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# Generalized linear sampling methods for locally perturbed periodic layers

Nouha Jenhani

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# THÈSE DE DOCTORAT

présentée par

**Nouha JENHANI**

pour l'obtention du

**DIPLÔME NATIONAL DE DOCTORAT**

**en Mathématiques Appliquées**

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## **Generalized linear sampling methods for locally perturbed periodic layers**

---

Soutenue le 23 Décembre 2023 devant le jury composé de :

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Thèse préparée au sein du Laboratoire de Modélisation Mathématique et Numérique dans les Sciences de l'Ingénieur (LAM SIN)



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# Introduction

The developments presented in this PhD thesis are related to the study of the inverse scattering problem for an unbounded periodic medium with defects. This area has attracted much attention in the recent mathematical literature as it is related to many applicative areas such as: chemistry, mechanics, optics, antenna arrays, etc.

A typical example of a targeted application is the non-destructive testing of the so-called metamaterials, which are human-engineered materials with electromagnetic properties modified from their nature (may have a negative refractive index). Initially designed to control electromagnetic waves, these metamaterials have been developed for other technological applications such as improving the properties of optical devices, noise control,... In general, the metamaterials are designed to be periodically printed on several layers of substrate (See Figure 1.1).

We are interested in considering the case where these periodic layers admit defects. Our problematic is then to identify the location and the shape of these defects. In order to do so, an electromagnetic incident field is applied to this structures by some given source and the scattered field is measured at some receivers. Collecting these measurements for different source locations we would like to reconstruct the defect using some specific non iterative inversion methods that will be introduced below. The main difficulty we face in addressing this inverse problem comes from the fact that the defect breaks the periodicity, and then we cannot deal with the problem in one period cell. This leads to

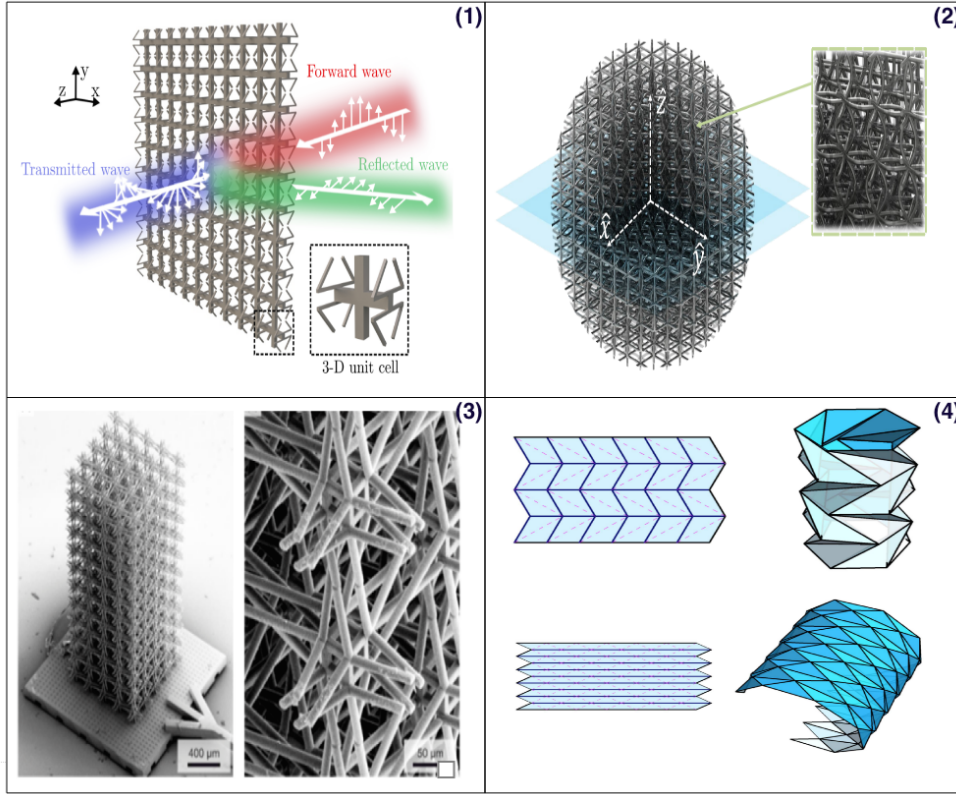


FIGURE 1.1: (1): Illustration of a 3D Metamaterial allows controlling the passage of light [1]. (2): Example of 3D electromagnetic metamaterials (synthesized meta-lens)[57]. (3): Example of mechanical metamaterial micro-structure [52]. (4): Illustration of an Antenna metamaterial [27].

treat an inverse scattering problem in an unbounded domain. We refer for the studies of scattering problems from unbounded surfaces to [15, 16, 18, 19, 46], and for the case of wave guides we refer to [9, 10, 25, 33, 40].

In the literature, several works have considered the inverse scattering problem from perturbed periodic layers by assuming the knowledge of the periodic structure. However, in real applications, we may not have access to this information or we obtain it in an inaccurate way. The first challenge we raise in this work is to solve the inverse problem assuming that we have no information about the periodic structure except the periodicity length. A technical assumption was made in [30] to deal with this problem where they assumed that the perturbation itself is periodically distributed (with larger periodicity scale). This allows them to reduce the problem to a large and bounded period. The main objective in this work is to use a similar approach in [30] but get rid of this technical assumption.

We first give an overview of the theoretical and numerical studies made in the literature for the direct scattering problem. Under the assumption that the refractive index has a positive imaginary part the well-posed character of our problem is proved in [39]. The idea is based in applying the so-called Floquet-Bloch transform (for a detailed introduction of this operator, its properties and its application to scattering problems, we refer to [24, 41, 43]). Indeed, the application of this operator allows to transform the problem into coupled quasi-periodic problems in one bounded period. This allows to exploit the well-posedness of the non-perturbed quasi-periodic problem to prove the well-posedness of the perturbed case. We shall extend the analysis in [39] by considering solutions that are continuous with respect to the Floquet-Bloch variable. This analysis when the refractive index has only a non negative imaginary part is done in [38]. On the numerical side, a volume integral method was introduced in [44, 47] to solve the direct problem for periodic domain. This method has been extended in [29, 49] to the locally perturbed periodic case. We shall present the main idea of this method and give some numerical examples for the solution of our scattering problem. One of the perspectives of this work is to consider the inverse problem for the anisotropic case. This is why we also extend the well-posedness of the direct problem to the anisotropic case.

In order to solve the inverse problem, we employ sampling methods as introduced in [30]. Indeed, these methods have many advantages. They do not require solving a series of forward problems. This makes the numerical implementation of the method easier and faster compared to iterative methods. They do not require a linearization hypothesis. Moreover, although we have no information about the physical characteristics of the scatterer, we still can identify its location and shape. On the other side, one of the disadvantages of these methods is the requirement of a large number of measured multi-static data.

The idea of these methods is based on the construction of an indicator function that allows to indicate for a grid of sampling points in the probed domain whether a sampling point is inside or outside the obstacle. The Linear Sampling Method (LSM) was first introduced in [20], (also [11, 22]). The theoretical justification of this method admits some weak points as it does not give a regularization scheme to construct the indicator

function. A new formulation called Generalized Linear Sampling Method (GLSM) was proposed in [7, 16] to deal with the weak point of the LSM. It gives an explicit indicator function that provides an exact characterization for the scatterer geometry.

Several research works have applied sampling methods to solve the inverse problem from a periodic medium without defects [2, 3, 4, 31, 32, 42, 48, 53, 56]. For the case with defects that assumes the knowledge of the periodic background, we refer to [9, 23, 26, 28, 39, 45, 54]. Sampling methods applied to our case (which do not assume the knowledge of the periodic background) were considered in [30, 49] but as mentioned before, they assumed that the defect is also periodic (with periodicity an integer multiple of the background periodicity). The first main goal considered in this thesis is to provide a theoretical justification of the GLSM method by getting rid of this technical assumption. We consider first the GLSM method for a single Floquet-Bloch mode where quasi-periodic incident fields are used. We introduce the near field operator for fixed Floquet-Bloch variable and we prove that it admits a factorization similar to the classical one encountered in the literature. However, this doesn't allow us to apply the abstract framework of the factorization method as introduced in [36, 37]. This explains the reason for which the penalty term used for the application of the GLSM method in this work is different from the one used in the literature. In a second step we consider the case where non-periodic incident fields are applied. We introduce the full near-field operator and its factorizations, and we give the theoretical justification of the GLSM method in this context. The inversion method applied in this two cases provides two indicator functions, the first one for the intersection of the domain with the first period (distributed periodically), and the second one for the whole domain. These two results were obtained under the assumption that the defect does not intersect the periodic domain, which allows us to introduce the so-called interior transmission problem for the periodic background and the one associated with the perturbation separately. Therefore, we can obtain in this case their well-posed character directly from the literature [11, 12, 13, 34, 55]. The study of the interior transmission problem for the case where the intersection is not empty is one of the perspectives of this thesis. Some preliminary results are obtained in the last chapter.

A Differential Linear Sampling Method (DLSM) was first proposed in [6] that allows identifying defects from a complex domain using differential measurements, i.e it requires measurements for the setting without and with defects. A new differential sampling method inspired from [6] has been introduced in [30] for locally perturbed periodic layers. It does not use differential measurements and provides an indicator function to directly construct the defect using only a priori knowledge of the periodicity length (but still assuming that the defect is also periodic). We revisit this differential sampling method as presented in [30] without assuming that the local perturbation is also periodic. The principle of this method consists in observing that the GLSM setting for single Floquet-Bloch mode still holds if we change the periodicity of the background to an integer multiple of the original periodicity. Then one can construct a criterion depending on the GLSM indicator functions established for different lengths of periodicity that allows to directly construct the defect. We also study the GLSM for non-quasi periodic sources and show that one can reconstruct the full domain without knowledge of this periodicity length. This can also be exploited for DLSM.

One of our main perspectives is to consider this inverse problem in the case where the perturbation intersects the periodic background. We are then interested in the analysis of the interior transmission problem (ITP) in this case. In the case of non-quasi periodic sources, this problem is difficult to solve because it arises in an unbounded domain. Then Fredholm's alternative cannot be applied as in the case of bounded domains treated in the literature, see for instance [11]. Similarly to the solvability of the direct problem, we apply the Floquet-Bloch transform and we analyze first the quasi-periodic interior transmission problem. However, we face a difficulty to follow the idea proposed in [39]. Indeed, contrary to the direct problem, after applying the Floquet-Bloch transform to the variational formulation of the problem (ITP) we still have a coupled term w.r.t to the Floquet-Bloch variable which although it has a compact support, its compactness property cannot be proven and therefore Fredholm's theory cannot be applied as in the idea proposed in [39]. In order to solve this problem, we consider the case for which the perturbation is contained in the periodic background and

we perform a discretization with respect to the Floquet-Bloch variable. We prove first the solvability of the discretized interior transmission problem with the help of the well-posed character of the quasi-periodic one. Finally, we perform a convergence analysis to prove that the discretized solution converges to a solution of ITP. For uniqueness, we still have to assume that the imaginary part of the periodic background is positive in an open domain.

## Organization of the manuscript

The manuscript is organized as follows:

**Chapter 2:** This chapter is dedicated to a presentation of some results on the well-posedness of the direct scattering problem from unbounded locally perturbed periodic layers. We first present and complement some theoretical results obtained from the literature for the case of isotropic media. We then extend these results to prove the well-posed character of the anisotropic inhomogeneous problem. We then give a brief outline of a numerical method used to generate the solutions for the direct problem and we present some numerical examples.

**Chapter 3:** An analysis of sampling methods is given in this chapter. More precisely, two theoretical results are presented. In the first one, we consider quasi-periodic incident fields and we give a theoretical justification of the Generalized Linear Sampling Method (GLSM) for a single Floquet-Bloch mode. Then based on this result, we give a justification of the GLSM method for the case of full measurements. The theoretical framework presented in this chapter is extracted from the published paper [8], and some numerical examples are added in the end of the chapter.

**Chapter 4:** We revisit in this chapter the differential linear sampling method (DLSM) as introduced in [30] based on the two results of the GLSM method obtained in Chapter 3. We first reintroduce the setting of the inverse problem in an integer multiple of the original periodicity and then give two indicator functions that allow a direct reconstruction of the defect. We conclude this chapter with some numerical experiments.

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**Chapter 5:** This chapter is devoted to the analysis of the interior transmission problem in the case where the intersection between the periodic background and the defect is not empty. We first present the solvability of the quasi-periodic interior transmission problem. We then consider the case where the defect is contained in the periodic domain. We apply the Floquet-Bloch transform and make a discretization with respect to the Floquet-Bloch variable which brings us to the study of the solvability of the semi-discretized problem. Finally, a convergence analysis is performed in order to construct a solution of the problem. Uniqueness is proved under the assumption that the refractive index has positive imaginary part.



# The direct scattering problem from locally perturbed periodic medium

**Abstract:** *In this chapter we present and complement some results obtained in the literature for the theoretical and numerical studies of the direct scattering problem from unbounded periodic layers with compactly supported defects. Then, we extend the study of the solvability of the direct problem to the case of anisotropic media.*

## Contents

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## 2.1 Introduction

We introduce in this chapter the direct scattering problem from unbounded periodic layers with defects. We consider the problem in the upper half space  $U^0 := \mathbb{R} \times \mathbb{R}_+$  with Dirichlet condition on  $\Gamma^0 := \mathbb{R} \times \{0\}$ . We first discuss the well-posedness of this problem in isotropic media. We shall assume the existence of a small absorption zone in the periodic layers in order to obtain the uniqueness of the solution. The presence of the defect prevents us from transferring the problem from an unbounded periodic domain to a bounded period and from there comes the main difficulty since it is not easy to use Fredholm's theory in an unbounded domain. This problem has been solved in [39]. The main idea is to first consider the periodic problem without defects such that we can write using the Floquet-Bloch transform an equivalent coupled quasi-periodic problems with respect to the Floquet-Bloch variable in one bounded period. By proving that this equivalent problem is uniquely solvable one can deduce the well-posedness of the perturbed case. We present also in this chapter the main idea of the method proposed in [29] to generate solutions for our considered direct problem and we give some numerical examples. Finally we extend the well-posedness of the direct problem for the case of anisotropic media.

The chapter is organized as follows. In Section 2.2 we present the solvability of the problem in isotropic media as proposed in [39] with some complementary results. Section 2.3 is dedicated to presenting some numerical illustrations of the solution to direct problem. In Section 2.4 we follow the same idea proposed in [39] to prove the well-posedness of the problem in anisotropic media.

## 2.2 Solvability of the direct scattering problem in isotropic Media

### 2.2.1 Setting of the problem

We denote by  $k$  the real valued positive wavenumber and we denote by  $n \in L^\infty(U^0)$  the refractive index satisfying  $n = n_p$  outside a bounded domain  $\tilde{D}$ , with  $n_p \in L^\infty(U^0)$  a  $2\pi$ -periodic refractive index. Through this chapter, a  $2\pi$ -periodic functions refer to functions

that are periodic with respect to the first variable  $x_1$ , with period  $2\pi$ . Moreover,  $n$  and  $n_p$  are assumed to have a non negative imaginary parts.

For  $R > R_0 > 0$ , we denote by  $D^p$  a periodic domain included in an infinite band  $\Omega^R := \mathbb{R} \times [0, R]$  such that  $n_p = 1$  outside  $D^p$ . We set  $\Gamma^R := \mathbb{R} \times \{R\}$ , and we denote by  $D := D^p \cup \tilde{D}$  where  $\tilde{D}$  is assumed to be included in the period  $\Omega_0^R := [0, 2\pi] \times [0, R]$  as shown in Figure 2.1. Given  $f \in L^2(U^0)$ , we consider the following problem

$$\begin{cases} \Delta u + k^2 n u = f & \text{in } U^0, \\ u = 0 & \text{on } \Gamma^0, \end{cases} \quad (2.1)$$

and we assume that  $u$  satisfies the angular spectrum representation

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix \cdot \xi + i\sqrt{k^2 - |\xi|^2}(x_2 - R)} \hat{u}(\xi, R) d\xi, \quad \text{for } x_2 > R, \quad (2.2)$$

with  $\hat{u}(\xi, R)$  is the Fourier transform of  $u|_{\Gamma^R}$  defined as  $\hat{u}(\xi, R) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_1} u(x_1, R) dx_1$ .

Using (2.2) we conclude that problem (2.1)-(2.2) can be reformulated as: seek  $u \in H_{loc}^2(\Omega^R) \cap H^1(\Omega^R)$  such that

$$\begin{cases} \Delta u + k^2 n u = f & \text{in } \Omega^R, \\ u = 0 & \text{on } \Gamma^0, \\ \frac{\partial u}{\partial x_2}(\cdot, R) = T^R(u|_{\Gamma^R}) & \text{on } \Gamma^R. \end{cases} \quad (2.3)$$

where  $T^R : H^{1/2}(\Gamma^R) \longrightarrow H^{-1/2}(\Gamma^R)$  is the exterior Dirichlet-to-Neumann map defined by

$$T^R(\varphi) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - |\xi|^2} e^{ix_1 \cdot \xi} \hat{\varphi}(\xi) d\xi, \quad (2.4)$$

with  $\hat{\varphi}$  is the Fourier transform of  $\varphi$  and the square root is the one with non negative imaginary part.

## 2.2.2 The case of periodic layers without defects

In this section we discuss the well-posedness of the direct scattering problem from periodic domain without presence of defects, i.e  $\tilde{D} = \emptyset$  and  $n = n_p$ . Let us first introduce

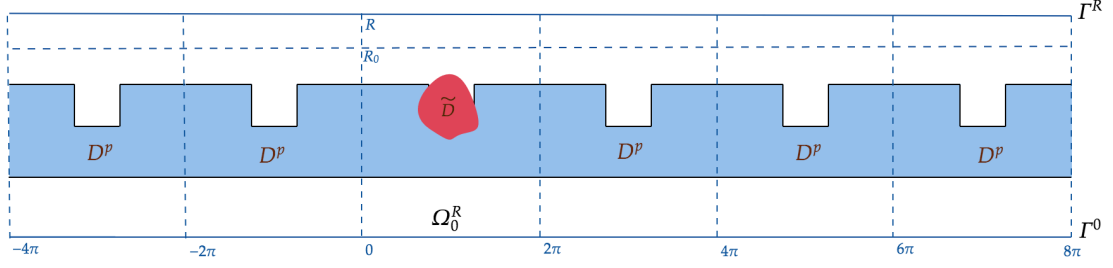


FIGURE 2.1: Sketch of the domain

some definitions and functional spaces.

**Definition 1.** For  $\xi \in \mathbb{R}$ , we say that a regular function  $u$  is  $\xi$ -quasi-periodic of period  $2\pi$  if it satisfies

$$u(x_1 + 2\pi j, x_2) = e^{i2\pi\xi \cdot j} u(x_1, x_2) \quad \forall j \in \mathbb{Z} \text{ and } (x_1, x_2) \in \mathbb{R}^2. \quad (2.5)$$

In what follows, we do not indicate the quasi-periodicity length since it is fixed to  $2\pi$ . Moreover, we introduce for a real valued  $m \geq 1$  the following functional spaces

- $\check{H}^m(\Omega^R) := \{u \in H^m(\Omega^R); u = 0 \text{ on } \Gamma^0\}$
- $H_\xi^m(\Omega^R) := \{u \in H_{loc}^m(\overline{\Omega^R}); u \text{ is } \xi\text{-quasi-periodic with period } 2\pi\},$
- $\check{H}_\xi^m(\Omega^R) := \{u_\xi \in H_\xi^m(\Omega^R); u_\xi = 0 \text{ on } \Gamma^0\},$
- $H_\xi^1(\Omega_0^R) = \{u|_{\Omega_0^R}; u \in H_\xi^1(\Omega^R)\}.$
- $L_\xi^2(\Omega^R) := \{u \in L_{loc}^2(\Omega^R); u \text{ is } \xi\text{-quasi-periodic with period } 2\pi\},$

Finally, for a regular function  $\phi \in C_0^\infty(U^0)$  we define the one dimensional Floquet-Bloch transform as

$$\mathcal{J}\phi(\xi, x_1, x_2) = \sum_{j \in \mathbb{Z}} \phi(x_1 + 2\pi j, x_2) e^{-i2\pi\xi \cdot j}, \quad \xi \in I, (x_1, x_2) \in U^0. \quad (2.6)$$

We recall that for a  $2\pi$ -periodic function  $n_p$ , the Floquet-Bloch transform satisfies

$$\mathcal{J}(n_p u) = n_p \mathcal{J}(u). \quad (2.7)$$

Therefore, using the periodicity of  $n_p$ , we can reduce the scattering problem to a bounded period by applying the Floquet-Bloch transform to problem (2.1) with  $n$  replaced by  $n_p$ .

Therefore, (2.1) lead to the following quasi-periodic problem for  $u_\xi = \mathcal{J}(u)(\xi, \cdot)$

$$\begin{cases} \Delta u_\xi + k^2 n_p u_\xi = f_\xi & \text{in } U^0, \\ u_\xi = 0 & \text{on } \Gamma^0, \end{cases} \quad (2.8)$$

where  $\xi \in I := [0, 1]$  is the Floquet-Bloch variable,  $f_\xi = \mathcal{J}(f)(\xi, \cdot) \in L_\xi^2(U^0)$  and  $L_\xi^2(U^0)$  denotes the space of  $\xi$ -quasi-periodic functions in  $L^2(U^0)$ , with period  $2\pi$ . Let us denote by  $\Gamma_0^R := [0, 2\pi] \times \{R\}$  and we prove in the following Lemma that  $u_\xi$  satisfies the Rayleigh radiation condition

$$u_\xi(x) := \sum_{j \in \mathbb{Z}} \widehat{(u_\xi|_{\Gamma_0^R})}(j) e^{i\alpha_\xi(j)x_1 + i\beta_\xi(j)(x_2 - R)} \quad \text{for } x_2 > R, \quad (2.9)$$

where  $\alpha_\xi(j) := \xi + j$ ,  $\beta_\xi(j) := \sqrt{k^2 - |\xi + j|^2}$  such that  $\Im m(\beta_\xi(j)) \geq 0$  for  $j \in \mathbb{Z}$  and  $\widehat{u}_\xi(j)$  is the  $j$ -th Fourier coefficient of  $e^{-i\xi x_1} u(x_1, R)$  defined as

$$\widehat{u}_\xi(j) := \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha_\xi(j)x_1} u(x_1, R) dx_1.$$

The proof of the following Lemma can also be found in [41, 39].

**Lemma 2.** *Let  $u \in H^1(\Omega^R)$  be the solution of (2.1) satisfying the angular spectrum representation (2.8). Then for  $\xi$  fixed we have  $\mathcal{J}(u)(\xi, \cdot) \in H_\xi^1(\Omega_0^R)$  solution of (2.8) satisfies the Rayleigh radiation condition given by (2.9).*

*Proof.* Let  $x = (x_1, x_2)$ , we observe that  $\psi(\xi, x) := \mathcal{J}(\varphi)(\xi, x)$  is a  $\xi$ -quasi-periodic function w.r.t  $x_1$ , then we fix  $\xi \in \mathbb{R}$  and we use the Fourier series to expand  $\psi$  as

$$\psi(\xi, x) = \sum_{j \in \mathbb{Z}} a_\xi(j, x_2) e^{i\alpha_\xi(j)x_1}, \quad (2.10)$$

with

$$\begin{aligned} a_\xi(j, x_2) &= \frac{1}{2\pi} \int_0^{2\pi} \psi(\xi, x) e^{-i\alpha_\xi(j)x_1} dx_1, \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{\ell \in \mathbb{Z}} \varphi(x_1 + 2\pi\ell, x_2) e^{-i2\pi\xi \cdot \ell} e^{-i\alpha_\xi(j)x_1} dx_1. \end{aligned}$$

Taking  $z_1 := x_1 + 2\pi\ell$  we conclude that

$$\begin{aligned} a_\xi(j, x_2) &= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(z_1) e^{-i2\pi\xi \cdot \ell} e^{-i\alpha_\xi(j)(z_1 - 2\pi\ell)} dz_1 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(z_1) e^{-i\alpha_\xi(j)z_1} dz_1 = \mathcal{F}(\varphi)(\xi + j), \end{aligned} \quad (2.11)$$

with  $\mathcal{F}$  denoting the Fourier transform. Therefore, from (2.10)-(2.11) we deduce that the Floquet-Bloch transform  $\mathcal{J}$  can be rewritten as

$$\mathcal{J}\varphi(\xi, x_1, x_2) = \sum_{j \in \mathbb{Z}} \mathcal{F}(\varphi)(\xi + j) e^{i\alpha_\xi(j)x_1}, \quad \text{for } \xi \in \mathbb{R}, (x_1, x_2) \in U^0. \quad (2.12)$$

Recall that  $\mathcal{J}$  is an isomorphism between  $L^2(\Gamma^R)$  and  $L^2(I, H_\xi^m(\Gamma_0^R))$ , with  $H_\xi^m(\Gamma_0^R)$  denotes the restriction to  $\Gamma_0^R$  of  $\xi$ -quasi periodic functions in  $H_{loc}^m(\Gamma^R)$ , with norm

$$\|\varphi\|_{H_\xi^m(\Gamma_0^R)}^2 = \sum_{j \in \mathbb{Z}} (1 + j^2)^m |\hat{\varphi}_\xi(j)|^2.$$

Consider  $\mathcal{G} : \xi \longrightarrow \mathcal{F}^{-1} \left( e^{i\sqrt{k^2 - |\xi|^2}(x_2 - R)} \mathcal{F}(u|_{\Gamma^R})(\xi) \right) \in L^2(\mathbb{R})$ . Then, applying the Floquet-Bloch transform given by the expression (2.12) to the radiation condition (2.2) we get

$$\begin{aligned} \mathcal{J}(u|_{\Gamma^R})(\xi, x) &= \sum_{j \in \mathbb{Z}} \mathcal{F}(\mathcal{G})(\xi + j) e^{i\alpha_\xi(j)x_1} \\ &= \sum_{j \in \mathbb{Z}} \mathcal{F}(u|_{\Gamma^R})(\xi + j) e^{i(\alpha_\xi(j)x_1 + \sqrt{k^2 - |\xi + j|^2}(x_2 - R))}. \end{aligned} \quad (2.13)$$

Consequently, since we have  $\mathcal{F}(u|_{\Gamma^R})(\xi + j) = \widehat{(u_\xi|_{\Gamma_0^R})}(j)$  with  $u_\xi := \mathcal{J}(u)(\xi, \cdot)$ , we deduce that (2.13) coincides with the Rayleigh radiation condition given by (2.9).  $\square$

Using the radiation condition (2.9) we can define the exterior quasi-periodic Dirichlet-to-Neumann map  $T_\xi^R : H_\xi^{1/2}(\Gamma_0^R) \longrightarrow H_\xi^{-1/2}(\Gamma_0^R)$  as

$$T_\xi^R(\varphi)(x_1) = \frac{\partial u_\xi}{\partial x_2} \Big|_{\Gamma_0^R} = i \sum_{j \in \mathbb{Z}} \beta_\xi(j) \hat{\varphi}_\xi(j) e^{i\alpha_\xi(j) \cdot x_1}, \quad (2.14)$$

where with  $\hat{\varphi}_\xi(j)$  is the  $j$ -th Fourier coefficient of  $e^{-i\xi x_1} \varphi(x_1, R)$ . Therefore, (2.8)-(2.9) can be reformulated as: seek  $u_\xi \in H_\xi^1(\Omega_0^R)$  such that

$$\begin{cases} \Delta u_\xi + k^2 n_p u_\xi = f_\xi & \text{in } \Omega^R, \\ u_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial u_\xi}{\partial x_2}(\cdot, R) = T_\xi^R(u_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R. \end{cases} \quad (2.15)$$

We have the following Lemma, which a straightforward consequence of the expression of  $T_\xi^R$ .

**Lemma 3.** *The Dirichlet-to-Neumann operator  $T_\xi^R$  is bounded and satisfies*

$$\Re \langle T_\xi^R u, u \rangle_{\Gamma_0^R} \leq 0 \quad \text{and} \quad \Im \langle T_\xi^R u, u \rangle_{\Gamma_0^R} \geq 0 \quad \forall u \in H_\xi^{1/2}(\Gamma^R),$$

where the notation  $\langle \cdot, \cdot \rangle_{\Gamma_0^R}$  refers to the  $H^{-1/2}(\Gamma_0^R) - H^{1/2}(\Gamma_0^R)$  duality product that coincides with the  $L^2(\Gamma_0^R)$  scalar product for regular functions.

### 2.2.2.1 Well posedness of the problem

In order to prove the well posedness of the problem (2.15) we make in what follows the following assumption.

**Assumption 4.** *Assume that the set  $\{\Im(n_p) > 0\}$  contains a non empty open set  $\mathcal{O}$ .*

The variational formulation equivalent to problem (2.15) is given as: seek  $u_\xi \in \check{H}_\xi^1(\Omega_0^R)$  such that

$$\int_{\Omega_0^R} (\nabla u_\xi \cdot \nabla \overline{\psi_\xi} - k^2 n_p u_\xi \overline{\psi_\xi}) dx - \langle T_\xi^R u_\xi, \psi_\xi \rangle_{\Gamma_0^R} = \int_{\Omega_0^R} f_\xi \overline{\psi_\xi}, \quad \forall \psi_\xi \in \check{H}_\xi^1(\Omega_0^R). \quad (2.16)$$

We define the sesquilinear form

$$b_\xi(u_\xi, \psi_\xi) := \int_{\Omega_0^R} (\nabla u_\xi \cdot \nabla \overline{\psi_\xi} - k^2 n_p u_\xi \overline{\psi_\xi}) dx - \langle T_\xi^R u_\xi, \psi_\xi \rangle_{\Gamma_0^R}. \quad (2.17)$$

Let  $B_\xi : \check{H}_\xi^1(\Omega_0^R) \longrightarrow \check{H}_\xi^1(\Omega_0^R)$  be the operator defined by the Riesz representation theorem such that

$$(B_\xi u_\xi, \psi_\xi)_{H^1(\Omega_0^R)} := b_\xi(u_\xi, \psi_\xi) \quad \forall \psi_\xi, u_\xi \in \check{H}_\xi^1(\Omega_0^R). \quad (2.18)$$

**Theorem 5.** Consider  $f_\xi \in L_\xi^2(\Omega^R)$ . Assume that Assumption 4 holds. Then there exists a unique solution  $u_\xi \in H_\xi^1(\Omega_0^R)$  to problem (2.15).

*Proof.* We define by the Riesz representation theorem  $B_\xi^{(1)}, B_\xi^{(2)} : \check{H}_\xi^1(\Omega_0^R) \longrightarrow \check{H}_\xi^1(\Omega_0^R)$  such that  $B_\xi = B_\xi^{(1)} + B_\xi^{(2)}$

$$\begin{aligned} (B_\xi^{(1)} u_\xi, \psi_\xi)_{H_\xi^1(\Omega_0^R)} &:= \int_{\Omega_0^R} \nabla u_\xi \cdot \nabla \overline{\psi_\xi} dx + \int_{\Omega_0^R} u_\xi \cdot \overline{\psi_\xi} dx - \langle T_\xi^R u_\xi, \psi_\xi \rangle_{\Gamma_0^R}, \\ (B_\xi^{(2)} u_\xi, \psi_\xi)_{H_\xi^1(\Omega_0^R)} &:= - \int_{\Omega_0^R} (k^2 n_p + 1) u_\xi \overline{\psi_\xi} dx, \end{aligned}$$

for all  $u_\xi, \psi_\xi \in \check{H}_\xi^1(\Omega_0^R)$ . Clearly  $B_\xi^{(1)}$  is a coercive operator since  $\Re \langle T_\xi^R u_\xi, u_\xi \rangle_{\Gamma_0^R} \leq 0$ . Moreover, since the period  $\Omega_0^R$  is bounded then one can use the compact injection of  $\check{H}_\xi^1(\Omega_0^R)$  into  $L_\xi^2(\Omega_0^R)$  to prove that  $B_\xi^{(2)}$  is compact. Consequently, the operator  $B_\xi$  is a Fredholm operator of index zero. Assume now that

$$b_\xi(u_\xi, \psi_\xi) = 0 \quad \text{for all } \psi_\xi \in \check{H}_\xi^1(\Omega_0^R).$$

Form Lemma 3 we have  $\Im m \langle T_\xi^R u_\xi, u_\xi \rangle_{\Gamma_0^R} \geq 0$ . Therefore, Assumption 4 implies in particular that

$$0 = \Im m |b_\xi(u_\xi, \psi_\xi)| \geq k^2 \int_{\Omega_0^R} \Im m(n_p) |u_\xi|^2 dx \geq 0,$$

Consequently, we have  $u_\xi$  vanishes in the open ball  $\mathcal{O}$ . Using the unique continuation principle we deduce that  $u_\xi = 0$  everywhere in  $\Omega_0^R$  which gives the injectivity of the operator  $B_\xi$ .  $\square$



In order to deal with the case that considers the presence of defects in the next section, we prove in the following theorem some uniform bound with respect to the Floquet-Bloch variable for the operator  $B_\xi^{-1}$ .

**Theorem 6.** *Let  $B_\xi : \check{H}_\xi^1(\Omega_0^R) \longrightarrow \check{H}_\xi^1(\Omega_0^R)$  be the operator defined by (2.18). There exists a constant  $c > 0$  independent of  $\xi$  such that  $\|B_\xi^{-1}\| \leq c$ .*

*Proof.* Fix  $\xi \in I$  and define for  $u_\xi$  and  $\varphi_\xi \in H_\xi^1(\Omega_0^R)$ ,

$$u_\xi^\# := e^{-i\xi \cdot x_1} u_\xi, \quad \text{and} \quad \psi_\xi^\# := e^{-i\xi \cdot x_1} \psi_\xi.$$

We have  $u_\xi^\#, \psi_\xi^\# \in H_\#^1(\Omega_0^R)$ , where  $H_\#^1(\Omega_0^R)$  denotes the space  $H_\xi^1(\Omega_0^R)$  for  $\xi = 0$ . We have

$$\frac{\partial u_\xi}{\partial x_1} = e^{i\xi \cdot x_1} \left( i\xi u_\xi^\# + \frac{\partial u_\xi^\#}{\partial x_1} \right) \quad \text{and} \quad \frac{\partial \psi_\xi}{\partial x_1} = e^{i\xi \cdot x_1} \left( i\xi \psi_\xi^\# + \frac{\partial \psi_\xi^\#}{\partial x_1} \right).$$

Then, replacing  $u_\xi$  and  $\psi_\xi$  respectively by  $u_\xi^\# e^{ix_1\xi}$  and  $\psi_\xi^\# e^{ix_1\xi}$  in (2.16) we get

$$\int_{\Omega_0^R} (\nabla u_\xi^\# \cdot \overline{\nabla \psi_\xi^\#} + i u_\xi^\# \xi \cdot \frac{\partial \psi_\xi^\#}{\partial x_1} - i \xi \cdot \frac{\partial u_\xi^\#}{\partial x_1} \overline{\psi_\xi^\#} + (|\xi|^2 - k^2 n_p) u_\xi^\# \overline{\psi_\xi^\#}) dx - \left\langle \tilde{T}_\xi^R u_\xi^\#, \psi_\xi^\# \right\rangle_{\Gamma_0^R} =: b_\xi^\#(u_\xi^\#, \psi_\xi^\#)$$

with

$$\left\langle \tilde{T}_\xi^R u_\xi^\#, u_\xi^\# \right\rangle_{\Gamma_0^R} := i \sum_{j \in \mathbb{Z}} \beta_\xi(j) \widehat{(u_\xi^\#|_{\Gamma_0^R})} \overline{\widehat{(\psi_\xi^\#|_{\Gamma_0^R})}}.$$

Fix  $\eta > 0$  and consider  $\xi_1, \xi_2 \in I$  such that  $|\xi_1 - \xi_2| < \eta$ . For all  $u^\#, \psi^\# \in H_\#^1(\Omega^R)$  we have

$$\begin{aligned} \left| b_{\xi_1}^\#(u^\#, \psi^\#) - b_{\xi_2}^\#(u^\#, \psi^\#) \right| &\leq (|\xi_1|^2 - |\xi_2|^2 + 2|\xi_1 - \xi_2|) \|u^\#\|_{H^1(\Omega_0^R)} \|\psi^\#\|_{H^1(\Omega_0^R)} \\ &\quad + \sum_{j \in \mathbb{Z}} |\beta_{\xi_1}(j) - \beta_{\xi_2}(j)| \left| \widehat{(u^\#|_{\Gamma_0^R})} \overline{\widehat{(\psi^\#|_{\Gamma_0^R})}} \right| \\ &\leq \gamma C_{\xi_1, \xi_2}(j) \|u^\#\|_{H^1(\Omega_0^R)} \|\psi^\#\|_{H^1(\Omega_0^R)}, \end{aligned}$$

with  $\gamma$  is the continuity constant of the trace operator from  $H_\#^1(\Omega_0^R)$  into  $L^2(\Gamma_0^R)$ . Moreover, we have

$$\sup_{j \in \mathbb{Z}} C_{\xi_1, \xi_2}(j) \leq \beta |\xi_2 - \xi_1|^{1/2} \longrightarrow 0 \quad \text{for } \xi_1 \rightarrow \xi_2, \quad (2.19)$$

where  $\beta$  is a constant independent from  $j$ ,  $\xi_1$  and  $\xi_2$ . The proof of (2.19) is given in the proof of Theorem 21 in the next chapter. Consequently,  $\left| b_{\xi_1}^\#(u^\#, \psi^\#) - b_{\xi_2}^\#(u^\#, \psi^\#) \right|$  goes to zero as  $\xi_1 \rightarrow \xi_2$ , which implies that  $\xi \rightarrow B_\xi$  is uniformly continuous from  $I$  into  $\mathcal{L}(\check{H}_\xi^1(\Omega^R))$ .

Now, let  $\xi_0 \in \bar{I}$ , and set  $\epsilon = \frac{1}{2\|B_{\xi_0}^{-1}\|}$ . Then there exists  $\eta_{\xi_0} > 0$  such that for  $|\xi - \xi_0| < \eta_{\xi_0}$ , we have

$$\|B_\xi - B_{\xi_0}\| \leq \epsilon.$$

Therefore  $\|B_{\xi_0}^{-1}(B_\xi - B_{\xi_0})\| \leq \frac{1}{2}$  which implies that

$$\|(I + B_{\xi_0}^{-1}(B_\xi - B_{\xi_0}))^{-1}\| \leq 2.$$

Consequently, by observing that  $B_\xi = B_{\xi_0}(I + B_{\xi_0}^{-1}(B_\xi - B_{\xi_0}))$  we deduce that

$$\|B_\xi^{-1}\| \leq 2\|B_{\xi_0}^{-1}\|, \quad \forall \xi \in \mathcal{B}_{\eta_{\xi_0}},$$

where  $\mathcal{B}_{\eta_{\xi_0}}$  denotes the ball of radius  $\eta_{\xi_0}$  centered at  $\xi_0$ . Since  $\bar{I}$  is a compact set and there exists a finite set  $J \subset \bar{I}$  for which  $\bar{I} = \bigcup_{\xi \in J} \{\xi_0 \in \bar{I}; |\xi - \xi_0| < \eta_{\xi_0}\}$ . Therefore, for all  $\xi \in I$  we have

$$\|B_\xi^{-1}\| \leq 2 \sup_{\xi_0 \in J} \|B_{\xi_0}^{-1}\| = c.$$

□

### 2.2.3 The case of periodic layers with defects

In this section we come back to the case of the locally perturbed periodic domain, i.e.  $\tilde{D} \neq \emptyset$  and we use the well posedness of the quasi-periodic problem given in the previous section to prove the well posedness of (2.3).

#### 2.2.3.1 Well posedness of (2.3)

We define the sesquilinear form

$$b(u, \psi) := \int_{\Omega^R} (\nabla u \cdot \nabla \bar{\psi} - k^2 n u \bar{\psi}) dx - \langle T^R u, \psi \rangle_{\Gamma^R}. \quad (2.20)$$

The variational formulation equivalent to problem (2.3) is given as: seek  $u \in \check{H}^1(\Omega^R)$  such that

$$b(u, \psi) = \int_{\Omega^R} f \bar{\psi} dx, \quad \forall \psi \in H^1(\Omega^R). \quad (2.21)$$

In order to prove the well-posedness of (2.21), we shall first prove in the following theorem that problem (2.21) with  $n = n_p$  is equivalent to a coupled quasi-periodic problems with respect to the Floquet-Bloch variable.

**Theorem 7.** *Seeking a solution  $u \in \check{H}^1(\Omega^R)$  for problem (2.21) with  $n$  replaced by  $n_p$  is equivalent to seek  $\hat{u} := \mathcal{J}u \in L^2(I; \check{H}_\xi^1(\Omega_0^R))$  satisfying*

$$\int_I b_\xi(u_\xi, \psi_\xi) d\xi = \int_I \int_{\Omega_0^R} f_\xi \bar{\psi}_\xi dx d\xi, \quad \forall \hat{\psi} \in L^2(I, \check{H}_\xi^1(\Omega_0^R)), \quad (2.22)$$

with the notation  $u_\xi = \hat{u}(\xi, \cdot)$  and  $\psi_\xi = \hat{\psi}(\xi, \cdot)$ .

*Proof.* Let us first recall in addition to (2.7) that for  $m \in \mathbb{R}$  positive,  $\mathcal{J}$  is an isomorphism between  $H^m(\Omega^R)$  and  $L^2(I; H_\xi^m(\Omega_0^R))$  and that  $\mathcal{J}^*$  the adjoint of  $\mathcal{J}$  coincides with the inverse of  $\mathcal{J}$  (see [24, 43]). Consequently, we have

$$\begin{aligned} \int_{\Omega^R} (\nabla u \cdot \nabla \bar{\psi} - k^2 n_p u \bar{\psi}) dx &= \int_{\Omega^R} (\nabla u \cdot \overline{\mathcal{J}^* \mathcal{J}(\nabla \psi)} - k^2 n_p u \overline{\mathcal{J}^* \mathcal{J}(\psi)}) dx, \\ &= \int_I \int_{\Omega_0^R} (\nabla u_\xi \nabla \bar{\psi}_\xi - k^2 n_p u_\xi \bar{\psi}_\xi) dx d\xi. \end{aligned} \quad (2.23)$$

On the same way we have that

$$\int_{\Omega^R} f \bar{\psi} dx = \int_I \int_{\Omega_0^R} f_\xi \bar{\psi}_\xi dx d\xi.$$

Moreover, since  $\mathcal{J}(u|_{\Gamma^R})(\xi, x_1) = ((\mathcal{J}u)(\xi, x_1))|_{\Gamma_0^R}$  we deduce from Lemma 2 that

$$\mathcal{J}(T^R u)(\xi, \cdot) = T_\xi^R(\mathcal{J}u)(\xi, \cdot),$$

and therefore

$$\int_{\Gamma^R} T^R(u|_{\Gamma^R}) \overline{\mathcal{J}^* \mathcal{J}(\psi|_{\Gamma^R})} ds = \int_I \int_{\Gamma_0^R} T_\xi^R(u_\xi|_{\Gamma_0^R}) \overline{(\psi_\xi|_{\Gamma_0^R})} ds d\xi. \quad (2.24)$$

which ends the proof.  $\square$

**Lemma 8.** *Assume that Assumption 4 holds. Then there exists a unique solution for problem (2.22).*

*Proof.* From the well posedness of problem (2.15) we consider  $u_\xi \in \check{H}_\xi^1(\Omega^R)$  solution of (2.16) for all  $\xi \in I$  with  $f_\xi = \mathcal{J}f(\xi, \cdot)$ . Consequently we have thanks to Theorem 6 that  $\hat{u}$  defined by  $\hat{u}(\xi) = u_\xi \in L^\infty(I, \check{H}_\xi^1(\Omega_0^R)) \subset L^2(I, \check{H}_\xi^1(\Omega_0^R))$  and verifies (2.22). Then it remains to prove that (2.22) is uniquely solvable.

We consider  $\hat{u}$  solution to the homogeneous problem (2.22). Then, we have in particular that

$$\int_I b_\xi(u_\xi, \psi_\xi) d\xi = 0, \quad \forall \hat{\psi} \in L^2(I, \check{H}_\xi^1(\Omega_0^R)). \quad (2.25)$$

Using the same arguments as in the proof of uniqueness in Theorem 5 we get

$$k^2 \int_I \int_{\Omega_0^R} \Im m(n_p) |u_\xi|^2 dx = 0.$$

Therefore  $u_\xi$  vanishes in the open ball  $\mathcal{O}$  almost everywhere in  $I$ . Since  $u_\xi$  satisfies  $\Delta u_\xi + k^2 n_p u_\xi = 0$  in  $\Omega_0^R$  we deduce by the unique continuation principle that  $u_\xi = 0$  in  $\Omega_0^R$  and almost everywhere with respect to  $\xi$ .  $\square$

**Theorem 9.** *Assume that Assumption 4 holds and that  $\Im m(n) \geq 0$ . Consider  $f \in L^2(\Omega^R)$ , then there exists a unique solution  $u \in H^1(\Omega^R)$  to problem (2.3) such that*

$$\|u\|_{H^1(\Omega^R)} \leq c \|f\|_{L^2(\Omega^R)},$$

with  $c > 0$  is a constant independent from  $f$ .

*Proof.* We observe that (2.20) can be decomposed as

$$b(u, \psi) = \int_{\Omega^R} \nabla u \cdot \nabla \bar{\psi} - k^2 n_p u \bar{\psi} dx - \langle T^R u, \psi \rangle_{\Gamma^R} - k^2 \int_{\Omega_0^R} (n - n_p) u \bar{\psi} dx. \quad (2.26)$$

and we define using the Riesz representation Theorem the operators  $B, B^p, \tilde{B} : \check{H}^1(\Omega^R) \longrightarrow \check{H}^1(\Omega^R)$  such that

$$\begin{aligned} (Bu, \psi)_{H^1(\Omega^R)} &:= b(u, \psi), \\ (B^p u, \psi)_{H^1(\Omega^R)} &:= \int_{\Omega^R} \nabla u \cdot \nabla \bar{\psi} - k^2 n_p u \bar{\psi} dx - \langle T^R u, \psi \rangle_{\Gamma^R}, \\ (\tilde{B}u, \psi)_{H^1(\Omega^R)} &:= -k^2 \int_{\Omega^R} (n - n_p) u \bar{\psi} dx, \end{aligned} \tag{2.27}$$

for all  $u, \psi \in \check{H}^1(\Omega^R)$ . We have

$$(B^p u, \psi)_{H^1(\Omega^R)} = (\mathcal{J}(B^p u), \mathcal{J}(\psi))_{L^2(I, H^1(\Omega_0^R))} = \int_I (B_\xi u_\xi, \psi_\xi)_{H^1(\Omega_0^R)} d\xi.$$

Using Theorems 5, 6 and 7 we deduce that  $B^p$  is invertible. Moreover, since  $(n - n_p)$  is compactly supported. Then we can use the compact injection of  $H^1(\tilde{D})$  into  $L^2(\tilde{D})$  to prove the compactness of the operator  $\tilde{B}$ . This implies that  $B$  is of Fredholm type of index zero. It remains to prove that  $B$  is injective. Assume that

$$(Bu, \psi)_{H^1(\Omega^R)} = 0 \quad \forall \psi \in \check{H}^1(\Omega^R).$$

Since it holds that  $\Im m(T^R u, u) \geq 0$ , then we have in particular that

$$0 = k^2 \int_{\Omega^R} \Im m(n) |u|^2 \geq 0,$$

consequently  $u = 0$  in  $\mathcal{O}$  by Assumption 4. The unique continuation principle implies that  $u = 0$  in  $\Omega^R$ . This ends the proof.  $\square$

## 2.3 Numerical examples

We present in this section some numerical examples obtained from the volume integral method introduced in [29] for solving the direct scattering problem from locally perturbed periodic domain.

