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Vukašin Stojisavljević

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T E L A V I V U N I V E R S I T Y

The Raymond and Beverley Sackler Faculty of Exact Sciences

School of Mathematical Science

Persistence Modules in Geometry and Dynamics

Thesis submitted for the degree “Doctor of Philosophy”

by

Vukašin Stojisavljević

Prepared under the supervision of

Prof. Leonid Polterovich

and

Prof. Yaron Ostrover

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Chapter 1

Introduction

1.1 Summary of the thesis

The subject of this thesis are applications of persistence modules and barcodes to geometry and dynamics. More precisely, we present applications to symplectic topology following [122,144] and to topological function theory¹ and spectral geometry following [113].

The theory of persistence modules and barcodes emerged inside topological data analysis community in the early 2000s, see [50, 167], with certain precursors, most notably in the work of Barannikov [16]. Since then it witnessed a rapid expansion and development both inside and outside topological data analysis, see [49, 70, 30, 159, 19, 20] for surveys, [48, 106, 35] for detailed treatments of the theory and [120] for an exposition focused on applications to pure mathematics. Recently, the technique of persistence modules and barcodes has been successfully used in symplectic and contact topology. For instance, [121], [150], [166], [122], [143], [81] and [139] used persistence modules constructed from Floer homology to study questions in Hamiltonian dynamics, while [26], [89], [137], [80] and [138] applied persistence techniques in the framework of C^0 -symplectic topology. On the other hand, in [63], persistence modules defined using generating function homology were considered, while [47] used barcodes to deduce displacement energy bounds for Legendrian submanifolds. Persistence modules coming from filtered symplectic homology were considered in [144] and [148].

There exist different definitions of a persistence module in the literature, depending on the level of generality. For us, a persistence module over a field \mathbb{K} will be given by a family of finite-dimensional \mathbb{K} -vector spaces V^t , $\dim_{\mathbb{K}} V^t < \infty$, indexed by a real parameter $t \in \mathbb{R}$, together with a family of linear structure maps $\pi_{s,t} : V^s \rightarrow V^t$ for

¹The term “topological function” theory was taken from [120]. It refers to the study of properties of functions which are invariant under the action of diffeomorphisms or homeomorphisms of the underlying space, such as the C^0 -norm.

all $s \leq t$. Moreover we ask that

$$\pi_{t,t} = \mathbb{1}_{V^t} \text{ and } \pi_{s,t} \circ \pi_{r,s} = \pi_{r,t} \text{ for all } r \leq s \leq t.$$

A typical example of a persistence module is given using a Morse function f on a closed manifold M . We define $V_k^t(f) = H_k(\{f < t\}; \mathbb{K})$, structure maps $\pi_{s,t}$ being induced by inclusions $\{f < s\} \subset \{f < t\}$. Informally, this persistence module encodes how topology of the sublevel set $\{f < t\}$ changes as we vary parameter t . Other examples of interest for us will be given by filtered Floer homology, filtered symplectic homology and filtered Morse-Bott homology. In these examples, M is replaced by the free loop space of a symplectic or Riemannian manifold, while the role of f is played by symplectic action functional or Riemannian energy functional, see Chapters 2, 3 and 4 for details.

A barcode is a collection of intervals in \mathbb{R} , called bars, each bar appearing finitely many times, possibly more than once. As such, barcode is a combinatorial object. However, the structure theorem for persistence modules, Theorem 2.1.8, tells us that to each isomorphism class of persistence modules corresponds a unique barcode. In order to get the intuition about this correspondence, it is instructive to consider the above mentioned example of a persistence module associated to a Morse function. Each bar in the corresponding barcode represents a lifespan of a homology class of the sublevel set as level parameter increases. A left endpoint of a bar represents a “birth” of a homology class, i.e., the value of a function at which the homology class first appears in the sublevel set. Similarly, a right endpoint of a bar represents a “death” of a homology class, i.e., the value of a function at which the homology class disappears². Since topology of the sublevel set changes only upon passing through a critical value, all endpoints of all bars are critical values of the function.

One may define a distance on the space of all barcodes, called the bottleneck distance and denoted by d_{bottle} . Informally, d_{bottle} measures the minimal value ε we need in order to bijectively match the bars in the two barcodes with error at most ε at the endpoints. One of the major results of the theory of persistence modules and barcodes is the stability theorem of Cohen-Steiner, Edelsbrunner and Harer proven in [38]. It states that for two Morse functions f and g on a closed manifold M and any integer k it holds

$$d_{bottle}(\mathcal{B}_k(f), \mathcal{B}_k(g)) \leq d_{C^0}(f, g), \tag{1.1}$$

where $\mathcal{B}_k(\cdot)$ denotes the barcode of the persistence module $V_k(\cdot)$ and $d_{C^0}(f, g) = \|f - g\|_{C^0} = \max_{x \in M} |f(x) - g(x)|$. One may interpret (1.1) as saying that barcodes behave in a stable way with respect to C^0 -perturbations of functions. Moreover, the stability theorem tells us that barcode can be used as an invariant to quantitatively study the space of functions equipped with the C^0 -distance.

The stability theorem has a far-reaching, abstract generalization called the isometry theorem. Namely, one may define, in purely algebraic terms, a distance between

²This value might be equal to $+\infty$ if the class never disappears, i.e., if it is a homology class of the underlying manifold.

two persistence modules, called the interleaving distance and denoted by d_{inter} . The isometry theorem states that for any two persistence modules V and W with corresponding barcodes $\mathcal{B}(V)$ and $\mathcal{B}(W)$ it holds

$$d_{bottle}(\mathcal{B}(V), \mathcal{B}(W)) = d_{inter}(V, W). \quad (1.2)$$

In order to deduce (1.1) from (1.2) it is enough to show that $d_{inter}(V_k(f), V_k(g)) \leq d_{C^0}(f, g)$. This can be done in a straightforward manner using functorial properties of homology, see Section 2.1 for details. Due to the abstract nature of the isometry theorem, the same idea may be applied in different contexts and stability results similar to (1.1) can be proven systematically. Indeed, an important ingredient in [121] and in our paper [122] (which builds on [121]) is a stability result similar to (1.1), with d_{C^0} replaced by Hofer's metric and with barcodes coming from filtered Floer homology. On the other hand, in [144] we use a stability result analogous to (1.1) with d_{C^0} replaced by the recently defined symplectic Banach-Mazur distance and with barcodes coming from filtered symplectic homology. These results allow us to quantitatively study Hofer's metric and symplectic Banach-Mazur distance using barcodes. Let us now briefly describe the content of [122, 144, 113].

Morse and Floer homology carry product structures given by the intersection product and the pair-of-pants product respectively. In [122] we made a first step towards integrating these products in the framework of persistence modules. To this end, we introduced the notion of a persistence module with an operator and extended certain elements of persistence theory to this framework. As an application, we generalized the main result of Polterovich and Shelukhin from [121] about Hofer's distance to full p -th powers of Hamiltonian diffeomorphisms. As another illustration of our technique we provided an example of two Morse functions which can not be distinguished using standard persistence modules, but can be distinguished using persistence modules with operators. Along the way we also proved a version of the Künneth formula for persistence modules.

In [144] we studied the recently defined symplectic Banach-Mazur distance using persistence techniques. In general, symplectic Banach-Mazur distance, denoted by d_{SBM} , is a distance on the space of Liouville domains. Roughly speaking, it measures how much we need to rescale the domains so that we can symplectically embed them into one another. In [144] we restricted ourselves to fiberwise star-shaped domains in the cotangent bundle of a fixed closed manifold M . Denote the space of all such domains by \mathcal{C}_M , (\mathcal{C}_M, d_{SBM}) being a pseudo-metric space. Our main result states that if M is a closed, orientable surface of positive genus, then (\mathcal{C}_M, d_{SBM}) is large in the sense that there exists a quasi-isometric embedding from $(\mathbb{R}^n, |\cdot|_\infty)$ to (\mathcal{C}_M, d_{SBM}) for all $n \in \mathbb{N}$. The proof of this statement relies on a stability result analogous to (1.1) with d_{SBM} playing the role of d_{C^0} and barcodes coming from filtered symplectic homology. In order to construct the quasi-isometric embeddings, we analyse closed geodesics on certain surfaces of revolution, which gives us information about the relevant barcodes. The domains from \mathcal{C}_M realizing the embedding are then given

as unit codisc bundles of these metrics of revolution. As another application, using similar methods, we obtained a result about stability of closed geodesics on a general closed Riemannian manifold with respect to perturbations of the Riemannian metric.

The main object we studied in [113] is a functional, Φ_1 , on the space of Morse functions on a fixed manifold M , given by

$$\Phi_1(f) = \max f - \min f + \text{total length of all finite bars in } \mathcal{B}(f).$$

This functional can be thought of as a measure of total oscillation of a function. We proved that when M is a compact, orientable surface with a Riemannian metric, $\Phi_1(f)$ admits an upper bound in terms of the L^2 -norms of derivatives of f up to order two. The proof relies on a similar estimate for the integral of the Banach indicatrix due to Polterovich and Sodin - [123] and some elementary topological considerations. We should note that this result of Polterovich and Sodin is a part of a classical subject of estimating the integral of the Banach indicatrix, see [87, 153, 162]. As a direct corollary of the bound on Φ_1 , we obtained an upper bound on the number of “significant” critical values of a Morse function. Other corollaries include an inverse result about C^0 -approximations of a function, possibly after a change of variables, by a linear combination of Laplace-Beltrami eigenfunctions, as well as an upper bound on the average length of a finite bar in $\mathcal{B}(f)$, f being a Morse function on a 2-dimensional torus.

Organization of the thesis: In the rest of the chapter we present the main results of the thesis in greater detail. Precisely, the results of [122], [144] and [113] are described in Sections 1.2, 1.3, and 1.4 respectively. Chapter 2 is divided in two sections. The first one contains necessary background about persistence modules and barcodes, while the second one gives a brief overview of symplectic topology and Floer theory. Chapters that follow, namely Chapters 3, 4 and 5, are based on [122], [144] and [113] respectively. They provide a detailed treatment of the results announced in Sections 1.2, 1.3 and 1.4, as well as some additional results.

1.2 Hofer’s geometry and distance to powers

Let (M, ω) be a closed symplectic manifold, $\text{Ham}(M)$ the group of Hamiltonian diffeomorphisms of M and d the Hofer’s metric on $\text{Ham}(M)$. For an integer $p \geq 2$, denote by

$$\text{Powers}_p(M) = \{\phi \in \text{Ham}(M) \mid \exists \psi \in \text{Ham}(M), \phi = \psi^p\},$$

the set of all p -th powers of Hamiltonian diffeomorphisms and let

$$\text{powers}_p(M) = \sup_{\phi \in \text{Ham}(M)} d(\phi, \text{Powers}_p(M)).$$

The following question was considered by Polterovich and Shelukhin in [121].

Question: Is it true that $\text{powers}_p(M) = +\infty$?

In other words, we are asking if there exist Hamiltonian diffeomorphisms arbitrary far away in Hofer's metric from the set of p -th powers. There are many different motivations to ask this question. For example, the set of autonomous Hamiltonian diffeomorphisms is contained in $\text{Powers}_p(M)$ and thus a positive answer to the above question would imply that there exist Hamiltonian diffeomorphisms arbitrarily far away from the autonomous ones. Even for $M = S^2$, the question of existence of such Hamiltonian diffeomorphisms is a well-known open problem posed in 2006 by Kapovich and Polterovich, see [97, Chapter 14, Problems 21 and 22]. For other motivations and more context surrounding this question we refer the reader to [121]. The main result of [121] is the following.

Theorem 1.2.1 ([121]). *Let (Σ, σ) be a closed oriented surface of genus at least 4, endowed with an area form, and let (N, ω_N) be either a point, or a closed symplectically aspherical symplectic manifold. Then for each $p \in \mathbb{Z}, p \geq 2$ there exists a sequence $\phi_j \in \text{Ham}(\Sigma, \sigma)$ such that*

$$d(\phi_j \times \mathbb{1}_N, \text{Powers}_p(\Sigma \times N)) \xrightarrow{j \rightarrow \infty} \infty.$$

In particular $\text{powers}_p(\Sigma \times N) = +\infty$.

In order to prove Theorem 1.2.1, Polterovich and Shelukhin described a general framework for tackling the above question. Namely, on a symplectically aspherical manifold, they considered filtered Hamiltonian Floer homology as a persistence module, called *Floer persistence module*, and denoted by $HF_*^t(H)_\alpha$. More precisely, for a homotopy class of free loops α and a non-degenerate Hamiltonian H , $HF_*^t(H)_\alpha$ is the homology of a Floer chain complex of H , generated by closed orbits in class α of index $*$ with action less than t . As explained in [121], when (M, ω) is symplectically aspherical, $HF_*^t(H)_\alpha$ is a persistence module which depends only on the time-one diffeomorphism generated by the Hamiltonian flow of H . In other words, Floer persistence module $HF_*^t(\phi)_\alpha$ is well defined for a non-degenerate $\phi \in \text{Ham}(M)$.

The first thing which was noticed in [121] is that the stability of barcodes holds for Floer persistence modules, i.e.

$$d(\mathcal{B}(HF_k(\phi)_\alpha), \mathcal{B}(HF_k(\theta)_\alpha)) \leq d(\phi, \theta),$$

for every degree k , every homotopy class α and all non-degenerate $\phi, \theta \in \text{Ham}(M)$. This inequality follows from the isometry theorem for persistence modules combined with standard action estimates for continuation maps in Floer theory. We give its proof in Subsection 2.2.8.

The second thing noticed in [121] is that Floer persistence module $HF_*(\phi^p)_\alpha$ can be endowed with a \mathbb{Z}_p -action by acting with ϕ on all the elements in the construction of the Floer chain complex. Moreover, if $\phi \in \text{Powers}_p(M)$, this action can be upgraded

to a \mathbb{Z}_{p^2} -action. Using these algebraic structures, Polterovich and Shelukhin defined an invariant, called *multiplicity-sensitive spread*, which can be used to distinguish a Hamiltonian diffeomorphism from $\text{Powers}_p(M)$ in a Hofer-robust way.

Finally, to prove Theorem 1.2.1, a sequence of Hamiltonian diffeomorphisms ϕ_j was constructed as time-one maps of the family of *egg-beater flows*. The properties of the egg-beater flows were analysed directly, which allowed for the estimate of the multiplicity-sensitive spread.

Our work [122] can be seen, to a large extent, as a continuation of [121]. In [122], we worked with symplectically monotone manifolds and aimed to include the product structure, given by the pair-of-pants or quantum product, in the framework of Floer persistence modules. To this end, we introduced the notion of a *persistence module with an operator*. This notion encompasses the previously considered \mathbb{Z}_p -action as well as intersection, pair-of-pants or quantum product with a fixed class in the ambient homology. Algebraic considerations of these structures yield a refinement of Theorem 1.2.1, which is the main result of [122]. Let us formulate this result precisely.

Let $p \geq 2$ be an integer, (N, ω) , $\dim N = 2n$, a monotone symplectic manifold, i.e. $\omega|_{\pi_2(N)} = \kappa c_1|_{\pi_2(N)}$ with $\kappa > 0$, and denote by c_N the minimal Chern number³ of N . Let \mathbb{K} be a field whose characteristic does not divide p and which contains all p -th roots of unity, i.e. $x^p - 1 \in \mathbb{K}[x]$ splits over \mathbb{K} . Moreover, assume that for a primitive p -th root of unity ζ_p and an integer k , the equation $x^p - (\zeta_p)^k = 0$ has no solutions in \mathbb{K} unless $p|k$. An example of such a field is the splitting field \mathbb{Q}_p of $x^p - 1$ over \mathbb{Q} . Let

$$\Lambda_{\mathbb{K}} = \left\{ \sum_{i \in \mathbb{Z}} a_i q^i \mid a_i \in \mathbb{K}, (\exists i_0 \in \mathbb{N}) a_i = 0 \text{ for } i \geq i_0 \right\},$$

be the field of bounded from above Laurent series in a formal variable q with coefficients in \mathbb{K} . Denote by

$$QH(N) = H(N; \mathbb{K}) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}},$$

the quantum homology of N , $\Lambda_{\mathbb{K}}$ serving as a Novikov field. $QH(N)$ is given a \mathbb{Z} -grading by declaring that $\deg q = 2c_N$. Fixing a homogenous element $e \in QH(M)$, quantum product yields a \mathbb{K} -linear map

$$e* : QH_r(N) \rightarrow QH_{r-2n+\deg e}(N) \tag{1.3}$$

for every $r \in \mathbb{Z}$. We define the r -th Betti number associated to e as $b_r(e) = \dim_{\mathbb{K}}(\text{im}(e*))$, where $e*$ has domain $QH_r(N)$ as in 1.3. These Betti numbers satisfy $b_r(e) = b_{r+2c_N}(e)$ for every $r \in \mathbb{Z}$ because multiplication by q induces an isomorphism

$$q : QH_r(N) \rightarrow QH_{r+2c_N}(N),$$

which commutes with $e*$. Thus, there are at most $2c_N$ different ones, namely $b_0(e), \dots, b_{2c_N-1}(e)$. The following theorem is the main result of [122].

³Minimal Chern number is defined as $c_N > 0$ s.t. $c_1(\pi_2(N)) = c_N \cdot \mathbb{Z}$.

Theorem 1.2.2 (Theorem 3.1.2). *Let (Σ, σ) be a closed oriented surface of genus at least 4, endowed with an area form and (N, ω_N) a closed symplectically monotone manifold. If there exists a homogenous $e \in QH(N)$ such that $p \nmid b_r(e)$ for some $r \in \{0, \dots, 2c_N - 1\}$ then*

$$\text{powers}_p(\Sigma \times N) = +\infty.$$

A symplectically aspherical manifold can be considered to be symplectically monotone with $c_N = +\infty$. Theorem 1.2.2 also applies to symplectically aspherical manifolds, in which case quantum homology is replaced by the usual homology and quantum product is replaced by the intersection product. Since the fundamental class $[N]$ is the unit for the intersection product, it follows that $b_k([N]) = \dim_{\mathbb{K}}(H_k(N; \mathbb{K}))$ for all degrees k . Taking $k = 0$ we get $b_0([N]) = 1$ and hence Theorem 1.2.2 implies Theorem 1.2.1. Other concrete examples to which Theorem 1.2.2 applies are discussed in Subsection 3.1.2.

Remark 1.2.3. Another generalization of Theorem 1.2.1 was obtained in [166], see Subsection 3.1.2 for the formulation of the result and comparison with Theorem 1.2.2.

Theorem 1.2.2 is the main application of the framework of persistence modules with operators introduced in [122]. In order to prove it, we analysed persistence modules coming from tensor products of pairs of filtered Floer chain complexes. To this end, we also proved a Künneth type formula for persistence modules, see Subsection 3.2.2. To further illustrate the utility of our method, we provided an example of two Morse functions on a surface of genus two, whose associated barcodes and spectral invariants are equal, but which can be distinguished using operators coming from the intersection product. This example is discussed in Subsection 3.2.4.

1.3 Symplectic Banach-Mazur distance

Let M be a closed, orientable manifold, T^*M its cotangent bundle, λ_{can} the canonical Liouville 1-form on T^*M and $\omega_{can} = d\lambda_{can}$ the canonical symplectic form on T^*M . The Liouville vector field X is given by $\omega_{can}(X, \cdot) = \lambda_{can}$ and is expressed as $X = \sum_{i=1}^{\dim M} p_i \frac{\partial}{\partial p_i}$ in Darboux coordinates (q, p) , $p = (\frac{\partial}{\partial q})^*$. We call $U \subset T^*M$ an *admissible domain* if it is a smooth, fiberwise star-shaped⁴, domain with boundary such that $X \lrcorner \partial U$. This condition renders $(\partial U, \lambda_{can}|_{\partial U})$ into a contact manifold and is necessary in order for our method of filtered symplectic homology to work, see Section 4.3 for more details. Admissible domains of particular importance for us are unit codisc bundles $U_g^*M \subset T^*M$ of Riemannian metrics g on M .

Denote by \mathcal{C}_M the set of all admissible domains in T^*M . We wish to define a distance on \mathcal{C}_M and study its large-scale geometry. To this end, we must first

⁴By star-shaped we mean star-shaped with center at 0. In particular $0_M \subset U$.

introduce a couple of notions. Let $U, V \in \mathcal{C}_M$. A $\tilde{\pi}_1$ -trivial Liouville embedding $\phi : U \rightarrow V$ is an exact symplectic embedding, i.e. an embedding such that $\phi^*\lambda_{can} - \lambda_{can}$ is exact, which acts as identity on free homotopy classes of loops in T^*M . Assume that $U \subset V$ and $\phi : U \rightarrow V$ is a $\tilde{\pi}_1$ -trivial Liouville embedding. We call ϕ *strongly unknotted* if there exists a homotopy between ϕ and the inclusion $i : U \rightarrow V$ which goes through exact symplectic embeddings of U to V . It can be easily shown that such a homotopy can be generated by a Hamiltonian flow, i.e. $\phi(U)$ is a Hamiltonian deformation of U . We denote by $U \xrightarrow{\phi} V$ the $\tilde{\pi}_1$ -trivial Liouville embedding $\phi : U \rightarrow V$. For $U \xrightarrow{\phi} V$ and $C > 0$ define $\phi(C) : CU \rightarrow CV$ as $\phi(C)(\cdot) = C\phi(\frac{1}{C}\cdot)$ where multiplication is fiberwise $C(q, p) = (q, Cp)$. It is easy to check that $\phi(C)$ is also a $\tilde{\pi}_1$ -trivial Liouville embedding.

Definition 1.3.1. Let $U, V \in \mathcal{C}_M$. Define *symplectic Banach-Mazur distance* between U and V as

$$d_{SBM}(U, V) = \inf \left\{ \ln C \mid \begin{array}{l} \exists \frac{1}{C}U \xrightarrow{\phi} V \xrightarrow{\psi} CU \text{ (and hence } \frac{1}{C}V \xrightarrow{\psi(C^{-1})} U \xrightarrow{\phi(C)} CV) \\ \text{s.t. } \psi \circ \phi \text{ and } \phi(C) \circ \psi(C^{-1}) \text{ are strongly unknotted} \end{array} \right\}$$

As we already mentioned, we are interested in the large-scale geometry of (\mathcal{C}_M, d_{SBM}) . Since symplectic maps preserve volume, we immediately see that the diameter of (\mathcal{C}_M, d_{SBM}) is infinite if there are no restrictions on the volume of domains. Hence, we denote

$$\bar{\mathcal{C}}_M = \left\{ U \in \mathcal{C}_M \mid \text{Vol}(U) = \int_U \frac{(\omega_{can})^{\wedge n}}{n!} = 1 \right\},$$

and focus on the large-scale geometry of $(\bar{\mathcal{C}}_M, d_{SBM})$. Recall that a map $\Phi : (X_1, d_1) \rightarrow (X_2, d_2)$ between pseudo-metric spaces is called a *quasi-isometric embedding* if there exist $A \geq 1, B \geq 0$ such that

$$\frac{1}{A}d_1(x, y) - B \leq d_2(\Phi(x), \Phi(y)) \leq Ad_1(x, y) + B,$$

for all $x, y \in X_1$. The main result of [144] is the following.

Theorem 1.3.2 (Corollary 4.1.14). *If $M = \Sigma$ is a closed, orientable surface whose genus is at least 1 then for every $N \in \mathbb{N}$ there exists a quasi-isometric embedding*

$$\Phi : (\mathbb{R}^N, |\cdot|_\infty) \rightarrow (\bar{\mathcal{C}}_M, d_{SBM}).$$

If $M = S^2$, then there exists a quasi-isometric embedding

$$\Phi : ([0, \infty), |\cdot|) \rightarrow (\bar{\mathcal{C}}_M, d_{SBM}).$$

Intuitively, Theorem 1.3.2 states that for any $N \in \mathbb{N}$ there exist N linearly independent unbounded directions inside $(\bar{\mathcal{C}}_\Sigma, d_{SBM})$ and there exists an unbounded ray inside $(\bar{\mathcal{C}}_{S^2}, d_{SBM})$. The proof of this theorem can, roughly speaking, be divided

in two parts. The first part consists of proving stability of barcodes coming from filtered symplectic homology with respect to d_{SBM} . This is a general result which holds for any base manifold M . We should note that the definition of d_{SBM} as well as this stability result were first suggested by Ostrover and Polterovich - [117, 118, 105]. The importance of unknottedness-condition was noticed by Gutt and Usher in [75]. Finally, precise statements and proofs of stability appeared, in various contexts, around the same time in [120, 144, 148].

The second part of the proof consists of constructing domains which realize the quasi-isometric embeddings in Theorem 1.3.2. For $\vec{x} \in \mathbb{R}^N$ we construct a domain $\Phi(\vec{x})$ as the unit codisc bundle of a Riemannian metrics $g_{\vec{x}}$, which roughly looks as in Figure 1.1. On the cut-out cylindrical part the metrics are metrics of revolution

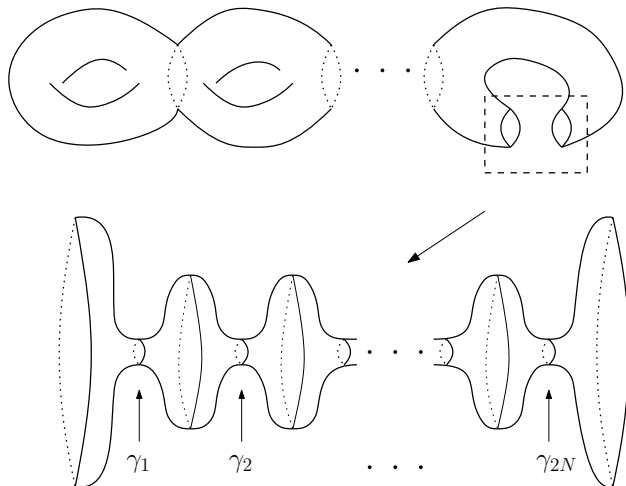


Figure 1.1. Riemannian metric $g_{\vec{x}}$

and different directions in $(\bar{\mathcal{C}}_M, d_{SBM})$ are obtained by “shrinking the necks”. To make this precise we use the above mentioned stability result, together with the fact that filtered symplectic homology of U_g^*M is isomorphic, as a persistence module, to the homology of the free loop space $\mathcal{L}(M)$ filtered by energy, see Section 4.4. More precisely, computing the barcode of filtered symplectic homology of U_g^*M is equivalent to computing the barcode of $V^t = H_*(\{E_g \leq t\})$ where

$$E_g : \mathcal{L}(M) \rightarrow \mathbb{R}, \quad E_g(\gamma) = \int_0^1 \frac{\|\dot{\gamma}(\tau)\|^2}{2} d\tau.$$

Critical points of E_g are closed geodesics on (M, g) and hence the endpoints of bars in the barcode of $H_*(\{E_g \leq t\})$ are energies of certain closed geodesics. So, in order to describe the barcode of $H_*(\{E_g \leq t\})$, we need to analyse closed geodesics on (M, g) . This is possible in the example shown in the figure, due to the fact that geodesic flow on a surface of revolution is an integrable system, see Section 4.6 for details.

Remark 1.3.3. In [148], Usher proved a result analogous to Theorem 1.3.2 in the case of star-shaped domains in \mathbb{R}^{2n} . Roughly speaking, in a similar fashion as above,

one defines the space of admissible star-shaped domains in \mathbb{R}^{2n} , denoted by \mathcal{S}_{2n} , as well as symplectic Banach-Mazur distance, d_{SBM} , on this space. Usher proved that for $n \geq 2$ and any $N \in \mathbb{N}$ there exists a quasi-isometric embedding from $(\mathbb{R}^N, |\cdot|_\infty)$ to $(\mathcal{S}_{2n}, d_{SBM})$. While the formulations of the two results are similar, the constructions of the quasi-isometric embeddings are rather different.

In a somewhat different direction, we exploit the fact that energies of closed geodesics correspond to the endpoints of bars in the barcode of $H_*(\{E_g \leq t\})$ in order to study stability of closed geodesics with respect to perturbations of a metric. Given two Riemannian metrics g_1 and g_2 on M , we use $g_1 \preceq g_2$ to denote the fact that $\|v\|_{g_1} \leq \|v\|_{g_2}$ for every $v \in TM$. In [144] we proved the following theorem.

Theorem 1.3.4 (Theorem 4.1.19). *Let g_1, g_2 be two bumpy metrics on a closed, orientable manifold M such that $\frac{1}{C_1}g_1 \preceq g_2 \preceq C_2g_1$. If there exists a bar $[x, y)$ in the barcode of $H_*(\{E_{g_1} \leq t\})$ such that $\frac{y}{x} > C_1C_2$ then there exist closed geodesics γ_1 and γ_2 of (M, g_2) , whose energies satisfy*

$$\frac{1}{C_1}x \leq E_{g_2}(\gamma_1) \leq C_2x, \quad \frac{1}{C_1}y \leq E_{g_2}(\gamma_2) \leq C_2y,$$

and the barcode of $H_*(\{E_{g_2} \leq t\})$ contains the bar $[E_{g_2}(\gamma_1), E_{g_2}(\gamma_2))$. In the case of an infinite bar $[x, +\infty)$, there exists a closed geodesic γ_1 of (M, g_2) such that

$$\frac{1}{C_1}x \leq E_{g_2}(\gamma_1) \leq C_2x,$$

and the barcode of $H_*(\{E_{g_2} \leq t\})$ contains the bar $[E_{g_2}(\gamma_1), +\infty)$.

The statement about infinite bars can be considered a reformulation of the following result which can be traced back to Birkhoff. If $g_2 \preceq g_1$ then $l(g_2) \leq l(g_1)$, where $l(g)$ denotes the length of the shortest non-constant and “homologically visible” closed geodesic of g . Indeed, endpoints of infinite bars correspond to energies of “homologically visible” closed geodesic and the mentioned result is a statement about the smallest endpoint of an infinite bar, see Corollary 4.1.21 for more details. On the other hand, finite bars in the barcode of $H_*(\{E_g \leq t\})$ belong to the realm of Gromov’s quantitative homotopy theory, as noticed by Weinberger, see Remark 4.1.20. In order to illustrate the appearance of finite bars, in Example 4.1.23 we computed the full barcode of a specific metric of revolution on \mathbb{T}^2 .

1.4 Oscillations of Laplace eigenfunctions

Let M be a compact, orientable, surface, possibly with boundary, equipped with a Riemannian metric g . Denote by Δ the positive definite Laplace-Beltrami operator and by $\|\cdot\|$ the L^2 -norm induced by g . Let $f : M \rightarrow \mathbb{R}$ be a Morse function, which we

assume to vanish on ∂M if $\partial M \neq \emptyset$. Denote by $\mathcal{B}'(f)$ the multiset of all finite bars⁵ in the barcode of the persistence module $V_*^t(f) = H_*(\{f < t\}; \mathbb{R})$. Let $u : \mathbb{R} \rightarrow [0, +\infty)$ be a continuous function. Define

$$\Phi_u(f) = \begin{cases} \int_{\min f}^{\max f} u(t) dt + \sum_{I \in \mathcal{B}'(f)} \int_I u(t) dt & \text{if } \partial M = \emptyset, \\ \int_{\min f}^0 u(t) dt + \sum_{I \in \mathcal{B}'(f)} \int_I u(t) dt & \text{if } \partial M \neq \emptyset. \end{cases} \quad (1.4)$$

The main result of [113] is the following.

Theorem 1.4.1 (Theorem 5.1.7⁶). *In the above setup it holds*

$$\Phi_u(f) \leq \kappa_g(\|f\| + \|\Delta f\|) \cdot \|u \circ f\|,$$

where κ_g is a constant which depends only on (M, g) .

Remark 1.4.2. By slightly abusing the notation, we denote all constants which depend only on (M, g) by κ_g .

Let us explain the intuition and give some context behind Theorem 1.4.1. We focus on the case $u \equiv 1$, since Φ_1 has a relatively transparent meaning. Φ_u for arbitrary u can be considered a u -weighted version of Φ_1 . Related functionals have been earlier considered in [39] and certain upper bounds in the spirit of Theorem 1.4.1 were proven, see Remark 5.1.19 and a discussion in [120].

Firstly, note that $\Phi_1(f)$ can be defined on an arbitrary manifold using the same expression (1.4) and that it equals the sum of lengths of all the finite bars in the barcode of f and the length of the range of f (or $\min f$ if $\partial M \neq \emptyset$). Now, assume that $M = S^1$. A Morse function on S^1 has the same number, denote it by N , of local minima x_1, \dots, x_N and local maxima y_1, \dots, y_N and they appear in an alternating order on S^1 as follows

$$x_1, y_1, \dots, x_N, y_N, x_1.$$

Basic properties of barcodes (see Lemma 2.1.10) now imply that

$$\Phi_1(f) = \sum_{i=1}^N (f(y_i) - f(x_i)) = \frac{1}{2} \text{Var}(f),$$

where $\text{Var}(f)$ denotes the total variation of f , see Example 5.1.15 for more details. Thus, we might consider $\Phi_1(f)$ to be a measure of the total variation of f . To give further evidence in support of this claim, let us consider what happens in higher

⁵By this we mean the union over all degrees k of all finite bar in the barcode of $H_k(\{f < t\}; \mathbb{R})$.

⁶Strictly speaking, there is a slight difference between Theorem 1.4.1 and Theorem 5.1.7. However, the proof of Theorem 1.4.1 can be easily extracted from the proof of Theorem 5.1.7.

dimensions. To this end, denote by $\beta(t, f)$ the number of connected components of $f^{-1}(t)$. Function $\beta(t, f)$ is called the *Banach indicatrix of f* . It is easy to see that

$$\mathbf{I}(f) := \int_{-\infty}^{+\infty} \beta(t, f) dt = \text{Var}(f),$$

for a Morse function f on a closed interval. In fact the analogous statement, known as the *Banach indicatrix theorem*, was proven by Banach for every continuous function on a closed interval. Based on this fact, the integral of the Banach indicatrix $\mathbf{I}(f)$ was considered a measure of the total variation of f in higher dimensions, see [153]. We showed in [113], see Proposition 5.3.1, that on a compact orientable surface, possibly with boundary, it holds

$$\Phi_1(f) \leq \mathbf{I}(f). \tag{1.5}$$

Moreover, on the sphere (1.5) becomes an equality. Thus, Theorem 1.4.1 can be interpreted as an upper bound on the total variation of f in terms of it's $W^{2,2}$ -Sobolev norm. We should note that bounding $\mathbf{I}(f)$ from above by various norms of f is a classical topic in analysis, see [87, 153, 162]. Due to (1.5), any such bound yields an upper bound for $\Phi_1(f)$. In particular, Theorem 1.4.1 follows from a u -weighted version of (1.5) and an upper bound for a u -weighted version of $\mathbf{I}(f)$ due to Polterovich and Sodin - [123].

Remark 1.4.3. Proposition 5.3.1 which asserts (1.5) actually gives a similar estimate in arbitrary dimension. However, the results of [123] hold only in dimension two.

We now present certain applications, obtained in [113], of Theorem 1.4.1 to the study of linear combinations of eigenfunction of Δ . From now on we always assume that M is a compact orientable surface, possibly with boundary.

For $\lambda > 0$, let $\mathcal{F}_\lambda = \{f \in C^\infty(M) \mid \|f\| = 1, \|\Delta f\| \leq \lambda\}$. This set contains normalized linear combinations of eigenfunction of Δ with eigenvalue $\leq \lambda$. It follows from Theorem 1.4.1 that if $f \in \mathcal{F}_\lambda$ is Morse then

$$\Phi_1(f) \leq \kappa_g(\lambda + 1). \tag{1.6}$$

We call a critical value of f δ -significant if it is an endpoint of a bar of length $\geq \delta$. Denote by $\mathcal{N}_\delta(f)$ the total number of δ -significant critical values of f (counted with multiplicities if there are multiple bars with this endpoint). The following is a direct corollary of (1.6).

Corollary 1.4.4 (Corollary 5.1.11). *If $f \in \mathcal{F}_\lambda$ is a Morse function, then*

$$\mathcal{N}_\delta(f) \leq \frac{\kappa_g}{\delta}(\lambda + 1).$$

It is easy to see that the above inequality does not hold if δ -significant condition is dropped, see Example 5.1.12.

Another application of (1.6) is to C^0 -approximation theory. Denote by

$$\text{approx}_\lambda(f) = \inf_{\varphi \in \text{Diff}(M)} d_{C^0}(f \circ \varphi, \mathcal{F}_\lambda).$$

This quantity measures how well can a function f be approximated by functions from \mathcal{F}_λ , after a change of variables. C^0 -approximation of a function after a change of variables is a classical topic in Fourier analysis, see [132, 103, 71]. Since barcodes of f and $f \circ \phi$ coincide for every $\phi \in \text{Diff}(M)$, invariants coming from barcodes, such as Φ_1 are naturally adapted to these kind of questions. In particular we may prove the following.

Proposition 1.4.5 (Proposition 5.2.1). *For every Morse function $f : M \rightarrow \mathbb{R}$, vanishing on the boundary, the following inequality holds*

$$\text{approx}_\lambda(f) \geq \begin{cases} \frac{1}{2 \cdot (|\mathcal{B}'(f)|+1)} \left(\Phi_1(f) - \kappa_g(\lambda + 1) \right) & \text{for } \partial M = \emptyset \\ \frac{1}{2|\mathcal{B}'(f)|+1} \left(\Phi_1(f) - \kappa_g(\lambda + 1) \right) & \text{for } \partial M \neq \emptyset \end{cases} \quad (1.7)$$

where $|\mathcal{B}'(f)|$ denotes the number of finite bars.

The lower bound in Proposition 1.4.5 becomes negative as $\lambda \rightarrow +\infty$. This means that this proposition is not of asymptotic nature, i.e. it concerns fixed, possibly large, values of λ . We refer the reader to Subsection 5.2.1 for further discussion.

Finally, we want to mention that, in the case of a flat torus $M = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, Proposition 1.4.5 becomes an inverse statement about C^0 -approximation of a function by trigonometric polynomials. Juxtaposition of this result with a direct result due to Yudin - [163], yields an upper bound on the average length of a finite bar in the barcode of a function on \mathbb{T}^2 in terms of it's modulus of continuity, see Subsection 5.2.2.

Chapter 2

Background

This chapter covers some basic aspects of the theory of persistence modules and barcodes, symplectic topology and Floer theory. Its purpose is to provide a background on these subjects needed in the rest of the thesis.

2.1 Persistence modules and barcodes

In this section we briefly review the theory of persistence modules and barcodes. For detailed treatments of the theory we refer the reader to [48, 106, 35, 120]. During the exposition we assume that the reader is familiar with basic Morse theory. A brief survey of Morse homology can be found in Subsection 2.2.1.

2.1.1 Basics and the structure theorem

Let \mathbb{K} be a field.

Definition 2.1.1. A *persistence module* (V, π) over \mathbb{K} consists of a family of finite-dimensional \mathbb{K} -vector spaces V^t , indexed by a real parameter $t \in \mathbb{R}$, together with linear *structure maps* $\pi_{s,t} : V^s \rightarrow V^t$ for all $s \leq t$ which satisfy

1. For all $t \in \mathbb{R}$, $\pi_{t,t} = \mathbb{1}_{V^t}$;
2. For any $r, s, t \in \mathbb{R}$ such that $r \leq s \leq t$ it holds

$$\pi_{s,t} \circ \pi_{r,s} = \pi_{r,t}.$$

We sometimes abbreviate and write V for the persistence module (V, π) .

Remark 2.1.2. The object defined by Definition 2.1.1 is sometimes referred to as a “pointwise finite-dimensional persistence module indexed over \mathbb{R} ”. This comes from

the fact that one may consider similar object, by allowing some V^t to have infinite dimension or by taking t in a different partially ordered set. We should note that not all of the theory presented here carries over to these, more general, setups. All persistence modules we consider in this thesis fit in the framework of Definition 2.1.1.

Example 2.1.3. Let M be a closed manifold and $f : M \rightarrow \mathbb{R}$ a Morse function. For $k \in \mathbb{Z}$ define a persistence module $(V_k(f), \pi)$ by

$$V_k^t(f) = H_k(\{f < t\}; \mathbb{K}) \text{ for } t \in \mathbb{R},$$

structure maps $\pi_{s,t} : V_k^s(f) \rightarrow V_k^t(f)$ being induced by inclusions of sublevel sets

$$\{f < s\} \subset \{f < t\} \text{ for } s \leq t.$$

Since topology of sublevel sets only changes upon passing through a critical value, one readily sees that $\pi_{s,t}$ is an isomorphism if $[s, t]$ contains no critical values of f . We denote by $V(f) = \bigoplus_k V_k(f)$ the persistence module coming from total homology. $V(f)$ is a persistence module of \mathbb{Z} -graded vector spaces.

Definition 2.1.4. Let $(V, \pi^V), (W, \pi^W)$ be two persistence modules over \mathbb{K} . A *morphism of persistence modules* or *persistence module morphism* $f : (V, \pi^V) \rightarrow (W, \pi^W)$ is a family of linear maps $f_t : V^t \rightarrow W^t$ for each $t \in \mathbb{R}$ which satisfy

$$\pi_{s,t}^W \circ f_s = f_t \circ \pi_{s,t}^V \text{ for all } s \leq t.$$

One defines $\ker f$ and $\text{im } f$ as persistence modules given by

$$(\ker f)^t = \ker f^t \subset V^t, \quad (\text{im } f)^t = \text{im } f^t \subset W^t,$$

structure maps $\pi^{\ker f}$ and $\pi^{\text{im } f}$ being the restrictions of π^V and π^W to $\ker f$ and $\text{im } f$. *Direct sum* $(V \oplus W, \pi^{V \oplus W})$ of persistence modules is given by

$$(V \oplus W)^t = V^t \oplus W^t, \quad \pi_{s,t}^{V \oplus W} = \pi_{s,t}^V \oplus \pi_{s,t}^W.$$

Given these definitions, persistence modules form an abelian category.

Remark 2.1.5. Let (\mathbb{R}, \leq) be a poset category, i.e. a category whose objects are real numbers and which has a unique morphisms between s and t whenever $s \leq t$. A persistence module may be defined in more categorical terms as a functor from the poset category (\mathbb{R}, \leq) to the category of finite-dimensional vector spaces. In this language, a morphism of persistence modules is a natural transformation of functors.

Definition 2.1.6. A *barcode* $\mathcal{B} = \{(I, m_I)\}$ is a multiset¹ of intervals $I \subset \mathbb{R}$, called *bars*, with finite multiplicities m_I .

¹Recall that a multiset is a set in which elements may appear more than once. The number of times an element appears in the multiset is called the multiplicity of that element.

As mentioned in Section 1.1, the structure theorem for persistence modules tells us that to each isomorphism class of persistence modules corresponds a unique barcode. To formulate this theorem precisely we will need the following abstract example of a persistence module associated to an interval.

Example 2.1.7. Let \mathbb{K} be a field and $I \subset \mathbb{R}$ an interval. Define an *interval persistence module* \mathbb{K}_I by

$$\mathbb{K}_I^t = \begin{cases} \mathbb{K}, & \text{for } t \in I \\ 0, & \text{otherwise} \end{cases}, \quad \pi_{s,t}^{\mathbb{K}_I} = \begin{cases} \mathbb{1}_{\mathbb{K}}, & \text{for } s, t \in I \\ 0, & \text{otherwise} \end{cases}.$$

Interval modules are building blocks for all persistence modules, i.e. the following holds.

Theorem 2.1.8 (The structure theorem). *To every persistence module (V, π) over \mathbb{K} corresponds a barcode $\mathcal{B}(V)$, unique up to isomorphisms of persistence modules, such that*

$$(V, \pi) \cong \bigoplus_{(I, m_I) \in \mathcal{B}(V)} ((\mathbb{K}_I, \pi^{\mathbb{K}_I}))^{m_I}.$$

The term barcode was first used in [167] and a version of Theorem 2.1.8 for persistence modules indexed over \mathbb{N} was proven. Analogous objects in the framework of filtered chain complexes were previously considered in [16] and a theorem analogous to Theorem 2.1.8 was proven. Theorem 2.1.8, in the level of generality as stated here, was proven in [44].

Let us now elaborate on the persistence module-to-barcode correspondence in the case of persistence modules coming from Morse functions, introduced in Example 2.1.3. For a Morse function f , denote by $\mathcal{B}_k(f) = \mathcal{B}(V_k(f))$ the *degree k barcode* of f and by $\mathcal{B}(f) = \mathcal{B}(V(f)) = \cup_k \mathcal{B}_k(f)$ the *full barcode* of f . We start by a concrete example of a Morse function on S^1 .

Example 2.1.9. Let $f : S^1 \rightarrow \mathbb{R}$ be a height function on a deformed circle (see Figure 2.1). Critical values of f are a, b, c and d , and for $t \in \mathbb{R}$, the sublevel sets $f^{-1}((-\infty, t))$ are homeomorphic to:

$$f^{-1}((-\infty, t)) = \begin{cases} \emptyset, & \text{for } t \leq a \\ I, & \text{for } a < t \leq b \\ I \sqcup I, & \text{for } b < t \leq c \\ I, & \text{for } c < t \leq d \\ S^1, & \text{for } d < t \end{cases}$$

where I stands for an open interval. Degree 1 barcode is now easily seen to contain one infinite bar $\mathcal{B}_1(f) = \{(d, +\infty)\}$, while degree 0 barcode contains one infinite and one finite bar $\mathcal{B}_0(f) = \{(a, +\infty), (b, c]\}$. The finite bar $(b, c]$ corresponds to the fact that for $b < t \leq c$, $f^{-1}((-\infty, t))$ has two connected components which merge for $t > c$. The full barcode is given by $\mathcal{B}(f) = \{(a, +\infty), (d, +\infty), (b, c]\}$.

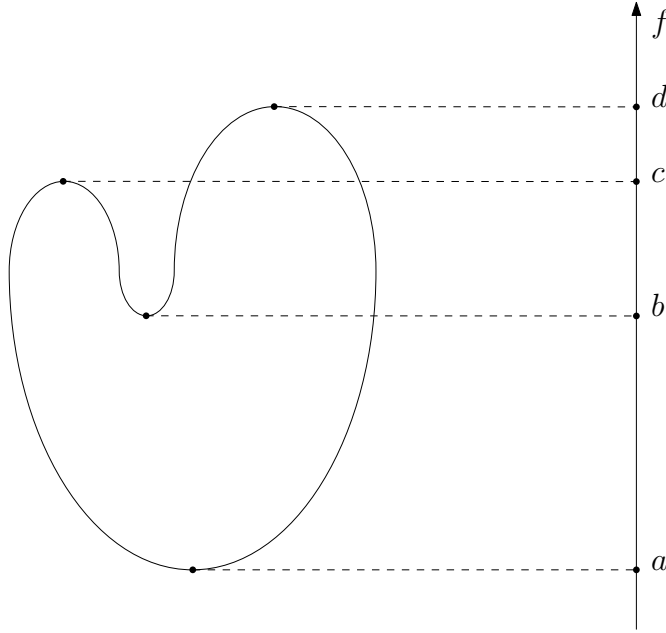


Figure 2.1. Height function on a deformed circle.

Firstly, we notice that all the bars in Example 2.1.9 have the form $(a, b]$ or $(a, +\infty)$ for $a, b \in \mathbb{R}$. This is true for any Morse function f . Indeed, notice that when $t \leq \min f$, $\{f < t\} = \emptyset$ and hence $V^t(f) = 0^2$. Thus, all bars in $\mathcal{B}(f)$ are left-bounded. The fact that they have left endpoints open and right endpoints closed comes from the choice of strict inequality in the definition of sublevel sets $\{f < t\}$. If we would consider $\{f \leq t\}$ instead, the situation would be exactly the opposite, i.e. the left endpoints of bars would be closed and right endpoints would be open.

Secondly, notice that the number of infinite bars, i.e. rays of the form $(a, +\infty)$, in $\mathcal{B}_k(f)$ equals $\dim H_k(M; \mathbb{K})$. Again, this is true for Morse functions in general. To see this, note that when $t > \max f$, $\{f < t\} = M$ and thus $V_k^t(f)$ stabilizes, meaning that $\pi_{s,t}$ are isomorphisms for $s, t > \max f$. Now, for $t > \max f$, $\dim V_k^t(f)$ is on the one hand equal to the number of infinite bars in $\mathcal{B}_k(f)$ and on the other hand equal to $\dim H_k(\{f < t\}; \mathbb{K}) = \dim H_k(M; \mathbb{K})$.

Finally, we notice that endpoints of bars in $\mathcal{B}(f)$ correspond to critical values of f . As we already mentioned in Example 2.1.3, homology of sublevel sets only changes upon passing through a critical value, hence the endpoints of bars in $\mathcal{B}(f)$ must be equal to critical values of f . Moreover, we claim that each critical point of f of index k and with critical value c either starts a bar in $\mathcal{B}_k(f)$ at c or ends a bar in $\mathcal{B}_{k-1}(f)$ at c . More formally, the following holds.

Lemma 2.1.10. *Let f be a Morse function on a closed manifold M and c its critical value. The number of critical points of f with critical value c and index k is equal to*

²Here and in the rest of the thesis we use a convention that homology of the empty set is zero.

the combined number of bars in $\mathcal{B}_k(f)$ starting at c and bars in $\mathcal{B}_{k-1}(f)$ ending at c .

Proof. All considerations work for homology over an arbitrary field \mathbb{K} , so we omit \mathbb{K} from the notation. For $t \in \mathbb{R}$ denote by $M^t = \{f < t\}$. Assume that there are m critical points of f of index k at critical level $f^{-1}(c)$ and assume that $\varepsilon > 0$ is such that $[c - \varepsilon, c + \varepsilon]$ contains no critical values other than c . It is a standard fact of Morse theory that in this case

$$\dim H_k(M^{c+\varepsilon}, M^{c-\varepsilon}) = m.$$

Consider the following part of the long exact sequence of the pair $(M^{c+\varepsilon}, M^{c-\varepsilon})$:

$$H_k(M^{c-\varepsilon}) \xrightarrow{i} H_k(M^{c+\varepsilon}) \xrightarrow{p} H_k(M^{c+\varepsilon}, M^{c-\varepsilon}) \xrightarrow{\partial} H_{k-1}(M^{c-\varepsilon}) \xrightarrow{j} H_{k-1}(M^{c+\varepsilon}).$$

The number of bars in $\mathcal{B}_k(f)$ starting at c is equal to $\dim H_k(M^{c+\varepsilon}) - \dim(\text{im } i)$, while the number of bars in $\mathcal{B}_{k-1}(f)$ ending at c is equal to $\dim(\ker j)$. From the exactness of the above sequence we have

$$\dim H_k(M^{c+\varepsilon}) - \dim(\text{im } i) = \dim H_k(M^{c+\varepsilon}) - \dim(\ker p) = \dim(\text{im } p) = \dim(\ker \partial),$$

as well as

$$\dim(\ker j) = \dim(\text{im } \partial).$$

Since $\dim(\ker \partial) + \dim(\text{im } \partial) = \dim H_k(M^{c+\varepsilon}, M^{c-\varepsilon}) = m$ the claim follows. \square

A couple of remarks are in order. Firstly, we wish to point out that Lemma 2.1.10 does not hold for non-Morse functions, as can be seen from an example of a function on S^1 with three critical points.

Secondly, the proof of Lemma 2.1.10 does not construct a canonical bijection between critical points of f and endpoints of bars in $\mathcal{B}(f)$. In general, there is no reason to expect the existence of a canonical bijection between these two sets. However, one may construct a different basis of the Morse chain complex, in which the boundary operator has a particularly simple form and for which there is a canonical bijection between basis vectors and the endpoints of bars. The existence of such a basis was used in [16] to prove a theorem analogous to Theorem 2.1.8 in the case of filtered chain complexes. For more details on this approach see [120] and references therein.

Lastly, we want to point out that barcodes of f and $f \circ \phi$ coincide for any diffeomorphism $\phi \in \text{Diff}(M)$. Indeed for every $t \in \mathbb{R}$ restricting ϕ to $\{f \circ \phi < t\}$ gives a homeomorphism

$$\phi : \{f \circ \phi < t\} \rightarrow \{f < t\},$$

which for every $k \in \mathbb{Z}$ induces an isomorphism

$$\phi_* : H_k(\{f \circ \phi < t\}; \mathbb{K}) \rightarrow H_k(\{f < t\}; \mathbb{K}).$$

It is easy to see that this is an isomorphism of persistence modules $V_k(f)$ and $V_k(f \circ \phi)$ and hence $\mathcal{B}_k(f) = \mathcal{B}_k(f \circ \phi)$ for every $k \in \mathbb{Z}$.

2.1.2 Stability and isometry theorems

In this subsection we introduce a quantitative perspective on persistence modules and barcodes. To this end, we define the distance on the space of all barcodes, called the *bottleneck distance* as well as its algebraic counterpart, the *interleaving distance* on the space of all persistence modules. The highlight of the discussion are the stability theorem, Theorem 2.1.17, and the isometry theorem, Theorem 2.1.21. For a detailed treatment of the isometry theorem, see [19, 20] and references therein.

Definition 2.1.11. A *matching* σ , between multisets X and Y is a bijection between subsets of X and Y called, respectively, the *coimage* and the *image* of σ and denoted by $\text{coim } \sigma$ and $\text{im } \sigma$. In other words

$$\sigma : \text{coim } \sigma \xrightarrow{\text{bijection}} \text{im } \sigma, \text{ for some } \text{coim } \sigma \subset X, \text{im } \sigma \subset Y.$$

Remark 2.1.12. In Definition 2.1.11, the terms subset and bijection refer to a subset and a bijection in the sense of multisets. This means that $\text{coim } \sigma$ and $\text{im } \sigma$ are multisets whose elements have multiplicities not greater than their original multiplicities in X and Y . Similarly, a bijection σ must match each copy of an element from $\text{coim } \sigma$ with (a copy of) an element in $\text{im } \sigma$. In particular, the total multiplicities of all elements in $\text{coim } \sigma$ and $\text{im } \sigma$ are equal.

For a barcode \mathcal{B} and $\varepsilon \geq 0$, denote by $\mathcal{B}^\varepsilon \subset \mathcal{B}$ the subset of all bars of length $\geq \varepsilon$. We use notation $\langle a, b \rangle$ to denote any of the intervals (a, b) , $[a, b)$, $(a, b]$ or $[a, b]$.

Definition 2.1.13. Let $\mathcal{B}_1, \mathcal{B}_2$ be two barcodes and let $\varepsilon \geq 0$. A matching σ , between \mathcal{B}_1 and \mathcal{B}_2 , is called an ε -*matching* if

- 1) $\mathcal{B}_1^{2\varepsilon} \subset \text{coim } \sigma, \mathcal{B}_2^{2\varepsilon} \subset \text{im } \sigma$;
- 2) If $\sigma(\langle a, b \rangle) = \langle c, d \rangle$ then $|a - c| \leq \varepsilon$ and $|b - d| \leq \varepsilon$.

We say that \mathcal{B}_1 and \mathcal{B}_2 are ε -*matched* if there exists an ε -matching between them.

Remark 2.1.14. In property 2) of Definition 2.1.13 some endpoints may be infinite. In this case we take $|-\infty - (-\infty)| = |+\infty - (+\infty)| = 0$, i.e. infinite endpoints have to match exactly. In particular, in order to have an ε -matching between \mathcal{B}_1 and \mathcal{B}_2 , we need the numbers of left-infinite rays, right-infinite rays and bars $(-\infty, +\infty)$ in \mathcal{B}_1 and \mathcal{B}_2 to be equal.

If σ is an ε -matching, $\mathcal{B}_1 \setminus \text{coim } \sigma$ and $\mathcal{B}_2 \setminus \text{im } \sigma$ only contain bars of length $< 2\varepsilon$. Informally, we may think of these bars as being erased, or matched with empty intervals. Thus, an ε -matching may be thought of as a bijection up to an error of ε at the endpoints of bars.

Definition 2.1.15. Let $\mathcal{B}_1, \mathcal{B}_2$ be two barcodes. The *bottleneck distance* between \mathcal{B}_1 and \mathcal{B}_2 is given by

$$d_{bottle}(\mathcal{B}_1, \mathcal{B}_2) = \inf\{\varepsilon \mid \mathcal{B}_1 \text{ and } \mathcal{B}_2 \text{ are } \varepsilon\text{-matched}\}.$$

If there exists no ε -matching between \mathcal{B}_1 and \mathcal{B}_2 for any $0 \leq \varepsilon < +\infty$, we set $d_{bottle}(\mathcal{B}_1, \mathcal{B}_2) = +\infty$.

One readily checks that the bottleneck distance satisfies all axioms for a genuine metric except that two different barcodes may be at bottleneck distance zero. Indeed, for $a, b \in \mathbb{R}, a < b$ each two of the following barcodes

$$\mathcal{B}_1 = \{(a, b)\}, \mathcal{B}_2 = \{[a, b)\}, \mathcal{B}_3 = \{(a, b]\}, \mathcal{B}_4 = \{[a, b]\},$$

are on bottleneck distance zero. Since bottleneck distance can also be equal to $+\infty$, it is an *extended pseudo-metric*.

Example 2.1.16. Barcodes $\mathcal{B}_1 = \{(0, 2], [0, 1]\}, \mathcal{B}_2 = \{(0, 2)\}, \mathcal{B}_3 = \emptyset, \mathcal{B}_4 = \{[-100, +\infty)\}$ satisfy

$$d_{bottle}(\mathcal{B}_1, \mathcal{B}_2) = \frac{1}{2}, d_{bottle}(\mathcal{B}_1, \mathcal{B}_3) = d_{bottle}(\mathcal{B}_2, \mathcal{B}_3) = 1, d_{bottle}(\mathcal{B}_i, \mathcal{B}_4) = +\infty$$

for $i = 1, 2, 3$.

In [38], Cohen-Steiner, Edelsbrunner and Harer proved that barcodes associated to Morse functions behave in a stable way with respect to C^0 -perturbations of a function. Namely, let M be a closed manifold. For two functions f and g on M denote $d_{C^0}(f, g) = \|f - g\|_{C^0} = \max_{x \in M} |f(x) - g(x)|$ and recall from Example 2.1.3 that $\mathcal{B}_k(f)$ denotes the barcode associated to $V_k^t(f) = H_k(\{f < t\}; \mathbb{K})$.

Theorem 2.1.17 (The stability theorem - [38]). *Let M be a closed manifold and f and g two Morse functions on M . For every $k \in \mathbb{Z}$ it holds*

$$d_{bottle}(\mathcal{B}_k(f), \mathcal{B}_k(g)) \leq d_{C^0}(f, g).$$

In particular, it also holds

$$d_{bottle}(\mathcal{B}(f), \mathcal{B}(g)) \leq \max_k d_{bottle}(\mathcal{B}_k(f), \mathcal{B}_k(g)) \leq d_{C^0}(f, g).$$

As we mentioned in Subsection 1.1, stability results in the spirit of Theorem 2.1.17 will be crucial for our applications of barcodes to symplectic topology. In order to prove results of this sort, we will use an abstract generalization of Theorem 2.1.17 called the isometry theorem. Let us now formulate this theorem.

Definition 2.1.18. Let (V, π^V) be a persistence module. For $\delta \in \mathbb{R}$ define a δ -shifted module $(V[\delta], \pi^{V[\delta]})$ by

$$(V[\delta])^t = V^{t+\delta}, \quad \pi_{s,t}^{V[\delta]} = \pi_{s+\delta, t+\delta}^V.$$

If $f : V \rightarrow W$ is a morphism of persistence modules, we denote by $f[\delta] : V[\delta] \rightarrow W[\delta]$ the induced morphism given by $f[\delta]_t = f_{t+\delta}$.

Remark 2.1.19. The barcode $\mathcal{B}(V[\delta])$ is equal to the barcode $\mathcal{B}(V)$ translated by $-\delta$.

Definition 2.1.20. Let $\delta \geq 0$. We say that persistence modules (V, π^V) and (W, π^W) are δ -interleaved if there exists a pair of morphisms of persistence modules

$$f : V \rightarrow W[\delta], \quad g : W \rightarrow V[\delta],$$

such that for all $t \in \mathbb{R}$ it holds

$$(g[\delta] \circ f)_t = \pi_{t, t+2\delta}^V, \quad (f[\delta] \circ g)_t = \pi_{t, t+2\delta}^W.$$

A pair of morphisms f and g as above is called a δ -interleaving between V and W . The *interleaving distance* between V and W is defined as

$$d_{inter}(V, W) = \inf\{\delta \mid V \text{ and } W \text{ are } \delta\text{-interleaved}\}.$$

Similarly to d_{bottle} on barcodes, d_{inter} defines an extended pseudo-metric on the space of persistence modules. An interested reader may check that the interleaving distance between any two of the following interval modules equals to zero:

$$\mathbb{K}_{(a,b)}, \mathbb{K}_{[a,b)}, \mathbb{K}_{(a,b]}, \mathbb{K}_{[a,b]}, \text{ for } a, b \in \mathbb{R}, a < b.$$

Theorem 2.1.21 (The isometry theorem - [38, 34, 90]). *For any two persistence modules V and W it holds*

$$d_{inter}(V, W) = d_{bottle}(\mathcal{B}(V), \mathcal{B}(W)).$$

We refer the reader to [19, 20] for a detailed historical account of the isometry theorem. Let us now show how Theorem 2.1.21 implies Theorem 2.1.17.

Proof of Theorem 2.1.17. Denote by $\delta = d_{C^0}(f, g)$. From the definition of d_{C^0} it follows that for each $t \in \mathbb{R}$ it holds

$$\{f < t\} \subset \{g < t + \delta\} \text{ and } \{g < t\} \subset \{f < t + \delta\}.$$

For each $k \in \mathbb{Z}$, these inclusions induce maps in k -th homology

$$H_k(\{f < t\}; \mathbb{K}) \rightarrow H_k(\{g < t + \delta\}; \mathbb{K}) \text{ and } H_k(\{g < t\}; \mathbb{K}) \rightarrow H_k(\{f < t + \delta\}; \mathbb{K}),$$

and one readily checks that these maps constitute a δ -interleaving between $V_k(f)$ and $V_k(g)$. Thus

$$d_{inter}(V_k(f), V_k(g)) \leq \delta = d_{C^0}(f, g),$$

and Theorem 2.1.21 implies that

$$d_{bottle}(\mathcal{B}_k(f), \mathcal{B}_k(g)) \leq d_{C^0}(f, g).$$

□

We shall see in Chapters 3 and 4 how the idea of the proof of Theorem 2.1.17 can be applied to obtain different stability results in the spirit of Theorem 2.1.17. Our motto in these chapters will be to construct interleavings using functorial properties of Floer and symplectic homology and then analyze barcodes in a more combinatorial fashion.

2.1.3 Persistence modules of finite and locally finite type

The theory presented in Subsections 2.1.1 and 2.1.2 applies to a wide class of persistence modules given by Definition 2.1.1. However, for many applications (in particular for purposes of this thesis) it is enough to consider a smaller class of persistence modules called *persistence modules of finite or locally finite type*. We give a quick review of these persistence modules and refer the reader to [120] for a more detailed study.

Definition 2.1.22. A persistence module (V, π) is said to be of *finite type* if it satisfies the following properties:

- 1) For all but a finite number of points $r \in \mathbb{R}$, there is a neighbourhood $U \ni r$ such that $\pi_{s,t}$ are isomorphisms for all $s < t$ with $s, t \in U$;
- 2) There exists t_- such that $V^t = 0$ for $t < t_-$;
- 3) For every $t \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $\pi_{s,t}$ are isomorphisms for all $t - \varepsilon < s \leq t$.

Exceptional points in 1), i.e. points $r \in \mathbb{R}$ for which there does not exist a neighbourhood $U \ni r$ such that $\pi_{s,t}$ are isomorphisms for all $s, t \in U$, are called *spectral* and the set of all such points is called the *spectrum* of (V, π) and denoted by $\mathcal{S}(V)$. One readily checks that if $[s, t] \cap \mathcal{S}(V) = \emptyset$ then $\pi_{s,t}$ is an isomorphism. In other words, V^t only changes upon passing through a spectral point.

In terms of the barcode, $\mathcal{S}(V)$ is the set of all endpoints of bars in $\mathcal{B}(V)$ and 1) is equivalent to saying that $\mathcal{B}(V)$ is finite. Property 2) is equivalent to saying that all the bars in $\mathcal{B}(V)$ are left-bounded. Property 3) forces bars to have left endpoints open and right endpoints closed, i.e. to be of the form $(a, b]$ or $(a, +\infty)$ for $a, b \in \mathbb{R}$.

Example 2.1.23. Persistence module $V(f)$ from Example 2.1.3, associated to a Morse function f on a closed manifold M , is of finite type. Indeed, as we discussed at the end of Subsection 2.1.1, $\mathcal{S}(V(f))$ consists of critical values of f . Since f is Morse its critical points are isolated and since M is compact there are only finitely many of them which proves 1). To verify 2) it is enough to take $t_- = \min f$. Finally, 3) comes from the choice of a strict inequality in $\{f < t\}$.

Remark 2.1.24. Let (V, π) be a persistence module of finite type. Since it's spectrum is finite, there exists t_+ such that $\pi_{s,t}$ are isomorphisms for all $t_+ \leq s \leq t$. Thus, for t large enough, all V^t are canonically isomorphic to $V^\infty := \varinjlim V^t$. Let f be a Morse function on a closed manifold and for $k \in \mathbb{Z}$ let $V_k(f) = H_k(\{f < t\}; \mathbb{K})$ be a persistence module considered in Examples 2.1.3 and 2.1.23. In this case, there exists a natural isomorphism $\Psi : V_k^\infty(f) \rightarrow H_k(M; \mathbb{K})$ and given $a \in H_k(M; \mathbb{K})$, $a \neq 0$, we can define a number

$$c(a, f) := \inf\{t \in \mathbb{R} \mid \Psi^{-1}(a) \in \text{im}(V_k^t(f) \rightarrow V_k^\infty(f))\}.$$

This number is called a *spectral invariant associated to f and a* , and has many remarkable properties. One can prove that for each $a \neq 0$, $c(a, f)$ is a starting point of an infinite bar in the barcode of $V_*(f)$, and each such starting point can be obtained in this way.

In Chapter 3, we will need the following lemma whose proof is straightforward.

Lemma 2.1.25. *Let V_1, \dots, V_l and W_1, \dots, W_l be persistence modules of finite type. Then*

$$\mathcal{S}\left(\bigoplus_{r=1}^l V_r\right) = \bigcup_{r=1}^l \mathcal{S}(V_r), \quad \mathcal{B}\left(\bigoplus_{r=1}^l V_r\right) = \sum_{r=1}^l \mathcal{B}(V_r),$$

and

$$d_{\text{bottle}}\left(\mathcal{B}\left(\bigoplus_{r=1}^l V_r\right), \mathcal{B}\left(\bigoplus_{r=1}^l W_r\right)\right) \leq \max_r d_{\text{bottle}}(\mathcal{B}(V_r), \mathcal{B}(W_r)).$$

Here Σ denotes multiset sum, that is, union of elements, adding up multiplicities.

All persistence modules we will consider in Chapters 3 and 5 will be of finite type. For purposes of Chapter 4, a slightly more general framework is necessary. According to [120], a persistence module is said to be of *locally finite type* if it satisfies property 3) from Definition 2.1.22, as well as a modified property 1), namely the following:

- 1') The set of exceptional points $\mathcal{S}(V)$, as defined in 1), is a closed, discrete, bounded from below subset of \mathbb{R} .

Persistence modules of locally finite type are not required to satisfy 2), even though from 1') it follows that $\pi_{s,t}$ are isomorphisms for s, t small enough.

Remark 2.1.26. Morally speaking, persistence modules we consider in Chapter 4 are of locally finite type. However, there is a slight ambiguity, as persistence modules appearing in Chapter 4 satisfy property 1'), but not property 3). Namely, property 3) forces all bars in the barcode to have left endpoints open and right endpoints closed, while in Chapter 4 we use the opposite conventions, i.e., bars have left endpoints closed and right endpoints open. Since we are interested in metric properties of persistence modules and barcodes, this difference is rather superficial. Indeed, by modifying the

endpoints of bars in a barcode in any way, one obtains a barcode whose bottleneck distance from the original one is zero. We leave it as an exercise to the reader to modify 3) to obtain conventions we use in Chapter 4.

Example 2.1.27. Let (M, g) be a closed Riemannian manifold. Define the *energy* of a loop $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow M$ as

$$E_g(\gamma) = \int_0^1 \frac{\|\dot{\gamma}(\tau)\|_g^2}{2} d\tau.$$

For a generic metric g , E_g is a Morse-Bott function on the (appropriately chosen) free loop space and its critical points are closed geodesics. In this case, persistence module given by

$$V_k^t = H_k(\{E_g < t\}; \mathbb{Z}_2), \quad \pi_{s,t} \text{ being induced by inclusions } \{E_g < s\} \subset \{E_g < t\},$$

is of locally finite type and endpoints of bars in the barcode $\mathcal{B}(V_k^t)$ are energies of certain closed geodesics.

By switching from $\{E_g < t\}$ to $\{E_g \leq t\}$ in the above example one obtains a persistence module with opposite conventions on the endpoints of bars in the barcode, as discussed in Remark 2.1.26. In Chapter 4 we use this persistence module to compute filtered symplectic homology of unit codisc bundles, as well as to study the existence and stability of closed geodesics with respect to perturbations of a Riemannian metric.

2.2 Basics of Floer theory

In this section we discuss (Hamiltonian) Floer theory and Floer persistence modules. These are the main technical tools used in our work in symplectic topology - [122,144].

2.2.1 Morse homology on closed manifolds

Floer theory is an infinite-dimensional version of Morse theory on the loops space of a symplectic manifold. We start our exposition of Floer theory by presenting an approach to Morse homology on a closed, finite-dimensional, manifold which mimics the construction of Floer homology. The material covered in this subsection should be considered a toy-model whose purpose is to provide intuition about the infinite-dimensional (Floer) case. To this end, we try to keep the exposition brief and technically non-demanding and omit most of the details of proofs. These details are standard and can be found in various places in the literature, see [134, 13, 125].

Let us recall basic notions of Morse theory. Let N be a smooth manifold and $f : N \rightarrow \mathbb{R}$ a smooth function. For a critical point x of f define the *Hessian of f at*

x as a bilinear form

$$Hess_f(x) : T_x N \times T_x N \rightarrow \mathbb{R}$$

given by

$$Hess_f(x)(X_x, Y_x) = X_x(Y(f)),$$

where X and Y are vector fields defined in a neighborhood of x . Since $df(x) = 0$, we have that

$$X_x(Y(f)) - Y_x(X(f)) = df(x)([X, Y]) = 0,$$

and hence $X_x(Y(f)) = Y_x(X(f))$. From here one concludes that $Hess_f(x)$ only depends on X_x and Y_x and not on their extensions to the neighborhood of x , as well as that it is a symmetric bilinear form. In local coordinates the Hessian is expressed by a matrix of second derivatives of f . Denote by $\text{Crit}(f)$ the set of all critical points of f .

Definition 2.2.1. A critical point $x \in \text{Crit}(f)$ is called *non-degenerate* if $Hess_f(x)$ is a non-degenerate bilinear form. In this case, the number of negative eigenvalues of $Hess_f(x)$ is called the *index* or *Morse index* of x and is denoted by $\text{ind}(x)$. If all critical points of f are non-degenerate f is called *Morse*. For a Morse function f , we denote by $\text{Crit}_k(f)$ the set of all critical points of f of index k .

Example 2.2.2. Let $N = \mathbb{R}^n$, $c \in \mathbb{R}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by

$$f(x_1, \dots, x_n) = c - \sum_{i=1}^k x_i^2 + \sum_{i=k+1}^n x_i^2.$$

We have that

$$df(x_1, \dots, x_n) = -2 \sum_{i=1}^k x_i dx_i + 2 \sum_{i=k+1}^n x_i dx_i,$$

and hence 0 is the only critical point of f . With respect to the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$, $Hess_f(0)$ is given by the matrix of second derivatives

$$Hess_f(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j=1,\dots,n} = \begin{pmatrix} -2\mathbb{1}_k & 0 \\ 0 & 2\mathbb{1}_{n-k} \end{pmatrix}.$$

We conclude that f is Morse and that $\text{ind}(0) = k$. In other words $\text{Crit}(f) = \text{Crit}_k(f) = \{0\}$.

The above example describes the local form of any function near a non-degenerate critical point. Namely, if N is any smooth manifold and $x \in \text{Crit}(f)$ a non-degenerate critical point of index k then there exist local coordinates (x_1, \dots, x_n) in a neighborhood of x s.t.

$$f(x_1, \dots, x_n) = f(x) - \sum_{i=1}^k x_i^2 + \sum_{i=k+1}^n x_i^2.$$

This statement is called *Morse lemma*. It implies, in particular, that non-degenerate critical points are isolated. Thus, if f is a Morse function on a closed manifold, it has only finitely many critical points.

