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# Finite-time control and estimation of some classes of PDEs

Salim Zekraoui

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**CENTRALE LILLE**

**THESE**

Présentée en vue  
d'obtenir le grade de

**DOCTEUR**

En

Spécialité : Automatique

Par

**Salim ZEKRAOUI**

**DOCTORAT DÉLIVRÉ PAR CENTRALE LILLE**

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**Titre de la thèse :**  
**Contrôle et estimation en temps fini de  
certaines classes d'EDP**

---

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Thèse préparée dans le Centre de Recherche en Informatique,  
Signal et Automatique de Lille (CRISStAL)  
École Doctorale MADIS 631





CENTRALE LILLE

THESIS

Presented in order  
to become a

**DOCTOR**

In

Speciality: Automatic control

By

**Salim ZEKRAOUI**

PHD AWARDED BY CENTRALE LILLE

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**Title of the thesis:**  
**Finite-time control and estimation of some  
classes of PDEs**

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Defended on 31 October 2023 in front of the examination board:

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# Contents

<b>Acknowledgements</b>	<b>i</b>
<b>Acronyms</b>	<b>xi</b>
<b>Notation</b>	<b>xiii</b>
<b>Résumé long</b>	<b>1</b>
<b>1 Introduction</b>	<b>5</b>
1.1 Context and Motivation . . . . .	5
1.2 Non-Asymptotic concepts . . . . .	8
1.3 Problems addressed in the thesis, structure and main contributions . . . . .	10
1.4 Publications . . . . .	13
<b>2 Preliminaries of Non-Asymptotic concepts and control design in finite and infinite-dimensional settings</b>	<b>15</b>
2.1 Introduction . . . . .	15
2.2 Mathematical background and main tools . . . . .	18
2.3 Non-Asymptotic concepts in an infinite-dimensional setting . . . . .	35
2.4 Positioning of our contributions with respect to results for systems which are of the same class as those addressed in this thesis . . . . .	43
<b>I Non-smooth tools for finite/fixed-time stabilization and estimation in infinite-dimensional settings</b>	<b>45</b>
<b>3 Finite/fixed-time estimation of LTI systems with pointwise input delay</b>	<b>47</b>
3.1 Introduction . . . . .	47
3.2 Problem Statement . . . . .	48
3.3 Stability analysis . . . . .	53
3.4 Simulation . . . . .	67

<b>4</b>	<b>Finite/fixed-time stabilization of LTI systems with pointwise input delay</b>	<b>79</b>
4.1	Introduction . . . . .	79
4.2	Problem statement . . . . .	80
4.3	Finite/fixed-time predictor-based controller via PDE-based backstepping approach	81
4.4	Some characterizations of $\mathcal{F}$ for the design of finite/fixed-time predictor-based controllers . . . . .	89
4.5	Simulations . . . . .	91
<b>5</b>	<b>Finite/fixed-time boundary stabilization of Parabolic PDEs</b>	<b>95</b>
5.1	Introduction . . . . .	95
5.2	Problem statement: . . . . .	96
5.3	Stability analysis . . . . .	96
5.4	Application to finite/fixed-time stabilization . . . . .	98
5.5	Simulations . . . . .	99
5.6	Comments on possible extensions of our approach . . . . .	99
5.7	Limitation of our approach . . . . .	102
<b>II</b>	<b>Time-varying feedbacks for prescribed-time stabilization in infinite-dimensional settings</b>	<b>105</b>
<b>6</b>	<b>Prescribed-time stabilization of LTI systems with distributed input delay</b>	<b>107</b>
6.1	Introduction . . . . .	107
6.2	Problem statement . . . . .	108
6.3	Prescribed-time predictor-based control: an ODE-PDE setting and Reduction-based and backstepping-forwarding transformations . . . . .	108
6.4	Simulations . . . . .	125
<b>7</b>	<b>Prescribed-time boundary output-feedback stabilization of a class of reaction-diffusion with input delay</b>	<b>127</b>
7.1	Introduction . . . . .	128
7.2	Problem statement . . . . .	129
7.3	Prescribed-time boundary stabilization by full-state feedback in the delay-free case	130

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7.4	Prescribed-time stabilization by full-state feedback for the input delay case . . . . .	131
7.5	Prescribed-time output boundary feedback stabilization in the delay-free case . . . . .	139
7.6	Prescribed-time stabilization by output feedback for the input delay case . . . . .	141
7.7	Numerical results . . . . .	148
<b>8</b>	<b>Conclusion and perspectives</b>	<b>153</b>
<b>A</b>	<b>Appendix</b>	<b>157</b>
A.1	Standard homogeneity . . . . .	157
A.2	Weighted Homogeneity . . . . .	158
A.3	Linear homogeneity . . . . .	160
	<b>Bibliography</b>	<b>176</b>



# List of Figures

1.1	Toricelli's law . . . . .	9
2.1	The time evolution of the state of a scalar <b>FTS</b> ordinary differential equation . . . . .	20
2.2	The time evolution of the state of a scalar <b>FxTS</b> ordinary differential equation . . . . .	21
2.3	The time evolution of the state of a scalar <b>PdTS</b> ordinary differential equation . . . . .	21
2.4	The time evolution of the state of a scalar <b>PTS</b> ordinary differential equation . . . . .	22
3.1	The evolution of the states of (3.4)-(3.7) alongside the corresponding estimated states. . . . .	67
3.2	The evolution of the error states of the <b>FxTS</b> ODE-PDE system (3.14)-(3.16). . . . .	68
3.3	The evolution of the norm of the <b>FxTS</b> error system (3.14)-(3.16) alongside the evolution of the output and the estimated output. . . . .	68
3.4	A diagram of a two-links manipulator . . . . .	69
3.5	The evolution of the states of (3.194)-(3.197) alongside the estimated states without noise measurement. . . . .	72
3.6	The evolution of the error states of the <b>FTS</b> ODE-PDE subsystem (3.201)-(3.204) without noise measurement. . . . .	73
3.7	The evolution of the error states of the <b>FTS</b> ODE-PDE subsystem (3.205)-(3.208) without noise measurement. . . . .	73
3.8	The evolution of the norm of the two error subsystems (3.201)-(3.204) and (3.205)-(3.208) without noise measurement alongside the evolution of the outputs and the estimated outputs. . . . .	74
3.9	The evolution of the states of (3.194)-(3.197) alongside the estimated states with the presence of noise measurement. . . . .	75
3.10	The evolution of the error states of the <b>FTS</b> ODE-PDE subsystem (3.201)-(3.204) with the presence of noise measurement. . . . .	76
3.11	The evolution of the error states of the <b>FTS</b> ODE-PDE subsystem (3.205)-(3.208) with the presence of noise measurement. . . . .	76
3.12	The evolution of the norm of the two error subsystems (3.201)-(3.204) and (3.205)-(3.208) with the presence of noise measurement alongside the evolution of the outputs and the estimated outputs. . . . .	77
4.1	The evolution of the states of the <b>FTS</b> closed-loop system (4.93)-(4.96) with (4.97) and (4.75). . . . .	92



4.2	The evolution of the norm of the <b>FTS</b> closed-loop system (4.87)-(4.88) alongside the <b>FxTS</b> closed-loop system (4.87)-(4.88) for 3 different initial conditions. . . . .	93
4.3	The evolution of the norm of the <b>FTS</b> closed-loop system (4.87)-(4.88) for 3 different delays alongside the evolution of the a <b>FTS</b> predictor-based control (4.97) with (4.75) for 3 different initial conditions. . . . .	93
5.1	The evolution of the state and the norm of the state the <b>FTS</b> closed-loop reaction-diffusion system (5.1)-(5.4) alongside, the evolution of the applied <b>FTS</b> boundary control (5.20) with $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))^\alpha$ . . . . .	100
5.2	The evolution of the state and the norm of the state of the <b>FxTS</b> closed-loop reaction-diffusion system (5.1)-(5.4) alongside, the evolution of the applied <b>FxTS</b> boundary control (5.20) with $\mathcal{K}(V(z(t, \cdot))) = c_1V(z(t, \cdot))^\alpha + c_2V(z(t, \cdot))^\beta$ . . . . .	101
6.1	The evolution of the state of the scalar <b>PTS</b> closed-loop system (6.3) alongside its norm for 3 different initial conditions. . . . .	126
7.1	The evolution of the states of the <b>PTS</b> closed-loop reaction-diffusion systems (7.20)-(7.22) and (7.132)-(7.134) alongside the evolution of the <b>FxTS</b> transport equation (7.53)-(7.54). . . . .	149
7.2	The evolution of the norm of the <b>PTS</b> closed-loop-reaction diffusion system (7.20)-(7.24) for 3 different delays. . . . .	150
7.3	A comparison between the evolution of the norm of the <b>PTS</b> closed-loop-reaction diffusion system (7.20)-(7.24) with (7.161) and the classical exponentially stable closed-loop system (7.20)-(7.24) with (7.164), alongside the evolution of the applied <b>PTS</b> predictor-based control (7.161). . . . .	151

# List of Tables

2.1	Positioning of our contributions with respect to existing Non-Asymptotic stabilization/estimation results and open problems for some classes of differential equations	43
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# Acronyms

- a.e. Almost Everywhere
- FT Finite-time
- FxT Fixed-time
- PT Prescribed-time
- AS Asymptotically Stable/Stability
- ES Exponentially Stable/Stability
- FTS Finite-time Stable/Stability
- FxTS Fixed-time Stable/Stability
- PTS Prescribed-time Stable/stability
- DEs Differential Equations
- ODEs Ordinary Differential Equations
- PDEs Partial Differential Equations
- DDEs Delay Differential Equations
- PIDEs Partial Integro-Differential Equations
- TDS time delay systems
- CLF Control Lyapunov function
- ILF Implicit Lyapunov function



# Notation

Some notations and preliminary definitions used throughout the thesis are as follows:

- $\mathbb{R}$  denotes the set of real numbers;
- $\mathbb{R}_+$  denotes the set of non-negative real numbers;
- $\mathbb{N}$  denotes the set of natural numbers including zero (non-negative integers);
- $\mathbb{N}^*$  denotes the set of natural numbers excluding zero (positive integers);
- $\mathbb{Z}$  denotes the set of all integers including zero;
- $\mathbb{R}^n$  denotes the real coordinate space of dimension  $n \in \mathbb{N}^*$ , that is the set of the  $n$ -tuples of real numbers, with the canonical basis  $\{e_1, \dots, e_n\}$  where  $e_i = [\delta_{1,i}, \delta_{2,i}, \dots, \delta_{n,i}]^\top$ , with  $\delta_{ij}$  is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases}$$

- $\mathbb{R}_+^n$  denotes the non-negative real coordinate space of dimension  $n \in \mathbb{N}^*$ , that is the set of the  $n$ -tuples of non-negative real numbers;
- $|\cdot|$  denotes the absolute value of a real number;
- $\|\cdot\|_2$  denotes the Euclidean vector norm and is defined for any  $z \in \mathbb{R}^n$  as

$$\|z\|_2 = \sqrt{\sum_{i=1}^n |z_i|^2};$$

- $\text{sign}(\cdot)$  denotes the sign of a real number;
- For all  $a \in \mathbb{R}_+$  and all  $x \in \mathbb{R}$  we define the signed power  $a$  of  $x$  by  $\{x\}^a = \text{sign}(x)|x|^a$ ;
- We denote by  $\mathbb{1}_{\{x>0\}} : \mathbb{R} \rightarrow \{0, 1\}$  the function defined by  $\mathbb{1}_{\{x>0\}}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $\mathcal{C}([a, b])$  the set of all continuous functions defined on the closed interval  $[a, b]$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $\mathcal{C}^n([a, b])$ ,  $n \in \mathbb{N}^*$ , the set of functions that have continuous derivatives up to the  $n$ -th order on the closed interval  $[a, b]$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $\mathcal{M}_{n,m}(\mathbb{R})$ ,  $n, m \in \mathbb{N}^*$ , the set of all  $m$ -by- $n$  real matrices with the norm  $\|M\|_2 = \sup \{\|Mz\|_2 : z \in \mathbb{R}^2 \text{ with } \|z\|_2 = 1\}$ ;
- $\|\cdot\|_F$  denotes the Frobenius norm and is defined for any  $M \in \mathcal{M}_{n,m}(\mathbb{R})$  as  $\|M\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m |M_{i,j}|^2}$ ;

- For all  $m \in \mathbb{N}^*$  and  $n \in \mathbb{N}^*$ , we denote by  $0_{m \times n}$  is the  $(m, n)$ -zero matrix which is the matrix with all entries equal to zero;
- For all  $n \in \mathbb{N}^*$ , we denote by  $\bar{I}_n$  be the identity matrix of dimension  $n$ ;
- For all  $m \in \mathbb{Z}$ ,  $I_m(\cdot)$ ,  $J_m(\cdot)$ , denote the modified Bessel and (nonmodified) Bessel functions of the first kind;
- For all  $n \in \mathbb{N}^*$ , we denote by  $L_n^{(\alpha)}(\cdot)$  the generalized Laguerre polynomials;
- For all  $n \in \mathbb{N}^*$ , we denote by  $\bar{\sigma}_n(\cdot)$  the elementary symmetric polynomials;
- For all  $a$  and  $b$  in  $\mathbb{R}_+$ , we denote by  $\hat{\Gamma}(a) = \int_0^\infty t^{a-1} e^{-t} dt$  the Gamma Function;
- For all  $a$  and  $b$  in  $\mathbb{R}_+$ , we denote by  $I(a, b, r) = \frac{\int_0^r t^{a-1} (1-t)^{b-1} dt}{\int_0^1 t^{a-1} (1-t)^{b-1} dt}$  the regularized Incomplete Beta Function which is defined for all  $r \in [0, 1]$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $L^1(a, b)$  the set  $L^1(a, b) := \{f : [a, b] \rightarrow \mathbb{R} : \int_a^b |f(x)| dx < \infty\}$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $L_{\text{loc}}^1(a, b)$  the set  $L_{\text{loc}}^1(a, b) := \{f : [a, b] \rightarrow \mathbb{R} : \int_c^d |f(x)| dx < \infty, \forall (c, d) \in \mathbb{R}^2 : a \leq c \leq d \leq b\}$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $L^2(a, b)$  the set  $L^2(a, b) := \{f : [a, b] \rightarrow \mathbb{R} : \int_a^b |f(x)|^2 dx < \infty\}$  with the scalar product  $\langle f, g \rangle_{L^2(a, b)} := \int_a^b f(x)g(x) dx$ , and the norm  $\|f\|_{L^2(a, b)} := (\int_a^b f(x)^2 dx)^{\frac{1}{2}}$ ;
- For all  $a \in \mathbb{R}$  and all  $b \geq a$ , we denote by  $H^1(a, b)$  the set  $H^1(a, b) := \{f \in L^2(a, b); f' \in L^2(a, b)\}$ , with the scalar product  $\langle f, g \rangle_{H^1(a, b)} := \langle f, g \rangle_{L^2(a, b)} + \langle f', g' \rangle_{L^2(a, b)}$  and with the norm  $\|f\|_{H^1(a, b)} := (\|f\|_{L^2(a, b)}^2 + \|f'\|_{L^2(a, b)}^2)^{\frac{1}{2}}$ ;
- Let us also denote by  $f_t(t, x)$  (resp.  $f_x(t, x)$ ) the partial derivative of a function  $f$  with respect to the time (resp. space) variable  $t$  (resp.  $x$ );
- Let us also denote by  $f_{xx}(t, x)$  the second partial derivative of a function  $f$  with respect to the space variable  $x$ ;
- A function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be a class- $\mathcal{K}$  function if it is continuous, zero at zero, and strictly increasing;
- A function  $\gamma : \mathbb{R}_+ \rightarrow [0, 1)$  is said to be a class- $\mathcal{K}^1$  function if it is continuous, zero at zero, strictly increasing, and  $\lim_{r \rightarrow +\infty} \gamma(r) = 1$ ;
- A class- $\mathcal{K}$  function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be a class- $\mathcal{K}_\infty$  function if it is unbounded with its argument;
- A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class- $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}_\infty$  for each fixed  $t \in \mathbb{R}_+$ , and  $\beta(r, \cdot)$  is decreasing and  $\lim_{t \rightarrow +\infty} \beta(r, t) = 0$  for each fixed  $r \in \mathbb{R}_+$ ;
- A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be a *generalized class- $\mathcal{KL}$  function* ( $\mathcal{GKL}$ -function) if
  - i) the mapping  $r \mapsto \beta(r, 0)$  is a class- $\mathcal{K}$  function;
  - ii) for each fixed  $r \geq 0$  the mapping  $t \mapsto \beta(r, t)$  is continuous, decreases to zero and there exists some  $\tilde{T}(r) \in [0, +\infty)$  such that  $\beta(r, t) = 0$  for all  $t \geq \tilde{T}(r)$ ;

# Résumé long

Cette thèse est consacrée à l'étude des problèmes d'estimation d'états non-asymptotique et de stabilisation non-asymptotique (en temps fini, fixe et prescrit) pour certaines classes de systèmes dynamiques de dimension infinie. Principalement, nous étudions les équations différentielles ordinaires (EDO) linéaires, invariantes dans le temps, affectées par des retards d'entrée ou de sortie (ponctuels ou distribués), ainsi que les équations aux dérivées partielles (EDP) à une dimension de type réaction-diffusion avec et sans retards d'entrée. Pour résoudre les problèmes d'estimation et de stabilisation, nous étendrons les outils et résultats approfondis développés dans le contexte des systèmes de dimension finie aux cas des systèmes de dimension infinie. Plus précisément, nous nous focaliserons sur deux types de résultats : les résultats d'*homogénéité* qui produisent des commandes par retour d'état non linéaires (non régulières), et sur les résultats des commandes par retour d'état qui utilisent dans leurs formulations des fonctions gains qui divergent en temps fini et qui aident à la stabilisation en temps prescrit. Dans ce contexte, nous abordons tout d'abord le problème de la compensation, en temps fini et fixe, des retards d'entrée ou de sortie pour les systèmes linéaires invariantes dans le temps. En reformulant cette classe de systèmes en système cascade EDO-EDP, où la partie EDP correspond à une équation de transport modélisant l'effet du retard sur la sortie, nous utiliserons des techniques des systèmes de dimension infinie. En effet, nous exploiterons l'approche du *backstepping* maintenant dans un contexte non linéaire ou temps variant. Ensuite, nous aborderons le problème de la stabilisation frontière en temps fini ou fixe d'une classe d'EDP de type réaction-diffusion. À notre connaissance, les tentatives existantes de stabilisation frontière pour cette classe de systèmes utilisent des commandes qui dépendent de l'état du système, mais aussi d'une fonction du temps qui diverge en temps fini. Cependant, l'utilisation de commandes par retours d'état dépendants de l'état seul (éventuellement non linéaires ou non régulières) n'a pas été considérée jusqu'à présent (comme nous l'expliquerons au chapitre 2). Nous aborderons ce problème complexe à l'aide de méthodes classiques liées aux *Fonctions de Lyapunov de Contrôle (CLF)*. À la fin, nous soulignerons les avantages et les limitations de cette approche et donnerons quelques indications sur l'extension de cette méthode au problème de stabilisation entrée-état (ISS) et au problème du suivi en temps fini ou fixe pour les EDP de réaction-diffusion.

Ce manuscrit est divisé en deux grandes parties. La première partie concerne les outils non linéaires (non réguliers) qui permettent d'obtenir une stabilisation et une estimation en temps fini ou fixe pour des systèmes de dimension infinie. La deuxième partie est consacrée à l'exploitation d'outils basés sur l'utilisation de gains variant dans le temps et divergeant en temps prescrit dans des contextes de dimension infinie. Nous commencerons, au chapitre 2, par rappeler certains résultats notables sur la stabilisation non-asymptotique pour certaines classes de systèmes de dimension infinie. Nous introduirons ensuite les notions nécessaires et les principaux outils (y compris les différentes notions de stabilité et leurs caractérisations avec les fonctions de Lyapunov) qui sont essentiels pour élaborer nos contributions. Enfin, nous positionnerons nos contributions par rapport aux travaux les plus pertinents dans la littérature. Le reste de la thèse sera divisé comme suivant:

## Partie I

- C1) Dans le chapitre 3, nous présenterons notre première contribution qui vise à résoudre le problème d'estimation en temps fini ou fixe d'états d'une classe des systèmes linéaire invariants



dans le temps soumis à un retard de mesure:

$$\begin{cases} \dot{z}(t) = Az(t) + BU(t), & t \geq \bar{t}_0, \\ Y(t) = Cz(t - D), & t \geq \bar{t}_0, \end{cases} \quad (1)$$

où  $\bar{t}_0 \geq 0$  est le temps initial,  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$ ,  $n \in \mathbb{N}^*$ , est l'état du systèmes  $U(t) \in \mathbb{R}$  est la commande,  $Y(t) \in \mathbb{R}$  est la partie mesurée, et  $D > 0$  est un retard constant connu à l'avance.

Pour résoudre ce problème, nous reformulerons d'abord le système initial en un système cascade EDO-EDP, où la partie EDP correspond à une équation de transport modélisant l'effet du retard sur la sortie. Afin de synthétiser un observateur assurant une estimation en temps fini ou fixe, nous construirons les gains de l'observateur de manière non linéaire en fonction de l'erreur entre la sortie mesurée et la sortie estimée. L'idée est de choisir les gains pour garantir que le système d'erreur soit stable en temps fini ou fixe. Pour faire cela, nous utiliserons une transformation *backstepping* non linéaire pour convertir le système d'erreur en un système cible qui est stable en temps fini ou fixe grâce au outils d'homogénéité. Enfin, nous appliquerons la transformation *backstepping* inverse pour transférer la stabilité en temps fini ou fixe au système d'erreur.

- C2)** Dans le chapitre 4, nous présenterons notre deuxième contribution qui vise à résoudre le problème de stabilisation en temps fini ou fixe d'une classe des systèmes linéaire invariants dans le temps soumis à un retard d'entrée :

$$\dot{z}(t) = Az(t) + BU(t - D), \quad t \geq \bar{t}_0, \quad (3)$$

où  $\bar{t}_0 \geq 0$  est le temps initial,  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$ ,  $n \in \mathbb{N}^*$ , est l'état du systèmes  $U(t) \in \mathbb{R}$  est la commande, et  $D > 0$  est un retard constant connu à l'avance. Ici, la matrice du système et le vecteur d'entrée sont données respectivement par  $A := \{A_{i,j}\} \in \mathcal{M}_n(\mathbb{R})$ , où pour  $i, j \in \{1, \dots, n\}$ ,  $A_{i,j} = 1$  si  $j = i + 1$  et  $A_{i,j} = 0$  sinon,  $B = e_n := [0 \ \dots \ 0 \ 1]^\top$ .

L'idée de cette contribution est similaire à notre première contribution. Pour être précis, nous commencerons par reformuler la chaîne d'intégrateurs avec un retard en un système cascade ODE-PDE (une cascade d'une EDP de transport linéaire modélisant, l'effet du retard sur l'entrée, avec la chaîne d'intégrateurs). Cependant, dans cette situation, plusieurs défis supplémentaires s'ajoutent, car nous utiliserons une transformation *backstepping non linéaire* pour convertir le système cascade en un système cible stable en temps fini ou fixe. Nous effectuerons l'analyse de stabilité sur le système cible en utilisant des outils classiques non asymptotiques tels que l'homogénéité et les fonctions  $\mathcal{GKL}$ . Enfin, nous utiliserons la transformation inverse pour transférer la propriété de stabilité au système en boucle fermée.

- C3)** Dans le chapitre 5, nous présenterons notre troisième contribution qui vise à résoudre le problème de stabilisation frontière en temps fini ou fixe d'une classe des EDP linéaire de réaction-diffusion :

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \geq \bar{t}_0, x \in [0, 1], \\ z(t, 0) = 0, & t \geq \bar{t}_0, \\ z(t, 1) = U(t), & t \geq \bar{t}_0, \end{cases} \quad (4)$$

où  $\bar{t}_0 \geq 0$  est le temps initial,  $\lambda \in \mathbb{R}$  est le terme de réaction,  $z(t, x) \in \mathbb{R}$  représente l'état du système,  $U(t) \in \mathbb{R}$  est la commande.

Pour résoudre ce problème, nous utiliserons une approche basée sur les *Fonctions de Lyapunov de Contrôle (CLF)*. L'idée de l'approche consiste à utiliser la "norme  $L^2$  avec poids" en

tant que fonction de Lyapunov pour construire une commande non linéaire garantissant que le système en boucle fermée est stable non-asymptotiquement. À la fin, nous soulignerons les avantages et les limitations de cette approche et donnerons quelques indications sur l'extension de cette méthode au problème de stabilisation entrée-état (ISS) et au problème du suivi en temps fini ou fixe pour les EDP de réaction-diffusion.

## Partie II

- C4)** Dans le chapitre 6, nous présenterons notre quatrième contribution visant à résoudre le problème de stabilisation en temps prescrit d'une classe des systèmes linéaires invariants dans le temps et soumis à un retard distribué d'entrée :

$$\dot{z}(t) = Az(t) + \int_0^D B(D - \sigma)U(t - \sigma)d\sigma, \quad t \in [\bar{t}_0, \bar{t}_0 + T), \quad (7)$$

où  $\bar{t}_0 \geq 0$  est le temps initial,  $T > 0$  est un temps prescrit à l'avance,  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$  représente l'état du système,  $U(t) \in \mathbb{R}$  est la commande,  $D > 0$  est un retard constant connu à l'avance,  $A$  et  $B$  sont respectivement la matrice système et le vecteur d'entrée de dimensions appropriées. Le vecteur d'entrée  $B(\cdot)$  est une fonction vectorielle continue à valeurs réelles définie sur  $[0, D]$ .

L'idée centrale de cette contribution est basée sur la technique du *backstepping* avec des gains variant dans le temps. Pour mettre en œuvre cette approche, nous commencerons par modéliser l'effet de retard distribué par une EDP de transport, puis nous reformulerons le problème original sous la forme d'un système en cascade ODE-EDP. En parallèle avec la transformation *backstepping*, nous introduirons des transformations de réduction des systèmes de dimension finie. Cela nous permettra de convertir le système en cascade en un système cible composé d'une équation de transport qui s'annule en un temps fixe (égal au retard  $D$ ) et d'une EDO linéaire avec une matrice variant dans le temps. Cette matrice est basée sur les développements récents concernant les matrices de Vandermonde et les polynômes généralisés de Laguerre. Ensuite, nous démontrerons que la propriété de stabilité en temps prescrit est transférée au système cascade en utilisant les transformations inverses.

- C5)** Finalement, dans le chapitre 7, nous présenterons notre cinquième contribution qui vise à résoudre le problème de stabilisation frontière d'une classe des systèmes de réaction-diffusion à une dimension (1D) soumis à un retard d'entrée :

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \in [\bar{t}_0, \bar{t}_0 + T), x \in [0, 1], & (8) \\ z(t, 0) = 0, & t \in [\bar{t}_0, \bar{t}_0 + T), & (9) \\ z(t, 1) = U(t - D), & t \in [\bar{t}_0, \bar{t}_0 + T), & (10) \\ Y(t) = z_x(t, 1), & t \in [\bar{t}_0, \bar{t}_0 + T), & (11) \end{cases}$$

où  $\bar{t}_0 \geq 0$  est le temps initial,  $T > 0$  est un temps prescrit à l'avance,  $\lambda \in \mathbb{R}$  est le terme de réaction,  $z(t, x) \in \mathbb{R}$  représente l'état du système,  $U(t) \in \mathbb{R}$  est la commande (avec  $U(\bar{t}_0 + s) = 0$  pour tout  $s \in [-D, 0]$ ),  $Y(t) \in \mathbb{R}$  est la partie mesurée, et  $D > 0$  est un retard constant connu à l'avance.

Pour résoudre ce problème, nous commencerons par reformuler le système sous la forme d'un système en cascade EDP-EDP, consistant en une cascade d'une EDP de transport linéaire avec une EDP de réaction-diffusion linéaire. Ensuite, nous appliquerons une transformation *backstepping* linéaire dépendant des gains variant dans le temps et des prédicteurs (généralisés pour le cas de dimension infinie) pour convertir le système en cascade en un système cible stable en temps prescrit. L'analyse de stabilité sera effectuée sur le système cible, et

la propriété de stabilité souhaitée sera transférée au système en boucle fermée en utilisant la transformation inverse. Une attention particulière sera accordée à l'analyse de la croissance dans le temps des gains par rapport à la décroissance dans le temps des solutions, afin d'assurer la bornitude et la convergence du contrôle construit, tout en garantissant la convergence des solutions en boucle fermée.

# Introduction

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## 1.1 Context and Motivation

Science has always been driven by the desire to not only understand and describe complex phenomena through mathematical models but also to influence and control them. In engineering applications, the need for suitable mathematical models becomes more critical. Indeed, by expressing the system dynamics through mathematical equations, one can analyze and understand how inputs, outputs, and states interact, allowing for a deep understanding of how a system behaves and how it responds to different inputs. This understanding can be later used for designing control strategies that effectively manipulate the system to achieve desired outcomes. Control theory provides then suitable tools for the modeling and control of complex engineering systems. To better understand this, let us take the vehicle dynamics system as an example. In this context, we aim to move a vehicle (a car, for instance) from one location to another smoothly and safely. To control the car's movements, we use the gas pedal, the brake, and the steering wheel. From an engineering point of view, these variables serve as control inputs. As the goal is to move the car, the variables of interest -or the states- are the position of the car, its velocity, the orientation of the car, and its associated velocity. Additionally, the vehicle's position can be measured using, for example, a GPS. This measurement serves as the output. In this particular application, without a mathematical model, it would be sufficiently easy to control the movement of the car using the inputs. However, having a mathematical model provides a precise description of the relationship between the different variables of the system which allows us to anticipate and examine the behavior of the vehicle under different scenarios without risks and facilitate the design and optimization of control strategies to reach the desired outcome. To further elaborate on the role of mathematical models, let us consider another example, this time, related to option pricing in financial markets where an option is a financial contract that gives the holder the right to buy or sell an underlying asset at a predetermined price. In this complex scenario, mathematical models (e.g. the Black-Scholes model [1], [2]) aim to determine the fair value of options and effectively manage their associated risks. Note that, we can draw parallels between controlling a vehicle's movements and pricing options. Instead of a gas pedal, brake, and steering wheel, the control inputs in the model consist of variables such as the underlying asset's price (the current market value of the asset), the option's strike price (the price at which you can buy or sell the underlying asset if you decide to exercise the option), the risk-free interest rate (the theoretical rate of return of an investment with zero risk), and the option's time to expiration. These inputs guide the valuation and pricing strategies for options and lead the options' prices to converge to specific reference values. Similar to the position, velocity, angle, and angular velocity of a vehicle serving as the states of the vehicle model, the states in this economical example are the option's price and other relevant market variables. These states capture the dynamics of the financial market and provide insights into the option's value. Just as a GPS can measure the position of a vehicle, market data in the financial model, such as the current prices of the underlying asset and the option, serve as the output. These measurements help analysts monitor the market conditions and adjust their pricing strategies, accordingly.

Differential equations are examples of mathematical models that can be used to express the relation between the different physical components of systems. Differential Equations can be classified according to their properties. For instance, they can be Ordinary Differential Equations (in short ODEs), Partial Differential Equations (in short PDEs), or Time-Delay systems (in short TDS).

On the one hand, ODEs are equations containing derivatives of different orders of one or multiple required unknown functions with respect to one independent variable (generally thought of as time). These functions are usually referred to as the states of the system, and they evolve on a finite-dimensional space ( $\mathbb{R}^n$  for instance) which means that they can be described by a finite number of degrees of freedom. Due to this fact, ODEs are generally referred to as *finite-dimensional systems*. Sometimes, ODEs contain some additional functions that can be freely adjusted. In control theory, these functions are referred to as inputs and they serve as a tool to adjust the systems to achieve desired outcomes or what we call outputs (e.g. stabilization of the process, attenuation/rejection of uncertainties, optimization of a performance criterion, tracking a reference trajectory ...). When ODEs involve a delay term in their formulation, we talk about TDS. This class of systems is ubiquitous in physics, biology [3], epidemiological processes [4], [5] and engineering (robotics [6], networked control systems [7], [8], intelligent transportation systems [9], [10],...) among many other disciplines [11], [12]. In most of these applications, particularly those related to networks, time delays occur during the transmission of information, energy, or/and material through the network which may lead to performance degradation or even instability of the system [13]–[15].

On the other hand, PDEs can model, in a much more accurate manner, vast real applications and phenomena. In particular, they are useful in modeling transport phenomena such as the propagation of waves through a medium [16] or physical networks of different nature: hydraulic [17], [18], road traffic [19], gas pipeline [20], electrical lines [21], data/communication [22] networks. Besides, PDEs can model electrochemical systems including fuel cells or batteries, chemical or biochemical tubular bioreactors [23], thermal systems [24], flexible structures in aerospace applications [25]), fluid dynamics ([26], fusion reactors including tokamak plasma [27], large-scale networks of Multi-Agent Systems (MAS) [28], epidemiological applications [29]–[31], and more general diffusion processes (e.g. diffusion of chemicals in a fluid [32], the dispersal of pollutants in the atmosphere [33], the diffusion of heat in a solid material, the spread of infectious diseases within a population [34], [35], ...) among many others. Overall, the most relevant classes of PDEs modeling those applications are Hyperbolic and Parabolic PDEs—in one or higher dimension of space. Both TDS and PDEs are of *infinite-dimensional* nature due to their solutions evolving on an infinite-dimensional space<sup>1</sup> or in other terms having an infinite number of degrees of freedom.

The dynamic operation of all aforementioned classes of systems essentially relies on incorporating suitable *control and estimation* strategies to influence the system dynamics while leading the system to behave as desired. Regarding the estimation design, it is known that the knowledge of the whole state of the infinite-dimensional system is in general not available nor realistic (e.g., no access to the whole state or putting sensors everywhere is expensive). We can access information through the boundaries or very point-wise locations in the domain. Therefore, there is a need to reconstruct or estimate the state from a few available measurements. For that, it is common to use observers. As for the control design of PDEs, in particular, two main ways of acting on these systems can be highlighted: *boundary* and *in-domain control*. Although in several applications, in-domain control is hard to achieve due to many reasons: first, it is hard to, physically, access the interior of the system to control it; secondly, for in-domain control often requires significant

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<sup>1</sup>Consider for instance the space  $L^2(0, 1) := \{f : [0, 1] \rightarrow \mathbb{R} : \int_0^1 |f(x)|^2 dx < \infty\}$  which is an infinite-dimensional space because it contains infinitely many elements that are pairwise linearly independent; which means that we can not find a finite basis that can generate the entire space.

financial and maybe logistical resources to be effectively applied. On the other hand, acting on the boundaries of the systems is usually much easier and feasible. Moreover, as boundary control targets only specific parts of the domain, it requires fewer resources compared to in-domain control. Due to these factors, boundary controls are often a good physical choice.

For boundary control, the most used and powerful methods are the *Lyapunov techniques* ([17]) and *the backstepping* control design ([36]). The latter consists in transforming the unstable PDE, by using an invertible Volterra and/or Fredholm type transformation, into another PDE system of the same type, called the target system, satisfying a desired stability property. Then, using the inverse transformation the desired stability property is transferred back to the original PDE system. The method has been used to deal with the boundary stabilization of broader classes of PDEs such that: Slender Timoshenko Beam equations [37], Navier-Stokes equations [38] in 2006, Schrodinger equations [39], [40] in 2007, Burgers equations [41], Euler-Bernoulli beam equations [42] and hyperbolic wave equations in [43] in 2008, ... Moreover, this approach has been exploited to deal with the problem of delay compensation for LTI systems with input delay in [44], where the key idea was to notice that the input delay can be expressed using a transport equation for which it is possible to apply the backstepping approach and the resulting control is the classical predictor feedback obtained by the Artstein's reduction approach [45]. This last result was later generalized for LTI systems with time-varying input delay in [46], for nonlinear ODEs with arbitrary long constant input delays in [47], for state-dependent delay in [48], for time-varying input delay saturation in [49], [50] and delay-adaptive control [51]–[53].

This method has been widely extended to other parabolic systems (see [54]–[60] and the references therein). In [56], for instance, exponential stabilization of a class of coupled reaction-diffusion PDEs with different input and output delays was solved using an observer-based boundary feedback law based on an invertible Fredholm backstepping transformation. Note that an alternative method for stabilization of parabolic PDEs with input delay is the *modal decomposition* method (see [61], [62]). The method consists of splitting the unstable PDE into a stable infinite-dimensional part and a finite-dimensional unstable part. Then, using classical predictor approaches, the unstable finite-dimensional system is stabilized. Recently, this approach was applied in [63], [64] to construct a new finite-dimensional observer design for a class of parabolic PDEs, which in turn was used in [63], [64] to construct an observed-based control that stabilizes the PDE system.

For most of the mentioned mathematical models, the control and estimation designs usually achieve convergence in an infinite amount of time (asymptotically or exponentially). However, in many applications, the need to meet time constraints and increase temporal performance is crucial. To clarify this, let us take as an example the situation of the spatial spreading of an epidemic in a population. As the disease evolves in time and space, this process can be modeled by, e.g., reaction-diffusion PDEs [34], [65]. In this situation, the control input includes all the measures that help in controlling the spreading of the disease in the population (through control actions such as lock-downs, travel bans, and mass testing and vaccination projects...). Using asymptotic controllers (i.e., those whose action can only be appreciated over a long period) may effectively reduce the spread of the epidemic to a manageable level or eliminate it, but this may lead to prolonging the impact of the epidemic on the population and the economy. How can we deal with this situation? Can we reduce the time it takes to control the spread of the epidemic? Moreover, can we make this time to be finite?

All these questions are hard to answer, even in other general situations (or applications other than epidemics). If one considers finite-time (Non-Asymptotic) mitigation measures, it would be possible to reduce the impact of the epidemic on the population and the economy and avoid situations where the epidemic escalates and evolves to become a severe threat to the population, such as becoming a highly contagious or deadly disease. Moreover, what about if delays are present in the epidemiological process or in the control actions (e.g., a lock-down policy whose positive

effects can only be perceived some weeks or months later)? Can we perfectly compensate for the delay within a finite amount of time?

All of this calls for advanced control and estimation methods that accelerate the convergence and improve the performance (e.g., transient processes to occur within a finite time) while also accounting for the effects of the perturbation and the delays when present. The controllers allowing to achieve this type of better performance will be referred to as *Non-Asymptotic* controllers (with some specificities to discuss later on). It is then worth recalling and further motivating Non-Asymptotic concepts for dynamical systems, particularly in the context of infinite-dimensional systems, as we discuss next (as well as in Chapter 2).

## 1.2 Non-Asymptotic concepts

Let us first recall that the stability notion is one of the most interesting mathematical notions for studying mathematical models including differential equations. In fact, this notion was introduced to help in understanding the behavior of systems over time, even when the exact solutions are not known explicitly. In engineering, this notion can be used as a tool to analyze mathematical models subject to control laws to make sure that the system achieves the desired outcome. Mathematically speaking, the stability of a system means that if the initial condition of a system is close to the equilibrium point, then the system's trajectories will tend to remain close to this equilibrium point. When such a condition holds, the system is called *stable*; otherwise, it is called *unstable*.

Depending on the time of convergence, the stability notion can be divided into two main categories: *Asymptotic* and *Non-Asymptotic*. Asymptotic stability refers to the situation where a system's trajectories approach the system's equilibrium when time tends to infinity. In other words, it means that after a sufficiently large amount of time, all trajectories will approach the equilibrium, but they will never reach it. When, in addition, the rate of convergence to the equilibrium is exponential, we talk about *exponential stability*. The asymptotic and exponential stability properties of dynamical systems have been widely studied in the control community (see [36], [66]–[69],...).

Non-Asymptotic stability, on the other hand, refers to the situation where a system is stable with trajectories approaching the equilibrium in a finite amount of time (referred to as the settling time or the terminal time). It is worth mentioning that in some dynamical systems, such as the one depicted in Figure 1.1 and governed by Torricelli's law, the Non-Asymptotic stability concept appears naturally. Torricelli's law is a theorem in fluid dynamics relating the speed of fluid flowing from an orifice  $v$  to the height of fluid above the opening  $h$ , assuming no air resistance, viscosity, or other hindrance to the fluid flow, (see Figure 1.1) by the formula  $v = \sqrt{2gh}$ . This theorem is an application of Bernoulli's principle. In this application, the rate of change of the water level height can be described by the following equation:

$$\frac{A}{a} \frac{dh}{dt} = -\sqrt{2gh}, \quad (1.1)$$

where  $A$  and  $a$  are respectively the cross sections of the container and the tube,  $g$  is the gravitational constant, and  $h$  is the water level height that can be computed, by solving the previous equation, to get  $h(t) = h(0) \left(1 - \frac{t}{T}\right)^2$ . By analyzing this last expression, it can be shown that the total time to empty the container (i.e.,  $h(T) = 0$ ) is finite and is given by  $T = \frac{A}{a} \sqrt{\frac{2}{g} h(0)} < \infty$ .

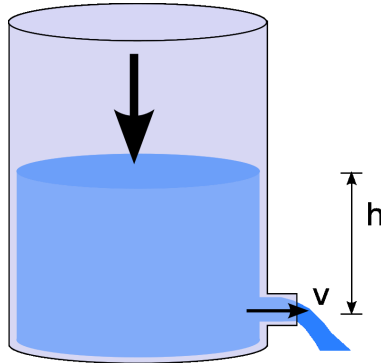


Figure 1.1: Torricelli's law (taken from Wikipedia)

### 1.2.1 Finite-time stability

When the settling time is a function of the initial condition, the stability is referred to as *finite-time stability* (**FTS**). Historically, finite-time stability was early studied by Zubov [70] in 1958, then by Roxin [71] in 1966, and by Haimo [72] in 1986. It was not until the end of the 90s for this theory developed with Bhat and Bernstein in [73], [74], where they showed in [74] that under some homogeneity property, the asymptotic stability is equivalent to the finite-time stability. Later on, in [75], [76], they proved the first converse theorem adding a new page to this theory. In 2003, Moulay and Perruquetti took up this work and extended the results to different system classes including non-autonomous systems in [77]. This last result was extended in 2019 by Zimenko et al. to output finite-time stability in [78], [79].

It is worth mentioning the works of Amato in [80], Lazarevic in [81], Weiss and Infante in [82], Dorato in [83], Amato et al. [84], which considered the notion of practical finite-time stability.

### 1.2.2 Fixed-time stability

The notion of **FTS** can be upgraded to define what we call *fixed-time stability* (**FxTS**), which refers to the case when the settling-time function of an **FTS** system is bounded by a finite constant value that does not depend on the initial conditions. This property implies that all the system's trajectories will converge exactly to zero before a specified time regardless of the system's initial conditions. To the best of our knowledge, **FxTS** was introduced, for the first time, in [85, Corollary 2.24] using the notion of homogeneity in *bi-limit*. However, this result did not allow for adjusting or even estimating the settling time of the closed-loop system. It was not until 2012, in the work of Polyakov [86], that **FxTS** was formally defined and a new Lyapunov-based stability result, that provides fixed-time stability and overcomes the shortcomings of [85], was given using what he called a special modification of "nested" (terminal) second-order sliding mode control algorithm [87]. In 2015, the **FTS** and **FxTS** Lyapunov stability results were extended by Polyakov et al. in [88] using Implicit Lyapunov Functions (ILF) which are, as their name indicates, Lyapunov functions defined, implicitly, as solutions to some algebraic equation. In Lu et al. [89] in 2016, new results were obtained concerning sufficient conditions for **FxTS**. In 2018, Lopez-Ramirez et al. provided in [90], [91] some necessary and sufficient conditions for **FxTS** of continuous autonomous systems that take into account the regularity of the settling-time function. Recently, some sufficient optimality conditions for **FxTS** in terms of the optimal control theory were provided by Michalak and Nowakowski in [92].



### 1.2.3 Predefined-time stability

A more recent stability concept called predefined-time stability (**PdTS**) was introduced in 2014 by Sánchez-Torres et al. [93], [94] to overcome the shortcomings and the difficulties of **FxTS** concept in tuning the control parameters to ensure a convergence within a finite time upper-bounded by a desired predefined constant. These difficulties are mainly due to the unclear relationship between the control parameters and the upper bound of the settling time. In order to overcome these difficulties, the **PdTS** concept introduces explicitly the upper bound for the settling time as an adjustable control parameter. In 2015, Sánchez-Torres et al. [94], [95] introduced a Lyapunov-based characterization of the **PdTS** concept. Another characterization was given by Aldana-Lopez et al. [96] in 2019 and a more generalized one was given later on by Jiménez-Rodríguez et al. [97].

### 1.2.4 Prescribed-time stability

Note that the **PdTS** concept can be also improved in a sense where the settling time itself can be chosen a priori independently of initial conditions and the system's parameters. This type of stability is called *prescribed-time stability* (**PTS**) and was introduced by Song et al. [98] in 2017 where a Lyapunov-based characterization of the **PTS** concept was given. The key idea behind this characterization is the use of time-varying tools, in particular, time-dependent gains that grow to infinity as time tends toward the prescribed settling time. The resulting controllers that allow to achieve this Non-Asymptotic property are time-varying feedbacks (or what we call prescribed-time controllers). Because of the blow-up characteristic of the time-varying gains, the **PTS** concept is limited to processes that take effect within a finite time interval (e.g. missile guidance [99]–[101]). For more details about this concept, we refer to the most recent survey on this topic: [102].

One of the most striking features of time-varying prescribed-time controllers that use growing gains is not only to achieve convergence in desired time but to reduce state peaking during stabilization and to also reduce the control effort by distributing it more evenly over the time interval of convergence.