**Numerical method:** Recall that the problem can be reformulated as : Given  $f \in$

$L^2(U^0)$ , seek  $\hat{u} := \mathcal{J}u \in L^2(I; H_\xi^2(\Omega_0))$  satisfying

$$\begin{cases} \Delta \hat{u}(\xi, \cdot) + k^2 n_p \hat{u}(\xi, \cdot) + k^2(n - n_p) \mathcal{J}^{-1}(\hat{u}) = f_\xi & \text{in } \Omega_0, \quad \xi \in I, \\ \mathcal{J}^{-1}(\hat{u}) := \int_I \hat{u}(\xi, \cdot) d\xi & \text{in } \Omega_0, \end{cases} \quad (2.28)$$

with  $f_\xi := \mathcal{J}f(\xi, \cdot)$ . We define  $\Omega_0 := [-\pi, \pi] \times \mathbb{R}$ , and we set for numerics the period  $\Omega_0^R = [-\pi, \pi] \times [0, R]$  and  $I = [-\frac{1}{2}, \frac{1}{2}]$ . The main idea of the method proposed in [29] is to first consider a discretization with respect to the Floquet-Bloch variable, using a uniform partition of  $I$  into sub-domains of size  $\Delta\xi = \frac{1}{M}$ , where  $M$  denotes the number of discretization points. This allows to write a discretized problem in the form: seek  $\hat{u}_M(\cdot, \xi_j) \in H_{\xi_j}^2(\Omega_0)$  satisfying

$$\begin{cases} \Delta \hat{u}_M(\xi_j, \cdot) + k^2 n_p \hat{u}_M(\xi_j, \cdot) + k^2(n - n_p) u_M = f_{\xi_j} & \text{in } \Omega_0, \quad [-\frac{M}{2}] + 1 \leq j \leq [\frac{M}{2}], \\ u_M := \frac{1}{M} \sum_{j=[-\frac{M}{2}]+1}^{[\frac{M}{2}]} \hat{u}_M(\xi_j, \cdot) & \text{in } \Omega_0. \end{cases} \quad (2.29)$$

An equivalent volume integral problem to (2.29) was given as follows

$$\begin{cases} u_M(\xi_j, \cdot) - k^2 V_{\xi_j}((n_p - 1)\hat{u}_M(\xi_j, \cdot)) + k^2 V_{\xi_j}((n - n_p)u_M) = V_{\xi_j}(f_{\xi_j}) & \text{in } L^2(\Omega_0^R), \\ u_M := \frac{1}{M} \sum_{j=[-\frac{M}{2}]+1}^{[\frac{M}{2}]} \hat{u}_M(\xi_j, \cdot) \end{cases} \quad (2.30)$$

where  $V_\xi$  is the volume potential defined in  $\Omega_0^R$  as

$$V_\xi g_\xi(x) := \int_{\Omega_0^R} G_\xi(x - y) g_\xi(y) dy \quad x \in \mathbb{R},$$

and  $G_\xi$  is the  $\xi$ -quasi-periodic Green function satisfying

$$\Delta G_\xi + k^2 G_\xi = -\delta_0, \quad \text{in } \Omega_0.$$

Finally, a periodization with respect to  $x_2$  variable for (2.30) was considered in order to discretize the problem using a spectral Fourier basis.

**Numerical examples:** We present now some numerical examples of the solution calculated using the MATLAB code developed to solve the direct problem using the method outlined above. This code was developed by [39] and we just slightly adapted it for point sources. We consider in our numerical examples an incident field generated by a point source  $y := (y_1, R)$  situated on  $\Gamma^R$  with Dirichlet condition on  $\Gamma^0$  given by the following expression:

$$v_{\xi_j}(y, \cdot) := -\frac{i}{4\pi} \sum_{\ell \in \mathbb{Z}} e^{-i\alpha_{\xi_j}(\ell)(y_1 - x_1)} \left[ e^{-i\beta_{\xi_j}(\ell)R} \left( \frac{e^{i\beta_{\xi_j}(\ell)x_2} - e^{-i\beta_{\xi_j}(\ell)x_2}}{\beta_{\xi_j}(\ell)} \right) \right],$$

for  $x_1 \in \Omega_0$  and  $x_2 \leq R$ . We define

$$v_M := \frac{1}{M} \sum_{j=[-\frac{M}{2}] + 1}^{[\frac{M}{2}]} v_{\xi_j}.$$

The  $\xi_j$ -quasi periodicity implies in particular that

$$v_M(x_1 + 2\pi m, x_2) := \frac{1}{M} \sum_{j=[-\frac{M}{2}] + 1}^{[\frac{M}{2}]} e^{i2\pi m \xi_j} v_{\xi_j}(x_1, x_2) \quad \text{for } x \in \Omega_0,$$

$$u_M(x_1 + 2\pi m, x_2) := \frac{1}{M} \sum_{j=[-\frac{M}{2}] + 1}^{[\frac{M}{2}]} e^{i2\pi m \xi_j} \hat{u}((x_1, x_2), \xi_j) \quad \text{for } x \in \Omega_0,$$

for  $m \in \mathbb{N}$ . In the examples below we vary the wavenumber  $k$  and the number of periods  $M$  from an example to another and we fix the following physical parameters:

$$\lambda = \frac{2\pi}{|k|}, \quad R = 2.5\lambda, \quad n_p = 5, \quad n = 3, \quad N_x = 400, \quad N_y = 400,$$

where  $N_x$  and  $N_y$  denotes the number of discretization points respectively with respect to  $x_1$  and  $x_2$  directions. In all the following tests we represent the solution  $u_M$  in  $\Omega_M^R := [(M - \frac{1}{2})2\pi, (m + \frac{1}{2})2\pi] \times [-R, R]$  and in  $\Omega_0$ , the incident field  $v_M$ , and the total field  $u_M + v_M$  in the sub-figures respectively numbered (1) (2), (3) and (4).

**Example 1:** We present in the first example the domain  $D^p$  formed by periodic balls centered at  $(0, 0.8\lambda)$  with radius  $0.35\lambda$  and a rectangle with sides of width  $0.25\lambda$  for fixed  $M = 3$  and  $k = \frac{3, 5\pi}{3.14} + 0.7i$ .

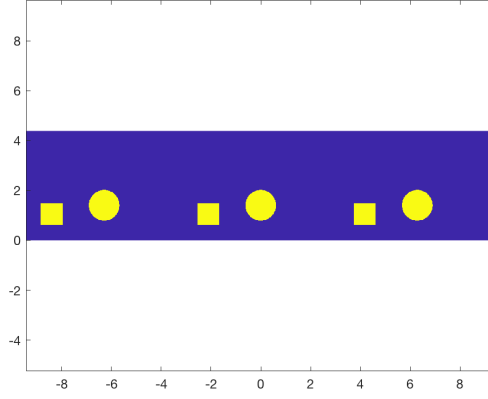


FIGURE 2.2: The exact geometry

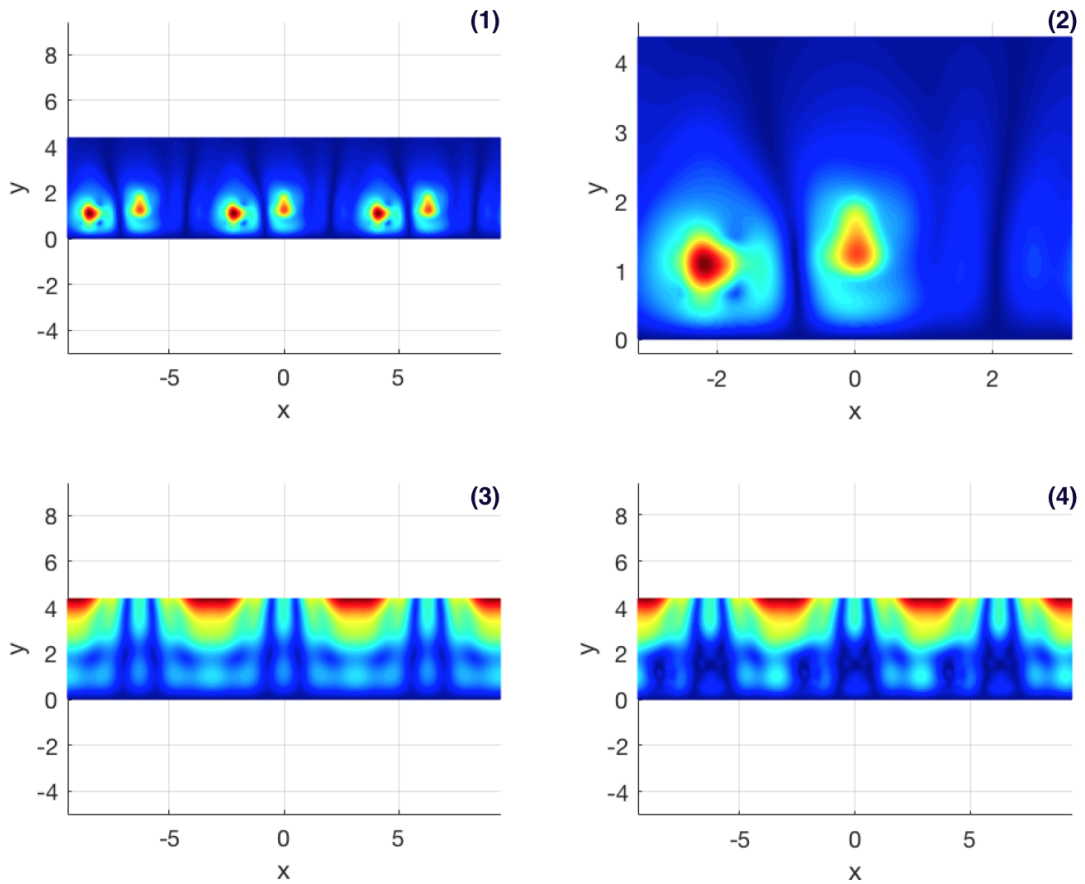


FIGURE 2.3: (1) The scattered field  $u_M$  in  $\Omega_M^R$ . (2) The scattered field  $u_M$  in  $\Omega_0^R$ . (3) The incident field  $v_M$  in  $\Omega_M^R$ . (4) The total field  $u_M + v_M$  in  $\Omega_M^R$ .



**Example 2:** We consider the same periodic domain given in the previous example and we add a perturbation given by a ball centered at  $(0.85\lambda, 1.1\lambda)$  with radius  $R_D = 0.3\lambda$  for fixed  $M = 3$  and  $k = \frac{3,5\pi}{3.14} + 0.7i$ .

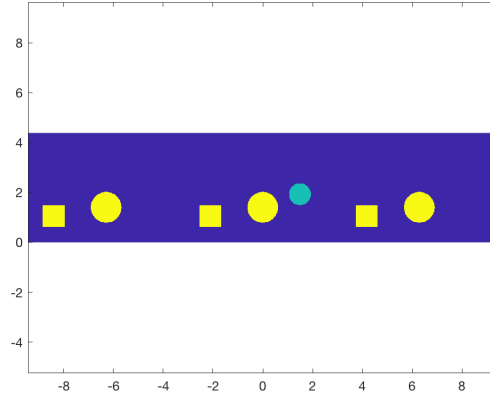


FIGURE 2.4: The exact geometry

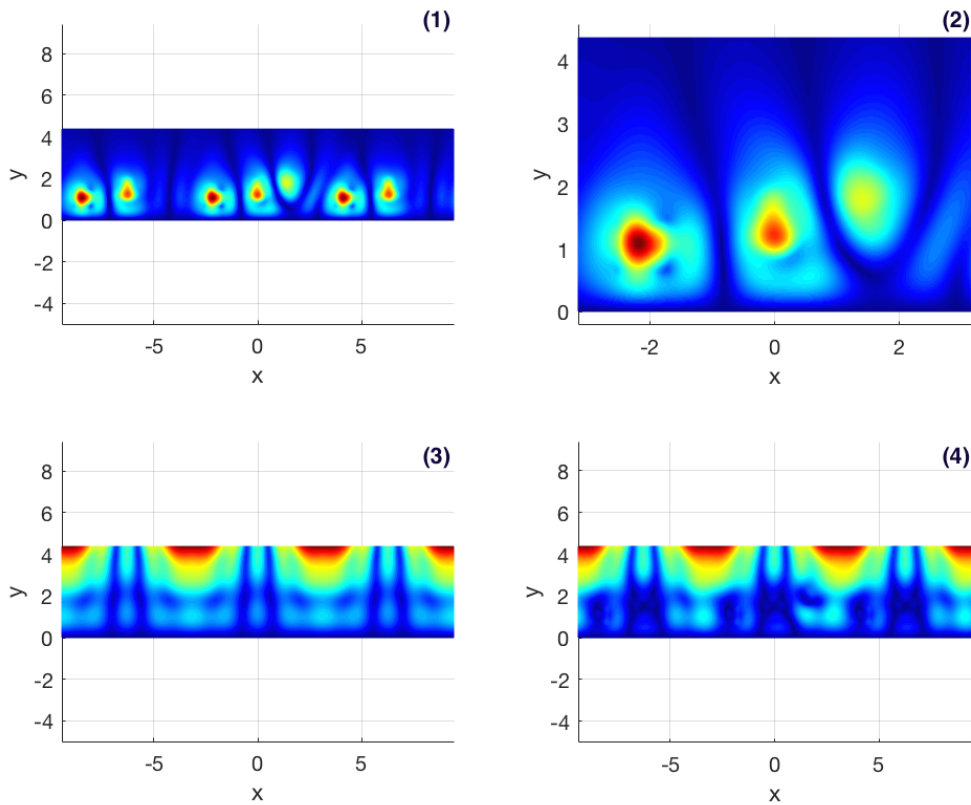


FIGURE 2.5: (1) The scattered field  $u_M$  in  $\Omega_M^R$ . (2) The scattered field  $u_M$  in  $\Omega_0^R$ . (3) The incident field  $v_M$  in  $\Omega_M^R$ . (4) The total field  $u_M + v_M$  in  $\Omega_M^R$ .

**Example 3:** We consider in this example a more complex periodic domain as shown in the Figure 2.6 for fixed  $M = 4$  and  $k = \frac{3, 5\pi}{3.14} + 0.5i$ .

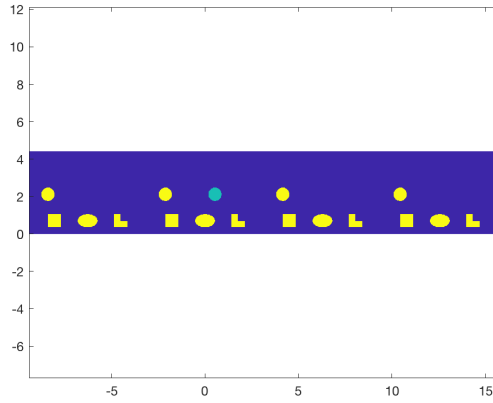


FIGURE 2.6: Left: The exact geometry

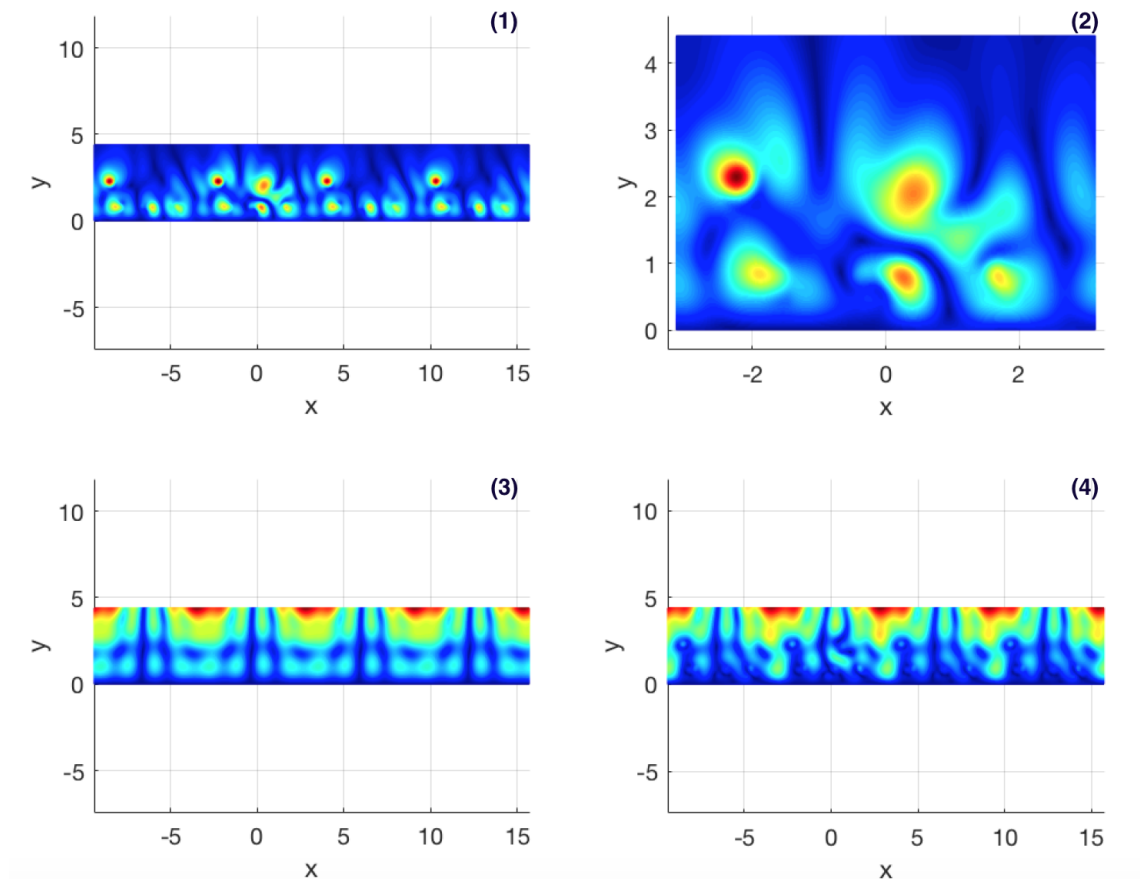


FIGURE 2.7: (1) The scattered field  $u_M$  in  $\Omega_M^R$ . (2) The scattered field  $u_M$  in  $\Omega_0^R$ . (3) The incident field  $v_M$  in  $\Omega_M^R$ . (4) The total field  $u_M + v_M$  in  $\Omega_M^R$ .

**Example 4:** We consider in the last example  $D^p$  and  $\tilde{D}$  as an  $L$ -shape geometry as shown in Figure 2.9 and we present the case for which  $\tilde{D}$  intersects  $D^p$  for fixed  $M = 3$  and  $k = \frac{3,5\pi}{3.14} + 0.2i$ .

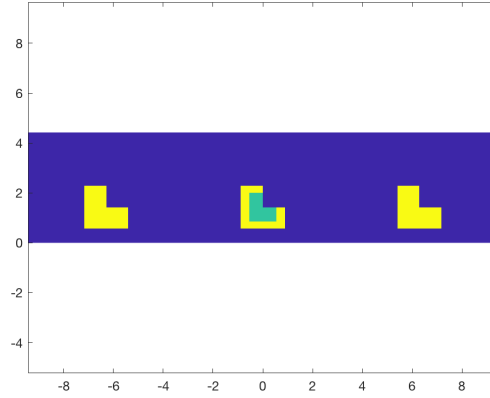


FIGURE 2.8: The exact geometry

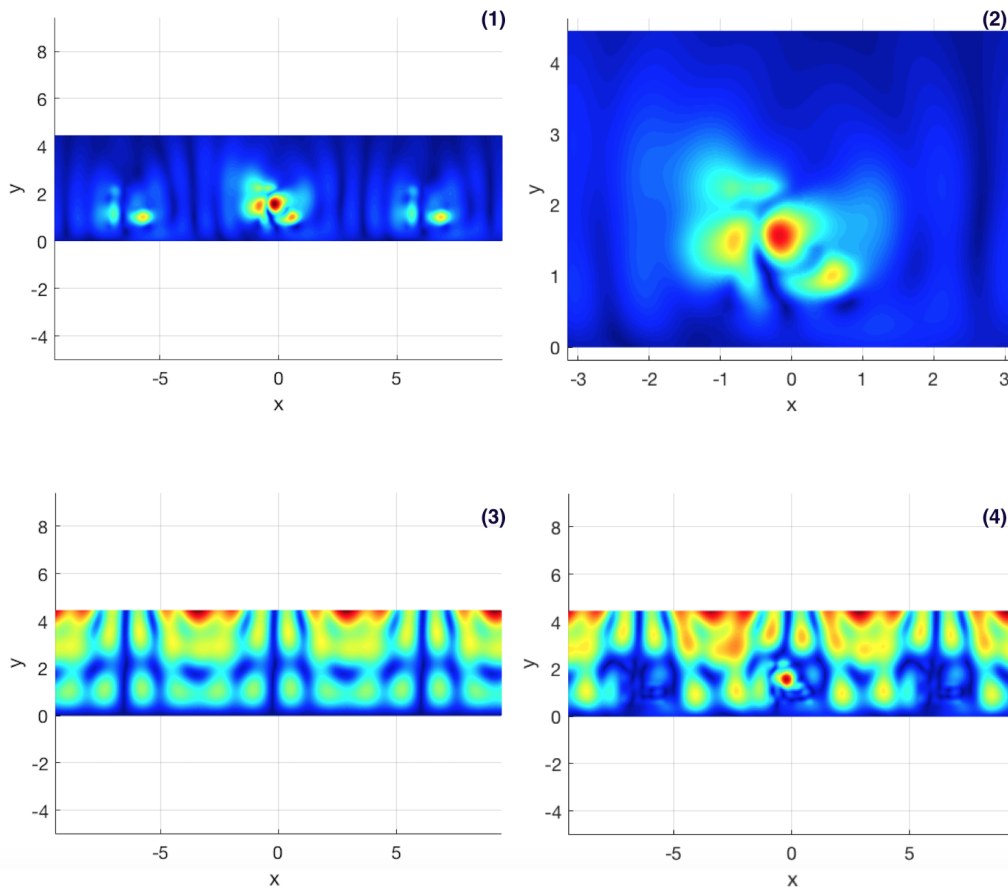


FIGURE 2.9: (1) The scattered field  $u_M$  in  $\Omega_M^R$ . (2) The scattered field  $u_M$  in  $\Omega_0^R$ . (3) The incident field  $v_M$  in  $\Omega_M^R$ . (4) The total field  $u_M + v_M$  in  $\Omega_M^R$

## 2.4 Solvability of the direct scattering problem in anisotropic Media

### 2.4.1 Setting of the problem

Let  $A_p$  be a  $2 \times 2$  symmetric and  $2\pi$ -periodic matrix with  $L^\infty(U^0)$ -entries. For  $a > 0$  we assume that  $A_p$  satisfies:

$$\bar{\zeta} \cdot \Re(A_p)\zeta \geq a|\zeta|^2 \quad \text{and} \quad \bar{\zeta} \cdot \Im(A_p)\zeta \leq 0, \quad (2.31)$$

for all  $\zeta \in \mathbb{C}^2$ , and that  $A_p = Id$  outside  $D^p$ , where  $Id$  denotes the  $2 \times 2$  identity matrix. We consider for given  $(f, g) \in H^1(\Omega^R) \times L^2(\Omega^R)$  the inhomogeneous anisotropic problem as follows: seek  $u \in H^1(\Omega^R)$  satisfying

$$\begin{cases} \nabla \cdot (A_p \nabla u) + k^2 n u = \nabla \cdot ((Id - A_p) \nabla f) + k^2(1 - n)g & \text{in } U^0, \\ u = 0 & \text{on } \Gamma^0, \end{cases} \quad (2.32)$$

and we impose the radiation condition (2.2). Then (2.32)-(2.2) can be reformulated as follows:

$$\begin{cases} \nabla \cdot (A_p \nabla u) + k^2 n u = \nabla \cdot ((Id - A_p) \nabla f) + k^2(1 - n)g & \text{in } U^0, \\ u = 0 & \text{on } \Gamma^0, \\ \frac{\partial u}{\partial x_2}(\cdot, R) = T^R(u|_{\Gamma^R}) & \text{on } \Gamma^R, \end{cases} \quad (2.33)$$

Similarly to the previous section we consider first the case without defects, i.e  $n = n_p$ . This leads us to the study of the quasi-periodic problem that will be the topic of the next subsection.

### 2.4.2 The case of periodic layers without defects

The inhomogeneous anisotropic quasi-periodic problem can be formulated as: seek  $u_\xi = \mathcal{J}(u)(\xi, \cdot) \in H_\xi^1(\Omega_0^R)$  such that

$$\begin{cases} \nabla \cdot (A_p \nabla u_\xi) + k^2 n_p u_\xi = \nabla \cdot ((Id - A_p) \nabla f_\xi) + k^2 (1 - n_p) g_\xi & \text{in } \Omega^R, \\ u_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial u_\xi}{\partial x_2}(\cdot, R) = T_\xi^R(u_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases} \quad (2.34)$$

with  $(f_\xi, g_\xi) := (\mathcal{J}(f)(\xi, \cdot), \mathcal{J}(g)(\xi, \cdot)) \in H_\xi^1(\Omega^R) \times L_\xi^2(\Omega^R)$ . The variational formulation equivalent to problem (2.34) is given as: seek  $u_\xi \in \check{H}_\xi^1(\Omega_0^R)$  such that

$$h_\xi(u_\xi, \psi_\xi) = \int_{\Omega_0^R} (Id - A_p) \nabla f_\xi \cdot \overline{\nabla \psi_\xi} + k^2 (n_p - 1) g_\xi \overline{\psi_\xi}, \quad \forall \psi_\xi \in \check{H}_\xi^1(\Omega_0^R), \quad (2.35)$$

where  $h_\xi$  being the sesquilinear form

$$h_\xi(u_\xi, \psi_\xi) := \int_{\Omega_0^R} (A_p \nabla u_\xi \cdot \nabla \overline{\psi_\xi} - k^2 n_p u_\xi \overline{\psi_\xi}) dx - \langle T_\xi^R u_\xi, \psi_\xi \rangle_{\Gamma_0^R}. \quad (2.36)$$

Let  $\mathcal{H}_\xi : \check{H}_\xi^1(\Omega_0^R) \longrightarrow \check{H}_\xi^1(\Omega_0^R)$  be the operator defined by the Riesz representation theorem such that

$$(\mathcal{H}_\xi u_\xi, \psi_\xi)_{H^1(\Omega_0^R)} := h_\xi(u_\xi, \psi_\xi) \quad \forall \psi_\xi, u_\xi \in \check{H}_\xi^1(\Omega_0^R). \quad (2.37)$$

**Theorem 10.** Consider  $(f_\xi, g_\xi) \in H_\xi^1(\Omega^R) \times L_\xi^2(\Omega^R)$ . Assume that Assumption 4 holds. Then there exists a unique solution  $u_\xi \in H_\xi^1(\Omega_0^R)$  to problem (2.34).

*Proof.* We define by the Riesz representation theorem  $\mathcal{H}_\xi^{(1)}, \mathcal{H}_\xi^{(2)} : \check{H}_\xi^1(\Omega_0^R) \longrightarrow \check{H}_\xi^1(\Omega_0^R)$  such that  $\mathcal{H}_\xi = \mathcal{H}_\xi^{(1)} + \mathcal{H}_\xi^{(2)}$

$$\begin{aligned} (\mathcal{H}_\xi^{(1)} u_\xi, \psi_\xi)_{H_\xi^1(\Omega_0^R)} &:= \int_{\Omega_0^R} A_p \nabla u_\xi \cdot \nabla \overline{\psi_\xi} dx + \int_{\Omega_0^R} u_\xi \cdot \overline{\psi_\xi} dx - \langle T_\xi^R u_\xi, \psi_\xi \rangle_{\Gamma_0^R}, \\ (\mathcal{H}_\xi^{(2)} u_\xi, \psi_\xi)_{H_\xi^1(\Omega_0^R)} &:= - \int_{\Omega_0^R} (k^2 n_p + 1) u_\xi \overline{\psi_\xi} dx, \end{aligned}$$

for all  $u_\xi, \psi_\xi \in \check{H}_\xi^1(\Omega_0^R)$ . Clearly, the operator  $\mathcal{H}_\xi^{(1)}$  is coercive since  $\Re \langle T_\xi^R u_\xi, u_\xi \rangle_{\Gamma_0^R} \leq 0$  and  $\bar{\xi} \cdot \Re(A_p)\xi \geq a|\xi|^2$ . Moreover, since the period  $\Omega_0^R$  is bounded then one can use the compact injection of  $\check{H}_\xi^1(\Omega_0^R)$  into  $L_\xi^2(\Omega_0^R)$  to prove that  $\mathcal{H}_\xi^{(2)}$  is compact. Consequently,  $\mathcal{H}_\xi$  is a Fredholm operator of index zero. Assume now that

$$h_\xi(u_\xi, \psi_\xi) = 0 \quad \text{for all } \psi_\xi \in \check{H}_\xi^1(\Omega_0^R).$$

From Lemma 3 we have  $\Im \langle T_\xi^R u_\xi, u_\xi \rangle_{\Gamma_0^R} \geq 0$ . Therefore (2.31) and Assumption 4 implies in particular that

$$\begin{aligned} 0 &= \left| \int_{\Omega_0^R} (-\Im(A_p)|\nabla u_\xi|^2 + k^2 \Im(n_p)|u_\xi|^2) dx + \Im \langle T_\xi^R u_\xi, u_\xi \rangle_{\Gamma_0^R} \right| \\ &\geq k^2 \int_{\Omega_0^R} \Im(n_p)|u_\xi|^2 dx \geq 0, \end{aligned} \quad (2.38)$$

consequently, we have  $u_\xi$  vanishes in the open ball  $\mathcal{O}$  almost everywhere in  $I$ . By the unique continuation principle we conclude that  $u_\xi = 0$  everywhere in  $\Omega_0^R$  and almost everywhere with respect to  $\xi$ .  $\square$

**Theorem 11.** *Let  $\mathcal{H}_\xi : \check{H}_\xi^1(\Omega_0^R) \rightarrow \check{H}_\xi^1(\Omega_0^R)$  be the operator defined by (2.37). There exists a constant  $c > 0$  independent of  $\xi$  such that  $\|\mathcal{H}_\xi^{-1}\| \leq c$ .*

*Proof.* The proof follows the same arguments as the proof of Theorem 6.  $\square$

### 2.4.3 The case of periodic layers with defects

We consider now the locally perturbed periodic problem (2.33). Define the sesquilinear form

$$h(u, \psi) := \int_{\Omega^R} (A_p \nabla u \cdot \overline{\nabla \psi} - k^2 n u \overline{\psi}) dx - \langle T^R u, \psi \rangle_{\Gamma^R}. \quad (2.39)$$

The variational formulation equivalent to problem (2.33) is given as: seek  $u \in \check{H}^1(\Omega^R)$  such that

$$h(u, \psi) = \int_{\Omega^R} (Id - A_p) \nabla f \cdot \overline{\nabla \psi} + k^2 (n - 1) g \overline{\psi} dx \quad \forall \psi \in \check{H}^1(\Omega^R). \quad (2.40)$$

**Theorem 12.** Seeking a solution  $u \in \check{H}^1(\Omega^R)$  for problem (2.40) with  $n$  replaced by  $n_p$  is equivalent to seek  $\hat{u} := \mathcal{J}u \in L^2(I; \check{H}_\xi^1(\Omega_0^R))$  satisfying

$$\int_I h_\xi(u_\xi, \psi_\xi) d\xi = \int_I \int_{\Omega_0^R} (Id - A_p) \nabla f_\xi \cdot \overline{\nabla \psi_\xi} + k^2(n_p - 1) g_\xi \overline{\psi_\xi} dx d\xi, \quad (2.41)$$

for all  $\hat{\psi} \in L^2(I, \check{H}_\xi^1(\Omega_0^R))$ , with the notations  $u_\xi = \hat{u}(\xi, \cdot)$  and  $\psi_\xi = \hat{\psi}(\xi, \cdot)$ .

*Proof.* We have  $n_p$  and  $A_p$  are both  $2\pi$ -periodic, then similarly to the the proof of Theorem 7 one can use the properties of the Floquet-Bloch transform to prove that

$$\int_{\Omega^R} (A_p \nabla u \cdot \nabla \bar{\psi} - k^2 n_p u \bar{\psi}) dx = \int_I \int_{\Omega_0^R} (A_p \nabla u_\xi \cdot \overline{\nabla \psi_\xi} - k^2 n_p u_\xi \overline{\psi_\xi}) dx d\xi, \quad (2.42)$$

$$\begin{aligned} \int_{\Omega^R} (Id - A_p) \nabla f \cdot \overline{\nabla \psi} + k^2(n_p - 1) g \bar{\psi} dx d\xi \\ = \int_I \int_{\Omega_0^R} (Id - A_p) \nabla f_\xi \cdot \overline{\nabla \psi_\xi} + k^2(n_p - 1) g_\xi \overline{\psi_\xi} dx d\xi \end{aligned} \quad (2.43)$$

Therefore, from (2.42), (2.43) and (2.24) we obtain (2.41).  $\square$

**Lemma 13.** Assume that Assumption 4 holds. Then there exists a unique solution for problem (2.41).

*Proof.* From the well posedness of problem (2.34) we consider  $u_\xi \in H_\xi^1(\Omega^R)$  solution of (2.35) for all  $\xi \in I$  with  $(f_\xi, g_\xi) \in H_\xi^1(\Omega^R) \times L_\xi^2(\Omega^R)$ . Consequently we have thanks to Theorem 11 that  $\hat{u}$  defined by  $\hat{u}(\xi) = u_\xi \in L^\infty(I, \check{H}_\xi^1(\Omega^R)) \subset L^2(I, \check{H}_\xi^1(\Omega^R))$  and verifies (2.41). Then it remains to prove that (2.41) is uniquely solvable.

We consider  $\hat{u}$  solution to the homogeneous problem (2.41). Then, we have in particular that

$$\int_I h_\xi(u_\xi, \psi_\xi) d\xi = 0 \quad \forall \hat{\psi} \in L^2(I, \check{H}_\xi^1(\Omega_0^R)). \quad (2.44)$$

Using (2.38) we get

$$k^2 \int_I \int_{\Omega_0^R} \Im m(n_p) |u_\xi|^2 dx = 0.$$

Therefore, using the unique continuation principle we deduce that  $u_\xi = 0$  in  $\Omega_0^R$  and almost everywhere with respect to  $\xi$ .  $\square$

**Theorem 14.** *Assume that Assumption 4 holds and that  $\Im m(n) \geq 0$ . Consider  $(f, g) \in H^1(\Omega^R) \times L^2(\Omega^R)$ , then there exists a unique solution  $u \in H^1(\Omega^R)$  to problem (2.33) such that*

$$\|u\|_{H^1(\Omega^R)} \leq c \left( \|f\|_{H^1(\Omega^R)} + \|g\|_{L^2(\Omega^R)} \right),$$

with  $c > 0$  is a constant independent from  $f$  and  $g$ .

*Proof.* We observe that (2.39) can be decomposed as

$$h(u, \psi) = \int_{\Omega^R} (A_p \nabla u \cdot \nabla \bar{\psi} - k^2 n_p u \bar{\psi} - k^2 (n - n_p) u \bar{\psi}) dx - \langle T^R u, \psi \rangle_{\Gamma^R}, \quad (2.45)$$

and we define using the Riesz representation Theorem the operators  $\mathcal{H}, \mathcal{H}^p, \tilde{\mathcal{H}} : \check{H}^1(\Omega^R) \rightarrow \check{H}^1(\Omega^R)$  such that

$$\begin{aligned} (\mathcal{H}u, \psi)_{H^1(\Omega^R)} &:= h(u, \psi), \\ (\mathcal{H}^p u, \psi)_{H^1(\Omega^R)} &:= \int_{\Omega^R} A_p \nabla u \cdot \nabla \bar{\psi} - k^2 n_p u \bar{\psi} dx - \langle T^R u, \psi \rangle_{\Gamma^R}, \\ (\tilde{\mathcal{H}}u, \psi)_{H^1(\Omega^R)} &:= - \int_{\Omega^R} k^2 (n - n_p) u \bar{\psi} dx, \end{aligned} \quad (2.46)$$

for all  $u, \psi \in \check{H}^1(\Omega^R)$ .

Using Theorems 10, 11 and 12 we deduce that  $\mathcal{H}^p$  is invertible. Moreover, since  $(n - n_p)$  is compactly supported. Then we can use the compact injection of  $H^1(\check{D})$  into  $L^2(\check{D})$  to prove the compactness of the operator  $\tilde{\mathcal{H}}$ . This implies that  $\mathcal{H}$  is of Fredholm type of index zero. It remains to prove that  $\mathcal{H}$  is injective. Assume that

$$(\mathcal{H}u, \psi)_{H^1(\Omega^R)} = 0 \quad \forall \psi \in \check{H}^1(\Omega^R).$$



Since it holds that  $\Im m(T^R u, u) \geq 0$ , and  $\zeta \cdot \Im m(A_p)\zeta \leq 0$ , then we have in particular that

$$\begin{aligned} 0 &= \left| \int_{\Omega^R} (-\Im m(A_p)|\nabla u|^2 + k^2 \Im m(n)|u|^2) dx + \Im m \langle T^R u, u \rangle_{\Gamma^R} \right| \\ &\geq k^2 \int_{\Omega^R} \Im m(n)|u|^2 dx \geq 0, \end{aligned} \quad (2.47)$$

consequently  $u = 0$  in  $\mathcal{O}$  by Assumption 4. The unique continuation principle implies that  $u = 0$  in  $\Omega^R$ . This ends the proof.  $\square$

# Sampling methods for imaging a periodic layer and its defects

**Abstract:** *The theoretical content of this chapter is extracted from the published paper [8]. We analyze sampling methods for the identification of an unbounded periodic domain and a local perturbation. We consider first the case where quasi-periodic incident fields are applied. We give a theoretical justification of the GLSM method for a single Floquet-Bloch mode. We then analyze the application of the GLSM method when non-periodic incident fields are used. In addition to the theoretical part extracted from [8] we give some numerical examples for the case of data associated with a single Floquet-Bloch mode.*

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### 3.1 Introduction

We consider in this chapter the inverse scattering problem for the identification of a periodic domain and its defects from near field measurements at fixed frequency. This problem has connections with many practical applications, such as non-destructive testing of photonic structures, antenna arrays... The presence of the perturbation does not allow us to reduce the problem to one-period cell and makes the analysis more challenging. We would like to develop so-called sampling methods to address the inverse problem of identifying the geometry of the defect. For the non perturbed inverse periodic problem we refer to [3, 31, 42, 56] and references therein. For the perturbed case, it is frequently assumed that the periodic background is known a priori. We refer for those cases for instance to [9, 26, 28, 39, 45, 54]. However, for some applications, this information is not available or cannot be obtained in an exact way. This is what we would like to consider in this work. An analysis of sampling methods was given in [30] by assuming that the defect is also periodic with a larger periodicity (equals to an integer multiple of the background periodicity). Our goal here is to perform an analysis that removes this technical assumption on the defect. In order to do so, we analyze the scattering problem in spaces that include continuity with respect to the Floquet-Bloch variable. This allows for instance to consider the scattering problem at a fixed Floquet-Bloch mode. We first provide the theoretical justification of the so-called Generalized Linear Sampling Method (GLSM) [6, 7] for quasi-periodic incident waves. We remark that although a classical factorization of the near field operator can be obtained in this case, we are not able to apply the abstract framework of the factorization method as introduced in [37]. This is why for the GLSM method seems to be more adapted and this is why the penalty term that we use in our theory is different from the one used in the literature [7, 11]. For the justification of the method we assume that the local perturbation does not intersect the periodic background. The case where this intersection is not empty requires the study of an interior transmission problem that has a non standard structure similar to the one considered in [14]. For the sake of conciseness we leave this to future investigations.

We also provide a justification of a GLSM method using the whole near field operator associated with point sources. This method needs in particular a specific result related

to the denseness of a single layer operator in the space of solutions to the Helmholtz equation that have continuous dependence with respect to the Floquet-Bloch variable. This is what mainly justifies the consideration of the scattering problem in half plane with Dirichlet boundary condition at the interface. Our analysis also assumes that the periodic index of refraction has a positive imaginary part in at least some open domain of the periodic background. We believe that this assumption can be removed using the analysis of the direct problem as in [35, 38]. Considering this case will be subject of future work.

The chapter is organized as follows. Section 3.2 is dedicated to the introduction of the direct problems (with point source incident waves or quasi-periodic point source incident waves). In Section 3.3, we study the GLSM method for quasi-periodic incident waves. Section 3.4 is dedicated to the analysis of the GLSM for non quasi-periodic incident waves. We end the chapter with some numerical examples for data associated with a single Floquet-Bloch mode.

## 3.2 Setting of the direct problem

In this section, we introduce the direct scattering problem for a locally perturbed two dimensional periodic medium and the corresponding quasi-periodic problems.

### 3.2.1 The locally perturbed periodic scattering problem

Let  $U^0$  be the upper half-space  $\mathbb{R} \times \mathbb{R}_+$  in  $\mathbb{R}^2$ . We set  $\Omega^R := \mathbb{R} \times [0, R]$  the domain delimited by  $\Gamma^0 := \mathbb{R} \times \{0\}$  and  $\Gamma^R := \mathbb{R} \times \{R\}$ , with  $R > R_0 > 0$  as shown in Figure 3.1. Let  $n_p \in L^\infty(U^0)$  be the refractive index with non negative imaginary part,  $2\pi$ -periodic with respect to the first component  $x_1$  such that  $n_p = 1$  outside a  $2\pi$  periodic domain  $D^p$  included in  $\Omega^R$ .

We consider  $D := D^p \cup \tilde{D}$  where  $\tilde{D}$  is a bounded domain included in  $\Omega_0^R := [0, 2\pi] \times [0, R]$ . We assume that the complement of  $D$  in  $\mathbb{R}^2$  is connected. Let  $n \in L^\infty(U^0)$  be the perturbed refractive index with non negative imaginary part verifying  $n = n_p$  outside  $\tilde{D}$ .

Consider an incident field  $v \in L^2(D)$ . The direct scattering problem we are interested

in can be formulated as: seek a scattered field  $w \in H_{loc}^2(\Omega^R)$  verifying

$$(\mathcal{P}) \begin{cases} \Delta w + k^2 n w = k^2(1-n)v & \text{in } \Omega^R, \\ w = 0 & \text{on } \Gamma^0, \\ \frac{\partial w}{\partial x_2}(\cdot, R) = T^R(w|_{\Gamma^R}) & \text{on } \Gamma^R, \end{cases} \quad (3.1)$$

where  $T^R : H^{1/2}(\Gamma^R) \longrightarrow H^{-1/2}(\Gamma^R)$  is exterior Dirichlet-to-Neumann map defined by

$$T^R(\varphi) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - |\xi|^2} e^{ix_1 \cdot \xi} \widehat{\varphi}(\xi) d\xi, \quad (3.2)$$

with  $\widehat{\varphi}$  is the Fourier transform defined as  $\widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_1} \varphi(x_1, R) dx_1$  for  $L^1$  functions on  $\Gamma^R$ .

**Assumption 15.** Assume that in addition to the assumptions above, the set  $\{\Im m(n_p) > 0\}$  contains a non empty open set  $\mathcal{O}$  and  $\Im m(n - n_p) \geq 0$ .

Under this assumption the above stated direct scattering problem has been studied in Chapter 2 and we summarize here the main theorem.

**Theorem 16.** If assumption 15 holds then there exists a unique solution  $w \in H_{loc}^2(\Omega^R) \cap H^1(\Omega^R)$  satisfying  $(\mathcal{P})$  and continuously depend on  $v \in L^2(D)$ .

**Remark 17.** Given the solution  $w$  to problem  $(\mathcal{P})$  we extend  $w$  for  $|x_2| \geq R$  by

$$w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix \cdot \xi + i\sqrt{k^2 - |\xi|^2}(x_2 - R)} \widehat{w}(\xi, R) d\xi, \quad \text{for } x_2 > R. \quad (3.3)$$

This provides the solution satisfying

$$\Delta w + k^2 n w = k^2(1-n)v \quad \text{in } U^0. \quad (3.4)$$

The scattering problem  $(\mathcal{P})$  can be equivalently formulated as (3.3)-(3.4) and the boundary conditions on  $\Gamma^0$ .

Let  $\Phi(\cdot, y)$  be the fundamental solution of the homogeneous problem associated with  $(\mathcal{P})$  given by

$$\Phi(\cdot, y) := \frac{i}{4} \left[ H_0^{(1)}(k|\cdot - y|) - H_0^{(1)}(k|\cdot - y'|) \right], \quad (3.5)$$

where  $y := (y_1, y_2)$ ,  $y' := (y_1, -y_2)$  and  $y_2 > 0$ .