We assume from now on that N is a closed manifold of dimension n and $f : N \rightarrow \mathbb{R}$ a Morse function. Let g be a Riemannian metric on N . We are interested in the flow $\phi_t : N \rightarrow N$ of the negative gradient vector field

$$\frac{d}{dt}\phi_t(x) = -\nabla f(\phi_t(x)).$$

Since N is compact, this flow is defined for all times $t \in \mathbb{R}$ and the *negative gradient trajectory* or *negative gradient flow line* is a curve $\gamma : \mathbb{R} \rightarrow N$ s.t.

$$\dot{\gamma}(t) = -\nabla f(\gamma(t)). \quad (2.1)$$

One shows that $\lim_{t \rightarrow \pm\infty} \gamma(t)$ exist and are critical points of f . This means that each negative gradient trajectory connects two critical points. For $x, y \in \text{Crit}(f)$ define the *space of connecting trajectories*

$$\mathcal{M}(x, y) = \{\gamma : \mathbb{R} \rightarrow N \mid \dot{\gamma}(t) = -\nabla f(\gamma(t)), \gamma(-\infty) = x, \gamma(+\infty) = y\}.$$

Theorem 2.2.3. *Let N be a closed manifold and $f : N \rightarrow \mathbb{R}$ a Morse function. For a generic Riemannian metric g the following holds. For every two $x, y \in \text{Crit}(f)$, $\mathcal{M}(x, y)$ has a structure of a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y)$.*

Let us comment on the formulation and the proof of this theorem. Firstly, we notice that $\mathcal{M}(x, y)$ can be identified with the following subset of N

$$W(x, y) = \{q \in N \mid \lim_{t \rightarrow -\infty} \phi_t(q) = x, \lim_{t \rightarrow +\infty} \phi_t(q) = y\}.$$

Indeed, mapping

$$\mathcal{M}(x, y) \ni \gamma \rightarrow \gamma(0) \in W(x, y)$$

is a bijection due to the existence and uniqueness of solutions of the ordinary differential equation (2.1). Secondly, we see that

$$W(x, y) = W^u(x) \cap W^s(y),$$

where

$$W^u(x) = \{q \in N \mid \lim_{t \rightarrow -\infty} \phi_t(q) = x\}, \quad W^s(y) = \{q \in N \mid \lim_{t \rightarrow +\infty} \phi_t(q) = y\}$$

are the *stable manifold of x* and the *unstable manifold of y* . One may show that these sets are indeed manifolds whose dimensions are

$$\dim W^u(x) = \text{ind}(x), \quad \dim W^s(y) = n - \text{ind}(y).$$

Metric g in Theorem 2.2.3 should be chosen so that stable and unstable manifolds of every two critical points intersect transversally. This condition is satisfied for a generic Riemannian metric. It implies that $W(x, y)$, and hence also $\mathcal{M}(x, y)$, is a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y)$ as a transverse intersection of $W^u(x)$ and $W^s(y)$. Now notice that for every $\gamma \in \mathcal{M}(x, y)$ and $t_0 \in \mathbb{R}$ the curve $t_0 \cdot \gamma$ given by $(t_0 \cdot \gamma)(\cdot) = \gamma(t_0 + \cdot)$ also belongs to $\mathcal{M}(x, y)$. Assuming $x \neq y$, this gives us a free \mathbb{R} -action on $\mathcal{M}(x, y)$ and we denote by $\bar{\mathcal{M}}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$ the *space of unparametrized trajectories*. This space is a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y) - 1$. When $\text{ind}(x) = \text{ind}(y) + 1$, $\bar{\mathcal{M}}(x, y)$ is 0-dimensional and it turns out that it is also compact, i.e. it consists of finitely many points.

We now have all ingredients necessary to define Morse homology with \mathbb{Z}_2 -coefficients. Let $f : N \rightarrow \mathbb{R}$ be a Morse function and (f, g) a generic pair in the sense of Theorem 2.2.3. For $k \in \mathbb{Z}$, define *Morse chain complex of (f, g) in degree k* as

$$CM_k(f, g) = \text{Span}_{\mathbb{Z}_2}(\text{Crit}_k(f)).$$

The differential

$$\partial : CM_k(f, g) \rightarrow CM_{k-1}(f, g)$$

is a \mathbb{Z}_2 -linear map given on $x \in \text{Crit}_k(f)$ by

$$\partial x = \sum_{y \in \text{Crit}_{k-1}(f)} \#_2 \bar{\mathcal{M}}(x, y) y,$$

$\#_2$ denoting cardinality modulo two.

Theorem 2.2.4. *It holds $\partial^2 = 0$.*

Sketch of a proof. Let $x \in \text{Crit}_k(f), z \in \text{Crit}_{k-2}(f)$. In this case $\bar{\mathcal{M}}(x, z)$ is a 1-dimensional manifold which can be compactified to a 1-dimensional manifold with boundary $\widetilde{\mathcal{M}}(x, z)$ such that

$$\partial \widetilde{\mathcal{M}}(x, z) = \bigcup_{y \in \text{Crit}_{k-1}(f)} \bar{\mathcal{M}}(x, y) \times \bar{\mathcal{M}}(y, z).$$

Intuitively, elements of $\bigcup_{y \in \text{Crit}_{k-1}(f)} \bar{\mathcal{M}}(x, y) \times \bar{\mathcal{M}}(y, z)$ are “broken” unparametrized negative gradient trajectories connecting x and y . In other words failure of compactness of $\mathcal{M}(x, y)$ can be described by a “breaking mechanism” which we illustrate in Example 2.2.7. Now, we compute

$$\begin{aligned} \partial(\partial x) &= \partial \left(\sum_{y \in \text{Crit}_{k-1}(f)} \#_2 \bar{\mathcal{M}}(x, y) y \right) = \sum_{y \in \text{Crit}_{k-1}(f)} \#_2 \bar{\mathcal{M}}(x, y) \partial y \\ &= \sum_{y \in \text{Crit}_{k-1}(f)} \left(\#_2 \bar{\mathcal{M}}(x, y) \sum_{z \in \text{Crit}_{k-2}(f)} \#_2 \bar{\mathcal{M}}(y, z) z \right) \\ &= \sum_{z \in \text{Crit}_{k-2}(f)} \sum_{y \in \text{Crit}_{k-1}(f)} (\#_2 \bar{\mathcal{M}}(x, y) \cdot \#_2 \bar{\mathcal{M}}(y, z)) z \\ &= \sum_{z \in \text{Crit}_{k-2}(f)} \#_2 \partial \widetilde{\mathcal{M}}(x, z) z = 0. \end{aligned}$$

The last equality follows because $\#_2 \partial \widetilde{\mathcal{M}}(x, z) = 0$ since compact 1-dimensional manifolds have an even number of boundary components. \square

Theorem 2.2.4 implies that $(CM_*(f, g), \partial)$ is a chain complex. The homology of this chain complex is called *Morse homology of (f, g)* and is denoted by $HM_*(f, g)$. It turns out that Morse homologies of different generic pairs (f, g) are isomorphic and are moreover isomorphic to singular homology of N with \mathbb{Z}_2 -coefficients.

Example 2.2.5. Let $f : S^1 \rightarrow \mathbb{R}$ be a height function (y -coordinate) on the unit circle in the plane. More precisely, $S^1 = \mathbb{R}/\mathbb{Z}$ embeds in \mathbb{R}^2 via $t \rightarrow (\cos 2\pi t, \sin 2\pi t)$ and $f(t) = \sin 2\pi t$. This function has two critical points, $x = \frac{1}{4}, y = \frac{3}{4}$ which are it's maximum and minimum, see Figure 2.2.

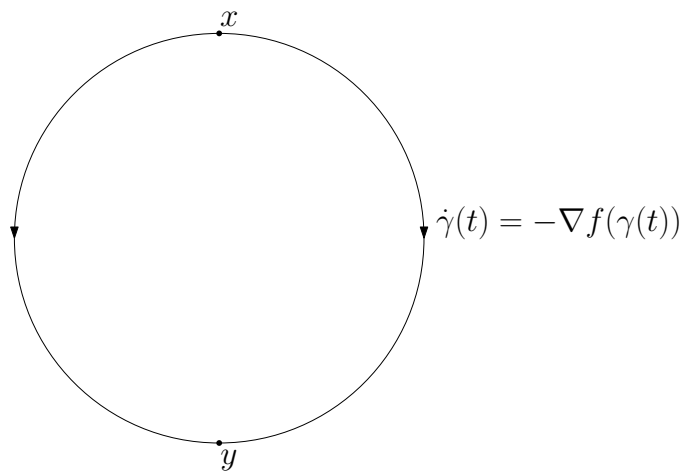


Figure 2.2. Negative gradient trajectories of f

The Hessian is given by

$$Hess_f(x) = f''\left(\frac{1}{4}\right) = -4\pi^2, \quad Hess_f(y) = f''\left(\frac{3}{4}\right) = 4\pi^2$$

and hence x and y are non-degenerate, i.e. f is Morse, and $\text{ind}(x) = 1, \text{ind}(y) = 0$. Let g be a metric on S^1 induced from the standard flat metric on \mathbb{R}^2 . One readily computes that there are two negative gradient trajectories connecting x to y and hence $\bar{\mathcal{M}}(x, y)$ consists of two points, as shown in Figure 2.2. We may now compute Morse homology of (f, g) . We have that

$$CM_0(f, g) = \mathbb{Z}_2 \cdot y, \quad CM_1(f, g) = \mathbb{Z}_2 \cdot x, \quad \partial(y) = 0, \quad \partial(x) = \#_2 \bar{\mathcal{M}}(x, y)y = 0.$$

Thus

$$HM_k(f, g) = \begin{cases} \mathbb{Z}_2, & \text{for } k = 0, 1 \\ 0, & \text{otherwise} \end{cases},$$

as we already could predict from the isomorphism with singular homology.

Example 2.2.6. For $n \geq 2$, define a height function on the n -dimensional sphere

$$f : S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\} \rightarrow \mathbb{R}$$

as

$$f(x_1, \dots, x_{n+1}) = x_{n+1}.$$

A direct computation shows that f has two critical points, a maximum $x = (0, \dots, 0, 1)$ and a minimum $y = (0, \dots, 0, -1)$, both of them are non-degenerate and $\text{ind}(x) = n, \text{ind}(y) = 0$. Thus

$$CM_0(f, g) = \mathbb{Z}_2 \cdot y, \quad CM_n(f, g) = \mathbb{Z}_2 \cdot x,$$

and since the index difference between x and y is greater than one it follows that $\partial x = \partial y = 0$. Hence

$$HM_k(f, g) = \begin{cases} \mathbb{Z}_2, & \text{for } k = 0, n \\ 0, & \text{otherwise} \end{cases}.$$

Example 2.2.7 (Example 10.2.3 in [119]). Let $\mathbb{T}^2 = \mathbb{R}^2 / (2\pi \cdot \mathbb{Z})^2$ and $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ given by $f(s, t) = \sin s + \sin t$, $(s, t) \in \mathbb{R}^2$. Critical points of f are

$$x = \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad z = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), \quad w = \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right).$$

The Hessian is given by the matrix of second derivatives $Hess_f(s, t) = \begin{pmatrix} -\sin s & 0 \\ 0 & -\sin t \end{pmatrix}$ and hence we see that x, y, z, w are all non-degenerate and $\text{ind}(x) = 2, \text{ind}(y) = \text{ind}(z) = 1, \text{ind}(w) = 0$. Let $g = ds^2 + dt^2$ be the flat metric on \mathbb{T}^2 . It follows that $-\nabla f(s, t) = (-\cos s, -\cos t)$ and we may draw the negative gradient trajectories as in Figure 2.3.

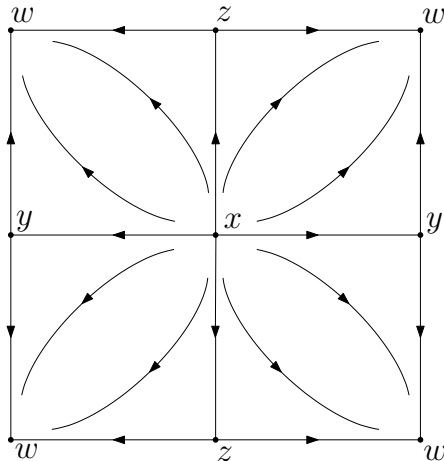


Figure 2.3. Negative gradient trajectories on \mathbb{T}^2

It follows that each of the spaces $\bar{\mathcal{M}}(x, y)$, $\bar{\mathcal{M}}(x, z)$, $\bar{\mathcal{M}}(y, w)$, $\bar{\mathcal{M}}(z, w)$ contains exactly two points and hence all coefficients of the Morse differential are zero. This gives us that $\partial = 0$ and hence

$$CM_k(f, g) = HM_k(f, g) = \begin{cases} \mathbb{Z}_2, & \text{for } k = 0, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{for } k = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Finally, notice that the breaking mechanism described in the proof of Theorem 2.2.4 can be clearly seen on Figure 2.3. Indeed, $\bar{\mathcal{M}}(x, w)$ consists of four 1-dimensional families of unparametrized trajectories, one in each of the four squares. As we approach y , these trajectories break into two concatenated trajectories, one from x to y , the other from y to w . These concatenated trajectories belong to $\partial\widetilde{\mathcal{M}}(x, w)$. Similar breaking is seen as we approach z .

Remark 2.2.8. We defined Morse homology only with \mathbb{Z}_2 -coefficients. Similar construction can be carried for arbitrary coefficients. The main difference is that we would have to count trajectories with appropriate signs which requires additional considerations regarding orientations.

2.2.2 Basics of Symplectic topology

In this subsection we introduce basic notions of symplectic topology.

A differential 2-form ω on a smooth manifold M is called a *symplectic form* if it is non-degenerate and closed. By non-degenerate we mean that for every $x \in M$ and $v \in T_x M$ there exists $w \in T_x M$ such that $\omega(v, w) \neq 0$. A smooth manifold equipped with a symplectic form is called a *symplectic manifold*. A linear algebra argument implies that, due to non-degeneracy of ω , the dimension of a symplectic manifold M has to be even. Denoting $\dim M = 2n$, non-degeneracy of ω is equivalent to $\omega^{\wedge n}$ being a volume form on M . In particular, symplectic manifolds are orientable.

Example 2.2.9. Let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be standard coordinates on \mathbb{R}^{2n} . Define a 2-form

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

It is easy to see that ω_0 is non-degenerate and closed, i.e. $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic manifold. This form is called the *standard symplectic form on \mathbb{R}^{2n}* . In the case of \mathbb{R}^2 , ω_0 is the standard area form.

Example 2.2.9 gives a local model for all symplectic manifolds. Namely, the famous Darboux's theorem states that every point on a symplectic manifold has a neighborhood and local coordinates in this neighborhood in which symplectic form equals to ω_0 . These coordinates are called *Darboux coordinates* and local charts $\varphi : (U, \omega) \rightarrow (\mathbb{R}^{2n}, \omega_0)$ for which $\varphi^* \omega_0 = \omega$ are called *Darboux charts*.

Example 2.2.10. Using Example 2.2.9 we may define a symplectic form on the $2n$ -dimensional torus \mathbb{T}^{2n} . Indeed, define $\omega = \omega_0$ in the local coordinates inherited from the quotient $\mathbb{T}^{2n} = \mathbb{R}^{2n}/(2\pi\mathbb{Z})^{2n}$. These coordinates are clearly Darboux coordinates in a neighborhood of every point on \mathbb{T}^{2n} .

Example 2.2.11. A closed, orientable, surface with an area form (Σ, σ) is a symplectic manifold. Indeed, σ is non-degenerate as an area form and closed because $\dim \Sigma = 2$.

Example 2.2.12. Let $(M_1, \omega_1), (M_2, \omega_2)$ be two symplectic manifolds. The product $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is also a symplectic manifold. This construction gives another way to define a symplectic form on \mathbb{T}^{2n} by viewing $\mathbb{T}^{2n} = (\mathbb{T}^2)^n$ and using Example 2.2.11.

Example 2.2.13. Let N be any smooth manifold and denote by T^*N the cotangent bundle of N . Denote by $\pi : T^*N \rightarrow N$ the canonical projection. Every point $p \in T^*N$ is a covector on the tangent space $T_{\pi(p)}N$. Define a canonical 1-form λ_{can} on T^*N , called the *canonical Liouville form*, by

$$\lambda_{can}(p)(\xi_p) = p(\pi_*(p)(\xi_p)),$$

where $\xi_p \in T_p T^*N$ is any tangent vector and $\pi_*(p) : T_p T^*N \rightarrow T_{\pi(p)}N$ is the differential of π at p . Canonical Liouville form can also be characterized as a unique 1-form on T^*N such that for every 1-form α on N it holds $\alpha^* \lambda_{can} = \alpha$. Here α plays both the role of a 1-form on N and of a section of T^*N over N .

We claim that $\omega_{can} = d\lambda_{can}$ is a symplectic form on T^*N . To prove this, we describe local Darboux coordinates for ω_{can} on T^*N as follows. Let (q_1, \dots, q_n) be any local coordinates on N and denote by $((\frac{\partial}{\partial q_1})^*, \dots, (\frac{\partial}{\partial q_n})^*)$ the dual frame to $(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n})$, i.e. $(\frac{\partial}{\partial q_i})^*(\frac{\partial}{\partial q_j}) = \delta_{ij}$. Define coordinates (p_1, \dots, p_n) on the fibers by expressing each covector p as $p = \sum_{i=1}^n p_i (\frac{\partial}{\partial q_i})^*$. A direct computation shows that in these coordinates $\lambda_{can} = \sum_{i=1}^n p_i dq_i$ and hence $\omega_{can} = d\lambda_{can} = \sum_{i=1}^n dp_i \wedge dq_i$.

Example 2.2.9 can be seen as a special case of Example 2.2.13, after identifying $\mathbb{R}^{2n} = T^*\mathbb{R}^n$. In both of these examples the symplectic form ω has a primitive 1-form λ , $\omega = d\lambda$, which means that ω is not only closed, but also exact. Symplectic manifolds with an exact symplectic form are called *exact*. Since $\omega^{\wedge n}$ is a volume form one readily sees that closed symplectic manifolds can not be exact, i.e. an exact symplectic manifold has to be non-compact.

A smooth map $\phi : (M_1, \omega_1) \rightarrow (M_2, \omega_2)$ between symplectic manifolds is called *symplectic* if $\phi^* \omega_2 = \omega_1$. In case the manifolds are exact, $\omega_1 = d\lambda_1, \omega_2 = d\lambda_2$ condition $\phi^* \omega_2 = \omega_1$ is equivalent to $\phi^* \lambda_2 - \lambda_1$ being closed. If moreover this form is exact, symplectic map ϕ is called *exact*. A symplectic diffeomorphism is called a *symplectomorphism*. Let us give a couple of examples of symplectomorphisms.

Example 2.2.14. Let ω_0 be the standard symplectic form on \mathbb{R}^{2n} defined in Example 2.2.9. The group of *symplectic matrices* is defined as

$$\text{Sp}(2n) = \{A \in \text{GL}(2n, \mathbb{R}) \mid A^* \omega_0 = \omega_0\}.$$

Multiplication by a symplectic matrix is a linear symplectomorphism. Denote by $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ the matrix representing the standard complex structure on \mathbb{R}^{2n} (we identify \mathbb{R}^{2n} with \mathbb{C}^n by $z_j = x_j + iy_j$). The standard scalar product $\langle \cdot, \cdot \rangle$, J_0 and ω_0 are related in the following way

$$(\forall v, w \in \mathbb{R}^{2n}) \langle v, w \rangle = \omega_0(v, J_0 w). \quad (2.2)$$

Using (2.2) one checks that the condition $A^* \omega_0 = \omega_0$ is equivalent to $A^T J_0 A = J_0$. It is also not difficult to prove that after identifying \mathbb{C}^n with (\mathbb{R}^{2n}, J_0) it holds $U(n) \subset Sp(2n)$, see [97].

Example 2.2.15. By definition, every volume preserving diffeomorphism of a surface with a volume form is a symplectomorphism.

Example 2.2.16. Let N be a smooth manifold. A diffeomorphism $\phi \in \text{Diff}(N)$ defines a map $\phi^* : T^*N \rightarrow T^*N$. A direct computation shows that $(\phi^*)^* \lambda_{can} = \lambda_{can}$ where λ_{can} is the canonical Liouville form on T^*N defined in Example 2.2.13. This implies that ϕ^* is a symplectomorphism which is moreover exact.

A class of symplectomorphisms of particular interest for us can be constructed using flows of Hamiltonian systems. The construction goes as follows. First note that non-degeneracy of ω implies that a map $I_\omega : TM \rightarrow T^*M$ given by $I_\omega(v) = \omega(v, \cdot)$ is an isomorphism of vector bundles. This means that for every 1-form α there exists a unique vector field $X = (I_\omega)^{-1}(\alpha)$ such that $\omega(X, \cdot) = \alpha(\cdot)$.

Definition 2.2.17. A function $H : M \times [0, 1] \rightarrow \mathbb{R}$ is called a *time-dependent Hamiltonian function on M* . A time-dependent vector field X_{H_t} given by

$$\omega(X_{H_t}, \cdot) = -dH_t(\cdot), \text{ for all } t \in [0, 1],$$

is called a *Hamiltonian vector field*. Integrating X_{H_t} we obtain the *Hamiltonian flow of H* , denoted by ϕ_t^H . In other words, $\phi_t^H : M \rightarrow M$ is a family of diffeomorphisms given as a solution of the ordinary differential equation

$$\frac{d}{dt}(\phi_t^H(x)) = X_{H_t}(\phi_t^H(x)), \quad \phi_0^H = \mathbb{1}_M.$$

Time-one map ϕ_1^H of this flow is called a *Hamiltonian diffeomorphism*. We will sometimes abbreviate X_{H_t} to X_H .

Using Cartan's formula we get that

$$\frac{d}{dt}((\phi_t^H)^* \omega) = (\phi_t^H)^*(d(i_{X_{H_t}} \omega) + i_{X_{H_t}} d\omega) = (\phi_t^H)^*(-ddH_t) = 0,$$

and hence $(\phi_t^H)^* \omega = \omega$ for all $t \in [0, 1]$. In particular $(\phi_1^H)^* \omega = \omega$, which means that every Hamiltonian diffeomorphism is a symplectomorphism. If H does not

depend on t it is called *autonomous*. In this case X_H also does not depend on t and hence $\phi_s^H \circ \phi_t^H = \phi_{s+t}^H$, i.e. $\{\phi_t^H\}_{t \in \mathbb{R}}$ is a 1-parameter subgroup³ of Hamiltonian diffeomorphisms. Hamiltonian flow and Hamiltonian diffeomorphism generated by an autonomous Hamiltonian are also called *autonomous*. Since for autonomous flows

$$\frac{d}{dt}(H(\phi_t^H(x))) = dH(X_H(\phi_t^H(x))) = -\omega(X_H, X_H) = 0,$$

the level sets of H are invariant under the flow of ϕ_t^H .

Example 2.2.18. Let $H : \mathbb{R}^{2n+2} \rightarrow \mathbb{R}$ be given by $H(z) = \frac{1}{2}\|z\|^2$. Now $dH = \sum_{j=1}^n (x_j dx_j + y_j dy_j)$ and hence $\omega(X_H, \cdot) = -dH$ implies that

$$X_H(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = (-y_1, \dots, -y_{n+1}, x_1, \dots, x_{n+1}).$$

Using the complex coordinates $z_j = x_j + iy_j$ we get that $X_H(z) = iz$ and hence

$$\phi_t^H(z) = e^{it}z.$$

Restricting the flow to the level $H^{-1}(\frac{1}{2}) = S^{2n+1}$, we get a free S^1 -action on S^{2n+1} , which foliates S^{2n+1} into invariant circles. Now, for every $z \in S^{2n+1}$, the symplectic complement

$$(T_z S^{2n+1})^\omega = \{v \in \mathbb{R}^{2n+2} \mid \omega(v, w) = 0 \text{ for all } w \in T_z S^{2n+1}\}$$

is spanned by $X_H(z)$. Indeed, by a linear algebra argument we can show that $(T_z S^{2n+1})^\omega$ is 1-dimensional and for every $v \in T_z S^{2n+1}$ it holds

$$\omega_0(X_H, v) = -\omega_0(v, X_H) = \omega_0(v, J_0 J_0 X_H) = \langle v, J_0 X_H \rangle = 0,$$

because $J_0 X_H = iX_H \perp S^{2n+1}$. Thus, ω induces a symplectic form on the quotient $S^{2n+1}/S^1 = \mathbb{C}P^n$. This form is called the *Fubini-Study* form⁴ and is denoted by ω_{FS} . The method of defining a symplectic form on a quotient which we used can be applied in more general contexts and is called *symplectic* or *Marsden-Weinstein reduction*.

Remark 2.2.19. As explained in Arnol'd's [12], an important invariant formulation of the equations of motion of classical mechanics involves a symplectic manifold (M, ω) , the *phase space*, and a smooth, possibly time-dependent, Hamiltonian function $H : [0, 1] \times M \rightarrow \mathbb{R}$ on M , the *total energy* of the system. The dynamics on the symplectic manifold is then described by the Hamiltonian flow of H . In this language, the fact that level sets of an autonomous Hamiltonian are invariant with respect to the Hamiltonian flow corresponds to the law of conservation of energy.

³Hamiltonian diffeomorphisms form a group as we will explain at the end of this subsection.

⁴A different normalizations might appear in the literature, i.e. the Fubini-Study form may be defined to be a multiple of the form we just defined.

Denote by $\text{Ham}(M, \omega)$ the set of all Hamiltonian diffeomorphisms of M . It turns out that $\text{Ham}(M, \omega)$ is actually a group with respect to the composition. To prove this one first proves (by a straightforward computation) the following formula for the symplectic change of coordinates:

$$\phi_* X_{H_t} = X_{H_t \circ \phi^{-1}}, \quad (2.3)$$

ϕ being any symplectomorphism. Using this property, one now proves that the Hamiltonian

$$(H \# F)_t = H_t + F_t \circ (\phi_t^H)^{-1}$$

generates the flow $\phi_t^H \circ \phi_t^F$. Time one map of this flow is $\phi_1^H \circ \phi_1^F$ and hence $\text{Ham}(M, \omega)$ is closed with respect to compositions. One proves, in a similar fashion, that if $\phi \in \text{Ham}(M, \omega)$ then $\phi^{-1} \in \text{Ham}(M, \omega)$ and hence $\text{Ham}(M, \omega)$ is a group. We leave the details of these computations to an interested reader.

Remark 2.2.20. It follows from the definitions that for any smooth function $a : [0, 1] \rightarrow [0, 1]$ with $a(0) = 0$, Hamiltonian $a'(t)H_{a(t)}$ generates the flow $\phi_{a(t)}^H$. In particular, by taking $a(t) = ct, c \in [0, 1]$ we see that for each $t \in [0, 1]$, ϕ_t^H is a Hamiltonian diffeomorphism. On the other hand, by taking $a = 0$ in a neighborhood of 0 and $a = 1$ in a neighborhood of 1, we conclude that every Hamiltonian diffeomorphism can be generated by a 1-periodic Hamiltonian function.

2.2.3 Hofer's metric

We now wish to introduce a quantitative perspective on the group of Hamiltonian diffeomorphisms. In order to do this, we will define a bi-invariant metric on $\text{Ham}(M, \omega)$ called *Hofer's metric*. The existence and non-degeneracy of Hofer's metric were major discoveries in symplectic topology in the 1990s. Today, the study of Hofer's metric and notions derived from it are some of the classical topics in symplectic topology, see [116, 119, 120] and references therein for a detailed treatment of the subject. We will use stability of barcodes coming from Floer homology with respect to Hofer's metric to study the Hofer geometry of $\text{Ham}(M, \omega)$.

Let (M, ω) be a closed⁵ symplectic manifold. In order to define Hofer's metric, we wish to consider $\text{Ham}(M, \omega)$ as an infinite-dimensional Lie group. The procedure we use is formal and applies to any Lie group. It goes as follows. Firstly, we define a norm on the Lie algebra. This norm can be extended to a Finsler metric on the Lie group in two ways, using either left or right translations. If, however, the norm on the Lie algebra was invariant with respect to the adjoint action of the group, these two Finsler metrics coincide. In this case we define lengths of curves on the group using this Finsler metric and then define the distance between two elements of the group as

⁵Non-compact case is treated similarly. We will not discuss it, since all our results about Hofer's metric concern closed manifolds.

the infimum of lengths of curves connecting them. The distance on the group defined in this way will automatically be bi-invariant.

Following this procedure, we start by describing the Lie algebra of $\text{Ham}(M, \omega)$. Let $h_t, t \in [0, \varepsilon), h_0 = \mathbb{1}_M$ be a smooth path in $\text{Ham}(M, \omega)$. By a result of Banyaga - [15], it follows that h_t is generated by a time-dependent Hamiltonian function, i.e. $h_t = \phi_t^H$ for some H . By pointwise differentiation we conclude that $\frac{d}{dt}|_{t=0} h_t$ can be identified with X_{H_0} . In other words, the Lie algebra of $\text{Ham}(M, \omega)$ consists of Hamiltonian vector fields X_{H_0} given as solutions of

$$\omega(X_{H_0}, \cdot) = -dH_0(\cdot). \quad (2.4)$$

Since dH_0 determines H_0 up to a constant, X_{H_0} also determines H_0 up to a constant. We say that a Hamiltonian $H : M \times [0, 1] \rightarrow \mathbb{R}$ is *normalized* if $\int_M H_t \omega^n = 0$ for all $t \in [0, 1]$. Now, solutions of (2.4) bijectively correspond to normalized (time-independent!) function on M . In other words, the Lie algebra of $\text{Ham}(M, \omega)$ is given by

$$\mathfrak{ham}(M, \omega) = \left\{ F : M \rightarrow \mathbb{R} \mid \int_M F \omega^n = 0 \right\}.$$

Let $F \in \mathfrak{ham}(M, \omega)$ and denote by f_t the Hamiltonian flow of F . Using the formula for symplectic change of coordinates (2.3), we see that for every $\phi \in \text{Ham}(M, \omega)$ the adjoint action is given by

$$\text{Ad}_\phi(F) = \frac{d}{dt} \Big|_{t=0} (\phi \circ f_t \circ \phi^{-1}) = X_{F \circ \phi^{-1}},$$

which after identification of vector fields and functions gives

$$\text{Ad}_\phi(F) = F \circ \phi^{-1}.$$

So, in order to obtain a bi-invariant metric on $\text{Ham}(M, \omega)$, we need a norm on $\mathfrak{ham}(M, \omega)$, i.e. on normalized functions, such that $\|F\| = \|F \circ \phi^{-1}\|$ for every $\phi \in \text{Ham}(M, \omega)$. Since ϕ preserves symplectic volume, natural choices for these norms are the L^p -norms for $p \in [1, +\infty]$. Eliashberg and Polterovich showed in [52] that for $p \in [1, +\infty)$ the metric on $\text{Ham}(M, \omega)$ obtained from the L^p -norm is identically equal to zero. However, the case of $p = +\infty$ gives a non-degenerate bi-invariant metric on $\text{Ham}(M, \omega)$ called *Hofer's metric*.

Definition 2.2.21. *Hofer's distance* between Hamiltonians F_t and G_t on $M, t \in [0, 1]$ is given by

$$\mathcal{E}(F - G) = \int_0^1 \left(\max_{x \in M} (F_t(x) - G_t(x)) - \min_{x \in M} (F_t(x) - G_t(x)) \right) dt.$$

Hofer's metric on $\text{Ham}(M, \omega)$ is given by

$$d(f, g) = \inf \mathcal{E}(F - G),$$

where the infimum runs over all the F, G such that $\phi_1^F = f, \phi_1^G = g$. Hofer's pseudo-metric on the universal cover $\widetilde{\text{Ham}}(M, \omega)$ of $\text{Ham}(M, \omega)$ is defined as

$$\tilde{d}(\tilde{f}, \tilde{g}) = \inf \mathcal{E}(F - G),$$

where the infimum runs over all the F, G such that $[\{\phi_t^F\}] = \tilde{f}, [\{\phi_t^G\}] = \tilde{g}$ in $\widetilde{\text{Ham}}(M, \omega)$.

We should note that all axioms for the definition of a metric immediately follow from the construction, except for non-degeneracy. Proving non-degeneracy of Hofer's metric turned out to be a difficult problem which was solved by a combined work of different researchers - [77, 151, 114, 88]. To this day there exists no elementary proof of this fact. While significant progress in understanding Hofer's geometry of $\text{Ham}(M, \omega)$ has been made, many questions still remain open. For example, it is not known whether the Hofer-diameter of $\text{Ham}(M, \omega)$ is infinite in general (even though it has been proven to be infinite in many cases). We discussed certain questions about Hofer's geometry in Section 1.2 and will deal with them in more detail in Chapter 3.

2.2.4 Almost complex structures

An *almost complex structure* J on a smooth manifold M is a smooth family of linear maps $J_x : T_x M \rightarrow T_x M$ such that $J_x^2 = -\mathbb{1}_{T_x M}$ for all $x \in M$. Assume now that (M, ω) is symplectic. We call an almost complex structure J *compatible with ω* if $g(v, w) := \omega(v, Jw)$ is a Riemannian metric on M .

Example 2.2.22. Recall from Examples 2.2.9 and 2.2.14 that the standard symplectic form ω_0 , the standard complex structure J_0 and the standard scalar product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^{2n} satisfy

$$(\forall v, w \in \mathbb{R}^{2n}) \langle v, w \rangle = \omega_0(v, J_0 w).$$

This means that J_0 is compatible with ω_0 .

Example 2.2.23. If M is a complex manifold, its tangent bundle is a complex vector bundle and multiplication by i induces an almost complex structure on M . If (M, ω) is also symplectic and this almost complex structure is compatible with ω , M is called *Kähler*. As explained in Example 2.2.10 the standard symplectic form ω_0 on \mathbb{R}^{2n} induces a symplectic form on the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/(2\pi\mathbb{Z})^{2n}$. Similarly, J_0 induces a complex structure on \mathbb{T}^{2n} which is compatible with the symplectic form, as in Example 2.2.22. Thus, \mathbb{T}^{2n} is a Kähler manifold. Another example of a Kähler manifold is the complex projective space with Fubini-Study form $(\mathbb{C}P^n, \omega_{FS})$ introduced in Example 2.2.18. Indeed, ω_{FS} and the complex structure on $\mathbb{C}P^n$ are inherited from ω_0 and J_0 via symplectic reduction and since J_0 is compatible with ω_0 , $(\mathbb{C}P^n, \omega_{FS})$ is Kähler.

One might show that on every symplectic manifold (M, ω) there exists an almost complex structure compatible with ω . Moreover, the space of all compatible almost

complex structures, denoted by $\mathcal{J}(M, \omega)$, is contractible, see [97]. Due to this fact, we may define the *first Chern class* of (M, ω) , $c_1(M, \omega) \in H^2(M; \mathbb{Z})$, as the first Chern class of (TM, J) for any $J \in \mathcal{J}(M, \omega)$ and the definition will not depend on a choice of J . First Chern class of (M, ω) plays an important role in Floer theory, especially in the definition of grading in Floer homology. We will now define two classes of symplectic manifolds, called *symplectically aspherical* and *symplectically monotone*, for which the definition of Floer homology is the simplest possible. Nevertheless, these two cases are sufficient for all results presented in this thesis.

Using Hurewicz homomorphism $h_* : \pi_2(M) \rightarrow H_2(M; \mathbb{Z})$ we consider ω and c_1 to be maps from $\pi_2(M)$ to \mathbb{R} .

Definition 2.2.24. Symplectic manifold (M, ω) is called *symplectically aspherical* if $\omega = c_1 = 0$ as maps from $\pi_2(M)$.

Example 2.2.25. Every symplectic manifold for which $\pi_2(M) = 0$ is symplectically aspherical. This includes $(\mathbb{R}^{2n}, \omega_0)$, surfaces of positive genus with a volume form and products of these.

Example 2.2.26. Cotangent bundle of a closed manifold with the canonical symplectic form $(T^*M, \omega_{can} = d\lambda_{can})$ is symplectically aspherical. Since $\omega_{can} = d\lambda_{can}$ it follows from Stokes' theorem that $\omega_{can}|_{\pi_2(M)} = 0$. The proof that $c_1|_{\pi_2(M)} = 0$ is more technical and we omit it here.

Definition 2.2.27. Symplectic manifold (M, ω) is called *monotone* or *symplectically monotone* if $\omega|_{\pi_2(M)} = \kappa c_1|_{\pi_2(M)}$ for some $\kappa > 0$. κ is called the *monotonicity constant*.

Example 2.2.28. Complex projective space with the Fubini-Study form is a monotone symplectic manifold. To prove this, first notice that

$$\text{im}(h_* : \pi_2(M) \rightarrow H_2(M; \mathbb{Z})) = H_2(M; \mathbb{Z}) = \mathbb{Z} \cdot [\mathbb{C}P^1],$$

where $[\mathbb{C}P^1]$ is the homology class of a submanifold

$$\{[z_0 : z_1 : 0 : \dots : 0] \mid z_0, z_1 \in \mathbb{C}\} \subset \mathbb{C}P^n.$$

Since $\omega_{FS}([\mathbb{C}P^1]) > 0$ it is enough to show that $c_1([\mathbb{C}P^1]) > 0$. Indeed, it holds

$$c_1([\mathbb{C}P^1]) = n + 1.$$

This can be shown by decomposing

$$T\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong \underbrace{(\gamma_{1,n})^* \oplus \dots \oplus (\gamma_{1,n})^*}_{n+1 \text{ times}},$$

where $\underline{\mathbb{C}}$ is a trivial complex line bundle and $(\gamma_{1,n})^*$ is the dual of the tautological line bundle over $\mathbb{C}P^n$.

Compatible almost complex structures were used by Gromov in his revolutionary work - [72] in order to develop a theory of holomorphic curves on symplectic manifolds. Let Σ be a complex curve (real dimension two) and $j : T\Sigma \rightarrow T\Sigma$, $j^2 = -\mathbb{1}$ induced from multiplication by i . Given a symplectic manifold (M, ω) with a compatible almost complex structure J , a map $u : \Sigma \rightarrow M$ which satisfies

$$du \circ j = J \circ du$$

is called a *J-holomorphic* or *pseudo-holomorphic curve*. The study of pseudo-holomorphic curves and their moduli spaces lead to a discovery of striking rigidity phenomena in symplectic topology. Even though we do not explicitly discuss Gromov's theory in this thesis, it is present in the background of our discussion of Floer theory. For an extensive treatment of the subject we refer the reader to [98].

2.2.5 The Arnol'd conjecture and Floer homology

Before we begin a detailed presentation of Floer theory we give a short historic outlook on Arnol'd's conjecture and briefly describe Floer's work and it's relation to Arnol'd's conjecture in the simplest setting of symplectically aspherical manifolds.

Fixed points of the Hamiltonian diffeomorphism $\phi = \phi_1^H$ correspond to 1-periodic orbits of the flow $\{\phi_t^H\}_{t \in [0,1]}$. In the 1960's Arnol'd has proposed a famous conjecture [10, 11] which states that, essentially, the number of fixed points of $\phi \in \text{Ham}(M, \omega)$ should satisfy the same lower bounds as the number of critical points of a smooth function f on M . The most common interpretation of this conjecture states that under the nondegeneracy⁶ assumption on H , the number of fixed points of ϕ_1^H is bounded from below by the sum of the rational Betti numbers of M . This conjecture has been a major driving force for the development of the field of symplectic topology. It was first proven in dimension 2 by Eliashberg [51], for tori of arbitrary dimension by Conley and Zehnder [41], and on complex projective spaces by Fortune and Weinstein [61, 62]. The decisive breakthrough on this question was achieved by Floer [55, 56, 58], who combined the variational methods of Conley-Zehnder and Gromov's then-recent discovery of the theory of pseudo-holomorphic curves on symplectic manifolds [72], to construct a homology theory on the loop space of M which parallels the more classical Morse homology (in turn originating in Witten's interpretation of Morse theory [161]; cf. [134]). Floer's idea for proving Arnol'd's conjecture goes as follows.

Assuming that M is symplectically aspherical, that is, $\omega|_{\pi_2(M)} = 0$, $c_1(M, \omega)|_{\pi_2(M)} = 0$, one can define the *action functional* \mathcal{A}_H on the space $\mathcal{L}_{[pt]}M$ of contractible loops on M by setting

$$\mathcal{A}_H(z) = \int_0^1 H_t(z(t)) dt - \int_{D^2} \bar{z}^* \omega,$$

⁶Meaning that $\det(\mathbb{1} - d\phi_1^H(x)) \neq 0$ for every fixed point x of ϕ_1^H , i.e., the graph of ϕ_1^H intersects the diagonal in $M \times M$ transversely.

where $z : [0, 1] \rightarrow M$, $z(0) = z(1)$ and $\bar{z} : D^2 \rightarrow M$, $\bar{z}(e^{2\pi it}) = z(t)$. Indeed by the asphericity assumption this value depends only on z , and not on \bar{z} .

Periodic orbits of the Hamiltonian flow coincide with critical points of the action functional \mathcal{A}_H , which serves as a Morse function in the construction of Floer homology. Since critical points are generators of Morse chain complexes, in analogy to Morse homology, if H is non-degenerate, Floer chain complex $CF_*(H)$ is generated by contractible periodic orbits of H . Floer homology turns out to be isomorphic to singular homology of M with rational coefficients, and hence there must be at least

$$\dim CF_*(H) \geq \dim HF_*(H) = \sum_k \dim H_k(M; \mathbb{Q}),$$

such periodic orbits in M . This solves the rational homological version of the Arnol'd conjecture. The detailed construction of Floer homology is rather involved and it has been developed in increasing generality over the years by a combined work of many people (cf. [65, 66, 111, 92, 127, 107]), in particular proving the above statement for general M (without assumptions on $\pi_2(M)$). However, various other interpretations of the conjecture are still open in general, with only partial results currently achieved (see for example [58, 128, 57, 76, 46, 104, 17]).

In the rest of this section we will focus on explaining the construction of Floer homology with \mathbb{Z}_2 -coefficients on symplectically aspherical and monotone symplectic manifolds, as well as how it fits in the framework of persistence modules.

2.2.6 Conley-Zehnder index

We now briefly discuss Conley-Zehnder index of a path of symplectic matrices. In Floer theory, this index (up to constants and signs) plays the role of the Morse index and is used to define grading on Floer homology. For a detailed exposition of the subject, see [41, 131, 126].

Definition 2.2.29. Let $A : [0, 1] \rightarrow \mathrm{Sp}(2n)$ be a path of symplectic matrices such that $A(0) = \mathbb{1}$, $\det(A(1) - \mathbb{1}) \neq 0$. $t_* \in [0, 1]$ is called a *crossing* if $\det(A(t_*) - \mathbb{1}) = 0$. *Crossing form* $\Gamma(A, t_*) : \ker(A(t_*) - \mathbb{1}) \rightarrow \mathbb{R}$ is a quadratic form given by

$$\Gamma(A, t_*)v = \omega_0(v, \dot{A}(t_*)v). \quad (2.5)$$

Crossing t_* is called *regular* if $\Gamma(A, t_*)$ is non-degenerate and A is a *regular path* if all its crossings are regular.

Crossing form is symmetric and we may compute its matrix in the following way. Recall from Example 2.2.14 that since $A(t) \in \mathrm{Sp}(2n)$ it holds $A^T(t)J_0A(t) = J_0$. Differentiating this expression with respect to t implies that $S(t) = -J_0\dot{A}(t)A^{-1}(t)$ is symmetric for every $t \in [0, 1]$ and hence

$$\dot{A}(t) = J_0S(t)A(t), \quad (2.6)$$

where S is a path of symmetric matrices. Using this equality we have that for $v \in \ker(A(t_*) - \mathbb{1})$ it holds

$$\Gamma(A, t_*)v = \omega_0(v, \dot{A}(t_*)v) = \omega_0(v, J_0 S(t_*)A(t_*)v) = \omega_0(v, J_0 S(t_*)v) = \langle v, S(t_*)v \rangle,$$

and hence $S(t_*)$ is the symmetric matrix which represents the crossing form $\Gamma(A, t_*)$.

One proves that regular crossings are isolated and hence a regular path has only finitely many crossings. Thus, we may give the following definition.

Definition 2.2.30. The *Conley-Zehnder index* of a regular path $A : [0, 1] \rightarrow \text{Sp}(2n)$ is defined as

$$\text{ind}_{\text{CZ}}(A) = \frac{1}{2} \text{sgn} \Gamma(A, 0) + \sum_{t_*\text{-crossing}} \text{sgn} \Gamma(A, t_*).$$

Here $\text{sgn} \Gamma(A, t) = \text{sgn} S(t)$ denotes the signature of a symmetric form, i.e. the difference between the number of positive and negative eigenvalues.

Example 2.2.31. Let $a \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $z_j = x_j + iy_j$ and define $A : [0, 1] \rightarrow \text{Sp}(2n)$ by

$$A(t)v = e^{ait}v, \text{ for every } v \in \mathbb{R}^{2n}.$$

We see that $\ker(A(t) - \mathbb{1}) \neq 0$ when $at \in 2\pi\mathbb{Z}$ and hence the crossings of A are $\frac{2\pi}{a}\mathbb{Z} \cap [0, 1]$. For every crossing $t_* \in \frac{2\pi}{a}\mathbb{Z} \cap [0, 1]$ we have that $A(t_*) = \mathbb{1}$ and hence $\ker(A(t_*) - \mathbb{1}) = \mathbb{R}^{2n}$. Since

$$S(t_*) = -J_0 \dot{A}(t_*)A^{-1}(t_*) \text{ and } \dot{A}(t_*) = a \cdot J_0 A(t_*),$$

it follows that $S(t_*) = a \cdot \mathbb{1}$. Thus, for every crossing t_* it holds

$$\text{sgn} \Gamma(A, t_*) = \text{sgn}(a \cdot \mathbb{1}) = 2n \cdot \text{sign}(a).$$

This finally gives us

$$\text{ind}_{\text{CZ}}(A) = \text{sign}(a) \left(n + 2n \left\lfloor \frac{|a|}{2\pi} \right\rfloor \right),$$

where $\lfloor \frac{|a|}{2\pi} \rfloor$ denotes the largest integer not greater than $\frac{|a|}{2\pi}$.

Example 2.2.32. Previous example naturally appears when considering the following autonomous Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$:

$$H(z) = a \cdot \|z\|^2, \quad a \in \mathbb{R} \setminus \pi\mathbb{Z}.$$

By definition, the Hamiltonian vector field X_H is computed from $\omega_0(X_H, \cdot) = -dH$ to be

$$X_H(x, y) = 2a \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right).$$

In complex coordinates $X_H(z) = 2aiz$ and the Hamiltonian flow of H is given by

$$\phi_t^H(z) = e^{2ait}z.$$

The origin is a critical point of H and hence also a fixed point of ϕ_t^H . By identifying $T_0\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ we obtain

$$d\phi_t^H(0) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}, \quad d\phi_t^H(0)v = e^{2ait}v.$$

The previous example gives us

$$\text{ind}_{\text{CZ}}(d\phi_t^H(0)) = \text{sign}(a) \left(n + 2n \left\lfloor \frac{|a|}{\pi} \right\rfloor \right).$$

Now, if $|a| < \pi$ it follows that

$$\text{ind}_{\text{CZ}}(d\phi_t^H(0)) = n \cdot \text{sign}(a) = n - \text{ind}(0),$$

where $\text{ind}(0)$ denotes the Morse index of 0 as a critical point of H .

Let $A, B : [0, 1] \rightarrow \text{Sp}(2n)$ be two regular paths, $A(0) = B(0) = \mathbb{1}$, $A(1) = B(1)$ and $\det(A(1) - \mathbb{1}) = \det(B(1) - \mathbb{1}) \neq 0$. One may show that if A and B are homotopic relative to the endpoints then $\text{ind}_{\text{CZ}}(A) = \text{ind}_{\text{CZ}}(B)$. Moreover, every path $C : [0, 1] \rightarrow \text{Sp}(2n)$ s.t. $C(0) = \mathbb{1}, \det(C(1) - \mathbb{1}) \neq 0$ is homotopic relative to the endpoints to a regular path and hence the definition of the Conley-Zehnder index extends to all such paths. This and many other properties of the Conley-Zehnder index, as well as generalizations and relations to other indices (such as the Maslov index) are discussed in detail in [41, 131, 126]. Here, we only state one such property in the lemma that follows, in order to clarify our definition of grading in Floer homology in the next subsection.

Lemma 2.2.33. *Let (M, ω) be a closed symplectic manifold and $H : M \rightarrow \mathbb{R}$ a C^2 -small function, which is non-degenerate as a Hamiltonian (this implies that H is Morse). For every $z \in \text{Crit}(H)$ it holds*

$$\text{ind}(z) = n - \text{ind}_{\text{CZ}}(d\phi_t^H(z)),$$

where $\text{ind}(z)$ denotes the Morse index of z and ind_{CZ} is computed with respect to a symplectic trivialization $(T_zM, \omega) \rightarrow (\mathbb{R}^{2n}, \omega_0)$.

Example 2.2.32 is a special case of this lemma. Indeed, the statement of the lemma clearly localizes to neighborhoods of critical points, i.e. the global structure of M plays no role in the lemma.

2.2.7 Floer homology in the symplectically aspherical case

We will now present the construction of Floer homology with \mathbb{Z}_2 -coefficients on a closed symplectically aspherical manifold. Even though this is the simplest case, the details of the construction are still technically involved and greatly surpass the scope of this thesis. Our treatment will focus on describing the structure and properties of

Floer homology, having persistence perspective in mind. Hence, most of the proofs will be omitted and some statements will be taken as “black boxes”. We refer the reader to [98, 13, 129] for detailed treatments of the theory. Expositions more similar to ours and well suited to our context can be found in [119, 120].

Let (M, ω) be a $2n$ -dimensional, closed, symplectically aspherical manifold and $H : M \times [0, 1] \rightarrow \mathbb{R}$ a time-dependent Hamiltonian. We will assume that $H_{t+1} = H_t$, i.e. $H : M \times S^1 \rightarrow \mathbb{R}$, which is inessential due to Remark 2.2.20. As before, denote by ϕ_t^H the Hamiltonian flow and by ϕ_1^H the Hamiltonian diffeomorphism generated by H . Periodic orbits⁷ of the Hamiltonian flow

$$z : S^1 \rightarrow M, \quad \dot{z}(t) = \phi_t^H(z(0)),$$

correspond bijectively to fixed points of ϕ_1^H via $z \rightarrow z(0)$. We call a periodic orbit z and a fixed point $z(0)$ *non-degenerate* if $d\phi_1^H : T_{z(0)}M \rightarrow T_{z(0)}M$ does not have 1 as an eigenvalue. Hamiltonian H and diffeomorphism ϕ_1^H are called *non-degenerate* if all periodic orbits of the corresponding flow are non-degenerate. Equivalently, H is non-degenerate if $\text{graph}(\phi_1^H) \subset M \times M$ is transversal to the diagonal $\Delta \subset M \times M$. From compactness of M it follows that the Hamiltonian flow of a non-degenerate H has only finitely many periodic orbits.

Denote by $\mathcal{L}_{[pt]}M$ the space of contractible loops in M and define the *action functional*

$$\mathcal{A}_H : \mathcal{L}_{[pt]}M \rightarrow \mathbb{R}, \quad \mathcal{A}_H(z) = \int_0^1 H_t(z(t))dt - \int_{D^2} \bar{z}^* \omega,$$

where $\bar{z} : D^2 \rightarrow M$, $\bar{z}|_{S^1} = z$ is a capping disc of z . Since $\omega|_{\pi_2(M)} = 0$ it follows that the value of $\mathcal{A}_H(z)$ does not depend on the choice of a capping disc. Action functional plays the role of a Morse function in our construction. Let us formally calculate the differential of \mathcal{A}_H . A curve $u : (-\varepsilon, \varepsilon) \rightarrow \mathcal{L}_{[pt]}M$ can be thought of as a map $u : (-\varepsilon, \varepsilon) \times S^1 \rightarrow M$. We always denote the parameter of u as a path in $\mathcal{L}_{[pt]}M$ by s , $s \in (-\varepsilon, \varepsilon)$ and the loop parameter by t , $t \in S^1$. Now, if $z(t) = u(0, t)$, we have that $\frac{\partial u}{\partial s}(0, t) \in T_{z(t)}M$ is a vector field along z and we identify $T_z(\mathcal{L}_{[pt]}M)$ with the space of vector fields along z . Let $\xi \in T_z(\mathcal{L}_{[pt]}M)$ and take u such that $\frac{\partial u}{\partial s}(0, t) = \xi(t)$. We compute

$$d\mathcal{A}_H(z)(\xi) = \frac{d}{ds}\Big|_{s=0} \mathcal{A}_H(u) = \int_0^1 \frac{d}{ds}\Big|_{s=0} H_t(u(s, t))dt - \int_{D^2} \frac{d}{ds}\Big|_{s=0} \bar{u}_s^* \omega,$$

where \bar{u}_s denotes a capping disc of $u(s, \cdot)$. A short computation using Cartan’s formula and Stokes’ theorem shows that

$$\int_{D^2} \frac{d}{ds}\Big|_{s=0} \bar{u}_s^* \omega = \int_0^1 \omega(\xi(t), \dot{z}(t))dt.$$

From the definition of the Hamiltonian vector field X_H , we get

$$d\mathcal{A}_H(z)(\xi) = \int_0^1 \omega(\xi(t), X_H(z(t)) - \dot{z}(t))dt. \tag{2.7}$$

⁷By a periodic orbit we mean a 1-periodic orbit.

Since ω is non-degenerate, it follows from (2.7) that $d\mathcal{A}_H(z) = 0$ if and only if $\dot{z}(t) = X_H(z(t))$. In other words, critical points of \mathcal{A}_H are periodic orbits of the flow of H and we denote the set of these orbits by $\text{Crit}(\mathcal{A}_H)$. As in the Morse case, elements of $\text{Crit}(\mathcal{A}_H)$ will be generators of the Floer chain complex. Let us now define the analogue of the Morse index in Floer theory.

Let $z \in \text{Crit}(\mathcal{A}_H)$ and let $\bar{z} : D^2 \rightarrow M$ be a capping disc of z . Let

$$\Psi : (\bar{z}^*TM, \bar{z}^*\omega) \rightarrow D^2 \times (\mathbb{R}^{2n}, \omega_0),$$

be a symplectic trivialization, which exists since D^2 is contractible. Denoting by Ψ_t the restriction of Ψ to $T_{z(t)}M$, we obtain a path of symplectic matrices

$$A : [0, 1] \rightarrow \text{Sp}(2n), \quad A(t) = \Psi_t \circ d\phi_t^H \circ \Psi_0^{-1}.$$

It follows from properties of the Conley-Zehnder index that $\text{ind}_{\text{CZ}}(A)$ does not depend on the choice of a trivialization Ψ . Hence, we may define $\text{ind}_{\text{CZ}}(\bar{z})$ as well as

$$\text{ind}(\bar{z}) = n - \text{ind}_{\text{CZ}}(\bar{z}).$$

To see that $\text{ind}(\bar{z})$ only depends on the orbit z and not on the capping, we use the following relation between the Conley-Zehnder index and the first Chern class, see [97]. Let \bar{z} and \bar{z}' be two discs capping z . It holds

$$\text{ind}_{\text{CZ}}(\bar{z}) - \text{ind}_{\text{CZ}}(\bar{z}') = 2c_1(\bar{z}\#(-\bar{z}')), \quad (2.8)$$

where $\#$ denotes gluing of discs along the boundary. Since our manifold is symplectically aspherical, $c_1(\bar{z}\#(-\bar{z}')) = 0$ and we may define $\text{ind}_{\text{CZ}}(z)$ as well as

$$\text{ind}(z) = n - \text{ind}_{\text{CZ}}(z).$$

Now, we wish to consider negative gradient flow lines of \mathcal{A}_H . To this end, let $\{J_t\}_{t \in S^1}$ be an S^1 -family of almost complex structures compatible with ω . For $z \in \mathcal{L}_{[pt]}M$ define a bilinear form on $T_z\mathcal{L}_{[pt]}M$ by

$$\langle \xi(t), \eta(t) \rangle = \int_0^1 \omega(\xi(t), J_t \eta(t)) dt, \quad \xi, \eta \in T_z\mathcal{L}_{[pt]}M.$$

This bilinear form will play the role of a Riemannian metric on $\mathcal{L}_{[pt]}M$. The gradient of \mathcal{A}_H is defined by

$$d\mathcal{A}_H(z)(\xi) = \langle \xi, \nabla \mathcal{A}_H \rangle$$

and using (2.7) we obtain

$$\nabla \mathcal{A}_H(z(t)) = J_t(\dot{z}(t) - X_H(z(t))). \quad (2.9)$$

By looking at $u : \mathbb{R} \rightarrow \mathcal{L}_{[pt]}M$ as a map $u : \mathbb{R} \times S^1 \rightarrow M$, the negative gradient flow equation of \mathcal{A}_H becomes the following first-order PDE

$$\frac{\partial u}{\partial s} + J_t \left(\frac{\partial u}{\partial t} - X_H(u(s, t)) \right) = 0. \quad (2.10)$$

This equation is called *Floer equation*. In general, the flow of Floer equation on $\mathcal{L}_{[pt]}M$ fails to be defined even for short times, see [119] for an explicit example. This prevents us from directly applying Morse-theoretic techniques to \mathcal{A}_H . However, a revolutionary idea, due to Floer, is to consider the solutions of the following boundary value problem

$$u : \mathbb{R} \times S^1 \rightarrow M, \quad u(-\infty, t) = z_-(t), \quad u(+\infty, t) = z_+(t), \quad u \text{ satisfies (2.10)}, \quad (2.11)$$

where $z_-, z_+ \in \text{Crit}(\mathcal{A}_H)$ and $u(\pm\infty, \cdot)$ denotes the uniform limit. Define the *space of Floer trajectories*

$$\mathcal{M}(z_-, z_+) = \{u : \mathbb{R} \times S^1 \rightarrow M \mid u \text{ solves (2.11)}\}.$$

The following theorem is one of the first major results of Floer theory.

Theorem 2.2.34. *Let (M, ω) be a closed, symplectically aspherical manifold and $H : M \times S^1 \rightarrow \mathbb{R}$ a non-degenerate Hamiltonian. For a generic choice of $\{J_t \in \mathcal{J}(M, \omega)\}_{t \in S^1}$ the following holds. For any $z_-, z_+ \in \text{Crit}(\mathcal{A}_H)$ the space $\mathcal{M}(z_-, z_+)$ has a structure of a smooth manifold of dimension $\text{ind}(z_-) - \text{ind}(z_+)$.*

Remark 2.2.35. Let us comment on our use of the word “generic” and the definition of a smooth manifold structure on $\mathcal{M}(z_-, z_+)$. Both of these come from the use of Sard-Smale transversality theorem for Fredholm maps of Banach manifolds. Namely, $\mathcal{M}(z_-, z_+)$ arise as inverse images of zero⁸ under the map

$$u \rightarrow \frac{\partial u}{\partial s} + J_t \left(\frac{\partial u}{\partial t} - X_H(u(s, t)) \right),$$

while $\mathcal{J}(M, \omega)$ serves as a space of parameters used to achieve transversality.

Note that if $u \in \mathcal{M}(z_-, z_+)$ and $s_0 \in \mathbb{R}$ then $s_0 \cdot u = u(s_0 + \cdot, \cdot)$ also belongs to $\mathcal{M}(z_-, z_+)$ since Floer equation is invariant under translations of the s -parameter. If $z_- \neq z_+$, these translations define a free \mathbb{R} -action on $\mathcal{M}(z_-, z_+)$ and we denote by $\bar{\mathcal{M}}(z_-, z_+) = \mathcal{M}(z_-, z_+)/\mathbb{R}$. This space is a smooth manifold of dimension $\text{ind}(z_-) - \text{ind}(z_+) - 1$. Following the analogy with the Morse case, the differential should count (modulo two) the elements of $\bar{\mathcal{M}}(z_-, z_+)$ for $\text{ind}(z_-) = \text{ind}(z_+) + 1$. In order for this to make sense, $\bar{\mathcal{M}}(z_-, z_+)$ should be a finite set. Since it is a 0-dimensional manifold, this is equivalent to it being compact. This turns out to be true, see Remark 2.2.36. We are now in a position to define Floer homology.

For $k \in \mathbb{Z}$ denote by $\text{Crit}_k(\mathcal{A}_H) = \{z \in \text{Crit}(\mathcal{A}_H) \mid \text{ind}(z) = k\}$ and by $CF_k(H, J) = \text{Span}_{\mathbb{Z}_2}(\text{Crit}_k(\mathcal{A}_H))$. Define the differential

$$\partial : CF_k(H, J) \rightarrow CF_{k-1}(H, J)$$

as a \mathbb{Z}_2 -linear map whose value on a generator $z_- \in \text{Crit}_k(\mathcal{A}_H)$ is given by

$$\partial(z_-) = \sum_{z_+ \in \text{Crit}_{k-1}(\mathcal{A}_H)} \#_2 \bar{\mathcal{M}}(z_-, z_+) z_+,$$

⁸More precisely of the zero section of a certain Banach bundle.

where $\#_2$ denotes cardinality modulo 2. One may prove that $\partial^2 = 0$, see Remark 2.2.36, and hence $(CF(H, J), \partial)$ is a chain complex over \mathbb{Z}_2 , called *Floer chain complex*. The homology of this complex is called *Floer homology* of (H, J) and is denoted by $HF_*(H, J)$. A generic pair (H, J) for which the presented construction of $HF_*(H, J)$ works is called *regular*. It turns out that $HF_*(H, J)$ do not depend on a choice of a regular pair (H, J) . Moreover, it holds

$$HF_*(H, J) \cong H_*(M; \mathbb{Z}_2),$$

where H_* denotes singular homology. Let us sketch the construction of these isomorphisms.

Isomorphisms between Floer homologies of different pairs (H, J) are constructed as follows. Let $F, G : M \times S^1 \rightarrow \mathbb{R}$ be two non-degenerate Hamiltonians and $J^{(1)}, J^{(2)} : S^1 \rightarrow \mathcal{J}(M, \omega)$ such that pairs $(F, J^{(1)})$, $(G, J^{(2)})$ are regular. Let $H : \mathbb{R} \times S^1 \times M \rightarrow \mathbb{R}$ and $J : \mathbb{R} \times S^1 \rightarrow \mathcal{J}(M, \omega)$ be homotopies between F, G and $J^{(1)}, J^{(2)}$ respectively, i.e.

$$H(s, t, x) = \begin{cases} F(t, x), & \text{for } s \leq -1 \\ G(t, x), & \text{for } s \geq 1 \end{cases} \quad J_{s,t} = \begin{cases} J_t^{(1)}, & \text{for } s \leq -1 \\ J_t^{(2)}, & \text{for } s \geq 1 \end{cases}.$$

Consider the parametric version of the Floer equation given as follows

$$\frac{\partial u}{\partial s} + J_{s,t} \left(\frac{\partial u}{\partial t} - X_{H_s}(u(s, t)) \right) = 0. \quad (2.12)$$

Let $z_- \in \text{Crit}(\mathcal{A}_F)$, $z_+ \in \text{Crit}(\mathcal{A}_G)$ and consider the following boundary value problem

$$u : \mathbb{R} \times S^1 \rightarrow M, \quad u(-\infty, t) = z_-(t), \quad u(+\infty, t) = z_+(t), \quad u \text{ satisfies (2.12)}. \quad (2.13)$$

Similarly to Theorem 2.2.34, for a generic homotopy (H, J) each of the spaces

$$\mathcal{M}^{par}(z_-, z_+) = \{u : \mathbb{R} \times S^1 \rightarrow M \mid u \text{ solves (2.13)}\}$$

can be given a structure of a smooth manifold of dimension $\text{ind}(z_-) - \text{ind}(z_+)$. Since (2.12) is not invariant under the translations of the s -parameter, there is no \mathbb{R} -action on \mathcal{M}^{par} as in the case of \mathcal{M} . However, if we take $\text{ind}(z_-) = \text{ind}(z_+)$, $\mathcal{M}^{par}(z_-, z_+)$ is compact, i.e. it consists of finitely many points. Thus, for every $k \in \mathbb{Z}$, we may define a \mathbb{Z}_2 -linear map

$$C_{chain} : CF_k(F, J^{(1)}) \rightarrow CF_k(G, J^{(2)})$$

given on $z_- \in \text{Crit}_k(\mathcal{A}_F)$ by

$$C_{chain}(z_-) = \sum_{z_+ \in \text{Crit}_k(\mathcal{A}_G)} \#_2 \bar{\mathcal{M}}^{par}(z_-, z_+) z_+.$$

This is a chain map and hence it induces a map on HF , called *continuation map*, which we denote by $C(F, G)$. Continuation maps do not depend on a generic choice of a homotopy (H, J) , but rather on the regular pairs $(F, J^{(1)})$, $(G, J^{(2)})$, see Remark

2.2.36. Now, if $(F, J^{(1)}) = (G, J^{(2)})$ then $C(F, G) = \mathbb{1}$, which can be proven by taking a constant homotopy. Moreover, for any three regular pairs $(H^{(i)}, J^{(i)})_{i=1,2,3}$ the following diagram commutes

$$\begin{array}{ccc}
 HF_*(H^{(1)}, J^{(1)}) & \xrightarrow{C(H^{(1)}, H^{(3)})} & HF_*(H^{(3)}, J^{(3)}) \\
 & \searrow^{C(H^{(1)}, H^{(2)})} & \nearrow_{C(H^{(2)}, H^{(3)})} \\
 & HF_*(H^{(2)}, J^{(2)}) &
 \end{array} \tag{2.14}$$

By taking $(H^{(1)}, J^{(1)}) = (H^{(3)}, J^{(3)})$ we conclude that continuation maps are isomorphisms and hence Floer homologies of different regular pairs are isomorphic. In other words, we may define $HF_*(M, \omega)$.

Finally, to see that $HF_*(M, \omega) \cong H_*(M; \mathbb{Z}_2)$ one should choose H to be a C^2 -small function and $J_t \equiv J$ not dependent on t . One proves that in this case the only fixed points of ϕ_1^H are critical points of H , non-degeneracy of H as a Hamiltonian is equivalent to it being Morse and the above defined index coincides with the Morse index (see Lemma 2.2.33). Now, $CF_*(H, J)$ essentially degenerates into the Morse complex $CM_*(H, \omega(\cdot, J\cdot))$ and since Morse homology is isomorphic to singular homology, the claim follows. The precise statements and details of the proof can be found in [13].

Remark 2.2.36. Statements that $\bar{\mathcal{M}}(z_-, z_+)$, $\bar{\mathcal{M}}^{par}(z_-, z_+)$ are compact when they are 0-dimensional, that $\partial^2 = 0$, as well as that C_{chain} is a chain map and that C is independent of the choice of a homotopy are similar in nature. Namely, their proofs follow from analysing compactifications of spaces of Floer trajectories such as \mathcal{M} and \mathcal{M}^{par} (we already used this idea to prove that $\partial^2 = 0$ in the Morse case, see the proof of Theorem 2.2.4). Assuming that (M, ω) is symplectically aspherical, the compactifications appearing in the proofs of these statements can be given structures of manifolds with boundaries. If non-empty, their boundaries consist of the so-called ‘‘broken Floer trajectories’’. The main technical ingredients used to construct these compactifications are Gromov’s compactness theorem and Floer’s gluing technique. In general, the boundary of a space of Floer trajectories might be much more complicated due to the possible appearance of pseudo-holomorphic bubbles. This is excluded in our case by the asphericity assumption. For a detailed discussion, we refer the reader to [98] and references therein.

Remark 2.2.37. The construction of Floer homology described here can be applied to some non-compact symplectically aspherical manifolds, such as cotangent bundles. In this case, we assume that, outside of a compact set, H is linear and J has a particularly simple form. This guarantees that all Hamiltonian periodic orbits, as well as all Floer trajectories remain inside a compact set and the construction of HF_* follows as in the closed case. We should note that in this setup $HF_*(H, J)$ depends on the slope of H at infinity. In order to define invariants independent of (H, J) one usually takes a limit of HF_* over a certain set of Hamiltonians. An invariant defined in

this way, called *symplectic homology*, plays a crucial role in our work [144], presented in Chapter 4. In Chapter 4, we elaborate on the definition of Floer homology of cotangent bundles and describe construction and properties of (filtered) symplectic homology.

2.2.8 Morse and Floer persistence modules

In this subsection we will define Morse and Floer persistence modules and prove stability of respective barcodes with respect to d_{C^0} and Hofer's metric. Let us start by recalling the example of a persistence module considered in Section 2.1.1. Let N be a closed manifold and $f : N \rightarrow \mathbb{R}$ a Morse function. For each $k \in \mathbb{Z}$ we defined a persistence module $V_k(f)$ as

$$V_k^t(f) = H_k(\{f < t\}; \mathbb{K}), \quad \pi_{s,t} \text{ induced by inclusions } \{f < s\} \subset \{f < t\} \text{ for } s \leq t.$$

We can give an alternative definition of $V_k^t(f)$, using the Morse chain complex of f , as follows. Let $\mathbb{K} = \mathbb{Z}_2$ and let g be a Riemannian metric such that (f, g) is a regular pair. For $t \in \mathbb{R}$, define the *filtered Morse complex in degree k*

$$CM_k^t(f, g) = \text{Span}_{\mathbb{Z}_2}(\{x \in \text{Crit}_k(f) \mid f(x) < t\}).$$

Now, notice that f decreases along the negative gradient flow lines. Indeed, if $x, y \in \text{Crit}(f)$, $\gamma \in \mathcal{M}(x, y)$, we have that

$$\frac{d}{dt}(f(\gamma(t))) = df(\gamma(t))(\dot{\gamma}(t)) = -\|\nabla f(\gamma(t))\|_g^2 \leq 0,$$

which after integration yields

$$f(x) - f(y) = \int_{-\infty}^{+\infty} \|\nabla f(\gamma(t))\|_g^2 dt \leq 0. \quad (2.15)$$

This implies that ∂ maps $CM_k^t(f, g)$ to $CM_{k-1}^t(f, g)$ and we define *filtered Morse homology* $HM_*^t(f, g)$ as homology of the chain complex $(CM_*^t(f, g), \partial)$. Notice that for $s \leq t$ it holds $CM_*^s(f, g) \subset CM_*^t(f, g)$ and these inclusion induce maps on homology

$$\pi_{s,t}^{HM} : HM_*^s(f, g) \rightarrow HM_*^t(f, g).$$

Together with these maps filtered Morse homology is a persistence module called *Morse persistence module* of (f, g) . It is a standard result of Morse theory that $(V_k(f), \pi)$ and $(HM_k(f, g), \pi^{HM})$ are isomorphic persistence modules for every degree $k \in \mathbb{Z}$. Thus, Theorem 2.1.17 implies that

$$d_{\text{bottle}}(\mathcal{B}(HM_k(f_1, g_1)), \mathcal{B}(HM_k(f_2, g_2))) \leq d_{C^0}(f_1, f_2),$$

for any two regular pairs $(f_1, g_1), (f_2, g_2)$ and any degree k .

Remark 2.2.38. From the definition of Morse persistence module it immediately follows that the endpoints of bars in $\mathcal{B}(HM_*(f, g))$ are equal to certain critical values of f . Since $HM_*(f, g) \cong V_*(f)$, we know from Lemma 2.1.10 that in fact every critical value appears as an endpoint of a bar. One can also prove this fact by applying the same arguments as in the proof of Lemma 2.1.10 to the Morse chain complex, without relying on an isomorphism with $V_*(f)$.

The construction of Morse persistence module translates to Floer homology in a straightforward manner. Let (M, ω) be a closed, symplectically aspherical manifold, $H : M \times S^1 \rightarrow \mathbb{R}$ a non-degenerate Hamiltonian and $\{J_t\}_{t \in S^1}$ a family of compatible almost complex structure such that (H, J) is regular. For $k \in \mathbb{Z}, t \in \mathbb{R}$ define *filtered Floer complex in degree k*

$$CF_k^t(H, J) = \text{Span}_{\mathbb{Z}_2}(\{z \in \text{Crit}_k(\mathcal{A}_H) \mid \mathcal{A}_H(z) < t\}).$$

A computation similar to the one in the Morse case shows that for $z_-, z_+ \in \text{Crit}(\mathcal{A}_H)$, $u \in \mathcal{M}(z_-, z_+)$, it holds

$$\mathcal{A}_H(z_-) - \mathcal{A}_H(z_+) = \int_{-\infty}^{+\infty} \int_0^1 \left\| \frac{\partial u}{\partial s}(s, t) \right\|_{J_t}^2 dt ds, \quad (2.16)$$

where $\|\cdot\|_{J_t}$ denotes the norm induced by the metric $\omega(\cdot, J_t \cdot)$. Again, as in the Morse case, (2.16) implies that

$$\partial : CF_k^t(H, J) \rightarrow CF_{k-1}^t(H, J),$$

and we define *filtered Floer homology* $HF_*^t(H, J)$ as the homology of the chain complex $(CF_*^t(H, J), \partial)$. Inclusions $CF_*^s(H, J) \subset CF_*^t(H, J)$ for $s \leq t$ induce maps on homology

$$\pi_{s,t} : HF_*^s(H, J) \rightarrow HF_*^t(H, J).$$

Taking these maps as structure maps we obtain *Floer persistence module of (H, J)* , $(HF_*(H, J), \pi)$.

Remark 2.2.39. As in the Morse case, one readily sees that the endpoints of bars in $\mathcal{B}(HF_*(H, J))$ are equal to $\mathcal{A}_H(z)$ for some $z \in \text{Crit}(\mathcal{A}_H)$. Moreover, by adapting, in a straightforward manner, the proof of Lemma 2.1.10 one proves an analogous statement for Floer persistence modules.