It is essential to say that Non-Asymptotic concepts (that, in this thesis, we will mostly refer to as either finite-time, fixed-time, or prescribed-time stability) discussed above have been extensively studied for finite-dimensional systems. The question that arises now is, what about infinite-dimensional systems? Although there are few results in the literature (that we further detail in Chapter 2), too much research on this area is still needed. This motivates and shapes the core content of the thesis. However, as we will see, we limit the scope of the thesis to some Non-Asymptotic results for some classes of infinite-dimensional systems that we discuss in the sequel.

## 1.3 Problems addressed in the thesis, structure and main contributions

This thesis provides Non-Asymptotic estimation and stabilization results for some classes of infinite-dimensional systems, namely LTI systems subject to input/sensor delays and 1D reaction-diffusion PDEs. Inspired by the comprehensive tools and results in the context of finite-dimensional systems, we extend *state-dependent and time-varying feedbacks* to infinite-dimensional settings. We start with the problem of input and sensor delay compensation in finite/fixed time of LTI

systems within a PDE backstepping perspective, which reformulates the problem as stabilization of a cascade ODE-PDE system. Indeed, we exploit the backstepping approach (now in a nonlinear, non-smooth, and time-varying context), which brings additional challenges and issues. On the other hand, for reaction-diffusion PDEs, the question of boundary finite/fixed-time stabilization has remained open in the literature. As we will see (more particularly in Chapter 2), the seminal attempts for boundary stabilization in finite-time use time-varying tools, but using state-dependent feedbacks (possibly non-smooth) have not been considered so far in the literature to the best of our knowledge. We attempt not only to tackle this challenging problem using old ideas in relation to *Control Lyapunov Functions* but also to point out the advantages and limitations of our approach. Finally, we consider the problem of input delay compensation of reaction-diffusion systems in prescribed time by output feedback. This problem is challenging, as one deals with observer and control design with time-varying gains. This requires introducing novel infinite time-varying backstepping transformations in conjunction with advanced predictor-based concepts, now for parabolic PDEs. We believe that addressing this problem has paved the way to consider Non-Asymptotic stabilization for more complex dynamics like coupled parabolic-hyperbolic PDEs and other classes of PDE systems.

The thesis is divided into two parts. The first part concerns non-smooth tools, which allow for obtaining finite/fixed-time stabilization and estimation in infinite-dimensional settings. The second part of the thesis is devoted to exploiting time-varying tools in infinite-dimensional settings.

We start, in Chapter 2, by recalling some noteworthy results on finite/fixed/prescribed-time stabilization for some classes of infinite-dimensional systems. We introduce then the necessary background and main tools -including the stability concepts, some important Lyapunov-based stability results, and main techniques- that are instrumental to working out our contributions. Finally, we position our contributions with respect to the most relevant works in the literature.

Our main contributions are summarized in order:

## Part I

- C1)** In Chapter 3, we present our first contribution that deals with the problem of finite/fixed-time estimation of LTI systems in the observable form with measurement delay:

$$\begin{cases} \dot{z}(t) = Az(t) + BU(t), & t \geq \bar{t}_0, \\ Y(t) = Cz(t - D), & t \geq \bar{t}_0, \end{cases} \quad (1.2)$$

$$(1.3)$$

where the initial time  $\bar{t}_0 \geq 0$ , the state  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$ ,  $n \in \mathbb{N}^*$ , the control  $U(t) \in \mathbb{R}$ , the output  $Y(t) \in \mathbb{R}$ , the delay  $D > 0$  is a known constant.

To accomplish this, we reformulate the original system into a cascade ODE-PDE system where the PDE part is a transport equation that models the effect of the delay on the output. We construct suitable nonlinear (non-smooth) gains in a way that ensures the error system is finite-time stable. To prove this, we use an invertible backstepping transformation to convert the error system into a target system which is shown to be finite-time stable using Lyapunov-based analysis and homogeneity tools. We use the inverse transformation to transfer this property to the error system.

- C2)** In Chapter 4, we present our second contribution that deals with the problem of finite/fixed-time stabilization of a chain of integrators with pointwise input delay:

$$\dot{z}(t) = Az(t) + BU(t - D), \quad t \geq \bar{t}_0, \quad (1.4)$$

where the initial time  $\bar{t}_0 \geq 0$ , the state  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$ ,  $n \in \mathbb{N}^*$ , the control  $U(t) \in \mathbb{R}$ ,  $D > 0$  is a known constant delay, the system matrix and input vector of appropriate dimensions and are given respectively by  $A := \{A_{i,j}\} \in \mathcal{M}_n(\mathbb{R})$ , where, for  $i, j \in \{1, \dots, n\}$ ,  $A_{i,j} = 1$  if  $j = i + 1$  and  $A_{i,j} = 0$  otherwise,  $B = e_n := [0 \ \dots \ 0 \ 1]^\top$ .

The idea of this contribution is similar to our first contribution. To be precise, we first reformulate the chain of integrators with input delay as a cascade ODE-PDE system (i.e., a cascade of a linear transport PDE with the chain of integrators) where the transport equation models the effect of the delay on the input as we explained before. However, more challenges arise in this problem, as we use a nonlinear infinite-dimensional backstepping transformation to convert the cascade system to a suitable target system that is chosen to be finite-time or fixed-time stable. We perform the stability analysis on the target system by means of classical Non-Asymptotic concepts and tools such as linear homogeneity and "generalized  $\mathcal{KL}$ " functions. Then, we use the inverse transformation to transfer back the stability property to the closed-loop system. Finally, we give some characterizations of finite/fixed time predictor-based controllers followed by numerical simulations.

- C3)** In Chapter 5, we present our third contribution that deals with the problem of finite/fixed-time stabilization of a class of reaction-diffusion PDEs with boundary control

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \geq \bar{t}_0, x \in [0, 1], & (1.5) \\ z(t, 0) = 0, & t \geq \bar{t}_0, & (1.6) \\ z(t, 1) = U(t), & t \geq \bar{t}_0, & (1.7) \end{cases}$$

with the initial time  $\bar{t}_0 \geq 0$ , the reaction term  $\lambda \in \mathbb{R}$ , the state  $z(t, x) \in \mathbb{R}$ , and the control  $U(t) \in \mathbb{R}$ .

As aforementioned, we tackle this problem by using a Lyapunov-based approach. The idea of the approach is to use a "spatially weighted  $L^2$ -norm" as a Lyapunov functional to design a nonlinear controller that ensures that the closed-loop system is stable with any desired convergence including (finite/fixed-time convergences). As an application, we focus on the finite/fixed-time notions for which we give some particular explicit controllers. We provide some hints on how we can extend the approach to Input-to-state stability (ISS) results as well as the problem of tracking. We point out limitations to observer design.

## Part II

- C4)** In Chapter 6, we present our fourth contribution that deals with the problem of prescribed-time stabilization of a class of controllable linear systems with distributed input delay:

$$\dot{z}(t) = Az(t) + \int_0^D B(D - \sigma)U(t - \sigma)d\sigma, \quad t \in [\bar{t}_0, \bar{t}_0 + T], \quad (1.8)$$

where the initial time  $\bar{t}_0 \geq 0$ , the prescribed-time  $T > 0$ , the state  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$ ,  $n \in \mathbb{N}^*$ , the control  $U(t) \in \mathbb{R}$ ,  $D > 0$  is a known constant delay,  $A, B$  are respectively the system matrix and input vector of appropriate dimensions. The input vector  $B(\cdot)$  is a continuous real-valued vector function defined on  $[0, D]$ .

The idea of this contribution is based on the backstepping technique, we start by modeling the input delay as a transport PDE and reformulating the original problem as a cascade ODE-PDE system while accounting for the infinite dimensionality of the actuator. We introduce reduction-based and backstepping-forwarding transformations to convert (1.8) into a target system composed of a homogeneous transport equation that vanishes in a fixed time equal to

the delay  $D$  and a Linear Time-Varying ODE that relies on the recent developments which include a polynomial-based Vandermonde matrix and the generalized Laguerre polynomials. Then, we prove the bounded invertibility of the transformations and hence we show that the prescribed-time stability property is transferred to (1.8).

- C5) Finally, in Chapter 7, our fifth contribution deals with the problem of boundary stabilization of one-dimensional (1D) reaction-diffusion systems with boundary input delay of the form

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \in [\bar{t}_0, \bar{t}_0 + T], x \in [0, 1], & (1.9) \\ z(t, 0) = 0, & t \in [\bar{t}_0, \bar{t}_0 + T], & (1.10) \\ z(t, 1) = U(t - D), & t \in [\bar{t}_0, \bar{t}_0 + T], & (1.11) \\ Y(t) = z_x(t, 1), & t \in [\bar{t}_0, \bar{t}_0 + T], & (1.12) \end{cases}$$

with the initial time  $\bar{t}_0 \geq 0$ , the prescribed-time  $T > 0$ , the reaction term  $\lambda \in \mathbb{R}$ , the state  $z(t, x) \in \mathbb{R}$ , the control  $U(t) \in \mathbb{R}$ , with the following initial condition:  $U(\bar{t}_0 + s) = 0$  for all  $s \in [-D, 0]$ , the collocated output  $Y(t) \in \mathbb{R}$ , and  $D > 0$  is a known constant delay.

To deal with this problem, we first reformulate the system as a PDE-PDE cascade system (i.e., a cascade of a linear transport PDE with a linear reaction-diffusion PDE), where the transport equation represents the effect of the input delay. We then apply a time-varying infinite-dimensional backstepping transformation, in conjunction with predictor-based concepts to convert the cascade system into a prescribed-time stable (PTS) target system. The stability analysis is conducted on the target system, and the desired stability property is transferred back to the closed-loop system using the inverse transformation. We pay special attention to the analysis of the growth-in-time of the gains relative to the decreasing-in-time of solutions while ensuring convergence of the closed-loop solutions and boundedness of the resulting output feedback controllers.

## 1.4 Publications

The contributions presented in this thesis have given rise to the following publications:

- **Journal papers**

- [J4]. S. Zekraoui, N. Espitia, W. Perruquetti, and M. Krstic. **Output-feedback stabilization in prescribed time of a class of reaction-diffusion PDEs with boundary input delay**, under review in Transactions on Automatic Control (TAC), 2024.
- [J3]. S. Zekraoui, N. Espitia, and W. Perruquetti. **Lyapunov-based nonlinear boundary control design with predefined convergence for a class of 1D linear reaction-diffusion equations**, European Journal of Control (EJC), 2023.
- [J2]. S. Zekraoui, N. Espitia, and W. Perruquetti. **Finite/fixed-time stabilization of a chain of integrators with input delay via PDE-based nonlinear backstepping approach**, Automatica, Vol 155, 2023.
- [J1]. S. Zekraoui, N. Espitia, and W. Perruquetti. **Finite-time estimation of second-order linear time-invariant systems in the presence of delayed measurement**, International Journal of Robust and Nonlinear Control, 2023.

- **Conference papers**

- [C3]. S. Zekraoui, N. Espitia, W. Perruquetti, and M. Krstic. **Predictor-based prescribed-time output feedback for a parabolic PDE**, submitted to the 2024 American Control Conference (ACC24), 2024.
- [C2]. S. Zekraoui, N. Espitia, and W. Perruquetti. **Lyapunov-based nonlinear boundary control design with predefined convergence for a class of 1D linear reaction-diffusion equations**, 6th European Control Conference (ECC23), 2023.
- [C1]. S. Zekraoui, N. Espitia, and W. Perruquetti. **Prescribed-time predictor control of LTI systems with distributed input delay**, 60th IEEE Conference on Decision and Control (CDC21), 2021.

# Preliminaries of Non-Asymptotic concepts and control design in finite and infinite-dimensional settings

## Contents

<b>2.1</b>	<b>Introduction</b>	<b>15</b>
2.1.1	Non-Asymptotic concepts of TDS	16
2.1.2	Non-Asymptotic concepts for PDEs	16
<b>2.2</b>	<b>Mathematical background and main tools</b>	<b>18</b>
2.2.1	Lyapunov-based approach for ODEs	24
2.2.2	Non-smooth tools in finite-dimensional setting	31
2.2.3	Time-Varying tools in finite-dimensional setting	33
<b>2.3</b>	<b>Non-Asymptotic concepts in an infinite-dimensional setting</b>	<b>35</b>
2.3.1	Lyapunov-based approach for PDEs	36
2.3.2	Finite/prescribed-time boundary stabilization key results for 1D reaction-diffusion PDEs	40
<b>2.4</b>	<b>Positioning of our contributions with respect to results for systems which are of the same class as those addressed in this thesis</b>	<b>43</b>

In this chapter, we start by discussing some of the most relevant results on finite/fixed/prescribed-time stabilization for some classes of infinite-dimensional systems. We then provide the main background and tools that are instrumental in the thesis by recalling first the Non-Asymptotic stability concepts for finite-dimensional and infinite-dimensional systems (Section 2.2 and 2.3), all of this accompanied by several examples and remarks. We finally position our contributions (summarized in Chapter 1) with respect to the most relevant works in the literature (Section 2.4).

Notice that additional helpful results related to the homogeneity theory are left to Appendix A) for the sake of brevity.

## 2.1 Introduction

In line with the discussion given in the previous chapter, we recall that Non-Asymptotic concepts have been extensively studied within the framework of stabilization of linear and nonlinear ODEs. Numerous noteworthy contributions can be cited for finite/fixed/predefined/prescribed-time stability based on: the backstepping approach for ODEs (see e.g. [85], [103]–[108]), the desingularization technique [109], the control Lyapunov functions (CLF) (see e.g. [110], [111]),

the weighted homogeneity concept (see e.g. [74], [75], [86], [97], [112]–[122]), the generalized homogeneity concept (see e.g. [123], [124]), the implicit Lyapunov functions (ILF) (see e.g. [88], [125]–[128]), the time-varying tools (see e.g. [98], [102], [129]–[133]), a mix of homogeneity tools and time-varying tools (see e.g. [134]–[138]), or periodic delay feedbacks (see e.g. [139], [140]), ...

### 2.1.1 Non-Asymptotic concepts of TDS

In the framework of finite/fixed/prescribed-time stabilization of TDS, very few results can be found even for finite-dimensional systems with input delay. For finite-time stabilization, one may refer to some of the pioneering contributions starting with the work of Karafyllis [141] (2006) for triangular time-varying systems described by retarded functional differential equations and finishing with the work of Moulay et al. [142] (2008) for linear time-delay systems using the Artstein’s transformation and weighted homogeneity. For fixed-time stabilization, one can mention two key recent contributions [143] and [144] dealing with general LTI systems with input delay. The former builds upon the Artstein–Kwon–Pearson reduction transformation and uses weighted homogeneity in a similar fashion to [142]. The latter proposes a novel strategy based on act-and-wait predictor-based control and opens new research avenues on fixed-time control design.

Prescribed-time stabilization [98] and predictor feedback for compensation of input delay [69] are a perfect match because both techniques deal with finite-time dynamics. Intuitively, by applying predictor feedback to prescribed-time feedback (for either an ODE or PDE plant), the former feedback being time-varying and the latter infinite-dimensional, one should be able to obtain convergence in a time that is the sum of the prescribed time and the input delay. And yet, already six years after [98], one can still find very few results that combine prescribed-time feedback with predictors. In an exception from this slow progress, in [145], an ODE-PDE cascade representation of a class of LTI systems with input delay is provided and a time-varying backstepping-based approach is used to design a predictor feedback that ensures delay compensation in prescribed time. An extension of the methodology was carried out in [146] (to a class of 1D reaction-diffusion PDE with boundary input delay), and after that in [147] (to a class of first-order semi-linear hyperbolic PDE that is bidirectionally interconnected with nonlinear ODEs).

Finite/fixed/prescribed-time stabilization of time-delay systems still remains sparse and constitutes challenging topics.

### 2.1.2 Non-Asymptotic concepts for PDEs

#### 2.1.2.1 1D hyperbolic PDEs

The first result related to this topic for hyperbolic PDEs has been achieved, to the best of our knowledge, by Coron et al. based on the backstepping approach in [148] where they mainly proved exponential output-feedback stabilization, followed by the finite-time convergence for a class of  $2 \times 2$  quasi-linear hyperbolic systems with single boundary control. In 2016, a similar approach [149] was used to design output feedback laws for a system of  $n \times n$  coupled first-order hyperbolic linear PDEs achieving finite-time convergence to zero. Later on, this result was optimized in [150] in the sense of the target system was slightly modified to minimize the settling time to fit with the theoretical optimal finite-time of convergence (see e.g. [151]). In 2017, this last result was revised by Coron et al. in [152] and the result was re-proved in a simplified way. The idea of the approach was to use an invertible Volterra transformation of the second kind, followed by a Fredholm backstepping

transformation to convert the coupled system into a much simpler target system from which the finite-time stabilization with the theoretical optimal finite-time of convergence can be recovered. In 2018, an interesting finite-time stabilization result was provided by Anfinsen et al. in [153] for a class of 1D linear hyperbolic PDEs using some backstepping transformations that make use of time-dependent kernels. In 2021, an extension of all the previous results to time-dependent systems was presented by Coron et al. in [154] in the same fashion as in [153]. In addition to these previous references, some recent related noteworthy results can be also cited starting from [155] that solved the problem of finite-time stabilization of a  $2 \times 2$  hyperbolic systems (and in particular for the Saint-Venant equations) using some boundary control that satisfies a finite-time stable nonlinear ODE; [156] in which Coron and Nguyen managed to solve the problem null-controllability of a general linear hyperbolic system in one space dimension using boundary controls on one side; [157], [158] which study the problem of finite-time stabilization of a hyperbolic wave PDE coupled with a second order ODE; and [159], [160] that deals with finite-time stabilization in optimal time for homogeneous linear and quasi-linear hyperbolic systems.

### 2.1.2.2 1D parabolic PDEs

For 1D parabolic PDEs, the backstepping approach has continued to dominate in solving the problem of boundary stabilization in finite time. The first result related to this topic was introduced in 2017 by Coron and Nguyen [161] where they solved not only the problem of null controllability but also the problem of finite-time stabilization for a general class of 1D parabolic PDEs with variable coefficients in space, by making use of some periodic time-varying feedback laws. In 2017, Polyakov et al. [162] used also the backstepping method to solve the problem of finite-time stabilization of the heat equation using state-dependent switching boundary control. Later on, with the emergence of the prescribed-time stability concept in [98], and inspired by [161] and [162], Espitia et al were able to take the backstepping approach to another level while solving the problem of prescribed-time stabilization of a class of 1D reaction-diffusion in [163]–[165]. The key idea is to use of a time-varying backstepping transformations and choose target system with the desired prescribed-stability property. This last result was later generalized to the case of output feedback stabilization of the same class of systems in [166], then for a pair of coupled reaction–diffusion equations in [167]; and since then has opened new research avenues for prescribed-time control for PDEs. More technical details are given in Section 2.3.2.

It is worth mentioning that as an alternative to the backstepping approach, as soon as one deals with in-domain control, the notion of generalized homogeneity (introduced in [168]) as in [169], [170] and Lyapunov techniques have been instrumental for the design on controllers achieving finite-time stabilization for this class of PDE.

### 2.1.2.3 Other classes of PDEs

For other classes of PDEs, one can refer to some pioneering results, that are also based on the backstepping approach: [171] for local finite-time stabilization of the Viscous Burgers equations; [167], [172] for prescribed-time stabilization of the linearized Schrödinger equation; [173] for prescribed-time stabilization of LTI systems with diffusive actuator dynamics.



## 2.2 Mathematical background and main tools

Now that we have a better understanding of asymptotic and finite/fixed/predefined/prescribed-time stability concepts, let us proceed first to define mathematically each stability notion for ODEs.

Let us consider the domain  $\Omega \subset \mathbb{R}^n$  as an open connected set containing the origin, with  $n \in \mathbb{N}^*$ , and  $\partial\Omega$  its boundary. Consider the following ODE system:

$$\dot{z} = \mathcal{K}(z), \quad z \in \Omega, \quad (2.1)$$

where  $\mathcal{K} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function and  $\mathcal{K}(0) = 0$ . Let us assume that  $\mathcal{K}$  is such that (2.1) has the property of existence and uniqueness of solutions in forward time outside the origin.

### Definition 2.1

The origin of the system (2.1) is said to be:

- **stable** if and only if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for every initial time  $\bar{t}_0 \geq 0$  and any initial condition  $z_0 \in \Omega$ ,

$$(\|z_0\|_2 \leq \delta) \implies (\|z(t)\|_2 \leq \varepsilon),$$

for all  $t \geq \bar{t}_0$ ;

- **asymptotically stable (AS)** if and only if it is stable and

$$\lim_{t \rightarrow +\infty} \|z(t)\|_2 = 0,$$

for any  $z_0 \in \Omega$ ;

- **exponentially stable (ES)** if and only if it is AS and there exist  $M > 0$  and  $\lambda > 0$  such that

$$\|z(t)\|_2 \leq M \|z_0\|_2 e^{-\lambda t},$$

for all  $t \geq \bar{t}_0 \geq 0$  and any  $z_0 \in \Omega$ , ( $\lambda$  is referred to as the rate of the exponential convergence of the system);

- **finite-time stable (FTS)[73]** if and only if it is stable and for any  $z_0 \in \Omega$  there exists  $0 < T^{z_0} < +\infty$  such that  $\|z(t)\|_2 = 0$  for all  $t \geq \bar{t}_0 + T^{z_0}$ . The functional  $T(z_0) = \inf \{T^{z_0} \geq 0 : \|z(t)\|_2 = 0, \forall t \geq \bar{t}_0 + T^{z_0}\}$  defines the settling time of the system (2.1);

- **nearly fixed-time stable (nearly FxTS)** if and only if it is stable and for any  $\varrho > 0$  there exists  $0 < T_\varrho < +\infty$  such that  $\|z(t)\|_2 \leq \varrho$  for all  $t \geq \bar{t}_0 + T_\varrho$  and all  $z_0 \in \Omega$ ;

- **fixed-time stable (FxTS)[86]** if and only if it is FTS and  $\sup_{z_0 \in \Omega} T(z_0) < +\infty$ ;

- **predefined-time stable (PdTS)[97]** if and only if it is FxTS and  $\sup_{z_0 \in \Omega} T(z_0) \leq T_{\max}$  where  $T_{\max}$  is a tuning parameter chosen, in advance, independently of  $z_0 \in \Omega$ ;

- **prescribed-time stable (PTS)[98]** if and only if it is PdTS and  $T$  is a tuning parameter chosen, in advance, independently of  $z_0 \in \Omega$ .

The set  $\Omega$  is called the domain of stability/attraction. If  $\Omega = \mathbb{R}^n$ , then the corresponding properties are global.

In the above definitions “uniformly” with respect to initial time  $\bar{t}_0$  has been omitted for the sake of brevity.

**Remark 2.1**

Notice that when studying the prescribed-time stability notion later on, we will need to deal with a non-autonomous version of (2.1) of the form:

$$\dot{z} = \mathcal{K}(t, z), \quad (t, z) \in [\bar{t}_0, \bar{t}_0 + \bar{T}) \times \Omega. \quad (2.2)$$

Here,  $\bar{T} > 0$  is a positive constant and  $\mathcal{K} : [\bar{t}_0, \bar{t}_0 + \bar{T}) \times \Omega \rightarrow \mathbb{R}^n$  is a continuous function such that  $\mathcal{K}(t, 0) = 0$  for all  $t \in [\bar{t}_0, \bar{t}_0 + \bar{T})$  and such that (2.2) has the property of existence and uniqueness of solutions in forward time outside the origin. However, given that we are interested in “uniform” stability properties with respect to the initial time  $\bar{t}_0$ , the previously introduced definitions remain unchanged (as seen in, for example, [86, Definitions 1 and 2] for uniform **F****T****S**/**F****x****T****S** definitions and in [68, Definition 4.4] for both uniform stability and uniform asymptotic stability). For this reason and to avoid unnecessarily complicating the notations, we will only invoke the non-autonomous system (2.2) when studying the **P****T****S** concept. It is worth noting that alternative non-uniform stability notions could be employed in this Thesis with some modifications following the lines of [68, Definition 4.4], [112, Definitions 2.3, 2.4, and 2.5], and [174].

To illustrate these notions, let us give some preliminary examples accompanied by some numerical simulations,

**Example 2.1 (ES)**

Let us consider the following scalar system:

$$\begin{cases} \dot{z}(t) = -cz(t), & c > 0, t \geq \bar{t}_0 \geq 0, \\ z(\bar{t}_0) = z_0, \end{cases} \quad (2.3)$$

$$(2.4)$$

whose solution is given by

$$z(t) = z_0 e^{-c(t-\bar{t}_0)}. \quad (2.5)$$

It is clear from (2.3) that the origin is the equilibrium point. In addition, from (2.5) we can see first that the system is stable ( $|z(t)| \leq |z_0| : \forall t \geq \bar{t}_0$ ). Furthermore, from (2.5) we recover that when  $t$  tends to infinity, the trajectories of (2.3) converge to the origin. Thus, the origin of (2.3) is **ES** (and thus also **AS**).

**Example 2.2 (F****T****S**)

Next, let us consider the following scalar system:

$$\begin{cases} \dot{z}(t) = -c\{z(t)\}^\alpha, & c > 0, \alpha \in (0, 1), t \geq \bar{t}_0 \geq 0, \\ z(\bar{t}_0) = z_0, \end{cases} \quad (2.6)$$

$$(2.7)$$

whose solution is

$$z(t) = \begin{cases} \text{sign}(z_0) [|z_0|^{1-\alpha} - c(1-\alpha)(t-\bar{t}_0)]^{\frac{1}{1-\alpha}}, & t < \bar{t}_0 + \frac{|z_0|^{1-\alpha}}{c(1-\alpha)}, \\ 0, & t \geq \bar{t}_0 + \frac{|z_0|^{1-\alpha}}{c(1-\alpha)}. \end{cases} \quad (2.8)$$

One can see that (2.6) has an equilibrium at the origin and from (2.8) we can see first that the system is stable (always because  $|z(t)| \leq |z_0| : \forall t \geq \bar{t}_0$ ). Furthermore, from (2.8) we recover that the trajectories of (2.6) converge to the equilibrium in a finite time  $T(z_0) = \bar{t}_0 + \frac{|z_0|^{1-\alpha}}{c(1-\alpha)}$  as shown in Figure 2.1. Notice that the settling time depends on the initial condition  $z_0$  which means that the system is **F****T****S** but not **F****x****T****S**.

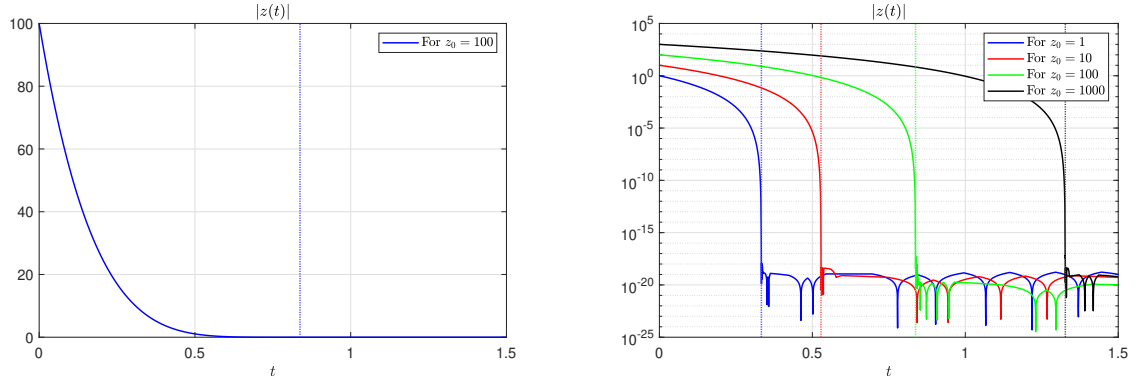


Figure 2.1: The evolution of (2.8) (in logarithmic scale on the right) for  $\bar{t}_0 = 0$ ,  $\alpha = 0.8$ ,  $c = 15$ , and 4 different initial conditions in blue using  $z_0 = 1$ , in red using  $z_0 = 10$ , in green using  $z_0 = 100$ , and in black using  $z_0 = 1000$ . We can observe that the larger the initial condition, the larger the settling time (i.e. the times of convergence depend on the initial condition).

**Remark 2.2**

Note that the oscillations of the solutions (in the neighborhood of  $10^{-20}$ ) shown on the right-hand side of Figure 2.1 are due to our ODE solver reaching the maximum precision of the machine, and thus should be discarded.

**Example 2.3 (FxTS)**

Now, let us consider the following scalar system:

$$\begin{cases} \dot{z}(t) = -c\{z(t)\}^{\alpha_1} - c\{z(t)\}^{\alpha_2}, & c > 0, \alpha_1 \in (0, 1), \alpha_2 > 1, t \geq \bar{t}_0 \geq 0, \\ z(\bar{t}_0) = z_0. \end{cases} \quad (2.9)$$

$$(2.10)$$

Notice that solutions of (2.9) can only be computed explicitly for some particular cases. For instance, when  $\alpha_2 = 2 - \alpha_1$ , the solutions are given by

$$z(t) = \begin{cases} \text{sign}(z_0) \left( \tan \left[ \arctan(|z_0|^{1-\alpha_1}) - c(1-\alpha_1)(t-\bar{t}_0) \right] \right)^{\frac{1}{1-\alpha_1}}, & t < \bar{t}_0 + \frac{\arctan(|z_0|^{1-\alpha_1})}{c(1-\alpha_1)}, \\ 0, & t \geq \bar{t}_0 + \frac{\arctan(|z_0|^{1-\alpha_1})}{c(1-\alpha_1)}. \end{cases} \quad (2.11)$$

As before, we can clearly see that (2.9) has an equilibrium at the origin and from (2.11) that the system is stable and the trajectories of (2.9) converge to the equilibrium in a finite time  $T(z_0) = \bar{t}_0 + \frac{\arctan(|z_0|^{1-\alpha_1})}{c(1-\alpha_1)}$  upper-bounded by  $T_{\max} = \bar{t}_0 + \frac{\pi}{2c(1-\alpha_1)}$  which means that the system is **FxTS** (see Figure 2.2).

**Example 2.4 (PdTS)**

Now, let us consider the following scalar system:

$$\begin{cases} \dot{z}(t) = -\frac{\pi}{2(1-\alpha_1)T_c} (\{z(t)\}^{\alpha_1} + \{z(t)\}^{2-\alpha_1}), & T_c > 0, \alpha_1 \in (0, 1), t \geq \bar{t}_0 \geq 0, \\ z(\bar{t}_0) = z_0. \end{cases} \quad (2.12)$$

$$(2.13)$$

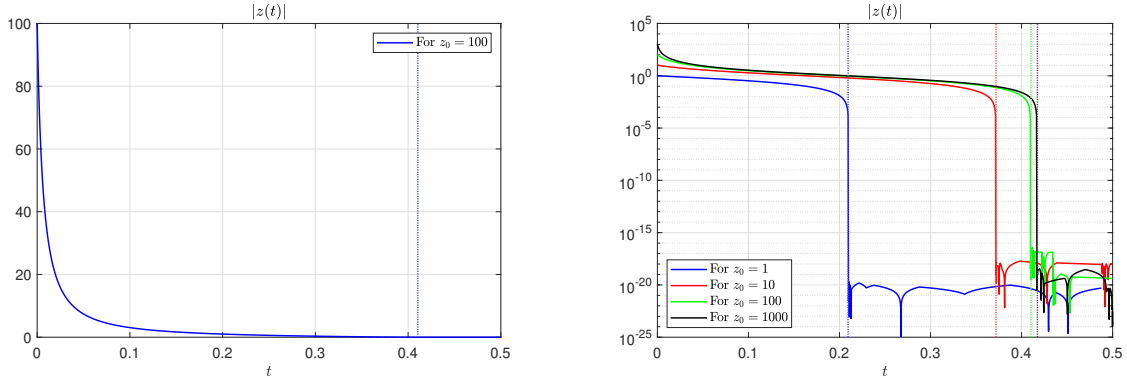


Figure 2.2: The evolution of (2.11) (in logarithmic scale on the right) for  $\bar{t}_0 = 0$ ,  $\alpha_1 = 0.25$ ,  $\alpha_2 = 1.75$ ,  $c_1 = c_2 = 5$ , and 4 different initial conditions in blue using  $z_0 = 1$ , in red using  $z_0 = 10$ , in green using  $z_0 = 100$ , and in black using  $z_0 = 1000$ . We can observe that the settling time is upper bounded by a constant that does not depend on the initial conditions (i.e. the time of the convergence does not depend on the initial conditions).

whose solutions of (2.12) are given by

$$z(t) = \begin{cases} \text{sign}(z_0) \left( \tan \left[ \arctan(|z_0|^{1-\alpha_1}) - \frac{\pi}{2T_c}(t - \bar{t}_0) \right] \right)^{\frac{1}{1-\alpha_1}}, & t < \bar{t}_0 + \frac{2T_c \arctan(|z_0|^{1-\alpha_1})}{\pi}, \\ 0, & t \geq \bar{t}_0 + \frac{2T_c \arctan(|z_0|^{1-\alpha_1})}{\pi}. \end{cases} \quad (2.14)$$

As before, we can clearly see that (2.12) has an equilibrium at the origin and from (2.14) that the system is stable and the trajectories of (2.12) converge to the equilibrium in a finite time  $T(z_0) = \bar{t}_0 + \frac{2T_c \arctan(|z_0|^{1-\alpha_1})}{\pi}$  upper-bounded by  $T_{\max} = \bar{t}_0 + T_c$  which means that the system is **PdTS** (see Figure 2.3).

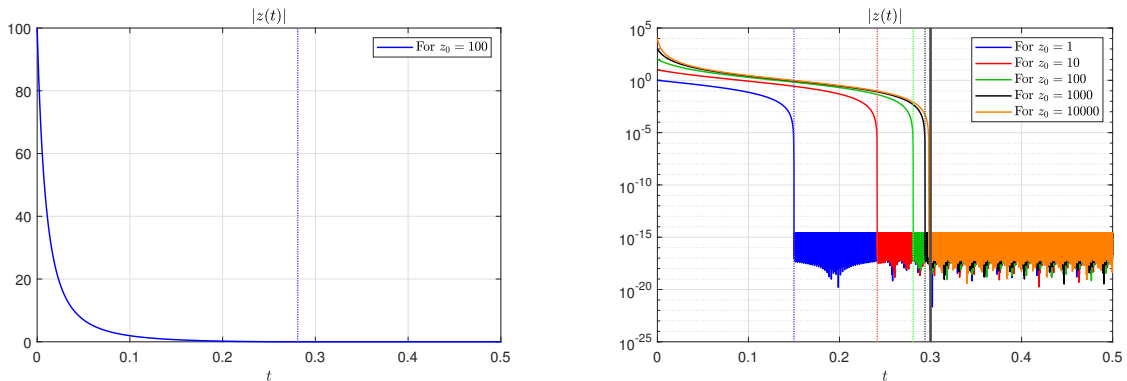


Figure 2.3: The evolution of (2.14) (in logarithmic scale on the right) for  $\bar{t}_0 = 0$ ,  $\alpha_1 = 0.5$ ,  $T_c = 0.3$ , and 5 different initial conditions in blue using  $z_0 = 1$ , in red using  $z_0 = 10$ , in green using  $z_0 = 100$ , in black using  $z_0 = 1000$ , and in orange using  $z_0 = 10000$ . We can observe that the settling time is upper bounded by a constant that is chosen a priori independently of the initial conditions.

**Example 2.5 (PTS)**

Let us now consider the following scalar system:

$$\begin{cases} \dot{z}(t) = -\left(\frac{cT}{\bar{t}_0 + T - t}\right)^2 z(t), & c > 0, t \in [\bar{t}_0, \bar{t}_0 + T), \bar{t}_0 \geq 0, T > 0, \\ z(\bar{t}_0) = z_0, \end{cases} \quad (2.15)$$

whose solution is given by

$$z(t) = z_0 e^{c^2 T} e^{-\frac{c^2 T^2}{\bar{t}_0 + T - t}}. \quad (2.17)$$

In this case, we can clearly see that (2.15) has an equilibrium at the origin and from (2.17) that the system is stable (when it is defined) and the trajectories of (2.15) converge to the equilibrium in a finite time  $\bar{t}_0 + T$  prescribed independently of the initial conditions as shown in Figure 2.4. Thus, (2.15) is **PTS**.

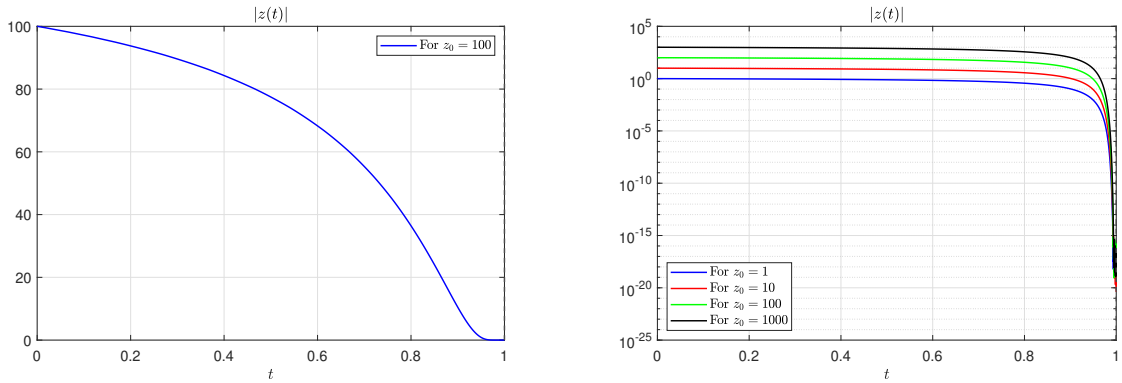


Figure 2.4: The evolution of (2.17) (in logarithmic scale on the right) for  $\bar{t}_0 = 0$ ,  $T = 1$ ,  $c = 1$ , and 4 different initial conditions in blue using  $z_0 = 1$ , in red using  $z_0 = 10$ , in green using  $z_0 = 100$ , and in black using  $z_0 = 1000$ .

It is possible to use the comparison functions to give an alternative definition to some of the notions in the previous Definition 2.1:

**Definition 2.2**

The origin of the system (2.1) is said to be:

- **stable** if and only if there exists a class  $\mathcal{K}$  function  $\sigma$  such that

$$\|z(t)\|_2 \leq \sigma(\|z_0\|_2), \quad (2.18)$$

for all  $t \geq \bar{t}_0$  and any initial condition  $z_0 \in \Omega$ ;

- **asymptotically stable (AS)** if and only if there exists a class  $\mathcal{KL}$  function  $\beta$  such that

$$\|z(t)\|_2 \leq \beta(\|z_0\|_2, t - \bar{t}_0), \quad (2.19)$$

for all  $t \geq \bar{t}_0$  and any  $z_0 \in \Omega$ ;

- **finite-time stable (FTS)** if and only if there exist  $T(\cdot)$  a continuous function with  $T(0) = 0$  and a class  $\mathcal{KL}$  function  $\beta$  with  $\beta(r, t - \bar{t}_0) \equiv 0$  when  $t \geq \bar{t}_0 + T(r)$  ( $\beta$  is called a generalized  $\mathcal{KL}$  function or  $\mathcal{GKL}$  function) such that

$$\|z(t)\|_2 \leq \beta(\|z_0\|_2, t - \bar{t}_0), \quad (2.20)$$

for every  $t \geq \bar{t}_0$  and any  $z_0 \in \Omega$ ;

- **fixed-time stable (FxTS)** if and only if it is **FTS** and  $\sup_{r \in \Omega} T(r) < +\infty$ ;
- **prescribed-time stable (PTS)** if and only if there exist a class  $\mathcal{KL}$  function  $\beta$  and a function  $\mu : [\bar{t}_0, \bar{t}_0 + T) \rightarrow \mathbb{R}_+$  that tends to infinity as  $t$  goes to a prescribed time  $\bar{t}_0 + T$ , such that

$$\|z(t)\|_2 \leq \beta\left(\|z_0\|_2, \mu(t - \bar{t}_0)\right), \quad (2.21)$$

for every  $t \in [\bar{t}_0, \bar{t}_0 + T)$ , any  $z_0 \in \Omega$ .

### Remark 2.3

Following similar arguments to the ones given in Remark 2.1, the previously introduced alternative definitions of the stability concepts remain unchanged when dealing with non-autonomous systems of the form (2.2).

### Remark 2.4

Note that a simple characterization of the stability concepts for the scalar case of (2.1) when  $\mathcal{K} \in L^1_{loc}(\Omega)$  (i.e.  $\mathcal{K} : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$ ) can be formulated using the Landau notation<sup>1</sup>. More precisely, the origin of (2.1) is

- **asymptotically stable (AS)** if and only if  $\mathcal{K} \in S := \{\mathcal{K} \in L^1_{loc}(\Omega) : \mathcal{K}(z) = 0 \Leftrightarrow z = 0, z\mathcal{K}(z) > 0, \forall z \in \Omega \setminus \{0\}\}$ ;
- **finite-time stable (FTS)** if and only if  $\mathcal{K} \in \mathcal{E}_{k_0, a_0} = \{\mathcal{K} \in S : \mathcal{K}(z) \underset{z \rightarrow 0}{\sim} k_0\{z\}^{a_0}, \lim_{|z| \rightarrow \infty} \mathcal{K}(z) \neq 0\}$ , where  $\{\cdot\}$  is the signed power function (see the Notation Section),  $a_0 \in [0, 1)$ , and  $k_0 > 0$ ;
- **fixed-time stable (FxTS)** if and only if  $\mathcal{K} \in \mathcal{E}_{k_0, a_0}^{k_\infty, a_\infty} = \{\mathcal{K} \in \mathcal{E}_{k_0, a_0} : \mathcal{K}(z) \underset{|z| \rightarrow \infty}{\sim} k_\infty\{z\}^{a_\infty}\}$ ; where  $a_0 \in [0, 1)$ ,  $a_\infty > 1$ ,  $k_0 > 0$ , and  $k_\infty > 0$ .

### Example 2.6

Let  $a_0 \in [0, 1)$ ,  $a_\infty > 1$ ,  $k_0 > 0$ ,  $k_1 > 0$ ,  $k_2 > 0$ ,  $k_\infty > 0$  and  $\psi$  be any continuous positive function which is zero at 0 and at  $\infty$

$$\mathcal{K}_1(z) = k_0\{z\}^{a_0}(1 + \psi(z)), \quad (2.22)$$

$$\mathcal{K}_2(z) = (k_0\{z\}^{a_0} + k_\infty\{z\}^{a_\infty})(1 + \psi(z)), \quad (2.23)$$

$$\mathcal{K}_3(z) = k_0\{z\}^{a_0} \left(1 + k_1|z|^{\frac{a_\infty - a_0}{k_2}}\right)^{k_2}, \quad (2.24)$$

then  $\mathcal{K}_1 \in \mathcal{E}_{k_0, a_0}$ ,  $\mathcal{K}_2 \in \mathcal{E}_{k_0, a_0}^{k_\infty, a_\infty}$ ,  $\mathcal{K}_3 \in \mathcal{E}_{k_0, a_0}^{k_\infty, a_\infty}$  where in this last case  $k_\infty = k_0 k_1^{k_2}$ .

<sup>1</sup>  $f(z) \underset{x \rightarrow a}{\sim} g(z)$  if and only if  $\frac{f(z) - g(z)}{g(z)} \xrightarrow{x \rightarrow a} 0$ .

### 2.2.1 Lyapunov-based approach for ODEs

The use of Definitions 2.1 and 2.2 requires to know the explicit solution of (2.1) which can be hard to achieve for general classes of differential equations (nonlinear ODEs for instance or even for PDEs). Therefore, Lyapunov analysis is a powerful tool that can provide sufficient conditions to determine the stability of general nonlinear systems without the need of computing their solutions explicitly. The key idea behind this technique is to use and analyze the properties of a scalar function, known as a *Lyapunov function*, which maps the system's state to a real number (some kind of "distance" to the origin), to determine whether the system is stable or not.

In this subsection, we recall the main Lyapunov analysis results for autonomous ODEs starting with the well-known Lyapunov stability theorem.

We recall that  $\Omega \subset \mathbb{R}^n$  is an open connected set containing the origin, with  $n \in \mathbb{N}^*$ .

**Theorem 2.1** (Lyapunov Stability Theorem [68, Theorem 4.1])

Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ , ( $V$  is positive-definite);
2.  $\dot{V}(z) \leq 0$  in  $\Omega$ , ( $\dot{V}(z)$  is negative-semidefinite).

Then, the origin of (2.1) is Lyapunov stable on  $\Omega$ . Moreover, if we replace condition 2 by

3.  $\dot{V}(z) < 0$  in  $\Omega \setminus \{0\}$ , ( $\dot{V}(z)$  is negative-definite),

then, the origin of (2.1) is **AS** on  $\Omega$ .

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded (i.e.  $V(z) \rightarrow +\infty$  as  $\|z\|_2 \rightarrow +\infty$ ,  $z \in \mathbb{R}^n$ ) then the stability properties are global.

**Remark 2.5**

When the function  $V$  satisfies the first two conditions,  $V$  is referred to as a *Lyapunov function*, whereas when  $V$  satisfies conditions 1 and 3,  $V$  is called a *strict Lyapunov function*.

**Remark 2.6**

Notice that the fact that  $V$  is continuously differentiable on  $\Omega$ , implies that the time derivative of  $V$  along the trajectories of (2.1) is given by

$$\dot{V}(z) = \frac{\partial V}{\partial z} \mathcal{K}(z) = \sum_{i=1}^n \frac{\partial V}{\partial z_i} \mathcal{K}_i(z), \quad z \in \Omega \setminus \{0\}, \quad (2.25)$$

where we recall that  $\mathcal{K} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function with  $\mathcal{K}(0) = 0$  given in (2.1).