**Theorem 18.** *For  $v \in L^2(D)$ , the solution  $w \in H_{loc}^2(\Omega^R)$  of problem  $(\mathcal{P})$  can be represented as*

$$w(x) = k^2 \int_D \Phi(x, y)(n-1)(w+v)(y)dy. \quad (3.6)$$

*Proof.* Consider  $N \in \mathbb{N}$  sufficiently large and let  $D^N := ([-N, N] \times \mathbb{R}_+) \cap D$ . Let us define a cut-off function  $\chi_N \in C^\infty(U^0)$  such that  $\chi_N(y) = 1$  for  $y \in D^N$  and  $\chi_N(y) = 0$  for  $y \notin D^{N+1}$  and  $\chi_N(y)$  only depends on  $y_1$ . We set

$$w_N := k^2 \int_D \Phi(x, y)(n-1)(w+v)\chi_N(y)dy \quad \text{for } x \in U^0,$$

with  $w \in H_{loc}^2(\Omega^R)$  being the solution of  $(\mathcal{P})$ . Using the properties of the volume potentials [21] we deduce that  $w_N \in H_{loc}^2(\Omega^R)$  satisfies

$$\begin{cases} \Delta w_N + k^2 w_N = k^2(1-n)(w+v)\chi_N(y) & \text{in } \Omega^R, \\ w_N = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_N}{\partial x_2}(\cdot, R) = T^R(w_N|_{\Gamma^R}) & \text{on } \Gamma^R. \end{cases} \quad (3.7)$$

We set  $u_N := w - w_N$ , then  $u_N \in H_{loc}^2(\Omega^R)$  satisfies (3.7) with the right hand side of the first equation is replaced by  $k^2(1-n)(w+v)(1-\chi_N)(y)$ . Since  $(1-n)(w+v) \in L^2(\Omega^R)$ , then

$$k^2(1-n)(w+v)(1-\chi_N) \xrightarrow{N \rightarrow \infty} 0 \quad \text{in } L^2(\Omega^R).$$

Hence  $\lim_{N \rightarrow \infty} u_N = 0$  in  $H_{loc}^2(\Omega^R)$ . On the other hand, since  $(n-1)(w+v)(y) \in L^2(\Omega^R)$  and  $\Phi(\cdot, y) \in L^2(\Omega^R)$  [17], then we have

$$\lim_{N \rightarrow \infty} w_N(x) = k^2 \int_D \Phi(x, y)(n-1)(w+v)(y)dy,$$

almost everywhere in  $\Omega^R$  by Lebesgue's dominated convergence theorem. Therefore  $w \in H_{loc}^2(\Omega^R)$  satisfies (3.6) by the uniqueness of the limit.  $\square$

For the study of the inverse problem we shall consider quasi-periodic solutions obtained by applying the Floquet-Bloch transform to the solution of  $(\mathcal{P})$ . We will need to restrict

the set of admissible solutions to those with some continuity property with respect to the Floquet-Bloch variable. This is the subject of the following subsection.

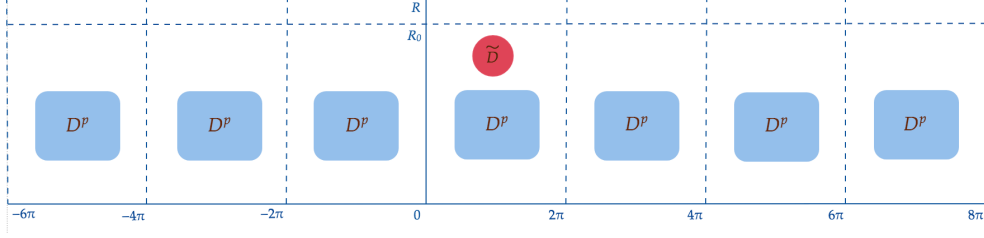


FIGURE 3.1: Sketch of the domain

### 3.2.2 The quasi-periodic scattering problem

A function  $u$  is called  $\xi$ -quasi-periodic with period  $2\pi$  for some  $\xi \in \mathbb{R}$  if it verifies

$$u(x_1 + 2\pi j, x_2) = e^{i2\pi\xi j} u(x_1, x_2) \quad \text{for all } j \in \mathbb{Z}, \quad (3.8)$$

and  $(x_1, x_2) \in \mathbb{R}^2$ . In the sequel we shall skip indicating the periodicity length  $2\pi$  since it is kept fixed and periodicity or quasi-periodicity only apply to the first variable  $x_1$ .

In the following we denote by  $L_\xi^2(\Omega^R)$  the set of  $\xi$ -quasi periodic functions in  $L_{loc}^2(\Omega^R)$  and by  $H_\xi^m(\Omega^R)$  the set of  $\xi$ -quasi periodic functions in  $H_{loc}^m(\overline{\Omega^R})$ . For  $m \geq 1$  we denote by  $\check{H}_\xi^m(\Omega^R)$  the subspace of functions in  $H_\xi^m(\Omega^R)$  that vanish on  $\Gamma^0$ . We define  $H_\xi^1(\Omega_0^R)$  as the restriction to  $\Omega_0^R$  of functions in  $H_\xi^1(\Omega^R)$ . In order to avoid notation confusion we denote by  $H_\#^1(\Omega_0^R)$  the space  $H_\xi^1(\Omega_0^R)$  for  $\xi = 0$ . We finally define  $H_\xi^s(\Gamma_0^R)$  to be the restriction to  $\Gamma_0^R$  of  $\xi$ -quasi periodic functions in  $H_{loc}^s(\Gamma^R)$ .

Let  $\xi \in I := [0, 1]$  and  $\Gamma_0^R := [0, 2\pi] \times \{R\}$ . Consider an incident field  $v_\xi \in L_\xi^2(\Omega^R)$ , the quasi-periodic direct scattering problem is formulated as: seek a scattered field  $w_\xi \in H_\xi^1(\Omega^R)$  verifying

$$\begin{cases} \Delta w_\xi + k^2 n_p w_\xi = -k^2 (n_p - 1) v_\xi & \text{in } \Omega^R, \\ w_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_\xi}{\partial x_2}(\cdot, R) = T_\xi^R(w_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases} \quad (3.9)$$

where  $T_\xi^R : H_\xi^{1/2}(\Gamma_0^R) \longrightarrow H_\xi^{-1/2}(\Gamma_0^R)$  is the exterior quasi-periodic Dirichlet-to-Neumann map defined by

$$T_\xi^R(\varphi)(x_1) = i \sum_{j \in \mathbb{Z}} \beta_\xi(j) \hat{\varphi}_\xi(j) e^{i\alpha_\xi(j) \cdot x_1}, \quad (3.10)$$

where

$$\alpha_\xi(j) := \xi + j, \quad \beta_\xi(j) := \sqrt{k^2 - |\xi + j|^2}, \quad \Im m(\beta_\xi(j)) \geq 0, \quad \text{for } j \in \mathbb{Z},$$

and  $\hat{\varphi}_\xi(j)$  is the  $j$ -th Fourier coefficient of  $e^{-i\xi x_1} \varphi(x_1, R)$  defined as

$$\hat{\varphi}_\xi(j) := \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha_\xi(j)x_1} \varphi(x_1, R) dx_1.$$

For the norm in  $H_\xi^s(\Gamma_0^R)$  we shall use the following definition

$$\|\varphi\|_{H_\xi^s(\Gamma_0^R)}^2 = \sum_{j \in \mathbb{Z}} (1 + j^2)^s |\hat{\varphi}_\xi(j)|^2.$$

Multiplying the first equation of (3.9) with  $\overline{\psi}_\xi \in \check{H}_\xi^1(\Omega_0^R)$ , integrating by parts and using the boundary conditions and the quasi periodicity we obtain the variational formulation given as

$$\int_{\Omega_0^R} (\nabla w_\xi \cdot \nabla \overline{\psi}_\xi - k^2 n_p w_\xi \overline{\psi}_\xi) dx - \langle T_\xi^R w_\xi, \psi_\xi \rangle_{\Gamma_0^R} = k^2 \int_{\Omega_0^R} (n_p - 1) v_\xi \overline{\psi}_\xi, \quad \forall \psi_\xi \in \check{H}_\xi^1(\Omega_0^R). \quad (3.11)$$

where the notation  $\langle \cdot, \cdot \rangle_{\Gamma_0^R}$  refers to the  $H^{-1/2}(\Gamma_0^R) - H^{1/2}(\Gamma_0^R)$  duality product. Using the Riesz representation theorem we can define the operator  $A_\xi : \check{H}_\xi^1(\Omega_0^R) \longrightarrow \check{H}_\xi^1(\Omega_0^R)$  such that

$$(A_\xi w_\xi, v_\xi)_{\check{H}_\xi^1(\Omega_0^R)} := \int_{\Omega_0^R} (\nabla w_\xi \cdot \nabla \overline{\psi}_\xi - k^2 n_p w_\xi \overline{\psi}_\xi) dx - \langle T_\xi^R w_\xi, \psi_\xi \rangle_{\Gamma_0^R} \quad \forall \psi_\xi, w_\xi \in \check{H}_\xi^1(\Omega_0^R).$$

We recall here the result proved in Chapter 2 (Theorems 5 and 6).

**Theorem 19.** *Assume that  $\{\Im m(n_p) > 0\}$  in a non empty open set of  $\Omega_0^R$ , then problem (3.9) is well posed. Moreover,  $\|A_\xi^{-1}\| \leq c$  with  $c$  is a constant independent of  $\xi$ .*



**Remark 20.** Given the solution  $w_\xi$  to problem (3.9) we extend  $w_\xi$  for  $|x_2| \geq R$  by

$$w_\xi(x) := \sum_{j \in \mathbb{Z}} \widehat{(w_\xi|_{\Gamma_0^R})}(j) e^{i\alpha_\xi(j) \cdot x_1 + i\beta_\xi(j)(x_2 - R)} \quad \text{for } x_2 > R. \quad (3.12)$$

This provides the solution satisfying

$$\Delta w_\xi + k^2 n_p w_\xi = k^2 (1 - n_p) v_\xi \quad \text{in } U^0. \quad (3.13)$$

The scattering problem (3.9) can be equivalently formulated as (3.12)-(3.13) and the boundary conditions on  $\Gamma^0$ .

Defining now for  $\phi \in C_0^\infty(U^0)$  the one dimensional Floquet-Bloch transform as the following

$$\mathcal{J}\phi(\xi, x_1, x_2) = \sum_{j \in \mathbb{Z}} \phi(x_1 + 2\pi j, x_2) e^{-i2\pi\xi \cdot j}, \quad \xi \in I, (x_1, x_2) \in U^0. \quad (3.14)$$

Recall that the Floquet-Bloch transform is an isomorphism between  $H^s(\Omega^R)$  (respectively  $H^s(\Gamma^R)$ ) and  $L^2(I, H_\xi^s(\Omega_0^R))$  (respectively  $L^2(I, H_\xi^s(\Gamma_0^R))$ ). Then, for  $0 \leq \alpha < 1$ , we denote by

$$C_\#^{0,\alpha}(I, H_\xi^s(\Omega_0^R)) := \{\varphi \in L^2(I, H_\xi^s(\Omega_0^R)); e^{-i\xi \cdot x_1} \varphi \in C_\#^{0,\alpha}(I, H_\xi^s(\Omega_0^R))\},$$

the space of periodic and  $\alpha$  Hölderian functions on  $I$  with values in  $H_\xi^s(\Omega_0^R)$ . The norm of  $\varphi \in C_\#^{0,\alpha}(I, H_\xi^s(\Omega_0^R))$  is defined as

$$\sup_{\xi \in I} \|\varphi(\xi, \cdot)\|_{H^s(\Omega_0^R)} + \sup_{\xi_1 \neq \xi_2 \in I} \left( \frac{\|\tilde{\varphi}(\xi_1, \cdot) - \tilde{\varphi}(\xi_2, \cdot)\|_{H^s(\Omega_0^R)}}{|\xi_1 - \xi_2|^\alpha} \right),$$

with  $\tilde{\varphi} := e^{-i\xi \cdot x_1} \varphi$ . We then set

$$\tilde{H}^{s,\alpha}(\Omega^R) := \left\{ u \in H^s(\Omega^R) / \mathcal{J}u \in C_\#^{0,\alpha}(I, H_\xi^s(\Omega_0^R)) \right\}, \quad (3.15)$$

$$\tilde{H}^{s,\alpha}(\Gamma^R) := \left\{ u \in H^s(\Gamma^R) / \mathcal{J}u \in C_\#^{0,\alpha}(I, H_\xi^s(\Gamma_0^R)) \right\}. \quad (3.16)$$

Then we have the following theorem complementing the result of Theorem 19.

**Theorem 21.** Assume that hypothesis of Theorem 19 holds and consider  $v \in \tilde{L}^{2,\alpha}(\Omega^R)$  for  $0 \leq \alpha < 1$ . Let  $w_\xi \in H_\xi^1(\Omega_0^R)$  be the solution of (3.9) with  $v_\xi = (\mathcal{J}v)(\xi, \cdot)$ . Then  $w = \mathcal{J}^{-1}w_\xi := \int_I w_\xi d\xi$  belongs to  $\tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$  with  $\tilde{\alpha} = \min(\alpha, \frac{1}{2})$  and

$$\|w\|_{\tilde{H}^{1,\tilde{\alpha}}(\Omega^R)} \leq c\|v\|_{\tilde{L}^{2,\alpha}(D^p)}$$

with  $c$  independent from  $v$ .

*Proof.* Set  $\tilde{w}_\xi := e^{-i\xi \cdot x_1} w_\xi$  and  $\check{v}_\xi = -k^2(n_p - 1)v_\xi$ . Then we have  $\tilde{w}_\xi \in H_\#^1(\Omega_0^R)$  and verifies

$$\Delta \tilde{w}_\xi = e^{-i\xi \cdot x_1} \check{v}_\xi - 2i\xi \frac{\partial \tilde{w}_\xi}{\partial x_1} + \xi^2 \tilde{w}_\xi - k^2 n_p \tilde{w}_\xi \quad \text{in } \Omega^R.$$

Let  $\xi_1, \xi_2 \in I$ , and set  $e := e^{i\xi_1 \cdot x_1}(\tilde{w}_{\xi_1} - \tilde{w}_{\xi_2})$ . Then  $e \in H_{\xi_1}^1(\Omega_0^R)$  and

$$\Delta e + k^2 n_p e = L_{\xi_1, \xi_2} \quad \text{in } \Omega^R, \quad (3.17)$$

with

$$L_{\xi_1, \xi_2} := e^{i\xi_1 \cdot x_1} \left[ (e^{-i\xi_1 \cdot x_1} - e^{-i\xi_2 \cdot x_1}) v_{\xi_1} + e^{-i\xi_2 \cdot x_1} (\check{v}_{\xi_1} - \check{v}_{\xi_2}) - 2i(\xi_1 - \xi_2) \frac{\partial \tilde{w}_{\xi_2}}{\partial x_1} + (\xi_1^2 - \xi_2^2) \tilde{w}_{\xi_2} \right].$$

By the Cauchy-Schwartz inequality and  $\check{v} \in L^{2,\alpha}(\Omega_0^R)$  we get the existence of a constant  $c > 0$  independent from  $\xi_1$  and  $\xi_2$  such that

$$\|L_{\xi_1, \xi_2}\|_{L^2(\Omega_0^R)} \leq c|\xi_1 - \xi_2|^\alpha \left( \|v\|_{\tilde{L}^{2,\alpha}(D^p)} + \|w_{\xi_2}\|_{H^1(\Omega_0^R)} \right). \quad (3.18)$$

On the other hand, using the third equation in (3.9) we have on  $\Gamma^R$

$$\frac{\partial e}{\partial x_2} = T_{\xi_1}^R(w_{\xi_1}) - e^{i(\xi_1 - \xi_2)x_1} T_{\xi_2}^R(w_{\xi_2}) = T_{\xi_1}^R(e) + i \sum_{j \in \mathbb{Z}} (\beta_{\xi_1}(j) - \beta_{\xi_2}(j)) \hat{w}_{\xi_2}(j) e^{i\alpha_{\xi_1}(j)}. \quad (3.19)$$

The variational formulation of (3.17)-(3.19) can be written as

$$\int_{\Omega_0^R} (\nabla e \cdot \nabla \bar{\psi} - k^2 n_p e \bar{\psi}) - \langle T_{\xi_1}^R e, \psi \rangle_{\Gamma_0^R} = \int_{\Omega_0^R} L_{\xi_1, \xi_2} \bar{\psi} dx - \langle g_{\xi_1, \xi_2}, \psi \rangle_{\Gamma_0^R}, \quad \forall \psi \in \check{H}_{\xi_1}^1(\Omega_0^R), \quad (3.20)$$

with  $g_{\xi_1, \xi_2} := \sum_{j \in \mathbb{Z}} (\beta_{\xi_1}(j) - \beta_{\xi_2}(j)) \widehat{w}_{\xi_2}(j) e^{i\alpha_{\xi_1}(j)}$ . Using the definition of  $H^{-1/2}(\Gamma_0^R)$  norm in terms of Fourier coefficients we get

$$\|g_{\xi_1, \xi_2}(x_1)\|_{H_{\xi_1}^{-1/2}(\Gamma_0^R)}^2 = \sum_{j \in \mathbb{Z}} (1 + |j|^2)^{1/2} |\widehat{w}_{\xi_2}(j)|^2 C_{\xi_1, \xi_2}^2(j),$$

with  $C_{\xi_1, \xi_2}(j) := \frac{|\beta_{\xi_1}(j) - \beta_{\xi_2}(j)|}{(1 + |j|^2)^{1/2}}$ . For  $j \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$  such that  $\beta_{\xi_1}(j) = 0$ , i.e  $k^2 = |\xi_1 + j|^2$

$$C_{\xi_1, \xi_2}(j) = \frac{||j + \xi_1|^2 - |j + \xi_2|^2|^{1/2}}{(1 + |j|^2)^{1/2}} \leq (k + 3)^{1/2} |\xi_1 - \xi_2|^{1/2},$$

since  $|j| \leq |j + \xi_1| + |\xi_1| \leq k + 1$ . For the case  $j = 0$ , if  $(k^2 - \xi_2^2)(k^2 - \xi_1^2) \leq 0$

$$C_{\xi_1, \xi_2}(0) = |\sqrt{k^2 - \xi_2^2} - i\sqrt{\xi_1^2 - k^2}| = |\xi_1^2 - \xi_2^2|^{1/2} \leq 2|\xi_1 - \xi_2|^{1/2},$$

while if  $(k^2 - \xi_2^2)(k^2 - \xi_1^2) > 0$

$$C_{\xi_1, \xi_2}(0) = \frac{|\xi_1^2 - \xi_2^2|}{|\sqrt{k^2 - \xi_2^2} + \sqrt{k^2 - \xi_1^2}|^{1/2}} \leq \sqrt{2} |\xi_1 - \xi_2|^{1/2}.$$

Finally for the case where  $k^2 \neq |\xi_1 + j|^2$  and  $j \neq 0$  we write

$$C_{\xi_1, \xi_2}(j) = \frac{||\xi_2 + j|^2 - |\xi_1 + j|^2|}{(1 + |j|^2)^{1/2} |\sqrt{k^2 - |\xi_1 + j|^2} + \sqrt{k^2 - |\xi_2 + j|^2}|}.$$

Since  $|\beta_{\xi_1}(j)| > 0$ , then there exists  $\delta > 0$  independent of  $j$  such that  $|\beta_{\xi_1}(j) + \beta_{\xi_2}(j)| \geq \delta$ , therefore

$$\sup_{j \in \mathbb{Z}^*, k^2 \neq |\xi_1 + j|^2} C_{\xi_1, \xi_2}(j) \leq \sup_{j \in \mathbb{Z}^*, k^2 \neq |\xi_1 + j|^2} \frac{|\xi_2 - \xi_1| |\xi_2 + \xi_1 + 2j|}{|j| \delta} \leq \frac{4}{\delta} |\xi_2 - \xi_1|.$$

Summarizing, there exist a constant  $\beta$  independent from  $j$ ,  $\xi_1$  and  $\xi_2$  such that

$$\sup_{j \in \mathbb{Z}} C_{\xi_1, \xi_2}(j) \leq \beta |\xi_2 - \xi_1|^{1/2}.$$

Consequently

$$\|g_{\xi_1, \xi_2}\|_{H_{\xi_1}^{-1/2}(\Gamma_0^R)}^2 \leq \beta^2 |\xi_1 - \xi_2| \|w_{\xi_2}\|_{H_{\xi_2}^{1/2}(\Gamma_0^R)}^2. \quad (3.21)$$

We observe that

$$\|w_{\xi_2}\|_{H_{\xi_2}^{1/2}(\Gamma_0^R)} \leq c \|e^{-i\xi_2 \cdot x_1} w_{\xi_2}\|_{H_{\sharp}^1(\Omega_0^R)} \leq 2c \|w_{\xi_2}\|_{H^1(\Omega_0^R)},$$

with  $c$  being the continuity constant of the trace operator on  $H_{\sharp}^1(\Omega_0^R)$ . From Theorem 19 we have that

$$\|w_{\xi_2}\|_{H^1(\Omega_0^R)} \leq \tilde{c} \|v_{\xi_2}\|_{L^2(\Omega_0^R)} \leq \|v\|_{L^{2,\alpha}(D^p)}, \quad (3.22)$$

where  $\tilde{c}$  is independent of  $\xi_2$ . Applying Theorem 19 to the variational formulation (3.20) proves, using (3.18), (3.21) and (3.22), the existence of a constant  $c$  independent of  $\xi_1$  and  $\xi_2$  such that

$$\|e\|_{H^1(\Omega_0^R)} \leq c |\xi_1 - \xi_2|^{\tilde{\alpha}} \|v\|_{L^{2,\alpha}(D^p)},$$

which ends the proof.  $\square$

**Theorem 22.** *Assume that hypothesis 15 holds and that  $v \in \tilde{L}^{2,\alpha}(\Omega^R)$  for  $0 \leq \alpha < 1$ . Then the solution  $w \in H_{loc}^2(\Omega^R)$  of problem (P) belongs to  $\tilde{H}^{1,\alpha}(\Omega^R)$  and*

$$\|w\|_{\tilde{H}^{1,\alpha}(\Omega^R)} \leq c (\|v\|_{\tilde{L}^{2,\alpha}(D^p)} + \|v\|_{L^2(\bar{D})}),$$

with  $c$  independent from  $v$ .

*Proof.* Let  $w_{\xi} := (\mathcal{J}w)(\xi, \cdot)$ . Since the support of  $n - n_p$  is included in  $\Omega_0^R$ , then we have  $w_{\xi} \in H_{\xi}^1(\Omega^R)$  and satisfies

$$\begin{cases} \Delta w_{\xi} + k^2 n_p w_{\xi} = -k^2 (n_p - 1) v_{\xi} - k^2 \mathcal{R}_{\xi}((n - n_p)(w + v)) & \text{in } \Omega^R, \\ w_{\xi} = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_{\xi}}{\partial x_2}(\cdot, R) = T_{\xi}^R(w_{\xi}|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases}$$

with  $v_{\xi} := (\mathcal{J}v)(\xi, \cdot)$  and  $\mathcal{R}_{\xi}(\varphi)$  for a function  $\varphi$  compactly supported in  $\Omega_0^R$  denotes the extension by  $\xi$ -quasi periodicity of  $\varphi$  to all of  $\Omega^R$ . Indeed  $\mathcal{R}_{\xi}(\varphi) = \mathcal{J}(\varphi \chi_{\Omega_0^R})$  where  $\chi_{\Omega_0^R}$  indicates the indicator function of the domain  $\Omega_0^R$ . We decompose  $w_{\xi}$  as

$$w_{\xi} = w_{\xi}^p + \tilde{w}_{\xi},$$

with  $w_\xi^p$  being the solution of (3.9) and  $\check{w}_\xi := w_\xi - w_\xi^p$  satisfying

$$\begin{cases} \Delta \check{w}_\xi + k^2 n_p \check{w}_\xi = -k^2 \mathcal{R}_\xi((n - n_p)(w + v)) & \text{in } \Omega^R, \\ \check{w}_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial \check{w}_\xi}{\partial x_2}(\cdot, R) = T_\xi^R(\check{w}_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R. \end{cases}$$

Denoting by  $w^p := \mathcal{J}^{-1}w_\xi^p$ . From Theorem 21 we have that  $w^p \in \tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$ . Moreover, since  $\mathcal{R}_\xi((n - n_p)(w + v)) \in C_\#^{0,\alpha}(I, L_\xi^2(\Omega_0^R))$ , then using Theorem 21 we deduce that  $\check{w} := \mathcal{J}^{-1}\check{w}_\xi \in \tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$  and then  $w = w^p + \check{w} \in \tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$ . The estimate follows also from application of Theorem 21.  $\square$

For the sake of studying the inverse problem, for fixed  $\xi_0 \in I$ , we consider  $v_{\xi_0} \in L_{\xi_0}^2(D)$ , where

$$L_{\xi_0}^2(D) := \{v \in L_{loc}^2(D) / v|_{D^p} \in L_{\xi_0}^2(D^p)\},$$

where  $L_{\xi_0}^2(D^p)$  denotes the set of  $\xi_0$  quasi-periodic functions that are in  $L_{loc}^2(D^p)$ . We would like to define a solution  $w_{\xi_0}$  to problem (P) associated with  $v = v_{\xi_0}$ . Indeed, since  $v_{\xi_0} \notin L^2(D)$ , the solution can not be defined as in Theorem 16. We rather define the solution in this case as

$$w_{\xi_0} := w_{\xi_0}^p + \tilde{w}_{\xi_0}, \quad (3.23)$$

with  $w_{\xi_0}^p \in H_{\xi_0}^1(\Omega_0^R)$  verifying

$$\begin{cases} \Delta w_{\xi_0}^p + k^2 n_p w_{\xi_0}^p = k^2(1 - n_p)v_{\xi_0} & \text{in } \Omega^R, \\ w_{\xi_0}^p = 0 & \text{on } \Gamma_0^R, \\ \frac{\partial w_{\xi_0}^p}{\partial x_2}(\cdot, R) = T_\xi^R(w_{\xi_0}^p|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases} \quad (3.24)$$

and  $\tilde{w}_{\xi_0} \in \tilde{H}^1(\Omega^R)$  satisfying

$$\begin{cases} \Delta \tilde{w}_{\xi_0} + k^2 n \tilde{w}_{\xi_0} = k^2(n_p - n)(v_{\xi_0} + w_{\xi_0}^p) & \text{in } \Omega^R, \\ \tilde{w}_{\xi_0} = 0 & \text{on } \Gamma^0, \\ \frac{\partial \tilde{w}_{\xi_0}}{\partial x_2}(\cdot, R) = T^R(\tilde{w}_{\xi_0}|_{\Gamma^R}) & \text{on } \Gamma^R. \end{cases} \quad (3.25)$$

The solutions of (3.24) and (3.25) are respectively defined by Theorems 19 and 22. Multiplying the first equation of (3.25) with  $\bar{\psi} \in \tilde{H}^1(\Omega^R)$ , integrating by parts and using the boundary conditions we obtain the variational formulation given as

$$\int_{\Omega^R} (\nabla \tilde{w}_{\xi_0} \cdot \nabla \bar{\psi} - k^2 n \tilde{w}_{\xi_0} \bar{\psi}) dx - \langle T^R \tilde{w}_{\xi_0}, \psi \rangle_{\Gamma^R} = k^2 \int_{\Omega^R} (n - n_p)(v_{\xi_0} + w_{\xi_0}^p) \bar{\psi}, \quad \forall \psi \in \tilde{H}^1(\Omega^R), \quad (3.26)$$

where the notation  $\langle \cdot, \cdot \rangle_{\Gamma^R}$  refers to the  $H^{-1/2}(\Gamma^R) - H^{1/2}(\Gamma^R)$  duality product.

**Remark 23.** Let  $v \in \tilde{L}^2(\Omega^R)$  and set  $v_\xi = \mathcal{J}(v)(\xi, \cdot)$ . We define  $w_\xi^p \in H_\xi^1(\Omega_0^R)$  verifying (3.24) and  $\tilde{w}_\xi \in \tilde{H}^1(\Omega^R)$  the solution of (3.25). Then  $w^p := \int_I w_\xi^p d\xi$  is solution to (P) with  $n = n_p$  and  $\tilde{w} := \int_I \tilde{w}_\xi d\xi \in \tilde{H}^1(\Omega^R)$  is solution of

$$\begin{cases} \Delta \tilde{w} + k^2 n_p \tilde{w} = k^2 (n_p - n)(w^p + v + \tilde{w}) & \text{in } \Omega^R, \\ \tilde{w} = 0 & \text{on } \Gamma^0, \\ \frac{\partial \tilde{w}}{\partial x_2}(\cdot, R) = T^R(\tilde{w}|_{\Gamma^R}) & \text{on } \Gamma^R. \end{cases} \quad (3.27)$$

Consequently  $\tilde{w} + w^p \in \tilde{H}^1(\Omega^R)$  and is the solution of problem (P).

### 3.3 The inverse problem for quasi-periodic incident fields

#### 3.3.1 Setting for the inverse problem

Consider  $\xi_0 \in I$  fixed, and let  $\Phi_{\xi_0}(x, y) := (\mathcal{J}\Phi(\cdot, y))(\xi_0, x)$  be the  $\xi_0$ -quasi-periodic Green function having the following expression [45]

$$\Phi_{\xi_0}(x, y) := \frac{i}{4\pi} \sum_{j \in \mathbb{Z}} e^{i\alpha_{\xi_0}(j)(x_1 - y_1)} \theta_{\xi_0}(j, x_2, y_2), \quad y_2 < x_2, \quad (3.28)$$

with

$$\theta_{\xi_0}(j, x_2, y_2) := e^{i\beta_{\xi_0}(j)x_2} \left[ \frac{e^{-i\beta_{\xi_0}(j)y_2} - e^{i\beta_{\xi_0}(j)y_2}}{\beta_{\xi_0}(j)} \right]. \quad (3.29)$$

Let  $y \in \Gamma_0^R$ . We define  $u_{\xi_0}^s(\cdot, y) = w_{\xi_0}$  given by (3.23) with  $v_{\xi_0} = \overline{\Phi_{\xi_0}(y, \cdot)} \in L_{\xi_0}^2(D)$ . From (3.23) we decompose  $u_{\xi_0}^s(\cdot, y) = u_{\xi_0}^{s,p}(\cdot, y) + \tilde{u}_{\xi_0}^s(\cdot, y)$  with  $u_{\xi_0}^{s,p}(\cdot, y) = w_{\xi_0}^p$  solution of (3.24) and  $\tilde{u}_{\xi_0}^s(\cdot, y) = \tilde{w}_{\xi_0}$  solution of (3.25). We introduce the  $\xi_0$ -quasi periodic near

field operator  $N_{\xi_0} : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(\Gamma^R)$  as

$$N_{\xi_0} g_{\xi_0}(x) := \int_{\Gamma_0^R} g_{\xi_0}(y) u_{\xi_0}^{s,p}(x, y) ds(y) + \int_{\Gamma_0^R} g_{\xi_0}(y) \mathcal{J}(\tilde{u}_{\xi_0}^s(\cdot, y))(\xi_0, x) ds(y). \quad (3.30)$$

Define  $S_{\xi_0} : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(D)$  as

$$S_{\xi_0} g_{\xi_0}(x) := \int_{\Gamma_0^R} g_{\xi_0}(y) \overline{\Phi_{\xi_0}(y, x)} ds(y). \quad (3.31)$$

Then, obviously the operator  $N_{\xi_0}$  can be decomposed as

$$N_{\xi_0} = G_{\xi_0}(S_{\xi_0}), \quad (3.32)$$

where  $G_{\xi_0} : L_{\xi_0}^2(D) \longrightarrow L_{\xi_0}^2(\Gamma^R)$  is the operator defined by

$$G_{\xi_0}(v_{\xi_0}) = (w_{\xi_0}^p + \tilde{w}_{\xi_0}^p)|_{\Gamma_0^R}, \quad (3.33)$$

with  $w_{\xi_0}^p$  being the solution of (3.24) and  $\tilde{w}_{\xi_0}^p = \mathcal{J}(\tilde{w}_{\xi_0})(\xi_0, \cdot)$  with  $\tilde{w}_{\xi_0}$  is the solution of (3.25). We observe that  $\tilde{w}_{\xi_0}^p \in H_{\xi_0}^1(\Omega_0^R)$  and verifies

$$\begin{cases} \Delta \tilde{w}_{\xi_0}^p + k^2 n_p \tilde{w}_{\xi_0}^p = k^2 (n_p - n)(w_{\xi_0}^p + v_{\xi_0} + \tilde{w}_{\xi_0}) & \text{in } \Omega^R, \\ \tilde{w}_{\xi_0}^p = 0 & \text{on } \Gamma_0^R, \\ \frac{\partial \tilde{w}_{\xi_0}^p}{\partial x_2}(\cdot, R) = T_{\xi}^R(\tilde{w}_{\xi_0}^p|_{\Gamma_0^R}) & \text{on } \Gamma_0^R. \end{cases} \quad (3.34)$$

Multiplying the first equation of (3.34) with  $\overline{\psi_{\xi_0}} \in \check{H}_{\xi_0}^1(\Omega_0^R)$ , integrating by parts and using the boundary conditions we obtain the variational formulation given as

$$\int_{\Omega_0^R} \left( \nabla \tilde{w}_{\xi_0}^p \cdot \nabla \overline{\psi_{\xi_0}} - k^2 n_p \tilde{w}_{\xi_0}^p \overline{\psi_{\xi_0}} \right) dx - \left\langle T_{\xi_0}^R \tilde{w}_{\xi_0}^p, \psi_{\xi_0} \right\rangle_{\Gamma_0^R} = k^2 \int_{\Omega_0^R} (n - n_p)(w_{\xi_0}^p + v_{\xi_0} + \tilde{w}_{\xi_0}) \overline{\psi_{\xi_0}}, \quad (3.35)$$

for all  $\psi_{\xi_0} \in \check{H}_{\xi_0}^1(\Omega_0^R)$ . For later use we decompose  $N_{\xi_0} = N_{\xi_0}^p + \tilde{N}_{\xi_0}^p$  where  $N_{\xi_0}^p : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(\Gamma^R)$  and  $\tilde{N}_{\xi_0}^p : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(\Gamma^R)$  are respectively defined as

$$N_{\xi_0}^p g_{\xi_0}(x) := \int_{\Gamma_0^R} g_{\xi_0}(y) u_{\xi_0}^{s,p}(x, y) ds(y), \quad \tilde{N}_{\xi_0}^p g_{\xi_0}(x) := \int_{\Gamma_0^R} g_{\xi_0}(y) \mathcal{J}(\tilde{u}_{\xi_0}^s(\cdot, y))(\xi_0, x) ds(y). \quad (3.36)$$

**Lemma 24.** *The operators  $N_{\xi_0}^p$  and  $\tilde{N}_{\xi_0}^p$  can be respectively factorized as*

$$N_{\xi_0}^p = S_{\xi_0}^* T_{\xi_0}^p S_{\xi_0} \quad \text{and} \quad \tilde{N}_{\xi_0}^p = S_{\xi_0}^* \tilde{T}_{\xi_0}^p S_{\xi_0}, \quad (3.37)$$

with  $T_{\xi_0}^p : L_{\xi_0}^2(D^p) \longrightarrow L_{\xi_0}^2(D^p)$  and  $\tilde{T}_{\xi_0}^p : L_{\xi_0}^2(D) \longrightarrow L_{\xi_0}^2(D)$  are respectively defined by

$$T_{\xi_0}^p v_{\xi_0} = k^2(1 - n_p)(v_{\xi_0} + w_{\xi_0}^p), \quad (3.38)$$

$$\tilde{T}_{\xi_0}^p v_{\xi_0} = k^2(1 - n_p)\tilde{w}_{\xi_0}^p + k^2(n_p - n)(w_{\xi_0}^p + v_{\xi_0} + \tilde{w}_{\xi_0}), \quad (3.39)$$

where  $w_{\xi_0}^p$  being the solution of (3.24) and  $\tilde{w}_{\xi_0}^p = \mathcal{J}(\tilde{w}_{\xi_0})(\xi_0, \cdot)$  with  $\tilde{w}_{\xi_0}$  is the solution of (3.25).

*Proof.* The proof of (3.37) is classical and we here outline the main steps. The solution  $w_{\xi_0}^p$  of (3.24) with  $v_{\xi_0} = S_{\xi_0} g_{\xi_0}$  can be represented as [29]

$$w_{\xi_0}^p(x) = \int_{D_0^p} k^2 \Phi_{\xi_0}(x, y)(1 - n_p)(w_{\xi_0}^p + S_{\xi_0} g_{\xi_0})(y) dy \quad \text{for } x \in \Omega_0^R.$$

Then

$$\begin{aligned} (N_{\xi_0}^p g_{\xi_0}, \tilde{g}_{\xi_0})_{L^2(\Gamma_0^R)} &= \int_{D_0^p} k^2(1 - n_p)(w_{\xi_0}^p + S_{\xi_0} g_{\xi_0})(y) \int_{\Gamma_0^R} \Phi_{\xi_0}(x, y) \overline{\tilde{g}_{\xi_0}(x)} ds(x) dy, \\ &= \int_{D_0^p} k^2(1 - n_p)(w_{\xi_0}^p + S_{\xi_0} g_{\xi_0})(y) \overline{S_{\xi_0} \tilde{g}_{\xi_0}(y)} dy, \\ &= (T_{\xi_0}^p S_{\xi_0} g_{\xi_0}, S_{\xi_0} \tilde{g}_{\xi_0})_{L^2(D_0^p)}, \end{aligned}$$

which proves the first of factorization in (3.37).

The second factorization is obtained in the same way based on the fact that  $w_{\xi_0}^p$  solution of (3.34) can be represented as

$$\tilde{w}_{\xi_0}^p = \int_{D_0^p} k^2 \Phi_{\xi_0}(\cdot, y)(1 - n_p)\tilde{w}_{\xi_0}^p(y) dy + \int_D k^2 \Phi_{\xi_0}(\cdot, y)(n_p - n)(w_{\xi_0}^p + S_{\xi_0} g_{\xi_0} + \tilde{w}_{\xi_0}) dy.$$

□



From Lemma 24 we conclude the following factorization

$$N_{\xi_0} = S_{\xi_0}^* T_{\xi_0} S_{\xi_0}$$

with  $T_{\xi_0} : L_{\xi_0}^2(D) \longrightarrow L_{\xi_0}^2(D)$  is defined by  $T_{\xi_0}(v_{\xi_0}) = T_{\xi_0}^p(v_{\xi_0}|_{D^p}) + \tilde{T}_{\xi_0}^p(v_{\xi_0})$  or equivalently

$$T_{\xi_0} v_{\xi_0} = k^2(1 - n_p)(w_{\xi_0}^p + v_{\xi_0} + \tilde{w}_{\xi_0}^p) + k^2(n_p - n)(w_{\xi_0}^p + v_{\xi_0} + \tilde{w}_{\xi_0}). \quad (3.40)$$

### 3.3.2 Some properties of the operators defined in the previous section

In order to study the inverse problem we need to prove some properties of the operators defined in the previous sections.

**Lemma 25.** *The operator  $S_{\xi_0} : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(D)$  is injective. The closure of its range is*

$$H_{\xi_0}^{inc}(D) := \{v \in L_{\xi_0}^2(D), \Delta v + k^2 v = 0 \text{ in } D\}. \quad (3.41)$$

*Proof.* Let  $g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R)$  such that  $S_{\xi_0} g_{\xi_0} = 0$  in  $D_0$ , where  $D_0 := \Omega_0^R \cap D$ . Using the unique continuation principle we obtain  $S_{\xi_0} g_{\xi_0} = 0$  in  $\Omega_0^R$ . Let  $U^R := \mathbb{R} \times [R, \infty[$ . Using the continuity and regularity of single layer potentials we have that  $S_{\xi_0} \in H_{loc}^2(U^R)$ , is  $\xi_0$ -quasi periodic, verifies

$$\begin{cases} \Delta S_{\xi_0} g_{\xi_0} + k^2 S_{\xi_0} g_{\xi_0} = 0 & \text{in } U^R, \\ S_{\xi_0} g_{\xi_0} = 0 & \text{on } \Gamma^R, \end{cases} \quad (3.42)$$

and the upper going radiation condition (3.12) with  $\Gamma^R$  replaced by  $\Gamma^{R'}$  with  $R' > R$ . The uniqueness for this Dirichlet quasi-periodic scattering problem [39] implies that  $S_{\xi_0} g_{\xi_0} = 0$  in  $U^R$ . Therefore, using the jump relations for the normal derivative of  $S_{\xi_0}$  we obtain  $g_{\xi_0} = 0$  which proves the injectivity of  $S_{\xi_0}$ .

Let  $S_{\xi_0}^* : L_{\xi_0}^2(D) \longrightarrow L_{\xi_0}^2(\Gamma^R)$  be the adjoint of  $S_{\xi_0}$  given by

$$S_{\xi_0}^* v_{\xi_0}(y) := \int_{D_0} \Phi_{\xi_0}(y, x) v_{\xi_0}(x) dx.$$

Let  $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$ , we set

$$w_{\xi_0} := \int_{D_0} \Phi_{\xi_0}(\cdot, x) v_{\xi_0}(x) dx.$$

Using the properties of the volume potential we deduce that  $w_{\xi_0} \in H_{\xi_0}^2(\Omega^R)$  satisfies

$$\begin{cases} \Delta w_{\xi_0} + k^2 w_{\xi_0} = -v_{\xi_0} & \text{in } D_0, \\ \Delta w_{\xi_0} + k^2 w_{\xi_0} = 0 & \text{in } \Omega_0^R \setminus D_0, \\ \frac{\partial w_{\xi_0}}{\partial x_2}(\cdot, R) = T_{\xi}^R(w_{\xi_0}|_{\Gamma_0^R}) & \text{on } \Gamma_0^R. \end{cases} \quad (3.43)$$

Assume that  $w_{\xi_0} = 0$  on  $\Gamma_0^R$ . Then  $w_{\xi_0}$  vanishes in  $U^R$ . Using the unique continuation principle we obtain that  $w_{\xi_0} = 0$  in  $\Omega_0^R \setminus D_0$ . We then have  $w_{\xi_0} \in H_0^2(D_0)$ . Therefore, since  $\Delta w_{\xi_0} + k^2 w_{\xi_0} = 0$  in  $D$ ,

$$0 = \int_{D_0} v_{\xi_0}(y) \overline{(\Delta w_{\xi_0} + k^2 w_{\xi_0})(y)} dy = -\|v_{\xi_0}\|_{L^2(D_0)}^2. \quad (3.44)$$

This proves that  $v_{\xi_0} = 0$  and  $S_{\xi_0}$  has a dense range in  $H_{\xi_0}^{inc}(D)$ .  $\square$

For the analysis below we need to assume the well posedness of the following two Interior Transmission Problems (ITP).

**(ITP1)**: Seek  $(u, v) \in L_{\xi_0}^2(D^p) \times L_{\xi_0}^2(D^p)$  such that  $(u - v) \in H_{\xi_0}^2(D^p)$  satisfying

$$\begin{cases} \Delta u + k^2 n_p u = 0 & \text{in } D_0^p, \\ \Delta v + k^2 v = 0 & \text{in } D_0^p, \\ u - v = \varphi & \text{on } \partial D_0^p, \\ \frac{\partial(u - v)}{\partial \nu} = \psi & \text{on } \partial D_0^p, \end{cases} \quad (3.45)$$

for given  $(\varphi, \psi) \in H_{\xi_0}^{3/2}(\partial D^p) \times H_{\xi_0}^{1/2}(\partial D^p)$ . The spaces  $H_{\xi_0}^m(D^p)$  and  $H_{\xi_0}^s(D^p)$  are defined similarly to  $H_{\xi_0}^m(\Omega^R)$  and  $H_{\xi_0}^s(\Gamma^R)$ .

**(ITP2):** Seek  $(u, v) \in L^2(\tilde{D}) \times L^2(\tilde{D})$  such that  $(u - v) \in H^2(\tilde{D})$  satisfying

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } \tilde{D}, \\ \Delta v + k^2 v = 0 & \text{in } \tilde{D}, \\ u - v = \varphi & \text{on } \partial \tilde{D}, \\ \frac{\partial(u - v)}{\partial \nu} = \psi & \text{on } \partial \tilde{D}, \end{cases} \quad (3.46)$$

for given  $(\varphi, \psi) \in H^{3/2}(\partial \tilde{D}) \times H^{1/2}(\partial \tilde{D})$ .

