient nécessaires dans une telle syntaxe basée sur $\mu\mathbf{LL}$. However, Olivier Laurent in [Lau02] introduced a “polarized” linear logic LLP which relaxes the use of structural rules on so-called negative formulas. Donc, nous profitons de cet avantage de

la règle structurelle implicite dans LLP, et nous appelons LLP_{imp} pour la logique linéaire polarisée avec des règles structurelles implicites.

Nous étudions $\mu\text{LLP}_{\text{imp}}$, une extension de LLP_{imp} avec plus petit et plus grand points fixes, en introduisant et en étudiant un système L calculus [CH00, CM10, Mun13, Mun09] pour une version polarisée de μLL . La recherche d’une simplicité syntaxique maximale a guidé notre conception de ce calcul. Nous utilisons la lettre grecque κ pour le nom binder au lieu du plus traditionnel $\mu/\tilde{\mu}$ [Par92] qui conduirait à des confusions avec la notation standard associée au plus petit et au plus grand points fixes (μ/ν). Lié à cette simplicité syntaxique, un terme négatif ou une commande peut être typé par un séquent négatif ou positif de sorte qu’il existe en fait cinq types de jugements de typage et que ce partitionnement est pris en compte par la sémantique. La polarisation des points fixes signifie que les plus petits points fixes permettent de définir des types de données (entiers, listes, arbres *etc.*) tandis que les plus grands points fixes permettent de définir des co-types de données, c’est-à-dire des types de consommateurs de données². Nous nous référons à [APTS13] pour une discussion détaillée de la dualité computationnelle entre les données et les co-types de données. Le calcul typé introduit de cette manière s’appelle $\kappa\mu\text{LLP}$. Il a une construction de promotion positive $s^!$ pour mettre un terme négatif s dans une boîte (ou *thunk*) qui peut être utilisée comme une donnée et une déréluction négative $\text{der } p$ qui permet pour ouvrir une telle boîte. Tout comme le $\lambda\mu$ -calcul, $\kappa\mu\text{LLP}$ a des noms $\alpha, \beta \dots$ associés aux formules négatives d’un séquent³. Puisqu’au plus une formule positive peut apparaître dans un séquent LLP unilatéral, nous n’avons besoin que d’une seule variable que nous notons \bullet . Il existe plusieurs “binders” pour les noms : un classeur général $\kappa\alpha.c$ qui permet de sélectionner une formule négative dans le contexte, et les autres sont associés à \perp , \mathfrak{A} et la règle de Park. Tous ces liants produisent un terme négatif dont le type est une formule négative rendue active pour des utilisations ultérieures. Il existe également un “binder” $\tilde{\kappa}.c$ associé à la variable unique \bullet qui produit un terme positif. Une caractéristique cruciale de \bullet est qu’il ne peut apparaître que *linéairement* dans une commande ou un terme négatif. Encore une fois, cela est dû au fait que toutes les formules dans le cadre d’une promotion doivent être négatives. Notez également que \bullet ne peut pas apparaître libre dans un terme positif en raison du fait qu’un séquentiel positif a exactement une formule positive. Tous ces liants s’appliquent aux *commandes* qui sont des coupures $s \star p$ entre un terme négatif s et un terme positif p . Notre sémantique opérationnelle ne fournit que des relations de réduction pour les commandes et peut être considérée comme décrivant les interactions entre les constructeurs positifs et les destructeurs négatifs. Une commande critique spécifique est $\kappa\alpha.c \star \tilde{\kappa}.d$ qui pourrait *a priori* conduire à $c[\tilde{\kappa}.d/\alpha]$ ou $d[\text{Kappa}\alpha c/\bullet]$; nous choisissons la deuxième option rendant notre sémantique de réduction déterministe ; nous définissons en fait une sorte de machine abstraite dont les états sont des commandes $s \star p$ où s est le programme et p est la pile (il n’y a pas d’environnements car les substitutions sont exécutées immédiatement).