To illustrate this result, let us look at the following example:

**Example 2.7**

Consider the following second-order ODE:

$$\begin{cases} \dot{z}_1(t) = z_2(t), & t \geq \bar{t}_0 \geq 0, \\ \dot{z}_2(t) = -c_1 z_1(t)^{\frac{1}{3}} - c_2 z_2(t)^{\frac{1}{3}}, & t \geq \bar{t}_0 \geq 0, \end{cases} \quad (2.26)$$

$$t \geq \bar{t}_0 \geq 0, \quad (2.27)$$

with  $z(t) = [z_1(t), z_2(t)]^\top \in \mathbb{R}^2$ , and  $c_1, c_2 > 0$ .

Let  $V$  be given by the following function:

$$V(z) = c_1 \frac{5}{6} |z_1|^{\frac{6}{5}} + \frac{1}{2} z_2^2, \quad (2.28)$$

It is easy to check that  $V$  satisfies Condition 1:  $V(0) = 0$  and  $V(z) = c_1 \frac{5}{6} |z_1|^{\frac{6}{5}} + \frac{1}{2} |z_2|^2 > 0$  for all  $z \neq [0, 0]$ . Moreover, by computing the time derivative of  $V$  along the trajectories of (2.26)-(2.27), we get

$$\dot{V}(z) = \frac{\partial V(z)}{\partial z_1} \dot{z}_1 + \frac{\partial V(z)}{\partial z_2} \dot{z}_2, \quad (2.29)$$

$$= \frac{\partial V(z)}{\partial z_1} z_2 + \frac{\partial V(z)}{\partial z_2} \left( -c_1 z_1^{\frac{1}{5}} - c_2 z_2^{\frac{1}{3}} \right) \quad (2.30)$$

$$= c_1 z_1^{\frac{1}{5}} z_2 + z_2 \left( -c_1 z_1^{\frac{1}{5}} - c_2 z_2^{\frac{1}{3}} \right), \quad (2.31)$$

$$= -c_2 |z_2|^{\frac{4}{3}}, \quad (2.32)$$

$$= -c_2 \left( \sqrt[3]{|z_2|} \right)^4 \leq 0, \quad (2.33)$$

Then,  $V$  satisfies Condition 2. and by Theorem 2.1 the system (2.26)-(2.27) is Lyapunov stable.

Note that a reformulation and useful version of Theorem 2.1 was introduced by Clarke et al in [175].

**Theorem 2.2** ([175, Theorem 1.2])

Let  $V : \Omega \rightarrow \mathbb{R}$  and  $W : \Omega \setminus \{0\} \rightarrow \mathbb{R}$  be two continuously differentiable functions such that

1.  $V(0) = 0$ ,  $V(z) > 0$  and  $W(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\dot{V}(z) \leq 0$  in  $\Omega$ .

Then, the origin of (2.1) is Lyapunov stable on  $\Omega$ . Moreover, if we replace condition 2 by

3.  $\dot{V}(z) \leq -W(z)$  in  $\Omega \setminus \{0\}$ ,

then, the origin of (2.1) is **AS** on  $\Omega$ .

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded then the origin is globally **AS**.

**Remark 2.7**

Note that if in addition,  $W(z) = -cV(z)$  for  $c > 0$ , then the origin of (2.1) is (globally) **ES**.

**Remark 2.8**

It is worth mentioning that the Lyapunov-based characterization of stability and asymptotic stability, previously introduced, can be seamlessly adapted to fit the framework of non-autonomous ODEs of the form (2.2). To do that, we start first by making the function of interest  $V$  depend explicitly on the time variable  $t$  in a continuously differentiable way (i.e.  $V : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$  is a continuously differentiable function on  $[\bar{t}_0, +\infty) \times \Omega$ ); Then we replace conditions 1. and 3. given in Theorem 2.2, respectively, by the following two alternative conditions as elaborated in [68, Theorems 4.8-4.10]:

- 1'.  $W_1(z) \leq V(t, z) \leq W_2(z)$  in  $[\bar{t}_0, +\infty) \times \Omega$ ,



$$3'. \dot{V}(t, z) = \frac{\partial V(t, z)}{\partial t} + \frac{\partial V(t, z)}{\partial z} \mathcal{K}(t, z) \leq -W_3(z), \text{ in } [\bar{t}_0, +\infty) \times \Omega \setminus \{0\},$$

where  $W_1$ ,  $W_2$ , and  $W_3$  are continuous positive definite functions on  $\Omega$ .

### 2.2.1.1 Lyapunov-based characterization of finite/fixed/prescribed-time concepts

Previous Lyapunov results have only addressed asymptotic stability, while it was not until 1995, to the best of our knowledge, that the first Lyapunov-based result contributing to finite-time stability (**FTS**) was introduced by S.P. Bhat and D.S. Bernstein in their paper [73], which was later on refined in [76] where in addition an estimate of the settling time was provided. In this subsection, we revisit this pivotal result and explore related contributions within the same framework, focusing particularly on Lyapunov-based characterization of fixed-time, predefined-time, and prescribed-time stability concepts.

we recall this important result alongside some other related results that contributed in this framework (mainly for fixed-time and prescribed-time and predefined-time)

We recall that  $\Omega \subset \mathbb{R}^n$  is an open connected set containing the origin, with  $n \in \mathbb{N}^*$ .

**Theorem 2.3** ([76, Theorem 4.2].)

Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\dot{V}(z) \leq -cV(z)^\alpha$  in  $\Omega$  for some real numbers  $c > 0$  and  $\alpha \in (0, 1)$ .

Then, the origin of (2.1) is **FTS** and the settling time  $T$  is a continuous function on  $\Omega$  that satisfies

$$T(z) \leq \frac{1}{c(1-\alpha)} V(z)^{1-\alpha}, \quad \forall z \in \Omega. \quad (2.34)$$

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded, then the origin of (2.1) is a globally **FTS**.

To illustrate this, let us give an example.

**Example 2.8 (FTS)**

Consider the following second-order ODE:

$$\begin{cases} \dot{z}_1(t) = z_2(t) - \frac{c}{2} \{z_1(t)\}^{2\alpha-1}, & t \geq \bar{t}_0 \geq 0, \\ \dot{z}_2(t) = -z_1(t) - \frac{c}{2} \{z_2(t)\}^{2\alpha-1}, & t \geq \bar{t}_0 \geq 0, \end{cases} \quad (2.35)$$

$$\quad (2.36)$$

with  $z(t) = [z_1(t), z_2(t)]^\top \in \mathbb{R}^2$ ,  $c > 0$ ,  $\alpha \in (\frac{1}{2}, 1)$ .

The conditions of Theorem 2.3 are satisfied for the following choice of function  $V$ :

$$V(z) = |z_1|^2 + |z_2|^2. \quad (2.37)$$

Indeed,  $V$  satisfies Condition 1 (i.e.  $V(0) = 0$  and  $V(z) > 0$  for all  $z \neq [0, 0]$ ) and condition 2.

$$\dot{V}(z) = \frac{\partial V(z)}{\partial z_1} \dot{z}_1 + \frac{\partial V(z)}{\partial z_2} \dot{z}_2, \quad (2.38)$$

$$= \frac{\partial V(z)}{\partial z_1} \left( z_2 - \frac{c}{2} \{z_1\}^{2\alpha-1} \right) + \frac{\partial V(z)}{\partial z_2} \left( -z_1 - \frac{c}{2} \{z_2\}^{2\alpha-1} \right), \quad (2.39)$$

$$= 2z_1 \left( z_2 - \frac{c}{2} \{z_1\}^{2\alpha-1} \right) + 2z_2 \left( -z_1 - \frac{c}{2} \{z_2\}^{2\alpha-1} \right), \quad (2.40)$$

$$= -c (z_1^{2\alpha} + z_2^{2\alpha}), \quad (2.41)$$

$$\leq -c (|z_1|^2 + |z_2|^2)^\alpha, \quad (2.42)$$

$$= -cV(z)^\alpha. \quad (2.43)$$

Consequently, (2.35)-(2.36) is **FTS**.

This result was later generalized by Moulay et al. in [174], as given in the following:

**Theorem 2.4** ([174, Proposition 4.1])

Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function and  $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous positive definite function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\int_0^\varepsilon \frac{ds}{r(s)} < +\infty$  for some  $\varepsilon > 0$ ;
3.  $\dot{V}(z) \leq -r(V(z))$  in  $\Omega$ .

Then, the origin of (2.1) is **FTS** and the settling-time function  $T$  is continuous on  $\Omega$  and satisfies the inequality:

$$T(z) \leq \int_0^{V(z)} \frac{ds}{r(s)}, \forall z \in \Omega. \quad (2.44)$$

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded then the origin is globally **FTS**.

The following two results extend Theorems 2.3 and 2.4 to the case of fixed-time stability.

**Theorem 2.5** ([86, Lemma 1])

Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\dot{V}(z) \leq -c_1 V(z)^{\alpha_1} - c_2 V(z)^{\alpha_2}$  in  $\Omega$  for some real numbers  $c_1 > 0$ ,  $c_2 > 0$ ,  $\alpha_1 \in (0, 1)$ , and  $\alpha_2 > 1$ .

Then, the origin of (2.1) is **FxTS** and the settling time  $T$  is a continuous function on  $\Omega$  that satisfies

$$T(z) \leq \frac{1}{c_1(1-\alpha_1)} + \frac{1}{c_2(\alpha_2-1)}, \quad \forall z \in \Omega. \quad (2.45)$$

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded, then the origin of (2.1) is a globally **FxTS**.

**Theorem 2.6** ([91, Theorem 5])

Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function and  $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous positive definite function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\int_0^{\sup_{z \in \Omega} V(z)} \frac{ds}{r(s)} < +\infty$  for some  $\varepsilon > 0$
3.  $\dot{V}(z) \leq -r(V(z))$  in  $\Omega$ .

Then, the origin of (2.1) is **FxTS** and the settling-time function  $T$  is continuous on  $\Omega$  and satisfies the inequality:

$$T(z) \leq \int_0^{\sup_{z \in \Omega} V(z)} \frac{ds}{r(s)}, \forall z \in \Omega. \quad (2.46)$$

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded then the origin is globally **FxTS**.

Consider the following example:

**Example 2.9 (FxTS)**

Consider the following second-order ODE:

$$\begin{cases} \dot{z}_1(t) = z_2(t) - \frac{c_1}{2} \{z_1(t)\}^{2\alpha_1-1} - \frac{c_2}{2} \{z_1(t)\}^{2\alpha_2-1}, & t \geq \bar{t}_0 \geq 0, \\ \dot{z}_2(t) = -z_1(t) - \frac{c_1}{2} \{z_2(t)\}^{2\alpha_1-1} - \frac{c_2}{2} \{z_2(t)\}^{2\alpha_2-1}, & t \geq \bar{t}_0 \geq 0, \end{cases} \quad (2.47)$$

$$\quad (2.48)$$

with  $z(t) = [z_1(t), z_2(t)]^\top \in \mathbb{R}^2$ ,  $c_1, c_2 > 0$ ,  $\alpha_1 \in (\frac{1}{2}, 1)$ , and  $\alpha_2 > 1$ . Similarly to Example 2.8, we can prove that the conditions of Theorem 2.5 are satisfied for the same choice of the function  $V$  given in (2.37). As a result, we conclude that, (2.47)-(2.48) is **FxTS**.

Next, we present a Lyapunov-based characterization of the predefined-time stability (**PdTS**) concept.

**Proposition 2.1** ([97, Theorem 1])

Let  $T_{\max} > 0$  be chosen a priori and assume that (2.1) depends on some tunable parameters  $\rho \in \mathbb{R}^l$ ,  $l \in \mathbb{N}$ . Let  $\kappa \in \mathcal{K}^1$  (a class- $\mathcal{K}^1$  function) be differentiable in  $\mathbb{R} \setminus \{0\}$  and depending on  $\rho$ , and let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\dot{V}(z) \leq -\frac{1}{(1-p)T_{\max}} \frac{\kappa(V(z))^p}{\kappa'(V(z))}$  in  $\Omega \setminus \{0\}$  for some  $0 \leq p < 1$ .

Then, the origin of is **PdTS** in a time  $\bar{t}_0 + T(z_0)$  where  $\sup_{z_0 \in \Omega} T(z_0) \leq T_{\max}$ . Moreover, if 2) is replaced by an equality, then  $\sup_{z_0 \in \Omega} T(z_0) = T_{\max}$ .

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded, then the origin of (2.1) is a globally **PdTS**.

This last proposition yields the following Corollary:

**Corollary 2.1** ([96, Theorem 2])

Let  $T_{\max} > 0$  be chosen a priori and assume that (2.1) depends on some tunable parameters  $\rho = [\rho_1, \dots, \rho_6]^\top \in \mathbb{R} \times \mathbb{R}_+^5$ . Let  $V : \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $V(0) = 0$  and  $V(z) > 0$  in  $\Omega \setminus \{0\}$ ;
2.  $\dot{V}(z) \leq -\frac{\rho_1}{T_{\max}} (\rho_2 V(z)^{\rho_3} + \rho_4 V(z)^{\rho_5})^{\rho_6}$  in  $\Omega \setminus \{0\}$  for  $\rho$  is chosen such that  $\rho_6 \rho_3 < 1$ ,  $\rho_6 \rho_5 > 1$  and

$$\rho_1 = \frac{\hat{\Gamma}\left(\frac{1-\rho_6\rho_3}{\rho_5-\rho_3}\right) \hat{\Gamma}\left(\frac{\rho_6\rho_5-1}{\rho_5-\rho_3}\right)}{\alpha^{\rho_6} \hat{\Gamma}(\rho_6)(\rho_5-\rho_3)} \left(\frac{\rho_2}{\rho_4}\right)^{\frac{1-\rho_6\rho_3}{\rho_5-\rho_3}} \quad (2.49)$$

where  $\hat{\Gamma}$  is the Gamma Function (see Notation section).

Then, the origin of (2.1) is **PdTS** in a time  $\bar{t}_0 + T(z_0)$  where  $\sup_{z_0 \in \Omega} T(z_0) \leq T_{\max}$ . Moreover, if 2) is replaced by an equality, then  $\sup_{z_0 \in \Omega} T(z_0) = T_{\max}$ .

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded, then the origin of (2.1) is a globally **PdTS**.

*Proof.* The proof is a direct application of Proposition 2.1 with the particular selections of  $\kappa(r) = I\left(\frac{1-\rho_6\rho_3}{\rho_5-\rho_3}, \frac{\rho_6\rho_5-1}{\rho_5-\rho_3}, \frac{\rho_4 r^{\rho_5-\rho_3}}{\beta r^{\rho_5-\rho_3} + \rho_2}\right)$ , with  $p = 0$  (see [97, Remark 3]) where  $I$  is the regularized Incomplete Beta Function (see Notation section). □

### Remark 2.9

Building upon Remark 2.8, the previously introduced Lyapunov-based characterizations of **FTS**, **FxTS**, and **PdTS** can be generalized to non-autonomous ODEs (2.2) by making the function on interest  $V$  depend explicitly on the time variable  $t$  in a continuously differentiable way (i.e.  $V : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$  is a continuously differentiable function on  $[\bar{t}_0, +\infty) \times \Omega$ ); Then we replace conditions 1. given in each result by the following alternative condition as shown for example in [102], [174]:

- 1'.  $W_1(z) \leq V(t, z) \leq W_2(z)$  in  $[\bar{t}_0, +\infty) \times \Omega$ , where  $W_1$  and  $W_2$  are continuous positive definite functions on  $\Omega$ .

As highlighted in Remarks 2.1-2.9, dealing with prescribed-time stability (**PTS**), we will need to deal with a non-autonomous ODEs of the form (2.2). For this reason, we finish this subsection, by a Lyapunov-based characterization of the prescribed-time stability (**PTS**) concept for this class of systems.

### Proposition 2.2

Let  $T > 0$  be chosen a priori and let  $V : [\bar{t}_0, \bar{t}_0 + T) \times \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $V(t, 0) = 0$  and  $V(t, z) > 0$  in  $[\bar{t}_0, \bar{t}_0 + T) \times \Omega \setminus \{0\}$ ;
2.  $\dot{V}(t, z) \leq -c(t - \bar{t}_0)V(t, z)$  in  $\Omega$  for some continuous function  $c : [\bar{t}_0, \bar{t}_0 + T) \rightarrow \mathbb{R}_+$  such that  $\int_{\bar{t}_0}^t c(s - \bar{t}_0) ds$  is finite for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$  and becomes unbounded as  $t$  approaches  $\bar{t}_0 + T$ .

Then, the origin of (2.1) is **PTS** in a time  $\bar{t}_0 + T$ .

If in addition,  $\Omega = \mathbb{R}^n$ , and  $V$  is radially unbounded, then the origin of (2.1) is a globally **PTS**.

*Proof.* The proof follows directly from Theorem 2.1 and from Grönwall's lemma on condition 2). Indeed, applying Grönwall's lemma, we get

$$V(t, z(t)) \leq e^{-\int_{\bar{t}_0}^t c(s-\bar{t}_0)ds} V(\bar{t}_0, z(\bar{t}_0)). \quad (2.50)$$

from which we can deduce that  $V(t, z(t)) \rightarrow 0$  as  $t \rightarrow \bar{t}_0 + T$  and in turn  $\|z(t)\|_2 \rightarrow 0$  as  $t \rightarrow \bar{t}_0 + T$ .  $\square$

**Example 2.10 (PTS)**

Based on [145], [176], consider the following second-order ODE:

$$\begin{cases} \dot{z}_1(t) = z_2(t), & t \in [\bar{t}_0, \bar{t}_0 + T), \\ \dot{z}_2(t) = - \left[ (r_1 + r_2)\gamma_2(t - \bar{t}_0) - \frac{2}{\gamma_{2,0}T} \sqrt{\gamma_2(t - \bar{t}_0)} \right] z_2(t) \\ \quad - r_1 r_2 \gamma_2(t - \bar{t}_0)^2 z_1(t), & t \in [\bar{t}_0, \bar{t}_0 + T), \end{cases} \quad (2.51)$$

with  $z(t) = [z_1(t), z_2(t)]^\top \in \mathbb{R}^2$ ,  $\bar{t}_0 \geq 0$ ,  $r_1, r_2 > 0$  such  $r_1 \neq r_2$ ,  $T > 0$  is a priori fixed, and the function  $\gamma_2(\cdot)$  is chosen as follows:

$$\gamma_2(s) := \frac{\gamma_{2,0}^2 T^2}{(T - s)^2}, \quad s \in [0, T), \quad (2.53)$$

for  $\gamma_{2,0} > 0$  where

$$\int_{\bar{t}_0}^t \gamma_2(s - \bar{t}_0) ds = \left[ \gamma_{2,0} T \sqrt{\gamma_2(s - \bar{t}_0)} \right]_{\bar{t}_0}^t, \quad (2.54)$$

$$= \gamma_{2,0} T (\sqrt{\gamma_2(t - \bar{t}_0)} - \gamma_{2,0}) \longrightarrow +\infty, \quad (2.55)$$

as  $t \rightarrow \bar{t}_0 + T$ . Let  $V$  be given by the following function:

$$V(t, z) = V_1(t, z) + V_2(t, z), \quad (2.56)$$

where

$$V_i(t, z) = [r_i z_1 + \gamma_2(t - \bar{t}_0)^{-1} z_2]^2, \quad i \in \{1, 2\}. \quad (2.57)$$

It easy to check that  $V$  satisfies Condition 1:  $V(t, 0) = 0$  and  $V(t, z) > 0$  for all  $t \in [\bar{t}_0, \bar{t}_0 + T]$  and all  $z \neq [0, 0]$ . Moreover, from the time derivative of  $V_i$  along the trajectories of (2.51)-(2.52),

we get

$$\dot{V}_i(t, z) = \frac{\partial V_i(t, z)}{\partial z_1} \dot{z}_1 + \frac{\partial V_i(t, z)}{\partial z_2} \dot{z}_2 + \frac{\partial V_i(t, z)}{\partial t}, \quad (2.58)$$

$$\begin{aligned} &= \frac{\partial V_i(t, z)}{\partial z_2} \left( -r_1 r_2 \gamma_2 (t - \bar{t}_0)^2 z_1 - \left[ (r_1 + r_2) \gamma_2 (t - \bar{t}_0) - \frac{2}{\gamma_{2,0} T} \sqrt{\gamma_2 (t - \bar{t}_0)} \right] z_2 \right) \\ &\quad + \frac{\partial V_i(t, z)}{\partial z_1} z_2 + \frac{\partial V_i(t, z)}{\partial t}, \end{aligned} \quad (2.59)$$

$$\begin{aligned} &= 2 \left[ r_i z_1 + \gamma_2 (t - \bar{t}_0)^{-1} z_2 \right] \left[ r_i z_2 + \gamma_2 (t - \bar{t}_0)^{-1} \left( -r_1 r_2 \gamma_2 (t - \bar{t}_0)^2 z_1 \right. \right. \\ &\quad \left. \left. - \left[ (r_1 + r_2) \gamma_2 (t - \bar{t}_0) - \frac{2}{\gamma_{2,0} T} \sqrt{\gamma_2 (t - \bar{t}_0)} \right] z_2 \right) - \frac{2}{\gamma_{2,0} T} \gamma_2 (t - \bar{t}_0)^{\frac{-1}{2}} z_2 \right], \end{aligned} \quad (2.60)$$

$$= 2 \left[ r_i z_1 + \gamma_2 (t - \bar{t}_0)^{-1} z_2 \right] \left[ -r_{2-i} z_2 - r_i r_{2-i} \gamma_2 (t - \bar{t}_0) z_1 \right], \quad (2.61)$$

$$= -2r_{2-i} \gamma_2 (t - \bar{t}_0) \left[ r_i z_1 + \gamma_2 (t - \bar{t}_0)^{-1} z_2 \right]^2, \quad (2.62)$$

$$= -2r_{2-i} \gamma_2 (t - \bar{t}_0) V_i(t, z), \quad (2.63)$$

$$\leq -2r_{\min} \gamma_2 (t - \bar{t}_0) V_i(t, z), \quad (2.64)$$

which yields

$$\dot{V}(t, z) \leq -2r_{\min} \gamma_2 (t - \bar{t}_0) V(t, z). \quad (2.65)$$

Thus,  $V$  satisfies Condition 2. and by Proposition 2.2 the system is (2.51)-(2.52) is **PTS**.

Notice that using Grönwall's lemma, we obtain

$$V(t, z(t)) \leq e^{-2r_{\min} \int_{\bar{t}_0}^t \gamma_2(s - \bar{t}_0) ds} V(\bar{t}_0, z(\bar{t}_0)). \quad (2.66)$$

Moreover, since  $V$  can be rewritten as follows:

$$V(t, z) = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} r_1 & r_2 \\ \gamma_2 (t - \bar{t}_0)^{-1} & \gamma_2 (t - \bar{t}_0)^{-1} \end{bmatrix} \begin{bmatrix} r_1 & \gamma_2 (t - \bar{t}_0)^{-1} \\ r_2 & \gamma_2 (t - \bar{t}_0)^{-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad (2.67)$$

$$= \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} r_1^2 + r_2^2 & (r_1 + r_2) \gamma_2 (t - \bar{t}_0)^{-1} \\ (r_1 + r_2) \gamma_2 (t - \bar{t}_0)^{-1} & 2\gamma_2 (t - \bar{t}_0)^{-2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad (2.68)$$

$$:= z^\top \bar{W}(\gamma_2(t - \bar{t}_0)) z, \quad (2.69)$$

we get the following coercivity property:

$$\lambda_{\min}(\bar{W}(\gamma_2(t - \bar{t}_0))) \|z\|_2^2 \leq V(t, z) \leq \lambda_{\max}(\bar{W}(\gamma_2(t - \bar{t}_0))) \|z\|_2^2. \quad (2.70)$$

By applying this property on (2.66), we obtain

$$\|z(t)\|_2^2 \leq \lambda_{\max}(\bar{W}(\gamma_{2,0})) \lambda_{\min}(\bar{W}(\gamma_2(t - \bar{t}_0)))^{-1} e^{-2r_{\min} \int_{\bar{t}_0}^t \gamma_2(s - \bar{t}_0) ds} \|z(\bar{t}_0)\|_2^2, \quad (2.71)$$

from this, we can clearly see that  $\|z(t)\|_2^2 \rightarrow 0$  as  $t \rightarrow \bar{t}_0 + T$ .

### 2.2.2 Non-smooth tools in finite-dimensional setting

In this section, we will recall some results of finite-time stabilization of nonlinear systems starting by [103] (2002) where Hong managed to use a constructive iterative method to solve the problem

of finite-time stabilization of a power chain of power-integrators of the form:

$$\begin{cases} \dot{z}_1(t) = z_2(t)^{m_1}, & t \geq \bar{t}_0 \geq 0, & (2.72) \\ \vdots & & \\ \dot{z}_{n-1}(t) = z_n(t)^{m_{n-1}}, & t \geq \bar{t}_0 \geq 0, & (2.73) \\ \dot{z}_n(t) = U(t), & t \geq \bar{t}_0 \geq 0, & (2.74) \\ z(\bar{t}_0) = z_0, & & (2.75) \end{cases}$$

where  $z(t) = [z_1(t), \dots, z_n(t)]^T \in \mathbb{R}^n$  and  $U(t) \in \mathbb{R}$  are the system's state and control, and  $m_i > 0$ ,  $i \in \{1, \dots, n\}$  are odd integers,  $n \in \mathbb{N}^*$ .

This recursive approach was based on the paper of Coron and Praly [177] for desingularization, which basically consists in using, for each step  $j$  of the induction, a homogeneous  $C^1$ -Lyapunov function  $V_j$  and a virtual controller  $U_j$  defined for the chain of power-integrators of the size  $j$  while depending recursively on the previous induction steps. More precisely, for each step  $j$  of the induction, the virtual controller  $U_j$  is constructed in terms of previous virtual controllers, to make the closed-loop chain of power integrators of size  $j$  homogeneous with a negative degree. After that, the closed-loop chain of power integrators of size  $j$  is proved to be **AS** on some given compact set (i.e.  $\dot{V}_j$  is definite negative on some compact set). This last property is then extended to all the domain using the homogeneity property of  $\dot{V}_j$ , and the closed-loop system (2.72)-(2.75), with the control  $U(t) = U_n(z_1(t), \dots, z_n(t))$ , is deduced to be **FTS**, in the light of Theorem A.3, where the last virtual controller  $U_n(z_1(t), \dots, z_n(t))$  is given in terms of all the previous virtual controllers  $U_i(z_1(t), \dots, z_i(t))$ ,  $i \in \{1, \dots, n-1\}$  which are defined as follows:

$$\begin{cases} U_1(z_1(t)) = 0, \quad i \in \{1, \dots, n-1\} & (2.76) \\ U_{i+1}(z_1(t), \dots, z_{i+1}(t)) = -l_{i+1} \left\{ \{z_{i+1}(t)\}^{m_i \alpha_i} - \{U_i(z_1(t), \dots, z_i(t))\}^{\frac{\bar{r}_i + 1 + k}{\bar{r}_i + 1 m_i \alpha_i}} \right\} & (2.77) \end{cases}$$

with  $\bar{r}_i, \alpha_{i-1}, i \in \{1, \dots, n\}$  and  $k$  be constants satisfying the following given inequalities:

$$\begin{aligned} \bar{r}_1 = 1, \dots, \bar{r}_i = \frac{\bar{r}_{i-1} + k}{m_{i-1}}, \bar{r}_i > -k > 0, \quad i \in \{1, \dots, n\}, \\ \alpha_0 = \bar{r}_2, (\alpha_i m_i + 1) \bar{r}_{i+1} \geq (\alpha_{i-1} m_{i-1} + 1) \bar{r}_i > 0, \quad i \in \{1, \dots, n-2\}, \alpha_{n-1} > 0, \end{aligned}$$

for  $m_0 = 1$  and for some positive constants  $l_i, i \in \{1, \dots, n\}$ .

### Remark 2.10

Note that for small  $n$ , the recursive approach is easy to handle and the resulting controller is simple. However, when  $n$  is large (e.g.  $n \geq 4$ ), the recursive approach becomes hard to use and the controller becomes complicated.

### Remark 2.11

It is possible to replace the odd integer exponents  $m_i, i \in \{1, \dots, n\}$ , by some positive real exponents  $\bar{m}_i \in \mathbb{R}_+, i \in \{1, \dots, n\}$ . However, for that, we need to replace the terms  $z_i(t)^{m_i-1}$  by  $\{z_i(t)\}^{m_i-1} = \text{sign}(z_i(t))|z_i(t)|^{m_i-1}$ .

#### 2.2.2.1 A homogeneity-based method

It is worth mentioning that if all the exponents  $m_i, i \in \{1, \dots, n\}$ , are equal to 1, then the previous result still holds. Note that even though this system looks easy to stabilize using the Lyapunov

analysis, sadly, if we take

$$U(t) = \sum_{j=1}^n k_j \{z_j(t)\}^{\alpha_j}, \quad (2.78)$$

for  $k_j < 0$ ,  $\alpha_j \in (0, 1)$ ,  $j \in \{1, \dots, n\}$ , we can not find an explicit Lyapunov function, when  $n > 2$  without using the recursive method. For the particular case of  $n = 2$  with  $\alpha_1 = \frac{\alpha_2}{2 - \alpha_2}$ , one can use the following Lyapunov function

$$V(z_1, z_2) = \frac{-k_1(2 - \alpha_2)}{2} |z_1|^{\frac{2}{2 - \alpha_2}} + \frac{z_2^2}{2}. \quad (2.79)$$

which is continuously differentiable,  $\mathbf{r}$ -homogeneous of degree 2 (see Appendix A) and  $\dot{V}(z) = k_2 |z_2|^{1 + \alpha_2}$ . Since  $k_1 < 0$  and  $k_2 < 0$ , the function  $V$  is positive-definite, and  $\dot{V}$  is negative-semidefinite. A direct application of the LaSalle invariance principle shows that the origin of the double chain of integrators is globally **AS**. Being homogeneous of negative degree implies that the system is globally **FTS** (see Appendix A). If we want to prove the asymptotic stability without relying on the LaSalle principle, we have to change the Lyapunov function as Bernuau et al. showed in [118]. More precisely, if the control parameters  $k_1$  and  $k_2$  are taken such that:

$$k_2 < \frac{M}{M^2(1 - \alpha_2) - 1}, \quad (2.80)$$

and

$$k_1 \in \left[ \frac{1}{2} + \frac{(2 - \alpha_2)k_2}{2M(3 - \alpha_2)} - \frac{\sqrt{\Delta}}{2(3 - \alpha_2)}, M(1 - \alpha_2)k_2 \right], \quad (2.81)$$

with

$$\Delta = \left( (3 - \alpha_2) + \frac{(2 - \alpha_2)k_2}{M} \right)^2 + 4(1 - \alpha_2)(3 - \alpha_2)k_2^2, \quad (2.82)$$

and

$$M = \frac{(2 - \alpha_2)^{\frac{1 - \alpha_2}{2}} \alpha_2^{\frac{\alpha_2}{2}}}{2}, \quad (2.83)$$

then, one can ensure stabilization of the double chain of integrators in finite time. Moreover, the proof of stability property can be checked directly using a different Lyapunov function.

### 2.2.3 Time-Varying tools in finite-dimensional setting

In this section, we recall the first key contribution of prescribed-time stabilization for nonlinear systems which was achieved by Song et al. [98] (2017). In this result, Song et al. proved that one can design a control  $U(t)$ , by means of time-varying feedback, to ensure prescribed-time stabilization of the following class of triangular systems:

$$\begin{cases} \dot{z}_1(t) = z_2(t), & t \in [\bar{t}_0, \bar{t}_0 + T), & (2.84) \\ \vdots & \\ \dot{z}_{n-1}(t) = z_n(t), & t \in [\bar{t}_0, \bar{t}_0 + T), & (2.85) \\ \dot{z}_n(t) = f(z(t), t) + b(z(t), t)U(t), & t \in [\bar{t}_0, \bar{t}_0 + T), & (2.86) \\ z(\bar{t}_0) = z_0, & & (2.87) \end{cases}$$

where the state  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$ , the control  $U(t) \in \mathbb{R}$ , the initial condition  $z_0 \in \mathbb{R}^n$ , the initial time  $\bar{t}_0$ , the prescribed-time  $T \geq 0$ , and the functions  $b$ ,  $f$  are possible uncertain and non-vanishing such that:



- $\underline{b} \leq |b(\bar{z}, t)| < \infty$  for some known  $\underline{b} > 0$ , and for all  $\bar{z} \in \mathbb{R}^n$ ,  $t \in \mathbb{R}_+$ .
- $|f(\bar{z}, t)| \leq d(t)\psi(\bar{z})$ , where  $d(t)$  is a disturbance with an unknown bound  $\|d\|_{[\bar{t}_0, t]} := \sup_{\tau \in [\bar{t}_0, t]} |d(\tau)|$  and  $\psi(\bar{z}) \geq 0$  is a known scalar-valued continuous function.

It is worth mentioning that when  $b(\bar{z}, t)$  and  $f(\bar{z}, t)$  are known, Song et al., provided a similar proof in [98].

The main idea of the proof is the use of a scaling transformation involving a blow-up function that grows to infinity as the time gets closer to the settling time  $\bar{t}_0 + T$ . This function is given explicitly as follows :

$$\mu_1(s) = \frac{T}{T-s}, \quad s \in [0, T). \quad (2.88)$$

The following step is to design a controller that stabilizes the system in the scaled state representation. To conclude, they used the inverse transformation to transport the stability property from the scaled state representation to the original one.

To better illustrate the idea of Song et al. [98], let us focus on the scalar case,

$$\begin{cases} \dot{z}(t) = f(z(t), t) + b(z(t), t)U(t), & t \in [\bar{t}_0, \bar{t}_0 + T), \\ z(\bar{t}_0) = z_0, \end{cases} \quad (2.89)$$

$$(2.90)$$

where  $z(t) \in \mathbb{R}$  and  $U(t) \in \mathbb{R}$ . In this case, the scaling transformation is given by

$$\omega(t) = \mu(t - \bar{t}_0) z(t), \quad (2.91)$$

with

$$\mu(t - \bar{t}_0) = \frac{T^{1+m}}{(\bar{t}_0 + T - t)^{1+m}} = \mu_1(t - \bar{t}_0)^{1+m} = \nu_1(t - \bar{t}_0)^{-1-m} = \nu(t - \bar{t}_0)^{-1}. \quad (2.92)$$

where  $m > 0$  is a positive constant and  $\nu$  is a monotonically decreasing function with the properties that  $\nu(0) = 1$  and  $\nu(T) = 0$ .

Using this transformation, the original scalar system (2.89)-(2.90) is converted into

$$\dot{\omega}(t) = \dot{\mu}(t - \bar{t}_0) \nu(t - \bar{t}_0) \omega(t) + \mu(t - \bar{t}_0) \left( f(\nu(t - \bar{t}_0) \omega(t), t) + b(\nu(t - \bar{t}_0) \omega(t), t) U(t) \right). \quad (2.93)$$

This last system is proven, using the Lyapunov function  $V(\omega(t)) = \frac{|\omega(t)|^2}{2}$ , to be prescribed-time input-to-state stable in time  $\bar{t}_0 + T$  and converge to zero under the following control (see [98, Proof of Theorem 4]):

$$U(t) = -\frac{1}{\underline{b}} \left( k + \lambda \psi(\nu(t - \bar{t}_0) \omega(t))^2 + \frac{1+m}{T} \right) \omega(t), \quad (2.94)$$

for some positive constants  $k > 0$  and  $\lambda > 0$ . Using the inverse transformation the control input is then expressed in terms of the original state  $z(t)$  as follows:

$$U(t) = -\frac{1}{\underline{b}} \left( k + \lambda \psi(z(t))^2 + \frac{1+m}{T} \right) \mu(t - \bar{t}_0) z(t), \quad (2.95)$$

and the prescribed-time input-to-state stability and the convergence to zero properties are transferred to (2.89)-(2.90). Moreover, the following inequality is recovered

$$|z(t)| \leq \nu(t - \bar{t}_0) \left( e^{-\frac{kT}{m+n-1}} (\mu_1(t - \bar{t}_0)^{m+n-1}) |z(\bar{t}_0)| + \frac{\|d\|_{[\bar{t}_0, t]}}{2\sqrt{k\lambda}} \right), \quad (2.96)$$

for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$ .

## 2.3 Non-Asymptotic concepts in an infinite-dimensional setting

In the framework of infinite-dimensional systems, all the definitions mentioned earlier in Subsection 2.2 can be adapted by substituting the Euclidean norm with an appropriate spatial norm, as we will explain below. In this section, we will focus on one-dimensional partial differential equations of the first order with respect to time. This class of PDEs can be described using an evolution equation of the form:

$$z_t(t, \cdot) = \mathcal{A}z(t, \cdot), \quad (2.97)$$

with  $t \geq \bar{t}_0 \geq 0$ , where  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(a, b) \rightarrow L^2(a, b)$  ( $a, b \in \mathbb{R}$ ,  $a \leq b$ ) is a (possibly unbounded) linear operator,  $\bar{t}_0$  is the initial time, and  $z_0$  will denote the initial condition.

### Example 2.11

Consider the following heat equation with homogeneous Dirichlet-type boundary conditions

$$\begin{cases} z_t(t, x) = z_{xx}(t, x), & t \geq \bar{t}_0, x \in [0, 1], \\ z(t, 0) = z(t, 1) = 0, & t \geq \bar{t}_0, \end{cases} \quad (2.98)$$

with  $\bar{t}_0 \geq 0$ . For this example, we can clearly rewrite (2.98)-(2.99) as in (2.97) by taking the linear operator  $\mathcal{A}$  as  $\mathcal{A} = \partial_{xx}$ , which corresponds to the second-order partial derivative with respect to the variable  $x$ , defined on the domain  $\mathcal{D}(\mathcal{A}) = \{z \in H^1(0, 1) : z_{xx} \in L^2(0, 1), z(0) = z(1) = 0\} := H^2(0, 1) \cap H_0^1(0, 1)$ .

Now, let us give the following global stability definition:

### Definition 2.3

The origin of system (2.97) is said to be

- **stable** [154, Definition 1.1.] if and only if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for every  $\bar{t}_0 \geq 0$  and  $z_0 \in L^2(a, b)$ ,

$$\left( \|z_0\|_{L^2(a,b)} \leq \delta \right) \implies (\|z(t, \cdot)\|_{L^2(a,b)} \leq \varepsilon),$$

for any  $t \geq \bar{t}_0$ ;

- **asymptotically stable (AS)**<sup>2</sup> if and only if it is stable and

$$\lim_{t \rightarrow +\infty} \|z(t, \cdot)\|_{L^2(a,b)} = 0,$$

for any  $z_0 \in L^2(a, b)$ ;

- **exponentially stable (ES)** if and only if it is AS and there exist  $M > 0$  and  $\beta > 0$  such that

$$\|z(t, \cdot)\|_{L^2(a,b)} \leq M \|z_0\|_{L^2(a,b)} e^{-\beta t},$$

for any  $t \geq \bar{t}_0$  and any  $z_0 \in L^2(a, b)$ ;

- **finite-time stable (FTS)** if and only if it is stable and for any  $z_0 \in L^2(a, b)$  there exists  $0 < T^{z_0} < +\infty$  such that  $\|z(t, \cdot)\|_{L^2(a,b)} = 0$  for all  $t \geq \bar{t}_0 + T^{z_0}$ . The functional  $T(z_0) = \inf \{T^{z_0} \geq 0 : \|z(t, \cdot)\|_{L^2(a,b)} = 0, \forall t \geq \bar{t}_0 + T^{z_0}\}$  defines the settling time of the system (2.97);

<sup>2</sup>one can use  $\mathcal{KL}$ -function (see [178, Definition 2.8.])

- *nearly fixed-time stable (nearly FxTS)* if and only if it is stable and for any  $\varrho > 0$  there exists  $0 < T_\varrho < +\infty$  such that  $\|z(t, \cdot)\|_{L^2(a,b)} \leq \varrho$  for all  $t \geq \bar{t}_0 + T_\varrho$  and all  $z_0 \in L^2(a, b)$ ;
- *fixed-time stable (FxTS)* if and only if it is **FTS** and  $\sup_{z_0 \in L^2(a,b)} T(z_0) < +\infty$ ;
- *prescribed-time stable (PTS)* if and only if it is **FxTS** and the settling time  $T$  is chosen in advance independently of the initial condition  $z_0 \in L^2(a, b)$ .

In the above definitions “uniformly” with respect to initial time  $\bar{t}_0$  and “global” with respect to initial conditions  $z_0$  have been omitted for the sake of brevity.

**Remark 2.12**

Using the comparison functions, it is possible to give equivalent definitions to Definition 2.3.

**Remark 2.13**

Similarly to the finite-dimensional case (see Remark 2.1), when studying the prescribed-time stability (**PTS**) notion, we will have to deal with a non-autonomous version of (2.97) of the form (see [179, Chapter 5]):

$$z_t(t, \cdot) = \mathcal{A}(t)z(t, \cdot), \tag{2.100}$$

where in this case  $\mathcal{A}(t) : \mathcal{D}(\mathcal{A}(t)) \subset L^2(a, b) \rightarrow L^2(a, b)$  ( $a, b \in \mathbb{R}, a \leq b$ ) is a time-dependent (possibly unbounded) linear operator (e.g.  $\mathcal{A}(t) = \partial_{xx} + a(t)\text{Id}$  where  $a : [\bar{t}_0, +\infty) \mapsto \mathbb{R}$  is a continuous function). As mentioned in Remark 2.1, we are interested, in this Thesis, in “uniform” stability properties with respect to  $\bar{t}_0$ . In this situation, the previously introduced stability definitions remain unchanged as presented for example in [154, Definition 1.1]). To avoid unnecessarily complicating the notations, we will only switch to non-autonomous systems when dealing with the **PTS** concept.

### 2.3.1 Lyapunov-based approach for PDEs

In this section, we present some sufficient conditions for the stability notions in the framework of infinite-dimensional systems.

Let  $\Omega \subset \mathcal{D}(\mathcal{A})$  is an open connected set containing the origin.

**Definition 2.4**

A continuous function  $V : \Omega \subset \mathcal{D}(\mathcal{A}) \rightarrow \mathbb{R}_+$  is said to be coercive if and only if there exist two class- $\mathcal{K}_\infty$  functions  $\sigma_1$  and  $\sigma_2$  such that

$$\sigma_1(\|z\|_{L^2(a,b)}) \leq V(z) \leq \sigma_2(\|z\|_{L^2(a,b)}), \tag{2.101}$$

for any  $z \in \Omega$ .

Based on Definition 2.3, the following results generalize the Lyapunov stability Theorem 2.1 for infinite-dimensional systems.

**Proposition 2.3** ([178, Proposition 3.2] )

Let  $V : \Omega \subset \mathcal{D}(\mathcal{A}) \rightarrow \mathbb{R}_+$  be a continuous function, continuously differentiable on  $\Omega \setminus \{0\}$ , satisfying the coercivity condition, such that

$$1. \dot{V}(z) = \left\langle \frac{\partial V(z)}{\partial z}, Az \right\rangle_{L^2(a,b)} \leq 0 \text{ in } \Omega.$$

Then, the origin of (2.97) is Lyapunov stable. Moreover, if there exists a class- $\mathcal{K}_\infty$  function  $\sigma_3$  such that

$$2. \dot{V}(z) \leq -\sigma_3(\|z\|_{L^2(a,b)}) \text{ in } \Omega,$$

then, the origin of (2.97) is asymptotically stable (**AS**).

### Corollary 2.2

Let  $V : \Omega \subset \mathcal{D}(A) \rightarrow \mathbb{R}_+$  be a continuous function, continuously differentiable on  $\Omega \setminus \{0\}$ , satisfying the coercivity condition, such that

$$1. \dot{V}(z) \leq -cV(z) \text{ in } \Omega, \text{ with some real positive constant } c > 0.$$

Then, the origin of (2.97) is exponentially stable (**ES**).

*Proof.* The proof is a direct application of Proposition 2.3 and the coercivity property of  $V$ .  $\square$

### Example 2.12

Consider the following parabolic reaction-diffusion PDE with a homogeneous Dirichlet-type boundary

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) - (\lambda_0 - \lambda)z(t, x), & t \geq \bar{t}_0, x \in [0, 1], \\ z(t, 0) = z(t, 1) = 0, & t \geq \bar{t}_0, \end{cases} \quad (2.102)$$

$$(2.103)$$

where the initial time  $\bar{t}_0 \geq 0$ , the reaction term  $(\lambda_0 - \lambda) > 0$  with  $\lambda, \lambda_0 \in \mathbb{R}$ .

Let  $V$  be given by

$$V(z(t, \cdot)) = \|z(t, \cdot)\|_{L^2(0,1)}^2 = \int_0^1 |z(t, x)|^2 dx. \quad (2.104)$$

Clearly,  $V$  is a continuous function, continuously differentiable, coercive, and positive-definite. Moreover, by computing the time derivative of  $V$  along the solution of (2.102)-(2.103), we get

$$\frac{d}{dt} V(z(t, \cdot)) = \int_0^1 z(t, x) (z_{xx}(t, x) - (\lambda_0 - \lambda)z(t, x)) dx, \quad (2.105)$$

$$\leq -(\lambda_0 - \lambda)V(z(t, \cdot)), \quad (2.106)$$

where the term  $\int_0^1 z(t, x) z_{xx}(t, x) dx$  can be proven to be less or equal to 0 by double integration by parts. Thus,  $V$  satisfies the conditions of Corollary 2.2 and (2.102)-(2.103) is **ES**.