**Assumption 26.** Assume that  $k, n_p$  are such as the **(ITP1)** is well posed.

**Assumption 27.** Assume that  $k, n_p$  and  $n$  are such that **(ITP2)** is well posed.

Moreover, we need first to prove the following Lemma

**Lemma 28.** For all  $v_{\xi_0}^1, v_{\xi_0}^2 \in L_{\xi_0}^2(D)$  we have

$$(T_{\xi_0} v_{\xi_0}^1, \overline{v_{\xi_0}^2})_{L^2(D_0)} = (T_{\xi_0} v_{\xi_0}^2, \overline{v_{\xi_0}^1})_{L^2(D_0)}. \quad (3.47)$$

*Proof.* For  $i = 1, 2$ , consider  $w_{\xi_0}^{p,i} \in \check{H}_{\xi_0}^1(\Omega_0^R)$  solution of (3.11) with  $\xi = \xi_0$  and  $v_\xi = v_{\xi_0}^i$ . Define  $\tilde{w}_{\xi_0}^{p,i} := \mathcal{J}(\tilde{w}_{\xi_0}^i)(\xi_0, \cdot)$  where  $\tilde{w}_{\xi_0}^i \in \check{H}^1(\Omega^R)$  satisfies (3.25) with  $v_{\xi_0} = v_{\xi_0}^i$ . We set

$$w_{\xi_0}^i := w_{\xi_0}^{p,i} + \tilde{w}_{\xi_0}^{p,i}$$

which verifies

$$\int_{\Omega_0^R} (\nabla w_{\xi_0}^i \cdot \nabla \overline{\psi_{\xi_0}} - k^2 n_p w_{\xi_0}^i \overline{\psi_{\xi_0}}) dx - \langle T_{\xi_0}^R w_{\xi_0}^i, \psi_{\xi_0} \rangle_{\Gamma_0^R} = -L_{\xi_0}^i(\psi_{\xi_0}) \quad (3.48)$$

for all  $\psi_{\xi_0} \in \check{H}_{\xi_0}^1(\Omega_0^R)$  with

$$L_{\xi_0}^i(\psi_{\xi_0}) := \int_{D_0^p} k^2 (1 - n_p) v_{\xi_0}^i \overline{\psi_{\xi_0}} dy + k^2 \int_{\Omega_0^R} (n_p - n) (w_{\xi_0}^{p,i} + v_{\xi_0}^i + \tilde{w}_{\xi_0}^i) \overline{\psi_{\xi_0}}.$$

Taking  $\overline{\psi_{\xi_0}} = w_{\xi_0}^2$  and  $\overline{\psi_{\xi_0}} = w_{\xi_0}^1$  respectively in the variational formulation (3.48) satisfied by  $w_{\xi_0}^1$  and  $w_{\xi_0}^2$  we obtain by taking the difference

$$\begin{aligned} \langle T_{\xi_0}^R w_{\xi_0}^1, \overline{w_{\xi_0}^2} \rangle_{\Gamma_0^R} - \langle T_{\xi_0}^R w_{\xi_0}^2, \overline{w_{\xi_0}^1} \rangle_{\Gamma_0^R} &= \int_{D_0^p} k^2(1-n_p)v_{\xi_0}^1 w_{\xi_0}^2 dy \\ &+ \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,1} + \tilde{w}_{\xi_0}^1 + v_{\xi_0}^1)w_{\xi_0}^2 dy \\ &- \int_{D_0^p} k^2(1-n_p)v_{\xi_0}^2 w_{\xi_0}^1 dy - \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^2 + v_{\xi_0}^2)w_{\xi_0}^1 dy. \end{aligned}$$

Since  $T_{\xi_0}^R$  is symmetric, the left hand side in the previous equality vanishes and therefore

$$\begin{aligned} \int_{D_0^p} k^2(1-n_p)v_{\xi_0}^2 (w_{\xi_0}^{p,1} + \tilde{w}_{\xi_0}^{p,1} + v_{\xi_0}^1) dy &= \int_{D_0^p} k^2(1-n_p)v_{\xi_0}^1 (w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^{p,2} + v_{\xi_0}^2) dy \\ &+ \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,1} + \tilde{w}_{\xi_0}^1 + v_{\xi_0}^1)(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^{p,2}) dy \\ &- \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^2 + v_{\xi_0}^2)(w_{\xi_0}^{p,1} + \tilde{w}_{\xi_0}^{p,1}) dy. \end{aligned} \quad (3.49)$$

On the other hand, taking  $\overline{\psi_{\xi_0}} = \tilde{w}_{\xi_0}^{p,2}$  and  $\overline{\psi_{\xi_0}} = \tilde{w}_{\xi_0}^{p,1}$  respectively in the variational formulations (3.35) satisfied by  $\tilde{w}_{\xi_0}^{p,1}$  and  $\tilde{w}_{\xi_0}^{p,2}$  we obtain by taking the difference and the symmetry of  $T_{\xi_0}^R$

$$\int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,1} + v_{\xi_0}^1 + \tilde{w}_{\xi_0}^1)\tilde{w}_{\xi_0}^{p,2} dy = \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,2} + v_{\xi_0}^2 + \tilde{w}_{\xi_0}^2)\tilde{w}_{\xi_0}^{p,1} dy. \quad (3.50)$$

Moreover, taking  $\overline{\psi_{\xi_0}} = \tilde{w}_{\xi_0}^2$  and  $\overline{\psi_{\xi_0}} = \tilde{w}_{\xi_0}^1$  respectively in the variational formulation (3.26) satisfied by  $\tilde{w}_{\xi_0}^1$  and  $\tilde{w}_{\xi_0}^2$  we obtain after taking the difference and using the symmetry of the operator  $T^R$

$$\int_{\tilde{D}} k^2(n_p-n)v_{\xi_0}^2 \tilde{w}_{\xi_0}^1 dy = - \int_{\tilde{D}} k^2(n_p-n)w_{\xi_0}^{p,2} \tilde{w}_{\xi_0}^1 dy + \int_{\tilde{D}} k^2(n_p-n)\tilde{w}_{\xi_0}^2 (v_{\xi_0}^1 + w_{\xi_0}^{p,1}) dy. \quad (3.51)$$

Now, using (3.40) we have

$$T_{\xi_0} v_{\xi_0}^i = k^2(1-n_p)(w_{\xi_0}^{p,i} + v_{\xi_0}^i + \tilde{w}_{\xi_0}^{p,i}) + k^2(n_p-n)(w_{\xi_0}^{p,i} + v_{\xi_0}^i + \tilde{w}_{\xi_0}^i). \quad (3.52)$$

Using (3.49) to substitute the first term in the right hand side of (3.52) we get

$$\begin{aligned}
(T_{\xi_0} v_{\xi_0}^1, \overline{v_{\xi_0}^2})_{L^2(D_0)} &= \int_{D_0^p} k^2(1-n_p)v_{\xi_0}^1(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^{p,2} + v_{\xi_0}^2) \\
&+ \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,1} + \tilde{w}_{\xi_0}^1 + v_{\xi_0}^1)(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^{p,2}) \\
&- \int_{\tilde{D}} k^2(n_p-n)(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^2 + v_{\xi_0}^2)(w_{\xi_0}^{p,1} + \tilde{w}_{\xi_0}^{p,1}) \\
&+ \int_{\tilde{D}} k^2(n_p-n)v_{\xi_0}^2(w_{\xi_0}^{p,1} + v_{\xi_0}^1 + \tilde{w}_{\xi_0}^1).
\end{aligned} \tag{3.53}$$

Finally, using (3.50), (3.51) to simplify the previous expression we obtain

$$\begin{aligned}
(T_{\xi_0} v_{\xi_0}^1, \overline{v_{\xi_0}^2})_{L^2(D_0)} &= \int_{D_0^p} k^2(1-n_p)v_{\xi_0}^1(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^{p,2} + v_{\xi_0}^2)dy \\
&+ \int_{\tilde{D}} k^2(n_p-n)v_{\xi_0}^1(w_{\xi_0}^{p,2} + \tilde{w}_{\xi_0}^2 + v_{\xi_0}^2)dy,
\end{aligned} \tag{3.54}$$

where the right hand side coincides with the expression of  $(T_{\xi_0} v_{\xi_0}^2, \overline{v_{\xi_0}^1})_{L^2(D_0)}$ . This ends the proof.  $\square$

**Lemma 29.** *Assume that the assumptions of Theorem 19 hold and that Assumptions 26, 27 hold. Assume in addition that  $D_0^p \cap \tilde{D} = \emptyset$ . Then, the operator  $G_{\xi_0} : H_{\xi_0}^{inc}(D) \rightarrow L_{\xi_0}^2(\Gamma^R)$  given by (3.33) is injective with dense range.*

*Proof.* Let  $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$  such that  $G_{\xi_0} v_{\xi_0} = 0$  on  $\Gamma_0^R$ , i.e

$$w_{\xi_0} := w_{\xi_0}^p + \tilde{w}_{\xi_0}^p = 0 \quad \text{on } \Gamma_0^R,$$

where  $w_{\xi_0}^p$  is the solution of (3.24) and  $\tilde{w}_{\xi_0}^p = \mathcal{J}(\tilde{w}_{\xi_0})(\xi_0, \cdot)$  with  $\tilde{w}_{\xi_0}$  being the solution of (3.25). Therefore,  $w_{\xi_0} \in H_{loc}^2(U^R)$ , is  $\xi_0$ -quasi periodic and verifies

$$\begin{cases} \Delta w_{\xi_0} + k^2 w_{\xi_0} = 0 & \text{in } U^R, \\ w_{\xi_0} = 0 & \text{on } \Gamma^R, \end{cases}$$

and the upper going radiation condition (3.12) with  $\Gamma^R$  replaced by  $\Gamma^{R'}$  with  $R' > R$ . The uniqueness for this Dirichlet quasi-periodic scattering problem implies that  $w_{\xi_0} = 0$  in  $U^R$ . Using the unique continuation principle we obtain that  $w_{\xi_0} = 0$  in  $U^0 \setminus D_0$ .

Moreover,  $w_{\xi_0} \in H_{\xi_0}^2(\Omega^R)$  and satisfies

$$\Delta w_{\xi_0} + k^2 n_p w_{\xi_0} + k^2(n - n_p)(w_{\xi_0}^p + \tilde{w}_{\xi_0}) = k^2(1 - n)v_{\xi_0} \quad \text{in } \Omega_0^R. \quad (3.55)$$

Since  $D_0^p \cap \tilde{D} = \emptyset$ , then we have in particular  $w_{\xi_0} \in H_0^2(D_0^p)$  and

$$\Delta w_{\xi_0} + k^2 n_p w_{\xi_0} = k^2(1 - n_p)v_{\xi_0} \quad \text{in } D_0^p.$$

Setting  $u_{\xi_0} := w_{\xi_0} + v_{\xi_0}$  we observe that the couple  $(u_{\xi_0}, v_{\xi_0})$  verifies **(ITP1)** with zero data. Therefore,  $w_{\xi_0}|_{D_0^p} = v_{\xi_0}|_{D_0^p} = 0$ . The latter implies in particular that  $w_{\xi_0}^p = 0$  by well posedness of the periodic direct scattering problem. On the other hand, since  $n_p = 1$  in  $\tilde{D}$ , then  $w_{\xi_0} \in H_0^2(\tilde{D})$  and satisfies

$$\Delta w_{\xi_0} + k^2 w_{\xi_0} + k^2(n - 1)\tilde{w}_{\xi_0} = k^2(1 - n)v_{\xi_0} \quad \text{in } \tilde{D}. \quad (3.56)$$

Since  $w_{\xi_0} \in H_0^2(\tilde{D})$ , then we have

$$\int_{\tilde{D}} (\Delta w_{\xi_0} + k^2 w_{\xi_0})\theta = 0 \quad \text{for all } \theta \in H^{inc}(\tilde{D}),$$

where  $H^{inc}(\tilde{D}) := \{v \in L^2(\tilde{D}) / \Delta v + k^2 v = 0 \text{ in } \tilde{D}\}$ . Therefore, taking the  $L^2$  scalar product of (3.56) with  $\theta$  we get

$$\int_{\tilde{D}} (k^2(1 - n)v_{\xi_0} + k^2(1 - n)\tilde{w}_{\xi_0})\bar{\theta} = 0 \quad \forall \theta \in H^{inc}(\tilde{D}). \quad (3.57)$$

From Theorem 18 we have that  $\tilde{w}_{\xi_0} \in \tilde{H}^1(\Omega^R)$  can be represented as

$$\tilde{w}_{\xi_0}(x) = k^2 \int_{D^p} \Phi(x, y)(1 - n_p)\tilde{w}_{\xi_0}(y)dy + k^2 \int_{\tilde{D}} \Phi(x, y)((1 - n)v_{\xi_0} + (1 - n)\tilde{w}_{\xi_0})dy \quad \text{for } x \in \Omega^R.$$

Since  $y \rightarrow \Phi(x, y) \in H^{inc}(\tilde{D})$  for  $x \notin \tilde{D}$ , we obtain from (3.57) that

$$\tilde{w}_{\xi_0}(x) = \int_{D^p} k^2(1 - n_p)\tilde{w}_{\xi_0}(y)\Phi(x, y)dy \quad \text{for } x \notin \tilde{D}.$$

Let us define  $\check{w}_{\xi_0} \in H^2(\tilde{D})$  by

$$\check{w}_{\xi_0}(x) := \int_{D^p} k^2(1 - n_p)\check{w}_{\xi_0}(y)\Phi(x, y)dy \quad \text{for } x \in \tilde{D}.$$

We set  $u_{\xi_0} := \tilde{w}_{\xi_0} + v_{\xi_0}$ . Then the couple  $(u_{\xi_0}, v_{\xi_0})$  satisfies **(ITP2)** with  $(\varphi, \psi) = (\check{w}_{\xi_0}, \frac{\partial \check{w}_{\xi_0}}{\partial \nu})$ . Moreover, since

$$\Delta \check{w}_{\xi_0} + k^2 \check{w}_{\xi_0} = 0 \quad \text{in } \tilde{D},$$

and **(ITP2)** is well posed then  $v_{\xi_0} + \check{w}_{\xi_0} = 0$  and  $\tilde{w}_{\xi_0} + v_{\xi_0} = 0$  in  $\tilde{D}$ . We then deduce that  $\tilde{w}_{\xi_0} = \check{w}_{\xi_0}$  in  $\tilde{D}$ . Consequently

$$\tilde{w}_{\xi_0}(x) = \int_{D^p} k^2(1 - n_p)\tilde{w}_{\xi_0}(y)\Phi(x, y)dy \quad \text{for } x \in \Omega^R,$$

which implies that  $\tilde{w}_{\xi_0}$  satisfies  $(\mathcal{P})$  with  $n = n_p$  and  $v = 0$ . We then conclude that  $\tilde{w}_{\xi_0} = 0$  by uniqueness of the solution to problem  $(\mathcal{P})$  with  $n = n_p$ . Therefore  $v_{\xi_0} = 0$  in  $\tilde{D}$  which, together with  $v_{\xi_0} = 0$  in  $D^p$  prove the injectivity of  $G_{\xi_0}$ .

Now, we prove the denseness of the range of  $G_{\xi_0}$ . Let  $g_{\xi_0} \in \overline{R(G_{\xi_0})}^\perp$ , then

$$(G_{\xi_0}v_{\xi_0}, g_{\xi_0})_{L^2(D_0)} = 0 \quad \forall v_{\xi_0} \in H_{\xi_0}^{inc}(D).$$

Let  $f_{\xi_0} \in L_{\xi_0}^2(\Gamma^R)$  and consider  $v_{\xi_0} = \overline{S_{\xi_0}f_{\xi_0}}$ . Using Lemma 24 we have

$$(T_{\xi_0} \overline{(S_{\xi_0}f_{\xi_0})}, S_{\xi_0}g_{\xi_0}) = 0, \quad \forall f_{\xi_0} \in L_{\xi_0}^2(\Gamma^R). \quad (3.58)$$

Moreover, using Lemma 28 we get

$$(T_{\xi_0} \overline{(S_{\xi_0}f_{\xi_0})}, S_{\xi_0}g_{\xi_0}) = (T_{\xi_0} \overline{(S_{\xi_0}g_{\xi_0})}, S_{\xi_0}f_{\xi_0}) \quad \forall f_{\xi_0} \in L_{\xi_0}^2(\Gamma^R).$$

Therefore, (3.58) implies that

$$(G_{\xi_0} \overline{(S_{\xi_0}g_{\xi_0})}, f_{\xi_0})_{L^2(\Gamma_0^R)} = 0, \quad \forall f_{\xi_0} \in L_{\xi_0}^2(\Gamma^R).$$

Then  $G_{\xi_0}(\overline{S_{\xi_0} g_{\xi_0}}) = 0$ . The injectivity of  $G_{\xi_0}$  gives that  $S_{\xi_0} g_{\xi_0} = 0$  and then  $g_{\xi_0} = 0$  by Lemma 25.  $\square$

**Lemma 30.** *Under the same assumptions of Lemma 29 we have that*

$$(z \in D_0) \iff (\Phi_{\xi_0}(\cdot, z) \in \text{Range}(G_{\xi_0})).$$

*Proof.* Let  $z \in D^p$ . We consider  $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$  such that  $v_{\xi_0}|_D = -\Phi_{\xi_0}(\cdot, z)$ . Let  $(u_{\xi_0}, v_{\xi_0}) \in L_{\xi_0}^2(D^p) \times L_{\xi_0}^2(D^p)$  be the solution of **(ITP1)** with  $(\varphi, \psi) = (\Phi_{\xi_0}(\cdot, z), \frac{\partial \Phi_{\xi_0}(\cdot, z)}{\partial \nu})$ .

We set

$$w_{\xi_0}^p := \begin{cases} u_{\xi_0} - v_{\xi_0} & \text{in } D^p, \\ \Phi_{\xi_0}(\cdot, z) & \text{in } \Omega^R \setminus D^p. \end{cases}$$

We observe that  $w_{\xi_0}^p \in H_{\xi_0}^2(\Omega^R)$  and satisfies (3.24). Moreover, let  $\tilde{w}_{\xi_0}$  be the solution of (3.25). Since  $(n_p - n)(w_{\xi_0}^p + v_{\xi_0}) = 0$ , then  $\tilde{w}_{\xi_0} = 0$  and consequently  $G_{\xi_0}(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$ .

Consider now the case where  $z \in \tilde{D}$ . Since  $\tilde{D} \cap D^p = \emptyset$  Recall that the Green function  $\Phi(\cdot, z)$  defined by (3.5) belongs to  $L^2(\Omega^R)$  [17]. Let  $u \in H_{loc}^2(\Omega^R)$  be the solution of  $(\mathcal{P})$  with  $n = n_p$  and  $v = \Phi(\cdot, z)$ . Let us define  $\Phi_{n_p}(\cdot, z) := u + \Phi(\cdot, z)$  that satisfies in particular,  $\Phi_{n_p}(\cdot, z) \in L^2(\Omega^R) \cap H_{loc}^2(\Omega^R \setminus \tilde{D})$

$$\Delta \Phi_{n_p}(\cdot, z) + k^2 n_p \Phi_{n_p}(\cdot, z) = -\delta_z \quad \text{in } \Omega^R$$

together with the upper going radiation condition (3.3). Consider  $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$  such that  $v_{\xi_0}|_{D^p} = 0$  and let  $(\tilde{u}_{\xi_0}, v_{\xi_0}) \in L^2(\tilde{D}) \times L^2(\tilde{D})$  be the solution of **(ITP2)** with  $(\varphi, \psi) = (\Phi_{n_p}(\cdot, z), \frac{\partial \Phi_{n_p}(\cdot, z)}{\partial \nu})$ . We set

$$\tilde{w}_{\xi_0} := \begin{cases} \tilde{u}_{\xi_0} - v_{\xi_0} & \text{in } \tilde{D}, \\ \Phi_{n_p}(\cdot, z) & \text{in } \Omega^R \setminus \tilde{D}. \end{cases}$$

We observe that  $\tilde{w}_{\xi_0} \in H_{loc}^2(\Omega^R)$  satisfies (3.25). Moreover, since  $v_{\xi_0}|_{D^p} = 0$ , then the right hand side of the first equation of (3.24) vanishes and therefore the solution  $w_{\xi_0}^p$  of



(3.24) vanishes. Consequently

$$G_{\xi_0}(v_{\xi_0}) = \mathcal{J}(\Phi_{n_p}(\cdot, z))(\xi_0) =: \Phi_{n_p, \xi_0}(\cdot, z).$$

On the other hand, we consider  $u_{\xi_0} := \mathcal{J}(u)(\xi_0, \cdot) \in H_{\xi_0}^1(\Omega_0^R)$ . Then  $u_{\xi_0}$  satisfies (3.24) with  $v_{\xi_0} = \Phi_{\xi_0}(\cdot, z)$ . We set

$$\tilde{v}_{\xi_0} := \begin{cases} \Phi_{\xi_0}(\cdot, z) & \text{in } D^p, \\ -u_{\xi_0} & \text{in } \tilde{D}. \end{cases}$$

Let  $w_{\xi_0}^p$  and  $\tilde{w}_{\xi_0}$  be respectively the solutions of (3.24) and (3.25) with  $v_{\xi_0} = \tilde{v}_{\xi_0}$ . By uniqueness of the solution of problem (3.24) we have  $w_{\xi_0}^p = u_{\xi_0}$ . Moreover,  $\tilde{w}_{\xi_0} = 0$  since  $k^2(n_p - n)(\tilde{v}_{\xi_0} + u_{\xi_0}) = 0$ . Therefore

$$G_{\xi_0}(\tilde{v}_{\xi_0}) = u_{\xi_0}.$$

Consequently

$$G_{\xi_0}(v_{\xi_0} - \tilde{v}_{\xi_0}) = \Phi_{\xi_0}(\cdot, z).$$

Consider finally the case where  $z \notin D_0$ . Assume that there exists  $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$  such that  $G_{\xi_0}(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$ . Using the unique continuation principle we get  $w_{\xi_0} := w_{\xi_0}^p + \tilde{w}_{\xi_0}^p = \Phi_{\xi_0}(\cdot, z)$  in  $U^0 \setminus D_0$  which is a contradiction since  $w_{\xi_0} \in H_{\xi_0}^2(\Omega^R \setminus D_0)$  while  $\Phi_{\xi_0}(\cdot, z) \notin H_{\xi_0}^2(\Omega^R \setminus D_0)$ .  $\square$

Let us define now the following norm for  $g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R)$

$$I_{\xi_0} g_{\xi_0} := \left| (N_{\xi_0}^p g_{\xi_0}, g_{\xi_0})_{L^2(\Gamma_0^R)} \right| + \left| (\tilde{N}_{\xi_0}^p g_{\xi_0}, g_{\xi_0})_{L^2(\Gamma_0^R)} \right|. \quad (3.59)$$

From the factorizations (3.37) we have the equivalent expression

$$I_{\xi_0} g_{\xi_0} := \left| (T_{\xi_0}^p S_{\xi_0} g_{\xi_0}, S_{\xi_0} g_{\xi_0})_{L_{\xi_0}^2(D^p)} \right| + \left| (\tilde{T}_{\xi_0}^p S_{\xi_0} g_{\xi_0}, S_{\xi_0} g_{\xi_0})_{L_{\xi_0}^2(D)} \right|. \quad (3.60)$$

**Lemma 31.** *Assume that Assumption 15 and 27 hold. Assume in addition that  $\Re(1 - n_p) \geq \gamma > 0$  in  $D^p$  and  $\Re(n_p - n) \geq \gamma_1 > 0$  in  $\tilde{D}$  or  $\Re(1 - n_p) \leq -\gamma < 0$  in  $D^p$  and*

$\Re(n_p - n) \leq -\gamma_1 < 0$ . Then, there exists a constant  $c > 0$  independent from  $\xi_0$  such that

$$I_{\xi_0} g_{\xi_0} \geq c \left( \|S_{\xi_0} g_{\xi_0}\|_{L^2(D_0^p)}^2 + \|S_{\xi_0} g_{\xi_0}\|_{L^2(\tilde{D})}^2 \right) \quad \forall g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R),$$

with  $I_{\xi_0} g_{\xi_0}$  is the norm given by (3.59).

*Proof.* We prove the Lemma in the case  $\Re(1 - n_p) \geq \gamma > 0$  in  $D^p$  and  $\Re(n_p - n) \geq \gamma_1 > 0$  in  $\tilde{D}$ . The other case can be proved in the same way. We shall use a contradiction argument. We consider  $g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R)$  and we denote by  $v_{\xi_0} = S_{\xi_0} g_{\xi_0}$ . Assume that there exists a sequence  $v_{\xi_0}^\ell$  such that

$$\frac{1}{\ell} \|v_{\xi_0}^\ell\|_{L^2(D_0)}^2 \geq \left| (T_{\xi_0}^p v_{\xi_0}^\ell, v_{\xi_0}^\ell)_{L^2(D_0^p)} \right| + \left| (\tilde{T}_{\xi_0}^p v_{\xi_0}^\ell, v_{\xi_0}^\ell)_{L^2(D_0)} \right|. \quad (3.61)$$

We set  $\hat{v}_{\xi_0}^\ell = \frac{v_{\xi_0}^\ell}{\|v_{\xi_0}^\ell\|_{L^2(D_0)}}$ . Let  $w_{\xi_0}^{p,\ell}$  be the solution of (3.11) and  $\tilde{w}_{\xi_0}^{p,\ell} = \mathcal{J}(\tilde{w}_{\xi_0}^\ell)(\xi_0, \cdot)$  with  $v_\xi = \hat{v}_{\xi_0}^\ell$  and  $\tilde{w}_{\xi_0}$  is the solution of (3.26). Since  $\|\hat{v}_{\xi_0}^\ell\|_{L^2(D_0)}$  is bounded, then we can extract a subsequence (that we still denote the same)  $\hat{v}_{\xi_0}^\ell$  that converge weakly to  $\hat{v}_{\xi_0}$  in  $L^2(D_0)$ . Moreover,  $w_{\xi_0}^{p,\ell}$  and  $\tilde{w}_{\xi_0}^{p,\ell}$  converge weakly in  $H_{\xi_0}^1(\Omega_0^R)$  and strongly in  $L^2(D_0)$  respectively to some  $w_{\xi_0}^p$  and  $\tilde{w}_{\xi_0}^p \in H_{\xi_0}^1(\Omega_0^R)$ . On the other hand, taking  $\psi_{\xi_0} = w_{\xi_0}^{p,\ell}$  in the variational formulation (3.11) satisfied by  $w_{\xi_0}^{p,\ell}$  we obtain

$$\int_{\Omega_0^R} (|\nabla w_{\xi_0}^{p,\ell}|^2 - k^2 |w_{\xi_0}^{p,\ell}|^2) dy = -k^2 \int_{D_0^p} (1 - n_p) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell) \overline{w_{\xi_0}^{p,\ell}} dy + \left\langle T_{\xi_0}^R w_{\xi_0}^{p,\ell}, w_{\xi_0}^{p,\ell} \right\rangle_{\Gamma_0^R}.$$

Therefore, decomposing  $(\hat{v}_{\xi_0}^\ell + w_{\xi_0}^{p,\ell}) \overline{\hat{v}_{\xi_0}^\ell} = |\hat{v}_{\xi_0}^\ell + w_{\xi_0}^{p,\ell}|^2 - (\hat{v}_{\xi_0}^\ell + w_{\xi_0}^{p,\ell}) \overline{w_{\xi_0}^{p,\ell}}$  we get

$$\begin{aligned} (T_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0^p)} &= \int_{D_0^p} k^2 (1 - n_p) |\hat{v}_{\xi_0}^\ell + w_{\xi_0}^{p,\ell}|^2 dy + \int_{\Omega_0^R} (|\nabla w_{\xi_0}^{p,\ell}|^2 - k^2 |w_{\xi_0}^{p,\ell}|^2) dy \\ &\quad - \left\langle T_{\xi_0}^R w_{\xi_0}^{p,\ell}, w_{\xi_0}^{p,\ell} \right\rangle_{\Gamma_0^R}. \end{aligned}$$

Taking the imaginary part we obtain

$$\Im m (T_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0^p)} = - \int_{D_0^p} \Im m (n_p) |\hat{v}_{\xi_0}^\ell + w_{\xi_0}^{p,\ell}|^2 dy - \Im m \left\langle T_{\xi_0}^R w_{\xi_0}^{p,\ell}, w_{\xi_0}^{p,\ell} \right\rangle_{\Gamma_0^R}.$$

From (3.61) we have

$$\left| \Im m (T_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0^p)} \right| \xrightarrow{\ell \rightarrow \infty} 0.$$

Therefore, using the fact that  $\Im m \langle T_{\xi_0}^R w_{\xi_0}^{p,\ell}, w_{\xi_0}^{p,\ell} \rangle_{\Gamma_0^R} \geq 0$ ,

$$\int_{D_0^p} \Im m(n_p) |\hat{v}_{\xi_0}^\ell + w_{\xi_0}^{p,\ell}|^2 dy \xrightarrow{\ell \rightarrow \infty} 0 \geq \int_{D_0^p} \Im m(n_p) |\hat{v}_{\xi_0} + w_{\xi_0}^p|^2 dy.$$

Since  $\Im m(n_p) > 0$  in  $\mathcal{O}$  we obtain that  $u_{\xi_0}^p := \hat{v}_{\xi_0} + w_{\xi_0}^p = 0$  in  $\mathcal{O}$ . Observing that  $\Delta u_{\xi_0}^p + k^2 n_p u_{\xi_0}^p = 0$  in  $D_0^p$ , by unique continuation principle we deduce that  $u_{\xi_0}^p = 0$  in  $D_0^p$ . Therefore,  $w_{\xi_0}^p$  satisfies (3.24) with  $n_p = 1$  and  $\hat{v}_{\xi_0} = 0$ , which implies that  $w_{\xi_0}^p$  vanishes in  $\Omega_0^R$ . Moreover, since  $w_{\xi_0}^{p,\ell}$  converges strongly to  $w_{\xi_0}^p$  in  $L^2(D_0^p)$ , then we have

$$\int_{D_0^p} k^2 (1 - n_p) w_{\xi_0}^{p,\ell} \overline{\hat{v}_{\xi_0}^\ell} dy \longrightarrow \int_{D_0^p} k^2 (1 - n_p) w_{\xi_0}^p \overline{\hat{v}_{\xi_0}} dy = 0.$$

On the other hand, we have that

$$\left| (T_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0^p)} \right| \geq k^2 \left| \int_{D_0^p} (1 - n_p) |\hat{v}_{\xi_0}^\ell|^2 dy \right| - k^2 \left| \int_{D_0^p} k^2 (1 - n_p) w_{\xi_0}^{p,\ell} \overline{\hat{v}_{\xi_0}^\ell} dy \right|.$$

From the hypothesis  $\Re e(1 - n_p) \geq \gamma > 0$  in  $D^p$  we then have

$$\lim_{\ell \rightarrow \infty} \left| (T_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0^p)} \right| \geq \gamma \|\hat{v}_{\xi_0}\|_{L^2(D_0^p)}^2.$$

Therefore  $\hat{v}_{\xi_0}|_{D_0^p} = 0$ .

Now, taking  $\psi = \tilde{w}_{\xi_0}^\ell$  in the variational formulation (3.26) satisfied by  $\tilde{w}_{\xi_0}^\ell$  we obtain

$$\begin{aligned} -k^2 \int_{\tilde{D}} (n_p - n) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell) \overline{\tilde{w}_{\xi_0}^\ell} dy &= k^2 \int_{\Omega^R} (1 - n_p) |\tilde{w}_{\xi_0}^\ell|^2 dy + \\ &\int_{\Omega^R} (|\nabla \tilde{w}_{\xi_0}^\ell|^2 - k^2 |\tilde{w}_{\xi_0}^\ell|^2) dy - \langle T^R \tilde{w}_{\xi_0}^\ell, \tilde{w}_{\xi_0}^\ell \rangle_{\Gamma^R}. \end{aligned} \quad (3.62)$$

Moreover, taking  $\psi = \tilde{w}_{\xi_0}^{p,\ell}$  in the variational formulation (3.35) satisfied by  $\tilde{w}_{\xi_0}^{p,\ell}$  we get

$$\begin{aligned} -k^2 \int_{\tilde{D}} (n_p - n) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^{p,\ell}) \overline{\tilde{w}_{\xi_0}^{p,\ell}} dy &= \int_{\Omega_0^R} (|\nabla \tilde{w}_{\xi_0}^{p,\ell}|^2 - k^2 |\tilde{w}_{\xi_0}^{p,\ell}|^2) dy \\ &+ k^2 \int_{D_0^p} (1 - n_p) |\tilde{w}_{\xi_0}^{p,\ell}|^2 dy - \langle T_{\xi_0}^R \tilde{w}_{\xi_0}^{p,\ell}, \tilde{w}_{\xi_0}^{p,\ell} \rangle_{\Gamma_0^R}. \end{aligned} \quad (3.63)$$

On the other hand, we have

$$\begin{aligned} (\tilde{T}_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0)} &= \int_{\Omega_0^R} k^2(1-n_p) \tilde{w}_{\xi_0}^{p,\ell} \overline{\hat{v}_{\xi_0}^\ell} dy + \int_{\Omega_0^R} k^2(n_p-n) |w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell|^2 dy \\ &\quad - \int_{\Omega^R} k^2(n_p-n) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell) \overline{\tilde{w}_{\xi_0}^{p,\ell}} dy \\ &\quad - \int_{\Omega^R} k^2(n_p-n) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell) \overline{\tilde{w}_{\xi_0}^\ell} dy. \end{aligned}$$

Then, using (3.62) and (3.63) we obtain by taking the imaginary part

$$\begin{aligned} \Im m(\tilde{T}_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0)} &= \int_{\Omega_0^R} k^2 \Im m(n_p-n) |w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell|^2 dy - k^2 \int_{\Omega^R} \Im m(n_p) |\tilde{w}_{\xi_0}^\ell|^2 dy \\ &\quad + k^2 \int_{\Omega_0^R} \Im m((1-n_p) \tilde{w}_{\xi_0}^{p,\ell} \overline{\hat{v}_{\xi_0}^\ell}) dy - \int_{\Omega^R} k^2 \Im m((n_p-n) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell) \overline{w_{\xi_0}^{p,\ell}}) dy \\ &\quad - \Im m \langle T^R \tilde{w}_{\xi_0}^\ell, \tilde{w}_{\xi_0}^\ell \rangle_{\Gamma^R}. \end{aligned} \quad (3.64)$$

On the other hand, the application  $v_{\xi_0} \rightarrow (1-n_p) \tilde{w}_{\xi_0}^p$  with  $\tilde{w}_{\xi_0}^p$  is solution of (3.34) is compact from  $L^2(D_0)$  into  $L^2(D_0^p)$  using the compactness of the injection of  $H_{\xi_0}^1(\Omega_0^R)$  into  $L^2(D_0^p)$ . Similarly, the application  $v_{\xi_0} \rightarrow (n_p-n) w_{\xi_0}^p$  with  $w_{\xi_0}^p$  is solution of (3.24) is compact from  $L^2(D_0)$  into  $L^2(\tilde{D})$ . Therefore we have, using that  $\hat{v}_{\xi_0}|_{D_0^p} = 0$  and  $w_{\xi_0}^p = 0$ ,

$$\begin{aligned} k^2 \int_{\Omega_0^R} \Im m((1-n_p) \tilde{w}_{\xi_0}^{p,\ell} \overline{\hat{v}_{\xi_0}^\ell}) dy &\xrightarrow{\ell \rightarrow \infty} 0, \\ \int_{\Omega^R} k^2 \Im m((n_p-n) (w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell) \overline{w_{\xi_0}^{p,\ell}}) dy &\xrightarrow{\ell \rightarrow \infty} 0. \end{aligned} \quad (3.65)$$

From (3.61) we have

$$\left| \Im m(\tilde{T}_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0)} \right| \xrightarrow{\ell \rightarrow \infty} 0. \quad (3.66)$$

We observe that  $\Im m \langle T^R \tilde{w}_{\xi_0}^\ell, \tilde{w}_{\xi_0}^\ell \rangle_{\Gamma^R} \geq 0$ . Consequently, using (3.64), (3.65) and (3.66)

we get

$$\int_{\tilde{D}} \Im m(n-n_p) |w_{\xi_0}^{p,\ell} + \hat{v}_{\xi_0}^\ell + \tilde{w}_{\xi_0}^\ell|^2 dy + \int_{D^p} \Im m(n_p) |\tilde{w}_{\xi_0}^\ell|^2 dy \xrightarrow{\ell \rightarrow \infty} 0.$$

In particular, by Assumption 15 we deduce that  $\int_{\mathcal{O}} \Im m(n_p) |\tilde{w}_{\xi_0}^\ell|^2 dy \xrightarrow{\ell \rightarrow \infty} 0$ . This implies that  $\tilde{w}_{\xi_0} = 0$  in  $\mathcal{O}$ . We remark that  $w_{\xi_0}^p = 0$  implies in particular that  $\Delta \tilde{w}_{\xi_0} + k^2 n \tilde{w}_{\xi_0} = 0$  in  $\Omega^R \setminus \tilde{D}$ . Consequently  $\tilde{w}_{\xi_0} = 0$  in  $\Omega^R \setminus \tilde{D}$  by unique continuation principle. This proves

that the couple  $(\tilde{w}_{\xi_0} + \hat{v}_{\xi_0}, \hat{v}_{\xi_0}) \in L^2(\tilde{D}) \times L^2(\tilde{D})$  is solution of **(ITP2)** with zero data. Hence  $\tilde{w}_{\xi_0} = 0$  in  $\tilde{D}$ . On the other hand, the application  $v_{\xi_0} \longrightarrow (n_p - n)\tilde{w}_{\xi_0}$  with  $\tilde{w}_{\xi_0}$  is solution of (3.25) is compact from  $H_{\xi_0}^{inc}(D)$  into  $L^2(\tilde{D})$  thanks to the compactness of the injection of  $H^1(\tilde{D})$  into  $L^2(\tilde{D})$ . Therefore

$$\int_{D_0^p} k^2(1 - n_p)\tilde{w}_{\xi_0}^{p,\ell}\overline{\hat{v}_{\xi_0}^\ell} + \int_{\tilde{D}} k^2(n_p - n)(w_{\xi_0}^{p,\ell} + \tilde{w}_{\xi_0}^\ell)\overline{\hat{v}_{\xi_0}^\ell} dy \longrightarrow 0.$$

Moreover, we have

$$\begin{aligned} \left| (\tilde{T}_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell)_{L^2(D_0)} \right| &\geq \left| \int_{\tilde{D}} k^2(n_p - n)|\hat{v}_{\xi_0}^\ell|^2 \right| - \left| \int_{D_0^p} k^2(1 - n_p)\tilde{w}_{\xi_0}^{p,\ell}\overline{\hat{v}_{\xi_0}^\ell} \right. \\ &\quad \left. + \int_{\tilde{D}} k^2(n_p - n)(w_{\xi_0}^{p,\ell} + \tilde{w}_{\xi_0}^\ell)\overline{\hat{v}_{\xi_0}^\ell} dy \right|. \end{aligned}$$

Using the hypothesis  $\Re(n_p - n) \geq \gamma_1 > 0$  in  $\tilde{D}$  we conclude

$$0 = \lim_{\ell \rightarrow \infty} \left| (\tilde{T}_{\xi_0}^p \hat{v}_{\xi_0}^\ell, \hat{v}_{\xi_0}^\ell) \right| \geq \gamma_1 \|\hat{v}_{\xi_0}\|_{L^2(\tilde{D})}^2,$$

which gives  $\hat{v}_{\xi_0} = 0$  in  $\tilde{D}$ . Combined with the result above we have that  $\hat{v}_{\xi_0} = 0$  in  $D_0$  which contradicts with  $\|\hat{v}_{\xi_0}\|_{L^2(D_0)} = 1$ .  $\square$

### 3.3.3 Application of the Generalized Linear Sampling Method (GLSM)

We present the free noise version of the GLSM. For fixed  $\xi_0 \in I$ , introducing the functional  $J_{\xi_0}^\alpha : L_{\xi_0}^2(\Gamma^R) \longrightarrow \mathbb{R}$  given as

$$J_{\xi_0}^\alpha(\phi; g_{\xi_0}) = \alpha I_{\xi_0}(g_{\xi_0}) + \|(N_{\xi_0}^p + \tilde{N}_{\xi_0}^p)g_{\xi_0} - \phi\|^2,$$

We denote by  $j_{\xi_0}^\alpha(\phi) = \inf_{g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R)} J_{\xi_0}^\alpha(\phi; g_{\xi_0})$ . Moreover, let  $c(\alpha) > 0$  verifying  $\frac{c(\alpha)}{\alpha} \rightarrow 0$  as  $\alpha \rightarrow 0$ .

**Theorem 32.** *Assume that Assumptions 26 and 27 hold. Assume in addition that the hypothesis of Theorem 19 and Lemma 31 hold. Consider  $z \in \Omega^R$ , and let  $g_{\xi_0}^\alpha \in L_{\xi_0}^2(\Gamma^R)$  such that*

$$J_{\xi_0}^\alpha(\Phi_{\xi_0}(\cdot, z), g_{\xi_0}^\alpha(z)) \leq j_{\xi_0}^\alpha(\Phi_{\xi_0}(\cdot, z)) + c(\alpha),$$

then

$$z \in D_0 \iff \lim_{\alpha \rightarrow 0} I_{\xi_0}(g_{\xi_0}^\alpha(z)) < \infty.$$

Moreover, if  $z \in D_0$  then  $S_{\xi_0}g_{\xi_0}^\alpha|_{\tilde{D}}$  converges to some  $\tilde{v}$  in  $L^2(\tilde{D})$  and  $S_{\xi_0}g_{\xi_0}^\alpha|_{D_0^p}$  converges to some  $v_{\xi_0}$  in  $L^2(D_0^p)$  where  $\tilde{v}$  is solution of **(ITP2)** and  $v_{\xi_0}$  is solution of **(ITP1)**.

*Proof.* The proof of this theorem is an application of the abstract framework of GLSM given by Theorem 2.7 in [11] and the series of Lemmas (24)-(31). Lemma 24, Lemma 25 and Lemma 29 prove that the operator  $N_{\xi_0} = N_{\xi_0}^p + \tilde{N}_{\xi_0}^p$  can be factorized as

$$N_{\xi_0} = G_{\xi_0}S_{\xi_0} = S_{\xi_0}^*T_{\xi_0}S_{\xi_0}, \quad (3.67)$$

and has dense range. Moreover, we need to verify that the norm  $I_{\xi_0}g_{\xi_0}$  is an equivalent norm to  $\|S_{\xi_0}g_{\xi_0}\|_{L^2(D_0)}$  for all  $g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R)$ . Theorem 19, Theorem 22 and the expression of the operator  $T_{\xi_0}$  prove the existence of a constant  $c_1 > 0$  (independent from  $\xi_0$ ) such that

$$I_{\xi_0}g_{\xi_0} \leq c_1 \left( \|S_{\xi_0}g_{\xi_0}\|_{L^2(D_0^p)}^2 + \|S_{\xi_0}g_{\xi_0}\|_{L^2(\tilde{D})}^2 \right) \quad \forall g_{\xi_0} \in L_{\xi_0}^2(\Gamma^R). \quad (3.68)$$

Therefore, Lemma 31 and (3.68) prove this norm equivalence. The results of the theorem are then a straightforward application of Theorem 2.7 in [11] and Lemma 30.  $\square$

## 3.4 Inverse problem for non-periodic incident fields

### 3.4.1 Setting of the inverse problem

Let  $y \in \Gamma^R$ . One can deduce from (3.81)-(3.85) that  $\Phi(\cdot, y) \in \tilde{L}^2(\Omega^R)$  (see also [18, 58]). We then define  $u^s(\cdot, y) \in \tilde{H}_{loc}^1(\Omega^R)$  the scattered field solution of  $(\mathcal{P})$  with  $v(\cdot, y) = \overline{\Phi(\cdot, y)}$ . We introduce the near field operator  $N : \tilde{L}^2(\Gamma^R) \rightarrow \tilde{L}^2(\Gamma^R)$  as

$$Ng(x) = \int_{\Gamma^R} u^s(x, y)g(y)ds(y).$$

Define  $S : \tilde{L}^2(\Gamma^R) \rightarrow \tilde{L}^2(D)$  as

$$Sg(x) = \int_{\Gamma^R} \overline{\Phi(x, y)}g(y)ds(y).$$

Then, the operator  $N$  can be decomposed as

$$N = GS,$$

where  $G : \tilde{L}^2(D) \longrightarrow \tilde{L}^2(\Gamma^R)$  is the operator defined by

$$G(v) = w|_{\Gamma^R},$$

with  $w$  being the solution of  $(\mathcal{P})$ .

**Link between  $N$  and  $\xi$ -quasi periodic solutions:** For  $\xi \in I$ , we denote by  $g_\xi := (\mathcal{J}g)(\xi, \cdot)$  and we observe that

$$Sg_\xi = S_\xi g_\xi(x), \quad (3.69)$$

with  $S_\xi$  being the operator given by (3.31), in fact

$$\begin{aligned} Sg_\xi(x) &= \int_{\mathbb{R}} \overline{\Phi(x, (y_1, R))} g_\xi(y_1, R) ds(y_1) = \sum_{l \in \mathbb{Z}} \int_{2\pi l}^{2\pi(l+1)} \overline{\Phi(x, (y_1, R))} g_\xi(y_1, R) ds(y_1), \\ &= \sum_{l \in \mathbb{Z}} \int_0^{2\pi} \overline{\Phi(x, (y_1 + 2\pi l, R))} g_\xi(y_1 + 2\pi l) ds(y_1) = \int_{\Gamma_0^R} \overline{\Phi_\xi(y, x)} g_\xi(y) ds(y), \\ &= S_\xi g_\xi(x), \end{aligned}$$

therefore

$$\int_I S_\xi g_\xi(x) d\xi = \int_{\Gamma^R} \overline{\Phi(x, y)} \int_I g_\xi(y) d\xi = Sg(x). \quad (3.70)$$

Let  $\tilde{N}_\xi : L_\xi^2(\Gamma^R) \longrightarrow L_{loc}^2(\Gamma^R)$  be the operator defined by

$$\tilde{N}_\xi g_\xi(x) := \int_{\Gamma_0^R} g_\xi(y) u_\xi^{s,p}(x, y) ds(y) + \int_{\Gamma_0^R} g_\xi(y) \tilde{u}_\xi^s(x, y) ds(y),$$

with  $u_\xi^{s,p}$  and  $\tilde{u}_\xi^s$  are respectively the solutions of (3.24) and (3.25) with  $v_\xi = \overline{\Phi_\xi(y, \cdot)}$ .

Then

$$\begin{aligned} \int_I (\tilde{N}_\xi g_\xi)(x) d\xi &= \int_I \int_{\Gamma_0^R} g_\xi(y) u_\xi^{s,p}(x, y) ds(y) + \int_I \int_{\Gamma_0^R} g_\xi(y) \tilde{u}_\xi^s(x, y) ds(y) \\ &= \int_I w_\xi^p(x) d\xi + \int_I \tilde{w}_\xi(x) d\xi, \end{aligned} \quad (3.71)$$

with  $w_\xi^p$  and  $\tilde{w}_\xi$  are respectively solutions of (3.24) and (3.25) with  $v_\xi = S_\xi g_\xi$ . We denote by

$$w^p(x) := \int_I w_\xi^p(x) d\xi \quad \text{and} \quad \tilde{w}(x) := \int_I \tilde{w}_\xi(x) d\xi. \quad (3.72)$$

Using (3.70) we observe that  $w^p$  satisfies  $(\mathcal{P})$  with  $n = n_p$  and  $v = Sg$ . Moreover,  $\tilde{w}$  satisfies

$$\begin{cases} \Delta \tilde{w} + k^2 n_p \tilde{w} = k^2 (n_p - n) (w^p + v + \tilde{w}) & \text{in } \Omega^R, \\ \tilde{w} = 0 & \text{on } \Gamma^0, \\ \frac{\partial \tilde{w}}{\partial x_2}(\cdot, R) = T^R(\tilde{w}|_{\Gamma^R}) & \text{on } \Gamma^R, \end{cases} \quad (3.73)$$

Therefore,  $w := w^p + \tilde{w}$  satisfies  $(\mathcal{P})$  with  $v = Sg$ . Hence, we can equivalently define the operator  $N : \tilde{L}^2(\Gamma^R) \longrightarrow \tilde{L}^2(\Gamma^R)$  as

$$Ng(x) = \int_I (\tilde{N}_\xi g_\xi)(x) d\xi. \quad (3.74)$$

We finally observe that  $N_\xi g_\xi$  is not equal in general to  $\mathcal{J}(Ng)(\xi)$  since the latter corresponds with the scattered field  $w_\xi \in H_\xi^1(\Omega_0^R)$  satisfying

$$\Delta w_\xi + k^2 n_p w_\xi = k^2 (1 - n_p) v_\xi + k^2 (n_p - n) (w + v) \quad \text{in } \Omega_0^R,$$

where  $w$  satisfies  $(\mathcal{P})$  with  $v = Sg$  and  $v_\xi = S_\xi g_\xi$ . This equation is different from (3.55) that corresponds with the scattered field associated with  $N_\xi g_\xi$ . The main difference comes from the term  $k^2 (n_p - n) v$  in the right hand side.

