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# Surfaces in 4-manifolds: piecewise-linear, smooth, symplectic, and complex

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# **Surfaces in 4-manifolds**

**Piecewise-linear, smooth, symplectic, and complex**

Marco Golla



Thèse d'habilitation à diriger la recherche soutenue à

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## *Contents*

# 1 Introduction

The main goal of this manuscript is to present my contribution to the study of surfaces in 4-manifolds, with an emphasis on symplectic surfaces in symplectic 4-manifolds and their comparison to complex curves in complex surfaces. We will especially work with surfaces with conical singularities in the 4-sphere and in the complex projective plane. We will look at obstructions to the existence of surfaces with certain properties, using a mixture of classical and modern invariants, and at concrete examples, obtained using both local and global constructions.

## Smooth 4-manifolds are wild

From the viewpoint of differential topology, dimension 4 is a unique source of all sorts of bizarre phenomena. It is the smallest dimension in which exotica appear: there are classes of smooth, closed 4-manifolds that are all homeomorphic, but not pairwise diffeomorphic [Don87]. It is the only dimension in which these classes are allowed to be infinite [OVdV86, FM88b, FM88a]. If we pass to non-compact manifolds, 4 is the only dimension in which  $\mathbb{R}^d$  admits exotic structures (see Casson’s notes in [GM86], together with Freedman’s theory [Fre82] and Donaldson’s work [Don87])—not only that, but it admits *continuously many* of them [Tau87].

One way of studying 4-manifolds is by looking at the surfaces they contain. For instance, the key to untangling exotica in higher dimension is the Whitney trick, which notably fails in dimension 4. The main character in the Whitney trick is a smoothly immersed 2-disc. We know *why* the Whitney trick fails in dimension 4 (namely, because a self-transverse 2-disc need not be embedded), but understanding *by how much* it fails can provide rich information about the topology of smooth 4-manifolds.

On the other hand, surfaces give us a useful way of probing or exhibiting exotica in dimension 4. Consider two homeomorphic smooth 4-manifolds  $X$  and  $X'$  and fix a homology class  $A \in H_2(X)$  that is represented by a smoothly embedded genus- $g$  surface. Suppose that we are able to prove that for all isomorphism  $\Phi: H_2(X) \rightarrow H_2(X')$  (as lattices endowed with a quadratic form),  $\Phi(A) \in H_2(X')$  cannot be represented by a smoothly embedded genus- $g$  surface. Then  $X$  and  $X'$  cannot be diffeomorphic. This observation motivates the definition of the *minimal genus*  $g(A)$  of a class  $A \in H_2(X)$  and the *minimal genus problem*, which asks to compute the function  $g: H_2(X) \rightarrow \mathbb{Z}_{\geq 0}$ .

Constructing “interesting” 4-manifolds is, in many ways, hard. Constructing “interesting” surfaces in 4-manifolds is also quite hard. This is where complex algebraic geometry comes to rescue, providing us with plenty of examples both of 4-manifolds (non-singular complex surfaces) and of surfaces therein (complex curves). These can be often used as “seeds” for surgery operations to take place and produce new



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4-manifolds (e.g. via Fintushel and Stern’s knot surgery [FS98]) or new surfaces in 4-manifolds (e.g. via Fintushel and Stern’s rim surgery [FS98]).

Detecting exotica is a different matter entirely: thanks to the pioneering work of Donaldson [Don83], we understood that the study of moduli spaces of solutions of certain PDEs are a very powerful, if hard-to-handle, tool to study smooth structures on 4-manifolds: it was the birth of (mathematical) gauge theory. Since then, gauge theory has expanded and the techniques have been refined and, to some extent, simplified: among the many names that I should list here, let me mention Fintushel, Stern, Kronheimer, Mrowka, Ozsváth, Szabó, Frøyshov, and Taubes.

It soon became clear that complex structures interact well with gauge theory, and it was later realised that *symplectic structures* also do. At the same time, symplectic geometry allows for a lot more flexibility than complex algebraic geometry does in many aspects: surgery operations and local constructions can be always be performed in symplectic geometry, while they are sometimes obstructed in algebraic geometry. This way, symplectic geometry gives us a larger playground where we have interesting constructions, both of closed 4-manifolds and of surfaces therein, as well as a toolset to detect exotica.

The recurring theme of this manuscript is the comparison between these three different worlds: smooth topology, symplectic topology, and complex algebraic geometry. Another recurring theme is how smooth or symplectic topology can inform algebraic geometry, very much in the spirit of the work of Zariski, van Kampen, Chisini, Moishezon, Ruberman, Catanese, and many, many more.

To be very concrete, it is rather striking that we do not yet understand how non-singular symplectic surfaces compare to non-singular complex curves in the “smallest” possible example, namely the complex projective plane  $\mathbb{C}P^2$ . This is the challenge posed by the *symplectic isotopy problem*: is every smoothly embedded symplectic surface in  $\mathbb{C}P^2$  isotopic through symplectic surfaces in  $\mathbb{C}P^2$ ? The answer is known for surfaces of self-intersection at most  $17^2$  (i.e. curves of degree at most 17) [Gro85, Sik03, She00, ST05], but not beyond that.

## A roadmap

In this manuscript, I tried to organise some of my past work on surfaces in 4-manifolds within a coherent narrative, without aiming for historical or chronological accuracy. This narrative has emerged naturally over time, rather than being my initial motivation for thinking about certain problems. Therefore, the spirit of this summary does not necessarily agree with that of the papers I am presenting.

Rather than giving all the preliminaries in a single batch at the beginning, I preferred to introduce some of the concepts as I went along.

In Chapter 2 I will present some results concerning piecewise-linear (PL) surfaces in 4-manifolds, with a strong focus on (arguably) the two simplest 4-manifolds, namely the 4-sphere and the complex projective plane. Besides a very quick introduction to the subject of surfaces in 4-manifolds, the chapter is roughly divided into three parts.

First we discuss Heegaard Floer homology, and especially correction terms, with some input from [BG18]. We then proceed to discuss PL surfaces in  $S^4$  or, more or

less equivalently, smooth cobordisms between knots in the 3-sphere, mostly based on [BCG17]. Finally, we introduce PL surfaces in  $\mathbb{C}\mathbb{P}^2$ , and present work of [AG17] and [AGLL20] for the case of spheres and of [BCG16] for the case of surfaces of higher genus. We also lay the ground for the next chapter by stating some classical results in this specialised context and sketching some of the proofs.

In Chapter 3 I will present some results about singular symplectic surfaces in 4-manifolds, essentially only focusing on the case of  $\mathbb{C}\mathbb{P}^2$ . We start with a gentle introduction to the aspects of this very broad subject which are relevant in the rest of the chapter, especially to pseudo-holomorphic techniques, and define the central object of investigation: symplectic curves.

We then present some results obtained in [GS22] (and partly in [GK23]) about the existence and uniqueness (up to isotopy) of symplectic rational cuspidal curves in  $\mathbb{C}\mathbb{P}^2$ , with an eye towards the algebro-geometric aspects of the questions we are investigating. As these are in particular PL spheres, ideas from Chapter 2 are quite ubiquitous here. We explain in some detail the ideas entering the proof, but prefer to prove things by example rather than in detail.

To close things up, we take a peek at the relative case: we both look at surfaces with boundary, and especially *symplectic hats* [EG22], as well as at caps and fillings of contact 3-manifolds, presenting some of the ideas developed in [EG22] and in [GS22, GS21].

A final word of caution for the reader: proofs will only be sketched and sometimes omitted entirely. Some statements are classical, but I decided to give an argument where I felt the statements in the literature were too general and the proofs simplified in the cases at hand.

## Summary of results

The results I am presenting can be broadly split into two classes: results about piecewise-linear surfaces in 4-manifolds, and results about symplectic 4-manifolds and their symplectic submanifolds. As mentioned above, the common thread is the relationship with complex algebraic geometry, and especially the comparison between PL surfaces in smooth 4-manifolds, singular symplectic surfaces in symplectic 4-manifolds, and singular curves in complex surfaces. This extends also to the realm of singularity theory and deformations of singularities in dimension 1 (and sometimes 2).

### Piecewise-linear surfaces

A surface  $F$  in a smooth 4-manifold  $X$  is piecewise-linear if there is a compatible triangulation of  $X$  such that  $F$  is a simplicial surface. In practice, this means that  $F$  is smooth away from a finite number of points that are not locally-flat. At these non-locally-flat points,  $F$  has a *conical singularity*, i.e.  $F$  is a cone over a knot in the 3-sphere: we call this knot the *type* (or sometimes the *link*) of the singularity of  $F$ , or the type of  $F$  at that point. Examples of such surfaces are given by complex curves with irreducible singularities. (One can also look at the case where  $F$  is a

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cone over a link, which approximately corresponds to passing from embeddings to immersions in the case of smooth surfaces.)

Often times, the minimal genus problem for PL surfaces is rather dull: for instance, in  $\mathbb{C}\mathbb{P}^2$  every homology class is represented by a PL 2-sphere (in fact, by a singular complex curve). It becomes a lot more interesting if one restricts the possible types of singularities of  $F$ .

The main result we present in Chapter 2 is obtained in collaboration with Aceto, Larson, and Lecuona<sup>1</sup> [AGLL20].

**Theorem 1.1.** *There is an explicit list of sixteen families and four sporadic cases of PL spheres in a rational homology  $\mathbb{C}\mathbb{P}^2$  with a unique non-locally-flat point whose link is a positive torus knot.*

The precise statement and the exact list are in Theorem 2.28 below. The question was motivated by the analogous result in the case of rational cuspidal curves, i.e. complex curves in  $\mathbb{C}\mathbb{P}^2$  that are PL-embedded and with a unique singularity whose link is a torus knot [FLMN07]. (In this case the link is automatically a positive torus knot.)

Theorem 1.1 result is based, among others, on earlier work of Borodzik and Livingston [BL14] and on work in collaboration with Aceto [AG17].

In collaboration with Bodnár and Celoria [BCG16], we also studied the case of surfaces of higher genus. We give a non-technical version here, and defer to Theorem 2.32 below for the precise statement.

**Theorem 1.2.** *There are strong restrictions on the genus of a PL surface in a homology  $\mathbb{C}\mathbb{P}^2$ , once we fix its homology class and the types of its singularities. These restrictions can be rephrased in terms of the Alexander polynomial if the singularity types are positive torus knots (or, more generally, links of plane curve singularities).*

It turns out that these obstructions are *extremely* effective if we look at cuspidal curves in  $\mathbb{C}\mathbb{P}^2$  (i.e. complex curves which are PL-embedded).

In [BCG17], we used similar techniques (more on this below) for studying cobordisms between knots. In hindsight, these are just PL surfaces in  $S^4$  with two singularities. We state here a simplified (and vague) version of the statement for links of complex curve singularities (see Theorem 2.21 below).

**Theorem 1.3.** *Let  $K_0$  and  $K_1$  be two knots in  $S^3$  that are the link of two complex curve singularities in  $\mathbb{C}^2$ . Their Alexander polynomials give restrictions on the genera of cobordisms from  $K_0$  to  $K_1$ , or equivalently two PL surfaces whose only two singularities have types  $K_0$  and  $m(K_1)$ . (Here  $m$  denotes mirroring.)*

All three results mentioned here rely on Heegaard Floer homology and their proofs are based on computations of correction terms for certain 3-manifolds obtained as surgeries along knots in  $S^3$  or in connected sums of  $S^1 \times S^2$ . While introducing these invariants and stating some of their relevant properties, below, we mention a result obtained in collaboration with Behrens [BG18] about correction terms. In [BG18],

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<sup>1</sup>Note that the paper is still under review. The results have been computationally verified for degrees up to 500.

we define fully twisted correction terms in Heegaard Floer homology and prove some of their properties. Unlike ‘classical’ correction terms, or their ‘bottom-most’ and ‘top-most’ versions, fully twisted correction terms are defined uniformly for all 3-manifolds equipped with a torsion  $\text{spin}^c$  structure.

The chapter ends with some more classical remarks about PL surfaces in  $\mathbb{C}\mathbb{P}^2$ , using branched covers and the Arf invariant. These will make an appearance in the following chapter. The part using branched covers is essentially due to Gilmer [Gil81] (see also [CG78]), and is close to some work in collaboration with Aceto and Lecuona [AGL17]. Since I have not found the statement given here in the literature and since extracting it from Gilmer’s work is not evident, I have spelled it out explicitly and given a fairly detailed sketch of the proof.

## Symplectic curves

If  $X$  is a complex projective surface (or, more generally, a Kähler 4-manifold), then it admits a symplectic structure  $\omega$ , i.e. a closed, non-degenerate 2-form, which is compatible with the complex structure. A consequence of the compatibility is the fact that non-singular complex curves in  $X$  are embedded surfaces on which  $\omega$  restricts to an area form—they are *symplectic* submanifolds.

Looking at the simplest possible complex surface,  $\mathbb{C}\mathbb{P}^2$ , we have the Fubini–Study symplectic form  $\omega_{\text{FS}}$ . Complex curves in  $\mathbb{C}\mathbb{P}^2$  are automatically  $\omega_{\text{FS}}$ -symplectic, so one can ask whether *all* symplectic surfaces in  $\mathbb{C}\mathbb{P}^2$  arise in this way. Since being symplectic is a  $C^1$ -open condition and being complex is not, this cannot be true on the nose. There is hope, however, that this is true up to *symplectic isotopy*, i.e. up to an isotopy through symplectically embedded surfaces. This is the context of the symplectic isotopy problem, mentioned above.

The bulk of the work about this question presented in this manuscript comes from my collaboration with Starkston [GS22, GS21]. In [GS22] we propose a definition of singular symplectic submanifolds of symplectic 4-manifolds, which we call *symplectic curves*, and we lay the foundations for their study. This class comprises singular symplectic surfaces that locally look like complex curves.

I say ‘*a* definition’ since there are many other definitions available, depending on which singularities we want to allow; see Section 3.7 for a more detailed discussion of this issue. We called them ‘curves’ to emphasise that we are thinking of them as relatives of (possibly singular) curves in complex surfaces.

In [GS22] we formulate a singular version of the symplectic isotopy problem for these curves and collect examples which show that there are two classes of symplectic curves for which the problem is not known to be false: the non-singular case (i.e. the ‘classical’ symplectic isotopy problem), and the case of *rational cuspidal curves* (i.e. curves which are PL-embedded spheres).

The singular version of the symplectic isotopy problem had been considered in the more restricted setting of curves with nodes and simple cusps (i.e. transverse double points and curves locally modelled on  $\{x^2 - y^3 = 0\}$ ) by several authors [Moi94, Bar00, She04, Aur06, Fra05], especially in connection with Auroux’s work on branched covers of  $\mathbb{C}\mathbb{P}^2$  [Aur00, ADK03].

It turns out that symplectic curves can be equivalently described as  $J$ -holomorphic

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curves for some almost-complex structure  $J$  compatible with the symplectic structure. We choose to emphasise the symplectic, rather than the pseudo-holomorphic, aspects, since this is a more natural framework for the symplectic isotopy problem.

In [GS22, GK23], we provide evidence for the symplectic isotopy problem for rational cuspidal curves.

**Theorem 1.4.** *The symplectic isotopy problem for rational cuspidal curves is true in degrees up to 7.*

The study of symplectic curves was motivated partly by the symplectic isotopy problem, and partly by the question of the existence of exotic  $\mathbb{C}\mathbb{P}^2$ s. It turns out that rational cuspidal curves cannot help us find an exotic  $\mathbb{C}\mathbb{P}^2$ .

**Theorem 1.5.** *If  $X$  is a closed symplectic 4-manifold that has the same rational homology as  $\mathbb{C}\mathbb{P}^2$  and that contains a symplectic rational cuspidal curve, then  $X$  is diffeomorphic to  $\mathbb{C}\mathbb{P}^2$ .*

In the last part of the chapter we present some results about symplectic 4-manifolds with boundary and their symplectic submanifolds.

In [GS21] we focus on the contact and symplectic aspects of symplectic curves, by shifting the perspective from symplectic isotopies to *divisorial contact structures*, which are contact structures associated to symplectic curves of positive self-intersection, and their fillings. The question we ask (and very partially answer) is: given an *abstract* symplectic curve, that we think of as the data of its genus, its singularity types, and its self-intersection, which symplectic 4-manifolds can it live in? The answer was known for non-singular curves of sufficiently positive self-intersection, due to work of McDuff for curves of genus 0 [McD90] and Kütle for curves of positive genus [Küt21]. A sample result from [GS21], which we can phrase in terms of fillings of certain divisorial contact structures, is the following.

**Theorem 1.6.** *No closed symplectic 4-manifold can contain a symplectic surface of genus 0 with a simple cusp and self-intersection larger than 9.*

*If  $X$  is a closed minimal symplectic 4-manifold that contains a symplectic curve  $C$  of genus 0 with a simple cusp and self-intersection 9, then  $X$  is  $\mathbb{C}\mathbb{P}^2$  and  $C$  is symplectically isotopic to the cuspidal cubic  $\{x^2z - y^3 = 0\}$ .*

In [EG22] we look at properly embedded symplectic surfaces in symplectic 4-manifolds with boundary, especially when the boundary is *concave* (i.e. the case of *symplectic caps*). We focus on the case of  $P = \mathbb{C}\mathbb{P}^2 \setminus D$ , the complex projective plane with a small open Darboux ball removed. The contact boundary of this 4-manifold is the standard contact structure on the 3-sphere,  $(S^3, \xi_{\text{st}})$ , and we look at symplectic surfaces in  $P$  whose boundary is a transverse knot or link in the 3-sphere. The main result of the paper is the following existence result.

**Theorem 1.7.** *Every transverse link in  $(S^3, \xi_{\text{st}})$  bounds a properly embedded symplectic surface in  $P$ . Moreover, every transverse knot bounds a properly immersed disc in  $P$  with positive double points and a properly embedded disc in a blow-up of  $P$ .*

This is in stark contrast with the existence of symplectic surfaces in any filling of  $(S^3, \xi_{\text{st}})$ : very few knots admit one. (Knots which do are called *quasipositive*.) We then proceed to study the complexity of these surface for certain classes of knots, and to use these surfaces to study restrictions on fillings of contact structures obtained as branched covers of transverse knots.

Here we present an example which already showcases most of the general ideas (and a bit more). Recall that the Brieskorn sphere  $\Sigma(2, 3, 7)$  is the link of the Brieskorn–Pham singularity  $\{x^2 + y^3 + z^7 = 0\} \subset \mathbb{C}^3$  and let  $\xi_{\text{can}}$  be the associated canonical contact structure.

**Theorem 1.8.** *A Stein filling  $W$  of  $(\Sigma(2, 3, 7), \xi_{\text{can}})$  always has  $H_1(W) = 0$  and either it is spin and  $H_2(W) \cong E_8 \oplus 2H$ , or  $H_2(W) \cong \langle -1 \rangle$ . Both cases occur. By contrast,  $\xi_{\text{can}}$  has strong symplectic fillings with arbitrarily large  $b_2^+$ .*

## Looking ahead (and around)

As mentioned above, 4-manifolds are wild, which is exactly what makes them so fascinating. In a way, most of my research is focused on the comparison between three categories of 4-manifolds: smooth, symplectic, and complex. Part of my future research is very much in continuity with the collaboration with Starkston, and deals with various aspects of symplectic curves. This in turn leads naturally to studying complex manifolds in broader generality, and to look at different algebro-geometric objects, like singular surfaces with cyclic quotient singularities or singularities of complex surfaces and their deformations. I am also thinking about special metrics on 4-manifolds and associated decompositions (akin to the JSJ decomposition in dimension 3).

In the next three sections, I will talk about some projects about symplectic curves, embeddings of lens spaces in  $\mathbb{C}\mathbb{P}^2$ , and some more geometric questions on 4-manifolds.

## More symplectic curves

The true goal of this quest is the original symplectic isotopy problem. Singular symplectic curves are a potential source of counterexamples for the symplectic isotopy problem via their *smoothings*: given a singular symplectic curve  $C$ , we can smooth out all of its singularities by surgering out neighbourhoods of their singularities and replacing them by their Milnor fibres. Certain configurations of cusps on a symplectic rational cuspidal curve would lead to a counterexample to the symplectic isotopy problem, if we could prove that these curves exist. I am actively working on this, by trying to construct these curves.

Singular symplectic curves also have the potential to detect exotic  $\mathbb{C}\mathbb{P}^2$ : if one could produce a curve that exists in  $X$ , a symplectic homotopy  $\mathbb{C}\mathbb{P}^2$  (equipped with some symplectic structure) but that cannot exist in the genuine  $\mathbb{C}\mathbb{P}^2$  (equipped with the Fubini–Study form), then an aforementioned theorem of Taubes [Tau96] would show that  $X$  is not diffeomorphic to  $\mathbb{C}\mathbb{P}^2$ . Theorem 1.5 above, however, tells us that rational cuspidal curves cannot do the trick. At the moment, I do not know of an

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effective way of constructing “interesting” examples of non-rational or non-cuspidal symplectic curves. It would be interesting to explore these avenues in a farther future.

More modestly, it would be very interesting to know whether rational cuspidal curves can be somewhat exotic. For instance, complex rational cuspidal curves in  $\mathbb{C}\mathbb{P}^2$  are known to have at most four cusps—in fact there exists exactly *one* complex rational cuspidal curve in  $\mathbb{C}\mathbb{P}^2$  with exactly four cusps, up to biholomorphism [KP22]. Is this also true in the symplectic category? Are there interesting differences between equisingular isotopy classes of rational cuspidal curves, when comparing the symplectic and the complex category? (This is related to Zariski pairs of curves, mentioned below.) Can we classify symplectic rational cuspidal curves? This seems to require developing a suitable logarithmic minimal model program, à la Palka–Peřka [PP17, PP20], which would also connect with the study of homology planes (affine varieties which have trivial homology) [Peř21].

In a different direction, with Hanine Awada we are studying fundamental groups of complements of symplectic curves in  $\mathbb{C}\mathbb{P}^2$ , generalising ideas of Zariski, van Kampen, Chisini, Moishezon, and Libgober. In particular, we are working on proving Libgober’s divisibility conditions on the Alexander polynomial of a symplectic curve [Lib82]. Libgober had defined the Alexander polynomial of a (say affine) plane complex curve and proved that this polynomial always divides the product of Alexander polynomials of the singularities of the curve (which are links in  $S^3$ , so they classically have an Alexander polynomial) as well as the Alexander polynomial of the *link at infinity* of the curve (i.e. the intersection of the curve with a very large sphere, which is again a link in  $S^3$ ). As a corollary, for instance, the roots of the Alexander polynomial of a curve are roots of 1.

I am also working on configurations of curves, especially of lines, trying to generalise ideas of Ruberman and Starkston [RS19] to study more symplectic line arrangements. It is especially interesting to expand the types of covers one uses—Ruberman and Starkston focused their efforts to a very clever use of cyclic covers: this is a line of investigation that Hirzebruch exploited in the 80s, especially using Kummer covers (which are a type of Abelian, non-cyclic covers). Hirzebruch then uses the Bogomolov–Miyaoka–Yau inequality to give very general restrictions on the combinatorics of line arrangements: in the opposite direction, producing “interesting” symplectic line arrangements (or curve configurations) would lead to counterexamples to the Bogomolov–Miyaoka–Yau inequality for symplectic 4-manifolds. This is a very exciting line of research that I am thinking about.

### **Codimension-1 embeddings in $\mathbb{C}\mathbb{P}^2$**

I have been interested in embeddings of 3-manifolds in 4-manifolds since, in collaboration with Aceto and Larson, we studied embeddings of 3-manifolds in spin 4-manifolds [AGL17]. With Brendan Owens we are studying embeddings of 3-manifolds in  $\mathbb{C}\mathbb{P}^2$ , with a focus on lens spaces. This also comes up in the project with Aceto [AG17] and Aceto, Larson, and Lecuona [AGLL20], where in a way we are answering the question for 3-manifolds that are obtained as positive integer surgeries along torus knots.

The first, quite standard obstruction is the following: if a lens space  $L$  embeds in  $\mathbb{C}\mathbb{P}^2$ , then  $L$  bounds a rational homology ball. (Lisca has classified lens spaces that bound rational homology balls in [Lis07].) We have developed some further obstructions, based on Donaldson’s diagonalisation theorem, on Heegaard Floer homology, on Pin(2)-Seiberg–Witten Floer homology, and constructed a number of examples.

We are especially interested in studying embeddings of pairwise disjoint rational homology balls with lens space boundary in  $\mathbb{C}\mathbb{P}^2$ . This is related to a theorem in algebraic geometry (due to Hacking and Prokhorov [HP10]) and its symplectic counterpart (due to Evans and Smith [ES18]), asserting that if these embeddings come from (complex or symplectic) surface singularities (which are cyclic quotients, since we are talking about lens spaces), then one cannot embed more than three lens spaces. Moreover, all embeddings of this type are classified, and they are related to solutions of the Markov equation  $x^2 + y^2 + z^2 = 3xyz$ . We know that not all of this rigidity is preserved when passing to the topological category, but so far we have no example of an embedding of four disjoint lens spaces in  $\mathbb{C}\mathbb{P}^2$ , nor do we have a proof that such an embedding cannot exist.

In turn, this is related to the Montgomery–Yang problem on circle actions on  $S^4$  and to Seifert fibred homology 3-spheres that bound homology balls. (The connection is far from being apparent.)

## An eye towards geometry

In collaboration with Luca Di Cerbo, I have started to think about ‘special’ metrics on 4-manifolds, and especially complex-hyperbolic surfaces and Einstein 4-manifolds.

Complex-hyperbolic surfaces are obtained as quotients of the 4-ball (equipped with the Bergmann metric) by a discrete and free group action. The 4-ball is akin to Poincaré’s disc model for the hyperbolic plane, from which the name of “complex-hyperbolic manifolds”. They are very interesting objects from the complex-geometric perspective, since they are on the threshold of the Bogomolov–Miyaoka–Yau inequality.