Notre objectif dans ce chapitre est double. D’une part, nous proposons une catégorie sémantique du $\kappa\mu\text{LLP}$ sur celui fourni au Chapitre 1. Étant donné un modèle $(\mathcal{L}, \vec{\mathcal{L}})$ (noté simplement \mathcal{L}) de μLL , l’idée principale est standard : interpréter une formule fermée positive P comme un objet de la catégorie d’Eilenberg-Moore $\mathcal{L}^!$. Cela nécessite cependant de traiter

²This suggère fortement que les listes et les flux ne sont pas de même nature, les flux ne sont pas des données mais des consommateurs de données.

³En effet dans le $\lambda\mu$ -calcul tout les formules sont négatives et les noms sont associés aux formules apparaissant sur le côté droit d’un séquent tandis que les variables sont associées aux formules sur la gauche.

également les contacts positifs ouverts. formules : on les prend en compte en introduisant la notion de *foncteurs positifs* qui sont des foncteurs “strong” (comme spécifié par \mathcal{L}) munis d’une loi distributive par rapport à la comonade $!$, ils sont une généralisation fonctorielle de la notion de $!$ -coalgèbre. Nous illustrons cette sémantique dans les modèles concrets **Rel** et **Nuts**.

D’autre part, nous proposons également une forme de normalisation (élimination des coupures) pour $\kappa\mu\text{LLP}$. Cela s’est avéré étonnamment difficile. La solution est venue de comprendre que prouver une propriété de terminaison par rapport à la sémantique relationnelle — en disant très grossièrement que si une commande a une sémantique relationnelle non vide sa réduction se termine — serait possible car les points du modèle relationnel sont des arbres *finis* sur lesquels l’induction est possible. Concrètement cela signifie que l’on associe des ensembles de termes à des points des modèles relationnels et ces ensembles sont facilement définissables par induction. Pour rendre plus naturelle cette preuve de normalisation nous donnons également une présentation de la sémantique relationnelle de $\kappa\mu\text{LLP}$ comme système de typage d’intersection. Pour dériver de cette normalisation relationnelle une normalisation standard propriété, il suffit de prouver que, dans suffisamment de situations significatives, lorsque $c \vdash \alpha : N$, la commande c a une sémantique relationnelle non vide. Nous faisons cela pour $N = ?\text{nat}$ où nat est un type d’entiers défini comme une formule de moindre point fixe. Le modèle **Nuts** des espaces *totalité* non uniformes nous donne précisément cette information : tout sous-ensemble total de l’interprétation relationnelle de $?\text{nat}$ est non vide et donc, en particulier, l’interprétation de c est non vide.

Chapitre 3

Dans le chapitre 3, nous étudierons la logique linéaire non bien fondée [Dou17, BDS16] d’un point de vue sémantique dénotationnel. Dans [Dou17, BDS16], le système μMALL_∞ , multiplicatif et additif non bien fondée LL avec deux règles de dépliage des points fixes, est étudié. Ils ont défini une notion syntaxique de validité sur les preuves afin de distinguer les preuves solides des preuves “unsound”. On considère une extension de μMALL_∞ , appelée μLL_∞ , avec des exponentielles. Nous récapitulons d’abord le langage et les règles d’inférence de μLL_∞ . Ensuite, nous fournirons une interprétation pour les preuves μMALL_∞ dans **Rel**. Cependant, les arbres de preuve sont peut-être non bien fondée. Ainsi, nous considérons, comme c’est standard, toutes les approximations finies d’une preuve, puis prenons le supremum de leur interprétation. Pour définir la définition précise de cette approximation, considérons une extension de μLL_∞ avec cette règle : $\overline{\vdash \Gamma}^{(\Omega)}$ pour toute séquence Γ . Nous interprétons cette règle comme l’ensemble vide, et cela nous aide à considérer tout sous-arbre fini d’un μLL_∞ donné comme une preuve dans cette extension de μLL_∞ . Comme la procédure d’élimination des coupures dans μLL_∞ est un processus infini, nous considérerons une notion de limite en définissant une métrique sur l’ensemble de toutes les preuves finies de μLL_∞ . A partir de cette notion de métrique, nous définissons une relation d’équivalence sur l’ensemble de toutes les suites de Cauchy. Nous allons montrer que la complétion métrique de la collection de toutes les séquences de Cauchy est isomorphe à l’ensemble de tous (potentiellement infini) μLL_∞ preuve (Bien que ce qui est fourni jusqu’à présent soit standard dans la littérature, nous les fournissons dans ce chapitre dans un souci d’autonomie du manuscrit). Nous avons ensuite prouvé la procédure de préservation par élimination de coupures, qui peut être éventuellement