Let us now focus on proving the stability of Floer persistence modules with respect to Hofer's metric. Recall from Subsection 2.2.3 that for two Hamiltonians $F, G : M \times S^1 \rightarrow \mathbb{R}$

$$\mathcal{E}(F - G) = \int_0^1 \left(\max_{x \in M} (F_t(x) - G_t(x)) - \min_{x \in M} (F_t(x) - G_t(x)) \right) dt.$$

and that Hofer's metric between $f = \phi_1^F$ and $g = \phi_1^G$ is given by

$$d(f, g) = \inf \mathcal{E}(F - G),$$

where infimum runs over all Hamiltonians F, G which generate f, g . Note that adding a constant $c \in \mathbb{R}$ to the Hamiltonian F does not change the flow ϕ_t^F nor the value of $\mathcal{E}(F - G)$. However, the action functional \mathcal{A}_F shifts by c , which results in a shift by c of the Floer persistence module. In order to deal with this ambiguity, we will consider normalized Hamiltonians. Recall from Subsection 2.2.3 that a Hamiltonian H is called normalized if $\int_M H_t \omega^{\wedge n} = 0$ for all $t \in S^1$. In order to prove stability of barcodes with respect to d , we will need the following lemma, which is a parametric version of (2.16). Its proof is similar to the proofs of (2.15) and (2.16).

Lemma 2.2.40. *Let $(F, J^{(1)}), (G, J^{(2)})$ be two regular pair and (H, J) a homotopy between them. For $z_- \in \text{Crit}(\mathcal{A}_F)$, $z_+ \in \text{Crit}(\mathcal{A}_G)$ and $u \in \mathcal{M}^{\text{par}}(z_-, z_+)$ it holds*

$$\mathcal{A}_F(z_-) - \mathcal{A}_G(z_+) = \int_{-\infty}^{+\infty} \int_0^1 \left\| \frac{\partial u}{\partial s}(s, t) \right\|_{J_t}^2 dt ds - \int_{-\infty}^{+\infty} \int_0^1 \frac{\partial H}{\partial s}(u(s, t)) dt ds.$$

To prove stability, we use continuation maps as interleavings between Floer persistence modules. Lemma 2.2.40 gives an estimate of the size of such an interleaving. More precisely, we have two following corollary.

Corollary 2.2.41. *Let F and G be two non-degenerate, normalized Hamiltonians and $J^{(1)}, J^{(2)}$ such that $(F, J^{(1)}), (G, J^{(2)})$ are regular. For every $k \in \mathbb{Z}$, Floer persistence modules $HF_k(F, J^{(1)})$ and $HF_k(G, J^{(2)})$ are $\mathcal{E}(F - G)$ -interleaved.*

Proof. Since $\int_{-\infty}^{+\infty} \int_0^1 \left\| \frac{\partial u}{\partial s}(s, t) \right\|_{J_t}^2 \geq 0$, Lemma 2.2.40 implies that for any homotopy (H, J) between $(F, J^{(1)})$ and $(G, J^{(2)})$, $z_- \in \text{Crit}(\mathcal{A}_F)$, $z_+ \in \text{Crit}(\mathcal{A}_G)$ and $u \in \mathcal{M}^{\text{par}}(z_-, z_+)$, it holds

$$\mathcal{A}_G(z_+) \leq \mathcal{A}_F(z_-) + \int_{-\infty}^{+\infty} \int_0^1 \frac{\partial H}{\partial s}(u(s, t)) dt ds. \quad (2.17)$$

Let $\beta : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\beta|_{(-\infty, -1]} \equiv 0$, $\beta|_{[1, +\infty)} \equiv 1$ and β is increasing on $[-1, 1]$. Now consider a homotopy

$$H_{s,t}(x) = (1 - \beta(s))F_t(x) + \beta(s)G_t(x).$$

We have that

$$\frac{\partial H}{\partial s}(x) = \beta'(s)(G_t(x) - F_t(x)) \leq \beta'(s) \max_{x \in M} (G_t(x) - F_t(x)) = -\beta'(s) \min_{x \in M} (F_t(x) - G_t(x)),$$

which together with (2.17) implies

$$\mathcal{A}_G(z_+) \leq \mathcal{A}_F(z_-) - \int_0^1 \min_{x \in M} (F_t(x) - G_t(x)) dt. \quad (2.18)$$

Since F and G are normalized we have that $\max_{x \in M} (G_t(x) - F_t(x)) \geq 0$, which together with (2.18) gives us

$$\mathcal{A}_G(z_+) \leq \mathcal{A}_F(z_-) + \mathcal{E}(F - G).$$

Now, as in the case of continuation maps, after choosing a generic J , one defines for every $t \in \mathbb{R}$ a map

$$C_{chain} : CF_*^t(F, J^{(1)}) \rightarrow CF_*^{t+\mathcal{E}(F-G)}(H, J^{(2)})$$

by counting, modulo two, the solutions of the equation (2.12). Adapting the arguments in the definition of continuation maps to the filtered setting yields

$$C(F, G) : HF_*^t(F, J^{(1)}) \rightarrow HF_*^{t+\mathcal{E}(F-G)}(H, J^{(2)}).$$

Since C_{chain} clearly commutes with inclusions of subcomplexes $CF^s \subset CF^t$ for $s \leq t$, it follows that $C(F, G)$ is a morphism of the following persistence modules

$$C(F, G) : HF_*(F, J^{(1)}) \rightarrow HF_*(H, J^{(2)})[\mathcal{E}(F - G)].$$

Similarly we define

$$C(G, F) : HF_*(G, J^{(2)}) \rightarrow HF_*(F, J^{(1)})[\mathcal{E}(F - G)].$$

A filtered version of the diagram (2.14) implies that these maps define an $\mathcal{E}(F - G)$ -interleaving which finishes the proof. \square

A priori, Floer persistence module depends on a choice of a regular pair $(F, J^{(1)})$. However, one might prove that if F and G generate the same Hamiltonian diffeomorphism ϕ , $\phi = \phi_1^F = \phi_1^G$, then Floer persistence modules $HF_*(F, J^{(1)})$ and $HF_*(G, J^{(2)})$ are isomorphic. This is a non-trivial fact which relies on the assumption that (M, ω) is symplectically aspherical, see a discussion in [121]. In other words, if (M, ω) is a closed, symplectically aspherical manifold and $\phi \in \text{Ham}(M, \omega)$ a Hamiltonian diffeomorphism whose fixed points are non-degenerate, we may define Floer persistence module of ϕ , denoted by $HF_*(\phi)$. The following theorem was proven in [121].

Theorem 2.2.42. *Let (M, ω) be a closed, symplectically aspherical manifold and $\phi, \psi \in \text{Ham}(M, \omega)$ such that all their fixed points are non-degenerate. Then for every $k \in \mathbb{Z}$ it holds*

$$d_{bottle}(\mathcal{B}(HF_k(\phi)), \mathcal{B}(HF_k(\psi))) \leq d(\phi, \psi).$$

Proof. It follows from Corollary 2.2.41 that for every F and G such that $\phi = \phi_1^F, \psi = \phi_1^G$ it holds

$$d_{inter}(HF_k(\phi), HF_k(\psi)) \leq \mathcal{E}(F - G).$$

After taking the infimum over F and G we get that

$$d_{inter}(HF_k(\phi), HF_k(\psi)) \leq d(\phi, \psi).$$

Now, the isometry theorem, Theorem 2.1.21, yields the statement. \square

Remark 2.2.43. One may use Theorem 2.2.42 to extend certain invariants derived from the barcode from non-degenerate Hamiltonian diffeomorphisms to all of $\text{Ham}(M, \omega)$. It is also possible to extend the definition of $HF_*^t(H, J)$, $t \in \mathbb{R}$ to all Hamiltonians H via a limit procedure, see a discussion in [121]. However, the object obtained in this way is not necessarily a persistence module in the sense of Definition 2.1.1.

Remark 2.2.44. Instead of relying on an isomorphism with $V_*(f)$, we could also prove stability of Morse persistence modules with respect to C^0 -distance using the same method we used for Floer persistence modules, i.e. via continuation maps.

Remark 2.2.45. In this chapter we only considered Floer persistence modules defined using contractible loops in M . One may define, in an analogous way, Floer chain complex generated by periodic orbits in a fixed, not necessarily trivial, free homotopy class α . Using this chain complex we may again define the *Floer persistence module of (H, J) in class α* , denoted by $HF_*^t(H, J)_\alpha$, and stability given by Theorem 2.2.42 continues to hold, see [121] for details. Unfiltered Floer homology in a non-trivial class α is equal to zero, as can be seen by taking a C^2 -small autonomous Morse Hamiltonian which has no non-constant periodic orbits.

2.2.9 Floer theory in the monotone case

As the last piece of background we explain how to extend the definitions of Floer homology and Floer persistence module to monotone symplectic manifolds.

Recall that a symplectic manifold is called monotone if $\omega|_{\pi_2(M)} = \kappa c_1|_{\pi_2(M)}$ for some $\kappa > 0$. Let (M, ω) be a closed, monotone symplectic manifold, which we assume not to be symplectically aspherical⁹, i.e. $c_1|_{\pi_2(M)} \neq 0$. The first issue we run into when defining Floer homology in the monotone case is that the value of the action functional

$$\mathcal{A}_H(z) = \int_0^1 H_t(z(t))dt - \int_{D^2} \bar{z}^* \omega$$

depends on the capping disc \bar{z} . Thus, in order to define \mathcal{A}_H we need to keep track of cappings of orbits. To this end, denote by

$$\bar{\pi}_2(M) := \pi_2(M) / \ker(\omega) = \pi_2(M) / \ker(c_1),$$

the last equality following from monotonicity. Let

$$\widetilde{\mathcal{L}_{[pt]}M} = \{(z, u) \mid z \in \mathcal{L}_{[pt]}M, u : D^2 \rightarrow M, u|_{\partial D^2} = z\} / \sim$$

where \sim is the equivalence relation given by

$$(z_1, u_1) \sim (z_2, u_2) \Leftrightarrow z_1 = z_2, [u_1 \# (-u_2)] = 0 \in \bar{\pi}_2(M),$$

⁹We exclude symplectically aspherical manifolds in order to avoid ambiguity in some definitions and statements. Symplectically aspherical case is simpler and has been discussed in Subsections 2.2.7 and 2.2.8.

denoting gluing of discs along the boundary. Now, we may define $\mathcal{A}_H : \widetilde{\mathcal{L}_{[pt]}M} \rightarrow \mathbb{R}$ by

$$\mathcal{A}_H([z, u]) = \int_0^1 H_t(z(t))dt - \int_{D^2} u^* \omega.$$

By the definition of $\bar{\pi}_2(M)$ the value of \mathcal{A}_H only depends on the equivalence class $[z, u]$ of (z, u) . The same calculation as in the symplectically aspherical case shows that

$$\text{Crit}(\mathcal{A}_H) = \{[z, u] \mid z \text{ is a periodic orbit of } \phi_t^H\}.$$

Assuming H is non-degenerate, the definition of index of an element of $\text{Crit}(\mathcal{A}_H)$ directly carries over to the monotone setting. Indeed, for $[z, u] \in \text{Crit}(\mathcal{A}_H)$ we may compute the Conley-Zehnder index of $d\phi_t^H(z(0))$ using any symplectic trivialization of $(u^*TM, u^*\omega)$. The result will only depend on the equivalence class of $[z, u]$ due to the relation between the first Chern class and the Conley-Zehnder index given by formula (2.8). As before, we define

$$\text{ind}([z, u]) = n - \text{ind}_{\text{CZ}}([z, u]).$$

Denote by $\text{Crit}_k(\mathcal{A}_H) \subset \text{Crit}(\mathcal{A}_H)$ the subset of critical points of index k . We claim that this set is finite. Indeed, the non-degeneracy assumption on H implies that there are only finitely many 1-periodic orbits z of ϕ_t^H . Furthermore, if $[z, u_1] \neq [z, u_2]$ we have that $c_1(u_1 \# (-u_2)) \neq 0$ and formula (2.8) implies that $\text{ind}([z, u_1]) \neq \text{ind}([z, u_2])$. Thus, for a fixed k , to each periodic orbit z corresponds at most one class $[z, u] \in \text{Crit}(\mathcal{A}_H)$ such that $\text{ind}([z, u]) = k$. We modify the definition of the space of Floer trajectories $\mathcal{M}(z_-, z_+)$ as follows

$$\mathcal{M}([z_-, u_-], [z_+, u_+]) = \{u : \mathbb{R} \times S^1 \rightarrow M \mid u \text{ solves (2.11), } [u_- \# u \# (-u_+)] = 0\},$$

where $[u_- \# u \# (-u_+)] = 0 \in \bar{\pi}_2(M)$. The analogue of Theorem 2.2.34 holds, i.e. $\mathcal{M}([z_-, u_-], [z_+, u_+])$ is a smooth manifold of dimension $\text{ind}([z_-, u_-]) - \text{ind}([z_+, u_+])$ for a (generic) regular pair (H, J) . The space $\bar{\mathcal{M}}([z_-, u_-], [z_+, u_+])$ is defined as in the symplectically aspherical case and satisfies the same properties. The rest of the construction is analogous to the one in the aspherical case. Namely, we define for $k \in \mathbb{Z}$ the Floer chain complex in degree k as

$$CF_k(H, J) = \text{Span}_{\mathbb{Z}_2}(\text{Crit}_k(\mathcal{A}_H)).$$

Since $\text{Crit}_k(\mathcal{A}_H)$ is finite, this is a finite-dimensional \mathbb{Z}_2 -vector space. As before, the differential

$$\partial : CF_k(H, J) \rightarrow CF_{k-1}(H, J)$$

is a \mathbb{Z}_2 -linear map given by

$$\partial(x) = \sum_{y \in \text{Crit}_{k-1}(\mathcal{A}_H)} \#_2 \bar{\mathcal{M}}(x, y)y,$$

for $x = [z, u] \in \text{Crit}_k(\mathcal{A}_H)$. It again follows that $\partial^2 = 0$ and we define Floer homology $HF_*(H, J)$ as the homology of the Floer chain complex. The theory of continuation maps verbatim translates to the monotone case by modifying the definition of \mathcal{M}^{par} in the same way we modified the definition of \mathcal{M} . In other words, we may define $HF_*(M, \omega)$.

Remark 2.2.46. The arguments used while constructing compactifications of spaces of trajectories (see Remark 2.2.36) are slightly different in the monotone and in the symplectically aspherical case. Namely, the appearance of pseudo-holomorphic bubbles in the monotone case is excluded via index considerations.

In the symplectically aspherical case each periodic orbit z gave rise to exactly one generator of $CF_*(H, J)$ and, since there were finitely many of them, $CF_*(H, J)$ was finite dimensional. In the monotone case, each periodic orbit gives rise to a countable set of generators, obtained from different cappings of the orbit, which all have different indices. Thus, while for each $k \in \mathbb{Z}$, $HF_k(H, J)$ is finite dimensional, there might be infinitely many k for which $HF_k(H, J) \neq 0$. This is indeed the case and moreover $HF_*(H, J)$ are periodic in degree. Namely, if $c_1(\pi_2(M)) = c_M \cdot \mathbb{Z}$, $c_M > 0$ being the *minimal Chern number*, we claim that for every $k \in \mathbb{Z}$

$$HF_k(H, J) \cong HF_{k+2c_M}(H, J).$$

Let us explain the origin of this periodicity. Notice that $\bar{\pi}_2(M)$ acts on the Floer chain complex as follows. For $v : S^2 \rightarrow M$, $[v] \in \bar{\pi}_2(M)$ define

$$[v] \cdot [z, u] = [z, u \# (-v)]$$

and extend this action to $CF_*(H, J)$ by \mathbb{Z}_2 -linearity. Formula (2.8) implies that

$$\text{ind}([v] \cdot [z, u]) = \text{ind}([z, u]) + 2c_1([v]). \quad (2.19)$$

Now $c_1 : \bar{\pi}_2(M) \rightarrow c_M \cdot \mathbb{Z}$ is by definition an isomorphism and we denote by $Q \in \bar{\pi}_2(M)$ the generator such that $c_1(Q) = c_M$. It follows from (2.19) that, for every $k \in \mathbb{Z}$, the action of Q is an isomorphism

$$Q : CF_k(H, J) \rightarrow CF_{k+2c_M}(H, J).$$

Moreover, for every $x, y \in \text{Crit}(\mathcal{A}_H)$, it holds $\mathcal{M}(x, y) = \mathcal{M}(Q(x), Q(y))$ and hence $Q \circ \partial = \partial \circ Q$. This implies that, for every $k \in \mathbb{Z}$, Q induces an isomorphism of Floer homologies (which we denote again by Q):

$$Q : HF_k(H, J) \rightarrow HF_{k+2c_M}(H, J).$$

Now recall that in symplectically aspherical case $HF_*(H, J)$ was isomorphic to singular homology $H_*(M; \mathbb{Z}_2)$. In the monotone case Floer homology is isomorphic to another invariant, called *quantum homology*, which is defined as follows. Let

$$\Lambda_{\mathbb{Z}_2} = \left\{ \sum_{i \in \mathbb{Z}} a_i q^i \mid a_i \in \mathbb{Z}_2, (\exists i_0 \in \mathbb{N}) a_i = 0 \text{ for } i \geq i_0 \right\},$$

be the \mathbb{Z}_2 -vector space of Laurent series in a formal variable q with \mathbb{Z}_2 -coefficients. One readily checks that $\Lambda_{\mathbb{Z}_2}$ is also a field with respect to usual addition and multiplication of power series, which we call the *Novikov field*. *Quantum homology with \mathbb{Z}_2 -coefficients* is a finite-dimensional $\Lambda_{\mathbb{Z}_2}$ -vector space

$$QH(M) = H(M; \mathbb{Z}) \otimes_{\mathbb{Z}_2} \Lambda_{\mathbb{Z}_2}.$$

We may define grading on $QH(M)$ by declaring that $\deg q = 2c_M$. This gives us, for each $k \in \mathbb{Z}$, a finite dimensional \mathbb{Z}_2 -vector space $QH_k(M)$. Moreover, multiplication by q defines an isomorphism

$$q \cdot : QH_k(M) \rightarrow QH_{k+2c_M}(M).$$

On the other hand, by defining $q \cdot x = Q(x)$ for every $x \in CF_*(H, J)$ we turn $HF_*(H, J)$ into a finite dimensional $\Lambda_{\mathbb{Z}_2}$ -vector space. Due to (2.19), it holds $\deg q = 2c_M$ in Floer homology as well. A similar argument as in the aspherical case (taking C^2 -small Morse Hamiltonian) implies that

$$HF_*(M, \omega) \cong QH_*(M, \omega)$$

both as graded \mathbb{Z}_2 -vector spaces and as $\Lambda_{\mathbb{Z}_2}$ -vector spaces.

Finally, we wish to say that by fixing a degree k , one defines Floer persistence module $(HF_k(H, J), \pi)$ in the same way as in the symplectically aspherical case. In contrast to the symplectically aspherical case, in the monotone case $(HF_k(H, J), \pi)$ depends on the class of the path $[\phi_t^H] \in \widetilde{\text{Ham}}(M, \omega)$ in the universal cover of $\text{Ham}(M, \omega)$. The same proof as the proof of Theorem 2.2.42 shows that for every $k \in \mathbb{Z}$ it holds

$$d_{\text{bottle}}(\mathcal{B}(HF_k(\tilde{\phi})), \mathcal{B}(HF_k(\tilde{\psi}))) \leq \tilde{d}(\tilde{\phi}, \tilde{\psi})$$

where $\tilde{\phi}, \tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega)$ and diffeomorphisms $\tilde{\phi}_1, \tilde{\psi}_1 \in \text{Ham}(M, \omega)$ are non-degenerate. The same remains true for Floer persistence modules in a non-trivial free homotopy class α , see Remark 2.2.45.

Chapter 3

Persistence modules with operators in Morse and Floer theory

3.1 Introduction

In a recent paper [121] (extended in [166]) it was observed that the arithmetical properties of barcodes are pertinent to such questions on Hofer's geometry as the study of the minimal Hofer's norm of a perturbation of a given Hamiltonian diffeomorphism necessary to make it autonomous, or more generally - to admit a root of order $p \geq 2$. In this chapter, we introduce and discuss the notion of persistence modules with operators, which allows us to use operators of intersection with cycles in the ambient (quantum) homology to further control the multiplicities of bars in the barcode. This provides new results on Hofer's geometry, and can be shown to provide strictly new information, as compared with traditional persistent homology (including spectral invariants), about the C^0 -geometry of Morse functions.

3.1.1 Persistence modules with operators

As before, denote the C^0 -distance between two smooth functions f and g on a compact manifold by

$$d_{C^0}(f, g) = |f - g|_{C^0} = \max_{x \in X} |f(x) - g(x)|,$$

and Hofer's distance between Hamiltonians F_t and G_t , $t \in [0, 1]$ on (M, ω) by

$$\mathcal{E}(F - G) = \int_0^1 \left(\max_{x \in M} (F_t(x) - G_t(x)) - \min_{x \in M} (F_t(x) - G_t(x)) \right) dt.$$

Recall from Subsection 2.2.3 that the Hofer's metric on $\text{Ham}(M, \omega)$ is given by

$$d(f, g) = \inf \mathcal{E}(F - G),$$

where the infimum runs over all the F, G such that $\phi_1^F = f, \phi_1^G = g$. Similarly, Hofer's pseudo-metric on the universal cover $\widetilde{\text{Ham}}(M, \omega)$ of $\text{Ham}(M, \omega)$ is defined as

$$\tilde{d}(\tilde{f}, \tilde{g}) = \inf \mathcal{E}(F - G),$$

where the infimum runs over all the F, G such that $[\{\phi_t^F\}] = \tilde{f}, [\{\phi_t^G\}] = \tilde{g}$ in $\widetilde{\text{Ham}}(M, \omega)$. As explained in Chapter 2, barcodes coming from Morse and Floer persistence modules are stable with respect to d_{C^0} , d and \tilde{d} . Stability of Floer barcodes with respect to d was a crucial property used in [121].

In this chapter we primarily investigate an additional structure on Morse and Floer persistence modules coming from the ambient homology. Our main observation is that the ambient homology acts on the persistence module by intersecting cycles in the sublevel sets of functions (and a similar picture holds in the Floer case). We consider this action as a particular case of the notion of a persistence module with an operator. Namely, we consider pairs (V, A) where $A : V^t \rightarrow V^{t+c_A}$ is a persistence module morphism, as main objects of interest and define morphisms between these objects to be usual persistence module morphisms which commute with the corresponding operators. We may now define operator interleaving as an interleaving in this new category, i.e., an interleaving which commutes with the operators. The fact that (V, A) and (W, B) are c -operator interleaved will immediately imply that $\text{im } A$ and $\text{im } B$ (as well as $\ker A$ and $\ker B$) are c -interleaved (see Section 3.2.3 for a discussion of persistence modules with operators).

In the Morse and the Floer case, fixing a (quantum) homology class a , we obtain an operator $a*$ induced by intersection (or quantum) product. Continuation maps commute with this operator, hence constitute morphisms of persistence modules with operators and induce operator interleavings. Finally, they provide both $\text{im}(a*)$ and $\ker(a*)$ for two functions f and g or two Hamiltonians F and G , with c -interleavings, for $c = |f - g|_{C^0}$ or $c = \mathcal{E}(F - G)$ respectively. This means that we may bound these values from below by using barcodes associated to $\text{im}(a*)$ or $\ker(a*)$. Following this line of reasoning, we show that there exists a pair f, g of Morse function on a manifold (even of dimension 2) such that all their spectral invariants, as well as their barcodes coincide, and yet the corresponding $\text{im}(a*)$ modules are at a positive (computable) interleaving distance c . We conclude that the two functions must be at C^0 -distance at least c (see Section 3.2.4 for an example).

Finally, we present an application to Hofer's geometry, by proving new cases of the conjecture that on any closed symplectic manifold and for any integer $p \geq 2$, there exist Hamiltonian diffeomorphisms which are arbitrarily far away, in Hofer's metric, from having a root of order p . First results of this kind were obtained in [121], and were then extended to certain other cases in [166] (for p a sufficiently large prime number). In our situation, the multiplication with classes in ambient homology allows to adjust multiplicities of certain long bars, the number theoretic properties of which are crucial to the argument, and allows to extend the class of p 's for which the result holds, yielding Theorem 3.1.2 (see Section 3.1.2).

3.1.2 Hofer's distance to p -th powers

We will now introduce some of the notations used in the rest of the chapter and recall Theorem 1.2.2 from Section 1.2, see Theorem 3.1.2 below. We will also give concrete examples to which this result applies.

Let $p \geq 2$ be any integer. As in Section 1.2, in this chapter we shall assume that the ground field \mathbb{K} has characteristic which does not divide p , $\text{char}(\mathbb{K}) \nmid p$, \mathbb{K} contains all p -th roots of unity¹, and fixing a primitive p -th root of unity ζ_p and an integer q , the equation $x^p - (\zeta_p)^q = 0$ has no solutions in \mathbb{K} unless $p|q$. An example of such a field is the splitting field \mathbb{Q}_p over \mathbb{Q} of $x^p - 1 \in \mathbb{Q}[x]$ (see Lemma 3.2.17 below).

Let (M, ω) be a closed symplectic manifold, and denote by $\text{Powers}_p(M, \omega) \subset \text{Ham}(M, \omega)$, where p is an integer, the set of all diffeomorphisms in $\text{Ham}(M, \omega)$, admitting a root of order p (in the same group). Denote by $\text{powers}_p(M)$ the supremum of the Hofer's distance to p -th powers in $\text{Ham}(M)$. That is, for each $\phi \in \text{Ham}(M)$ let $d(\phi, \text{Powers}_p(M)) = \inf_{\theta \in \text{Powers}_p(M)} d(\phi, \theta)$ and define

$$\text{powers}_p(M) := \sup_{\phi \in \text{Ham}(M)} d(\phi, \text{Powers}_p(M)).$$

Conjecture 3.1.1. *For every closed symplectic manifold M and every integer $p \geq 2$ it holds*

$$\text{powers}_p(M) = +\infty.$$

Let (Σ, σ) be a closed oriented surface of genus at least 4 equipped with a volume form. In [121], Polterovich and Shelukhin solved a special case of Conjecture 3.1.1, when $M = \Sigma \times N$, N being a symplectically aspherical manifold, see Theorem 1.2.1. Theorem 1.2.2, announced in the introduction, covers some instances of Conjecture 3.1.1 not covered by Theorem 1.2.1. For completeness, we will now formulate Theorem 1.2.2 again, see Theorem 3.1.2 below. This theorem is the main result of this chapter.

Assume that N is a monotone symplectic manifold, fix an integer $p \geq 2$ and denote by $\Lambda_{\mathbb{K}}$ the field of Laurent series in a formal variable q with coefficients in \mathbb{K} ,

$$\Lambda_{\mathbb{K}} = \left\{ \sum_{i \in \mathbb{Z}} a_i q^i \mid a_i \in \mathbb{K}, (\exists i_0 \in \mathbb{N}) a_i = 0 \text{ for } i \geq i_0 \right\}.$$

The quantum homology of N with \mathbb{K} -coefficients is the vector space $H_*(N; \mathbb{K}) \otimes_{\mathbb{K}} \Lambda_{\mathbb{K}}$ over $\Lambda_{\mathbb{K}}$, which we denote by $QH(N)$. Assuming that $\deg q = 2c_N$, where c_N is the minimal Chern number of N , $QH(N)$ has a natural \mathbb{Z} -grading, that is, we can define $QH_r(N)$ for $r \in \mathbb{Z}$, which will be vector spaces over the base field \mathbb{K} . We also have

¹That is the polynomial $x^p - 1 \in \mathbb{K}[x]$, which is separable by the assumption $\text{char}(\mathbb{K}) \nmid p$, splits over \mathbb{K} .

that $QH_{r+2c_N}(N) \cong QH_r(N)$ for every $r \in \mathbb{Z}$, where the isomorphism is given by multiplication by q . Let $e \in QH(N)$ be a homogeneous element and define a map

$$e* : QH(N) \rightarrow QH(N), \quad (e*)a = e * a,$$

where $*$ denotes the quantum product. This map is a linear morphism between vector spaces over $\Lambda_{\mathbb{K}}$ which restricts to a linear morphism between vector spaces over \mathbb{K} after fixing the grading:

$$e* : QH_r(N) \rightarrow QH_{r-2n+\deg e}(N), \quad \text{for } r \in \mathbb{Z}.$$

Now $E := e*(QH(N)) \subset QH(N)$ is a vector space over $\Lambda_{\mathbb{K}}$ and

$$E_r := e*(QH_r(N)) \subset QH_{r-2n+\deg e}(N),$$

are vector spaces over \mathbb{K} which satisfy $E_r \cong E_{r+2c_N}$, the isomorphism being induced by multiplication by q . These spaces give us $2c_N$ Betti numbers associated to a homogeneous element $e \in QH(N)$, which we define as

$$b_r(e) = \dim_{\mathbb{K}} E_r, \quad r = 0, \dots, 2c_N - 1.$$

Now, we can state the result regarding Hofer's geometry.

Theorem 3.1.2 (Theorem 1.2.2). *If there exists $e \in QH(N)$ such that $p \nmid b_r(e)$ for some $r \in \{0, \dots, 2c_N - 1\}$ then*

$$\text{powers}_p(\Sigma \times N) = +\infty.$$

To prove this result we describe the Floer theoretical setup that fits into our algebraic framework of equivariant persistence modules with operators, and then make a concrete computation in the case of the egg-beater flow, which yields the result.

Example 3.1.3. Taking N to be any monotone or symplectically aspherical manifold and $e \in QH(N)$, $r \in \{0, \dots, 2c_N - 1\}$ such that $b_r(e) \neq 0$ (for example $r = 0$ and $e = [N]$ the fundamental class), we have $p \nmid b_r(e)$ for large enough p . This means that for large enough p

$$\text{powers}_p(\Sigma \times N) = +\infty.$$

Since autonomous Hamiltonian diffeomorphisms have p -th roots for all p , we in particular have that Hofer's distance to autonomous flows in $\text{Ham}(\Sigma \times N)$ is unbounded.

Example 3.1.4. Let N be connected, $\dim N = 2n$ and assume $c_N \geq n + 1$. We now have that $b_0([N]) = b_0(N) = 1$, where $[N]$ is the fundamental class and $b_0(N)$ classical Betti number, and hence

$$\text{powers}_p(\Sigma \times N) = +\infty, \quad \text{for all } p.$$

This is for example the case for $N = \mathbb{C}P^n$. Connected symplectically aspherical N fall in this class of manifolds, with $c_N = +\infty$.

Example 3.1.5. Let $N = S^2 \times S^2$ and denote by $P = [pt]$, $A = [S^2 \times pt]$, $B = [pt \times S^2]$ and by $[N]$ the fundamental class. These four classes form a basis of $QH(N)$ over $\Lambda_{\mathbb{K}}$ and multiplication is completely described by the relations

$$A * B = P, \quad A^2 = B^2 = q^{-1}[N].$$

We calculate

$$(A + B) * A = (A + B) * B = P + q^{-1}[N] \in QH_0(N),$$

as well as

$$(A + B) * [N] = A + B \in QH_2(N), \quad (A + B) * P = q^{-1}(A + B) \in QH_{-2}(N),$$

and hence $b_0(A + B) = b_2(A + B) = 1$. This implies that

$$\text{powers}_p(\Sigma \times N) = +\infty, \quad \text{for all } p.$$

Note that in this example it is crucial that $A + B$ is not invertible. Otherwise, multiplication would be an isomorphism of $QH(N)$ and all the Betti numbers would be equal to 2, so we would have to assume $p \geq 3$.

Remark 3.1.6. A different extension of [121, Theorem 1.3], using different methods, was obtained recently by Zhang in [166]. In the setup of that paper p is assumed to be a prime number. The result refers to a more general manifold, namely the product $\Sigma \times N$, where N is any symplectic manifold (not necessarily monotone or aspherical) and gives a condition on p in terms of quantum Betti numbers for $\text{powers}_p(\Sigma \times N)$ to be infinite. The k -th quantum Betti number is defined as

$$qb_k(N) = \sum_{s \in \mathbb{Z}} b_{k+2c_N \cdot s}(N),$$

where $b_i(N)$ are classical Betti numbers. The main theorem of [166] states that if

$$p \nmid qb_p(N) + 2qb_0(N) + qb_{-p}(N),$$

then

$$\text{powers}_p(\Sigma \times N) = +\infty.$$

One immediately sees that when N is monotone, $qb_k(N) = b_k([N])$, thus in this case our theorem implies Zhang's result. The above examples of $N = \mathbb{C}P^n$ and $N = S^2 \times S^2$ show that our criterion covers some new cases of N and p , e.g., when $p = 2$.

3.2 Persistence modules

3.2.1 Conventions

In this chapter we will always work with persistence modules of finite type assuming conventions which guarantee that all bars in the barcode are of the form $(a, b]$ or $(a, +\infty)$ for $a, b \in \mathbb{R}, a < b$. Precisely, we require that our persistence modules (V, π) also satisfy

- 1) For all but a finite number of points $r \in \mathbb{R}$, there is a neighbourhood $U \ni r$ such that $\pi_{s,t}$ are isomorphisms for all $s < t$ with $s, t \in U$;
- 2) There exists t_- such that $V^t = 0$ for $t < t_-$;
- 3) For every $t \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $\pi_{s,t}$ are isomorphisms for all $t - \varepsilon < s \leq t$.

The category of all such persistence modules will be denoted by **pmod** and we will simply refer to them as persistence modules.

3.2.2 Künneth formula for persistence modules

As we mentioned before, **pmod** is an abelian category, and we wish to define a monoidal structure \otimes and its derived functors in this category in a similar fashion to the situation which we have for \mathbb{Z} modules (similar constructions, yet with different aims and applications, appeared in [31, 32, 45, 154, 155]). Let (V^s, π^V) and (W^t, π^W) be two persistence modules and define vector spaces

$$X^r = \bigoplus_{t+s=r} V^s \otimes W^t, \text{ and } Y^r \subset X^r \text{ for every } r \in \mathbb{R},$$

given by

$$Y^r = \left\langle \left\{ (\pi_{\alpha, s_1}^V v_\alpha) \otimes (\pi_{\beta, t_1}^W w_\beta) - (\pi_{\alpha, s_2}^V v_\alpha) \otimes (\pi_{\beta, t_2}^W w_\beta) \right\} \right\rangle,$$

where $\langle S \rangle$ stands for vector space over \mathbb{K} generated by the set S and indices $s_1, s_2, t_1, t_2, \alpha$ and β satisfy $s_1 + t_1 = s_2 + t_2 = r, \alpha \leq \min\{s_1, s_2\}, \beta \leq \min\{t_1, t_2\}$. We may now define $(V \otimes W)^r = X^r / Y^r$ and maps $\pi^V \otimes \pi^W$ on X^r induce maps $\pi^{V \otimes W}$ on $V \otimes W$, which give this space the structure of persistence module. We call this module *the tensor product* of persistence modules (V, π^V) and (W, π^W) . Another way to think of $V \otimes W$ is that $(V \otimes W)^r$ is the colimit in the category of (finite-dimensional, as is easy to see) vector spaces over our ground field of the diagram with objects $\{V^s \otimes W^t\}_{s+t \leq r}$ and maps $\pi_{s_1, s_2} \otimes \pi_{t_1, t_2} : V^{s_1} \otimes W^{t_1} \rightarrow V^{s_2} \otimes W^{t_2}$ for $s_1 \leq s_2$ and $t_1 \leq t_2$ (we use the convention that $\pi_{t,t} = \mathbb{1}_{V^t}$).

It is easy to see that we can also define the tensor product $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ of persistence module morphisms $f : V \rightarrow V'$ and $g : W \rightarrow W'$ by setting $f \otimes g([v_\alpha \otimes w_\beta]) = [f(v_\alpha) \otimes g(w_\beta)]$.

Fixing a persistence module W we get a functor $\otimes W : \mathbf{pmod} \rightarrow \mathbf{pmod}$ which acts on objects and morphisms by

$$\otimes W(V) = V \otimes W, \quad \otimes W(f) = f \otimes \mathbb{1}_W.$$

One can check that $\otimes W$ is a right exact functor and in order to define its derived functors we need to construct a projective resolution of every persistence module V . In the simplest case when V is an interval module $\mathbb{K}_{(a,b]}$ we have the following projective resolution of V of length two:

$$0 \rightarrow \mathbb{K}_{(b,+\infty)} \rightarrow \mathbb{K}_{(a,+\infty)} \rightarrow \mathbb{K}_{(a,b]} \rightarrow 0,$$

where arrows denote obvious maps. Note that we used the fact that $\mathbb{K}_{(a,+\infty)}$ is projective object for every $a \in \mathbb{R}$. One may also check that in fact a persistence module V is projective if and only if its barcode contains no finite bars. Using this fact together with Theorem 2.1.8 we may construct a projective resolution of length two of every persistence module V in the same manner as we did for the interval module. Recall that (classical) derived functors of $\otimes W$ applied to V are computed as homologies of the sequence

$$\dots \rightarrow P_2 \otimes W \xrightarrow{f_2 \otimes \mathbb{1}} P_1 \otimes W \xrightarrow{f_1 \otimes \mathbb{1}} P_0 \otimes W \rightarrow 0,$$

where $\dots \rightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} V \rightarrow 0$ is a projective resolution of V . Since every persistence module has a projective resolution of length two, there is only one non-trivial derived functor of $\otimes W$, which we denote by $Tor(\cdot, W)$. Both \otimes and Tor are symmetric in the sense that $V \otimes W \cong W \otimes V$ and $Tor(V, W) \cong Tor(W, V)$ and it immediately follows that if either V or W is projective $Tor(V, W) = 0$.

Example 3.2.1. Let $V = \mathbb{K}_{(a,b]}$, $W = \mathbb{K}_{(c,d]}$ be two interval persistence modules. It follows directly from the definition of \otimes that

$$V \otimes W = \mathbb{K}_{(a,b]} \otimes \mathbb{K}_{(c,d]} = \mathbb{K}_{(a+c, \min\{a+d, b+c\}]}$$

In order to compute $Tor(\mathbb{K}_{(a,b]}, \mathbb{K}_{(c,d]})$, let us take the following projective resolution of $\mathbb{K}_{(a,b]}$:

$$0 \rightarrow \mathbb{K}_{(b,+\infty)} \rightarrow \mathbb{K}_{(a,+\infty)} \rightarrow \mathbb{K}_{(a,b]} \rightarrow 0.$$

After applying $\otimes \mathbb{K}_{(c,d]}$ to this resolution we get

$$0 \rightarrow \mathbb{K}_{(b+c, b+d]} \rightarrow \mathbb{K}_{(a+c, a+d]} \rightarrow \mathbb{K}_{(a+c, \min\{a+d, b+c\}]} \rightarrow 0,$$

and hence after calculating homology we get

$$Tor(\mathbb{K}_{(a,b)}, \mathbb{K}_{(c,d)}) = \mathbb{K}_{(\max\{a+d, b+c\}, b+d)}.$$

Our goal is to establish a Künneth type formula for filtered homology groups using \otimes and Tor . Let us first recall the following definition.

Definition 3.2.2. We say that chain complex $(C_k, \partial_k), \partial_k : C_k \rightarrow C_{k-1}, k \in \mathbb{Z}$ of finite-dimensional vector spaces over a field \mathbb{K} is *filtered by function* ν if $\nu : C_* \rightarrow \mathbb{R} \cup \{-\infty\}$ and

- 1) $\nu(x) = -\infty$ if and only if $x = 0$;
- 2) For all $\lambda \in \mathbb{K}, \lambda \neq 0$ it holds $\nu(\lambda x) = \nu(x)$;
- 3) For all $x, y \in C_*$ it holds $\nu(x + y) \leq \max\{\nu(x), \nu(y)\}$;
- 4) For all $x \in C_*$ it holds $\nu(\partial_* x) \leq \nu(x)$.

Remark 3.2.3. This definition of a chain complex filtered by a function is the special case of the definition of a *Floer-type complex* over the Novikov field $\Lambda^{\mathbb{K}, \Gamma}$ given in [150] in case of $\Gamma = \{0\}$ and trivial valuation on \mathbb{K} .

The main examples of filtered chain complexes of interest to us are the Morse chain complex $CM_*(f)$ for a Morse function f , where f also serves as a filtration function and the Floer chain complex $CF_*(H)_\alpha$ filtered by the action functional \mathcal{A}_H , where H is a Hamiltonian function and α is an atoroidal or toroidally monotone class of free loops (see Section 3.3.1 for details).

Now if (C_*, ∂_*, ν) is a chain complex with filtration function ν , we may define $C_*^t = \{x \in C_* \mid \nu(x) < t\}$ for every $t \in \mathbb{R}$ and by property 4) we have that $\partial_* : C_*^t \rightarrow C_{*-1}^t$. This implies that $(C_*^t, \partial|_{C_*^t})$ is a new chain complex and we denote its homology by $H_*^t(C)$ and refer to it as *filtered homology*. Since $C^s \subset C^t$ for $s \leq t$, inclusions induce maps $\pi_{s,t} : H_*^s(C) \rightarrow H_*^t(C)$, which render $(H_*^t(C), \pi)$ into a persistence module. In order to obtain the Künneth formula for filtered homology, we must examine the product of two filtered chain complexes. Let us start with an example.

Example 3.2.4. Let $(C_*^1, \partial_*^1, \nu^1)$ and $(C_*^2, \partial_*^2, \nu^2)$ be two filtered chain complexes given by

$$C_0^1 = \langle x \rangle, C_1^1 = \langle y \rangle, \partial^1 x = 0, \partial^1 y = x, \nu^1(x) = a, \nu^1(y) = b,$$

and

$$C_0^2 = \langle z \rangle, C_1^2 = \langle w \rangle, \partial^2 z = 0, \partial^2 w = z, \nu^2(z) = c, \nu^2(w) = d.$$

We have that

$$H_0^t(C^1) = \mathbb{K}_{(a,b)}^t, H_0^t(C^2) = \mathbb{K}_{(c,d)}^t, H_*^t(C^1) = H_*^t(C^2) = 0 \text{ for } * \neq 0.$$

The product complex $(C^1 \otimes C^2, \partial, \nu = \nu^1 + \nu^2)$ (with usual product differential) is given by

$$(C^1 \otimes C^2)_0 = \langle x \otimes z \rangle, (C^1 \otimes C^2)_1 = \langle \{x \otimes w, y \otimes z\} \rangle, (C^1 \otimes C^2)_2 = \langle y \otimes w \rangle,$$

$$\partial(x \otimes z) = 0, \partial(x \otimes w) = \partial(y \otimes z) = x \otimes z, \partial(y \otimes w) = x \otimes w - y \otimes z,$$

with filtration $\nu(x \otimes w) = a + d, \nu(y \otimes z) = b + c, \nu(x \otimes z) = a + c, \nu(y \otimes w) = b + d$. It readily follows that

$$H_0^t(C^1 \otimes C^2) = \mathbb{K}_{(a+b, \min\{a+d, b+c\}}^t = (H_0(C^1) \otimes H_0(C^2))^t,$$

$$H_1^t(C^1 \otimes C^2) = \mathbb{K}_{(\max\{a+d, b+c\}, b+d]}^t = (Tor(H_0(C^1), H_0(C^2)))^t,$$

and $H_2^t = 0$.

Remark 3.2.5. Note that the *Tor* functor naturally appears even in the simplest case of product of interval modules. As already mentioned, in this case torsion comes from finite bars in the barcode and hence is unavoidable even when we work with fields and vector spaces. This becomes more transparent if one adopts the definition of persistence modules as modules over the polynomial ring $\mathbb{K}[t]$ (see [167]). In this case the action of structure maps comes from multiplication by formal variable t and thus finite bars correspond to homology classes x for which $t^n \cdot x = 0$ for some $n \in \mathbb{N}$, which are torsion elements. It seems likely that, by carefully analysing how the two approaches correspond to one another, one may obtain the results of this subsection from the universal coefficient theorem for complexes of modules over $\mathbb{K}[t]$.

We may now formulate the full statement.²

Proposition 3.2.6 (Künneth formula for filtered homology). *Let (C^1, ∂^1, ν^1) and (C^2, ∂^2, ν^2) be two filtered chain complexes and let $(C^1 \otimes C^2, \partial, \nu = \nu_1 + \nu_2)$ be their product complex. Then for every $k \in \mathbb{Z}$ there exists a short exact sequence of persistence modules*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=k} (H_i(C^1) \otimes H_j(C^2))^t &\xrightarrow{K} H_k^t(C^1 \otimes C^2) \rightarrow \\ &\rightarrow \bigoplus_{i+j=k-1} (Tor(H_i(C^1), H_j(C^2)))^t \rightarrow 0, \end{aligned}$$

which splits. Here, K denotes the canonical map given by

$$K([\sum_i \lambda_i x_i] \otimes [\sum_j \mu_j y_j]) = [\sum_{i,j} \lambda_i \mu_j x_i \otimes y_j].$$

²Some form of this statement was already known to experts, in particular to Michael Usher and Jun Zhang.

Sketch of the proof. We already saw in Example 3.2.4 that the statement holds when C^1 and C^2 have the following form

$$\dots \rightarrow 0 \rightarrow \langle y \rangle \rightarrow \langle x \rangle \rightarrow 0 \rightarrow \dots$$

It readily follows that the statement is also true if we allow C^1 and C^2 to be of the following form

$$\dots \rightarrow 0 \rightarrow \langle x \rangle \rightarrow 0 \rightarrow \dots$$

By Remark 3.2.3 we may look at our complexes as a special case of the definition given in [150] and we may use the existence of singular value decomposition of operator ∂ proven there. This theorem essentially states that every filtered chain complex decomposes into a direct sum of the simple complexes which have one of the two forms described above. Now, the general case follows by reduction to the two simple ones and considerations about interval modules. \square

Remark 3.2.7. Essentially the same computation of the product of chain complexes as one presented in Example 3.2.4 and in the proof of Proposition 3.2.6 appears in [166]. The context is, however, slightly different, since we eventually work on the level of homology, while the author of [166] works on chain level. One may also try to prove Proposition 3.2.6 directly, without referring to much more general machinery developed in [150].

3.2.3 Persistence modules with operators

The methods we use, which are of independent interest, have to do with persistence modules endowed with an additional structure, and their equivariant version.

Consider the category **pmod-op** with objects pairs (V, A) with $V \in \mathbf{pmod}$, and $A : V \rightarrow V[c_A]$, for certain $c_A \in \mathbb{R}$, a morphism of persistence modules. Morphisms between (V, A) and (W, B) , when $c_A = c_B$ consist of morphisms $F : V \rightarrow W$ of persistence modules such that $F[c_A] \circ A = B \circ F$, and if $c_A \neq c_B$, only of the zero morphism $V \rightarrow W$.

Examples

Example 3.2.8. (Shift operator) For each $\delta \geq 0$, each $V \in \mathbf{pmod}$ comes with a canonical shift operator $sh(\delta) : V \rightarrow V[\delta]$. For $\delta = 0$, this is simply the identity operator. For $\delta > 0$, $sh(\delta)_t : V^t \rightarrow V^{t+\delta}$ is defined as the persistence structure map $\pi_{t,t+\delta}$ of V . Hence $(V, sh(\delta))$ is an object of **pmod-op**.

Example 3.2.9. (\mathbb{Z}_p -action) Fix an integer $p \geq 2$. Given a $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ -representation in **pmod**, the action of the cyclic generator $1 \in \mathbb{Z}_p$ gives an operator $A : V \rightarrow V$, with $c_A = 0$ (that satisfies $A^p = \mathbb{1}_V$).

Example 3.2.10. (Product map) Consider a Morse function $f : X \rightarrow \mathbb{R}$ on a closed finite-dimensional manifold X of dimension $\dim X = m$. It defines a \mathbb{Z} -graded persistence module by $V_*(f)^s = H_*(\{f \leq s\}; \mathbb{K}) = H_*(\{f < s\}; \mathbb{K})$, for s a regular value of f . Let $p_s : H_*(X) \rightarrow H_*(X, \{f \geq s\}) = H_*(\{f \leq s\}, \{f = s\})$ be the natural map. Taking a class $a \in H_r(X)$, the intersection product with $p_s(a)$, $(p_s(a) \cap) : V_*(f)^s \rightarrow V_{*+r-m}(f)^s$ defines an operator $(a \cap) : V_*(f) \rightarrow V_*(f)$, with $c_{a \cap} = 0$, that shifts the grading by $r - m$.

Key estimate

For two objects (V, A) and (W, B) of **pmod-op** with $c_A = c_B$, and $\delta \geq 0$, define an operator- δ -interleaving between them to be a δ -interleaving $f : V \rightarrow W[\delta]$, $g : W \rightarrow V[\delta]$ that commutes with the operators A and B , that is

$$f[c_A] \circ A = B[\delta] \circ f, \quad g[c_B] \circ B = A[\delta] \circ g.$$

Define the operator-interleaving distance between them by

$$d_{\text{op-inter}}((V, A), (W, B)) = \inf\{\delta \geq 0 \mid \text{there exists a } \delta\text{-operator-interleaving}\}.$$

Proposition 3.2.11. *For all (V, A) , (W, B) in **pmod-op** with $c_A = c_B$,*

$$d_{\text{inter}}(\text{im}(A), \text{im}(B)) \leq d_{\text{op-inter}}((V, A), (W, B)).$$

Put $c := c_A = c_B$. The proof is an immediate diagram chase in the following diagram (and its analogue with f, g interchanged):

$$\begin{array}{ccccccc} V^t & \xrightarrow{f} & W[\delta]^t & \xrightarrow{g[\delta]} & V[2\delta]^t & & \\ A_t \downarrow & & B_t[\delta] \downarrow & & A_t[2\delta] \downarrow & & (3.1) \\ V[c]^t & \xrightarrow{f[c]} & W[c + \delta]^t & \xrightarrow{g[c + \delta]} & V[c + 2\delta]^t & & \end{array}$$

Discussion

While Proposition 3.2.11 is elementary, it turns out to be useful already in the more basic examples.

Example 3.2.12. (Shift operator) Proposition 3.2.11 applied to the example of persistence shift maps, reduces to the following statement. If V, W are δ -interleaved, then $V' = \text{im } sh(c)_V$, $W' = \text{im } sh(c)_W$ are δ -interleaved for every $c \in \mathbb{R}$. The reason is that with respect to shift operators, operator- δ -interleaving is the same as δ -interleaving, so V, W being δ -interleaved implies that they are also operator- δ -interleaved.

Example 3.2.13. (Intersection product) In Section 3.2.4 we give examples of two Morse functions f, g on a surface Σ_2 of genus 2 with identical barcodes, and identical spectral invariants, the images of whose persistence modules under the intersection product with a class in $H_1(\Sigma_2; \mathbb{K})$ are, however, at a positive interleaving distance $c > 0$. We conclude, by Proposition 3.2.11, that any two functions in the respective orbits of f, g under the identity component of the diffeomorphism group are at C^0 -distance $c > 0$. Indeed for such a diffeomorphism $\psi \in \text{Diff}_0(\Sigma_2)$, $(V(f \circ \psi), a \cap)$ and $(V(f), a \cap)$ are isomorphic objects in **pmod-op**, and $d_{\text{op-inter}}((V(f), a \cap), (V(g), a \cap)) \leq |f - g|_{C^0}$.

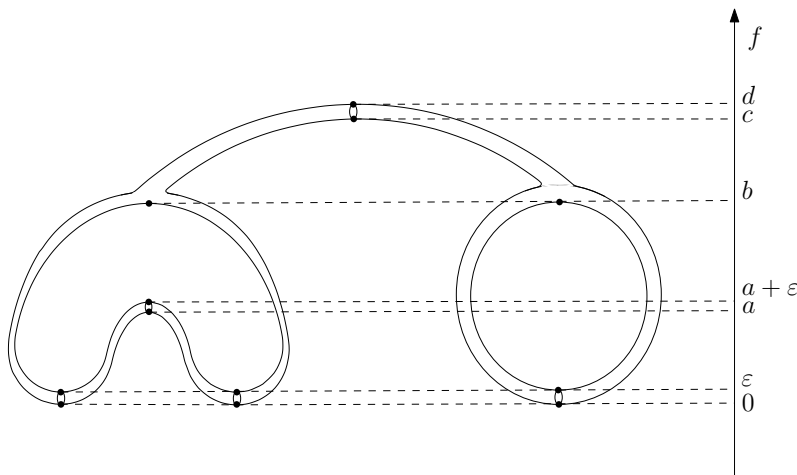
3.2.4 Example of a Morse function on $\mathbb{T}^2 \sharp \mathbb{T}^2$

We present an example in Morse homology illustrating the effect of a product on the Floer persistence module which we will define later and we also justify claims of Example 3.2.13.

Adopting the setup of Example 2.1.3 and Example 3.2.10, to a Morse function f on a closed manifold X of dimension m we associate a persistence module $(V_*^t(f), \pi)$ by taking $V_*^t(f) = H_*(\{f < t\}; \mathbb{K})$, the structure maps $\pi_{s,t}$ being induced by inclusion of sublevel sets. Alternatively, we may consider the Morse chain complex induced by the critical points whose critical value is less than t . Now, $a \in H_*(M)$ acts on $V_*^t(M)$ by intersecting cycles (or by counting Y-shaped configurations of gradient flow lines in the Morse picture) and we get a map:

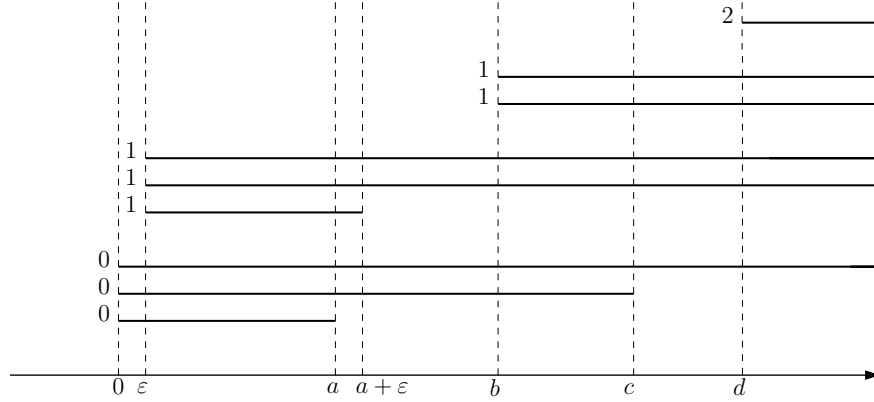
$$a \cap : V_*^t(f) \rightarrow V_{*+\text{deg } a - m}^t(f).$$

Let Σ_2 be a surface of genus 2. We construct two Morse functions on Σ_2 which have same barcodes and same spectral invariants associated to every homology class, but their intersection barcodes with a fixed class differ by a finite bar. First, observe that $\Sigma_2 = \mathbb{T}^2 \sharp \mathbb{T}^2$ and hence $H_1(\Sigma_2) \cong H_1(\mathbb{T}^2) \oplus H_1(\mathbb{T}^2)$, where generators are given by standard generators of $\mathbb{T}^2 = S^1 \times S^1$, namely two circles. We consider a Morse function $f : \Sigma_2 \rightarrow \mathbb{R}$ given by the height function on the following picture:

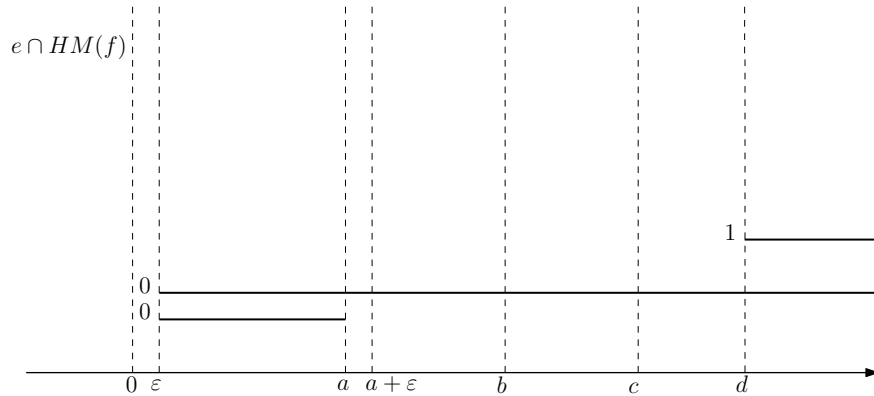


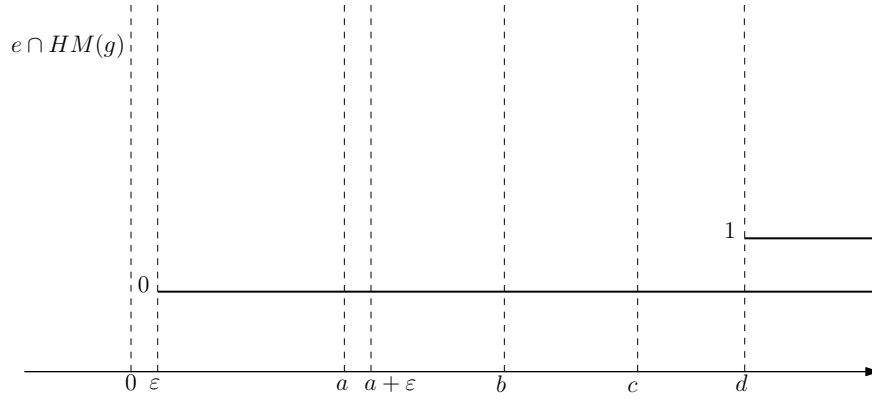
We observe that $H_1(\Sigma_2)$ is generated by four homology classes represented by embedded circles, two of which have spectral invariants associated to f equal to ε and the other two with spectral invariants equal to b .

The other function we consider is the height function g on the same picture with left and right reversed. More precisely, $g = f \circ \varphi$, where $\varphi : \Sigma_2 \rightarrow \Sigma_2$ is a diffeomorphism which interchanges two copies of $\mathbb{T}^2 \setminus D^2$ which we glue together to form Σ_2 . Since $g = f \circ \varphi$, the barcodes of f and g are the same and they look as follows:



One also readily checks that for every $z \in H_*(M)$, $c(z, f) = c(z, g)$, where $c(z, f)$, $c(z, g)$ are spectral invariants associated to functions f and g and a homology class z (see Remark 2.1.24). This means that standard methods, namely barcodes and spectral invariants fail to distinguish between f and g . However, after intersecting with one of the two big circles (for example the one on the left in the above picture), which corresponds to the homology class e with spectral invariants $c(e, f) = c(e, g) = b$, we get the following intersection barcodes:





These barcodes differ by a finite bar $(\varepsilon, a]$. Thus, by using the product structure in homology and analysing its effect on the barcode we are able to make a distinction between f and g . Note also that the bar $(\varepsilon, a]$ did not exist in the original barcode.

It would be interesting to find a general formula for the image persistence module of the intersection by homology class a . Examples show that this is not a trivial question.

3.2.5 Equivariant version

In order to study the question of Hofer's distance to autonomous Hamiltonian diffeomorphisms and more generally to full p -th powers in Ham , persistence modules with additional $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ action were used in [121]. A \mathbb{Z}_p persistence module (V, π, T) is a persistence module (V, π) together with an automorphism $T : (V, \pi) \rightarrow (V, \pi)$ which satisfies $T^p = \mathbb{1}$. This definition immediately implies that $T_t : V^t \rightarrow V^t$ is a linear operator whose eigenvalues are p -th roots of unity. Hence, for $\zeta^p = 1$, $\pi_{s,t}$ maps a ζ -eigenspace of T_s to ζ -eigenspace of T_t and we can define a ζ -eigenspace of T to be a persistence module obtained by restricting π to ζ -eigenspaces of each T_t .

We require the following immediate statement.

Lemma 3.2.14. *Let (V_r, T_r) , $r \in 1, \dots, l$ be \mathbb{Z}_p persistence modules, $(V, T) = (\bigoplus_{r=1}^l V_r, \bigoplus_{r=1}^l T_r)$ and denote by L_ζ , ζ -eigenspace of T , where $\zeta^p = 1$, $\zeta \neq 1$. Then*

$$L_\zeta = \bigoplus_{r=1}^l L_\zeta^r,$$

where L_ζ^r are ζ -eigenspaces of V_r .

Interleavings between \mathbb{Z}_p persistence modules which commute with the \mathbb{Z}_p action are called *equivariant*. Again, taking infimum over all $\delta > 0$ such that V and W are equivariantly δ -interleaved gives us an *equivariant interleaving distance*, which we

denote by $\widehat{d}_{inter}(V, W)$. It immediately follows that

$$\widehat{d}_{inter}(V, W) \geq d_{inter}(V, W) \quad \text{and} \quad \widehat{d}_{inter}(V, W) \geq d_{inter}(L_\zeta, K_\zeta), \quad (3.2)$$

where L_ζ and K_ζ are the ζ -eigenspaces of T_V and T_W respectively.

Applying our new method to the equivariant situation is tantamount to studying \mathbb{Z}_p persistence modules with an operator $A : V \rightarrow V[c_A]$, which moreover commutes with the \mathbb{Z}_p -action. Examples of such operators will come from a version of the pair-of-pants product in Floer homology.

Definition 3.2.15. A \mathbb{Z}_p persistence module with an operator is a pair (V, A) where V is a \mathbb{Z}_p persistence module and $A : V \rightarrow V[c_A]$ is a morphism of persistence modules that commutes with the \mathbb{Z}_p -action.

Let (V, A) and (W, B) be two \mathbb{Z}_p persistence modules with operators with $c = c_A = c_B$, and suppose that $f : V \rightarrow W[\delta]$ and $g : W \rightarrow V[\delta]$ is an equivariant δ -interleaving. We say that this interleaving is *op-equivariant* if it respects the operator actions, that is,

$$B(\delta) \circ f = f(c) \circ A, \quad A(\delta) \circ g = g(c) \circ B.$$

Taking infimum over all δ such that V and W are op-equivariantly δ -interleaved gives us a new distance, which we denote $\widehat{d}_{op-inter}(V, W)$. Since A and B are \mathbb{Z}_p persistence module morphisms we have that $\text{im}(A) \subset V^a$ and $\text{im}(B) \subset W^a$ are \mathbb{Z}_p persistence submodules of V^a and W^a . Every op-equivariant interleaving between V and W induces an equivariant interleaving between $\text{im}(A)$ and $\text{im}(B)$, which in particular implies

$$\widehat{d}_{op-inter}((V, A), (W, B)) \geq \widehat{d}_{inter}(\text{im}(A), \text{im}(B)). \quad (3.3)$$

Note however that in general this may not be an equality.

Remark 3.2.16. The situation we encounter when working with singular, Morse or Floer homology is not exactly the same as described above since our product map may change the degree and not just the filtration. One can overcome this ambiguity by giving a slightly more general definition analogous to the one given above, where $A : V^t \rightarrow \bar{V}^{t+a}$ for different persistence modules V and \bar{V} or by considering graded vector spaces.

In order to tackle the problem of Hofer's distance to full powers in Ham a numerical invariant $\mu_p(W)$ called *multiplicity-sensitive spread* was defined in [121]. We recall the definitions and properties of μ_p and an auxiliary invariant $\mu_{p,\zeta}$, which we use later (see [121] for proofs).

For an interval $I = (a, b]$ or $I = (a, +\infty)$ and $c \in \mathbb{R}$ let $I^c = (a + c, b - c]$, when $b - a > 2c$, or $I^c = (a + c, +\infty)$. Let \mathcal{B} be a barcode, I an interval and denote by $m(\mathcal{B}, I)$ the number of bars in \mathcal{B} containing I (counted with multiplicities). We will

write $\mu_p(\mathcal{B})$ for a supremum of those $c \geq 0$ for which there exists an interval I of length greater than $4c$ such that $m(\mathcal{B}, I) = m(\mathcal{B}, I^{2c}) = l$ with l not divisible by p . Using this notation we define $\mu_{p,\zeta}$ as

$$\mu_{p,\zeta}(W) = \mu_p(\mathcal{B}(L_\zeta)),$$

where L_ζ is ζ -eigenspace of T . Now μ_p is defined as

$$\mu_p(W) = \max_{\zeta} \mu_{p,\zeta}(W).$$

We have that

$$|\mu_p(\mathcal{B}(L_\zeta)) - \mu_p(\mathcal{B}(K_\zeta))| = |\mu_{p,\zeta}(V) - \mu_{p,\zeta}(W)| \leq d_{\text{bottle}}(\mathcal{B}(L_\zeta), \mathcal{B}(K_\zeta)), \quad (3.4)$$

where L_ζ and K_ζ are the ζ -eigenspaces of T_V and T_W respectively.

A \mathbb{Z}_p persistence module (W, T) is called a *full p -th power* if $T = S^p$ for some morphism $S : W \rightarrow W$.

From now on we impose the same assumption on the ground field \mathbb{K} as in Section 3.1.2. An important property of μ_p for such a ground field \mathbb{K} is that $\mu_p(W) = 0$ given that W is a full p -th power. The proof of this fact is the same as in the case when p is a prime number and has been carried out in [121]. However, the proof of the fact that splitting field \mathbb{Q}_p of $x^p - 1$ over \mathbb{Q} satisfies the required assumptions is slightly different and we present it here for completeness. The following lemma is the substitution for Lemma 4.14 in [121].

Lemma 3.2.17. *Let $p \geq 2$ be any integer, $\zeta_p \in \mathbb{Q}_p$ a primitive p -th root of unity and assume that equation*

$$x^p - (\zeta_p)^q = 0,$$

has a solution in \mathbb{Q}_p for some integer q . Then $p|q$.

Proof. For an integer $k \geq 2$, we denote by ζ_k a fixed k -th primitive root of unity and by $\mathbb{Q}_k = \mathbb{Q}(\zeta_k)$ the cyclotomic extension of \mathbb{Q} by ζ_k . We may embed $\mathbb{Q}_p \subset \mathbb{Q}_{p^2} \subset \overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of \mathbb{Q} and we may also assume that $(\zeta_{p^2})^p = \zeta_p$. Now if x is a solution of the above equation, for some $m \in \mathbb{Z}$ it holds that $x = (\zeta_{p^2})^q \cdot (\zeta_p)^m$ inside $\overline{\mathbb{Q}}$ and hence if x is in \mathbb{Q}_p so is $(\zeta_{p^2})^q$. If p does not divide q , we have that $\gcd(p, q) = d < p$ and $p = nd$ for some natural number n . Taking $u, v \in \mathbb{Z}$ such that $uq + vp = d$, we get that

$$(\zeta_{p^2})^d = (\zeta_{p^2})^{uq} \cdot (\zeta_{p^2})^{vp} \in \mathbb{Q}_p.$$

However, $(\zeta_{p^2})^d = \zeta_{pn}$ and hence $\mathbb{Q}_p = \mathbb{Q}_{pn}$, which is only possible when p is odd and $n = 2$ (see [96, Corollary 3]). This cannot happen because $n|p$. \square

3.3 Floer theory and Hofer's geometry

3.3.1 Product map on Floer persistence module

Let (M, ω) be a closed symplectic manifold, and denote by $c_1(TM)$ the first Chern class of the tangent bundle, equipped with any ω -compatible almost complex structure. Take a homotopy class of free loops $\alpha \in \pi_0(\mathcal{L}M)$ and denote by $\mathcal{L}_\alpha M$ all loops in class α . We say that (M, ω) is α -toroidally monotone if there exists $\kappa > 0$ such that

$$\langle [\omega], A \rangle = \kappa \cdot \langle c_1(TM), A \rangle,$$

for all $A \in \text{Im}(\Psi)$, where $\Psi : \pi_1(\mathcal{L}_\alpha M) \rightarrow H_2(M; \mathbb{Z})$ sends a loop $\beta \in \pi_1(\mathcal{L}_\alpha M)$, regarded as a map $\beta : \mathbb{T}^2 \rightarrow M$, to $\beta_*([\mathbb{T}^2])$. It readily follows that M is also spherically monotone with same monotonicity constant κ , that is,

$$[\omega] = \kappa \cdot c_1(TM),$$

where both $[\omega]$ and $c_1(TM)$ are regarded as functionals on $\pi_2(M)$. Assuming M is α -toroidally monotone³, to every element $\tilde{f} \in \widetilde{\text{Ham}}(M)$ of the universal cover of $\text{Ham}(M)$ that is non-degenerate in class α , we associate a Floer persistence module $HF_*^t(\tilde{f})_\alpha$ with parameter t (see [121, 147]).

Remark 3.3.1. Here, and later on in Section 3.3.2, we deal with degenerate Hamiltonian diffeomorphisms $\mathbb{1} \in \text{Ham}(M)$ and $\phi \times \mathbb{1} \in \text{Ham}(\Sigma \times N)$ (for Σ a closed symplectic surface of higher genus, and N a monotone symplectic manifold), which, however, are of Morse-Bott degeneracy in the appropriate classes of orbits. Associating a persistence module to this situation can be done in two different ways. First, we may perturb $\mathbb{1}$ and obtain a persistence module as a limit when perturbations tend to zero. More precisely, we fix a Morse function h , replace $\mathbb{1}$ with the flow of $\delta \cdot h$, and look at the appropriate persistence modules as $\delta \rightarrow 0$. Using standard action estimates, one sees that these modules converge in interleaving distance to a well-defined genuine persistence module (which is uniquely determined up to isomorphism by this property). In our cases, the spectrum of this persistence module is discrete, since the set of the critical values of the action functional of the zero Hamiltonian is discrete. This approach is essentially the same as the one described in [121, Definition 2.8]. The second way to proceed is by using Frauenfelder's approach of cascades [64] to the Morse-Bott case, which readily yields a persistence module by the same procedure as in [121, 147].

Let us now describe the product structure which we will be using. First note that the Novikov field $\Lambda_{\mathbb{K}}$ admits a non-Archimedean valuation

$$\nu : \Lambda_{\mathbb{K}} \rightarrow \mathbb{R} \cup \{-\infty\}, \quad \sum_{n \in \mathbb{Z}} a_n q^n \mapsto \max\{n \cdot (\kappa c_N) \mid a_n \neq 0\}.$$

³All the considerations in this section also apply to α which is symplectically atoroidal, meaning $\omega = c_1 = 0$ on $\pi_1(\mathcal{L}_\alpha M)$.

This valuation naturally extends to $QH(M)$, by declaring that $\nu(x) = 0$ for all non-zero $x \in H_*(M; \mathbb{K}) \otimes 1$. Now, in a fixed degree $r \in \mathbb{Z}$, $QH_r^t(M)$ is defined as $QH_r^t(M) = \{x \in QH_r(M) \mid \nu(x) < t\}$. The product map

$$* : QH_*^t(M) \otimes HF_*^s(\tilde{f})_\alpha \rightarrow HF_*^{t+s}(\tilde{f})_\alpha,$$

has a direct description via intersection of cycles representing elements in $QH_*^t(N)$ with Floer trajectories. This product structure is sometimes referred to as the *quantum module structure* since in the filtration-free setting it describes Floer homology as a module over quantum homology (see [111, Example 3.4] or [136]).

Another way to define this product is by exploiting the fact that $HF_*^t(\mathbb{1})_{pt} = QH_*^t(M)$, where $HF_*^t(\mathbb{1})_{pt}$ is understood in the sense of Remark 3.3.1 and $QH_*^t(M)$ is made into a persistence module by defining structure maps on it to be the obvious inclusions. Now the product map

$$* : HF_*^t(\mathbb{1})_{pt} \otimes HF_*^s(\tilde{f})_\alpha \rightarrow HF_*^{t+s}(\tilde{f})_\alpha \quad (3.5)$$

has a different description depending on which of the two definitions of $HF_*^t(\mathbb{1})_{pt}$ we adopt. In the perturbative setting, it is defined by counting pairs of pants on the chain level and the action estimates follow from [135, Section 4.1], [53], [102, Section 6.2], while in the Morse-Bott setting, the product takes the form of counting "spiked cylinders", quite similar to the definition of the PSS map [111] (see for example [33] and references therein for details on the more complicated, Lagrangian, version).

Let us examine some of the properties of this product.

Denote by \tilde{d} the Hofer's pseudo-distance on $\widetilde{\text{Ham}}(M)$ and by d the Hofer's distance on $\text{Ham}(M)$. We write $\tilde{f} \in \widetilde{\text{Ham}}(M)$ for a homotopy class of paths relative endpoints in $\text{Ham}(M)$ and $f \in \text{Ham}(M)$ for its endpoint. Let $\nu : QH(M) \rightarrow \mathbb{R}$ be the natural valuation. Fixing homogeneous $a \in QH(M)$ we obtain a map

$$a* : HF_r^t(\tilde{f})_\alpha \rightarrow HF_{r-2n+\deg a}^{t+\nu(a)}(\tilde{f})_\alpha.$$

The map $a*$ is a persistence module morphism between $V_r^t = HF_r^t(\tilde{f})_\alpha$ and $\tilde{V}_r^t = HF_{r-2n+\deg a}^{t+\nu(a)}(\tilde{f})_\alpha$. Moreover, it follows from standard considerations in Floer theory that $a*$ commutes with continuation maps

$$C(F, G) : HF_r^t(F)_\alpha \rightarrow HF_r^{t+\mathcal{E}^+(G-F)}(G)_\alpha,$$

where $\mathcal{E}^+(G - F) = \int_0^1 \max_M (G_t - F_t) dt$.