Einstein metrics are a generalisation of metrics of constant sectional curvature on 3-manifolds. As the name suggest, they originate in general relativity, but they have been studied for decades in more abstract context. For instance, in dimension 4 they are the unique fixed points of the (renormalised) Ricci flow, hence they are a natural class to study from the viewpoint of Thurston’s and Perelman’s geometrisation in dimension 4.

With Di Cerbo, we are trying to study these metrics from a more topological perspective. In [DCG23] we considered the problem of whether Einstein metrics can be obtained as fillings of complex-hyperbolic metrics, much like hyperbolic 3-manifolds can be produced by Dehn filling hyperbolic knot complements [Thu80], and Einstein 4-manifolds can be obtained by Dehn filling hyperbolic 2-torus complements [And06]. In [DCG23] we show that the answer is negative.

We are now exploring the connection between asphericity and signature. It is conjectured that the ratio signature/Euler characteristic is bounded by 3 for closed aspherical 4-manifolds. Kotschick [Kot98] proved this for surface bundles over sur-



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faces which admit a complex structure (see also earlier work of Atiyah [Ati69] and Kodaira [Kod67]) and for geometric 4-manifolds. The broader context is that of the Singer conjecture on the  $L^2$  Betti numbers of aspherical manifolds. More specifically, we look at 4-manifolds with a geometric decomposition (in the sense of Hillman [Hil98]), which are a generalisation of higher graph manifolds, defined by Frigerio, Lafont, and Sisto [FLS15].

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## 2 Piecewise-linear surfaces

In this chapter, we will look at piecewise-linear surfaces in 4-manifolds, and especially in the 4-sphere and in the complex projective plane,  $\mathbb{C}\mathbb{P}^2$ . The prototypical example is a complex curve in  $\mathbb{C}\mathbb{P}^2$  with irreducible singularities. The main question will be similar to the more famous minimal genus problem, which we will recall shortly: if we know the singularities of a piecewise-linear surface in  $\mathbb{C}\mathbb{P}^2$ , what can we say about its genus? In this chapter we will mostly focus on obstructions, and some constructions can be found in the next chapter. The main tools in this chapter come from Heegaard Floer homology [OSz04b, OSz04a, OSz03], Donaldson's diagonalisation theorem [Don83], and branched covers.

After introducing the objects in Section 2.1 and the tools in Section 2.2, in Section 2.3 we will present some results obtained in collaborations with Józsi Bodnár and Daniele Celoria [BCG17] on knot cobordisms. Finally, in Section 2.4 will deal with PL-embedded surfaces in homology  $\mathbb{C}\mathbb{P}^2$ s, presenting some results obtained in collaboration with Paolo Aceto [AG17], with Paolo Aceto, Kyle Larson, and Ana García Lecuona [AG17, AGLL20], and with Józsi Bodnár and Daniele Celoria [BCG16].

### 2.1 Surfaces in 4-manifolds, a primer

Let  $X$  be a smooth 4-manifold and  $j: F \hookrightarrow X$  be a  $C^0$  embedding. We gently abuse the notation by denoting by  $F$  the image of  $j$  in  $X$ , as well as the source of  $j$ . We say that  $F$  is:

- *locally-flat* if for every point  $x \in F$ , there exists a neighbourhood  $U \ni x$  and a homeomorphism  $(U, F \cap U) \rightarrow (\mathbb{R}^4, \mathbb{R}^2 \times \{0\})$ ;
- *piecewise-linear*, or *PL* for short, if there exists a triangulation of  $X$  such that  $F$  is isotopic to a triangulated submanifold of  $X$ ;
- *smooth* if  $j$  is a  $C^\infty$  immersion (i.e.  $dj$  has rank 2 at each point of  $F$ ).

A smooth surface is obviously both PL and locally-flat (by the implicit function theorem), but the converse is not true, and there are no implications between being locally-flat and PL.

One of the main questions in low-dimensional topology is the *minimal genus problem*. Given a homology class  $A \in H_2(X)$ , it is well-known that there exists a smoothly embedded surface in  $X$  whose homology class is  $A$ . The minimal genus problem asks to find the minimal genus among all such representatives. This is a notoriously hard question whose answer is known only in a handful of cases. The most notable example is the minimal genus problem in  $\mathbb{C}\mathbb{P}^2$ , also known as the Thom conjecture, which was established by Kronheimer and Mrowka [KM94].

**Theorem 2.1** (Kronheimer, Mrowka). *When  $d \neq 0$ , the minimal genus of the class  $dh \in H_2(\mathbb{C}\mathbb{P}^2)$  is  $\frac{1}{2}(|d| - 1)(|d| - 2)$ .*

PL surfaces are very closely related to smooth surfaces, but they allow for non-locally-flat points. Let  $K \subset S^3$  be a knot. We say that a PL surface  $F$  in  $X$  has a singularity of type  $K$  at  $p$  if there exists a neighbourhood  $U$  of  $p$  and a PL homeomorphism  $(U, F \cap U, p) \rightarrow (B^4, C(K), 0)$ , where we view  $B^4$  as the cone over  $S^3$  and  $C(K)$  is the cone over  $K$ . As mentioned above, complex curves with irreducible singularities provide the first examples of such objects. In analogy with complex algebraic geometry we will denote with  $\text{Sing}(F)$  the set of non-locally-flat points of  $F$ .

*Remark 2.2.* The minimal genus problem for PL surfaces in  $\mathbb{C}\mathbb{P}^2$  is much easier than the smooth or locally-flat one: every homology class is represented by a PL sphere. For instance, we can take  $F$  to be the complex curve defined by the equation  $\{x^{d-1}z + y^d = 0\}$ , which is a PL sphere (it is parametrised by  $\mathbb{C}\mathbb{P}^1 \cong S^2$ ) with a singularity of type  $T(d - 1, d)$ .

Interestingly, there are examples of (open) 4-manifolds that are homotopy equivalent to  $S^2$  but whose generator of the second homology is not represented by a PL sphere. These examples were first found by Levine and Lidman [LL19] (see also [GL20]).

With the notation and terminology set up, let us formulate the driving question of the chapter.

**Question 2.3.** Given a class  $A \in H_2(X)$ , a non-negative integer  $g$ , and a collection of knots  $K_1, \dots, K_\nu$ , does there exist a PL surface in the homology class  $A$  of genus  $g$  and with singularities of types  $K_1, \dots, K_\nu$ ?

Note that the corresponding algebro-geometric question for complex curves with irreducible singularities in a complex surface has a long and rich history, and some rather surprising answers: only to cite a few modern references [Ore02, FLMN07, BL14, KP17, KP22].

A PL surface  $F$  in  $X$  has a *regular neighbourhood*  $N(F)$ , which is the PL analogue of a tubular neighbourhood of a smoothly (or globally-flatly) embedded surface. This is defined from the PL perspective by taking a sufficiently fine subdivision of a triangulation of  $X$  in which  $F$  is polygonal, and taking the union of 4-simplices having a facet on  $F$ . In practice, this means that  $N(F)$  is a union of small balls, each centred at a singularity of  $F$ , and a regular neighbourhood of the locally-flatly embedded part of  $F$ . This viewpoint allows to give an explicit description of  $N(F)$  as a 4-manifold. Let  $H_m$  be the boundary connected sum of  $m$  copies of  $S^1 \times D^3$  or, equivalently, the result of attaching  $m$  1-handles to  $B^4$ .

**Proposition 2.4** (Borodzik, Hedden, Livingston [BHL17]; Bodnár, Celoria, G.). *If  $F$  has genus  $g$ , singularities  $K_1, \dots, K_\nu$ , and self-intersection  $s = F \cdot F$ ,  $N(F)$  is diffeomorphic to the 4-dimensional handlebody obtained by attaching a 2-handle to  $H_{2g}$  along the knot  $B_g \# K$  with framing  $s$ , where  $B_g$  is the Borromean knot in  $\#^{2g} S^1 \times S^2$  (see Figure 2.1) and  $K = K_1 \# \dots \# K_\nu$ .*

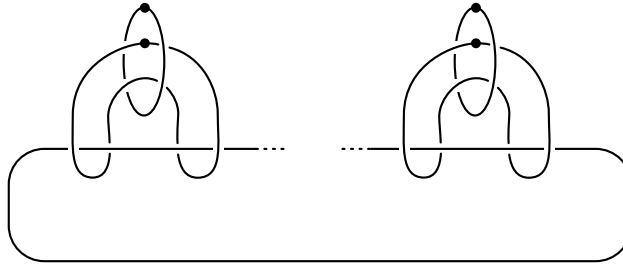


Figure 2.1: The Borromean knot in  $B_g \subset \#^{2g} S^1 \times S^2$ , in dotted handle notation. (There are  $2g$  dotted handles in the picture.)

The knot  $K_1 \# \dots \# K_\nu$  will appear quite frequently, so we feel like it deserves its own notation. Therefore, given a PL-embedded surface, we denote the connected sum of its singularity types by  $K_F$ .

In particular, note that if the PL surface  $F$  has genus 0, its regular neighbourhood is the *trace* of the surgery along  $K$ , i.e. it is the 4-ball with a single  $s$ -framed 2-handle attached along  $K$ .

The idea of the proof is rather simple. We sketch it for the case  $g(F) = 0$ ,  $\nu = 1$ , i.e. when  $F$  is a PL sphere with a unique singularity. In that case, a regular neighbourhood of  $F$  is given by a regular neighbourhood of its singular point, which is a 4-ball  $D$ , together with a regular neighbourhood of the rest of the curve, which is a 2-disc. This neighbourhood is therefore a 4-ball  $D'$ , which is attached to the boundary of  $B$  along a neighbourhood of  $F \cap \partial D'$ , i.e. precisely along  $K$ . That is,  $N(F)$  is obtained by attaching  $D'$  to  $D$  as a 2-handle. The framing is then determined by the self-intersection of  $F$ .

The general case is very similar. We choose a cell decomposition of  $F$  that has the singularities of  $F$  as 0-cells (we pick any cell decomposition if  $F$  is PL and locally-flat). A regular neighbourhood of  $F$  is a thickening of this cell decomposition, which is built by adding handles of the same index of the dimension of the corresponding cell.

## 2.2 Some ideas from gauge theory

Quite indisputably, the birth of modern 4-dimensional topology dates to the beginning of the 80s, when two major results have been proven, one by Freedman in the topological category [Fre82, BKK<sup>+</sup>21], and one by Donaldson in the smooth category [Don83]. We will not dwell on Freedman's result, but rather talk about Donaldson's work and one of its descendants, Heegaard Floer homology.

### The diagonalisation theorem

First off, let us recall what the intersection form on a 4-manifold is. If  $X$  is a closed and oriented (topological) 4-manifold, we can use Poincaré duality, the cup product, and the orientation class to define a symmetric bilinear product  $Q_X$  on  $H_2(X)/\text{Tor}$ , with values in  $\mathbb{Z}$ :

$$Q_X(A, B) = \langle \text{PD}(A) \cup \text{PD}(B), [X] \rangle \in \mathbb{Z}.$$

If  $X$  is smooth, we can interpret  $Q_X$  as counting the (signed) intersection between two transverse representatives of  $A$  and  $B$ . (Here the assumption that  $X$  is smooth is added for convenience, but one can make sense of the same statement for topological 4-manifolds. We also note that the definition can be extended to oriented 4-manifolds that are not closed by first embedding  $H_2(X)$  into  $H_2(X, \partial X)$  and then using Poincaré–Lefschetz duality.)

We can tensor  $Q_X$  with  $\mathbb{R}$  and obtain a symmetric bilinear form over the reals, which therefore has a well-defined *signature*, given by the difference between the count of its positive and negative eigenvalues. We call this signature the *signature* of  $Q_X$  and of  $X$ , and we denoted it by  $\sigma(X)$ . We say that a 4-manifold is *negative definite* (respectively, *positive definite*) if its intersection form is, that is if we have  $\sigma(X) = -\text{rk } H_2(X) = -b_2(X)$  (resp.  $\sigma(X) = \text{rk } H_2(X) = b_2(X)$ ).

We say that  $Q_X$  is *diagonalisable* (over  $\mathbb{Z}$ ) if there exists an integer orthonormal basis of  $H_2(X)/\text{Tor}$ . Note that there are many examples of non-diagonalisable forms over the integers, like the hyperbolic form  $Q_{S^2 \times S^2}$  (which is indefinite) or the  $E_8$ -form (which is definite) [MH73].

We can now state Donaldson’s diagonalisation theorem.

**Theorem 2.5** (Donaldson). *If  $X$  is a closed, smooth, negative definite 4-manifold, then  $Q_X$  is diagonalisable.*

Note that, as a corollary to Freedman’s results, the diagonalisation theorem fails miserably when  $X$  is only assumed to be a topological 4-manifold.

Donaldson’s proof was based on the study of *instantons*, solutions to a certain PDE on  $X$ . Since Donaldson’s pioneering work, the tools of *gauge theory* have been refined and extended, and form now a rich, almost indispensable part of 3- and 4-dimensional topology. We will meet in the next section a grandchild of instanton theory.

## Heegaard Floer homology: rational homology spheres

Heegaard Floer homology was devised by Ozsváth and Szabó [OSz04b, OSz04a] as a way to make Seiberg–Witten invariants of 3-manifolds more computable. In turn, Seiberg–Witten theory [Wit94, KM07] was born as a friendlier cousin of instanton theory [Don83].

It is an invariant of 3-manifolds which (almost) gives a  $(3+1)$ -dimensional TQFT. Via Heegaard diagrams and Lagrangian Floer homology, it assigns to each  $\text{spin}^c$  3-manifold  $(Y, \mathfrak{t})$  four  $\mathbb{Q}$ -graded, relatively- $\mathbb{Z}$ -graded module,  $\text{HF}^\circ(Y, \mathfrak{t})$ , where  $\circ$  stands for  $+$ ,  $-$ ,  $\infty$ , or for a hat. We will focus on the case of  $\circ = +$ , when we obtain the so-called ‘plus’ flavour of Heegaard Floer homology,  $\text{HF}^+(Y, \mathfrak{t})$ . When talking about Heegaard Floer homology, we will always work with coefficients over  $\mathbb{F} = \mathbb{F}_2$ , the field with two elements.

The theory is a bit simpler when  $Y$  is a *rational homology sphere*, i.e. when  $b_1(Y) = 0$ , so we will start by discussing this case. For the rest of this section,  $Y$  will therefore be a rational homology sphere. We introduce here also a class of “small” 4-manifolds with boundary: a *rational homology 4-ball* is an orientable compact 4-manifold  $W$  with  $\tilde{H}_*(W; \mathbb{Q}) = 0$ .

Under this assumption,  $\mathrm{HF}^+(Y, \mathfrak{t})$  fits in a short exact sequence of  $\mathbb{F}[U]$ -modules:

$$0 \rightarrow \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U] \rightarrow \mathrm{HF}^+(Y, \mathfrak{t}) \rightarrow \mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{t}) \rightarrow 0,$$

where  $\mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{t})$  has finite dimension over  $\mathbb{F}$ . Let us consider the grading: the module on the left,  $\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$  is also graded and each monomial  $U^k$  is homogeneous, so the image of the injection has a well-defined minimal grading. This minimal grading is called the *correction term*, or *d-invariant*, of  $(Y, \mathfrak{t})$ , and is denoted by  $d(Y, \mathfrak{t})$ . The instanton and Seiberg–Witten analogues of correction terms were first studied by Frøyshov [Frø96], and were shown to have many useful properties with respect to cobordisms. The Heegaard Floer version was introduced by Ozsváth and Szabó in [OSz03] and shown to agree with their Seiberg–Witten counterparts by Ramos [Ram18].

**Theorem 2.6** (Ozsváth, Szabó). *Heegaard Floer correction terms satisfy the following:*

- (i)  $d(S^3, \mathfrak{t}) = 0$ ;
- (ii)  $d(Y, \mathfrak{t}) = d(Y, \bar{\mathfrak{t}})$ , where the bar denotes conjugation of  $\mathrm{spin}^c$  structures;
- (iii)  $d(Y \# Y', \mathfrak{t} \# \mathfrak{t}') = d(Y, \mathfrak{t}) + d(Y', \mathfrak{t}')$ ;
- (iv) if  $(W, \mathfrak{s})$  is a negative definite  $\mathrm{spin}^c$  cobordism from  $(Y, \mathfrak{t})$  to  $(Y', \mathfrak{t}')$ :

$$d(Y, \mathfrak{t}) + \frac{c_1^2(\mathfrak{s}) - b_2(W)}{4} \leq d(Y', \mathfrak{t}'),$$

and in particular  $d(Y, \mathfrak{t})$  is a  $\mathrm{spin}^c$  rational homology cobordism invariant;

- (v) combining the last two properties, we also have that  $d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t})$ .

We separately state a corollary of point (v) above. We will use this corollary over and over again in this chapter.

**Corollary 2.7.** *If the 3-manifold  $Y$  bounds a rational homology ball  $W$ , then, for each  $\mathrm{spin}^c$  structure  $\mathfrak{t}$  on  $Y$  that extends to  $W$ ,  $d(Y, \mathfrak{t}) = 0$ .*

The advantage of Heegaard Floer homology over its older cousins is computability. A nice example of this feature is the following theorem due to Rasmussen [Ras04] and Ni and Wu [NW15], who computed the correction term of positive, integer surgeries along a knot  $K$ . We denote  $n$ -surgery along a null-homologous knot  $K$  in a 3-manifold  $Y$  by  $Y_n(K)$ , and the unknot in  $Y$  by  $\mathcal{O}_Y$  (or simply by  $\mathcal{O}$  if there is no ambiguity).

**Theorem 2.8** (Rasmussen; Ni, Wu). *Let  $K$  be a knot in  $S^3$ . There exists a sequence  $\{V_i(K)\}_{i \geq 0}$  of non-negative integers such that for every positive integer  $n$ :*

$$d(S_n^3(K), \mathfrak{t}_i) = d(S_n^3(\mathcal{O}), \mathfrak{t}_i) - 2 \max\{V_i(K), V_{n-i}(K)\}.$$

Moreover, the sequence  $\{V_i(K)\}_{i \geq 0}$  is weakly decreasing,  $V_i(K) - V_{i+1}(K) \leq 1$ , and  $V_i(K) = 0$  whenever  $i \geq g(K)$ .



## 2 Piecewise-linear surfaces

The first instance where the invariants  $V_i(K)$  start vanishing is called  $\nu^+(K)$ : formally,  $\nu^+(K) = \min\{k \mid V_k(K) = 0\}$  [HW16].

In the statement we are implicitly using Ozsváth and Szabó's labelling of spin<sup>c</sup> structures on  $S_n^3(K)$ :  $\mathfrak{t}_i$  is the restriction of the spin<sup>c</sup> structure  $\mathfrak{s}_i$  on  $X_n(K)$ , the trace of the surgery (i.e.  $B^4$  with an  $n$ -framed 2-handle attached along  $K$ ), and  $\mathfrak{s}_i$  is defined by its Chern class,  $\langle c_1(\mathfrak{s}_i), G \rangle = n - 2i$ , where  $G \in H_2(X_n(K))$  is the homology class of the PL 2-sphere obtained by capping off the cone over  $K$  with the core of the 2-handle<sup>1</sup>.

The first summand in the formula above is independent of the knot  $K$ , and is easily computed [OSz04a]:

$$d(S_n^3(\mathcal{O}), \mathfrak{t}_i) = -\frac{1}{4} + \frac{(n - 2i)^2}{4n}.$$

The second term is computable for certain families of knots (including alternating knots and links of complex curve singularities), and is in principle combinatorially computable for all knots. We state explicitly the calculation for positive torus knots, and a bit less explicitly for all algebraic knots. (See Chapter 3 for more details.)

**Proposition 2.9** (Borodzik, Livingston [BL14]). *Let  $p, q > 1$  be coprime integers. We write  $\Gamma_{p,q}$  for the semigroup generated by  $p$  and  $q$ :  $\Gamma_{p,q} = \{hp + kq \mid h, k \in \mathbb{Z}_{\geq 0}\}$ . For each integer  $n$ , we define  $R_{p,q}(n) = \#\{\Gamma_{p,q} \cap (-\infty, n]\}$  and  $\delta_{p,q} = \frac{1}{2}(p-1)(q-1)$ . (Note that  $R_{p,q}(n) = 0$  if  $n < 0$  and that  $\delta_{p,q}$  is the genus of the  $(p, q)$ -torus knot.) Then:*

$$V_i(T(p, q)) = R_{p,q}(i + \delta_{p,q}) - i.$$

*More generally, if  $K$  is the link of (the germ of) a complex curve singularity  $(C, 0)$ , then the same statement holds with  $\Gamma_{p,q}$  replaced by the semigroup  $\Gamma_{(C,0)}$  of the singularity.*

Recall that the semigroup of a complex curve singularity  $(C, 0)$  is the set of multiplicities of intersection of germs of curve multi-branches with  $C$  at 0: more formally,

$$\Gamma_{(C,0)} = \{(D \cdot C)_0 \mid D \text{ is a union of curve branches}\} \subset \mathbb{Z}_{\geq 0}.$$

This is a semigroup of  $\mathbb{Z}_{\geq 0}$  since it is closed under addition (just take the union  $D \cup D'$  of two multi-branches). One can directly check that for  $(C, 0) = (\{x^p + y^q = 0\}, (0, 0))$  this is the semigroup  $\Gamma_{p,q}$  defined in the proposition.

We will be interested in understanding which rational homology 3-spheres bound rational homology 4-balls. This is a question that Casson posed, and is now a problem in Kirby's list. We do not expect to be able to give a reasonable characterisation of these 3-manifolds, but we will study the case of surgeries. A first, well-known obstruction for a rational homology 3-sphere  $Y$  to bound a rational homology 4-ball  $W$  is that the order of  $H_1(Y)$  has to be a square. This follows directly from inspecting the long exact sequence in homology for the pair  $(W, Y)$ . If  $Y$  is  $n$ -surgery along a knot  $K$ , this translates into  $n$  being a square.

<sup>1</sup>Strictly speaking, this requires orienting  $K$ . Changing the orientation corresponds to changing sign to  $G$ , which in turn corresponds to conjugating the spin<sup>c</sup> structure. This has no effect on correction terms, so we can afford to be a bit sloppy here.

In light of Corollary 2.7, we need to understand which  $\text{spin}^c$  structures on a 3-manifold potentially extend to a rational homology ball. The following proposition shows that, if the 3-manifold is an integer surgery along a knot in  $S^3$ , then the  $\text{spin}^c$  structures that extend are independent of the rational ball.

**Proposition 2.10** (Borodzik, Livingston [BL14], Aceto, G. [AG17]). *Suppose that  $S_{m_2}^3(K)$  bounds a rational homology ball  $W$ . The  $\text{spin}^c$  structure  $\mathfrak{t}_i$  extends to  $W$  if and only if  $i = \frac{1}{2}m(m - 2k - 1)$  for some integer  $k$  satisfying  $-\frac{m}{2} \leq k \leq \frac{m}{2}$ .*

## Heegaard Floer homology: other 3-manifolds, I

The structure of Heegaard Floer homology becomes more complicated when  $b_1$  is positive, i.e. when the 3-manifold is not a rational homology sphere. For simplicity, we will make two assumptions:

- we will assume that  $\mathfrak{t}$  is a *torsion*  $\text{spin}^c$  structure, i.e.  $c_1(\mathfrak{t}) \in H^2(Y)$  is torsion;
- we will assume that the triple cup product on  $Y$  is trivial; that is, for each triple  $\alpha, \beta, \gamma \in H^1(Y)$ ,  $\alpha \cup \beta \cup \gamma = 0$ .

Note that both assumptions are automatic when, for example,  $Y$  is a rational homology sphere.

$\text{HF}^+(Y, \mathfrak{t})$  has the structure of a module over  $\Lambda[U] = \mathbb{F}[U] \otimes_{\mathbb{Z}} \bigwedge^* H^1(Y)$ . (If  $Y$  is a rational homology sphere,  $\Lambda[U] = \mathbb{F}[U]$ ).

Under the assumptions above,  $\text{HF}^+(Y, \mathfrak{t})$  fits into an exact sequence:

$$0 \rightarrow M \rightarrow \mathbb{F}[U, U^{-1}] \otimes_{\mathbb{F}[U]} \Lambda[U] \rightarrow \text{HF}^+(Y, \mathfrak{t}) \rightarrow \text{HF}_{\text{red}}(Y, \mathfrak{t}) \rightarrow 0,$$

where  $\text{HF}_{\text{red}}(Y, \mathfrak{t})$  has finite dimension over  $\mathbb{F}$ , and  $M$  is an  $\Lambda[U]$ -submodule of  $\mathbb{F}[U, U^{-1}] \otimes_{\mathbb{F}[U]} \Lambda[U]$  that contains  $U^N \cdot \Lambda[U]$  for some integer  $N$ . That is to say, in  $\text{HF}^+(Y, \mathfrak{t})$  every element is  $U$ -torsion. Inside  $\mathbb{F}[U, U^{-1}] \otimes_{\mathbb{F}[U]} \Lambda[U]$  we have two distinguished copies of  $\mathbb{F}[U, U^{-1}]$ , one corresponding to  $\bigwedge^0 H^1(Y) \cong \mathbb{Z}$  and another to  $\bigwedge^{b_1(Y)} H^1(Y) \cong \mathbb{Z}$ . We call these copies  $T_{\text{top}}$  and  $T_{\text{bot}}$ , respectively.

We can now use the degree to define two numerical invariants associated to  $(Y, \mathfrak{t})$ , called the *top-most* and *bottom-most correction term* of  $(Y, \mathfrak{t})$ :

$$d_{\text{top}}(Y, \mathfrak{t}) = \min_{x \in T_{\text{top}}} \deg x;$$

$$d_{\text{bot}}(Y, \mathfrak{t}) = \min_{x \in T_{\text{bot}}} \deg x.$$

Note that these definitions agree when  $b_1(Y) = 0$ , and that they both agree with the definition of  $d(Y, \mathfrak{t})$ . An important difference is the lack of symmetry with respect to orientation-reversal: it is *not* true that  $d_{\text{bot}}(Y, \mathfrak{t}) = -d_{\text{bot}}(-Y, \mathfrak{t})$ . For instance  $d_{\text{bot}}(S^1 \times S^2) = -\frac{1}{2}$ .