### Example 2.13

Consider the following hyperbolic transport PDE with a homogeneous Dirichlet-type boundary

$$\begin{cases} z_t(t, x) = z_x(t, x), & t \geq \bar{t}_0, x \in [0, 1], \\ z(t, 1) = 0, & t \geq \bar{t}_0, \end{cases} \quad (2.107)$$

$$(2.108)$$

with  $\bar{t}_0 \geq 0$ . Let  $V$  be given by (see [180]):

$$V(z(t, \cdot)) = \int_0^1 e^{\sigma x} |z(t, x)|^2 dx, \quad \sigma > 0. \quad (2.109)$$

$V$  is a continuous function, continuously differentiable, coercive, positive-definite, and the time derivative of  $V$  along the solution of (2.107)-(2.108) satisfies,

$$\frac{d}{dt}V(z(t, \cdot)) = 2 \int_0^1 e^{\sigma x} z(t, x) z_x(t, x) dx, \quad (2.110)$$

$$= \int_0^1 e^{\sigma x} \frac{\partial |z(t, x)|^2}{\partial x} dx, \quad (2.111)$$

$$= -|z(t, 0)|^2 - \sigma \int_0^1 e^{\sigma x} |z(t, x)|^2 dx, \quad (2.112)$$

$$\leq -\sigma V(z(t, \cdot)), \quad (2.113)$$

$$\leq -\sigma \|z(t, \cdot)\|_{L^2(0,1)}^2, \quad (2.114)$$

where we integrated by parts and then used the coercivity of  $V$ . Then, from Proposition 2.3, (2.107)-(2.108) is **AS**. (Notice that from inequality (2.113), we can deduce that (2.107)-(2.108) is **ES**).

Now, let us give some Lyapunov characterization for finite/fixed/prescribed-time stability of PDEs:

**Proposition 2.4**

Let  $V : \Omega \subset \mathcal{D}(A) \rightarrow \mathbb{R}_+$  be a continuous function, continuously differentiable on  $\Omega \setminus \{0\}$ , satisfying the coercivity condition, such that

1.  $\dot{V}(z) \leq 0$  in  $\Omega$ .
2. there exists  $0 \leq T^{V(z_0)} < +\infty$  such that  $V(z(t, \cdot)) = 0$  for all  $t \geq T^{V(z_0)}$ .

Then, the origin of (2.97) is **FTS** and the settling time  $T(V(z_0))$  is defined similarly as in Definition 2.3. Moreover, if

3.  $\sup_{z_0 \in \Omega} T(V(z_0)) < +\infty$ .

Then, the origin of (2.97) is **FxTS**.

*Proof.* The proof of Proposition 2.4 is a direct application of Definition 2.4. □

**Remark 2.14**

Note that if one can find a suitable coercive Lyapunov function then, using the Comparison principle, one may reduce the complexity of the stability analysis to the study of a simple scalar ODE (2.1) (i.e.  $\dot{z} = -\mathcal{K}(z)$ ,  $z \in \mathbb{R}$  where  $\mathcal{K} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function with  $\mathcal{K}(0) = 0$  given in (2.1)).

In light of Remarks 2.4 and 2.14, we present the following result:

**Proposition 2.5**

Under the conditions of Proposition 2.4, if there exists a continuous function  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that

$$\frac{d}{dt}V(z(t, \cdot)) \leq -\mathcal{K}(V(z(t, \cdot))), \quad (2.115)$$

and  $\mathcal{K} \in \mathcal{E}_{k_0, a_0}$  (resp.  $\mathcal{K} \in \mathcal{E}_{k_0, a_0}^{k_\infty, a_\infty}$ ), given in Remark 2.4, with  $a_0 \in [0, 1)$  (resp.  $a_0 \in [0, 1), a_\infty > 1$ ), then the origin of (2.97) is **FTS** (resp. **FxTS**).

**Example 2.14 (FTS)**

Inspired from [169], [181]–[183], consider the following reaction-diffusion PDE with a homogeneous Dirichlet-type boundary

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) - \frac{c}{2} \frac{z(t, x)}{\|z(t, \cdot)\|_{L^2(0,1)}^{2-2\alpha}}, & t \geq \bar{t}_0, x \in [0, 1], \\ z(t, 0) = z(t, 1) = 0, & t \geq \bar{t}_0, \end{cases} \quad (2.116)$$

with  $\bar{t}_0 \geq 0$ ,  $c > 0$  and  $\alpha \in (0, 1)$ . Let  $V$  be given by

$$V(z(t, \cdot)) = \|z(t, \cdot)\|_{L^2(0,1)}^2 = \int_0^1 |z(t, x)|^2 dx. \quad (2.118)$$

Clearly,  $V$  is a continuous function, continuously differentiable, coercive, and positive-definite. Moreover, by computing the time derivative of  $V$  along the solution of (2.116)–(2.117), we get

$$\frac{d}{dt} V(z(t, \cdot)) = 2 \int_0^1 z(t, x) \left( z_{xx}(t, x) - \frac{c}{2} \frac{z(t, x)}{\|z(t, \cdot)\|_{L^2(0,1)}^{2-2\alpha}} \right) dx, \quad (2.119)$$

$$= 2 \int_0^1 z(t, x) z_{xx}(t, x) dx - c \frac{1}{\|z(t, \cdot)\|_{L^2(0,1)}^{2-2\alpha}} \int_0^1 |z(t, x)|^2 dx, \quad (2.120)$$

$$= 2 \int_0^1 z(t, x) z_{xx}(t, x) dx - c \|z(t, \cdot)\|_{L^2(0,1)}^{2\alpha}, \quad (2.121)$$

$$\leq -cV(z(t, \cdot))^\alpha, \quad (2.122)$$

where the  $\int_0^1 z(t, x) z_{xx}(t, x) dx$  can be proven to be less or equal to 0 by double integration by parts. Then,  $V$  satisfies the conditions of Proposition 2.5 for  $\mathcal{K} \in \mathcal{E}_{k_0, a_0} : s \mapsto cs^\alpha$  which means that (2.116)–(2.117) is **FTS**.

To finish this section, let us give a Lyapunov characterization of prescribed-time stability for non-autonomous PDEs of the form (2.100),

**Proposition 2.6**

Let  $T > 0$  be chosen a priori and let  $V : [\bar{t}_0, \bar{t}_0 + T) \times \Omega \rightarrow \mathbb{R}$  be a continuously differentiable function such that

1.  $\dot{V}(t, z) \leq -c(t - \bar{t}_0)V(t, z)$  in  $[\bar{t}_0, \bar{t}_0 + T) \times \Omega$  for some continuous function  $c : [\bar{t}_0, \bar{t}_0 + T) \rightarrow \mathbb{R}_+$  such that  $\int_{\bar{t}_0}^t c(s - \bar{t}_0) ds$  is finite for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$  and becomes unbounded as  $t$  approaches  $\bar{t}_0 + T$ .

Then, the origin of (2.97) is **PTS** in a time  $\bar{t}_0 + T$ .

*Proof.* The proof is a direct consequence of Grönwall's lemma applied to Condition 2,

$$V(t, z) \leq V(\bar{t}_0, z_0) e^{-\int_{\bar{t}_0}^t c(s - \bar{t}_0) ds}. \quad (2.123)$$

□

**Remark 2.15**

Note that, in Example 2.13, (2.107)–(2.108) can be proven to be **FxTS** either by computing the

explicit solution via the characteristic method or using a Lyapunov function (see [160] for more details). However, finding a Lyapunov function satisfying the conditions of Proposition 2.5 is still an open problem.

**Example 2.15 (PTS)**

Inspired from [165], consider the following reaction-diffusion PDE with a homogeneous Dirichlet-type boundary and time-varying damping term:

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) - (\gamma_m(t - \bar{t}_0) - \lambda)z(t, x), & t \in [\bar{t}_0, \bar{t}_0 + T), x \in [0, 1], \\ z(t, 0) = z(t, 1) = 0, & t \in [\bar{t}_0, \bar{t}_0 + T), \end{cases} \quad (2.124)$$

$$(2.125)$$

where  $\bar{t}_0 \geq 0$ ,  $T > 0$  is a priori fixed and the function  $\gamma_m(\cdot)$  is chosen as follows:

$$\gamma_m(t - \bar{t}_0) := \frac{\gamma_{m,0}^m T^m}{(\bar{t}_0 + T - t)^m}, \quad m \in \mathbb{N}^*, \quad m \neq 1, \quad (2.126)$$

for  $\gamma_{m,0} > 0$ . Let  $V$  be given by

$$V(z(t, \cdot)) = \|z(t, \cdot)\|_{L^2(0,1)}^2 = \int_0^1 |z(t, x)|^2 dx. \quad (2.127)$$

Clearly,  $V$  is a continuous function, continuously differentiable, coercive, and positive-definite. Moreover, by computing the time derivative of  $V$  along the solution of (2.124)-(2.125), we get

$$\frac{d}{dt}V(z(t, \cdot)) = 2 \int_0^1 z(t, x) (z_{xx}(t, x) - (\gamma_m(t - \bar{t}_0) - \lambda)z(t, x)) dx, \quad (2.128)$$

$$\leq -2(\gamma_m(t - \bar{t}_0) - \lambda)V(z(t, \cdot)), \quad (2.129)$$

where the term  $\int_0^1 z(t, x)z_{xx}(t, x)dx$  can be proven to be less or equal to 0 by double integration by parts. Moreover, we have

$$\int_{\bar{t}_0}^t (\gamma_m(s - \bar{t}_0) - \lambda) ds = \left[ \frac{\gamma_{m,0} T}{m-1} \gamma_m(s - \bar{t}_0)^{\frac{m-1}{m}} - \lambda s \right]_{\bar{t}_0}^t, \quad (2.130)$$

$$= \frac{\gamma_{m,0} T}{m-1} \left( \gamma_m(t - \bar{t}_0)^{\frac{m-1}{m}} - \gamma_{m,0}^{m-1} \right) - \lambda(t - \bar{t}_0) \longrightarrow +\infty, \quad (2.131)$$

as  $t \rightarrow \bar{t}_0 + T$ . Thus,  $V$  satisfies the conditions of Proposition 2.6 and (2.124)-(2.125) is **PTS**.

### 2.3.2 Finite/prescribed-time boundary stabilization key results for 1D reaction-diffusion PDEs

In this subsection, we present key results on finite-time and prescribed-time boundary stabilization for 1D reaction-diffusion PDEs. The main employed method is the backstepping approach. Before we enter into details, it is important to remember that the backstepping approach for PDEs has gained a lot of popularity in recent years due to its effectiveness in addressing the challenging problem of stabilizing PDEs. This technique provides a systematic and structured approach to designing feedback control laws that stabilize PDE systems by transforming the original PDEs into a suitable target system that satisfies the chosen stability property. To better illustrate this approach, let us briefly study first the problem of exponential boundary stabilization (inspired

from [184]) of 1D parabolic reaction-diffusion PDEs with boundary control:

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \geq \bar{t}_0, x \in [0, 1], & (2.132) \\ z(t, 0) = 0, & t \geq \bar{t}_0, & (2.133) \\ z(t, 1) = U(t), & t \geq \bar{t}_0, & (2.134) \\ z(\bar{t}_0, x) = z_0(x), & & (2.135) \end{cases}$$

with initial time  $\bar{t}_0 \geq 0$ , reaction term  $\lambda \in \mathbb{R}$ , state  $z(t, \cdot) \in \mathcal{D}(\mathcal{A}) := \{\bar{z} \in H^1(0, 1) : \bar{z}_{xx} \in L^2(0, 1), \bar{z}(0) = 0, \bar{z}(1) = U(t)\}$  where the operator  $\mathcal{A} = \frac{\partial^2}{\partial x^2} + \lambda \text{Id}$  with  $\frac{\partial^2}{\partial x^2}$  is the second-order partial derivative with respect to the space variable  $x$ ,  $\text{Id}$  is the identity operator, the control  $U(t) \in \mathbb{R}$ , and the initial condition  $z_0 \in \mathcal{D}(\mathcal{A})$ .

Notice that the open-loop system (2.132)-(2.135) (i.e.  $z(t, 1) = 0$ ) is unstable for any  $\lambda > \pi^2$ . Let us assume that  $\lambda > \pi^2$ , otherwise, there is no reason to study the stabilization problem.

We recall that the key steps of the Backstepping approach are as follows:

- The first and the most important step of this approach is to choose a suitable target system of the same type of (2.132)-(2.135). For instance, one can choose the following exponentially stable system (see Example 2.12):

$$\begin{cases} \omega_t(t, x) = \omega_{xx}(t, x) - \lambda_0 \omega(t, x), & t \geq \bar{t}_0, x \in [0, 1], & (2.136) \\ \omega(t, 0) = 0, & t \geq \bar{t}_0, & (2.137) \\ \omega(t, 1) = 0, & t \geq \bar{t}_0, & (2.138) \end{cases}$$

for  $\lambda_0 > 0$ .

- The next step is to use an invertible Volterra-type transformation to link (2.132)-(2.135) to the target system (2.136)-(2.138)

$$\omega(t, x) = z(t, x) - \int_0^x K^{\text{exp}}(x, y) z(t, y) dy. \quad (2.139)$$

This transformation makes use of some function  $K^{\text{exp}}$  called the kernel and it satisfies some conditions (see Remark 7.3 for more details about  $K^{\text{exp}}$ ). To find these conditions it is sufficient to substitute (2.139) into the target system (2.136)-(2.138) and to use of the original system (2.132)-(2.135).

The inverse transformation is given similarly by

$$z(t, x) = \omega(t, x) + \int_0^x L^{\text{exp}}(x, y) \omega(t, y) dy. \quad (2.140)$$

- The final step is to recover the expression of the control either from the direct transformation (2.139), for  $x = 1$ , as follows:

$$U_{\text{exp}}(t) = \int_0^1 K^{\text{exp}}(1, y) z(t, y) dy, \quad (2.141)$$

or from the inverse transformation, for  $x = 1$ , as follows:

$$U_{\text{exp}}(t) = \int_0^1 L^{\text{exp}}(1, y) \omega(t, y) dy, \quad (2.142)$$

and use the direct transformation (2.139) and inverse transformation (2.140) to recover some equivalence inequalities between the norm of the target system (2.136)-(2.138) and the original system (2.132)-(2.135).



From finite-time to prescribed-time stabilization of 1D reaction-diffusion PDEs

As we discussed earlier on, for parabolic PDEs, the backstepping approach has contributed significantly to solving the problem of prescribed-time stabilization. The first result related to this topic was introduced in 2017 by Coron and Nguyen [161] where they solved the problems of null controllability followed by the problem of finite-time stabilization for a general class of parabolic PDEs with variable coefficients in space, by making use of some periodic time-varying feedback laws.

To illustrate the key points of their contribution, let us focus on the reaction-diffusion equation (2.132)-(2.135) defined on  $[\bar{t}_0, \bar{t}_0 + T)$  where  $\bar{t}_0 + T$  will be considered as the settling time. For the null controllability problem, the idea is to consider a strictly increasing sequence of real numbers  $(t_j)_{j \in \mathbb{N}}$  defined such as  $\lim_{j \rightarrow +\infty} t_j = \bar{t}_0 + T$  (e.g.  $t_0 = \bar{t}_0, t_j = (\bar{t}_0 + T) \left(1 - \frac{1}{2^j}\right)$  for  $j \in \mathbb{N}^*$ ) and to apply the backstepping approach (whose steps we recalled above), on each time interval  $[t_j, t_{j+1})$ . To be precise, we choose the target system to be the same as (2.136)-(2.138). The only difference is the term  $\lambda_0$  is replaced by  $\lambda_{0,j}$  on each time interval where the sequence  $(\lambda_{0,j})_{j \in \mathbb{N}}$  is chosen to be an increasing sequence of positive numbers converging to infinity (e.g.  $\lambda_{0,0} = \lambda_0, \lambda_{0,j} = (j + 1)^8$  for  $j \in \mathbb{N}^*$ ). After that, one uses backstepping transformations (2.139) and (2.140) where we replace, respectively, the kernels  $K^{\text{exp}}(x, y)$  and  $L^{\text{exp}}(x, y)$  by  $K_j^{\text{FTS}}(x, y)$  and  $L_j^{\text{FTS}}(x, y)$  on each time interval. Making use of these kernels, the boundary control is defined as a sequence of controllers

$$U_{\text{FTS}}(t) = \int_0^1 K_j^{\text{FTS}}(1, y)z(t, y)dy, \tag{2.143}$$

for every  $t \in [t_j, t_{j+1})$ ,  $j \in \mathbb{N}$ . Under this boundary control, the authors were able to prove that the solution of the closed-loop system (2.132)-(2.135) satisfies the following property:

$$\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow 0, \tag{2.144}$$

as  $t \rightarrow T$ , for any initial condition  $z_0 \in L^2(0, 1)$  with  $\|z_0\|_{L^2(0,1)} \leq M$  for some  $M > 0$  (see [161, Theorem 1]). In particular, they proved that

$$|U_{\text{FTS}}(t)| \rightarrow 0, \tag{2.145}$$

as  $t \rightarrow T$ .

Furthermore, the authors proved that using the following modified version of (2.143):

$$U_{\text{FTS}}(t) = \begin{cases} \int_0^1 K_j^{\text{FTS}}(1, y)z(t, y)dy, & t \in [t_j, t_{j+1}), j \in \mathbb{N}, j \leq N - 1, \\ \phi_j(\|z(t, \cdot)\|_{L^2(0,1)}) \int_0^1 K_j^{\text{FTS}}(1, y)z(t, y)dy, & t \in [t_j, t_{j+1}), j \in \mathbb{N}, j \geq N. \end{cases} \tag{2.146}$$

the original system (2.132)-(2.135) is **FTS** (see [161, Theorem 2]), where  $N$  is a fixed chosen positive large integer, and  $(\phi_j)_{j \in \mathbb{N}}$  is a sequence of class  $C^1(\mathbb{R})$ -functions such that  $0 \leq \phi_j \leq 1$ ,  $\phi_j(s) = 1$  for  $s \leq \mu_j$  and  $\phi_j(s) = 0$  if  $s \geq 2\mu_j$  with  $(\mu_j)_{j \in \mathbb{N}}$  be defined by  $\mu_n := e^{-j^\alpha}, \forall j \in \mathbb{N}$  for some real number  $\alpha \in (4, 5)$ .

Inspired by this result and with the emergence of the prescribed-time stability concept in [98], Espitia et al. [165] solved the problem of prescribed-time stabilization of (2.132)-(2.135) (restricted to  $[\bar{t}_0, \bar{t}_0 + T) \times [0, 1]$ ) using a time-varying backstepping approach. This approach proposes to replace the constant reaction term in the target system (2.136)-(2.138) by a strictly increasing time-varying reaction term  $t \rightarrow \gamma_m(t - \bar{t}_0)$  that goes to infinity when  $t$  approaches the terminal time  $\bar{t}_0 + T$  (see Example 2.15) and to replace the kernels  $K^{\text{exp}}(x, y)$  and  $L^{\text{exp}}(x, y)$  by

$K_2^{\text{PTS}}(1, y, t - \bar{t}_0)$  and  $L_2^{\text{PTS}}(1, y, t - \bar{t}_0)$  in the backstepping transformations (2.139) and (2.140). By doing this, the closed-loop system (2.132)-(2.135) with the resulting boundary time-varying feedback

$$U(t) = \int_0^1 K_2^{\text{PTS}}(1, s, t - \bar{t}_0)z(t, s)ds, \quad (2.147)$$

or equivalently

$$U(t) = \int_0^1 L_2^{\text{PTS}}(1, s, t - \bar{t}_0)\omega(t, s)ds. \quad (2.148)$$

is ensured to be **PTS** (see [165, Theorem 3]) provided that

$$2\gamma_{2,0}T > 1. \quad (2.149)$$

In particular,  $\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$ , and  $|U(t)| \rightarrow 0$  when  $t \rightarrow \bar{t}_0 + T$ .

## 2.4 Positioning of our contributions with respect to results for systems which are of the same class as those addressed in this thesis

Having completed a literature review on the most relevant works on Non-Asymptotic concepts for some classes of infinite-dimensional systems, and having presented the background and the main tools that we rely on, we end this chapter by positioning our contributions (summarized in Chapter 1, Section 1.3) with respect to up-to-now-known results of finite/fixed/prescribed-time stabilization/estimation for 3 different classes of differential equations including LTI systems with input delay or output delay, 1D reaction-diffusion PDE with boundary control, and 1D reaction-diffusion PDE with boundary input delay. In the table below, we refer to our contributions using ‘**Ci**’, and to extensions that can be possibly achieved using our contributions using the ‘✓’ symbol. The open problems are left ‘blank’.

Type of system	Type of results	FTS	FxTS	PTS
LTI systems subject to input/sensor delay	Stabilization	[142] <b>C2)</b>	[143], [144] <b>C2)</b>	[145], [176] <b>C4)</b>
	Estimation	<b>C1)</b>	<b>C1)</b>	[185]
	Output-feedback stabilization	✓	✓	✓
One-dimensional reaction-diffusion PDE with boundary control	Stabilization	[161], [162] <b>C3)</b>	<b>C3)</b>	[165]
	Estimation			[166]
	Output-feedback stabilization			[166]
One-dimensional reaction-diffusion PDE with boundary input delay	Stabilization	✓	✓	<b>C5)</b>
	Estimation			[166]
	Output-feedback stabilization			<b>C5)</b>

Table 2.1: Positioning of our contributions with respect to existing Non-Asymptotic stabilization/estimation results and open problems for some classes of differential equations.



## Part I

# Non-smooth tools for finite/fixed-time stabilization and estimation in infinite-dimensional settings



# Finite/fixed-time estimation of LTI systems with pointwise input delay

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## Contents

<b>3.1</b>	<b>Introduction</b>	<b>47</b>
<b>3.2</b>	<b>Problem Statement</b>	<b>48</b>
3.2.1	Characterisation of the observer-gain functions for second-order LTI systems	51
3.2.2	Characterisation of the observer-gain functions for nth-order LTI systems	51
<b>3.3</b>	<b>Stability analysis</b>	<b>53</b>
<b>3.4</b>	<b>Simulation</b>	<b>67</b>
3.4.1	An academic example	67
3.4.2	Application to a simplified robot manipulator	67

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In this chapter, we design a nonlinear observer for linear time-invariant (LTI) systems in the observable form with measurement delay. The design guarantees the convergence of the error state to the origin within a finite time that depends on the initial conditions. To accomplish this, we reformulate the original system into a cascade ODE-PDE system where the PDE part is a transport equation that models the effect of the delay on the output. We construct the nonlinear gains in a way that ensures the error system is finite-time stable (FTS). To prove this, we use an invertible backstepping transformation to convert the error system into a target system which is shown to be finite-time stable using Lyapunov-based analysis and homogeneity tools. We use the inverse transformation to transfer this property to the error system.

## 3.1 Introduction

State estimation is one of the most important topics in control theory. Indeed, usually, the full state is not available and/or sensors costs are prohibitive. Moreover, in engineering applications, delays are often in different parts of the system. In particular, the delay can be caused by transmitting the state's measurements via a communication network. Furthermore, finite-time observation becomes very desired in several applications that require the transient process to finish in a finite amount of time.

For instance, in Teleoperation, the surgery is performed using a robotic hand controlled by the doctor using a remote control. The robotic hand consists of a number of rigid links connected with joints. Using the Euler-Lagrange equation the dynamic model of the robotic hand is given by

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = U(t - D), \quad (3.1)$$

where  $U$  is the applied torque generated by the actuators,  $q \in \mathbb{R}^n$ ,  $\dot{q} \in \mathbb{R}^n$  and  $\ddot{q} \in \mathbb{R}^n$  are respectively vectors of joint positions, velocities, and accelerations,  $M(q)$  is the inertia matrix,  $C(q, \dot{q})$  the vector of Coriolis and centrifugal forces,  $G(q)$  the vector of gravitational forces, and  $D$  is the delay. In this situation, transmitting the measurements through a communication channel causes a time delay in reception and response. In addition, using a finite-time observer in this situation is critical, as the surgery requires as much precision in the control objective, and also it ends in a finite period.

In this chapter, we design a robust nonlinear observer which compensates for the delay and that is able to estimate the sensor dynamics (infinite-dimensional), as well. Indeed, we study the problem of finite-time observation for LTI systems in the observable form with a delay in the output measurement. To solve this problem, we first rewrite the original system into an ODE-PDE form, where the PDE part models the effect of the delay on the output. Next, we propose an observer with nonlinear injection terms, that reconstruct the system's states in a finite time. To achieve this, we use an invertible backstepping transformation to transform the error system into a finite-time stable target system. Finally, we use the inverse transformation to transfer this property to the error system.

This chapter is organized as follows: In Section 3.2, we introduce the main class of systems that we are interested in (LTI systems with output delay), and we rewrite it into an ODE-PDE setting. Next, we present our nonlinear finite-time observer from which we recover the error system. Then, we use a suitable choice of transformation to link this error system to a well-chosen target system. In Section 3.3, we make use of the transformation and the fact that the target system is **F**T**S** to prove the main result of this chapter. In Section 3.4 we consider a numerical example to illustrate the results.

## 3.2 Problem Statement

Let us consider an  $n$ th-order linear time-invariant system with delayed output. We assume that delay-free system is observable. Then one can rewrite the system in the form

$$\begin{cases} \dot{z}(t) = Az(t) + BU(t), & t \geq t_0, \\ Y(t) = Cz(t - D), & t \geq t_0, \end{cases} \quad (3.2)$$

where

$$A = \begin{bmatrix} a_{n-1} & 1 & 0 & \cdots & 0 \\ a_{n-2} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C = [1 \ 0 \ \cdots \ 0],$$

are the system's matrices ( $[a_0, \dots, a_{n-1}]^T \in \mathbb{R}^n$ ),  $z(t) = [z_1(t), \dots, z_n(t)]^T \in \mathbb{R}^n$  ( $n \in \mathbb{N}^*$ ) is the state with initial condition  $z(\theta) = z_0(\theta) = [z_{1,0}(\theta), \dots, z_{n,0}(\theta)]^T$  for all  $\theta \in [t_0 - D, t_0]$ , where  $t_0 \geq 0$  is the initialization time,  $U(t) \in \mathbb{R}$  is the input signal, and  $Y(t) \in \mathbb{R}$  is the measurement which is delayed by  $D > 0$  units of time.

Our goal is to design an observer for (3.2)-(3.3) which guarantees that the error state converges in a finite time to zero. To do so, we start by rewriting (3.2)-(3.3) as an ODE-PDE cascade setting,

from [69, Chapter 3], as follows:

$$\begin{cases} \dot{z}(t) = Az(t) + BU(t), & t \geq t_0, & (3.4) \\ u_t(t, x) = u_x(t, x), & t \geq t_0, x \in [0, D], & (3.5) \\ u(t, D) = Cz(t), & t \geq t_0, & (3.6) \\ Y(t) = u(t, 0), & t \geq t_0, & (3.7) \end{cases}$$

where the initial condition of (3.5)-(3.6) is taken as  $u(t_0, x) = Cz(t_0 - D + x)$  for all  $t_0 \geq 0$ .

Next, we propose the following observer system for (3.4)-(3.7):

$$\begin{cases} \dot{\hat{z}}(t) = A\hat{z}(t) + BU(t) - e^{AD}\mathcal{L}(Y(t) - \hat{u}(t, 0)), & t \geq t_0, & (3.8) \\ \hat{u}_t(t, x) = \hat{u}_x(t, x) - Ce^{Ax}\mathcal{L}(Y(t) - \hat{u}(t, 0)), & t \geq t_0, x \in [0, D], & (3.9) \\ \hat{u}(t, D) = C\hat{z}(t), & t \geq t_0, & (3.10) \end{cases}$$

where the initial condition is given by  $\hat{z}(\theta) = \hat{z}_0(\theta) = [\hat{z}_{1,0}(\theta), \dots, \hat{z}_{n,0}(\theta)]^\top$  for all  $\theta \in [t_0 - D, t_0]$  and the observer gain function  $\mathcal{L}(\cdot) = [\mathcal{L}_1(\cdot), \dots, \mathcal{L}_n(\cdot)]^\top$  will be designed later. Note that usually in practice, we take  $\hat{z}(\theta) = 0$ .

### Remark 3.1

By using classical predictor techniques, we can replace our observer (3.8)-(3.10) with the following observer (see [69, Chapter 3, page 46]):

$$\dot{\hat{z}}(t) = A\hat{z}(t) + BU(t) - e^{AD}\bar{\mathcal{L}}\left(Y(t) - Ce^{-AD}\hat{z}(t) + C \int_{t-D}^t e^{A(t-D-\theta)} BU(\theta) d\theta\right), \quad (3.11)$$

where the nonlinear function  $\bar{\mathcal{L}} = [\bar{\mathcal{L}}_1(\cdot), \dots, \bar{\mathcal{L}}_n(\cdot)]^\top$  can be designed in the same way as in (3.8)-(3.10). Such classical predictor-based techniques have the following drawbacks:

1. a predictor-based observer is a reduced-order observer (in the sense of [69, Chapter 3, page 46]) because it does not estimate the sensor state  $Y(t+x)$ ,  $x \in [0, D]$ , with  $D$  is the delay, whereas our observer (3.8)-(3.10) ensures finite-time estimation of the sensor state  $Y(t+x)$ ,  $x \in [0, D]$ : this is an important disadvantage because it is well known that in contrast to full-order observers, reduced-order observers are overly sensitive to measurement noise. [69, Chapter 3, page 46]
2. predictor-based observers do not have a straightforward extension to systems with time-varying delays, state-dependent delays, or distributed delays, whereas our formulation opens a path for extending the obtained results to these kinds of systems and more general ones.
3. lastly, predictor-based observers do not provide a construction of the Lyapunov Krassovski functional, which allows us to assess directly the desired **FTS** property of the error system and to estimate the time of convergence (the settling time), whereas our observer helps in establishing an explicit Lyapunov-based stability for the error system.

Our goal is to prove that, for a well-designed vector function  $\mathcal{L}$ ,  $[\hat{z}(t), \hat{u}(t, \cdot)]$  solution of the observer (3.8)-(3.10) converges in a finite-time to  $[z(t), u(t, \cdot)]$  solution of (3.4)-(3.7). To achieve this, we consider the following error variables:

$$\begin{cases} \tilde{z}(t) = e^{-AD}(z(t) - \hat{z}(t)), & (3.12) \\ \tilde{u}(t, x) = u(t, x) - \hat{u}(t, x), & (3.13) \end{cases}$$



from which the following error system is derived:

$$\begin{cases} \dot{\tilde{z}}(t) = A\tilde{z}(t) + \mathcal{L}(\tilde{u}(t, 0)), & t \geq t_0, & (3.14) \\ \tilde{u}_t(t, x) = \tilde{u}_x(t, x) + Ce^{Ax}\mathcal{L}(\tilde{u}(t, 0)), & t \geq t_0, x \in [0, D], & (3.15) \\ \tilde{u}(t, D) = Ce^{AD}\tilde{z}(t), & t \geq t_0, & (3.16) \end{cases}$$

with initial condition  $\tilde{z}(\theta) = \tilde{z}_0(\theta) = [\tilde{z}_{1,0}(\theta), \dots, \tilde{z}_{n,0}(\theta)]^\top$  for all  $\theta \in [t_0 - D, t_0]$ .

Next, we consider the following transformation:

$$\tilde{\omega}(t, x) = \tilde{u}(t, x) - Ce^{Ax}\tilde{z}(t), \quad (3.17)$$

from [69, Chapter 3] which transforms (3.14)-(3.16) into the following target system:

$$\begin{cases} \dot{\tilde{z}}(t) = A\tilde{z}(t) + \mathcal{L}(\tilde{\omega}(t, 0) + C\tilde{z}(t)), & t \geq t_0, & (3.18) \\ \tilde{\omega}_t(t, x) = \tilde{\omega}_x(t, x), & t \geq t_0, x \in [0, D], & (3.19) \\ \tilde{\omega}(t, D) = 0, & t \geq t_0, & (3.20) \end{cases}$$

which is equivalent to the following system:

$$\begin{cases} \dot{\tilde{z}}_1(t) = a_{n-1}\tilde{z}_1(t) + \tilde{z}_2(t) + \mathcal{L}_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), & t \geq t_0, & (3.21) \\ \vdots & & \\ \dot{\tilde{z}}_{n-1}(t) = a_1\tilde{z}_1(t) + \tilde{z}_n(t) + \mathcal{L}_{n-1}(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), & t \geq t_0, & (3.22) \\ \dot{\tilde{z}}_n(t) = a_0\tilde{z}_1(t) + \mathcal{L}_n(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), & t \geq t_0, & (3.23) \\ \tilde{\omega}_t(t, x) = \tilde{\omega}_x(t, x), & t \geq t_0, x \in [0, D], & (3.24) \\ \tilde{\omega}(t, D) = 0, & t \geq t_0. & (3.25) \end{cases}$$

In order to characterize transformation (3.17), let us compute its time and spatial derivatives and the value of  $\tilde{\omega}(t, D)$ .

$$\tilde{\omega}_t(t, x) = \tilde{u}_t(t, x) - Ce^{Ax}\dot{\tilde{z}}(t), \quad (3.26)$$

$$= \tilde{u}_x(t, x) + Ce^{Ax}\mathcal{L}(\tilde{u}(t, 0)) - Ce^{Ax}A\tilde{z}(t) - Ce^{Ax}\mathcal{L}(\tilde{u}(t, 0)), \quad (3.27)$$

$$= \tilde{u}_x(t, x) - Ce^{Ax}A\tilde{z}(t). \quad (3.28)$$

$$(3.29)$$

On the other hand,

$$\tilde{\omega}_x(t, x) = \tilde{u}_x(t, x) - CAe^{Ax}\tilde{z}(t), \quad (3.30)$$

$$= \tilde{\omega}_t(t, x), \quad (3.31)$$

where we have used the fact that  $Ae^{Ax} = e^{Ax}A$ . Using  $\tilde{u}(t, D) = Ce^{AD}\tilde{z}(t)$  in (3.17) at  $x = D$ , we obtain

$$\tilde{\omega}(t, D) = \tilde{u}(t, D) - Ce^{AD}\tilde{z}(t), \quad (3.32)$$

$$= Ce^{AD}\tilde{z}(t) - Ce^{AD}\tilde{z}(t), \quad (3.33)$$

$$= 0. \quad (3.34)$$

### 3.2.1 Characterisation of the observer-gain functions for second-order LTI systems

Choosing  $\mathcal{L}_1$  and  $\mathcal{L}_2$  respectively as follows:

$$\mathcal{L}_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) = -k_1 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_1} - a_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \quad (3.35)$$

and

$$\mathcal{L}_2(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) = -k_2 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_2} - a_0(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \quad (3.36)$$

gives us

$$\begin{cases} \dot{\tilde{z}}_1(t) = \tilde{z}_2(t) - k_1 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_1} - a_1 \tilde{\omega}(t, 0), & t \geq t_0, & (3.37) \\ \dot{\tilde{z}}_2(t) = -k_2 \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_2} - a_0 \tilde{\omega}(t, 0), & t \geq t_0, & (3.38) \\ \tilde{\omega}_t(t, x) = \tilde{\omega}_x(t, x), & t \geq t_0, x \in [0, D], & (3.39) \\ \tilde{\omega}(t, D) = 0, & t \geq t_0, & (3.40) \end{cases}$$

where  $k_1 > 0$ ,  $k_2 > 0$ ,  $\alpha_1 \in (\frac{1}{2}, 1)$ , and  $\alpha_2 \in (0, 1)$ . Note that  $\tilde{\omega}(t, 0) + \tilde{z}_1(t) = Y(t)$  thus  $\mathcal{L}_1(\cdot)$  and  $\mathcal{L}_2(\cdot)$  are only functions of  $Y(t)$ .

#### Remark 3.2

Note that, the idea behind choosing  $\mathcal{L}_1$  and  $\mathcal{L}_2$  as in (3.35)-(3.36) is to make the ODE part of the target error system (i.e. (3.21)-(3.23)) **FTS** when  $\tilde{\omega}(t, 0)$  becomes zero. Similarly, if we want to ensure fixed-time stability (**FxTS**) of (3.21)-(3.23) it is sufficient to replace  $\mathcal{L}_1$  and  $\mathcal{L}_2$  respectively by

$$\begin{aligned} \mathcal{L}_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) &= -k_{1,0} \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_{1,0}} - k_{1,\infty} \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_{1,\infty}} \\ &\quad - a_1(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \end{aligned} \quad (3.41)$$

and

$$\begin{aligned} \mathcal{L}_2(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) &= -k_{2,0} \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_{2,0}} - k_{2,\infty} \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_{2,\infty}} \\ &\quad - a_0(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \end{aligned} \quad (3.42)$$

for some  $k_{1,0} > 0$ ,  $k_{2,0} > 0$ ,  $k_{1,\infty} > 0$ ,  $k_{2,\infty} > 0$ ,  $\alpha_{1,0} \in (\frac{1}{2}, 1)$ ,  $\alpha_{2,0} \in (0, 1)$ ,  $\alpha_{1,\infty} > 1$ , and  $\alpha_{2,\infty} > 1$ .

### 3.2.2 Characterisation of the observer-gain functions for nth-order LTI systems

In the general case, we can choose the observer-gain functions in a similar fashion to the second-order case:

$$\mathcal{L}_i(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) = -k_i \{\tilde{\omega}(t, 0) + \tilde{z}_1(t)\}^{\alpha_i} - a_{n-i}(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), i \in \{1, \dots, n\}. \quad (3.43)$$

for some positive coefficients  $k_i > 0$ ,  $i \in \{1, \dots, n\}$  and some constants  $\alpha_i \in (0, 1)$ ,  $i \in \{1, \dots, n\}$ . However, in this case, the finite-time stability analysis of the target system when  $\omega \equiv 0$  will be hard. To avoid this, we can propose instead (see [186, Theorem 6]):

$$\begin{aligned} \mathcal{L}(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) &= (\tilde{p} |\tilde{\omega}(t, 0) + \tilde{z}_1(t)|)^{\frac{\nu-\epsilon}{\epsilon}} \mathbf{d} \left( \ln \left( \tilde{p} |\tilde{\omega}(t, 0) + \tilde{z}_1(t)|^{\frac{1}{\epsilon}} \right) \right) L_{FT}(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) \\ &\quad + L_0(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \end{aligned} \quad (3.44)$$

### Chapter 3. Finite/fixed-time estimation of LTI systems with pointwise input delay

where  $\mathbf{d}$  is a dilation with the generator  $G_{\mathbf{d}}$  (see Appendix A),  $v \in [-1, 0)$  and  $L_0 = (H + \bar{I}_n)^{-1} X_0 \in \mathbb{R}^n$  are chosen such that the system of matrix equations and inequalities

$$\begin{cases} AH - HA - A - X_0 C = 0, & (3.45) \end{cases}$$

$$\begin{cases} CH = 0, & (3.46) \end{cases}$$

$$\begin{cases} H + H^T + 2\bar{I}_n > 0, & (3.47) \end{cases}$$

is feasible with  $H \in \mathcal{M}_{n,n}(\mathbb{R})$ ,  $X_0 \in \mathbb{R}^n$ , and  $\tilde{p} = X_1^{\frac{1}{2}} \in \mathbb{R}$ ,  $L_{FT} = P^{-1} X_2 \in \mathbb{R}^n$ ,  $\epsilon \in \mathbb{R}_+$ ,  $G_{\mathbf{d}} = vH + \epsilon \bar{I}_n$  are parameters chosen such that for some  $\xi > \tau > 0$ ,  $\theta \in \mathbb{R}_+$  the system of matrix inequalities

$$\left\{ \begin{array}{l} \begin{bmatrix} PA + A^T P + PL_0 C + C^T L_0^T P + X_2 C + C^T X_2^T + \xi P & P \\ P & -\bar{X}_1 \end{bmatrix} \leq 0, & (3.48) \\ P > 0, \bar{X}_1 > 0, X_1 > 0, & (3.49) \end{array} \right.$$

$$\left\{ \begin{array}{l} \begin{bmatrix} \tau X_1 & X_2^T \\ X_2 & \theta P \end{bmatrix} \geq 0 & (3.50) \\ P - C^T X_1 C \geq 0, & (3.51) \end{array} \right.$$

$$\left\{ \begin{array}{l} vPH + vH^T P + 2\epsilon P > 0, & (3.52) \end{array} \right.$$

$$\left\{ \begin{array}{l} \forall \lambda \in [0, 1] : \lambda^{2\epsilon} \left( e^{(vH+v\bar{I}_n)\ln\lambda} - \bar{I}_n \right)^T \bar{X}_1 \left( e^{(vH+v\bar{I}_n)\ln\lambda} - \bar{I}_n \right) \leq \frac{1}{\theta} P, & (3.53) \end{array} \right.$$

is feasible with  $P \in \mathcal{M}_{n,n}(\mathbb{R})$ ,  $\bar{X}_1 \in \mathcal{M}_{n,n}(\mathbb{R})$ ,  $X_1 \in \mathbb{R}$ ,  $X_2 \in \mathbb{R}^n$ .

Moreover, in this case, if, in addition, the following inequality is feasible:

$$-\frac{\lambda_{\min}}{2\nu} (\nu H + n u H^T + 2\epsilon \bar{I}_n) < 1 \quad (3.54)$$

with  $\lambda_{\min} > 0$ , then the (3.21)-(3.23) is not only **FTS** but also **ISS** with respect to input and output disturbances (see [186, Corollary 3]).

#### Remark 3.3

Based on [186], a similar representation of the observer-gain functions can be given to ensure that (3.21)-(3.23) is **FxTS** and **ISS** (see [186, Theorem 7 and Corollary 4]):

$$\begin{aligned} \mathcal{L}_{FxT}(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) &= \frac{1}{2} \left[ \tilde{p}_1^{\frac{v-\epsilon}{\epsilon}} |\tilde{\omega}(t, 0) + \tilde{z}_1(t)|^{\frac{v-\epsilon}{\epsilon}} \mathbf{d} \left( \ln \left( \tilde{p}_1^{\frac{1}{\epsilon}} |\tilde{\omega}(t, 0) + \tilde{z}_1(t)|^{\frac{1}{\epsilon}} \right) \right) \right. \\ &+ \tilde{p}_2^{\frac{\epsilon-\nu}{\epsilon}} |\tilde{\omega}(t, 0) + \tilde{z}_1(t)|^{\frac{\epsilon-\nu}{\epsilon}} \mathbf{d} \left( -\ln \left( \tilde{p}_2^{\frac{1}{\epsilon}} |\tilde{\omega}(t, 0) + \tilde{z}_1(t)|^{\frac{1}{\epsilon}} \right) \right) \left. \right] L_{FxT}(\tilde{\omega}(t, 0) + \tilde{z}_1(t)) \\ &+ L_0(\tilde{\omega}(t, 0) + \tilde{z}_1(t)), \end{aligned} \quad (3.55)$$

where  $\mathbf{d}$  is a dilation with the generator  $G_{\mathbf{d}} = vH + \epsilon \bar{I}_n$ ,  $L_0 = (H + \bar{I}_n)^{-1} X_0 \in \mathbb{R}^n$ , and  $v \in [-1, 0)$  are chosen such that the system of matrix equations

$$\begin{cases} AH - HA - A - X_0 C = 0, & (3.56) \end{cases}$$

$$\begin{cases} CH = 0, & (3.57) \end{cases}$$

$$\begin{cases} H + H^T + 2\bar{I}_n > 0, & (3.58) \end{cases}$$

$$\begin{cases} 2\epsilon \bar{I}_n - vH - vH^T > 0, & (3.59) \end{cases}$$

is feasible with  $H \in \mathcal{M}_{n,n}(\mathbb{R})$ ,  $X_0 \in \mathbb{R}^n$ , and  $\epsilon \in \mathbb{R}_+$ . In addition, the parameters  $\tilde{p}_i = \delta_i X_1^{\frac{1}{2}} \in \mathbb{R}$ ,  $i \in \{1, 2\}$ ,  $L_{FxT} = P^{-1} X_2 \in \mathbb{R}^n$ ,  $\epsilon \in \mathbb{R}_+$  are chosen such that for some  $\xi_i > \tau > 0$ ,  $\delta_i \in \mathbb{R}_+$ ,

$i \in \{1, 2\}$ ,  $\theta \in \mathbb{R}_+$ , the system of matrix inequalities

$$\left\{ \begin{array}{l} \begin{bmatrix} \tau X_1 & X_2^T \\ X_2 & \theta P \end{bmatrix} \geq 0, \quad (3.60) \\ P - C^T X_1 C \geq 0, \quad (3.61) \\ vPH + vH^T P + 2\epsilon P > 0, \quad (3.62) \\ \begin{bmatrix} PA + A^T P + PL_0 C + C^T L_0^T P + X_2 C + C^T X_1^T + \xi_i P & P \\ P & -\bar{X}_i \end{bmatrix} \leq 0, \quad (3.63) \\ P > 0, \quad \bar{X}_i > 0, \quad X_1 > 0, \quad (3.64) \\ \forall [\lambda_1, \lambda_2] \in [0, 1]^2 : \frac{1}{4} \lambda_1^{2\epsilon} \left[ (\delta_1 \lambda_1 \lambda_2^{1-i})^{v-\epsilon} \mathbf{d}(\ln(\delta_1 \lambda_1 \lambda_2^{1-i})) \right. \\ \quad \left. + (\delta_2 \lambda_1 \lambda_2^{2-i})^{\epsilon-v} \mathbf{d}(-\ln(\delta_2 \lambda_1 \lambda_2^{2-i})) - 2\bar{I}_n \right]^T \\ \quad \times \bar{X}_i \left[ (\delta_1 \lambda_1 \lambda_2^{1-i})^{v-\epsilon} \mathbf{d}(\ln(\delta_1 \lambda_1 \lambda_2^{1-i})) \right. \\ \quad \left. + (\delta_2 \lambda_1 \lambda_2^{2-i})^{\epsilon-v} \mathbf{d}(-\ln(\delta_2 \lambda_1 \lambda_2^{2-i})) - 2\bar{I}_n \right] \leq \frac{1}{\theta} P, \quad (3.65) \end{array} \right.$$

is feasible with  $P \in \mathcal{M}_{n,n}(\mathbb{R})$ ,  $\bar{X}_i \in \mathcal{M}_{n,n}(\mathbb{R})$ ,  $i \in \{1, 2\}$ ,  $X_1 \in \mathbb{R}$ , and  $X_2 \in \mathbb{R}^n$ .