Now, let $g \in \text{Ham}(M)$ and define a map

$$P(g) : HF_*^t(\tilde{f})_\alpha \rightarrow HF_*^t(g \circ \tilde{f} \circ g^{-1})_\alpha,$$

by acting with g on all the objects appearing in the construction of Floer chain complex. More precisely, on the chain level $P(g)$ defines an isomorphism of filtered chain complexes

$$P(g) : (CF(H, J)_\alpha, \mathcal{A}_H) \rightarrow ((CF(H \circ g^{-1}, g_*(J))_\alpha, \mathcal{A}_{H \circ g^{-1}})),$$

by sending a periodic orbit $z(t)$ of H to a periodic orbit $g(z(t))$ of $H \circ g^{-1}$. This map is called *the push-forward map* (see [121] for a detailed treatment of push-forward maps). One can check that $P(g)$ and a_* commute.

Our objects of interest are Floer persistence modules of the form $HF_*^t(\tilde{f}^p)_\alpha$ for $\tilde{f} \in \widetilde{\text{Ham}}(M)$. In this case $P(f) : HF_*^t(\tilde{f}^p)_\alpha \rightarrow HF_*^t(\tilde{f}^p)_\alpha$ defines a \mathbb{Z}_p action on $HF_*^t(\tilde{f}^p)_\alpha$ and we get a \mathbb{Z}_p Floer persistence module. Since $P(f)$ and a_* commute, a_* is a \mathbb{Z}_p persistence module morphism and we wish to treat it as an operator on $HF_*^t(\tilde{f}^p)_\alpha$ and apply considerations from Section 3.2.5. To do so, define a \mathbb{Z}_p persistence module

$$W_r^t(a, \tilde{f}^p) = \text{im}(a_*) = (a_*)(HF_r^t(\tilde{f}^p)_\alpha) \subset HF_{r-2n+\text{deg } a}^{t+\nu(a)}(\tilde{f}^p)_\alpha,$$

with \mathbb{Z}_p action given by $P(f)$. Denote by F_t and G_t normalized 1-periodic Hamiltonians generating paths in $\text{Ham}(M)$ which represent classes of \tilde{f} and \tilde{g} in $\widetilde{\text{Ham}}(M)$ respectively and by $F_t^{(p)} = pF_{pt}$ and $G_t^{(p)} = pG_{pt}$ normalized 1-periodic Hamiltonians generating paths which represent \tilde{f}^p and \tilde{g}^p . Continuation maps

$$HF_r^t(F^{(p)})_\alpha \xrightarrow{C(F^{(p)}, G^{(p)})} HF_r^{t+p \cdot \mathcal{E}^+(G-F)}(G^{(p)})_\alpha,$$

and

$$HF_r^{t+p \cdot \mathcal{E}^+(G-F)}(G^{(p)})_\alpha \xrightarrow{C(G^{(p)}, F^{(p)})} HF_r^{t+p \cdot (\mathcal{E}^+(G-F) - \mathcal{E}^-(G-F))}(F^{(p)})_\alpha,$$

induce a $p \cdot (\mathcal{E}^+(G-F) - \mathcal{E}^-(G-F))$ op-equivariant interleaving between $HF_*^t(\tilde{f}^p)_\alpha$ and $HF_*^t(\tilde{g}^p)_\alpha$, where $\mathcal{E}^-(G-F) = \int_0^1 \min_M(G_t - F_t) dt$. Taking infimum over all F and G generating $\tilde{f}, \tilde{g} \in \widetilde{\text{Ham}}(M)$ we get that

$$p \cdot \tilde{d}(\tilde{f}, \tilde{g}) \geq \widehat{d}_{\text{op-inter}}(HF_*^t(\tilde{f}^p), HF_*^t(\tilde{g}^p)),$$

which together with (3.3) gives us

$$p \cdot \tilde{d}(\tilde{f}, \tilde{g}) \geq \widehat{d}_{\text{op-inter}}(HF_*^t(\tilde{f}^p), HF_*^t(\tilde{g}^p)) \geq \widehat{d}_{\text{inter}}(W_*^t(a, \tilde{f}^p), W_*^t(a, \tilde{g}^p)). \quad (3.6)$$

Remark 3.3.2. Let $f \in \text{Ham}(M)$ and fix a lift $\tilde{f} \in \widetilde{\text{Ham}}(M)$ of f . We can use $W_r^t(a, \tilde{f}^p)$ to estimate Hofer's distance from f to p -th powers inside $\text{Ham}(M)$. Indeed, denote by $\text{Powers}_p(M) \subset \text{Ham}(M)$ the set of all p -th powers of Hamiltonian diffeomorphisms and by $\widetilde{\text{Powers}}_p(M) \subset \widetilde{\text{Ham}}(M)$ the set of all lifts of elements from $\text{Powers}_p(M)$. In other words $\widetilde{\text{Powers}}_p(M) = \pi^{-1}(\text{Powers}_p(M))$ under the natural projection $\pi : \widetilde{\text{Ham}}(M) \rightarrow \text{Ham}(M)$. For $\tilde{g} \in \widetilde{\text{Powers}}_p(M)$, we have that $W_r^t(a, \tilde{g}^p)$ is

a full p -th power persistence module because $g = \phi^p$ implies $P(\phi)^p = P(g)$ and $P(\phi)$ restricts to $W_r^t(a, \tilde{g}^p)$ because $P(\phi)$ and a^* commute. It follows that $\mu_p(W_r^t(a, \tilde{g}^p)) = 0$ and thus by (3.4)

$$|\mu_p(W_r^t(a, \tilde{f}^p))| = |\mu_p(W_r^t(a, \tilde{f}^p)) - \mu_p(W_r^t(a, \tilde{g}^p))| \leq d_{\text{bottle}}(\mathcal{B}(L_\zeta), \mathcal{B}(K_\zeta)),$$

where L_ζ and K_ζ denote ζ -eigenspaces of $P(f)$ and $P(g)$ inside $W_r^t(a, \tilde{f}^p)$ and $W_r^t(a, \tilde{g}^p)$ respectively. Now, combining (3.2) with the isometry theorem yields

$$|\mu_p(W_r^t(a, \tilde{f}^p))| \leq d_{\text{bottle}}(\mathcal{B}(L_\zeta), \mathcal{B}(K_\zeta)) \leq \widehat{d}_{\text{inter}}(W_r^t(a, \tilde{f}^p), W_r^t(a, \tilde{g}^p)),$$

which together with (3.6) gives us

$$|\mu_p(W_r^t(a, \tilde{f}^p))| \leq \widehat{d}_{\text{inter}}(W_r^t(a, \tilde{f}^p), W_r^t(a, \tilde{g}^p)) \leq p \cdot \tilde{d}(\tilde{f}, \tilde{g}).$$

Finally, we have

$$d(f, \text{Powers}_p(M)) = \tilde{d}(\tilde{f}, \widetilde{\text{Powers}_p(M)}) \geq \frac{1}{p} \cdot |\mu_p(W_r^t(a, \tilde{f}^p))|. \quad (3.7)$$

3.3.2 Stabilization and the egg-beater example

We now turn to a manifold M of the form $M = \Sigma \times N$, where Σ is surface of genus at least 4 and N is a spherically monotone symplectic manifold with monotonicity constant κ . The element $\tilde{\psi}_\lambda \in \widetilde{\text{Ham}}(M)$ which we consider is

$$\tilde{\psi}_\lambda = \tilde{\varphi}_\lambda^p \times \mathbb{1}, \quad \tilde{\varphi}_\lambda \in \widetilde{\text{Ham}}(\Sigma), \quad \mathbb{1} \in \widetilde{\text{Ham}}(N),$$

where $\tilde{\varphi}_\lambda$ is given by the *egg-beater flow* on Σ , with mixing parameter λ . Construction and detailed analysis of the egg-beater flow are carried out in [8, 121]. What we will use is that there exists a family of Hamiltonian flows $\tilde{\varphi}_\lambda$ on Σ , depending on an unbounded increasing real parameter λ , along with a family of classes of free loops α_λ on Σ which satisfy:

- 1) $\tilde{\varphi}_\lambda^p$ has exactly 2^{2p} p -tuples of fixed points with same indices and actions $\{z, \varphi_\lambda(z), \dots, \varphi_\lambda^{p-1}(z)\}$, for sufficiently large λ ;
- 2) If z_1 and z_2 belong to different p -tuples their action differences satisfy

$$|\mathcal{A}(z_1) - \mathcal{A}(z_2)| \geq c_0 \lambda + O(1);$$

- 3) The indices of all fixed points are bounded by a constant which does not depend on λ .

The class $\bar{\alpha}_\lambda \in \pi_0(\mathcal{LM})$ which we consider is a product of classes

$$\bar{\alpha}_\lambda = \alpha_\lambda \times pt, \quad \alpha_\lambda \in \pi_0(\mathcal{L}\Sigma),$$

Σ being symplectically α_λ -atoroidal. Our manifold M will be $\bar{\alpha}_\lambda$ -toroidally monotone with same monotonicity constant κ . We will leave out these classes from the notation and write $HF_*^t(\tilde{\varphi}_\lambda^p \times \mathbb{1})$ and $HF_*^t(\tilde{\varphi}_\lambda^p)$ for $HF_*^t(\tilde{\varphi}_\lambda^p \times \mathbb{1})_{\bar{\alpha}_\lambda}$ and $HF_*^t(\tilde{\varphi}_\lambda^p)_{\alpha_\lambda}$. Let us now work out the example which proves Theorem 3.1.2.

Proposition 3.3.3. *Let $\tilde{\varphi}_\lambda$ be the egg-beater flow and assume $e \in QH(N)$ satisfies assumptions of Theorem 3.1.2. There exists $k \in \mathbb{Z}$ such that*

$$\mu_p(W_k^t([\Sigma] \otimes e, \tilde{\varphi}_\lambda^p \times \mathbb{1})) \geq c\lambda + O(1),$$

for some $c > 0$, when $\lambda \rightarrow +\infty$. Here $[\Sigma] \otimes e \in QH(M) = QH(\Sigma \times N)$.

Proof. Let α_1, α_2 be two toroidally monotone classes of free loops in symplectic manifolds M_1 and M_2 , with the same monotonicity constant κ (we may also take one of both of them to be atoroidal) and let $\tilde{\phi} \in \widetilde{\text{Ham}}(M_1), \tilde{\psi} \in \widetilde{\text{Ham}}(M_2)$. The manifold $M_1 \times M_2$ is symplectic and the class $\alpha_1 \times \alpha_2$ is toroidally monotone with the same monotonicity constant κ . Now, we apply Proposition 3.2.6 for general filtered homologies to Floer chain complexes filtered by the action functional and Floer persistence modules to get the short exact sequence:

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=k} (HF_i(\tilde{\phi})_{\alpha_1} \otimes HF_j(\tilde{\psi})_{\alpha_2})^t &\xrightarrow{K} HF_k^t(\tilde{\phi} \times \tilde{\psi})_{\alpha_1 \times \alpha_2} \rightarrow \\ &\rightarrow \bigoplus_{i+j=k-1} (Tor(HF_i(\tilde{\phi})_{\alpha_1}, HF_j(\tilde{\psi})_{\alpha_2}))^t \rightarrow 0, \end{aligned}$$

for $K([\sum_i \lambda_i x_i] \otimes [\sum_j \mu_j y_j]) = [\sum_{i,j} \lambda_i \mu_j x_i \otimes y_j]$.

In our case $\tilde{\phi} = \tilde{\varphi}_\lambda^p$, $\alpha_1 = \alpha_\lambda$, $\tilde{\psi} = \mathbb{1} \in \widetilde{\text{Ham}}(N), \alpha_2 = \{pt\}$ and we have $HF_*^t(\mathbb{1})_{\{pt\}} = QH_*^t(N)$, where $QH_*^t(N) = \{x \in QH_*(N) \mid \nu(x) < t\}$ is a persistence module with trivial structure maps given by $\pi_{s,t}(x) = x$ since $QH_*^s(N) \subset QH_*^t(N)$ for $s \leq t$.

This readily gives us that the barcode of $QH_*^t(N)$ has only infinite bars and thus $QH_*^t(N)$ is a projective persistence module and $Tor(HF_i(\tilde{\varphi}_\lambda^p), QH_j(N)) = 0$ for all $j \in \mathbb{Z}$, which implies that

$$K : \bigoplus_{i+j=k} (HF_i(\tilde{\varphi}_\lambda^p) \otimes QH_j(N))^t \rightarrow HF_k^t(\tilde{\varphi}_\lambda^p \times \mathbb{1}),$$

is an isomorphism. Moreover, it holds that $P(\varphi_\lambda \times \mathbb{1}) \circ K = K \circ (P(\varphi_\lambda) \otimes \mathbb{1})$ (see [121] for a proof in the atoroidal case, the proof in the toroidally monotone case is the same) and thus K is also an isomorphism of \mathbb{Z}_p persistence modules. Now,

consider multiplication by e as a persistence module morphism $(e^*) : QH_r^t(N) \rightarrow QH_{r-2n+\deg e}^{t+\nu(e)}(N)$ between $QH_r^t(N)$ and shifted module $QH_{r-2n+\deg e}^{t+\nu(e)}(N) = QH_{r-2n+\deg e}^t(N)[\nu(e)]$, for every $r \in \mathbb{Z}$. Our product map splits on the components of the product, i.e., it enters the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{i+j=k} (HF_i(\tilde{\varphi}_\lambda^p) \otimes QH_j(N))^t & \xrightarrow{K} & HF_k^t(\tilde{\varphi}_\lambda^p \times \mathbb{1}) \\ \downarrow \mathbb{1} \otimes (e^*) & & \downarrow ([\Sigma] \otimes e)^* \\ \bigoplus_{i+j=k} (HF_i(\tilde{\varphi}_\lambda^p) \otimes QH_{j-2n+\deg e}(N)[\nu(e)])^t & \xrightarrow{K} & HF_{k-2n+\deg e}^t(\tilde{\varphi}_\lambda^p \times \mathbb{1})[\nu(e)] \end{array}$$

where each arrow represents a \mathbb{Z}_p persistence module morphism. Using this diagram we calculate

$$W_k^t = W_k^t([\Sigma] \otimes e, \tilde{\varphi}_\lambda^p \times \mathbb{1}) = \bigoplus_{r \in I_k} (HF_{k-r}(\tilde{\varphi}_\lambda^p) \otimes (e^*)(QH_r(N)))^t,$$

where I_k is the set of all r such that there exists a fixed point of φ_λ^p of index $k-r$ and $\text{im}(e^*)^t = (e^*)(QH_r^t(N)) \subset QH_{r-2n+\deg e}^t(N)[\nu(e)]$. Let us describe the barcode of $\text{im}(e^*)^t$ explicitly.

First, note that we have inclusions of all $QH_r^t(N)$ into the full quantum homology $QH_r(N) = QH_r^{+\infty}(N)$ and moreover for $s \leq t$, $QH_r^s(N) \subset QH_r^t(N) \subset QH_r(N)$ and structure maps act as $\mathbb{1}$ under these inclusions. Now, $E_r = e^*(QH_r(N)) \subset QH_{r-2n+\deg e}(N)$ is the image of the full quantum homology group $QH_r(N)$ and by definition $\dim_{\mathbb{K}} E_r = b_r(e)$.

We may also look at E_r as a persistence submodule of the shifted module $E_r^t \subset QH_{r-2n+\deg e}^t(N)[\nu(e)]$ and $(e^*) : QH_r^t(N) \rightarrow E_r^t$ is a persistence module morphism. Since the structure maps on E_r^t are restrictions of structure maps on $QH_{r-2n+\deg e}(N)[\nu(e)]$, we again have that they act as $\mathbb{1}$ under the inclusions to the full quantum homology group $QH_{r-2n+\deg e}(N)$ and the same holds for $\text{im}(e^*)^t$. This implies that the barcode of $\text{im}(e^*)^t$ contains no finite bars. Now, if we denote $a_r = \min\{\nu(x) | x \in QH_r(N)\}$ and $A_r = \max\{\nu(x) | x \in QH_r(N)\}$, it follows that $(e^*)(QH_r^t(N)) = 0$ for $t \leq a_r$ and $(e^*)(QH_r^t(N)) = E_r$ for $t > A_r$ and thus the barcode of $\text{im}(e^*)^t$ consists of bars $(c_{r,1}, +\infty), \dots, (c_{r,b_r(e)}, +\infty)$ where $a_r \leq c_{r,1} \leq \dots \leq c_{r,b_r(e)} \leq A_r$. Moreover, since the \mathbb{Z}_p action on $QH_r^t(N)$ is trivial for all r we have that

$$\text{im}(e^*)^t = \bigoplus_{i=1}^{b_r(e)} (\mathbb{K}_{(c_{r,i}, +\infty)}^t, \mathbb{1}),$$

as \mathbb{Z}_p persistence modules, which together with the above diagram gives us

$$(W_k^t, P(\varphi_\lambda \times \mathbb{1})) \cong \bigoplus_{r \in I_k} \left(\left(HF_{k-r}(\tilde{\varphi}_\lambda^p) \otimes \bigoplus_{i=1}^{b_r(e)} \mathbb{K}_{(c_{r,i}, +\infty)} \right)^t, P(\varphi_\lambda) \otimes \mathbb{1} \right),$$

the isomorphism being given by K . Elementary calculations on interval persistence modules now imply

$$(W_k^t, P(\varphi_\lambda \times \mathbb{1})) \cong \bigoplus_{r \in I_k} \bigoplus_{i=1}^{b_r(e)} \left(HF_{k-r}^{t-c_{r,i}}(\tilde{\varphi}_\lambda^p)_{\alpha_\lambda}, P(\varphi_\lambda) \right).$$

Denoting the ζ -eigenspace of $(HF_k^t(\varphi_\lambda^p)_{\alpha_\lambda}, P(\varphi_\lambda))$ by $L_{k,\zeta}^t$ and ζ -eigenspace of $(W_k^t, P(\varphi_\lambda \times \mathbb{1}))$ by L_ζ^t we have by Lemma 3.2.14

$$L_\zeta^t \cong \bigoplus_{r \in I_k} \bigoplus_{i=1}^{b_r(e)} L_{k-r,\zeta}^{t-c_{r,i}}.$$

The indices of fixed points of the egg-beater map are uniformly bounded (the bound does not depend on λ) and thus we have $|r| < M$ for $r \in I_k$ for some constant M not depending on λ . This also gives us that there exists a constant $C > 0$ independent of λ such that $|a_r| < C$ and $|A_r| < C$ for all $r \in I_k$ and thus $|c_{r,i}| < C$ for all $r \in I_k, i = 1, \dots, b_r(e)$. By Lemma 2.1.25 we have that

$$d_{bottle} \left(\mathcal{B}(L_\zeta^t), \mathcal{B} \left(\bigoplus_{r \in I_k} (L_{k-r,\zeta}^t)^{b_r(e)} \right) \right) < C,$$

and hence by Lipschitz property of μ_p we have

$$\mu_p(W_k^t) \geq \mu_{p,\zeta}(W_k^t) = \mu_p \left(\bigoplus_{r \in I_k} \bigoplus_{i=1}^{b_r(e)} L_{k-r,\zeta}^{t-c_{r,i}} \right) \geq \mu_p \left(\bigoplus_{r \in I_k} (L_{k-r,\zeta}^t)^{b_r(e)} \right) - C.$$

Assume now that $p \nmid b_{r_0}(e)$ and that the index of a fixed point z_0 of φ_λ^p with minimal action $A = A(z_0)$ in class α_λ is d_0 . Taking $k = d_0 + r_0$ we have that

$$\bigoplus_{r \in I_k} (L_{k-r,\zeta}^t)^{b_r(e)} = (L_{d_0,\zeta}^t)^{b_{r_0}(e)} \oplus \bigoplus_{r \neq r_0} (L_{k-r,\zeta}^t)^{b_r(e)}.$$

If z is a fixed point of φ_λ^p with action $A(z) \neq A$ it follows that $A(z) \geq B = A + c_0\lambda + O(1)$ and we have that

$$m \left(\mathcal{B} \left(\bigoplus_{r \in I_k} (L_{k-r,\zeta}^t)^{b_r(e)} \right), (A, B] \right) = b_{r_0}(e).$$

Now, $p \nmid b_{r_0}(e)$ and thus

$$\mu_p \left(\mathcal{B} \left(\bigoplus_{r \in I_k} (L_{k-r,\zeta}^t)^{b_r(e)} \right) \right) \geq \frac{c_0}{4}\lambda + O(1),$$

which gives us $\mu_p(W_k^t) \geq c\lambda + O(1)$ as claimed. \square

The proof of Theorem 3.1.2 follows directly from Proposition 3.3.3 and Remark 3.3.2.

3.3.3 Erratum: behavior of μ_p under stabilization in the aspherical case.

This erratum is written in order to correct a mistake in Theorem 4.24 in [121]. The main theorem (which this mistake could potentially affect), [121, Theorem 1.3], holds still. See Theorem E1 and the update to the proof of [121, Theorem 1.3] below.

Alternatively, as noted in Example 3.1.4, [121, Theorem 1.3] holds as a special case of the main theorem, Theorem 3.1.2 of the current chapter, and its proof extends the proof of [121, Theorem 1.3].

In fact the estimate $\mu_p(\phi) \leq \mu_p(\phi \times \mathbb{1})$ cannot be expected to hold, as can be seen by elementary examples. The error in the proof of Theorem 4.24 is contained in the implication "Thus we are left with $i = 0 \dots$ " because the barcodes $\mathcal{B}_{r-i}(\phi)_\zeta$ for $i > 0$ can have I and I^{2c} with different μ multiplicities, thus affecting the value of $\mu_{p,\zeta}(r, \phi \times \mathbb{1}_N)$.

Denote

$$\gamma_{p,\zeta}(r, \phi) = \frac{1}{2} \max_{i>0} \beta(\mathcal{B}_{r-i}(\phi)_\zeta).$$

By (26), and the remarks on the Künneth formula in the proof of Theorem 4.24, it is immediate that

$$\mu_{p,\zeta}(r, \phi \times \mathbb{1}_N) \geq \mu_{p,\zeta}(r, \phi) - \gamma_{p,\zeta}(r, \phi).$$

Indeed

$$d_{\text{bottle}}(\mathcal{B}_r(\phi)_\zeta, \mathcal{B}_r(\phi \times \mathbb{1}_N)_\zeta) \leq \gamma_{p,\zeta}(r, \phi),$$

which can be seen by erasing all intervals corresponding to $(b_i(N)$ -copies of) the barcode $\mathcal{B}_{r-i}(\phi)_\zeta$ (recall that $\beta(\mathcal{B})$ is the maximal length of a finite bar in the barcode \mathcal{B}).

Thus denoting

$$\mu_{p,\zeta}^{\text{reduced}}(r, \phi) = \mu_{p,\zeta}(r, \phi) - \gamma_{p,\zeta}(r, \phi),$$

and

$$\mu_p^{\text{reduced}}(r, \phi) = \max_{\zeta} \mu_{p,\zeta}^{\text{reduced}}(r, \phi)$$

we replace Theorem 4.24 by the following.

Theorem E1. For $\phi \in \text{Ham}(M)$, $\alpha \in \pi_0(\mathcal{L}M)$, and any closed connected symplectically aspherical manifold N , consider the stabilization $\phi \times \mathbb{1} \in \text{Ham}(M \times N)$ of ϕ . Then we have

$$\mu_p^{\text{reduced}}(\phi) \leq \mu_p(\phi \times \mathbb{1}_N) \leq \mu_p(\phi),$$

the value $\mu_p(\phi \times \mathbb{1}_N)$ being computed in the class $\alpha \times pt_N$ in $\pi_0(\mathcal{L}(M \times N))$.

Now we turn to Section 5.1 and show how to adapt the proof of Theorem 1.3 in view of the corrected Theorem E1 above. The necessary changes are:

- The sentence

”Further, among the 2^{2p} p -tuples of fixed points of ϕ_λ^p in the class α_λ choose the p -tuple, say $\{z, \phi_\lambda(z), \dots, (\phi_\lambda)^{p-1}(z)\}$ with the minimal action. Let r be the index of z .”

should be corrected to

”Further, among the 2^{2p} p -tuples of fixed points of ϕ_λ^p in the class α_λ choose the p -tuple, say $\{z, \phi_\lambda(z), \dots, (\phi_\lambda)^{p-1}(z)\}$ with the minimal index r , and minimal action among p -tuples of this index.”

- The passage

”By the definition of the multiplicity-sensitive spread, we conclude that $\mu_p(\phi_\lambda) \geq \lambda(c - 2\varepsilon)/4$ ”

should read

”By the definition of the multiplicity-sensitive spread and the observation that $\gamma_{p,\zeta}(r, \phi_\lambda) = 0$, we conclude that $\mu_p^{\text{reduced}}(\phi_\lambda) \geq \lambda(c - 2\varepsilon)/4$ ”

Chapter 4

Persistence modules, symplectic Banach-Mazur distance and Riemannian metrics

4.1 Introduction

In this chapter, we will consider persistence modules coming from filtered symplectic homology in order to study fiberwise star-shaped domains in the cotangent bundle of a fixed manifold in a quantitative fashion.

4.1.1 The metrical set-up

The quantitative perspective which we wish to adopt has its roots in the concept of Banach-Mazur distance, initially appearing in functional analysis with the aim of comparing convex bodies. Let M be a closed, orientable manifold of dimension n . Its cotangent bundle T^*M is equipped with a canonical symplectic form $\omega_{can} = d\lambda_{can}$, where λ_{can} is the Liouville form, and a canonical vector field X given by $i_X(\omega_{can}) = \lambda_{can}$ called Liouville vector field. We call a domain $U \subset T^*M$ *admissible* if it is a compact, fiberwise star-shaped domain, centered at the zero section $0_M \subset U \subset T^*M$, whose boundary ∂U is smooth and such that $X \pitchfork \partial U$. Restriction of the Liouville form to the boundary of an admissible domain is a contact form, i.e. $(\partial U, \lambda_{can}|_{\partial U})$ is a contact manifold. Denote

$$\mathcal{C}_M = \{\text{admissible domains } U \text{ in } T^*M\}.$$

For two admissible domains $U, V \in \mathcal{C}_M$, an embedding $\phi : U \rightarrow V$ satisfying $\phi^*\lambda_{can} - \lambda_{can} = df$ for some smooth function $f : U \rightarrow \mathbb{R}$ is called a *Liouville embedding*. Denote the set of homotopy classes of free loops in M by $\tilde{\pi}_1(M)$. Notice that any

$U \in \mathcal{C}_M$ deformation retracts to the zero section 0_M of T^*M , and the projection $\pi : T^*M \rightarrow M$ restricted to U induces a homotopy equivalence $\pi|_U : U \rightarrow M$. Thus, any Liouville embedding $\phi : U \rightarrow V$ between two admissible domains in T^*M induces a map ϕ_* on $\tilde{\pi}_1(M)$. Majority of maps which we will consider in this chapter will be a special type of Liouville embeddings which are defined as follows.

Definition 4.1.1. Given two admissible domains $U, V \in \mathcal{C}_M$, a Liouville embedding $\phi : U \rightarrow V$ is $\tilde{\pi}_1$ -trivial if $\phi_*\alpha = \alpha$ for all $\alpha \in \tilde{\pi}_1(M)$. We adopt the notation $U \xrightarrow{\phi} V$ for a $\tilde{\pi}_1$ -trivial Liouville embedding $\phi : U \rightarrow V$.

One readily checks that the composition of two $\tilde{\pi}_1$ -trivial Liouville embeddings is again a $\tilde{\pi}_1$ -trivial Liouville embedding. The following definition modifies a key definition from [75].

Definition 4.1.2. Let $U \subset V$ be two admissible domains in T^*M and $\phi : U \rightarrow V$ a Liouville embedding. We call ϕ *strongly unknotted* if there exists an isotopy $\{\phi_t\}_{t \in [0,1]}$ such that each $\phi_t : U \rightarrow V$ is a Liouville embedding and $\phi_0 = i_U$, $\phi_1 = \phi$, i_U being the inclusion $i_U : U \rightarrow V$.

Let us illustrate these concepts on an example coming from Riemannian geometry. This example is also going to be the main example considered in this chapter.

Example 4.1.3. Let (M, g) be a closed, orientable Riemannian manifold with induced norm $\|\cdot\|_g : TM \rightarrow \mathbb{R}$ and denote the unit ball at a point q by $B_1(g)_q = \{x \in T_qM \mid \|x\|_g \leq 1\}$. The dual norm $\|\cdot\|_{g^*}$ on T^*M is given by $\|\xi_q\|_{g^*} = \max\{\xi_q(x) \mid x \in B_1(g)_q\}$ and the unit coball $B_1^*(g^*)_q = \{p \in T_q^*M \mid \|p\|_{g^*} \leq 1\}$ defines a convex set in T_q^*M . Denoting the unit codisc bundle (union of unit coballs over all points of the manifold) by U_g^*M , we have that U_g^*M is an admissible domain in T^*M . The boundary ∂U_g^*M is the unit cosphere bundle and the Reeb flow on $(\partial U_g^*M, \lambda_{can}|_{\partial U_g^*M})$ is the cogeodesic flow of g . Now, if for every $q \in M$ and every $x \in T_qM$, it holds $\|x\|_{g_1} \leq \|x\|_{g_2}$, we have that $U_{g_1}^*M \subset U_{g_2}^*M$ and inclusion $i : U_{g_1}^*M \rightarrow U_{g_2}^*M$ is a $\tilde{\pi}_1$ -trivial Liouville embedding. Obviously, this embedding is also strongly unknotted.

We will now define the distance which we wish to consider.

Definition 4.1.4. (Ostrover, Polterovich, Gutt, Usher [117,118,105,120,75,148]) For $U, V \in \mathcal{C}_M$, we define *symplectic Banach-Mazur distance* $d_{SBM}(U, V)$ by

$$d_{SBM}(U, V) = \inf \left\{ \ln C \mid \begin{array}{l} \exists \frac{1}{C}U \xrightarrow{\phi} V \xrightarrow{\psi} CU \text{ (and hence } \frac{1}{C}V \xrightarrow{\psi(C^{-1})} U \xrightarrow{\phi(C)} CV) \\ \text{s.t. } \psi \circ \phi \text{ and } \phi(C) \circ \psi(C^{-1}) \text{ are strongly unknotted} \end{array} \right\}$$

Here multiplication CU applies on the covector component, i.e. for any $(q, p) \in U$, $C(q, p) = (q, Cp)$. Moreover, $\phi(C)$ is defined as $\phi(C)(q, p) = C\phi(q, p/C)$, for $(q, p) \in U$, where again multiplication acts on the covector component and $\psi(C^{-1})$ is defined similarly.

In order to study unit codisc bundles of different Riemannian metrics with respect to d_{SBM} , we will need an auxiliary distance defined on the space of Riemannian metrics on M . Denote by

$$\mathcal{G}_M = \{\text{Riemannian metrics } g \text{ on } M\}.$$

Similarly to d_{SBM} on \mathcal{C}_M , we have

Definition 4.1.5. For $g_1, g_2 \in \mathcal{G}_M$, we define *Riemannian Banach-Mazur distance* denoted by $d_{RBM}(g_1, g_2)$ as follows,

$$d_{RBM}(g_1, g_2) = \inf \left\{ \ln C \in [0, \infty) \mid \exists \phi \in \text{Diff}_0(M) \text{ s.t. } \frac{1}{C}g_1 \preceq \phi^*g_2 \preceq Cg_1 \right\},$$

where $g_1 \preceq g_2$ means that for any $q \in M$ and any $x \in T_qM$, $\|x\|_{g_1} \leq \|x\|_{g_2}$. $\text{Diff}_0(M)$ stands for the identity component of $\text{Diff}(M)$.

Remark 4.1.6. Riemannian Banach-Mazur distance can be considered a smooth isotopy version of the well-known Lipschitz distance, see Remark 4.2.9.

As we saw in Example 4.1.3, every Riemannian metric g defines a domain $U_g^*M \in \mathcal{C}_M$ and thus \mathcal{G}_M can be naturally identified with a subset of \mathcal{C}_M . With this in mind, d_{RBM} and d_{SBM} are two pseudo-metrics on \mathcal{G}_M which turn out to be comparable. More precisely, the following inequality is proven in Proposition 4.2.8

$$2 \cdot d_{SBM}(U_{g_1}^*M, U_{g_2}^*M) \leq d_{RBM}(g_1, g_2). \quad (4.1)$$

Recall that given a contact manifold (Y, μ) with Reeb flow φ_t^μ , a periodic Reeb orbit $\varphi_t^\mu(x)$, $\varphi_T^\mu(x) = x$ of period T is called *non-degenerate* if $\det(d\varphi_T^\mu|_{\ker \mu(x)} - \mathbb{1}_{\ker \mu(x)}) \neq 0$. If all periodic Reeb orbits are non-degenerate, contact manifold (Y, μ) is called *non-degenerate*. In the light of this definition an admissible domain U is called *non-degenerate* if $(\partial U, \lambda_{can}|_{\partial U})$ is a non-degenerate contact manifold. A classical tool used to study admissible domains is symplectic homology, denoted by $\text{SH}_*(U; \alpha)$, for $U \in \mathcal{C}_M$ and α a homotopy class of free loops in M . Assuming U is non-degenerate, a filtered version of symplectic homology $\text{SH}_*^a(U; \alpha)$, $a > 0$ can be viewed as a persistence module which we denote by $\mathbb{S}\mathbb{H}_{*,\alpha}(U)$, see Subsection 4.3.1. Multiplying the domain by $C > 0$ results in the proportional scaling of the filtration, that is $\text{SH}_*^{C^a}(CU; \alpha) = \text{SH}_*^a(U; \alpha)$ for all $a > 0$. In accordance with Definition 4.1.4 we introduce the logarithmic version of $\text{SH}_*(U; \alpha)$,

$$S_*^t(U; \alpha) = \text{SH}_*^{e^t}(U; \alpha), \quad t \in \mathbb{R},$$

which satisfies $S_*^{t+\ln C}(CU; \alpha) = S_*^t(U; \alpha)$. The resulting persistence module is denoted by $\mathbb{S}_{*,\alpha}(U)$. We are able to estimate d_{RBM} and d_{SBM} from below using the associated barcodes, namely the following stability property holds.

Theorem 4.1.7. For $U, V \in \mathcal{C}_M$ non-degenerate, denote the barcodes of $\mathbb{S}_{*,\alpha}(U)$ and $\mathbb{S}_{*,\alpha}(V)$ by $\mathbb{B}_{*,\alpha}(U)$ and $\mathbb{B}_{*,\alpha}(V)$ respectively. Then

$$d_{bottle}(\mathbb{B}_{*,\alpha}(U), \mathbb{B}_{*,\alpha}(V)) \leq d_{SBM}(U, V).$$

In particular, when $U = U_{g_1}^* M$ and $V = U_{g_2}^* M$, it follows from (4.1) that

$$2 \cdot d_{bottle}(\mathbb{B}_{*,\alpha}(U), \mathbb{B}_{*,\alpha}(V)) \leq d_{RBM}(g_1, g_2).$$

Precise definitions and a proof of this theorem are given in Section 4.3. Different versions of Theorem 4.1.7 for star-shaped domains in \mathbb{R}^{2n} can be found in [120] and [148].

Remark 4.1.8. (*Alternative definition*) One may give a definition of symplectic Banach-Mazur distance different from Definition 4.1.4 as follows.

Definition 4.1.9. Let $U, V \in \mathcal{C}_M$ and

$$\rho(U, V) = \inf \left\{ \ln C \in [0, \infty) \mid \begin{array}{l} \exists \frac{1}{C}U \xrightarrow{\phi} V \xrightarrow{\psi} CU \\ \text{s.t. } \psi \circ \phi \text{ is strongly unknotted} \end{array} \right\}.$$

We define $d'_{SBM}(U, V) = \max\{\rho(U, V), \rho(V, U)\}$.

One may prove that d'_{SBM} defines a pseudo-metric on \mathcal{C}_M in a similar way to the proof of Proposition 4.2.1. However, in order to prove the stability of d_{bottle} with respect to d'_{SBM} , i.e. an analogue of Theorem 4.1.7, one needs a stronger version of the classical isometry theorem for barcodes which was communicated to us by M. Usher, [149]. Quantities ρ and d'_{SBM} can be considered analogous to δ_f and d_f defined in [148], as explained in Subsection 1.2 of [148].

Remark 4.1.10. Throughout this chapter, we assume that the base manifold M is orientable. This is done in order to simplify considerations regarding the grading in symplectic homology, see Subsection 4.3.1. It seems likely that, using the results from [156], one may apply similar arguments and obtain analogous results in the non-orientable case.

Remark 4.1.11. All the persistence modules considered in this chapter are defined using conventions which guarantee that all the intervals in the corresponding barcodes have left endpoints closed and right endpoints open. In other words, all the bars are either equal to $(-\infty, +\infty)$ or of the form $[a, b)$ for $a < b \leq +\infty$ with finite a . Moreover, we sometimes use the set of parameters $t \in \mathbb{R}^+ = (0, +\infty)$ instead of $t \in \mathbb{R}$ in the definition of our persistence modules. This difference is non-essential because the two sets of parameters can be related by an order-preserving bijection, for example $\ln : \mathbb{R}^+ \rightarrow \mathbb{R}$.

4.1.2 Large-scale geometry of the space of Riemannian metrics

Recall that a map $\Phi : (X_1, d_1) \rightarrow (X_2, d_2)$ between two (pseudo-)metric spaces is called *quasi-isometric embedding* if there exist constants $A \geq 1, B \geq 0$ s.t.

$$\frac{1}{A}d_1(x, y) - B \leq d_2(\Phi(x), \Phi(y)) \leq Ad_1(x, y) + B,$$

for all $x, y \in X_1$.

From a general perspective, given a (pseudo-)metric space (X, d) , we wish to ask the following questions with the flavor of large-scale geometry.

- (A) What is the diameter of (X, d) ?
- (B) If $\text{diam}(X, d) = +\infty$, how many unbounded linearly independent directions are there in X ? More precisely, for which N does there exist a quasi-isometric embedding of \mathbb{R}^N into (X, d) ?

Our goal is to give partial answer to these questions for the space of admissible domains in T^*M , i.e. when $(X, d) = (\mathcal{C}_M, d_{SBM})$. In the case of Hofer's metric, i.e. $(X, d) = (\text{Ham}(M, \omega), d_{\text{Hofer}})$, these questions have been studied and partially answered using advanced tools from symplectic topology (see, for instance, [115] and [147]).

Before we state the main results we wish to point out that without imposing additional assumptions on spaces (\mathcal{C}_M, d_{SBM}) and (\mathcal{G}_M, d_{SBM}) it is easy to see that both of their diameters are infinite. This follows from the fact that d_{SBM} satisfies $d_{SBM}(U, CU) = \ln C$ for any $U \in \mathcal{C}_M$ and $C \geq 1$. Indeed, for any $C \geq 1$ it readily follows that $d_{SBM}(U, CU) \leq \ln C$ simply by taking ϕ and ψ in the definition of d_{SBM} to be inclusions. On the other hand, if there would exist some $C' < C$ such that $U/C' \hookrightarrow CU \hookrightarrow C'U$, the second embedding would contradict preservation of volume and hence $d_{SBM}(U, CU) = \ln C$. Thus, in order to make question (A) meaningful we must introduce certain normalizations. To this end we define

$$\bar{\mathcal{C}}_M = \{\text{admissible domains } U \text{ in } T^*M \text{ s.t. } \text{Vol}(U) = V_n\},$$

where $\text{Vol}(U) = \int_U \frac{(d\lambda_{can})^{\wedge n}}{n!} = \int_U \frac{(\omega_{can})^{\wedge n}}{n!}$ and V_n denotes the volume of the n -dimensional unit ball. Similarly, we define

$$\bar{\mathcal{G}}_M = \{\text{Riemannian metrics } g \text{ on } M \text{ s.t. } \text{Vol}_g(M) = 1 \text{ and } \text{diam}(M, g) \leq 100\}.$$

Note that when $U = U_g^*M$, one has $\text{Vol}(U) = V_n \cdot \text{Vol}_g(M)$ and hence we may include $\bar{\mathcal{G}}_M$ in $\bar{\mathcal{C}}_M$ via the map $g \rightarrow U_g^*M$. Slightly abusing the notation we write $\bar{\mathcal{G}}_M \subset \bar{\mathcal{C}}_M$.

Remark 4.1.12. We also wish to explain the restriction that we put on the diameter of (M, g) . Assume that $g_2 \preceq C^2 g_1$, for a constant $C \geq 1$. Now, for every smooth curve

γ in M it holds that $L_{g_2}(\gamma) \leq C \cdot L_{g_1}(\gamma)$, L_{g_i} denoting the length with respect to g_i , and thus $\text{diam}(M, g_2) \leq C \cdot \text{diam}(M, g_1)$. If $\text{diam}(M, g_1)$ is fixed and $\text{diam}(M, g_2) \rightarrow +\infty$, we see that $C \rightarrow +\infty$ and since $\text{diam}(M, g_2) = \text{diam}(M, \phi^*g_2)$ for all diffeomorphisms ϕ , we have that $d_{RBM}(g_1, g_2) \rightarrow +\infty$. This means that if there was no restriction on the diameter of (M, g) , the space (\mathcal{G}_M, d_{RBM}) would trivially have infinite diameter even if we fix the volume of M .

The following theorem is the main result of the chapter.

Theorem 4.1.13. *If $M = S^2$, then there exists a quasi-isometric embedding*

$$\Phi : ([0, \infty), |\cdot|) \rightarrow (\bar{\mathcal{G}}_M, d_{SBM}).$$

If $M = \Sigma$ is a closed, orientable surface whose genus is at least 1, then for every $N \in \mathbb{N}$ there exists a quasi-isometric embedding

$$\Phi : (\mathbb{R}^N, |\cdot|_\infty) \rightarrow (\bar{\mathcal{G}}_M, d_{SBM}).$$

Both statements remain true if we replace d_{SBM} by d_{RBM} .

Since $\bar{\mathcal{G}}_M \subset \bar{\mathcal{C}}_M$ we immediately obtain the following.

Corollary 4.1.14 (Theorem 1.3.2). *If $M = S^2$, then there exists a quasi-isometric embedding*

$$\Phi : ([0, \infty), |\cdot|) \rightarrow (\bar{\mathcal{C}}_M, d_{SBM}).$$

If $M = \Sigma$ is a closed, orientable surface whose genus is at least 1 then for every $N \in \mathbb{N}$ there exists a quasi-isometric embedding

$$\Phi : (\mathbb{R}^N, |\cdot|_\infty) \rightarrow (\bar{\mathcal{C}}_M, d_{SBM}).$$

Corollary 4.1.14 readily implies that if M is any closed, orientable surface it holds $\text{diam}(\bar{\mathcal{C}}_M, d_{SBM}) = +\infty$, which answers question **(A)**. However, regarding question **(B)**, we observe a sharp contrast between cases of a sphere and of higher genus surfaces. Indeed, when $M = \Sigma$ is a closed, orientable surface of positive genus, Corollary 4.1.14 proves the existence of many unbounded directions inside $(\bar{\mathcal{C}}_M, d_{SBM})$, namely there exist N unbounded directions inside $(\bar{\mathcal{C}}_M, d_{SBM})$ for any $N \in \mathbb{N}$. On the other hand when $M = S^2$ it provides only one unbounded direction. This contrast ultimately comes from the fact that $\pi_1(S^2) = 0$ while $\pi_1(\Sigma) \neq 0$. Nevertheless, we pose the following conjecture.

Conjecture 4.1.15. *For every $N \in \mathbb{N}$, there exists a quasi-isometric embedding*

$$\Phi : (\mathbb{R}^N, |\cdot|_\infty) \rightarrow (\bar{\mathcal{C}}_{S^2}, d_{SBM}).$$

We break down the proof of Theorem 4.1.13 into the following two propositions.

Proposition 4.1.16. *For any $\varepsilon > 0$, there exists a map $\Phi : [0, \infty) \rightarrow \bar{\mathcal{G}}_{S^2}$ such that for any $x, y \in [0, \infty)$,*

$$|x - y| - \varepsilon \leq 2d_{SBM}(U_{\Phi(x)}^*M, U_{\Phi(y)}^*M) \leq d_{RBM}(\Phi(x), \Phi(y)) \leq 2|x - y| + \varepsilon. \quad (4.2)$$

Proposition 4.1.17. *Let Σ be a closed, orientable surface of genus at least 1. Then for any $N \in \mathbb{N}$ and any $\varepsilon > 0$, there exists a map $\Phi : \mathbb{R}^N \rightarrow \bar{\mathcal{G}}_\Sigma$ such that for all $\vec{x}, \vec{y} \in \mathbb{R}^N$*

$$\frac{1}{4} \cdot |\vec{x} - \vec{y}|_\infty - \varepsilon \leq 2d_{SBM}(U_{\Phi(\vec{x})}^*M, U_{\Phi(\vec{y})}^*M) \leq d_{RBM}(\Phi(\vec{x}), \Phi(\vec{y})) \leq 4N \cdot |\vec{x} - \vec{y}|_\infty + \varepsilon.$$

The lower bounds are the most significant parts of Propositions 4.1.16 and 4.1.17. Their proofs use the technique of barcodes and occupy the entire Section 4.5. The upper bounds in both theorems are proven simultaneously in Subsection 4.6.3.

In order to construct quasi-isometric embeddings Φ as above, we consider geometric models which we call *bulked spheres* and *multi-bulked surfaces*. A bulked sphere is a surface of revolution. Roughly speaking, it is obtained as a connected sum of two spheres through a very narrow “neck” as shown in Figure 4.8. We analyze closed geodesics on a bulked sphere in Section 4.6. More precisely, in Subsection 4.6.1 we analyze the shortest non-constant closed geodesic, coming from the connecting neck, and its iterates, while in Subsection 4.6.2 we analyze the rest of the closed geodesics. By shrinking the neck we produce the desired direction going to infinity in terms of d_{SBM} . On the other hand, a *multi-bulked surface* is a closed, orientable surface of genus at least one which has a part that looks like a connected sum of $N + 1$ spheres through N “narrow necks”, see Figure 4.10. By shrinking different necks, we obtain different unbounded directions. The behaviour of closed geodesics in a multi-bulked surface is also discussed in Subsections 4.6.1 and 4.6.2. Finally, in order to exclude multiple covers of the same loop from our considerations, we work with symplectic homology in the non-trivial class of loops α . This explains the significance of the condition on the genus of Σ , since every loop in S^2 is contractible.

Remark 4.1.18. A theorem similar to Theorem 4.1.13 was proven by M. Usher in [148] in the context of star-shaped domains in \mathbb{C}^n . Even though the general setup is similar, the constructions of the quasi-isometric embeddings as well as the arguments used in the proofs of these two results are fundamentally different.

4.1.3 Applications to the study of closed geodesics

If g is bumpy, homologies $H_*(\mathcal{L}_\alpha^\lambda(M); \mathbb{Z}_2)$ form a persistence module with parameter $\lambda \in \mathbb{R}$, which we denote by $\mathbb{H}_{*,\alpha}(M, g)$ (here the homology of the empty set is taken to be zero). Structure maps

$$\iota_{\lambda,\eta} : H_*(\mathcal{L}_\alpha^\lambda(M); \mathbb{Z}_2) \rightarrow H_*(\mathcal{L}_\alpha^\eta(M); \mathbb{Z}_2)$$

for $\lambda \leq \eta$ are given by the inclusions $\{\gamma \mid E_g(\gamma) \leq \lambda\} \subset \{\gamma \mid E_g(\gamma) \leq \eta\}$. Now, Morse-Bott theory of E_g implies that the endpoints of bars in the barcode $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$ are equal to energies of certain closed geodesics. This allows us to use persistence modules to study closed geodesics, namely, in Subsection 4.7.2, we prove the following result.

Theorem 4.1.19 (Theorem 1.3.4). *Let g_1, g_2 be two bumpy metrics on a closed, orientable manifold M such that $\frac{1}{C_1}g_1 \preceq g_2 \preceq C_2g_1$. If there exists a bar $[x, y) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1))$ such that $\frac{y}{x} > C_1C_2$ then there exist closed geodesics γ_1 and γ_2 of (M, g_2) in homotopy class α , whose energies satisfy*

$$\frac{1}{C_1}x \leq E_{g_2}(\gamma_1) \leq C_2x, \quad \frac{1}{C_1}y \leq E_{g_2}(\gamma_2) \leq C_2y,$$

and furthermore $[E_{g_2}(\gamma_1), E_{g_2}(\gamma_2)) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_2))$.

In the case of an infinite bar $[x, +\infty) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1))$, there exists a closed geodesic γ_1 of g_2 such that

$$\frac{1}{C_1}x \leq E_{g_2}(\gamma_1) \leq C_2x,$$

and we have that $[E_{g_2}(\gamma_1), +\infty) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_2))$.

Remark 4.1.20. Infinite rays in $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$, i.e. bars of the form $[x, +\infty)$, have transparent meaning. Indeed, the total number of left endpoints of such bars below any $\lambda \in \mathbb{R}$ equals to $\text{rank}(H_*(\mathcal{L}_\alpha^\lambda(M); \mathbb{Z}_2) \rightarrow H_*(\mathcal{L}_\alpha(M); \mathbb{Z}_2))$. On the other hand, finite bars in $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$ are a meaningful invariant from the viewpoint of Gromov's quantitative homotopy theory, see [73]. Namely, Gromov considered the following question:

For $l > 0$ define $f(l)$ as the minimal L such that every closed contractible curve of length $\leq l$ can be contracted to a point by a homotopy passing through closed curves of length $\leq L$. Can one estimate $f(l)$?

The same question can be posed by replacing the length by the Lipschitz constant or the energy. A connection between this question and the barcode of $\mathbb{H}_{0,[pl]}(M, g)$ was, to the best of our knowledge, first observed in [159], see also [160, Section 4]. As noted in [159, 160], a bar $[x, y) \in \mathcal{B}(\mathbb{H}_{0,[pl]}(M, g))$ signifies the fact that there exists a closed geodesic of energy x such that every homotopy connecting this geodesic to a curve of energy less than x must pass through a curve of energy at least y . For a detailed study of similar ideas see [74, 158, 160] and references therein.

In [145], for a fixed Finsler metric F , the quantity $l(F)$ was introduced as the length of the shortest non-constant and ‘‘homologically visible’’ closed geodesic γ_0 . Assume that F comes from a Riemannian bumpy metric g and denote by L_g the length with respect to g . Since γ_0 has constant speed we have $E_g(\gamma_0) = \frac{L_g(\gamma_0)^2}{2} = \frac{(l(F))^2}{2}$. In the

language of barcodes $\frac{l(F)^2}{2}$ is equal to the smallest non-zero endpoint of an infinite bar in $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$, where smallest means smallest among all such endpoints for all α . Now Theorem 4.1.19 implies the following.

Corollary 4.1.21 (Theorem 1.10 in [145] - Bumpy metric case¹). *Let g_1, g_2 be two bumpy metrics on a closed, orientable manifold M such that $g_2 \preceq g_1$. Then $l(g_2) \leq l(g_1)$ and in particular there exists a non-constant “homologically visible” closed geodesic γ of g_2 such that $L_{g_2}(\gamma) \leq l(g_1)$.*

Proof. Since M is compact, there exists C_1 such that $\frac{1}{C_1}g_1 \preceq g_2 \preceq g_1$. For some α we have that

$$\left[\frac{l(g_1)^2}{2}, +\infty \right) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1)),$$

and thus Theorem 4.1.19 implies that there exists a “homologically visible” closed geodesic γ of g_2 such that

$$\frac{1}{\sqrt{C_1}}l(g_1) \leq L_{g_2}(\gamma) \leq l(g_1),$$

which finishes the proof. □

Remark 4.1.22. In view of Corollary 4.1.21, Theorem 4.1.19 may be considered a quantitative version of Theorem 1.10 in [145] (in the case of bumpy metrics on an orientable manifold). Indeed, it provides estimates for the energies (or equivalently lengths) of the closed geodesics in terms of constants C_1 and C_2 which are used to measure the discrepancy between g_1 and g_2 . Another benefit of our method is that it allows us to study “homologically invisible” closed geodesics, i.e. finite bars in $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$.

Let us illustrate the appearance of finite bars in $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$ on a concrete example of metrics of revolution on \mathbb{T}^2 .

Example 4.1.23. Let $A > 0$, $f : [-A, A] \rightarrow (0, +\infty)$ a smooth, even function, strictly increasing on $[-A, 0]$ and hence strictly decreasing on $[0, A]$ with unique maximum at 0 and two minima at $\pm A$. Moreover, assume that f extends $2A$ -periodically to a smooth function on \mathbb{R} and let g be a Riemannian metric on $\mathbb{T}^2 = \mathbb{R}/2A\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ induced by an embedding into \mathbb{R}^3 given by

$$(x, \theta) \rightarrow (x, f(x) \cos \theta, f(x) \sin \theta).$$

Define a change of variable $X(x) = \int_0^x \sqrt{1 + (f'(t))^2} dt$, $x \in [-A, A]$ and denote by $T = \int_0^A \sqrt{1 + (f'(t))^2}$, so that $X \in [-T, T]$. In X variable we denote $F(X) = f(x(X))$.

In Subsection 4.8.3 we give a detailed analysis of the geodesic flow of g . In particular, this analysis shows that parallel circles $X = \pm T$ and $X = 0$ (i.e. $x = \pm A$

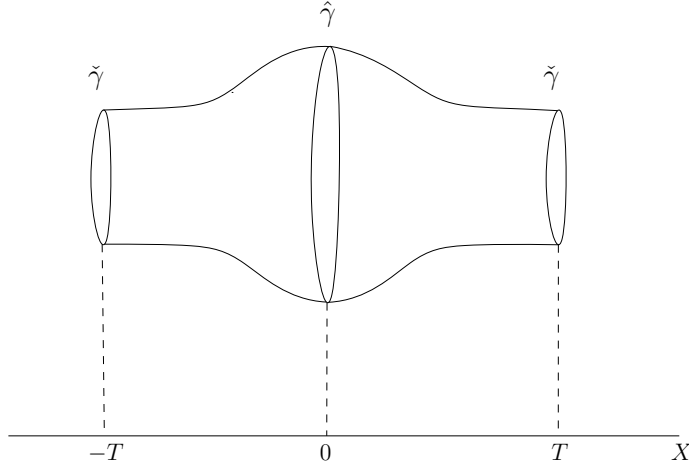


Figure 4.1. Metric of revolution on \mathbb{T}^2

and $x = 0$) are closed geodesics which we denote by $\tilde{\gamma}$ and $\hat{\gamma}$ respectively, see Figure 4.1.

Let α be the homotopy class of loops $\tilde{\gamma}$ and $\hat{\gamma}$ oriented in the direction in which θ -coordinate increases. The function F which we wish to consider is defined² on $[-1, 1]$ as a C^0 -small smoothing of $\frac{1}{\sqrt{kX^2+m}}$ for $k, m > 0$, $\sqrt{k} < m$. Namely in Subsection 4.8.3 we prove the following:

Lemma 4.1.24. *Let $k, m > 0$, $\sqrt{k} < m$. For small enough $\varepsilon > 0$ there exists $F_\varepsilon : [-1, 1] \rightarrow (0, +\infty)$ as above such that $F_\varepsilon(X) = \frac{1}{\sqrt{kX^2+m}}$ for $X \in [-1 + \varepsilon, 1 - \varepsilon]$ and there are no closed geodesics of metric g induced by F_ε in class α other than $\tilde{\gamma}$ and $\hat{\gamma}$. Moreover, $\tilde{\gamma}$ and $\hat{\gamma}$ are non-degenerate and $\text{ind } \tilde{\gamma} = 0, \text{ind } \hat{\gamma} = 1$. Furthermore, $F_\varepsilon \xrightarrow{C^0} \frac{1}{\sqrt{kX^2+m}}$ as $\varepsilon \rightarrow 0$.*

Using Lemma 4.1.24 we may compute the whole barcode $\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g))$. To this end, first note that

$$H_*(\mathcal{L}_\alpha(\mathbb{T}^2); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & * = 0, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & * = 1 \\ 0, & \text{otherwise} \end{cases} \quad (4.3)$$

Indeed, using the group action of \mathbb{T}^2 on $\mathcal{L}(\mathbb{T}^2)$ one proves that $\mathcal{L}(\mathbb{T}^2) \cong \mathbb{T}^2 \times \Omega\mathbb{T}^2$, $\Omega\mathbb{T}^2$ being the based loop space of \mathbb{T}^2 , and hence $\mathcal{L}_\alpha(\mathbb{T}^2) \simeq \mathbb{T}^2$. For another computation of $H_*(\mathcal{L}_\alpha(\mathbb{T}^2); \mathbb{Z}_2)$, using symplectic homology, see Section 5 in [24].

¹A version of this corollary can be traced back to Birkhoff.

²In principal, one should define f in order to define g , however, in this case one may show that f is implicitly defined by F as explained in Subsection 4.8.3.

In order to compute $\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g))$ we use filtered (by $\lambda \in \mathbb{R}$) Morse-Bott chain complex³ $CMB_{*,\alpha}^\lambda(E_g)$, described in detail in Subsection 4.4.1. Since $\tilde{\gamma}$ is non-degenerate, it contributes two generators to $CMB_{*,\alpha}^\lambda(E_g)$, one in degree $\text{ind } \tilde{\gamma} = 0$, the other in degree $\text{ind } \tilde{\gamma} + 1 = 1$, both on filtration level $E_{min} = E_g(\tilde{\gamma}) = \frac{2\pi^2}{k+m} + o(\varepsilon)$. Similarly, $\hat{\gamma}$ contributes two generators to $CMB_{*,\alpha}^\lambda(E_g)$, one in degree 1, the other in degree 2, both on filtration level $E_{max} = E_g(\hat{\gamma}) = \frac{2\pi^2}{m}$. The general rules for computing the barcode of $HMB_{*,\alpha}^\lambda(E_g) = H_*(\mathcal{L}_\alpha^\lambda(M); \mathbb{Z}_2)$ are the following (see Lemma 2.1.10 for the case of Morse functions or [150] for a much more general version of this procedure):

Each generator corresponds to an endpoint of a bar equal to its filtration level. Moreover, the generator in degree d corresponds either to a left endpoint of a bar in $\mathcal{B}(\mathbb{H}_{d,\alpha}(M, g))$ or to a right endpoint of a bar in $\mathcal{B}(\mathbb{H}_{d-1,\alpha}(M, g))$. All infinite bars are of the form $[\cdot, +\infty)$ and they correspond to linearly independent homology classes in $H_*(\mathcal{L}_\alpha(M); \mathbb{Z}_2)$. Hence, the number of infinite bars is equal to the total dimension of $H_*(\mathcal{L}_\alpha(M); \mathbb{Z}_2)$.

In our case, in particular, there are 4 infinite bars in $\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g))$ by (4.3). Since there are only 4 generators of $CMB_{*,\alpha}^\lambda(E_g)$ each of them contributes an infinite bar or in other words we have

$$\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g)) = \begin{cases} \{[E_{min}, +\infty)\}, * = 0 \\ \{[E_{min}, +\infty), [E_{max}, +\infty)\}, * = 1 \\ \{[E_{max}, +\infty)\}, * = 2 \end{cases} \quad (4.4)$$

Let us now define for each $n \in \mathbb{N}$ a metric of revolution g_n , on $\mathbb{T}^2 = \mathbb{R}/2n\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ by stacking n copies of a profile function F next to each other, see Figure 4.2.

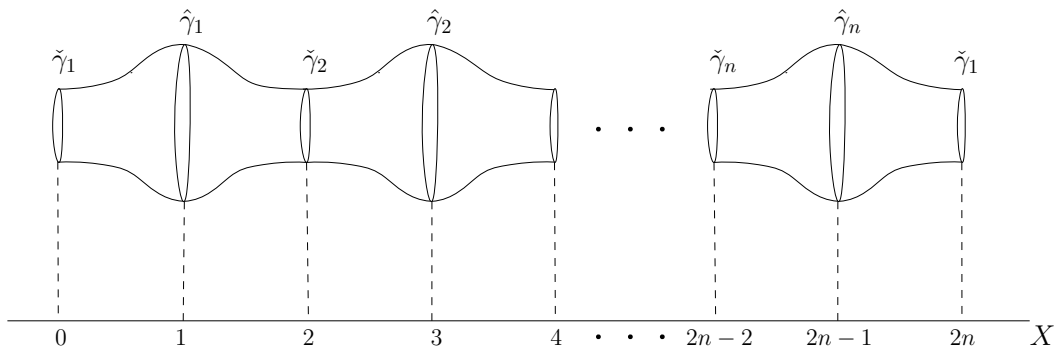


Figure 4.2. Metric of revolution g_n on \mathbb{T}^2

Same analysis as in the proof of Lemma 4.1.24 shows that $\tilde{\gamma}_i, \hat{\gamma}_i, i = 1, \dots, n$ are the only closed geodesics of g_n in class α as well as that they are non-degenerate and

³Here we omit the auxiliary height function h on S^1 from the notation.

ind $\check{\gamma}_i = 0$, ind $\hat{\gamma}_i = 1$ for all i . Similarly to the previous situation, we conclude that in filtered Morse-Bott chain complex $CMB_{*,\alpha}^\lambda(E_{g_n})$ on filtration level E_{min} there are n generators in degree 0 and n generators in degree 1 coming from $\check{\gamma}_i$. On the other hand, on filtration level E_{max} there are n generators in degree 1 and n generators in degree 2 coming from $\hat{\gamma}_i$. From (4.3) we know that one of degree-0 generators contributes a bar $[E_{min}, +\infty)$, one of degree-2 generators contributes a bar $[E_{max}, +\infty)$, and there are two more infinite bars coming from degree-1 generators. Hence, the remaining $4n - 4$ generators correspond to endpoints of finite bars, i.e. there are $2n - 2$ finite bars in $\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g_n))$, $n - 1$ of them in $\mathcal{B}(\mathbb{H}_{0,\alpha}(\mathbb{T}^2, g_n))$ and $n - 1$ of them in $\mathcal{B}(\mathbb{H}_{1,\alpha}(\mathbb{T}^2, g_n))$. Since differential ∂_{CMB} strictly lowers filtration, there are no degenerate bars, meaning bars of length zero, in $\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g_n))$. Thus all finite bars are equal to $[E_{min}, E_{max})$ and it readily follows that

$$\mathcal{B}(\mathbb{H}_{*,\alpha}(\mathbb{T}^2, g_n)) = \begin{cases} \{[E_{min}, +\infty), [E_{min}, E_{max}) \times (n - 1)\}, * = 0 \\ \{[E_{min}, +\infty), [E_{max}, +\infty), [E_{min}, E_{max}) \times (n - 1)\}, * = 1 \\ \{[E_{max}, +\infty)\}, * = 2 \end{cases} \quad (4.5)$$

Remark 4.1.25. One may define a “non-symmetric” version of d_{RBM} in the following way. We say that g_1, g_2 are (C_1, C_2) -equivalent if there exists a diffeomorphism $\phi \in \text{Diff}_0(M)$ such that $C_1 g_1 \preceq \phi^* g_2 \preceq C_2 g_1$. Define *equivalence-distance* d_{EQ} on \mathcal{G}_M by

$$d_{EQ}(g_1, g_2) = \inf \left\{ \ln \frac{C_2}{C_1} \mid \begin{array}{l} \text{there exist } C_2 \geq C_1 > 0 \text{ such that} \\ (g_1, g_2) \text{ are } (C_1, C_2)\text{-equivalent} \end{array} \right\}.$$

One readily checks that $d_{EQ}(g_1, g_2) \geq 0$, $d_{EQ}(g_1, g_2) = d_{EQ}(g_2, g_1)$ and $d_{EQ}(g_1, g_2) + d_{EQ}(g_2, g_3) \geq d_{EQ}(g_1, g_3)$. On the other hand, if for some $\phi \in \text{Diff}_0(M)$ it holds $\phi^* g_2 = C g_1$ then $d_{EQ}(g_1, g_2) = 0$. It follows from the definitions that

$$d_{EQ}(g_1, g_2) \leq 2d_{RBM}(g_1, g_2).$$

Finally, we wish to mention that by taking $C_1 = C_2$ in Theorem 4.1.19, we obtain Corollary 4.1.26 given below. This corollary also has a direct proof using stability of barcodes, i.e. Theorem 4.1.7, which we present in Section 4.7.

Corollary 4.1.26. *Let M be a closed, orientable manifold, α a homotopy class of free loops in M , g_1 a bumpy metric on M and suppose that $[a^2/2, b^2/2) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1))$ for some $0 < a < b$. For any bumpy metric g_2 on M , such that $0 \leq d_{RBM}(g_1, g_2) < \ln(b/a)$, there exist non-constant closed geodesics γ_1, γ_2 of g_2 in homotopy class α such that*

$$\max \left\{ \left| \ln \left(\frac{E_{g_2}(\gamma_1)}{a^2/2} \right) \right|, \left| \ln \left(\frac{E_{g_2}(\gamma_2)}{b^2/2} \right) \right| \right\} \leq d_{RBM}(g_1, g_2).$$

In the case of an infinite bar $[a^2/2, +\infty) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1))$, there exists a “homologically visible” closed geodesic γ_1 of g_2 which satisfies

$$\left| \ln \left(\frac{E_{g_2}(\gamma_1)}{a^2/2} \right) \right| \leq d_{RBM}(g_1, g_2).$$

4.2 Basic properties of d_{SBM} and d_{RBM}

4.2.1 Symplectic Banach-Mazur distance

Symplectic Banach-Mazur distance d_{SBM} is defined in [120] in the setting of general Liouville manifolds and it is proven to be a pseudo-metric. Our case of admissible domains in the cotangent bundle is a special case of this situation. For completeness, we include the proof of the following proposition

Proposition 4.2.1. d_{SBM} defines a pseudo-metric on \mathcal{C}_M .

We start the proof with a lemma.

Lemma 4.2.2. Let $\frac{1}{C}U \xrightarrow{\phi} V \xrightarrow{\psi} CU$, $U, V \in \mathcal{C}_M$, $C > 1$ be $\tilde{\pi}_1$ -trivial Liouville embeddings such that $\psi \circ \phi$ is strongly unknotted. Then for any $D > 1$ and embeddings

$$\frac{1}{CD}U \xrightarrow{\phi(D^{-1})} V \xrightarrow{\psi(D)} CDU,$$

$\psi(D) \circ \phi(D^{-1})$ is strongly unknotted

Proof. For $t \in [0, 1]$ consider the following maps

$$\frac{1}{C(1+(D-1)t)}U \xrightarrow{\phi((1+(D-1)t)^{-1})} V \xrightarrow{\psi(1+(D-1)t)} C(1+(D-1)t)U. \quad (4.6)$$

Since $D-1 > 0$ we have that $C(1+(D-1)t) \geq C$ and hence

$$\frac{1}{CD}U \subset \frac{1}{C(1+(D-1)t)}U, \quad C(1+(D-1)t)U \subset CDU.$$

Composing (4.6) with these inclusions, we get

$$\frac{1}{CD}U \xrightarrow{\psi(1+(D-1)t) \circ \phi((1+(D-1)t)^{-1})} CDU,$$

Denoting

$$\beta_t = \psi(1+(D-1)t) \circ \phi((1+(D-1)t)^{-1}), \quad \beta_t : \frac{1}{CD}U \hookrightarrow CDU,$$

we get $\beta_0 = \psi \circ \phi|_{\frac{1}{CD}U}$, $\beta_1 = \psi(D) \circ \phi(D^{-1})$. Since $\psi \circ \phi$ is strongly unknotted, there exists $\alpha_t : \frac{1}{C}U \hookrightarrow CU$ such that $\alpha_0 = i_{\frac{1}{C}U}$, $\alpha_1 = \psi \circ \phi$. Restricting α_t to $\frac{1}{CD}U$ and composing with the inclusion $CU \xrightarrow{i} CDU$, we get that $\alpha'_t = i \circ (\alpha_t|_{\frac{1}{CD}U})$ satisfies $\alpha'_0 = i_{\frac{1}{CD}U}$, $\alpha'_1 = \beta_0$. Concatenation of α' and β gives the desired isotopy. \square

Proof. (Proof of Proposition 4.2.1) It readily follows that $d_{SBM}(U, U) = 0$ by taking $C = 1$, $\phi = \psi = \mathbb{1}_U$ in the definition of d_{SBM} .

To prove symmetry and triangle inequality we need the following two properties which can be proven by direct calculations. If $U \xrightarrow{\phi} V \xrightarrow{\psi} W$ and $C, D > 0$ then

$$(*) \quad (\phi \circ \psi)(C) = \phi(C) \circ \psi(C),$$

$$(**) \quad (\phi(C))(D) = \phi(CD).$$

Now, if $U/C \xrightarrow{\phi} V \xrightarrow{\psi} CU$, $V/C \xrightarrow{\psi(C^{-1})} U \xrightarrow{\phi(C)} CV$, are such that $\psi \circ \phi$ and $\phi(C) \circ \psi(C^{-1})$ are strongly unknotted, $(*)$ implies that so are

$$\frac{1}{C}V \xrightarrow{\psi(C^{-1})} U \xrightarrow{\phi(C)} CV, \quad \frac{1}{C}U \xrightarrow{\phi(C)(C^{-1})} V \xrightarrow{\psi(C^{-1})(C)} CU.$$

This proves that $d_{SBM}(U, V) = d_{SBM}(V, U)$.

Thus, we are left to prove the triangle inequality. Given $U/C \xrightarrow{\phi} V \xrightarrow{\psi} CU$ and $V/D \xrightarrow{\theta} W \xrightarrow{\xi} DV$ with strongly unknotted compositions, we claim that the composition of the following maps

$$\frac{1}{CD}U \xrightarrow{\phi(D^{-1})} \frac{1}{D}V \xrightarrow{\theta} W \xrightarrow{\xi} DV \xrightarrow{\psi(D)} CDU$$

is also strongly unknotted. Indeed, denote by $\alpha_t : \frac{1}{D}V \hookrightarrow DV$ the isotopy such that $\alpha_0 = i_{\frac{1}{D}V}$, $\alpha_1 = \xi \circ \theta$, given by the unknottedness of $\xi \circ \theta$ and by $\beta_t : \frac{1}{CD}U \hookrightarrow CDU$ the isotopy such that $\beta_0 = i_{\frac{1}{CD}U}$, $\beta_1 = \psi(D) \circ \phi(D^{-1})$, given by the unknottedness of $\psi \circ \phi$ and Lemma 4.2.2. Now, the isotopy $\gamma_t : \frac{1}{CD}U \hookrightarrow CDU$ given by

$$\gamma_t = \begin{cases} \beta_{2t} & \text{for } 0 \leq t \leq 1/2 \\ \psi(D) \circ \alpha_{2t-1} \circ \phi(D^{-1}) & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

satisfies $\gamma_0 = i_{\frac{1}{CD}U}$, $\gamma_1 = \psi(D) \circ \xi \circ \theta \circ \psi(D^{-1})$ which proves the claim.

This way, we proved that the composition of maps

$$\frac{1}{CD}U \xrightarrow{\theta \circ \phi(D^{-1})} W \xrightarrow{\psi(D) \circ \xi} CDU$$

is strongly unknotted. What is left is to prove that the composition of maps

$$\frac{1}{CD}W \xrightarrow{(\psi(D) \circ \xi)((CD)^{-1})} U \xrightarrow{(\theta \circ \phi(D^{-1}))(CD)} CDW \quad (4.7)$$

is strongly unknotted. Using (*) and (**), we reformulate (4.7) as

$$\frac{1}{CD}W \xrightarrow{(\xi(D^{-1}))(C^{-1})} \frac{1}{C}V \xrightarrow{\psi(C^{-1})} U \xrightarrow{\phi(C)} CV \xrightarrow{(\theta(D))(C)} CDW.$$

Now the same construction of the isotopy as the one we used for γ applies, only starting from $W/D \xrightarrow{\xi(D^{-1})} V \xrightarrow{\theta(D)} DW$ and $W/C \xrightarrow{\psi(C^{-1})} V \xrightarrow{\phi(C)} CW$. \square

Remark 4.2.3. Note that d_{SBM} is not a genuine metric. Indeed if $U, V \in \mathcal{C}_M$ are exactly symplectomorphic via a $\tilde{\pi}_1$ -trivial map, i.e. there exists a diffeomorphism $\phi : U \rightarrow V$ such that 1-form $\phi^* \lambda_{can} - \lambda_{can}$ is exact and $\phi_* = \mathbb{1}_{\tilde{\pi}_1(M)}$, we have

$$U \xrightarrow{\phi} V \xrightarrow{\phi^{-1}} U \quad \text{and} \quad V \xrightarrow{\phi^{-1}} U \xrightarrow{\phi} V$$

and thus $d_{SBM}(U, V) = 0$. This is, for example, the case when $V = \phi(U)$ and $\phi \in \text{Ham}_c(T^*M)$.

Remark 4.2.4. Using (*) and (**) one easily checks that for all $C > 0$ and all $U, V \in \mathcal{C}_M$ it holds $d_{SBM}(CU, CV) = d_{SBM}(U, V)$ as well as $d_{SBM}(U, CU) = \ln C$.

4.2.2 Riemannian Banach-Mazur distance

We begin with the following statement

Proposition 4.2.5. d_{RBM} defines a pseudo-metric on \mathcal{G}_M .