Nevertheless, the top-most and bottom-most correction term of  $Y$  satisfy properties that analogous to those of the ordinary correction terms. We give two statements that mirror Theorem 2.6(iv) and Corollary 2.7.

**Theorem 2.11** (Ozsváth, Szabó). *Let  $(W, \mathfrak{s})$  be a  $\text{spin}^c$  4-manifold with boundary  $(Y, \mathfrak{t})$  such that:*

## 2 Piecewise-linear surfaces

- $W$  is negative semidefinite;
- the restriction map  $H^1(W) \rightarrow H^1(Y)$  is the zero map;
- $\mathfrak{t}$  is torsion;
- $Y$  has vanishing triple cup product.

Then the following inequality holds:

$$c_1^2(\mathfrak{s}) + b_2^-(W) \leq 4d_{\text{bot}}(Y, \mathfrak{t}) + 2b_1(Y).$$

Note that the third and fourth assumptions are necessary to even ensure that the inequality makes sense: if  $\mathfrak{t}$  is not torsion,  $c_1^2(\mathfrak{s})$  is not defined, and if  $Y$  has non-zero triple cup product or  $\mathfrak{t}$  is not torsion, then  $d_{\text{bot}}(Y, \mathfrak{t})$  is not defined either.

We say that a 4-manifold has *null* intersection form if it is both negative semi-definite and positive semi-definite. In other words, if it contains no surfaces of non-zero self-intersection. Note that 4-manifolds with the same rational homology as a connected sum of  $S^1 \times S^3$  are the only *closed* 4-manifolds with this property.

**Corollary 2.12.** *Let  $W$  be a compact, orientable 4-manifold with null intersection form, whose boundary  $\partial W = Y$  has vanishing triple cup product. Suppose furthermore that the map  $H^1(W) \rightarrow H^1(Y)$  induced by the inclusion is the zero map. Then for each torsion  $\text{spin}^c$  structure  $\mathfrak{t}$  on  $Y$  that extends to  $W$ ,*

$$d_{\text{bot}}(\pm Y, \mathfrak{t}) + 2b_1(Y) \geq 0$$

We now give some relevant examples of 3-manifolds that have (or do not have) vanishing triple cup product.

*Example 2.13.* If  $F$  is an orientable surface of positive genus, then  $F \times S^1$  does *not* have vanishing triple cup product. This is easily seen, since the orientation class in  $H^2(F)$  is a wedge of two classes in  $H^1(F)$ , and the generator of  $H^1(S^1)$  pairs non-trivially with the orientation class in  $H^2(F)$ .

*Example 2.14.* If  $Y$  is a 3-manifold with vanishing triple cup product and  $K \subset Y$  is a null-homologous knot, then  $Y_n(K)$  has vanishing triple cup product for each  $n \neq 0$ . In fact, up to torsion,  $H^*(Y_n(K))$  and  $H^*(Y)$  are isomorphic as rings. This can be seen, for instance, by considering intersections of triples of surfaces in  $Y_n(K)$ : since the dual knot  $\tilde{K}$  to  $K$  in  $Y_n(K)$  is rationally null-homologous, each surface in  $Y_n(K)$  can be made disjoint from  $\tilde{K}$  by tubing (an operation that does not affect the homology class), so each class in  $H_2(Y_n(K))$  is represented by a surface which lives in  $Y_n(K) \setminus \tilde{K} = Y \setminus K$ , and triples of surfaces in  $Y$  intersect trivially (with signs).

## Heegaard Floer homology: other 3-manifolds, II

The assumption on the triple cup product of the 3-manifold being trivial is a rather annoying one, for instance because it excludes products of circles and surfaces, as we have seen in Example 2.13. In terms of neighbourhoods of PL surfaces, more generally we are excluding embedded surfaces of positive genus and self-intersection 0.

This gap in the literature was fixed in a collaboration with Behrens [BG18] by passing to *fully-twisted coefficients*. (We note that other refinements of bottom-most and top-most correction terms have been developed by Levine and Ruberman [LR14].) That is to say, instead of working with coefficients in  $\mathbb{F}[U]$ , we work over  $R_Y[U]$ , where  $R_Y = \mathbb{F}_2[H_2(Y; \mathbb{Z})]$ . The resulting Heegaard Floer chain complexes come with a (non-trivial!) action by  $H_2(Y; \mathbb{Z})$ , and their homology groups are the so-called *fully-twisted* Heegaard Floer homology groups of  $Y$ , denoted with  $\underline{\text{HF}}^\circ(Y, \mathfrak{t})$ . Ozsváth and Szabó first considered these groups and proved that  $\underline{\text{HF}}^+(Y, \mathfrak{t})$  fits into a short exact sequence of  $R_Y[U]$ -modules:

$$0 \rightarrow \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U] \rightarrow \underline{\text{HF}}^+(Y, \mathfrak{t}) \rightarrow \underline{\text{HF}}_{\text{red}}(Y, \mathfrak{t}) \rightarrow 0,$$

where the  $R_Y$ -action on the left-hand side is trivial (i.e.  $h$  acts as the identity for every  $h \in H_2(Y; \mathbb{Z}) \subset R_Y^*$ ) and the group  $\underline{\text{HF}}_{\text{red}}(Y, \mathfrak{t})$  is annihilated by  $U^M$  for some  $M \geq 0$ . (Note that this does not imply that the latter group is finite-dimensional over  $\mathbb{F}$ !) As in the case of ordinary correction terms, the image of the left-most map has a minimal degree.

**Definition 2.15** (Behrens, G.). The minimal grading of any element in the image of the map  $\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U] \rightarrow \underline{\text{HF}}^+(Y, \mathfrak{t})$  is called the *fully-twisted correction term*, or  *$\underline{d}$ -invariant*, of  $(Y, \mathfrak{t})$ , and is denoted by<sup>2</sup>  $\underline{d}(Y, \mathfrak{t})$ .

This invariant turns out to be additive, like the ordinary correction terms for rational homology spheres, but not orientation-reversal symmetric, like the bottom-most correction terms for 3-manifolds with trivial triple cup product. The key result we proved in [BG18] is an analogue of Theorem 2.6(iv) for fully-twisted correction terms.

**Theorem 2.16** (Behrens, G.). *Let  $(W, \mathfrak{s})$  be a negative semidefinite  $\text{spin}^c$  cobordism between torsion  $\text{spin}^c$  3-manifolds  $(Y, \mathfrak{t})$  and  $(Y', \mathfrak{t}')$  such that the inclusion  $Y \hookrightarrow W$  induces an injection  $H_1(Y; \mathbb{Q}) \rightarrow H_1(W; \mathbb{Q})$ . Then*

$$c_1^2(\mathfrak{s}) + b_2^-(W) \leq 4\underline{d}(Y', \mathfrak{t}') - 4\underline{d}(Y, \mathfrak{t}) + 2b_1(Y') - 2b_1(Y).$$

*In particular,  $\underline{d}$  is a  $\text{spin}^c$  rational homology cobordism invariant.*

We also computed the invariant in the case of 0-surgeries along knots in  $S^3$  and of products.

**Proposition 2.17.** *Let  $K$  be a knot in  $S^3$ . Then*

$$\underline{d}(S_0^3(K), \mathfrak{t}_0) = d_{\text{bot}}(S_0^3(K), \mathfrak{t}_0) = -\frac{1}{2} - 2V_0(-K).$$

*If  $F$  is a closed orientable surface of genus  $g$ , then*

$$\underline{d}(S^1 \times F) = \begin{cases} -\frac{1}{2} & \text{if } g \text{ is even,} \\ \frac{1}{2} & \text{if } g \text{ is odd.} \end{cases}$$

---

<sup>2</sup>Rather unfortunately, the notation  $\underline{d}$  is also used for one of the two correction terms of rational homology 3-spheres coming from involutive Heegaard Floer homology [HM17]. We keep the notation as in [BG18], which will certainly not cause any confusion in this manuscript.

As a byproduct, for instance, one can give strong restrictions on  $Q_W$  whenever  $W$  is a negative semi-definite 4-manifold whose boundary is  $T^3$  or  $F \times S^1$ , where  $F$  is a surface of genus 2. For instance, if  $W$  is additionally supposed to be spin, then  $Q_W / \ker Q_W$  is either  $-E_8$  or trivial.

## 2.3 PL surfaces in $S^4$ and knot cobordisms

We start by analysing PL surfaces in the simplest 4-manifold: the 4-sphere. We phrase results mostly in terms of knot cobordisms, which one can think of as what remains when one removes the singularities of a PL surface in  $S^4$ . (This is essentially precise when the PL surface has two singularities, less so when it has one or at least three.) Most results we state will actually hold in (rational or integral) 4-spheres, but we state them in the simpler context of  $S^4$ .

### Knot cobordisms

A *smooth cobordism* between two oriented knots  $K_0, K_1 \subset S^3$  is a properly embedded, oriented compact surface  $B \subset S^3 \times [0, 1]$  whose boundary is  $K_1 \times \{1\} \cup K_0^r \times \{0\}$  (here  $r$  means orientation reversal). We say that two knots are *smoothly concordant* if there exists a smooth cobordism of genus 0 between them. Smooth concordance is an equivalence relation, and the set of knots up to smooth concordance, endowed with the connected sum of knots  $\#$ , is an Abelian group  $\mathcal{C}$  known as the *smooth concordance group*. The inverse of a knot  $K$  is then the mirror of  $K$  with the reversed orientation, which we denote with  $-K$ ; that is,  $(S^3, -K)$  is orientation-preserving diffeomorphic to  $-(S^3, K^r)$ .

We will drop the adjective ‘smooth’ and only talk about cobordism and concordance. (The locally-flat counterparts of the concepts are usually called *locally-flat cobordisms* and *topological concordance*, and they are a rich area of study of their own as well as in comparison with their smooth cousins.)

The basic question we try to address here is the following version of the minimal genus problem.

**Question 2.18.** Given two knots  $K_0, K_1$ , what is the minimal genus of a cobordism between them?

The minimal genus of a cobordism induces a distance  $d_{\mathcal{C}}$  on  $\mathcal{C}$ , called the *cobordism distance*:

$$d_{\mathcal{C}}(K_0, K_1) = \min\{g(F) \mid F \text{ is a smooth cobordism from } K_0 \text{ to } K_1\}.$$

It is easy to see that  $d_{\mathcal{C}}$  is indeed a distance on  $\mathcal{C}$ , whereas it is only a pseudo-distance if viewed as a function on the set of isotopy classes of knots. It is sub-additive with respect to the connected sum.

The connection between cobordisms and Section 2.1 is given in the following statement.

**Proposition 2.19.** *The cobordism distance between  $K_0$  and  $K_1$  is  $g$  if and only if there exists a genus- $g$  PL surface in  $S^4$  with two singularities whose links are  $K_0$  and  $-K_1$  respectively.*

The proof is evident<sup>3</sup>: given a cobordism, we can cap off its ends with the cones over the two knots, and given a PL surface as in the statement we can remove a regular neighbourhood of its singularities. The mirror of  $K_1$  appears because we are viewing  $S^3 \times \{1\}$  as the boundary of  $S^3 \times [0, 1]$  when looking at a cobordism, and as the boundary of the regular neighbourhood of the singular point when looking at a PL surface, and the two orientations are opposite.

We have already presented, albeit without explicit mention, a number of concordance invariants and lower bounds for genera of cobordisms. A special place among concordance invariants is occupied by Levine–Tristram signatures, which are both very efficient and easily computable. From our perspective, their only shortcoming is that they are locally-flat obstructions, so they cannot see the difference between the topological and smooth concordance groups.

Once again, cobordisms between knots naturally arise in complex algebraic geometry, this time in the context of *deformations of singularities*, which we define presently.

Let  $\{F_s\}_s$  be an analytic family of polynomials in  $\mathbb{C}[x, y]$  parametrised by  $s$  in the unit disc  $\Delta \subset \mathbb{C}$  and let  $C_s = \{F_s = 0\} \subset \mathbb{C}^2$ . Suppose that  $F_s(0, 0) = 0$  for every  $s$ . For each  $s \ll 1$  we say that  $(C_s, 0)$  is a *deformation* of  $(C_0, 0)$ .

**Proposition 2.20.** *If  $(C_s, 0)$  is a deformation of  $(C_0, 0)$ , there is a cobordism of Euler characteristic  $\mu(C_0, 0) - \mu(C_s, 0)$  between the links of  $(C_s, 0)$  and  $(C_0, 0)$ .*

Here  $\mu(C_s, 0)$  denotes the *Milnor number* of the curve singularity  $(C_s, 0)$ : this is defined as  $b_1$  of the Milnor fibre of  $(C_s, 0)$ , which in turn is  $M = \{F_s = \varepsilon\} \cap B_\eta^4$  for  $\varepsilon \ll \eta \ll 1$ . Milnor studied isolated higher-dimensional hypersurface singularities. He proved that for a hypersurface singularity in  $\mathbb{C}^{n+1}$ , the Milnor fibre  $M$  (defined as in the curve case we just mentioned) has the homotopy type of a wedge of  $n$ -spheres [Mil68], so its middle-dimensional Betti number  $\mu$ , called the Milnor number of the singularity, is the only homotopy invariant needed to describe  $M$ . (Note that this is obvious for curve singularities in  $\mathbb{C}^2$ , since open surfaces are automatically homotopy-equivalent to wedges of circles.)

*Sketch of proof.* Choose a radius  $\varepsilon > 0$  such that  $S_\varepsilon^3 \subset \mathbb{C}^2$  is transverse to the curve  $F_0$  and the intersection  $S_\varepsilon^3 \cap C_0$  defines the link of the singularity of  $(C_0, 0)$ .

Choose another radius  $\delta > 0$  such that for every  $|s| < \delta$ ,  $C_s$  is also transverse to  $S_\varepsilon^3$ . In particular,  $S_\varepsilon^3 \cap C_s$  is isotopic to  $S_\varepsilon^3 \cap C_0$ .

Now, for each  $s$  choose  $\eta < \varepsilon$  such that  $S_\eta^3$  is transverse to<sup>4</sup>  $C_s$ . By (suitably) perturbing the family  $F_s$  if necessary, we can make  $C_s$  smooth inside the cylinder  $B_\varepsilon^4 \setminus B_\eta^4$ . Alternatively, we can replace the link  $K$  of each singularity of  $C_s$  (away from the origin) with  $M_K$ , to obtain a surface  $F$  in the cylinder  $B_\varepsilon^4 \setminus B_\eta^4$ . The surface  $F$  is the desired cobordism.  $\square$

When the links of  $(C_0, 0)$  and  $(C_s, 0)$  above are two knots,  $K_0$  and  $K_1$ , the proposition produces for us a cobordism of genus  $g(K_0) - g(K_1)$ . As a corollary of the proof of the Thom conjecture (Theorem 2.1), this cobordism has minimal genus.

<sup>3</sup>Modulo sweeping under the rug the fact that every locally-flat PL surface can be smoothed, which is far from evident, but classical.

<sup>4</sup>We could choose a value of  $\eta$  which works for every  $|s| < \delta$ , but we do not need it here.

The driving question of my joint work with Bodnár and Celoria [BCG17] was to compute the minimal genus of cobordisms between algebraic knots and determine when this minimal genus is  $|g(K_0) - g(K_1)|$ —these cobordisms are sometimes called *optimal*. In particular, if there is no optimal cobordism between two algebraic knots, then there can be no deformation of one of the two singularities to the other.

## Cobordisms and correction terms

The following theorem gives a lower bound on the cobordism distance between two knots, expressed in terms of the invariants  $V_i$  of Theorem 2.8.

**Theorem 2.21** (Bodnár, Celoria, G. [BCG17]). *If there is a genus- $g$  cobordism between two knots  $K_0$  and  $K_1$ , then for each  $k \geq 0$ :*

$$V_{k+g}(K_0) \leq V_k(K_1) \quad \text{and} \quad V_{k+g}(K_1) \leq V_k(K_0).$$

We draw several corollaries from this statement. First, given that for the link of a singularity the  $V_i$ s are computed in terms of the semigroup of the singularity, we can extract from the theorem a lower bound for the genus of a cobordism of two algebraic knots in terms of their semigroups. We phrase it in terms of the enumeration function of the semigroup: given a set  $\Gamma \subset \mathbb{Z}_{\geq 0}$ , the function  $\Gamma: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  is the unique strictly increasing function whose image is  $\Gamma$ . (We give the function the same name as the semigroup, but this will not create any confusion.)

**Corollary 2.22.** *Let  $K_0$  and  $K_1$  be the links of two complex curve singularities with semigroups  $\Gamma_0$  and  $\Gamma_1$ , respectively. Then*

$$d(K_0, K_1) \geq g(K_0) - g(K_1) + \max_{n \geq 0} \{\Gamma_1(n) - \Gamma_0(n)\}.$$

*The same inequality holds by swapping the roles of  $K_0$  and  $K_1$ .*

Note that the expression on the right-hand side makes sense:  $\Gamma_i(n) = g(K_i) + n$  for  $n \geq g(K_i)$ , so the last summand is bounded.

We also give more knot-theoretic applications to the cobordism distance and the Gordian distance, via the invariant  $\nu^+$  defined above. The latter measures how many crossing changes one has to do to pass from (a projection of) a knot  $K_0$  to (that of) a knot  $K_1$ .

**Corollary 2.23.** *The invariant  $\nu^+$  can be used to give lower bounds on the concordance distance*

$$|\nu^+(K_0) - \nu^+(K_1)| \leq d_c(K_0, K_1)$$

*and on the signed Gordian distance:*

$$\nu^+(K') \leq \nu^+(K) \leq \nu^+(K') + 1$$

*whenever  $K$  is obtained from  $K'$  by changing a negative crossing of a projection of  $K'$  into a positive one.*

Contrarily to many other Floer-theoretic knot invariants,  $\nu^+$  is *not* a homomorphism, and does *not* change sign when taking the inverse in concordance. So it could (and does) happen that one gets interesting information by considering a knot and its mirror (or a pair of knots and their mirrors) in the theorem and in the corollaries above. (However, see below for a constraint on the behaviour of  $\nu^+$  under connected sums.)

*Sketch of proof of Theorem 2.21.* Suppose that  $F$  is a genus- $g$  cobordism between  $K_0$  and  $K_1$ . The main idea is to construct a sequence of PL surfaces in a sequence of 4-manifolds and apply the Ozsváth–Szabó machinery to extract an inequality on correction terms.

The 4-manifolds we use are the traces  $X_n(K_1)$  of surgeries along  $K_1$ . Recall that this manifold is obtained from the 4-ball  $B^4$  by attaching a single 2-handle along  $K_1 \subset S^3$ , with framing  $n$ .

Within  $X_n(K_1)$  we find a PL surface of genus  $g$ , self-intersection  $n$ , with a singularity of type  $K_0$ . Indeed, the cobordism  $F$  can be capped off with a cone over  $(S^3, K_0)$ , and give a surface in  $B^4$  of genus  $g$  and with a singularity of type  $K_0$ , whose boundary is  $K_1$ . When we attach the 2-handle to obtain  $X_n(K_1)$ , we can cap off this latter surface with the core of the 2-handle, obtaining a surface that we call  $F_n$ .

The complement of  $F_n$  has null intersection form and torsion  $H_1$ , so we can apply the Ozsváth–Szabó inequality of Theorem 2.11. Some bookkeeping yields the desired inequality.  $\square$

In the same paper, we also proved the following result about  $\nu^+$ .

**Theorem 2.24** (Bodnár, Celoria, G. [BCG17]). *The concordance invariant  $\nu^+$  is subadditive. That is,*

$$\nu^+(K_0 \# K_1) \leq \nu^+(K_0) + \nu^+(K_1).$$

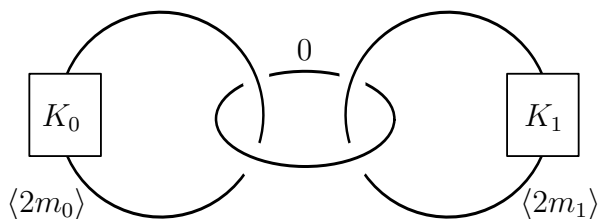


Figure 2.2: The surgery diagram for the upside-down cobordism  $\overline{W}$  from  $S_{2m_0}^3(K_0) \# S_{2m_1}^3(K_1)$  to  $S_{2(m_0+m_1)}^3(K_0 \# K_1)$ . The coefficients in brackets represent the negative boundary  $\partial_- W$ .

*Sketch of proof.* Once again, we construct a suitable cobordism  $W$  between two 3-manifolds associated to  $K_0$  and  $K_1$ : if we turn it upside-down, we get the cobordism  $\overline{W}$  shown in Figure 2.2.  $W$  goes from the surgeries along  $K_0$  and  $K_1$  with framings  $2m_0$  and  $2m_1$  to the surgery along  $K_0 \# K_1$  with framing  $2m_0 + 2m_1$ . A careful analysis of the inequality induced on the correction terms by a *spin* structure on  $W$



yields the inequality  $V_{m_0+m_1}(K_0\#K_1) \leq V_{m_0}(K_0) + V_{m_1}(K_1)$ , from which the statement readily follows. Interestingly, for the case  $m_0 = m_1 = 0$  the inequalities given above do not suffice, and we need to work with fully-twisted coefficients and apply Theorem 2.16. (The case where  $m_0$  and  $m_1$  are both positive uses Theorem 2.6(iv), while the case where only one of them vanishes uses Theorem 2.11.)  $\square$

## 2.4 PL surfaces in $\mathbb{C}\mathbb{P}^2$

The next simplest 4-manifold after  $S^4$  is probably  $\mathbb{C}\mathbb{P}^2$ . While our primary objective is the study of PL surfaces in  $\mathbb{C}\mathbb{P}^2$ , we will emphasise when statements hold for 4-manifolds with the same integer, rational, or  $\mathbb{Z}/2\mathbb{Z}$ -homology as  $\mathbb{C}\mathbb{P}^2$ .

### PL spheres

In this section, we will focus on PL spheres in smooth 4-manifolds, and especially in the complex projective plane  $\mathbb{C}\mathbb{P}^2$ . This is motivated by the study of *rational cuspidal curves* in algebraic geometry, which we will discuss in more detail in the next chapter.

Given a rational homology  $\mathbb{C}\mathbb{P}^2$ ,  $X$ , and a closed, orientable surface  $F$  embedded in  $X$ , we say that  $F$  has *degree*  $d$  if  $F$  has self-intersection  $d^2$ . Equivalently,  $F$  has degree  $d$  if and only if there exist a class  $H \in H_2(X)$  that generates  $H_2(X)/\text{Tor}$  and a class  $\tau \in \text{Tor}(H_2(X))$  such that  $[F] = dH + \tau$ . This notion generalises the notion of degree for complex curves in  $\mathbb{C}\mathbb{P}^2$ .

The first observation we make is the following.

**Proposition 2.25** (Borodzik, Livingston [BL14]). *Let  $X$  be a rational homology  $\mathbb{C}\mathbb{P}^2$  and  $F \subset X$  a PL sphere of degree  $d > 0$ . Then  $S_{d^2}^3(K_F)$  bounds a rational homology ball.*

Recall that  $K_F$  is the connected sum  $K_1\#\dots\#K_\nu$ , where  $K_1, \dots, K_\nu$  are the types of the singularities of  $F$ .

*Proof.* By definition, since  $F$  has degree  $d$ ,  $F \cdot F = d^2$ . By Proposition 2.4,  $F$  has a regular neighbourhood diffeomorphic to the knot trace  $T = X_{d^2}(K_F)$ . By the long exact sequence of the pair  $(X, T)$ , we get:

$$\begin{aligned} 0 &\rightarrow H_4(X; \mathbb{Q}) \rightarrow H_4(X, T; \mathbb{Q}) \rightarrow 0, \\ 0 &\rightarrow H_3(X, T; \mathbb{Q}) \rightarrow H_2(T; \mathbb{Q}) \rightarrow H_2(X; \mathbb{Q}) \rightarrow H_2(X, T; \mathbb{Q}) \rightarrow 0, \\ &0 \rightarrow H_1(X, T; \mathbb{Q}) \rightarrow 0. \end{aligned}$$

From these and from the fact that the map  $H_2(T; \mathbb{Q}) \rightarrow H_2(X; \mathbb{Q})$  is non-zero (since  $[F] \neq 0 \in H_2(X; \mathbb{Q})$ ), we quickly see that  $H_*(X, T; \mathbb{Q}) = \mathbb{Q}_{(4)}$ . By excision, if we let  $W$  to be the closure of  $X \setminus T$ ,  $H_*(W, \partial W; \mathbb{Q}) = H_*(X, T; \mathbb{Q}) = \mathbb{Q}_{(4)}$ , and by Poincaré–Lefschetz duality  $W$  is a rational homology ball. The boundary of  $-W$  is the boundary of  $T$ , which is exactly  $S_{d^2}^3(K_F)$ .  $\square$

An immediate corollary of the proposition and of Corollary 2.7 is the following.

**Theorem 2.26** (Borodzik, Livingston [BL14]; Aceto, G. [AG17]). *Let  $X$  be a rational homology  $\mathbb{C}\mathbb{P}^2$  and  $F \subset X$  a PL sphere of degree  $d > 0$ . Then for every  $0 \leq k < \frac{d}{2}$ :*

$$V_{\frac{d(d-2k-1)}{2}}(K) = \frac{k(k+1)}{2}.$$

**Corollary 2.27** (Aceto, G. [AG17]). *For each knot  $K$ , there exist at most two values of  $d > 0$  such that there exists a PL sphere of degree  $d$  in a homology  $\mathbb{C}\mathbb{P}^2$  with a unique singularity of type  $K$ .*

The corollary is easily proven using the invariant  $\nu^+(K)$ . From Theorem 2.26, choosing  $k = 0$  and  $k = 1$  one sees that  $\frac{d(d-3)}{2} < \nu^+(K) \leq \frac{d(d-1)}{2}$ . One easily proves that there are at most two values of  $d$  satisfying both inequalities, and furthermore if there are two, they are consecutive. (Whenever  $\nu^+(K)$  is not a triangular number, there exists a unique solution to the two inequalities, and hence at most one value of  $d$ .)