### 3.3 Stability analysis

In this section, we present and prove the main result of this chapter. To better illustrate, we focus on the case of second-order LTI systems for which we prove that the error state of (3.14)-(3.16) goes to zero within a finite time. To prove this result we need to establish some intermediate results. First, we prove that the solutions of the target system are bounded for all  $t \in [t_0, t_0 + D]$ . Next, we estimate, in two different ways, the solution of the  $\tilde{z}$ -dynamics of the target system at time  $t_0 + D$  by a function that depends on the initial condition of the target system. Then, we prove that these solutions converge to zero within a finite time that can be estimated.

#### Lemma 3.1

Let  $t_0 \geq 0$ ,  $\tilde{\omega}_0 \in L^2$  and bounded, and  $\tilde{z}_0 = [\tilde{z}_{1,0}, \tilde{z}_{2,0}]^T \in \mathbb{R}^2$ . Then for any  $t \in [t_0, t_0 + D]$  and any  $x \in [0, D]$ ,  $(\tilde{z}_1, \tilde{z}_2, \tilde{\omega})$  the solution of the target system (3.37)-(3.40) is bounded. Moreover, the solution of the  $\tilde{z}$ -dynamics satisfies the following inequalities:

$$\sup_{t \in [t_0, t_0 + D]} |\tilde{z}_1(t)|^2 \leq M_1 \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + M_2 (\|\tilde{z}_0\|_2), \quad (3.66)$$

$$\sup_{t \in [t_0, t_0 + D]} |\tilde{z}_2(t)|^2 \leq N_1 \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + N_2 \sup_{t \in [t_0, t_0 + D]} |\tilde{z}_1(t)|^2 + N_3 (\|\tilde{z}_0\|_2), \quad (3.67)$$

where

$$\begin{aligned} M_1 &= \frac{4D [2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2]}{1 - 4D^2\epsilon^2 (4D^2k_2^2 + k_1^2)}, \\ M_2(\|\tilde{z}_0\|_2) &= \frac{8D^2 \left[ 4D^2k_2^2 \left(1 + \epsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \epsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2}{1 - 4D^2\epsilon^2 (4D^2k_2^2 + k_1^2)}, \quad (3.68) \\ N_1 &= 2D(k_2 + |a_0|)^2, \\ N_2 &= 4D^2k_2^2\epsilon^2, \\ N_3(\|\tilde{z}_0\|_2) &= 8 \left[ D^2k_2^2 \left(1 + \epsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + \|\tilde{z}_0\|_2^2 \right], \end{aligned}$$

with  $\varepsilon \in (0, 1)$  is chosen small enough so that  $(1 - 4D^2\varepsilon^2(4D^2k_2^2 + k_1^2))$  is positive.

*Proof.* Using the initial condition  $\tilde{\omega}(t_0, x) = \tilde{\omega}_0(x)$  and by the method of characteristics, we have that  $\tilde{\omega}(t, x) = \tilde{\omega}_0(x + t - t_0)$  for any  $t \leq t_0 + D - x$  and  $\tilde{\omega}(t, x) = 0$  for any  $t \geq t_0 + D - x$ . Thus  $\tilde{\omega}$  is bounded for all  $t \in [t_0, t_0 + D]$ .

Next, integrating (3.38) from  $t_0$  to  $t$ , gives us

$$\tilde{z}_2(t) = -k_2 \int_{t_0}^t \{\tilde{\omega}(s, 0) + \tilde{z}_1(s)\}^{\alpha_2} ds - a_0 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{2,0}. \quad (3.69)$$

Using the fact that  $t_0 \leq t \leq t_0 + D$ , we get

$$|\tilde{z}_2(t)| \leq k_2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0) + \tilde{z}_1(s)|^{\alpha_2} ds + |a_0| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{2,0}|. \quad (3.70)$$

Using the fact that  $\forall(a, b) \in \mathbb{R}_+^2, \forall \beta \in (0, 1) : (a + b)^\beta \leq a^\beta + b^\beta$  and since  $\alpha_2 \in (0, 1)$ , we obtain

$$|\tilde{z}_2(t)| \leq k_2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^{\alpha_2} ds + k_2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^{\alpha_2} ds + |a_0| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{2,0}|. \quad (3.71)$$

Now, using the fact that  $\forall a \in \mathbb{R}_+, \forall \beta \in (0, 1) : a^\beta \leq \beta a + (1 - \beta)$  on the first term and the fact that  $\forall a \in \mathbb{R}_+, \forall \varepsilon \in (0, 1), \forall \beta \in (0, 1) : a^\beta \leq \varepsilon a + (1 - \beta) \left(\frac{\beta}{\varepsilon}\right)^{\frac{\beta}{1-\beta}}$  on the second term, we get

$$\begin{aligned} |\tilde{z}_2(t)| &\leq \alpha_2 k_2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + (1 - \alpha_2) D k_2 + |a_0| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds \\ &\quad + k_2 \varepsilon \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds + k_2 (1 - \alpha_2) D \left(\frac{\alpha_2}{\varepsilon}\right)^{\frac{\alpha_2}{1-\alpha_2}} + |\tilde{z}_{2,0}|, \end{aligned} \quad (3.72)$$

$$\begin{aligned} &\leq k_2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + D k_2 + |a_0| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds \\ &\quad + k_2 \varepsilon \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds + k_2 D \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} + |\tilde{z}_{2,0}|, \end{aligned} \quad (3.73)$$

$$\begin{aligned} &\leq (k_2 + |a_0|) \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + k_2 \varepsilon \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds \\ &\quad + \left[ D k_2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right) + |\tilde{z}_{2,0}| \right]. \end{aligned} \quad (3.74)$$

By squaring this last inequality, we find

$$\begin{aligned} |\tilde{z}_2(t)|^2 &\leq 2(k_2 + |a_0|)^2 \left( \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds \right)^2 + 4k_2^2 \varepsilon^2 \left( \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds \right)^2 \\ &\quad + 4 \left[ D k_2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right) + |\tilde{z}_{2,0}| \right]^2. \end{aligned} \quad (3.75)$$

$$(3.76)$$

Finally, using the Jensen inequality on each integral and Young's inequality on the last term, we get

$$\begin{aligned} |\tilde{z}_2(t)|^2 &\leq 2D(k_2 + |a_0|)^2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^2 ds + 4Dk_2^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds \\ &\quad + 8 \left[ D^2 k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + |\tilde{z}_{2,0}|^2 \right]. \end{aligned} \quad (3.77)$$

Using the value of  $\tilde{\omega}(t, 0)$  in the previous inequality, gives us

$$|\tilde{z}_2(t)|^2 \leq 2D(k_2 + |a_0|)^2 \int_{t_0}^{t_0+D} |\tilde{\omega}_0(s - t_0)|^2 ds + 4Dk_2^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 8 \left[ D^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right], \quad (3.78)$$

$$\leq 2D(k_2 + |a_0|)^2 \int_0^D |\tilde{\omega}_0(x)|^2 dx + 4Dk_2^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 8 \left[ D^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right], \quad (3.79)$$

$$\leq 2D(k_2 + |a_0|)^2 \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + 4Dk_2^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 8 \left[ D^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + |\tilde{z}_{2,0}|^2 \right], \quad (3.80)$$

which implies that

$$|\tilde{z}_2(t)|^2 \leq 2D(k_2 + |a_0|)^2 \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + 4D^2 k_2^2 \varepsilon^2 \sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 + 8 \left[ D^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + \|\tilde{z}_0\|_2^2 \right]. \quad (3.81)$$

Then,

$$\sup_{t \in [t_0, t_0+D]} |\tilde{z}_2(t)|^2 \leq N_1 \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + N_2 \sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 + N_3 (\|\tilde{z}_0\|_2), \quad (3.82)$$

where

$$N_1 = 2D(k_2 + |a_0|)^2, \quad (3.83)$$

$$N_2 = 4D^2 k_2^2 \varepsilon^2, \quad (3.84)$$

$$N_3 (\|\tilde{z}_0\|_2) = 8 \left[ D^2 k_2^2 \left( 1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}} \right)^2 + \|\tilde{z}_0\|_2^2 \right]. \quad (3.85)$$

Similarly, integrating (3.37) from  $t_0$  to  $t$ , gives us

$$\tilde{z}_1(t) = \int_{t_0}^t \tilde{z}_2(s) ds - k_1 \int_{t_0}^t \{\tilde{\omega}(s, 0) + \tilde{z}_1(s)\}^{\alpha_1} ds - a_1 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{1,0}. \quad (3.86)$$

Using the fact that  $t_0 \leq t \leq t_0 + D$ , we get

$$|\tilde{z}_1(t)| \leq \int_{t_0}^{t_0+D} |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0) + \tilde{z}_1(s)|^{\alpha_1} ds + |a_1| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|. \quad (3.87)$$

Using the fact that  $\forall [a, b] \in \mathbb{R}_+^2, \forall \beta \in (0, 1) : (a + b)^\beta \leq a^\beta + b^\beta$  and since  $\alpha_1 \in (0, 1)$ , we obtain

$$|\tilde{z}_1(t)| \leq \int_{t_0}^{t_0+D} |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^{\alpha_1} ds + k_1 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^{\alpha_1} ds + |a_1| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|. \quad (3.88)$$

### 56 Chapter 3. Finite/fixed-time estimation of LTI systems with pointwise input delay

Now, using the fact that  $\forall a \in \mathbb{R}_+, \forall \beta \in (0, 1) : a^\beta \leq \beta a + (1 - \beta)$  on the second term and fact that  $\forall a \in \mathbb{R}_+, \forall \varepsilon \in (0, 1), \forall \beta \in (0, 1) : a^\beta \leq \varepsilon a + (1 - \beta) \left(\frac{\beta}{\varepsilon}\right)^{\frac{\beta}{1-\beta}}$  on the third term of the previous inequality, we get

$$\begin{aligned} |\tilde{z}_1(t)| &\leq \int_{t_0}^{t_0+D} |\tilde{z}_2(s)| ds + \alpha_1 k_1 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + Dk_1(1 - \alpha_1) \\ &\quad + k_1 \varepsilon \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds + Dk_1(1 - \alpha_1) \left(\frac{\alpha_1}{\varepsilon}\right)^{\frac{\alpha_1}{1-\alpha_1}} \\ &\quad + |a_1| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|, \end{aligned} \quad (3.89)$$

$$\begin{aligned} &\leq \int_{t_0}^{t_0+D} |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + Dk_1 + k_1 \varepsilon \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds \\ &\quad + Dk_1 \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}} + |a_1| \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + |\tilde{z}_{1,0}|, \end{aligned} \quad (3.90)$$

$$\begin{aligned} &\leq \int_{t_0}^{t_0+D} |\tilde{z}_2(s)| ds + (k_1 + |a_1|) \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds + k_1 \varepsilon \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds \\ &\quad + \left[ Dk_1 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right) + |\tilde{z}_{1,0}| \right]. \end{aligned} \quad (3.91)$$

Squaring this last inequality gives us

$$\begin{aligned} |\tilde{z}_1(t)|^2 &\leq 4 \left( \int_{t_0}^{t_0+D} |\tilde{z}_2(s)| ds \right)^2 + 4(k_1 + |a_1|)^2 \left( \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)| ds \right)^2 \\ &\quad + 4k_1^2 \varepsilon^2 \left( \int_{t_0}^{t_0+D} |\tilde{z}_1(s)| ds \right)^2 + 4 \left[ Dk_1 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right) + |\tilde{z}_{1,0}| \right]^2. \end{aligned} \quad (3.92)$$

Applying the Jensen's inequality on each integral and the Young's inequality on the last term, we obtain

$$\begin{aligned} |\tilde{z}_1(t)|^2 &\leq 4D \int_{t_0}^{t_0+D} |\tilde{z}_2(s)|^2 ds + 4D(k_1 + |a_1|)^2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^2 ds \\ &\quad + 4Dk_1^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 8 \left[ D^2 k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 + |\tilde{z}_{1,0}|^2 \right]. \end{aligned} \quad (3.93)$$

Using the inequality (3.77) on the last inequality, gives us

$$\begin{aligned} |\tilde{z}_1(t)|^2 &\leq 4D^2 \left[ 2D(k_2 + |a_0|)^2 \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^2 ds + 4Dk_2^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds \right. \\ &\quad \left. + 8 \left[ D^2 k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + \|\tilde{z}_0\|_2^2 \right] \right] + 4D(k_1 + |a_1|)^2 \\ &\quad \times \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^2 ds + 4Dk_1^2 \varepsilon^2 \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds \\ &\quad + 8 \left[ D^2 k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 + \|\tilde{z}_0\|_2^2 \right]. \end{aligned} \quad (3.94)$$

By developing this last inequality and using the value of  $\tilde{\omega}(t, 0)$ , we get for all  $t \in [t_0, t_0 + D]$

$$\begin{aligned} |\tilde{z}_1(t)|^2 &\leq 4D\varepsilon^2 (4D^2k_2^2 + k_1^2) \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 4D (2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \\ &\quad \times \int_{t_0}^{t_0+D} |\tilde{\omega}(s, 0)|^2 ds + 8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] \\ &\quad + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2, \end{aligned} \quad (3.95)$$

$$\begin{aligned} &\leq 4D\varepsilon^2 (4D^2k_2^2 + k_1^2) \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 4D (2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \\ &\quad \times \int_{t_0}^{t_0+D} |\tilde{\omega}_0(s - t_0)|^2 ds + 8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] \\ &\quad + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2, \end{aligned} \quad (3.96)$$

$$\begin{aligned} &\leq 4D\varepsilon^2 (4D^2k_2^2 + |k_1|^2) \int_{t_0}^{t_0+D} |\tilde{z}_1(s)|^2 ds + 4D (2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \\ &\quad \times \int_0^D |\tilde{\omega}_0(x)|^2 dx + 8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] \\ &\quad + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2, \end{aligned} \quad (3.97)$$

$$\begin{aligned} &\leq 4D^2\varepsilon^2 (4D^2k_2^2 + k_1^2) \sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 + 4D (2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \\ &\quad \times \|\tilde{\omega}_0\|_{L^2(0, D)}^2 + 8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] \\ &\quad + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2, \end{aligned} \quad (3.98)$$

which implies that

$$\begin{aligned} \sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 &\leq 4D^2\varepsilon^2 (4D^2k_2^2 + k_1^2) \sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 + 4D (2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \\ &\quad \times \|\tilde{\omega}_0\|_{L^2(0, D)}^2 + 8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] \\ &\quad + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2. \end{aligned} \quad (3.99)$$

Then :

$$\begin{aligned} (1 - 4D^2\varepsilon^2 (4D^2k_2^2 + k_1^2)) \sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 &\leq 4D (2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2) \|\tilde{\omega}_0\|_{L^2(0, D)}^2 \\ &\quad + 8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] \\ &\quad + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2. \end{aligned} \quad (3.100)$$

Finally, choosing  $\varepsilon$  small enough so that  $(1 - 4D^2\varepsilon^2 (4D^2k_2^2 + k_1^2))$  is positive, (for instance, we can take  $\varepsilon = [8D^2(4D^2k_2^2 + k_1^2)]^{-\frac{1}{2}}$ ), gives us

$$\sup_{t \in [t_0, t_0+D]} |\tilde{z}_1(t)|^2 \leq M_1 \sup_{t \in [t_0, t_0+D]} |\tilde{\omega}(t, 0)|^2 + M_2(\|\tilde{z}_0\|_2), \quad (3.101)$$

where

$$M_1 = \frac{4D [2D^2(k_2 + |a_0|)^2 + (k_1 + |a_1|)^2]}{1 - 4D^2\varepsilon^2 (4D^2k_2^2 + k_1^2)}, \quad (3.102)$$

$$M_2(\|\tilde{z}_0\|_2) = \frac{8D^2 \left[ 4D^2k_2^2 \left(1 + \varepsilon^{-\frac{\alpha_2}{1-\alpha_2}}\right)^2 + k_1^2 \left(1 + \varepsilon^{-\frac{\alpha_1}{1-\alpha_1}}\right)^2 \right] + 8 [4D^2 + 1] \|\tilde{z}_0\|_2^2}{1 - 4D^2\varepsilon^2 (4D^2k_2^2 + k_1^2)}. \quad (3.103)$$



### Chapter 3. Finite/fixed-time estimation of LTI systems with pointwise input delay

Looking at (3.82) and (3.101), we can clearly see that the solution of the  $\tilde{z}$ -dynamics is also bounded for all  $t \in [t_0, t_0 + D]$ .  $\square$

#### Lemma 3.2

Let  $\tilde{\omega}_0 \in L^2$  and bounded, and  $\hat{z}_0, z_0 \in \mathbb{R}^2$ . For any  $t_0, D > 0$ , we have

$$\|\tilde{z}(t_0 + D)\|_2^2 \leq G_1(\|\tilde{\omega}_0\|_{L^2(0,D)}, \|z_0 - \hat{z}_0\|_2), \quad (3.104)$$

where  $G_1$  is the function given by

$$G_1(s_1, s_2) = (N_1 + N_2 M_1) s_1^2 + [N_2 M_2 (\|e^{-AD}\|_2 s_2) + N_3 (\|e^{-AD}\|_2 s_2)], \quad (3.105)$$

where  $M_1, M_2, N_1, N_2$ , and  $N_3$  are given in (3.68).

*Proof.* Using Lemma 3.1, we can clearly see that

$$\|\tilde{z}(t)\|_2^2 \leq (N_1 + N_2 M_1) \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + [N_2 M_2 (\|\tilde{z}_0\|_2) + N_3 (\|\tilde{z}_0\|_2)] \text{ for all } t \in [t_0, t_0 + D], \quad (3.106)$$

where  $M_1, M_2, N_1, N_2$ , and  $N_3$  are given in (3.68).

Using (3.12)-(3.13), we obtain

$$\|\tilde{z}(t)\|_2^2 \leq (N_1 + N_2 M_1) \|\tilde{\omega}_0\|_{L^2(0,D)}^2 + [N_2 M_2 (\|e^{-AD}\|_2 \|z_0 - \hat{z}_0\|_2) + N_3 (\|e^{-AD}\|_2 \|z_0 - \hat{z}_0\|_2)]. \quad (3.107)$$

In particular for  $t = t_0 + D$ , we have

$$\|\tilde{z}(t_0 + D)\|_2^2 \leq G_1(\|\tilde{\omega}_0\|_{L^2(0,D)}, \|z_0 - \hat{z}_0\|_2). \quad (3.108)$$

where  $G_1$  is the function given by (3.105).  $\square$

#### Lemma 3.3

For any  $t_0, D > 0$  and any  $\hat{z}_0$  in  $\mathbb{R}^2$ , we have

$$\|\tilde{z}(t_0 + D)\|_2^2 \leq G_2(\|z_0 - \hat{z}_0\|_2), \quad (3.109)$$

where  $G_2$  is a function defined for any  $s \geq 0$  by

$$G_2(s) = 4e^{2D(|a_0|+|a_1|+1)} \left[ \|e^{-AD}\|_2^2 s^2 + D^2(|a_0| + |a_1|)^2 \tilde{M}^2 + 2D^2(k_1 + k_2)^2 (\tilde{M}^{2\alpha_1} + \tilde{M}^{2\alpha_2}) \right], \quad (3.110)$$

with  $\tilde{M}$  is such that

$$\sup_{t \in [t_0, t_0 + D]} |\tilde{u}(t, 0)| \leq \tilde{M}. \quad (3.111)$$

*Proof.* By integrating the second equation of (3.37)-(3.40), we obtain

$$\tilde{z}_2(t) = -k_2 \int_{t_0}^t \{\tilde{\omega}(s, 0) + \tilde{z}_1(s)\}^{\alpha_2} ds - a_0 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{2,0}. \quad (3.112)$$

Using the transformation (3.17) at  $x = 0$ , we get

$$|\tilde{z}_2(t)| \leq k_2 \int_{t_0}^t |\tilde{u}(s, 0)|^{\alpha_2} ds + |a_0| \int_{t_0}^t |\tilde{u}(s, 0)| ds + |a_0| \int_{t_0}^t |\tilde{z}_1(s)| ds + |\tilde{z}_{2,0}|, \quad (3.113)$$

for any  $t \geq t_0$ .

On the other hand, integrating the first equation of (3.37)-(3.40), gives us

$$\tilde{z}_1(t) = \int_{t_0}^t \tilde{z}_2(s) ds - k_1 \int_{t_0}^t \{\tilde{\omega}(s, 0) + \tilde{z}_1(s)\}^{\alpha_1} ds - a_1 \int_{t_0}^t \tilde{\omega}(s, 0) ds + \tilde{z}_{1,0}. \quad (3.114)$$

Using the transformation (3.17) at  $x = 0$ , we get

$$|\tilde{z}_1(t)| \leq \int_{t_0}^t |\tilde{z}_2(s)| ds + k_1 \int_{t_0}^t |\tilde{u}(s, 0)|^{\alpha_1} ds + |a_1| \int_{t_0}^t |\tilde{u}(s, 0)| ds + |a_1| \int_{t_0}^t |\tilde{z}_1(s)| ds + |\tilde{z}_{1,0}|, \quad (3.115)$$

for any  $t \geq t_0$ .

Summing the two inequalities (3.113) and (3.115), we obtain

$$\begin{aligned} |\tilde{z}_1(t)| + |\tilde{z}_2(t)| &\leq (|a_0| + |a_1| + 1) \int_{t_0}^t (|\tilde{z}_1(s)| + |\tilde{z}_2(s)|) ds + |\tilde{z}_{1,0}| + |\tilde{z}_{2,0}| + (k_1 + k_2) \\ &\quad \times \int_{t_0}^{t_0+D} (|\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1}) ds + (|a_0| + |a_1|) \int_{t_0}^{t_0+D} |\tilde{u}(s, 0)| ds. \end{aligned} \quad (3.116)$$

Using the Grönwall Lemma, we find

$$\begin{aligned} |\tilde{z}_1(t)| + |\tilde{z}_2(t)| &\leq e^{(|a_0|+|a_1|+1)(t-t_0)} \left[ |\tilde{z}_{1,0}| + |\tilde{z}_{2,0}| + (k_1 + k_2) \int_{t_0}^{t_0+D} (|\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1}) ds \right. \\ &\quad \left. + (|a_0| + |a_1|) \int_{t_0}^{t_0+D} |\tilde{u}(s, 0)| ds \right], \end{aligned} \quad (3.117)$$

$$\begin{aligned} &\leq e^{(|a_0|+|a_1|+1)D} \left[ |\tilde{z}_{1,0}| + |\tilde{z}_{2,0}| + (k_1 + k_2) \int_{t_0}^{t_0+D} (|\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1}) ds \right. \\ &\quad \left. + (|a_0| + |a_1|) \int_{t_0}^{t_0+D} |\tilde{u}(s, 0)| ds \right]. \end{aligned} \quad (3.118)$$

Squaring this inequality gives us

$$\begin{aligned} \|\tilde{z}(t)\|_2^2 &\leq (|\tilde{z}_1(t)| + |\tilde{z}_2(t)|)^2 \leq e^{2D(|a_0|+|a_1|+1)} \left[ 2(|\tilde{z}_{1,0}| + |\tilde{z}_{2,0}|)^2 + 4(|a_0| + |a_1|)^2 \right. \\ &\quad \times \left( \int_{t_0}^{t_0+D} |\tilde{u}(s, 0)| ds \right)^2 + 4(k_1 + k_2)^2 \\ &\quad \left. \times \left( \int_{t_0}^{t_0+D} (|\tilde{u}(s, 0)|^{\alpha_2} + |\tilde{u}(s, 0)|^{\alpha_1}) ds \right)^2 \right]. \end{aligned} \quad (3.119)$$

Using Jensen's inequality and Young's inequality, we find

$$\begin{aligned} \|\tilde{z}(t)\|_2^2 &\leq e^{2D(|a_0|+|a_1|+1)} \left[ 4\|\tilde{z}_0\|_2^2 + 4D(|a_0| + |a_1|)^2 \int_{t_0}^{t_0+D} |\tilde{u}(s,0)|^2 ds \right. \\ &\quad \left. + 8D(k_1 + k_2)^2 \int_{t_0}^{t_0+D} \left( |\tilde{u}(s,0)|^{2\alpha_2} + |\tilde{u}(s,0)|^{2\alpha_1} \right) ds \right], \end{aligned} \quad (3.120)$$

$$\begin{aligned} &\leq e^{2D(|a_0|+|a_1|+1)} \left[ 4\|\tilde{z}_0\|_2^2 + 4D^2(|a_0| + |a_1|)^2 \sup_{t \in [t_0, t_0+D]} |\tilde{u}(t,0)|^2 \right. \\ &\quad \left. + 8D^2(k_1 + k_2)^2 \sup_{t \in [t_0, t_0+D]} |\tilde{u}(t,0)|^{2\alpha_2} + 8D^2(k_1 + k_2)^2 \right. \\ &\quad \left. \times \sup_{t \in [t_0, t_0+D]} |\tilde{u}(t,0)|^{2\alpha_1} \right]. \end{aligned} \quad (3.121)$$

Using the transformation (3.17) and Lemma 3.1, we have  $\tilde{\omega}$  being bounded for all  $t \in [t_0, t_0 + D]$ , then, there exists a positive constant  $\tilde{M}$  such that

$$\sup_{t \in [t_0, t_0+D]} |\tilde{u}(t,0)| \leq \tilde{M}. \quad (3.122)$$

Using this property in the previous inequality, we get

$$\|\tilde{z}(t)\|_2^2 \leq 4e^{2D(|a_0|+|a_1|+1)} \left[ \|\tilde{z}_0\|_2^2 + D^2(|a_0| + |a_1|)^2 \tilde{M}^2 + 2D^2(k_1 + k_2)^2 (\tilde{M}^{2\alpha_1} + \tilde{M}^{2\alpha_2}) \right]. \quad (3.123)$$

This inequality is true for all  $t \in [t_0, t_0 + D]$ , in particular for  $t = t_0 + D$ , we have :

$$\|\tilde{z}(t_0 + D)\|_2^2 \leq 4e^{2D(|a_0|+|a_1|+1)} \left[ \|\tilde{z}_0\|_2^2 + D^2(|a_0| + |a_1|)^2 \tilde{M}^2 + 2D^2(k_1 + k_2)^2 (\tilde{M}^{2\alpha_1} + \tilde{M}^{2\alpha_2}) \right]. \quad (3.124)$$

Employing (3.12)-(3.13), we find

$$\|\tilde{z}(t_0 + D)\|_2^2 \leq G_2(\|z_0 - \hat{z}_0\|_2). \quad (3.125)$$

where  $G_2$  is the function defined by (3.110).  $\square$

**Remark 3.4**

Note that if we replace the finite-time observer-gain functions (3.35)-(3.36) by the fixed-time observer-gain functions (3.35)-(3.36), the previous Lemmas can be easily adapted.

**Remark 3.5**

For the general case and in both cases of the finite-time observer-gain function (3.44) and the fixed-time observer-gain function (3.55), Lemmas 3.1-3.3 can be deduced using the ISS property of the (3.21)-(3.23) (see [186, Corollaries 3 and 4]) by seeing the term  $\tilde{\omega}(t,0)$  as a bounded disturbance that vanishes after a certain amount of time.

**Lemma 3.4**

For any  $t_0, D \geq 0$ , the target system (3.37)-(3.40) is **FTS**. Moreover, there exist positive scalar parameters  $\delta, P_{min}$  and  $P_{max}$  such that for  $t \geq t_0 + D$

$$\|\tilde{z}(t)\|_2^2 \leq F(t - t_0 - D, \|\tilde{z}(t_0 + D)\|_2) + F(t - t_0 - D, \|\tilde{z}(t_0 + D)\|_2)^{\frac{2}{1+\alpha}}, \quad (3.126)$$

where  $F$  is an increasing function with respect to the second variable defined for any  $s \geq 0$  by

$$F(t - t_0 - D, s) = \frac{1}{P_{min}} \left[ -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - D) + (P_{max})^{1 - \frac{\nu''}{\nu'}} (s^{\alpha+1} + s^2)^{1 - \frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}}, \quad (3.127)$$

with  $\alpha \in (0, 1)$ ,  $\nu' = \alpha + 1$ ,  $\nu'' = \frac{3\alpha+1}{2}$ .

In particular, when  $t$  tends to the settling time  $t_0 + D + T_{\max}(\|\tilde{z}(t_0 + D)\|_2)$ , the function  $F$  and the norm  $\|\tilde{z}\|_2^2$  go to zero, where  $T_{\max}$  is given for any  $s \geq 0$  by

$$T_{\max}(s) = \frac{\nu'}{\delta(\nu' - \nu'')} (P_{\max})^{1 - \frac{\nu''}{\nu'}} (s^{\alpha+1} + s^2)^{1 - \frac{\nu''}{\nu'}}. \quad (3.128)$$

*Proof.* By the method of the characteristics, the solution of the  $\tilde{\omega}$ -dynamics in (3.37)-(3.40) for  $t \geq t_0 + D$  is equal to zero, then  $\tilde{\omega}(t, 0) = 0$  for all  $t \geq t_0 + D$ . Hence, the target system (3.37)-(3.40) becomes

$$\begin{cases} \dot{\tilde{z}}_1(t) = \tilde{z}_2(t) - k_1 \{\tilde{z}_1(t)\}^{\alpha_1}, & t \geq t_0, \\ \dot{\tilde{z}}_2(t) = -k_2 \{\tilde{z}_1(t)\}^{\alpha_2}, & t \geq t_0. \end{cases} \quad (3.129)$$

To prove that this system is **FTS**, we choose  $\alpha_1$  and  $\alpha_2$  so that the system (3.129)-(3.130) is  $r$ -homogeneous of degree  $\nu$ , where  $r = [r_1, r_2]^\top \in \mathbb{R}^2$  and  $\nu < 0$  satisfy

$$\begin{cases} r_1 + \nu = r_2 = r_1 \alpha_1, \\ r_2 + \nu = r_1 \alpha_2. \end{cases} \quad (3.131)$$

$$(3.132)$$

Let  $\alpha_2 = \alpha \in (0, 1)$  and  $r_1 = 1$ . Then we find

$$\nu = \frac{(\alpha - 1)}{2} < 0, \quad (3.133)$$

$$\alpha_1 = r_2 = \frac{(\alpha + 1)}{2} \in \left(\frac{1}{2}, 1\right). \quad (3.134)$$

Next, we will prove that the system is asymptotically stable. In order to do that, let  $\varphi$  be a function defined by

$$\varphi(\tilde{z}_1, \tilde{z}_2) = \left[ \{\tilde{z}_1\}^{\frac{\alpha+1}{2}}, \tilde{z}_2 \right]. \quad (3.135)$$

We can clearly see that  $\varphi$  is  $(1, \frac{\alpha+1}{2})$ -homogeneous of degree  $\mu = \frac{\alpha+1}{2}$ . Now, let  $V$  be a Lyapunov candidate function defined by (as in [118])

$$V(\tilde{z}_1, \tilde{z}_2) = \frac{1}{2} \varphi(\tilde{z}_1, \tilde{z}_2) \bar{P} \varphi(\tilde{z}_1, \tilde{z}_2)^\top, \quad (3.136)$$

with  $\bar{P} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ , where for any  $k_1, k_2 > 0$ , the parameters  $a$ ,  $b$ , and  $c$  are given as follows:

$$\begin{aligned} a &= \frac{(-2ck_2 + 1)}{(\alpha + 1)k_1} > 0, \\ b &= \frac{(\alpha + 1)(a - ck_1)}{2k_2} > 0, \\ c &= -1 < 0. \end{aligned} \quad (3.137)$$

The function  $V$  is  $(1, \frac{\alpha+1}{2})$ -homogeneous of degree  $\nu' = \alpha + 1$  and positive-definite because the

### Chapter 3. Finite/fixed-time estimation of LTI systems with pointwise input delay

matrix  $\bar{P}$  is also positive definite. In fact, the principal minors of  $\bar{P}$  are positive

$$a = \frac{(-2ck_2 + 1)}{(\alpha + 1)k_1} > 0, \quad (3.138)$$

$$ab - c^2 = \frac{(a - ck_1)(-2ck_2 + 1) - 2k_1k_2c^2}{2k_1k_2}, \quad (3.139)$$

$$= \frac{(a + k_1)(1 + 2k_2) - 2k_1k_2}{2k_1k_2}, \quad (3.140)$$

$$= \frac{a + k_1 + 2ak_2 + 2k_1k_2 - 2k_1k_2}{2k_1k_2}, \quad (3.141)$$

$$= \frac{a + k_1 + 2ak_2}{2k_1k_2} > 0. \quad (3.142)$$

On the other hand, the time derivative of  $V$  is given for all  $t \geq t_0 + D$  as follows:

$$\dot{V}(\tilde{z}_1, \tilde{z}_2) = \left[ \{\tilde{z}_1\}^{\frac{\alpha+1}{2}}, \tilde{z}_2 \right] \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} \frac{\alpha+1}{2} \dot{\tilde{z}}_1 |\tilde{z}_1|^{\frac{\alpha-1}{2}} \\ \dot{\tilde{z}}_2 \end{bmatrix}, \quad (3.143)$$

$$= \left[ a \{\tilde{z}_1\}^{\frac{\alpha+1}{2}} + c\tilde{z}_2, \quad c \{\tilde{z}_1\}^{\frac{\alpha+1}{2}} + b\tilde{z}_2 \right] \begin{bmatrix} \frac{\alpha+1}{2} |\tilde{z}_1|^{\frac{\alpha-1}{2}} (\tilde{z}_2 - k_1 \{\tilde{z}_1\}^{\frac{\alpha+1}{2}}) \\ -k_2 \{\tilde{z}_1\}^\alpha \end{bmatrix}, \quad (3.144)$$

$$= \frac{1}{2} [-2bk_2 + (\alpha + 1)(a - ck_1)] \{\tilde{z}_1\}^\alpha \tilde{z}_2 + \frac{1}{2} [-ak_1(\alpha + 1) - 2ck_2] |\tilde{z}_1|^{\frac{3\alpha+1}{2}} \\ + \frac{1}{2} c(\alpha + 1) |\tilde{z}_1|^{\frac{\alpha-1}{2}} \tilde{z}_2^2, \quad (3.145)$$

$$= \frac{1}{2} |\tilde{z}_1|^{\frac{\alpha-1}{2}} \tilde{z}_2^2 \left( [-2bk_2 + (\alpha + 1)(a - ck_1)] \frac{\{\tilde{z}_1\}^{\frac{\alpha+1}{2}}}{\tilde{z}_2} + [-ak_1(\alpha + 1) - 2ck_2] \right. \\ \left. \times \left( \frac{\{\tilde{z}_1\}^{\frac{\alpha+1}{2}}}{\tilde{z}_2} \right)^2 + c(\alpha + 1) \right), \quad (3.146)$$

$$= \frac{1}{2} |\tilde{z}_1|^{\frac{\alpha-1}{2}} \tilde{z}_2^2 ([-2bk_2 + (\alpha + 1)(a - ck_1)] E + [-ak_1(\alpha + 1) - 2ck_2] E^2 + c(\alpha + 1)), \quad (3.147)$$

$$= \frac{|\tilde{z}_1|^{\frac{3\alpha+1}{2}}}{2E^2} ([-2bk_2 + (\alpha + 1)(a - ck_1)] E + [-ak_1(\alpha + 1) - 2ck_2] E^2 + c(\alpha + 1)), \quad (3.148)$$

where  $E = \frac{\{\tilde{z}_1\}^{\frac{\alpha+1}{2}}}{\tilde{z}_2}$ . We can clearly see that  $\dot{V}$  is  $(1, \frac{\alpha+1}{2})$ -homogeneous of degree  $\nu'' = \frac{3\alpha+1}{2}$ . Let  $Q$  be the polynomial function given by

$$Q(E) = \bar{a}E^2 + \bar{b}E + \bar{c} = \bar{a} \left[ \left( E + \frac{\bar{b}}{2\bar{a}} \right)^2 - \frac{\bar{b}^2 - 4\bar{a}\bar{c}}{4\bar{a}^2} \right], \quad (3.149)$$

where  $\bar{a}$ ,  $\bar{b}$ , and  $\bar{c}$  are given as follows:

$$\bar{a} = [-ak_1(\alpha + 1) - 2ck_2], \quad (3.150)$$

$$\bar{b} = [-2bk_2 + (\alpha + 1)(a - ck_1)], \quad (3.151)$$

$$\bar{c} = c(\alpha + 1). \quad (3.152)$$

Our goal is to prove that  $\dot{V}$  is definite negative, which is equivalent to proving that  $Q$  is negative, or more precisely it is equivalent to proving that  $\bar{a}$  and  $\bar{b}^2 - 4\bar{a}\bar{c}$  are negative. In fact using the

value of  $a$ ,  $b$ , and  $c$  from (3.137), we obtain

$$\bar{a} = -ak_1(\alpha + 1) - 2ck_2, \quad (3.153)$$

$$= \frac{-2k_2 - 1}{k_1(\alpha + 1)}k_1(\alpha + 1) + 2k_2, \quad (3.154)$$

$$= -2k_2 - 1 + 2k_2, \quad (3.155)$$

$$= -1 < 0, \quad (3.156)$$

and

$$\bar{b}^2 - 4\bar{a}\bar{c} = [-2bk_2 + (\alpha + 1)(a - ck_1)]^2 - 4c(\alpha + 1)[-ak_1(\alpha + 1) - 2ck_2], \quad (3.157)$$

$$= \left[ -2k_2 \frac{(\alpha + 1)(a - ck_1)}{2k_2} + (\alpha + 1)(a - ck_1) \right]^2 - 4(\alpha + 1), \quad (3.158)$$

$$= -4(\alpha + 1) < 0. \quad (3.159)$$

Thus, we proved that  $\dot{V}$  is definite negative and that the system (3.129)-(3.130) is asymptotically stable. Combining this result with the homogeneity of the system (3.129)-(3.130) with degree  $\nu < 0$ , implies that the system (3.129)-(3.130) is **FTS**. (Note that by making use of Lemma A.1, we can give some similar computations to ensure **FxTS** instead).

Now, let  $\mathcal{W}$  be a compact defined as follows (see e.g. [187]):

$$\mathcal{W} := \{(\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^2 : V(\tilde{z}_1, \tilde{z}_2) = 1\}, \quad (3.160)$$

and set the constant  $\delta$  as follows:

$$\delta = - \max_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}} \{\dot{V}(\tilde{z}_1, \tilde{z}_2)\} = \min_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}} \{-\dot{V}(\tilde{z}_1, \tilde{z}_2)\} > 0. \quad (3.161)$$

Using the homogeneity of  $\dot{V}$  for  $\lambda = \left( V(\tilde{z}_1, \tilde{z}_2)^{\frac{-1}{\nu}} \right)$ , we have

$$\dot{V}(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2) = \frac{\dot{V}(\tilde{z}_1, \tilde{z}_2)}{V(\tilde{z}_1, \tilde{z}_2)^{\frac{\nu''}{\nu}}}. \quad (3.162)$$

On the other hand, we have

$$\{(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2) : (\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^2 \setminus \{[0, 0]\}\} = \mathcal{W}, \quad (3.163)$$

because

$$V(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2) = \left( V(\tilde{z}_1, \tilde{z}_2)^{\frac{-1}{\nu}} \right)^{\nu'} V(\tilde{z}_1, \tilde{z}_2) = 1. \quad (3.164)$$

Therefore,

$$\sup_{(\tilde{z}_1, \tilde{z}_2) \neq [0, 0]} \frac{\dot{V}(\tilde{z}_1, \tilde{z}_2)}{[V(\tilde{z}_1, \tilde{z}_2)]^{\frac{\nu''}{\nu}}} = \sup_{(\tilde{z}_1, \tilde{z}_2) \neq [0, 0]} \dot{V}(\lambda^{r_1} \tilde{z}_1, \lambda^{r_2} \tilde{z}_2), \quad (3.165)$$

$$= \sup_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}} \dot{V}(\tilde{z}_1, \tilde{z}_2), \quad (3.166)$$

$$= -\delta. \quad (3.167)$$

Hence it follows that:

$$\dot{V}(\tilde{z}_1, \tilde{z}_2) \leq -\delta V(\tilde{z}_1, \tilde{z}_2)^{\frac{\nu''}{\nu}}. \quad (3.168)$$

Integrating this inequality between  $t_0 + D$  and  $t$ , leads us to

$$V(\tilde{z}_1(t), \tilde{z}_2(t))^{1-\frac{\nu''}{\nu'}} \leq -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) + V(\tilde{z}_1(t_0 + D), \tilde{z}_2(t_0 + D))^{1-\frac{\nu''}{\nu'}}. \quad (3.169)$$

Then, we get

$$V(\tilde{z}_1(t), \tilde{z}_2(t)) \leq \left[ -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) + V(\tilde{z}_1(t_0 + D), \tilde{z}_2(t_0 + D))^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}}. \quad (3.170)$$

Moreover, we obtain

$$\begin{aligned} |\tilde{z}_1(t)|^{\alpha+1} + |\tilde{z}_2(t)|^2 &\leq \frac{1}{P_{min}} \left[ -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) + (P_{max})^{1-\frac{\nu''}{\nu'}} (|\tilde{z}_1(t_0 + D)|^{\alpha+1} \right. \\ &\quad \left. + |\tilde{z}_2(t_0 + D)|^2)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}}, \end{aligned} \quad (3.171)$$

$$\begin{aligned} &\leq \frac{1}{P_{min}} \left[ -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) + (P_{max})^{1-\frac{\nu''}{\nu'}} \left( \|\tilde{z}(t_0 + D)\|_2^{\alpha+1} \right. \right. \\ &\quad \left. \left. + \|\tilde{z}(t_0 + D)\|_2^2 \right)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}}, \end{aligned} \quad (3.172)$$

where we have used the following property:

$$P_{min} (|\tilde{z}_1|^{\alpha+1} + |\tilde{z}_2|^2) \leq V(\tilde{z}_1, \tilde{z}_2) \leq P_{max} (|\tilde{z}_1|^{\alpha+1} + |\tilde{z}_2|^2), \quad (3.173)$$

with  $P_{min}$  and  $P_{max}$  are the eigenvalues of  $\bar{P}$  with

$$0 < P_{min} \leq P_{max}.$$

Finally, using the fact that

$$\begin{aligned} |\tilde{z}_1(t)|^2 &\leq \left[ \frac{1}{P_{min}} \left[ -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) + (P_{max})^{1-\frac{\nu''}{\nu'}} \left( \|\tilde{z}(t_0 + D)\|_2^{\alpha+1} \right. \right. \right. \\ &\quad \left. \left. + \|\tilde{z}(t_0 + D)\|_2^2 \right)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}} \right]^{\frac{2}{1+\alpha}}, \end{aligned} \quad (3.174)$$

we get

$$\begin{aligned} \|\tilde{z}(t)\|_2^2 &\leq \frac{1}{P_{min}} \left[ -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) + (P_{max})^{1-\frac{\nu''}{\nu'}} \left( \|\tilde{z}(t_0 + D)\|_2^{\alpha+1} \right. \right. \\ &\quad \left. \left. + \|\tilde{z}(t_0 + D)\|_2^2 \right)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}} + \left( \frac{1}{P_{min}} \left[ -\delta \left(1 - \frac{\nu''}{\nu'}\right) (t - t_0 - D) \right. \right. \\ &\quad \left. \left. + (P_{max})^{1-\frac{\nu''}{\nu'}} \left( \|\tilde{z}(t_0 + D)\|_2^{\alpha+1} + \|\tilde{z}(t_0 + D)\|_2^2 \right)^{1-\frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}} \right)^{\frac{2}{1+\alpha}}. \end{aligned} \quad (3.175)$$

Then,

$$\|\tilde{z}(t)\|_2^2 \leq F(t - t_0 - D, \|\tilde{z}(t_0 + D)\|_2) + F(t - t_0 - D, \|\tilde{z}(t_0 + D)\|_2)^{\frac{2}{1+\alpha}}, \quad (3.176)$$

where  $F$  is the increasing function defined by (3.127).

In particular, when  $t$  tend to the settling time  $t_0 + D + T_{\max}(\|\tilde{z}(t_0 + D)\|_2)$ , the function  $F$  and the norm  $\|\tilde{z}(t)\|_2$  goes to zero, where  $T_{\max}$  is given by (3.128).  $\square$

**Remark 3.6**

For the general case, we can recover a similar inequality to (3.126) from [186, Theorem 6] to get **FTS** or from [186, Theorem 7] to get **FxTS**.