Proof. By taking $C = 1$ and $\phi = \mathbb{1}_M$ one readily concludes that $d_{RBM}(g, g) = 0$. On the other hand, for $\phi \in \text{Diff}_0(M)$ and $C \geq 1$, we have that $(1/C)g_1 \preceq \phi^*g_2 \preceq Cg_1$ if and only if

$$\frac{1}{C}g_2 \preceq (\phi^{-1})^*g_1 \preceq Cg_2.$$

This implies $d_{RBM}(g_1, g_2) = d_{RBM}(g_2, g_1)$.

Finally, for $\phi_1, \phi_2 \in \text{Diff}_0(M)$ and $C, D \geq 1$, the relations $(1/C)g_1 \preceq \phi_1^*g_2 \preceq Cg_1$ and $(1/D)g_2 \preceq \phi_2^*g_3 \preceq Dg_2$ imply

$$\frac{1}{CD}g_1 \preceq \frac{1}{D}(\phi_1^*g_2) \preceq \phi_1^*(\phi_2^*g_3) \preceq D(\phi_1^*g_2) \preceq CDg_1.$$

Setting $\phi = \phi_2 \circ \phi_1$ gives $d_{RBM}(g_1, g_3) \leq \ln C + \ln D$ and thus taking infimum over C and D gives $d_{RBM}(g_1, g_3) \leq d_{RBM}(g_1, g_2) + d_{RBM}(g_2, g_3)$. \square

Remark 4.2.6. Similarly to d_{SBM} , d_{RBM} is also not a genuine metric. Indeed, if there exists some $\phi \in \text{Diff}_0(M)$ such that $g_1 = \phi^*g_2$, taking $C = 1$ we have

$$g_1 \preceq \phi^*g_2 = g_1 \preceq g_1.$$

This implies $d_{RBM}(g_1, g_2) = 0$.

Remark 4.2.7. One readily checks that it holds $d_{RBM}(Cg_1, Cg_2) = d_{RBM}(g_1, g_2)$ as well as $d_{RBM}(g_1, Cg_1) = \ln C$ for all $C > 0$ and all $g_1, g_2 \in \mathcal{G}_M$.

As we saw in Example 4.1.3, \mathcal{G}_M can be identified with a subset of \mathcal{C}_M via inclusion $g(\in \mathcal{G}_M) \mapsto U_g^* M (\in \mathcal{C}_M)$. Therefore, it makes sense to compare d_{SBM} with d_{RBM} on \mathcal{G}_M and we have

Proposition 4.2.8. *Let M be a closed, orientable manifold and $g_1, g_2 \in \mathcal{G}_M$. Then*

$$2 \cdot d_{SBM}(U_{g_1}^* M, U_{g_2}^* M) \leq d_{RBM}(g_1, g_2).$$

Proof. For any $\varepsilon > 0$, there exists some $C^2 > 1$ and $\phi \in \text{Diff}_0(M)$ such that $\ln(C^2) \leq d_{RBM}(g_1, g_2) + \varepsilon$ and $(1/C^2)g_1 \preceq \phi^* g_2 \preceq C^2 g_1$. Since $B_1(g_2) \subset B_1(g_1)$ if $g_1 \preceq g_2$ we have that

$$U_{\frac{1}{C^2}g_1}^* M \subset U_{\phi^* g_2}^* M \subset U_{C^2 g_1}^* M. \quad (4.8)$$

One also readily checks that $U_{\frac{1}{C^2}g_1}^* M = \frac{1}{C} U_{g_1}^* M$ and $U_{C^2 g_1}^* M = C U_{g_1}^* M$.

On the other hand, $\phi \in \text{Diff}_0(M)$ lifts to a symplectomorphism $\phi^\#$ of T^*M , given by $\phi^\#(p, \xi) = (\phi(p), (\phi_{\phi(p)}^{-1})^* \xi)$. Since ϕ is isotopic to $\mathbb{1}_M$, the lift $\phi^\#$ is isotopic to $\mathbb{1}_{T^*M}$ and in particular, $\phi^\#$ acts trivially on $\tilde{\pi}_1(M)$. Moreover, one may check that $\phi^\#$ is exact as well as that $\phi^\#(U_{\phi^* g}^* M) = U_g^* M$. Therefore (4.8) can be rewritten as

$$\frac{1}{C} U_{g_1}^* M \subset (\phi^{-1})^\#(U_{g_2}^* M) \subset C U_{g_1}^* M$$

which implies

$$\frac{1}{C} U_{g_1}^* M \xrightarrow{(\phi^\#)|_{\frac{1}{C} U_{g_1}^* M}} U_{g_2}^* M \xrightarrow{(\phi^{-1})^\#|_{U_{g_2}^* M}} C U_{g_1}^* M.$$

The above maps are $\tilde{\pi}_1$ -trivial Liouville embeddings and their composition is the inclusion, thus strongly unknotted. We are left to check that the composition

$$\frac{1}{C} U_{g_2}^* M \xrightarrow{(\phi^{-1})^\#|_{U_{g_2}^* M(C^{-1})}} U_{g_1}^* M \xrightarrow{(\phi^\#)|_{\frac{1}{C} U_{g_1}^* M(C)}} C U_{g_2}^* M.$$

is strongly unknotted. This follows from the fact that $\phi^\#(C) = \phi^\#$ for all $\phi \in \text{Diff}_0(M)$ and all $C \geq 1$.

Therefore, by the definition of d_{SBM} , we obtain

$$d_{SBM}(U_{g_1}^* M, U_{g_2}^* M) \leq \ln C \leq \frac{1}{2} d_{RBM}(g_1, g_2) + \frac{\varepsilon}{2}.$$

Since the inequality holds for every $\varepsilon > 0$, the conclusion follows. \square

Remark 4.2.9. Recall that, for $C \geq 1$, a homeomorphism $\phi : (X, d_X) \rightarrow (Y, d_Y)$ between two metric spaces is called C -bi-Lipschitz if it holds

$$\frac{1}{C} d_X(x, y) \leq d_Y(\phi(x), \phi(y)) \leq C d_X(x, y),$$

for every two $x, y \in X$. The Lipschitz distance (see [27]) between two bi-Lipschitz homeomorphic metric spaces (X, d_X) and (Y, d_Y) is defined as

$$d_{Lip}(X, Y) = \inf\{\ln C \in [0, +\infty) \mid \exists \phi : (X, d_X) \rightarrow (Y, d_Y), \phi \text{ is } C\text{-bi-Lipschitz}\}.$$

The following lemma shows a connection between d_{RBM} and d_{Lip} .

Lemma 4.2.10. *Let g_1, g_2 be two Riemannian metrics on a closed manifold M and let $\phi \in \text{Diff}(M)$. Then*

$$\frac{1}{C^2}g_2 \preceq \phi^*g_2 \preceq C^2g_1 \Leftrightarrow \phi : (M, g_1) \rightarrow (M, g_2) \text{ is } C\text{-bi-Lipschitz}.$$

Proof. We will prove the two directions separately.

\Rightarrow : Let $x, y \in M$ and let $\gamma : [0, 1] \rightarrow M$ be a curve s.t. $\gamma(0) = x, \gamma(1) = y$. We estimate

$$L_{g_2}(\phi \circ \gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\phi^*g_2} dt \leq C \int_0^1 \|\dot{\gamma}(t)\|_{g_1} dt = CL_{g_1}(\gamma).$$

Taking infimum over γ yields $d_{g_2}(\phi(x), \phi(y)) \leq Cd_{g_1}(x, y)$. Inequality $\frac{1}{C}d_{g_1}(x, y) \leq d_{g_2}(\phi(x), \phi(y))$ is proven similarly.

\Leftarrow : Take any $x \in M$ and $v \in T_xM$. Let $U \ni x$ be a convex normal neighborhood of x with respect to g_1 and $V \ni \phi(x)$ a convex normal neighborhood of $\phi(x)$ with respect to g_2 . Let $\gamma : (-\varepsilon, \varepsilon) \rightarrow U$ be a curve such that $\gamma(0) = x, \dot{\gamma}(0) = v$ and $\text{im}(\phi \circ \gamma) \subset V$. Since U and V are convex and normal it holds

$$d_{g_1}(x, \gamma(t)) = \|(exp_x^{g_1})^{-1}(\gamma(t))\|_{g_1},$$

as well as

$$d_{g_2}(\phi(x), \phi \circ \gamma(t)) = \|(exp_{\phi(x)}^{g_2})^{-1}(\phi \circ \gamma(t))\|_{g_2}.$$

Thus, from the bi-Lipschitz condition we have that

$$\frac{1}{C} \|(exp_x^{g_1})^{-1}(\gamma(t))\|_{g_1} \leq \|(exp_{\phi(x)}^{g_2})^{-1}(\phi \circ \gamma(t))\|_{g_2} \leq C \|(exp_x^{g_1})^{-1}(\gamma(t))\|_{g_1}.$$

Restricting to $t > 0$ and dividing all inequalities by t gives us

$$\frac{1}{C} \left\| \frac{(exp_x^{g_1})^{-1}(\gamma(t))}{t} \right\|_{g_1} \leq \left\| \frac{(exp_{\phi(x)}^{g_2})^{-1}(\phi \circ \gamma(t))}{t} \right\|_{g_2} \leq C \left\| \frac{(exp_x^{g_1})^{-1}(\gamma(t))}{t} \right\|_{g_1}.$$

Taking $\lim_{t \rightarrow 0^+}$ we get that

$$\begin{aligned} \frac{1}{C} \left\| \frac{d}{dt} \Big|_{t=0} (exp_x^{g_1})^{-1}(\gamma(t)) \right\|_{g_1} &\leq \left\| \frac{d}{dt} \Big|_{t=0} (exp_{\phi(x)}^{g_2})^{-1}(\phi \circ \gamma(t)) \right\|_{g_2} \\ &\leq C \left\| \frac{d}{dt} \Big|_{t=0} (exp_x^{g_1})^{-1}(\gamma(t)) \right\|_{g_1}. \end{aligned}$$

Since $\exp_*(0) = \mathbb{1}$ it follows that

$$\frac{1}{C} \|\dot{\gamma}(0)\|_{g_1} \leq \|\phi_*(\dot{\gamma}(0))\|_{g_2} \leq C \|\dot{\gamma}(0)\|_{g_1},$$

i.e.

$$\frac{1}{C} \|v\|_{g_1} \leq \|v\|_{\phi^*g_2} \leq C \|v\|_{g_1},$$

which completes the proof. \square

One might modify the definition of the Lipschitz distance by considering only bi-Lipschitz homeomorphisms which are homotopic to the identity. Denote the distance defined in this way by d_{Lip_0} . In light of the above lemma, the definitions of $d_{RBM}(g_1, g_2)$ resembles that of $d_{Lip_0}((M, g_1), (M, g_2))$ (up to a factor of two), the difference being in the choice of diffeomorphisms versus bi-Lipschitz homeomorphisms. It would be interesting to compare d_{RBM} and d_{Lip_0} .

4.3 Symplectic homology as a persistence module

4.3.1 Background on symplectic homology

Symplectic homology has been developed in the 90's by the work of many people, see [59, 60, 36, 37, 152]. There exist different versions of the theory, depending on the class of manifolds and admissible Hamiltonians which are considered. We will use the version developed in [24] and [157] (with different signs from [24]).

Throughout this chapter, all Floer homologies as well as symplectic homology are taken with \mathbb{Z}_2 -coefficients. As a result all persistence modules will also be persistence modules over \mathbb{Z}_2 .

We start by briefly recalling the setup of [24] and [157]. For a fixed homotopy class α of free loops in M , consider the following space

$$\mathcal{L}_\alpha(T^*M) = \left\{ z : S^1 \rightarrow T^*M \mid z = (x, y), x : S^1 \rightarrow M \text{ s.t. } [x] = \alpha \text{ and } y(t) \in T_{x(t)}^*M \right\}.$$

Recall that $(T^*M, \omega_{can} = d\lambda_{can})$ is a symplectic manifold and given a Hamiltonian function $H : \mathbb{R}/\mathbb{Z} \times T^*M \rightarrow \mathbb{R}$ we may define its Hamiltonian vector field X_H by $i_{X_H}\omega_{can} = -dH$. The collection of all Hamiltonian 1-periodic orbits of H in $\mathcal{L}_\alpha(T^*M)$ is denoted by $\mathcal{P}(H; \alpha)$. Recall also that the symplectic action functional \mathcal{A}_H is given by

$$\mathcal{A}_H(z) = \int_0^1 H_t(z(t)) dt - \int_{S^1} z^* \lambda_{can},$$

for any loop $z : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow T^*M$. The action spectrum of H in class α is

$$\text{Spec}(H; \alpha) = \{ \mathcal{A}_H(z) \mid z \in \mathcal{P}(H; \alpha) \}.$$

Since T^*M is not compact, in order to define Floer homology, we need to impose certain restrictions on the Hamiltonian H . The standard assumption in this situation is that H is linear⁴ outside a compact subset of T^*M . For simplicity, in this section we only consider compactly supported Hamiltonians, i.e. linear outside of a compact set with slope equal to zero, as this class of Hamiltonians suffices to define symplectic homology. However, in the proof of Theorem 4.4.6 we will need to work with non-zero slopes and we will review the relevant setup in Subsection 4.4.2.

Now, to a given compactly supported Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times T^*M \rightarrow \mathbb{R}$, a class α of free loops in M and real numbers $a < b$ not belonging to $\text{Spec}(H; \alpha)$ (and also $0 \notin [a, b]$ if $\alpha = [pt]$) we associate Hamiltonian Floer homology in action window $[a, b]$ denoted by $HF_*^{[a,b]}(H)_\alpha$. This is homology of a Floer chain complex generated by elements of $\mathcal{P}(H; \alpha)$ with action in $[a, b]$ (see [24, 157] for details).

Grading on Floer chain complex is defined using the Lagrangian distribution of vertical subspaces $T^v T^*M \subset TT^*M$ given by $T_x^v T^*M = \ker d\pi(x)$ where $\pi : T^*M \rightarrow M$ is projection. Namely, let $z \in \mathcal{P}(H; \alpha)$ and let $\Phi : S^1 \times \mathbb{R}^{2n} \rightarrow z^*(TT^*M)$ be a symplectic trivialization such that $\Phi_t(0 \times \mathbb{R}^n) = T_{z(t)}^v T^*M$ for all $t \in S^1$. The existence of such a trivialization follows from orientability of $z^*(T^v T^*M)$ (which follows from orientability of M), see, for example, Lemma 1.2 in [2]. Now, if ϕ_t^H is the Hamiltonian flow of H , $\phi_t^H(z(0)) = z(t)$, we have a path of symplectic matrices

$$P(t) = \Phi_t^{-1} \circ d\phi_t^H(z(0)) \circ \Phi_0, \quad t \in [0, 1],$$

and we define $\text{ind}_{HF}(z) = \text{ind}_{\text{CZ}}(P)$, where ind_{CZ} stands for the Conley-Zehnder index. It is easy to check that $\text{ind}_{HF}(z)$ does not depend on the choice of Φ as above, see Lemma 1.3 in [2]. Moreover, our conventions for the Conley-Zehnder index are chosen in such a way that isomorphism in Theorem 4.4.6 preserves grading, see [156, 2] and references therein.

Remark 4.3.1. The definition of Floer homology also includes an auxiliary choice of an almost complex structure. Since $HF_*^{[a,b]}(H)_\alpha$ does not depend on this choice, we omit it from the notation. We should also mention that one first defines $HF_*^{[a,b]}(H)_\alpha$ for Hamiltonians H whose periodic orbits are non-degenerate. Floer homology of a general Hamiltonian H as above is then defined, roughly speaking, as $HF_*^{[a,b]}(H')_\alpha$, where H' is a C^∞ -small perturbation of H whose periodic orbits are non-degenerate.

We now focus on defining symplectic homology of an admissible domain $U \in \mathcal{C}_M$. Denote by \mathcal{H}_U the set of all functions on $S^1 \times T^*M$ compactly supported in $S^1 \times \text{Int}(U)$. Given $a < b$ (with a, b possibly being $\pm\infty$), let

$$\mathcal{H}_{U,a,b} = \{H \in \mathcal{H}_U \mid a, b \notin \text{Spec}(H; \alpha) \text{ and } 0 \notin [a, b] \text{ if } \alpha = [pt]\}$$

⁴Linearity of H in this context can be understood as $H_t(x, y) = \beta\|y\|_{g^*} + \beta'$ for some $\beta, \beta' \in \mathbb{R}$ and a fixed Riemannian metric g on M . More generally, if (U, λ) is a Liouville domain, linearity is understood as linearity with respect to the radial coordinate in the completion of (U, λ) and the previously described linearity corresponds to the case $(U, \lambda) = (U_g^*M, \lambda_{can})$.

If $b = +\infty$, we denote $\mathcal{H}_{U,a,+\infty}$ by $\mathcal{H}_{U,a}$.

Define a partial order on $\mathcal{H}_{U,a,b}$ as follow. For $H_1, H_2 \in \mathcal{H}_{U,a,b}$, $H_1 \preceq H_2$ if and only if $H_1(t, z) \geq H_2(t, z)$ for all $(t, z) \in S^1 \times U$. If $H_1 \preceq H_2$ then there exists a smooth homotopy $\tau \rightarrow H_\tau$ from H_1 to H_2 such that $\partial_\tau H_\tau \leq 0$. We call such a homotopy *monotone*. Every monotone homotopy induces a \mathbb{Z}_2 -linear continuation map

$$\sigma_{12} : HF_*^{[a,b]}(H_1)_\alpha \rightarrow HF_*^{[a,b]}(H_2)_\alpha.$$

Moreover, the map σ_{12} does not depend on the choice of the monotone homotopy. In general, σ_{12} may not be an isomorphism. However if there exists a monotone homotopy $\tau \rightarrow H_\tau$ such that $H_\tau \in \mathcal{H}_{U,a,b}$ for every τ , then σ_{12} is an isomorphism, see Proposition 4.5.1 in [24]. Such a monotone homotopy is called *action-regular*. For a detailed treatment of maps induced by monotone homotopies, see [24, 157] and references therein. We will need the following statement (see, for example, Lemma 2.7 in [157]).

Lemma 4.3.2. *Let $U \in \mathcal{C}_M$. For any three functions H_1, H_2 and H_3 in $\mathcal{H}_{U,a,b}$ with $H_1 \preceq H_2 \preceq H_3$, the induced maps on Hamiltonian Floer homologies in action window $[a, b]$ satisfy $\sigma_{13} = \sigma_{23} \circ \sigma_{12}$.*

Note that Lemma 4.3.2 together with the fact that continuation map is independent of the choice of the monotone homotopy implies that if $H_1 \preceq H_2 \preceq H_3$ and $H_1 \preceq H_4 \preceq H_3$, $H_i \in \mathcal{H}_{U,a,b}$, the following diagram commutes

$$\begin{array}{ccc} HF_*^{[a,b]}(H_2)_\alpha & \xrightarrow{\sigma_{23}} & HF_*^{[a,b]}(H_3)_\alpha \\ \sigma_{12} \uparrow & & \uparrow \sigma_{43} \\ HF_*^{[a,b]}(H_1)_\alpha & \xrightarrow{\sigma_{14}} & HF_*^{[a,b]}(H_4)_\alpha. \end{array} \quad (4.9)$$

We will use this in the proof of Theorem 4.4.6.

Now notice that $(\mathcal{H}_{U,a,b}, \preceq)$ is a *downward directed partially ordered system*, i.e. for any $H_2, H_3 \in \mathcal{H}_{U,a,b}$, there exists some $H_1 \in \mathcal{H}_{U,a,b}$ such that $H_1 \preceq H_2$ and $H_1 \preceq H_3$. Lemma 4.3.2 implies that Floer homologies $HF_*^{[a,b]}(H)_\alpha$, $H \in \mathcal{H}_{U,a,b}$ together with continuation maps $\sigma_{12} : HF_*^{[a,b]}(H_1)_\alpha \rightarrow HF_*^{[a,b]}(H_2)_\alpha$ for $H_1 \preceq H_2$, define an inverse system of \mathbb{Z}_2 -vector spaces over $(\mathcal{H}_{U,a,b}, \preceq)$. Thus we can take the inverse limit of such an inverse system, which leads to the definition of symplectic homology. For a general background on inverse system and inverse limit, see subsection 4.6 in [24].

As we mentioned in the introduction, if $U \in \mathcal{C}_M$ then $(\partial U, \lambda_{can}|_{\partial U})$ is a contact manifold. We denote by $\text{Spec}(\partial U)$ the set of periods of all periodic Reeb orbits of $(\partial U, \lambda_{can}|_{\partial U})$. Recall that U is called non-degenerate if all periodic Reeb orbits of $(\partial U, \lambda_{can}|_{\partial U})$ are non-degenerate. In this case for every $T > 0$ there are finitely many Reeb orbits of period less than T and in particular $\text{Spec}(\partial U)$ is discrete.

Definition 4.3.3. For a homotopy class α of free loops in M , $U \in \mathcal{C}_M$ non-degenerate and $a > 0, a \notin \text{Spec}(\partial U)$ define the *filtered symplectic homology of U* by

$$\text{SH}_*^a(U; \alpha) = \varprojlim_{H \in \mathcal{H}_{U,a}} HF_*^{[a, \infty)}(H)_\alpha.$$

The *symplectic persistence module of U in class α* is given by the collection of data

$$\text{SH}_{*,\alpha}(U) = \{ \{ \text{SH}_*^a(U; \alpha) \}_{a \in \mathbb{R}_{>0}}; \{ \iota_{a,b} : \text{SH}_*^a(U; \alpha) \rightarrow \text{SH}_*^b(U; \alpha) \}_{a \leq b} \}$$

with $\iota_{a,b}$ being induced from corresponding filtered Hamiltonian Floer homologies.

There are two points which we wish to clarify regarding Definition 4.3.3. Firstly, if $0 < a \leq b$ and $H \in \mathcal{H}_{U,a} \cap \mathcal{H}_{U,b}$ the map $\iota_{a,b}^H : HF_*^{[a, +\infty)}(H)_\alpha \rightarrow HF_*^{[b, +\infty)}(H)_\alpha$ is induced by the inclusion of Floer chain complexes⁵. This map commutes with continuation maps and hence induces a map $\iota_{a,b} : \text{SH}_*^a(U; \alpha) \rightarrow \text{SH}_*^b(U; \alpha)$ for $a, b \notin \text{Spec}(\partial U)$.

Secondly, we only defined $\text{SH}_*^a(U; \alpha)$ for $a \notin \text{Spec}(\partial U)$. However, due to non-degeneracy of U we may extend the definition of $\text{SH}_*^a(U; \alpha)$ to all $a > 0$ by asking for all the bars in the barcode of $\text{SH}_{*,\alpha}(U)$ to have left endpoints closed and right endpoints open. Indeed, one may show that for each $b > 0, b \notin \text{Spec}(\partial U)$, $\text{SH}_*^b(U; \alpha)$ is a finite dimensional \mathbb{Z}_2 -vector space as well as that if $[b, c] \cap \text{Spec}(\partial U) = \emptyset$ then $\iota_{b,c}$ is an isomorphism, see Remark 4.4.5 for the case of unit codisc bundles. Since $\text{Spec}(\partial U)$ is discrete, for every $a \in \text{Spec}(\partial U)$ there exists $\varepsilon > 0$ such that $[a - \varepsilon, a + \varepsilon]$ contains no other points from $\text{Spec}(\partial U)$. Now, we define $\text{SH}_*^a(U; \alpha)$ by asking for $\iota_{a, a+\varepsilon}$ to be an isomorphism, or, more formally, by setting $\text{SH}_*^a(U; \alpha) = \varprojlim_{t \in (a, a+\varepsilon]} (\text{SH}_*^t(U; \alpha), \iota)$. An interested reader may check that all the bars in the barcode of a symplectic persistence module defined this way have endpoints in $\text{Spec}(\partial U)$, each point in $\text{Spec}(\partial U)$ is an endpoint of finitely many bars and all bars have left endpoints closed and right endpoints open.

We are mainly interested in the functorial properties of filtered symplectic homology, which are expressed by the following proposition.

Proposition 4.3.4. *Let $U, V, W \in \mathcal{C}_M$ be non-degenerate.*

- (1) *If $U \xrightarrow{\phi} V$ (recall that this means there exists a $\tilde{\pi}_1$ -trivial Liouville embedding from U to V), then there exists a persistence module morphism $\mathfrak{h}_\phi : \text{SH}_{*,\alpha}(V) \rightarrow \text{SH}_{*,\alpha}(U)$. Moreover, if $U \xrightarrow{\phi} V \xrightarrow{\psi} W$, the following diagram commutes*

$$\begin{array}{ccccc} \text{SH}_{*,\alpha}(W) & \xrightarrow{\mathfrak{h}_\psi} & \text{SH}_{*,\alpha}(V) & \xrightarrow{\mathfrak{h}_\phi} & \text{SH}_{*,\alpha}(U) \\ & & & \searrow \mathfrak{h}_{\psi \circ \phi} & \\ & & & & \end{array}$$

⁵Here we think of a Floer chain complex $\text{CF}_*^{[a, +\infty)}(H, \alpha)$ as a quotient $\text{CF}_*^{[a, +\infty)}(H, \alpha) = \text{CF}_*^{(-\infty, +\infty)}(H, \alpha) / \text{CF}_*^{(-\infty, a)}(H, \alpha)$.

- (2) For $C, a > 0$, there exists a canonical persistence module isomorphism $r_C : \mathrm{SH}_*^a(U; \alpha) \xrightarrow{\cong} \mathrm{SH}_*^{Ca}(CU; \alpha)$. These isomorphisms satisfy $(r_C)^{-1} = r_{\frac{1}{C}}$ for all $C > 0$. Moreover for $C \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{SH}_*^{Ca}(CU; \alpha) & \xleftarrow[r_C]{\cong} & \mathrm{SH}_*^a(U; \alpha) \\ & \searrow \mathfrak{h}_i^{Ca} & \swarrow \iota_{a, Ca}^{\mathrm{SH}_*, \alpha(U)} \\ & \mathrm{SH}_*^{Ca}(U; \alpha) & \end{array}$$

where \mathfrak{h}_i^{Ca} is the persistence module morphism induced by inclusion $U \xrightarrow{i} CU$ and $\iota_{a, Ca}^{\mathrm{SH}_*, \alpha(U)}$ is the structure map of the persistence module $\mathrm{SH}_{*, \alpha}(U)$.

Similarly, for $C \leq 1$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{SH}_*^{Ca}(CU; \alpha) & \xleftarrow[r_C]{\cong} & \mathrm{SH}_*^a(U; \alpha) \\ & \swarrow \mathfrak{h}_i^{Ca} & \searrow \iota_{Ca, a}^{\mathrm{SH}_*, \alpha(U)} \\ & \mathrm{SH}_*^{Ca}(U; \alpha) & \end{array}$$

where \mathfrak{h}_i^{Ca} is the persistence module morphism induced by inclusion $CU \xrightarrow{i} U$ and $\iota_{Ca, a}^{\mathrm{SH}_*, \alpha(U)}$ is the structure map of the persistence module $\mathrm{SH}_{*, \alpha}(U)$.

- (3) If $\phi : U \rightarrow V$ is a $\tilde{\pi}_1$ -trivial Liouville embedding, then for any positive C and a , it holds $\mathfrak{h}_{\phi(C)}^{Ca} \circ r_C = r_C \circ \mathfrak{h}_\phi^a$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SH}_*^a(V; \alpha) & \xrightarrow{\mathfrak{h}_\phi^a} & \mathrm{SH}_*^a(U; \alpha) \\ r_C \downarrow \cong & & \cong \downarrow r_C \\ \mathrm{SH}_*^{Ca}(CV; \alpha) & \xrightarrow{\mathfrak{h}_{\phi(C)}^{Ca}} & \mathrm{SH}_*^{Ca}(CU; \alpha) \end{array}$$

- (4) Let $U \subset V$ and suppose a Liouville embedding $\phi : U \rightarrow V$ is isotopic to inclusion i_U through Liouville embeddings, i.e. strongly unknotted. Then $\mathfrak{h}_\phi = \mathfrak{h}_{i_U}$.

The proof of Proposition 4.3.4 can be derived from Definition 4.3.3 and is left to an interested reader. The proof is analogous to the proof in the case of star-shaped domains which is treated in [120]. The main difference between the two cases is in the way the grading is defined. Indeed, in the case of star-shaped domains this is done using a symplectic trivialization of the tangent bundle over a disc capping the orbit, while, as explained above, we use the vertical Lagrangian distribution. Hence, we should prove that in our case \mathfrak{h}_f preserves grading. We do this in the lemma that follows.

Lemma 4.3.5. *Let M be a closed, orientable manifold, $U, V \in \mathcal{C}_M$ non-degenerate domains and let $f : U \rightarrow V$ be a $\tilde{\pi}_1$ -trivial Liouville embedding. Then \mathfrak{h}_f preserves grading.*

Proof. For $H \in \mathcal{H}_U$, denote by $f_*H \in \mathcal{H}_V$ the extension by zero of $H \circ f^{-1}$. Recall from [120] that \mathfrak{h}_f is induced by the map f_* between Floer chain complexes of H and f_*H . This map sends a periodic orbit $z \in \mathcal{P}(H; \alpha)$ to a periodic orbit $f \circ z \in \mathcal{P}(f_*H; \alpha)$. Hence, we need to prove that $\text{ind}_{HF}(z) = \text{ind}_{HF}(f \circ z)$ where both indices are calculated using the vertical Lagrangian distribution. To this end, let $\Phi : S^1 \times \mathbb{R}^{2n} \rightarrow z^*(TT^*M)$, $\Psi : S^1 \times \mathbb{R}^{2n} \rightarrow (f \circ z)^*(TT^*M)$ be symplectic trivializations such that for all $t \in S^1$ it holds

$$\Phi_t(0 \times \mathbb{R}^n) = T_{z(t)}^v T^*M, \quad \Psi_t(0 \times \mathbb{R}^n) = T_{f(z(t))}^v T^*M.$$

By definition

$$\text{ind}_{HF}(z) = \text{ind}_{CZ}(\Phi_t^{-1} \circ d\phi_t^H(z(0)) \circ \Phi_0),$$

as well as

$$\text{ind}_{HF}(f \circ z) = \text{ind}_{CZ}(\Psi_t^{-1} \circ d\phi_t^{f_*H}(f(z(0))) \circ \Psi_0).$$

Since f is a symplectic embedding, we have that $d\phi_t^H = (df)^{-1} \circ d\phi_t^{f_*H} \circ df$ and thus

$$\begin{aligned} \text{ind}_{HF}(z) &= \text{ind}_{CZ}(\Phi_t^{-1} \circ (df)^{-1} \circ d\phi_t^{f_*H} \circ df \circ \Phi_0) \\ &= \text{ind}_{CZ}(\Phi_t^{-1} \circ (df)^{-1} \circ \Psi_t \circ \Psi_t^{-1} \circ d\phi_t^{f_*H} \circ \Psi_0 \circ \Psi_0^{-1} \circ df \circ \Phi_0) \\ &= \text{ind}_{CZ}(\theta(t) \circ \Phi_0^{-1} \circ (df)^{-1} \circ \Psi_0 \circ \Psi_t^{-1} \circ d\phi_t^{f_*H} \circ \Psi_0 \circ \Psi_0^{-1} \circ df \circ \Phi_0), \end{aligned}$$

where $\theta(t) = \Phi_t^{-1} \circ (df)^{-1} \circ \Psi_t \circ \Psi_0^{-1} \circ df \circ \Phi_0$ is a loop of symplectic matrices. One readily checks that $\theta(0) = \theta(1) = \mathbb{1}$ and hence, using loop and naturality properties of the Conley-Zehnder index, we have that

$$\begin{aligned} \text{ind}_{HF}(z) &= \text{ind}_{CZ}(\Phi_0^{-1} \circ (df)^{-1} \circ \Psi_0 \circ \Psi_t^{-1} \circ d\phi_t^{f_*H} \circ \Psi_0 \circ \Psi_0^{-1} \circ df \circ \Phi_0) + 2\mu(\theta) \\ &= \text{ind}_{CZ}(\Psi_t^{-1} \circ d\phi_t^{f_*H} \circ \Psi_0) + 2\mu(\theta) \\ &= \text{ind}_{HF}(f \circ z) + 2\mu(\theta), \end{aligned}$$

where μ denotes the Maslov index. Thus, our goal is to show that $\mu(\theta) = 0$.

Fix a Lagrangian subspace $V_0 \subset \mathbb{R}^{2n}$ given by $V_0 = \Phi_0^{-1}((df)^{-1}(T_{f(z(0))}^v T^*M))$. Loop θ induces a loop of Lagrangian subspaces $\Lambda(t) = \theta(t)V_0$ and from properties of Φ and Ψ it follows that $\Lambda(t) = \Phi_t^{-1}((df)^{-1}(T_{f(z(t))}^v T^*M))$. Let $G : [0, 1] \times S^1 \rightarrow U$ be a homotopy such that $G_0(t) = z(t)$, $G_1(t) = \bar{z}(t)$ for $\bar{z} : S^1 \rightarrow M \subset U$ (such G exists because U is fiberwise star-shaped). Let $\tilde{\Phi} : [0, 1] \times S^1 \times \mathbb{R}^{2n} \rightarrow G^*(TT^*M)$ be a symplectic trivialization such that for all $t \in S^1$, $\tilde{\Phi}_{0,t} = \Phi_t$ and for all $s \in [0, 1]$, $t \in S^1$ it holds⁶ $\tilde{\Phi}_{s,t}(0 \times \mathbb{R}^n) = T_{G(s,t)}^v T^*M$. Denote by $\bar{\Phi}_t = \tilde{\Phi}_{1,t} : S^1 \times \mathbb{R}^{2n} \rightarrow \bar{z}^*(TT^*M)$. Now, $\tilde{\Phi}_{s,t}^{-1}((df)^{-1}(T_{f(G(s,t))}^v T^*M))$ provides a homotopy between loops $\Lambda(t)$ and $\bar{\Lambda}(t) := \bar{\Phi}_t^{-1}((df)^{-1}(T_{f(\bar{z}(t))}^v T^*M))$ of Lagrangian subspaces of \mathbb{R}^{2n} . Thus,

$$\begin{aligned} \mu(\theta) &= \mu(\Lambda) = \mu(\bar{\Lambda}) = \mu(\bar{\Phi}_t^{-1}((df)^{-1}(T_{f(\bar{z}(t))}^v T^*M)), 0 \times \mathbb{R}^n) \\ &= \mu(\tilde{\Phi}_t^{-1}((df)^{-1}(T_{f(\bar{z}(t))}^v T^*M)), \tilde{\Phi}_t^{-1}(T_{\bar{z}(t)}^v T^*M)), \end{aligned}$$

⁶The existence of such a trivialization $\tilde{\Phi}$ follows from an argument similar to the proof of Lemma 1.7 in [2]. In [2], $\tilde{\Phi}_{1,t}$ is predetermined and hence $\tilde{\Phi}_{s,t}(0 \times \mathbb{R}^n) = T_{G(s,t)}^v T^*M$ only holds for $s = 0, 1$. One may notice that this weaker condition would also be sufficient for our purposes.

where $\mu(\cdot, \cdot)$ denotes the relative Maslov index for the pair of Lagrangian paths, see [126].

Now, notice that $(df)^{-1}(T^v T^* M)$ and $T^v T^* M$ are Lagrangian subbundles of the symplectic vector bundle $TT^* M$ over M and we may define their Maslov class, $\mu_{T^v T^* M, (df)^{-1}(T^v T^* M)} \in H^1(M; \mathbb{Z})$, see [14] for the definition and properties of the Maslov class of a pair of Lagrangian subbundles. Moreover, it holds

$$\mu(\bar{\Phi}_t^{-1}((df)^{-1}(T_{f(\bar{z}(t))}^v T^* M)), \bar{\Phi}_t^{-1}(T_{\bar{z}(t)}^v T^* M)) = \mu_{T^v T^* M, (df)^{-1}(T^v T^* M)}([\bar{z}]). \quad (4.10)$$

Since, $T^v T^* M$ and TM are fiberwise transversal Lagrangian subbundles of $TT^* M$, we have that $\mu(TM, T^v T^* M) = 0$ and hence

$$\mu_{T^v T^* M, (df)^{-1}(T^v T^* M)} = \mu_{TM, T^v T^* M} + \mu_{T^v T^* M, (df)^{-1}(T^v T^* M)} = \mu_{TM, (df)^{-1}(T^v T^* M)}. \quad (4.11)$$

Using df to identify $TT^* M$ and $f^*(TT^* M)$ as symplectic vector bundles over M , we have that

$$\mu_{TM, (df)^{-1}(T^v T^* M)} = \mu_{TM, f^*(T^v T^* M)} = \mu_f, \quad (4.12)$$

where μ_f denotes the Maslov class of a Lagrangian immersion $f : M \rightarrow T^* M$. Since f is actually an exact Lagrangian embedding, it follows from [86, Appendix E] that $\mu_f = 0$ and we have $\mu(\theta) = \mu_f([\bar{z}]) = 0$, which proves that $\text{ind}_{HF}(z) = \text{inf}_{HF}(f \circ z)$. \square

As explained in the introduction, in order to use standard (additive) parametrization of persistence modules, we also consider a logarithmic version of $\text{SH}_{*,\alpha}(U)$.

Definition 4.3.6. For $t \in \mathbb{R}$, let $S_*^t(U; \alpha) = \text{SH}_*^{e^t}(U; \alpha)$. Define a logarithmic version of the symplectic persistence module associated to $U \in \mathcal{C}_M$ as

$$\mathbb{S}_{*,\alpha}(U) = \left\{ \{S_{*,\alpha}^t(U)\}_{t \in \mathbb{R}}; \{\iota_{s,t}^{\text{SH}} = \iota_{e^s, e^t}^{\text{SH}} : S_*^s(U; \alpha) \rightarrow S_*^t(U; \alpha)\}_{s \leq t} \right\}.$$

4.3.2 Proof of Theorem 4.1.7

Proof. The second inequality directly follows from the first one and Proposition 4.2.8. Thus we will prove the first inequality. By Definition 4.1.4, for any $\varepsilon > 0$, there exists $C \geq 1$ with $\ln C \leq d_{SBM}(U, V) + \varepsilon$ such that

$$(*) \quad U/C \xrightarrow{\phi} V \xrightarrow{\psi} CU \text{ and } \psi \circ \phi \text{ is strongly unknotted;}$$

$$(**) \quad V/C \xrightarrow{\psi(C^{-1})} U \xrightarrow{\phi(C)} CV \text{ and } \phi(C) \circ \psi(C^{-1}) \text{ is strongly unknotted.}$$

Then, (1) and (4) in Proposition 4.3.4 together with (*) imply for any positive a , $\mathfrak{h}_\phi^a \circ \mathfrak{h}_\psi^a = \mathfrak{h}_{\psi \circ \phi}^a = \mathfrak{h}_i^a$ where i is the inclusion $i : U/C \rightarrow CU$. Moreover, (2) in

Proposition 4.3.4 implies that the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{SH}_*^a(CU; \alpha) & \xrightarrow{\mathfrak{h}_\psi^a} & \mathrm{SH}_*^a(V; \alpha) & \xrightarrow{\mathfrak{h}_\phi^a} & \mathrm{SH}_*^a(U/C; \alpha) \\
\uparrow r_C \simeq & & \searrow \mathfrak{h}_{i'}^a & \nearrow \mathfrak{h}_{i''}^a & \downarrow r_C \simeq \\
& & \mathrm{SH}_*^a(U; \alpha) & & \\
\mathrm{SH}_*^{\frac{a}{C}}(U; \alpha) & \xrightarrow[\iota_{a/C,a}^{\mathrm{SH}_*,\alpha(U)}]{} & \mathrm{SH}_*^a(U; \alpha) & \xrightarrow[\iota_{a,Ca}^{\mathrm{SH}_*,\alpha(U)}]{} & \mathrm{SH}_*^{Ca}(U; \alpha)
\end{array} \tag{4.13}$$

where $\mathfrak{h}_{i'}$ is induced by the inclusion $i' : U \rightarrow CU$ and $\mathfrak{h}_{i''}$ is induced by the inclusion $i'' : U/C \rightarrow U$. Now set

- $\Psi := \mathfrak{h}_\psi \circ r_C$ where $\Psi^{a/C} = \mathfrak{h}_\psi^a \circ r_C : \mathrm{SH}_*^{\frac{a}{C}}(U; \alpha) \rightarrow \mathrm{SH}_*^a(V; \alpha)$;
- $\Phi := r_C \circ \mathfrak{h}_\phi$ where $\Phi^a = r_C \circ \mathfrak{h}_\phi^a : \mathrm{SH}_*^a(V; \alpha) \rightarrow \mathrm{SH}_*^{Ca}(U; \alpha)$.

For any positive a , (4.13) implies

$$\begin{aligned}
\Phi^a \circ \Psi^{a/C} &= (r_C \circ \mathfrak{h}_\phi^a) \circ (\mathfrak{h}_\psi^a \circ r_C) \\
&= r_C \circ \mathfrak{h}_{i''}^a \circ \mathfrak{h}_{i'}^a \circ r_C \\
&= \iota_{a,Ca}^{\mathrm{SH}_*,\alpha(U)} \circ \iota_{a/C,a}^{\mathrm{SH}_*,\alpha(U)} = \iota_{a/C,Ca}^{\mathrm{SH}_*,\alpha(U)}.
\end{aligned}$$

Similarly to (4.13), (1), (2) and (4) in Proposition 4.3.4 together with (**) give a commutative diagram

$$\begin{array}{ccccc}
\mathrm{SH}_*^a(CV; \alpha) & \xrightarrow{\mathfrak{h}_{\phi(C)}^a} & \mathrm{SH}_*^a(U; \alpha) & \xrightarrow{\mathfrak{h}_{\psi(C^{-1})}^a} & \mathrm{SH}_*^a(V/C; \alpha) \\
\uparrow r_C \simeq & & \searrow \mathfrak{h}_{j'}^a & \nearrow \mathfrak{h}_{j''}^a & \downarrow r_C \simeq \\
& & \mathrm{SH}_*^a(V; \alpha) & & \\
\mathrm{SH}_*^{\frac{a}{C}}(V; \alpha) & \xrightarrow[\iota_{a/C,a}^{\mathrm{SH}_*,\alpha(V)}]{} & \mathrm{SH}_*^a(V; \alpha) & \xrightarrow[\iota_{a,Ca}^{\mathrm{SH}_*,\alpha(V)}]{} & \mathrm{SH}_*^{Ca}(V; \alpha)
\end{array} \tag{4.14}$$

where \mathfrak{h}_j , $\mathfrak{h}_{j'}$ and $\mathfrak{h}_{j''}$ are induced by the inclusions $j : V/C \rightarrow CV$, $j' : V \rightarrow CV$ and $j'' : V/C \rightarrow V$ respectively. Moreover, applied to Ψ and Φ which we defined above, (3) in Proposition 4.3.4 gives

$$\Phi^{a/C} = r_C \circ \mathfrak{h}_\phi^{a/C} = \mathfrak{h}_{\phi(C)}^a \circ r_C \quad \text{and} \quad \Psi^a = \mathfrak{h}_\psi^{Ca} \circ r_C = r_C \circ \mathfrak{h}_{\psi(C^{-1})}^a.$$

Then commutative diagram (4.14) implies

$$\begin{aligned}
\Psi^a \circ \Phi^{a/C} &= (\mathfrak{h}_\psi^{Ca} \circ r_C) \circ (r_C \circ \mathfrak{h}_\phi^{a/C}) \\
&= r_C \circ \mathfrak{h}_{\psi(C^{-1})}^a \circ \mathfrak{h}_{\phi(C)}^a \circ r_C \\
&= r_C \circ \mathfrak{h}_{j''}^a \circ \mathfrak{h}_{j'}^a \circ r_C \\
&= \iota_{a,Ca}^{\mathrm{SH}_*,\alpha(V)} \circ \iota_{a/C,a}^{\mathrm{SH}_*,\alpha(V)} = \iota_{a/C,Ca}^{\mathrm{SH}_*,\alpha(V)}.
\end{aligned}$$

Therefore, passing to the logarithmic version of symplectic persistence modules defined in Definition 4.3.6, the existence of the pair (Φ, Ψ) implies that $\mathbb{S}_{*,\alpha}(U)$ and $\mathbb{S}_{*,\alpha}(V)$ are $(\ln C)$ -interleaved. Hence, by the isometry theorem (Theorem 2.1.21),

$$d_{bottle}(\mathbb{B}_{*,\alpha}(U), \mathbb{B}_{*,\alpha}(V)) = d_{inter}(\mathbb{S}_{*,\alpha}(U), \mathbb{S}_{*,\alpha}(V)) \leq \ln C \leq d_{SBM}(U, V) + \varepsilon.$$

We draw the conclusion by letting $\varepsilon \rightarrow 0$. □

4.4 Filtered homology of the free loop space

In this section we review basic notions about the homology of the free loop space filtered by energy and show how this filtered homology relates to symplectic homology of the unit codisc bundle.

4.4.1 Morse-Bott perspective

Let (M, g) be a closed, orientable, Riemannian manifold, α a homotopy class of free loops in M and $\mathcal{L}_\alpha(M)$ the space of smooth loops in M in class α . Recall that the energy functional $E_g : \mathcal{L}_\alpha(M) \rightarrow \mathbb{R}$ is defined as $E_g(\gamma) = \int_0^1 \frac{\|\dot{\gamma}\|_g^2}{2} dt$ for any $\gamma \in \mathcal{L}_\alpha(M)$. This functional is never Morse, but rather Morse-Bott in a generic situation. In this subsection we briefly review some basic notions of Morse-Bott homology in the context of E_g . Our exposition mostly follows Section 4 in [3], which is based on [1] and [64]. For other treatments of this topic, see [84, 101].

Let $f : W \rightarrow \mathbb{R}$ be a smooth function on a Hilbert manifold W and assume that $\text{Crit}(f)$ consists of a disjoint union of closed submanifolds of W . Hessian, $\text{Hess}(f)_p$, at a point $p \in \text{Crit}(f)$ is a bilinear form on $T_p W$, and we have that $T_p \text{Crit}(f) \subset \ker(\text{Hess}(f)_p)$. Let $N \subset \text{Crit}(f)$ be a connected component and $p \in N$ a critical point.

We define *nullity* of p to be equal to $\dim(\ker(\text{Hess}(f)_p)) - 1$ and *index* of p to be the maximal dimension of a subspace of $T_p W$ on which $\text{Hess}(f)_p$ is negative definite. Both index and nullity are constant along N and hence we may define index and nullity of N as index and nullity of any point in N . N is said to be a *non-degenerate critical submanifold of f* if $\ker(\text{Hess}(f)_p) = T_p \text{Crit}(f)$ for all $p \in N$ or equivalently if nullity of N equals $\dim N - 1$.

We consider E_g as a functional on the space $W^{1,2}(S^1, M) \supset \mathcal{L}_\alpha(M)$ which is a Hilbert manifold. Critical points of E_g are closed geodesics (this includes constant loops too). By a closed geodesic, we mean a closed curve $\gamma : S^1 \rightarrow M$ such that $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. In particular, $\|\dot{\gamma}\|_g^2 = \text{constant}$. Constant geodesics form a critical submanifold diffeomorphic to M . This critical submanifold is always non-degenerate and has index equal to 0 (see Proposition 2.4.6 in [84]). On the other hand, any non-constant closed geodesic appears in an S^1 -family corresponding to reparameterizations. More

precisely, if γ is a non-constant closed geodesic of constant speed, so is $s \cdot \gamma$, $s \in S^1$ given by $(s \cdot \gamma)(t) = \gamma(t + s)$. We say that a non-constant closed geodesic γ is non-degenerate if $S^1 \cdot \gamma$ is a non-degenerate critical submanifold, i.e. nullity of $S^1 \cdot \gamma$ is zero.

Definition 4.4.1 ([6]). A metric g is called *bumpy* if all of its closed geodesics are non-degenerate. One may check that this definition is equivalent to (U_g^*M, λ_{can}) being a non-degenerate domain.

Remark 4.4.2. A generic Riemannian metric is bumpy, see [6, 9] for a precise statement.

Remark 4.4.3. In certain cases index and nullity of a closed geodesic γ can be computed in a more direct way by analyzing Poincare return map and Jacobi vector fields along γ . We will make this precise in Section 4.6 and use it to carry out calculations for the bulked spheres and multi-bulked surfaces.

If $\text{Crit}(E_g)$ consists only of non-degenerate critical submanifolds E_g is called *Morse-Bott*. For a bumpy g , E_g is Morse-Bott and one may use it to define Morse-Bott homology. There are different approaches to constructing Morse-Bott homology (see [78] and references therein for finite dimensional cases) and we focus on the one described in [64] which uses moduli spaces of flow lines with cascades. Let g be a bumpy metric and pick an auxiliary Morse function h on $\text{Crit}(E_g)$, meaning Morse on each connected component of $\text{Crit}(E_g)$. If $x \in \text{Crit}(h)$, it follows that $x \in N$, where $N \subset \text{Crit}(E_g)$ is a connected critical submanifold of E_g and we define *total index* of x as

$$\text{ind}_{E_g, h}(x) = \text{ind}_{E_g}(N) + \text{ind}_h(x),$$

where $\text{ind}_{E_g}(N)$ denotes the index of N as a critical submanifold and $\text{ind}_h(x)$ denotes the standard Morse index. Slightly abusing the notation, throughout this chapter we will write just ind when it is clear what is the index in question. Morse-Bott k -th chain group in homotopy class α is defined as

$$CMB_{k, \alpha}(E_g, h) = \text{Span}_{\mathbb{Z}_2}(\{x \in \text{Crit}(h) \mid [x] = \alpha, \text{ind}_{E_g, h}(x) = k\}).$$

In order to define the differential, we introduce moduli spaces of flow lines with cascades. Fix two regular metrics⁷, one on $W^{1,2}(S^1, M)$, the other one on $\text{Crit}(E_g)$ and denote by ∇E_g and ∇h the gradient vector fields corresponding to these metrics. For $x, y \in \text{Crit}(h)$ let

$$\mathcal{M}_0^{cas}(x, y) = \{u : \mathbb{R} \rightarrow W^{1,2}(S^1, M) \mid \dot{u} = -\nabla h(u), u(-\infty) = x, u(+\infty) = y\}.$$

Note that $\mathcal{M}_0^{cas}(x, y)$ can only be non-empty if x and y belong to the same connected component of $\text{Crit}(E_g)$. For $k \geq 1$ define $\mathcal{M}_k^{cas}(x, y)$ as the set of pairs (\mathbf{u}, \mathbf{t}) where $\mathbf{u} = (u_1, \dots, u_k)$ is a k -tuple of negative gradient flow lines $u_i : \mathbb{R} \rightarrow W^{1,2}(S^1, M)$,

$$\dot{u}_i = -\nabla E_g(u_i),$$

⁷Regular means such that transversality is achieved in the definition of all the moduli spaces which appear. Such choice of metrics is generic.

and $\mathbf{t} = (t_1, \dots, t_{k-1})$ a $(k-1)$ -tuple of non-negative numbers $t_i \geq 0$ such that

1. $u_1(-\infty) \in W^u(x)$, $u_k(+\infty) \in W^s(y)$, where $W^u(x)$ and $W^s(y)$ denote respectively unstable and stable manifolds of x and y with respect to the flow of $-\nabla h$.
2. For every $1 \leq i \leq k-1$ there exists a negative gradient flow line $v_i : \mathbb{R} \rightarrow \text{Crit}(E_g)$, $\dot{v}_i = -\nabla h(v_i)$ such that

$$v_i(0) = u_i(+\infty), \quad v_i(t_i) = u_{i+1}(-\infty).$$

Now, \mathbb{R} acts freely on each of u_i by translations and thus \mathbb{R}^k acts freely on $\mathcal{M}_k^{cas}(x, y)$ and we denote

$$\bar{\mathcal{M}}_k^{cas}(x, y) = \mathcal{M}_k^{cas}(x, y)/\mathbb{R}^k \quad \text{and} \quad \bar{\mathcal{M}}^{cas}(x, y) = \bigcup_{k \geq 0} \bar{\mathcal{M}}_k^{cas}(x, y).$$

Regularity of the choice of metrics implies that $\bar{\mathcal{M}}^{cas}(x, y)$ is a smooth manifold of dimension $\text{ind}_{E_g, h}(x) - \text{ind}_{E_g, h}(y) - 1$. When $\text{ind}_{E_g, h}(x) = \text{ind}_{E_g, h}(y) + 1$ this manifold is zero-dimensional and compact, i.e. it is a finite set of points, and we denote by $n(x, y)$ the number of points in $\bar{\mathcal{M}}^{cas}(x, y)$ modulo 2. The Morse-Bott differential $\partial : CMB_{k, \alpha}(E_g, h) \rightarrow CMB_{k-1, \alpha}(E_g, h)$, is given by

$$\partial x = \sum_{y, \text{ind } y = k-1} n(x, y)y.$$

It satisfies $\partial^2 = 0$ and the resulting Morse-Bott homology does not depend on the regular choices of two metrics, h or E_g . In fact, we have that

$$HMB_{k, \alpha}(E_g, h) \cong H_k(\mathcal{L}_\alpha(M); \mathbb{Z}_2).$$

For our purposes it is essential to consider Morse-Bott chain complex together with the filtration by energy, i.e., we define

$$CMB_{k, \alpha}^\lambda(E_g, h) = \text{Span}_{\mathbb{Z}_2}(\{x \in \text{Crit}(h) \mid [x] = \alpha, \text{ind}_{E_g, h}(x) = k, E_g(x) \leq \lambda\}).$$

Since E_g decreases along the flow lines of $-\nabla E_g$, ∂ restricts to $CMB_{k, \alpha}^\lambda(E_g, h)$ and we may define filtered Morse-Bott homology $HMB_{k, \alpha}^\lambda(E_g, h)$. In this case it holds

$$HMB_{k, \alpha}^\lambda(E_g, h) \cong H_k(\mathcal{L}_\alpha^\lambda(M); \mathbb{Z}_2), \quad (4.15)$$

and this isomorphism commutes with the maps induced from inclusions of sublevel sets $\{E_g \leq \lambda\}$.

If g is bumpy, then for all $\lambda \geq 0$ there are finitely many critical submanifolds in the sublevel $\{E_g \leq \lambda\}$ and hence $CMB_{k, \alpha}^\lambda(E_g, h)$ is finitely generated (see Theorem 3.5 in [101] and references therein). This, together with (4.15) implies that the collection of data

$$\mathbb{H}_{*, \alpha}(M, g) = \left\{ \begin{array}{l} \{H_*(\mathcal{L}_\alpha^\lambda(M, g); \mathbb{Z}_2)\}_{\lambda \in \mathbb{R}_{>0}}; \\ \{\iota_{\lambda, \eta} : H_*(\mathcal{L}_\alpha^\lambda(M, g); \mathbb{Z}_2) \rightarrow H_*(\mathcal{L}_\alpha^\eta(M); \mathbb{Z}_2)\}_{\lambda \leq \eta} \end{array} \right\} \quad (4.16)$$

forms a persistence module with \mathbb{Z}_2 -coefficients, where $\iota_{\lambda,\eta}$ are induced by inclusion $\mathcal{L}_\alpha^\lambda(M, g) \hookrightarrow \mathcal{L}_\alpha^\eta(M, g)$ when $\lambda \leq \eta$. Moreover, since the endpoints of bars in the barcode $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$ come from generators of $CMB_{k,\alpha}^\lambda(E_g, h)$, the barcode $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g))$ has only finitely many endpoints of bars below every fixed $\lambda \geq 0$. These endpoints are equal to the energies of certain closed geodesics.

Remark 4.4.4. For isomorphism (4.15) to hold it is enough that all closed geodesics of energy not greater than λ are non-degenerate. Indeed, one may apply the same considerations as above directly to $\mathcal{L}_\alpha^\lambda(M)$.

Let us sum up the important features of the above construction. Firstly, every non-constant, non-degenerate closed geodesic gives rise to a critical submanifold of E_g diffeomorphic to S^1 . There exists a function on S^1 which has exactly 2 critical points of Morse index 0 and 1 (for example the standard height function). By picking the auxiliary function h to be equal to such a function on each of the S^1 -critical submanifolds, we obtain that to each non-constant, non-degenerate closed geodesic γ correspond two critical points of h whose total indices are equal to $\text{ind } \gamma$ and $\text{ind } \gamma + 1$. In other words, γ produces two generators of the chain complex $CMB_{*,\alpha}(E_g, h)$, one in degree $\text{ind } \gamma$ and the other one in degree $\text{ind } \gamma + 1$. On the other hand, critical submanifold of constant geodesics is diffeomorphic to M and has index equal to 0. Hence it gives rise to critical points of h whose total indices are equal to their Morse indices with respect to h . In other words, if we view h as a function on M , each critical points of Morse index k produces a generator of $CMB_{k,pt}(E_g, h)$. Finally the differential counts certain broken trajectories in $\mathcal{L}_\alpha M$. Each broken trajectory can be viewed as a tuple of maps from a cylinder to M connecting different closed geodesics.

4.4.2 The isomorphism with symplectic homology

In this subsection, we will elaborate a result which enables us to transfer computations from symplectic homology to the homology of the loop space. It states that, under certain parametrizations, filtered versions of these homologies are isomorphic as persistence modules, see Theorem 4.4.6. We will use this result to describe the barcode of the symplectic persistence module associated to the unit cotangent bundle of metrics coming from our main geometric constructions, see Sections 4.5 and 4.6.

The isomorphism between symplectic homology or Floer homology of the cotangent bundle and the homology of the loop space first appeared in [152]. Other versions of this isomorphism, constructed using different methods, have appeared in [130, 157, 2, 4, 5]. The version which fits our conventions is the one from [157] and we give a short exposition of it below. Let us recall some notions first.

Given a closed, orientable, Riemannian manifold (M, g) and a homotopy class α of free loops, denote the loop space of M in class α by $\mathcal{L}_\alpha(M)$. We define the *length spectrum* of g in class α , denoted by Λ_α , to be the set of lengths of all closed geodesics

in class α . Recall that if g is bumpy there are finitely many closed geodesics below any fixed energy level and hence Λ_α is discrete.

Remark 4.4.5. One may check that all the endpoints of all bars in $\mathcal{B}(\mathrm{SH}_*(U_g^*M; \alpha))$ belong to Λ_α . Indeed, it is enough to check that

$$\iota_{a,b} : \mathrm{SH}_*^a(U_g^*M; \alpha) \rightarrow \mathrm{SH}_*^b(U_g^*M; \alpha)$$

is an isomorphism if $0 < a \leq b$ are such that $[a, b] \cap \Lambda_\alpha = \emptyset$. To prove this one considers a radially symmetric Hamiltonian $H(\xi) = h(\|\xi\|_{g^*})$ for $h : [0, +\infty) \rightarrow \mathbb{R}$. If h is a decreasing function such that $h|_{[0, 1-\varepsilon]} = C$, and $h|_{[1, +\infty)} = 0$, taking C large enough (namely $C > b$) and ε small enough one sees that there are no periodic orbits of H in the action window $[a, b]$. This implies that

$$\iota_{a,b} : HF_*^{[a, \infty)}(H)_\alpha \rightarrow HF_*^{[b, \infty)}(H)_\alpha$$

is an isomorphism and by taking $C \rightarrow +\infty, \varepsilon \rightarrow 0$ we get the desired conclusion. For more details see [157].

Recall that

$$\mathcal{L}_\alpha^\lambda(M, g) = \{\gamma \in \mathcal{L}_\alpha(M) \mid E_g(\gamma) \leq \lambda\}.$$

We have explained in (4.16), that if g is a bumpy metric then $\{H_*(\mathcal{L}_\alpha^\lambda(M, g); \mathbb{Z}_2)\}_{\lambda \in \mathbb{R}_{>0}}$ form a persistence module $\mathbb{H}_{*, \alpha}(M, g)$ such that $\mathcal{B}(\mathbb{H}_{*, \alpha}(M, g))$ has finitely many endpoints of bars below every fixed λ . Moreover the endpoints of bars in $\mathcal{B}(\mathbb{H}_{*, \alpha}(M, g))$ are equal to energies of certain closed geodesics. We are now ready to state the result.

Theorem 4.4.6. (*[152, 130, 157, 2, 4, 5]*) *Let (M, g) be a closed, orientable, Riemannian manifold with bumpy metric g and α a homotopy class of free loops in M . There exists a family of isomorphisms*

$$\Phi_a : \mathrm{SH}_*^a(U_g^*M; \alpha) \rightarrow H_*(\mathcal{L}_\alpha^{a^2/2}(M); \mathbb{Z}_2),$$

for $a > 0$, which commute with structure maps. In other words, under suitable parameterizations, Φ is a persistence module isomorphism.

For every $a \in \mathbb{R}_{>0} \setminus \Lambda_\alpha$, the isomorphism Φ_a as in Theorem 4.4.6 has been constructed in Theorem 3.1 in [157]. Our goal is to show that this Φ commutes with structure maps and then to extend it to a persistence module isomorphism for all $a \in \mathbb{R}_{>0}$. For reader's convenience, let us review the construction of Φ_a from [157] first.

According to Definition 4.3.3, in order to study symplectic homology, we need to understand the associated Hamiltonian Floer homologies. In Subsection 4.3.1 we explained how to associate Floer homology $HF_*^{[a, b]}(H)_\alpha$ to $H \in \mathcal{H}_{U, a, b}$. In an analogous fashion, instead of considering compactly supported Hamiltonians, one may look at the sets

$$\mathcal{K}_{U_g^*M} = \{H : S^1 \times T^*M \rightarrow \mathbb{R} \mid \exists \beta > 0, \beta' \in \mathbb{R} \text{ s.t. } H_t(\xi) = -\beta \|\xi\|_{g^*} + \beta' \text{ for } \|\xi\|_{g^*} \geq 1\},$$

and

$$\mathcal{K}_{U_g^*M,a,b} = \{H \in K_{U_g^*M} \mid a, b \notin \text{Spec}(H; \alpha) \text{ and either } \beta \notin \Lambda_\alpha \text{ or } \beta' \notin [a, b]\}.$$

Hamiltonians compactly supported inside U_g^*M constitute the case $\beta = \beta' = 0$ and thus $\mathcal{H}_{U_g^*M,a,b} \subset \mathcal{K}_{U_g^*M,a,b}$. As in the compactly supported case if $H_1 \preceq H_2$ one may define a continuation map associated to a monotone homotopy (manifestly if $H_1 \preceq H_2$ slopes satisfy $-\beta_{H_1} \geq -\beta_{H_2}$). Analogously, a monotone homotopy H_τ such that $H_\tau \in \mathcal{K}_{U_g^*M,a,b}$ for every τ is called *action-regular*. Continuation maps will have the same properties as before, in particular the existence of an action-regular monotone homotopy will imply that the corresponding continuation map is an isomorphism, see [157] for more details.

We are now ready to define Φ_a . Fix $a \in \mathbb{R}_{>0} \setminus \Lambda_\alpha$ and take a radially symmetric Hamiltonian $H \in \mathcal{H}_{U_g^*M}$, $H(\xi) = h(\|\xi\|_{g^*})$ with $h : [0, +\infty) \rightarrow \mathbb{R}$ such that $h|_{[0,1-\varepsilon]} = C > a$, h is decreasing and $h|_{[1,+\infty)} = 0$. Moreover consider a monotone homotopy, shown in Figure 4.3, from H to a new Hamiltonian \tilde{H}_a which we obtain by making the “tail” of H linear with slope $-a$. More precisely, one sees that there are exactly

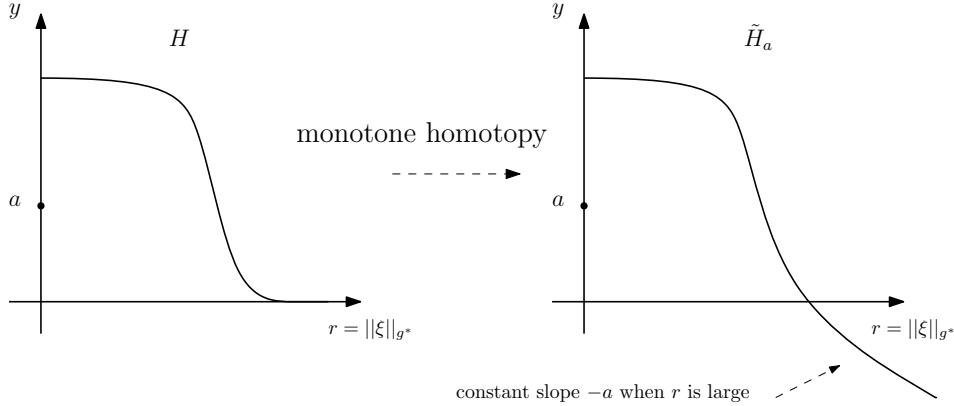


Figure 4.3. Monotone homotopy 1

two points $r_1, r_2 \in [0, 1]$ for which $h'(r_i) = -a$ for $i = 1, 2$. If we label them by $r_1 \leq r_2$ then $h(r_1) \approx C$ and $h(r_2) \approx 0$. Up to a small smoothing at $\|\xi\|_{g^*} = r_2$, Hamiltonian \tilde{H}_a is equal to H for $\|\xi\|_{g^*} \in [0, r_2]$ and is linear with slope $-a$ for $\|\xi\|_{g^*} \in [r_2, +\infty)$.

The monotone homotopy from H to \tilde{H}_a gives the isomorphism

$$c_a : HF_*^{[a,\infty)}(H)_\alpha \xrightarrow{\cong} HF_*^{[a,\infty)}(\tilde{H}_a)_\alpha \quad (4.17)$$

because no Hamiltonian 1-periodic orbit with action in the action window $[a, \infty)$ appears during the homotopy, i.e. it is action-regular.

Moreover, we construct the third Hamiltonian, denoted by H_a , in the similar fashion to the construction of \tilde{H}_a . Namely, up to a small smoothing at $\|\xi\|_{g^*} = r_1$, H_a coincides with H on the set $\|\xi\|_{g^*} \in [0, r_1]$ and is linear with slope $-a$ for $\|\xi\|_{g^*} \in [r_1, +\infty)$.

Now, there exists another action-regular monotone homotopy from H_a to \tilde{H}_a , see Figure 4.4, which provides another isomorphism

$$s_a : HF_*^{[a,\infty)}(H_a)_\alpha \xrightarrow{\cong} HF_*^{[a,\infty)}(\tilde{H}_a)_\alpha. \quad (4.18)$$

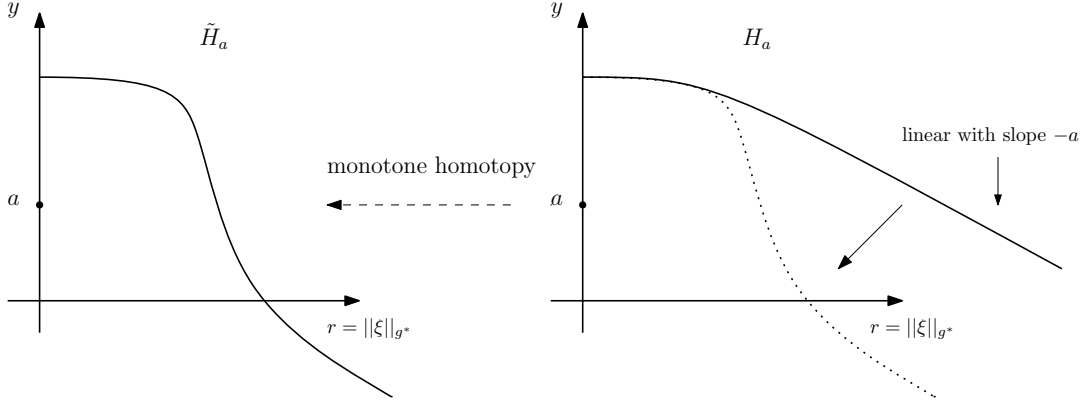


Figure 4.4. Monotone homotopy 2

Following [157], for a radially symmetric Hamiltonian function $H = h(r)$, $r = \|\xi\|_{g^*}$ with $h''(r) \leq 0$, define for any $\lambda \in \mathbb{R}_{\geq 0}$,

$$C(H, \lambda) = \lambda r_* + h(r_*) \quad \text{where} \quad h'(r_*) = -\lambda, \quad (4.19)$$

if such r_* exists. Observe that $C(H, \lambda)$ is the y -intercept of the line passing through point $(r_*, h(r_*))$ with slope $-\lambda$. Since the Hamiltonian function H_a in Figure 4.4 is concave with respect to r , value $C(H_a, \lambda)$ is well-defined for all $\lambda \in [0, a]$. Now, the advantage of considering Hamiltonian H_a is that it does not have any Hamiltonian 1-periodic orbit of action less than a . On the other hand, the maximal action of the Hamiltonian 1-periodic orbit of H_a is less than $C(H_a, a)$. Therefore, one gets the following isomorphisms

$$HF_*^{(-\infty, C(H_a, a))}(H_a)_\alpha \xrightarrow[\cong]{i_{H_a}^a} HF_*^{(-\infty, \infty)}(H_a)_\alpha \xrightarrow[\cong]{\pi_{H_a}^a} HF_*^{[a, \infty)}(H_a)_\alpha \quad (4.20)$$

from the equality on the chain level. For more details regarding all the constructions see Section 3 in [157].

Finally, Theorem 2.9 in [157] claims that there exists an isomorphism

$$\psi_{H_a}^a : HF_*^{(-\infty, C(H_a, a))}(H_a)_\alpha \xrightarrow{\cong} H_*(\mathcal{L}_\alpha^{a^2/2}(M); \mathbb{Z}_2). \quad (4.21)$$

Map $\psi_{H_a}^a$ essentially comes from the main result in [130] which compares the symplectic action functional with a certain energy functional on the loop space. Combining all the above defined isomorphisms together, one obtains the following isomorphism,

$$\Phi_{H,a} : HF_*^{[a,\infty)}(H)_\alpha \rightarrow H_*(\mathcal{L}_\alpha^{a^2/2}(M); \mathbb{Z}_2) \quad (4.22)$$

where $\Phi_{H,a} = \psi_{H_a}^a \circ (i_{H_a}^a)^{-1} \circ (\pi_{H_a}^a)^{-1} \circ s_a^{-1} \circ c_a$. The desired isomorphism Φ_a is then given by $\Phi_a = \varprojlim_{H \in \mathcal{H}_{U_g^* M}} \Phi_{H,a}$.

Proof. (Proof of Theorem 4.4.6) It follows from the definitions that both persistence modules $\mathbb{SH}_{*,\alpha}(U_g^*M)$ and $\mathbb{H}_{*,\alpha}(M, g)$ are such that all the bars in their barcodes have left endpoints closed and right endpoints open. Moreover, by Remark 4.4.5, the endpoints of bars in the barcode of $\mathbb{SH}_{*,\alpha}(U_g^*M)$ belong to Λ_α and since g is bumpy, for every fixed $\lambda > 0$, $\Lambda_\alpha \cap [0, \lambda]$ is finite. Thus, it is enough to prove Theorem 4.4.6 for $a \in \mathbb{R}_{>0} \setminus \Lambda_\alpha$ and afterwards extend Φ_a to $a \in \Lambda_\alpha$ by continuity.

From the definition of symplectic homology, it readily follows that for any $a < b$, there exists a single $H \in \mathcal{H}_{U_g^*M, \{a,b\}}$ such that $\iota_{a,b}^{\mathbb{SH}} : \mathbb{SH}_*^a(U_g^*M; \alpha) \rightarrow \mathbb{SH}_*^b(U_g^*M; \alpha)$ can be seen as

$$\iota_{a,b}^{HF} : HF_*^{[a,\infty)}(H)_\alpha \rightarrow HF_*^{[b,\infty)}(H)_\alpha.$$

The example of such H which we consider is a radially symmetric Hamiltonian, shown in Figure 4.5, such that $\max H \geq b$ (thus also $\max H \geq a$), H is equal to $\max H$ for $\|\xi\|_{g^*} = r \leq 1 - \varepsilon$ with some small $\varepsilon > 0$ and is decreasing in r .