### 3.4.2 Some properties of the operator $S$

The goal of this section is to prove that the operator  $S : \tilde{L}^2(\Gamma^R) \longrightarrow L^2(D)$  is injective and characterize its range. The main difficulty here comes from the required continuity with respect to the Floquet-Bloch variable. This is why we first prove the uniform continuity of  $\xi \longrightarrow S_\xi$  formalized in the technical Lemma 33.

Let  $\chi \in C^0(I)$  and  $\psi \in L_\#^2(\Gamma^R)$ . We consider  $g \in \tilde{L}^2(\Gamma^R)$  such that

$$g_\xi(x) := \mathcal{J}(g)(\xi, x) = e^{i\xi \cdot x_1} \psi(x) \chi(\xi), \quad \text{for } (\xi, x) \in I \times \Gamma_0^R. \quad (3.75)$$



Denoting

$$\widehat{\psi}_j = \frac{1}{2\pi} \int_0^{2\pi} e^{-ijy_1} \psi(y_1) dy_1,$$

we observe that the operator  $S_\xi$  given by (3.31) verifies for  $x_2 < R$

$$\begin{aligned} S_\xi g_\xi(x) &= \int_0^{2\pi} \overline{\frac{i}{4\pi} \sum_{j \in \mathbb{Z}} e^{i\alpha_\xi(j)(y_1-x_1)} \theta_\xi(j, R, x_2) g_\xi(y_1)} dy_1, \\ &= \frac{i}{4\pi} \sum_{j \in \mathbb{Z}} e^{-i\alpha_\xi(j)x_1} \theta_\xi(j, R, x_2) \int_0^{2\pi} e^{-i\alpha_\xi(j)y_1} g_\xi(y_1) dy_1. \end{aligned}$$

Therefore,

$$S_\xi g_\xi(x) = -\frac{i}{2} \sum_{j \in \mathbb{Z}} \widehat{\psi}_j \overline{\chi(\xi) \theta_\xi(j, R, x_2)} e^{i\alpha_\xi(j)x_1}. \quad (3.76)$$

Let  $0 < R_0 < R$ , we define  $\tilde{S}_\xi : L^2_{\#}(\Gamma^R) \longrightarrow L^2_{\#}(\Omega^{R_0})$  as

$$\tilde{S}_\xi \psi := e^{-i\xi \cdot x_1} S_\xi(e^{i\xi \cdot x_1} \psi). \quad (3.77)$$

Using (3.76) we have

$$\tilde{S}_\xi \psi(x) = -\frac{i}{2} \sum_{j \in \mathbb{Z}} \widehat{\psi}_j \overline{\theta_\xi(j, R, x_2)} e^{ijx_1}. \quad (3.78)$$

**Lemma 33.** *For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for  $|\xi' - \xi| < \delta$  we have*

$$\|\tilde{S}_\xi \psi - \tilde{S}_{\xi'} \psi\|_{L^2(\Omega_0^{R_0})}^2 \leq c\epsilon^2 \|\tilde{S}_\xi \psi\|_{L^2(\Omega_0^{R_0})}^2 \quad \forall \psi \in L^2_{\#}(\Gamma^R), \quad (3.79)$$

where  $c > 0$  is a constant independent of  $\xi$  and  $\xi' \in I$ .

*Proof.* Let  $\psi \in L^2_{\#}(\Gamma^R)$ . From (3.76) we have

$$\begin{aligned} \|\tilde{S}_\xi \psi\|_{L^2(\Omega_0^{R_0})}^2 &= \frac{\pi}{2} \sum_{j \in \mathbb{Z}} |\widehat{\psi}_j|^2 \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2, \\ \|(\tilde{S}_\xi - \tilde{S}_{\xi'}) \psi\|_{L^2(\Omega_0^{R_0})}^2 &= \frac{\pi}{2} \sum_{j \in \mathbb{Z}} |\widehat{\psi}_j|^2 \int_0^{R_0} |\theta_\xi(j, R, x_2) - \theta_{\xi'}(j, R, x_2)|^2 dx_2. \end{aligned}$$

Therefore, to prove (3.79) we prove for all  $\epsilon > 0$  the existence of a constant  $c > 0$  independent from  $\xi$  and  $\xi'$  such that

$$\int_0^{R_0} |\theta_\xi(j, R, x_2) - \theta_{\xi'}(j, R, x_2)|^2 dx_2 \leq c\epsilon^2 \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2.$$

Consider first  $j \in \mathbb{Z}$  such that  $k^2 \geq \alpha_\xi^2(j)$ , i.e  $\beta_\xi(j) = \sqrt{k^2 - |\xi + j|^2}$ . There exists only a finite number of  $j$  for which this holds. Then

$$|\theta_\xi(j, R, x_2)|^2 = \frac{|e^{i\beta_\xi(j)R}|^2}{|\beta_\xi(j)|^2} |2 \sin(\beta_\xi(j)x_2)|^2 = \frac{2}{|\beta_\xi(j)|^2} |1 - \cos(2\beta_\xi(j)x_2)|.$$

For  $\beta_\xi(j) = 0$ , we have

$$\lim_{\beta_\xi(j) \rightarrow 0} \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 = \frac{4R_0^3}{3} =: c_1, \quad (3.80)$$

while if  $\beta_\xi(j) > 0$ , we have

$$\int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 = 8R_0^3 \left[ \frac{y - \sin(y)}{y^3} \right], \quad (3.81)$$

with  $y = 2R_0|\beta_\xi(j)|$ . Since  $y \rightarrow \frac{y - \sin(y)}{y^3} > 0$  for  $y \geq 0$  and since  $\beta_\xi(j) > 0$  is bounded for  $j \in \mathbb{Z}$  such that  $k^2 \geq \alpha_\xi^2(j)$ , then there exists a constant  $c'_1 > 0$  independent of  $\xi$  and  $j$  such that

$$\int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 \geq c'_1. \quad (3.82)$$

Consider  $\epsilon > 0$ , since  $\theta_\xi(j, R, x_2)$  is continuous on the compact set  $\bar{I}$ . Then there exists  $\delta > 0$  such that for  $|\xi' - \xi| < \delta$  we have

$$|\theta_{\xi'}(j, R, x_2) - \theta_\xi(j, R, x_2)| \leq \epsilon. \quad (3.83)$$

Consequently, using (3.80)-(3.82) we get

$$\int_0^{R_0} |\theta_{\xi'}(j, R, x_2) - \theta_\xi(j, R, x_2)|^2 dy_2 \leq c''_1 R_0 \epsilon \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2, \quad (3.84)$$

with  $c''_1 := 1/\min(c_1, c'_1)$ .

Consider the case where  $k^2 < \alpha_\xi^2(j)$  for which  $\beta_\xi(j) = i\sqrt{|\xi + j|^2 - k^2}$ . Assume in addition that  $|\beta_\xi(j)| \leq 1$ . There exists only a finite number of  $j$  for which this holds.

We have

$$|\theta_\xi(j, R, x_2)|^2 = \frac{e^{-2|\beta_\xi(j)|R}}{|\beta_\xi(j)|^2} \sinh(|\beta_\xi(j)|x_2)^2.$$

Therefore

$$\int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 = \frac{2R_0 e^{-2|\beta_\xi(j)|R}}{|\beta_\xi(j)|^2} \left[ \frac{\sinh(2|\beta_\xi(j)|R_0)}{2R_0|\beta_\xi(j)|} - 1 \right] \geq \frac{4R_0^3}{3} e^{-2|\beta_\xi(j)|R}, \quad (3.85)$$

where we used the inequality  $\frac{\sinh(x)}{x} - 1 \geq \frac{x^2}{6}$  for  $x \geq 0$ . Since  $|\beta_\xi(j)| \leq 1$ , then we get

$$\int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 \geq \frac{4R_0^3}{3} e^{-2R} =: c'_2. \quad (3.86)$$

Consider  $\epsilon > 0$ , since  $\theta_\xi(j, R, x_2)$  is continuous on the compact set  $\bar{I}$ , then there exists  $\delta > 0$  such that for  $|\xi' - \xi| < \delta$  we have

$$\int_0^{R_0} |\theta_{\xi'}(j, R, x_2) - \theta_\xi(j, R, x_2)|^2 dy_2 \leq c'_2 R_0 \epsilon \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2. \quad (3.87)$$

Consider now  $k^2 \leq \alpha_\xi^2(j)$  such that  $|\beta_\xi(j)| > 1$ . Let  $\delta \geq 0$ ,  $0 \leq \delta_0 < \delta$  such that  $\xi' := \xi + \delta_0$ , then we have

$$(\theta_{\xi+\delta_0} - \theta_\xi)(j, R, x_2) = \int_\xi^{\xi+\delta_0} \frac{\partial \theta_\xi}{\partial \xi} d\xi. \quad (3.88)$$

By the Cauchy-Schwartz inequality we get

$$\int_0^{R_0} |\theta_\xi(j, R, x_2) - \theta_{\xi+\delta_0}(j, R, x_2)|^2 dx_2 \leq \delta_0 \int_\xi^{\xi+\delta_0} \left( \int_0^{R_0} \left| \frac{\partial \theta_\xi}{\partial \xi}(j, R, x_2) \right|^2 dx_2 \right) d\xi. \quad (3.89)$$

On the other hand, let us denote by  $\gamma_0 := 2i(\xi + j) \frac{e^{-|\beta_\xi(j)|R}}{|\beta_\xi(j)|^2}$  and  $\gamma_1 := R + \frac{1}{|\beta_\xi(j)|}$ , we have that

$$\frac{\partial \theta_\xi}{\partial \xi}(j, R, x_2) = \gamma_0 [\gamma_1 \sinh(|\beta_\xi(j)|x_2) - x_2 \cosh(|\beta_\xi(j)|x_2)] \quad (3.90)$$

Using that  $\sinh(2y) = 2 \sinh(y) \cosh(y)$ ,  $\cosh^2(y) - \sinh^2(y) = 1$  and  $2 \sinh^2(y) = \cosh(2y) - 1$  for all  $y \in \mathbb{R}$ , we get

$$\left| \frac{\partial \theta_\xi}{\partial \xi}(j, R, x_2) \right|^2 = |\gamma_0|^2 \left[ \left( \frac{\gamma_1^2 + x_2^2}{2} \right) \cosh(2|\beta_\xi(j)|x_2) - \gamma_1 x_2 \sinh(2|\beta_\xi(j)|x_2) - \frac{\gamma_1^2}{2} + \frac{x_2^2}{2} \right]. \quad (3.91)$$

Therefore

$$\begin{aligned} \int_0^{R_0} \left| \frac{\partial \theta_\xi}{\partial \xi}(j, R, x_2) \right|^2 dx_2 &= |\gamma_0|^2 \left[ R_0^2 \frac{\sinh(2|\beta_\xi(j)|R_0)}{4|\beta_\xi(j)|} - \left( \gamma_1 + \frac{1}{2|\beta_\xi(j)|} \right) \frac{R_0 \cosh(2|\beta_\xi(j)|R_0)}{2|\beta_\xi(j)|} \right. \\ &\quad + \frac{\gamma_1^2 \sinh(2|\beta_\xi(j)|R_0)}{4|\beta_\xi(j)|} - \frac{\gamma_1^2 R_0}{2} + \frac{R_0^3}{6} \\ &\quad \left. + \frac{1}{4|\beta_\xi(j)|^2} \left( \gamma_1 + \frac{1}{2|\beta_\xi(j)|} \right) \sinh(2|\beta_\xi(j)|R_0) \right]. \end{aligned}$$

In relation to (3.85) and the previous identity we consider the following functions defined for  $x \in \mathbb{R}$  such that  $x > 1$  as

$$\begin{aligned} f(x) &:= \frac{2e^{-2xR}}{x^2} \left[ \frac{\sinh(2xR_0)}{2x} - R_0 \right], \tag{3.92} \\ g(x) &:= 4 \left| \frac{\sqrt{x^2 + k^2} e^{-xR}}{x^2} \right|^2 \left[ \frac{R_0^2 \sinh(2xR_0)}{4x} - \left( R + \frac{3}{2x} \right) \frac{R_0 \cosh(2xR_0)}{2x} \right. \\ &\quad \left. + \frac{(R + \frac{1}{x})^2 \sinh(2xR_0)}{4x} - \left( R + \frac{1}{x} \right)^2 \frac{R_0}{2} + \frac{R_0^3}{6} + \frac{1}{4x^2} \left( R + \frac{3}{2x} \right) \sinh(2xR_0) \right], \end{aligned}$$

and we prove the existence of a constant  $\alpha > 0$  such that  $g(x) < \alpha f(x)$  for all  $x > 1$ . Let  $M \in \mathbb{R}$  sufficiently large and consider first the case  $1 < x \leq M$ . Since  $f$  and  $g$  are continuous functions and  $f(x) > 0$ , then we have

$$g(x) < \alpha' f(x) \quad \text{for } 1 < x \leq M, \tag{3.93}$$

with  $\alpha' := \frac{\max_{1 \leq x \leq M} g(x)}{\min_{1 \leq x \leq M} f(x)} > 0$ . For the case  $x > M$ , we compare  $f$  and  $g$  at infinity. We have

$$\begin{aligned} g(x) &= \frac{(x^2 + k^2)e^{-2xR}}{2x^4} \left( \frac{e^{2xR_0}}{x} \right) \left[ (R_0 - R)^2 + O\left(\frac{1}{x}\right) \right], \\ f(x) &= \frac{e^{-2xR}}{2x^2} \left( \frac{e^{2xR_0}}{x} \right) \left[ 1 + O\left(\frac{1}{x}\right) \right]. \end{aligned}$$

Therefore  $\frac{g(x)}{f(x)}$  is equivalent to  $(R - R_0)^2$  at infinity. Using (3.93) we deduce the existence of a constant  $\alpha > 0$  such that

$$g(x) \leq \alpha f(x), \quad \text{for all } x > 1,$$

which implies

$$g(|\beta_\xi(j)|) \leq \alpha f(|\beta_\xi(j)|),$$

for all  $j \in \mathbb{Z}$  such that  $|\beta_\xi(j)| > 1$ . Then we deduce that

$$\int_0^{R_0} \left| \frac{\partial \theta_\xi}{\partial \xi}(j, R, x_2) \right|^2 dx_2 \leq \alpha \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2. \quad (3.94)$$

Therefore, using (3.89)-(3.94) we have

$$\int_0^{R_0} |\theta_\xi(j, R, x_2) - \theta_{\xi+\delta_0}(j, R, x_2)|^2 dx_2 \leq \delta_0 \alpha \int_\xi^{\xi+\delta_0} \left( \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 \right) d\xi. \quad (3.95)$$

On the other hand, we show that  $x \rightarrow f(x)$  decreases in  $\mathbb{R}_+$ . Indeed, taking  $y = 2xR_0$

and using that  $\sinh(y) = \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!}$  we get

$$f'(y) = R_0^3 e^{-\frac{R}{R_0}y} h(y),$$

with

$$h(y) := \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k+3)!} \left( \frac{-R}{R_0} + \frac{2k+1}{(2k+4)(2k+5)} y \right).$$

Since  $\cosh(y) := \sum_{k=0}^{\infty} \frac{y^{2k}}{2k!}$  we observe that

$$y^3 h(y) \leq - \sum_{k=0}^{\infty} \frac{y^{2k+3}}{(2k+3)!} + \sum_{k=0}^{\infty} \frac{y^{2k+4}}{(2k+4)!} = -\sinh(y) + y + \cosh(y) - 1 - \frac{y^2}{2} \leq 0$$

for  $y \geq 0$ . Therefore  $f'(y) \leq 0$  for all  $y \geq 0$ . Since  $\xi \rightarrow |\beta_\xi(j)|$  increases in  $\bar{I}$  and  $x \rightarrow f(x)$  decreases in  $\mathbb{R}_+$  we infer that  $\xi \rightarrow \int_0^{R_0} |\theta_\xi(j, R, x_2)| dx_2$  decreases in  $\bar{I}$ .

Therefore, from (3.95) we finally obtain that

$$\int_0^{R_0} |\theta_\xi(j, R, x_2) - \theta_{\xi+\delta_0}(j, R, x_2)|^2 dx_2 \leq \alpha \delta_0^2 \left( \int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 \right),$$

which ends the proof.  $\square$

**Lemma 34.** *The operator  $S : \tilde{L}^2(\Gamma^R) \longrightarrow \tilde{L}^2(D)$  is injective. The closure of its range is*

$$H^{inc}(D) := \{v \in L^2(D); v|_{D^p} \in \tilde{L}^2(D^p); \Delta v + k^2 v = 0 \text{ in } D\}.$$

*Proof.* Let  $g \in \tilde{L}^2(\Gamma^R)$  such that  $Sg = 0$  in  $D$ . Using the unique continuation principle we obtain  $Sg = 0$  in  $\Omega^R$ . Using the continuity and regularity of single layer potentials we have that  $S \in H_{loc}^2(U^R)$  and verifies

$$\begin{cases} \Delta Sg + k^2 Sg = 0 & \text{in } U^R, \\ Sg = 0 & \text{on } \Gamma^R, \end{cases}$$

and the upper going radiation condition (3.3) with  $\Gamma^R$  replaced by  $\Gamma^{R'}$  with  $R' > R$ . Therefore  $Sg = 0$  in  $U^R$ . Using the jump relations for the normal derivative of  $S$  we obtain  $g = 0$  which proves the injectivity of  $S$ .

We prove now the denseness of the range of  $S$ . Let  $v \in \tilde{L}^2(D)$  and we denote by  $v_\xi := (\mathcal{J}v)(\xi, \cdot)$ . Fix  $\epsilon > 0$  and consider a uniform partition of  $I$  into sub-domains  $I_j^N := \bigcup_{j=1}^N I_j$  of size  $\delta = \frac{1}{N}$ . Using Lemma 25 we have, for all  $\xi_j^N \in I_j^N$  there exists  $\tilde{\psi}_j^N := e^{-i\xi_j^N \cdot x} \psi_{\xi_j^N} \in L_{\sharp}^2(\Gamma^R)$  such that

$$\|\tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N} - \tilde{v}_{\xi_j^N}\|_{L^2(\Omega_0^{R_0})} \leq \frac{\epsilon}{4}, \quad (3.96)$$

with  $\tilde{v}_{\xi_j^N} = e^{-i\xi_j^N \cdot x} v_{\xi_j^N}$ , where  $\tilde{S}_\xi$  is defined by (3.77). We introduce the hat functions  $\chi_j^N \in C^0(I)$  that are affine on each domain  $I_l$  and verifies  $\chi_j^N(\xi_l) = \delta_{jl}$ . We then define

$$\tilde{\psi}_\xi^N := \sum_{1 \leq j \leq N} \tilde{\psi}_{\xi_j^N} \chi_j^N(\xi), \quad \hat{v}_\xi^N := \sum_{1 \leq j \leq N} \tilde{v}_{\xi_j^N} \chi_j^N(\xi), \quad \hat{S}_\xi^N := \sum_{1 \leq j \leq N} (\tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N}) \chi_j^N(\xi),$$

for  $1 \leq j \leq N$ . Then, we have

$$\|\tilde{S}_\xi \tilde{\psi}_\xi^N - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} \leq \|\tilde{S}_\xi \tilde{\psi}_\xi^N - \hat{S}_\xi^N\|_{L^2(\Omega_0^{R_0})} + \|\hat{S}_\xi^N - \hat{v}_\xi^N\|_{L^2(\Omega_0^{R_0})} + \|\hat{v}_\xi^N - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})}. \quad (3.97)$$

Since  $\tilde{S}_\xi(\tilde{\psi}_{\xi_j^N} \chi_j^N(\xi)) = (\tilde{S}_\xi \tilde{\psi}_{\xi_j^N}) \chi_j^N(\xi)$ , then the first term in the right hand side of (3.97) verifies

$$\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \hat{S}_\xi^N = \sum_{1 \leq j \leq N} (\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N}) \chi_j^N(\xi).$$

Therefore

$$\|\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \hat{S}_\xi^N\|_{L^2(\Omega_0^{R_0})} \leq \sup_{1 \leq j \leq N} \sup_{\xi \in [\xi_{j-1}^N, \xi_{j+1}^N]} \|\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N}\|_{L^2(\Omega_0^{R_0})}.$$

Consider  $\epsilon' := \frac{\epsilon}{4\sqrt{c} \left( \sup_{\xi \in I} \|v_\xi\|_{L^2(\Omega_0^{R_0})} + 1 \right)}$ , using Lemma 33 we chose  $\delta > 0$  for which

$$\|\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N}\|_{L^2(\Omega_0^{R_0})} \leq \sqrt{c} \epsilon' \|\tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N}\|_{L^2(\Omega_0^{R_0})},$$

therefore

$$\sup_{1 \leq j \leq N} \sup_{\xi \in [\xi_{j-1}^N, \xi_{j+1}^N]} \|\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N}\|_{L^2_\#(\Omega_0^{R_0})} \leq \sqrt{c} \epsilon' \left( \sup_{\xi \in I} \|v_\xi\|_{L^2(\Omega_0^{R_0})} + 1 \right) = \frac{\epsilon}{4}. \quad (3.98)$$

On the other hand, using (3.96) we deduce that the second term in the right hand side of (3.97) verifies

$$\|\hat{S}_\xi^N - \hat{v}_\xi^N\|_{L^2(\Omega_0^{R_0})} = \left\| \sum_{1 \leq j \leq N} (\tilde{S}_{\xi_j^N} \tilde{\psi}_{\xi_j^N} - \tilde{v}_{\xi_j^N}) \chi_j^N(\xi) \right\|_{L^2(\Omega_0^{R_0})} \leq \frac{\epsilon}{4}. \quad (3.99)$$

Moreover, since  $v_\xi \in C_\#^0(I, L_\xi^2(D))$  then  $N$  could have been chosen from the beginning sufficiently large so that

$$\|\hat{v}_\xi^N - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} \leq \frac{\epsilon}{2}. \quad (3.100)$$

Finally, using (3.98), (3.99) and (3.100) we get

$$\sup_{\xi \in I} \|\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} = \sup_{1 \leq j \leq N} \sup_{\xi \in [\xi_{j-1}^N, \xi_{j+1}^N]} \|\tilde{S}_\xi \tilde{\psi}_{\xi_j^N} - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} \leq \epsilon, \quad (3.101)$$

for sufficiently large  $N$ . This proves the denseness of the range of the operator  $S$ .  $\square$

For the analysis below we need to assume the well posedness of the following Interior

Transmission Problem.

**(ITP3)**: Seek  $(u, v) \in \tilde{L}^2(D) \times \tilde{L}^2(D)$  such that  $(u - v) \in \tilde{H}^2(D)$  satisfying

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ u - v = \varphi & \text{on } \partial D \\ \frac{\partial(u - v)}{\partial \nu} = \psi & \text{on } \partial D \end{cases} \quad (3.102)$$

for given  $(\varphi, \psi) \in \tilde{H}^{3/2}(\partial D) \times \tilde{H}^{1/2}(\partial D)$ . This problem has been extensively studied in the literature in the case of bounded domains  $D$  see for instance [11]. Indeed the results for bounded domain  $D$  extend easily to the case where  $D$  is unbounded but is the (infinite) union of disjoint bounded domains. This corresponds for instance to our case when  $D^p$  is the union of disjoint bounded domains. Consider the following assumption

**Assumption 35.** Assume that  $k$ ,  $n_p$  and  $n$  are such as **(ITP3)** is well posed.

**Lemma 36.** Assume that Assumptions 15 and 35 hold. Then the operator  $G$  given by (3.69) is injective with dense range. Moreover,

$$(z \in D) \iff (\Phi(\cdot, z) \in \text{Range}(G)).$$

*Proof.* consider  $v \in H^{inc}(D)$  such that  $G(v) = 0$ , i.e

$$w = 0 \quad \text{on } \Gamma^R,$$

with  $w \in H_{loc}^2(\Omega^R)$  being the solution of  $(\mathcal{P})$ . Therefore,  $w \in H_{loc}^2(U^R)$  and verifies

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } U^R, \\ w = 0 & \text{on } \Gamma^R, \end{cases}$$

and the upper going radiation condition (3.3) with  $\Gamma^R$  replaced by  $\Gamma^{R'}$  with  $R' > R$ . Then  $w = 0$  in  $U^R$ . Using the unique continuation principle we obtain that  $w = 0$  in  $\Omega^R \setminus D$ . Setting  $u := w + v$  we observe that the couple  $(u, v)$  verifies **(ITP3)** with zero



data. Therefore we deduce that  $v = 0$  and then the injectivity of  $G$ .

Now we prove the denseness of the range of  $G$ . Let  $g \in \overline{\mathcal{R}(G)}^\perp$ , then

$$(G(v), g)_{L^2(D)} = 0 \quad \text{for all } v \in H^{inc}(D).$$

Let  $f \in \tilde{L}^2(\Gamma^R)$  and consider  $v = \overline{Sf}$ . We then have

$$(G(\overline{Sf}), g)_{L^2(\Gamma^R)} = 0 \quad \text{for all } f \in \tilde{L}^2(\Gamma^R).$$

On the other hand, consider  $w(f)$  and  $w(g)$  solution of  $(\mathcal{P})$  associated respectively to  $v = \overline{Sf}$  and  $v = \overline{Sg}$ . Using similar arguments as in the proof of Lemma 2.5 one can prove that

$$\int_D (1-n)w(f)\overline{Sg}dy = \int_D (1-n)w(g)\overline{Sf}dx. \quad (3.103)$$

Therefore, from Theorem 18 and (3.103) we get

$$\begin{aligned} (G(\overline{Sf}), g)_{L^2(\Gamma^R)} &= \int_D k^2(1-n)(w(f) + \overline{Sf})\overline{Sg}dy = \int_D k^2(1-n)(w(g) + \overline{Sg})\overline{Sf}dy \\ &= (G(\overline{Sg}), f)_{L^2(\Gamma^R)}. \end{aligned}$$

Consequently

$$(G(\overline{Sg}), f)_{L^2(\Gamma^R)} = 0 \quad \text{for all } f \in \tilde{L}^2(\Gamma^R)$$

which implies that  $G(\overline{Sg}) = 0$ . The injectivity of  $G$  gives that  $Sg = 0$  and then  $g = 0$  by Lemma 34.

Consider  $z \in D$ . We have that  $\chi\Phi(\cdot, z) \in \tilde{L}^2(D)$ , where  $\chi$  is a regular cutoff function that vanishes in a neighborhood of  $z$ . Since  $\Phi(\cdot, z)$  satisfies the Helmholtz equation outside  $z$ , elliptic regularity results applied to each component of  $D^p$  separately implies that  $\chi\Phi \in \tilde{H}^2(D)$ . Trace theorems then imply  $(\Phi(\cdot, z), \frac{\partial\Phi(\cdot, z)}{\partial\nu}) \in \tilde{H}^{3/2}(\partial D) \times \tilde{H}^{1/2}(\partial D)$ . We then consider  $(u, v) \in \tilde{L}^2(D) \times \tilde{L}^2(D)$  to be the solution of **(ITP3)** with  $(\varphi, \psi) =$

$(\Phi(\cdot, z), \frac{\partial \Phi(\cdot, z)}{\partial \nu})$ . We set

$$w := \begin{cases} u - v & \text{in } D \\ \Phi(\cdot, z) & \text{in } \Omega^R \setminus D. \end{cases}$$

We observe that  $w \in H_{loc}^2(\Omega^R)$  and satisfies  $(\mathcal{P})$ . Hence  $G(v) = \Phi(\cdot, z)$ .

Consider now the case where  $z \in \Omega^R \setminus D$ . Assume that there exists  $v \in H^{inc}(D)$  such that  $G(v) = \Phi(\cdot, z)$ . By unique continuation principle we obtain that  $w = \Phi(\cdot, z)$  in  $\Omega^R \setminus D$ , which is a contradiction since  $w \in H_{loc}^2(\Omega^R \setminus D)$  while  $\Phi(\cdot, z) \notin H_{loc}^2(\Omega^R \setminus D)$ .  $\square$

### 3.4.3 Application of the Generalized Linear Sampling Method (GLSM)

Let us consider the functional  $J_\alpha(\phi, \cdot) : \tilde{L}^2(\Gamma^R) \rightarrow \mathbb{R}$

$$J_\alpha(\phi; g) := \alpha \check{I}(g) + \|Ng - \phi\|^2, \quad \text{for all } g \in \tilde{L}^2(\Gamma^R)$$

where

$$\check{I}(g) := \sup_{\xi_0 \in I} I_{\xi_0}(\mathcal{J}g(\xi_0, \cdot)). \quad (3.104)$$

We denote by  $j_\alpha(\phi) = \inf_{g \in \tilde{L}^2(\Gamma^R)} J_\alpha(\phi; g)$ . Let  $c(\alpha) > 0$  verifying  $\frac{c(\alpha)}{\alpha} \rightarrow 0$  as  $\alpha \rightarrow 0$

**Theorem 37.** *Assume that Assumptions 15 and 35 hold. Assume in addition that the hypothesis of Lemma 31 holds. Consider  $z \in \Omega^R$ , and let  $g^\alpha \in \tilde{L}^2(\Gamma^R)$  such that*

$$J_\alpha(\Phi(\cdot, z), g^\alpha(z)) \leq j_\alpha(\Phi(\cdot, z)) + c(\alpha),$$

then

$$z \in D \iff \lim_{\alpha \rightarrow 0} \check{I}(g^\alpha(z)) < \infty. \quad (3.105)$$

Moreover, if  $z \in D$  then  $Sg^\alpha|_D$  converges to some  $v$  in  $L^2(D)$  where  $v$  is solution of (ITP3).

*Proof.* The proof of this theorem is an application of the abstract framework of GLSM given by Theorem 2.7 in [11] and Lemma 34, Lemma 36 and Lemma 31. Lemma 34 and

Lemma 36 prove that the operator

$$N = GS,$$

has dense range. Moreover, we need to verify that the norm  $\check{I}g$  is an equivalent norm to  $\sup_{\xi \in I} \|S\mathcal{J}g(\xi, \cdot)\|_{L^2(D_0)}$  for all  $g \in \check{L}^2(\Gamma^R)$ . Lemma 31 and Theorem 32 prove this norm equivalence. The results of the theorem are then a straightforward application of Theorem 2.7 in [11] and Lemma 36.  $\square$

### 3.5 Numerical examples

In this section we will test numerically the results of the Theorem 32.

**Numerical scheme:** Instead of solving the near field equation point-wise, we equivalently solve the equation in the Fourier domain. We are then led to construct numerically for a grid of sampling points  $z$  a nearby solution satisfying

$$\check{N}_{\xi_0} g_{\xi_0}^\alpha(\ell, z) \simeq \widehat{\Phi}_{\xi_0}(\ell, z),$$

where

$$\check{N}_{\xi_0} g_{\xi_0}^\alpha(\ell, z) := \sum_{\ell \in \mathbb{Z}} g_{\xi_0}^\alpha(\ell) (\widehat{w}_{\xi_0}^p + \widehat{w}_{\xi_0}^p)(\ell, z),$$

with  $\widehat{w}_{\xi_0}^p$  and  $\widehat{w}_{\xi_0}^p$  are respectively the Rayleigh sequences of  $w_\xi^p$  solution of (3.24) and  $\widehat{w}_{\xi_0}^p$  solution of (3.34), and  $\widehat{\Phi}_{\xi_0}(\ell, z)$  is the Rayleigh coefficient of the  $\xi_0$ -quasi-periodic green function given by (3.28). We use the Tikhonov regularization to determine a nearby solution

$$(\alpha + \check{N}_{\xi_0}^* \check{N}_{\xi_0}) g_{\xi_0}^\alpha(z) = \check{N}_{\xi_0}^* (\widehat{\Phi}_{\xi_0}(\ell, z)). \quad (3.106)$$

where  $\alpha$  is a constant determined by Morozov's principle and  $\check{N}_{\xi_0}^*$  is the adjoint of  $\check{N}_{\xi_0}$ . Let  $(\mathbf{e}_1, \mathbf{e}_2)$  be the canonical basis of  $\mathbb{R}^2$ . We denote by  $N_x$  and  $N_y$  respectively the number of discretization points with respect to  $\mathbf{e}_1$  and  $\mathbf{e}_2$  directions. We consider in practice the period  $\Omega_0^R = [-\pi, \pi] \times [0, R]$ . Let  $X_N$  be the matrix of size  $N_x \times N_x$  containing the discretization points of the interval  $]-\pi, \pi[$  and by  $Y_N$  be the matrix of

size  $N_y \times N_y$  containing the discretization points of the interval  $]0, R[$ . Moreover, for  $\epsilon = 10^{-10}$  we consider a uniform partition to the period  $[-\pi, \pi - \epsilon]$  into sub-domains  $Y_1^j := [y_1^j, y_1^{j+1}]$  of size  $\Delta x := \frac{\lambda}{10}$ , with  $\lambda := \frac{2\pi}{|k|}$ . We denote by  $N^{inc} = \frac{2\pi}{\Delta x}$  the number of discretization points ( $y_1^j$ ) and we let

$$\mathcal{Y}_j := \{y_j := (y_1^j, R); \quad 1 \leq j \leq N^{inc}\}, \quad (3.107)$$

the set of point sources and we define by  $L_{inc} := \{-N_L, \dots, N_L\}$  the set of the Rayleigh coefficients where  $N_L := \left\lfloor \frac{1}{2} \left( \left\lceil \frac{2\pi}{\Delta x} \right\rceil - 1 \right) \right\rfloor$  such that the number of components of  $L_{inc}$  is equal to  $N^{inc}$ . We then consider for every  $1 \leq j \leq N^{inc}$  and for  $\xi_0$  fixed

$$\begin{aligned} v_{\xi_0}^j &= \overline{\Phi_{\xi_0}(y_1^j, (X_N, Y_N))} \\ &= -\frac{i}{4\pi} \sum_{j \in \mathbb{Z}} e^{-i\alpha_{\xi_0}(j)(y_1^j - X_N)} \left[ e^{-i\beta_{\xi_0}(j)R} \left( \frac{e^{i\beta_{\xi_0}(j)Y_N} - e^{-i\beta_{\xi_0}(j)Y_N}}{\beta_{\xi_0}(j)} \right) \right], \end{aligned} \quad (3.108)$$

and use the numerical method presented in [29] to solve the scattering problem and generate for all  $1 \leq j \leq N^{inc}$  the solution

$$(w_{\xi_0}^p)_j + (\tilde{w}_{\xi_0}^p)_j,$$

where  $(w_{\xi_0}^p)_j$  is the solution of (3.24) and  $(\tilde{w}_{\xi_0}^p)_j = \mathcal{J}(\tilde{w}_{\xi_0})_j(\xi_0, \cdot)$  with  $(\tilde{w}_{\xi_0})_j$  being the solution of (3.25) with  $v_{\xi_0} = v_{\xi_0}^j$ . The discretized near field operator is then given by the matrix of size  $N^{inc} \times N^{inc}$  defined as

$$\check{N}_{\xi_0}(j, \ell) := ((\hat{w}_{\xi_0}^p)_j + (\tilde{w}_{\xi_0}^p)_j)(\ell), \quad \text{for } 1 \leq j \leq N^{inc} \text{ and } \ell \in L_{inc}. \quad (3.109)$$

We consider a complex-valued random matrix  $K$  of size  $N^{inc} \times N^{inc}$  such that the real and the imaginary parts are in  $[-1, 1]^2$ . The discretized noisy data are then given by

$$\check{N}_{\xi_0}^\delta(j, \ell) := \check{N}_{\xi_0}(j, \ell)(1 + \delta K(j, \ell)) \quad \forall 1 \leq j \leq N^{inc} \text{ and } \ell \in L_{inc}, \quad (3.110)$$

where  $\delta$  being the level of noise fixed at 1%. We use the following indicator function for the reconstruction of  $D_0^p$

$$\mathcal{I}_{\xi_0}(z) := \frac{1}{\|g_{\xi_0}^\alpha(z)\|}. \quad (3.111)$$

In all the following examples the period length is fixed at  $2\pi$  and the number of discretization points are fixed at  $N_x = N_y = 300$  and  $\lambda = \frac{2\pi}{|k|}$  for  $k$  fixed, where we fix  $k$  with a positive imaginary part.

**Remark 38.** *We obtain numerical results comparable to the results presented in the following examples when we use the indicator function defined as follows:*

$$\check{\mathcal{I}}_{\xi_0}(z) := \frac{1}{|(\check{N}_{\xi_0}^\delta g_{\xi_0}(z), g_{\xi_0}(z)) + \delta \|\check{N}_{\xi_0}^\delta\| \|g_{\xi_0}^\alpha(z)\|^2|}. \quad (3.112)$$

**Example 1:** In the first example we fix

$$k := \frac{3.5}{3.14\pi} + 0.7i \quad M = 4, \quad \text{and} \quad \xi_0 = \frac{1}{4}$$

and we consider a simple geometry given by a periodic ball centered at  $(0, \lambda)$  with radius  $R_{D^p} := 0.45\lambda$  without any perturbation as shown in Figure 3.2 (Left). Figure 3.2 (right) presents the numerical reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

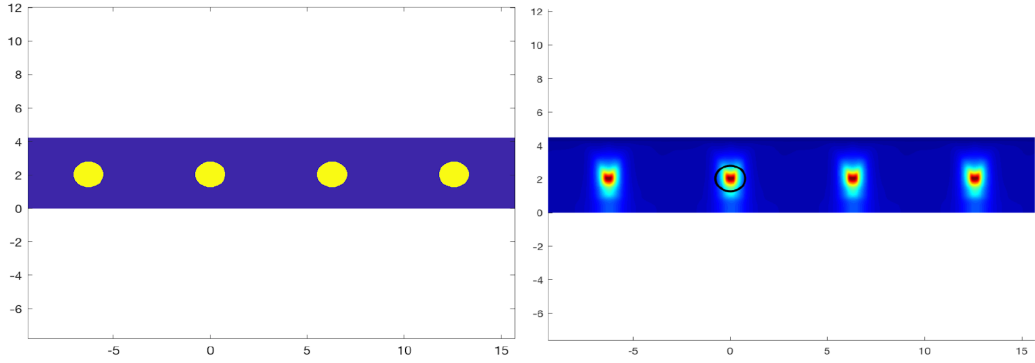


FIGURE 3.2: Left: exact geometry. Right: The reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

**Example 2:** We consider in this example the same physical parameters and periodic geometry, and we add a perturbation given by a ball centered at  $(1.2\lambda, 0.4\lambda)$  with radius  $R_{\bar{D}} := 0.33\lambda$  as shown in Figure 3.3 (Left). Figure 3.3 (right) presents the numerical reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

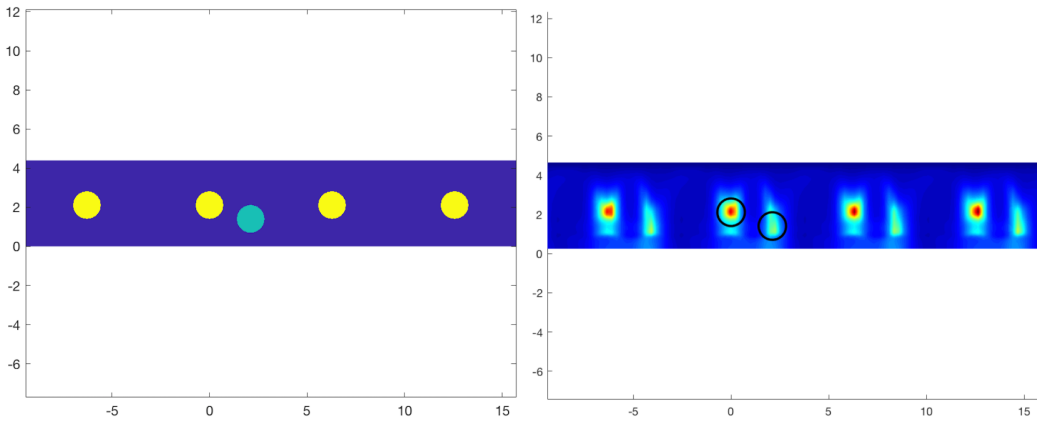


FIGURE 3.3: Left: exact geometry. Right: The reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

**Example 3:** We consider also for  $k := \frac{3.5}{3.14\pi} + 0.4i$ ,  $M = 6$ , and  $\xi_0 = \frac{1}{6}$ , and we consider the geometry of  $D^p$  and  $\tilde{D}$  given as L-shape as shown in Figure 3.4 (Left).

Figure 3.4 (right) presents the numerical reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

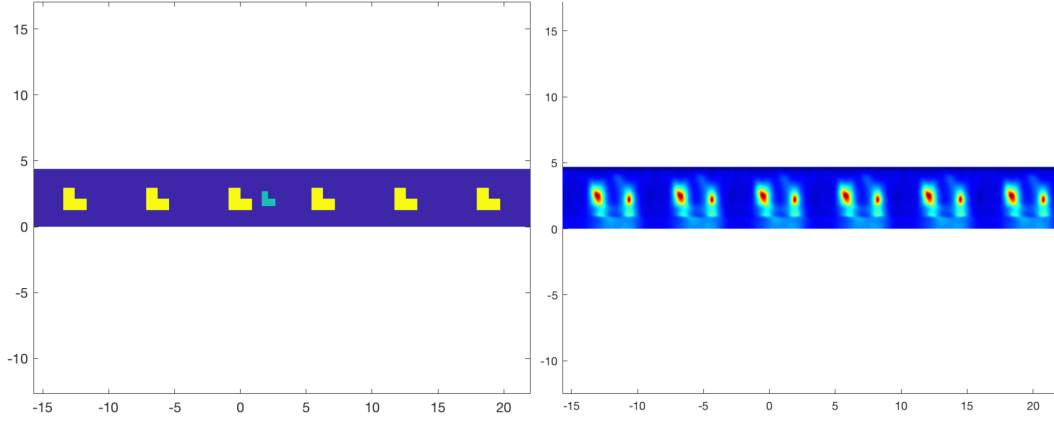


FIGURE 3.4: Left: exact geometry. Right: The reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

**Example 4:** We fix in the last example  $k := \frac{3.5}{3.14\pi} + 0.8i$ ,  $M = 3$ , and  $\xi_0 = \frac{1}{3}$ , and we consider  $D^p$  as a periodic ball centered at  $(0, 1.2\lambda)$  with radius  $R_{D^p} := 0.4\lambda$  and a periodic square centered at  $(0.85\lambda, 1.6\lambda)$  with sides  $0.25\lambda$ . The perturbation is given by the ball centered at  $(0.85\lambda, 1.6\lambda)$  with radius  $R_{\tilde{D}} := 0.25\lambda$  as shown in Figure 3.5 (Left).

Figure 3.5 (right) presents the numerical reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

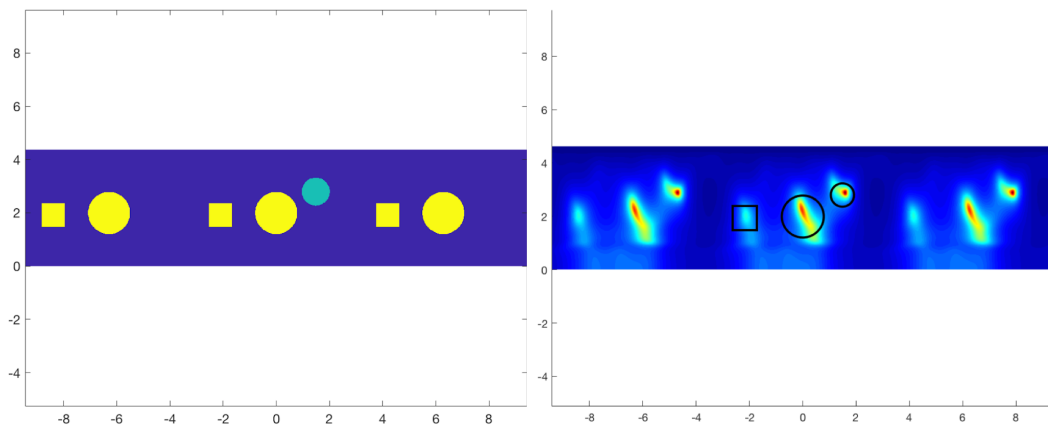


FIGURE 3.5: Left: exact geometry. Right: The reconstruction obtained using  $\mathcal{I}_{\xi_0}$ .

**Conclusion:** In the examples above, we first tested the code with an unperturbed periodic domain with a simple geometry given in Figure 3.2 (left). Using the indicator function defined by (3.111) we obtain the reconstruction of the periodic domain given in the Figure 3.2 (right).

Next, we tested our code for the case of a periodic domain containing a perturbation using different geometries, such as balls, L-shapes and squares as shown respectively in the Figures 3.3 (left), 3.4 (left) and 3.5 (left).

The Figures 3.3 (right), 3.4 (right) and 3.5 (right) respectively show the reconstruction obtained using the indicator function  $\mathcal{I}_{\xi_0}$  which provides the geometry of medium present in the first period distributed periodically with period  $2\pi$ . We also tested the code using the indicator function given by (3.112) and we obtained reconstructions similar to the one presented in the Figures above.

These results show that the indicator function that we introduced in the theory indeed plays the role of an indicator function for the geometry of the domain present in the first period.



# Application to differential imaging

**Abstract:** *We revisit the differential sampling method introduced in [30] for the identification of a periodic domain and some local perturbation. We provide a theoretical justification of the method that avoids assuming that the local perturbation is also periodic. We exploit the results obtained in the previous chapter extended to a domain of periodicity length equal to an integer multiple of original periodicity. This allows us to introduce two indicator functions that allow to directly reconstruct the defect. The theoretical analysis of this chapter is extracted from the published paper [8] and we give in the end some numerical examples.*

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## 4.1 Introduction

In this chapter we are interested in studying the so-called differential sampling method (DLSM) introduced in [30] for the reconstruction of a periodic domain with defects from near fields measurements at fixed frequency. For this algorithm only the periodicity size of the background is assumed to be known a priori. The theoretical developments of the previous chapter allow us to provide a theoretical justification of the algorithm proposed in [30] that provides an indicator function for the defect independently from periodic background. This justification does not assume that the perturbation is also periodic (with a larger periodicity) which was the case in [30]. Indeed, combining sampling methods for a single Floquet-Bloch mode and the sampling method using the full measurement operator, one is able to design an indicator function that separates the perturbation from the periodic background. The principle consists in observing that we do not change the scattering problem if we redefine the periodicity of the background as an integer multiple of the original periodicity. The differential indicator function is build using a comparison of the GLSM indicator function when we use these different definitions of the periodicity of the background. The analysis of GLSM for quasi-periodic incident waves is by itself sufficient to derive an indicator function in the spirit of the differential linear sampling method of [30]. We introduce two indicator functions, one depending only on the indicator function associated with the single Floquet-Bloch mode, and the other depending on the one associated with the full measurement operator.