un chemin de réduction infini, en utilisant un résultat montrant que l'interprétation de toute suite de Cauchy est obtenue par limsup de l'interprétation de chaque preuve dans la suite. Ensuite, nous relierons la condition de validité et la totalité de **Nuts** en prouvant que chaque preuve valide μLL_∞ sera interprétée comme un élément total dans **Nuts**. Notre méthode de preuve est similaire à la preuve de cohérence de LKID^ω dans [Bro06]. Cependant le système de [Bro06] est une logique classique avec des définitions inductives, et cette preuve est pour une sémantique "Tarskian". Nous devons donc adapter cette preuve en deux aspects : en considérant μLL_∞ au lieu de LKID^ω , et en essayant de traiter une sémantique dénotationnelle au lieu d'une sémantique "Tarskian". L'adaptation pour μLL_∞ se fait en quelque sorte dans [Dou17], puisqu'il existe un théorème de cohérence pour μMALL_∞ par rapport à la sémantique de vérité tronquée (une sémantique "Tarskian"). Donc, fondamentalement, le point principal de notre preuve est de transformer un théorème de cohérence de "Tarskian" en un théorème de correction dénotationnel. Enfin, nous terminons ce chapitre en prouvant que la sémantique des preuves est préservée via l'opération de traduction qui envoie toute preuve μLL vers une preuve μLL_∞ .

Chapitre 4

Enfin, dans le dernier chapitre, nous revenons à la question qui s'est posée dans le chapitre 0 concernant la prise en compte des exponentielles dans μLL . Dans [Bae12], il n'a pas considéré les exponentielles dans son système de μMALL , et l'une des raisons est que l'utilisation d'un encodage $!A = 1 \& A \& (!A \otimes !A)$ de $!A$, on peut piloter toutes les règles structurelles des exponentielles dans μMALL . Cependant, cela n'est pas satisfaisant d'un point de vue dénotationnel. Bien que cette exponentielle codée $!^t$ soit fonctorielle et ait également une structure de comonade, elle ne remplit pas toutes les exigences catégorielles telles que stipulées dans la définition d'une catégorie Seely, comme cela est également mentionné dans [Laf88b, Laf88a]. Plus précisément, on n'aura pas l'iso de Seely avec cette exponentielle $!$ (la proposition 6 de [Laf88b] n'est pas iso mais équivalence). En d'autres termes, la catégorie de Kleisli associée $\mathcal{L}_!$ n'est pas un CCC. Dans ce chapitre, nous étudierons une construction catégorielle générale pour cette exponentielle codée. Sur la base de la notion de comagma dans une catégorie \mathcal{L} avec un foncteur binaire, nous transformons la classe des comagmas sur un objet donné en une catégorie. Nous définissons ensuite la notion de *quasi-exponentielle magmatique libre* (FMQE) comme une opération qui associe, à chaque objet X de \mathcal{L} , un objet terminal de la catégorie comagmas sur X . Nous montrerons ensuite la fonctorialité, la comonadicité et la monoïdalité d'une FMQE. Et nous terminerons la construction catégorielle de $!^t$ en fournissant des morphismes de Seely.