**Remark 3.7**

Note that the problem of finite-time stability of the target system (3.129)-(3.130) is studied in the presence of some disturbances in [188] using an implicit Lyapunov-based approach and in [189] using a strict explicit Lyapunov-based approach. In the absence of the disturbance in [188], the settling-time is then estimated by the following formulas:

$$T_{\max}(s_1, s_2) = \frac{3 + \alpha}{\bar{\delta}(1 - \alpha)} V_0(s_1, s_2)^{\frac{1-\alpha}{3+\alpha}}, \quad \text{for any } s_1, s_2 \in \mathbb{R}^2, \quad (3.177)$$

where  $\bar{\delta} = \min_{(\tilde{z}_1, \tilde{z}_2) \in \mathcal{W}_1} \left\{ l_1 |\tilde{z}_1|^{\frac{2}{2-\alpha}} + l_2 \tilde{z}_2^2 \right\}$ ,  $\mathcal{W}_1 = \{(s_1, s_2) \in \mathbb{R}^2 : V_0(s_1, s_2) = 1\}$ ,  $l_1 = \bar{C} \frac{(3+\alpha)k_1}{2(1+\alpha)^{\frac{1-\alpha}{2+2\alpha}}}$ ,  $-\sqrt{k_2} - \frac{k_1^2}{\sqrt{k_2}}$ ,  $l_2 = \frac{1}{2\sqrt{k_2}}$ , and  $V_0(s_1, s_2) = \bar{C} \left( \frac{|s_1|^{\alpha+1}}{\alpha+1} + \frac{s_2^2}{2k_2} \right)^{\frac{3+\alpha}{2+2\alpha}} - \frac{s_1 s_2}{\sqrt{k_2}}$ , with  $\alpha = \alpha_2$ ,  $k_1$ , and  $k_2$  are given in (3.37)-(3.40), and with some constant  $\bar{C} > \left( \frac{2+2\alpha}{3+\alpha} \right)^{\frac{3+\alpha}{2+2\alpha}}$  such that  $l_1 > 0$  and  $V_0(s_1, s_2) > 0$ . This estimation is quite similar to our estimation (3.128), but may be less accurate than our estimation or even far from the real settling-time as we will see in the simulation later.

Now, let us state the main result of this chapter.

**Theorem 3.1**

Let the matrices  $A$ ,  $B$ , and  $C$  be as in (3.2)-(3.3), the vector function  $\mathcal{L}(\cdot) = [(\mathcal{L}_1(\cdot), \mathcal{L}_2(\cdot))]$  be as in (3.35) and (3.36), the function  $F$  be as in (3.127), the function  $G_1$  be as in (3.105) and  $G_2$  as in (3.110),  $\delta$  be given as in (3.161),  $P_{\min}$  and  $P_{\max}$  be as in (3.173). Let  $t_0$ ,  $D > 0$ . Then, for any initial conditions  $z_0$ ,  $\hat{z}_0$  the observer system guarantees that  $[\hat{z}, \hat{u}]$  converges to  $[z, u]$  within a finite time. Moreover the quantity  $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2$  is bounded for all  $t \in [t_0, t_0 + D]$ , and for all  $t \geq t_0 + D$

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2 \leq M \left[ F(t - t_0 - D, G(\|z_0 - \hat{z}_0\|_2)) + F(t - t_0 - D, G(\|z_0 - \hat{z}_0\|_2))^{\frac{2}{1+\alpha}} \right], \quad (3.178)$$

with  $M = \left( 2\|C e^{A \cdot}\|_{L^2(0,D)}^2 + 1 \right)$  and  $G = G_1$  (or  $G = G_2$ ).

In particular, the norm  $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2 \rightarrow 0$ , as  $t \rightarrow t_0 + D + \bar{T}_{\max}(\|z_0 - \hat{z}_0\|_2)$ , where  $\bar{T}_{\max}$  is given for any  $s \geq 0$  by

$$\bar{T}_{\max}(s) = \frac{\nu'}{\delta(\nu' - \nu'')} (P_{\max})^{1 - \frac{\nu''}{\nu'}} \left( G(s)^{\alpha+1} + G(s)^2 \right)^{1 - \frac{\nu''}{\nu'}}, \quad (3.179)$$

with  $\alpha \in (0, 1)$ ,  $\nu' = \alpha + 1$ ,  $\nu'' = \frac{3\alpha+1}{2}$ .

*Proof.* Using the transformation (3.17) and the Cauchy-Schwartz inequality, we obtain

$$|\tilde{u}(t, x)| \leq |\tilde{\omega}(t, x)| + \|C e^{Ax}\| \|\tilde{z}(t)\|. \quad (3.180)$$



Squaring this inequality and integrating with respect to the variable  $x$  between 0 and  $D$ , gives us

$$\|\tilde{u}(t, \cdot)\|_{L^2(0,D)}^2 \leq 2\|\tilde{\omega}(t, \cdot)\|_{L^2(0,D)}^2 + 2\|Ce^{A\cdot}\|_{L^2(0,D)}^2 \|\tilde{z}(t)\|_2^2. \quad (3.181)$$

Adding the norm  $\|z(t) - \hat{z}(t)\|_2^2$  in both sides and using (3.12)-(3.13), we get

$$\begin{aligned} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2 &\leq 2\|\tilde{\omega}(t, \cdot)\|_{L^2(0,D)}^2 + 2\|Ce^{A\cdot}\|_{L^2(0,D)}^2 \|\tilde{z}(t)\|_2^2 \\ &\quad + \|z(t) - \hat{z}(t)\|_2^2, \end{aligned} \quad (3.182)$$

$$\begin{aligned} &\leq 2\|\tilde{\omega}(t, \cdot)\|_{L^2(0,D)}^2 + \left(2\|Ce^{A\cdot}\|_{L^2(0,D)}^2 + \|e^{AD}\|_2^2\right) \\ &\quad \times \|\tilde{z}(t)\|_2^2. \end{aligned} \quad (3.183)$$

Using Lemma 3.1 and the last inequality, we see that  $\|\tilde{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|\tilde{z}(t)\|_2^2$  is bounded for all  $t \in [t_0, t_0 + D]$ .

On the other hand, for all  $t \geq t_0 + D$ , employing Lemma 3.4 and the fact that  $\tilde{\omega}(t, \cdot) \equiv 0$  for all  $t \geq t_0 + D$ , we get

$$\begin{aligned} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2 &\leq M \left[ F(t - t_0 - D, \|\tilde{z}(t_0 + D)\|_2) \right. \\ &\quad \left. + F(t - t_0 - D, \|\tilde{z}(t_0 + D)\|_2)^{\frac{2}{1+\alpha}} \right], \end{aligned} \quad (3.184)$$

where  $M = 2\|Ce^{A\cdot}\|_{L^2(0,D)}^2 + \|e^{AD}\|_2^2$  and  $F$  is given for any  $s \geq 0$  in (3.127) as follows:

$$F(t - t_0 - D, s) = \frac{1}{P_{min}} \left[ -\delta \left( 1 - \frac{\nu''}{\nu'} \right) (t - t_0 - D) + (P_{max})^{1 - \frac{\nu''}{\nu'}} (s^{\alpha+1} + s^2)^{1 - \frac{\nu''}{\nu'}} \right]^{\frac{\nu'}{\nu' - \nu''}}. \quad (3.185)$$

Employing Lemma 3.2 and the fact that  $F$  is increasing with respect to the second variable, we obtain

$$\begin{aligned} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2 &\leq M \left[ F(t - t_0 - D, G(\|z_0 - \hat{z}_0\|_2)) \right. \\ &\quad \left. + F(t - t_0 - D, G(\|z_0 - \hat{z}_0\|_2))^{\frac{2}{1+\alpha}} \right], \end{aligned} \quad (3.186)$$

where  $G = G_1$  and  $G_1$  is the function defined by (3.105). In addition, note that using Lemma 3.3 instead of Lemma 3.2 we find the same inequality as (3.186) with  $G = G_2$  and  $G_2$  is defined by (3.110).

Thus, we finally conclude that the norm  $\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2 + \|z(t) - \hat{z}(t)\|_2^2 \rightarrow 0$ , as  $t \rightarrow t_0 + D + \bar{T}_{max}(\|z_0 - \hat{z}_0\|_2)$ , where  $\bar{T}_{max}$  is given for any  $s \geq 0$  by

$$\bar{T}_{max}(s) = T_{max}(G(s)) = \frac{\nu'}{\delta(\nu' - \nu'')} (P_{max})^{1 - \frac{\nu''}{\nu'}} (G(s)^{\alpha+1} + G(s)^2)^{1 - \frac{\nu''}{\nu'}}. \quad (3.187)$$

□

**Remark 3.8**

*Notice that it is possible to use some change of variables (i.e.  $\bar{z}(t) = z(t - D)$ ) to pass the delay (given in (3.3)) to the input (appearing in (3.2)). In this situation, the observer design is simplified as we are only required to design a finite-dimensional observer. This task can be achieved using classical techniques based, for instance, on the Homogeneity theory (see e.g. [186]). However, passing the delay to the input will yield difficulties for the control design. To deal with it, we can use similar techniques based on the backstepping approach. Solving this problem will be the main objective of our next chapter.*

## 3.4 Simulation

In this section, we illustrate our results using some numerical examples.

### 3.4.1 An academic example

Let us start with a simple academic example. Consider (3.4)-(3.7) with  $a_0 = a_1 = 1$ . For our simulation, we consider 5 different initial conditions:  $z(t_0) := z_0 = [3, 8]^\top$ ,  $10z_0$ ,  $100z_0$ ,  $1000z_0$  and  $10000z_0$ . We take the initial time  $t_0 = 0$ , the control  $U(t) = -7z_1(t) - 5z_2(t)$  and the nonlinear observer (3.8)-(3.10) with  $n = 2$  and the **FxTS** observer gain functions (3.41)-(3.41). We take the parameters of the observer gain function  $\mathcal{L}$  given in (3.41)-(3.42) as follows:  $k_1 = k_3 = 1.5$ ,  $k_2 = k_4 = 2.5$ ,  $\alpha_2 = 0.7$ ,  $\alpha_1 = \frac{1+\alpha_2}{2}$ ,  $\beta_2 = 2$ ,  $\beta_1 = \frac{1+\beta_2}{2}$  and the delay  $D = 1$ s.

Figure 3.1, we plot the states  $z_1$  and  $z_2$  of the system (3.4)-(3.7) in solid blue lines and the estimated states  $\hat{z}_1$  and  $\hat{z}_2$  of (3.8)-(3.10) in dashed red lines, with feedback  $U(t) = -7z_1(t) - 5z_2(t)$ , delay  $D = 1$ s, and initial condition  $z(0) = [3, 8]^\top$ . Figure 3.2, we have on the left the evolution of the error states of the ODE part of the error system (3.14) multiplied by  $e^{AD}$  with feedback  $U(t) = -7z_1(t) - 5z_2(t)$  and delay  $D = 1$ s. On the right hand, the evolution of the error state of the PDE part of the error system (3.15)-(3.16). Finally, Figure 3.3 presents on the left, the evolution of the norm  $\|z(t) - \hat{z}(t)\|_2^2 + \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2$  of the error system, shown in a logarithmic scale, with feedback  $U(t) = -7z_1(t) - 5z_2(t)$  and delay  $D = 1$ s, where we can observe that the solutions converge to the origin in a finite-time upper bounded by  $t_0 + D + T_{max} = 10.6$ s (numerically estimated) for all the initial conditions. On the right, we observe the evolution of the outputs  $Y(t) = z_1(t - D) = u(t, 0)$  in a solid line and the estimated outputs  $\hat{u}(t, 0)$  in a dashed line with feedback  $U(t) = -7z_1(t) - 5z_2(t)$  and delay  $D = 1$ s

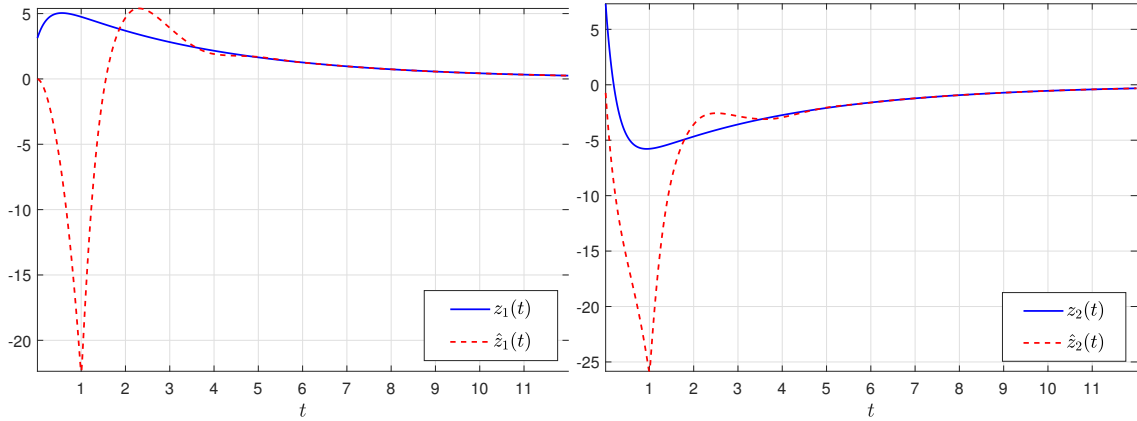


Figure 3.1: The evolution of the states  $z_1$  and  $z_2$  of the system (3.4)-(3.7) in solid blue lines and the estimated states  $\hat{z}_1$  and  $\hat{z}_2$  of (3.8)-(3.10) in dashed red lines, with feedback  $U(t) = -7z_1(t) - 5z_2(t)$ , delay  $D = 1$ s, and initial condition  $z(0) = [3, 8]^\top$ .

### 3.4.2 Application to a simplified robot manipulator

Now, let us focus on a simplified robot manipulator described in Figure 3.4, where for  $i \in \{1, 2\}$ ,  $q_i$  and  $m_i$  are, respectively, the angle and the mass of the  $i^{\text{th}}$  joint of the robot. Let us assume that

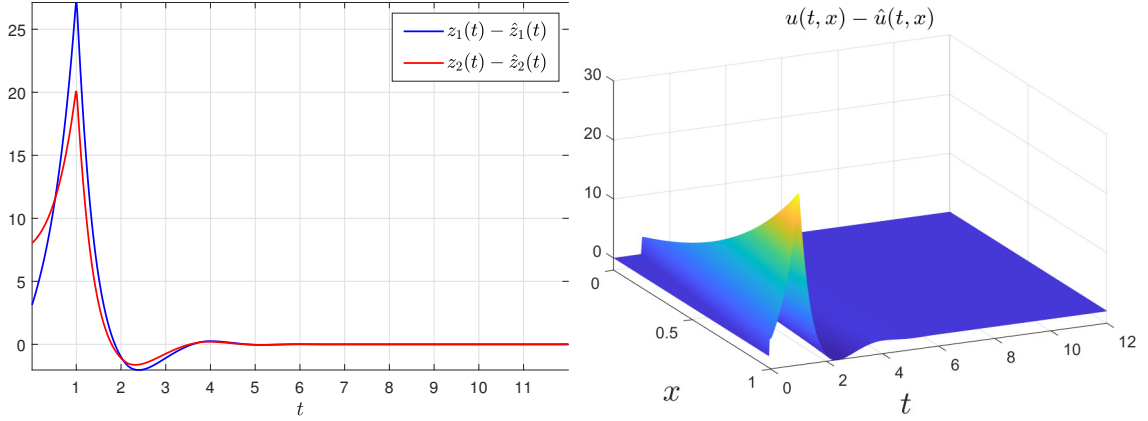


Figure 3.2: On the left hand, we have the evolution of the error states of the ODE part of the error system (3.14) multiplied by  $e^{AD}$  with feedback  $U(t) = -7z_1(t) - 5z_2(t)$  and delay  $D = 1$ s. On the right hand, the evolution of the error state of the PDE part of the error system (3.15)-(3.16).

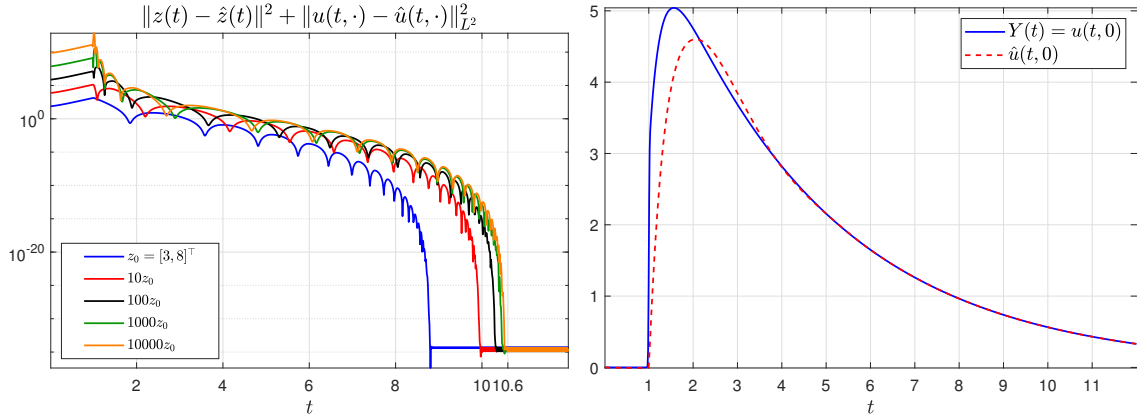


Figure 3.3: On the left, the evolution of the norm  $\|z(t) - \hat{z}(t)\|_2^2 + \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0,D)}^2$  of the error system, shown in a logarithmic scale, with feedback  $U(t) = -7z_1(t) - 5z_2(t)$  and delay  $D = 1$ s, for 5 different initial conditions  $z_0 = [3, 8]^\top$ ,  $10z_0$ ,  $100z_0$ ,  $1000z_0$ , and  $10000z_0$ . On the right, the evolution of the outputs  $Y(t) = z_1(t - D) = u(t, 0)$  in a solid line and the estimated outputs  $\hat{u}(t, 0)$  in a dashed line with feedback  $U(t) = -7z_1(t) - 5z_2(t)$  and delay  $D = 1$ s.

the manipulator is fully actuated, i.e., the number of actuators is equal to 2 (degree of freedom), and that the masses of the links are neglected with respect to the masses of the motors and payload ( $m_2$ ). Then, the model is (3.1) ( i.e.,  $M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$ ), with

$$\begin{aligned}
 C(q, \dot{q}) &= m_2 l_2 l_1 \sin(q_2) \begin{bmatrix} -\dot{q}_2^2 - 2\dot{q}_1 \dot{q}_2 \\ \dot{q}_1^2 \end{bmatrix}, \\
 G(q) &= g \begin{bmatrix} m_2 l_2 \sin(q_1 + q_2) + (m_1 + m_2) l_1 \sin(q_1) \\ m_2 l_2 \sin(q_1 + q_2) \end{bmatrix}, \\
 M(q) &= \begin{bmatrix} (m_1 + m_2) l_1^2 + 2m_2 l_1 l_2 \cos(q_2) + m_2 l_2^2 & m_2 l_2^2 + m_2 l_1 l_2 \cos(q_2) \\ m_2 l_2^2 + m_2 l_1 l_2 \cos(q_2) & m_2 l_2^2 \end{bmatrix},
 \end{aligned}$$

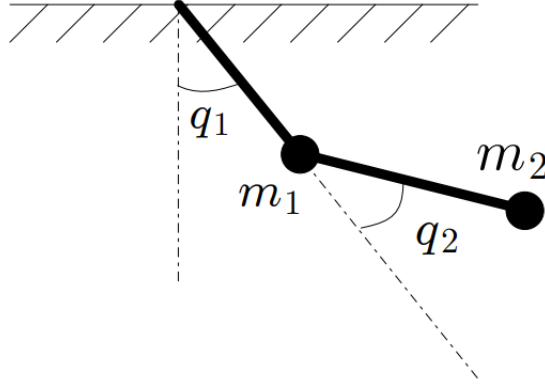


Figure 3.4: Two-links manipulator

where for any  $i \in \{1, 2\}$ ,  $l_i$  is the  $i^{\text{th}}$  length of the link. Next, (3.1) with feedback law

$$\tau = M(q)W + C(q, \dot{q}) + G(q), \quad (3.188)$$

where  $W = [w_1, w_2]^T \in \mathbb{R}^2$  is the new input, gives the following closed-loop system:

$$\ddot{q}_i(t) = w_i(t), \quad t \geq t_0, \quad i \in \{1, 2\}, \quad (3.189)$$

(where we have used the fact that  $M(q)$  is a symmetric positive-definite matrix). However, since  $q$  and  $\dot{q}$  are not directly available by measurements we can not use feedback (3.188) but instead, we can use feedback

$$\hat{\tau} = M(\hat{q})W + C(\hat{q}, \hat{\dot{q}}) + G(\hat{q}). \quad (3.190)$$

Thus, the system (3.189) should be replaced by a similar perturbed model,

$$\ddot{q}_i(t) = w_i(t) + \delta_i \left( W(t), q(t), \dot{q}(t), \hat{q}(t), \hat{\dot{q}}(t) \right), \quad t \geq t_0, \quad i \in \{1, 2\}, \quad (3.191)$$

where, for  $i \in \{1, 2\}$ ,  $\delta_i$  is a disturbance reflecting mismatched models given by

$$\left[ \delta_1 \left( W, q, \dot{q}, \hat{q}, \hat{\dot{q}} \right), \delta_2 \left( W, q, \dot{q}, \hat{q}, \hat{\dot{q}} \right) \right]^T = M(q)^{-1} [\hat{\tau}(\hat{q}, \hat{\dot{q}}) - M(q)W - C(q, \dot{q}) - G(q)]. \quad (3.192)$$

This fact explains why such a finite-time robust state estimation has to be performed as seen in the simulations below. Using the following change of coordinates:

$$z_1 = q_1, \quad z_2 = \dot{q}_1, \quad z_3 = q_2, \quad z_4 = \dot{q}_2, \quad (3.193)$$

the system (3.191) is transformed into the following two perturbed chains of double integrators:

$$\begin{cases} \dot{z}_1(t) = z_2(t), & t \geq t_0, & (3.194) \\ \dot{z}_2(t) = w_1(t) + \delta_1 \left( W(t), z(t), \bar{z}(t), \hat{z}(t), \hat{\bar{z}}(t) \right), & t \geq t_0, & (3.195) \\ \dot{z}_3(t) = z_4(t), & t \geq t_0, & (3.196) \\ \dot{z}_4(t) = w_2(t) + \delta_2 \left( W(t), z(t), \bar{z}(t), \hat{z}(t), \hat{\bar{z}}(t) \right), & t \geq t_0, & (3.197) \end{cases}$$

where  $z = [z_1, z_2]^T$ ,  $\bar{z} = [z_3, z_4]^T$ , and for  $i \in \{1, 2\}$ ,  $\delta_i$  is now given by

$$\left[ \delta_1 \left( W, z, \bar{z}, \hat{z}, \hat{\bar{z}} \right), \delta_2 \left( W, z, \bar{z}, \hat{z}, \hat{\bar{z}} \right) \right]^T = M(z_3)^{-1} [\hat{\tau}(\hat{z}, \hat{\bar{z}}) - M(z_3)W - C(z_2, z_3, z_4) - G(z_1, z_3)], \quad (3.198)$$

where  $\hat{z} = [\hat{z}_1, \hat{z}_2]^\top$ ,  $\hat{\bar{z}} = [\hat{z}_3, \hat{z}_4]^\top$ .

Then, we take the initialization time  $t_0 = 0$ s, the initial positions  $[z_1(\theta), z_3(\theta)]^\top = [0, 0]^\top rad$  for all  $\theta \in [t_0 - D, t_0)$  and  $[z_1(t_0), z_3(t_0)]^\top = [1, 2]^\top rad$ , and the initial velocities  $[z_2(\theta), z_4(\theta)]^\top = [0, 0]^\top rad/s$  for all  $\theta \in [t_0 - D, t_0]$ . In addition, we assume that  $z_1$  and  $z_3$  are available by delayed measurements, i.e.,

$$\begin{cases} Y_1(t) = z_1(t - D), & t \geq t_0, \\ Y_2(t) = z_3(t - D), & t \geq t_0, \end{cases} \quad (3.199)$$

$$\begin{cases} Y_1(t) = z_1(t - D), & t \geq t_0, \\ Y_2(t) = z_3(t - D), & t \geq t_0, \end{cases} \quad (3.200)$$

with a delay of  $D$  units of time.

Next, by following the same steps as in Section 3.3, we rewrite the two subsystems into an ODE-PDE cascade system,

$$\begin{cases} \dot{z}(t) = Az(t) + B[w_1(t) + \delta_1(W(t), z(t), \bar{z}(t), \hat{z}(t), \hat{\bar{z}}(t)))] , & t \geq t_0, \\ u_{1,t}(t, x) = u_{1,x}(t, x), & t \geq t_0, x \in [0, D], \\ u_1(t, D) = Cz(t), & t \geq t_0, \\ Y_1(t) = u_1(t, 0), & t \geq t_0, \end{cases} \quad (3.201)$$

$$\begin{cases} u_{1,t}(t, x) = u_{1,x}(t, x), & t \geq t_0, x \in [0, D], \\ u_1(t, D) = Cz(t), & t \geq t_0, \\ Y_1(t) = u_1(t, 0), & t \geq t_0, \end{cases} \quad (3.202)$$

$$\begin{cases} u_1(t, D) = Cz(t), & t \geq t_0, \\ Y_1(t) = u_1(t, 0), & t \geq t_0, \end{cases} \quad (3.203)$$

$$\begin{cases} Y_1(t) = u_1(t, 0), & t \geq t_0, \end{cases} \quad (3.204)$$

and

$$\begin{cases} \dot{\bar{z}}(t) = A\bar{z}(t) + B[w_2(t) + \delta_2(W(t), z(t), \bar{z}(t), \hat{z}(t), \hat{\bar{z}}(t)))] , & t \geq t_0, \\ u_{2,t}(t, x) = u_{2,x}(t, x), & t \geq t_0, x \in [0, D], \\ u_2(t, D) = C\bar{z}(t), & t \geq t_0, \\ Y_2(t) = u_2(t, 0), & t \geq t_0, \end{cases} \quad (3.205)$$

$$\begin{cases} u_{2,t}(t, x) = u_{2,x}(t, x), & t \geq t_0, x \in [0, D], \\ u_2(t, D) = C\bar{z}(t), & t \geq t_0, \\ Y_2(t) = u_2(t, 0), & t \geq t_0, \end{cases} \quad (3.206)$$

$$\begin{cases} u_2(t, D) = C\bar{z}(t), & t \geq t_0, \\ Y_2(t) = u_2(t, 0), & t \geq t_0, \end{cases} \quad (3.207)$$

$$\begin{cases} Y_2(t) = u_2(t, 0), & t \geq t_0, \end{cases} \quad (3.208)$$

where  $A$ ,  $B$ , and  $C$  are given in (3.2)-(3.3). The observers are then given by,

$$\begin{cases} \dot{\hat{z}}(t) = A\hat{z}(t) + Bw_1(t) - e^{AD}\mathcal{L}(Y_1(t) - \hat{u}_1(t, 0)), & t \geq t_0, \\ \hat{u}_{1,t}(t, x) = \hat{u}_{1,x}(t, x) - Ce^{Ax}\mathcal{L}(Y_1(t) - \hat{u}_1(t, 0)), & t \geq t_0, x \in [0, D], \\ \hat{u}_1(t, D) = C\hat{z}(t), & t \geq t_0, \end{cases} \quad (3.209)$$

$$\begin{cases} \hat{u}_{1,t}(t, x) = \hat{u}_{1,x}(t, x) - Ce^{Ax}\mathcal{L}(Y_1(t) - \hat{u}_1(t, 0)), & t \geq t_0, x \in [0, D], \\ \hat{u}_1(t, D) = C\hat{z}(t), & t \geq t_0, \end{cases} \quad (3.210)$$

$$\begin{cases} \hat{u}_1(t, D) = C\hat{z}(t), & t \geq t_0, \end{cases} \quad (3.211)$$

and

$$\begin{cases} \dot{\hat{\bar{z}}}(t) = A\hat{\bar{z}}(t) + Bw_2(t) - e^{AD}\mathcal{L}(Y_2(t) - \hat{u}_2(t, 0)), & t \geq t_0, \\ \hat{u}_{2,t}(t, x) = \hat{u}_{2,x}(t, x) - Ce^{Ax}\mathcal{L}(Y_2(t) - \hat{u}_2(t, 0)), & t \geq t_0, x \in [0, D], \\ \hat{u}_2(t, D) = C\hat{\bar{z}}(t), & t \geq t_0, \end{cases} \quad (3.212)$$

$$\begin{cases} \hat{u}_{2,t}(t, x) = \hat{u}_{2,x}(t, x) - Ce^{Ax}\mathcal{L}(Y_2(t) - \hat{u}_2(t, 0)), & t \geq t_0, x \in [0, D], \\ \hat{u}_2(t, D) = C\hat{\bar{z}}(t), & t \geq t_0, \end{cases} \quad (3.213)$$

$$\begin{cases} \hat{u}_2(t, D) = C\hat{\bar{z}}(t), & t \geq t_0, \end{cases} \quad (3.214)$$

where  $\mathcal{L}$  is given in (3.35) and (3.36).

In this case, since we do not design a predictor feedback, one can only ensure the robustness of the controller to small delays (although the finite-time convergence of the observer is guaranteed for larger delays).

To simulate the systems (3.201)-(3.204), (3.205)-(3.208), (3.209)-(3.211), and (3.212)-(3.214), we first discretize them using the two-step variant of the Lax-Friedrichs numerical method introduced in [190], then we use its corresponding solver in Matlab. For simplicity, we chose the parameters of  $\mathcal{L}$  as follows:  $k_1 = k_2 = 4$ ,  $\alpha_2 = 0.8$  and  $\alpha_1 = 0.9$  for both observers, and the delay  $D = 0.04$ s. The rest of the parameters are estimated numerically as follows: the eigenvalues of the matrix  $P$  in (3.136),  $P_{min} = 0.2150$ ,  $P_{max} = 2.2162$ , the coefficient  $\delta = 5.9525$  in

(3.161) which is calculated in Matlab using the function *fmincon*, the state  $z - \hat{z}$  at time  $t_0 + D$ :  $z(t_0 + D) - \hat{z}(t_0 + D) = [0.9933, -0.3206, 2.0048, 0.2360]^\top$ . Therefore, the settling-time of each subsystem is numerically estimated using (3.128):  $T_{1,max}(\|z(t_0 + D) - \hat{z}(t_0 + D)\|_2) = 3.2996s$  and  $T_{2,max}(\|z(t_0 + D) - \hat{z}(t_0 + D)\|_2) = 3.5380s$ .

Next, in Figure 3.5 we give the evolution of the states  $z_1, z_2, z_3,$  and  $z_4$  of the system (3.194)-(3.197) in solid blue lines and the estimated states  $\hat{z}_1, \hat{z}_2, \hat{z}_3,$  and  $\hat{z}_4$  in dashed red lines with the delay  $D = 0.04s$ , the feedback  $\hat{\tau}$  in (3.190) instead of  $\tau$  and the new inputs  $w_1(t) = 2 \sin(10t)$  and  $w_2(t) = 1.4 \sin(20t)$ . Then, in Figure 3.6 we show the evolution of the error states  $[\tilde{z}_1, \tilde{z}_2]^\top$  of the ODE part of the error system linked to the subsystem (3.201)-(3.204) where we have used the feedback  $\hat{\tau}$  (3.188), multiplied by  $e^{AD}$ . In addition, we can observe that the solutions converge to the origin in a finite-time less than  $t_0 + D + T_{1,max} = 3.3396s$  (numerical estimation using (3.128)): the log plot (not shown) confirmed that  $t_0 + D + T_{1,max} \approx 3.31s$ . On the right-hand side, we show the evolution of the error state of the PDE part of the error system. Similarly, Figure 3.7 shows the evolution of the error states  $[\tilde{z}_3, \tilde{z}_4]^\top$  of the ODE part of the error system linked to the subsystem (3.205)-(3.208) with the feedback  $\hat{\tau}$  (3.188), multiplied by  $e^{AD}$ . In addition, we can observe that the solutions converge to the origin in a finite-time less than  $t_0 + D + T_{2,max} = 3.5780s$  (numerical estimation using (3.128)): the log plot (not shown) confirmed that  $t_0 + D + T_{2,max} \approx 3.5s$ . On the right hand, we show the evolution of the error state of the PDE part of the error system. Finally, Figure 3.8 present on the left, the evolution of the norm  $\|z(t) - \hat{z}(t)\|_2^2 + \|\bar{z}(t) - \hat{\bar{z}}(t)\|_2^2 + \sum_{i=1}^2 \|u_i(t, \cdot) - \hat{u}_i(t, \cdot)\|_{L^2(0,D)}^2$  of the error systems (in a logarithmic scale), where is shown to converge to zero in finite-time less than  $t_0 + D + T_{max} \approx 3.5s$  (compare to 3.578s obtained using (3.128)). On the right, we observe the evolution of the outputs  $Y_1(t) = z_1(t-D) = u_1(t, 0)$  and  $Y_2(t) = z_3(t-D) = u_2(t, 0)$  in solid lines and the estimated outputs  $\hat{u}_1(t, 0)$  and  $\hat{u}_2(t, 0)$  in dashed lines.

In Figures 3.9, 3.10, 3.11, and 3.12, we add noise measurement of power 0.0001 and we give the same simulations as before and verify the impact of noise to the convergence of the closed-loop. As anticipated, the observer is robust with respect to the modeled uncertainties and the noise measurement. However, we only ensure a finite-time convergence to a neighborhood of the origin characterized by the magnitude of the noise.

### Remark 3.9

Now, following Remark 3.7 with  $\bar{C} = 1.7$ ,  $\alpha$ ,  $k_1$ , and  $k_2$  as chosen in the above simulations (i.e.,  $k_1 = k_2 = 4$ ,  $\alpha = \alpha_2 = 0.8$ ), one can numerically estimate (3.177) as follows:  $T_{1,max} = 16.9744s$  and  $T_{2,max} = 18.4054s$ . Indeed, to get these estimates, we have  $l_1 = 2.5049$ ,  $l_2 = 0.25$ ,  $V_{0,1}(z(t_0 + D) - \hat{z}(t_0 + D)) = 0.6883$ ,  $V_{0,2}(z(t_0 + D) - \hat{z}(t_0 + D)) = 3.2034$  and  $\bar{\delta} = 1.075$  which is calculated using the Matlab function *fmincon*. We can clearly see that these two values are far from the settling time observed in the simulations (see Figures 3.6, 3.7 and 3.8) and our estimates using (3.128) ( $T_{1,max} = 3.3396s$  and  $T_{2,max} = 3.5380s$ ).

## Conclusion

In this chapter, we dealt with the problem of finite-time estimation of linear time-invariant (LTI) systems, in the observable form, with delayed output. The main ideas relied on rewriting the system into an ODE-PDE cascade setting, where the PDE part modeled the effect of the delay on the output. The nonlinear gains were designed such that the error observer system is either **FTS** or **FxTS**. To achieve this, we used the backstepping approach where we chose a suitable

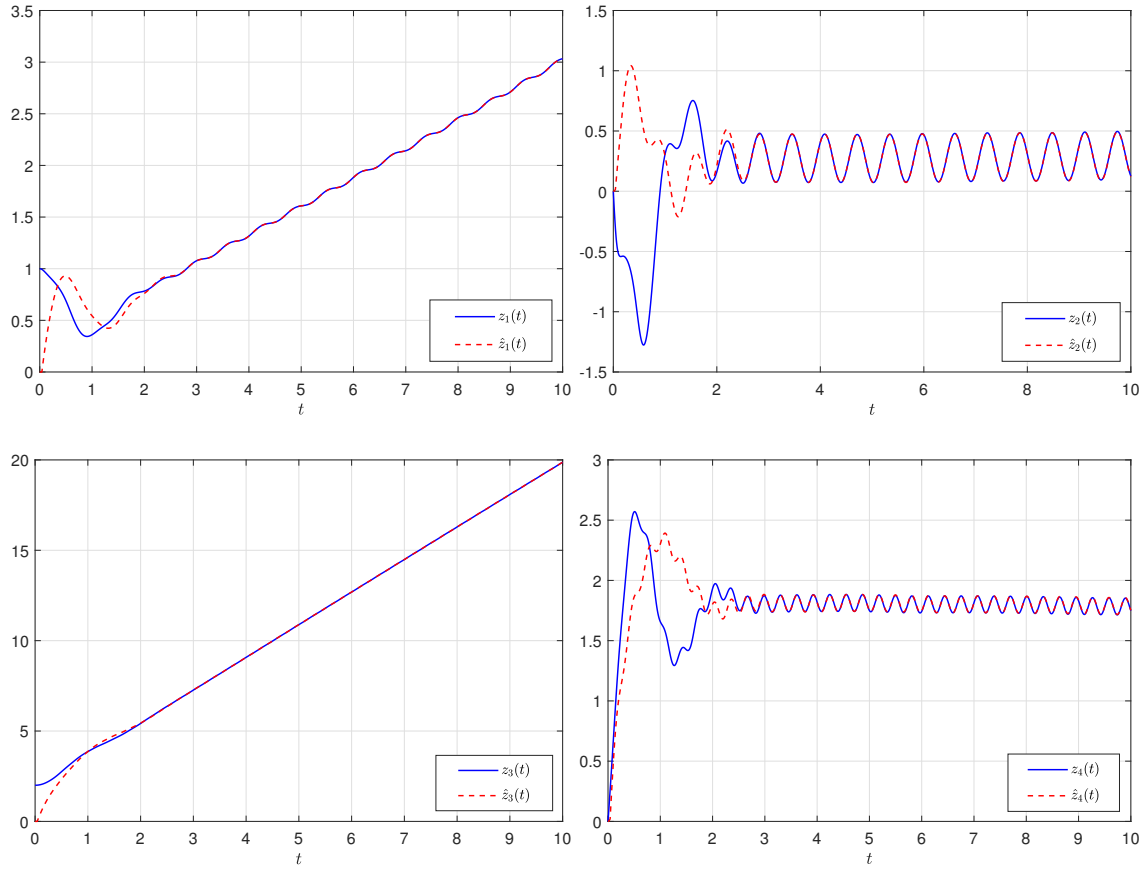


Figure 3.5: The evolution of the states  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  of the system (3.194)-(3.197) in solid blue lines and the estimated states  $\hat{z}_1$ ,  $\hat{z}_2$ ,  $\hat{z}_3$ , and  $\hat{z}_4$  in dashed red lines, with feedback  $\hat{\tau}$  in (3.190) and without noise measurement, where the delay  $D = 0.04\text{s}$ , the initial positions  $[z_1(0), z_3(0)]^\top = [1, 2]^\top$ , the initial velocities  $[z_2(0), z_4(0)]^\top = [0, 0]^\top \text{m/s}$ , and the input  $w_1(t) = 2 \sin(10t)$  and  $w_2(t) = 1.4 \sin(20t)$ .

nonlinear target system satisfying a chosen finite/fixed-time convergence property. Finally, we used the invertibility of the backstepping transformation to pass this property to the error system.

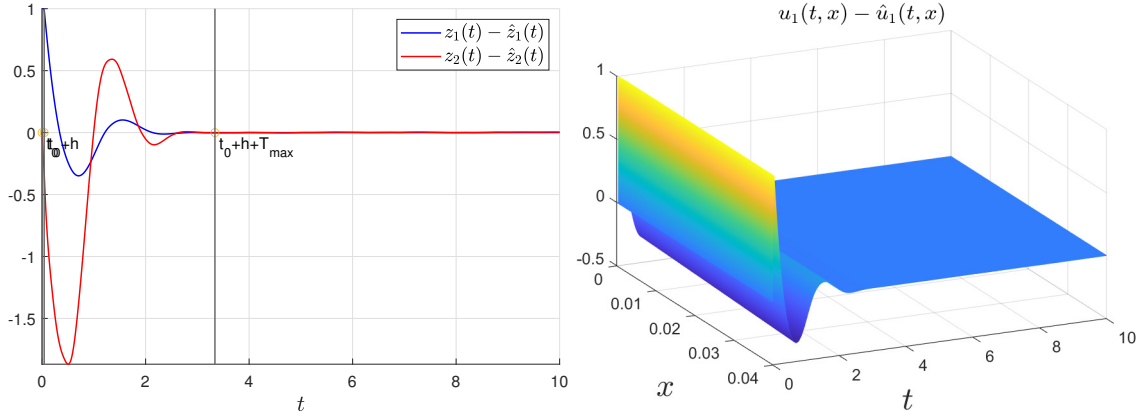


Figure 3.6: On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (3.201)-(3.204) multiplied by  $e^{AD}$  with feedback  $\hat{\tau}$  in (3.190) and without noise measurement, where  $D = 0.04s$  and  $T_{1,max} = 3.3396s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (3.190) and without noise measurement.

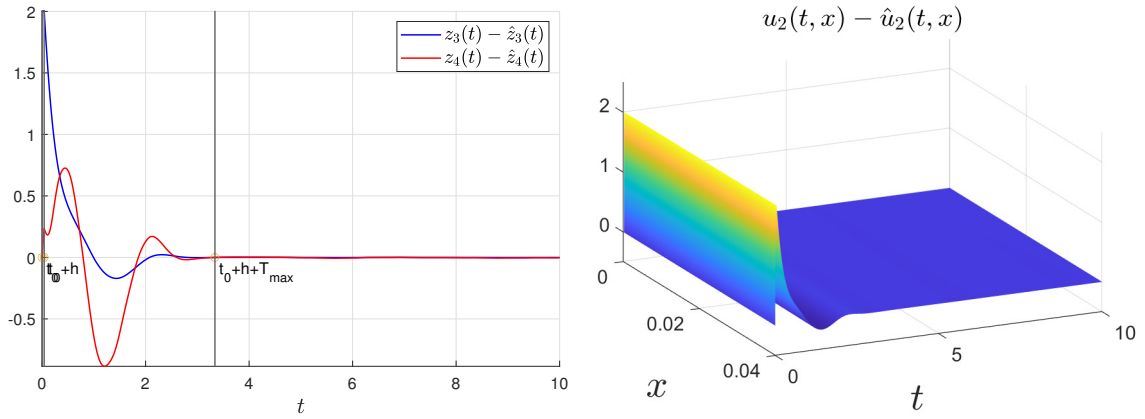


Figure 3.7: On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (3.205)-(3.208) multiplied by  $e^{AD}$  with feedback  $\hat{\tau}$  in (3.190) and without noise measurement, where  $D = 0.04s$  and  $T_{2,max} = 3.5380s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (3.190) and without noise measurement.



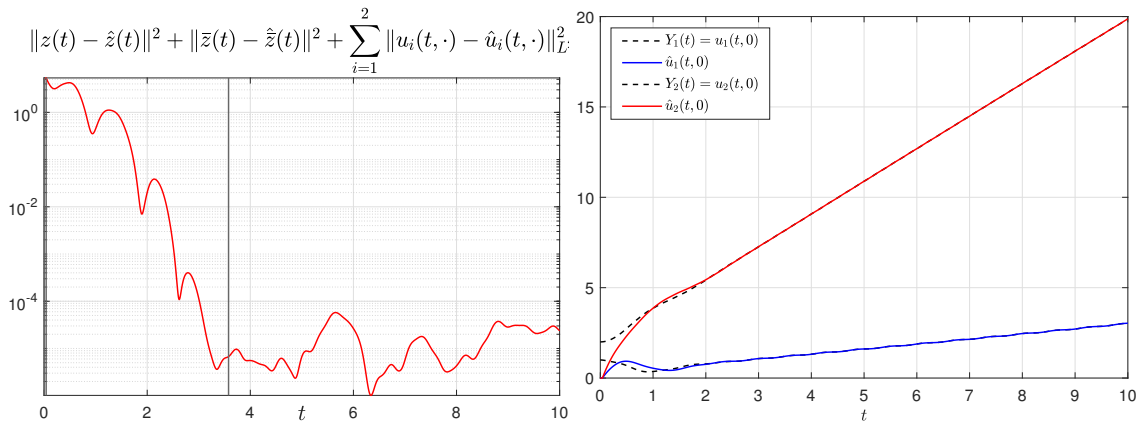


Figure 3.8: On the left, the evolution of the norm  $\|z(t) - \hat{z}(t)\|_2^2 + \|\bar{z}(t) - \hat{\bar{z}}(t)\|_2^2 + \sum_{i=1}^2 \|u_i(t, \cdot) - \hat{u}_i(t, \cdot)\|_{L^2(0,D)}^2$  of the two error systems (plotted in logarithmic scale) with feedback  $\hat{\tau}$  in (3.190) and without noise measurement, with the delay  $D = 0.04$ , and  $T_{\max} = 3.5380$ s. On the right, the evolution of the outputs  $Y_1(t) = z_1(t - D) = u_1(t, 0)$  and  $Y_2(t) = z_3(t - D) = u_2(t, 0)$  in solid lines and the estimated outputs  $\hat{u}_1(t, 0)$  and  $\hat{u}_2(t, 0)$  in dashed with feedback  $\hat{\tau}$  in (3.190) and lines without noise measurement.