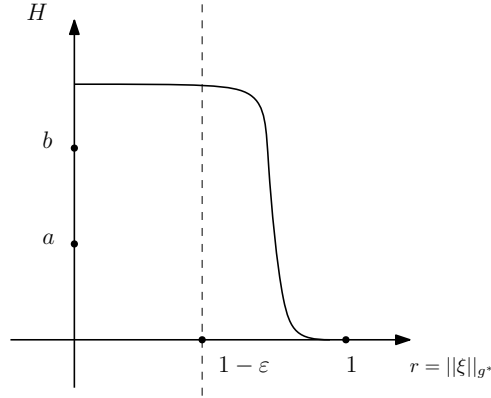


Figure 4.5. A radially symmetric Hamiltonian which computes symplectic homology

Using this H , we can carry out monotone homotopies as described above and shown in Figure 4.3 and Figure 4.4 for both slopes a and b . This way, we obtain new Hamiltonian functions \tilde{H}_a, \tilde{H}_b as well as H_a, H_b , see Figure 4.6. We claim that the following diagram commutes.

$$\begin{array}{ccccc}
& & HF_*^{[a,\infty)}(H)_\alpha & \xrightarrow{\iota_{a,b}^{HF}} & HF_*^{[b,\infty)}(H)_\alpha \\
& \swarrow c_a & \downarrow c'_b & & \downarrow c_b \\
HF_*^{[a,\infty)}(\tilde{H}_a)_\alpha & \xrightarrow{\tilde{c}} & HF_*^{[a,\infty)}(\tilde{H}_b)_\alpha & \xrightarrow{inc_*} & HF_*^{[b,\infty)}(\tilde{H}_b)_\alpha \\
\uparrow s_a \downarrow s_a^{-1} & & \uparrow s'_b & & \uparrow s_b \downarrow s_b^{-1} \\
HF_*^{[a,\infty)}(H_a)_\alpha & \xrightarrow{c} & HF_*^{[a,\infty)}(H_b)_\alpha & \xrightarrow{inc_*} & HF_*^{[b,\infty)}(H_b)_\alpha
\end{array}$$

Commutativity comes from the following arguments:

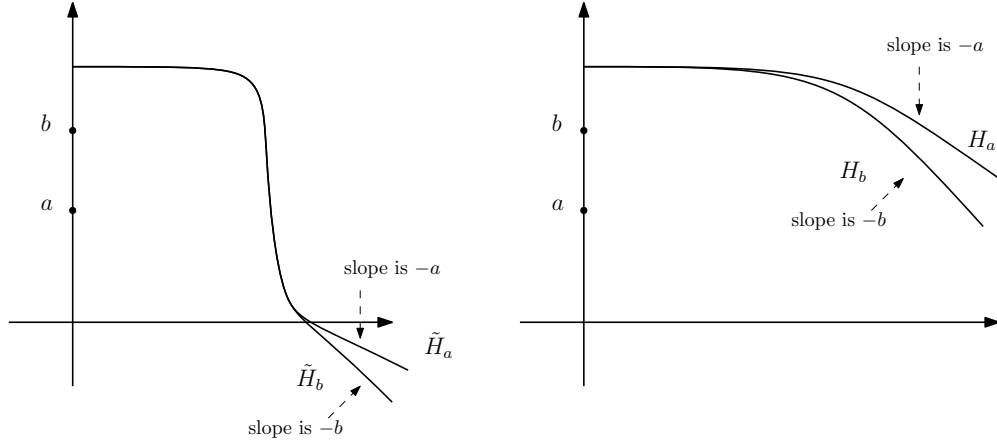


Figure 4.6. New Hamiltonian functions coming from monotone homotopies

- In the upper-left triangle, c'_b is induced from a monotone homotopy from H to \tilde{H}_b and \tilde{c} is induced from a monotone homotopy from \tilde{H}_a to \tilde{H}_b . Because $H \preceq \tilde{H}_a \preceq \tilde{H}_b$, $c'_b = \tilde{c} \circ c_a$ comes from Lemma 4.3.2.
- In the lower-left rectangle, s'_b is induced from a monotone homotopy from H_b to \tilde{H}_b and c is induced from a monotone homotopy from H_a to H_b . Because $H_a \preceq \tilde{H}_a \preceq \tilde{H}_b$ and $H_a \preceq H_b \preceq \tilde{H}_b$, from (4.9) we get $\tilde{c} \circ s_a = s'_b \circ c$, which implies $\tilde{c} = s'_b \circ c \circ s_a^{-1}$ where s_a^{-1} is the inverse of s_a (s_a is an isomorphism by (4.18)).
- The upper-right rectangle trivially commutes because we may take monotone homotopy inducing c'_b to be the same as the monotone homotopy inducing c_b and hence the maps count the same Floer trajectories.
- The lower-right rectangle trivially commutes by the same reason as above, which implies $inc_* = s_b^{-1} \circ inc_* \circ s'_b$.

Finally, we also claim that the following diagram commutes.

$$\begin{array}{ccccc}
HF_*^{[a,\infty)}(H_a)_\alpha & \xrightarrow{c} & HF_*^{[a,\infty)}(H_b)_\alpha & \xrightarrow{inc_*} & HF_*^{[b,\infty)}(H_b)_\alpha \\
\pi_{H_a}^a \updownarrow (\pi_{H_a}^a)^{-1} & & \pi_{H_b}^a \up & & \pi_{H_b}^b \updownarrow (\pi_{H_b}^b)^{-1} \\
HF_*^{(-\infty,\infty)}(H_a)_\alpha & \xrightarrow{c} & HF_*^{(-\infty,\infty)}(H_b)_\alpha & \xrightarrow{\mathbb{1}} & HF_*^{(-\infty,\infty)}(H_b)_\alpha \\
i_{H_a}^a \updownarrow (i_{H_a}^a)^{-1} & & i_{H_b}^a \up & & i_{H_b}^b \updownarrow (i_{H_b}^b)^{-1} \\
HF_*^{(-\infty,C(H_a,a))}(H_a)_\alpha & \xrightarrow{c} & HF_*^{(-\infty,C(H_b,a))}(H_b)_\alpha & \xrightarrow{inc_*} & HF_*^{(-\infty,C(H_b,b))}(H_b)_\alpha \\
& \searrow \psi_{H_a}^a & \downarrow \psi_{H_b}^a & & \downarrow \psi_{H_b}^b \\
& & H_*(\mathcal{L}_\alpha^{a^2/2}(M)) & \xrightarrow{t_{a^2/2,b^2/2}^{\mathbb{H}}} & H_*(\mathcal{L}_\alpha^{b^2/2}(M))
\end{array}$$

The only non-trivial commutativity is of the lower-left triangle and the lower-right rectangle. The former comes from the second proposition of Theorem 2.9 in [157] while the latter comes from the third proposition of Theorem 2.9 in [157]. Notice that maps $\pi_{H_a}^a, i_{H_a}^a$ and $\pi_{H_b}^b, i_{H_b}^b$ are all isomorphisms by (4.20), but $i_{H_b}^a$ is not an isomorphism. Denote by $\iota_{a^2/2, b^2/2}^{\mathbb{H}}$ the persistence structure map from filtration level $a^2/2$ to filtration level $b^2/2$ of persistence module $\mathbb{H}_{*,\alpha}(M, g)$. Using the definition of $\Phi_{H,a}$ given by (4.22) and the two commutative diagrams above we obtain $\iota_{a^2/2, b^2/2}^{\mathbb{H}} \circ \Phi_{H,a} = \Phi_{H,b} \circ \iota_{a,b}^{HF}$, which finished the proof. \square

4.5 Proofs of Proposition 4.1.16 and Proposition 4.1.17 (lower bounds)

In this section, we prove lower bounds in Propositions 4.1.16 and 4.1.17. To this end, we will describe two classes of Riemannian metrics which realize quasi-isometric embeddings in Propositions 4.1.16 and 4.1.17. The first class of metrics will be defined on S^2 and metrics in this class will be called *bulked sphere metrics* on S^2 . The other class will be defined on a closed, orientable surface Σ of genus at least 1, and metrics in this class will be called *multi-bulked metrics* on Σ . The way we construct these metrics enables us to precisely analyze closed geodesics and prove that they have various nice properties, see Propositions 4.5.4 and 4.5.8. Then, using Theorem 4.4.6, we are able to describe parts of the barcodes of the corresponding symplectic persistence modules. Finally, the lower bounds in both Proposition 4.1.16 and Proposition 4.1.17 comes from the stability property - Theorem 4.1.7 and a combinatorial result - Lemma 4.5.1, which we will now prove.

4.5.1 A combinatorial lemma

The following combinatorial lemma says that a particular shape of barcodes can help us get a lower bound on the bottleneck distance.

Lemma 4.5.1. *Let \mathbb{B}_1 and \mathbb{B}_2 be two barcodes. Let $a_1 \geq \dots \geq a_n$ be the n smallest left endpoints of bars in \mathbb{B}_1 and denote by $[a_1, C_{a_1}), \dots, [a_n, C_{a_n}) \in \mathbb{B}_1$ the corresponding bars. Similarly let $b_1 \geq \dots \geq b_n$ be the n smallest left endpoints of bars in \mathbb{B}_2 with corresponding bars $[b_1, C_{b_1}), \dots, [b_n, C_{b_n}) \in \mathbb{B}_2$. Assume that*

$$\min\{C_{a_1}, \dots, C_{a_n}, C_{b_1}, \dots, C_{b_n}\} > \max\{a_1, b_1\}.$$

Then it holds

$$\frac{1}{2} |\vec{a} - \vec{b}|_{\infty} \leq d_{\text{bottle}}(\mathbb{B}_1, \mathbb{B}_2),$$

where $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$. The statement remains true if some of the C_{a_i} or C_{b_j} are equal to $+\infty$.

Proof. Let k be such that $|\vec{a} - \vec{b}|_\infty = |a_k - b_k|$ and assume without loss of generality that $a_k \leq b_k$. Further assume that there exists a δ -matching $\sigma : \mathbb{B}_1 \rightarrow \mathbb{B}_2$. It is enough to prove that $2\delta \geq b_k - a_k = |\vec{a} - \vec{b}|_\infty$. We split the proof in three cases.

- Case 1 - One of the bars $[a_k, C_{a_k}), \dots, [a_n, C_{a_n})$ is erased.

Denote by l the index of the erased bar. Since $a_l \leq a_k$ and $C_{a_l} > b_1$ we have

$$2\delta \geq C_{a_l} - a_l \geq C_{a_l} - a_k > b_1 - a_k \geq b_k - a_k.$$

- Case 2 - None of the bars $[a_k, C_{a_k}), \dots, [a_n, C_{a_n})$ are erased, but at least one of them is matched with a bar different from $[b_k, C_{b_k}), \dots, [b_n, C_{b_n})$.

Let $l, k \leq l \leq n$ be such that $[a_l, C_{a_l})$ is not matched with any of the bars $[b_k, C_{b_k}), \dots, [b_n, C_{b_n})$ and let $\sigma([a_l, C_{a_l})) = [x, y)$. By the assumption of the theorem, we have that $x \geq b_k$ and hence

$$\delta \geq x - a_l \geq b_k - a_l \geq b_k - a_k.$$

- Case 3 - Bars $[a_k, C_{a_k}), \dots, [a_n, C_{a_n})$ are all matched with bars $[b_k, C_{b_k}), \dots, [b_n, C_{b_n})$.

Let $l, k \leq l \leq n$ be such that $\sigma([a_l, C_{a_l})) = [b_k, C_{b_k})$. We have

$$\delta \geq b_k - a_l \geq b_k - a_k,$$

and the proof is finished. □

4.5.2 Proof of Proposition 4.1.16

We start with the definition of a bulked sphere.

Definition 4.5.2. A *bulkied sphere* $S \subset \mathbb{R}^3$ is a surface of revolution obtained by rotating a *profile function* $r : [-L, L] \rightarrow [0, \infty)$ around axis l as shown in Figure 4.7.

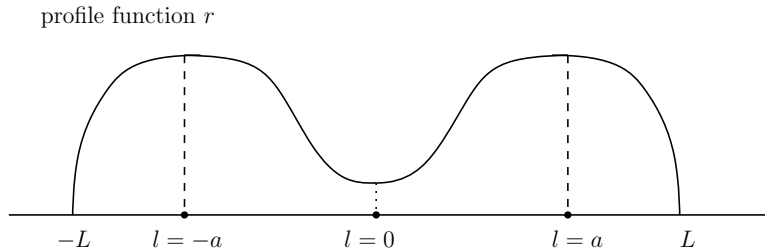


Figure 4.7. Profile function of a bulkied sphere S

We ask for r to satisfy the following properties.

- $r(l)$ is a smooth even function on $(-L, L)$ and $r(l) = 0$ exactly at $l = L$ and $-L$.
- $r(l)$ has only three critical points at $l = -a, 0, a$ and r attains global maximum at $l = a, -a$ and local minimum at $l = 0$.
- $r''(0) > 0$.

Figure 4.8 shows a general picture of a bulked sphere. A *bulked sphere metric* g is a metric on S^2 induced from the standard metric on \mathbb{R}^3 .

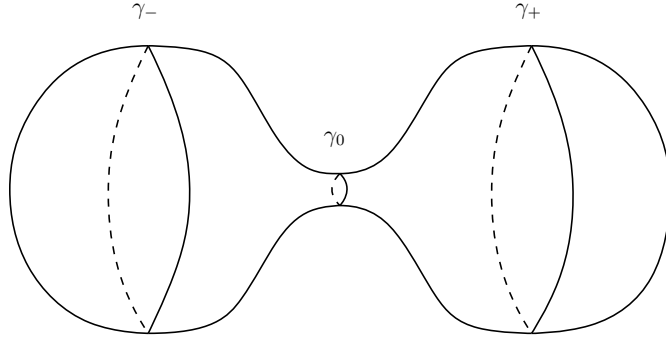


Figure 4.8. A general picture of a bulked sphere

A parallel circle is a geodesic if and only if it passes through a local extremum. In other words, we have three non-constant geodesic parallel circles of a bulked sphere metric, which we denote by γ_-, γ_0 and γ_+ as shown in Figure 4.8.

Lemma 4.5.3. *For $m \in \mathbb{N}$, denote by γ_0^m the m -times iteration of a closed geodesic γ_0 and by γ_0^{-m} the m -times iteration of γ_0 in the opposite direction. For every $m \in \mathbb{N}$, $\gamma_0^{\pm m}$ are non-degenerate and $\text{Ind}(\gamma_0^{\pm m}) = 0$.*

The proof of Lemma 4.5.3 comes from a direct computation which we carry out in Subsection 4.6.1. The following proposition is crucial for our proof of Proposition 4.1.16 (lower bound).

Proposition 4.5.4. *Given any $0 < \varepsilon_0 < 1$, there exists a positive $\delta_0 \ll 1$ such that for every $x \in [0, \infty)$, there exists a bulked sphere metric $g_x \in \mathcal{G}_{S^2}$ satisfying the following properties.*

- (1) Closed geodesic γ_0 has energy $E_{g_x}(\gamma_0) = \frac{\delta_0^2}{2} e^{-2x}$.
- (2) Any closed geodesic γ of (S^2, g_x) different from $\gamma_0^{\pm m}$, $m \in \mathbb{N}$ has energy $E_{g_x}(\gamma) > \frac{\delta_0^2}{2}$.
- (3) There exists a constant $R_x \in \left[\sqrt{\frac{1}{1+\varepsilon_0}}, \sqrt{\frac{1}{1-\varepsilon_0}} \right]$ such that $R_x \cdot g_x \in \bar{\mathcal{G}}_{S^2}$.

Part (2) of Proposition 4.5.4 is proven in Subsection 4.6.2. Roughly speaking, it comes from a fact that every closed geodesic γ different from $\gamma_0^{\pm m}$ has to exit the “narrow neck” and enter the two “spherical regions”, i.e. regions where $l \notin [-a, a]$. By making these regions sufficiently large we get that the length of γ must be large compared to the length of γ_0 . Finally, in order to prove (1) and (3) in Proposition 4.5.4, we need an explicit parametrization of S , see Subsection 4.8.1 in the Appendix.

Remark 4.5.5. Metrics g_x in Proposition 4.5.4 are not bumpy⁸ due to the existence of a rotational symmetry. However, they can be perturbed, by a C^∞ -small perturbation, to a bumpy metric which still satisfies all the properties from Lemma 4.5.3 and Proposition 4.5.4 (up to a small difference in logarithms of energies), see [9]. Since C^∞ -small perturbations create small differences in d_{SBM} , we ignore this point in the proof that follows, for the sake of clarity.

We are now ready to give a proof of the lower bound in Proposition 4.1.16.

Proof. (Proof of Proposition 4.1.16 (lower bound)) Define $\tilde{\Phi} : [0, \infty) \rightarrow \mathcal{G}_{S^2}$ as $\tilde{\Phi}(x) = g_x$ where g_x is the metric given by Propostion 4.5.4.

Recall that

$$\mathcal{L}_{pt}^\lambda(S^2, g_x) = \{\gamma \in \mathcal{L}_{pt}(S^2) \mid E_{g_x}(\gamma) \leq \lambda\}.$$

and also that $\mathbb{H}_{*,pt}(S^2, g_x)$ denotes the persistence module given by $\mathbb{H}_{*,pt}^\lambda(S^2, g_x) = H_*(\mathcal{L}_{pt}^\lambda(S^2, g_x); \mathbb{Z}_2)$, structure maps being induced by inclusions of sublevel sets. Our goal is to describe the barcode $\mathcal{B}(\mathbb{H}_{*,pt}(S^2, g_x))$. By Proposition 4.5.4 all closed geodesics of energy $\leq \frac{\delta_0^2}{2}$ are iterations of γ_0 and they are all non-degenerate. Thus, we may use Morse-Bott techniques described in Subsection 4.4.1, namely the identity (4.15), see also Remark 4.4.4.

As explained in Subsection 4.4.1, constant geodesics will produce two generators $p_0 \in CMB_{0,pt}(E_g, h)$ and $p_2 \in CMB_{2,pt}(E_g, h)$ corresponding to two critical points of a height function on S^2 . On the other hand, by Lemma 4.5.3, every $\gamma_0^{\pm m}$ satisfies $\text{Ind}(\gamma_0^{\pm m}) = 0$ and hence every $\gamma_0^{\pm m}$ produces two generators $p_{\pm m}^0 \in CMB_{0,pt}(E_g, h)$ and $p_{\pm m}^1 \in CMB_{1,pt}(E_g, h)$. These two generators correspond to minimum and maximum of a height function on S^1 -critical submanifold $S^1 \cdot \gamma_0^{\pm m}$.

Furthermore $E_{g_x}(p_0) = E_{g_x}(p_2) = 0$ while

$$E_{g_x}(p_{\pm m}^0) = E_{g_x}(p_{\pm m}^1) = mE_{g_x}(\gamma_0) = m \cdot \frac{\delta_0^2}{2} e^{-2x}.$$

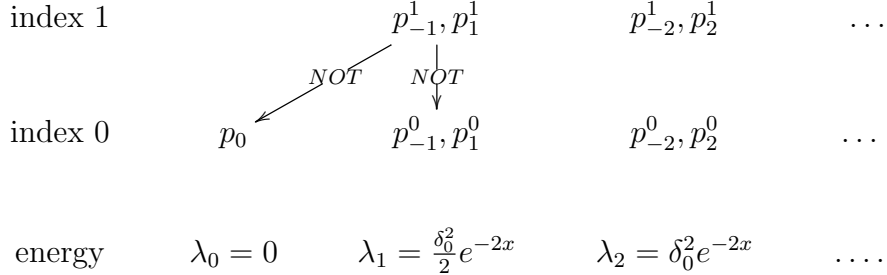
The boundary operator does not increase energy and thus we have that

$$\partial p_1^1 = n(p_1^1, p_0)p_0 + n(p_1^1, p_1^0)p_1^0 + n(p_1^1, p_{-1}^0)p_{-1}^0,$$

where $n(p_1^1, p_0)$ equals the number of flow lines with cascades connecting p_1^1 to p_0 , and same for $n(p_1^1, p_1^0)$, $n(p_1^1, p_{-1}^0)$, see Subsection 4.4.1. Since p_1^1 and p_1^0 belong to

⁸They may be thought of as ”bumpy below energy level $\frac{\delta_0^2}{2}$ ”.

the same S^1 -critical submanifold we have that $n(p_1^1, p_1^0) = 2 = 0 \pmod 2$. On the other hand, p_1^1 and p_{-1}^0 have the same energy, but belong to different S^1 -critical submanifolds, which implies that there are no flow lines with cascades connecting them, i.e. $n(p_1^1, p_{-1}^0) = 0$. Finally, as the global minimum, p_0 represents the homology class of a point which is not zero, i.e. $\partial p_1^1 \neq p_0$ and we conclude that $\partial p_1^1 = 0$. The same argument shows that $\partial p_{-1}^1 = 0$. Thus, we may schematically present boundary relations with the following diagram.



Since $\gamma_0^{\pm m}$ do not produce any critical points of index 2, (2) in Proposition 4.5.4 guarantees that $[p_{-1}^1], [p_1^1] \in H_1(\mathcal{L}_{pt}^\lambda(S^2, g_x); \mathbb{Z}_2)$ are non-zero for all $\lambda \leq \frac{\delta_0^2}{2}$. In other words $\mathcal{B}(\mathbb{H}_{1,pt}(S^2, g_x))$ contains a bar $[E_{g_x}(\gamma_0), C_x]$ with $C_x \geq \frac{\delta_0^2}{2}$ (in fact it contains two such bars). Moreover $E_{g_x}(\gamma_0)$ is the smallest left endpoint in $\mathcal{B}(\mathbb{H}_{1,pt}(S^2, g_x))$.

Recall that $\mathbb{B}_{1,pt}(U_{g_x}^* S^2)$ denotes the barcode of a symplectic persistence module with logarithmic parametrization in degree one and homotopy class of a point. Theorem 4.4.6 implies that

$$\left[\ln \sqrt{2E_{g_x}(\gamma_0)}, \ln \sqrt{2C_x} \right] = \left[\ln \delta_0 - x, \ln \sqrt{2C_x} \right] \in \mathbb{B}_{1,pt}(U_{g_x}^* S^2).$$

By (2) in Proposition 4.5.4 we also have that $\ln \sqrt{2C_x} \geq \ln \delta_0 - y$ for any $y \geq 0$. Hence, for any $x, y \in [0, \infty)$ Lemma 4.5.1 gives

$$\frac{1}{2}|x - y| \leq d_{bottle}(\mathbb{B}_{1,pt}(U_{g_x}^* S^2), \mathbb{B}_{1,pt}(U_{g_y}^* S^2)),$$

which together with Theorem 4.1.7 implies $\frac{1}{2}|x - y| \leq d_{SBM}(U_{g_x}^* S^2, U_{g_y}^* S^2)$.

Now define the desired embedding $\Phi : [0, \infty) \rightarrow \bar{\mathcal{G}}_{S^2}$ by

$$\Phi(x) = R_x \cdot \tilde{\Phi}(x) = R_x \cdot g_x,$$

where R_x is the rescaling factor given by (3) in Proposition 4.5.4. From Remark 4.2.4 it follows that

$$\begin{aligned} d_{SBM}(U_{\Phi(x)}^* S^2, U_{\Phi(y)}^* S^2) &= d_{SBM}(\sqrt{R_x} U_{g_x}^* S^2, \sqrt{R_y} U_{g_y}^* S^2) \\ &= d_{SBM}(\sqrt{R_x/R_y} U_{g_x}^* S^2, U_{g_y}^* S^2) \\ &\geq d_{SBM}(U_{g_x}^* S^2, U_{g_y}^* S^2) - d_{SBM}(U_{g_x}^* S^2, \sqrt{R_x/R_y} U_{g_x}^* S^2) \\ &= d_{SBM}(U_{g_x}^* S^2, U_{g_y}^* S^2) - \frac{1}{2} |\ln R_x - \ln R_y| \\ &\geq \frac{1}{2}|x - y| - \frac{1}{2} |\ln R_x - \ln R_y|. \end{aligned}$$

For any $\varepsilon > 0$, take $\varepsilon_0 = \frac{e^{2\varepsilon}-1}{e^{2\varepsilon}+1}$ in Proposition 4.5.4. Then the range of R_x given by (3) in Proposition 4.5.4 implies $|\ln R_x - \ln R_y| \in [0, \varepsilon]$. Thus, we get the desired lower bound. \square

4.5.3 Proof of Proposition 4.1.17

Let us give the definition of a multi-bulked surface first. Let Σ be a closed, orientable surface of genus at least 1.

We call a subset of \mathbb{R}^3 a *cylindrical segment* if it can be obtained as an open surface of revolution with a constant profile function $r : (L_-, L_+) \rightarrow \mathbb{R}$ on some interval (L_-, L_+) .

For $N \geq 1$ an *open chain of $N - 1$ spheres*, denoted by $O(N)$, is an open surface of revolution with a smooth profile function $r : (L_-, L_+) \rightarrow \mathbb{R}$ which satisfies the following properties:

- $r(\ell)$ has N local minima a_1, \dots, a_N and $N - 1$ local maxima b_1, \dots, b_{N-1} .
- $r''(a_i) > 0$ for all $i = 1, \dots, N$.

Profile function of an open chain of $N - 1$ spheres is illustrated in Figure 4.9.

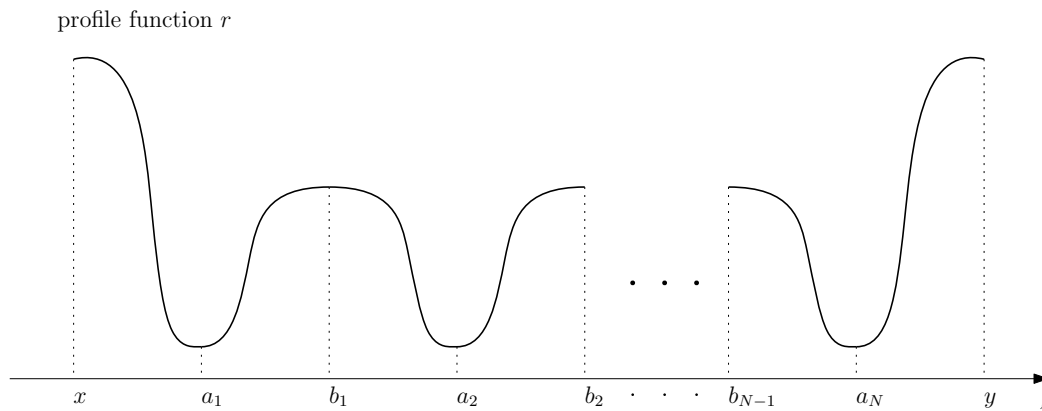


Figure 4.9. Profile function of an open chain of $N - 1$ spheres

Definition 4.5.6. Fix an embedding $\phi : \Sigma \rightarrow \mathbb{R}^3$ such that $\text{im } \phi$ contains a cylindrical segment. A *multi-bulked surface* $S \subset \mathbb{R}^3$ is obtained by cutting out the cylindrical segment from $\text{im } \phi$ and inserting $O(N)$. A general picture of a multi-bulked surface is shown in Figure 4.10. A *multi-bulked metric* g is a metric on Σ induced by the standard metric on \mathbb{R}^3 . If we want to emphasize the role of N , we will also use terms *an N -bulk surface* and *an N -bulk metric*.

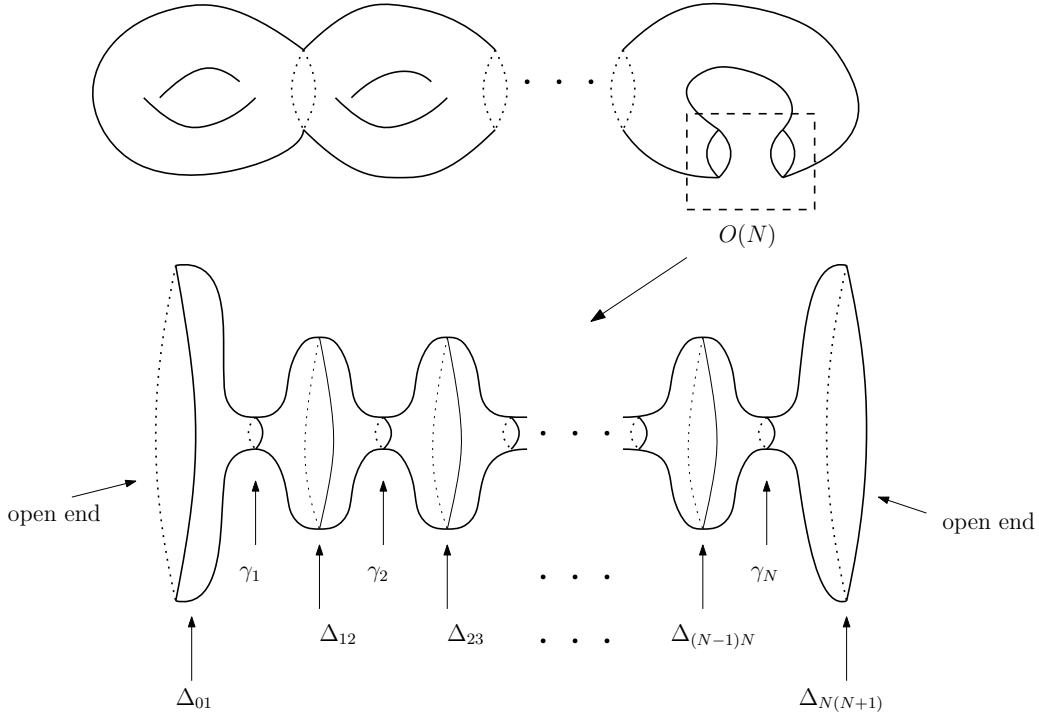


Figure 4.10. A general picture of a multi-bulked surface

Denote the short simple closed geodesics coming from the “narrow necks” in $O(N)$ by $\gamma_1, \dots, \gamma_N$ from left to right as in Figure 4.10. All of γ_i belong to the same free homotopy class, which we denote by α . Denote the long simple closed geodesics from the “spherical parts” in $O(N)$ by $\Delta_{12}, \dots, \Delta_{(N-1)N}$ from left to right and the boundary curves of $O(N)$ by Δ_{01} and $\Delta_{N(N+1)}$ as in Figure 4.10. Moreover, we put the following requirements.

- Lengths of $\Delta_{i(i+1)}$ satisfy $L_g(\Delta_{12}) = \dots = L_g(\Delta_{(N-1)N})$.
- Energies of γ_i satisfy $E_g(\gamma_1) \leq \dots \leq E_g(\gamma_N)$.

For any $N \in \mathbb{N}$, let

$$\mathcal{T}(N) = \{ \vec{x} = (x_1, \dots, x_N) \in [0, \infty)^N \mid x_1 \leq x_2 \leq \dots \leq x_N \}.$$

Similarly to Lemma 4.5.3, we have the following result.

Lemma 4.5.7. *Each closed geodesic γ_i is non-degenerate and $\text{Ind}(\gamma_i) = 0$ for $i \in \{1, \dots, N\}$.*

Lemma 4.5.7 is proven in Subsection 4.6.1. Similarly to Proposition 4.5.4, we have the following result.

Proposition 4.5.8. *Let Σ be a closed, orientable surface of genus at least 1. For any $N \in \mathbb{N}$ and $0 < \varepsilon_0 < 1$, there exists a positive $\delta_0 \ll 1$ such that for any $\vec{x} = (x_1, \dots, x_N) \in \mathcal{T}(N)$, there exists an N -bulked metric $g_{\vec{x}} \in \mathcal{G}_\Sigma$ satisfying the following properties.*

- (1) *Each closed geodesic γ_i has energy $E_{g_{\vec{x}}}(\gamma_i) = \frac{\delta_0^2}{2} e^{-2x_i}$ for $i \in \{1, \dots, N\}$.*
- (2) *Any closed geodesic γ on $(\Sigma, g_{\vec{x}})$ different from $\gamma_1, \dots, \gamma_N$ and their iterates has energy $E_{g_{\vec{x}}}(\gamma) > \frac{\delta_0^2}{2}$.*
- (3) *Every cylinder connecting γ_i and γ_j for $i \neq j$ must pass through a loop with energy greater than $\frac{\delta_0^2}{2}$.*
- (4) *There exists some constant $R_{\vec{x}} \in \left[\sqrt{\frac{1}{1+\varepsilon_0}}, \sqrt{\frac{1}{1-\varepsilon_0}} \right]$ such that $R_{\vec{x}} \cdot g_{\vec{x}} \in \bar{\mathcal{G}}_\Sigma$.*

Properties (1), (2) and (3) in Proposition 4.5.8 can be confirmed by the same argument as (1), (2) and (4) in Proposition 4.5.4. Property (3) in Proposition 4.5.8 essentially comes from the fact that curves $\Delta_{i(i+1)}$ are very long compared to γ_j .

The quasi-isometric embedding of $(\mathbb{R}^N, |\cdot|_\infty)$ into $\bar{\mathcal{G}}_\Sigma$ which we construct to prove Proposition 4.1.17 will be realized as a composition of two quasi-isometric embeddings according to the following scheme

$$(\mathbb{R}^N, |\cdot|_\infty) \xrightarrow{Q} (\mathcal{T}(2N), |\cdot|_\infty) \xrightarrow{\Psi} \bar{\mathcal{G}}_\Sigma.$$

To this end, in Subsection 4.8.2 we prove the following lemma.

Lemma 4.5.9. *Fix $N \in \mathbb{N}$. There exists a map $Q : (\mathbb{R}^N, |\cdot|_\infty) \rightarrow (\mathcal{T}(2N), |\cdot|_\infty)$ such that for any $\vec{x}, \vec{y} \in (\mathbb{R}^N, |\cdot|_\infty)$,*

$$\frac{1}{4} |\vec{x} - \vec{y}|_\infty \leq |Q(\vec{x}) - Q(\vec{y})|_\infty \leq (2N) \cdot |\vec{x} - \vec{y}|_\infty.$$

Remark 4.5.10. Similarly to metrics g_x in Proposition 4.5.4, metrics $g_{\vec{x}}$ in Proposition 4.5.8 may not be bumpy. As explained in Remark 4.5.5, they can be perturbed by a d_{SBM} -small perturbation to a bumpy metric which still satisfies all the properties from Lemma 4.5.7 and Proposition 4.5.8 (up to a small difference in logarithms of energies). Again, we ignore this point in the proof that follows, for the sake of clarity.

We are now in a position to give a proof of the lower bound in Proposition 4.1.17.

Proof. (Proof of Proposition 4.1.17 (lower bound)) Define a map $\tilde{\Psi} : \mathcal{T}(2N) \rightarrow \mathcal{G}_\Sigma$ as $\tilde{\Psi}(\vec{x}) = g_{\vec{x}}$, where $\vec{x} = (x_1, \dots, x_{2N}) \in \mathcal{T}(2N)$ and $g_{\vec{x}}$ is a $2N$ -bulked metric given by Proposition 4.5.8. The short geodesics in the $O(2N)$ part are labelled from the longest to the shortest by $\gamma_1, \dots, \gamma_{2N}$. Denote the homotopy class of these geodesics by $\alpha = [\gamma_1] = \dots = [\gamma_{2N}]$.

For this α , reversed loops γ_i^{-1} as well as iterations $\gamma_i^{\pm m}$ for $m \geq 2$ are all not in α . Constant loops are also not in α and thus (3) in Proposition 4.5.8 implies that the only closed geodesics in class α with energy less or equal to $\frac{\delta_0^2}{2}$ are γ_i , $i = 1, \dots, 2N$. Lemma 4.5.7 guarantees that all γ_i are non-degenerate and thus we may use Morse-Bott techniques introduced in Subsection 4.4.1, namely the identity (4.15), see also Remark 4.4.4.

As in the proof of Proposition 4.1.16 each γ_i , $i = 1, \dots, 2N$ produces two generators of the Morse-Bott chain complex, $p_i^0 \in CMB_{0,\alpha}(E_{g_{\vec{x}}}, h)$ and $p_i^1 \in CMB_{1,\alpha}(E_{g_{\vec{x}}}, h)$. Moreover these are the only generators of $CMB_{*,\alpha}^\lambda(E_{g_{\vec{x}}}, h)$ for $\lambda \leq \frac{\delta_0}{2}$.

In terms of the boundary operator we have that for all $i = 1, \dots, 2N$ it holds $\partial p_i^0 = 0$ as well as $n(p_i^1, p_i^0) = 0$ because p_i^1 and p_i^0 belong to the same S^1 -critical submanifold. We claim that also $n(p_i^1, p_j^0) = 0$ when $i \neq j$. Indeed, assume that there exists a flow line with cascades $(u_1, \dots, u_k, t_1, \dots, t_{k-1})$ connecting p_i^1 and p_j^0 . Since $\gamma_i \neq \gamma_j$, we must have $k \geq 1$ and one of the flow lines u_l would have to start at a critical submanifold $S^1 \cdot \gamma_{i_1}$ and end at a critical submanifold $S^1 \cdot \gamma_{i_2}$ with $i_1 \neq i_2$. However, this would mean that $\text{im } u_l \subset \Sigma$ defines a cylinder which connects γ_{i_1} and γ_{i_2} and which passes only through loops of energy no greater than $\lambda \leq \frac{\delta_0^2}{2}$. Existence of such a cylinder is ruled out by (3) in Proposition 4.5.8 and hence $n(p_i^1, p_j^0) = 0$ for all i, j . This means that for $\lambda \leq \frac{\delta_0^2}{2}$, $\partial = 0$ on $CMB_{*,\alpha}^\lambda(E_{g_{\vec{x}}}, h)$.

Using (4.15) we conclude that $\mathcal{B}(\mathbb{H}_{1,\alpha}(\Sigma, g_{\vec{x}}))$ contains bars $[E_{g_{\vec{x}}}(\gamma_i), C_i(\vec{x})]$ for $i = 1, \dots, 2N$, with $C_i(\vec{x}) \geq \frac{\delta_0^2}{2}$ (possibly $C_i(\vec{x}) = \infty$). Moreover, $E_{g_{\vec{x}}}(\gamma_i)$ are the $2N$ smallest left endpoints of bars in $\mathcal{B}(\mathbb{H}_{1,\alpha}(\Sigma, g_{\vec{x}}))$.

Recall that $\mathbb{B}_{1,\alpha}(U_{g_{\vec{x}}}^* \Sigma)$ denotes the barcode of a symplectic persistence module with logarithmic parametrization in degree one and homotopy class α . Theorem 4.4.6 implies that, for any $i \in \{1, \dots, 2N\}$,

$$\left[\ln \sqrt{2E_{g_{\vec{x}}}(\gamma_i)}, \ln \sqrt{2C_i(\vec{x})} \right) = \left[\ln \delta_0 - x_i, \ln \sqrt{2C_i(\vec{x})} \right) \in \mathbb{B}_{1,\alpha}(U_{g_{\vec{x}}}^* \Sigma).$$

Similar conclusion holds for any $\vec{y} \in \mathcal{T}(2N)$. Moreover, $\ln \sqrt{2C_i(\vec{x})} \geq \ln \delta_0 - y_j$ for any $y_j \in [0, \infty)$. Hence, Lemma 4.5.1 implies

$$\frac{1}{2} |\vec{x} - \vec{y}|_\infty \leq d_{\text{bottle}}(\mathbb{B}_{1,\alpha}(U_{g_{\vec{x}}}^* \Sigma), \mathbb{B}_{1,\alpha}(U_{g_{\vec{y}}}^* \Sigma)).$$

Theorem 4.1.7 then yields $\frac{1}{2} |\vec{x} - \vec{y}|_\infty \leq d_{SBM}(U_{g_{\vec{x}}}^* \Sigma, U_{g_{\vec{y}}}^* \Sigma)$.

Now Lemma 4.5.9 provides an embedding $\tilde{\Phi} := \tilde{\Psi} \circ Q : \mathbb{R}^N \rightarrow \mathcal{G}_\Sigma$, which satisfies

$$\frac{1}{8} |\vec{x} - \vec{y}|_\infty \leq \frac{1}{2} |Q(\vec{x}) - Q(\vec{y})|_\infty \leq d_{SBM}(U_{\tilde{\Phi}(\vec{x})}^* \Sigma, U_{\tilde{\Phi}(\vec{y})}^* \Sigma),$$

for any $\vec{x}, \vec{y} \in \mathbb{R}^N$.

Finally, $\Phi : \mathbb{R}^N \rightarrow \bar{\mathcal{G}}_\Sigma$ is defined by setting $\Phi(\vec{x}) = R_{Q(\vec{x})} \cdot \tilde{\Phi}(\vec{x}) = R_{Q(\vec{x})} \cdot g_{Q(\vec{x})}$, where $R_{Q(\vec{x})}$ is the rescaling factor given by (4) in Proposition 4.5.8, associated to

vector $Q(\vec{x}) \in \mathcal{T}(2N)$. The same argument as in the proof of Proposition 4.1.16 (lower bound) implies

$$\frac{1}{8}|\vec{x} - \vec{y}|_\infty - \frac{1}{2}|\ln R_{Q(\vec{x})} - \ln R_{Q(\vec{y})}| \leq d_{SBM}(U_{\Phi(\vec{x})}^* \Sigma, U_{\Phi(\vec{y})}^* \Sigma).$$

For any $\varepsilon > 0$, take $\varepsilon_0 = \frac{e^{2\varepsilon} - 1}{e^{2\varepsilon} + 1}$ in Proposition 4.5.8. Then (4) in Proposition 4.5.8 implies the desired lower bound. \square

4.6 Bulkled sphere and multi-bulkled surface

4.6.1 Analyzing short geodesics

The goal of this subsection is to prove Lemmas 4.5.3 and 4.5.7. All considerations in this subsection are local and hence apply equally to both propositions. Let us focus on γ_0 on a bulkled sphere S .

Lemma 4.6.1. *The geodesic γ_0 on a bulkled sphere is hyperbolic.*

We start with some necessary background. Let (M, g) be an n -dimensional Riemannian manifold. Recall that a vector field J along the geodesic path $\gamma : [0, 1] \rightarrow M$ is called *Jacobi field* if it satisfies the Jacobi equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J + R(J, \dot{\gamma})\dot{\gamma} = 0, \quad (4.23)$$

where $R(\cdot, \cdot)$ stands for the curvature tensor associated to g . Jacobi fields are tangent to the space of geodesic paths with free endpoints. When γ is a closed geodesic, they can be used to calculate index and nullity of γ . To this end, first notice that Jacobi field is uniquely determined by two initial conditions $J(0)$ and $\nabla_{\dot{\gamma}} J(0)$. Moreover, we may choose these two vectors freely, which means that the space of Jacobi fields is $2n$ -dimensional. The two initial conditions $J_0(0) = \dot{\gamma}(0)$, $\nabla_{\dot{\gamma}} J_0(0) = 0$ and $\bar{J}_0(0) = 0$, $\nabla_{\dot{\gamma}} \bar{J}_0(0) = \dot{\gamma}(0)$ yield Jacobi fields $J_0(t) = \dot{\gamma}(t)$ and $\bar{J}_0(t) = t\dot{\gamma}(t)$ which are tangent to γ . Let

$$E(t) = (T\gamma(t))^\perp \oplus (T\gamma(t))^\perp \subset T_{\gamma(t)}M \oplus T_{\gamma(t)}M$$

be the $(2n - 2)$ -dimensional vector bundle along γ , where $(T\gamma(t))^\perp$ denotes the orthogonal space to $\dot{\gamma}(t)$ inside $T_{\gamma(t)}M$. It is easy to check that if $J(0) \perp \dot{\gamma}(0)$ and $\nabla_{\dot{\gamma}(0)} J(0) \perp \dot{\gamma}(0)$ then $J(t) \perp \dot{\gamma}(t)$ and $\nabla_{\dot{\gamma}(t)} J(t) \perp \dot{\gamma}(t)$ for all $t \in [0, 1]$. This means that we may define a family of maps

$$P(t) : E(0) \rightarrow E(t)$$

by $P(t)(v, w) = (J(t), \nabla_{\dot{\gamma}} J(t))$ where J is the Jacobi field with the initial condition $(J(0), \nabla_{\dot{\gamma}} J(0)) = (v, w)$. In particular if γ is closed, i.e. $\gamma(t + 1) = \gamma(t)$, we have that

$$P(1) : E(0) \rightarrow E(0)$$

and this map is called the *linearized Poincare map*.

Definition 4.6.2. Closed geodesic γ is called *hyperbolic* if no eigenvalue of the linearized Poincare map has norm equal to 1.

Taking advantage of the geometry of a bulked sphere, the proof of Lemma 4.6.1 comes from a direct computation of eigenvalues of the linearized Poincare map.

Proof. (Proof of Lemma 4.6.1) Suppose that our bulked sphere S comes from rotating a profile function r around the x -axis and denote the radius of the circle $\gamma_0(t)$ by $r(0) := r$. Then

$$\gamma_0(t) = (0, r \cos(2\pi t), r \sin(2\pi t)), \quad t \in [0, 1]$$

and its velocity is given by

$$\dot{\gamma}_0(t) = (0, -2\pi r \sin(2\pi t), 2\pi r \cos(2\pi t)).$$

Gaussian curvature K_G along $\gamma_0(t)$ is constant and can be expressed using the formula for the Gaussian curvature of the surface of revolution. More precisely, we have

$$K_G = -\frac{r''(0)}{r},$$

which is negative by the third property in the definition of a bulked sphere, namely $r''(0) > 0$, see Definiton 4.5.2.

In order to calculate the linearized Poincare map, we are only interested in the Jacobi fields orthogonal to $\dot{\gamma}_0(t)$. Let $J = J(t)$ be such a Jacobi field, $J(t) \perp \dot{\gamma}_0(t)$ for all $t \in [0, 1]$. Since $\dim S = 2$, $J(t)$ and $\dot{\gamma}_0(t)$ span the tangent planes $T_{\gamma_0(t)}S$. On the other hand, the curvature tensor satisfies $\langle R(J, \dot{\gamma}_0)\dot{\gamma}_0, \dot{\gamma}_0 \rangle = 0$ and hence $R(J(t), \dot{\gamma}_0(t))\dot{\gamma}_0(t)$ is proportional to $J(t)$. We calculate

$$\begin{aligned} \langle R(J(t), \dot{\gamma}_0(t))\dot{\gamma}_0(t), J(t) \rangle &= |\dot{\gamma}_0(t)|^2 |J(t)|^2 \langle R(e_2, e_1)e_1, e_2 \rangle \quad \text{with } \{e_1, e_2\} \text{ orthonormal} \\ &= |\dot{\gamma}_0(t)|^2 |J(t)|^2 K_G \\ &= (2\pi r)^2 |J(t)|^2 \cdot \frac{-r''(0)}{r} \\ &= -4\pi^2 r r''(0) |J(t)|^2 \\ &= -4\pi^2 r r''(0) \langle J(t), J(t) \rangle. \end{aligned}$$

Denoting $K = -4\pi^2 r r''(0)$, (4.23) is simplified as

$$\nabla_{\dot{\gamma}_0} \nabla_{\dot{\gamma}_0} J + K \cdot J = 0. \quad (4.24)$$

Note that K is always negative because $r''(0) > 0$.

Now, since S is a surface of revolution with axis of rotation being the x -axis, and since $r'(0) = 0$, the tangent space to S at $\gamma_0(t)$ is generated by $\dot{\gamma}_0(t)$ and $(1, 0, 0)$. This means that a Jacobi field orthogonal to $\dot{\gamma}_0$ has the form $J(t) = (J_1(t), 0, 0)$. It follows that

$$\nabla_{\dot{\gamma}_0} J(t) = (\dot{J}_1(t), 0, 0), \quad (4.25)$$

as well as

$$\nabla_{\dot{\gamma}_0} \nabla_{\dot{\gamma}_0} J(t) = (\ddot{J}_1(t), 0, 0), \quad (4.26)$$

and (4.24) becomes a second order equations

$$\ddot{J}_1(t) + K \cdot J_1(t) = 0. \quad (4.27)$$

Two solutions of this equation are vector fields $J_+(t) = (e^{\sqrt{-K}t}, 0, 0)$ and $J_-(t) = (e^{-\sqrt{-K}t}, 0, 0)$. Moreover, initial vectors

$$(J_+(0), (\nabla_{\dot{\gamma}_0} J_+)(0)) = ((1, 0, 0), (\sqrt{-K}, 0, 0))$$

and

$$(J_-(0), (\nabla_{\dot{\gamma}_0} J_-)(0)) = ((1, 0, 0), (-\sqrt{-K}, 0, 0))$$

are linearly independent and hence generate $E(0) = (T\gamma_0(0))^\perp \oplus (T\gamma_0(0))^\perp$. In order to compute the eigenvalues of the linearized Poincare map $P : E(0) \rightarrow E(1)$ it is enough to notice that from (4.25) we have

$$(J_+(1), (\nabla_{\dot{\gamma}_0} J_+)(1)) = ((e^{\sqrt{-K}}, 0, 0), (\sqrt{-K}e^{\sqrt{-K}}, 0, 0))$$

as well as

$$(J_-(1), (\nabla_{\dot{\gamma}_0} J_-)(1)) = ((e^{-\sqrt{-K}}, 0, 0), (-\sqrt{-K}e^{-\sqrt{-K}}, 0, 0)).$$

Thus $((1, 0, 0), (\sqrt{-K}, 0, 0))$ and $((1, 0, 0), (-\sqrt{-K}, 0, 0))$ are eigenvectors of P with eigenvalues $\lambda_1 = e^{\sqrt{-K}}$ and $\lambda_2 = e^{-\sqrt{-K}}$. Since $K \neq 0$, neither one of these has norm one, which means that γ_0 is hyperbolic by definition. \square

Recall that a closed geodesics is non-degenerate if its nullity is zero. The following lemma is a direct consequence of the second variation formula, see, for example, Corollary 2.5.6 in [85].

Lemma 4.6.3. *Nullity of a closed geodesic γ is equal to the dimension of the space of periodic Jacobi fields along γ minus one. In particular, γ is non-degenerate, that is nullity of γ is 0, if and only if there are no periodic Jacobi fields along γ which are orthogonal to $\dot{\gamma}$.*

Note that ‘‘minus one’’ in Lemma 4.6.3 comes from the need to exclude the tangent Jacobi field $J_0(t) = \dot{\gamma}(t)$.

When a closed geodesic is hyperbolic, its index as well as the indices of all its iterations are particularly easy to compute. Let us recall some related formulas. When γ is hyperbolic, we have a splitting

$$E = E_s \oplus E_u$$

such that $P(t)|_{E_s}$ is contracting and $P(t)|_{E_u}$ is expanding as t goes from 0 to 1. Now, for each $t_* \in [0, 1]$, define a number $\iota(t_*)$ to be the dimension of the subspace of Jacobi fields $J(t)$ along γ such that $(J(t), (\nabla_{\dot{\gamma}} J(t))) \in E_s(t)$, for all $t \in [0, 1]$ and $J(t_*) = 0$. The number of points t_* for which $\iota(t_*) > 0$ is finite and the following holds.

Lemma 4.6.4 (Proposition 5, page 4 of [83]). *If γ is hyperbolic, then*

$$\text{Ind}(\gamma) = \sum_{t_* \in [0,1)} \iota(t_*).$$

One may regard this lemma as an analogue of the well-known Morse index theorem for a geodesic segment. A general result about the index of a closed geodesic (not necessarily hyperbolic) is worked out in [82]. Finally,

Lemma 4.6.5 (Corollary 3.2.15 in [85]). *For a hyperbolic γ it holds:*

$$\text{Ind}(\gamma^m) = m \cdot \text{Ind}(\gamma).$$

We are now ready to give the desired proofs.

Proof. (Proof of Lemmas 4.5.3 and 4.5.7) From the computations in the proof of Lemma 4.6.1, we know that the space of Jacobi fields orthogonal to $\dot{\gamma}_0$ is generated by the fields $J_+(t) = (e^{\sqrt{-K}t}, 0, 0)$ and $J_-(t) = (e^{-\sqrt{-K}t}, 0, 0)$. Since $e^{\sqrt{-K}t} \rightarrow +\infty$ and $e^{-\sqrt{-K}t} \rightarrow 0$ when $t \rightarrow +\infty$, no linear combination of J_+ and J_- can be periodic. Thus, by Lemma 4.6.3, we know that γ_0^m are non-degenerate for all $m \in \mathbb{N}$.

On the other hand, by (4.25) we have that for $t \in [0, 1]$

$$(J_+(t), (\nabla_{\dot{\gamma}_0} J_+)(t)) = ((e^{\sqrt{-K}t}, 0, 0), (\sqrt{-K}e^{\sqrt{-K}t}, 0, 0))$$

as well as

$$(J_-(t), (\nabla_{\dot{\gamma}_0} J_-)(t)) = ((e^{-\sqrt{-K}t}, 0, 0), (-\sqrt{-K}e^{-\sqrt{-K}t}, 0, 0)).$$

In other words contracting and expanding spaces in the splitting $E(t) = E_s(t) \oplus E_u(t)$ are generated by $(J_-(t), (\nabla_{\dot{\gamma}_0} J_-)(t))$ and $(J_+(t), (\nabla_{\dot{\gamma}_0} J_+)(t))$ respectively. Since for all $t \in [0, 1]$ it holds $J_-(t) \neq 0$, Lemma 4.6.4 implies that $\text{Ind}(\gamma_0) = 0$ and thus by Lemma 4.6.5 $\text{Ind}(\gamma_0^m) = m \cdot \text{Ind}(\gamma_0) = 0$. Finally, note that the direction of γ_0 played no role in this subsection, i.e. all statements apply equally to γ_0^{-m} . This completes the proof of Lemma 4.5.3. Since all the considerations are local, the proof of Lemma 4.5.7 follows in the same fashion. \square

Remark 4.6.6. One may also prove that $\text{Ind}(\gamma_0^m) = 0$ by a direct computation using Lemma 4.6.4, without relying on Lemma 4.6.5.

4.6.2 Analyzing long geodesics

In this subsection, we will prove (2) in Proposition 4.5.4 as well as (2) and (3) in Proposition 4.5.8. To this end, let us describe closed geodesics on a bulked sphere and a multi-bulked surface. We start with a bulked sphere.

Assume that our bulked sphere $S \subset \mathbb{R}^3$ is obtained by rotating a profile function r around an axis l as described in Subsection 4.5.2. For a point $p \in S$ we denote by $l(p)$ the coordinate of p on the l -axis and by $r(p)$ the value of the profile function at $l(p)$, i.e. $r(p)$ is the distance from p to the axis l .

Firstly, we notice that parallel circles given by $l = \text{const}$ are geodesics if and only if $r'(l) = 0$. This means that γ_- , γ_0 and γ_+ are the only geodesic parallel circles. In order to describe geodesics which are not parallel circles we evoke the well-known Clairaut's relation, see, for example, Proposition 4.4 in [140].

Theorem 4.6.7. *Suppose that $S(r)$ is a surface of revolution obtained by rotating a profile function r around a fixed axis. Then any geodesic on $S(r)$ satisfies the equation*

$$r \cos(\phi) = \text{constant} \tag{4.28}$$

where ϕ is the angle between the geodesic and the parallel circles. Conversely, any constant speed curve satisfying (4.28) which is not a parallel circle is a geodesic.

Using notations from Subsection 4.5.2, we call the part of the bulked sphere S where $l \in (-a, a)$ the *neck* of S and the part where $|l| \geq a$ the *spherical regions* of S . The next lemma claims that a closed geodesic different from $\gamma_0^{\pm m}$, $m \in \mathbb{N}$, can not be entirely contained in the neck.

Lemma 4.6.8. *Assume that $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow S$ is a closed geodesic different from $\gamma_0^{\pm m}$, $m \in \mathbb{N}$ such that $l(\gamma_{t_0}) \in (-a, a)$ for some $t_0 \in \mathbb{R}/\mathbb{Z}$. Then γ intersects either γ_- or γ_+ .*

Proof. Since $\gamma \neq \gamma_0^{\pm m}$, we have that γ is not a parallel circle and thus for some $T \in \mathbb{R}/\mathbb{Z}$, $\dot{\gamma}(T)$ is transverse to the parallel circle $P = \{p \in S \mid l(p) = a_0\}$ for some $0 \leq a_0 < a$. We may assume that $T = 0$ as well as that $\dot{\gamma}(0)$ points away from γ_0 and towards γ_+ . We can make this assumption because if $\dot{\gamma}(0)$ points towards γ_0 we may look at $\gamma^{-1}(t) = \gamma(-t)$ which defines the same curve as γ only with reversed direction. We may also assume that the angle $\phi(\gamma(0))$ between the parallel circle P and $\dot{\gamma}(0)$ satisfies $\phi(\gamma(0)) \in (0, \frac{\pi}{2})$, see Figure 4.11, the case $\phi(\gamma(0)) \in (\frac{\pi}{2}, \pi)$ is treated in the same manner.

Clairaut's relation implies that $r(\gamma(t)) \cos(\phi(\gamma(t))) = C_0 > 0$. For a small $\varepsilon > 0$ it holds $l(\gamma(-\varepsilon)) < a_0$, $l(\gamma(\varepsilon)) > a_0$ and since γ is closed, we have that for some $\tau > 0$, $l(\gamma(\tau)) < a_0$. This means that γ eventually exits the region $\{l > a\}$ and hence it must intersect P with a negative angle $-\phi(\gamma(0))$. Formally, there exists $\tau_1 > 0$ such that $\gamma(\tau_1) \in P$ and $\phi(\gamma(\tau_1)) = -\phi(\gamma(0)) < 0$. It follows that there exists $0 < \tau_0 < \tau_1$ such that $\phi(\tau_0) = 0$, and Clairaut's relation implies

$$C_0 = r(\gamma(0)) \cos(\phi(\gamma(0))) = r(\gamma(\tau_0)) \cos(\phi(\gamma(\tau_0))) = r(\gamma(\tau_0)).$$

Thus $r(\gamma(\tau_0)) < r(\gamma(0))$ and since r increases on the interval $[0, a]$ we have that $l(\gamma(\tau_0)) > a$, which proves the claim. \square

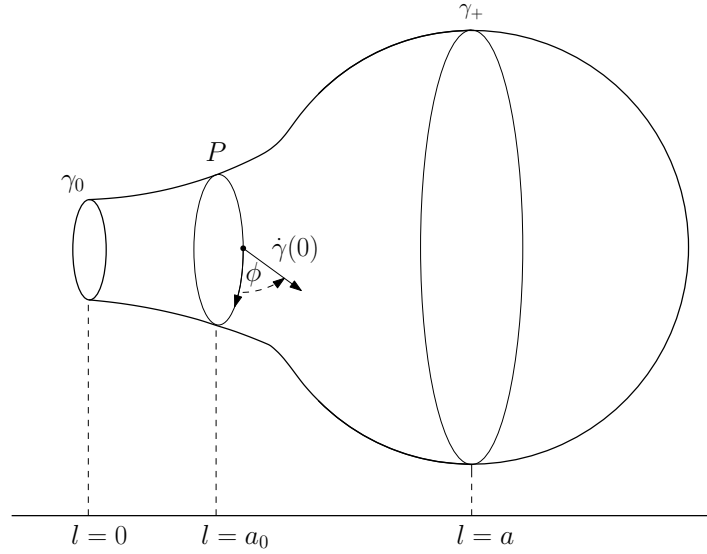


Figure 4.11. Geodesic γ intersecting parallel circle P

Now, if S is a multi-bulked surface defined in Subsection 4.5.3, we call the part of S between $\Delta_{(i-1)i}$ and $\Delta_{i(i+1)}$ *the neck of γ_i* . Using the same argument as in the proof of Lemma 4.6.8 we may prove the following.

Lemma 4.6.9. *Let γ be a closed geodesic on a multi-bulked surface S which enters the neck of γ_i . Then γ intersects either $\Delta_{(i-1)i}$ or $\Delta_{i(i+1)}$.*

Remark 4.6.10. One may also deduce Lemmas 4.6.8 and 4.6.9 from the analysis of the geodesic flow similar to the one presented in Subsection 4.8.3.

We are now in a position to give a proof of (2) in Proposition 4.5.4 as well as (2) and (3) in Proposition 4.5.8. However, before we proceed with the arguments, we wish to explain the general logic which these proofs follow.

In the case of a multi-bulked surface, firstly we fix the genus of the surface and the number of necks N (in the case of the bulked sphere these are automatically fixed).

Secondly, we fix an embedding $\phi : \Sigma \rightarrow \mathbb{R}^3$, of the surface and a cylindrical segment inside $\text{im } \phi$ which we wish to replace by an open chain of $N - 1$ spheres $O(N)$ as described in Subsection 4.5.3. After inserting $O(N)$ we obtain the multi-bulked surface $S \subset \mathbb{R}^3$.

Finally the major part of S remains fixed as we vary $g_{\vec{x}}$, for $\vec{x} \in \mathcal{T}(N)$ (or g_x , for $x \in [0, \infty)$ in the bulked sphere case). In fact, for different $\vec{x} \in \mathcal{T}(N)$, metrics $g_{\vec{x}}$ only differ in very small neighbourhoods of $\gamma_1, \dots, \gamma_N$ (or in a small neighbourhood of γ_0 in the bulked sphere case). Moreover, we have the freedom to define $g_{\vec{x}}$ in these neighbourhoods in such a way that the energies of $\gamma_1, \dots, \gamma_N$ (or the energy of γ_0) are equal to any sufficiently small numbers, see Subsection 4.8.1.

Now, proving the existence of δ_0 as in (2) in Proposition 4.5.4 and (2), (3) in Proposition 4.5.8 actually means providing δ_0 which only depends on the fixed part of S . In other words, δ_0 should not depend on the small change that we make in the neighbourhoods of short geodesics $\gamma_1, \dots, \gamma_N$ (or γ_0). Given such δ_0 , we may define $g_{\bar{x}}$ (or g_x) in the neighbourhoods of γ_i in such a way that (1) in Propositions 4.5.4 and 4.5.8 are satisfied, see Subsection 4.8.1.

Proof. (Proof of (2) in Proposition 4.5.4 and (2), (3) in Proposition 4.5.8)

We will prove properties (2) and (3) in Proposition 4.5.8. Property (2) in Proposition 4.5.4 is proven in the same way as (2) in Proposition 4.5.8. Let us start by giving a lower bound as in (3).

Assume that a cylinder $u : \mathbb{R} \times S^1 \rightarrow M$ connects γ_i and γ_j for $i < j$, that is $u(-\infty, t) = \gamma_i(t)$, $u(+\infty, t) = \gamma_j(t)$. Since γ_i and γ_j belong to different connected components of $\Sigma \setminus (\Delta_{01} \cup \Delta_{i(i+1)})$, we have that $\text{im}(u)$ must intersect either Δ_{01} or $\Delta_{i(i+1)}$. Assume first that it intersects $\Delta_{i(i+1)}$ and let $s_0 \in \mathbb{R}$ be such that curves $u_{s_0} = u(s_0, \cdot) : S^1 \rightarrow M$ and $\Delta_{i(i+1)}$ intersect. Take $p = u(s_0, t_0) \in u_{s_0} \cap \Delta_{i(i+1)}$ and let $B(p; \rho)$ be a disc of radius ρ around p , with respect to the distance induced by $g_{\bar{x}}$. If we take ρ to be smaller than the injectivity radius at p , $B(p; \rho)$ is embedded. Since curve u_{s_0} belongs to a non-trivial homotopy class α , it is not completely contained in $B(p; \rho)$, i.e. there exists t_1 such that $u(s_0, t_1) \notin B(p; \rho)$. Hence, two arcs from t_0 to t_1 on S^1 are mapped into two paths $u(\overline{t_0 t_1})$ and $u(\overline{t_1 t_0})$ which connect the center p of the disc $B(p; \rho)$ with the outside of the disc, see Figure 4.12. This implies that $L_{g_{\bar{x}}}(u_{s_0}) \geq 2\rho$.

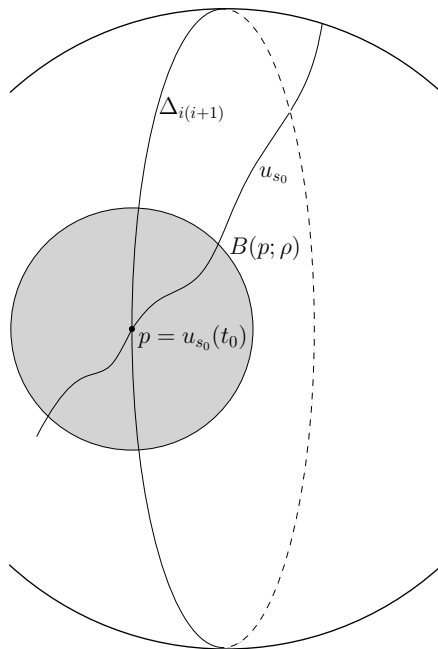


Figure 4.12. Disc centered at a point $p \in u_{s_0} \cap \Delta_{i(i+1)}$.

The exact same reasoning applies if we assume that $\text{im}(u)$ intersects Δ_{01} at some point p' , in which case we get that $L_{g_{\vec{x}}}(u_{s_0}) \geq 2\rho'$, where ρ' is smaller than the injectivity radius at p' . Note also that our metric $g_{\vec{x}}$ has S^1 -symmetry near each $\Delta_{k(k+1)}$, $0 \leq k \leq N$ because it was defined as an induced metric on a surface of revolution. This means that ρ and ρ' may be chosen to be the same for all $p \in \Delta_{i(i+1)}$ and all $p' \in \Delta_{01}$. Moreover, the neighbourhoods of $\Delta_{k(k+1)}$ for $1 \leq k \leq N-1$ are all isometric and hence ρ and ρ' can be chosen independently of i and j . By taking $\bar{\delta}_0 = \min\{\rho, \rho'\}$, we have that $L_{g_{\vec{x}}}(u_{s_0}) \geq \bar{\delta}_0$. Finally Cauchy-Schwarz inequality yields

$$E_{g_{\vec{x}}}(u_{s_0}) \geq \frac{L_{g_{\vec{x}}}(u_{s_0})^2}{2} \geq \frac{\bar{\delta}_0^2}{2},$$

which gives a lower bound as in (3).

In order to give a lower bound as in (2), first notice that by Lemma 4.6.9 every closed geodesics γ in class α , different than $\gamma_1, \dots, \gamma_N$, either intersects $\Delta_{i(i+1)}$ for some $i = 0, \dots, N$ or it is entirely contained in $S \setminus O(N)$. In the first case we get a lower bound

$$E_{g_{\vec{x}}}(\gamma) \geq \frac{\bar{\delta}_0^2}{2},$$

with the same $\bar{\delta}_0$ as above by applying the exact same argument to γ that we applied to u_{s_0} .

In the second case we have that γ is also a closed geodesic on $\text{im } \phi \subset \mathbb{R}^3$, where $\phi : \Sigma \rightarrow \mathbb{R}^3$ is the embedding we fixed in order to define a multi-bulked surface. This means that

$$E_{g_{\vec{x}}}(\gamma) \geq E_{\min},$$

with E_{\min} being the minimal energy of a closed geodesic in class α on $\text{im } \phi$. Finally, taking $\delta_0 < \min\{\bar{\delta}_0, \sqrt{2E_{\min}}\}$ finishes the proof. \square

4.6.3 Upper bounds in Proposition 4.1.16 and Proposition 4.1.17

In this subsection, we will explain how to prove the upper bounds in Propositions 4.1.16 and 4.1.17.

Recall that metrics g_x , $x \in [0, \infty)$ which we used to prove lower bound in Proposition 4.1.16 come from bulked spheres which are surfaces of revolution, see Proposition 4.5.4. On the other hand, metrics $g_{\vec{x}}$, $\vec{x} \in \mathcal{T}(N)$ which we used to prove lower bound in Proposition 4.1.17, come from multi-bulked surfaces, which contain $O(N)$ -part which is a surface of revolution, see Proposition 4.5.8. Moreover, for different $\vec{x} \in \mathcal{T}(N)$, metric $g_{\vec{x}}$ only differ in the $O(N)$ -part. Thus in order to compare different (multi)-bulk metrics, we will first explain how to compare metrics which come from surfaces of revolution in general.

Let $I \subset \mathbb{R}$ be an interval and $r : I \rightarrow [0, \infty)$ a smooth function. Using r as a profile function, we define a surface of revolution $S(r) \subset \mathbb{R}^3$ and, by pulling back the standard metric from \mathbb{R}^3 , we define a metric g^r on $I \times S^1$. We claim the following.

Lemma 4.6.11. *Let $r_1, r_2 : I \rightarrow [0, \infty)$ be two profile functions and fix $C > 0$. Then $g^{r_1} \preceq Cg^{r_2}$ if and only if*

$$(r_1(l))^2 \leq C(r_2(l))^2, \quad 1 + (r_1'(l))^2 \leq C(1 + (r_2'(l))^2)$$

for all $l \in I$.

Proof. Introduce local coordinates $(l, \theta) \in I \times S^1$. If $r : I \rightarrow [0, \infty)$ is a smooth profile function, a simple computation shows that the matrix of g^r expressed in coordinates $(\partial_l, \partial_\theta)$ satisfies

$$[g^r]_{(l, \theta)} = \begin{pmatrix} 1 + (r'(l))^2 & 0 \\ 0 & (r(l))^2 \end{pmatrix}.$$

By definition $g^{r_1} \preceq Cg^{r_2}$ is equivalent to $\|\cdot\|_{g^{r_1}} \leq \|\cdot\|_{Cg^{r_2}}$ at all points $(l, \theta) \in I \times S^1$, and hence the claim follows. \square

From Lemma 4.6.11 we obtain the following corollary.

Corollary 4.6.12. *Let $r_1, r_2 : I \rightarrow [0, \infty)$ be two profile functions and denote by⁹*

$$C = \max_{l \in I} \max \left\{ \frac{r_1(l)}{r_2(l)}, \frac{r_2(l)}{r_1(l)}, \frac{r_1'(l)}{r_2'(l)}, \frac{r_2'(l)}{r_1'(l)} \right\}.$$

If g^{r_1}, g^{r_2} are the induced Riemannian metrics on $I \times S^1$ it holds

$$\frac{1}{C^2}g^{r_1} \preceq g^{r_2} \preceq C^2g^{r_1}.$$

Proof. By Lemma 4.6.11 $g^{r_2} \preceq C^2g^{r_1}$ is equivalent to

$$(r_1(l))^2 \leq C^2(r_2(l))^2, \quad 1 + (r_1'(l))^2 \leq C^2(1 + (r_2'(l))^2).$$

The first inequality follows directly from the definition of C . Since $C \geq 1$, the second inequality follows from

$$1 + (r_1'(l))^2 \leq 1 + C^2(r_2'(l))^2 \leq C^2 + C^2(r_2'(l))^2.$$

Inequality $\frac{1}{C^2}g^{r_1} \preceq g^{r_2}$ is proven by the same argument. \square

Finally, we have the following proposition.

⁹Here we use the convention that $\frac{0}{0} = 1$.

Proposition 4.6.13. *Bulked sphere metrics g_x , $x \in [0, \infty)$, whose existence is guaranteed by Proposition 4.5.4, can be defined in such a way that their profile functions satisfy*

$$\max_{l \in I} \max \left\{ \frac{r_x(l)}{r_y(l)}, \frac{r_y(l)}{r_x(l)}, \frac{r'_x(l)}{r'_y(l)}, \frac{r'_y(l)}{r'_x(l)} \right\} = e^{|x-y|}$$

for all $x, y \in [0, \infty)$. Similarly, multi-bulked metrics $g_{\vec{x}}$, $\vec{x} \in \mathcal{T}(N)$ in Proposition 4.5.8 can be defined in such a way that the profile functions of the corresponding $O(N)$ -parts satisfy

$$\max_{l \in I} \max \left\{ \frac{r_{\vec{x}}(l)}{r_{\vec{y}}(l)}, \frac{r_{\vec{y}}(l)}{r_{\vec{x}}(l)}, \frac{r'_{\vec{x}}(l)}{r'_{\vec{y}}(l)}, \frac{r'_{\vec{y}}(l)}{r'_{\vec{x}}(l)} \right\} = e^{|\vec{x}-\vec{y}|_\infty}$$

for all $\vec{x}, \vec{y} \in \mathcal{T}(N)$.

In order to prove Proposition 4.6.13 one must specify precisely the profile functions which are used to define bulked spheres and multi-bulked surfaces. This is done in Subsection 4.8.1.

We are now ready to give a proof of the upper bounds.

Proof. (Upper bounds in Propositions 4.1.16 and 4.1.17) We will only prove the upper bounds in terms of d_{RBM} . The upper bounds in terms of d_{SBM} then follow from Proposition 4.2.8.

For any $x \in [0, \infty)$, let g_x be the bulked sphere metric given by Proposition 4.5.4. In order to prove the upper bound in Proposition 4.1.16, notice that Corollary 4.6.12 and Proposition 4.6.13 imply that, for $x, y \in [0, \infty)$, it holds

$$e^{-2|x-y|} g_x \preceq g_y \preceq e^{2|x-y|} g_x.$$

Taking $\phi = \mathbb{1}_M$ in the definition of d_{RBM} we get

$$d_{RBM}(g_x, g_y) \leq 2|x-y|.$$

Recall that the embedding $\Phi : [0, \infty) \rightarrow \bar{\mathcal{G}}_{S^2}$ is defined by $\Phi(x) = R_x \cdot g_x$ where R_x is the rescaling factor from (3) in Proposition 4.5.4. Now, Remark 4.2.7 implies

$$\begin{aligned} d_{RBM}(\Phi(x), \Phi(y)) &= d_{RBM}(R_x \cdot g_x, R_y \cdot g_y) \\ &= d_{RBM}((R_x/R_y) \cdot g_x, g_y) \\ &\leq d_{RBM}(g_x, (R_x/R_y) \cdot g_x) + d_{RBM}(g_x, g_y) \\ &\leq |\ln R_x - \ln R_y| + 2|x-y|. \end{aligned}$$

For any given $\varepsilon > 0$, using (3) in Proposition 4.5.4 with $\varepsilon_0 = \frac{e^{2\varepsilon} - 1}{e^{2\varepsilon} + 1}$ yields the desired upper bound in Proposition 4.1.16.