In [AG17] and [AGLL20] we used Theorem 2.26 in combination with Donaldson's diagonalisation theorem (Theorem 2.5) to completely classify for which torus knots  $T(p, q)$  there is a PL sphere in a homology  $\mathbb{C}\mathbb{P}^2$  whose unique singularity is of type  $T(p, q)$ . (This is a topological analogue of a unicuspidal curve with one Puiseux pair in  $\mathbb{C}\mathbb{P}^2$ , objects which were studied in [FLMN07] and which will be discussed in the next chapter.)

**Theorem 2.28.** *There is a PL sphere of self-intersection  $n \neq 0$  and with a unique singularity of type  $T(p, q)$  in some rational homology  $\mathbb{C}\mathbb{P}^2$ ,  $X$ , and degree  $d$ , if and only if the triple  $(p, q; d)$  belongs to a set of triples  $\mathcal{G} \cup \mathcal{R} \cup \mathcal{L}$ , defined below.*

We now define the three sets  $\mathcal{G}$ ,  $\mathcal{R}$ , and  $\mathcal{L}$ . Recall the definition of the Fibonacci sequence  $\{F_k\}$ :  $F_{k+1} = F_k + F_{k-1}$ , with  $F_0 = 0$  and  $F_1 = 1$ . We also define three auxiliary sequences<sup>5</sup>  $\{R_k\}$ ,  $\{S_k\}$ , and  $\{T_k\}$ . Note that we use the same recursive relation and only change the initial values.

$$\begin{cases} R_0 = 1, \\ R_1 = 3, \\ R_{k+1} = 6R_k - R_{k-1}; \end{cases} \quad \begin{cases} S_0 = 1, \\ S_1 = 1, \\ S_{k+1} = 6S_k - S_{k-1}; \end{cases} \quad \begin{cases} T_0 = 0, \\ T_1 = 1, \\ T_{k+1} = 6T_k - T_{k-1}. \end{cases}$$

The set  $\mathcal{G}$  contains the following four families and two exceptional cases:

- (1)  $(p, q; d) = (p, p+1; p^2)$  for some  $p \geq 2$ ;
- (2)  $(p, q; d) = (p, p+1; (p+1)^2)$  for some  $p \geq 2$ ;
- (3)  $(p, q; d) = (p, 4p \pm 1; (2p)^2)$  for some  $p \geq 2$ ;
- (4)  $(p, q; d) = (R_k, R_{k+1}; R_k R_{k+1} - 2)$  for some  $k \geq 1$ ;
- (5)  $(p, q; d) \in \{(3, 22; 64), (6, 43; 256)\}$ .

The set  $\mathcal{R}$  comprises the following seven families and two exceptional cases:

- (6)  $(p, q; d) = (r^2, (r+1)^2; r^2(r+1)^2)$  for some  $r \geq 2$ ;

---

<sup>5</sup>These sequences are in fact well-known:  $R_k$  is the  $2k^{\text{th}}$  half-companion Pell number;  $S_k$  is the  $(2k-1)^{\text{th}}$  Pell number;  $2T_k$  is the  $2k^{\text{th}}$  Pell number.

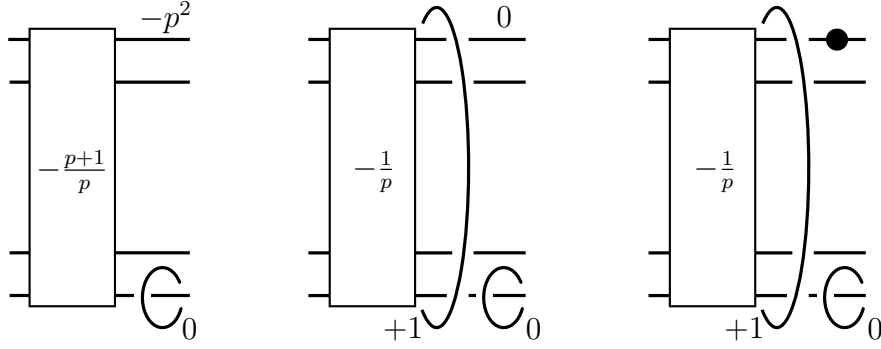


Figure 2.3: From left to right: the double of  $X_{-p^2}(-T(p, p + 1))$ , its (anti-holomorphic) blow-up, and a handle decomposition of  $\mathbb{C}\mathbb{P}^2$ , obtained by doing a zero-dot surgery in the blow-up  $X_{-p^2}(-T(p, p + 1))$ . Thus  $\mathbb{C}\mathbb{P}^2$  contains  $X_{p^2}(T(p, p + 1))$ .

- (7)  $(p, q; d) = (r^2, (2r \pm 1)^2; r^2(2r \pm 1)^2)$  for some  $r \equiv 2 \pmod{4}$  and  $r \geq 2$ ;
- (8)  $(p, q; d) = (r^2, (2r \pm 2)^2; r^2(2r \pm 2)^2)$  for some  $r \equiv \mp 3 \pmod{8}$  and  $r \geq 5$ ;
- (9)  $(p, q; d) = (F_k^2, F_{k+1}^2; F_k^2 F_{k+1}^2)$  for some  $k \geq 3$ ;
- (10)  $(p, q; d) = (F_{2k-1}^2, F_{2k+1}^2; F_{2k-1}^2 F_{2k+1}^2)$  for some  $k \geq 2$ ;
- (11)  $(p, q; d) = (S_k^2, 4T_k^2; 4T_k^2 S_k^2)$  for some  $k \geq 2$ ;
- (12)  $(p, q; d) = (4T_k^2, S_{k+1}^2; S_{k+1}^2 4T_k^2)$  for some  $k \geq 2$ ;
- (13)  $(p, q; d) \in \{(9^2, 14^2; 9^2 \cdot 14^2), (11^2, 14^2; 11^2 \cdot 14^2)\}$ .

Finally, the set  $\mathcal{L}$  comprises the following five families:

- (14)  $(p, q; d) = (2r - 1, 2r + 1; 4r^2)$  for some  $r \geq 2$ ;
- (15)  $(p, q; d) = (F_{2k}, F_{2k+2}; F_{2k} F_{2k+2} + 1)$  for some  $k \geq 1$ ;
- (16)  $(p, q; d) = (F_{2k+1}, F_{2k+3}; F_{2k+1} F_{2k+3} - 1)$  for some  $k \geq 1$ ;
- (17)  $(p, q; d) = (F_{2k+1}, F_{2k+5}; F_{2k+1} F_{2k+5} - 1)$  for some  $k \geq 1$ ;
- (18)  $(p, q; d) = (S_{k+1}, S_{k+2}; S_{k+1} S_{k+2} - 1)$  for some  $k \geq 1$ .

The subdivision into three families is due to the topology of the boundary of a tubular neighbourhood of  $F$ : triples in  $\mathcal{G}$  correspond to surgeries that are Seifert fibred 3-manifolds with three singular fibres, those in  $\mathcal{R}$  correspond to connected sums of two lens spaces, and those in  $\mathcal{L}$  correspond to lens space surgeries.

We can also say that *some* of the embeddings above can be realised for  $X = \mathbb{C}\mathbb{P}^2$ .

**Proposition 2.29.** *For  $(p, q; n)$  in families (1), (2), (3), and (5) above, the embedding can be realised in  $\mathbb{C}\mathbb{P}^2$ . That is, for those triples there exists a PL sphere of self-intersection  $n$  whose unique singularity is of type  $T(p, q)$ .*

In fact, families (2), (3) when the sign is negative, and (5) are known to be realised by rational cuspidal curves in  $\mathbb{C}\mathbb{P}^2$  [FLMN07]. In Figure 2.3 we draw an explicit handle decomposition of  $\mathbb{C}\mathbb{P}^2$  containing  $X_{p^2}(T(p, p + 1))$ .

The proof of Theorem 2.28 is lengthy and technical. We only briefly sketch some of the ideas that go into it.

*Sketchy sketch of proof.* Three main ideas go into the proof of this theorem. The first two serve to prove that the list is complete, while the third to show that every element in the list is realised.

Let us suppose that there exists  $F$  with a unique singularity of type  $T(p, q)$  and degree  $d$ .

The first idea is to use correction terms, and specifically combining Theorem 2.26 with the computation of Proposition 2.9, to obtain the inequality  $p < 8q$ . The inequality  $p < 9q$  was obtained in [AG17] and the refinement  $p < 8q$  was obtained in [AGLL20], building on the same ideas appearing in the proof of Corollary 2.27. By Corollary 2.27, we can also pin down the possible degrees  $d$  once we fix  $p$  and  $q$ .

The second idea is to use Donaldson's diagonalisation theorem, as pioneered by Lisca [Lis07]: the trace  $N(F)$  of an integer surgery along  $T(p, q)$  has the same boundary as a definite plumbing of spheres  $P$  (which can be positive or negative definite, depending on the exact values of  $d$ ,  $p$ , and  $q$ ). Surgering out  $N(F)$  and replacing it by  $P$  yields a smooth, closed, definite 4-manifold, whose intersection form contains the intersection form of  $P$  as an embedded sublattice. By Theorem 2.5, the intersection form of  $P$  embeds isometrically in a definite diagonal form. A lengthy combinatorial analysis can be carried out to exclude all triples that are not in the list above.

Finally, the third input is the construction of the rational homology balls that we can use to cap off the traces above. For the families  $\mathcal{L}$  and  $\mathcal{R}$ , the necessary rational homology balls were constructed by Lisca [Lis07], who had classified all lens spaces which bound rational homology balls (using Donaldson's diagonalisation theorem). As mentioned above, for the family  $\mathcal{G}$ , some examples were known from complex algebraic geometry (this also holds for families (10) and (17)). For the other, the lattice embeddings produced using Donaldson's theorem guided our handle-by-handle construction of the rational homology ball<sup>6</sup>.  $\square$

*Remark 2.30.* In fact, a posteriori we know that the use of Donaldson's theorem should not be necessary, and that the Heegaard Floer obstruction contains the same amount of information. This is because surgeries along torus knots are small Seifert fibred spaces or lens spaces or connected sums of lens spaces. Either way, they bound *sharp* 4-manifolds, and for these manifolds the two obstructions, the one coming from Donaldson's diagonalisation theorem and the other one coming from correction terms, are equivalent. This is spelled out in [Lar21].

## PL surfaces of higher genus

It turns out that many ideas presented in the previous section can be extended to surfaces of higher genus. As mentioned above, this was done, independently, by Borodzik, Hedden, and Livingston [BHL17] and in a collaboration with Bodnár and Celoria [BCG16].

We start with the analogue of Proposition 2.25, whose proof is essentially identical to the one given above.

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<sup>6</sup>This idea had been developed and advertised by Owens, see [OS21].

**Proposition 2.31** (Borodzik, Livingston [BL14]). *Let  $X$  be a rational homology  $\mathbb{C}\mathbb{P}^2$  and  $F \subset X$  a PL surface of degree  $d > 0$ . Then  $S_{d^2}^3(K_F)$  bounds a 4-manifold with vanishing  $b_1$  and null intersection form.*

Recall that having null intersection form means that the intersection form is identically zero. The 4-manifold complement, however, has positive  $b_2$  (in fact,  $b_2$  is  $2g(F)$ ). Also, observe that since  $b_1(X \setminus F) = 0$ , the map induced on cohomology by the inclusion  $H^1(\partial N(F)) \rightarrow H^1(X \setminus N(F))$  automatically vanishes.

After determining which  $\text{spin}^c$  structures on  $\partial N(F)$  extend to  $X \setminus N(F)$ , which is similar to Proposition 2.10 above, one readily obtains the following analogue of Theorem 2.26.

**Theorem 2.32** (Borodzik, Hedden, Livingston [BHL17]; Bodnár, Celoria, G.). *Let  $X$  be a rational homology  $\mathbb{C}\mathbb{P}^2$  and  $F \subset X$  a PL surface of degree  $d > 0$  and genus  $g$ . Then, for every  $h = -\frac{d-1}{2}, -\frac{d-3}{2}, \dots, \frac{d-1}{2}$  and every  $k = 0, \dots, g$  we have:*

$$-g \leq V_{g-2k+hd}(K_F) - \frac{(d-2h)^2 - 1}{8} - k \leq 0,$$

where  $K_F$  is the connected sum of all links of singularities of  $F$ .

It is quite remarkable that the condition above is *very* restrictive when combined with the adjunction formula (which we will discuss in the next chapter), which asserts that  $2g + 2g(K_F) = (d-1)(d-2)$ . We will call a PL surface in  $\mathbb{C}\mathbb{P}^2$  satisfying this formula *adjunctive*. For instance, we will see in the next chapter that complex and symplectic curves are adjunctive, which can be viewed as a strong justification for considering this class.

For every integer  $k > 0$ , let us define a sequence  $\{A_n^k\}_{n \in \mathbb{Z}}$  by

$$\begin{cases} A_{n+1}^k = 3A_n^k - A_{n-1}^k, \\ A_0^k = 1, \\ A_1^k = k. \end{cases}$$

(Note that the sequence is also defined backwards, for  $n < 0$ .)

**Theorem 2.33** (Bodnár, Celoria, G.). *Fix a positive integer  $g > 0$ . Let  $F \subset \mathbb{C}\mathbb{P}^2$  be an adjunctive PL surface of genus  $g$  and degree  $d \gg 0$ , with a unique singular point, which is of type  $T(a, b)$ . Then:*

$$a + b = 3d \quad \text{and} \quad (7b - 2a)^2 - 45b^2 = 36(2g - 1).$$

*It follows that if  $g \equiv 2, 4 \pmod{5}$ , there are only finitely possibilities for  $a, b$ , and  $d$ . If  $g = \frac{1}{2}k(k-1)$  for some  $k \equiv 2 \pmod{3}$  and  $2g - 1$  is a prime power, then the only possibilities for  $a, b$ , and  $d \gg 0$  are  $(a, b, d) = (A_{2n}^k, A_{2n+2}^k, A_{2n+1}^k)$ , and they are all realised.*

The first part of the theorem follows from a very careful analysis of the restrictions imposed by Theorem 2.32 and the combinatorics of the semigroup generated by  $a$  and  $b$  (which in turn controls the sequence  $\{V_k(T(a, b))\}_k$ ).

The second half is an analysis of the Pell equation in the statement, whose set of solutions depends exactly on the prime factorisation of  $2g - 1$ .

Finally the existence part is given by a generalisation of a construction due to Orevkov, which we will discuss in some detail in Section 3.6 below.

## Branched covers and signatures

There is a much more classical approach to studying PL surfaces in a 4-manifold  $X$ , which is especially effective when the 4-manifold  $X$  is an integer homology  $\mathbb{C}\mathbb{P}^2$  (that is,  $H_*(X) \cong H_*(\mathbb{C}\mathbb{P}^2)$  over the integers, and  $\sigma(X) = 1$ ). This makes most classes in  $H_2(X)$  divisible, which in turn allows one to use branched covers and signatures to study surfaces. This method was studied by Casson and Gordon [GM86] in the context of slice knots, and later by Gilmer [Gil81] in the context of surfaces in 4-manifolds. Here we only give a hint of their ideas in our more restricted setting.

The main construction is the topological analogue of the Milnor fibre of a curve singularity. Given a knot  $K$  in  $S^3$ , choose a surface  $M_K \subset B^4$  of minimal genus such that  $M_K$  is smoothly<sup>7</sup> and properly embedded, transverse to  $\partial B^4$ , and  $\partial M_K = K$ . This allows to *minimally smoothen* a PL surface in a 4-manifold.

**Definition 2.34.** A *smoothing*  $\text{sm}(F)$  of a PL surface  $F \subset X$  is obtained by replacing a regular neighbourhood  $(U_p, F \cap U_p) \cong (B^4, C(K_p))$  of the singular point  $p$  of  $F$  with  $(B^4, M_{K_p})$  for each singular point  $p$  of  $F$ .

Note that we say *a* smoothing and not *the* smoothing. In this context, without any restriction on  $K$ , we do not have a preferred choice for  $M_K$ , so the surface  $\text{sm}(F)$  is not unique. If  $K$  is an algebraic knot, then we *do* have a preferred choice, namely the Milnor fibre of the singularity whose link is  $K$ . If that is the case, we choose  $M_K$  to be the Milnor fibre, and if all the singularities of  $F$  are algebraic we have a preferred smoothing, unique up to isotopy. We keep the notation  $\text{sm}$  for this unique smoothing, but we invite the reader to keep this ambiguity in mind. (This preferred smoothing is going to be relevant in the next chapter.)

We now locally have a surface  $M_K$  in  $B^4$ , and we can consider its cyclic covers. We have a classical result of Viro [Vir75] (see also Kauffman and Taylor [KT76] for the case of double covers), asserting that the equivariant signatures of this branched cover (with respect to the cyclic group action given by its deck transformations) are the Levine–Tristram signatures of the knot  $K$ .

We first start with two baby cases, which showcase some phenomena that we will generalise later.

*Example 2.35.* We claim that there is no PL sphere of degree 6 in a  $\mathbb{Z}/2\mathbb{Z}$ -homology  $\mathbb{C}\mathbb{P}^2$   $X$ , with a unique singularity of type  $T(2, 21)$ . Suppose that such an  $F$  exist. Then its smoothing  $\text{sm } F$  has genus  $g(T(2, 21)) = 10$ , which incidentally minimises the genus in its homology class. From the long exact sequence of the pair  $(X, \text{sm } F)$  we deduce that the meridian of  $\text{sm } F$  generates  $H_1(X \setminus \text{sm } F; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ , so there exists a double cover  $\tilde{X} \rightarrow X$  branched over  $\text{sm } F$ . By a direct computation,  $\tilde{X}$  has Euler characteristic 24 and, since  $H_1(X \setminus \text{sm } F)$  is cyclic, the Goldschmidt lemma [HS71] implies that  $b_1(\tilde{X}) = 0$ . By the  $G$ -signature theorem (see below),  $\tilde{X}$  has signature  $-16$ . Since  $b_2(\tilde{X}) = 22$ , we obtain  $b_2^-(\tilde{X}) = 19$ .

However, the double cover of  $B^4$  branched over  $M_{T(2,21)}$  is the Milnor fibre  $M$  of the  $A_{20}$ -singularity, so it is negative definite, and therefore it has  $b_2^-(M) = 20$ . This gives a contradiction since  $M$  cannot embed in  $\tilde{X}$  as  $b_2^-(M) > b_2^-(\tilde{X})$ .

<sup>7</sup>Most of the discussion below works if we choose  $M_K$  to be locally-flat, but this requires a deep theorem of Quinn that ensures that locally-flat compact surfaces in 4-manifold are globally-flat [FQ90, Section 9.3], [Qui82, Theorem 2.5.1], [BKK<sup>+</sup>21, Section 21.4.8].

## 2 Piecewise-linear surfaces

*Example 2.36.* We claim that there is no PL sphere of degree 5 in a  $\mathbb{Z}/5\mathbb{Z}$ -homology  $\mathbb{C}\mathbb{P}^2$ ,  $X$ , with six singularities of type  $T(2, 3)$ . Suppose that such a surface  $F$  exists.

As above, using the long exact sequence of the pair  $(X, \text{sm } F)$  we can show that  $H_1(X \setminus \text{sm } F; \mathbb{Z}/5\mathbb{Z})$  is generated by the meridian of  $\text{sm } F$ , so there exists a 5-fold cover  $\tilde{X} \rightarrow X$  branched over  $F$ . The  $G$ -signature theorem tells us that  $\sigma(\tilde{X}) = -35$ . The Euler characteristic of  $\tilde{X}$  is easily calculated since  $g(\text{sm } F) = 6$ , and by the Goldschmidt lemma  $b_1(\tilde{X}) = 0$ , so that  $b_2(\tilde{X}) = 53$  and  $b_2^-(\tilde{X}) = 44$ . Now, the 5-fold cover  $M$  of  $B^4$  branched over  $M_{T(2,3)}$  is the Milnor fibre of the  $E_8$ -singularity, whose intersection form is  $-E_8$ , so we have  $b_2^-(M) = 8$ . But this implies that six copies of  $M$  cannot be embedded in  $\tilde{X}$ .

Before going on to the general case, we set up some notation. Given an action of the cyclic group  $C_d = \mathbb{Z}/d\mathbb{Z}$  on a 4-manifold  $M$ , we can look at the eigenspaces of the induced  $C_d$ -action on  $H_2(M; \mathbb{C})$ . More precisely, let  $g \in \text{Diff}^+(M)$  be a fixed generator of the action. Since  $g$  has order  $d$ , the eigenvalues are  $d^{\text{th}}$  roots of unity.  $H_2(M; \mathbb{C})$  has a Hermitian product  $Q_M^{\mathbb{C}}$  coming from  $Q_M$ , and the eigenspace decomposition of  $H_2(M; \mathbb{C})$  is  $Q_M^{\mathbb{C}}$ -orthogonal. We can therefore define the signature of  $Q_M^{\mathbb{C}}$  restricted to the  $\lambda$ -eigenspace for every  $d^{\text{th}}$  root of unity  $\lambda$ , which we denote with  $\sigma^\lambda(g, M)$ .

On the other hand, to a knot (or a link)  $K \subset S^3$  we can associate a Seifert form  $V$ , and we can define the Levine–Tristram signature of  $K$  as the function  $\sigma_K: S^1 \rightarrow \mathbb{Z}$  assigning to  $\lambda \in S^1$  the signature of  $(1 - \lambda)V + (1 - \bar{\lambda})V$ .

**Theorem 2.37** (Viro). *Fix a positive every integer  $d$ . Let  $K \subset S^3$  be a knot and  $F \subset B^4$  be a compact, smoothly and properly embedded surface whose boundary is  $K$ . Let  $g$  be the generator of the automorphism group of the cyclic branched cover  $p: \Sigma_d(B^4, F) \rightarrow B^4$  that acts as rotations by  $2\pi/d$  around  $p^{-1}(F)$ . Then, for every  $d^{\text{th}}$  root of unity  $\omega$ :*

$$\sigma^\omega(\Sigma_d(B^4, F), g^k) = \sigma_K(\omega^k).$$

Viro’s statement becomes very powerful when combined with the  $G$ -signature theorem. We state the  $G$ -signature theorem only in the case of cyclic actions whose fixed point set is a connected surface. Recall that if  $g \in \text{Diff}^+(X)$  is a diffeomorphism of order  $d$  whose fixed-point set is a connected surface  $F$ , then it acts by rotations in the normal direction (namely,  $dg_x$  is a rotation on  $T_x X/T_x F$  for every  $x \in F$ ). Call  $\phi(g)$  this angle of rotation. As above, we have a well-defined eigenspace decomposition of  $H_2(X; \mathbb{C})$  and we can define the  $g$ -signature of  $X$  to be:

$$\sigma(g, X) = \sum_{\omega^d=1} \omega \cdot \sigma^\omega(g, X)$$

**Theorem 2.38** (Atiyah, Singer [AS68]; Gordon [Gor86]). *Let  $g$  be a finite-order diffeomorphism of  $X$  whose fixed-point set is a connected surface  $S$ . Then*

$$\sigma(g, X) = \frac{S \cdot S}{\sin^2(\phi(g)/2)}$$

Combining Viro’s theorem and the  $G$ -signature theorem, we obtain the following.

**Theorem 2.39** (Gilmer). *Let  $F \subset X$  be a PL surface of degree  $d$  and genus  $g$  in a smooth homology  $\mathbb{C}\mathbb{P}^2$ . For a prime power  $q$  dividing  $d$ , let  $\omega_q = e^{2\pi i/q}$ . Then for every prime power  $q$  dividing  $d$  and any  $0 \leq r < q$  we have:*

$$\left| \sigma_{K_F}(\omega_q^r) - 1 + 2\left(\frac{d}{q}\right)^2 r(q-r) \right| \leq 1 + 2g.$$

*Sketch of proof.* The surface  $F$  induces a decomposition of  $X$  into a neighbourhood  $N$  of  $F$  and its complement,  $W$ . This also induces a decomposition of the cyclic  $q$ -fold cover  $\tilde{X}$  of  $X$ , ramified over  $\text{sm}(F)$  into the preimages  $\tilde{N}$  and  $\tilde{W}$  of  $N$  and  $W$ , respectively.

Since the cover has prime power order, we know that  $b_1(\tilde{X}) = b_1(\tilde{N}) = b_1(\tilde{W}) = 0$ ,  $b_1(\partial\tilde{W}) = 2g$ , and  $b_3(\tilde{X}) = b_3(\tilde{N}) = b_3(\tilde{W}) = 0$ . This is a corollary of the Goldschmidt lemma [HS71], for instance, but it also follows from looking at equivariant homology as in [TW74, Gil81].

Note that the cover  $\tilde{W} \rightarrow W$  is an honest (non-branched) cover, and in particular we know that the equivariant Euler characteristics of  $\tilde{W}$  agree with those of  $W$ . Since  $b_1(\tilde{W}) = b_1(W) = 0$ , and the equivariant  $b_0$  vanishes as soon as the action is non-trivial, we know that  $b_2(\tilde{W}) = (2g+1)q - 1 = (2g+1)(q-1) + 2g$  and that the (non-trivial) eigenspaces of the  $C_q$ -action of  $H_2(\tilde{W}; \mathbb{C})$  are all  $2g+1$ -dimensional. This gives a signature inequality

$$|\sigma^\omega(g, \tilde{W})| \leq 1 + 2g$$

for each  $\omega$ -eigenspace. To prove the statement, we just need to compute  $\sigma^\omega(g, \tilde{W})$ .