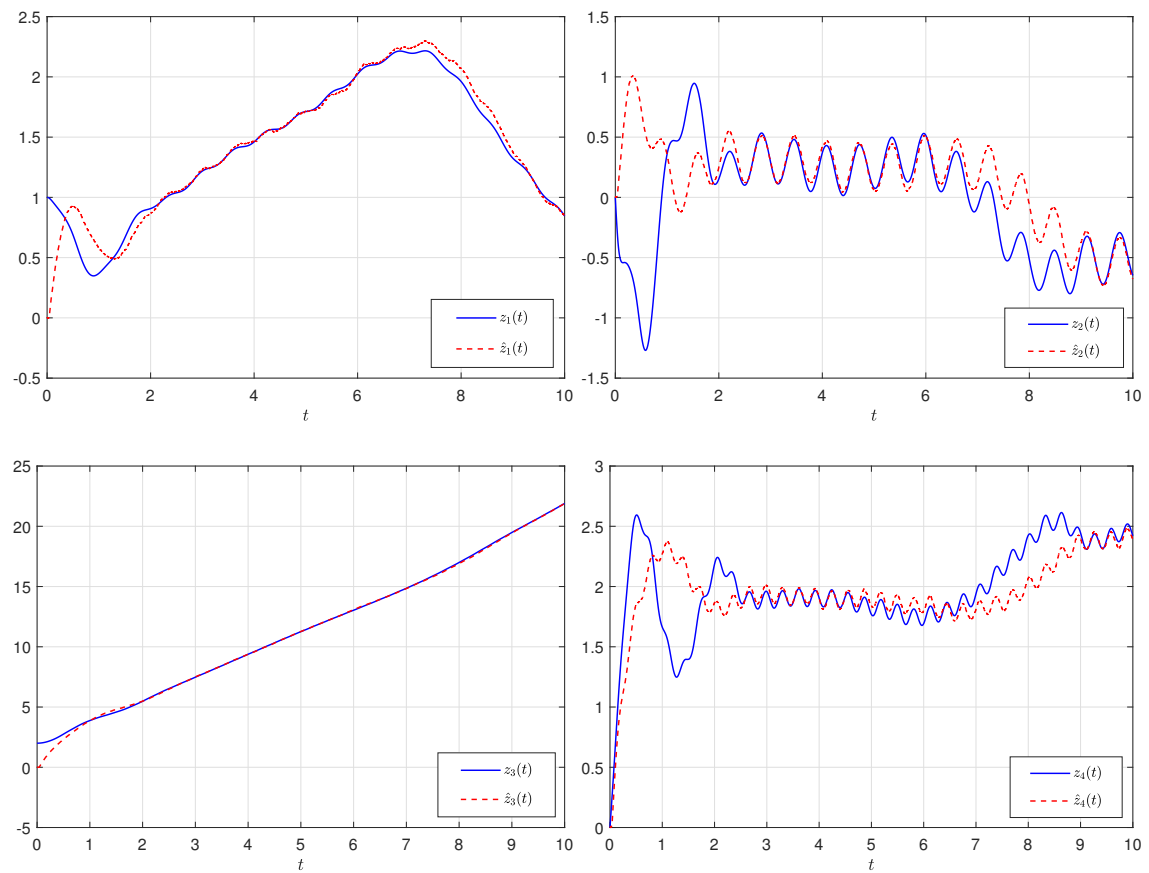


Figure 3.9: The evolution of the states  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  of the system (3.194)-(3.197) in blue lines and the estimated states  $\hat{z}_1$ ,  $\hat{z}_2$ ,  $\hat{z}_3$ , and  $\hat{z}_4$  in red lines, with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001, where the delay  $D = 0.04$ s, the initial positions  $[z_1(0), z_3(0)]^\top = [1, 2]^\top$ , the initial velocities  $[z_2(0), z_4(0)]^\top = [0, 0]^\top$  m/s, and the input  $w_1(t) = 2 \sin(10t)$  and  $w_2(t) = 1.4 \sin(20t)$ .

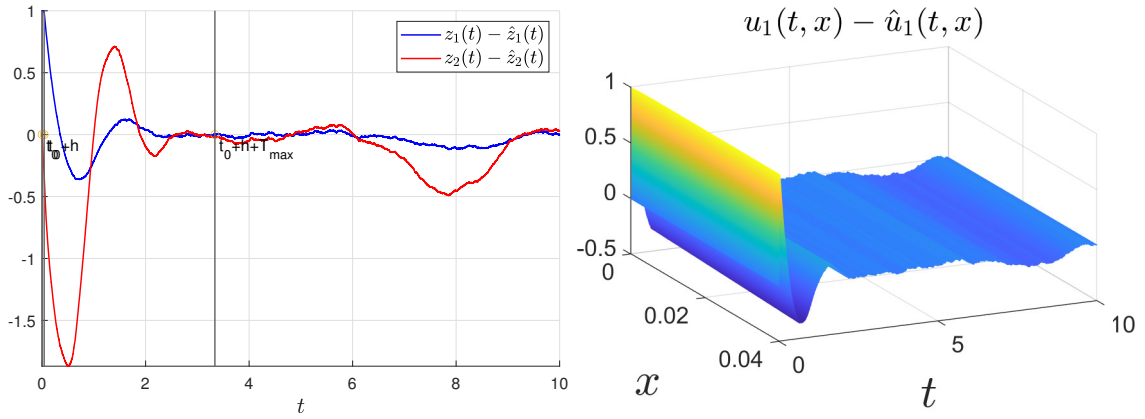


Figure 3.10: On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (3.201)-(3.204) multiplied by  $e^{AD}$  with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001, where  $D = 0.04s$  and  $T_{1,max} = 3.3024s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001.

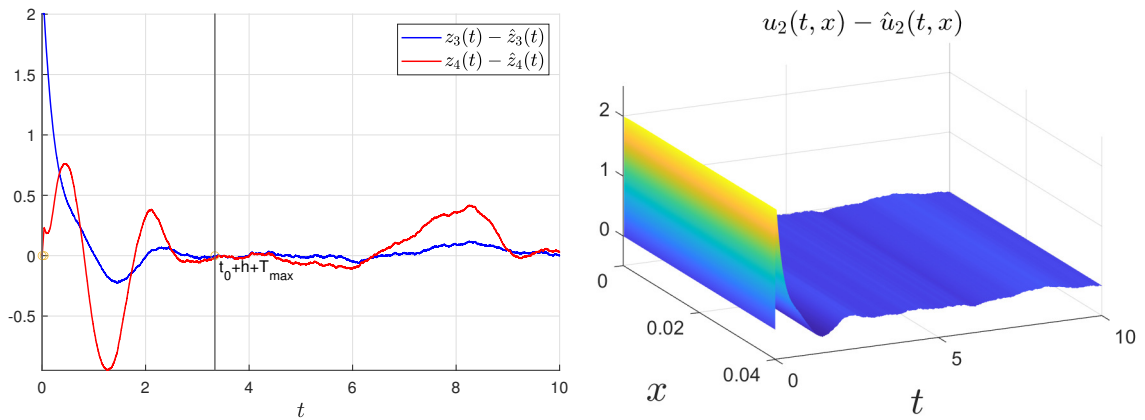


Figure 3.11: On the left hand, we have the evolution of the error states of the ODE part of the error system linked to the subsystem (3.205)-(3.208) multiplied by  $e^{AD}$  with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001, where  $D = 0.04s$  and  $T_{2,max} = 3.5381s$ . On the right hand, the evolution of the error state of the PDE part of the error system with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001.

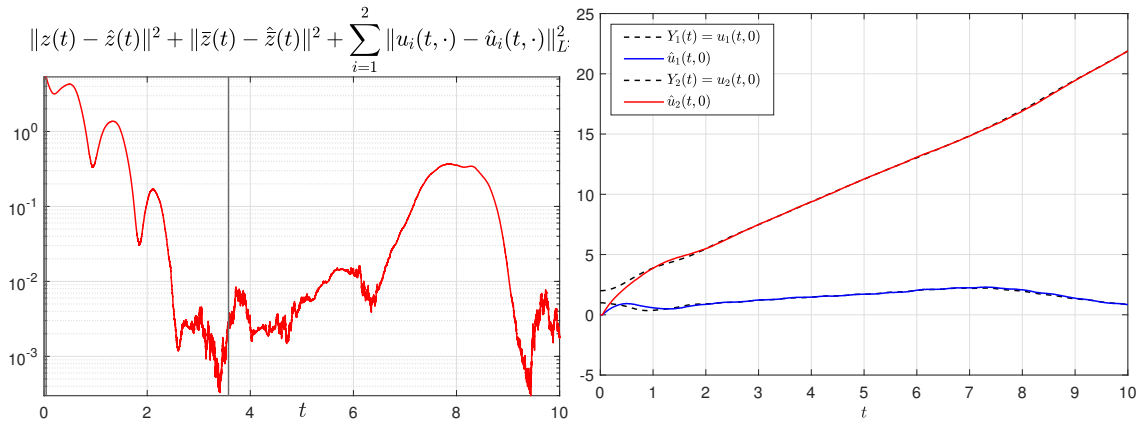


Figure 3.12: On the left, the evolution of the norm  $\|z(t) - \hat{z}(t)\|_2^2 + \|\bar{z}(t) - \hat{\bar{z}}(t)\|_2^2 + \sum_{i=1}^2 \|u_i(t, \cdot) - \hat{u}_i(t, \cdot)\|_{L^2(0, D)}^2$  of the two error systems (plotted in logarithmic scale) with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001, and the delay  $D = 0.04$ , and  $T_{\max} = 3.5381s$ . On the right, the evolution of the outputs  $Y_1(t) = z_1(t - D) = u_1(t, 0)$  and  $Y_2(t) = z_3(t - D) = u_2(t, 0)$  in solid lines and the estimated outputs  $\hat{u}_1(t, 0)$  and  $\hat{u}_2(t, 0)$  in dashed lines with feedback  $\hat{\tau}$  in (3.190) and noise of power 0.0001.



# Finite/fixed-time stabilization of LTI systems with pointwise input delay

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## Contents

<b>4.1</b>	<b>Introduction</b>	<b>79</b>
<b>4.2</b>	<b>Problem statement</b>	<b>80</b>
<b>4.3</b>	<b>Finite/fixed-time predictor-based controller via PDE-based backstepping approach</b>	<b>81</b>
4.3.1	Nonlinear infinite-dimensional backstepping transformation	81
4.3.2	Inverse transformation	83
4.3.3	On the selection of the finite/fixed-time predictor-based controller	83
4.3.4	Stability analysis	85
<b>4.4</b>	<b>Some characterizations of <math>\mathcal{F}</math> for the design of finite/fixed-time predictor-based controllers</b>	<b>89</b>
4.4.1	Explicit controllers for double chain of integrators	89
4.4.2	Implicit Controllers for chain of integrators	90
<b>4.5</b>	<b>Simulations</b>	<b>91</b>

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In this chapter, we present a general approach to studying the problem of finite-time and fixed-time stabilization of a chain of integrators with input delay. To accomplish this, we first reformulate the chain of integrators with input delay as a cascade ODE-PDE system (i.e., a cascade of a linear transport partial differential equation (PDE) with the chain of integrators) where the transport equation models the effect of the delay on the input. Next, we use a nonlinear infinite-dimensional backstepping transformation to convert the cascade system into a suitable target system that is chosen to be **FTS** or **FxTS**. We perform the stability analysis on the target system by means of classical Non-Asymptotic concepts and tools such as the linear homogeneity and "generalized  $\mathcal{KL}$ " functions. Then, we use the inverse transformation to transfer back the stability property to the closed-loop system. Finally, we give some characterizations of finite/fixed time predictor-based controllers followed by numerical simulations.

## 4.1 Introduction

In this chapter, we revisit the problem of finite/fixed stabilization of a chain of integrators with input delay and propose a more general approach for the design of finite/fixed-time state-dependent predictor-based controllers. We use a cascade ODE-PDE system (i.e., a cascade of a linear transport partial differential equation (PDE) with the chain of integrators) where the transport equation

models the effect of the delay on the input and builds on a nonlinear infinite-dimensional backstepping transformation inspired by [69]. Compared to [176]-which uses a linear transformation and time-varying tools to ensure the prescribed-time stability property for an LTI system with input delay- our approach uses a nonlinear transformation and nonsmooth tools to ensure a different (i.e. finite/fixed) stability property. Both methods bring different challenges and have specific issues. The approach in this chapter allows to perform the stability analysis on a suitable target system (chosen to exhibit the desired stability properties, i.e., either finite time or fixed time) while employing classical notions and tools such as Lyapunov-based characterization of finite/fixed-time stability property of ODEs, linear homogeneity [168] (as those discussed in Chapter 2) as well as the "generalized  $\mathcal{KL}$ " (in short " $\mathcal{GKL}$ ") functions [191]. Hence, we can provide some characterizations of the resulting finite/fixed-time predictor-based controllers.

It is worth mentioning that [142], [143] achieve similar results (finite/fixed-time stabilization of LTI systems with input delay) to ours using Artstein's model reduction. Nevertheless, no state estimates of the closed-loop solution are provided. The actuator dynamic is not identified throughout the analysis, either. Moreover, extensions of [142], [143] to complex infinite-dimensional systems (including cascaded systems) with constant/time-varying/state-dependent delays, distributed delays are not straightforward. In contrast, our approach does account for the infinite dimensionality of the input, and may allow possible extensions to more complex infinite-dimensional systems (e.g., 1D reaction-diffusion PDEs with delayed boundary) or when just cascading finite/fixed-time ISS subsystems.

This chapter is organized as follows. In Section 4.2, we give the problem statement in which we present the chain of integrators with input delay and its reformulation within an ODE-PDE setting. In Section 4.3, we give a general approach to stabilize the chain of integrators in a finite time or fixed time. We present the nonlinear backstepping transformation to transform the ODE-PDE setting to a suitable target system and to come up with a finite/fixed-time predictor-based control. Next, in Section 4.4 we apply our approach to different target systems to attain finite-time stability or fixed-time stability. Then, we give in Section 4.5 some numerical simulations to illustrate the results.

## 4.2 Problem statement

We consider the following chain of integrators with input delay:

$$\begin{cases} \dot{z}_j(t) = z_{j+1}(t), & j = 1, \dots, n-1, & t \geq t_0, & (4.1) \\ \dot{z}_n(t) = U(t-D), & & t \geq t_0, & (4.2) \end{cases}$$

where  $z(t) = [z_1(t), \dots, z_n(t)]^\top \in \mathbb{R}^n$  ( $n \in \mathbb{N} \setminus \{0\}$ ) is the instantaneous state of the system,  $U(t) \in \mathbb{R}$  is the control input, and  $D > 0$  is a known constant delay.

Our goal is to design a nonlinear predictor-based controller for the system (4.1)-(4.2) to achieve **FTS** or **FxTS**. To this end, the methodology developed in this chapter relies on representing the actuator delay as a linear transport PDE and builds upon the cascade ODE-PDE setting (i.e., a cascade of linear hyperbolic PDE with an LTI system) of [69].

### Remark 4.1

Note that in [192], the problem of exponential stabilization of the following class of strict-feedback

system with delayed integrators and delayed input

$$\begin{cases} \dot{z}_j(t) = \sum_{i=1}^j a_{ji} z_i(t) + z_{j+1}(t - D_j), & j = 1, \dots, n-1, & t \geq t_0, & (4.3) \\ \dot{z}_n(t) = \sum_{i=1}^n a_{ni} z_i(t) + U(t - D_n), & & t \geq t_0, & (4.4) \end{cases}$$

was solved, where the coefficients  $(a_{ji})_{1 \leq i \leq j \leq n}$  are real constants, and  $(D_j)_{1 \leq j \leq n}$  are positive known delays. This class of systems is more challenging compared to (4.1)-(4.2), but can be dealt with to achieve finite/fixed-time stabilization, as shown in details in Section 4.5, using the approach given in this chapter. The idea is to use the state transformation (52)-(53) introduced in [192] (with  $c_1, \dots, c_n = 0$ ) to get rid of the non-delayed terms from (4.3)-(4.4). Then, we use the following change of variables  $\bar{z}_{j+1}(t) = z_{j+1}\left(t - \sum_{i=1}^j D_i\right)$  for all  $j = 1, \dots, n$ , which moves the delays to the last equation. As a result, we obtain a similar system to (4.1)-(4.2) with some additional delayed and non-delayed terms in the last equation. Therefore, studying (4.3)-(4.4) comes down to studying (4.1)-(4.2). This motivates applying our approach to the simplest case (4.1)-(4.2) in order to better communicate the ideas of the approach.

We henceforth represent system (4.1)-(4.2) as

$$\begin{cases} \dot{z}_j(t) = z_{j+1}(t), & j = 1, \dots, n-1, & t \geq t_0, & (4.5) \\ \dot{z}_n(t) = u(t, 0), & & t \geq t_0, & (4.6) \\ u_t(t, x) = u_x(t, x), & & t \geq t_0, x \in [0, D] & (4.7) \\ u(t, D) = U(t), & & t \geq t_0, & (4.8) \end{cases}$$

with  $t \geq t_0 \geq 0$ ,  $x \in [0, D]$ , and  $u(t, \cdot)$  is the transport PDE state whose solution is given by

$$u(t, x) = \begin{cases} u_0(t + x - t_0), & t_0 \leq t + x \leq t_0 + D, \\ U(t + x - D), & t + x \geq t_0 + D, \end{cases} \quad (4.9)$$

where  $u_0$  is a bounded function in  $L^2(0, D)$ .

The objective of the first part of this chapter is to give a general approach to design a controller (predictor-type) for the system (4.1)-(4.2), to attain **FTS** and/or **FxTS**. We employ a nonlinear infinite-dimensional backstepping transformation. The key idea is to transform the original system into a suitable target system that is chosen to exhibit the **FTS** or **FxTS** properties.

## 4.3 Finite/fixed-time predictor-based controller via PDE-based backstepping approach

### 4.3.1 Nonlinear infinite-dimensional backstepping transformation

Inspired by [193] and [69, Chapter 10], we consider the following nonlinear infinite-dimensional backstepping transformation:

$$\omega(t, x) = u(t, x) - \mathcal{F}(\varphi_1(t, x), \dots, \varphi_n(t, x)), \quad (4.10)$$



where  $\mathcal{F}$  is a suitable nonlinear function to be characterized later on, and  $\varphi_1, \dots, \varphi_n$  are the solutions of

$$\begin{cases} \varphi_{j,x}(t, x) = \varphi_{j+1}(t, x), & j = 1, \dots, n-1, & (4.11) \\ \varphi_j(t, 0) = z_j(t), & j = 1, \dots, n-1, & (4.12) \\ \varphi_{n,x}(t, x) = u(t, x), & & (4.13) \\ \varphi_n(t, 0) = z_n(t). & & (4.14) \end{cases}$$

Notice that  $\varphi_i(t, x) = z_i(t+x)$  for all  $i \in \{1, \dots, n\}$ , all  $t \geq t_0$  and all  $x \in [0, D]$ . Then, by the variation of the constant formula, we obtain:

$$\varphi_i(t, x) = \int_0^x \frac{(x-y)^{n-i}}{(n-i)!} u(t, y) dy + \sum_{j=i}^n \frac{x^{j-i}}{(j-i)!} z_j(t). \quad (4.15)$$

The proof of (4.15) is as follows:

$$z(t+x) = e^{Ax} z(t) + \int_t^{t+x} e^{A(t+x-s)} B u(s, 0) ds, \quad (4.16)$$

with  $z = [z_1, \dots, z_n]^\top$ ,  $B = e_n := [0, \dots, 0, 1]^\top$ , and  $A := \{A_{ij}\} \in \mathcal{M}_{n,n}(\mathbb{R})$ , where  $A_{ij} = 1$  if  $j = i+1$  and  $A_{ij} = 0$  otherwise.

Next, using the change of variables  $y = s - t$ , we get

$$z(t+x) = e^{Ax} z(t) + \int_0^x e^{A(x-y)} B u(t+y, 0) dy, \quad (4.17)$$

$$= e^{Ax} z(t) + \int_0^x e^{A(x-y)} B u(t, y) dy. \quad (4.18)$$

Using the expression  $e^{Ax} = \sum_{k=0}^{n-1} \frac{x^k}{k!} A^k$ , we recover

$$z(t+x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} A^k z(t) + \sum_{k=0}^{n-1} \int_0^x \frac{(x-y)^k}{k!} A^k B u(t, y) dy. \quad (4.19)$$

Then, using the fact  $A^k z(t) = [z_{k+1}(t), \dots, z_n(t), 0, \dots, 0]^\top$  and  $A^k B = e_{n-k}$ , we obtain

$$z_i(t+x) = \sum_{k=0}^{n-i} \frac{x^k}{k!} z_{k+i}(t) + \int_0^x \frac{(x-y)^{n-i}}{(n-i)!} u(t, y) dy, \quad (4.20)$$

$$= \sum_{j=i}^n \frac{x^{j-i}}{(j-i)!} z_j(t) + \int_0^x \frac{(x-y)^{n-i}}{(n-i)!} u(t, y) dy. \quad (4.21)$$

Next, using (4.10) we transform the system (4.5)-(4.8) into the following nonlinear target system:

$$\begin{cases} \dot{z}_j(t) = z_{j+1}(t), & j = 1, \dots, n-1, & t \geq t_0, & (4.22) \\ \dot{z}_n(t) = \mathcal{F}(z_1(t), \dots, z_n(t)) + \omega(t, 0), & & t \geq t_0, & (4.23) \\ \omega_t(t, x) = \omega_x(t, x), & & t \geq t_0, x \in [0, D] & (4.24) \\ \omega(t, D) = 0, & & t \geq t_0, & (4.25) \end{cases}$$

with  $\omega : [0, +\infty) \times [0, D] \rightarrow \mathbb{R}$  is the transport PDE state. The nonlinear function  $\mathcal{F}$ , to be specified latter on (see Subsection 4.3.3), is suitably chosen to get **FTS**/**FxTS** of the target system when  $\omega(t, 0)$  becomes zero (this key feature of the transport PDE  $\omega$  is discussed in Subsection 4.3.4).

Note that using the fact that  $\varphi(t, x) = z(t+x)$  for all  $t \geq t_0$  and al  $x \in [0, D]$ , it is clear that the nonlinear transformation (4.10) satisfies the PDE part of (4.22)-(4.25) (i.e.  $\omega_t(t, x) = \omega_x(t, x)$ ).

### 4.3.2 Inverse transformation

The inverse transformation is given by,

$$u(t, x) = \omega(t, x) + \mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x)), \quad (4.26)$$

where  $\psi_1, \dots, \psi_n$  are the solutions of:

$$\begin{cases} \psi_{j,x}(t, x) = \psi_{j+1}(t, x), & j = 1, \dots, n-1, & (4.27) \\ \psi_j(t, 0) = z_j(t), & j = 1, \dots, n-1, & (4.28) \\ \psi_{n,x}(t, x) = \mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x)) + \omega(t, x), & & (4.29) \\ \psi_n(t, 0) = z_n(t). & & (4.30) \end{cases}$$

Similarly to the direct transformation, we recover from the inverse transformation:  $u_t(t, x) = u_x(t, x)$ .

### 4.3.3 On the selection of the finite/fixed-time predictor-based controller

In this section, we give an important assumption on the nonlinear function  $\mathcal{F}$  given in (4.22)-(4.25) to ensure **FTS** or **FxTS** properties. Then, we give the expression of our predictor-based controller  $U(t)$  using the transformation (4.10) or (4.26).

#### 4.3.3.1 An assumption on the nonlinear function $\mathcal{F}$

In order to ensure that the target system (4.22)-(4.25) is **FTS** (resp. **FxTS**), let us assume that  $\mathcal{F}$  satisfies the following assumption (for the ODE part of (4.22)-(4.25)):

##### Assumption 4.1

$\mathcal{F}$  is a continuous nonlinear function, differentiable everywhere except on

$$S = \bigcup_{i=1}^n \{s = [s_1, \dots, s_n] \in \mathbb{R}^n, s_i = 0\},$$

such that  $\mathcal{F}(0, \dots, 0) = 0$  and the origin of (4.1)-(4.2) with  $D = 0$ :

$$\begin{cases} \dot{z}_j(t) = z_{j+1}(t), & j = 1, \dots, n-1, & t \geq t_0, & (4.31) \\ \dot{z}_n(t) = U(t), & & t \geq t_0, & (4.32) \end{cases}$$

is **F<sub>T</sub>S** (resp. **F<sub>x</sub>T<sub>S</sub>**) i.e., there exists a class  $\mathcal{GKL}$  function  $\beta$  such that the solution of the previous closed-loop system with a well chosen feedback control  $U(t)$ , satisfies:

$$\|z(t)\|_2 \leq \beta(\|\bar{z}_0\|_2, t - \bar{t}_0), \quad \forall t \geq \bar{t}_0, \quad (4.33)$$

where  $\bar{z}_0 = [\bar{z}_{1,0}, \dots, \bar{z}_{n,0}]$  is the initial condition at time  $\bar{t}_0$ . Moreover, there exists an increasing function  $T(\cdot)$  such that  $\|z(t)\|_2 = 0$  when  $t \geq \bar{t}_0 + T(\|z(\bar{t}_0)\|_2)$ , (resp. a positive real constant  $T_{\max}$  such that  $\|z(t)\|_2 = 0$  when  $t \geq \bar{t}_0 + T_{\max}$ ).

**Remark 4.2**

Note that since  $\mathcal{F}$  is non-differentiable on  $S = \bigcup_{i=1}^n \{s = [s_1, \dots, s_n] \in \mathbb{R}^n, s_i = 0\}$ , equations (4.24) (i.e.  $\omega_t(t, x) = \omega_x(t, x)$ ) and (4.7) (i.e.  $u_t(t, x) = u_x(t, x)$ ) will not be defined everywhere (i.e. for all  $(t, x) \in [0, +\infty) \times [0, 1]$ ). However, they can be verified almost everywhere (except in a set of measure zero). This last fact is sufficient for us to define  $u$  and  $\omega$ , using the characteristic method, almost everywhere and then by using the continuity of  $\mathcal{F}$ , and the transformations (4.10) and (4.26), we recover  $u$  and  $\omega$  everywhere. The question now is how to prove that, for instance, (4.24) is satisfied almost everywhere. To do that, we need to first prove that  $(t, x) \mapsto \omega(t, x)$  is differentiable almost everywhere. From (4.10), this is equivalent to proving that  $(t, x) \mapsto \mathcal{F} \circ \varphi(t, x)$  is differentiable almost everywhere (where  $\varphi(t, x)$  is given by  $\varphi(t, x) = [\varphi_1(t, x), \dots, \varphi_n(t, x)]^\top \in \mathbb{R}^n$ ) and since  $[\varphi_1(t, x), \dots, \varphi_n(t, x)] = [z_1(t+x), \dots, z_n(t+x)]$ , it is equivalent to proving that  $t \mapsto \mathcal{F} \circ \varphi(t, 0)$  is differentiable almost everywhere (except on a countable set  $\eta$  of measure zero) and that the set  $S_\eta := \{(t, x) \in [0, +\infty) \times [0, 1] : t+x \in \eta\}$  is of measure zero ( $S_\eta$  is the set where  $(t, x) \mapsto \mathcal{F} \circ \varphi$  is not differentiable).

To better understand this, let us focus on the case  $n = 2$ . Let us recall  $t \mapsto \varphi(t, 0) = z(t)$  is absolutely continuous since it satisfies continuously an ODE with respect to  $t$  (see (4.5)-(4.6)). In addition, the mapping  $[s_1, s_2] \mapsto \mathcal{F}(s_1, s_2)$  is absolutely continuous w.r.t. each variable, continuously differentiable except on the set  $S = \bigcup_{i=1}^2 \{s = [s_1, s_2] \in \mathbb{R}^2, s_i = 0\}$  (of Lebesgue measure equal to zero). Notice, however, that the mapping  $t \mapsto \mathcal{F} \circ \varphi(t, 0)$  may not be assured to be absolutely continuous due to the lack of Lipschitzness of  $\mathcal{F}$  on  $S$ . Nevertheless, we can use the arguments of [194, Remark 1] to state that since the trajectories  $z(t)$  of (4.5)-(4.6) may eventually cross the surface  $S$  but cannot stay on it (due to the trajectories' oscillatory nature), unless we reach the equilibrium, that the  $t \mapsto \mathcal{F} \circ \varphi(t, 0)$  is differentiable for almost every  $t$  except on a (countable) set  $\eta$  of measure zero. Moreover, the set  $S_\eta := \{(t, x) \in [0, +\infty) \times [0, 1] : t+x \in \eta\}$  is of measure zero, since  $S_\eta$  can be represented as a countable union of the lines  $\{(t, x) \in [0, +\infty) \times [0, 1] : t+x = n_i, n_i \in \eta, i \in \mathbb{N}\}$  of measure zero. Finally, by the chain rule on (4.10) the following holds:

$$\omega_x(t, x) = u_x(t, x) - \frac{\partial \mathcal{F}(\varphi_1(t, x), \dots, \varphi_n(t, x))}{\partial x}, \quad (4.34)$$

$$= u_x(t, x) - \sum_{i=1}^n z_{i,x}(t+x) \frac{\partial \mathcal{F}(\varphi_1(t, x), \dots, \varphi_n(t, x))}{\partial \varphi_i}, \quad (4.35)$$

$$= u_t(t, x) - \sum_{i=1}^n z_{i,t}(t+x) \frac{\partial \mathcal{F}(\varphi_1(t, x), \dots, \varphi_n(t, x))}{\partial \varphi_i}, \quad (4.36)$$

$$= u_t(t, x) - \frac{\partial \mathcal{F}(\varphi_1(t, x), \dots, \varphi_n(t, x))}{\partial t}, \quad (4.37)$$

$$= \omega_t(t, x), \quad (4.38)$$

almost everywhere. In particular, using (4.10) at  $x = D$  alongside (4.8), we get  $\omega(t, D) = 0$ . This concludes the proof.

**Remark 4.3**

A construction of a Lyapunov function  $t \mapsto V(z(t), \omega(t, \cdot))$  such that one has an estimate of this type

$$\dot{V}(z(t), \omega(t, \cdot)) \leq -c_1 V^\alpha(z(t), \omega(t, \cdot)) - c_2 V^\beta(z(t), \omega(t, \cdot)), \quad c_1, c_2 > 0, \alpha \in (0, 1), \beta > 1, \quad (4.39)$$

could be an alternative yielding the finite/fixed time stability property to the target system and thereby the original one. However, unfortunately, for a cascade nonlinear ODE - transport PDE system (such as (4.22)-(4.25)), it is still unclear whether one can construct such a Lyapunov function (without even the PDE part). See also the open question discussed in Remark 2.15 in Chapter 2. This is one reason why, our approach relies on GK $\mathcal{L}$ -class functions  $\beta$  and estimates on the solutions.

**4.3.3.2 Finite/fixed-time predictor-based controller**

Under Assumption 4.1, and from (4.10) at  $x = D$ , and using (4.15), the boundary control is then,

$$U(t) = u(t, D) = \mathcal{F}(\varphi_1(t, D), \dots, \varphi_n(t, D)), \quad (4.40)$$

where

$$\varphi_i(t, D) = \int_0^D \frac{(D-y)^{n-i}}{(n-i)!} u(t, y) dy + \sum_{j=i}^n \frac{D^{j-i}}{(j-i)!} z_j(t). \quad (4.41)$$

Or from (4.26), at  $x = D$ ,

$$U(t) = u(t, D) = \mathcal{F}(\psi_1(t, D), \dots, \psi_n(t, D)), \quad (4.42)$$

where  $[\psi_1(t, x), \dots, \psi_n(t, x)]$  is solution of (4.27)-(4.30).

**4.3.4 Stability analysis**

In this subsection, we first perform the stability analysis on the target system (4.22)-(4.25). Then, we use the inverse transformation (4.26) to establish the boundedness of the state of the original system (4.5)-(4.8) and its convergence to zero in finite-time (resp. fixed-time) using a suitable norm equivalence.

**Proposition 4.1**

Let  $t \geq t_0 + D$ , there exists a class GK $\mathcal{L}$  function  $\beta_1$  such that the solution of system (4.27)-(4.30),  $\psi(t, x) = (\psi_1(t, x), \dots, \psi_n(t, x))$  satisfies

$$\|\psi(t, \cdot)\|_{L^2(0, D)} \leq \beta_1(\|z(t_0 + D)\|_2, t - t_0 - D). \quad (4.43)$$

Moreover,  $\|\psi(t, \cdot)\|_{L^2(0, D)} \rightarrow 0$  when  $t \rightarrow t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ),

where  $\|\psi(t, \cdot)\|_{L^2(0, D)}^2 = \sum_{j=1}^n \|\psi_j(t, \cdot)\|_{L^2(0, D)}^2$ ,  $T(\cdot)$  and  $T_{\max}$  are given in Assumption 4.1.

*Proof.* By the method of the characteristics, the solution of the  $\omega$ -dynamics of the target system (4.22)-(4.25) for any  $t \geq t_0 + D$  is zero (i.e.  $\omega(t, x) = 0$  for all  $x \in [0, D]$  and  $t \geq t_0 + D$ ). Then,

we can conclude using Assumption 4.1 (replacing " $t$ " by " $x$ ") that the solution of the system (4.27)-(4.30) satisfies for  $t \geq t_0 + D$

$$\|\psi(t, x)\|_2 \leq \beta(\|z(t)\|_2, x), \quad \forall x \in [0, D], \quad (4.44)$$

where  $\beta$  is a class  $\mathcal{GKL}$  function. Moreover, there exists an increasing function  $T(\cdot)$  such that  $\|\psi(t, x)\|_2 \rightarrow 0$  when  $x \rightarrow T(\|z(t)\|_2)$  (resp.  $x \rightarrow T_{\max}$ ). Furthermore, when  $t \rightarrow t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ )  $\|\psi(t, x)\|_2 \rightarrow 0$ .

Now, using the fact that  $\beta$  is decreasing with respect to the second variable  $x$ , we get,

$$\|\psi(t, x)\|_2 \leq \beta(\|z(t)\|_2, 0), \quad \forall x \in [0, D], \quad \forall t \geq t_0 + D. \quad (4.45)$$

By squaring and integrating with respect to  $x$  from 0 to  $D$ , then passing to the square roots, we find,

$$\|\psi(t, \cdot)\|_{L^2(0, D)} \leq \sqrt{D}\beta(\|z(t)\|_2, 0), \quad \forall t \geq t_0 + D. \quad (4.46)$$

Next, we use inequality (4.33) from Assumption 4.1 to obtain,

$$\|\psi(t, \cdot)\|_{L^2(0, D)} \leq \sqrt{D}\beta(\beta(\|z(t_0 + D)\|_2, t - t_0 - D), 0), \quad \forall t \geq t_0 + D. \quad (4.47)$$

Then,

$$\|\psi(t, \cdot)\|_{L^2(0, D)} \leq \beta_1(\|z(t_0 + D)\|_2, t - t_0 - D), \quad \forall t \geq t_0 + D, \quad (4.48)$$

where for any  $s, t \in \mathbb{R}_+$ ,  $\beta_1$  is a class  $\mathcal{GKL}$  function given by,  $\beta_1(s, t) = \sqrt{D}\beta(\beta(s, t), 0)$ . Furthermore, when  $t \rightarrow t_0 + D + T(s)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ),  $\beta(s, t) \rightarrow 0$ , and by continuity  $\beta_1(s, t) \rightarrow 0$ .  $\square$

### Proposition 4.2

There exists a class  $\mathcal{GKL}$  function  $\beta_2$  such that for any  $x \in [0, D]$ ,  $\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))$  satisfies for  $t \geq t_0 + D$ ,

$$|\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))| \leq \beta_2(\|z(t_0 + D)\|_2, t - t_0 - D), \quad (4.49)$$

and

$$\|\mathcal{F}(\psi_1(t, \cdot), \dots, \psi_n(t, \cdot))\|_{L^2(0, D)} \leq \sqrt{D}\beta_2(\|z(t_0 + D)\|_2, t - t_0 - D). \quad (4.50)$$

Moreover, for all  $x \in [0, D]$ ,  $|\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))| \rightarrow 0$  and  $\|\mathcal{F}(\psi_1(t, \cdot), \dots, \psi_n(t, \cdot))\|_{L^2(0, D)} \rightarrow 0$  when  $t \rightarrow t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ).

*Proof.* Let  $x \in [0, D]$ . We can see from Proposition 4.1 that  $\|\psi(t, x)\|_2 \rightarrow 0$  when  $t \rightarrow t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ), and  $\|\psi(t, x)\|_2 = 0$  when  $t \geq t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \geq t_0 + D + T_{\max}$ ). Next, by continuity of  $\mathcal{F}$ , we also have  $\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x)) \rightarrow 0$  when  $t \rightarrow t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ). Then, there exists a class  $\mathcal{GKL}$  function  $\beta_2$  such that  $\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))$  satisfies,

$$|\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))| \leq \beta_2(\|z(t_0 + D)\|_2, t - t_0 - D), \quad (4.51)$$

for all  $x \in [0, D]$  and  $t \geq t_0 + D$ .

Next, by squaring and integrating from 0 to  $D$  with respect to  $x$  and passing to the square roots, we find,

$$\|\mathcal{F}(\psi_1(t, \cdot), \dots, \psi_n(t, \cdot))\|_{L^2(0, D)} \leq \sqrt{D}\beta_2(\|z(t_0 + D)\|_2, t - t_0 - D), \quad (4.52)$$

for all  $t \geq t_0 + D$ . In addition, for all  $x \in [0, D]$ , we have,  $|\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))| \rightarrow 0$  and  $\|\mathcal{F}(\psi_1(t, \cdot), \dots, \psi_n(t, \cdot))\|_{L^2(0, D)} \rightarrow 0$  when  $t \rightarrow t_0 + D + T(\|z(t_0 + D)\|_2)$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ).  $\square$

**Proposition 4.3**

From the transformation (4.26), the following estimate holds for  $t \geq t_0 + D$ :

$$\|u(t, \cdot)\|_{L^2(0,D)} \leq \sqrt{2D}\beta_2(\|z(t_0 + D)\|_2, t - t_0 - D), \quad (4.53)$$

where  $\beta_2$  is a class  $\mathcal{GKL}$  function given in Proposition 4.2.

*Proof.* Using (4.26), we have

$$|u(t, x)| \leq |\omega(t, x)| + |\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))|, \forall t \geq t_0. \quad (4.54)$$

Next, squaring the previous inequality and using Young inequality, we get,

$$|u(t, x)|^2 \leq 2|\omega(t, x)|^2 + 2|\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))|^2. \quad (4.55)$$

Now, using  $\omega(t, x) = 0, \forall t \geq t_0 + D, \forall x \in [0, D]$ , we obtain

$$|u(t, x)|^2 \leq 2|\mathcal{F}(\psi_1(t, x), \dots, \psi_n(t, x))|^2 \quad \forall t \geq t_0 + D, \forall x \in [0, D]. \quad (4.56)$$

Finally, by integrating from 0 to  $D$  with respect to the space variable  $x$  and passing to the square roots, we get

$$\|u(t, \cdot)\|_{L^2(0,D)} \leq \sqrt{2}\|\mathcal{F}(\psi_1(t, \cdot), \dots, \psi_n(t, \cdot))\|_{L^2(0,D)}, \quad \forall t \geq t_0 + D. \quad (4.57)$$

Using inequality (4.50) from Proposition 4.2, we obtain,

$$\|u(t, \cdot)\|_{L^2(0,D)} \leq \sqrt{2D}\beta_2(\|z(t_0 + D)\|_2, t - t_0 - D), \quad \forall t \geq t_0 + D. \quad (4.58)$$

□

**Theorem 4.1**

Let the input initial condition  $U_{t_0} : s \in [-D, 0] \mapsto U(t_0 + s)$  be defined and bounded in  $L^2(-D, 0)$ . Let  $D > 0$  and  $t_0 \geq 0$ . Then, the solution of the closed-loop system (4.5)-(4.8) with finite-time (resp. fixed-time) predictor-based controller (4.40) (or (4.42)) is **F $T$ S** (resp. **F $x$ T $S$** ) in the following sense: For any initial condition  $z_0 \in \mathbb{R}^n$ , the quantity  $I(t) = \|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2$  remains bounded for  $t \in [t_0, t_0 + D]$ , and for all  $t \in [t_0 + D, t_0 + D + T(\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty))]$  (resp.  $t \in [t_0 + D, t_0 + D + T_{\max})$ ), there exists a class  $\mathcal{GKL}$  function  $\beta_3$  such that,

$$I(t) \leq \beta_3(\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty), t - t_0 - D), \quad (4.59)$$

with  $\mathcal{B}_D(s_1, s_2) = e^D s_1 + D e^{2D} s_2$  for any  $s_1, s_2 \geq 0$ .

In particular,  $I(t) \rightarrow 0$  and  $|U(t)| \rightarrow 0$ , as  $t \rightarrow t_0 + D + T(\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty))$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ).

*Proof.* Let us start by proving the boundedness of  $\|z(t)\|_2$  for all  $t \in [t_0, t_0 + D]$ . By the variation of the constant formula on (4.1)-(4.2) we recover,

$$z(t) = e^{A(t-t_0)} z_0 + \int_{t_0}^t e^{A(t-y)} B U(y - D) dy, \quad (4.60)$$

$$= e^{A(t-t_0)} \left[ z_0 + \int_{t_0}^t e^{A(t_0-y)} B U(y - D) dy \right]. \quad (4.61)$$

Using the change of variables  $s = y - D - t_0$ , we obtain

$$z(t) = e^{A(t-t_0)} \left[ z_0 + \int_{-D}^{t-t_0-D} e^{A(-s-D)} BU(t_0 + s) ds \right]. \quad (4.62)$$

Using  $\|e^{A(t-t_0)}\|_2 \leq e^D$  and  $\|e^{A(-s-D)}\|_2 \leq e^D$ , we get

$$\|z(t)\|_2 \leq e^D \|z_0\|_2 + e^{2D} \int_{-D}^{t-t_0-D} |U_{t_0}(s)| ds, \quad (4.63)$$

$$\leq e^D \|z_0\|_2 + e^{2D} \int_{-D}^0 |U_{t_0}(s)| ds, \quad (4.64)$$

$$\leq e^D \|z_0\|_2 + De^{2D} \|U_{t_0}\|_\infty, \quad (4.65)$$

$$= \mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty). \quad (4.66)$$

Then,  $\|z(\cdot)\|_2$  is bounded in  $[t_0, t_0 + D]$ .

Next, let us prove inequality (4.59). Let  $t \geq t_0 + D$ . Using (4.53) from Proposition 4.3, we have

$$I(t) \leq \|z(t)\|_2^2 + 2D\beta_2(\|z(t_0 + D)\|_2, t - t_0 - D)^2. \quad (4.67)$$

By inequality (4.33) from Assumption 4.1, we get

$$I(t) \leq \beta(\|z(t_0 + D)\|_2, t - t_0 - D)^2 + 2D\beta_2(\|z(t_0 + D)\|_2, t - t_0 - D)^2, \quad (4.68)$$

which leads to

$$I(t) \leq \beta_3(\|z(t_0 + D)\|_2, t - t_0 - D), \quad (4.69)$$

with  $\beta_3 = \beta^2 + 2D\beta_2^2$  is a class  $\mathcal{GKL}$  function.

Then, using inequality (4.66), we obtain,

$$I(t) \leq \beta_3(\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty), t - t_0 - D). \quad (4.70)$$

In particular, we recover that  $I(t) \rightarrow 0$  when  $t \rightarrow t_0 + D + T(\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty))$  (resp. when  $t \rightarrow t_0 + D + T_{\max}$ ) and that  $\|z(t)\|_2$  is bounded for all  $t \geq t_0$ .

Now, let us prove that  $\|u(t, \cdot)\|_{L^2(0,D)}$  is bounded for all  $t \in [t_0, t_0 + D]$ . Notice that the solution  $u$  is given by

$$u(t, x) = \begin{cases} u_0(t + x - t_0), & t \in [t_0, t_0 + D - x], \\ U(t + x - D), & t \in [t_0 + D - x, t_0 + D]. \end{cases} \quad (4.71)$$

From this last equation, it is easy to deduce the boundedness of  $\|u(t, \cdot)\|_{L^2(0,D)}$  using the transformation (4.26), the fact that  $|U(t + x - D)| \leq |\mathcal{F}(z_1(t + x), \dots, z_n(t + x))|$  and the boundedness of  $\|z(t + x)\|_2$  for all  $t + x \geq t_0$ . As a result,  $I(t)$  is bounded for all  $t \in [t_0, t_0 + D]$ .

Finally, let us prove that  $|U(t)| \rightarrow 0$  as  $t \rightarrow t_0 + D + T(\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty))$  (resp.  $t \rightarrow t_0 + D + T_{\max}$ ). From the equation (4.40), we have,

$$|U(t)| = |\mathcal{F}(\psi_1(t, D), \dots, \psi_n(t, D))|, \quad (4.72)$$

and that  $|U(t)|$  is bounded for all  $t \in [t_0, t_0 + D]$  (because  $\psi(t, D) = z(t + D)$  bounded for all  $t \geq t_0$ ).

By Proposition 4.2, we obtain from inequality (4.49),

$$|U(t)| \leq \beta_2(\|z(t_0 + D)\|_2, t - t_0 - D), \forall t \geq t_0 + D. \quad (4.73)$$

Then, using inequality (4.66), we get

$$|U(t)| \leq \beta_2 (\mathcal{B}_D(\|z_0\|_2, \|U_{t_0}\|_\infty), t - t_0 - D). \quad (4.74)$$

From where we deduce the desired property. □

## 4.4 Some characterizations of $\mathcal{F}$ for the design of finite/fixed-time predictor-based controllers

The previous section provides a general setting in which, as soon as one chooses  $\mathcal{F}$  satisfying Assumption 4.1, one can design a nonlinear predictor-based controller to stabilize the system (4.5)-(4.8) in finite time or in fixed time. In this section, let us give some characterizations of  $\mathcal{F}$ . For simplicity let us take  $t_0 = 0$ .