To prove the upper bound in Proposition 4.1.17, recall that $g_{\vec{x}}$ is the multi-bulked metric given by Proposition 4.5.8. Moreover, metrics $g_{\vec{x}}$ for different $\vec{x} \in \mathcal{T}(N)$ only differ in the $O(N)$ -part. Hence, Corollary 4.6.12 and Proposition 4.6.13 imply

$$e^{-2|\vec{x}-\vec{y}|_\infty} g_{\vec{x}} \preceq g_{\vec{y}} \preceq e^{2|\vec{x}-\vec{y}|_\infty} g_{\vec{x}},$$

for any $\vec{x}, \vec{y} \in \mathcal{T}(N)$. Taking $\phi = \mathbb{1}_M$ in the definition of d_{RBM} gives

$$d_{RBM}(g_{\vec{x}}, g_{\vec{y}}) \leq 2|\vec{x} - \vec{y}|_\infty. \quad (4.29)$$

Map $\Phi : \mathbb{R}^N \rightarrow \bar{\mathcal{G}}_M$ in the proof of Proposition 4.1.17 is defined by $\Phi(\vec{x}) = R_{Q(\vec{x})} \cdot g_{Q(\vec{x})}$ where $Q : \mathbb{R}^N \rightarrow \mathcal{T}(2N)$ is the quasi-isometric embedding given by Lemma 4.5.9 and $R_{Q(\vec{x})}$ is the rescaling factor from (4) in Proposition 4.5.8. The same argument as above together with Lemma 4.5.9 implies, for any $\vec{x}, \vec{y} \in \mathbb{R}^N$,

$$d_{RBM}(\Phi(\vec{x}), \Phi(\vec{y})) \leq |\ln R_{Q(\vec{x})} - \ln R_{Q(\vec{y})}| + 4N \cdot |\vec{x} - \vec{y}|_\infty.$$

For any $\varepsilon > 0$, using (4) in Proposition 4.5.8 with $\varepsilon_0 = \frac{e^{2\varepsilon}-1}{e^{2\varepsilon}+1}$ yields the desired upper bound in Proposition 4.1.17. \square

4.7 Quantitative existence of closed geodesics

The goal of this section is to prove Theorem 4.1.19 and Corollary 4.1.26. Since Corollary 4.1.26 immediately follows from Theorem 4.1.7, we give its proof first.

Proof. (Proof of Corollary 4.1.26) We will prove the claim in the case of a finite bar $[a^2/2, b^2/2)$, the case of an infinite ray is treated in the same fashion. Using the isomorphism of persistence modules provided by Theorem 4.4.6, we conclude that the barcode $\mathbb{B}_{*,\alpha}(U_{g_1}^* M)$ of $\mathbb{S}_{*,\alpha}(U_{g_1}^* M)$, contains the bar $[\ln a, \ln b)$. Theorem 4.1.7 and the assumptions give

$$d_{bottle}(\mathbb{B}_{*,\alpha}(U_{g_1}^* M), \mathbb{B}_{*,\alpha}(U_{g_2}^* M)) \leq \frac{1}{2} d_{RBM}(g_1, g_2) < \frac{1}{2}(\ln b - \ln a). \quad (4.30)$$

Denoting $D = d_{bottle}(\mathbb{B}_{*,\alpha}(U_{g_1}^* M), \mathbb{B}_{*,\alpha}(U_{g_2}^* M))$ we have that for every $0 < \varepsilon < \frac{1}{2}(\ln b - \ln a) - D$ there exists a $(D + \varepsilon)$ -matching between $\mathbb{B}_{*,\alpha}(U_{g_1}^* M)$ and $\mathbb{B}_{*,\alpha}(U_{g_2}^* M)$. Since $D + \varepsilon < \frac{1}{2}(\ln b - \ln a)$, the bar $[\ln a, \ln b) \in \mathbb{B}_{*,\alpha}(U_{g_1}^* M)$ is not erased in this matching but rather has a genuine match $[c_\varepsilon, d_\varepsilon) \in \mathbb{B}_{*,\alpha}(U_{g_2}^* M)$ such that

$$\max\{|\ln a - c_\varepsilon|, |\ln b - d_\varepsilon|\} \leq D + \varepsilon.$$

Since $\mathbb{S}_{*,\alpha}(U_{g_2}^* M)$ is pointwise finite dimensional, the fact that a bar $[c_\varepsilon, d_\varepsilon)$ as above exists for all $0 < \varepsilon < \frac{1}{2}(\ln b - \ln a) - D$ implies that there exists $[c_0, d_0) \in \mathbb{B}_{*,\alpha}(U_{g_2}^* M)$ such that

$$\max\{|\ln a - c_0|, |\ln b - d_0|\} \leq D. \quad (4.31)$$

Indeed, if this was not the case, by shrinking ε we would get infinitely many bars $[c_\varepsilon, d_\varepsilon) \in \mathbb{B}_{*,\alpha}(U_{g_2}^* M)$ which all contain the middle $\frac{\ln a + \ln b}{2}$ of the interval $[\ln a, \ln b)$. This would imply that $S_*^{\frac{\ln a + \ln b}{2}}(U_{g_2}^* M; \alpha)$ is not finite dimensional.

Finally, by Remark 4.4.5 we know that there exist closed geodesics γ_1, γ_2 of g_2 such that $c_0 = \ln \sqrt{2E_{g_2}(\gamma_1)}$, $d_0 = \ln \sqrt{2E_{g_2}(\gamma_2)}$ and the proof follows. \square

In the rest of the section, we focus on proving Theorem 4.1.19.

4.7.1 Lemmas about persistence modules

In order to prove Theorem 4.1.19, we will use a lemma about general persistence modules, see Lemma 4.7.2 below. We start with an auxiliary statement first.

Lemma 4.7.1. *Let $\mathbb{K}_{[a,b)}$ and $\mathbb{K}_{[c,d)}$ be two interval modules over field \mathbb{K} . Then a non-zero persistence module morphism*

$$\mathfrak{f} = \{f^t\}_{t \in \mathbb{R}} : \mathbb{K}_{[a,b)} \rightarrow \mathbb{K}_{[c,d)}$$

exists if and only if $c \leq a \leq d \leq b$. Similarly for $b = d = +\infty$ a non zero persistence module morphism

$$\mathfrak{f} = \{f^t\}_{t \in \mathbb{R}} : \mathbb{K}_{[a,+\infty)} \rightarrow \mathbb{K}_{[c,+\infty)}$$

exists if and only if $c \leq a$.

Proof. Firstly, note that if $c \leq a \leq d \leq b$ there exists a non-zero persistence module morphism \mathfrak{f} given by $f^t(1_{\mathbb{K}}) = 1_{\mathbb{K}}$ for $t \in [a, d)$ and $f^t = 0$ otherwise. This proves one direction of the equivalence. For the other direction one readily sees that $b > c$ and $d > a$ since otherwise $[a, b)$ and $[c, d)$ do not intersect. The rest of the proof follows from a case analysis in terms of the order of endpoints a, b, c, d . We will analyze one case, the other cases are treated in the same way. Assume, for example, that $c \leq a \leq b < d$ and let $\varepsilon > 0$ be such that $b < b + \varepsilon < d$. Now, if $f^t(1_{\mathbb{K}}) \neq 0$ for some $t \in [a, b)$, on the one hand we have

$$f^{b+\varepsilon}(\iota_{t,b+\varepsilon}(1_{\mathbb{K}})) = f^{b+\varepsilon}(0) = 0,$$

while on the other hand

$$f^{b+\varepsilon}(\iota_{t,b+\varepsilon}(1_{\mathbb{K}})) = \iota_{t,b+\varepsilon}(f^t(1_{\mathbb{K}})) = f^t(1_{\mathbb{K}}) \neq 0$$

which gives a contradiction. \square

Recall that if \mathbb{V} is a persistence module, for $A > 0$, shifted module $\mathbb{V}[A]$ is defined by $\mathbb{V}[A]^t = \mathbb{V}^{t+A}$ with structure maps $\iota_{s,t}^{\mathbb{V}[A]} = \iota_{s+A,t+A}^{\mathbb{V}}$. Barcode $\mathcal{B}(\mathbb{V}[A])$ is a shift of $\mathcal{B}(\mathbb{V})$ by A to the left, i.e. $[x, y) \in \mathcal{B}(\mathbb{V}[A])$ if and only if $[x + A, y + A) \in \mathcal{B}(\mathbb{V})$. The following lemma is the main combinatorial ingredient of the proof of Theorem 4.1.19.

Lemma 4.7.2. Let \mathbb{V}, \mathbb{W} be two persistence modules, $A, B \geq 0$ non-negative constants and

$$f : \mathbb{V} \rightarrow \mathbb{W}[A], \quad g : \mathbb{W}[A] \rightarrow \mathbb{V}[A + B]$$

persistence module morphisms such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{\iota_{t, t+A+B}^{\mathbb{V}}} & \mathbb{V}[A + B] \\ & \searrow f & \nearrow g \\ & \mathbb{W}[A] & \end{array}$$

If there exists a bar $[a, b) \in \mathcal{B}(\mathbb{V})$ such that $b - a > A + B$ then there exists a bar $[c, d) \in \mathcal{B}(\mathbb{W})$ such that

$$a - B \leq c \leq a + A, \quad b - B \leq d \leq b + A.$$

Proof. Fix a decomposition of \mathbb{V} given by the structure theorem (Theorem 2.1.8),

$$\mathbb{V} = \bigoplus_{I \in \mathcal{B}(\mathbb{V})} \mathbb{K}_I \quad (4.32)$$

and let

$$\mathbb{V}[A + B] = \bigoplus_{I \in \mathcal{B}(\mathbb{V})} \mathbb{K}_I[A + B] \quad (4.33)$$

be the induced decomposition of $\mathbb{V}[A + B]$. Since $[a, b) \in \mathcal{B}(\mathbb{V})$, $\mathbb{K}_{[a, b)}$ is a summand in (4.32) and $\mathbb{K}_{[a, b)}[A + B] = \mathbb{K}_{[a-A-B, b-A-B)}$ is a summand in (4.33). Denote by

$$\pi_{[a-A-B, b-A-B)} : \mathbb{V}[A + B] \rightarrow \mathbb{K}_{[a-A-B, b-A-B)}$$

the projection with respect to (4.33). By restricting f to $\mathbb{K}_{[a, b)}$ we obtain the following commutative diagram for all $t \in \mathbb{R}$:

$$\begin{array}{ccc} \mathbb{K}_{[a, b)}^t & \xrightarrow{\iota_{t, t+A+B}} & \mathbb{K}_{[a-A-B, b-A-B)}^t \\ & \searrow f^t & \nearrow (\pi_{[a-A-B, b-A-B)} \circ g)^t \\ & \mathbb{W}[A]^t & \end{array}$$

Condition $A + B < b - a$ implies that $[a, b) \cap [a - A - B, b - A - B) \neq \emptyset$ and hence

$$\iota_{t, t+A+B} : \mathbb{K}_{[a, b)}^t \rightarrow \mathbb{K}_{[a-A-B, b-A-B)}^t$$

is non-zero and is given by the obvious map equal to $1_{\mathbb{K}}$ when $t \in [a, b - A - B)$ and zero otherwise. Let us fix $t_0 \in (a, b - A - B)$ and $1^{t_0} \in \mathbb{K}_{[a, b)}^{t_0}$. By previous, It holds $\iota_{t_0, t_0+A+B}(1^{t_0}) \neq 0$.

Fix a decomposition of $\mathbb{W}[A]$,

$$\mathbb{W}[A] = \bigoplus_{I \in \mathcal{B}(\mathbb{W}[A])} \mathbb{K}_I \quad (4.34)$$

and assume that

$$f^{t_0}(1^{t_0}) = \sum_{i=1}^N \lambda_i 1_{I_i}^{t_0},$$

where $\lambda_i \in \mathbb{K}$, $\lambda_i \neq 0$ and $1_{I_i}^{t_0} \in \mathbb{K}_{I_i}^{t_0}$ for $I_i \in \mathcal{B}(\mathbb{W}[A])$. Since $(\pi_{[a-A-B, b-A-B]} \circ g)^{t_0} \circ f^{t_0} = \iota_{t_0, t_0+A+B}$ and $\iota_{t_0, t_0+A+B}(1^{t_0}) \neq 0$, we have that $(\pi_{[a-A-B, b-A-B]} \circ g)^{t_0}(\sum_{i=1}^N \lambda_i 1_{I_i}^{t_0}) \neq 0$ and hence there exists some $i_0 \in \{1, \dots, N\}$ such that $(\pi_{[a-A-B, b-A-B]} \circ g)^{t_0}(\lambda_{i_0} 1_{I_{i_0}}^{t_0}) \neq 0$, see Figure 4.13.

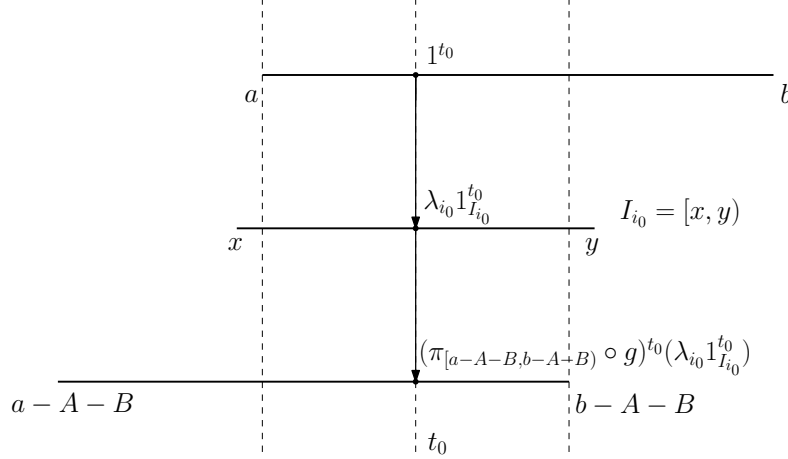


Figure 4.13. Morphisms at a point t_0 .

Let $I_{i_0} = [x, y)$. We claim that $a - A - B \leq x \leq a$ and $b - A - B \leq y \leq b$. Indeed, denote by

$$\pi_{I_{i_0}} : \mathbb{W}[A] \rightarrow \mathbb{K}_{I_{i_0}}$$

the projection with respect to (4.34). This projection is a morphism of persistence modules and we have that

$$\pi_{I_{i_0}} \circ \mathbf{f} : \mathbb{K}_{[a, b]} \rightarrow \mathbb{K}_{I_{i_0}}$$

is a non-zero persistence module morphism because $(\pi_{I_{i_0}} \circ \mathbf{f})^{t_0}(1^{t_0}) = \lambda_{i_0} 1_{I_{i_0}}^{t_0} \neq 0$. Thus, Lemma 4.7.1 implies that $x \leq a$ and $y \leq b$.

Similarly, restricting g to $\mathbb{K}_{I_{i_0}}$ gives

$$\pi_{[a-A-B, b-A-B]} \circ g : \mathbb{K}_{I_{i_0}} \rightarrow \mathbb{K}_{[a-A-B, b-A-B]}.$$

This morphism is non-zero because $(\pi_{[a-A-B, b-A-B]} \circ g)^{t_0}(\lambda_{i_0} 1_{I_{i_0}}^{t_0}) \neq 0$ and hence Lemma 4.7.1 implies that $a - A - B \leq x$ and $b - A - B \leq y$.

To finish the proof notice that $[x, y) \in \mathcal{B}(\mathbb{W}[A])$ and hence $[x + A, y + A) \in \mathcal{B}(\mathbb{W})$. For this bar it holds $a - B \leq x + A \leq a + A$ and $b - B \leq y + A \leq b + A$ and we may take $[c, d) = [x + A, y + A)$. \square

4.7.2 Proof of Theorem 4.1.19

Proof. (Proof of Theorem 4.1.19) We will prove only the case of a finite bar $[x, y] \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1))$, the other case is proved in the same manner.

It follows from the definition that $U_{Cg}^* M = \sqrt{C}U_g^* M$. From the assumption $\frac{1}{C_1}g_1 \preceq g_2 \preceq C_2g_1$, one concludes that

$$U_{\frac{1}{C_1}g_1}^* M \subset U_{g_2}^* M \subset U_{C_2g_1}^* M,$$

which is equivalent to

$$\frac{1}{\sqrt{C_1}}U_{g_1}^* M \subset U_{g_2}^* M \subset \sqrt{C_2}U_{g_1}^* M. \quad (4.35)$$

Applying contravariant functor $\mathrm{SH}_*^a(\cdot; \alpha)$ to (4.35) gives the following commutative diagram

$$\begin{array}{ccc} \mathrm{SH}_*^a(\sqrt{C_2}U_{g_1}^* M; \alpha) & \xrightarrow{\mathfrak{h}_{inc}} & \mathrm{SH}_*^a\left(\frac{1}{\sqrt{C_1}}U_{g_1}^* M; \alpha\right) \\ & \searrow \mathfrak{h}_{inc} & \nearrow \mathfrak{h}_{inc} \\ & \mathrm{SH}_*^a(U_{g_2}^* M; \alpha) & \end{array}$$

where \mathfrak{h}_{inc} denote maps induced by the respective inclusions. Applying (2) in Proposition 4.3.4 with $C = \frac{1}{\sqrt{C_1C_2}} \leq 1$ to the horizontal arrow gives us

$$\begin{array}{ccc} \mathrm{SH}_*^a(\sqrt{C_2}U_{g_1}^* M; \alpha) & \xrightarrow{\iota_{a, \sqrt{C_1C_2}a}} & \mathrm{SH}_*^{\sqrt{C_1C_2}a}(\sqrt{C_2}U_{g_1}^* M; \alpha) \\ & \searrow \mathfrak{h}_{inc} & \nearrow \left(r \frac{1}{\sqrt{C_1C_2}}\right)^{-1} \circ \mathfrak{h}_{inc} \\ & \mathrm{SH}_*^a(U_{g_2}^* M; \alpha) & \end{array}$$

where $\iota_{a, \sqrt{C_1C_2}a}$ denotes the persistence structure map of the symplectic persistence module $\mathbb{S}\mathbb{H}_{*,\alpha}(\sqrt{C_2}U_{g_1}^* M)$ and $r \frac{1}{\sqrt{C_1C_2}}$ is the isomorphism given by (2) in Proposition 4.3.4. In terms of the logarithmic version of symplectic persistence modules, setting $t = \ln a$ gives us

$$\begin{array}{ccc} S_*^t(\sqrt{C_2}U_{g_1}^* M; \alpha) & \xrightarrow{\iota_{t, t + \ln \sqrt{C_1} + \ln \sqrt{C_2}}} & S_*^{t + \ln \sqrt{C_1} + \ln \sqrt{C_2}}(\sqrt{C_2}U_{g_1}^* M; \alpha) \\ & \searrow \mathfrak{h}_{inc} & \nearrow \left(r \frac{1}{\sqrt{C_1C_2}}\right)^{-1} \circ \mathfrak{h}_{inc} \\ & S_*^t(U_{g_2}^* M; \alpha) & \end{array}$$

Since $[x, y] \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_1))$, $[\ln \sqrt{2x}, \ln \sqrt{2y}] \in \mathbb{B}_{*,\alpha}(U_{g_1}^* M)$ by Theorem 4.4.6, where $\mathbb{B}_{*,\alpha}(U_{g_1}^* M)$ denotes the barcode of persistence module $\mathbb{S}_{*,\alpha}(U_{g_1}^* M)$. Now, Proposition 4.3.4 implies that $[\ln \sqrt{2x} + \ln \sqrt{C_2}, \ln \sqrt{2y} + \ln \sqrt{C_2}] \in \mathbb{B}_{*,\alpha}(\sqrt{C_2}U_{g_1}^* M)$. The length

of this bar is $\ln \sqrt{\frac{y}{x}}$ and since $\ln \sqrt{\frac{y}{x}} > \ln \sqrt{C_1} + \ln \sqrt{C_2}$ by the assumption, we can apply Lemma 4.7.2 with $A = 0$ and $B = \ln \sqrt{C_1} + \ln \sqrt{C_2}$.

It follows that there exists a bar $(c, d] \in \mathbb{B}_{*,\alpha}(U_{g_2}^* M)$ such that

$$\ln \sqrt{2x} - \ln \sqrt{C_1} \leq c \leq \ln \sqrt{2x} + \ln \sqrt{C_2},$$

and

$$\ln \sqrt{2y} - \ln \sqrt{C_1} \leq d \leq \ln \sqrt{2y} + \ln \sqrt{C_2}.$$

By Theorem 4.4.6, bar $[\frac{1}{2}e^{2c}, \frac{1}{2}e^{2d}) \in \mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_2))$ and its endpoints satisfy

$$\frac{x}{C_1} \leq \frac{1}{2}e^{2c} \leq C_2x, \quad \frac{y}{C_1} \leq \frac{1}{2}e^{2d} \leq C_2y.$$

Endpoints of bars in $\mathcal{B}(\mathbb{H}_{*,\alpha}(M, g_2))$ correspond to the energies of closed geodesics and thus there exist closed geodesics γ_1, γ_2 of g_2 such that

$$E_{g_2}(\gamma_1) = \frac{1}{2}e^{2c}, \quad E_{g_2}(\gamma_2) = \frac{1}{2}e^{2d}.$$

This finishes the proof. □

4.8 Appendix

4.8.1 Precise parameterizations

In this subsection, we will give precise parameterizations of bulked spheres and multi-bulked surfaces announced in Sections 4.5 and 4.6. The metrics which we are going to define will satisfy all the properties that we used in these sections, namely we will prove (1) and (3) in Proposition 4.5.4, (1) and (4) in Proposition 4.5.8 as well as Proposition 4.6.13.

Parameterizations of bulked spheres

Let S be a union of two spheres with radius $A = \sqrt{\frac{1}{8\pi}}$ touching at point. The area of S is equal to 1 and S can be obtained as a (singular) surface of revolution. The graph of the profile function r which defines S is the union of two semicircles of radius A centered at $-A$ and A , see Figure 4.14.

Let $n \in \mathbb{N}$ and $B = 10^{-n}A$. We consider n to be a free parameter which will eventually be chosen large. The profile functions $r_x, x \in [0, \infty)$, which define bulked sphere metrics g_x , will all be even on $[-2A, 2A]$ and they will coincide with r on $[-2A, -B] \cup [B, 2A]$. On $[-B, B]$, each r_x will interpolate between two semicircles and will have a local minimum at 0. Let

$$\delta_0 = \sqrt{\frac{\pi}{8}} \cdot \frac{1}{2 \cdot 10^n + 1} \cdot \frac{3 - 10^{-2n}}{\sqrt{2 \cdot 10^n - 1}}. \tag{4.36}$$

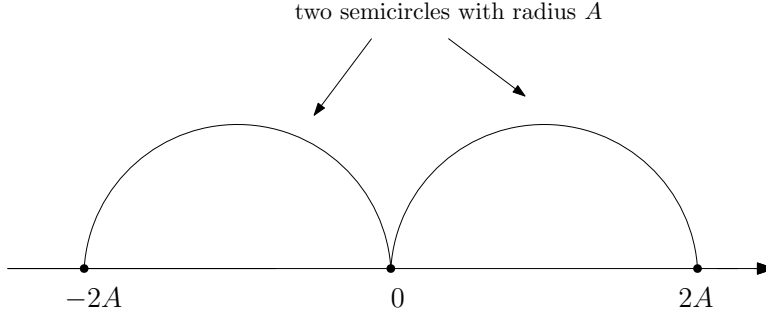


Figure 4.14. Profile function r

Since $\delta_0 = O(10^{-\frac{3n}{2}})$, by picking large enough n , we can make δ_0 arbitrarily small. The following proposition holds.

Proposition 4.8.1. *Given any sufficiently small $\varepsilon > 0$, for all sufficiently large n and A, B, δ_0 as above, there exists a family of profile functions r_x , $x \in [0, \infty)$, each of which defines a bulked sphere metric g_x such that*

- (1) $|\text{Vol}_{g_x}(S^2) - 1| \leq \varepsilon$ and $\text{diam}(S^2, g_x) \leq 100\sqrt{1 - \varepsilon}$.
- (2) r_x coincide outside $[-B, B]$ for all $x \in [0, \infty)$.
- (3) $r_x(0) = \frac{\delta_0}{2\pi}e^{-x}$.
- (4) For any $x, y \in [0, \infty)$,

$$\max_{l \in [-2A, 2A]} \max \left\{ \frac{r_x(l)}{r_y(l)}, \frac{r_y(l)}{r_x(l)}, \frac{r'_x(l)}{r'_y(l)}, \frac{r'_y(l)}{r'_x(l)} \right\} = e^{|x-y|}.$$

In (4), as before, we use convention that $\frac{0}{0} = 1$.

Proof. We ask for r_x to be even on $[-2A, 2A]$, and hence only give their definitions on $[0, 2A]$. Let $B' = 10^{-2n}A$, $h = \frac{3\delta_0}{2\pi}$, $h_x = \frac{\delta_0}{2\pi}e^{-x}$, and define

$$r_x(l) = \frac{h - h_x}{(B')^2}l^2 + h_x, \quad \text{for } l \in [0, B'].$$

It immediately follows that $r_x(0) = h_x = \frac{\delta_0}{2\pi}e^{-x}$ and hence (3) is satisfied. On the other hand, a simple computation shows that for all $x, y \in [0, \infty)$

$$\max_{l \in [0, B']} \max \left\{ \frac{r_x(l)}{r_y(l)}, \frac{r_y(l)}{r_x(l)}, \frac{r'_x(l)}{r'_y(l)}, \frac{r'_y(l)}{r'_x(l)} \right\} = e^{|x-y|}. \quad (4.37)$$

Notice also that $r_x(B') = h$ for all $x \in [0, \infty)$, i.e., the graphs of all r_x meet at the point (B', h) , see Figure 4.15 (a).

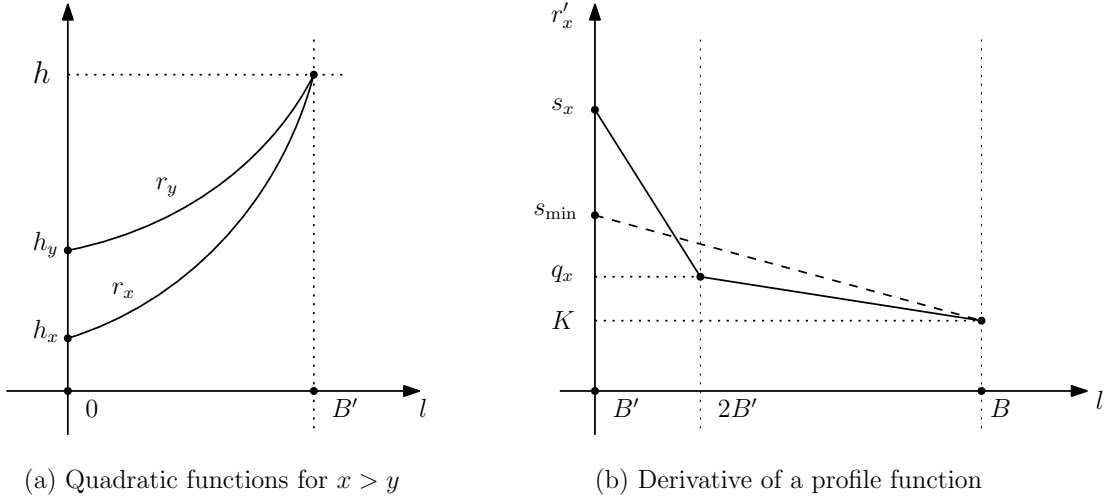


Figure 4.15. Parameterization of r_x in the region $[0, B]$

Graphs of profile functions r_x will connect (B', h) to $(B, \sqrt{A^2 - (A - B)^2})$ on a semicircle. To this end, let

$$s_x := r'_x(B') = \frac{2(h - h_x)}{B'} \quad \text{and} \quad K = \frac{A - B}{\sqrt{2AB - B^2}}.$$

For $x \in [0, \infty)$ we have $s_x \in [\frac{2\delta_0}{\pi B'}, \frac{3\delta_0}{\pi B'})$ and we denote $s_{\min} = s_0 = \frac{2\delta_0}{\pi B'}$. On the other hand K is the derivative at B of the function $y = \sqrt{A^2 - (A - l)^2}$, which defines a semicircle.

We now define r_x by giving its derivative on $[B', B]$. Let

$$q_x = \frac{s_{\min}(B - B') + KB' - s_x B'}{B - B'}.$$

On $[B', 2B']$, the derivative r'_x is by definition equal to a linear function whose graph connects (B', s_x) and $(2B', q_x)$. On $[2B', B]$, r'_x is equal to another linear function, whose graph connects $(2B', q_x)$ and (B, K) . It is easy to check that r'_0 is linear on $[B', B]$, i.e. (B', s_{\min}) , $(2B', q_0)$ and (B, K) are on the same line, as well as that $K < q_x \leq q_0$ for all $x \in [0, \infty)$, see Figure 4.15 (b). Explicitly r'_x is given by

$$r'_x(l) = \begin{cases} \frac{q_x - s_x}{B'}(l - B') + s_x & l \in [B', 2B'] \\ \frac{K - q_x}{B - 2B'}(l - B) + K & l \in [2B', B] \end{cases}. \quad (4.38)$$

Straightforward calculation shows that

$$r_x(B) = h + \int_{B'}^B r'_x(l) dl = \sqrt{A^2 - (A - B)^2},$$

and thus by setting $r_x(l) = \sqrt{A^2 - (A - l)^2}$ for $l \in [B, 2A]$, we obtain a C^1 -smooth function $r_x : [0, 2A] \rightarrow [0, \infty)$. Another straightforward calculation shows that for

$x, y \in [0, \infty)$

$$\max_{l \in [B', 2A]} \max \left\{ \frac{r_x(l)}{r_y(l)}, \frac{r_y(l)}{r_x(l)}, \frac{r'_x(l)}{r'_y(l)}, \frac{r'_y(l)}{r'_x(l)} \right\} \leq e^{|x-y|}. \quad (4.39)$$

Moreover, by making a C^1 -small perturbation near the points $B', 2B'$ and B , we can make sure that r_x are all smooth while (4.39) remains valid. Finally, we extend r_x to $[-2A, 2A]$ by setting $r_x(l) = r_x(-l)$. It is clear from the construction that property (2) holds.

Combining (4.37) and (4.39) proves property (4). By taking large enough n we can guarantee that property (1) holds, which finishes the proof. \square

We can now give a proof of (1) and (3) in Proposition 4.5.4.

Proof. (Proof of (1) and (3) in Proposition 4.5.4) Denote by g_x the metric induced from profile function r_x given by Proposition 4.8.1. By (2) in Proposition 4.8.1 we get that $L_{g_x}(\gamma_0) = 2\pi r_x(0) = \delta_0 e^{-x}$. Since γ_0 has constant speed

$$E_{g_x}(\gamma_0) = \frac{L_{g_x}^2(\gamma_0)}{2} = \frac{\delta_0^2}{2} e^{-2x},$$

which proves (1) in Proposition 4.5.4.

To prove (3) in Proposition 4.5.4, let $R_x = \frac{1}{\sqrt{\text{Vol}_{g_x} S^2}}$. Now $\text{Vol}_{R_x \cdot g_x} S^2 = 1$ and from (1) in Proposition 4.8.1 it follows that

$$\text{diam}(S^2, R_x \cdot g_x) = R_x \cdot \text{diam}(S^2, g_x) \leq \sqrt{\frac{1}{1-\varepsilon}} \cdot (100\sqrt{1-\varepsilon}) \leq 100,$$

as well as that

$$R_x^2(1-\varepsilon) \leq 1 \leq R_x^2(1+\varepsilon).$$

Thus $R_x \cdot g_x \in \bar{\mathcal{G}}_{S^2}$, and taking small enough ε finishes the proof. \square

Parameterizations of multi-bulked surfaces

Recall that a cylindrical segment is a surface of revolution with constant profile function $r : I \rightarrow [0, \infty)$ on an open interval I . Let Σ be a closed, orientable surface of genus at least one and fix $N \in \mathbb{N}$. Denote by g_{std} the standard Riemannian metric on \mathbb{R}^3 and let $0 < \tau \ll 1$ be a small number. We fix an embedding $\phi : \Sigma \rightarrow \mathbb{R}^3$ such that $\text{Vol}_{\phi^* g_{std}}(\Sigma) = 1$, $\text{diam}(\Sigma, \phi^* g_{std}) \leq 99$ and $\text{im } \phi$ contains a cylindrical segment C given by a profile function $r_{seq} : (L_-, L_+) \rightarrow \mathbb{R}$, $r_{seq} \equiv \tau$ with $L_+ - L_- = 2\tau$.

We construct our N -bulk surface by replacing C with an open chain of $N - 1$ spheres denoted by $O(N)$. Let $A_N = \frac{\tau}{N}$ and take a profile function $r : (L_-, L_+) \rightarrow [0, \infty)$ whose graph consists of $N - 1$ semicircles of radius A_N and two connecting ends. More precisely, on $[L_-, L_- + A_N]$, r is strictly decreasing, and moreover on

$[L_- + \frac{A_N}{2}, L_- + A_N]$ its graph coincides with a part of a semicircle with radius A_N centered at L_- . Similarly, r is strictly increasing on $[L_+ - A_N, L_+]$ and on $[L_+ - A_N, L_+ - \frac{A_N}{2}]$ its graph coincides with a semicircle with radius A_N centered at L_+ , see Figure 4.16.

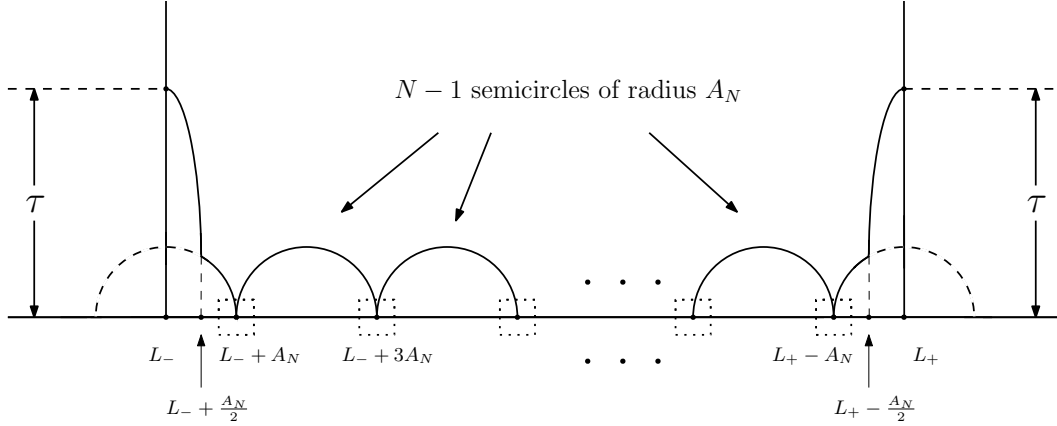


Figure 4.16. Profile function r of the chain of $N - 1$ spheres

Let g^r be the (singular) metric on Σ obtained from the standard metric on \mathbb{R}^3 after replacing C with the $O(N)$ given by the above profile function r . The area of $O(N)$ is of order τ^2 , while its diameter is of order τ . Thus, for any given $0 < \varepsilon < 1$, by taking τ small enough, we have that $|\text{Vol}_{g^r}(\Sigma) - 1| \leq \varepsilon$ and $\text{diam}(\Sigma, g^r) \leq 100\sqrt{1 - \varepsilon}$. Now, let $B_N = 10^{-n}A_N$, $\delta_0(N) = \delta_0 \cdot \frac{\varepsilon\sqrt{8\pi}}{N}$, δ_0 being given by (4.36). Notice also that B_N and $\delta_0(N)$ depend on a parameter n . By carrying out the same construction as in the bulked sphere case near each of the touching points $L_- + A_N, L_- + 3A_N, \dots, L_+ - 3A_N, L_+ - A_N$, we obtain the following proposition.

Proposition 4.8.2. *Given any sufficiently small $\varepsilon > 0$, for all sufficiently large n and $A_N, B_N, \delta_0(N)$ as above, there exists a family of profile functions $r_{\vec{x}}$, $\vec{x} \in \mathcal{T}(N)$, each of which defines an N -bulked metric $g_{\vec{x}}$ such that*

- (1) $|\text{Vol}_{g_{\vec{x}}}(\Sigma) - 1| \leq \varepsilon$ and $\text{diam}(\Sigma, g_{\vec{x}}) \leq 100\sqrt{1 - \varepsilon}$.
- (2) For different $\vec{x} \in \mathcal{T}(N)$, $r_{\vec{x}}$ coincide outside of B_N -neighbourhoods of $L_- + A_N, L_- + 3A_N, \dots, L_+ - 3A_N, L_+ - A_N$.
- (3) For $k = 1, \dots, N$, $r_{\vec{x}}(L_- + (2k - 1)A_N) = \frac{\delta_0}{2\pi} e^{-x_k}$, where $\vec{x} = (x_1, \dots, x_N)$.
- (4) For any $\vec{x}, \vec{y} \in \mathcal{T}(N)$,

$$\max_{l \in [L_-, L_+]} \max \left\{ \frac{r_{\vec{x}}(l)}{r_{\vec{y}}(l)}, \frac{r_{\vec{y}}(l)}{r_{\vec{x}}(l)}, \frac{r'_{\vec{x}}(l)}{r'_{\vec{y}}(l)}, \frac{r'_{\vec{y}}(l)}{r'_{\vec{x}}(l)} \right\} = e^{|\vec{x} - \vec{y}| \infty}.$$

In (4), as before, we use convention that $\frac{0}{0} = 1$.

We can now give a proof of (1) and (4) in Proposition 4.5.8.

Proof. (Proof of (1) and (4) in Proposition 4.5.8) Let $g_{\vec{x}}$ be the N -bulked metric from Proposition 4.8.2. By (2) in Proposition 4.8.1 it follows

$$L_{g_{\vec{x}}}(\gamma_1) = \delta_0(N)e^{-x_1}, \dots, L_{g_{\vec{x}}}(\gamma_N) = \delta_0(N)e^{-x_N}.$$

Since all γ_i have constant speed, we have

$$E_{g_{\vec{x}}}(\gamma_i) = \frac{L_{g_{\vec{x}}}^2(\gamma_i)}{2} = \frac{(\delta_0(N))^2}{2} e^{-2x_i}, \quad \text{for } i = 1, \dots, N,$$

which proves (1) in Proposition 4.5.8. The proof of (4) in Proposition 4.5.8 is exactly the same as the proof of (3) in Proposition 4.5.4 above. \square

Finally, the proof of Proposition 4.6.13 follows directly from (4) in Proposition 4.8.1 and (4) in Proposition 4.8.2.

4.8.2 Reduction of parameterization space

Recall that $\mathcal{T}(2N)$ is defined as

$$\mathcal{T}(2N) = \{ \vec{x} = (x_1, \dots, x_{2N}) \in [0, \infty)^{2N} \mid x_1 \leq x_2 \leq \dots \leq x_{2N} \}.$$

In this subsection, we will prove Lemma 4.5.9. It claims that for every $N \in \mathbb{N}$ there exists a quasi-isometric embedding $Q : (\mathbb{R}^N, |\cdot|_\infty) \rightarrow (\mathcal{T}(2N), |\cdot|_\infty)$. We construct Q as a composition of two quasi-isometric embeddings A and \mathbf{L} as follows

$$(\mathbb{R}^N, |\cdot|_\infty) \xrightarrow{\mathbf{L}} ([0, \infty)^{2N}, |\cdot|_\infty) \xrightarrow{A} (\mathcal{T}(2N), |\cdot|_\infty).$$

Construction of \mathbf{L}

Consider a map $L : \mathbb{R} \rightarrow [0, \infty)^2$, given by

$$L(x) = \begin{cases} (1, -x + 1) & \text{when } x < 0 \\ (1 + x, 1) & \text{when } x \geq 0 \end{cases}.$$

If we realize $[0, \infty)^2$ as the first quadrant of \mathbb{R}^2 , then map L gives an ‘‘L-shaped’’ embedding of \mathbb{R} with corner at $(1, 1)$. Now define a multi-dimensional version of L , that is $\mathbf{L} : \mathbb{R}^N \rightarrow [0, \infty)^{2N}$ by

$$\mathbf{L}(\vec{x}) = (L(x_1), \dots, L(x_N)).$$

We claim the following.

Lemma 4.8.3. For any $N \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^N$, it holds

$$\frac{1}{2}|\vec{x} - \vec{y}|_\infty \leq |\mathbf{L}(\vec{x}) - \mathbf{L}(\vec{y})|_\infty \leq |\vec{x} - \vec{y}|_\infty.$$

Proof. First consider the case when $N = 1$. When both x, y are negative or both x, y are non-negative, it is easy to see $|x - y| = |L(x) - L(y)|_\infty$. When $x < 0$ and $y \geq 0$,

$$\begin{aligned} |L(x) - L(y)|_\infty &= |(1, -x + 1) - (1 + y, 1)|_\infty \\ &= |(-y, -x)|_\infty = \max\{|x|, |y|\} \\ &\leq |x| + y = |x - y|. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2|L(x) - L(y)|_\infty &= 2 \max\{|x|, |y|\} \\ &\geq |x| + |y| \\ &= |x| + y = |x - y|. \end{aligned}$$

The same argument works for $x \geq 0$ and $y < 0$. Therefore, we get a bi-Lipschitz relation

$$|L(x) - L(y)|_\infty \leq |x - y| \leq 2|L(x) - L(y)|_\infty \quad (4.40)$$

Then

$$\begin{aligned} |\mathbf{L}(\vec{x}) - \mathbf{L}(\vec{y})|_\infty &= \max\{|L(x_1) - L(y_1)|_\infty, \dots, |L(x_N) - L(y_N)|_\infty\} \\ &\leq \max\{|x_1 - y_1|, \dots, |x_N - y_N|\} = |\vec{x} - \vec{y}|_\infty. \end{aligned}$$

and

$$\begin{aligned} 2|\mathbf{L}(\vec{x}) - \mathbf{L}(\vec{y})|_\infty &= \max\{2|L(x_1) - L(y_1)|_\infty, \dots, 2|L(x_N) - L(y_N)|_\infty\} \\ &\geq \max\{|x_1 - y_1|, \dots, |x_N - y_N|\} = |\vec{x} - \vec{y}|_\infty. \end{aligned}$$

Thus we get the conclusion. □

Construction of A

Consider the following map $A : [0, \infty)^{2N} \rightarrow \mathcal{T}(2N)$,

$$A(\vec{x}) = A(x_1, \dots, x_{2N}) = (x_1, x_1 + x_2, \dots, x_1 + \dots + x_{2N}).$$

We have

Lemma 4.8.4. For every $N \in \mathbb{N}$ and $\vec{x}, \vec{y} \in [0, \infty)^{2N}$, it holds

$$\frac{1}{2}|\vec{x} - \vec{y}|_\infty \leq |A(\vec{x}) - A(\vec{y})|_\infty \leq (2N) \cdot |\vec{x} - \vec{y}|_\infty.$$

Proof. Conclusion of Lemma 4.8.4 immediately follows from the following inequalities.

For $a_1, \dots, a_n \in \mathbb{R}$,

$$\frac{1}{2} \max\{|a_1|, \dots, |a_n|\} \leq \max\{|a_1|, |a_1 + a_2|, \dots, |a_1 + \dots + a_n|\} \leq n \cdot \max\{|a_1|, \dots, |a_n|\}.$$

The second inequality comes from the fact that for any $k \in \{1, \dots, n\}$,

$$|a_1 + \dots + a_k| \leq |a_1| + \dots + |a_k| \leq k \cdot \max\{|a_1|, \dots, |a_k|\} \leq n \cdot \max\{|a_1|, \dots, |a_n|\}.$$

For the first inequality, consider the two-term case first, that is

$$\max\{|a_1|, |a_1 + a_2|\} \geq \frac{1}{2} \max\{|a_1|, |a_2|\}. \quad (4.41)$$

If $|a_1| \geq |a_2|$, the inequality is obvious. If on the other hand, $|a_1| \leq |a_2|$, then

$$\begin{aligned} 2 \max\{|a_1|, |a_1 + a_2|\} &= 2 \max\{|a_1|, |a_1 - (-a_2)|\} \\ &\geq 2 \max\{|a_1|, ||a_1| - |a_2||\} \\ &= 2 \max\{|a_1|, |a_2| - |a_1|\} \\ &\geq |a_1| + |a_2| - |a_1| \\ &= |a_2| = \max\{|a_1|, |a_2|\}. \end{aligned}$$

This proves (4.41).

For the general case, assume that $\max\{|a_1|, |a_2|, \dots, |a_n|\} = |a_k|$. If $k = 1$, the inequality is obvious. If $k \geq 2$ then (4.41) implies that

$$\max\{|a_1 + \dots + a_{k-1}|, |a_1 + \dots + a_k|\} \geq \frac{1}{2} \max\{|a_1 + \dots + a_{k-1}|, |a_k|\} \geq \frac{1}{2} |a_k|,$$

and the claim follows. \square

Proof. (Proof of Lemma 4.5.9) Set $Q = A \circ \mathbf{L}$ and we get the conclusion. \square

4.8.3 Geodesic flow on a torus of revolution

We give a detailed analysis of the geodesic flow of the metric of revolution on \mathbb{T}^2 and in particular prove Lemma 4.1.24.

Recall from Example 4.1.23 that $f : [-A, A] \rightarrow (0, +\infty)$ was a smooth, even function, which extends $2A$ -periodically to a smooth function on \mathbb{R} . Moreover, f was strictly increasing on $[-A, 0]$ and hence strictly decreasing on $[0, A]$ with a unique maximum at 0 and two minima at $\pm A$. Using f as a profile function, we defined a metric of revolution g on $\mathbb{T}^2 = \mathbb{R}/2A\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. In other words, g is a pull back of the Euclidean metric on \mathbb{R}^3 via the embedding

$$(x, \theta) \rightarrow (x, f(x) \cos \theta, f(x) \sin \theta).$$

Recall also that we used a change of variable $X(x) = \int_0^x \sqrt{1 + (f'(t))^2} dt$, $x \in [-A, A]$. The new variable satisfies $X \in [-T, T]$ for $T = \int_0^A \sqrt{1 + (f'(t))^2}$ and we denoted $F(X) = f(x(X))$. A direct computation shows that in (X, θ) coordinates metric has the following form:

$$g_{(X,\theta)} = \begin{pmatrix} 1 & 0 \\ 0 & F^2(X) \end{pmatrix}. \quad (4.42)$$

The Lagrangian of the geodesic flow of g is given by

$$L(X, \theta, v_X, v_\theta) = \frac{1}{2} (v_X^2 + F^2(X)v_\theta^2).$$

while momenta are

$$p_X = \frac{\partial L}{\partial v_X} = v_X \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial v_\theta} = F^2(X)v_\theta.$$

We compute the Hamiltonian as a Legendre transform

$$H(X, \theta, p_X, p_\theta) = p_X v_X + p_\theta v_\theta - \frac{1}{2} (v_X^2 + F^2(X)v_\theta^2) = \frac{1}{2} \left(p_X^2 + \frac{p_\theta^2}{F^2(X)} \right). \quad (4.43)$$

Hamiltonian equations are

$$\left\{ \begin{array}{l} \dot{X} = \frac{\partial H}{\partial p_X} = p_X \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{F^2(X)} \\ \dot{p}_X = -\frac{\partial H}{\partial X} = \frac{F'(X)}{F^3(X)} p_\theta^2 \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0. \end{array} \right. \quad (4.44)$$

The above system is integrable with two integrals given by H and p_θ . Let us analyze the system on the energy level $H = \frac{1}{2}$ (this corresponds to unit speed geodesics) and let us assume that $p_\theta = \sqrt{C} \geq 0$. The case $p_\theta < 0$ is treated similarly (note that $(p_X, p_\theta) \rightarrow (-p_X, -p_\theta)$ corresponds to changing the direction of a geodesic). Now, (4.43) translates to

$$p_X^2 + \frac{C}{F^2(X)} = 1, \quad (4.45)$$

while Hamiltonian equations become

$$\begin{cases} \dot{X} = p_X \\ \dot{\theta} = \frac{\sqrt{C}}{F^2(X)} \\ \dot{p}_X = C \frac{F'(X)}{F^3(X)} \end{cases} \quad (4.46)$$

If $C = 0$, (4.45) and (4.46) imply that $\dot{\theta} = 0$, $\dot{X} = p_X = \pm 1$. Hence, in this case geodesics are given by $\theta(t) = \text{const}$, $X(t) = X(0) \pm t$.

If $C > 0$, (4.45) implies that $\sqrt{C} \leq \max F$ and we distinguish four cases.

1° $\sqrt{C} = \max F$:

In this case (4.45) implies that $X = 0$ and thus $p_X = 0$, $F'(X) = 0$. Now, (4.46) gives $\dot{X} = 0$, $\dot{p}_X = 0$, $\dot{\theta} = \frac{1}{\max F}$ and thus $X(t) = 0$, $p_X(t) = 0$, $\theta(t) = \theta(0) + \frac{t}{\max F}$. This solution describes a closed geodesic $\hat{\gamma}$, i.e. the parallel circle of radius $\max F$ at $X = 0$, and its iterations.

2° $\min F < \sqrt{C} < \max F$:

In this case the dynamics is constrained to the interval where $\sqrt{C} \leq F(X)$, i.e. on $[-\lambda_C(F), \lambda_C(F)]$ for $F^{-1}(\sqrt{C}) = \{-\lambda_C(F), \lambda_C(F)\}$. Moreover, on this interval it holds $p_X = \pm \sqrt{1 - \frac{C}{F^2(X)}}$ and the portrait of the system in (X, p_X) -plane looks as in Figure 4.17.

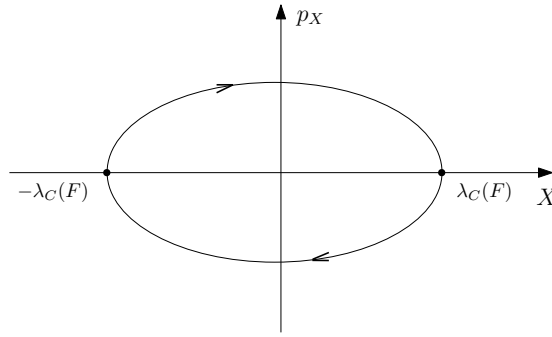


Figure 4.17. (X, p_X) -portrait when $\min F < \sqrt{C} < \max F$

3° $\sqrt{C} = \min F$:

In this case $\lambda_C(F) = T$ and $p_X = \pm \sqrt{1 - \frac{C}{F^2(X)}}$. The behaviour of the flow at $X = \pm T$ differs from the behaviour when $X \in (-T, T)$. Indeed, if $X = \pm T$, we have $p_X = F'(X) = 0$ and (4.46) becomes $\dot{X} = 0$, $\dot{p}_X = 0$, $\dot{\theta} = \frac{1}{\min F}$. Thus, we obtain a solution $X(t) = \pm T$, $p_X(t) = 0$, $\theta(t) = \theta(0) + \frac{t}{\min F}$, which describes a closed geodesic $\check{\gamma}$, i.e. the parallel circle of radius $\min F$ at $X = \pm T$, and its iterations. In

(X, p_X) -plane solutions with $X \in (-T, T)$ trace two curves which connect points $-T$ and T and the portrait looks as in Figure 4.18.

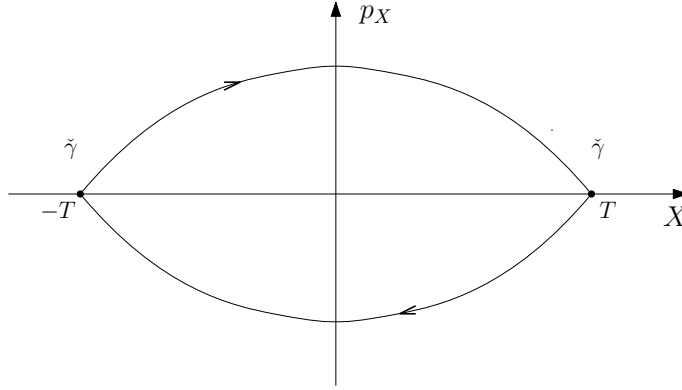


Figure 4.18. (X, p_X) -portrait when $\sqrt{C} = \min F$

4° $\sqrt{C} < \min F$:

In this case $1 - \frac{C}{F^2(X)} > 0$ for all $X \in [-T, T]$ and $p_X = \pm \sqrt{1 - \frac{C}{F^2(X)}}$. The portrait looks as in Figure 4.19.

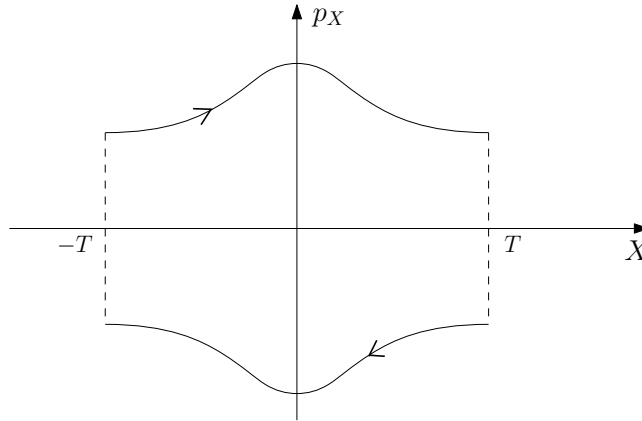


Figure 4.19. (X, p_X) -portrait when $\sqrt{C} < \min F$

Recall that α denotes the homotopy class of loops represented by a loop $\theta(t) = t, t \in [0, 2\pi], X = \text{const}$. Let us now focus on closed geodesics in homotopy class α . As we saw above, there are always two closed geodesics in this class, namely $\tilde{\gamma}$ and $\hat{\gamma}$. Their properties are summarized by the following lemma.

Lemma 4.8.5. *Assume that $F''(T) > 0$ as well as that $0 < -F(0)F''(0) < 1$. Then $\tilde{\gamma}$ and $\hat{\gamma}$ are non-degenerate closed geodesics and $\text{ind } \tilde{\gamma} = 0, \text{ind } \hat{\gamma} = 1$.*

Proof. First notice that $F(0) = f(0), F(\pm T) = f(\pm A), F'(0) = f'(0) = 0, F'(\pm T) = f'(\pm A) = 0$ and $F''(0) = f''(0), F''(\pm T) = f''(\pm A)$. This immediately follows after

differentiating (twice) the expression $f(x) = F(X(x))$, using that $X'(x) = \sqrt{1 + (f'(x))^2}$ as well as that $f'(0) = f'(\pm A) = 0$. Hence, $F''(\pm T) > 0$ implies $f''(\pm A) > 0$ and thus $\tilde{\gamma}$ is non-degenerate and $\text{ind } \tilde{\gamma} = 0$, as show in Subsection 4.6.1 (proof of Lemmas 4.5.3 and 4.5.7).

By Lemma 4.6.3, $\hat{\gamma}$ is non-degenerate if and only if there are no periodic Jacobi fields along $\hat{\gamma}$, orthogonal to $\dot{\hat{\gamma}}$. As in the case of $\tilde{\gamma}$, Jacobi fields are computed using (4.24), however in this case $K = -4\pi^2 f(0)f''(0) = -4\pi^2 F(0)F''(0) > 0$. Orthogonal Jacobi fields are of the form $J(t) = (J_1(t), 0, 0)$ and (4.24) translates to

$$\ddot{J}_1(t) + K \cdot J_1(t) = 0. \quad (4.47)$$

In other words, the space of orthogonal Jacobi fields is spanned by

$$J_{\sin}(t) = (\sin(\sqrt{K}t), 0, 0) \quad \text{and} \quad J_{\cos}(t) = (\cos(\sqrt{K}t), 0, 0).$$

By the assumption, $0 < -F(0)F''(0) < 1$ and hence $0 < \sqrt{K} < 2\pi$, which implies that no orthogonal Jacobi field is periodic, i.e. $\hat{\gamma}$ is non-degenerate.

Using that

$$(J_{\sin}(t), \dot{J}_{\sin}(t)) = ((\sin(\sqrt{K}t), 0, 0), (\sqrt{K} \cos(\sqrt{K}t), 0, 0))$$

and

$$(J_{\cos}(t), \dot{J}_{\cos}(t)) = ((\cos(\sqrt{K}t), 0, 0), (-\sqrt{K} \sin(\sqrt{K}t), 0, 0)),$$

we may express the linearized Poincare map $P : (T\hat{\gamma}(0))^\perp \oplus (T\hat{\gamma}(0))^\perp \rightarrow (T\hat{\gamma}(0))^\perp \oplus (T\hat{\gamma}(0))^\perp$ with respect to basis $((1, 0, 0), (0, 0, 0))^T, ((0, 0, 0), (\sqrt{K}, 0, 0))^T$ by the following matrix

$$P = \begin{pmatrix} \cos \sqrt{K} & \sin \sqrt{K} \\ -\sin \sqrt{K} & \cos \sqrt{K} \end{pmatrix}.$$

The eigenvalues of P are $e^{\pm i\sqrt{K}}$ and hence $\hat{\gamma}$ is not hyperbolic. Finally, in order to compute $\text{ind } \hat{\gamma}$ we use Lemma 3.4.2. from [85]. This lemma states that if a closed geodesic γ on an orientable surface is non-degenerate and not hyperbolic, then its index is an odd number equal to either m or $m + 1$, where m denotes the number of points $\gamma(t_*)$, $0 < t_* < 1$, conjugate¹⁰ to $\gamma(0)$ along γ . Since every orthogonal Jacobi field has the form

$$J(t) = (A \cos(\sqrt{K}t) + B \sin(\sqrt{K}t), 0, 0),$$

$J(0) = (0, 0, 0)$ is equivalent to $A = 0$ and since $0 < \sqrt{K} < 2\pi$ there can be at most one point conjugate to $\hat{\gamma}(0)$, namely $\hat{\gamma}(\frac{\pi}{\sqrt{K}})$. This means that $m = 0$ or $m = 1$ and thus $\text{ind } \hat{\gamma} = 1$. \square

¹⁰Recall that a point $\gamma(t_*)$ is called conjugate to $\gamma(0)$ along γ if there exists a Jacobi field J along γ such that $J(0) = J(t_*) = 0$. Since the space of Jacobi fields along γ is spanned by orthogonal Jacobi fields and $\dot{\gamma}$ and $t\dot{\gamma}$, one readily sees that $\gamma(t_*)$ is conjugate to $\gamma(0)$ if and only if there exists an orthogonal Jacobi field J along γ such that $J(0) = J(t_*) = 0$.

The above analysis of the portrait in (X, p_X) -plane shows that closed geodesics in class α other than $\tilde{\gamma}$ and $\hat{\gamma}$ can only appear when $\min F < \sqrt{C} < \max F$. In this case, for a fixed C , the flow is periodic in (X, p_X) -plane. Denote by $\Theta_F(C)$ the shift in θ -coordinate made by a flow line $\tilde{\gamma}$ by the time it makes a single turn from $(-\lambda_C, 0)$ back to $(-\lambda_C, 0)$ (we abbreviate $\lambda_C = \lambda_C(F)$).

Formally, let $\tilde{\gamma}(t) = (X(t), \theta(t), p_X(t), \sqrt{C})$ be a flow line of the Hamiltonian system (4.46), assume without loss of generality that $X(0) = -\lambda_C, p_X(0) = 0$ and let $t_0 > 0$ be the smallest time when $X(t_0) = -\lambda_C, p_X(t_0) = 0$ again. Define

$$\Theta_F(C) = \theta(t_0) - \theta(0).$$

As notation suggests, $\Theta_F(C)$ only depends on F and C . Indeed, using (4.45), (4.46) and the symmetry of F we calculate¹¹

$$\begin{aligned} \Theta_F(C) &= \int_0^{t_0} \dot{\theta}(t) dt = \int_0^{t_0} \frac{\sqrt{C}}{F^2(X(t))} dt = 2 \int_{-\lambda_C}^{\lambda_C} \frac{\sqrt{C}}{F^2(X)} \frac{dX}{\dot{X}} \\ &= 2 \int_{-\lambda_C}^{\lambda_C} \frac{\sqrt{C}}{F^2(X)} \frac{dX}{p_X} = 2\sqrt{C} \int_{-\lambda_C}^{\lambda_C} \frac{dX}{F(X)\sqrt{F^2(X) - C}}. \end{aligned} \quad (4.48)$$

We will define F_ε , described in Lemma 4.1.24, for which $\Theta_{F_\varepsilon}(C) > 2\pi$ for all $\min F_\varepsilon < \sqrt{C} < \max F_\varepsilon$. Since from (4.46) we have that $\dot{\theta} > 0$, $\theta(t)$ is increasing and hence $\Theta_{F_\varepsilon}(C) > 2\pi$ implies that any closed geodesic γ must make at least two full turns in θ -direction, i.e. it can not lie in the homotopy class α .

Everything we have done so far applies to any F satisfying the necessary conditions. Let us now focus on concrete examples and prove Lemma 4.1.24.

Proof. (Proof of Lemma 4.1.24) First, we note that F is implicitly defined by f and hence, it is not a priori clear that we may choose F freely. However, one can show that if $F : [-T, T] \rightarrow (0, +\infty)$ satisfies $|F'(X)| < 1$ for all $X \in [-T, T]$ then there exists $f : [-A, A] \rightarrow (0, +\infty)$, for some A , such that $F(X) = f(x(X))$. Indeed, by setting $x(X) = \int_0^X \sqrt{1 - (F'(\tau))^2} d\tau$, $A = \int_0^T \sqrt{1 - (F'(\tau))^2} d\tau$ and $f(x(X)) = F(X)$, one checks by a direct computation that f defines F . Moreover, since $|F'(X)| < 1$ for all $X \in [-T, T]$, $x(X)$ is a smooth function and $\frac{dx}{dX} > 0$ on $[-T, T]$. Thus, f is smooth if and only if F is smooth.

Fix $0 < \sqrt{k} < m$ and let us take $T = 1, F_0 : [-1, 1] \rightarrow (0, +\infty)$ given by $F_0(X) = \frac{1}{\sqrt{kX^2 + m}}$. One readily checks that $|F'(X)| < 1$ for all $X \in [-1, 1]$. For small enough $\varepsilon > 0$, F_ε will be a smoothing of F_0 near the points ± 1 . We start by analysing F_0 .

Denote by $\lambda_C^0 = \lambda_C(F_0) = \sqrt{\frac{\frac{1}{C} - m}{k}}$ and let $\tilde{\gamma}(t) = (X(t), \theta(t), p_X(t), \sqrt{C})$, $X(0) = -\lambda_C^0, p_X(0) = 0$ be a solution of the Hamiltonian system (4.45), (4.46) associated to

¹¹Compare to Proposition 2 in [7].

F_0 . From (4.46) we have

$$\ddot{X} = p_X = \frac{F'(X)}{F^3(X)}C = -CkX,$$

and hence $X(t) = a \cos(\sqrt{CK}t) + b \sin(\sqrt{CK}t)$. Initial conditions $X(0) = -\lambda_C^0$ and $\dot{X}(0) = p_X(0) = 0$ give us that $a = -\lambda_C^0$ and $b = 0$, i.e.

$$X(t) = -\sqrt{\frac{\frac{1}{C} - m}{k}} \cos(\sqrt{CK}t). \quad (4.49)$$

Using (4.46) and (4.49), a direct computation shows that

$$\begin{aligned} \Theta_{F_0}(C) &= \int_0^{\frac{2\pi}{\sqrt{Ck}}} \dot{\theta}(t) dt = \int_0^{\frac{2\pi}{\sqrt{Ck}}} \frac{\sqrt{C}}{F^2(X(t))} dt = \int_0^{\frac{2\pi}{\sqrt{Ck}}} \frac{\sqrt{C}}{\frac{1}{kX^2(t)+m}} dt \\ &= \sqrt{C} \int_0^{\frac{2\pi}{\sqrt{Ck}}} \left(\left(\frac{1}{C} - m \right) \cos^2(\sqrt{CK}t) + m \right) dt = \frac{\pi}{\sqrt{k}} \left(\frac{1}{C} + m \right). \end{aligned}$$

From $\sqrt{C} < \max F_0 = \frac{1}{\sqrt{m}}$ we have that $\frac{1}{C} > m$ and thus $\Theta_{F_0}(C) > 2\pi \frac{m}{\sqrt{k}}$. Since $m > \sqrt{k}$ it follows that $\Theta_{F_0}(C) > 2\pi$ for all $\min F_0 < \sqrt{C} < \max F_0$.

Finally, let us show that for $\varepsilon > 0$ we may smoothen F_0 on intervals $[-1, -1 + \varepsilon]$ and $[1 - \varepsilon, 1]$ in such a way that newly obtained F_ε also satisfies $\Theta_{F_\varepsilon}(C) > 2\pi$ for all $\min F_\varepsilon < \sqrt{C} < \max F_\varepsilon$. To this end, let F_ε be such that $F_\varepsilon|_{[-1+\varepsilon, 1-\varepsilon]} = F_0|_{[-1+\varepsilon, 1-\varepsilon]}$, $F_\varepsilon \geq F_0$ elsewhere, $|F'_\varepsilon(X)| \leq |F'_0(X)| < 1$ for all $X \in [-1, 1]$, F_ε extends 2-periodically to a smooth function on \mathbb{R} and $F_\varepsilon \xrightarrow{C^0} F_0$ as $\varepsilon \rightarrow 0$. Denote $F_\varepsilon^{-1}(\sqrt{C}) = \{-\lambda_C, \lambda_C\}$, $\lambda_C > 0$. Since $F_\varepsilon \geq F_0$ we have that $\lambda_C^0 \leq \lambda_C$. Now, note that if $\lambda_C^0 \leq 1 - \varepsilon$, it holds $\lambda_C = \lambda_C^0$ as well as $\Theta_{F_\varepsilon}(C) = \Theta_{F_0}(C) > 2\pi$, because two function coincide on $[-1 + \varepsilon, 1 - \varepsilon]$. If however $1 - \varepsilon < \lambda_C^0 \leq \lambda_C < 1$, from (4.48) we obtain

$$\Theta_{F_\varepsilon}(C) = 2\sqrt{C} \int_{-\lambda_C}^{\lambda_C} \frac{dX}{F_\varepsilon(X)\sqrt{F_\varepsilon^2(X) - C}} > 2\sqrt{C} \int_{-\lambda_{C+\varepsilon}}^{\lambda_{C-\varepsilon}} \frac{dX}{F_\varepsilon(X)\sqrt{F_\varepsilon^2(X) - C}}.$$

Since $\lambda_C^0 \leq \lambda_C$ we have

$$\begin{aligned} 2\sqrt{C} \int_{-\lambda_{C+\varepsilon}}^{\lambda_{C-\varepsilon}} \frac{dX}{F_\varepsilon(X)\sqrt{F_\varepsilon^2(X) - C}} &\geq 2\sqrt{C} \int_{-\lambda_{C+\varepsilon}^0}^{\lambda_{C-\varepsilon}^0} \frac{dX}{F_\varepsilon(X)\sqrt{F_\varepsilon^2(X) - C}} = \\ &= 2\sqrt{C} \int_{-\lambda_{C+\varepsilon}^0}^{\lambda_{C-\varepsilon}^0} \frac{dX}{F_0(X)\sqrt{F_0^2(X) - C}}. \end{aligned}$$

Same change of variables used in (4.48) gives us

$$\begin{aligned}
\Theta_{F_\varepsilon}(C) &> 2\sqrt{C} \int_{-\lambda_C^0 + \varepsilon}^{\lambda_C^0 - \varepsilon} \frac{dX}{F_0(X) \sqrt{F_0^2(X) - C}} \\
&= 2\sqrt{C} \int_{X^{-1}(-\lambda_C^0 + \varepsilon)}^{X^{-1}(\lambda_C^0 - \varepsilon)} \frac{dt}{F_0^2(X(t))} \\
&= 2\sqrt{C} \int_{\frac{\arccos\left(-1 + \frac{\varepsilon}{\lambda_C^0}\right)}{\sqrt{Ck}}}^{\frac{\arccos\left(1 - \frac{\varepsilon}{\lambda_C^0}\right)}{\sqrt{Ck}}} (kX^2(t) + m) dt \\
&> \frac{2m}{\sqrt{k}} \left(\arccos\left(-1 + \frac{\varepsilon}{\lambda_C^0}\right) - \arccos\left(1 - \frac{\varepsilon}{\lambda_C^0}\right) \right).
\end{aligned}$$

Since $\lambda_C^0 \geq 1 - \varepsilon$, it follows that $\frac{\varepsilon}{\lambda_C^0} \leq \frac{\varepsilon}{1 - \varepsilon} \rightarrow 0$ when $\varepsilon \rightarrow 0$, independently of C . Moreover, by the assumption $\sqrt{k} < m$, so for small enough ε we have that $\Theta_{F_\varepsilon}(C) > 2\pi$ for all $\min F_\varepsilon < \sqrt{C} < \max F_\varepsilon$. As explained above, this implies that the only closed geodesics in class α of a metric induced by F_ε are $\tilde{\gamma}$ and $\hat{\gamma}$ which together with Lemma 4.8.5 concludes the proof. \square

Chapter 5

Persistence barcodes and Laplace eigenfunctions on surfaces

5.1 Introduction and main results

5.1.1 Laplace-Beltrami eigenfunctions

The past fifteen years have witnessed a number of fascinating applications of the spectral theory of the Laplace-Beltrami operator to data analysis, such as dimensionality reduction and data representation [22,40] or shape segmentation in computer graphics [141,124]. In the present chapter we focus on this interaction the other way around and study persistence barcodes, a fundamental notion originated in topological data analysis, of the Laplace-Beltrami eigenfunctions and their linear combinations. Our main finding is a constraint on such barcodes in terms of the corresponding eigenvalues. This result turns out to have applications to approximation theory.

Let M be a compact n -dimensional Riemannian manifold, possibly with nonempty boundary. Let Δ be the (positive definite) Laplace-Beltrami operator on M ; if $\partial M \neq \emptyset$ we assume that the Dirichlet condition is imposed on the boundary. The spectrum of the Laplace-Beltrami operator on a compact Riemannian manifold is discrete, and the eigenvalues form a sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity. The corresponding eigenfunctions f_k , $\Delta f_k = \lambda_k f_k$, form an orthonormal basis in $L^2(M)$. The properties of Laplace-Beltrami eigenfunctions have fascinated researchers for more than two centuries, starting with the celebrated Chladni's experiments with vibrating plates. We refer to [79,164,165] for a modern overview of the subject. As the examples of trigonometric polynomials and spherical harmonics indicate, the shapes of the eigenfunctions are expected to have an increasingly complex structure as λ goes to infinity. At the same time, various restrictions on the behaviour of eigenfunctions can be formulated in terms of the corresponding eigenvalue. One of the basic facts about eigenfunctions

is Courant’s nodal domain theorem, stating that the number of nodal domains of an eigenfunction f_k is at most k (see [43]). There exist also bounds on the $(n - 1)$ -dimensional measure of the zero set of eigenfunctions (see [93–95] for most recent developments on this topic), on the distribution of nodal extrema ([123, 112]), on the growth of L^p -norms ([142]), and other related results.

In the present chapter we focus on topological properties of the sublevel sets of Laplace-Beltrami eigenfunctions, and, more generally, of the linear combinations of eigenfunctions with eigenvalues $\leq \lambda$. There has been a number of important recent advances in the study of topological properties of random linear combinations of Laplace eigenfunctions, with an emphasis on the nodal and critical sets (see, for instance, [99, 100, 68, 69, 133, 29]). Our approach is deterministic and is based on the study of *persistence barcodes*. In the probabilistic setting, some steps in this direction have been discussed in [28, Section 1.4.3], see also [108]. Roughly speaking, a persistence barcode is a collection of intervals in \mathbb{R} which encodes oscillation of a function (see Section 2.1 for a detailed overview). Our main result (Theorem 5.1.7) implies that the quantity $\Phi_1(f)$, the total length of the barcode of any such linear combination f with unit L^2 -norm, satisfies an upper bound $O(\lambda)$. This inequality is inspired by the ideas introduced in [123], where a similar bound was proved for the *Banach indicatrix* of f , another measure of oscillation which goes back to the works of Kronrod [87] and Yomdin [162]. Our central observation (see Proposition 5.3.1 below) is that the length of the barcode admits an upper bound via the Banach indicatrix, which together with [123] yields the main result.

We believe that discussing eigenfunctions and their linear combinations in the language of barcodes, which originated in topological data analysis, has a number of merits. First, there exists a well developed metric theory of barcodes which highlights their robustness with respect to perturbations of functions in the uniform norm. Some features of this robustness are inherited by the above-mentioned functional Φ_1 . This, in turn, paves the way for applications to the following question of approximation theory (see Section 5.2): given a function with unit L^2 -norm, how well one can approximate it by a linear combination of Laplace eigenfunctions with eigenvalues $\leq \lambda$? In particular, we show that a highly oscillating function does not admit a good uniform approximation of this kind unless λ is large enough, see Corollary 5.2.4. Second, our approximation results remain valid if a given function is composed with a diffeomorphism of the surface, see Proposition 5.2.1. Our approach yields it essentially for free, given that the barcodes are invariant with respect to compositions with diffeomorphisms. Note that the effect of a change of variables on analytic properties of functions is a classical theme in Fourier analysis, cf. the celebrated Bohr-Pál theorem [132]. Third, we conjecture that barcodes provide a right framework for a potential extension of our results to higher dimensions, see Conjecture 5.1.14 below.

In a different direction, we present an application to the problem of sorting finite bars of persistence barcodes. This task arises on a number of occasions in topology and data analysis. Our results allow to improve an estimate on the optimal running time

of a sorting algorithm for barcodes of linear combinations of Laplace eigenfunctions with eigenvalues $\leq \lambda$, see subsection 5.1.4.