The chapter is organized as follows. A new setting for the inverse problem in a domain with a periodicity length equal to an integer multiple of  $2\pi$  is given in section 4.2. Section 4.3 explains how the single Floquet-Bloch mode indicator function can provide an indicator function for the periodic background. We introduce in Section 4.4 two differential indicator functions for the defect. We give in the last section some numerical examples.

## 4.2 Setting of the problem and notations

Note that we keep here the notation of the previous chapter, and we keep referring by periodic functions (quasi-periodic functions) to those which are periodic (quasi-periodic) with respect to the first variable  $x_1$ . As mentioned above, the principle idea behind the DLSM method analyzed in this chapter is to consider the background as  $2\pi M$  periodic with  $M \in \mathbb{N}$  such that  $M \geq 1$  and combine the application of the framework given by the previous chapter to different values of  $M$ . Indeed, the refractive index  $n_p$  is also  $2\pi M$ -periodic with respect to the first component  $x_1$  (see Figure 4.1). Then, we can follow the same approach adopted in Section 3.3 of the previous chapter by taking.

$$\Omega_0^R = \Omega_0^{R,M} := [0, 2\pi M] \times [0, R] \quad \text{and} \quad D_0^p = D_0^{p,M} := \Omega_0^{R,M} \cap D^p,$$

in order to reconstruct  $D_0 = D_0^M := D_0^{p,M} \cup \tilde{D}$  using the GLSM method. To this end, we reformulate in this section the setting of our problem in  $2\pi M$ -periodic medium

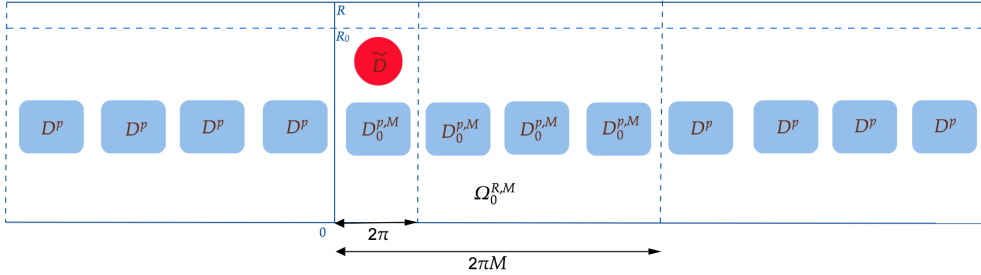


FIGURE 4.1: Sketch of the domain

In the following, for  $m \geq 0$ , the spaces  $L_{\xi,M}^2(\Omega^R)$ ,  $H_{\xi,M}^m(\Omega^R)$  and  $H_{\xi,M}^m(\Omega_0^{R,M})$  has respectively the same definition as  $L_{\xi,M}^2(\Omega^R)$ ,  $H_{\xi}^m(\Omega^R)$  and  $H_{\xi}^m(\Omega_0^R)$  in the previous chapter with period  $2\pi$  replaced by  $2\pi M$ , that is,  $L_{\xi,M}^2(\Omega^R)$  and  $H_{\xi,M}^m(\Omega^R)$  denotes respectively the spaces of  $\xi$ -quasi periodic functions in  $L_{loc}^2(\Omega^R)$  and  $H_{loc}^m(\overline{\Omega^R})$ , with period  $2\pi M$ . We denote  $H_{\xi,M}^m(\Omega_0^{R,M})$  the restriction to  $\Omega_0^{R,M}$  of functions in  $H_{\xi,M}^m(\Omega^R)$ , and  $H_{\#,M}^1(\Omega_0^{R,M})$  denotes the space  $H_{\xi,M}^1(\Omega_0^{R,M})$  for  $\xi = 0$ . Moreover, we define for  $\phi \in C_0^\infty(U^0)$  the one dimensional Floquet-Bloch transform with period  $2\pi M$  as

$$\mathcal{J}_M \phi(\xi, x_1, x_2) = \sum_{j \in \mathbb{Z}} \phi(x_1 + 2\pi M j, x_2) e^{-i2\pi M \xi \cdot j}, \quad \xi \in I^M := \left[0, \frac{1}{M}\right], \quad (x_1, x_2) \in U^0.$$

Fix  $\xi \in I$  and we denote by  $\xi_0 := \frac{\xi}{M} \in I^M$ . Consider  $v_{\xi_0}^M \in L_{\xi_0, M}^2(D)$ , where

$$L_{\xi_0, M}^2(D) := \{v \in L_{loc}^2(D) / v|_{D^p} \in L_{\xi_0, M}^2(D^p)\},$$

with  $L_{\xi_0, M}^2(D^p)$  denotes the set of  $\xi_0$  quasi-periodic functions in  $L_{loc}^2(D^p)$  with period  $2\pi M$ . Similarly to the previous chapter we define a solution  $w_{\xi_0}^M$  to problem  $(\mathcal{P})$  given by (3.1) associated with  $v = v_{\xi_0}^M$  as

$$w_{\xi_0}^M := w_{\xi_0}^{p, M} + \tilde{w}_{\xi_0}^M, \quad (4.1)$$

$$\begin{cases} \Delta w_{\xi_0}^{p, M} + k^2 n_p w_{\xi_0}^{p, M} = k^2(1 - n_p)v_{\xi_0}^M & \text{in } \Omega^R, \\ w_{\xi_0}^{p, M} = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_{\xi_0}^{p, M}}{\partial x_2}(\cdot, R) = T_{\xi_0}^{R, M}(w_{\xi_0}^{p, M}|_{\Gamma_0^{R, M}}) & \text{on } \Gamma_0^{R, M}, \end{cases} \quad (4.2)$$

and  $\tilde{w}_{\xi_0}^M \in \tilde{H}^1(\Omega^R)$  solution of

$$\begin{cases} \Delta \tilde{w}_{\xi_0}^M + k^2 n \tilde{w}_{\xi_0}^M = k^2(n_p - n)(v_{\xi_0}^M + w_{\xi_0}^{p, M}) & \text{in } \Omega^R, \\ \tilde{w}_{\xi_0}^M = 0 & \text{on } \Gamma^0, \\ \frac{\partial \tilde{w}_{\xi_0}^M}{\partial x_2}(\cdot, R) = T^R(\tilde{w}_{\xi_0}^M|_{\Gamma^R}) & \text{on } \Gamma^R, \end{cases} \quad (4.3)$$

with  $\Gamma_0^{R, M} := [0, 2\pi M] \times \{R\}$  and  $T_{\xi_0}^{R, M} : H_{\xi_0, M}^{1/2}(\Gamma_0^{R, M}) \longrightarrow H_{\xi_0, M}^{-1/2}(\Gamma_0^{R, M})$  is the Dirichlet-to-Neumann operator defined as (3.10) with  $\hat{\varphi}_\xi$  is replaced by

$\widehat{\varphi}_{\xi_0}^M(j) := \frac{1}{2\pi M} \int_0^{2\pi M} e^{-i(\xi_0 + j)x_1} \varphi(x_1, R) dx_1$  where the norm in  $H_{\xi_0}^s(\Gamma_0^{R, M})$  is defined as

$$\|\varphi\|_{H_{\xi_0}^s(\Gamma_0^{R, M})}^2 = \sum_{j \in \mathbb{Z}} (1 + j^2)^s |\widehat{\varphi}_{\xi_0}^M(j)|^2.$$

We consider  $\Phi_{\xi_0, M}(x, y) := (\mathcal{J}_M \Phi(\cdot, y))(\xi_0, x)$  the  $\xi_0 M$ -quasi-periodic Green function with period  $2\pi M$ . Let  $y \in \Gamma_0^{R, M}$ , we define  $u_{\xi_0}^{s, M}(\cdot, y) = w_{\xi_0}$  given by (4.1) with  $v_{\xi_0}^M = \overline{\Phi_{\xi_0, M}(y, \cdot)} \in L_{\xi_0, M}^2(D)$ . From (4.1) we decompose  $u_{\xi_0}^{s, M}(\cdot, y) = u_{\xi_0}^{s, p}(\cdot, y) + \tilde{u}_{\xi_0}^s(\cdot, y)$  with  $u_{\xi_0}^{s, p}(\cdot, y) = w_{\xi_0}^{p, M}$  solution of (4.2) and  $\tilde{u}_{\xi_0}^s(\cdot, y) = \tilde{w}_{\xi_0}^M$  solution of (4.3). We introduce

the  $\xi_0 M$ -quasi periodic near field operator  $N_{\xi_0}^M : L_{\xi_0, M}^2(\Gamma^R) \longrightarrow L_{\xi_0, M}^2(\Gamma^R)$  given as

$$\begin{aligned} N_{\xi_0}^{p, M} g_{\xi_0}^M(x) &:= \int_{\Gamma_0^{R, M}} g_{\xi_0}^M(y) u_{\xi_0, M}^{s, p}(x, y) ds(y), \\ \tilde{N}_{\xi_0}^{p, M} g_{\xi_0}^M(x) &:= \int_{\Gamma_0^{R, M}} g_{\xi_0}^M(y) \mathcal{J}_M(\tilde{u}_{\xi_0, M}^s(\cdot, y))(\xi_0, x) ds(y). \end{aligned}$$

Define  $S_{\xi_0}^M : L_{\xi_0, M}^2(\Gamma^R) \longrightarrow L_{\xi_0, M}^2(D)$  the operator given as

$$S_{\xi_0}^M g_{\xi_0}^M(x) := \int_{\Gamma_0^{R, M}} g_{\xi_0}^M(y) \overline{\Phi_{\xi_0, M}(y, x)} ds(y). \quad (4.4)$$

Then, as in (3.32) we can decompose the operator  $N_{\xi_0}^M$  as

$$N_{\xi_0}^M = G_{\xi_0}^M(S_{\xi_0}^M),$$

where  $G_{\xi_0}^M : L_{\xi_0, M}^2(D) \longrightarrow L_{\xi_0, M}^2(\Gamma^R)$  is the operator defined by

$$G_{\xi_0}^M(v_{\xi_0}^M) = (w_{\xi_0}^{p, M} + \tilde{w}_{\xi_0}^{p, M})|_{\Gamma_0^{R, M}},$$

with  $w_{\xi_0}^{p, M} \in H_{\xi_0, M}^1(\Omega_0^{R, M})$  being the solution of (4.2) and  $\tilde{w}_{\xi_0}^{p, M} = \mathcal{J}_M(\tilde{w}_{\xi_0}^M)(\xi_0, \cdot)$  with  $\tilde{w}_{\xi_0}^M \in \tilde{H}^1(\Omega^R)$  is the solution of (4.3). Moreover, Similarly to Lemma 24 we observe that the operators  $N_{\xi_0}^{p, M}$  and  $\tilde{N}_{\xi_0}^{p, M}$  can be respectively factorized as

$$N_{\xi_0}^{p, M} = (S_{\xi_0}^M)^* T_{\xi_0}^{p, M} S_{\xi_0}^M \quad \text{and} \quad \tilde{N}_{\xi_0}^{p, M} = (S_{\xi_0}^M)^* \tilde{T}_{\xi_0}^{p, M} S_{\xi_0}^M, \quad (4.5)$$

with  $T_{\xi_0}^{p, M} : L_{\xi_0, M}^2(D^p) \longrightarrow L_{\xi_0, M}^2(D^p)$  and  $\tilde{T}_{\xi_0}^{p, M} : L_{\xi_0, M}^2(D) \longrightarrow L_{\xi_0, M}^2(D)$  are respectively defined by

$$T_{\xi_0}^{p, M} v_{\xi_0}^M = k^2(1 - n_p)(v_{\xi_0}^M + w_{\xi_0}^{p, M}), \quad (4.6)$$

$$\tilde{T}_{\xi_0}^{p, M} v_{\xi_0}^M = k^2(1 - n_p)\tilde{w}_{\xi_0}^{p, M} + k^2(n_p - n)(w_{\xi_0}^{p, M} + v_{\xi_0}^M + \tilde{w}_{\xi_0}^M), \quad (4.7)$$

where  $w_{\xi_0}^{p, M}$  being the solution of (4.2) and  $\tilde{w}_{\xi_0}^{p, M} = \mathcal{J}_M(\tilde{w}_{\xi_0}^M)(\xi_0, \cdot)$  with  $\tilde{w}_{\xi_0}^M$  is the solution of (4.3). Finally let us define for  $g_{\xi_0}^M \in L_{\xi_0, M}^2(\Gamma^R)$  the norm

$$I_{\xi_0}^M g_{\xi_0}^M := \left| (N_{\xi_0}^{p, M} g_{\xi_0}^M, g_{\xi_0}^M)_{L^2(\Gamma_0^{R, M})} \right| + \left| (\tilde{N}_{\xi_0}^{p, M} g_{\xi_0}^M, g_{\xi_0}^M)_{L^2(\Gamma_0^{R, M})} \right|. \quad (4.8)$$

From the factorizations (4.5) we have the equivalent expression

$$I_{\xi_0}^M g_{\xi_0}^M := \left| (T_{\xi_0}^{p,M} S_{\xi_0}^M g_{\xi_0}^M, S_{\xi_0}^M g_{\xi_0}^M)_{L_{\xi_0, M}^2(D^p)} \right| + \left| (\tilde{T}_{\xi_0}^{p,M} S_{\xi_0}^M g_{\xi_0}^M, S_{\xi_0}^M g_{\xi_0}^M)_{L_{\xi_0, M}^2(D)} \right|. \quad (4.9)$$

### 4.3 Application to the GLSM method

#### 4.3.1 Application of the GLSM for the reconstruction of $D_0^M$ .

Following the same steps as in Section 3.3 we can present the free noise version of the GLSM. Introducing the functional  $J_{\xi_0}^{\alpha, M} : L_{\xi_0, M}^2(\Gamma^R) \rightarrow \mathbb{R}$  given as

$$J_{\xi_0}^{\alpha, M}(\phi; g_{\xi_0}) = \alpha I_{\xi_0}^M(g_{\xi_0}) + \|(N_{\xi_0}^{p, M} + \tilde{N}_{\xi_0}^{p, M})g_{\xi_0} - \phi\|^2.$$

We denote by  $j_{\xi_0}^{\alpha, M}(\phi) = \inf_{g_{\xi_0} \in L_{\xi_0, M}^2(D)} J_{\xi_0}^{\alpha, M}(\phi; g_{\xi_0})$ . Moreover, let  $c(\alpha) > 0$  verifying  $\frac{c(\alpha)}{\alpha} \rightarrow 0$  as  $\alpha \rightarrow 0$ .

**Theorem 39.** *Assume that Assumptions 26 and 27 hold. Assume in addition that the hypothesis of Theorem 19 and Lemma 31 hold. Consider  $z \in \Omega^R$ , and let  $g_{\xi_0}^\alpha \in L_{\xi_0, M}^2(\Gamma^R)$  such that*

$$J_{\xi_0}^{\alpha, M}(\Phi_{\xi_0, M}(\cdot, z), g_{\xi_0}^\alpha(z)) \leq j_{\xi_0}^{\alpha, M}(\Phi_{\xi_0, M}(\cdot, z)) + c(\alpha),$$

then

$$z \in D_0^M \iff \lim_{\alpha \rightarrow 0} I_{\xi_0}^M(g_{\xi_0}^\alpha(z)) < \infty.$$

*Proof.* The prove is similar to the proof of Theorem 32. □

#### 4.3.2 Application of the GLSM for the reconstruction of $D_0^{p, M}$

In this section we consider  $M \geq 2$  and we explain how one can reconstruct only the domain  $D_0^{p, M}$ . Fix  $\xi \in I$  and denote by  $\xi_0 := \frac{\xi}{M} \in I^M$ . We observe that the Green function  $\Phi_{\xi_0}(\cdot, z)$  is also  $\xi_0 M$ -quasi periodic with period  $2\pi M$ . Therefore, we can follow the same steps in the previous section by replacing  $\Phi_{\xi_0, M}(x, y)$  by  $\Phi_{\xi_0}(x, y)$  and we use that  $\Phi_{\xi_0}(\cdot, z)$  admits singular points in  $\Omega_0^{R, M}$  for  $z \in \tilde{D}$  to reconstruct only the periodic domain  $D_0^{p, M}$ .

**Lemma 40.** *Assume that the assumptions of Theorem 19 hold and that Assumptions 26 holds. Then we have that*

$$(z \in D_0^{p,M}) \iff (\Phi_{\xi_0}(\cdot, z) \in \text{Range}(G_{\xi_0}^M)).$$

*Proof.* Let  $z \in D^p$ . We consider  $v_{\xi_0} \in H_{\xi_0}^{\text{inc}}(D)$  such that  $v_{\xi_0}|_{\tilde{D}} = -\Phi_{\xi_0}(\cdot, z)$ . Let  $(u_{\xi_0}, v_{\xi_0}) \in L_{\xi_0}^2(D^p) \times L_{\xi_0}^2(D^p)$  be the solution of **(ITP1)** with  $(\varphi, \psi) = \left( \Phi_{\xi_0}(\cdot, z), \frac{\partial \Phi_{\xi_0}(\cdot, z)}{\partial \nu} \right)$ .

We set

$$w_{\xi_0}^p := \begin{cases} u_{\xi_0} - v_{\xi_0} & \text{in } D^p, \\ \Phi_{\xi_0}(\cdot, z) & \text{in } \Omega^R \setminus D^p. \end{cases}$$

We observe that  $v_{\xi_0} \in H_{\xi_0, M}^{\text{inc}}(D)$ , where

$$H_{\xi_0, M}^{\text{inc}}(D) := \{v \in L_{\xi_0, M}^2(D), \Delta v + k^2 v = 0 \text{ in } D\},$$

and  $w_{\xi_0}^p \in H_{\xi_0, M}^2(\Omega^R)$  satisfies (3.24). Moreover, let  $\tilde{w}_{\xi_0}$  be the solution of (3.25). Since  $(n_p - n)(w_{\xi_0}^p + v_{\xi_0}) = 0$ , then  $\tilde{w}_{\xi_0} = 0$  and consequently  $G_{\xi_0}^M(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$ .

Consider now the case where  $z := (z_1, z_2) \notin D^p$ . Assume that there exists  $v_{\xi_0} \in H_{\xi_0, M}^{\text{inc}}(D)$  such that  $G_{\xi_0}^M(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$ . By the unique continuation principle we get  $w_{\xi_0} := w_{\xi_0}^p + \tilde{w}_{\xi_0}^p = \Phi_{\xi_0}(\cdot, z)$  in  $U^0 \setminus D_0^M$ . Since  $\tilde{D}$  is not distributed periodically, then for  $z \in \tilde{D}$  there exists  $j \in \mathbb{Z}$  such that  $z_j := (z_1 + 2\pi j, z_2) \in \Omega_0^{R, M} \setminus D_0^M$ . Therefore,  $\Phi_{\xi_0}(\cdot, z) \notin H_{\xi_0, M}^2(\Omega^R \setminus D_0^M)$  for all  $z \in \Omega^R \setminus D_0^M$  while  $w_{\xi_0} \in H_{\xi_0, M}^2(\Omega^R \setminus D_0^M)$ , which is a contradiction.  $\square$

**Theorem 41.** *Assume that Assumptions 26 and 27 hold. Assume in addition that the hypothesis of Theorem 19 and Lemma 31 hold. Consider  $z \in \Omega^R$ , and let  $g_{\xi_0}^\alpha \in L_{\xi_0, M}^2(\Gamma^R)$  such that*

$$J_{\xi_0}^{\alpha, M}(\Phi_{\xi_0}(\cdot, z), g_{\xi_0}^\alpha(z)) \leq j_{\xi_0}^{\alpha, M}(\Phi_{\xi_0}(\cdot, z)) + c(\alpha),$$

then

$$z \in D_0^{p, M} \iff \lim_{\alpha \rightarrow 0} I_{\xi_0}^M(g_{\xi_0}^\alpha(z)) < \infty.$$

Moreover, if  $z \in D_0^{p, M}$  then  $S_{\xi_0}^M g_{\xi_0}^\alpha|_{D_0^{p, M}}$  converges to some  $v_{\xi_0}$  in  $L^2(D_0^{p, M})$  where  $v_{\xi_0}$  is solution of **(ITP1)**.

*Proof.* As in the proof of Theorem 32. By Lemma 24, Lemma 25 and Lemma 29 adopted to the  $2\pi M$  periodic case we prove that the operator  $N_{\xi_0}^M = N_{\xi_0}^{p,M} + \tilde{N}_{\xi_0}^{p,M}$  can be factorized as

$$N_{\xi_0}^M = G_{\xi_0}^M S_{\xi_0}^M = S_{\xi_0}^{*,M} T_{\xi_0}^M S_{\xi_0}^M, \quad (4.10)$$

and has dense range, with  $S_{\xi_0}^{*,M} : L_{\xi_0,M}^2(D) \rightarrow L_{\xi_0,M}^2(\Gamma^R)$  is the adjoint of the operator  $S_{\xi_0}^M$  and  $T_{\xi_0}^M$  is the operator defined as (3.40). Moreover, using Lemma 31 we prove that the norm  $I_{\xi_0}^M g_{\xi_0}$  is an equivalent norm to  $\|S_{\xi_0}^M g_{\xi_0}\|_{L^2(D_0^M)}$ . Therefore, the results of the theorem are then a straightforward application of Theorem 2.7 in [11] and Lemma 40.  $\square$

## 4.4 Application of the Differential Sampling Method

As in [30], we explain in this section how one directly reconstruct  $\tilde{D}$  using a differential indicator function. Consider  $M \geq 2$ , fix  $\xi \in I$ . We denote by  $\xi_0 := \frac{\xi}{M} \in I^M$ . Consider  $g^\alpha \in \tilde{L}^2(\Gamma^R)$ ,  $g_{\xi_0}^\alpha \in L_{\xi_0}^2(\Gamma^R)$  and  $g_{\xi_0}^{\alpha,M} \in L_{\xi_0,M}^2(\Gamma^R)$  satisfying

$$\begin{aligned} J_\alpha(\Phi(\cdot, z), g^\alpha(z)) &\leq j_\alpha(\Phi(\cdot, z)) + c(\alpha), \\ J_{\xi_0}^\alpha(\Phi_{\xi_0}(\cdot, z), g_{\xi_0}^\alpha(z)) &\leq j_{\xi_0}^\alpha(\Phi_{\xi_0}(\cdot, z)) + c(\alpha), \\ J_{\xi_0}^{\alpha,M}(\Phi_{\xi_0}(\cdot, z), g_{\xi_0}^{\alpha,M}(z)) &\leq j_{\xi_0}^{\alpha,M}(\Phi_{\xi_0}(\cdot, z)) + c(\alpha), \end{aligned}$$

for  $c(\alpha) > 0$  verifying  $\frac{c(\alpha)}{\alpha} \rightarrow 0$  as  $\alpha \rightarrow 0$ . Let us define the indicator function to identify  $\tilde{D}$  as

$$\mathcal{I}^\alpha(z) := \left[ \check{I}(g^\alpha) \left( 1 + \frac{\check{I}(g^\alpha)}{I_{\xi_0}^M(g_{\xi_0}^{\alpha,M} - \frac{1}{M}g_{\xi_0}^\alpha)} \right) \right]^{-1}, \quad (4.11)$$

with  $\check{I}$  is the norm defined by (3.104) and  $I_{\xi_0}^M$  is the norm given as (3.59) with  $N_{\xi_0}^p = N_{\xi_0}^{p,M}$ ,  $\tilde{N}_{\xi_0}^p = \tilde{N}_{\xi_0}^{p,M}$  and  $\Gamma_0^R = \Gamma_0^{R,M}$ .

**Theorem 42.** *Under the assumptions of Theorem 41 we have*

$$(z \in \tilde{D}) \iff \left( \lim_{\alpha \rightarrow 0} \mathcal{I}^\alpha > 0 \right).$$



*Proof.* Consider  $z \in D^p$ . By Theorem 32 and Theorem 41 we have that  $S_{\xi_0} g_{\xi_0}^\alpha$  and  $S_{\xi_0}^M g_{\xi_0}^{\alpha, M}$  converges respectively to  $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$  and  $v_{\xi_0}^M \in H_{\xi_0, M}^{inc}(D)$  verifying

$$G_{\xi_0}(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z) \quad \text{and} \quad G_{\xi_0}^M(v_{\xi_0}^M) = \Phi_{\xi_0}(\cdot, z). \quad (4.12)$$

Moreover, from Lemma 30 and Lemma 40 we observe that  $v_{\xi_0}$  and  $v_{\xi_0}^M$  are solutions of **(ITP1)** with  $(\varphi, \psi) = \left( \Phi_{\xi_0}(\cdot, z), \frac{\partial \Phi_{\xi_0}(\cdot, z)}{\partial \nu} \right)$ , then  $v_{\xi_0}$  coincides with  $v_{\xi_0}^M$ . On the other hand, from (3.60), Lemma 31 and Theorem 19 we have that  $I_{\xi_0}^M g_{\xi_0}$  is equivalent (uniformly with respect to  $\xi_0$ ) to  $\|S_{\xi_0}^M g_{\xi_0}\|_{L^2(D_0^M)}^2$ . In particular, there exists of constant  $c_1 > 0$  independent from  $\xi_0$  such that

$$I_{\xi_0}^M(g_{\xi_0}^{\alpha, M} - \frac{1}{M}g_{\xi_0}^\alpha) \leq c_1 \left\| S_{\xi_0}^M g_{\xi_0}^{\alpha, M} - \frac{1}{M}S_{\xi_0}^M g_{\xi_0}^\alpha \right\|_{L^2(D_0^M)}^2. \quad (4.13)$$

Moreover, we observe that  $S_{\xi_0}^M g_{\xi_0}^\alpha = M S_{\xi_0} g_{\xi_0}^\alpha$ . Therefore, the right hand side of (4.13) tends to zero as  $\alpha \rightarrow 0$ . On the other hand, from Theorem 37 we have that  $\check{I}(g^\alpha) < \infty$  as  $\alpha \rightarrow 0$ . Hence

$$\lim_{\alpha \rightarrow 0} \mathcal{I}^\alpha(z) = 0 \quad \text{for } z \in D^p.$$

Consider now the case where  $z \in \tilde{D}$ . From Theorem 32 and Theorem 41 we have that  $\|S_{\xi_0}^M g_{\xi_0}^{\alpha, M}\|_{L^2(D_0^M)} \rightarrow \infty$  and  $\|S_{\xi_0} g_{\xi_0}\|_{L^2(D_0)}$  is bounded as  $\alpha \rightarrow 0$ . Moreover, we have that

$$I_{\xi_0}^M(g_{\xi_0}^{\alpha, M} - \frac{1}{M}g_{\xi_0}^\alpha) \geq c_2 \|S_{\xi_0}^M g_{\xi_0}^{\alpha, M} - S_{\xi_0} g_{\xi_0}^\alpha\|_{L^2(D_0^M)}^2 \geq c_2 \|S_{\xi_0}^M g_{\xi_0}^{\alpha, M}\|_{L^2(D_0^M)}^2 - c_2 \|S_{\xi_0} g_{\xi_0}^\alpha\|_{L^2(D_0)}^2,$$

with  $c_2 > 0$  is a constant independent from  $\xi_0$ . therefore

$$I_{\xi_0}^M(g_{\xi_0}^{\alpha, M} - \frac{1}{M}g_{\xi_0}^\alpha) \rightarrow \infty \quad \text{as } \alpha \rightarrow 0,$$

which implies that

$$0 < \lim_{\alpha \rightarrow 0} \mathcal{I}^\alpha(z) < \infty \quad \text{for } z \in \tilde{D},$$

This ends the proof. □

**Remark 43.** Consider  $\hat{g}_{\xi_0}^{\alpha, M}(\Gamma^R)$  satisfying

$$J_{\xi_0}^{\alpha, M}(\Phi_{\xi_0, M}(\cdot, z), \hat{g}_{\xi_0}^{\alpha, M}(z)) \leq j_{\xi_0}^{\alpha, M}(\Phi_{\xi_0, M}(\cdot, z)) + c(\alpha),$$

and define

$$\mathcal{I}_M^\alpha(z) := \left[ I_{\xi_0}^M(\hat{g}_{\xi_0}^{\alpha, M}) \left( 1 + \frac{I_{\xi_0}^M(\hat{g}_{\xi_0}^{\alpha, M})}{I_{\xi_0}^M(g_{\xi_0}^{\alpha, M} - \frac{1}{M}g_{\xi_0}^\alpha)} \right) \right]^{-1}.$$

We observe that  $\mathcal{I}_M^\alpha$  can be considered also as an indicator function for the identification of  $\tilde{D}$  repeated  $2\pi M$  periodically, i.e Theorem 42 still holds if we change  $\mathcal{I}^\alpha$  by  $\mathcal{I}_M^\alpha$  and  $\tilde{D}$  by its  $2\pi M$  copies. The proof follows the same arguments but with applying Theorem 39 instead of Theorem 37.

## 4.5 Numerical examples

**Numerical scheme:** In this section we use the same procedure considered in Section 3.5 in the previous chapter such that we solve the equations in the Fourier domain, we also consider the same discretization scheme as in Section 3.5. We recall that  $X_N$  and  $Y_N$  denotes respectively the  $N_x \times N_x$  and  $N_y \times N_y$  matrices containing the discretization points of the intervals  $]-\pi, \pi[$  and  $]0, R[$ , where  $N_x$  and  $N_y$  are respectively the number of discretization points w.r.t the  $x_1$  and  $x_2$  directions, and  $\mathcal{Y}_j$  is the set of point sources of size  $N^{inc}$  defined by (3.107).

We consider in practice the period of size  $2\pi$  as  $\Omega_0^R = [-\pi, \pi] \times [0, R]$  and the period of size  $2\pi M$  as  $\Omega_0^{R, M} = [2\pi([\frac{-M}{2}] + \frac{1}{2}), 2\pi([\frac{M}{2}] + \frac{1}{2})] \times [0, R]$ .

Similarly to the theoretical analysis we also extend the numerical scheme analyzed in Section 3.5 to a  $2\pi M$ -periodic setting. In order to do so, we exploit the decomposition of  $2\pi M$ -periodic functions into the sum of  $M$   $2\pi$ -quasi-periodic functions [30], that is, for  $M' \in \mathbb{N}^*$  such that  $M' \geq M$  and  $\frac{M'}{M} \in \mathbb{N}^*$ , fixing

$$\xi_{j_0} := \frac{j_0}{M'}, \quad \text{for } \frac{-M}{2M'} + 1 \leq j_0 \leq \frac{M}{2M'},$$

considering

$$\tilde{\xi}_m := \xi_{j_0} + \frac{m}{M} = \frac{j_0}{M} + \frac{m}{M}, \quad \text{for } [\frac{-M}{2}] + 1 \leq m \leq [\frac{M}{2}],$$

and defining a  $\xi_{j_0}$   $2\pi M$ -periodic function  $w_{\xi_{j_0}, M} \in H_{\xi_{j_0}, M}^1(\Omega^R)$ , then the following decomposition holds

$$w_{\xi_{j_0}, M} = \frac{1}{M} \sum_{m=\lceil \frac{-M}{2} \rceil + 1}^{\lfloor \frac{M}{2} \rfloor} w_{\tilde{\xi}_m}, \quad (4.14)$$

where  $w_{\tilde{\xi}_m} \in H_{\tilde{\xi}_m}^1(\Omega_0^R)$ . Let us denote by  $N_{inc}^M := MN^{inc}$ , then for every  $\lceil \frac{-M}{2} \rceil + 1 \leq m \leq \lfloor \frac{M}{2} \rfloor$  we consider for  $\tilde{\xi}_m$  fixed and for all  $1 \leq j \leq N^{inc}$

$$v_{\tilde{\xi}_m}^j = \overline{\Phi_{\tilde{\xi}_m}(y_1^j, (X_N, Y_N))},$$

which has the same expression as (3.108) with  $\xi_0$  replaced by  $\tilde{\xi}_m$  and we generate the solutions

$$(w_{\tilde{\xi}_m}^p)_j + (\tilde{w}_{\tilde{\xi}_m}^p)_j,$$

where  $(w_{\tilde{\xi}_m}^p)_j$  is the solution of (3.24) and  $(\tilde{w}_{\tilde{\xi}_m}^p)_j = \mathcal{J}(\tilde{w}_{\tilde{\xi}_m})_j(\tilde{\xi}_m, \cdot)$  with  $(\tilde{w}_{\tilde{\xi}_m}^M)_j$  is the solution of (3.25) with  $v_{\xi_0} = v_{\tilde{\xi}_m}^j$ . Moreover, let us denote by  $(w_{\xi_{j_0}}^{p, M})_j$  the solution of (4.2) and  $(\tilde{w}_{\xi_{j_0}}^{p, M})_j = \mathcal{J}_M(\tilde{w}_{\xi_{j_0}}^M)_j(\xi_{j_0}, \cdot)$  with  $(\tilde{w}_{\xi_{j_0}}^M)_j$  is the solution of (4.3). The discretized near field operator for the  $2\pi M$  periodic setting is then given by the matrix of size  $N_{inc}^M \times N_{inc}^M$  defined as

$$\check{N}_{\xi_{j_0}, M}(j, \ell) := ((\hat{w}_{\tilde{\xi}_m}^{p, M})_j + (\tilde{w}_{\tilde{\xi}_m}^{p, M})_j)(\ell), \quad \text{for } \lceil \frac{-M}{2} \rceil + 1 \leq m \leq \lfloor \frac{M}{2} \rfloor, \quad 1 \leq j \leq N_{inc}^M \quad \text{and } \ell \in L_{inc}^M,$$

where  $L_{inc}^M$  is the space of Rayleigh coefficients which is defined as  $M$  copies of the space  $L_{inc}$  defined in Section 3.5. The discretized noisy data is given by the matrix  $\check{N}_{\xi_{j_0}, M}^\delta$  satisfying (3.110) with  $\check{N}_{\xi_{j_0}, M}$  is replaced by  $\check{N}_M$ , where the matrix of random values  $K$  is of size  $N_{inc}^M \times N_{inc}^M$ . For fixed  $\xi_{j_0}$ , we have  $\Phi_{\xi_{j_0}, M}(\cdot, z) \in H_{\xi_{j_0}, M}^1(\Omega^R)$ . Consequently, using the decomposition (4.14), one can decompose  $\Phi_{\xi_{j_0}, M}(\cdot, z)$  as follows

$$\Phi_{\xi_{j_0}, M}(\cdot, z) = \frac{1}{M} \sum_{m=\lceil \frac{-M}{2} \rceil + 1}^{\lfloor \frac{M}{2} \rfloor} \Phi_{\tilde{\xi}_m}(\cdot, z),$$

where  $\Phi_{\tilde{\xi}_m}(\cdot, z)$  is the  $\tilde{\xi}_m$ -quasi-periodic Green function with period  $2\pi$ . Consider a grid of sampling points  $z$  and we let first  $g_{\xi_{j_0}}^\alpha$  be the regularized solution of (3.106). Moreover,

we construct  $g_{\xi_{j_0}}^{\alpha, M}$  and  $g_M^\alpha$  satisfying

$$(\alpha + \check{N}_M^{\delta, *}\check{N}_M^\delta)g_{\xi_{j_0}}^{\alpha, M}(z) = \check{N}_M^*(\widehat{\Phi}_{\xi_{j_0}}(\ell, z)), \quad (4.15)$$

$$(\alpha + \check{N}_M^{\delta, *}\check{N}_M^\delta)g_M^\alpha(z) = \check{N}_M^*(\widehat{\Phi}_{\xi_{j_0}, M}(\ell, z)). \quad (4.16)$$

with  $\alpha$  is a constant determined by Morozov's principle. We then define

$$\mathcal{Z}_{\xi_{j_0}}^{\alpha, p} = \frac{1}{\|g_{\xi_{j_0}}^{\alpha, M}\|} \quad \text{and} \quad \mathcal{Z}_M^\alpha = \frac{1}{\|g_M^\alpha\|} \quad (4.17)$$

and we define the indicator function for the reconstruction of the defect as follows

$$\mathcal{Z}^\alpha := \left[ \|g_M^\alpha\| \left( 1 + \frac{\|g_M^\alpha\|}{\left| \left( \check{N}_M^\delta \left( g_{\xi_{j_0}}^{\alpha, M} - \frac{1}{M} g_{\xi_{j_0}}^\alpha \right), \left( g_{\xi_{j_0}}^{\alpha, M} - \frac{1}{M} g_{\xi_{j_0}}^\alpha \right) \right) \right|} \right) \right]^{-1}.$$

**Example 1:** In the first example we fix the physical and the geometrical parameters as follows  $k := \frac{3.5}{3.14\pi} + 0.4i$ ,  $\lambda = \frac{2\pi}{|k|}$ ,  $N_x = N_y = 300$ ,  $M = 4$ , and  $\xi_0 = 0$ . We consider  $D^p$  as a periodic ball centered at  $(0, 1.4\lambda)$  with radius  $R_{D^p} := 0.4\lambda$  and the perturbation  $\tilde{D}$  as a ball centered at  $(1.2\lambda, 0.8\lambda)$  with radius  $R_{\tilde{D}} := 0.4\lambda$  as shown in Figure 4.2 (1).

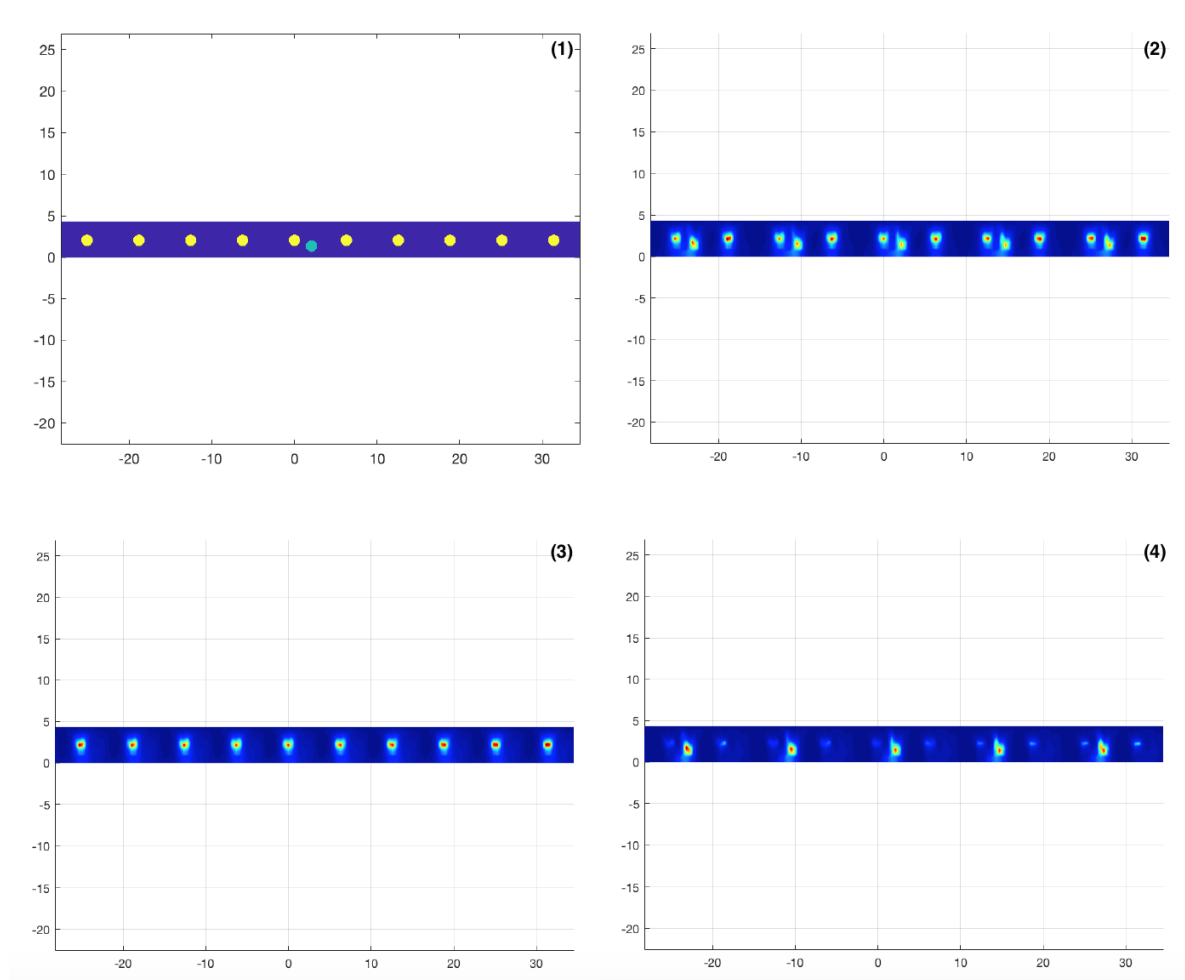


FIGURE 4.2: (1): The exact geometry. (2): The reconstruction obtained using  $\mathcal{Z}_M^\alpha$ . (3): The reconstruction obtained using  $\mathcal{Z}_{\xi_0}^{\alpha,p}$ . (4): The reconstruction obtained using  $\mathcal{Z}^\alpha$

**Example 2:** We fix in this example the physical and geometrical parameters as:  $k := \frac{3.5}{3.14\pi} + 0.7i$ ,  $\lambda = \frac{2\pi}{|k|}$ ,  $N_x = N_y = 300$  and  $M = 6$ , and  $\xi_0 = 0$ . We consider the periodic domain and the perturbation as L-shapes as shown in Figure 4.3 (1).

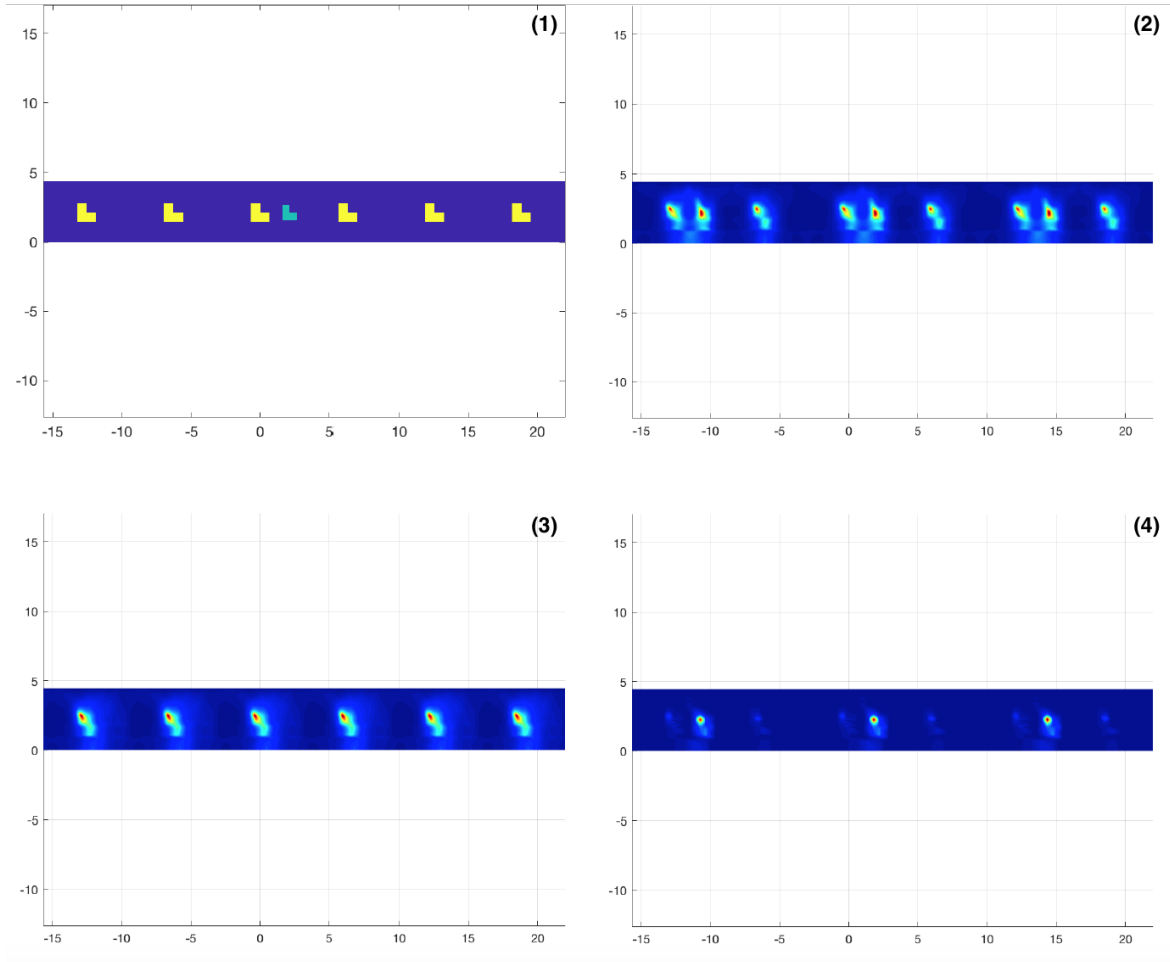


FIGURE 4.3: (1): The exact geometry. (2): The reconstruction obtained using  $\mathcal{Z}_M^\alpha$ . (3): The reconstruction obtained using  $\mathcal{Z}_{\xi_0}^{\alpha,p}$ . (4): The reconstruction obtained using  $\mathcal{Z}^\alpha$

**Example 3:** In the last example we fix  $k := \frac{3.5}{3.14\pi} + 1i$ ,  $\lambda = \frac{2\pi}{|k|}$ ,  $N_x = N_y = 300$ ,  $M = 2$ , and  $\xi_0 = 0$ , and we consider the periodic geometry and the perturbation as shown in the exact geometry given in Figure 4.4 (1)

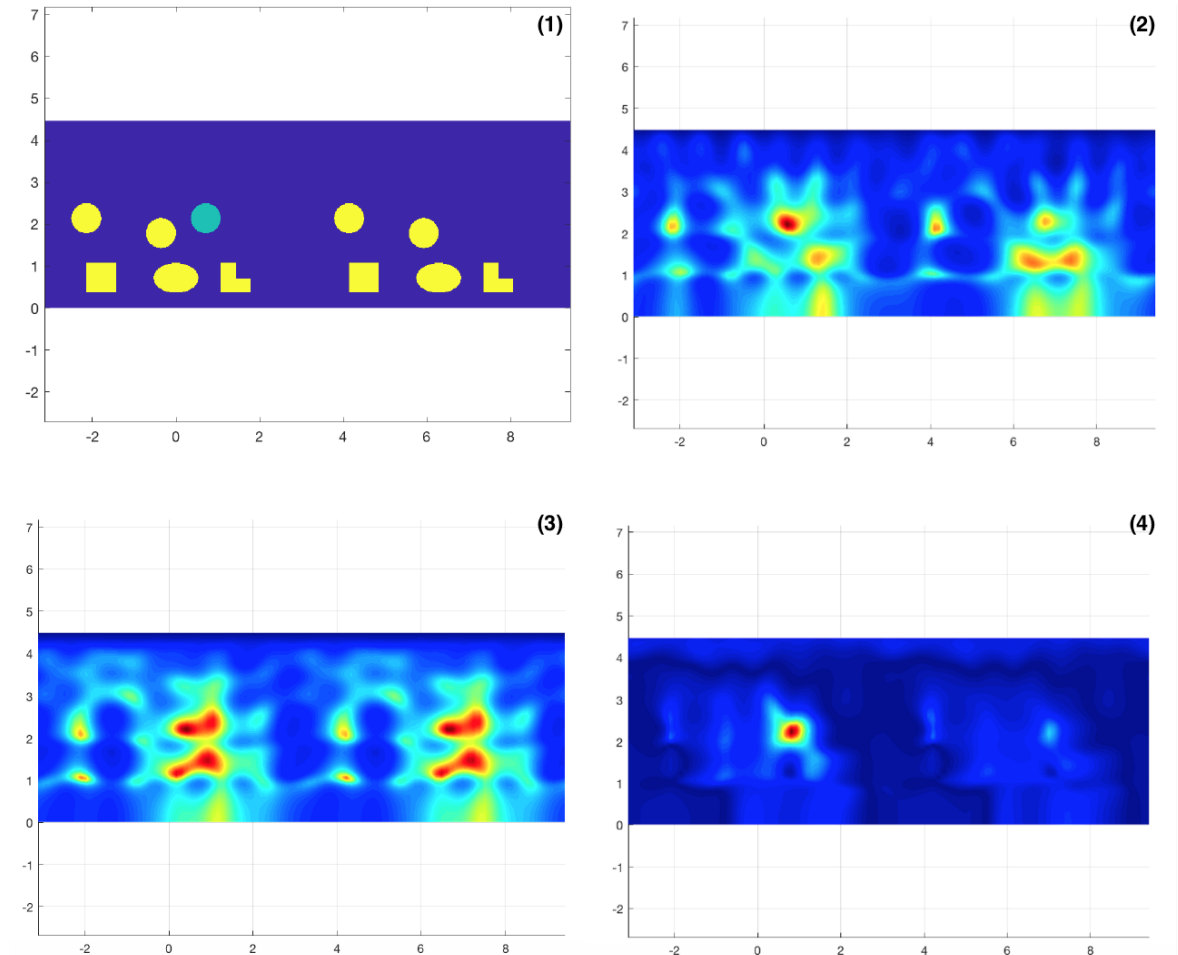


FIGURE 4.4: (1): The exact geometry. (2): The reconstruction obtained using  $\mathcal{Z}_M^\alpha$ . (3): The reconstruction obtained using  $\mathcal{Z}_{\xi_0}^{\alpha,p}$ . (4): The reconstruction obtained using  $\mathcal{Z}^\alpha$

**Conclusion:** As explained in the theory of the Differential Method, the reconstruction of the perturbation can be obtained from a criterion that depends on three indicator functions: one is associated with the periodic domain, and the others are associated with the domain present in the first period distributed periodically with the period  $2\pi$  and  $2\pi M$ , with  $M \geq 2$ . Then, in the examples above we presented the reconstruction obtained using all these indicator functions (The results of the reconstruction of the first period of length  $2\pi$  were presented in Chapter 3).