From the  $G$ -signature theorem, we know  $\sigma(g^s, \tilde{X})$  for each  $s$ . We can do an inverse discrete Fourier transform to use all these signatures to compute  $\sigma^{\omega_q^r}(g, \tilde{X})$  for each  $r$ . The computation is elementary, but tricky, [CG78] and yields:

$$\sigma^{\omega_q^r}(g, \tilde{X}) = 2\left(\frac{d}{q}\right)^2 r(q-r) + 1.$$

From Viro's theorem and Novikov's additivity of the (equivariant) signatures, we obtain:

$$\begin{aligned} \sigma^{\omega_q^r}(g, \tilde{W}) &= \sigma^{\omega_q^r}(g, \tilde{X}) - \sigma^{\omega_q^r}(g, \tilde{N}) \\ &= \left(\frac{d}{q}\right)^2 2r(q-r) + 1 - \sigma_{K_F}(\omega_q^r). \end{aligned}$$

Together with the signature inequality above, this proves the theorem.  $\square$

## The Arf invariant

There is another classical obstruction, also studied by Kervaire and Milnor in the context of embedded spheres, coming from the Arf (or Arf–Robertello) invariant [Rob65, Lic97]. This involves spin structures and only applies for surfaces of odd degrees. We only state it for PL spheres.

Suppose that  $X$  is now a  $\mathbb{Z}/2\mathbb{Z}$ -homology  $\mathbb{C}\mathbb{P}^2$ ; that is, the homology of  $X$  satisfies  $H_*(X; \mathbb{Z}/2\mathbb{Z}) = H_*(\mathbb{C}\mathbb{P}^2; \mathbb{Z}/2\mathbb{Z})$  and  $\sigma(X) = 1$ . Let  $F$  be a PL sphere in  $X$  of odd degree  $d$ . In this case  $W = X \setminus N(F)$  is a *spin* rational homology ball (see, for



## 2 Piecewise-linear surfaces

instance, [BHS18, Section 4]). Since  $W$  is a rational homology ball, it has signature 0. It follows that the Rokhlin invariant  $\mu(\partial N(F))$  of its boundary vanishes modulo<sup>8</sup> 16. Note that, since  $d$  is odd,  $H^1(\partial N(F); \mathbb{Z}/2\mathbb{Z}) = 0$  and thus  $\partial N(F)$  has a unique spin structure.

Since  $N(F)$  is the trace of  $d^2$ -surgery along a knot  $K$ , the Rokhlin invariant of its boundary is determined by  $d$  and by the *Arf invariant*  $\text{Arf}(K)$  of  $K$ . Recall that the Arf invariant is a knot invariant which takes values in  $\mathbb{Z}/2\mathbb{Z}$ , which encodes the type of the  $\mathbb{Z}/2\mathbb{Z}$ -valued quadratic form associated to any Seifert form of the knot. It is additive with respect to connected sums and it can be computed from the Alexander polynomial of the knot:  $\text{Arf}(K) \equiv 0$  if  $\Delta_K(-1) \equiv \pm 1 \pmod{8}$ , and  $\text{Arf}(K) \equiv 1$  if  $\Delta_K(-1) \equiv \pm 3 \pmod{8}$ . For instance, for torus knots  $T(p, q)$ :

$$\text{Arf}(T(p, q)) = \begin{cases} 0 & \text{if } p, q \text{ are both odd,} \\ 0 & \text{if } p \text{ is even and } q \equiv 1, 7 \pmod{8} \text{ or vice-versa,} \\ 1 & \text{if } p \text{ is even and } q \equiv 3, 5 \pmod{8} \text{ or vice-versa.} \end{cases}$$

Gordon proved in [Gor75] that  $\mu(S_n^3(K)) \equiv n - 1 + 8 \text{Arf}(K) \pmod{16}$ . In the case of embedded PL spheres, combining Gordon's Rokhlin invariant computation with the observation that  $\mu = 0$ , we obtain the following obstruction, which is both easily computable and surprisingly strong.

**Proposition 2.40.** *If  $F \subset X$  is a PL sphere of odd degree  $d$  in a  $\mathbb{Z}/2\mathbb{Z}$ -homology  $\mathbb{C}\mathbb{P}^2$ , then*

$$\text{Arf}(K_F) \equiv \frac{d^2 - 1}{8} \pmod{2}.$$

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<sup>8</sup>Here we think of the Rokhlin invariant as an element in  $\mathbb{Z}/16\mathbb{Z}$ , rather than an element in  $\mathbb{Z}/2\mathbb{Z}$ . This is more convenient when dealing with rational homology spheres as opposed to integer homology spheres.

## 3 Symplectic curves

In this chapter we will talk about a particular class of (configurations of) PL surfaces, namely *(singular) symplectic curves*, mostly in  $\mathbb{C}\mathbb{P}^2$  and its blow-ups. Most of the ideas in this chapter come from Gromov's theory of pseudo-holomorphic curves [Gro85] and from McDuff's work [McD90, McD92]. Wendl's book [Wen18] is a great compendium on the subject.

We start the chapter with some generalities on symplectic 4-manifolds and introduce symplectic curves and the (singular or non-singular) symplectic isotopy problems. In Section 3.2 we discuss some of Gromov's ideas on pseudo-holomorphic curves. In Section 3.3 we show how symplectic rational cuspidal curves do not help us finding exotic  $\mathbb{C}\mathbb{P}^2$ . In Sections 3.4, 3.5, and 3.6 we discuss the symplectic isotopy problem, starting with obstructions (in continuation with the previous chapter), the statements of the main results from [GS22, GK23], and some ideas coming from algebraic geometry which enter in the proofs.

Finally, Section 3.7 discusses the relative case, presenting ideas from [GS21] and [EG22]: symplectic fillings and caps of contact 3-manifolds, and symplectic cobordisms between transverse knots.

### 3.1 Symplectic 4-manifolds and their symplectic submanifolds

A *symplectic form* on a 4-manifold is a 2-form  $\omega$  on  $X$  that satisfies two properties:

- $\omega$  is closed:  $d\omega = 0$ ,
- $\omega$  is non-degenerate:  $(\omega \wedge \omega)_x \neq 0$  for each  $x \in X$ .

A 4-manifold equipped with a symplectic form is a *symplectic 4-manifold*. Note that a symplectic form  $\omega$  on a 4-manifold  $X$  induces an orientation via the volume form  $\omega \wedge \omega$ . Darboux's theorem asserts that the local model for all symplectic 4-manifolds is the standard  $\mathbb{R}^4$ , i.e.  $\mathbb{R}^4$  equipped with the symplectic form  $\omega_{\text{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ .

Cotangent bundles of real surfaces (with the derivative of the *Liouville form*  $q dp$ ) and non-singular complex projective surfaces (with the restriction of the *Fubini–Study form* of the  $\mathbb{C}\mathbb{P}^N$  in which they are embedded) are the first and most important examples of symplectic 4-manifolds. The main example we will consider is the complex projective plane,  $\mathbb{C}\mathbb{P}^2$ , endowed with the Fubini–Study symplectic form  $\omega_{\text{FS}}$ ; in complex coordinates, it is defined by:

$$\omega_{\text{FS}} = \partial\bar{\partial} \log(|z_0|^2 + |z_1|^2 + |z_2|^2).$$

### 3 Symplectic curves

A deep theorem of Taubes asserts that  $\omega_{\text{FS}}$  is the unique symplectic structure on  $\mathbb{C}\mathbb{P}^2$ , up to symplectomorphisms and rescaling [Tau96].

An immersion  $\iota: F \looparrowright X$  of a real surface into  $X$  is called  $\omega$ -*symplectic*, or just *symplectic* if there is no danger of confusion, if  $\iota^*\omega$  is an area form on  $F$ . In this case we also say that the image of  $\iota$  is an (*immersed*) *symplectic surface* in  $X$ .

Again, the prototypical example comes from complex algebraic geometry: if  $X$  is a complex projective surface and  $C \subset X$  is a nodal complex curve, then  $C$  is an immersed symplectic surface in  $X$ . The statement can be further generalised by considering complex curves in *Kähler surfaces*.

Let us focus on the case of embeddings, first. The main problem that remains open to this day is the *symplectic isotopy problem*.

**Question 3.1.** Is every embedded symplectic surface in  $\mathbb{C}\mathbb{P}^2$  symplectically isotopic to a complex curve?

Here by a *symplectic isotopy* we mean a smooth isotopy  $\{F_t\}_{t \in [0,1]}$  such that at each time  $t \in [0, 1]$  the surface  $F_t$  is symplectic. Since it is well-known that every two non-singular complex curves of the same degree are isotopic among complex curves<sup>1</sup>, the question asks whether any two symplectically embedded curves of the same degree in  $\mathbb{C}\mathbb{P}^2$  are isotopic. The answer is known for degrees up to 17: Gromov first proved the statement for  $d = 1, 2$  [Gro85], Sikorav for degree 3 [Sik03], Shevchishin pushed it to degree 6 [She00], and finally Siebert and Tian reached the current limit [ST05].

The main difference between generic immersed symplectic surfaces and nodal complex curves is the possible (local) signs of intersections: the sign of intersection of the two branches of a complex curve is always positive, whereas this is no longer true symplectically. Concretely, the two planes:

$$\{x_2 = y_2 = 0\}, \{x_2 = 2x_1, y_2 = -2y_1\} \subset (\mathbb{R}^4, \omega_{\text{st}})$$

are, by direct verification, both symplectic and they intersect transversely and negatively at the origin of  $\mathbb{R}^4$ .

If we want to define what a singular symplectic surface in a symplectic 4-manifold is, we are at a crossroad: what singularities should we allow? From the discussion above, positive and negative double points look reasonable, but then we weaken the connection with the complex world. It is also possible for a symplectic surface to have non-isolated singularities: the planes  $\{x_2 = y_2 = 0\}, \{x_2 = 0, y_2 = \frac{1}{2}y_1\}$ , are both symplectic, but they intersect along a line. On the other hand, if we restrict the singularities too much, then we might have a theory that is not sufficiently rich.

As we will see in Section 3.7, in a suitable sense, symplectic surfaces in  $\mathbb{C}\mathbb{P}^2$  can have isolated singularities of arbitrary type, where the *type* of a singularity is a transverse link in the standard contact 3-sphere  $(S^3, \xi_{\text{st}})$ .

A more restrictive definition would be to allow for *smoothable* singularities. These are the singularities for which there exists a *symplectic* smoothing  $M_K$ . In turn, this class of singularities corresponds to a well-studied class of (transverse) links in the

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<sup>1</sup>The key point of the argument is the following. The discriminant is a complex codimension-1 subvariety of the space of all degree  $d$ -curves. Since the latter is an irreducible, non-singular variety, the discriminant does not disconnect it.

### 3.1 Symplectic 4-manifolds and their symplectic submanifolds

standard contact 3-sphere: closures of quasipositive braids. We will talk about this class of knots and links in more detail in Section 3.7.

For the moment, we focus on the class of singular surfaces for which there is a singular analogue of the symplectic isotopy problem as stated above. To emphasise that we are restricting to a very special class of singular symplectic surfaces that mimics complex curves, we borrow the terminology from algebraic geometry and we will talk about symplectic *curves*. We will see in the next section an equivalent definition in pseudo-holomorphic terms.

**Definition 3.2** ([GS22]). A *symplectic curve* in a symplectic 4-manifold  $(X, \omega)$  is an immersed PL surface  $C \subset X$  such that:

- $C \setminus \text{Sing}(C)$  is a smooth symplectic surface in  $X$  (i.e.  $\omega$  restricts to an area form on  $C \setminus \text{Sing}(C)$ );
- for every  $q \in \text{Sing}(C)$  there are an open neighbourhood  $U \subset X$  and a symplectomorphism  $(U, C \cap U, q) \rightarrow (V, D \cap V, 0)$ , where  $V \subset \mathbb{C}^2$ , and  $D$  is a complex algebraic curve with a *non-trivial* singularity at 0.

As we did for the PL surfaces, we call each point in  $\text{Sing}(C)$  the a *singularity* (or *singular point*) of  $C$ .

We say that  $C$  is *cuspidal* if all its singularities are cusps (i.e. if the curve  $D$  in (b) above is locally irreducible at 0 for every  $q \in \text{Sing}(S)$ ) and *nodal* if all its singularities are transverse double points. A symplectic cuspidal curve is called *rational* if it is a PL-embedded 2-sphere.

The definition in particular implies that a symplectic cuspidal curve is a PL-embedded surface, whose singularity types are algebraic knots (i.e. links of irreducible curve singularities). As a symplectic object, the link is the unique transverse representative of that algebraic knot with maximal self-linking number. (More on this in Section 3.7.)

The definition above takes care, for instance, of the issues with negative double points, and it allows to formulate the singular version of Question 3.1.

**Question 3.3.** Is every symplectic curve in  $\mathbb{C}\mathbb{P}^2$  symplectically and equisingularly isotopic to a complex curve?

By a *symplectic equisingular* isotopy we mean the symplectic analogue of *equisingular deformations* in algebraic geometry: we want a family  $\{C_t\}_t$  of symplectic curves such that  $C_t$  has the same singularity types as  $C_{t'}$  for every  $t, t'$ .

I would like to stress that, as was mentioned in the introduction, similar versions of the question were considered by several authors in the case of curves with simple nodes (i.e. positive transverse double points) and simple cusps (i.e. singularities of type  $T(2, 3)$ ). We will shortly talk about Moishezon's work on the problem [Moi94], dating back to the 90s. More recently, a lot of interest on symplectic isotopy problem was sparked by Auroux's work on branched covers: in [Aur00], he proved that every symplectic 4-manifold is a cover of  $\mathbb{C}\mathbb{P}^2$  branched over a symplectic curve whose singularities are both positive and negative double points and simple cusps. Partly motivated by Auroux's work, Barraud [Bar00], Shevchishin [She04], and Francisco [Fra05] obtained several results on symplectic equisingular isotopies of nodal,

### 3 Symplectic curves

cuspidal, and nodal-cuspidal curves. In the last decade, Ruberman and Starkston also studied symplectic line arrangements [RS19]: their paper was a source of inspiration for part of the work I am presenting here.

In a sense, Question 3.3 is more multi-faceted than Question 3.1. On the one hand, it is no longer true that complex curves have a unique singular representative in each degree: for instance, the collection of types of singularities of  $C$  is not determined by the degree. On the other hand, we do not even have a classification of complex curves in all degrees, neither up to isotopy nor up to equisingularity.

An extra layer of complexity is that singular curves could be *reducible* (i.e.  $C$  could be a PL immersion of a disconnected surface), and their singularities could be *reducible* (i.e.  $C$  might not be PL-embedded, but just PL-immersed).

A final layer of complexity is the phenomenon of *Zariski pairs*, which shows that there exist equisingular complex curves of the same degree that are not isotopic. The first example is due to Zariski, who constructed two sextics, known as the *Zariski sextics*, whose complements are not even homeomorphic (they have non-isomorphic fundamental groups).

*Example 3.4* (Zariski sextics). In [Zar29], Zariski described two curves  $C, C'$  in  $\mathbb{C}\mathbb{P}^2$ , both of degree 6, genus 4, with six singular points that are simple cusps (i.e. their link is a right-handed trefoil,  $T(2, 3)$ ).

The first sextic,  $C$ , has its six singularities lying on a conic. In this case, the fundamental group of its complement is  $C_2 * C_3$ , which is non-Abelian. One can explicitly write down an equation in this case: take a generic cubic polynomial  $P$  and a generic quadratic polynomial  $Q$ , and let  $F = 27P^2 - 4Q^3$ . Then  $F$  passes through the six intersections of the cubic defined by  $P$  and the conic defined by  $Q$ , and has a simple cusp at each of these points.

The second sextic has its six singularity that do not line on the same conic. The fundamental group of its complement is  $C_6$ , which is clearly Abelian. So  $C'$  cannot be isotopic to  $C$ .

Question 3.3 is known to have a *negative* answer in many cases: if  $C$  is reducible, if  $C$  is irreducible of positive genus, and if  $C$  is irreducible, of genus 0, with reducible singularities. This leaves out the case of irreducible curves of genus 0 with irreducible singularities: these are *rational cuspidal curves*, and they will be the focus of this chapter.

*Example 3.5.* We get one example of a reducible symplectic curve (i.e. a symplectic configuration of curves) that is not equisingular to (and in particular not equisingularly isotopic to) a complex curve whenever we have a pseudo-line arrangement that does not have the combinatorial type of any complex arrangement [RS19]. For instance the pseudo-Pappus arrangement, shown in Figure 3.1, is obtained from the classical Pappus arrangement of nine lines in  $\mathbb{C}\mathbb{P}^2$  by  $C^\infty$ -perturbing one of the triple points of intersection. The classical Pappus theorem tells us that this configuration is not realised by a collection of complex lines. The fact that the symplectic condition and that positivity of intersections for *transverse* intersections are both open conditions guarantees that the arrangement we obtain is a symplectic line arrangement, which is therefore not isotopic to a complex one.

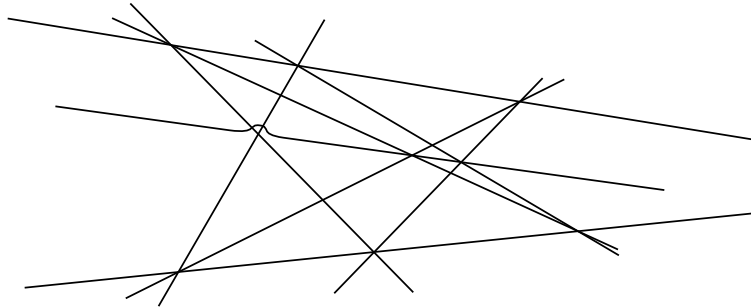


Figure 3.1: The pseudo-Pappus arrangement.

*Example 3.6.* In [Moi94], Moishezon produced infinite families of equisingular symplectic curves with nodal and simply-cuspidal singularities (i.e. their links are positive Hopf links and right-handed trefoils, respectively) in each degree in an infinite sequence, every two of which have complements with non-isomorphic fundamental groups. In every given degree  $d$  there are at most finitely many isotopy classes of complex curves, since the singularity types induce an algebraic stratification of the space of all degree- $d$  curves, which is a projective space. Therefore, infinitely many among Moishezon's example are not isotopic to a complex curve.

*Example 3.7.* Orevkov has an example of a rational symplectic curve with three irreducible singularities, each of which has the same link as the curve  $\{y(x^3 + y^5) = 0\}$  at the origin of  $\mathbb{C}^2$  [GS22]. This is obtained from a pseudo-Desargues configuration (similar to the pseudo-Pappus mentioned above) by means of birational transformation. Such a curve cannot exist over the complex numbers, since one could construct the pseudo-Desargues configuration starting from it, and the Desargues theorem forbids this.

## 3.2 Some pseudo-holomorphic ideas

As mentioned in the previous section, symplectic 4-manifolds are a more flexible version of complex projective surfaces. Not all symplectic 4-manifolds are complex: the first examples were produced by Thurston (the *Thurston–Kodaira manifold*) [Thu76], and then Gompf proved that every finitely presented group is the fundamental group of a symplectic 4-manifold [Gom95]. However, all symplectic 4-manifolds are *almost-complex*.

Recall that an almost-complex structure  $J$  on  $X$  is a vector bundle automorphism  $J: TX \rightarrow TX$  such that  $J^2 = -\text{id}_{TX}$ . An almost complex structure  $J$  is *tamed* by a symplectic structure  $\omega$  if  $\omega$  is positive on  $J$ -lines, i.e. if  $\omega(v, Jv) > 0$  for each non-zero  $v \in TX$ .  $J$  is *compatible* with  $\omega$  if furthermore  $\omega$  is  $J$ -equivariant, i.e.  $\omega(v, w) = \omega(Jv, Jw)$  for every  $v, w \in TX$ . (These notions appear naturally in Kähler geometry, where  $J$  is integrable.) It is a classical result that the set of almost-complex structures on  $X$  tamed by (or compatible) with a given  $\omega$  is contractible (and in particular non-empty).

Gromov's key insight was to use  $J$ -holomorphic curves in  $X$  for a non-integrable  $J$  compatible with  $\omega$  to probe  $(X, \omega)$ . This led to the development of Gromov–Witten

### 3 Symplectic curves

theory (enumerative geometry) and Floer homology/symplectic field theories. Here we will only scratch the surface of Gromov's ideas.

It turns out that symplectic curves, as defined in the previous section, are exactly  $J$ -holomorphic curves for some  $J$ .

**Proposition 3.8** (Micallef, White [MW95]; McDuff [McD92]; G., Starkston [GS22]). *Suppose  $C \subset (X, \omega)$  is a symplectic curve. Then there exists an almost-complex structure  $J$  on  $X$ , compatible with  $\omega$ , such that  $C$  is  $J$ -holomorphic. Conversely, if  $C$  is  $J$ -holomorphic then  $C$  is a symplectic curve.*

In particular,  $C = \iota(\Sigma)$  for some Riemann surface  $(\Sigma, j)$  and some map  $\iota$  which is  $(j, J)$ -holomorphic. We can suppose that  $\Sigma$  is *simple* (i.e. that  $\iota$  does not multiply cover  $C$ ), and in that case we call  $g(\Sigma)$  the *genus* of  $C$ . ( $\Sigma$  is what algebraic geometers call the *normalisation* of  $C$ , and  $g(\Sigma)$  is the *geometric genus* of  $C$ .) The bulk of the proof is the local characterisation of  $J$ -holomorphic singularities as actual complex curves singularities, which is due to Micallef and White and to McDuff. In [GS22] we simply adapted the classical, well-known non-singular version of the proposition, and globalised the construction.

A very useful corollary of the local part of the theorem is *positivity of intersections* for  $J$ -holomorphic curves.

**Theorem 3.9** (McDuff [McD94]). *Let  $C, C'$  be two symplectic curves. Then for every point  $p \in C \cap C'$  there is a positive integer  $(C \cdot C')_p$  such that*

$$C \cdot C' = \sum_{p \in C \cap C'} (C \cdot C')_p.$$

*In particular,  $|C \cap C'| \leq C \cdot C'$ . The quantity  $(C \cdot C')_p$  measures the number of signed intersections of a generic perturbation of  $C$  with  $C'$ , and in particular it is  $+1$  if and only if  $C$  and  $C'$  intersect transversely at  $p$ .*

We can also prove the *adjunction formula* for symplectic curves.

**Theorem 3.10.** *If  $C = \iota(\Sigma) \subset (X, \omega)$  is an irreducible symplectic curve and  $J$  is a compatible almost-complex structure with respect to which  $C$  is  $J$ -holomorphic, then the adjunction formula holds:*

$$\langle c_1(J), [C] \rangle - C \cdot C = \chi(\Sigma) - \sum_{p \in \text{Sing}(C)} \mu(C, p).$$

In the formula above,  $c_1(J)$  is the first Chern class of  $J$ , which is independent of  $J$  within the class of  $\omega$ -tame (or  $\omega$ -compatible) almost-complex structures, and so is a discrete topological invariant of  $\omega$ . The invariant  $\mu(C, p)$  is the *Milnor invariant* of the singularity of  $C$  at  $p$ , which is  $b_1$  of its Milnor fibre; for cuspidal singularities,  $\mu$  is twice the Seifert genus of its link. In particular, the genus of a non-singular symplectic curve only depends on its homology class. (Either side of the equation is what algebraic geometers refer to as the *arithmetic genus* of  $C$ .)

*Sketch of proof.* The smoothing  $\text{sm } C$  of  $C$  is a non-singular symplectic surface, since at each singular point we are gluing two symplectic surfaces, namely  $C \setminus \text{Sing}(C)$  and  $M_{K(C,p)}$ , along the same transverse knot. Since the difference in  $b_1$  between  $\Sigma$  and  $\text{sm } C$  is exactly  $\sum_{p \in \text{Sing}(C)} \mu(C,p)$ , it is enough to prove the formula for a non-singular surface.

If  $C$  is non-singular, then the restriction  $TX|_C$  of the tangent bundle of  $X$  splits as a complex rank-2 vector bundle as the sum of the tangent bundle to  $C$  (whose first Chern class is  $e(C)$ , so it evaluates on  $C$  as  $\chi(C)$ ) and the normal bundle (whose first Chern class evaluates on the orientation class as  $C \cdot C$ ). Using Whitney's formula for the first Chern class of a direct sum of bundles, we conclude the proof.  $\square$

We will not use it, but we mention here a generalisation of Kronheimer and Mrowka's solution to the Thom conjecture, Theorem 2.1 above.

**Theorem 3.11** (Morgan, Szabó, Taubes [MST96], Ozsváth, Szabó [OS00]). *A non-singular symplectic curve in a symplectic 4-manifold is genus-minimising in its homology class.*

Among the first, and for us most useful, striking results one can prove with pseudo-holomorphic techniques, we have Gromov's and McDuff's theorems on the complex projective plane. The first theorem is about lines in  $\mathbb{C}\mathbb{P}^2$ .

**Theorem 3.12** (Gromov [Gro85]). *Fix an almost complex structure  $J$  on  $\mathbb{C}\mathbb{P}^2$  compatible with  $\omega_{\text{FS}}$ . Given any two points on  $\mathbb{C}\mathbb{P}^2$ , there exists a unique  $J$ -holomorphic line passing through them. Given any point, the set of lines through the point forms a pencil.*

In particular, this implies that the answer to symplectic isotopy problem in degree 1 is positive.

**Corollary 3.13.** *Every symplectic curve in  $\mathbb{C}\mathbb{P}^2$  of degree 1 is symplectically isotopic to a complex line.*

Note that, crucially, when we have a symplectic isotopy, the almost-complex structure which is compatible with the curves along the isotopy is allowed to change with the parameter  $t$ , and we have to take that into account.

*Sketch of proof of the corollary.* Choose a symplectic curve  $L$  in degree 1 in  $\mathbb{C}\mathbb{P}^2$ . Then  $L$  is a sphere and it cannot be singular, by the adjunction formula. Choose a  $J_0$  that makes the curve  $J_0$ -holomorphic, and choose a path  $J_t$  of almost-complex structures from  $J_0$  to  $J_1 = J_{\text{st}}$ . Fix two points  $p \neq q$  on  $L$ . By Gromov's theorem above, for each  $J_t$  there exists a unique  $J_t$ -holomorphic line passing through  $p$  and  $q$ . One can prove that the space of degree-1 curves that are  $J_t$ -holomorphic and pass through  $p$  and  $q$  forms a manifold, which in this case has to be an interval (i.e. the solutions vary continuously with time). This gives an isotopy  $L_t$  from  $L$  to the unique  $J_{\text{st}}$ -holomorphic line through  $p$  and  $q$ .  $\square$

The second theorem, in a sense conversely, classifies symplectic 4-manifolds which can contain lines. Very strikingly, only the standard  $\mathbb{C}\mathbb{P}^2$  can. (This is a theorem of Castelnuovo in complex algebraic geometry, and follows quite easily from the Enriques–Kodaira classification of compact complex surfaces.) We will refer to it as *McDuff's theorem*.