5.1.2 A family of functionals on the space of barcodes

From now on, we assume that M is an orientable surface, possibly with boundary. Let us define, for every positive function $u \in C(\mathbb{R})$, a positive, lower semi-continuous functional Φ_u on the space of Morse functions on M . Let f be a Morse function, vanishing on the boundary if $\partial M \neq \emptyset$. Recall from Section 2.1 that $V_k^t(f) = H_k(\{f < t\}; \mathbb{R})$ was a persistence module associated to f in degree k , $\mathcal{B}_k(f)$ it's barcode, called the degree k barcode of f , and $\mathcal{B}(f) = \cup_k \mathcal{B}_k(f)$ the full barcode of f . Under our assumptions $\mathcal{B}(f)$ is finite, i.e. it consists of finitely many distinct intervals with finite multiplicities. Denote by $\mathcal{B}'(f) \subset \mathcal{B}(f)$ the multiset of all finite bars in the barcode $\mathcal{B}(f)$ and by $|\mathcal{B}'(f)|$ the total number of finite bars in $\mathcal{B}(f)$. Recall from Section 1.4 that a positive functional Φ_u on the set \mathcal{F}_{Morse} of all Morse functions (vanishing on the boundary) is defined by

$$\Phi_u(f) = \begin{cases} \int_{\min f}^{\max f} u(t) dt + \sum_{I \in \mathcal{B}'(f)} \int_I u(t) dt & \text{if } \partial M = \emptyset, \\ \int_{\min f}^0 u(t) dt + \sum_{I \in \mathcal{B}'(f)} \int_I u(t) dt & \text{if } \partial M \neq \emptyset. \end{cases} \quad (5.1)$$

In particular, $\Phi_1(f)$ is the sum of the lengths of all the finite bars in the barcode of f and the length of the range of f . A related functional has been earlier considered in [39], see Remark 5.1.19.

Lemma 5.1.1. *Let*

$$C(u, f) = 2 \cdot (|\mathcal{B}'(f)| + 1) \cdot \max_{[\min f, \max f]} u$$

in the case $\partial M = \emptyset$, or

$$C(u, f) = (2|\mathcal{B}'(f)| + 1) \cdot \max_{[\min f, \max f]} u$$

in the case $\partial M \neq \emptyset$. Then

$$\Phi_u(f) - \Phi_u(h) \leq C(u, f) \cdot d_{bottle}(\mathcal{B}(f), \mathcal{B}(h)). \quad (5.2)$$

Remark 5.1.2. If M has no boundary, then

$$2 \cdot (|\mathcal{B}'(f)| + 1) = |\text{Crit}(f)| - b_1(M), \quad (5.3)$$

where $|\text{Crit}(f)|$ stands for the number of critical points of f and $b_1 = \dim H_1(M; \mathbb{R})$. Morally speaking, each critical point of index i produces either a left endpoint of

a bar in degree i or a right endpoint of a bar in degree $i - 1$. This can be made precise in a number of ways (see Lemma 2.1.10 and the discussion surrounding it). Therefore, taking into account the critical points corresponding to infinite bars, we get $|\text{Crit}(f)| = 2|\mathcal{B}'(f)| + b_0(M) + b_1(M) + b_2(M)$, which implies (5.3). The same reasoning applies to general manifolds without boundary, where we have

$$|\text{Crit}(f)| = 2|\mathcal{B}'(f)| + \sum_{i=0}^{\dim M} b_i(M).$$

We prove Lemma 5.1.1 in subsection 5.4.1. Combining (5.2) with Theorem 2.1.17 yields

$$\Phi_u(f) - \Phi_u(h) \leq C(u, f) \cdot d_{C^0}(f, h), \quad (5.4)$$

where $d_{C^0}(f, h) = |f - h|_{C^0}$.

Proposition 5.1.3. *The functional Φ_u is lower semi-continuous both as a functional $\Phi_u : (\mathbf{B}, d_{\text{bottle}}) \rightarrow \mathbb{R}$ and as a functional $\Phi_u : (\mathcal{F}_{\text{Morse}}, d_{C^0}) \rightarrow \mathbb{R}$. Here \mathbf{B} stands for the set of all barcodes corresponding to functions in $\mathcal{F}_{\text{Morse}}$.*

Remark 5.1.4. We slightly abuse the notation here by looking at $\Phi_u(f) = \Phi_u(\mathcal{B}(f))$ as the function of barcode $\mathcal{B}(f)$. However, it is obvious that Φ_u depends only on $\mathcal{B}(f)$ and not on f itself. In the same spirit $\min f$ and $\max f$ should be replaced by the smallest and the largest endpoint of a bar in $\mathcal{B}(f)$.

Proof. Recall that a functional Φ defined on a metric space X is called lower semi-continuous at a point $f \in X$ if $\liminf_{h \rightarrow f} \Phi(h) \geq \Phi(f)$. This relation easily follows from the inequalities (5.2) and (5.4) for the functional Φ_u defined on the metric spaces $(\mathbf{B}, d_{\text{bottle}})$ and $(\mathcal{F}_{\text{Morse}}, d_{C^0})$, respectively. \square

The inequality (5.4) could be further strengthened. Let $\text{Diff}(M)$ denote the group of all smooth diffeomorphisms of the surface M (throughout this chapter, the term “smooth” stands for C^∞ -smooth).

Corollary 5.1.5. *We have*

$$\Phi_u(f) - \Phi_u(h) \leq C(u, f) \cdot d_{C^0}(f \circ \varphi, h \circ \psi), \quad (5.5)$$

for any two diffeomorphisms $\varphi, \psi \in \text{Diff}(M)$.

In particular, taking $\varphi = \psi = \mathbb{1}_M$ gives (5.4).

Proof. Indeed, for any diffeomorphism $\varphi : M \rightarrow M$, the barcodes $\mathcal{B}(f)$ and $\mathcal{B}(f \circ \varphi)$ are the same. Since Φ_u depends only on the barcode and not on the function itself, putting $f \circ \varphi$ and $h \circ \psi$ in (5.2) yields (5.5). \square

Let us now extend the functional Φ_u from \mathcal{F}_{Morse} to $C^0(M)$. First, we introduce a “cut-off version” of Φ_u . Define

$$\Phi_{u,k}(f) = \begin{cases} \int_{\min f}^{\max f} u(t) dt + \sum_{i=1}^k \int_{I_i} u(t) dt & \text{if } \partial M = \emptyset, \\ \int_{\min f}^0 u(t) dt + \sum_{i=1}^k \int_{I_i} u(t) dt & \text{if } \partial M \neq \emptyset. \end{cases} \quad (5.6)$$

where $I_i \in \mathcal{B}'(f)$ are finite intervals ordered by integral of u , i.e. we have

$$\int_{I_1} u(t) dt \geq \int_{I_2} u(t) dt \geq \dots$$

Lemma 5.1.6. *For every bounded function u the functional $\Phi_{u,k}$ is Lipschitz on \mathcal{F}_{Morse} with respect to d_{bottle} with Lipschitz constant $(2k + 2) \cdot \max u$ if $\partial M = \emptyset$ or with Lipschitz constant $(2k + 1) \cdot \max u$ if $\partial M \neq \emptyset$.*

The proof of Lemma 5.1.6 is given in subsection 5.4.2.

Assume now that $f \in C^0(M)$ is an arbitrary continuous function on M . Let $f_n \in \mathcal{F}_{Morse}$ such that $d_{C^0}(f, f_n) \rightarrow 0$ as $n \rightarrow \infty$. Set

$$\Phi_u(f) := \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \Phi_{u,k}(f_n) \quad (5.7)$$

Note that $\Phi_u(f_n)$ only depends on $u|_{[\min f_n, \max f_n]}$ and, since for sufficiently large n it holds $[\min f_n, \max f_n] \subset [\min f - 1, \max f + 1]$, we may restrict ourselves to this interval and argue as if u was bounded. Thus due to Lemma 5.1.6 and Theorem 2.1.17, the double limit on the right-hand side of (5.7) (which could be equal to $+\infty$) does not depend on the choice of the approximating sequence f_n . Therefore, the functional $\Phi_u(f)$ is well defined by (5.7). Moreover, it is easy to check that the right-hand sides of (5.7) and (5.1) coincide for $f \in \mathcal{F}_{Morse}$, and therefore (5.7) indeed defines an extension of (5.1) to $C^0(M)$.

5.1.3 Main results

As before, M is an orientable surface, possibly with boundary, equipped with a Riemannian metric g . Denote by $\|\cdot\|$ the L^2 -norm with respect to Riemannian area σ and by Δ the Laplace-Beltrami operator with respect to g . Slightly abusing the notation, throughout this chapter κ_g will denote various constants depending only on the Riemannian metric g .

Following⁴ [123], denote by \mathcal{F}_λ the set of all smooth functions on M (vanishing on the boundary if $\partial M \neq \emptyset$) which satisfy $\|f\| = 1$ and $\|\Delta f\| \leq \lambda$. One may check that

⁴Our definition is slightly different from the one in [123] since we do not assume that $\int_M f \sigma = 0$ if M has no boundary. However, this assumption is not needed for any of the results of [123] which we use.

\mathcal{F}_λ contains normalized linear combinations of eigenfunctions of Δ with eigenvalues $\lambda_i \leq \lambda$. If $\partial M \neq 0$, \mathcal{F}_λ contains also normalized eigenfunctions of the biharmonic clamped plate boundary value problem on M (see [123, Example 1.2]). Our main result is the following theorem.

Theorem 5.1.7 (Theorem 1.4.1). *Let $\lambda > 0$ be any positive real number, $u \in C(\mathbb{R})$ be a non-negative function and $f \in \mathcal{F}_\lambda$ be a function on an orientable surface (M, g) . Then there exists a constant $\kappa_g > 0$ such that*

$$\Phi_u(f) \leq \kappa_g(\lambda + 1)\|u \circ f\|. \quad (5.8)$$

In order to prove this theorem we compare both sides of inequality (5.8) with an intermediate quantity. Let $\beta(t, f)$ be the number of connected components of $f^{-1}(t)$. Function $\beta(t, f)$ is called the *Banach indicatrix* of f (see [87, 162]). In [123] it was proved that $\int_{-\infty}^{+\infty} u(t)\beta(t, f)dt \leq \kappa_g(\lambda + 1)\|u \circ f\|$ for $f \in \mathcal{F}_\lambda$. On the other hand, we show that $\Phi_u(f) \leq \int_{-\infty}^{+\infty} u(t)\beta(t, f)dt$, see Proposition 5.3.1. This proposition, which is of topological nature, constitutes the main technical result of this chapter.

Now, notice that taking $u \equiv 1$ in (5.8) we get the following corollary:

Corollary 5.1.8. *Let (M, g) be an orientable surface without boundary and let $f \in \mathcal{F}_\lambda$ be a Morse function on M . Denote by l_i the lengths of the finite bars of the barcode associated with f . Then*

$$\max f - \min f + \sum_i l_i \leq \kappa_g(\lambda + 1). \quad (5.9)$$

Example 5.1.9. The order of λ in inequality (5.9) is sharp. Indeed, consider the flat square torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi \cdot \mathbb{Z})^2$. We have a sequence $f_n(x, y) = \frac{1}{\pi} \sin(nx) \cos(ny)$, $n \in \mathbb{N}$ of eigenfunctions of Δ with eigenvalues $2n^2$. By analysing critical points of $f_1 = \sin x \cos x$ and using periodicity, one can compute that the full barcode of f_n contains

- An infinite bar $(-\frac{1}{\pi}, +\infty)$ and $2n^2 - 1$ copies of finite bar $(-\frac{1}{\pi}, 0]$ in degree 0;
- Two copies of infinite bar $(0, +\infty)$ and $2n^2 - 1$ copies of finite bar $(0, \frac{1}{\pi}]$ in degree 1;
- An infinite bar $(\frac{1}{\pi}, +\infty)$ in degree 2.

Putting these values in inequality (5.9) gives us

$$\frac{4}{\pi}n^2 \leq \kappa_g(2n^2 + 1),$$

which proves that the order of λ in (5.9) is sharp.

In order to present another application of Theorem 5.1.7 we need the following definition.

Definition 5.1.10. Let $f : M \rightarrow \mathbb{R}$ be a Morse function on a differentiable manifold M and let $\delta > 0$. We say that a critical value $\alpha \in \mathbb{R}$ of the function f is a δ -significant critical value of multiplicity m if the barcode of f contains m bars of length at least δ having α as one of the endpoints.

Given $\delta > 0$ and a Morse function f , let $\mathcal{N}_\delta(f)$ be the number of δ -significant critical values counted with multiplicities. Theorem 5.1.7 then immediately implies:

Corollary 5.1.11 (Corollary 1.4.4). *Let (M, g) be an orientable surface, possibly with boundary, and let $f \in \mathcal{F}_\lambda$ be a Morse function on M . Then*

$$\mathcal{N}_\delta(f) \leq \kappa_{g,\delta}(\lambda + 1) \tag{5.10}$$

for any $\delta > 0$.

The following example shows that the δ -significance condition for some $\delta > 0$ is essential in Corollary 5.1.11. For simplicity, we present it in one dimension, but it could be easily generalized to any dimension.

Example 5.1.12. Let $M = \mathbb{S}^1$ be a unit circle and let N_i be any sequence of natural numbers tending to infinity. Consider a sequence of functions on M :

$$f_i(x) = \frac{1}{\sqrt{\pi(2 + N_i^{-4})}} \left(1 + \frac{1}{N_i^2} \sin(N_i x) \right).$$

It is easy to check that $\|f_i\|_{L^2(M)} = 1$ and $f_i \in \mathcal{F}_\lambda$, $\lambda \leq \frac{1}{\sqrt{2}}$, for all $i = 1, 2, \dots$. At the same time, the number of critical points, and hence of critical values (counted with multiplicities) is equal to N_i , which goes to infinity and hence can not be controlled by λ . Note, however, that for any $\delta > 0$, the number of δ -significant critical values is bounded as $i \rightarrow \infty$.

Estimate (5.10) could be also compared to [100, Theorem 1.1], which shows that the expected value of the number of critical points of a random linear combination of Laplace eigenfunctions f_1, \dots, f_m on a Riemannian manifold satisfies an asymptotic expansion with the leading term of order m . Due to Weyl's law, for surfaces this is equivalent to having the number of critical points of order λ_m , which agrees with inequality (5.10). Inspired in part by this observation, we propose the following generalization of (5.10) to Riemannian manifolds of arbitrary dimension:

Conjecture 5.1.13. *Let (M, g) be a Riemannian manifold of dimension n , possibly with boundary, and let f be a L^2 -normalized linear combination of eigenfunctions of Δ with eigenvalues $\lambda_i \leq \lambda$. In addition, assume that f is Morse. Then*

$$\mathcal{N}_\delta(f) \leq \kappa_{g,\delta}(\lambda + 1)^{\frac{n}{2}} \tag{5.11}$$

for any $\delta > 0$.

Furthermore, for n -dimensional Riemannian manifolds, consider the following generalization of the functional Φ_u : it is defined for Morse functions by an analogue of (5.1), the sum being taken over all finite bars in $\mathcal{B}(f)$ in all degrees. Similarly to (5.7) it also could be extended to arbitrary functions in $C^0(M)$.

Conjecture 5.1.14. *Let $u \in C(\mathbb{R})$ be a non-negative function and f a L^2 -normalized linear combination of eigenfunctions of Δ with eigenvalues $\lambda_i \leq \lambda$ on a Riemannian manifold (M, g) . Then there exists a constant $\kappa_g > 0$ such that for any $\lambda > 0$,*

$$\Phi_u(f) \leq \kappa_g(\lambda + 1)^{\frac{n}{2}} \|u \circ f\|. \quad (5.12)$$

A possible approach to proving this conjecture is discussed in Remark 5.3.5.

Example 5.1.15. In order to provide intuition about Conjecture 5.1.14, let us examine what happens in dimension one (cf. [39, p. 137]). In this case, the notions coming from the barcode, such as the number or the total length of finite bars, have transparent meanings. Assume that $(M, g) = (\mathbb{S}^1, g_0) = (\mathbb{R}/(2\pi \cdot \mathbb{Z}), g_0)$ is the circle with the metric inherited from the standard length on \mathbb{R} , and $f : \mathbb{S}^1 \rightarrow \mathbb{R}$ is a Morse function. Since f is Morse, all critical points of f are either local minima or local maxima and they are located on \mathbb{S}^1 in an alternating fashion. More precisely, if there are N local minima x_1, \dots, x_N , there are also N local maxima y_1, \dots, y_N , and we may label them so that they are cyclically ordered as follows:

$$x_1, y_1, x_2, y_2, \dots, x_N, y_N, x_1.$$

Taking $u \equiv 1$, we have that $\Phi_1(f) = \max f - \min f +$ the total length of finite bars. All the finite bars appear in degree 0, and thus by Remark 5.1.2 we have N finite bars whose left endpoints are $f(x_1), \dots, f(x_N)$ and whose right endpoints are $f(y_1), \dots, f(y_N)$. From here it follows that

$$\Phi_1(f) = \sum_{i=1}^N (f(y_i) - f(x_i)).$$

On the other hand, the total variation of f satisfies

$$\text{Var}(f) = 2 \sum_{i=1}^N (f(y_i) - f(x_i)) = 2\Phi_1(f).$$

Furthermore, using Hölder's inequality and partial integration we have

$$\text{Var}(f) = \int_0^{2\pi} |f'(t)| dt \leq \sqrt{2\pi} \left(\int_0^{2\pi} (f'(t))^2 dt \right)^{\frac{1}{2}} = \sqrt{2\pi} \left| \int_0^{2\pi} f''(t) f(t) dt \right|^{\frac{1}{2}}.$$

From Cauchy-Schwarz inequality it follows

$$\text{Var}(f) \leq \sqrt{2\pi} \|f\|^{\frac{1}{2}} \|f''\|^{\frac{1}{2}}.$$

Finally, if $f \in \mathcal{F}_\lambda$, we have $\|f\|^{\frac{1}{2}} = 1$ and $\|f''\|^{\frac{1}{2}} \leq \lambda^{\frac{1}{2}}$ which gives

$$\frac{1}{2}\text{Var}(f) = \Phi_1(f) \leq \sqrt{\frac{\pi}{2}}\lambda^{\frac{1}{2}}, \quad (5.13)$$

as claimed by Conjecture 5.1.14. In order to extend the result to a general (not necessarily Morse) $f \in \mathcal{F}_\lambda$, observe that for every $\epsilon > 0$ there exists a sequence of Morse functions $f_n \in \mathcal{F}_{\lambda+\epsilon}$, such that $d_{C^0}(f, f_n) \rightarrow 0$ when $n \rightarrow \infty$. For all $k, n \geq 1$ it holds

$$\Phi_{1,k}(f_n) \leq \Phi_1(f_n) \leq \sqrt{\frac{\pi}{2}}(\lambda + \epsilon)^{\frac{1}{2}}.$$

Taking limits for $k, n \rightarrow \infty$ as in (5.7) and using the fact that $\epsilon > 0$ is arbitrary, we obtain the inequality (5.13) for any $f \in \mathcal{F}_\lambda$.

Example 5.1.16. The following example shows that the order of λ predicted by Conjecture 5.1.14 is sharp. Let $\mathbb{T}^n = \mathbb{R}^n / (2\pi \cdot \mathbb{Z})^n$ be the n -dimensional torus equipped with a Euclidean metric $ds^2 = \sum dx_i^2$. Define a sequence of functions

$$f_l(x_1, \dots, x_n) = \frac{\sqrt{2}}{n(2\pi)^{\frac{n}{2}}}(\sin lx_1 + \dots + \sin lx_n), \quad l \in \mathbb{N}.$$

It is easy to check that $\|f_l\| = 1$ and $\Delta f_l = l^2 f_l$. Thus $f_l \in \mathcal{F}_\lambda$ for $\lambda = l^2$.

Proposition 5.1.17. *There exist constants A_n and B_n such that*

$$\Phi_1(f_l) = A_n \lambda^{\frac{n}{2}} + B_n.$$

The proof of Proposition 5.1.17 uses the Künneth formula for persistence modules proven in Section 3.2.2 (originally from [122]), see subsection 5.4.3 for details.

Finally, we wish to emphasise that Conjecture 5.1.14 does not hold for functions in \mathcal{F}_λ in dimensions greater than two. This is illustrated by the following example due to Lev Buhovsky [25].

Example 5.1.18 (Buhovsky's example). For each $n \geq 3$, we provide a sequence of functions $F_k : \mathbb{T}^n \rightarrow \mathbb{R}$ on n -dimensional flat torus $\mathbb{T}^n = \mathbb{R}^n / (2\pi \cdot \mathbb{Z})^n$ such that $\|F_k\|$ and $\|\Delta F_k\|$ are uniformly bounded away from zero and infinity for all k , while $\Phi_1(F_k)$ grows as k^{n-2} . Such sequence violates inequality (5.12).

We define F_k as periodic functions on the cube $[-\pi, \pi]^n$ as follows. Let $h : [-1, 1]^n \rightarrow [0, 1]$ be a bump function. Divide $[-1, 1]^n$ into k^n smaller cubes by dividing each interval $[-1, 1]$ into k equal parts. Now $h(kx)$ is a bump function supported in $[-\frac{1}{k}, \frac{1}{k}]^n$ and we define auxiliary functions f_k to be equal to a copy of $\frac{1}{k^2}h(kx)$ inside each small cube. Since supports of different copies of $\frac{1}{k^2}h(kx)$ are disjoint, L^2 -orthogonality implies

$$\|f_k\|^2 = k^n \left\| \frac{1}{k^2}h(kx) \right\|^2 = k^{-4}\|h\|^2,$$

as well as that $\|\Delta f_k\|$ is bounded uniformly in k .

Finally, let¹ $F_k = f_k + 1$. This way we obtain a sequence of functions with $\|F_k\|$ and $\|\Delta F_k\|$ bounded away from zero and infinity. At the same time for $t \in (1, 1 + \frac{1}{k^2})$ the topology of sublevel sets $F_k^{-1}((-\infty, t))$ does not change and each sublevel set is homeomorphic to \mathbb{T}^n with k^n holes. This generates $\sim k^n$ bars of length $\frac{1}{k^2}$ in degree $n - 1$ and hence $\Phi_1(F_k) \gg k^{n-2}$, which contradicts (5.12) when $n \geq 3$ because $\frac{F_k}{\|F_k\|} \in \mathcal{F}_\lambda$ with bounded λ , but $\Phi_1\left(\frac{F_k}{\|F_k\|}\right)$ grows as k^{n-2} . A slight modification of this example also yields a counterexample to (5.11) in dimensions $n \geq 5$.

Remark 5.1.19. An example similar to Example 5.1.18 has been discussed in [39, Section 5]. In this paper, L^p -versions of functional Φ_1 , where the sum is taken over p -th powers of the lengths of bars, were considered. The results yield an upper bound for these L^p -functionals in terms of the Lipschitz constant of a function. However, for these bounds to hold, it is essential that p is at least the dimension of the base manifold, which can be seen from Example 5.1.18. As a consequence, while the results of [39] imply some spectral restrictions on the barcodes of Laplace eigenfunctions, they appear to be essentially different from the bounds on Φ_1 obtained in Theorem 5.1.7 and conjectured in Conjecture 5.1.14.

5.1.4 Sorting the finite bars of functions in \mathcal{F}_λ

Given a barcode \mathcal{B} , write the lengths of its finite bars in the descending order, $\beta_1 \geq \beta_2 \geq \dots$. The functions $\beta_i(\mathcal{B})$, which are Lipschitz with respect to the bottleneck distance, are important invariants of barcodes. For instance, β_1 , which was introduced by Usher in [146], is called *the boundary depth* and has various applications in Morse theory and symplectic topology. The functions β_i with $i \geq 2$ are sometimes used in order to distinguish barcodes, see e.g. [23]. Fix $\epsilon > 0$ and discard all bars of length $< \epsilon$, i.e., introduce the modified invariant

$$\beta_i^{(\epsilon)}(\mathcal{B}) := \max(\beta_i(\mathcal{B}), \epsilon).$$

Question 5.1.20. *Assume that the barcode \mathcal{B} contains N finite bars. What is the optimal (worst-case scenario) running time T of an algorithm which calculates the ordered sequence $\{\beta_i^{(\epsilon)}(\mathcal{B})\}$, $i \geq 1$?*

Since the sharp lower bound on the running time of any comparison sorting algorithm for an array of N real numbers is $O(N \log N)$ (see [42]), the answer to the above question for a general barcode is $O(N \log N)$. Interestingly enough, in some cases Corollary 5.1.8 enables one to reduce this running time when \mathcal{B} is a barcode of a function from \mathcal{F}_λ . More precisely, there exists a constant $c > 0$ such that for every $\epsilon > 0$, $\lambda > 0$ and any function $f \in \mathcal{F}_\lambda$ whose barcode contains exactly N finite bars,

¹Formally speaking, F_k should be a small perturbation of $f_k + 1$ in order to make it Morse, but we will ignore this detail for the sake of clarity.

one can find a sorting algorithm for all bars from $\mathcal{B}(f)$ of length $\geq \epsilon$ whose running time satisfies

$$T \leq N + c \cdot \frac{\kappa_g(\lambda + 1)}{\epsilon} \cdot \log \frac{\kappa_g(\lambda + 1)}{\epsilon} . \quad (5.14)$$

Indeed, consider the following algorithm. First compare the length of each bar with ϵ and pick only those bars whose length is $\geq \epsilon$. This takes time N . Denote by K the number of chosen bars. Next, perform the optimal sorting algorithm for these K bars. This takes time $O(K \log K)$. Finally, notice that by Corollary 5.1.8,

$$K \leq \frac{\kappa_g(\lambda + 1)}{\epsilon} ,$$

which proves (5.14). In certain regimes, the running time (5.14) is shorter than the generic bound $O(N \log N)$. For instance, if λ and ϵ are fixed and $N \rightarrow \infty$, we have $T = O(N)$.

Theorem 5.1.7 also has applications to questions regarding C^0 -approximations by functions from \mathcal{F}_λ , which is the subject of the next section.

5.2 Applications to approximations by eigenfunctions

5.2.1 An obstruction to C^0 -approximations

As before, let M be an orientable surface, possibly with boundary, endowed with the Riemannian metric g , denote by $\text{Diff}(M)$ the set of all diffeomorphisms of M and assume that $f : M \rightarrow \mathbb{R}$ is a Morse function (vanishing on ∂M). We are interested in the question of how well can f be approximated by functions from \mathcal{F}_λ in C^0 -sense. More precisely, we wish to find a lower bound for the quantity

$$d_{C^0}(f, \mathcal{F}_\lambda) = \inf \{ d_{C^0}(f, h) \mid h \in \mathcal{F}_\lambda \},$$

where $d_{C^0}(f, h) = \max_x |f(x) - h(x)|$ as before. In fact, we will study a more general question, namely we will give a lower bound for

$$\text{approx}_\lambda(f) = \inf_{\varphi \in \text{Diff}(M)} d_{C^0}(f \circ \varphi, \mathcal{F}_\lambda).$$

Taking $\varphi = \mathbb{1}_M$ one immediately sees that

$$d_{C^0}(f, \mathcal{F}_\lambda) \geq \text{approx}_\lambda(f).$$

We estimate $\text{approx}_\lambda(f)$ from below using the information coming from the barcode $\mathcal{B}(f)$. Recall that the functional $\Phi_1 : \mathcal{F}_{\text{Morse}} \rightarrow \mathbb{R}$ defined by (5.1) for $u \equiv 1$ gives the sum of the lengths of all finite bars in $\mathcal{B}(f)$.

Proposition 5.2.1 (Proposition 1.4.5). *For every Morse function $f : M \rightarrow \mathbb{R}$, vanishing on the boundary, the following inequality holds*

$$\text{approx}_\lambda(f) \geq \begin{cases} \frac{1}{2 \cdot (|\mathcal{B}'(f)| + 1)} \left(\Phi_1(f) - \kappa_g(\lambda + 1) \right) & \text{for } \partial M = \emptyset \\ \frac{1}{2|\mathcal{B}'(f)| + 1} \left(\Phi_1(f) - \kappa_g(\lambda + 1) \right) & \text{for } \partial M \neq \emptyset \end{cases} \quad (5.15)$$

Proof. From (5.5) and (5.8), with $\psi = \mathbb{1}_M$ we obtain

$$\kappa_g(\lambda + 1) \cdot \|u \circ h\| \geq \Phi_u(f) - C(u, f) \cdot d_{C^0}(f \circ \varphi, h),$$

for all Morse $h \in \mathcal{F}_\lambda$ and all diffeomorphisms $\varphi \in \text{Diff}(M)$, with constant $C(u, f)$ being equal to $2 \cdot (|\mathcal{B}'(f)| + 1) \cdot (\max_{[\min f, \max f]} u)$ or $(2|\mathcal{B}'(f)| + 1) \cdot (\max_{[\min f, \max f]} u)$ depending on whether M has a boundary. Putting $u \equiv 1$ we have

$$d_{C^0}(f \circ \varphi, h) \geq \frac{1}{2 \cdot (|\mathcal{B}'(f)| + 1)} \left(\Phi_1(f) - \kappa_g(\lambda + 1) \right),$$

if $\partial M = \emptyset$, or

$$d_{C^0}(f \circ \varphi, h) \geq \frac{1}{2|\mathcal{B}'(f)| + 1} \left(\Phi_1(f) - \kappa_g(\lambda + 1) \right),$$

if $\partial M \neq \emptyset$. Finally, taking infimum over all h and φ and using the fact that Morse functions in \mathcal{F}_λ are C^0 -dense in \mathcal{F}_λ , finishes the proof. \square

Remark 5.2.2. The inequality analogous to (5.15) can be proved for functions on the circle $\mathbb{S}^1 = \mathbb{R}/(2\pi \cdot \mathbb{Z})$ without referring to the language of barcodes. Taking into account (5.3) and (5.13), we can restate (5.15) as

$$\text{approx}_\lambda(f) \geq \frac{1}{|\text{Crit}(f)|} \left(\frac{1}{2} \text{Var}(f) - \sqrt{\frac{\pi}{2}} \lambda^{\frac{1}{2}} \right). \quad (5.16)$$

In order to prove (5.16) we proceed as in the proof of Proposition 5.2.1. One readily checks that

$$\frac{1}{2} \text{Var}(f) - \frac{1}{2} \text{Var}(h) \leq |\text{Crit}(f)| d_{C^0}(f, h). \quad (5.17)$$

Indeed, as in Example 5.1.15, if x_1, \dots, x_N are local minima and y_1, \dots, y_N are local maxima of f , we have that

$$\frac{1}{2} \text{Var}(f) = \sum_{i=1}^N (f(y_i) - f(x_i)).$$

On the other hand,

$$\frac{1}{2} \text{Var}(h) \geq \sum_{i=1}^N (h(y_i) - h(x_i)).$$

Subtracting the latter expression from the former yields (5.17), which together with (5.13) gives

$$\frac{1}{|\text{Crit}(f)|} \left(\frac{1}{2} \text{Var}(f) - \sqrt{\frac{\pi}{2}} \lambda^{\frac{1}{2}} \right) \leq d_{C^0}(f, \mathcal{F}_\lambda).$$

Since $|\text{Crit}(f)|$ and $\text{Var}(f)$ do not change when f is composed with a diffeomorphism, (5.16) follows.

Remark 5.2.3. The following analogue of Proposition 5.2.1 holds for any function $f \in C^0(M)$. For any $k = 1, 2, \dots$, we have

$$\text{approx}_\lambda(f) \geq \begin{cases} \frac{1}{2k+2} \left(\Phi_{1,k}(f) - \kappa_g(\lambda + 1) \right) & \text{for } \partial M = \emptyset \\ \frac{1}{2k+1} \left(\Phi_{1,k}(f) - \kappa_g(\lambda + 1) \right) & \text{for } \partial M \neq \emptyset \end{cases} \quad (5.18)$$

The proof is the same, with the constant $C(u, f)$ replaced by the Lipschitz constant from Lemma 5.1.6.

Corollary 5.2.4. *Let M be a surface without boundary and $f : M \rightarrow \mathbb{R}$ be a Morse function. Suppose that $\text{approx}_\lambda(f) \leq \epsilon$ for some $\epsilon > 0$, and the barcode $\mathcal{B}(f)$ contains N finite bars of length at least $L + 2\epsilon$ each, for some $L > 0$. Then*

$$\lambda \geq \frac{1}{\kappa_g} (N + 1)L - 1. \quad (5.19)$$

Proof. Indeed, it follows from the assumptions on the barcode of f that $\Phi_{1,N}(f) \geq (N + 1)(L + 2\epsilon)$, which together with Remark 5.2.3 yields

$$\epsilon \geq \frac{1}{2(N + 1)} \left((N + 1)(L + 2\epsilon) - \kappa_g(\lambda + 1) \right).$$

Rearranging this inequality we obtain (5.19). □

Remark 5.2.5. From (5.19) we see how λ , which is needed to uniformly approximate f by functions from \mathcal{F}_λ , grows with N and L . Informally speaking, one may think of N as a measure of how much f oscillates, while L gives a lower bound on the amplitude of these oscillations. The above inequality should then be understood as a quantitative version of the informal principle that the more the function oscillates and the bigger the oscillations, the larger eigenvalues of the Laplacian are needed to approximate it with a normalized linear combination of the corresponding eigenfunctions. We refer to [160] for other applications of persistence to approximation theory.

5.2.2 Modulus of continuity and average length of bars on \mathbb{T}^2

Proposition 5.2.1 gives an obstruction to approximating functions by functions from \mathcal{F}_λ . As we mentioned before, \mathcal{F}_λ contains normalized linear combinations of eigenfunctions

of Δ with eigenvalues not greater than λ . In the case of flat torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi \cdot \mathbb{Z})^2$ these eigenfunctions are trigonometric polynomials and Proposition 5.2.1 may be interpreted as an inverse statement about C^0 -approximations by trigonometric polynomials. A direct theorem about C^0 -approximations by trigonometric polynomials on n -dimensional flat torus was proved in [163] (theorems of this type are sometimes referred to as Jackson's theorems, see [110] for a survey), consequently giving an upper bound for $approx_\lambda(f)$ in terms of moduli of continuity and smoothness of f . We combine this result with Proposition 5.2.1 to obtain a relation between the average length of a bar in a barcode of a Morse function on \mathbb{T}^2 and its modulus of continuity which is defined below.

Assume $M = \mathbb{T}^2 = \mathbb{R}^2/(2\pi \cdot \mathbb{Z})^2$, let $f : M \rightarrow \mathbb{R}$ be a continuous function, $\delta > 0$ a real number and denote by

$$\omega_1(f, \delta) = \sup_{|t| \leq \delta} \max_x |f(x+t) - f(x)|,$$

the *modulus of continuity* of f and by

$$\omega_2(f, \delta) = \sup_{|t| \leq \delta} \max_x |f(x-t) - 2f(x) + f(x+t)|,$$

the *modulus of smoothness* of f . One readily checks that

$$\omega_2(f, \delta) \leq 2\omega_1(f, \delta). \tag{5.20}$$

Let

$$\mathcal{T}_\lambda = \left\langle \sin(v_1x + v_2y), \cos(v_1x + v_2y) \mid v_1^2 + v_2^2 \leq \lambda \right\rangle_{\mathbb{R}},$$

be the space of trigonometric polynomials on M whose eigenvalues (as eigenfunctions of Δ) are bounded by λ . The following proposition was proved in [163]:

Proposition 5.2.6. *For every continuous function $f : M \rightarrow \mathbb{R}$ it holds*

$$d_{C^0}(f, \mathcal{T}_\lambda) \leq 2\omega_2\left(f, \frac{C_0}{\sqrt{\lambda}}\right), \tag{5.21}$$

where $C_0 > 0$ is a constant.

By (5.20) we also have that

$$d_{C^0}(f, \mathcal{T}_\lambda) \leq 4\omega_1\left(f, \frac{C_0}{\sqrt{\lambda}}\right). \tag{5.22}$$

Remark 5.2.7. Constant C_0 is computed in [163] to be $C_0 = \sqrt{\mu_1(D^2(\frac{1}{2}))}$, where $\mu_1(D^2(\frac{1}{2}))$ is the first Dirichlet eigenvalue of Δ inside the 2-dimensional disk $D^2(\frac{1}{2})$ of radius $\frac{1}{2}$.

Our goal is to prove the following result which shows that the average bar length of a Morse function f on a flat torus M could be *uniformly* controlled by the L^2 -norm of f and the modulus of continuity of f on the scale $1/\sqrt{|\mathcal{B}'(f)|}$.

Theorem 5.2.8. *There exist constants $C_0, C_1 > 0$ such that for any $f \in \mathcal{F}_{Morse}$ on a flat torus $M = \mathbb{T}^2$,*

$$\frac{\Phi_1(f)}{|\mathcal{B}'(f)| + 1} \leq C_1 \|f\| + 8\omega_1 \left(f, \frac{C_0}{\sqrt{|\mathcal{B}'(f)|}} \right). \quad (5.23)$$

Proof. Inspecting the proof of Proposition 5.2.6 in [163], we observe that it relies on an explicit construction of a function h , depending on f and λ , which satisfies

$$d_{C^0}(f, h) \leq 2\omega_2 \left(f, \frac{C_0}{\sqrt{\lambda}} \right). \quad (5.24)$$

Our goal is to estimate $d_{C^0}(f, h)$ from below using Proposition 5.2.1. However, a priori we do not have any information about the L^2 -norm of h and Proposition 5.2.1 relates to distance from functions in \mathcal{F}_λ whose L^2 -norm is equal to one. In order to overcome this issue, we present the construction of the approximation-function h and prove that $\|h\| \leq \|f\|$.

For a vector $v \in \mathbb{Z}^2$ let

$$c_v(f) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x) e^{-i\langle v, x \rangle} dx,$$

be the corresponding Fourier coefficient of f . Take U to be the first Dirichlet eigenfunction of Δ inside the disk $D^2(\frac{1}{2})$ of radius $\frac{1}{2}$, normalized by $\|U\|_{L^2(D^2(\frac{1}{2}))} = 1$, and V its extension by zero to the whole plane, i.e.

$$V(x) = \begin{cases} U(x), & \text{if } x \in D^2(\frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$

If we denote by $W = V * V$ the convolution of V with itself, the desired approximation is given by the formula

$$h(x) = \sum_{\substack{v \in \mathbb{Z}^2 \\ |v| \leq \sqrt{\lambda}}} c_v(f) \cdot W\left(\frac{v}{\sqrt{\lambda}}\right) e^{i\langle v, x \rangle}, \quad (5.25)$$

where $|v|$ stands for the standard Euclidean norm on \mathbb{R}^2 . The function h defined by (5.25) is called the *multidimensional Korovkin's mean*. It satisfies (5.24), as proved in [163], and since $\sum_{v \in \mathbb{Z}^2} c_v(f) e^{i\langle v, x \rangle}$ is the Fourier expansion of f , we have that

$$\|h\| \leq \left(\max_{x \in \mathbb{R}^2} |W(x)| \right) \cdot \|f\|.$$

By using the Cauchy-Schwarz inequality we obtain

$$|W(x)| \leq \int_{\mathbb{R}^2} |V(t)| \cdot |V(x-t)| dt \leq \sqrt{\int_{\mathbb{R}^2} |V(t)|^2 dt} \cdot \sqrt{\int_{\mathbb{R}^2} |V(x-t)|^2 dt} = \|V\|^2 = 1,$$

which yields $\|h\| \leq \|f\|$.

We now proceed with analysing (5.24). First note that

$$\|h\| \cdot d_{C^0} \left(\frac{f}{\|h\|}, \mathcal{F}_\lambda \right) \leq \|h\| \cdot d_{C^0} \left(\frac{f}{\|h\|}, \frac{h}{\|h\|} \right) \leq 2\omega_2 \left(f, \frac{C_0}{\sqrt{\lambda}} \right),$$

because $\frac{h}{\|h\|} \in \mathcal{F}_\lambda$. The last inequality together with Proposition 5.2.1 gives

$$\frac{\|h\|}{2 \cdot (|\mathcal{B}'(\frac{f}{\|h\|})| + 1)} \left(\Phi_1 \left(\frac{f}{\|h\|} \right) - \kappa_0(\lambda + 1) \right) \leq 2\omega_2 \left(f, \frac{C_0}{\sqrt{\lambda}} \right). \quad (5.26)$$

Here $\kappa_0 = \kappa_g$ for g being the flat metric on M . Multiplying the function by a positive constant results in multiplying the endpoints of each bar in the barcode by the same constant. Thus, the total number of bars does not change after multiplication, while the lengths of finite bars scale with the same constant. In other words, we have that $\left| \mathcal{B}' \left(\frac{f}{\|h\|} \right) \right| = |\mathcal{B}'(f)|$ and $\|h\| \cdot \Phi_1 \left(\frac{f}{\|h\|} \right) = \Phi_1(f)$. Substituting these equalities in (5.26) and using $\|h\| \leq \|f\|$ we obtain

$$\frac{1}{4 \cdot (|\mathcal{B}'(f)| + 1)} \left(\Phi_1(f) - \kappa_0(\lambda + 1)\|f\| \right) \leq \omega_2 \left(f, \frac{C_0}{\sqrt{\lambda}} \right), \quad (5.27)$$

and by (5.20) also

$$\frac{1}{8 \cdot (|\mathcal{B}'(f)| + 1)} \left(\Phi_1(f) - \kappa_0(\lambda + 1)\|f\| \right) \leq \omega_1 \left(f, \frac{C_0}{\sqrt{\lambda}} \right).$$

Setting $\lambda = |\mathcal{B}'(f)|$ and $C_1 = \kappa_0$ in the last inequality completes the proof of the theorem. \square

Remark 5.2.9. As follows from Remark 5.2.3 and the proof of Theorem 5.2.8 above, for any $k \geq 1$ and any $f \in C^0(M)$ we have:

$$\frac{\Phi_{1,k}(f)}{k+1} \leq C_1\|f\| + 8\omega_1 \left(f, \frac{C_0}{\sqrt{k}} \right). \quad (5.28)$$

The left-hand side of (5.28) could be interpreted as the average length of a bar among the k longest bars in the barcode of f .

Remark 5.2.10. Note that formula (5.27) implies

$$\frac{\Phi_1(f)}{|\mathcal{B}'(f)| + 1} \leq C_1\|f\| + 4\omega_2 \left(f, \frac{C_0}{\sqrt{|\mathcal{B}'(f)|}} \right). \quad (5.29)$$

In fact, Theorem 5.2.8 admits the following generalization. Given a smooth function f on a flat torus $M = \mathbb{T}^2$, define its *modulus of smoothness of order m* by

$$\omega_m(f, \delta) = \sup_{|t| \leq \delta} \max_x \left| \sum_{j=0}^m (-1)^{(m-j)} \binom{m}{j} f(x + jt) \right|.$$

From the results of [67], it can be deduced that

$$d_{C^0}(f, \mathcal{T}_\lambda) \leq C_2(k) \omega_{2k} \left(f, \frac{C_0(k)}{\sqrt{\lambda}} \right)$$

for some positive constants $C_0(k), C_2(k)$ which depend on k . Similarly to the proof of Theorem 5.2.8, one then obtains

$$\frac{\Phi_1(f)}{|\mathcal{B}'(f)| + 1} \leq C_1(k) \|f\| + 2C_2(k) \omega_{2k} \left(f, \frac{C_0(k)}{\sqrt{|\mathcal{B}'(f)|}} \right).$$

Constants $C_0(k), C_1(k), C_2(k)$ could be computed explicitly.

Example 5.2.11. The following example shows that the choice of the scale in the modulus of continuity on the right-hand side of (5.28) is optimal. Take a unit disk B_1 inside the torus M and let χ be a smooth cut-off function supported in B and equal to one in $B_{\frac{1}{2}}$. Let $g_n(x, y) = \chi(x, y) \sin nx \cos ny$, $n \in \mathbb{N}$ be a sequence of functions on the torus. For any $0 < s < 1$, set $g_{n,s}(x, y) = g(\frac{x}{s}, \frac{y}{s})$. Let $\alpha \geq 1$ be some real number. Choose $k = n^2$ and $s = n^{\frac{1-\alpha}{2}}$. It suffices to verify that the inequality

$$\frac{\Phi_{1,n^2}(g_{n,s})}{n^2 + 1} \leq C_1 \|g_{n,s}\| + 8\omega_1 \left(g_{n,s}, \frac{C_0}{n^\alpha} \right). \quad (5.30)$$

holds for all n only for $\alpha = 1$. Indeed, take any $\alpha > 1$. Note that the left-hand side of (5.30) is bounded away from zero as $n \rightarrow \infty$, since the number of bars of unit length in the barcode of $g_{n,s}((x, y))$ is of order n^2 . At the same time, $\|g_{n,s}\| = s \|g_n\| \rightarrow 0$ as $n \rightarrow \infty$, since $s = n^{\frac{1-\alpha}{2}} \rightarrow 0$. Moreover, estimating the derivatives of $g_{n,s}$ one can easily check that

$$\omega_1 \left(g_{n,s}, \frac{C_0}{n^\alpha} \right) = O(n^{-\alpha}) O(n \cdot n^{\frac{\alpha-1}{2}}) = O \left(n^{\frac{1-\alpha}{2}} \right) = o(1)$$

for any $\alpha > 1$. Therefore, inequality (5.30) is violated for $\alpha > 1$ for n large, and hence the choice $\alpha = 1$ is optimal.

Note that, while the functions $g_{n,s}(x, y)$ are compactly supported and hence not Morse, they could be made Morse by adding a small perturbation. A similar argument would then yield optimality of the $1/\sqrt{|\mathcal{B}'(f)|}$ scale in the modulus of continuity on the right-hand side of inequality (5.23).

Remark 5.2.12. It would be interesting to generalize Theorem 5.2.8 to an arbitrary Riemannian surface. In order to do that we need an analogue of Proposition 5.2.6. For a different version of Jackson type approximation theorems on Riemannian manifolds see [109, Lemma 4.1] and [54, Lemma 9.1].

5.3 Barcodes and the Banach indicatrix

5.3.1 A topological bound on the Banach indicatrix

We proceed with some general topological considerations. Let M be a Riemannian manifold and assume that $f|_{\partial M} = 0$ if $\partial M \neq \emptyset$, 0 being a regular value. Let $t \neq 0$ be another regular value of f and denote by $M^t = f^{-1}((-\infty, t])$. $M^t \subset M$ is a submanifold with boundary $\partial M^t = f^{-1}(t)$ for $t < 0$ or $\partial M^t = f^{-1}(t) \sqcup \partial M$ for $t > 0$. Recall that Banach indicatrix $\beta(t, f)$ denotes the number of connected components of $f^{-1}(t)$. By description of ∂M^t , one sees that $\beta(t, f)$ essentially counts the number of the boundary components of M^t (up to the boundary components of the whole manifold M). We will exploit this fact to estimate $\beta(t, f)$ from below using information coming from barcode $\mathcal{B}(f)$. If we denote by χ_I the characteristic function of the interval I , the following proposition holds.

Proposition 5.3.1. *Let $f \in \mathcal{F}_{Morse}$ on a Riemannian manifold M of dimension n . Denote by $(x_i^{(k)}, y_i^{(k)}) \in \mathcal{B}_k(f)$ the finite bars in the degree k barcode of f and let $t \neq 0$ be a regular value. If $\partial M = \emptyset$ it holds*

$$\chi_{(\min f, \max f]}(t) + \sum_i \chi_{(x_i^{(0)}, y_i^{(0)})]}(t) + \sum_j \chi_{(x_j^{(n-1)}, y_j^{(n-1)})]}(t) \leq \beta(t, f), \quad (5.31)$$

and if $\partial M \neq \emptyset$ it holds

$$\chi_{(\min f, 0]}(t) + \sum_i \chi_{(x_i^{(0)}, y_i^{(0)})]}(t) + \sum_j \chi_{(x_j^{(n-1)}, y_j^{(n-1)})]}(t) \leq \beta(t, f). \quad (5.32)$$

We defer proving Proposition 5.3.1 and first deduce Theorem 5.1.7 using it.

Remark 5.3.2. One may easily check that if $M = S^2$ the inequality (5.31) becomes an equality.

Remark 5.3.3. If $\dim M = 2$, integrating inequalities in Proposition 5.3.1 gives an upper bound on the total length of the finite bars in the barcode of a function f in terms of the integral of its Banach indicatrix. The latter quantity admits an interpretation as the total length of the Reeb graph of a function f with respect to a natural metric incorporating the oscillations of f . It is likely that an analogue of the functional Φ_u defined in this setting is robust with respect to the distance on Reeb graphs introduced in [18, 21], and that this way one could get applications to approximation theory similar to the ones obtained in Section 5.2. We plan to explore this route elsewhere.

5.3.2 Proof of Theorem 5.1.7

Let us now restrict to the two-dimensional case and assume that M is an orientable surface, possibly with boundary, equipped with Riemannian metric g . First note

that it suffices to verify inequality (5.8) for Morse functions. Indeed, suppose that the inequality is proved for Morse functions in \mathcal{F}_λ for all $\lambda > 0$ and let $f \in \mathcal{F}_\lambda$ be arbitrary. One can easily check that for any $\epsilon > 0$ there exists a sequence of Morse functions $f_n \in \mathcal{F}_{\lambda+\epsilon}$ such that $d_{C^0}(f, f_n) \rightarrow 0$ as $n \rightarrow \infty$. For all $k, n \geq 1$ we have

$$\Phi_{u,k}(f_n) \leq \Phi_u(f_n) \leq \kappa_g(\lambda + 1 + \epsilon) \|u \circ f_n\|,$$

where the first inequality follows from the definition (5.6) of the functional $\Phi_{u,k}$ and the second inequality holds by the assumption that (5.8) is true for Morse functions. Taking the limits as k and n go to infinity in definition (5.7) and using the fact that $\epsilon > 0$ is arbitrary, we obtain that (5.8) holds for the function f .

It remains to prove inequality (5.8) when $f \in \mathcal{F}_\lambda$ is Morse. Denote by $\|\cdot\|$ the L^2 -norm with respect to Riemannian area σ and by Δ the Laplace-Beltrami operator with respect to g . The analytical tool that we are going to use is [123, Theorem 1.5] which gives us that for any continuous function $u \in C(\mathbb{R})$ and any smooth function f on M (which is assumed to be equal to zero on the boundary if $\partial M \neq \emptyset$), the following inequality holds

$$\int_{-\infty}^{+\infty} u(t)\beta(t, f) dt \leq \kappa_g(\|f\| + \|\Delta f\|) \cdot \|u \circ f\|, \quad (5.33)$$

where κ_g depends on the Riemannian metric g .

If we assume that $f \in \mathcal{F}_\lambda$ in (5.33), we immediately get

$$\int_{-\infty}^{+\infty} u(t)\beta(t, f) dt \leq \kappa_g(\lambda + 1) \cdot \|u \circ f\|. \quad (5.34)$$

Since f is Morse we can apply Proposition 5.3.1. Combining this proposition with inequality (5.34) immediately yields Theorem 5.1.7. \square

Remark 5.3.4. It follows from the proof of Theorem 5.1.7 that inequality (5.8) holds for any function in the closure of \mathcal{F}_λ in C^0 -topology.

Remark 5.3.5. The proof of Theorem 5.1.7 suggests the following approach to proving Conjecture 5.1.14. Recall that, by definition, the Banach indicatrix is given by $\beta(t, f) = b_0(f^{-1}(t))$. In view of Proposition 5.3.1, it is plausible that the following inequality holds in dimension $n \geq 3$:

$$\Phi_u(f) \leq \int_{-\infty}^{+\infty} \left(\sum_{i=0}^{n-2} b_i(f^{-1}(t)) \right) u(t) dt,$$

where b_i is the i -th Betti number. As follows from [162], the quantity on the right-hand side could be bounded from above using the uniform norm of the derivatives of f (see also [91] for related recent developments). In order to establish Conjecture 5.1.14, one would need to prove a higher-dimensional analogue of [123, Theorem 1.5], allowing to replace the uniform estimates by L^2 -bounds.

5.3.3 Proof of Proposition 5.3.1

Recall the notation previously introduced. For t regular value of f , we denote $M^t = f^{-1}((-\infty, t]) \subset M$. As mentioned before M^t is a submanifold with boundary $\partial M^t = f^{-1}(t)$ for $t < 0$ or $\partial M^t = f^{-1}(t) \sqcup \partial M$ for $t > 0$. Let $b_k(t) = \dim H_k(M^t; \mathbb{R})$ be the Betti numbers of M^t . We assume that $\min f < t < \max f$, since otherwise the inequalities obviously hold (both sides are equal to zero). We will always work with homologies with coefficients in \mathbb{R} and will omit the coefficients from the notation.

Let $j : f^{-1}(t) \rightarrow M^t$ be the inclusion and denote by $j_* : H_{n-1}(f^{-1}(t)) \rightarrow H_{n-1}(M^t)$ the induced map in homology. Since we work over \mathbb{R} , $j_*(H_{n-1}(f^{-1}(t))) \subset H_{n-1}(M^t)$ is a vector subspace. First, we claim that

$$\beta(t, f) = \dim H_n(M^t, f^{-1}(t)) + \dim(j_*(H_{n-1}(f^{-1}(t)))). \quad (5.35)$$

To prove this, we examine the following part of the long exact sequence of the pair $(M^t, f^{-1}(t))$:

$$H_n(M^t) \rightarrow H_n(M^t, f^{-1}(t)) \xrightarrow{\partial} H_{n-1}(f^{-1}(t)) \xrightarrow{j_*} H_{n-1}(M^t).$$

Since $f^{-1}(t)$ is an $(n-1)$ -dimensional orientable manifold, we have $\beta(t, f) = \dim H_{n-1}(f^{-1}(t))$, and by the Rank-nullity theorem

$$\beta(t, f) = \dim H_{n-1}(f^{-1}(t)) = \dim(\ker j_*) + \dim(j_*(H_{n-1}(f^{-1}(t)))).$$

By the exactness $\dim(\ker j_*) = \dim(\operatorname{im} \partial)$ and, because $H_n(M^t) = 0$, ∂ is an inclusion, which means that $\dim(\operatorname{im} \partial) = \dim H_n(M^t, f^{-1}(t))$ and (5.35) follows.

Second, we note that

$$\sum_j \chi_{(x_j^{(n-1)}, y_j^{(n-1)})}(t) = \dim(\ker i_*), \quad (5.36)$$

where $i : M^t \rightarrow M$ is the inclusion and $i_* : H_{n-1}(M^t) \rightarrow H_{n-1}(M)$ induced map on homology. This comes from the fact that finite bars in barcode of f correspond to homology classes which appear throughout filtration process, but do not exist in actual homology of M . Denote by $\widetilde{M}^t = f^{-1}([t, +\infty))$, $\widetilde{M}^t \cap M^t = f^{-1}(t)$, and by $\tilde{j} : f^{-1}(t) \rightarrow \widetilde{M}^t$ and $\tilde{i} : \widetilde{M}^t \rightarrow M$ the inclusions. We examine the following part of the Mayer-Vietoris sequence

$$H_{n-1}(f^{-1}(t)) \xrightarrow{(j_*, \tilde{j}_*)} H_{n-1}(M^t) \oplus H_{n-1}(\widetilde{M}^t) \xrightarrow{i_* - \tilde{i}_*} H_{n-1}(M).$$

From the exactness we have

$$\ker i_* \cong \ker(i_* - \tilde{i}_*) \cap (H_{n-1}(M^t), 0) = \operatorname{im}(j_*, \tilde{j}_*) \cap (H_{n-1}(M^t), 0),$$

while on the other hand

$$\begin{aligned} \text{im}(j_*, \tilde{j}_*) \cap (H_{n-1}(M^t), 0) &= \{(j_*(a), \tilde{j}_*(a)) | a \in H_{n-1}(f^{-1}(t)), \tilde{j}_*(a) = 0\} = \\ &= \{(j_*(a), 0) | a \in \ker \tilde{j}_* \subset H_{n-1}(f^{-1}(t))\} \cong j_*(\ker \tilde{j}_*), \end{aligned}$$

and thus

$$\ker i_* \cong j_*(\ker \tilde{j}_*).$$

However, since $\tilde{i}_* \circ \tilde{j}_* = i_* \circ j_*$, we see that $j_*(\ker \tilde{j}_*) \subset \ker i_*$ and thus

$$\ker i_* = j_*(\ker \tilde{j}_*). \quad (5.37)$$

From now on, we distinguish two cases.

Case I - $\partial M^t = f^{-1}(t)$

This case covers the situation when $\partial M = \emptyset$ and when $\partial M \neq \emptyset$, but $t < 0$. The left-hand sides of the inequalities (5.31) and (5.32) are equal for $t < 0$ and thus we need to prove (5.31). By the case-assumption, we have that

$$\dim H_n(M^t, f^{-1}(t)) = b_0(t),$$

and from the definition of barcode we know that

$$b_0(t) = \chi_{(\min f, \max f]}(t) + \sum_i \chi_{(x_i^{(0)}, y_i^{(0)}]}(t).$$

Combining these equalities with (5.35) renders the statement into

$$\sum_j \chi_{(x_j^{(n-1)}, y_j^{(n-1)}]}(t) \leq \dim(j_*(H_{n-1}(f^{-1}(t)))),$$

which after substituting (5.36) and (5.37) becomes

$$\dim(j_*(\ker \tilde{j}_*)) \leq \dim(j_*(H_{n-1}(f^{-1}(t)))).$$

This inequality is obvious because $\ker \tilde{j}_* \subset H_{n-1}(f^{-1}(t))$.

Case II - $\partial M^t = f^{-1}(t) \sqcup \partial M$, $\partial M \neq \emptyset$

This case covers the situation when $\partial M \neq \emptyset$ and $t > 0$. We need to prove (5.32), which for $t > 0$ becomes

$$\sum_i \chi_{(x_i^{(0)}, y_i^{(0)}]}(t) + \sum_j \chi_{(x_j^{(n-1)}, y_j^{(n-1)}]}(t) \leq \beta(t, f).$$

Denote by $\partial M = \Sigma_1 \sqcup \dots \sqcup \Sigma_l$ the boundary components of the whole manifold M , where Σ_i are connected, orientable, $(n-1)$ -dimensional manifolds. Now the boundary

of M^t is $\partial M^t = f^{-1}(t) \sqcup \Sigma_1 \sqcup \dots \sqcup \Sigma_l$. We may divide connected components of M^t into two groups, one of which consists of all the components whose boundary lies entirely in $f^{-1}(t)$ and the other one consists of all the components whose boundary contains at least one Σ_i (i.e. the boundary of these components is a mix of parts of ∂M and $f^{-1}(t)$). Denote by k the number of connected components of M^t whose boundary contains at least one Σ_i . Now, since

$$\dim H_n(M^t, f^{-1}(t) \sqcup \partial M) = b_0(t),$$

we have that

$$\dim H_n(M^t, f^{-1}(t)) = b_0(t) - k,$$

and thus by (5.35)

$$\beta(t, f) = b_0(t) - 1 + \dim(j_*(H_{n-1}(f^{-1}(t)))) - (k - 1).$$

Since M is connected

$$b_0(t) - 1 = \sum_i \chi_{(x_i^{(0)}, y_i^{(0)})}(t),$$

and hence we need to prove that

$$\sum_j \chi_{(x_j^{(n-1)}, y_j^{(n-1)})}(t) \leq \dim(j_*(H_{n-1}(f^{-1}(t)))) - (k - 1).$$

Using (5.36) we transform the statement into

$$\dim(\ker i_*) + k - 1 \leq \dim(j_*(H_{n-1}(f^{-1}(t)))).$$

In order to prove this inequality, we will find $k - 1$ linearly independent vectors in the quotient space $j_*(H_{n-1}(f^{-1}(t)))/\ker i_*$ (note that by (5.37) we have that $\ker i_* = j_*(\ker \tilde{j}_*) \subset j_*(H_{n-1}(f^{-1}(t)))$). Assume that $k \geq 2$ (because otherwise the statement is trivial) and denote by M_1^t, \dots, M_k^t the connected components of M^t whose boundary contains some Σ_i . We know that for $1 \leq i \leq k$, homology class $0 = [\partial M_i^t] \in H_{n-1}(M^t)$ decomposes as $0 = [\partial M_i^t] = d_i + e_i$, where $d_i = [\Sigma_{i_1}] + \dots + [\Sigma_{i_{m_i}}]$ for some $[\Sigma_{i_1}], \dots, [\Sigma_{i_{m_i}}]$ and $e_i \in j_*(H_{n-1}(f^{-1}(t)))$. Moreover, since M_i^t are disjoint, we have that

$$d_1 + \dots + d_k = [\Sigma_1] + \dots + [\Sigma_l],$$

and d_1, \dots, d_k partition the set $\{[\Sigma_1], \dots, [\Sigma_l]\}$. We have that

$$d_1, \dots, d_k \in j_*(H_{n-1}(f^{-1}(t))),$$

because $d_i = -e_i$, and let $[d_1], \dots, [d_k] \in j_*(H_{n-1}(f^{-1}(t)))/\ker i_*$ be the corresponding classes inside the quotient space. We claim that any $k - 1$ of $[d_1], \dots, [d_k]$ are linearly independent. Once we prove this, choosing any $k - 1$ of these gives us the $k - 1$ classes that we need.

First, we observe that $[\Sigma_1], \dots, [\Sigma_l] \in H_{n-1}(M^t)$ are linearly independent. In order to prove this, consider the following part of the long exact sequence of the pair $(M^t, \partial M)$:

$$H_n(M^t, \partial M) \rightarrow H_{n-1}(\partial M) \rightarrow H_{n-1}(M^t).$$

Note that $H_n(M^t, \partial M) = 0$. Indeed, if $H_n(M^t, \partial M) \neq 0$, then M^t contains a connected component N , such that $\partial N \subset \partial M$. However, this implies that $f^{-1}(t) \cap N = \emptyset$, or equivalently, $N \subset f^{-1}((-\infty, t))$. It is now easy to check that $N \subset M$ is both an open and a closed subset, which contradicts the fact that M is connected. Therefore, $H_n(M^t, \partial M) = 0$, and hence $H_{n-1}(\partial M) \rightarrow H_{n-1}(M^t)$ is an injection, i.e. $[\Sigma_1], \dots, [\Sigma_l]$ are linearly independent. This further implies that d_1, \dots, d_k are linearly independent.

Classes $i_*d_1, \dots, i_*d_k \in H_{n-1}(M)$ satisfy

$$i_*d_1 + \dots + i_*d_k = i_*[\Sigma_1] + \dots + i_*[\Sigma_l] = [\partial M] = 0.$$

By using the exactness of the following part of the long exact sequence of the pair $(M, \partial M)$

$$0 = H_n(M) \rightarrow H_n(M, \partial M) \rightarrow H_{n-1}(\partial M) \rightarrow H_{n-1}(M),$$

we conclude that

$$i_*[\Sigma_1] + \dots + i_*[\Sigma_l] = 0,$$

is the only relation which $i_*[\Sigma_1], \dots, i_*[\Sigma_l]$ satisfy. More formally, restriction of i_* to $\langle [\Sigma_1], \dots, [\Sigma_l] \rangle_{\mathbb{R}} \subset H_{n-1}(M^t)$ has the one-dimensional kernel given by

$$\ker(i_*|_{\langle [\Sigma_1], \dots, [\Sigma_l] \rangle_{\mathbb{R}}}) = \langle [\Sigma_1] + \dots + [\Sigma_l] \rangle_{\mathbb{R}}.$$

This readily implies the only relation which i_*d_1, \dots, i_*d_k satisfy is that their sum is zero, that is

$$\ker(i_*|_{\langle d_1, \dots, d_k \rangle_{\mathbb{R}}}) = \langle d_1 + \dots + d_k \rangle_{\mathbb{R}}.$$

Finally, combining the last equality with the fact that d_1, \dots, d_k are linearly independent immediately gives that any $k - 1$ of $[d_1], \dots, [d_k]$ are linearly independent. \square

5.4 Miscellaneous proofs

5.4.1 Proof of Lemma 5.1.1

It follows directly from the definitions and non-negativity of u that

$$\sum_{I \in \mathcal{B}'(h)} \int_I u(t) dt \geq \sum_{\tilde{I} \in \mathcal{B}'(f)} \int_{\tilde{I}} u(t) dt - 2|\mathcal{B}'(f)| \cdot \max_{[\min f, \max f]} u \cdot d_{bottle}(\mathcal{B}(f), \mathcal{B}(h)),$$

which means that we are left to prove that

$$\int_{\min f}^{\max f} u(t) dt - \int_{\min h}^{\max h} u(t) dt \leq 2 \cdot \max_{[\min f, \max f]} u \cdot d_{\text{bottle}}(\mathcal{B}(f), \mathcal{B}(h)), \quad (5.38)$$

if $\partial M = \emptyset$ and

$$\int_{\min f}^0 u(t) dt - \int_{\min h}^0 u(t) dt \leq \max_{[\min f, \max f]} u \cdot d_{\text{bottle}}(\mathcal{B}(f), \mathcal{B}(h)), \quad (5.39)$$

if $\partial M \neq \emptyset$. Let us prove (5.39). If $\min f \geq \min h$ the left-hand side of (5.39) is non-positive and hence the inequality trivially holds. If $\min f < \min h$ we need to prove that

$$\int_{\min f}^{\min h} u(t) dt \leq \max_{[\min f, \max f]} u \cdot d_{\text{bottle}}(\mathcal{B}(f), \mathcal{B}(h)).$$

However, in every $(d_{\text{bottle}}(\mathcal{B}(f), \mathcal{B}(h)) + \varepsilon)$ -matching the infinite bar $(\min f, +\infty) \in \mathcal{B}(f)$ has to be matched with some infinite bar $(a, +\infty) \in \mathcal{B}(h)$ and since $\min h$ is the smallest of all endpoints of all infinite bars in $\mathcal{B}(h)$, we have that

$$\min h - \min f \leq a - \min f \leq d_{\text{bottle}}(\mathcal{B}(f), \mathcal{B}(h)) + \varepsilon.$$

Since the above holds for all $\varepsilon > 0$ the inequality is proven. To prove (5.38) one proceeds in the similar fashion, by analysing cases depending on the relative position of $\min f, \min h$ and $\max f, \max h$. This completes the proof of the lemma. \square

5.4.2 Proof of Lemma 5.1.6

We prove the statement in the case of M without boundary, the other case is treated the same way. Let $\mathcal{B}(f)$ and $\mathcal{B}(\tilde{f})$ be two barcodes associated to two Morse functions and denote finite bars by $I_i \in \mathcal{B}(f)$, $\tilde{I}_j \in \mathcal{B}(\tilde{f})$ where intervals are sorted by integral of u as before. Assume that $\Phi_{u,k}(\mathcal{B}(f)) \geq \Phi_{u,k}(\mathcal{B}(\tilde{f}))$ and $\mu : \mathcal{B}(f) \rightarrow \mathcal{B}(\tilde{f})$ is an ε -matching between these barcodes (we add bars of length 0 if needed and assume that μ is a genuine bijection). For every finite bar $I \in \mathcal{B}(f)$ we have that the distance between endpoints of I and $\mu(I) \in \mathcal{B}(\tilde{f})$ is less or equal than ε and hence

$$\left| \int_I u(t) dt - \int_{\mu(I)} u(t) dt \right| \leq 2\varepsilon \max u.$$

Also $|\min f - \min \tilde{f}| \leq \varepsilon$ and $|\max f - \max \tilde{f}| \leq \varepsilon$ and hence

$$\left| \int_{\min f}^{\max f} u(t) dt - \int_{\min \tilde{f}}^{\max \tilde{f}} u(t) dt \right| \leq 2\varepsilon \max u.$$

Using these estimates and the fact that the integrals of u over \tilde{I}_j decrease with j we get

$$\begin{aligned}
0 \leq \Phi_{u,k}(\mathcal{B}(f)) - \Phi_{u,k}(\mathcal{B}(\tilde{f})) &= \int_{\min f}^{\max f} u(t) dt - \int_{\min \tilde{f}}^{\max \tilde{f}} u(t) dt + \sum_{i=1}^k \int_{I_i} u(t) dt - \\
&- \sum_{j=1}^k \int_{\tilde{I}_j} u(t) dt \leq 2\varepsilon \max u + \sum_{i=1}^k \int_{I_i} u(t) dt - \sum_{i=1}^k \int_{\mu(I_i)} u(t) dt \leq \varepsilon(2k+2) \max u.
\end{aligned}$$

Taking infimum over all ε -matchings finishes the proof. \square

5.4.3 Proof of Proposition 5.1.17

The barcode $\mathcal{B}(f_l)$ of the function

$$f_l(x_1, \dots, x_n) = \frac{\sqrt{2}}{n(2\pi)^{\frac{n}{2}}} (\sin lx_1 + \dots + \sin lx_n), \quad l \in \mathbb{N}.$$

can be computed using the Künneth formula for persistence modules. Below we briefly explain how to apply this formula and refer the reader to Section 3.2.2 or to [122] for a more detailed treatment.

Given two Morse functions $f : M_1 \rightarrow \mathbb{R}$ and $h : M_2 \rightarrow \mathbb{R}$ we define another Morse function $f \oplus h : M_1 \times M_2 \rightarrow \mathbb{R}$ by setting $f \oplus h(x_1, x_2) = f(x_1) + h(x_2)$. Barcode $\mathcal{B}(f \oplus h)$ may be computed from $\mathcal{B}(f)$ and $\mathcal{B}(h)$ via the following procedure:

- An infinite bar $(a, +\infty) \in \mathcal{B}_i(f)$ and an infinite bar $(c, +\infty) \in \mathcal{B}_j(h)$ produce an infinite bar $(a+c, +\infty) \in \mathcal{B}_{i+j}(f \oplus h)$.
- An infinite bar $(a, +\infty) \in \mathcal{B}_i(f)$ and a finite bar $(c, d] \in \mathcal{B}_j(h)$ produce a finite bar $(a+c, a+d] \in \mathcal{B}_{i+j}(f \oplus h)$. The same bar is produced if $(c, d] \in \mathcal{B}_i(f)$ and $(a, +\infty) \in \mathcal{B}_j(h)$.
- A finite bar $(a, b] \in \mathcal{B}_i(f)$ and a finite bar $(c, d] \in \mathcal{B}_j(h)$ produce two finite bars $(a+c, \min\{a+d, b+c\}] \in \mathcal{B}_{i+j}(f \oplus h)$ and $(\max\{a+d, b+c\}, b+d] \in \mathcal{B}_{i+j+1}(f \oplus h)$.

In order to compute $\mathcal{B}(f_l)$ it is enough to compute the barcode of $\sin lx_1 + \dots + \sin lx_n$ and rescale. In the light of the computational procedure described above we wish to look at $\sin lx : \mathbb{S}^1 \rightarrow \mathbb{R}$ and use $\mathbb{T}^n = (\mathbb{S}^1)^n$. One readily checks that

$$\mathcal{B}_0(\sin lx) = \{(-1, +\infty), (-1, 1] \times (l-1)\}, \quad \mathcal{B}_1(\sin lx) = \{(1, +\infty)\}$$

and hence

$$\mathcal{B}_0(\sin lx_1 + \sin lx_2) = \{(-2, +\infty), (-2, 0] \times (l^2 - 1)\},$$

$$\mathcal{B}_1(\sin lx_1 + \sin lx_2) = \{(0, +\infty) \times 2, (0, 2] \times (l^2 - 1)\},$$

$$\mathcal{B}_2(\sin lx_1 + \sin lx_2) = \{(2, +\infty)\}.$$

We claim that $\mathcal{B}(\sin lx_1 + \dots + \sin lx_n)$ contains 2^n infinite bars and $\frac{1}{2}((2l)^n - 2^n)$ finite bars. To prove the claim we use induction in n . We have already checked that the statement holds for $n = 1, 2$. To complete the induction step note that, in general, if $\mathcal{B}(f)$ contains k_1 infinite and m_1 finite bars, and $\mathcal{B}(h)$ contains k_2 infinite and m_2 finite bars, then $\mathcal{B}(f \oplus h)$ contains $k_1 k_2$ infinite and $k_1 m_2 + m_1 k_2 + 2m_1 m_2$ finite bars. Taking $k_1 = 2^n$, $m_1 = \frac{1}{2}((2l)^n - 2^n)$ and $k_2 = 2$, $m_2 = l - 1$ yields the proof.

Finally, notice that via the described procedure an infinite bar and a bar of length 2 produce a bar of length 2, as well as that two bars of length 2 produce two new bars of length 2. Since we start with $\mathcal{B}(\sin lx)$ for which all finite bars have length 2, we conclude that all finite bars in $\mathcal{B}(\sin lx_1 + \dots + \sin lx_n)$ have length 2, and thus

$$\Phi_1(\sin lx_1 + \dots + \sin lx_n) = 2n + (2l)^n - 2^n. \quad (5.40)$$

Rescaling (5.40) gives us

$$\Phi_1(f_l) = \frac{\sqrt{2}}{n(2\pi)^{\frac{n}{2}}} 2^n l^n + \frac{\sqrt{2}}{n(2\pi)^{\frac{n}{2}}} (2n - 2^n). \quad (5.41)$$

Since $l = \sqrt{\lambda}$ we have that

$$\Phi_1(f_l) = A_n \lambda^{\frac{n}{2}} + B_n,$$

with the constants A_n, B_n given explicitly by (5.41). This completes the proof of Proposition 5.1.17. \square

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אוניברסיטת תל אביב

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