In Example 1, we considered a simple (non-complex) geometry as shown in Figure 4.2 (1). We can clearly observe in Figure 4.2 (2) the reconstruction of domain in the first period distributed periodically with period  $2\pi M$  obtained by the indicator function  $\mathcal{Z}_M^\alpha$ . We can also observe the reconstruction of the periodic domain in Figure 4.2 (3) using  $\mathcal{Z}_{\xi_0}^{\alpha,p}$ . In Figure 4.2 (4), we obtain the reconstruction of the perturbation using  $\mathcal{Z}^\alpha$ .

In the second example we also considered a non-convex geometry given by L-shapes as shown in Figure 4.3 (1). Similarly to the first example, we obtained a good reconstruction for the  $2\pi M$ -period, the periodic domain, and the perturbation as shown respectively in Figures 4.3 (2), 4.3 (3) and 4.3 (4).

In the last example, we considered a complex multiply connected domain as shown in Figure 4.4 (1). In this example, we observe in Figure 4.4 (2) that we do not have a good reconstruction neither for the  $2\pi M$ -period nor for the periodic domain as shown in Figure 4.4 (3) due to the complexity of the domain. However, we observe in Figure 4.4 (4) that the reconstruction of the perturbation still can be obtained even in a very complex domain.



# Analysis of the Interior Transmission problem

**Abstract:** *We analyze in this chapter the interior transmission problem in a locally perturbed infinite periodic domain, considering the case where the perturbation intersects the periodic background. An equivalent coupled quasi-periodic problem is given by applying the Floquet-Bloch transform. We perform a discretization with respect to the Floquet-Bloch variable and we prove the well-posedness of the semi-discretized problem. This allows us to construct a sequence that converges to a solution of the problem.*

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## 5.1 Introduction

The inverse scattering problem from inhomogeneous media leads in general to the study of the so-called interior transmission problem (ITP). This problem has been extensively studied in the literature for the case of bounded domains, and we refer for instance to [11, 13, 12, 55, 34]. The ITP for the case of unbounded periodic domains without defects can always be reduced to a one period cell leading back to the study of the problem in a bounded domain. However, the case of perturbed periodic media (which does not suppose that the perturbation is also periodic) is more difficult since one is obliged to study the solvability of the problem in the whole unbounded domain. More precisely, the difficulty appears when the perturbation intersects the periodic background. In fact, for the case where the intersection is empty, one can decouple the problem into two distinct problems, one associated with the periodic domain and the other with the compact perturbation. Consequently, the study in this case can also be deduced from results on ITP for bounded domains (this is what we have considered in the two previous chapters).

We present in this chapter a study for the ITP problem when this intersection is not empty. In this case, one can not use Fredholm's alternative in the classical way considered in the literature. It also was not possible to follow the method in [39] (presented in Chapter 2) for the solvability of the direct problem to deal with this type of problems.

The main scheme that we propose to solve this problem is given as follows: we apply first the Floquet-Bloch transform to our considered problem and rewrite it in the form of an equivalent coupled quasi-periodic problems. This explains the reason for which we consider the well-posedness for the quasi-periodic ITP problems in Section 5.3 before considering our case. Next, we make a discretization with respect to the Floquet-Bloch variable and we prove that the semi-discretized problem is uniquely solvable. Finally, we perform a convergence analysis that allows us to prove that the solution of the discretized problem converges to the solution of initial problem, which provides an existence results for ITP. This analysis is done in the last Section. Our analysis assumes the same hypothesis made in Chapter 2 on the refractive indices  $n$  and  $n_p$ .

## 5.2 Setting of the problem

We consider the upper half space  $U^0 := \mathbb{R} \times \mathbb{R}_+$ . For  $R > R_0 > 0$  we denote by  $\Omega^R := \mathbb{R} \times [0, R]$  with boundary's  $\Gamma^0 := \mathbb{R} \times \{0\}$  and  $\Gamma^R := \mathbb{R} \times \{R\}$  (see Figure 5.1).

We consider the same domain notation as in Chapter 3:

- $D^p$  denotes a  $2\pi$ -periodic unbounded domain included in  $\Omega^R$ ,
- $n_p(\cdot, x_2)$ : the  $2\pi$ -periodic refractive index such that  $n_p = 1$  outside  $D^p$ ,
- $\tilde{D}$  denotes a bounded perturbation included in  $\Omega_0^R := [0, 2\pi] \times [0, R]$ ,
- $n \in L^\infty(U^0)$ : the perturbed refractive index verifying  $n = n_p$  outside  $\tilde{D}$ ,
- $D_0^p = D^p \cap \Omega_0^R$  and  $D = D^p \cup \tilde{D}$ ,

and we make the additional following notation:

$$\partial\check{D}_0^p := \partial D_0^p \setminus (\Gamma^0 \cup \{x_1 = 0\} \cup \{x_1 = 2\pi\}).$$

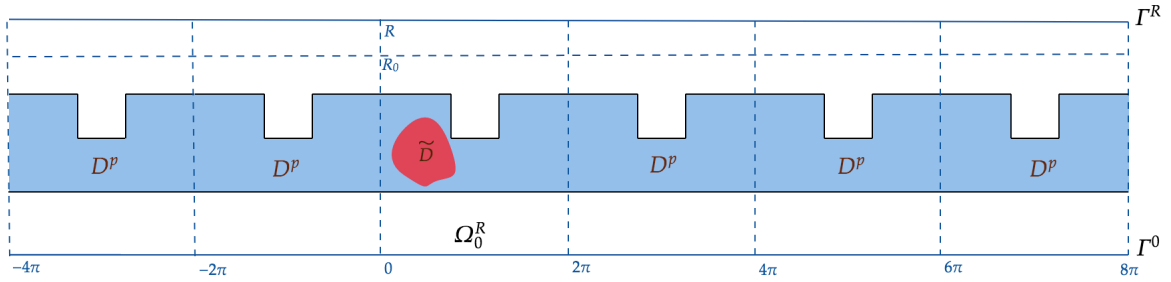


FIGURE 5.1: Sketch of the domain

The locally perturbed interior transmission problem is formulated as follows: seek  $(u, v) \in L^2(D) \times L^2(D)$  such that  $(u - v) \in H^2(D)$  satisfying

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ u - v = \varphi & \text{on } \partial D \setminus \Gamma^0, \\ \frac{\partial(u - v)}{\partial \nu} = \psi & \text{on } \partial D \setminus \Gamma^0, \\ u = v = 0 & \text{on } \Gamma^0, \end{cases} \quad (5.1)$$

for some boundary data  $(\varphi, \psi) \in H^{3/2}(\partial D \setminus \Gamma^0) \times H^{1/2}(\partial D \setminus \Gamma^0)$ . Let us first consider the case of periodic domain without perturbation, i.e as  $\check{D} = \emptyset$ . This brings us to the study of the well-posedness of the quasi-periodic Interior Transmission Problem which will be the subject of the subsection below.

## 5.3 The quasi-periodic Interior transmission problem

### 5.3.1 Setting of the problem

Fix  $\xi \in I := [0, 1]$ . The quasi-periodic interior transmission problem can be formulated as: Seek  $(u_\xi, v_\xi) \in L_\xi^2(D^p) \times L_\xi^2(D^p)$  such that  $(u_\xi - v_\xi) \in H_\xi^2(D^p)$  satisfying

$$\begin{cases} \Delta u_\xi + k^2 n_p u_\xi = 0 & \text{in } D_0^p, \\ \Delta v_\xi + k^2 v_\xi = 0 & \text{in } D_0^p, \\ u_\xi - v_\xi = \varphi_\xi & \text{on } \partial \check{D}_0^p, \\ \frac{\partial(u_\xi - v_\xi)}{\partial \nu} = \psi_\xi & \text{on } \partial \check{D}_0^p, \\ u_\xi = v_\xi = 0 & \text{on } \Gamma^0, \end{cases} \quad (5.2)$$

for given  $(\varphi_\xi, \psi_\xi) \in H_\xi^{3/2}(\partial \check{D}^p) \times H_\xi^{1/2}(\partial \check{D}^p)$ . The analysis will be done under the following assumption:

**Assumption 44.** *We assume that*

$$\Re(n_p(x)) - 1 \geq \alpha > 0 \quad \text{or} \quad 1 - \Re(n_p(x)) \geq \alpha > 0,$$

for almost every  $x \in D^p$  and for some  $\alpha > 0$ . Moreover, we assume that  $\Im(n_p) \geq 0$  and  $\{\Im(n_p) > 0\}$  contains a non empty open set  $\mathcal{O}$ .

We set

$$n_{p,*} := \inf_{D^p} \Re(n_p) \quad \text{and} \quad n^{p,*} := \sup_{D^p} \Re(n_p).$$

Under this assumption one can rewrite the problem (5.2) as: Seek  $w_\xi^0 := u_\xi - v_\xi \in H_\xi^2(D^p)$  satisfying

$$\begin{cases} (\Delta + k^2) \frac{1}{1 - n_p} (\Delta w_\xi^0 + k^2 n_p w_\xi^0) = 0 & \text{in } D_0^p, \\ w_\xi^0 = \varphi_\xi \text{ and } \frac{\partial w_\xi^0}{\partial \nu} = \psi_\xi & \text{on } \partial \check{D}_0^p, \\ w_\xi^0 = 0 & \text{on } \Gamma^0, \\ \frac{1}{1 - n_p} (\Delta w_\xi^0 + k^2 n_p w_\xi^0) = 0 & \text{on } \Gamma^0. \end{cases} \quad (5.3)$$

We set

$$\tilde{H}_\xi^2(D^p) := \{w_\xi \in H_\xi^2(D^p) / w_\xi = \frac{\partial w_\xi}{\partial \nu} = 0 \text{ on } \partial \check{D}_0^p \text{ and } w_\xi = 0 \text{ on } \Gamma^0\}.$$

We remark that  $w_\xi \rightarrow \sqrt{\|\Delta w_\xi\|_{L^2(D_0^p)}^2 + \|w_\xi\|_{L^2(D_0^p)}^2}$  defines an equivalent norm on  $\tilde{H}_\xi^2(D^p)$ .

Let  $\theta_\xi \in H_\xi^2(D^p)$  be a lifting function satisfying  $\theta_\xi = \varphi_\xi$  and  $\frac{\partial \theta_\xi}{\partial \nu} = \psi_\xi$  on  $\partial \check{D}_0^p$ , and  $\theta_\xi = 0$  on  $\Gamma^0$  and

$$\|\theta_\xi\|_{H^2(D_0^p)} \leq c(\|\varphi_\xi\|_{H^{3/2}(\partial \check{D}_0^p)} + \|\psi_\xi\|_{H^{1/2}(\partial \check{D}_0^p)}),$$

for some  $c > 0$ .

### 5.3.2 Well-posedness of the problem

Define the sesquilinear form

$$a_\xi(w_\xi, w'_\xi) := \int_{D_0^p} \frac{1}{n_p - 1} (\Delta w_\xi + k^2 n_p w_\xi) \overline{(\Delta w'_\xi + k^2 w'_\xi)} dx. \quad (5.4)$$

Then the variational formulation equivalent to (5.3) can be written as: Seek  $w_\xi := w_\xi^0 - \theta_\xi \in \tilde{H}_\xi^2(D^p)$  such that

$$a_\xi(w_\xi, w'_\xi) = - \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \theta_\xi + k^2 n_p \theta_\xi) \overline{(\Delta w'_\xi + k^2 w'_\xi)} dx, \quad (5.5)$$

for all  $w'_\xi \in \tilde{H}_\xi^2(D^p)$ . We observe that

$$\mathcal{L} : w'_\xi \rightarrow \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \theta_\xi + k^2 n_p \theta_\xi) \overline{(\Delta w'_\xi + k^2 w'_\xi)} dx,$$

is a bounded antilinear functional on  $\tilde{H}_\xi^2(D^p)$ . Using the Riesz representation theorem we consider  $f_\xi \in \tilde{H}_\xi^2(D^p)$  such that  $\mathcal{L}(w'_\xi) = (f_\xi, w'_\xi)_{H^2(D_0^p)}$  for all  $w'_\xi \in \tilde{H}_\xi^2(D^p)$  and

$$\|f_\xi\|_{H^2(D_0^p)} \leq c_1 \|\theta_\xi\|_{H^2(D_0^p)} \leq c_2 \left( \|\varphi_\xi\|_{H^{3/2}(\partial \check{D}_0^p)} + \|\psi_\xi\|_{H^{1/2}(\partial \check{D}_0^p)} \right),$$

with  $c_1, c_2$  are two positive constants independent from  $\xi$ . Thus, Problem (5.5) is equivalent to

$$\mathcal{A}_\xi w_\xi = f_\xi \quad \text{in } \tilde{H}_\xi^2(D^p),$$

where  $\mathcal{A}_\xi : \tilde{H}_\xi^2(D^p) \rightarrow \tilde{H}_\xi^2(D^p)$  is the operator defined by the Riesz representation Theorem as

$$(\mathcal{A}_\xi w_\xi, w'_\xi)_{H^2(D_0^p)} = a_\xi(w_\xi, w'_\xi) \quad \forall w_\xi, w'_\xi \in \tilde{H}_\xi^2(D^p), \quad (5.6)$$

**Theorem 45.** *Assume that Assumption 44 holds. Then the operator  $\mathcal{A}_\xi$  is a Fredholm operator of index zero.*

*Proof.* Let us define by the Riesz representation the following two operators  $\mathcal{A}_{\xi,0}, \mathcal{A}_{\xi,1} : \tilde{H}_\xi^2(D^p) \rightarrow \tilde{H}_\xi^2(D^p)$  satisfying

$$(\mathcal{A}_{\xi,0} w_\xi, w'_\xi)_{H^2(D_0^p)} = \int_{D_0^p} \frac{1}{n_p - 1} \left( \Delta w_\xi \overline{\Delta w'_\xi} + w_\xi \overline{w'_\xi} \right) dx, \quad (5.7)$$

$$(\mathcal{A}_{\xi,1} w_\xi, w'_\xi)_{H^2(D_0^p)} = \int_{D_0^p} \frac{1}{n_p - 1} \left( k^2 \Delta w_\xi \overline{w'_\xi} + k^2 n_p w_\xi \overline{\Delta w'_\xi} + (k^4 n_p - 1) w_\xi \overline{w'_\xi} \right) dx. \quad (5.8)$$

We observe that

$$\Re e(\mathcal{A}_{\xi,0} w, w)_{H^2(D_0^p)} \geq \alpha_0 \left( \|\Delta w_\xi\|_{L^2(D_0^p)}^2 + \|w_\xi\|_{L^2(D_0^p)}^2 \right),$$

with  $\alpha_0 := \frac{1}{n_{p,*} - 1}$  in the case  $n_{p,*} > 1$  and  $\alpha_0 := \frac{1}{n^{p,*} - 1}$  in the case  $0 < n^{p,*} < 1$ , which implies that  $\mathcal{A}_{\xi,0}$  is coercive in the case  $n_{p,*} > 1$  and  $-\mathcal{A}_{\xi,0}$  in the case  $0 < n^{p,*} < 1$ .

Now, let us denote by  $\mathcal{A}_{\xi,1}^{(1)}$  the part of the operator given by the first integral in (5.8). Then, taking  $w'_\xi = \mathcal{A}_{\xi,1}^{(1)}w_\xi$  we observe that

$$\|\mathcal{A}_{\xi,1}^{(1)}w_\xi\|_{H^2(D_0^p)}^2 = \left| \int_{D_0^p} \frac{k^2}{n_p - 1} \Delta w_\xi \overline{\mathcal{A}_{\xi,1}^{(1)}w_\xi} dx \right| \leq c' \|w_\xi\|_{H^2(D_0^p)} \|\mathcal{A}_{\xi,1}^{(1)}w_\xi\|_{L^2(D_0^p)}, \quad (5.9)$$

with  $c'$  is a positive constant that does not depend on  $\xi$ . Consider now a sequence  $w_\xi^\ell$  bounded in  $\tilde{H}_\xi^2(D^p)$ , then we can extract a subsequence that we still denote by  $w_\xi^\ell$  that converge weakly to  $w$  in  $\tilde{H}_\xi^2(D^p)$ . From the compact embedding of  $\tilde{H}_\xi^2(D^p)$  in  $L_\xi^2(D^p)$  we get that  $w_\xi^\ell$  converges strongly to  $w$  in  $L_\xi^2(D^p)$  and therefore  $\mathcal{A}_{\xi,1}^{(1)}w_\xi^\ell$  converges strongly to  $\mathcal{A}_{\xi,1}^{(1)}w$  in  $L_\xi^2(D^p)$ . Hence, from (5.9) we have

$$\|\mathcal{A}_{\xi,1}^{(1)}(w_\xi^\ell - w_\xi)\|_{H^2(D_0^p)}^2 \leq c' \|w_\xi^\ell - w_\xi\|_{H^2(D_0^p)} \|\mathcal{A}_{\xi,1}^{(1)}(w_\xi^\ell - w_\xi)\|_{L^2(D_0^p)} \longrightarrow 0 \quad \text{as } \ell \rightarrow \infty \quad (5.10)$$

Consequently,  $\mathcal{A}_{\xi,1}^{(1)}w_\xi^\ell$  converges strongly to  $\mathcal{A}_{\xi,1}^{(1)}w_\xi$  in  $\tilde{H}_\xi^2(D^p)$  which implies the compactness of  $\mathcal{A}_{\xi,1}^{(1)}$ . The compactness of the other parts in  $\mathcal{A}_{\xi,1}$  can be proved the same way.  $\square$

**Theorem 46.** *The operator  $\mathcal{A}_\xi$  is invertible.*

*Proof.* From Theorem 45 we have that  $\mathcal{A}_\xi$  is a Fredholm operator of index zero. Then, it remains to prove the injectivity of  $\mathcal{A}_\xi$ . We assume that

$$(\mathcal{A}_\xi w_\xi, w'_\xi)_{H^2(D_0^p)} = 0 \quad \text{for all } w'_\xi \in C_0^\infty(D^p),$$

then we get

$$(\Delta + k^2) \left( \frac{1}{n_p - 1} \right) (\Delta w_\xi + k^2 n_p w_\xi) = 0. \quad (5.11)$$

We set

$$v_\xi := -\frac{1}{k^2(n_p - 1)} (\Delta w_\xi + k^2 n_p w_\xi) \quad \text{and} \quad u_\xi := w_\xi + v_\xi. \quad (5.12)$$

Then,  $(u_\xi, v_\xi)$  satisfies the following equations

$$\begin{cases} \Delta u_\xi + k^2 n_p u_\xi = 0 & \text{in } D_0^p, \\ \Delta v_\xi + k^2 v_\xi = 0 & \text{in } D_0^p. \end{cases} \quad (5.13)$$

On the other hand, we observe

$$a(w_\xi, w_\xi) = \int_{D_0^p} \frac{1}{n_p - 1} |\Delta w_\xi + k^2 w_\xi|^2 - k^2 \int_{D_0^p} |\nabla w_\xi|^2 dx + k^4 \int_{D_0^p} |w_\xi|^2 dx.$$

Taking the imaginary part we get

$$\int_{D_0^p} \Im m \left( \frac{1}{n_p - 1} \right) |\Delta w_\xi + k^2 w_\xi|^2 dx = 0. \quad (5.14)$$

Therefore  $\Delta w_\xi + k^2 w_\xi = 0$  in  $\mathcal{O}$  with means  $u_\xi = 0$  in  $\mathcal{O}$ . The unique continuation principle implies that  $u_\xi = 0$  vanishes in  $D_0^p$ . Consequently we have  $\Delta w_\xi + k^2 w_\xi = 0$  in  $D_0^p$  and  $w_\xi = \frac{\partial w_\xi}{\partial \nu} = 0$  on  $\partial \check{D}^p$  which implies that  $w_\xi = v_\xi = 0$  in  $D_0^p$ . This proves the injectivity of the operator  $\mathcal{A}_\xi$ . Therefore  $\mathcal{A}_\xi$  is invertible.  $\square$

**Theorem 47.** *There exists a constant  $\gamma > 0$  independent of  $\xi$  such that*

$$\|\mathcal{A}_\xi^{-1}\| \leq \gamma. \quad (5.15)$$

*Proof.* Let us denote by  $w_\xi^\sharp := e^{-i\xi \cdot x_1} w_\xi$  and  $\tilde{w}_\xi^\sharp := e^{-i\xi \cdot x_1} w'_\xi$ . Then  $\tilde{w}_\xi^\sharp, w_\xi^\sharp \in H_\sharp^2(D^p)$ , where  $H_\sharp^2(D^p)$  is the space defined the same as  $\tilde{H}_\xi^2(D^p)$  for  $\xi = 0$ . We have

$$\Delta w_\xi = e^{i\xi \cdot x_1} ( -|\xi|^2 \tilde{w}_\xi + 2i\xi \nabla \tilde{w}_\xi + \Delta \tilde{w}_\xi ).$$

Then, replacing  $w_\xi$  and  $w'_\xi$  respectively by  $\tilde{w}_\xi^\sharp e^{ix_1 \xi}$  and  $\tilde{w}_\xi^{\sharp'} e^{ix_1 \xi}$  in (5.4) we get

$$\begin{aligned} & \int_{D_0^p} \left( \frac{1}{n_p - 1} \right) \left( \Delta w_\xi^\sharp + 2i\xi \nabla w_\xi^\sharp + (k^2 n_p - |\xi|^2) w_\xi^\sharp \right) \\ & \quad \times \left( \Delta w_\xi^{\sharp'} + 2i\xi \nabla w_\xi^{\sharp'} + (k^2 - |\xi|^2) w_\xi^{\sharp'} \right) dx := a_\xi^\sharp(w_\xi^\sharp, w_\xi^{\sharp'}). \end{aligned}$$



Fix  $\eta > 0$  and consider  $\xi, \xi_0 \in I$  such that  $|\xi - \xi_0| < \eta$ , we observe that

$$\begin{aligned} \left| a_\xi^\sharp(w^\sharp, w^{\sharp'}) - a_{\xi_0}^\sharp(w^\sharp, w^{\sharp'}) \right| &\leq C (|\xi|^4 - |\xi_0|^4 + |\xi^3 - \xi_0^3| + |\xi^2 - |\xi_0|^2| + |\xi - \xi_0|) \\ &\quad \times \|w^\sharp\|_{H_\sharp^2(D_0^p)} \|w^{\sharp'}\|_{H_\sharp^2(D_0^p)}, \quad \forall w^\sharp, w^{\sharp'} \in H_\sharp^2(D_0^p). \end{aligned} \quad (5.16)$$

with  $C$  is a positive constant that depends on  $k^2$ ,  $n_{p,*}$ ,  $n^{p,*}$  and does not depend of  $\xi$ . Therefore, the right hand side of (5.16) goes to zero as  $\xi \rightarrow \xi_0$ . Since  $w^\sharp \rightarrow w^\sharp e^{i\xi \cdot x_1}$  is an isomorphism between  $H_\sharp^2(D^p)$  and  $H_\xi^2(D^p)$  which is continuous with respect to  $\xi$ , we deduce that  $\xi \rightarrow \mathcal{A}_\xi$  is uniformly continuous from  $I$  into  $\mathcal{L}(\tilde{H}_\xi^2(D^p))$ .

Now, let  $\xi_0 \in \bar{I}$ , and set  $\epsilon = \frac{1}{2\|\mathcal{A}_{\xi_0}^{-1}\|}$ . Then there exists  $\eta_{\xi_0} > 0$  such that for  $|\xi - \xi_0| < \eta_{\xi_0}$ , we have

$$\|\mathcal{A}_\xi - \mathcal{A}_{\xi_0}\| \leq \epsilon.$$

Therefore  $\|\mathcal{A}_{\xi_0}^{-1}(\mathcal{A}_\xi - \mathcal{A}_{\xi_0})\| \leq \frac{1}{2}$  which implies that

$$\|(I + \mathcal{A}_{\xi_0}^{-1}(\mathcal{A}_\xi - \mathcal{A}_{\xi_0}))^{-1}\| \leq 2.$$

Consequently, observing that  $\mathcal{A}_\xi = \mathcal{A}_{\xi_0}(I + \mathcal{A}_{\xi_0}^{-1}(\mathcal{A}_\xi - \mathcal{A}_{\xi_0}))$  we deduce that

$$\|\mathcal{A}_\xi^{-1}\| \leq 2\|\mathcal{A}_{\xi_0}^{-1}\|, \quad \forall \xi \in B_{\eta_{\xi_0}},$$

where  $B_{\eta_{\xi_0}}$  denotes the ball of radius  $\eta_{\xi_0}$  centered at  $\xi_0$ . Since  $\bar{I}$  is a compact set and there exists a finite set  $J \subset \bar{I}$  for which  $\bar{I} = \bigcup_{\xi_0 \in J} \{\xi \in \bar{I}; |\xi - \xi_0| < \eta_{\xi_0}\}$ . Therefore, for all  $\xi \in I$  we have

$$\|\mathcal{A}_\xi^{-1}\| \leq 2 \sup_{\xi_0 \in J} \|\mathcal{A}_{\xi_0}^{-1}\| = \gamma.$$

□

## 5.4 The locally perturbed interior transmission problem

We analyze in this section the well-posedness of the interior transmission problem for the infinite periodic domain  $D^p$  when a local perturbation, namely the support of  $n_p - n$  lies inside  $D^p$ . In this case  $\tilde{D} \subset D^p$  and  $D = D^p$ . The interior transmission problem is formulated as follows: seek  $(u, v) \in L^2(D^p) \times L^2(D^p)$  such that  $(u - v) \in H^2(D^p)$  satisfying

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D^p, \\ \Delta v + k^2 v = 0 & \text{in } D^p, \\ u - v = \varphi & \text{on } \partial D^p \setminus \Gamma^0, \\ \frac{\partial(u - v)}{\partial \nu} = \psi & \text{on } \partial D^p \setminus \Gamma^0, \\ u = v = 0 & \text{on } \Gamma^0, \end{cases} \quad (5.17)$$

for some boundary data  $(\varphi, \psi) \in H^{3/2}(\partial D^p \setminus \Gamma_0) \times H^{1/2}(\partial D^p \setminus \Gamma_0)$ . The analysis will be done under the following assumption:

**Assumption 48.** *We assume that*

$$\Re(n(x)) - 1 \geq \alpha > 0 \quad \text{or} \quad 1 - \Re(n(x)) \geq \alpha > 0.$$

for almost every  $x \in D^p$  and some  $\alpha > 0$  and  $n = n_p$  outside  $\tilde{D} \subset D_0^p$ .

We set

$$n_* := \inf_{D^p} \Re(n) \quad \text{and} \quad n^* := \sup_{D^p} \Re(n).$$

Under this assumption, one can rewrite (5.17) as: seek  $w_0 := u - v \in H^2(D^p)$  satisfying

$$\begin{cases} (\Delta + k^2) \frac{1}{1 - n} (\Delta w_0 + k^2 n w_0) = 0 & \text{in } D^p, \\ w_0 = \varphi \quad \text{and} \quad \frac{\partial w_0}{\partial \nu} = \psi & \text{on } \partial D^p \setminus \Gamma^0, \\ w_0 = 0 & \text{on } \Gamma^0, \\ \frac{1}{1 - n} (\Delta w_0 + k^2 n w_0) = 0 & \text{on } \Gamma^0. \end{cases} \quad (5.18)$$

We set

$$\tilde{H}_0^2(D^p) := \{w_0 \in H^2(D^p) / w_0 = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial D^p \setminus \Gamma^0 \text{ and } w_0 = 0 \text{ on } \Gamma^0\}.$$

Let  $\theta \in H^2(D^p)$  be a lifting function satisfying  $\theta = \varphi$  and  $\frac{\partial \theta}{\partial \nu} = \psi$  on  $\partial D^p \setminus \Gamma^0$  and  $\theta = 0$  on  $\Gamma^0$  and define the sesquilinear form

$$a(w, w') := \int_{D^p} \frac{1}{n-1} (\Delta w + k^2 n w) \overline{(\Delta w' + k^2 w')} dx. \quad (5.19)$$

Then the variational formulation equivalent to (5.18) can be written as: Seek  $w := w_0 - \theta \in \tilde{H}_0^2(D^p)$  such that

$$a(w, w') = \int_{D^p} f \overline{(\Delta w' + k^2 w')} dx, \quad (5.20)$$

for all  $w' \in \tilde{H}_0^2(D^p)$  with

$$f = \frac{1}{n-1} (\Delta \theta + k^2 n \theta).$$

**Lemma 49.** *Problem (5.20) has at most one solution.*

*Proof.* The uniqueness of solutions for problem (5.20) can be shown following the same steps as the uniqueness result for problem (5.2) given by Theorem 46.  $\square$

To show the existence of solutions we shall rely on the rewriting of (5.20) using the Floquet-Bloch transform. Setting

$$\tilde{a}(w, w') := \int_{\tilde{D}} \left( \tilde{n} (\Delta w + k^2 n_p w) + k^2 \frac{(n - n_p)}{n-1} w \right) \overline{(\Delta w' + k^2 w')} dx, \quad (5.21)$$

with  $\tilde{n} := \left( \frac{1}{n-1} - \frac{1}{n_p-1} \right)$  we can decompose the sesquilinear form  $a(w, w')$  as

$$a(w, w') = \tilde{a}(w, w') + \int_{D^p} \frac{1}{n_p-1} (\Delta w + k^2 n_p w) \overline{(\Delta w' + k^2 w')} dx. \quad (5.22)$$

For  $\hat{w} \in L^2(I, \tilde{H}_\xi^2(D_0^p))$  and  $\hat{w}' \in L^2(I, \tilde{H}_\xi^2(D_0^p))$  we define

$$\hat{a}(\hat{w}, \hat{w}') := \int_I \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \hat{w}_\xi + k^2 n_p \hat{w}_\xi) \overline{(\Delta \hat{w}'_\xi + k^2 \hat{w}'_\xi)} dx d\xi + \tilde{a}(w, w') \quad (5.23)$$

with

$$w = \int_I \hat{w}(\xi, \cdot) d\xi \quad \text{and} \quad w' = \int_I \hat{w}'(\xi, \cdot) d\xi.$$

Then, using the properties of the Floquet-Bloch transform we deduce that

$$a(w, w') = \hat{a}(\mathcal{J}w, \mathcal{J}w'), \quad \forall w, w' \in \tilde{H}_0^2(D^p). \quad (5.24)$$

In the sequel, we shall use the short notation  $\hat{w} := \mathcal{J}w$  and  $\hat{w}_\xi := \mathcal{J}w(\xi, \cdot)$ . From (5.24) we deduce that the variational formulation (5.20) is equivalent to: Seek  $\hat{w} \in L^2(I, \tilde{H}_\xi^2(D_0^p))$  such that

$$\hat{a}(\hat{w}, \hat{w}') = \int_I \int_{D_0^p} \hat{f}_\xi \overline{(\Delta \hat{w}'_\xi + k^2 \hat{w}'_\xi)} dx d\xi \quad \forall \hat{w}' \in L^2(I, \tilde{H}_\xi^2(D_0^p)). \quad (5.25)$$

In order to simplify the exposition, we shall study the variational problem

$$\hat{a}(\hat{w}, \hat{w}') = \int_I \int_{D_0^p} \hat{f}_\xi \overline{\hat{w}'_\xi} dx d\xi \quad \forall \hat{w}' \in L^2(I, \tilde{H}_\xi^2(D_0^p)), \quad (5.26)$$

for  $\hat{f}_\xi \in H^1(I \times D_0^p)$  which is equivalent to the variational formulation

$$a(w, w') = \int_{D^p} f \overline{w'} dx \quad \forall w' \in \tilde{H}_0^2(D^p). \quad (5.27)$$

The case of right hand side as in (5.25) can be treated in the same way and does not change the result on the well posedness of the problem.

#### 5.4.1 Semi-discretized interior transmission problem

In order to prove the existence of solution to problem (5.26) we first prove existence of solution for semi-discretized version of this problem. We then construct a convergent sequence that provide a solution to problem (5.26).

Let  $N$  be the number of discretization points with respect to the Floquet-Bloch variable.

We consider a uniform partition of  $I$  into sub-domains  $I_j := [\xi_j, \xi_{j+1}]$  for  $0 \leq j \leq N - 1$

with  $\xi_j := \frac{j}{N}$ . We define the discrete space

$$X^N(D_0^p) := \{\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}); \hat{w}_{\xi_j} \in \tilde{H}_{\xi_j}^2(D^p) \text{ for } j = 0, \dots, N-1 \text{ and } \hat{w}_{\xi_N} = \hat{w}_{\xi_0}\},$$

equipped with the norm  $\|\hat{W}_N\|_{X^N(D_0^p)}^2 := \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{H^2(D_0^p)}^2$ . For  $\hat{W}_N := (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$  we denote by

$$\Sigma(\hat{W}_N) := \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j}.$$

Then a discretized version of (5.26) is given as the following: seek  $\hat{W}_N := (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$  such that

$$\begin{aligned} & \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta \hat{w}'_{\xi_j} + k^2 \hat{w}'_{\xi_j})} dx + \int_{\bar{D}} \frac{k^2(n - n_p)}{n - 1} w_N \overline{(\Delta w'_N + k^2 w'_N)} dx \\ & + \int_{\bar{D}} \tilde{n}(\Delta w_N + k^2 n_p w_N) \overline{(\Delta w'_N + k^2 w'_N)} dx = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx, \end{aligned} \quad (5.28)$$

for all  $\hat{W}'_N = (\hat{w}'_{\xi_0}, \dots, \hat{w}'_{\xi_N}) \in X^N(D_0^p)$ , where

$$w_N = \Sigma(\hat{W}_N) \quad \text{and} \quad w'_N = \Sigma(\hat{W}'_N).$$

With the short notation for the sesquilinear form

$$\tilde{a}(w, w') := \int_{\bar{D}} \left( \tilde{n}(\Delta w + k^2 n_p w) + k^2 \frac{(n - n_p)}{n - 1} w \right) \overline{(\Delta w' + k^2 w')} dx, \quad (5.29)$$

problem (5.28) can be synthetically written as

$$\frac{1}{N} \sum_{j=0}^{N-1} a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) + \tilde{a}(w_N, w'_N) = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx. \quad (5.30)$$

Here,  $\hat{f}_{\xi_j}$  can be seen as pointwise values of  $\hat{f}$  in  $C_0(\bar{I}, L^2(D_0^p))$  at  $\xi = \xi_j$ .

### 5.4.2 Well posedness of the discretized problem

We first prove uniqueness of solutions to problem (5.28). We proceed in two steps by proving first that  $w_N = 0$  and then deduce that  $\hat{w}_{\xi_j} = 0$ . Let us denote by

$$D_N^p := \bigcup_{j=0}^{N-1} \{D_0^p + 2\pi j \mathbf{e}_1\},$$

where  $(\mathbf{e}_1, \mathbf{e}_2)$  is the canonical basis of  $\mathbb{R}^2$ . Let  $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$ . Clearly, the functions  $\hat{w}_{\xi_j}$  extended by  $\xi_j$ -quasi periodicity in  $\mathbf{e}_1$  direction is in  $\tilde{H}_{\sharp}^2(D_N^p)$  for  $j = 0, \dots, N-1$ , with  $\tilde{H}_{\sharp}^2(D_N^p)$  being the subspace of  $\tilde{H}_{loc}^2(D^p)$  of  $2\pi N$ -periodic functions in the  $\mathbf{e}_1$  direction. Consider  $0 \leq j, j' \leq N-1$  and recall the following identity

$$\sum_{\ell=0}^{N-1} e^{i2\pi \xi_{\ell}(j-j')} = N \delta_j^{j'}. \quad (5.31)$$

where  $\delta_j^{j'}$  denotes the Kronecker symbol. Consequently, for  $\hat{w}_{\xi_j} \in L_{\xi}^2(D^p)$  and  $\hat{w}_{\xi_{j'}} \in L_{\xi'}^2(D^p)$  we have

$$\int_{D_N^p} \hat{w}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx = \sum_{\ell=0}^{N-1} e^{i2\pi \ell(\xi_j - \xi_{j'})} \int_{D_0^p} \hat{w}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx = N \delta_j^{j'} \int_{D_0^p} \hat{w}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx. \quad (5.32)$$

Similarly for  $\hat{w}_{\xi} \in H_{\xi}^2(D^p)$  and  $\hat{w}_{\xi'} \in H_{\xi'}^2(D^p)$  we have

$$(\hat{w}_{\xi}, \hat{w}_{\xi'})_{H^2(D_N^p)} = N \delta_j^{j'} (\hat{w}_{\xi}, \hat{w}_{\xi'})_{H^2(D_0^p)}. \quad (5.33)$$

This proves in particular that  $\tilde{H}_{\xi_j}^2(D^p)$  and  $\tilde{H}_{\xi_{j'}}^2(D^p)$  are two orthogonal subspaces of  $\tilde{H}_{\sharp}^2(D_N^p)$ . More precisely, we have the following decomposition of  $\tilde{H}_{\sharp}^2(D_N^p)$ .

**Lemma 50.** *We have the following orthogonal decomposition*

$$\tilde{H}_{\sharp}^2(D_N^p) = \bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D^p).$$

*Proof.* From the previous discussion we obviously have

$$\bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D^p) \subset \tilde{H}_{\#}^2(D_N^p).$$

To prove the reverse inclusion let us first consider the following space

$$S_{\#}(D_N^p) := \{\hat{w} \in \tilde{H}_{\#}^2(D_N^p) / \text{dist}(\text{supp}(\hat{w}), \partial \check{D}_0^p) > \delta, \text{ for some } \delta > 0\}.$$

This set is dense in  $\tilde{H}_{\#}^2(D_N^p)$ . We consider  $\hat{w} \in S_{\#}(D_N^p)$  that we extend by zero outside  $D_N^p$  which gives a function  $\hat{w} \in \tilde{H}_{\#}^2(\Omega_N^R)$ , with  $\Omega_N^R := [0, 2\pi N] \times [0, R]$ . Using Fourier series we have that  $\hat{w}$  can be expanded as

$$\hat{w}(x_1, x_2) = \sum_{n=0}^{N-1} a_n(x_2) e^{inx_1}, \quad (5.34)$$

with  $a_n(x_2)$  is defined by

$$a_n(x_2) = \frac{1}{2\pi N} \int_0^{2\pi N} \hat{w}(x_1, x_2) e^{-inx_1} dx_1.$$

Let  $0 \leq j \leq N-1$ , then for all  $n \in \mathbb{Z}$ , there exists a unique  $\ell \in \mathbb{Z}$  such that  $n = j + N\ell$ . therefore  $\hat{w}$  can be written as

$$\hat{w} = \sum_{j=0}^{N-1} \hat{w}_{\xi_j},$$

with  $\hat{w}_{\xi_j} = \sum_{\ell=0}^{N-1} a_{j+N\ell}(x_2) e^{i(j+N\ell)x_1}$  which belongs to  $H_{\xi_j}^2(\Omega^R)$  since for  $m \in \mathbb{Z}$  we have

$$e^{i(j+N\ell)(x_1+2\pi m)} = e^{i\xi_j m} e^{i(j+N\ell)x_1}.$$

Moreover,  $\hat{w} \in \tilde{H}_{\#}^2(D_N^p)$  then  $\hat{w}$  vanishes on  $\Gamma^0$ . Therefore,  $a_n(0) = 0$  and  $\hat{w}_{\xi_j} = 0$  on  $\Gamma^0$ . Let us now define a cut-off function  $\chi \in C^\infty(\mathbb{R}_+^2)$   $2\pi$ -periodic in the  $\mathbf{e}_1$  direction such that  $\chi = 1$  inside  $\text{supp}(\hat{w})$  and  $\chi = 0$  outside  $D_N^p$ . Therefore

$$\hat{w} = \chi \hat{w} = \sum_{j=0}^{N-1} \chi \hat{w}_{\xi_j},$$

where  $\chi \hat{w}_{\xi_j} \in \tilde{H}_{\sharp}^2(D_0^p)$ . Consequently  $S_{\sharp}(D_N^p) \subset \bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D^p)$  which implies that  $\bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D^p)$  is dense in  $\tilde{H}_{\sharp}^2(D_N^p)$ . Moreover, since the orthogonal sum is a closed subset of  $\tilde{H}_{\sharp}^2(D_N^p)$  then we deduce the equality between the two spaces.  $\square$

**Lemma 51.** *Let  $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_{N-1}}) \in X^N(D_0^p)$  be a solution of problem (5.28) and let  $w_N := \Sigma(\hat{W}_N)$ . Then  $w_N \in \tilde{H}_{\sharp}^2(D_N^p)$  and*

$$\int_{D_N^p} \frac{1}{n-1} (\Delta w_N + k^2 n w_N) \overline{(\Delta w' + k^2 w')} dx = \int_{D_N^p} f_N \overline{w'} dx, \quad \forall w' \in \tilde{H}_{\sharp}^2(D_N^p), \quad (5.35)$$

with  $f_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{f}_{\xi_j}$ .

*Proof.* Multiplying (5.28) by  $N$  and using (5.32) we get

$$\begin{aligned} & \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \frac{1}{n_p-1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta \hat{w}'_{\xi_j} + k^2 \hat{w}'_{\xi_j})} dx + N \tilde{a}(w_N, w'_N) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx. \end{aligned} \quad (5.36)$$

On the other hand, using again (5.32) we deduce that for  $j \neq j'$  we have

$$\int_{D_N^p} \frac{1}{n_p-1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta \hat{w}_{\xi_{j'}} + k^2 \hat{w}_{\xi_{j'}})} dx = 0, \quad (5.37)$$

$$\int_{D_N^p} \hat{f}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx = 0. \quad (5.38)$$

Let  $w' \in \tilde{H}_{\sharp}^2(D_N^p)$ . From Lemma 50 we can decompose it as  $w' = \sum_{j=0}^{N-1} \hat{w}'_{\xi_j} \in \bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D^p)$ .