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**Theorem 3.14** (McDuff [McD90]). *Suppose  $(X, \omega)$  is a symplectic 4-manifold that contains an embedded symplectic sphere  $S$  of self-intersection  $+1$ , and no embedded symplectic sphere of self-intersection  $-1$ . Then  $(X, \omega)$  is symplectomorphic to  $(\mathbb{C}\mathbb{P}^2, \lambda\omega_{\text{FS}})$  for some  $\lambda > 0$ . Moreover, we can choose the symplectomorphism so that  $S$  is sent to a complex line.*

McDuff proves something stronger in [McD90]: she proves that every symplectic 4-manifold  $X$  that contains a symplectically embedded sphere  $S$  of non-negative self-intersection is, up to blow-ups and blow-downs, either  $\mathbb{C}\mathbb{P}^2$  (or  $S^2 \times S^2$ ) or a symplectic  $S^2$ -bundle over a surface. Moreover, if the self-intersection is at least 5 and the 4-manifold is minimal (i.e. it contains no symplectic  $(-1)$ -spheres) then  $X$  fibres over  $S^2$  and  $S$  is a section.

I would like to mention here the main result from Fabien Kütle’s thesis, which is a generalisation of the latter statement above and of an algebro-geometric result of Hartshorne [Har69].

**Theorem 3.15** (Kütle [Küt21]). *If  $S$  is a symplectically embedded surface in  $(X, \omega)$  and  $(X, S)$  is relatively minimal (i.e.  $X \setminus S$  is minimal) and either  $S \cdot S > 4g(S) + 4$  and  $g(S) \neq 1$  or  $S \cdot S > 4g(S) + 5$ , then  $X$  is a fibre bundle over a surface of genus  $g(S)$  and  $S$  is a section.*

## 3.3 Rational cuspidal curves cannot detect an exotic $\mathbb{C}\mathbb{P}^2$

A useful probe to detect exotic smooth structures on a topological 4-manifold  $X$  is the *genus function*, assigning to a homology class in  $H_2(X)$  the minimal genus among its embedded representatives. This is an *effective* way of detecting exotica— for example, a minimal exotic  $\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$  has no class of self-intersection  $-1$  that is represented by a sphere.

More generally, one can wonder whether curves in a symplectic 4-manifolds can detect exotica. For instance: could there exist a symplectic 4-manifold  $(X, \omega)$  where  $X$  is homeomorphic to  $\mathbb{C}\mathbb{P}^2$ , but  $X$  contains a rational cuspidal curve that cannot be contained in  $\mathbb{C}\mathbb{P}^2$ ?

In [GS22] we proved that rational cuspidal curves cannot do this for us.

**Theorem 3.16** (G., Starkston [GS22]). *If  $X$  is a rational homology  $\mathbb{C}\mathbb{P}^2$  that contains a rational cuspidal curve  $C$ , then  $X$  is symplectomorphic to  $\mathbb{C}\mathbb{P}^2$ .*

*Remark 3.17.* There exist rational homology  $\mathbb{C}\mathbb{P}^2$ s that admit a symplectic structure. In fact, there are even complex projective surfaces that are rational homology  $\mathbb{C}\mathbb{P}^2$ s, and they are known as *fake projective planes*. The first example is due to Mumford [Mum79], and they have since been classified [PY07, CS10]. These are all ball quotients (i.e. quotients of the complex hyperbolic plane) [Yau77], so their universal cover is contractible, and in particular they cannot contain rational cuspidal curves. That is, the algebraic version of the theorem above was well-known.

*Sketch of proof.* If  $X$  is a rational homology  $\mathbb{C}\mathbb{P}^2$  with a symplectic structure, the existence of an almost-complex structure  $J$  implies that  $c_1(\omega)^2 = 9$ . Up to torsion,

there are now two possibilities for  $c_1(\omega)$ : either it intersects  $[\omega]$  positively, or it intersects it negatively.

If  $c_1(\omega) \cup [\omega] > 0$ , Taubes proved that  $X$  is the standard  $\mathbb{C}\mathbb{P}^2$  [Tau96]. If  $c_1(\omega) \cup [\omega] < 0$ , then we can compute the Euler characteristic of  $\text{sm } C$  from the adjunction formula:  $\chi(\text{sm } C) = -d^2 - 3d$ . This implies that  $g(K_C) = \frac{d^2+3d+2}{2}$ . Since  $K_C$  is a connected sum of algebraic knots,  $\nu^+(K_C) = g(K_C)$ .

Corollary 2.27 (or rather the exact computation in the paragraph below the statement) tells us that  $2\nu^+(K_C)$  would need to be in the interval  $(d(d-3), d(d-1)]$ , which gives us a contradiction.  $\square$

An interesting remark is that, thanks to a theorem of Taubes, every symplectic 4-manifold contains symplectic curves: more precisely the homology class (that is Poincaré dual to)  $c_1(\omega)$  or  $-c_1(\omega)$  is always represented by a symplectic curve. For a fake projective plane  $F\mathbb{P}^2$ ,  $c_1(F\mathbb{P}^2)^2 = 9$ . For many  $F\mathbb{P}^2$ s, it is currently not known whether a generator of  $H_2(F\mathbb{P}^2)/\text{Tor}$  can be represented by a complex curve. Note also that we currently do not know of any symplectic rational homology  $\mathbb{C}\mathbb{P}^2$  that does *not* admit a complex structure.

## 3.4 Pseudo-holomorphic obstructions for cuspidal curves

In the previous section we learnt that, if we are to look for symplectic rational cuspidal curves in a rational homology  $\mathbb{C}\mathbb{P}^2$ , we only have to look for them in the standard  $\mathbb{C}\mathbb{P}^2$ . Since a rational cuspidal curve is in particular an embedded PL sphere, all the obstructions that we developed in the previous section help us in our search for (mostly rational) cuspidal curves.

We are going to discuss three more obstructions here: one coming from the adjunction formula, one from branched covers of real surfaces and the Riemann–Hurwitz formula, and one from cobordisms and Levine–Tristram signatures. We also briefly discuss a potentially new obstruction coming from positivity of intersections and Bézout’s theorem, only to discover that it is in fact something we already knew.

These are all well-known obstructions in the complex algebro-geometric setting, and the proofs adapt almost verbatim to the setting of symplectic curves, as we observed in [GS22].

*Remark 3.18.* With some more technology, one can even recover Plücker formulae in the symplectic setup. The crucial point is that, contrarily to what happens in the complex case, there is no natural symplectic or almost-complex structure on the *dual* to an almost-complex projective plane. Nevertheless, Sikorav used Gromov’s *elliptic structures* on the dual projective plane to recover Plücker’s formulae [Sik04].

### Adjunction

This is the easiest one—in fact we already used it in the proof of Theorem 3.16.

Consider an irreducible (but not necessarily rational nor cuspidal) symplectic curve  $C = \iota(\Sigma) \subset (X, \omega)$ . Then from the adjunction formula we know that the sum

### 3 Symplectic curves

of the Milnor numbers of the singularities of  $C$  is determined by  $\chi(\Sigma)$  and  $[C]$ . More precisely,

$$\sum_{p \in \text{Sing}(C)} \mu(C, p) = C \cdot C + \chi(\Sigma) - \langle c_1(\omega), C \rangle.$$

Specialising to  $X = \mathbb{C}\mathbb{P}^2$  and calling  $d$  the degree of  $C$ , we obtain the following proposition.

**Proposition 3.19.** *If  $C$  is a singular symplectic curve in  $\mathbb{C}\mathbb{P}^2$  of degree  $d$  and genus  $g$ ,*

$$\sum_{p \in \text{Sing}(C)} \mu(C, p) = (d-1)(d-2) - 2g(\Sigma),$$

and further specialising to rational curves, we have:

$$\sum_{p \in \text{Sing}(C)} \mu(C, p) = (d-1)(d-2),$$

This imposes restrictions on rational cuspidal curves that are not purely topological. For instance, Family (1) in Theorem 2.28 violates this inequality: in this case, the PL surface  $F$  of degree  $p$  with a unique singular point of type  $(p, p+1)$  cannot be realised as a symplectic curve.

What is perhaps unexpected is that the only cases in Theorem 2.28 that satisfy adjunction *all* arise as symplectic (in fact, even complex algebraic) rational cuspidal curves. We will talk about this in more detail in the next section.

### Riemann–Hurwitz formulae

The Riemann–Hurwitz obstruction also uses symplectic information in global way. Fix an almost complex structure  $J$  on  $\mathbb{C}\mathbb{P}^2$  such that  $C = \iota(\Sigma)$  is  $J$ -holomorphic. Fix a point  $p_0 \in \mathbb{C}\mathbb{P}^2$ , and consider the pencil of  $J$ -holomorphic lines through  $p_0$ , and the associated projection  $\pi_0: \mathbb{C}\mathbb{P}^2 \setminus \{p_0\} \rightarrow \mathbb{P}(T_{p_0}\mathbb{C}\mathbb{P}^2) \cong \mathbb{C}\mathbb{P}^1$ . Restricting this projection to  $\pi_C: C \rightarrow \mathbb{C}\mathbb{P}^1$ , and pre-composing with the map  $\iota$  gives a ramified covering map  $\pi = \pi_C \circ \iota: \Sigma \rightarrow \mathbb{C}\mathbb{P}^1$ . A priori, if  $p_0 \in C$ , the map is not well-defined at  $p_0$ . If  $p_0$  is a cuspidal point in  $C$ ,  $\pi$  has a unique continuous extension defined by sending  $p_0$  to the image of the  $J$ -holomorphic line through  $p_0$  which is tangent to  $C$  at  $p_0$ .

The fact that the only singularities of  $\pi$  are complex ramification points follows from positivity of intersections between  $J$ -holomorphic curves and from McDuff’s and Micallef and White’s work on singularities of  $J$ -holomorphic curves [McD92, MW95]. Ramification points arise from tangencies between lines in the pencil with  $C$  and from singular points of  $C$ .

For a point  $q \neq p_0$ , the ramification index of  $\pi$  at  $q$  is equal to the multiplicity of intersection of the  $J$ -holomorphic line  $L_q$  through  $q$  with  $C$ . For a smooth point  $q$ , this is just the order of tangency between  $L_q$  and  $C$  at  $q$ . These numbers are bounded from below by the *multiplicity* and the *second multiplicity* of the singularity, which in turn are encoded in the *semigroup*  $\Gamma_{(C,p)}$  of the singularity of  $C$  at the point  $p$ . If there is no ambiguity, we drop  $C$  from the notation and just use the subscript  $p$ .

### 3.4 Pseudo-holomorphic obstructions for cuspidal curves

Recall that the semigroup of a singularity is the set of possible multiplicities of intersections of curves  $D$  with  $C$  at  $p$ :  $\Gamma_p$  always contains 0 (for a curve  $D$  that does not contain  $p$ ), then the first non-zero element is the multiplicity  $m_p$  of  $p$ , and the second non-zero element is  $m_p + m'_p$ .  $m_p$  is a lower bound for the multiplicity of intersection at  $p$  between  $C$  and a curve passing through  $p$ , whereas, if  $p$  is a cusp,  $m_p + m'_p$  is a lower bound for the multiplicity of intersection at  $p$  of  $C$  with a curve tangent to  $C$  at  $p$ .

The Riemann–Hurwitz formula is the calculation of the Euler characteristic of the branched covering in terms of the ramification indices and degree of the cover. If  $\pi: \Sigma \rightarrow \mathbb{CP}^1$  is a  $k$ -fold ramified cover with ramification points  $x_1, \dots, x_\ell$  and ramification indices  $e_\pi(x_j)$  then

$$\chi(\Sigma) = k(2 - \ell) + \sum_{j=1}^{\ell} (k + 1 - e_\pi(x_j)) = 2k - \sum_{j=1}^{\ell} (e_\pi(x_j) - 1).$$

Suppose  $d$  is the degree of  $C$ . If we choose  $p \notin C$ , then a generic line through  $p$  intersects  $C$ ,  $d$  times, so the degree of the cover is  $d$ . Therefore the above equation specialises to

$$2d - \chi(\Sigma) = \sum_{j=1}^{\ell} (e_\pi(x_j) - 1).$$

If instead, we choose  $p \in C$ , where  $p$  has multiplicity  $m_p$  (where  $m_p$  is the first entry of the multiplicity sequence if  $p$  is a singular point and is 1 if  $p$  is a smooth point of  $C$ ), then a generic line through  $p$  intersects  $C$  at  $d - m_p$  other points. Therefore  $\pi$  is a  $(d - m_p)$ -fold cover. This gives the following equation and inequality. The inequality is particularly useful as an obstruction to symplectically realising certain cuspidal curves in  $\mathbb{CP}^2$ .

$$2(d - m_p) - \chi(\Sigma) = \sum_{j=1}^{\ell} (e_\pi(x_j) - 1) \geq \sum_{q \neq p} (m_q - 1) + (m'_p - 1).$$

We summarise the discussion above in the following proposition.

**Proposition 3.20.** *If  $C$  is a symplectic cuspidal curve of degree  $d$  and genus  $g$ , for each  $p \in \text{Sing}(C)$  we have:*

$$2d - 2m_p + 2g \geq 2 + \sum_{q \neq p} (m_q - 1) + (m'_p - 1).$$

*In particular, if  $C$  is a rational cuspidal curve,*

$$2d - 2m_p \geq 2 + \sum_{q \neq p} (m_q - 1) + (m'_p - 1).$$

*Example 3.21.* Let us consider the configurations of cusps on a symplectic curve of degree 5 in  $\mathbb{CP}^2$ :  $\{(3, 5), 2(2, 3)\}$ ,  $\{(3, 4), 3(2, 3)\}$ , and  $\{(2, 5), 4(2, 3)\}$ . (We are using the shorthand notation  $k(2, 3)$  to denote  $k$  cusps of type  $(2, 3)$ .) We can apply

### 3 Symplectic curves

Riemann–Hurwitz to exclude all of them: In the first case, we project from the (3, 5)-cusp, and we obtain the following contradiction.

$$2 \cdot 5 - 2 \cdot 3 \geq 2 + 1 + 1 + 1.$$

In the second, we project from the (3, 4)-cusp:

$$2 \cdot 5 - 2 \cdot 3 \geq 2 + 1 + 1 + 1.$$

In the third case we project from the (2, 5)-cusp:

$$2 \cdot 5 - 2 \cdot 2 \geq 2 + 1 + 1 + 1 + 1 + 1.$$

Thus, in each of the three cases, we get a contradiction. The last example can also be obstructed smoothly by taking the 5-fold branched cover and computing the contributions to  $b_2^-$  coming from each of the five singularities, as we did in Section 2.4.

### Levine–Tristram signatures

We will derive an inequality based on the Levine–Tristram signature and nullity functions of the links,  $\sigma_\bullet, \eta_\bullet : S^1 \rightarrow \mathbb{Z}$ . It is related to, but not completely equivalent to, the algebro-geometric spectrum semicontinuity, formulated in terms of Hodge-theoretic data, and then recast in more topological terms by Borodzik and Némethi [BN12, Corollary 2.5.4].

**Proposition 3.22.** *Let  $C$  be a symplectic cuspidal curve of degree  $d$  and genus  $g$  in  $\mathbb{C}\mathbb{P}^2$ . Then for every  $\zeta$  in a dense subset  $S_!^1$  of  $S^1$  we have:*

$$|\sigma_{T(d,d)}(\zeta) - \sigma_K(\zeta)| + |\eta_{T(d,d)}(\zeta) - \eta_K(\zeta)| \leq 2g + d - 1.$$

From a  $J$ -holomorphic curve rational cuspidal  $C$ , a *generic*  $J$ -holomorphic line  $\ell$  gives us a genus-0 cobordism in  $S^3 \times [0, 1]$  from  $K_C$  to the torus link  $T(d, d)$ . Here by generic we mean that we want  $\ell$  to intersect  $C$  transversely—analogously to what happens in algebraic geometry, a  $J$ -holomorphic curve has finitely many tangent lines, so we have lots of choice for  $\ell$ .

This cobordism is obtained by removing a neighbourhood of a path connecting all singularities of  $C$ , and a neighbourhood of the line  $\ell$ . The fact that we obtain a cobordism to  $T(d, d)$  is a corollary of positivity of intersections, so we are indeed using the almost-complex structure  $J$ . If we did not have positivity of intersections, we would only obtain a cobordism to a link that (as an unoriented link) is isotopic to  $T(d + 2a, d + 2a)$ , but that has  $a$  components oriented in one direction and  $d + a$  in the other.

Suppose now that  $C$  is a cuspidal curve of genus  $g$  and degree  $d$ . Then  $K_C$  is a knot, and for every  $\zeta \in S_!^1 \subset S^1$  we have:

$$|\sigma_{T(d,d)}(\zeta) - \sigma_K(\zeta)| + |\eta_{T(d,d)}(\zeta) - \eta_K(\zeta)| \leq d - 1 + 2g.$$

We have a concrete description of the set  $S_!^1$ : it is the unit circle with all *Knoten-nullstellen* (roots of integer polynomials such that  $p(1) = 1$ ) removed; that is,

$$S_!^1 = S^1 \setminus \{\alpha \mid \exists p(t) \in \mathbb{Z}[t], p(1) = 1, p(\alpha) = 0\}.$$

### 3.4 Pseudo-holomorphic obstructions for cuspidal curves

The inequality above was essentially proved by Nagel and Powell [NP17], who also introduced the notation and terminology for  $S_1^1$  (see also [Con21, Theorem 2.12]).

To be concrete: all transcendental complex numbers of norm 1 belong to  $S_1^1$ , so the set of Knotennullstellen has measure 0. A root of unity belongs to  $S_1^1$  if and only if its order is a prime power; for this it suffices to evaluate cyclotomic polynomials at 1:  $\Phi_{p^r}(1) = p$  for every prime  $p$  and positive integer  $r$ , while  $\Phi_n(1) = 1$  whenever  $n$  has at least two distinct prime factors.

*Example 3.23.* The spectrum semicontinuity obstructs the existence of a symplectic quintic with two cusps of type  $(2, 7)$ : at  $\zeta = e^{2\pi i/6}$  we have  $\sigma_{T(6,6)}(\zeta) = -9$ ,  $\sigma_{T(2,7)}(\zeta) = -2$ ,  $\eta_{T(6,6)}(\zeta) = 4$ , and  $\eta_{T(2,7)}(\zeta) = 4$ , so

$$|\sigma_{T(6,6)}(\zeta) - 2\sigma_{T(2,7)}(\zeta)| + |\eta_{T(6,6)}(\zeta) - \eta_{T(2,7)}(\zeta)| = 9 \not\leq 5 = d - 1 + 2g.$$

*Remark 3.24.* There is a relation between the inequality above and the branched cover obstruction from the previous chapter. In the notation of Theorem 2.39, we let  $\omega_d = e^{2\pi i/d}$ . One can verify that

$$\sigma_{T(d,d)}(\omega_d^r) = 1 - 2r(d - r), \quad \text{and} \quad \eta_{T(d,d)} = d - 2.$$

If  $d/r$  is a prime power, then  $\omega_d^r \in S_1^1$ . If  $C$  is a rational cuspidal curve, the inequality from Proposition 3.22 at  $\omega_d^r$  reduces exactly to the inequality of Theorem 2.39. This should not be too surprising, since the inequality on the Levine–Tristram signature which underpins Proposition 3.22 can be interpreted, at roots of unity of prime power order, in terms of cyclic branched covers, à la Viro [Vir75]

## Bézout’s theorem

There is another observation of algebro-geometric nature, coming from Bézout’s theorem, that we can adapt to the pseudo-holomorphic setup and use as an obstruction. Suppose  $C$  is an irreducible symplectic curve in  $\mathbb{C}\mathbb{P}^2$  of degree  $d$ , and  $J$  a compatible almost-complex structure for which  $C$  is  $J$ -holomorphic. Then every other  $J$ -holomorphic curve  $D$  of degree  $d' < \deg C$  in  $\mathbb{C}\mathbb{P}^2$  intersects  $C$  in at most  $dd'$  points. Counting with multiplicity,  $C$  and  $D$  will intersect exactly  $dd'$  times. (Note that here we are using both assumptions:  $C$  has to be irreducible and  $d'$  has to be strictly less than  $d$ , for otherwise we could choose  $D$  to be a component of  $C$ .) By positivity of intersection, however, all intersections are *positive*, as they are in complex algebraic geometry.

Rather than trying to build the general theory, we give the most basic example. First we choose  $D$  to have degree 1, i.e. it is a line. We know that for every two singularities  $p, q \in \text{Sing}(C)$  the  $J$ -holomorphic line through  $p$  and  $q$  intersects  $C$  with multiplicity at least  $m_p$  at  $p$  and  $m_q$  at  $q$ , so by Bézout’s theorem we have that  $m_p + m_q \leq d$ . Similarly, the tangent line to a singular point  $p \in \text{Sing}(C)$  intersects  $C$  at  $p$  with multiplicity at least  $m_p + m'_p$ , so by Bézout’s theorem we have  $m_p + m'_p \leq d$ .

In complex algebraic geometry, we can look at curves of degree  $d'$  and impose  $\frac{1}{2}d'(d' + 1) - 1$  linear conditions at various points of the curve, and this gives an inequality that we can express in terms of the semigroup of the curve. For symplectic curves this is not as easy to do, since we do not have equations.

However, it turns out that this obstruction is *implied by* the Heegaard Floer obstruction of Theorem 2.26. This was observed by Borodzik [Bor17].

## 3.5 The symplectic isotopy problem for rational cuspidal curves

At last, we go back to Question 3.3: the symplectic isotopy problem for singular curves. Motivated by Examples 3.4–3.7 above, we restrict to the case of rational cuspidal curves. The question can be split into two sub-questions: the first concerns the equisingularity type, the second the isotopy classes within a given equisingularity class.

**Question 3.25.** Is every symplectic rational cuspidal curve in  $\mathbb{C}\mathbb{P}^2$  equisingular to a complex curve?

In plain words: suppose that  $C$  is a symplectic rational cuspidal curve, does there exist a complex curve  $C'$  of the same degree with the same singularities? If the answer is no, clearly  $C$  cannot be isotopic to any complex curve.

There are many (non-obvious) restrictions on the possible configurations of singularities of complex rational cuspidal curves. The most striking is probably the four-cusp theorem, proved by Koras and Palka [KP22].

**Theorem 3.26** (Koras, Palka). *A complex rational cuspidal curve in  $\mathbb{C}\mathbb{P}^2$  has at most four singularities. If it has four, then it is isomorphic to the unique 4-cuspidal quintic.*

Thus, a very simple way of showing that the answer to Question 3.25 is negative would be to produce a symplectic rational cuspidal curve with more than 4 cusps, or any curve of degree larger than 5 with four cusps.

As a side note, we had mentioned above that the equisingular classification of singular curves in  $\mathbb{C}\mathbb{P}^2$  is not known. For instance, we do not even know how many simple cusps a degree- $d$  curve can have (except when  $d$  is small). For complex rational cuspidal curves, there is a proposed classification result due to Palka and Pełka [PP17, PP20]. Their result depends on the *negativity conjecture* [Pal19], which is currently open.

The second, more subtle, question to come out of Question 3.3 is the isotopy problem properly said.

**Question 3.27.** Suppose  $C$  and  $C'$  are equisingular symplectic rational cuspidal curves in  $\mathbb{C}\mathbb{P}^2$ . Are they equisingularly symplectically isotopic?

As far as I know, the answer to the question is not even known to algebraic geometers if we restrict to rational cuspidal curves. That is, we do not know of any Zariski pair of rational cuspidal curves.

In collaboration with Starkston and with Kütle, we proved that the answer to the symplectic isotopy problem is true for small degrees and “small complexity”.

**Theorem 3.28** (G., Starkston [GS22] for  $\deg \leq 5$ , G., Kütle [GK23] for  $\deg = 6, 7$ ). *If  $C$  and  $C'$  are two equisingular symplectic rational cuspidal curves in  $\mathbb{C}\mathbb{P}^2$  of degree  $\deg C = \deg C' \leq 7$ , then  $C$  and  $C'$  are equisingularly symplectically isotopic, and their symplectic isotopy class contains a complex curve.*

### 3.5 The symplectic isotopy problem for rational cuspidal curves

We list here all singularity types that occur.

Degree 3	(2, 3)		
Degree 4	(3, 4) 3(2, 3)	(2, 7)	(2, 5), (2, 3)
Degree 5	(4, 5) (3, 4), (2, 5), (2, 3) (2, 7), 3(2, 3)	(3, 5), (2, 5) (2, 13) 3(2, 5)	(3, 4), (2, 7) (2, 9), (2, 5)
Degree 6	(5, 6) (2, 3; 2, 13), (2, 5) (4, 5), 2(2, 5) (3, 8), (3, 4)	(2, 3; 2, 17) (4, 5), (2, 9) (3, 11) (3, 7), (3, 5)	(2, 3; 2, 15), (2, 3) (4, 5), (2, 7), (2, 3) (3, 10), (2, 3)
Degree 7	(6, 7) (5, 6), (2, 9), (2, 3) (4, 5), (3, 10) (2, 3; 2, 13), (3, 7), (2, 3)	(5, 7), (2, 7) (5, 6), (2, 7), (2, 5) (2, 3; 2, 15), (3, 7) (3, 13), (2, 7)	(5, 6), (2, 11) (4, 7), (3, 7) (2, 3; 2, 13), (3, 8)

Before stating the second theorem, let us define what we mean by “small complexity”. We say that a curve  $C$  is *1-unicuspidal* if  $C$  has only one singularity and the link of this singularity is a torus knot  $T(p, q)$ . (The number 1 refers to the number of Puiseux pairs of the singularity or, equivalently, of cabling parameters of the link.)