Nous considérerons également certains modèles concrets comme des instances de cette structure catégorielle. En interprétant ce $!^t$ encodé comme un arbre binaire, nous montrerons que **Rel** et les espaces de cohérence sont bien deux cas concrets. Nous terminons ce chapitre en prouvant que l'interprétation d'une preuve logique linéaire dans des espaces de cohérence avec cette exponentielle arborescente est liée à son interprétation dans des espaces de cohérence non uniformes avec l'exponentielle de Boudes. Pour ce faire, nous utilisons la méthode des relations logiques, et ainsi, nous définissons une autre instance de notre cadre catégoriel fourni, et nous l'appelons *espaces de totalité locaux*. Un espace de totalité local (LTS) est un tuple généré par un espace de cohérence, un espace de cohérence non unifrom, une opération

envoyant une exponentielle arborescente à une exponentielle multi-ensemble notée ρ , et un gadget de totalité locale pour définir correctement la composition. Et nous prouverons que si nous appliquons l'opération ρ sur l'interprétation des preuves de logique linéaire dans des espaces de cohérence avec l'exponentielle arborescente, nous obtiendrons son interprétation dans des espaces de cohérence non uniformes.

Nous terminons ce manuscrit par un Chapitre 5 conclusif, dans lequel nous exposons les perspectives de notre travail.

$\mathcal{L} ? \otimes \wp \oplus \& \mu \nu !$

On peut résumer ce manuscrit en disant que il s'agit d'étudier le lien entre la théorie des points fixes, la logique linéaire et la correspondance de Curry-Howard-Lambek.

List of figures

1	Inference rules of LL	28
2	Inference rules of μ LL	34
3	Interpretation of LL rules	41
1.1	Monoidality and dig diagrams for strong functors	50
2.1	LLP inference rules	79
2.2	LLP _{imp} inference rules	79
2.3	Syntax of terms and typing rules of $\kappa\mu$ LLP	84
2.4	Reduction of commands	93
2.5	Syntax, size, typing rules and structural relation for points	103
2.6	Point deduction system — the rule $(\mathbf{i}\nu)$ is given in the body of Section 2.2.3.	105
2.7	Compatibility of $\tilde{\mathbb{P}}$ with \mathbf{der} , \mathbf{dig} , and $\hat{\mathbb{P}}$	117
2.8	Proof of Lemma 90	122
2.9	Interpretation of terms of $\kappa\mu$ LLP	124
2.10	Preservation of $\bar{\kappa}_{R^\perp, \zeta}\alpha.(c; s) \star \mathbf{fd}(p)$ reduction by semantic	129

Takeaway of this manuscript

Takeaway

- Chapter 1:
 - Axiomatize a notion of categorical model of μLL , and proof its soundness: Definition 17 and Theorem 19.
 - Provide some concrete models of μLL :
 - * **Rel**: Theorem 29, **Nuts**: Theorem 45, **Coh**: Theorem 54, and **CohT**: Theorem 62.
 - One can have similar constructions as what has been done in this chapter for the other concrete models such as Scott model (which based on the category of preorders and down-closed relations), hypercoherences, and probably of finiteness spaces. So, a general categorical setting taking into account all these situations would be quite useful (see Section 5.1 for some discussion about this).
- Chapter 2
 - Developing a polarized λ -calculus for LL with fixpoints: $\kappa\mu\text{LLP}$.
 - Prove interpretation lemma and a normalization theorem for $\kappa\mu\text{LLP}$: Theorem 84, 104, and 106.
 - Providing a categorical model for $\kappa\mu\text{LLP}$ and prove its soundness: Theorem 99.
 - Providing two concrete models of $\kappa\mu\text{LLP}$: **Rel** and **Nuts**.
- Chapter 3
 - Provide two denotational models for μLL_∞ in **Rel** and **Nuts**, and prove its soundness: Theorem 128.
 - Relate the syntactic validity conditions on μLL_∞ pre-proofs and totality notion of **Nuts**: Theorem 134.
 - The semantic is preserved via the translation of the μLL proofs into μLL_∞ proofs: Theorem 135.
- Chapter 4:
 - Provide a categorical setting for Tree exponentials: Section 4.1
 - Examine the mentioned categorical setting on two well-known models of LL, i.e, **Rel** and **Coh**: Propositions 140 and 141 for **Rel**. Propositions 143 and 144 for **Coh**.
 - Provide a new concrete instances of the mentioned categorical setting, i.e, Local Totality Spaces. Theorem 156, Propositions 163 and 165.
 - Relate interpretations of LL proofs in coherence spaces with the tree exponential and non-uniform coherence spaces: Theorem 166.