Using (5.37) and (5.38) we deduce from (5.36) that

$$\frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \frac{1}{n_p-1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta w' + k^2 w')} dx + \tilde{a}(w_N, w') = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \hat{f}_{\xi_j} \overline{w'} dx.$$

which gives (5.35) using the definition of  $\Sigma(\hat{W}_N)$ .  $\square$

**Lemma 52.** *Under Assumptions 44 and 48, problem (5.28) has at most one solution.*



*Proof.* Let  $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$  be a solution of the homogeneous problem (5.28). Then, by Lemma 51,  $w_N$  satisfies the  $2\pi N$ -periodic problem (5.35). Using uniqueness result for quasi-periodic interior transmission problems stated in Theorem 46 (replacing  $n_p$  by  $n$  and the periodicity  $2\pi$  by  $2\pi N$ ) we deduce that  $w_N = 0$ . Consequently, taking  $\hat{w}'_{\xi_j} = 0$  for  $j \neq \ell$  in (5.28) we get

$$\int_{D_0^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_\ell} + k^2 n_p \hat{w}_{\xi_\ell}) \overline{(\Delta \hat{w}'_{\xi_\ell} + k^2 \hat{w}'_{\xi_\ell})} dx = 0,$$

for all  $\hat{w}'_{\xi_\ell} \in \tilde{H}_{\xi_\ell}^2(D_0^p)$ . Thus,  $\hat{w}_{\xi_\ell} = 0$  by uniqueness of solutions to quasi-periodic interior transmission problems (Theorem 46).  $\square$

**Lemma 53.** *Under Assumptions 44 and 48, problem (5.28) is well posed.*

*Proof.* From the well posedness of the  $2\pi N$ -periodic problem given by Theorem 46 we consider  $w_N \in \tilde{H}_{\#}^2(D_N^p)$  that satisfies (5.35). Let  $0 \leq j \leq N - 1$ . Using Theorem 46 we define the solution  $\hat{w}_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p)$  satisfying

$$a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) = \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx - \tilde{a}(w_N, \hat{w}'_{\xi_j}) \quad \text{for all } \hat{w}'_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p). \quad (5.39)$$

Therefore

$$\frac{1}{N} \sum_{j=0}^{N-1} a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx - \frac{1}{N} \sum_{j=0}^{N-1} \tilde{a}(w_N, \hat{w}'_{\xi_j}). \quad (5.40)$$

Let us define

$$w' = \sum_{j=0}^{N-1} \hat{w}'_{\xi_j}, \quad \check{w}_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j}, \quad \text{and} \quad f_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{f}_{\xi_j}.$$

Then using (5.37) and (5.38) we deduce that (5.40) implies

$$\frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta w' + k^2 w')} dx = -\tilde{a}(w_N, w') + \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \hat{f}_{\xi_j} \overline{w'} dx.$$

Therefore  $\check{w}_N \in \tilde{H}_\#^2(D_N^p)$  and satisfies

$$\int_{D_N^p} \frac{1}{n_p - 1} (\Delta \check{w}_N + k^2 n_p \check{w}_N) \overline{(\Delta w' + k^2 w')} dx = -\tilde{a}(w_N, w') + \int_{D_N^p} f_N \overline{w'} dx, \quad (5.41)$$

for all  $w' \in \tilde{H}_\#^2(D_N^p)$ . Observe that  $w_N$  also satisfies (5.41). Consequently by uniqueness of solutions to  $2\pi N$ -periodic interior transmission problems (under Assumption 44) we deduce that  $\check{w}_N = w_N$ . In conclusion, setting  $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$ , we have  $w_N = \Sigma(\hat{W}_N)$  and  $\hat{W}_N$  satisfies problem (5.28).  $\square$

### 5.4.3 Convergence analysis

Let  $M > 0$  be a given constant. We prove the convergence for the right hand sides  $f \in \mathcal{F}_M$  where

$$\mathcal{F}_M := \{f \in L^2(D^p); \mathcal{J}f \in H^1(I, L^2(D_0^p)) \text{ and } \|\mathcal{J}f\|_{H^1(I, L^2(D_0^p))} \leq M\}.$$

Let  $f \in \mathcal{F}_M$  and denote  $\hat{f}_\xi := \mathcal{J}(f)(\xi, \cdot)$ ,  $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$  be the associated solution of problem (5.28) and set  $w_N := \Sigma(\hat{W}_N)$ . We define the  $\xi$ -quasi periodic function

$$\tilde{w}_\xi := e^{i\xi x_1} \hat{w}_{\xi_j} e^{-i\xi_j x_1} \quad \text{for } \xi \in [\xi_j, \xi_{j+1}[ \quad \text{and } j = 0, \dots, N-1, \quad (5.42)$$

and define on  $D^p$  the function

$$\tilde{w}_N = \int_I \tilde{w}_\xi d\xi. \quad (5.43)$$

Since the function  $\xi \mapsto \tilde{w}_\xi$  is in  $L^2(I, \tilde{H}_\xi^2(D_0^p))$ , then  $\tilde{w}_N \in \tilde{H}_0^2(D^p)$ . In order to prove existence of solution to the interior transmission problem (5.27) we shall prove that the sequence  $\tilde{w}_N$  weakly converges to a solution of problem (5.27). We observe that

$$\|\tilde{w}_N\|_{L^2(D^p)}^2 = \int_0^1 \|\tilde{w}_\xi\|_{L^2(D_0^p)}^2 d\xi = \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 = \|w_N\|_{L^2(D_N^p)}^2, \quad (5.44)$$

where the last equality is a consequence of (5.32). Clearly

$$\|\tilde{w}_\xi\|_{H^2(D^p)} \leq 4\|\hat{w}_{\xi_j}\|_{H^2(D_0^p)} \quad \text{for } \xi \in [\xi_j, \xi_{j+1}[ \quad \text{and } j = 0, \dots, N-1. \quad (5.45)$$

Therefore

$$\|\tilde{w}_N\|_{H^2(D^p)}^2 = \int_0^1 \|\tilde{w}_\xi\|_{H^2(D_0^p)}^2 d\xi \leq 4 \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{H^2(D_0^p)}^2 = 4 \|w_N\|_{H^2(D_N^p)}^2, \quad (5.46)$$

where the last equality is a consequence of Lemma 50. We also introduce the continuous and piecewise affine function with respect to the  $\xi$  variable

$$\check{w}_\xi := N e^{i\xi x_1} (\hat{w}_{\xi_j} e^{-i\xi_j x_1} (\xi_{j+1} - \xi) + \hat{w}_{\xi_{j+1}} e^{-i\xi_{j+1} x_1} (\xi - \xi_j)), \quad (5.47)$$

for  $\xi \in [\xi_j, \xi_{j+1}[$  and  $j = 0, \dots, N-1$ . We then define

$$\check{w}_N := \int_I \check{w}_\xi d\xi. \quad (5.48)$$

This function satisfies

$$\|\check{w}_N\|_{L^2(D^p)}^2 = \int_0^1 \|\check{w}_\xi\|_{L^2(D_0^p)}^2 d\xi \leq \frac{4}{3} \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 = \frac{4}{3} \|w_N\|_{L^2(D_N^p)}^2, \quad (5.49)$$

and

$$\|\check{w}_N\|_{H^2(D^p)}^2 = \int_0^1 \|\check{w}_\xi\|_{H^2(D_0^p)}^2 d\xi \leq 4 \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{H^2(D_0^p)}^2 = 4 \|w_N\|_{H^2(D_N^p)}^2. \quad (5.50)$$

We also have for  $\xi \in [\xi_j, \xi_{j+1}[$

$$\begin{aligned} \|\check{w}_\xi\|_{L^2(D_0^p)}^2 &\geq \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 (\xi - \xi_{j+1})^2 + \|\hat{w}_{\xi_{j+1}}\|_{L^2(D_0^p)}^2 (\xi - \xi_j)^2 \\ &\quad - 2 \left( (\xi_{j+1} - \xi)(\xi - \xi_j) \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \|\hat{w}_{\xi_{j+1}}\|_{L^2(D_0^p)} \right). \end{aligned}$$

Therefore,

$$\|\check{w}_N\|_{L^2(D^p)}^2 = \frac{1}{N} \sum_{j=0}^{N-1} \int_{\xi_j}^{\xi_{j+1}} \|\check{w}_\xi\|_{L^2(D_0^p)}^2 d\xi \geq \frac{1}{3} \|w_N\|_{L^2(D_N^p)}^2. \quad (5.51)$$

We shall prove that this sequence also converges to the solution of problem (5.27). It is introduced to be able to apply a compactness argument in the Floquet-Bloch spaces.

**Lemma 54.** Fix  $N_0 \in \mathbb{N}$  and consider  $N \geq N_0$ . Then there exists a constant  $C(N_0) > 0$  that only depends on  $N_0$  such that

$$\|w_N - \tilde{w}_N\|_{L^2(D_{N_0}^p)} \leq \frac{N_0 C(N_0)}{N} \left( \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 \right)^{1/2},$$

and

$$\|w_N - \check{w}_N\|_{L^2(D_{N_0}^p)} \leq \frac{N_0^2 C(N_0)}{12N^2} \left( \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \right)^{1/2}.$$

*Proof.* Taking the difference between  $w_N$  and  $\tilde{w}_N$  we get

$$w_N - \tilde{w}_N = \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j} - \sum_{j=0}^{N-1} \int_{\xi_j}^{\xi_{j+1}} e^{ix_1(\xi - \xi_j)} \hat{w}_{\xi_j} d\xi = \frac{1}{N} \sum_{j=0}^{N-1} \theta_N(x_1) \hat{w}_{\xi_j},$$

with

$$\theta_N(x_1) = \begin{cases} 1 - \frac{N}{ix_1} \left( e^{ix_1 \frac{1}{N}} - 1 \right) & \text{for } x_1 \neq 0, \\ 0 & \text{for } x_1 = 0. \end{cases} \quad (5.52)$$

We observe that  $\theta_N(x_1)$  have the following asymptotic form as  $N$  tends to infinity

$$\theta_N(x_1) = 1 - \frac{N}{ix_1} \left( \frac{ix_1}{N} + \frac{1}{2} \left( \frac{ix_1}{N} \right)^2 + \mathcal{O} \left( \frac{1}{N^3} \right) \right) = \frac{ix_1}{2N} + \mathcal{O} \left( \frac{1}{N^2} \right).$$

with  $x_1 \leq N_0$ . Therefore

$$\begin{aligned} \|w_N - \tilde{w}_N\|_{L^2(D_{N_0}^p)} &\leq \frac{1}{N} \sum_{j=0}^{N-1} \sup_{x_1 \in D_{N_0}^p} |\theta_N(x_1)| \|\hat{w}_{\xi_j}\|_{L^2(D_{N_0}^p)}, \\ &\leq \frac{N_0 C(N_0)}{N^2} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}, \end{aligned}$$

with  $C(N_0) > 0$  is constant that depends on  $N_0$  and does not depend on  $N$ , then using Cauchy-Schwartz inequality we get

$$\|w_N - \tilde{w}_N\|_{L^2(D_{N_0}^p)} \leq \frac{N_0 C(N_0)}{N} \left( \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 \right)^{1/2}. \quad (5.53)$$

Let us now take the difference between  $w_N$  and  $\check{w}_N$ . We have

$$\begin{aligned} w_N - \check{w}_N &= \sum_{j=0}^{N-1} \left( \frac{1}{N} \hat{w}_{\xi_j} - N \hat{w}_{\xi_{j+1}} \int_{\xi_j}^{\xi_{j+1}} e^{ix_1(\xi - \xi_{j+1})} (\xi - \xi_j) d\xi \right. \\ &\quad \left. + N \hat{w}_{\xi_j} \int_{\xi_j}^{\xi_{j+1}} e^{ix_1(\xi - \xi_j)} (\xi - \xi_{j+1}) d\xi \right) \\ &= \sum_{j=0}^{N-1} \left( \frac{1}{N} \hat{w}_{\xi_j} - N \hat{w}_{\xi_{j+1}} \left( \frac{1}{N} \frac{1}{ix_1} + \frac{1}{x_1^2} (1 - e^{-\frac{ix_1}{N}}) \right) \right. \\ &\quad \left. + N \hat{w}_{\xi_j} \left( \frac{1}{N} \frac{1}{ix_1} - \frac{1}{x_1^2} (1 - e^{\frac{ix_1}{N}}) \right) \right), \end{aligned}$$

Since  $\hat{w}_{\xi_0} = \hat{w}_{\xi_N}$  then we have

$$w_N - \check{w}_N = \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j} \left[ 1 - \frac{N^2}{x_1^2} \left( (1 - e^{-\frac{ix_1}{N}}) + (1 - e^{\frac{ix_1}{N}}) \right) \right] = \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j} \check{\theta}_N(x_1),$$

with

$$\check{\theta}_N(x_1) = \begin{cases} 1 - \frac{2N^2}{x_1^2} \left( 1 - \cos\left(\frac{x_1}{N}\right) \right) & \text{for } x_1 \neq 0, \\ 0 & \text{for } x_1 = 0. \end{cases} \quad (5.54)$$

We observe that  $\check{\theta}_N(x_1)$  have the following asymptotic form as  $N$  tends to infinity

$$\check{\theta}_N(x_1) = 1 - \frac{2N^2}{x_1^2} \left[ \frac{1}{2} \left( \frac{x_1}{N} \right)^2 - \frac{1}{4!} \left( \frac{x_1}{N} \right)^4 + \mathcal{O}\left(\frac{1}{N^6}\right) \right] = \frac{x_1^2}{12N^2} + \mathcal{O}\left(\frac{1}{N^4}\right),$$

for  $x_1 \leq N_0$ . Thus

$$\|w_N - \check{w}_N\|_{L^2(D_{N_0}^p)} \leq \frac{N_0^2 C(N_0)}{12N^3} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \leq \frac{N_0^2 C(N_0)}{12N^2} \left( \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \right)^{1/2},$$

which ends the proof.  $\square$

**Lemma 55.** *Let  $f \in \mathcal{F}_M$  and  $w_N$  as defined previously. Set  $f_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{f}_{\xi_j}$ . There exists a constant  $C > 0$  independent from  $f$  and  $N$  (that may depend on  $M$ ) such that*

$$\|w_N\|_{L^2(D_N^p)} \leq C \|f_N\|_{L^2(D_N^p)}. \quad (5.55)$$

*Proof.* We prove this Lemma using a contradiction argument. Assume that for all  $C \in \mathbb{N}$ , there exists  $f \in \mathcal{F}_M$  that depends on  $C$  and there exists  $N$  that also depends on  $C$  such that

$$\|w_N\|_{L^2(D_N^p)} \geq C \|f_N\|_{L^2(D_N^p)}.$$

One can choose the sequence  $N$  to be monotonically increasing with respect to  $C$ . Indexing  $C$  with respect to  $N$ , replacing  $f_N$  by  $\frac{f_N}{\|w_N\|_{L^2(D_N^p)}}$  and  $w_N$  by  $\frac{w_N}{\|w_N\|_{L^2(D_N^p)}}$ , the contradiction statement can be equivalently phrased as: there exists a monotonically increasing sequence  $N$  such that

$$\|f_N\|_{L^2(D_N^p)} \longrightarrow 0 \quad \text{as } N \rightarrow \infty, \quad (5.56)$$

and  $\|w_N\|_{L^2(D_N^p)} = 1$  where  $w_N$  is the solution of problem (5.35) associated with  $f_N$ . Taking  $w' = w_N$  in (5.35) we obtain

$$\int_{D_N^p} \frac{1}{n-1} |\Delta w_N|^2 dx = - \int_{D_N^p} (k^2 \Delta w_N \overline{w_N} + k^2 n w_N \overline{\Delta w_N} + k^2 n |w_N|^2) dx + \int_{D_N^p} f_N \overline{w_N} dx.$$

Therefore

$$\begin{aligned} \gamma_0 \|\Delta w_N\|_{L^2(D_N^p)}^2 &\leq \left( k^2 (n^* + 1) \|w_N\|_{L^2(D_N^p)} \|\Delta w_N\|_{L^2(D_N^p)} \right. \\ &\quad \left. + k^2 n^* \|w_N\|_{L^2(D_N^p)}^2 + \|f_N\|_{L^2(D_N^p)} \|w_N\|_{L^2(D_N^p)} \right), \end{aligned}$$

with  $\gamma_0 := \frac{1}{n^* - 1}$  in the case  $n_* > 1$  and  $\gamma_0 := \frac{1}{n_* - 1}$  in the case  $n^* < 1$ . This inequality implies in particular  $\Delta w_N$  is uniformly bounded with respect to  $N$  in  $L^2(D_N^p)$ . This implies in particular that  $w_N$  is uniformly bounded in  $H^2(\tilde{D})$  (using elliptic regularity).

Recall that

$$a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) = \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx - \tilde{a}(w_N, \hat{w}'_{\xi_j}) \quad \text{for all } w'_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p). \quad (5.57)$$

Using Lemma 47 we deduce that

$$\|\hat{w}_{\xi_j}\|_{H^2(D_0^p)} \leq C \left( \|w_N\|_{H^2(\tilde{D})} + \|\hat{f}_{\xi_j}\|_{L^2(D_0^p)} \right)$$

for some constant  $C$  independent from  $N$ . Therefore

$$\frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{H^2(D_0^p)}^2 \leq 2C^2 \left( \|w_N\|_{H^2(D)}^2 + \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{f}_{\xi_j}\|_{L^2(D_0^p)}^2 \right). \quad (5.58)$$

Using (5.46) and (5.50) we infer that  $\tilde{w}_N$  and  $\check{w}_N$  are bounded in  $\tilde{H}^2(D^p)$  and  $w_N$  is uniformly bounded in  $\tilde{H}^2(D_N^p)$ . Consequently, up to subsequence, we can assume that  $\tilde{w}_N$  and  $\check{w}_N$  weakly converge in  $\tilde{H}^2(D^p)$  and their limit is the same as a consequence of Lemma 54. Let us denote by  $w \in \tilde{H}^2(D^p)$  this limit. Let  $N_0 \in \mathbb{N}$  be a fixed parameter. According to Lemma 54 we can also assume (up to changing the subsequence) that  $w_N$  weakly converges to  $w$  in  $\tilde{H}^2(D_{N_0}^p)$ .

Consider now  $\varphi \in C_c^\infty(D_{N_0}^p \cup \Gamma^0)$  that we extend by zero in  $D_N^p$  and by  $2\pi N$  periodicity in the  $x_1$  direction. This extension give us a test function in  $\tilde{H}_\#^2(D_N^p)$  that we can use in (5.35) and get

$$\int_{D_{N_0}^p} \frac{1}{n-1} (\Delta w_N + k^2 n w_N) \overline{(\Delta \varphi + k^2 \varphi)} dx = \int_{D_{N_0}^p} f_N \bar{\varphi} dx. \quad (5.59)$$

Passing to the limit as  $N$  goes to infinity in (5.59) shows that  $w \in \tilde{H}^2(D_{N_0}^p)$  satisfies

$$\int_{D^p} \frac{1}{n-1} (\Delta w + k^2 n w) \overline{(\Delta \varphi + k^2 \varphi)} dx = 0, \quad \forall \varphi \in C_c^\infty(D_{N_0}^p \cup \Gamma^0). \quad (5.60)$$

Since  $N_0$  can be chosen arbitrarily large, the last equality holds for all  $\varphi \in C_c^\infty(D^p \cup \Gamma^0)$  and by denseness argument for all  $\varphi \in \tilde{H}^2(D^p)$ . Then (5.60) shows that  $w$  is a solution to the homogeneous interior transmission problem (5.20) and by the uniqueness result of Lemma 49,  $w = 0$ .

In order to obtain a contradiction, we shall prove that the sequence  $\check{w}_N$  strongly converges to  $w$  in  $L^2(D^p)$  which is equivalent to proving that  $\xi \rightarrow \check{w}_\xi$  is strongly convergent in  $L^2(I, L^2(D_0^p))$ . Since this sequence is bounded in  $L^2(I, H_\xi^2(D_0^p))$ , using the Rellich compactness embedding theorem it is sufficient to prove that  $\xi \rightarrow \check{w}_\xi$  is bounded in  $H^1(I \times D_0^p)$ .

From (5.50) we have that  $\check{w}_N$  is uniformly bounded with respect to  $N$  in  $\tilde{H}_0^2(D^p)$ . Therefore, there exists a subsequence  $\check{w}_N$  that weakly converges to zero in  $\tilde{H}_0^2(D^p)$ . Moreover,  $\check{w}_\xi$  is bounded in  $L^2(I, \tilde{H}_\xi^2(D_0^p))$ . Then it remains to prove that  $\xi \rightarrow \frac{\partial \check{w}_\xi}{\partial \xi}$  is

bounded in  $L^2(I \times D_0^p)$ . For  $\xi \in [\xi_j, \xi_{j+1}[$  we have

$$\frac{\partial \check{w}_\xi}{\partial \xi} = ix_1 \check{w}_\xi + Ne^{ix_1 \xi} (\hat{w}_{\xi_{j+1}}^\# - \hat{w}_{\xi_j}^\#),$$

with

$$\hat{w}_{\xi_j}^\# := e^{-ix_1 \xi_j} \hat{w}_{\xi_j} \quad \text{and} \quad \hat{w}_{\xi_{j+1}}^\# := e^{-ix_1 \xi_{j+1}} \hat{w}_{\xi_{j+1}}. \quad (5.61)$$

Let  $\hat{w}^\# \in \tilde{H}_\#^2(D^p)$  and set  $\hat{w}'_{\xi_j} = e^{ix_1 \xi_j} \hat{w}^\# \in \tilde{H}_{\xi_j}^2(D^p)$ . Then, replacing  $\hat{w}_{\xi_j}$  and  $\hat{w}'_{\xi_j}$  respectively by  $\hat{w}_{\xi_j}^\# e^{ix_1 \xi_j}$  and  $\hat{w}^\# e^{ix_1 \xi_j}$  in (5.57) we get

$$\begin{aligned} & \int_{D_0^p} \frac{1}{n_p - 1} \left( \Delta \hat{w}_{\xi_j}^\# + 2i\xi_j \frac{\partial \hat{w}_{\xi_j}^\#}{\partial x_1} + (k^2 n_p - |\xi_j|^2) \hat{w}_{\xi_j}^\# \right) \\ & \quad \times \overline{\left( \Delta \hat{w}^\# + 2i\xi_j \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 n_p - |\xi_j|^2) \hat{w}^\# \right)} dx \\ & \quad + \int_{\tilde{D}} \left( \tilde{n}(\Delta w_N + k^2 n_p w_N) + k^2 \frac{(n - n_p)}{n - 1} w_N \right) \\ & \quad \times \overline{\left( \Delta \hat{w}^\# + 2i\xi_j \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 n_p - |\xi_j|^2) \hat{w}^\# \right)} e^{ix_1 \xi_j} dx \\ & = \int_{D_0^p} \hat{f}_{\xi_j}^\# \overline{\hat{w}^\#} dx, \end{aligned} \quad (5.62)$$

with  $\hat{f}_{\xi_j}^\# := e^{-ix_1 \xi_j} \hat{f}_{\xi_j}$ . Define

$$e_{j+1} := N(\hat{w}_{\xi_{j+1}}^\# - \hat{w}_{\xi_j}^\#).$$

Consequently, taking the difference between equation (5.62) satisfied respectively for  $\xi_{j+1}$  and  $\xi_j$  we get

$$\begin{aligned} & \frac{1}{N} \int_{D_0^p} \frac{1}{n_p - 1} \left( \Delta e_{\xi_{j+1}} + 2i\xi_{j+1} \frac{\partial e_{\xi_{j+1}}}{\partial x_1} + (k^2 n_p - |\xi_{j+1}|^2) e_{\xi_{j+1}} \right) \\ & \quad \times \overline{\left( \Delta \hat{w}^\# + 2i\xi_{j+1} \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 n_p - |\xi_{j+1}|^2) \hat{w}^\# \right)} dx \\ & \quad + (\xi_{j+1} - \xi_j) \left( a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) + a_{p,2}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) + \tilde{a}_1(w_N, \hat{w}^\#) + \tilde{a}_2(w_N, \hat{w}^\#) \right) \\ & = \int_{D_0^p} (\hat{f}_{\xi_j}^\# - \hat{f}_{\xi_{j+1}}^\#) \overline{\hat{w}^\#} dx. \end{aligned} \quad (5.63)$$



with  $a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#)$  and  $a_{p,2}(\hat{w}_{\xi_j}^\#, \hat{w}^\#)$  are the sesquilinear forms defined by

$$\begin{aligned} a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) &:= \int_{D_0^p} \frac{1}{n_p - 1} \left( 2i \frac{\partial \hat{w}_{\xi_j}^\#}{\partial x_1} - (|\xi_{j+1}| + |\xi_j|) \hat{w}_{\xi_j}^\# \right) \\ &\quad \times \overline{\left( \Delta \hat{w}^\# + 2i \xi_j \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 - |\xi_j|^2) \hat{w}^\# \right)} dx, \\ a_{p,2}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) &:= \int_{D_0^p} \frac{1}{n_p - 1} \left( \Delta \hat{w}_{\xi_j}^\# + 2i \xi_{j+1} \frac{\partial \hat{w}_{\xi_j}^\#}{\partial x_1} + (k^2 n_p - |\xi_{j+1}|^2) \hat{w}_{\xi_j}^\# \right) \\ &\quad \times \overline{\left( 2i \frac{\partial \hat{w}^\#}{\partial x_1} - (|\xi_{j+1}| + |\xi_j|) \hat{w}^\# \right)} dx, \end{aligned}$$

and  $\tilde{a}_1(w_N, \hat{w}^\#)$ ,  $\tilde{a}_2(w_N, \hat{w}^\#)$  are defined by

$$\begin{aligned} \tilde{a}_1(w_N, \hat{w}^\#) &:= \int_{\tilde{D}} \left( \tilde{n}(\Delta w_N + k^2 n_p w_N) + k^2 \frac{(n - n_p)}{n - 1} w_N \right) \\ &\quad \times \overline{e^{i \xi_j x_1} \left( 2i \frac{\partial \hat{w}^\#}{\partial x_1} - (|\xi_j| + |\xi_{j+1}|) \hat{w}^\# \right)} dx \\ \tilde{a}_2(w_N, \hat{w}^\#) &:= \int_{\tilde{D}} \left( \tilde{n}(\Delta w_N + k^2 n_p w_N) + k^2 \frac{(n - n_p)}{n - 1} w_N \right) \\ &\quad \times \overline{N \left( e^{i x_1 \xi_{j+1}} - e^{i x_1 \xi_j} \right) \left( \Delta \hat{w}^\# + 2i \xi_{j+1} \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 - |\xi_{j+1}|^2) \hat{w}^\# \right)} dx. \end{aligned}$$

Setting  $\hat{e}_{\xi_{j+1}} := e^{i x_1 \xi_{j+1}} e_{j+1} \in \tilde{H}_{\xi_{j+1}}^2(D^p)$  and choosing  $w^\# = e^{-i x_1 \xi_{j+1}} \hat{w}'_{\xi_{j+1}}$  for some  $\hat{w}'_{\xi_{j+1}} \in \tilde{H}_{\xi_{j+1}}^2(D^p)$  we deduce that

$$\begin{aligned} a_{\xi_{j+1}}(\hat{e}_{\xi_{j+1}}, \hat{w}'_{\xi_{j+1}}) &= - \left( a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) + a_{p,2}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) + \tilde{a}_1(w_N, \hat{w}^\#) + \tilde{a}_2(w_N, \hat{w}^\#) \right) \\ &\quad + N \int_{D_0^p} \left( \hat{f}_{\xi_j}^\# - \hat{f}_{\xi_{j+1}}^\# \right) \overline{\hat{w}^\#} dx. \end{aligned}$$

Therefore, using Theorem 47 we conclude that

$$\|\hat{e}_{\xi_{j+1}}\|_{H^2(D_0^p)} \leq \gamma_1 \left( \|\hat{w}_{\xi_j}^\#\|_{H^2(D_0^p)} + \|w_N\|_{H^2(\tilde{D})} + N \left\| \hat{f}_{\xi_j}^\# - \hat{f}_{\xi_{j+1}}^\# \right\|_{L^2(D_0^p)} \right), \quad (5.64)$$

with  $\gamma_1 > 0$  is a constant independent from  $\xi_{j+1}$  and  $N$ . On the other hand, Denoting by  $\tilde{e}_\xi := e^{i \xi x_1} \hat{e}_{\xi_{j+1}} e^{-i x_1 \xi_{j+1}}$ , then we have

$$\frac{\partial \tilde{w}_\xi}{\partial \xi} = i x_1 \tilde{w}_\xi + \tilde{e}_\xi. \quad (5.65)$$

Using the same arguments as for (5.44) we observe that

$$\|\tilde{e}_\xi\|_{L^2(I, L^2(D_0^p))}^2 = \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{e}_{\xi_j}\|_{L^2(D_0^p)}^2, \quad (5.66)$$

since  $\hat{e}_{\xi_0} = \hat{e}_{\xi_N}$ . Consequently, from (5.65) and (5.66) we deduce that

$$\left\| \frac{\partial \check{w}_\xi}{\partial \xi} \right\|_{L^2(I, L^2(D_0^p))}^2 \leq 8\pi^2 \|\check{w}_N\|_{L^2(D^p)}^2 + 2 \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{e}_{\xi_j}\|_{L^2(D_0^p)}^2.$$

Then using (5.49), (5.64) and (5.58) we get

$$\begin{aligned} \left\| \frac{\partial \check{w}_\xi}{\partial \xi} \right\|_{L^2(I, L^2(D_0^p))}^2 &\leq \frac{16\pi^2}{3} \|w_N\|_{L^2(\bar{D})}^2 \\ &+ 8\gamma_1^2 \frac{1}{N} \sum_{j=0}^{N-1} \left( \|\hat{w}_{\xi_j}^\# \|_{H^2(D_0^p)}^2 + \|w_N\|_{H^2(\bar{D})}^2 + N^2 \|\hat{f}_{\xi_j}^\# - \hat{f}_{\xi_{j+1}}^\# \|_{L^2(D_0^p)}^2 \right), \end{aligned}$$

with  $\gamma_2 > 0$  is a constant independent from  $N$ . On the other hand,

$$\frac{1}{N} \sum_{j=0}^{N-1} N^2 \|\hat{f}_{\xi_j}^\# - \hat{f}_{\xi_{j+1}}^\# \|_{L^2(D_0^p)}^2 \leq C \|\mathcal{J}f(\cdot, \cdot)\|_{H^1(I \times D_0^p)}^2 \leq CM^2,$$

for some constant  $C > 0$ . Thus, combining this inequality with (5.58) we deduce that  $\xi \rightarrow \frac{\partial \check{w}_\xi}{\partial \xi}$  is bounded in  $L^2(I, L^2(D_0^p))$ . Therefore,  $\xi \rightarrow \check{w}_\xi$  is bounded in  $H^1(I \times D_0^p)$  which implies that  $\check{w}_N$  converges strongly to  $w = 0$  in  $L^2(D^p)$  and by . This contradicts  $\|\check{w}_N\|_{L^2(D^p)}^2 \geq \frac{1}{3}$  by (5.51).  $\square$

**Theorem 56.** *Assume that Assumption 48 is verified and let  $f \in \mathcal{F}_M$ . Then Problem (5.27) has a unique solution  $w \in \tilde{H}_0^2(D^p)$ . Moreover, there exists a constant  $C(M)$  independent from  $f$  such that*

$$\|w\|_{H^2(D^p)} \leq C(M) \|f\|_{L^2(D^p)}.$$

*Proof.* Let  $w_N$  and  $\check{w}_N$  be the sequences introduced above. From Lemma 55 we deduce that  $w_N$  is bounded in  $L^2(D_N^p)$ . Using (5.49) we infer that  $\check{w}_N$  is bounded in  $L^2(D^p)$ . The same reasoning as in the proof of Lemma 55 allows us to conclude that (up to a subsequence)  $w_N$  and  $\check{w}_N$  locally converge in  $L^2(D^p)$  to a same limit  $w$  in  $\tilde{H}_0^2(D^p)$  that

satisfies (5.27). Moreover, using the same arguments that lead to (5.58), Lemma 55, and (5.50) we infer the existence of a constant  $C(M)$  such that

$$\|\check{w}_N\|_{H^2(D^p)} \leq C(M) \|f_N\|_{L^2(D_N^p)}. \quad (5.67)$$

This shows in particular that we can choose the subsequence  $\check{w}_N$  to be weakly convergent to  $w$  in  $H^2(D^p)$  and passing to the limit in the inequality leads to (5.67) since  $\|w\|_{H^2(D^p)} \leq \liminf \|\check{w}_N\|_{L^2(D^p)}$  and

$$\|f_N\|_{L^2(D_N^p)}^2 = \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{f}_{\xi_j}\|_{L^2(D_0^p)}^2 \longrightarrow \int_I \|\hat{f}_{\xi}\|_{L^2(D_0^p)}^2 d\xi = \|f\|_{L^2(D^p)},$$

as  $N$  goes to infinity. This ends the proof.  $\square$

As explained before, the study of problem (5.25) follow the same lines as problem (5.26). The only minor difference is in the treatment of the right hand side of equation (5.62) that will contain an additional term depending on  $\xi_j$  but without any effect on the subsequent conclusions of Lemma 55. This leads to the following Theorem on the well posedness of the interior transmission problem (5.17).

**Theorem 57.** *Assume that Assumption 48 is verified and let  $M > 0$  be a given constant. Assume in addition that the boundary data  $\varphi$  and  $\psi$  are respectively the trace and the normal trace on  $\partial D^p \setminus \Gamma^0$  of some function  $\theta \in \tilde{H}_0^2(D^p)$  and such that  $\|\mathcal{J}\theta\|_{H^1(I, H^2(D_0^p))} \leq M$ . Then, Problem (5.18) has a unique solution  $w_0 \in H^2(D^p)$  satisfying*

$$\|w_0\|_{H^2(D^p)} \leq C(M) \left( \|\varphi\|_{H^{3/2}(\partial D^p \setminus \Gamma^0)} + \|\psi\|_{H^{1/2}(\partial D^p \setminus \Gamma^0)} \right),$$

with a constant  $C(M)$  independent from  $\varphi$  and  $\psi$ .

# Conclusion and perspectives

We presented in this PhD thesis a study of an inverse scattering problem for the reconstruction of defects in an unbounded periodic structures. By applying incident point sources and measuring the scattered field on a line parallel to the periodic layer, we proposed the method to reconstruct the shape of the defects without a knowledge of the periodic background.

In Chapter 2, we presented and complemented the results given in the literature on the well-posedness of the direct scattering problem from unbounded periodic layers. This problem was solved by applying the Floquet-Bloch transform which led us to study the quasi-periodic scattering problem without defects and then to deduce the well-posed character of the problem in the presence of defects. The uniqueness of the solution was proven by assuming that the periodic refractive index is positive at least in an open domain in the periodic background.

In Chapter 3, we employed the GLSM as introduced in [39] to solve the inverse scattering problem by removing the technical assumption made in [39], where it is assumed that the defect is also periodic with a longer period.

We first studied the inverse problem using quasi-periodic incident waves, we reintroduced the problem associated with this type of incident sources, and then introduced the quasi-periodic near field operator for fixed Floquet-Bloch variable. Next, we applied GLSM to solve the inverse problem for a single Floquet-Bloch mode and constructed an indicator function for reconstructing the defective period distributed periodically.

We then considered the case where non-quasi periodic incident sources are applied. The main difficulty here is that the Floquet-Bloch transform of the full near-field operator does not coincide with the quasi-periodic near field operator. In order to apply the

GLSM we assume known the single Floquet-Bloch near field operators. This allows us to construct a suitable penalty term for the GLSM and obtain an indicator function for reconstructing the full domain. These two results were obtained under the assumption that the defect does not intersect the periodic background. One of the main perspectives of this part is to extend the analysis to the case where the defect does intersect the periodic background.

The study of the inverse problem uses the results on the well-posedness of the forward problem where it is assumed that the refractive index has a positive imaginary part. Getting rid of this assumption is one of our perspectives that would rely on the analysis of the direct problem done in [35, 38].

In numerics, we used the code developed by [39] to generate solutions to the scattering problem from locally perturbed periodic layers, and we gave some numerical examples for the reconstruction obtained using the indicator function associated with the single Floquet-Bloch mode. Extending the numerics to 3D problems is one of our objectives in the near future.

In Chapter 4, we applied the DLMS as introduced in [39] also with removing the assumption that the defect is periodic. In order to do so, we reapplied the GLSM for a single Floquet-Bloch mode as in Chapter 3, but taking a period equal to an integer multiple of the original periodicity. Then, using the quasi-periodic indicator function associated with these two different periodicity lengths and the one associated with the complete domain, we introduced an indicator function allowing to directly reconstruct the defect. We also observed that applying GLSM for a single Floquet-Bloch mode is sufficient to derive an indicator function for the defect. We used the later to give some numerical examples for the application of the DLMS.

In Chapter 5, we presented a study for the well-posedness of the interior transmission problem (ITP) by considering one of the possible cases when the perturbation intersects the periodic background (when the defect is included in the periodic domain). The study

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was based on the application of the Floquet-Bloch transform, a discretization with respect to the Floquet-Bloch variable and a convergence analysis to construct a solution of the ITP. The uniqueness was also proved under the assumption that the refractive index has a positive imaginary part. Similarly to the direct problem, we would like to remove this assumption in a future work.

As a more broader perspective for this work we can mention the extension to other models for scattering problems such as Maxwell's equations or elastodynamics, which require in particular to consider the problem in a 3D setting.

# Bibliography

- [1] A. Alex-Amor, Á. Palomares-Caballero and C. Molero. *3-D metamaterials: Trends on applied designs, computational methods and fabrication techniques*. Electronics, 11(3), 410, (2022).
- [2] T. Arens. *Scattering by biperiodic layered media: The integral equation approach*. Habilitation Thesis, Universität Karlsruhe, (2010).
- [3] T. Arens and N. Grinberg. *A Complete Factorization Method for Scattering by Periodic Surfaces*. Computing 75(2), 111-132, (2005).
- [4] T. Arens and A. Kirsch. *The factorization method in inverse scattering from periodic structures*. Inverse Problems, 19(5), 1195, (2003).
- [5] L. Audibert, *Qualitative methods for heterogeneous media*, theses, Ecole Doctorale Polytechnique, (2015).
- [6] L. Audibert, A. Girard and H. Haddar, *Identifying defects in an unknown background using differential measurements*. Inverse Problems and Imaging, 9(3), 625, (2015).
- [7] L. Audibert and H. Haddar. *A generalized formulation of the linear sampling method with exact characterization of targets in terms of farfield measurements*. Inverse Problems, 30(3), 035011, (2014).
- [8] Y. Boukari, H. Haddar and N. Jenhani. *Analysis of sampling methods for imaging a periodic layer and its defects*. Inverse Problems, 39(5), 055001, (2023).
- [9] L. Bourgeois and S. Fliss. *On the identification of defects in a periodic waveguide from far field data*, Inverse Problems, 30(9), 095004, (2014).
- [10] L. Bourgeois, E. Lunéville,. *On the use of the linear sampling method to identify cracks in elastic waveguides*. Inverse problems, 29(2), 025017, (2013).

- [11] F. Cakoni, D. Colton and H. Haddar. *Inverse Scattering Theory and Transmission Eigenvalues*, 2nd Edition (Philadelphia, PA: Society for Industrial and Applied Mathematics) (<https://doi.org/10.1137/1.9781611977424>), (2022).
- [12] F. Cakoni and H. Haddar. On the existence of transmission eigenvalues in an inhomogeneous medium. *Applicable Analysis*, 88(4), 475-493, (2009).
- [13] F. Cakoni and H. Haddar. Transmission eigenvalues. *Inverse Problems*, 29(10), 100201, (2013).
- [14] F. Cakoni, H. Haddar and T. P. Nguyen. *New interior transmission problem applied to a single Floquet-Bloch mode imaging of local perturbations in periodic media*. *Inverse Problems*, 35(1), 015009, (2018).
- [15] S. N. Chandler-Wilde and J. Elschner. *Variational approach in weighted Sobolev spaces to scattering by unbounded rough surfaces*. *SIAM journal on mathematical analysis*, 42(6), 2554-2580, (2010).
- [16] S. N. Chandler-Wilde, E. Heinemeyer and R. Potthast. *Acoustic scattering by mildly rough unbounded surfaces in three dimensions*. *SIAM Journal on Applied Mathematics*, 66(3), 1002-1026, (2006).
- [17] S. N. Chandler-Wilde and C. R. Ross. *Scattering by rough surfaces: the Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane*. *Mathematical methods in the applied sciences*, 19(12), 959-976, (1996).
- [18] S. N. Chandler-Wilde, C. R. Ross and B. Zhang. *Scattering by infinite one-dimensional rough surfaces*. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 455(1990), 3767-3787, (1999).
- [19] S. N. Chandler-Wilde and B. Zhang. *A uniqueness result for scattering by infinite rough surfaces*. *SIAM Journal on Applied Mathematics*, 58(6), 1774-1790, (1998).
- [20] D. Colton, and A. Kirsch. *A simple method for solving inverse scattering problems in the resonance region*. *Inverse problems*, 12(4), 383, (1996).
- [21] D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd ed, Springer Verlag: Berlin, (2013).



- [22] D. Colton, M. Piana and R. Potthast. *A simple method using Morozov's discrepancy principle for solving inverse scattering problems*. Inverse Problems, 13(6), 1477, (1997).
- [23] J. Elschner and G. Hu . *Inverse scattering of elastic waves by periodic structures: uniqueness under the third or fourth kind boundary conditions*. Methods and Applications of Analysis, 18(2), 215-244, (2011).
- [24] S. Fliss. *Analyse mathématique et numérique de problèmes de propagation des ondes dans des milieux périodiques infinis localement perturbés*. PhD thesis, Ecole Polytechnique, (2009).
- [25] S. Fliss and P. Joly. *Solutions of the time-harmonic wave equation in periodic waveguides: asymptotic behaviour and radiation condition*. Archive for Rational Mechanics and Analysis, 219, 349-386, (2016).
- [26] T. Furuya. *The factorization and monotonicity method for the defect in an open periodic waveguide*. Journal of Inverse and Ill-posed Problems, 28(6), 783-796, (2020).
- [27] S. V. Georgakopoulos, C. L. Zekios, A. Sattar-Kaddour, M. Hamza, A. Biswas, B. Clark, ... and R. J. Lang. *Origami antennas*. IEEE Open Journal of Antennas and Propagation, 2, 1020-1043, (2021).
- [28] H. Haddar and A. Kirsch. *Factorization method for imaging a local perturbation in inhomogeneous periodic layers from far field measurements*. Inverse Problems and Imaging, 14(1), 133-152, (2020).
- [29] H. Haddar and T. P. Nguyen. *A volume integral method for solving scattering problems from locally perturbed infinite periodic layers*. Applicable Analysis, 96(1), 130-158, (2017).
- [30] H. Haddar and T.P. Nguyen. *Sampling methods for reconstructing the geometry of a local perturbation in unknown periodic layers*. Computers Mathematics with Applications, 74(11), 2831-2855, (2017).
- [31] G. Hu, Y. Lu and B. Zhang. *The factorization method for inverse elastic scattering from periodic structures*. Inverse Problems, 29(11), 115005, (2013).

- [32] G. Hu and B. Zhang. *The linear sampling method for the inverse electromagnetic scattering by a partially coated bi-periodic structure*. Mathematical methods in the applied sciences, 34(5), 509-519, (2011).
- [33] P. Joly, J. R. Li and S. Fliss. *Exact boundary conditions for periodic waveguides containing a local perturbation*. Commun. Comput. Phys, 1(6), 945-973, (2006).
- [34] A. Kirsch. *A note on Sylvester's proof of discreteness of interior transmission eigenvalues*. Comptes Rendus Mathematique, 354(4), 377-382, (2016).
- [35] A. Kirsch. *A scattering problem for a local perturbation of an open periodic waveguide*. Mathematical Methods in the Applied Sciences, (2022).
- [36] A. Kirsch. *Factorization of the far-field operator for the inhomogeneous medium case and an application in inverse scattering theory*. Inverse problems, 15(2), 413, (1999).
- [37] A. Kirsch and N. Grinberg. *The factorization method for inverse problems*. (Vol. 36). OUP Oxford, (2007).
- [38] A. Kirsch and A. Lechleiter. *The limiting absorption principle and a radiation condition for the scattering by a periodic layer*. SIAM Journal on Mathematical Analysis, 50(3), 2536-2565, (2018).
- [39] A. Kirsch and A. Lechleiter. *Reconstruction of a local perturbation in inhomogeneous periodic layers from partial near field measurements*. Inverse Problems, 35(11), 114006, (2019).
- [40] A. Lechleiter, *Imaging of periodic dielectrics*. BIT Numerical Mathematics, 50(1), 59-83, (2010).
- [41] A. Lechleiter. *The Floquet-Bloch transform and scattering from locally perturbed periodic surfaces*. Journal of Mathematical Analysis and Applications, 446(1), 605-627, (2017).
- [42] A. Lechleiter and D-L. Nguyen. *Factorization Method for Electromagnetic Inverse Scattering from Biperiodic Structures*, SIAM Journal on Imaging Sciences, 6(2), 1111-1139. (2013).

- [43] A. Lechleiter and D-L. Nguyen. *Scattering of Herglotz waves from periodic structures and mapping properties of the Bloch transform*. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 145(6), 1283-1311, (2015).
- [44] A. Lechleiter and D-L. Nguyen. *Volume integral equations for scattering from anisotropic diffraction gratings*. Mathematical Methods in the Applied Sciences, 36(3), 262-274, (2013).
- [45] A. Lechleiter and R. Zhang. *A convergent numerical scheme for scattering of aperiodic waves from periodic surfaces based on the Floquet-Bloch transform*. SIAM Journal on Numerical Analysis, 55(2), 713-736, (2017).
- [46] A. Lechleiter, R. Zhang. *Reconstruction of local perturbations in periodic surfaces*. Inverse Problems, 34(3), 035006, (2018)
- [47] D. L. Nguyen, *Spectral Methods for Direct and Inverse Scattering from Periodic Structures*, PhD thesis, (2012).
- [48] D. L. Nguyen, K. Stahl, and T. Truong. *A new sampling indicator function for stable imaging of periodic scattering media*. Inverse Problems, 39(6), 065013, (2023).
- [49] T. P. Nguyen. *Direct and inverse solvers for scattering problems from locally perturbed infinite periodic layers*, PhD thesis, (2017).
- [50] T. P. Nguyen. *Differential imaging of local perturbations in anisotropic periodic media*. Inverse Problems, 36(3), 034004, (2020).
- [51] T. P. Nguyen, F. Cakoni and H. Haddar. *Direct Imaging of Local Perturbations in a Unknown Bi-Periodic Layered Medium*, Preprint, (2022).
- [52] J. Reinbold, T. Frenzel, A. Münchinger and M. Wegener. *The rise of (chiral) 3D mechanical metamaterials*. Materials, 12(21), 3527, (2019).
- [53] K. Sandfort, *The factorization method for inverse scattering from periodic inhomogeneous media*, PhD thesis, Karlsruher Institut für Technologie, (2010).
- [54] J. Sun and C. Zheng. *Reconstruction of obstacles embedded in waveguides*. 8th Int. Conf. on Scientific Computing and Applications (University of Nevada, Las Vegas, Nevada, 1–4 April 2012) vol 586 pp 341–50, (2013).

- 
- [55] J. Sylvester. Discreteness of transmission eigenvalues via upper triangular compact operators. *SIAM Journal on Mathematical Analysis*, 44(1), 341-354, (2012).
- [56] J. Yang, B. Zhang and R. Zhang. *A sampling method for the inverse transmission problem for periodic media*. *Inverse Problems*, 28(3), 035004, (2012).
- [57] H. Wang, Q. Chen, O. Zetterstrom and O. Quevedo-Teruel. *Three-dimensional broadband and isotropic double-mesh twin-wire media for meta-lenses*. *Applied Sciences*, 11(15), 7153, (2021).
- [58] B. Zhang and S. N. Chandler-Wilde. *Acoustic Scattering by an Inhomogeneous Layer on a Rigid Plate*. *SIAM Journal on Applied Mathematics*, 58(6), 1931-1950, (1998).