**Theorem 3.29** (G., Starkston [GS22]). *If  $C$  and  $C'$  are two symplectic rational 1-unicuspidal curves in  $\mathbb{C}\mathbb{P}^2$ , then  $C$  and  $C'$  are equisingularly symplectically isotopic, and their symplectic isotopy class contains a complex curve.*

We can explicitly list the degrees and singularity types of the previous theorem:

- degree  $d$  and singularity of type  $(d - 1, d)$  (for  $d \geq 3$ );
- degree  $2m$  and singularity of type  $(m, 4m - 1)$  (for  $m \geq 2$ );
- degree  $F_{2k+1}$  and singularity of type  $(F_{2k-1}, F_{2k+4})$  (for  $k \geq 2$ );
- degree  $F_{2k-1}F_{2k+1}$  and singularity of type  $(F_{2k-1}^2, F_{2k+1}^2)$ ;
- degree 8 and singularity of type  $(3, 22)$ ;
- degree 16 and singularity of type  $(6, 43)$ .

These were known to algebraic geometers, thanks to fairly recent work of Fernández de Bobadilla, Luengo, Melle Hernández, and Némethi [FLMN07]. It is worth noting that they, too, used some topological ideas (spectrum semicontinuity, related to the Levine–Tristram signature obstruction described above) to heavily narrow down the possibilities. By contrast, we mostly used Heegaard Floer homology and correction terms: as observed in [BCG16] and [Liu14] the semigroup obstruction combined with the adjunction formula already suffices to pin down the list above.

Observe that this theorem is the symplectic counterpart of Theorem 2.28 in the next section: a rational 1-unicuspidal curve is a PL sphere with one singular point of type  $T(p, q)$  for some  $p$  and  $q$ .

The proofs of Theorems 3.28 and 3.29 are similar, and they both essentially involve a case-by-case analysis: first we list all unobstructed configurations of singularities



a curve can have (using the obstructions we discussed in this and in the previous chapter) and then we try to construct all the cases that are remaining.

One might want to proceed and classify curves degree by degree, extending the results of Theorem 3.28. Unfortunately, the number of possible configurations grows quite fast with the degree, and even after testing all of the obstructions we are left with a respectable number of constructions to attempt. In [GK23], when dealing with curves of degree 7, we had 718 possible configurations of singularities satisfying the adjunction formula, of which 28 passed all the obstructions we tested<sup>2</sup>, 11 of which are realised by symplectic curves. For the remaining 17, we had to work case by case. A driving reason to think about sextics and septic was not so much the classification in and of itself (although it *is* interesting to know that no exotic phenomena can appear in degree at most 7), but rather a way of testing the obstructions we know of and the constructions we could do. (Already from [GS22] to [GK23] we had found or rediscovered some more powerful invariants.)

The main idea underpinning the constructive part of both proofs is that of birational transformations, to which we dedicate the next section.

## 3.6 Birational transformations

Birational transformations are a well-known tool in algebraic geometry, used to compare varieties that are “mostly” isomorphic. Formally, a birational transformation is an isomorphism defined on a Zariski open subset of a variety onto an open subset of the target. It is the central equivalence relation used in the classification of algebraic varieties. In the case of complex surfaces, birational transformations amount to a sequence of blow-ups and blow-downs.

### In theory

A birational equivalence  $\phi: X \dashrightarrow Y$  between two varieties  $X$  and  $Y$  is a morphism  $\phi: U \rightarrow Y$  where  $U \subset X$  is Zariski open and  $\phi$  is an isomorphism onto a Zariski open subset of  $Y$ . As mentioned above, if  $X$  and  $Y$  are complex surfaces, then every birational equivalence factors into a compositions of blow-ups and blow-downs.

Suppose that  $X$  is non-singular and connected. Let  $\phi: X \dashrightarrow Y$  be a birational equivalence and choose a maximal open subset  $U \subset X$  such that  $\phi$  is a morphism on  $U$  and an  $\phi: U \rightarrow Y$  is an isomorphism onto its image. If  $C \subset X$  is a configuration of curves, then in  $Y$  we have two configurations of curves associated to  $\phi$  and  $C$ : one is the complement of  $\phi(U) \subset Y$ , and the other is the closure of  $\phi(C \cap U)$ .

We will now try to extend the definition of birational equivalence to the symplectic setting, and to consider what happens to symplectic curves under birational transformations.

First, we need to know that blow-ups and blow-downs exist in the symplectic category. This was essentially done by McDuff [McD91], ensuring that if  $X' = X \# \overline{\mathbb{C}\mathbb{P}^2}$  and  $b: X' \rightarrow X$  is the usual blow-down (i.e. contracting a  $-1$ -sphere  $E$

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<sup>2</sup>When writing [GK23], we were not entirely aware of Theorem 2.39 and had a weaker version of Proposition 3.22. With these more refined tools, there are 19 configurations that pass all obstructions.

generating  $H_2(\overline{\mathbb{C}\mathbb{P}^2}) \subset H_2(X')$  and  $\omega$  is a symplectic structure on  $X$ , then there is a symplectic structure  $\omega'$  on  $X'$  that “agrees with  $\omega$ ” and for which  $E$  is a symplectic sphere. This means that  $b^*\omega = \omega'$  outside the preimage of a ball in  $X$ , containing  $b(E)$ . A subtle difference between the blow-up in complex and symplectic geometry is that the complex blow-down contracts a curve to a single point, whereas the symplectic blow-down “contracts” a curve to a closed 4-ball.

This translates into some more work if one wants to carefully lift curves under blow-ups as one does in the complex set-up. This is what we do in [GS22], and that we can state loosely as follows.

**Proposition 3.30.** *If  $(C, 0)$  the germ of a complex curve in  $X = \mathbb{C}^2$ , then the the preimage  $b_{\mathbb{C}}^{-1}(C)$  of  $C$  in the complex blow-up  $b_{\mathbb{C}}: X'_{\mathbb{C}} \rightarrow X$  and  $b_{\omega}^{-1}(C)$  in the symplectic blow-up  $b_{\omega}: X'_{\omega} \rightarrow X$  (both at the origin) have deformation-equivalent neighbourhoods.*

We call  $b^{-1}(C)$  (in either setup) the *total transform* of  $C$ , denoted with  $\overline{C}$ , and  $\overline{b^{-1}(C \setminus \{0\})}$  the *proper transform* of  $C$ , denoted with  $\widetilde{C}$ .

The consequence of the proposition is that we can use (embedded) resolution of (curve) singularities in the symplectic setup, which will be crucial to our study.

**Definition 3.31** (G., Starkston [GS22]). Let  $C_0 \subset (X_0, \omega_0)$  and  $C_1 \subset (X_1, \omega_1)$  be two (configurations of) symplectic curves. We say that:

- they are *birationally equivalent* if there exist blow-ups  $X'_0 \rightarrow X_0$  and  $X'_1 \rightarrow X_1$  and a symplectic deformation-equivalence  $\Phi: X'_0 \rightarrow X'_1$  such that  $\Phi(\overline{C}_0)$  is isotopic to  $\overline{C}_1$ ;
- $C_1$  is *birationally derived* from  $C_0$  if there exists blow-ups  $X'_0 \rightarrow X_0$ ,  $b_1: X'_1 \rightarrow X_1$ , and a symplectomorphism  $\Phi: X'_0 \rightarrow X'_1$  such that  $C_1 = b_1(\Phi(\overline{C}_0))$ .

The key point is that birational transformations can (and often do) allow us to simplify a configuration, often reducing to configurations of lines and conics. The key results we proved in [GS22] are the following.

**Proposition 3.32.** *If  $C_0$  is a configuration in  $(X_0, \omega_0)$ , and  $C_1$  in  $(X_1, \omega_1)$  is birationally derived from  $C_0$ , then any subconfiguration of  $C_1$  symplectically embeds into  $(X_1, \omega_1)$ .*

This statement is immediate from the definition, but its usefulness comes from its contrapositive. Namely, we can show that certain configurations  $C_0$  cannot be symplectically realised in a closed symplectic manifold  $(X_0, \omega_0)$ , using the non-existence of a subconfiguration of symplectic curves that can be birationally derived from  $C_0$ .

**Proposition 3.33.** *Suppose a configuration  $C_1$  in  $(X_1, \omega_1)$  is birationally derived from  $C_0$  in  $(X_0, \omega_0)$ , and suppose  $C_1$  has a unique (non-empty) equisingular symplectic isotopy class in  $(X_1, \omega_1)$ . Then  $C_0$  also has a unique (non-empty) symplectic isotopy class in  $(X_0, \omega_0)$ . If  $(X_0, \omega_0) = (X_1, \omega_1) = (\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$  and  $C_1$  can be realised by a complex curve, then  $C_0$  can also be realised by a complex curve.*

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*Sketch of proof.* Suppose any two symplectic embeddings of  $C_1$  into  $(X_1, \omega_1)$  are symplectically isotopic. Let  $Q_0^0$  and  $Q_0^1$  be two symplectic embeddings of  $C_0$  into  $(X_0, \omega_0)$ . By definition of birational derivation, there is a sequence of blow-ups of  $Q_0^k$  to  $\overline{Q_0^k}$  and a sequence of blow-downs that contract  $\overline{Q_0^k}$  to  $Q_1^k$  for  $k = 0, 1$ , where  $Q_1^0$  and  $Q_1^1$  are symplectic realisations of  $C_1$  into  $(X_1, \omega_1)$ .

By assumption, there exists an equisingular isotopy  $\{Q_1^t\}_{t \in [0,1]}$  which connects  $Q_1^0$  and  $Q_1^1$ . For each  $Q_1^t$ , perform the sequence of blow-ups along the appropriate smooth or singular points in  $Q_1^t$  to obtain  $\overline{Q_1^t}$ . By definition of birational derivation, there is a distinguished subset of the components  $\overline{E}^t \subseteq \overline{Q_1^t}$  for  $t \in [0, 1]$ , agreeing with  $\overline{E}^0$  and  $\overline{E}^1$  for  $t = 0, 1$ . Contracting  $\overline{E}^t$  for each  $t \in [0, 1]$  gives the required equisingular symplectic embeddings  $Q_0^t$  of  $C_0$  into  $(M_0, \omega_0)$ .

Since complex curves are preserved under birational transformations, the last statement follows from the same proof.  $\square$

If  $C_0$  and  $C_1$  are birationally equivalent, they are each birationally derived from the other, yielding the following corollary.

**Corollary 3.34.** *Suppose  $C_0$  in  $(X_0, \omega_0)$  and  $C_1$  in  $(X_1, \omega_1)$  are birationally equivalent. There is a unique equisingular symplectic isotopy class for  $C_0$  in  $(X_0, \omega_0)$ , if and only if there is a unique equisingular symplectic isotopy class for  $C_1$  in  $(X_1, \omega_1)$ . If  $(X_0, \omega_0) = (X_1, \omega_1) = (\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$  and if the equisingular symplectic isotopy class contains complex representatives for one configuration, it contains complex representatives for the other.*

Finally, we state a proposition that allows to deduce the existence and uniqueness of more complex configurations starting from simpler ones. We call this the *add-a-line lemma*.

**Proposition 3.35.** *Suppose  $C_1$  is a configuration of singular symplectic curves in  $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$  obtained from  $C_0$  by adding a single symplectic line  $L$ . Suppose that in the configuration  $C_1$  either:*

1.  *$L$  intersects the curves of  $C_0$  transversally and the intersection points of  $L$  with  $C_0$  contain at most two singular points,  $P^i$  and  $P^j$ , of  $C_0$ , or*
2.  *$L$  has a simple tangency to a subset of the components of  $C_0$ ,  $\{B_{i_1}, \dots, B_{i_k}\}$  at a single point  $P^i$  in  $C_0$  ( $P^i$  may be either a smooth or singular point of  $B_{i_\ell}$  and it can be a singular point of  $C_0$  in which case it uses the existing label, or a smooth point of  $C_0$  in which case it takes a new label index) and all other intersections of  $L$  with  $C_0$  are transverse double points. Further assume in this case that in  $C_1$ , there are no other intersections of  $L$  with the components  $B_{i_1}, \dots, B_{i_k}$  outside of the tangent point (but  $L$  may intersect other components in transverse double points).*

*Then there is a bijection between the isotopy classes of realisations of  $C_0$  and those of  $C_1$ . In particular,  $C_0$  has a unique equisingular symplectic isotopy class if and only if  $C_1$  does.*

For instance, the pseudo-Pappus configuration is unique up to isotopy. This can be seen by adding lines in the order given by Figure 3.2.

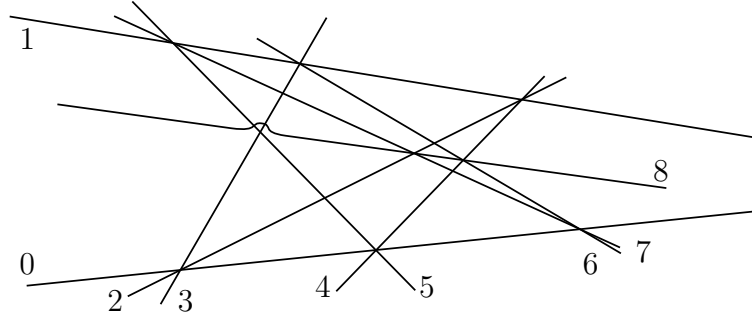


Figure 3.2: The pseudo-Pappus configuration is unique up to isotopy. Just add lines in the order indicated by the numbers and apply Proposition 3.35.

**In practice**

Instead of sketching the whole proof of the classification theorems stated above, we look at a few simple examples that showcase the main ideas of the proofs.

**Proposition 3.36.** *There is a symplectic rational cuspidal curve  $S$ , unique up to isotopy, of degree 4 with three simple cusps as singularities. This is isotopic to the complex curve known as the Steiner quartic.*

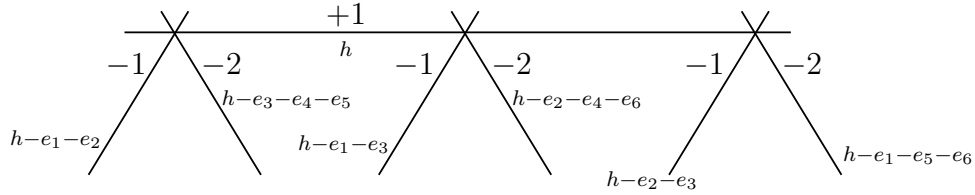


Figure 3.3: The total transform of the tricuspidal quartic and its homological embedding.

*Sketch of proof.* Blow up  $S$  twice at each of its singular points. The total transform  $\bar{S} \subset \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$  of  $S$  is shown in Figure 3.3. In particular, the proper transform  $\tilde{S}$  is a symplectically embedded  $+1$ -sphere. By McDuff’s theorem, there is a symplectic deformation auto-equivalence of  $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$  taking  $\tilde{S}$  to a symplectic line (i.e. to a line in  $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$  which is away from the blow-ups).

We want to analyse what becomes of the total transform  $\bar{S}$  under these blow-downs. The configuration we obtain is birationally derived from  $S$  in the sense of the previous section.

Once we impose that  $\tilde{S}$  is a line, i.e. it is in the homology class  $h \in H_2(\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2})$  with respect to the basis  $h_1, e_1, \dots, e_6$  given by the blow-ups, the homology classes of the components of  $\bar{S}$  are the ones exhibited in Figure 3.3. This is a bit trickier than it looks, as it requires using both positivity of intersections and the adjunction formula.

Blowing down the exceptional divisors in the homology classes  $e_1, \dots, e_6$  (whose existence is guaranteed by McDuff’s theorem, and whose uniqueness was also proved

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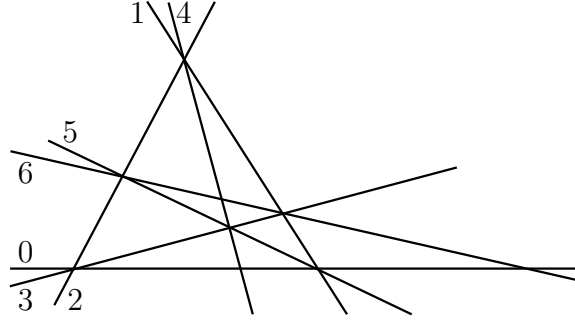


Figure 3.4: The configuration  $L$  of seven lines birationally derived from the Steiner quartic.

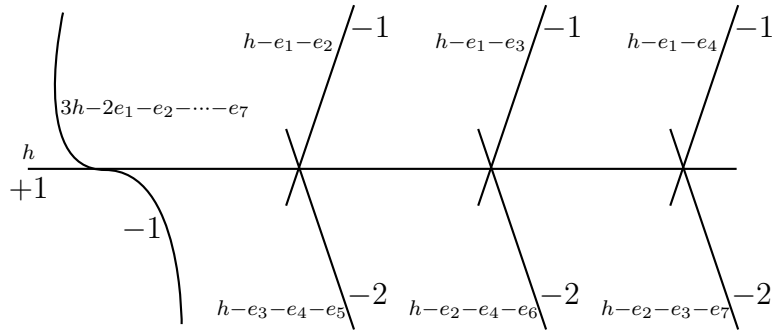


Figure 3.5: The unique homology classes of an embedding of the blow-up of a cuspidal curve with one cusp of type  $(3, 4)$  and three of type  $(2, 3)$ .

by McDuff [McD91]), one obtains a configuration  $L$  of seven lines meeting at six triple points (the three shown in the figure and the contractions of  $e_1$ ,  $e_2$ , and  $e_3$ ) and three double points (the contractions of  $e_4$ ,  $e_5$ , and  $e_6$ ), shown in Figure 3.4. This configuration can be shown to exist and to have a unique symplectic isotopy class thanks to Proposition 3.35, adding the lines one by one in the order given in Figure 3.4.

In summary, the quartic  $S$  birationally derives the configuration  $L$ , which has a unique symplectic isotopy class, so by Proposition 3.33  $S$  exists and is unique up to isotopy. Since  $L$  exists as a complex line arrangement, then  $S$  exists as a complex curve.  $\square$

**Proposition 3.37.** *There is no symplectic rational cuspidal curve of degree 5 in  $\mathbb{C}\mathbb{P}^2$  with one cusp of type  $(3, 4)$  and three of type  $(2, 3)$ .*

*Sketch of proof.* Suppose that there is such a curve  $C$ . Blow up once at the first singularity and twice at each simple cusp. The proper transform  $\tilde{C}$  of  $C$  is a symplectically embedded  $+1$ -sphere, and we can again apply McDuff's theorem and find the homological embedding shown in Figure 3.5.

Contracting the  $-1$ -spheres in the homology classes  $e_1, \dots, e_7$ , one obtains a cubic plus the *Fano configuration* as a configuration birationally derived from  $C$ . But the Fano configuration cannot be realised symplectically in  $\mathbb{C}\mathbb{P}^2$  [RS19], so we reach a contradiction.  $\square$

Ruberman and Starkston’s proof of the non-existence of the Fano configuration is a clever branched cover argument. One of the themes of [GK23] was to exploit birational transformations in conjunction with branched covers to give obstructions.

## Two birational transformations

The simplest birational transformation one can write down is the Cremona involution on  $\mathbb{CP}^2$ :  $\phi: (x : y : z) \mapsto (xy : yz : zx)$ . This is not well-defined at the three points  $(0 : 0 : 1)$ ,  $(0 : 1 : 0)$ , and  $(1 : 0 : 0)$ , and its image misses the three lines  $\{x = 0\}$ ,  $\{y = 0\}$ , and  $\{z = 0\}$ . It is a (birational) involution, since  $\phi(\phi(x : y : z)) = (\lambda x : \lambda y : \lambda z) = (x : y : z)$ , where  $\lambda = xyz$ . (In particular, it is defined on the complement of the three coordinate lines above and it is an isomorphism there.) Geometrically,  $\phi$  is a composition of three blow-ups (at  $(0 : 0 : 1)$ ,  $(0 : 1 : 0)$ , and  $(1 : 0 : 0)$ ) and three blow-downs (of the proper transforms of the three coordinate lines).

*Example 3.38.* It is very easy (and pretty) to prove the existence of the Steiner quartic using  $\phi$ : just take any non-singular conic  $Q$  that is tangent to the three coordinate lines (which in particular will avoid their pairwise intersections), and look at its image under  $\phi$ . We are blowing up away from  $Q$ , and we are contracting three exceptional divisors that are (simply) tangent to  $Q$ : this creates a simple cusp at each of the points that the lines contract to.

There is a very nice construction of Orevkov that uses a birational transformation  $\psi$  which starts from a nodal cubic (instead of a triangle, as we did for the Cremona involution above). Let  $C$  be a nodal cubic, and blow up at seven points infinitely close to the node of  $C$ . (Here we mean that we successively blow up at a node of the total transform of  $C$  that is on the proper transform of  $C$ , and always “on the same side”.) The proper transform of  $C$  is a  $-1$ -curve, that we can then contract, and we can keep contracting components of the total transform of  $C$  until we get back to  $\mathbb{CP}^2$ .

This birational transformation  $\psi$  is defined away from the cubic, and has as image the complement of another nodal cubic  $C'$  in  $\mathbb{CP}^2$ . It is an isomorphism  $\mathbb{CP}^2 \setminus C \rightarrow \mathbb{CP}^2 \setminus C'$ . We can then iterate the construction and play with it.

Here is Orevkov’s construction.

*Example 3.39 (Orevkov).* Suppose that  $C$  is a nodal cubic and  $D$  is a curve of degree  $d$  with a singularity of type  $T(a, b)$ , such that  $a + b = 3d$  and the multiplicity of intersections of  $D$  with the branches of  $C$  are  $a$  and  $b$ . (In particular,  $C$  intersects  $D$  only at its node, by Bézout’s theorem.)

Since  $\psi$  is an isomorphism away from  $C$ , and  $D$  only intersects  $C$  once, its restriction to  $D$  extends as a morphism to  $D$  which induces an isomorphism onto its image. That is,  $\psi(D)$  is again a curve with a cusp at the node of  $C'$ . Moreover, following the definition of  $\psi$  through, one sees that  $\psi(D)$  has a singularity of type  $(b, 7b - a)$  at the node of  $C'$ , and that it has degree  $(b + 7b - a)/3$ . (Note that if  $F_n$  are the Fibonacci numbers, then  $F_{n+4} = 7F_n - F_{n-4}$  and  $F_{n+2} = 3F_n - F_{n-2}$ .)

We can now play with  $D$ .

- If  $D$  is a line tangent to one of the branches of  $C$  at its node, then have  $d = 1 = F_1$ ,  $a = 1 = F_{-1}$ ,  $b = 2 = F_3$ , and iterating  $\psi$  starting from this  $D$  we

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get all rational cuspidal curves with a singularity of type  $T(F_{4k-1}, F_{4k+3})$  and of degree  $\frac{1}{3}(F_{4k-1} + F_{4k+3}) = F_{4k+1}$ .

- If  $D$  is a conic tangent to order 5 to one of the branches of  $C$  at its node, then have  $d = 2 = F_3$ ,  $a = 1 = F_1$ ,  $b = 5 = F_5$ , and iterating  $\psi$  starting from this  $D$  we get all rational cuspidal curves with a singularity of type  $T(F_{4k+1}, F_{4k+5})$  and of degree  $\frac{1}{3}(F_{4k+1} + F_{4k+5}) = F_{4k+3}$ .

That is, we retrieve all the Fibonacci rational cuspidal curves in the third family in Theorem 3.29 above.

In [BCG17] we generalised this construction to curves of *triangular* genus. (The case  $g = 1$  also appears in [BHL17].)

*Example 3.40* (Bodnár, Celoria, G.). We continue in the setup above, by taking curves of higher degree as the initial seed of the construction. First, we claim that for each  $d$  there exists a non-singular curve of degree  $d$  that intersects the two branches of a nodal cubic with multiplicities 1 and  $3d - 1$  respectively. We prove that such a curve exists by Bertini’s theorem on general elements in linear systems. First we produce a (possibly singular) curve that has the desired property, and then we take a pencil containing this curve. Bertini’s theorem says that most curves in this pencil are non-singular.

We therefore find the aforementioned examples of curves of triangular genus and a unique cusp of type defined by the recursion  $A_{n+1} = 7A_n - A_{n-1}$  with  $A_0 = 1$  and  $A_1 = 3d - 1$ .

## 3.7 The relative case

### Contact structures and symplectic cobordisms

A *contact structure* on an oriented 3-manifold  $Y$  is an orientable and co-orientable 2-plane field  $\xi$  that is maximally non-integrable. That is,  $\xi$  is the kernel of a 1-form  $\alpha$  that satisfies  $\alpha \wedge d\alpha > 0$ . We call the pair  $(Y, \xi)$  a *contact 3-manifold*<sup>3</sup>.

Suppose that  $Y$  is a boundary component of a 4-manifold  $W$ , equipped with a symplectic structure  $\omega$ . We say that  $Y$  is *strongly convex* (respectively, *strongly concave*) with respect to  $\omega$  if there exists a Liouville vector field on  $W$  defined near  $Y$  that is transverse to  $Y$  and pointing outwards (resp., inwards). Here, a *Liouville vector field* is a vector field  $V$  that expands the symplectic form exponentially:  $\mathcal{L}_V \omega = \omega$ , where  $\mathcal{L}$  is the Lie derivative. A Liouville vector field along a convex (respectively, concave) boundary component  $Y$  induces a contact structure  $\xi$  on  $Y$  (respectively, on  $-Y$ ). We drop the adverb strongly (and the adjective strong), as these are the only boundary conditions we will meet here.

We say that  $(W, \omega)$  is a *symplectic filling* (respectively, a *symplectic cap*) of  $(Y, \xi)$  if  $(Y, \xi)$  is the convex (resp., concave) boundary of  $(W, \omega)$ .  $(W, \omega)$  is a *symplectic*

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<sup>3</sup>Technically we have defined a co-orientable contact structure here, but these are the only contact structures we will encounter, so we drop the adjective “co-orientable” altogether.

*cobordism* from  $(Y_-, \xi_-)$  to  $(Y_+, \xi_+)$  if  $(Y_-, \xi_-)$  is the concave boundary<sup>4</sup> of  $W$  and  $(Y_+, \xi_+)$  is the convex boundary of  $(W, \omega)$ .

*Example 3.41.* The prototypical examples of a filling, a cap, and a cobordism are:

- The unit 4-ball in  $(\mathbb{R}^4, \omega_{\text{st}})$  is a filling of  $(S^3, \xi_{\text{st}})$ .
- The complement of a small Darboux ball in any closed symplectic 4-manifold is a cap of  $(S^3, \xi_{\text{st}})$ .
- The difference between two concentric 4-balls in  $(\mathbb{R}^4, \omega_{\text{st}})$  is a cobordism from  $(S^3, \xi_{\text{st}})$  to  $(S^3, \xi_{\text{st}})$ .

In all three cases, the radial vector field is a Liouville vector field witnessing the convexity/concavity of the boundary components.

More generally, given any contact structure  $\xi = \ker \alpha$  on  $Y$ , we can look at its *symplectisation*, i.e.  $\mathbb{R} \times Y$  endowed with the symplectic structure  $d(e^t \alpha)$ , where  $t$  is a coordinate on  $\mathbb{R}$ . Then  $\frac{\partial}{\partial t}$  is a Liouville vector field and any portion  $[a, b] \times Y \subset \mathbb{R} \times Y$  is a symplectic cobordism from  $(Y, \xi)$  to  $(Y, \xi)$ .

Convexity and concavity give the right gluing conditions for symplectic manifolds with boundary: if  $(Y, \xi)$  is a convex boundary component of  $(W_-, \omega_-)$  and a concave boundary component of  $(W_+, \omega_+)$ , then we can glue  $W_-$  and  $W_+$  along  $Y$ , and there is a symplectic structure on the gluing that agrees with  $\omega_-$  on  $W_-$  and with  $\lambda \omega_+$  on  $W_+$ , for some  $\lambda > 0$ .

Fillability is a strong restriction on the contact structure, and we now have many ways of obstructing it: for example, overtwisted discs [EG91], Giroux torsion [Gay06], contact invariants in gauge theory [OSz05]). However, there is no criterion that tells whether any given contact structure is fillable. By contrast, every contact 3-manifold has a symplectic cap [Etn04].

If  $(Y, \xi)$  is a fillable contact manifold, the classification of its fillings has potential applications to the construction of exotic 4-manifolds. (This is a consequence of the good interaction between gauge theory and symplectic/contact structure.) This idea was first pioneered by Fintushel and Stern with their rational blow-down operation [FS97].

## Divisorial contact structures

There is a second way of interpreting both McDuff's theorems (Theorem 3.14 and its generalisations from [McD90]) and the classification results for rational cuspidal curves (Theorems 3.28 and 3.29), which is in terms of *divisorial contact structures*.

Suppose that  $C$  is a symplectic curve in a symplectic, not necessarily closed, 4-manifold  $(X, \omega)$ . Suppose that  $C$  is connected (but possibly reducible) and that the subspace in  $H_2(X)$  generated by the components of  $C$  is *not* negative semi-definite—that is, there a linear combination of components of  $C$  whose self-intersection is strictly positive. Under these assumptions, we can resolve the singularities of  $C$  and apply a result of Li and Mak on plumbings of symplectic surfaces [LM19]: this gives

<sup>4</sup>The orientation on  $Y_-$  is opposite to the one induced as the boundary of  $W$ . This is consistent to make  $\xi_-$  oriented and co-oriented.



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us a family of neighbourhoods of  $C$  with concave boundary, with an induced contact structure  $\xi_C$  on the 3-manifold we call  $Y_C$ . These neighbourhoods all deformation-retract onto  $C$ . The 3-manifold  $Y_C$  is the boundary of a regular neighbourhood of  $C$ , and  $\xi_C$  is called the *divisorial contact structure* associated to  $C$ . Any concave neighbourhood of  $C$  in  $X$  given by the Li–Mak theorem is a cap of  $(Y_C, \xi_C)$ .

Moreover, the Li–Mak theorem mentioned above produces a symplectic cap for every possible *combinatorial type* of  $C$ , i.e. for every possible collection of singularities, (positive) self-intersection, and genus.

For instance, if the curve  $C$  is a sphere of self-intersection  $s > 0$ , then  $(Y_C, \xi_C)$  is the universally tight contact structure on  $L(s, 1)$ . When  $s = 1$ , this is just the standard contact structure on  $S^3$ . McDuff’s theorem can be rephrased as saying that, up to blow-ups and symplectic deformation,  $(S^3, \xi_{\text{st}})$  has a unique filling, which is the complement of a line in  $\mathbb{C}\mathbb{P}^2$ , i.e. a 4-ball. (This was known to Eliashberg [Eli90].)

**Theorem 3.42** (G., Starkston [GS22]; G., Kütle [GK23]). *Let  $C$  be a symplectic rational cuspidal curve that has the combinatorial type of a curve of degree at most 7 in  $\mathbb{C}\mathbb{P}^2$ . Then the associated divisorial contact 3-manifold  $(Y_C, \xi_C)$  has a rational homology ball filling if and only if  $C$  has the combinatorial type of a complex curve. Moreover, in this case the filling is unique up to symplectic deformation.*

This statement above is not quite equivalent to the that of Theorem 3.28 since it does not take into account the possible symmetries of  $(Y_C, \xi_C)$  nor those of  $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$ . That is, we are ignoring the contactomorphism group of  $(Y_C, \xi_C)$ , which measures the different ways one can glue the filling with the neighbourhood cap. The possible symmetries of  $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$ , on the other hand are not a problem: Gromov had proved that the symplectomorphism group of  $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$  deformation retracts onto  $PU(2)$ , which is connected. Translated into the setting of curves, if  $C$  and  $C'$  in  $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$  are symplectomorphic then they are also isotopic.

In a different direction, in [GS21] we turn the problem of finding curves on its head. More precisely, suppose that we fix the singularities and the genus of a curve, and we consider the self-intersection  $s \in \mathbb{Z}$  as a variable—recall that we had called the triple comprising the genus, the collection of singularities, and the self-intersection of a curve its combinatorial type. So, we have a family of combinatorial types of curves  $\{C_s\}_{s \in \mathbb{Z}_{>0}}$ . We ask whether  $C_s$  lives in a compact symplectic 4-manifold for all  $s$ . The answer is yes if  $C_s$  is non-singular, i.e. if the collection of singularity is empty: there are symplectic (in fact, even Kähler) *ruled* surfaces with sections of arbitrary self-intersections. These are  $\mathbb{C}\mathbb{P}^1$ -bundles over curves. Over curves of genus 0, these are known as *Hirzebruch surfaces*.

As mentioned above, a collection of results by Castelnuovo, Hartshorne [Har69] (in the complex setup), McDuff [McD90], and Kütle [Küt21] proved that, for  $s$  sufficiently large (with respect to the genus), ruled surfaces are the only possible examples (up to blow-ups). By contrast, the situation changes as soon as we introduce singularities, as the next theorem shows.

**Theorem 3.43** (G., Starkston [GS21]). *Let  $C$  be the combinatorial type of a singular symplectic curve. If the self-intersection  $C \cdot C$  is sufficiently large, then  $C$  does not embed in any closed symplectic 4-manifold.*

If  $C$  embeds in a closed symplectic 4-manifold, then all curves  $C'$  with the same singularities and genus as  $C$  and lower self-intersection embed as well.

The exact bound can be computed on a case-by-case analysis, and it can be expressed in terms of the singularities of  $C$  if  $g(C) = 0$ . We can choose a uniform bound for all curves: if  $C \cdot C > 4p_a(C) + 5$ , then  $C$  does not embed. (Here  $p_a$  is the arithmetic genus of  $C$ , i.e. the genus of the smoothing  $\text{sm } C$  of  $C$ .) Very concretely, for instance, a rational curve  $C$  with a singularity of type  $T(2, 3)$  does not embed in any closed symplectic 4-manifold if  $C \cdot C > 9$ .

The second part of the statement follows easily by blowing up at smooth points of  $C$ : the self-intersection of the proper transform decreases by 1 at each step.

The statement can be rephrased in terms of fillability of the associated cuspidal contact structure:  $\xi_C$  is not fillable whenever  $C \cdot C$  is too large<sup>5</sup>.

## Transverse knots and relative symplectic cobordisms

The knot cobordisms arising from deformations of complex surface singularities, mentioned in Section 2.3, automatically come as *symplectic* cobordisms: that is, those surfaces in  $\mathbb{C}^2$  are  $J_{\text{st}}$ -holomorphic, and hence  $\omega_{\text{st}}$ -symplectic. The boundary conditions are a bit delicate, so we spell out the definition in detail.

An oriented knot  $K$  in a 3-manifold  $Y$  is *transverse* to a contact structure  $\xi$  on  $Y$  if  $K$  intersects transversely and positively  $\xi$  at each point.

Suppose that  $(W, \omega)$  is a strong symplectic cobordism from  $(Y, \xi_-)$  to  $(Y, \xi_+)$ . Given two transverse knots  $K_{\pm}$  in  $(Y_{\pm}, \xi_{\pm})$ , we say they are *symplectically cobordant* if there are:

- (1) an  $\omega$ -symplectic surface  $\Sigma$  of  $W$  such that  $\partial\Sigma = K_-^r \cup K_+$  (where  $\Sigma$  is oriented by  $\omega$ , and  $K_-^r$  is  $K_-$  with its orientation reversed), and
- (2) there there are Liouville vector fields  $V_{\pm}$  for  $(W, \omega)$  near  $Y_{\pm}$  that restrict to be Liouville vector fields for  $(\Sigma, \omega|_{\Sigma})$  near  $K_{\pm}$ .

We call  $\Sigma$  a *relative symplectic cobordism*. We note that since the symplectic structure on  $\Sigma$  comes from the restriction of the symplectic structure on  $X$ , Condition (2) simply means that the Liouville vector fields for  $(X, \omega)$  are tangent to  $\Sigma$  near  $C_{\pm}$ . We note that while Condition (2) is convenient to include in the definition, it may be replaced with

- (2')  $\Sigma$  is transverse to the boundary of  $X$

if one is willing to deform the symplectic structure.

Much like contact structures (arising as strongly convex and strongly concave boundary components of symplectic 4-manifolds) are the correct boundary conditions for gluing symplectic cobordisms, transverse knots (arising as boundaries of relative symplectic cobordisms) glue well to give larger symplectic cobordisms, or closed symplectic surfaces.

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<sup>5</sup>Starkston and I suspect that these contact structures are tight, possibly even universally tight, so the fact that they are non-fillable is not trivial.

### 3 Symplectic curves

**Lemma 3.44** (Hayden [Hay21]; Etnyre, G. [EG22]). *If  $K$  is a transverse link in  $(Y, \xi)$  and a portion of  $K$  in a Darboux ball is as shown on the left of Figure 3.6, then there is a symplectic cobordism  $\Sigma$  in a piece of the symplectisation of  $(Y, \xi)$  from  $K$  to the knot  $K'$  obtained from  $K$  by replacing the tangle on the left of Figure 3.6 by the one on the right.*

*If  $K''$  is obtained by adding another positive crossing on the same two strands, then there is a symplectic cobordism  $\Sigma_+$  in a piece of the symplectisation of  $(Y, \xi)$  which is a union of annuli with a single positive double point.*



Figure 3.6: Front diagrams for transverse tangles in a Darboux ball.

We can apply Lemma 3.44 to construct cobordisms between transverse knots obtained as braid closures. Recall that if  $\beta$  is a braid, its closure  $K_\beta$  is a link in  $S^3$  that is transverse to the open book on  $S^3$  with disc pages. (The axis of the braid is the binding of the open book.) In particular, we can suppose that  $K_\beta$  is transverse to the standard contact structure on  $S^3$  (which is supported by the open book with disc pages), so we can associate to each braid a well-defined transverse knot. From this perspective, adding a positive generator to a braid gives a symplectic cobordism between the two braid closures, and adding the square of a generator gives an immersed, genus-0 cobordism.

### Hats, fillings, and singularities of symplectic surfaces

As in the case of (contact or smooth) manifolds, there are two interesting classes of relative symplectic cobordisms that are of special interest, namely those where one of the two ends is empty.

**Definition 3.45.** Let  $(W, \omega)$  be a symplectic 4-manifold whose convex boundary is  $(Y, \xi)$ . A *symplectic filling* of a transverse knot  $K \subset (Y, \xi)$  in  $W$  is a properly embedded  $\omega$ -symplectic surface properly embedded in  $W$  whose boundary is  $K$ . (This surface cannot intersect the concave boundary components of  $(W, \omega)$ , if there are any.)

Much like in the case of contact 3-manifolds and their symplectic fillings, fillability imposes strong restrictions on a transverse knot. In fact, transverse knots in  $(S^3, \xi_{\text{st}})$  that admit a symplectic filling in any filling of  $(S^3, \xi_{\text{st}})$  are classified: these are the closures of *quasipositive* braids. A braid is quasipositive if it is a product of conjugates of the standard generators.

In [EG22], we introduced the concave version of the previous definition.

**Definition 3.46.** Let  $(W, \omega)$  be a symplectic 4-manifold whose concave boundary is  $(Y, \xi)$ . A *symplectic hat* of a transverse knot  $K \subset (Y, \xi)$  in  $W$  is a properly embedded  $\omega$ -symplectic surface properly embedded in  $W$  whose boundary is  $K$ . (This surface cannot intersect the convex boundary components of  $(W, \omega)$ , if there are any.)

The main result in [EG22] is the existence of *projective hats* for transverse knots in  $(S^3, \xi_{\text{st}})$ , which is a relative counterpart of the existence of caps for symplectic structures [Etn04].

**Theorem 3.47.** *Every transverse knot  $K \subset (S^3, \xi_{\text{st}})$  has a symplectic hat in the cap  $(P, \omega) = (\mathbb{C}\mathbb{P}^2 \setminus D, \omega_{\text{FS}}|_{\mathbb{C}\mathbb{P}^2 \setminus D})$ , where  $D$  is a small Darboux ball. Every transverse knot has a symplectic hat of genus 0 in a blow-up of  $(P, \omega)$ .*

*Sketch of proof.* Given a transverse knot  $K$ , pick a braid  $\beta$  whose closure is transversely isotopic to  $K$ . We can stabilise the braid and add positive generators to  $\beta$  until we get to a full twist on  $d$  strands for some large  $d$ , whose closure is the torus link  $T(d, d)$  (more precisely, its transverse representative which is algebraic). This gives a symplectic cobordism from  $K$  to  $T(d, d)$ . We can cap off cobordism with a piece of a complex surface (e.g. by gluing in the neighbourhood of a complex line in  $\mathbb{C}\mathbb{P}^2$  intersecting a degree- $d$  curve transversely).

The second part of the statement follows by refining the strategy above: we can stabilise and add squares of generators until we get to a braid whose closure is the torus knot  $T(d-1, d)$  (more precisely, its transverse representative which is algebraic). This gives an immersed cobordism which we cap off with the neighbourhood of a line that is tangent to order  $d$  to a degree- $d$  curve. We then blow up at each of the double points of the cobordism to get an embedded surface.  $\square$

In [EG22] we then proceed to study the complexity of such cobordisms for some classes of knots. We refer to the paper for the precise results. We would like to point out a relation with singularities and symplectic curves as defined above instead. This connection/comparison was hinted at at the beginning of the chapter.

Suppose that  $(X^\circ, \omega^\circ)$  is a symplectic cap of  $(S^3, \xi_{\text{st}})$  and that  $H \subset (X^\circ, \omega^\circ)$  is a symplectic hat for  $K \subset (S^3, \xi_{\text{st}})$ . We can cap  $X^\circ$  off to a closed symplectic 4-manifold  $(X, \omega)$  by gluing in a 4-ball  $B$  with the standard symplectic structure<sup>6</sup>. If we cone off  $H$  into  $B$ , we obtain a PL surface  $F$  with a singularity of type  $K$  at a point  $p$  (the origin of  $B$ ). Moreover, this surface  $F$  is symplectic away from  $p$ , which is its unique non-locally-flat point.

In this sense, Theorem 3.47 above tells us that symplectic surfaces with isolated singular points can have *arbitrary links*, even in  $(\mathbb{C}\mathbb{P}^2, \omega_{\text{FS}})$ , where we think of the link as a *transverse knot* in  $(S^3, \xi_{\text{st}})$ . We do *not* call these surfaces symplectic curves.

There is another class of curves, however, which might deserve the name of symplectic curves. Suppose that  $F$  is a PL surface (possibly immersed), and that the link of its of each singularity of  $F$  is a quasipositive link (as a transverse link). Then  $F$  admits a (non-unique) smoothing  $\text{sm } F$  which is also symplectic. Perhaps this class would deserve the name of *smoothable (singular) symplectic surfaces*.

Going back to symplectic curves as we defined them in Section 3.1: they are smoothable, since links of algebraic singularities are closures of *positive* braids, but on top of that they are  $J$ -holomorphic for some compatible almost-complex structure  $J$ , and as such they admit a resolution by blow-ups. These are the most important features of symplectic curves that we used in all results above.

<sup>6</sup>The volume of this ball is determined by  $\omega^\circ$ . Equivalently, we can glue in the unit 4-ball up to rescaling.

## Hats and fillings of branched covers

As an application of hats, in [EG22] we considered fillings of branched covers. We will be interested in two types of fillings we have not defined yet.

A strong symplectic filling  $(W, \omega)$  of  $(Y, \xi)$  is *exact* if  $\omega$  is exact. It is *Stein* if there exist a proper holomorphic submanifold  $\widehat{W} \hookrightarrow \mathbb{C}^N$  for some integer  $N$ , such that  $\widehat{W}$  is transverse to the unit sphere,  $W = B_1^N \cap \widehat{W}$ , and  $\omega = \omega_{\text{st}}|_W$ . (In particular, Stein fillings are exact.)

If  $K$  is a transverse knot in  $(S^3, \xi_{\text{st}})$  and  $n$  is a positive integer, there is a contact structure on  $\Sigma_n(K)$ , the  $n$ -fold cyclic cover of  $S^3$  branched over  $K$  induced by  $K$  and  $\xi_{\text{st}}$  [Gon87]. This contact structure, that we denote with  $\xi_{K,n}$ , is isotopic to the pull-back of  $\xi_{\text{st}}$  away from  $K$ , and the preimage of  $K$  is transverse to it.

For instance, the canonical contact structure  $\xi_{\text{can}}$  on the link of the  $n$ -suspension  $\{z^n + f(x, y) = 0\}$  of a curve singularity  $\{f(x, y) = 0\}$  is the  $n$ -fold cyclic cover of  $\xi_{\text{st}}$  branched over the link of  $\{f(x, y) = 0\}$ .

We give two sample results. First, recall that the Brieskorn sphere  $\Sigma(2, 3, 7)$ , which is the link of the Brieskorn–Pham singularity  $\{x^2 + y^3 + z^7 = 0\} \subset \mathbb{C}^3$ , admits exactly two tight contact structures, namely the canonical one  $\xi_{\text{can}}$  (coming from the singularity) and its conjugate [MT18]. The following theorem concerns fillings of  $\xi_{\text{can}}$ , but note that each filling of  $\xi_{\text{can}}$  corresponds to one of  $\bar{\xi}_{\text{can}}$  (by changing sign to the symplectic form), so the statement actually holds for  $\bar{\xi}_{\text{can}}$  as well.

**Theorem 3.48.** *Let  $(W, \omega_W)$  be a Stein filling of  $(\Sigma(2, 3, 7), \xi_{\text{can}})$ . Then  $H_1(W) = 0$  and either  $W$  is spin and  $H_2(W) \cong E_8 \oplus 2H$ , or  $H_2(W) \cong \langle -1 \rangle$ , and both cases occur. By contrast,  $\xi_{\text{can}}$  has strong symplectic fillings with arbitrarily large  $b_2^+$ .*

Let us reverse the orientation and look at  $-\Sigma(2, 3, 7)$ . By work of Tosun [Tos20], on  $-\Sigma(2, 3, 7)$  there is exactly one tight contact structure up to isotopy. Let us denote it with  $\xi_0$ .

**Theorem 3.49.** *Let  $(W, \omega_W)$  be an exact filling of  $(-\Sigma(2, 3, 7), \xi_0)$ . Then  $W$  is spin,  $H_1(W) = 0$ , and  $Q_W = E_8 \oplus H$ . Moreover,  $\xi_0$  has strong symplectic fillings with arbitrarily large  $b_2^+$ .*

The proofs of Theorems 3.48 and 3.49 share many similarities, so we sketch them together.

*Sketch of the proofs of Theorems 3.48 and 3.49.* The Brieskorn manifold  $\Sigma(2, 3, 7)$  is the double cover of  $S^3$  branched over the positive torus knot  $T(3, 7)$ , and  $-\Sigma(2, 3, 7)$  is the double cover of  $S^3$  branched both over the *negative* torus knot  $m(T(3, 7))$  and over the pretzel knot  $P(-2, 3, 7) = 12n_{242}$ . Both knots  $T(3, 7)$  and  $P(-2, 3, 7)$  are closures of quasipositive braids, so they have transverse representatives whose cyclic branched covers are Stein fillable [Pla06]. By the classification of tight contact structures on  $\pm\Sigma$ , the contact structures  $\xi_{\text{can}}$  and  $\xi_0$  are both double covers of  $\xi_{\text{st}}$ .

Moreover, the double covers of  $B^4$  branched over the quasipositive surfaces for  $T(3, 7)$  and  $P(-2, 3, 7)$  give Stein fillings of  $\xi_{\text{can}}$  and  $\xi_0$  with the required topology. The other Stein filling of  $\Sigma(2, 3, 7)$  is the minimal resolution of the Brieskorn–Pham singularity, which is a regular neighbourhood of a rational curve with a simple cusp

and self-intersection  $-1$ . (The branched cover above is the Milnor fibre of the same singularity.) This proves that all cases in the statement are realised geometrically.

To show the last part of the statement, it suffices to exhibit a 4-manifold  $W$  with two convex boundary components, one of which is  $(Y, \xi)$ . (This is an observation originally due to McDuff.)  $-\Sigma(2, 3, 7)$  is the convex boundary of the  $E_{10}$ -plumbing  $W_0$  of Lagrangian spheres, as well as the boundary of the plumbing of  $E_{10}$ -plumbing of symplectic  $(-2)$ -spheres. We can pass from one to the other by deforming the contact structure in the interior. Since  $E_{10}$  is not negative definite, by Li and Mak's result we can find a concave neighbourhood  $C$  of the symplectic spheres within the plumbing  $W_0$ . The difference  $W_0 \setminus \text{Int}(C)$  is the desired 4-manifold  $W$ , whose boundary is actually  $(\Sigma(2, 3, 7), \xi_{\text{can}}) \sqcup (-\Sigma(2, 3, 7), \xi_0)$ . Capping off either boundary component (which we can do by adding arbitrarily large topology [Etn04]) gives a filling of the other with large topology.

In what follows,  $K$  will be the transverse closure of a quasipositive braid, and it will be of type either  $T(3, 7)$  or  $P(-2, 3, 7)$ ,  $Y$  will be the double cover of  $K$ , and  $\xi$  will be  $\xi_{\text{can}}$  if  $Y = \Sigma(2, 3, 7)$  or  $\xi_0$  if  $Y = -\Sigma(2, 3, 7)$ .

To give restrictions on fillings of  $(Y, \xi)$ , we will construct a suitable cap. In [EG22], we produced an explicit hat of  $K$  in  $P$  of degree 6. This means that, when we symplectically fill  $K$  in  $B^4$ , we obtain a symplectic surface in  $\mathbb{C}\mathbb{P}^2 = P \cup B^4$  of degree 6 (or, equivalently, of self-intersection 36). The cap  $(C, \omega)$  we consider is the double cover of  $P$  branched over this degree-6 hat.

We claim that  $(C, \omega)$  is a Calabi–Yau cap, in the sense of Li, Mak, and Yasui [LMY17], and that its intersection form is  $-E_8 \oplus H$  if  $K = T(3, 7)$  and  $-E_8 \oplus 2H$  if  $K = P(-2, 3, 7)$ . If  $(W, \omega)$  is an exact filling of  $(Y, \xi)$ , then  $X = C \cup W$  is a Calabi–Yau symplectic 4-manifold. Furthermore, if either  $K = P(-2, 3, 7)$  or  $W$  is *not* negative definite, then  $b_2^+(X) > 1$  and  $b_2^-(X) > 3$ . Calabi–Yau symplectic 4-manifolds have been classified by Li [Li06]: they are either  $T^2$ -bundles over  $T^2$ , or an Enriques surface, or a K3 surface. In this case, since  $b_2^+(X) > 1$  and  $b_2^-(X) > 3$ , the only possibility is that  $X$  is a K3 surface. Then  $W$  is spin and (since  $Y$  is a homology 3-sphere)  $H_1(W) = 0$ . The intersection form of  $W$  is then the orthogonal complement of the intersection form of  $C$  in  $X$ , which leaves as the only possibility the one given in the statement.

The only case that is left to analyse is the case when  $K = T(3, 7)$  and  $W$  is negative definite. This is where the Stein assumption comes in.  $W$  cannot be a rational homology ball: since a Stein filling induces a surjection  $H_1(Y) \rightarrow H_1(W)$ ,  $W$  would be an integral homology ball, but the Rokhlin invariant of  $Y$  is non-zero. So  $b_2(W) > 0$ . Using the contact invariant in Heegaard Floer homology and a result of Plamenevskaya [Pla04], one can prove that  $W$  cannot have  $b_2(W) > 1$ , by carefully analysing the behaviour of the cobordism maps  $\overline{W \setminus B}: -Y \rightsquigarrow -S^3$  on  $\text{HF}^+(Y)$ .  $\square$

### 3 *Symplectic curves*

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