### 4.4.1 Explicit controllers for double chain of integrators

For the target system (4.22)-(4.25) with  $n = 2$ , we propose the following characterization of  $\mathcal{F}$  inspired by [195]:

$$\mathcal{F}(z_1(t), z_2(t)) = -k_1\{z_1(t)\}^{\alpha_1} - k_2\{z_2(t)\}^{\alpha_2}, \quad (4.75)$$

which satisfies Assumption 4.1 as soon as  $k_1, k_2$  are any positive real numbers and  $\alpha_1, \alpha_2$  are selected so that weighted homogeneity of negative degree  $\kappa$  is obtained for (4.22)-(4.25) with  $n = 2$  and  $\omega \equiv 0$ : for example by selecting  $r > -2\kappa$  and

$$r_1 = r, r_2 = r + \kappa, \alpha_1 = \frac{r + 2\kappa}{r}, \alpha_2 = \frac{r + 2\kappa}{r + \kappa}.$$

Hence, we can realize the resulting nonlinear predictor-based controller  $U(t)$  (4.40), with  $\mathcal{F}$  having the structure in (4.75), stabilizing the system (4.5)-(4.8) in finite time.

Let us give now a characterization of  $\mathcal{F}$  to get a **FxTS** counterpart:

#### Proposition 4.4

The  $z$ -subsystem of the target system (4.22)-(4.25) with  $n = 2$  is **FxTS** when  $\mathcal{F}$  is selected as follows:

$$\mathcal{F}(z_1(t), z_2(t)) = -k_{1,0}\{z_1(t)\}^{\alpha_{1,0}} - k_{2,0}\{z_2(t)\}^{\alpha_{2,0}} - k_{1,\infty}\{z_1(t)\}^{\alpha_{1,\infty}} - k_{2,\infty}\{z_2(t)\}^{\alpha_{2,\infty}}, \quad (4.76)$$

where  $k_{1,0}, k_{2,0}, k_{1,\infty}$ , and  $k_{2,\infty}$  are any positive real numbers, and  $\alpha_{1,0} = \frac{r_0+2\kappa_0}{r_0}$ ,  $\alpha_{1,\infty} = \frac{r_\infty+2\kappa_\infty}{r_\infty}$ ,  $\alpha_{2,0} = \frac{r_0+2\kappa_0}{r_0+\kappa_0}$ ,  $\alpha_{2,\infty} = \frac{r_\infty+2\kappa_\infty}{r_\infty+\kappa_\infty}$  with  $\kappa_0 < 0$ ,  $\kappa_\infty > 0$ ,  $r_0 > -2\kappa_0$ ,  $r_\infty > 0$ .

*Proof.* Consider (4.22)-(4.25) with  $\omega \equiv 0$ . Using LaSalle invariance principle with

$$V(z) = \int_0^{z_1} \left( k_{1,0}\{s\}^{\alpha_{1,0}} + k_{1,\infty}\{s\}^{\alpha_{1,\infty}} \right) ds + \frac{z_2^2}{2}, \quad (4.77)$$

combined with [85, Corollary 2.24] ends the proof. □



### 4.4.2 Implicit Controllers for chain of integrators

For the  $z$ -subsystem of the target system (4.22)-(4.25), we can use the results from [120], [123], [168] to characterize a new  $\mathcal{F}$  from which we can subsequently design a nonlinear predictor-based controller  $U(t)$  achieving **F**TS or nearly **F****x**TS. However, for such a chain of integrators, it appears that  $G_{\mathbf{d}}$  has to be of the form  $G_{\mathbf{d}} = \text{diag}[r_1, \dots, r_n]$ ,  $r_i = r + (i-1)\kappa$ ,  $r > \max(0, -n\kappa)$ ,  $\kappa \in \mathbb{R}$  ( $\kappa$  is the degree of homogeneity) then

$$\mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \text{diag}[e^{r_1s}, \dots, e^{r_ns}]. \quad (4.78)$$

Note that  $AG_{\mathbf{d}} - G_{\mathbf{d}}A = \kappa A$  (the driftless part " $Az$ " is homogeneous) [88] can be rephrased as:

**Proposition 4.5** (see [88] for details)

Let  $a, b$  be chosen positive real numbers. For the  $z$ -subsystem of the target system (4.22)-(4.25), let

$$\mathcal{F}(z) = \|z\|_{\mathbf{d}}^{r+n\kappa} k \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z, \quad (4.79)$$

where  $\mathbf{d}$  is the dilation defined by (4.78) with  $r_i = r + (i-1)\kappa$ ,  $r > \max(0, -n\kappa)$ ,  $\|z\|_{\mathbf{d}}$  is its associated homogeneous norm and gain  $k = yP$  is derived from the solution  $X \in \mathbb{R}^{n \times n}$  ( $X = P^{-1}$ ),  $y \in \mathbb{R}^{1 \times n}$  of the LMIs:

$$\begin{cases} XA^\top + AX + y^\top B^\top + By + aX \leq 0, \\ X > 0, \quad bX \geq G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top > 0, \end{cases} \quad (4.80)$$

$$\quad (4.81)$$

where  $A = \left[ [0_{(n-1) \times 1}, \bar{I}_{n-1}]^\top, 0_{n \times 1} \right]^\top$ ,  $B = e_n := [0, \dots, 0, 1]^\top$ . Then, the  $z$ -subsystem of (4.22)-(4.25) with  $w \equiv 0$  is

- globally **F**TS for  $\kappa < 0$  and the settling time is given by

$$T(z_0) \leq \frac{b}{a(-\kappa)} \|z_0\|_{\mathbf{d}}^{-\kappa},$$

- globally uniformly **E**S for  $\kappa = 0$ ,
- globally nearly **F****x**TS for  $\kappa > 0$ .

Similarly, we get:

**Proposition 4.6** (see [88], [120], [123] for an equivalent formulation)

Select  $\kappa_0 < 0$ ,  $\kappa_\infty > 0$  and  $r_0 > -n\kappa_0$ ,  $r_\infty > 0$ . Let us define  $r_{i,0} = r_0 + (i-1)\kappa_0$ ,  $r_{i,\infty} = r_\infty + (i-1)\kappa_\infty$ . Set  $\mathbf{d}_0(s) = e^{G_{\mathbf{d}_0}s} = \text{diag}[e^{r_{1,0}s}, \dots, e^{r_{n,0}s}]$  and  $\mathbf{d}_\infty(s) = e^{G_{\mathbf{d}_\infty}s} = \text{diag}[e^{r_{1,\infty}s}, \dots, e^{r_{n,\infty}s}]$ . Let  $a_0, b_0, a_\infty, b_\infty$  be chosen positive reals. For the  $z$ -subsystem of the target system (4.22)-(4.25), let

$$\mathcal{F}(z) = \begin{cases} \|z\|_{\mathbf{d}_0}^{r_0+n\kappa_0} k_0 \mathbf{d}_0(-\ln \|z\|_{\mathbf{d}_0}) z & \text{for } \|z\| < 1 \\ \|z\|_{\mathbf{d}_\infty}^{r_\infty+n\kappa_\infty} k_\infty \mathbf{d}_\infty(-\ln \|z\|_{\mathbf{d}_\infty}) z & \text{for } \|z\| \geq 1 \end{cases} \quad (4.82)$$

where the gains  $k_0$  and  $k_\infty$  are such that the LMIs

$$\begin{cases} X_0A^\top + AX_0 + y_0^\top B^\top + By_0 + a_0X_0 \leq 0, \end{cases} \quad (4.83)$$

$$\begin{cases} X_0 > 0, \quad b_0X_0 \geq G_{\mathbf{d}_0}X_0 + XG_{\mathbf{d}_0}^\top > 0, \end{cases} \quad (4.84)$$

$$\begin{cases} X_\infty A^\top + AX_\infty + y_\infty^\top B^\top + By_\infty + a_\infty X_\infty \leq 0, \end{cases} \quad (4.85)$$

$$\begin{cases} X_\infty > 0, \quad b_\infty X_\infty \geq G_{\mathbf{d}_\infty} X_\infty + XG_{\mathbf{d}_\infty}^\top > 0, \end{cases} \quad (4.86)$$

have solution  $X_0$  and  $X_\infty$  in  $\mathbb{R}^{n \times n}$ ,  $y_0$  and  $y_\infty$  in  $\mathbb{R}^{1 \times n}$  (where  $k_0 = y_0 P_0, P_0 = X_0^{-1}, k_\infty = y_\infty P_\infty, P_\infty = X_\infty^{-1}$ ).

Then, the  $z$ -subsystem of (4.22)-(4.25) with  $w \equiv 0$  is globally **FxTS**.

## 4.5 Simulations

In this section, we focus on (4.3)-(4.4) for  $n = 2$ , i.e.

$$\begin{cases} \dot{z}_1(t) = a_{11}z_1(t) + z_2(t - D_1), & t \geq t_0, \\ \dot{z}_2(t) = a_{21}z_1(t) + a_{22}z_2(t) + U(t - D_2), & t \geq t_0, \end{cases} \quad (4.87)$$

where  $a_{11}$ ,  $a_{21}$  and  $a_{22}$  are real constants,  $D_1$  and  $D_2$  are positive known delays.

By combining the state transformations (52)-(53) introduced in [192] with the change of variables  $\tilde{z}_1(t) = z_1(t)$ ,  $\tilde{z}_2(t) = z_2(t - D_1)$ , we recover the following transformations:

$$\begin{cases} \bar{z}_1(t) = z_1(t), \\ \bar{z}_2(t) = z_2(t - D_1) + a_{11}z_1(t). \end{cases} \quad (4.89)$$

Thus, (4.87)-(4.88) is transformed into

$$\begin{cases} \dot{\bar{z}}_1(t) = \bar{z}_2(t), & t \geq t_0, \\ \dot{\bar{z}}_2(t) = -a_{11}a_{22}\bar{z}_1(t) + (a_{11} + a_{22})\bar{z}_2(t) + a_{21}\bar{z}_1(t - D_1) \\ \quad + U(t - D_1 - D_2), & t \geq t_0, \end{cases} \quad (4.91)$$

which is rewritten into

$$\begin{cases} \dot{\bar{z}}_1(t) = \bar{z}_2(t), & t \geq t_0, \\ \dot{\bar{z}}_2(t) = -a_{11}a_{22}\bar{z}_1(t) + (a_{11} + a_{22})\bar{z}_2(t) \\ \quad + a_{21}\bar{z}_1(t - D_1) + u(t, 0), & t \geq t_0, \\ u_t(t, x) = u_x(t, x), & t \geq t_0, \quad x \in [0, D_1 + D_2], \\ u(t, D_1 + D_2) = U(t), & t \geq t_0. \end{cases} \quad (4.92)$$

Remark that (4.91)-(4.92) is similar to (4.1)-(4.2) (except for the delayed term) and clearly the approach developed in this chapter can be adapted to it to obtain the following control:

$$\begin{aligned} U(t) = & \mathcal{F}(\varphi_1(t, D_1 + D_2), \varphi_2(t, D_1 + D_2)) - a_{21}\varphi_1(t, D_2) - (a_{11} + a_{22})\varphi_2(t, D_1 + D_2) \\ & + a_{11}a_{22}\varphi_1(t, D_1 + D_2), \end{aligned} \quad (4.97)$$

where  $\varphi_1$  and  $\varphi_2$  are solutions of

$$\begin{cases} \varphi_{1,x}(t, x) = \varphi_2(t, x), \\ \varphi_{2,x}(t, x) = -a_{11}a_{22}\varphi_1(t, x) + (a_{11} + a_{22})\varphi_2(t, x) + a_{21}\varphi_1(t, x - D_1) + u(t, x). \end{cases} \quad (4.98)$$

with  $x \in [0, D_1 + D_2]$  and  $u(t, x)$  the solution of the PDE part of (4.93)-(4.96).

Let us now give numerical simulations for the closed-loop system (4.93)-(4.96) with predictor-based controller  $U(t)$  in (4.97). First, using  $\mathcal{F}$  given in (4.75) to attain **FxTS** where we choose the

delays  $D_1 = 0.75\text{s}$ ,  $D_2 = 1\text{s}$ , and the parameters as follows:  $\kappa = -0.2$ ,  $r = 3$ ,  $k_1 = 10$  and  $k_2 = 11$ . Then, using  $\mathcal{F}$  given in (4.76) to attain **FxTTS** where we take the delays  $D_1 = 0.5\text{s}$ ,  $D_2 = 0.75\text{s}$ , and the parameters as follows:  $\kappa_0 = -0.5$ ,  $r_0 = 2$ ,  $k_{1,0} = 10$ ,  $k_{2,0} = 11$ ,  $\kappa_\infty = -0.2$ ,  $r_\infty = 3$ ,  $k_{1,\infty} = 11$ ,  $k_{2,\infty} = 10$ . Finally, we take the initial time  $t_0 = 0$ , the coefficients  $a_{11} = a_{21} = a_{22} = 1$  and we give the simulations for three different initial conditions:  $z_0 = [5, 3]^\top$ ,  $10z_0$  and  $100z_0$ .

Figure 4.1 shows on the left the evolution of the states  $z_1$  and  $z_2$  of the ODE part of the closed-loop system (4.93)-(4.96) with predictor-based controller  $U(t)$  in (4.97) (whose time evolution is described in Figure 4.3 alongside of the time evolution of the norm of (4.93)-(4.96) for different values of the delays, using the expression of  $\mathcal{F}$  in (4.75) to get **FTS**). On the right hand we can see the numerical solution  $u(t, x)$  of the PDE part of with respect the initial conditions  $z(t_0) = [5, 3]^\top$  and  $u(t_0, x) = 0$ ,  $x \in [0, D_1 + D_2]$ . Finally, Figure 4.2 shows in a logarithmic scale the evolution of the norm  $\|z(t)\|_2^2$  of the closed-loop system (4.87)-(4.88) with predictor-based controller  $U(t)$  in (4.97) on the left using the expression of  $\mathcal{F}$  in (4.75) and on the right using the expression of  $\mathcal{F}$  in (4.76). As it can be observed on the left, the times of convergence depend on the initial conditions (the larger the initial condition, the larger the settling time). On the right-hand side, we can observe that the times of the convergence do not depend on the initial conditions (the settling time is upper bounded by a constant independent of the initial conditions).

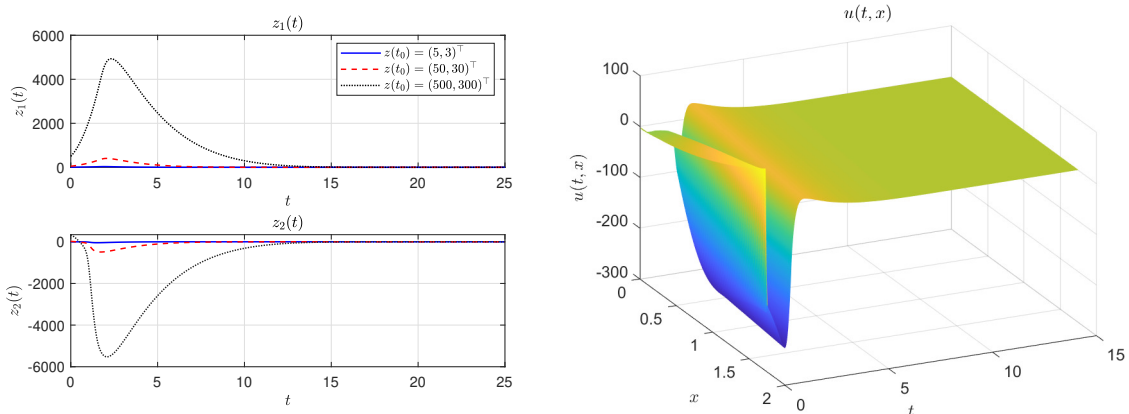


Figure 4.1: On the left: the evolution of the states  $z_1(t)$ ,  $z_2(t)$  of the ODE part of the closed-loop system (4.93)-(4.96) with predictor-based controller  $U(t)$  in (4.97) and using the expression of  $\mathcal{F}$  in (4.75) to get **FTS**, in blue solid lines for the initial condition  $z(t_0) = [5, 3]^\top$ , in red dashed lines for  $z(t_0) = [50, 30]^\top$ , and in black dotted lines for  $z(t_0) = [500, 300]^\top$ , with the delays  $D_1 = 0.75\text{s}$  and  $D_2 = 1\text{s}$ . On the right: the evolution of  $u(t, x)$  the state of the PDE part of (4.93)-(4.96) for only the initial condition  $z(t_0) = [5, 3]^\top$  and  $u(t_0, x) = 0$ ,  $x \in [0, D_1 + D_2]$ .

## Conclusion

In this Chapter, we solved the problem of finite/fixed-time stabilization of a chain of integrators with input delay. The chain of integrators was rewritten into an ODE-PDE setting, where the PDE part modeled the effect of the delay on the input. The predictor-based controller was designed using a nonlinear infinite-dimensional backstepping transformation that linked the ODE-PDE setting to the well-chosen finite/fixed-time stable target system. The finite/fixed-time stability property was transferred back to the original system by the inverse transformation and using  $\mathcal{GKL}$

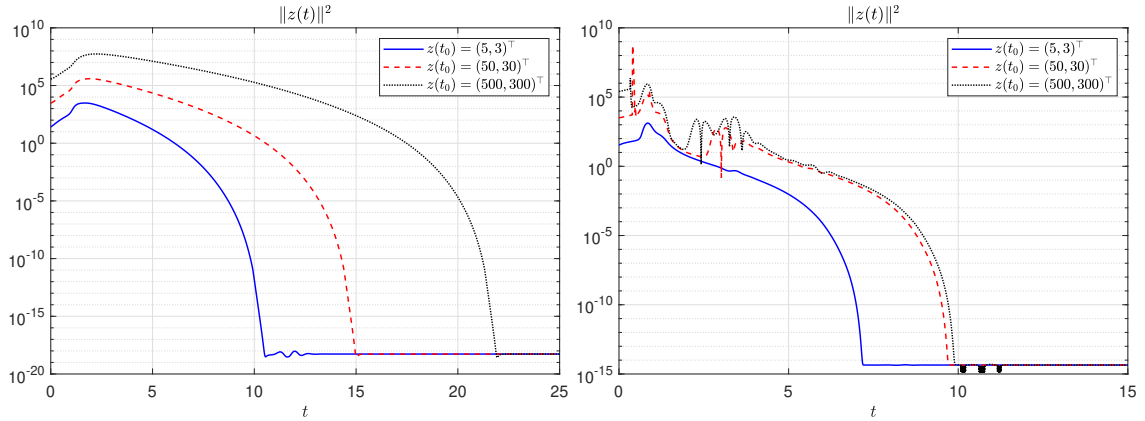


Figure 4.2: The evolution of the norm  $\|z(t)\|_2^2$  of the closed-loop system (4.87)-(4.88), with predictor-based controller  $U(t)$  in (4.97) and using on the left the expression of  $\mathcal{F}$  in (4.75) to get **FTS** with the delays  $D_1 = 0.75\text{s}$  and  $D_2 = 1\text{s}$ , and using on the right the expression of  $\mathcal{F}$  in (4.76) to get **FxTS** (logarithmic scale) with the delays  $D_1 = 0.5\text{s}$  and  $D_2 = 0.75\text{s}$ , in a blue solid line for the initial condition  $z(t_0) = [5, 3]^\top$ , in a red dashed line for  $z(t_0) = [50, 30]^\top$ , and in a black dotted line  $z(t_0) = [500, 300]^\top$ .

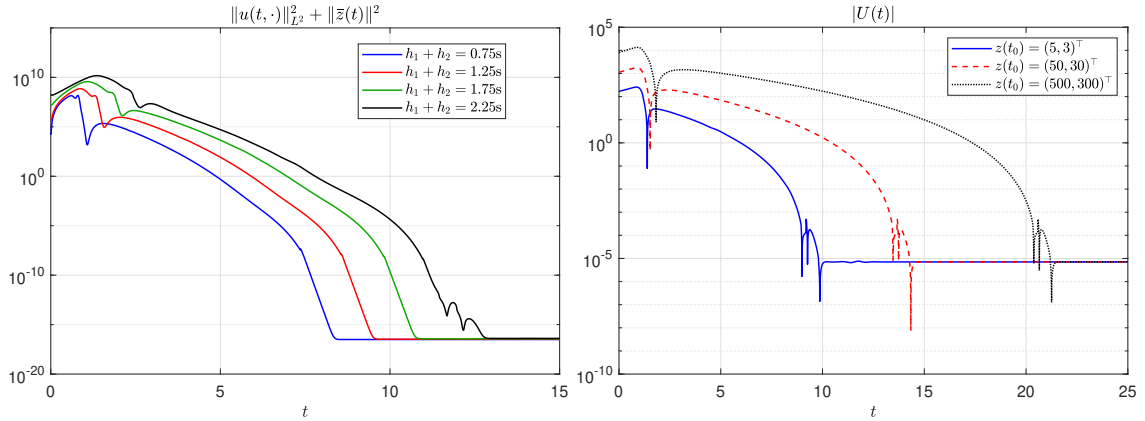


Figure 4.3: On the left, the evolution of the norm  $\|\bar{z}(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0, D)}^2$  of the closed-loop system (4.93)-(4.96) with predictor-based controller  $U(t)$  in (4.97) and using the expression of  $\mathcal{F}$  in (4.75) to get **FTS** (logarithmic scale) for the initial condition  $z(t_0) = [5, 3]^\top$  with different values for the delays  $D_1$  and  $D_2$ . On the right, the evolution of the applied predictor-based controller  $U(t)$  given in (4.97) using the expression of  $\mathcal{F}$  in (4.75) to get **FTS** in a blue solid line for the initial condition  $z(t_0) = [5, 3]^\top$ , in a red dashed line for  $z(t_0) = [50, 30]^\top$ , and in a black dotted line for  $z(t_0) = [500, 300]^\top$ , with the delays  $D_1 = 0.75\text{s}$  and  $D_2 = 1\text{s}$ .

functions.



# Finite/fixed-time boundary stabilization of Parabolic PDEs

## Contents

<b>5.1</b>	<b>Introduction</b>	<b>95</b>
<b>5.2</b>	<b>Problem statement:</b>	<b>96</b>
<b>5.3</b>	<b>Stability analysis</b>	<b>96</b>
<b>5.4</b>	<b>Application to finite/fixed-time stabilization</b>	<b>98</b>
<b>5.5</b>	<b>Simulations</b>	<b>99</b>
5.5.1	The case of finite-time stabilization	99
5.5.2	The case of fixed-time stabilization	99
<b>5.6</b>	<b>Comments on possible extensions of our approach</b>	<b>99</b>
5.6.1	Finite/fixed-time tracking of a trajectory	99
5.6.2	Attenuation/rejection of control-matched disturbances	100
<b>5.7</b>	<b>Limitation of our approach</b>	<b>102</b>

In this chapter, we treat the problem of Lyapunov-based nonlinear boundary stabilization of a class of 1D reaction-diffusion systems with any predefined convergence (asymptotic or Non-Asymptotic). As an application, we focus on the Non-Asymptotic notions (finite-time and fixed-time) for which we give some particular explicit control designs followed by some numerical simulations. The key idea of our approach is to use a “spatially weighted  $L^2$ -norm” as a Lyapunov functional to design a nonlinear controller and to ensure stability with any desired convergence.

## 5.1 Introduction

In this chapter, we revisit the problem of boundary control design for a class of linear 1D reaction-diffusion equations. In Section 5.2, we introduce the 1D reaction-diffusion system with Dirichlet actuation. In Section 5.3, we introduce and give some properties of the “spatially weighted  $L^2$ -norm” which is chosen as a Lyapunov functional (inspired from results in [180] and [69, Chapter 11, page 178] for hyperbolic systems). Next, we establish the Lyapunov stability analysis where we design a nonlinear controller that will ensure stability with a predefined convergence (asymptotic or Non-Asymptotic). Our approach is similar to the Control Lyapunov Function (CLF) approach, which has been investigated for parabolic PDEs in [196], [197], in the sense of using the Lyapunov function directly to design the boundary control. In Section 5.4, we provide some explicit control designs ensuring Non-Asymptotic stability (finite-time and fixed-time). In Section 5.5, we give some numerical simulations to illustrate the results of both the finite-time and fixed-time stabilization cases. Finally, in sections 5.6 and 5.7, we give some possible extensions of our approach and some limitations.

## 5.2 Problem statement:

We consider the following reaction-diffusion equation with constant reaction term and Dirichlet actuation:

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \geq t_0, x \in [0, 1], & (5.1) \\ z(t, 0) = 0, & t \geq t_0, & (5.2) \\ z(t, 1) = U(t), & t \geq t_0, & (5.3) \\ z(t_0, x) = z_0(x), & x \in [0, 1], & (5.4) \end{cases}$$

where  $t_0 \geq 0$  is the initial time,  $\lambda \in \mathbb{R}$  is the reaction coefficient,  $z(t, x) \in \mathbb{R}$  is the state,  $U(t) \in \mathbb{R}$  is the control,  $z_0 \in L^2(0, 1)$  is the initial condition.

The goal of this chapter is to design a nonlinear control  $U(t)$  and a Lyapunov functional  $V(z(t, \cdot))$  such that the time derivative of  $V$  along the solutions of (5.1)-(5.4) satisfies (2.115) in Proposition 2.5 for any continuous function  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that  $\mathcal{K}(0) = 0$ . As an application, we choose the function  $\mathcal{K}$  (satisfying  $\frac{d}{dt}V(z(t, \cdot)) \leq -\mathcal{K}(V(z(t, \cdot)))$  in (2.115)) such that the closed-loop system (5.1)-(5.4) with the control  $U(t)$  is **FTS** or **FxTS** in light of the notions presented in Chapter 2.

## 5.3 Stability analysis

In this section, we first introduce the Lyapunov functional candidate and we give some of its properties. Then, by computing its time derivative along the solutions of (5.1)-(5.4), we design a nonlinear control  $U(t)$  that will ensure inequality (2.115) for all  $t \geq t_0$  and all continuous function  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that  $\mathcal{K}(0) = 0$ .

Let us consider the following spatially weighted  $L^2$ -norm<sup>1</sup> as a Lyapunov function candidate:

$$V(z) = \int_0^1 e^{\sigma x} |z(x)|^2 dx, \quad \sigma > 0. \quad (5.5)$$

We can clearly see that  $V$  satisfies for any  $\sigma > 0$  the following property:

$$\|z(t, \cdot)\|_{L^2(0,1)}^2 \leq V(z(t, \cdot)) \leq e^\sigma \|z(t, \cdot)\|_{L^2(0,1)}^2. \quad (5.6)$$

Thus  $V$  is coercive (see Definition 2.4 in Chapter 2). Moreover, by computing the time derivative of  $V$  along the solutions (5.1)-(5.4), we can establish the following proposition

### Proposition 5.1

Let  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function such that  $\mathcal{K}(0) = 0$ . Then, the functional  $V$  given in (5.5) satisfies the following inequality for every  $t \geq t_0 \geq 0$  and every  $\sigma > 0$ :

$$\frac{d}{dt}V(z(t, \cdot)) \leq -2e^\sigma \left[ \frac{\sigma}{2} U(t)^2 - z_x(t, 1)U(t) - \mathcal{B}(V(z(t, \cdot))) + \frac{e^{-\sigma}}{2} \mathcal{K}(V(z(t, \cdot))) \right], \quad (5.7)$$

where  $\mathcal{B}(\cdot)$  is given by

$$\mathcal{B}(V(z(t, \cdot))) = e^{-\sigma} \max \left( 0, \lambda + \frac{\sigma^2}{2} \right) V(z(t, \cdot)) + \frac{e^{-\sigma}}{2} \mathcal{K}(V(z(t, \cdot))) \geq 0. \quad (5.8)$$

<sup>1</sup>similar functionals have been used in the framework of exponential stabilization (e.g. for linear conservation laws in [180] or for a transport PDE with a zero input at the boundary in [69, Chapter 11, page 178]).

*Proof.* Let us start by computing the time derivative of  $V$  in (5.5) along the solutions of (5.1)-(5.4),

$$\frac{d}{dt}V(z(t, \cdot)) = 2 \int_0^1 e^{\sigma x} z(t, x) z_t(t, x) dx, \quad (5.9)$$

$$= 2\lambda \int_0^1 e^{\sigma x} |z(t, x)|^2 dx + 2 \int_0^1 e^{\sigma x} z(t, x) z_{xx}(t, x) dx. \quad (5.10)$$

Next, by integration by parts on the last term, we get,

$$\begin{aligned} \frac{d}{dt}V(z(t, \cdot)) &= 2\lambda V(z(t, \cdot)) - 2 \int_0^1 e^{\sigma x} |z_x(t, x)|^2 dx - 2\sigma \int_0^1 e^{\sigma x} z(t, x) z_x(t, x) dx \\ &\quad + 2 [e^{\sigma x} z(t, x) z_x(t, x)]_0^1. \end{aligned} \quad (5.11)$$

Then, we get

$$\frac{d}{dt}V(z(t, \cdot)) \leq 2\lambda V(z(t, \cdot)) - \sigma \int_0^1 e^{\sigma x} \frac{\partial |z(t, x)|^2}{\partial x} dx + 2e^\sigma z(t, 1) z_x(t, 1) - 2z(t, 0) z_x(t, 0), \quad (5.12)$$

$$= 2\lambda V(z(t, \cdot)) + 2e^\sigma z_x(t, 1) U(t) - \sigma \int_0^1 e^{\sigma x} \frac{\partial |z(t, x)|^2}{\partial x} dx. \quad (5.13)$$

Now, by a second integration by parts on the last term, we obtain,

$$\begin{aligned} \frac{d}{dt}V(z(t, \cdot)) &\leq 2\lambda V(z(t, \cdot)) + 2e^\sigma z_x(t, 1) U(t) - \sigma [e^{\sigma x} |z(t, x)|^2]_0^1 \\ &\quad + \sigma^2 \int_0^1 e^{\sigma x} |z(t, x)|^2 dx, \end{aligned} \quad (5.14)$$

$$\begin{aligned} &= 2\lambda V(z(t, \cdot)) + 2e^\sigma z_x(t, 1) U(t) - \sigma e^\sigma |z(t, 1)|^2 + \sigma |z(t, 0)|^2 \\ &\quad + \sigma^2 \int_0^1 e^{\sigma x} |z(t, x)|^2 dx, \end{aligned} \quad (5.15)$$

$$= 2 \left( \lambda + \frac{\sigma^2}{2} \right) V(z(t, \cdot)) + 2e^\sigma z_x(t, 1) U(t) - \sigma e^\sigma U(t)^2, \quad (5.16)$$

$$= -2e^\sigma \left[ \frac{\sigma}{2} U(t)^2 - z_x(t, 1) U(t) - e^{-\sigma} \left( \lambda + \frac{\sigma^2}{2} \right) V(z(t, \cdot)) \right]. \quad (5.17)$$

Then, using the fact that  $a \leq \max(0, a)$  for any  $a \in \mathbb{R}$ , we get,

$$\frac{d}{dt}V(z(t, \cdot)) \leq -2e^\sigma \left[ \frac{\sigma}{2} U(t)^2 - z_x(t, 1) U(t) - e^{-\sigma} \max \left( 0, \lambda + \frac{\sigma^2}{2} \right) V(z(t, \cdot)) \right], \quad (5.18)$$

$$= -2e^\sigma \left[ \frac{\sigma}{2} U(t)^2 - z_x(t, 1) U(t) - \mathcal{B}(V(z(t, \cdot))) + \frac{e^{-\sigma}}{2} \mathcal{K}(V(z(t, \cdot))) \right], \quad (5.19)$$

with  $\mathcal{B}(\cdot)$  being given in (5.8).  $\square$

Let us now give the first main result of our chapter,

### Theorem 5.1

Let  $t_0 \geq 0$ ,  $\sigma > 0$ . Let  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function such that  $\mathcal{K}(0) = 0$ . Let  $\mathcal{B}(\cdot)$  be given as in (5.8). Then, under the following control:

$$U(t) = \frac{z_x(t, 1) - \sqrt{z_x(t, 1)^2 + 2\sigma \mathcal{B}(V(z(t, \cdot)))}}{\sigma}, \quad (5.20)$$

or

$$U(t) = \frac{z_x(t, 1) + \sqrt{z_x(t, 1)^2 + 2\sigma \mathcal{B}(V(z(t, \cdot)))}}{\sigma}, \quad (5.21)$$

the inequality (2.115) (i.e.  $\frac{d}{dt}V(z(t, \cdot)) \leq -\mathcal{K}(V(z(t, \cdot)))$ ) is satisfied for all  $t \geq t_0$ .



*Proof.* The proof of Theorem 5.1 is a direct application of the quadratic formula on the inequality (5.7), where we chose  $U(\cdot)$  to be the solution of the following second-degree equation:

$$\frac{\sigma}{2}U(t)^2 - z_x(t, 1)U(t) - \mathcal{B}(V(z(t, \cdot))) = 0. \quad (5.22)$$

□

**Remark 5.1**

Notice that by well choosing  $\mathcal{K}$  in inequality (2.115), one may be able to recover different types of stability including asymptotic (exponential, hyper-exponential,...) or Non-Asymptotic (finite-time, fixed-time, predefined-time, prescribed-time,...).

## 5.4 Application to finite/fixed-time stabilization

In this section, we use Theorem 5.1 to establish the second main result of our chapter which proves the **F**TS (resp. **F**x**T**S) of the closed-loop system (5.1)-(5.4) with the nonlinear control (5.20) for some well-chosen function  $\mathcal{K}$ .

**Theorem 5.2**

Let  $t_0 \geq 0$ ,  $\sigma > 0$ . Let  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function such that  $\mathcal{K}(0) = 0$ . Let  $\mathcal{B}(\cdot)$  be given as in (5.8). Then, if  $\mathcal{K}$  is in the set  $\mathcal{E}_{k_0, a_0}$  (resp. in the set  $\mathcal{E}_{k_0, a_0}^{k_\infty, a_\infty}$ ), then the closed-loop system (5.1)-(5.4) with the nonlinear control (5.20) is finite-time stable (resp. fixed-time stable).

Moreover, there exists a settling time  $T_{\max}(V(z_0))$  (upper bounded by a constant when **F**x**T**S) such that  $V(z(t, \cdot)) = 0$  when  $t \geq t_0 + T_{\max}(V(z_0))$ . By the coercivity condition  $\|z(t, \cdot)\|_{L^2(0,1)}^2 = 0$  when  $t \geq t_0 + T_{\max}(V(z_0))$ . In particular, if the control (5.20) is replaced by

$$U(t) = \frac{z_x(t, 1) - \text{sign}(z_x(t, 1))\sqrt{z_x(t, 1)^2 + 2\sigma\mathcal{B}(V(z(t, \cdot)))}}{\sigma}, \quad (5.23)$$

we have in addition,  $|U(t)| \rightarrow 0$  when  $t \rightarrow t_0 + T_{\max}(V(z_0))$  and  $|U(t)| = 0$  for any  $t \geq t_0 + T_{\max}(V(z_0))$ .

*Proof.* The proof of Theorem 5.2 is a straightforward application of Proposition 5.1 and Theorem 5.1. In fact, from Theorem 5.1, we have that (2.115) is satisfied for all  $t \geq t_0$  and any continuous function  $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that  $\mathcal{K}(0) = 0$ . In particular, for  $\mathcal{K} \in \mathcal{E}_{k_0, a_0}$  (resp.  $\mathcal{K} \in \mathcal{E}_{k_0, a_0}^{k_\infty, a_\infty}$ ). Then from Proposition 2.5, we conclude that the closed-loop system (5.1)-(5.4) with (5.20) (or (5.21)) is **F**TS (resp. **F**x**T**S).

Furthermore, if (5.20) is replaced by (5.23), we can prove that:

$$|U(t)|^2 = \frac{1}{\sigma^2} \left[ \sqrt{z_x(t, 1)^2 - \sqrt{z_x(t, 1)^2 + 2\sigma\mathcal{B}(V(z(t, \cdot)))}} \right]^2, \quad (5.24)$$

$$\leq \frac{1}{\sigma^2} \left[ \sqrt{z_x(t, 1)^2 + 2\sigma\mathcal{B}(V(z(t, \cdot)))} - z_x(t, 1) \right]^2, \quad (5.25)$$

$$= \frac{2}{\sigma} \mathcal{B}(V(z(t, \cdot))), \quad (5.26)$$

where we have used the fact that  $|\sqrt{a_1} - \sqrt{a_2}| \leq \sqrt{|a_1 - a_2|}$ , for any  $a_1, a_2 \geq 0$ . Using in addition the fact that  $V(z(t, \cdot)) = 0 \implies \mathcal{K}(V(z(t, \cdot))) = 0 \implies \mathcal{B}(V(z(t, \cdot))) = 0$ , we conclude that  $|U(t)| \rightarrow 0$  for any  $t \rightarrow t_0 + T_{\max}(V(z_0))$  and  $|U(t)| = 0$  for any  $t \geq t_0 + T_{\max}(V(z_0))$ . □

## 5.5 Simulations

In this section, we give numerical simulations for the closed-loop system (5.1)-(5.4) for three different initial conditions  $z_0 = x - x^2$ ,  $100z_0$ , and  $1000z_0$  with the following reaction coefficient  $\lambda = 20$ , the initial time  $t_0 = 0$ , and with the nonlinear control  $U(t)$  defined as in (5.20) first for  $\mathcal{K}(V(z(t, \cdot))) = \frac{c}{2}V(z(t, \cdot))^\alpha$ , with  $c = 0.5$ ,  $\alpha = 0.5$ , and  $\sigma = 2$  (given in (5.5)), to ensure **FTS** and then for  $\mathcal{K}(V(z(t, \cdot))) = c_1V(z(t, \cdot))^\alpha + c_2V(z(t, \cdot))^\beta$ , with  $c_1 = 0.5$ ,  $c_2 = 1$ ,  $\alpha = 0.5$ ,  $\beta = 2$ , and  $\sigma = 3$ , to ensure **FxTS**.

### 5.5.1 The case of finite-time stabilization

Figure 5.1 shows on top left the evolution of the state  $z(t, x)$  of the closed-loop system (5.1)-(5.4) with the control  $U(t)$  in (5.20) (whose time evolution for  $\mathcal{K}(V(z(t, \cdot))) = \frac{c}{2}V(z(t, \cdot))^\alpha$  is described at the bottom of Figure 5.1 for the initial condition  $z_0 = x - x^2$ ) with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))^\alpha$ , for the initial condition  $z_0 = x - x^2$ . Finally, Figure 5.1 shows on top right, in a logarithmic scale, the evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2$  of the closed-loop system (5.1)-(5.4) with the nonlinear control  $U(t)$  in (5.20) with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))^\alpha$  in solid lines and with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))$  in dashed lines, and for three different initial conditions:  $z_0 = x - x^2$  in blue lines,  $100z_0$  in red lines, and  $1000z_0$  in black lines. Hence, we can observe from the solid lines that the larger the initial condition, the larger the settling time (i.e. the times of convergence depend on the initial condition).

### 5.5.2 The case of fixed-time stabilization

Figure 5.2 shows on top left the evolution of the state  $z(t, x)$  of the closed-loop system (5.1)-(5.4) with the control  $U(t)$  in (5.20) with  $\mathcal{K}(V(z(t, \cdot))) = c_1V(z(t, \cdot))^\alpha + c_2V(z(t, \cdot))^\beta$  (whose time evolution is presented at the bottom of Figure 5.1 for the initial condition  $z_0 = x - x^2$ ), for the initial condition  $z_0 = x - x^2$ . Figure 5.2 shows in a logarithmic scale the evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2$  of the closed-loop system (5.1)-(5.4) with the nonlinear control  $U(t)$  in (5.20) with  $\mathcal{K}(V(z(t, \cdot))) = c_1V(z(t, \cdot))^\alpha + c_2V(z(t, \cdot))^\beta$  in solid lines and with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))$  in dashed lines, and for three different initial conditions:  $z_0 = x - x^2$  in blue solid lines,  $100z_0$  in red solid lines, and  $1000z_0$  in black solid lines. Hence, we can observe that the settling time is upper bounded by a constant that does not depend on the initial conditions (i.e. the time of the convergence does not depend on the initial conditions).

## 5.6 Comments on possible extensions of our approach

In this section, we will provide insightful comments regarding potential extensions of our approach.

### 5.6.1 Finite/fixed-time tracking of a trajectory

Using our approach it is also possible to design a control that ensures finite/fixed-time tracking of a given trajectory  $z^r(t, x)$  satisfying (5.1)-(5.4). In fact, by introducing an error variable  $\tilde{z}^r(t, x) = z(t, x) - z^r(t, x)$ , the study of the problem of tracking becomes equivalent to studying the stability

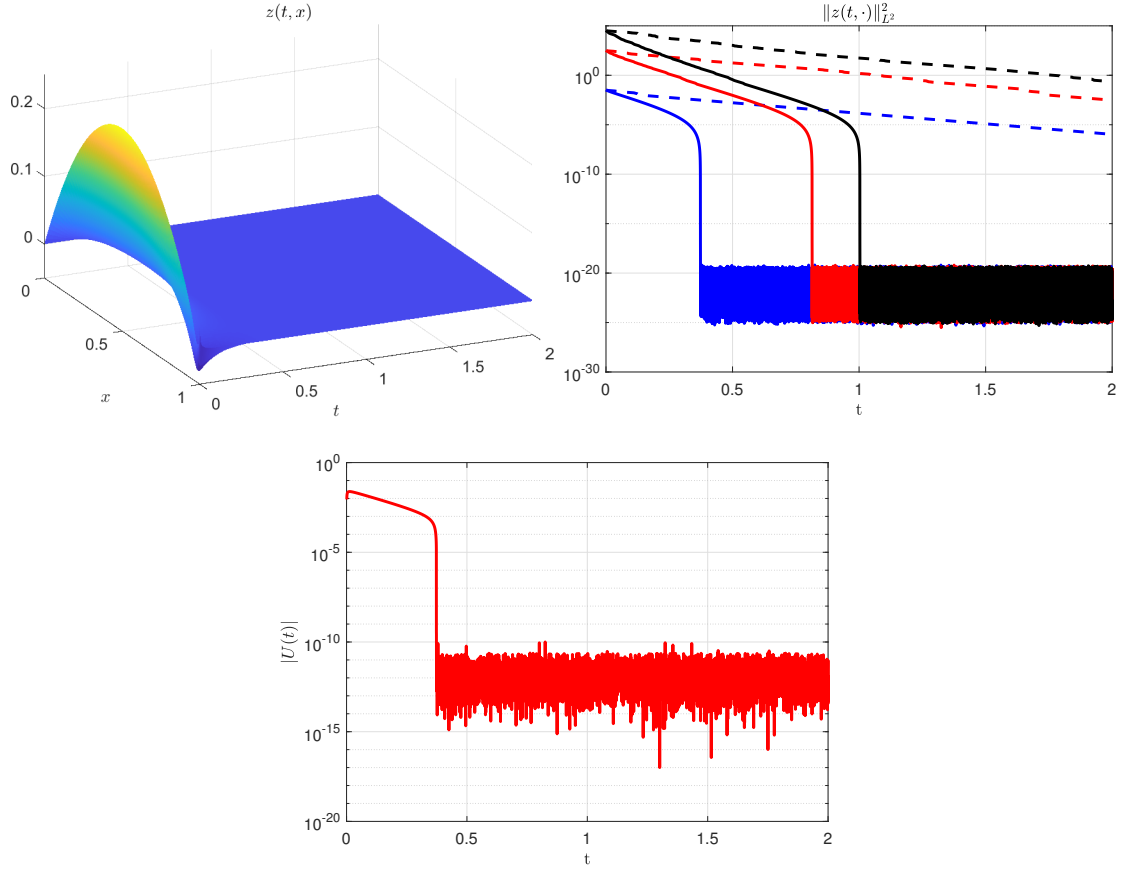


Figure 5.1: On top left: the evolution of the state  $z(t, x)$  of the closed-loop system (5.1)-(5.4) with the control  $U(t)$  in (5.20) with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))^\alpha$  (whose time evolution is given on the bottom) for the initial condition  $z(t_0, x) = x - x^2$ . On top right, the evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2$  of the closed-loop system (5.1)-(5.4) in a logarithmic scale in a blue line for the initial condition  $z(t_0, x) = x - x^2$ , in a red line for  $z(t_0, x) = 100(x - x^2)$ , and in a black line for  $z(t_0, x) = 1000(x - x^2)$ , where we used the nonlinear control  $U(t)$  given in (5.20) with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))^\alpha$  to get **FTS** as shown in solid lines, and with  $\mathcal{K}(V(z(t, \cdot))) = cV(z(t, \cdot))$  to get exponential stability shown in dashed lines.

of

$$\begin{cases} \tilde{z}_t^r(t, x) = \tilde{z}_{xx}^r(t, x) + \lambda \tilde{z}^r(t, x), & t \geq t_0, x \in [0, 1], & (5.27) \\ \tilde{z}^r(t, 0) = 0, & t \geq t_0, & (5.28) \\ \tilde{z}^r(t, 1) = U(t) - z^r(t, 1) := \tilde{U}(t), & t \geq t_0, & (5.29) \\ \tilde{z}^r(0, x) = z_0(x) - z^r(0, x) := \tilde{z}_0(x), & x \in [0, 1]. & (5.30) \end{cases}$$

## 5.6.2 Attenuation/rejection of control-matched disturbances

Another interesting feature of our approach is its ability to cope with a control-matched disturbance  $d(t) \in \mathbb{R}$  (i.e.  $z(t, 1) = U(t) + d(t)$ ). Remarkably, it guarantees either attenuation of this type of disturbance or rejection when the disturbance is bounded. Such a fact is possible by considering the control  $U(t) := \tilde{U}(t) - \tilde{M} \text{sign}(z_x(t, 1))$  for some  $\tilde{M} > 0$ . To design  $\tilde{U}(t)$ , we can

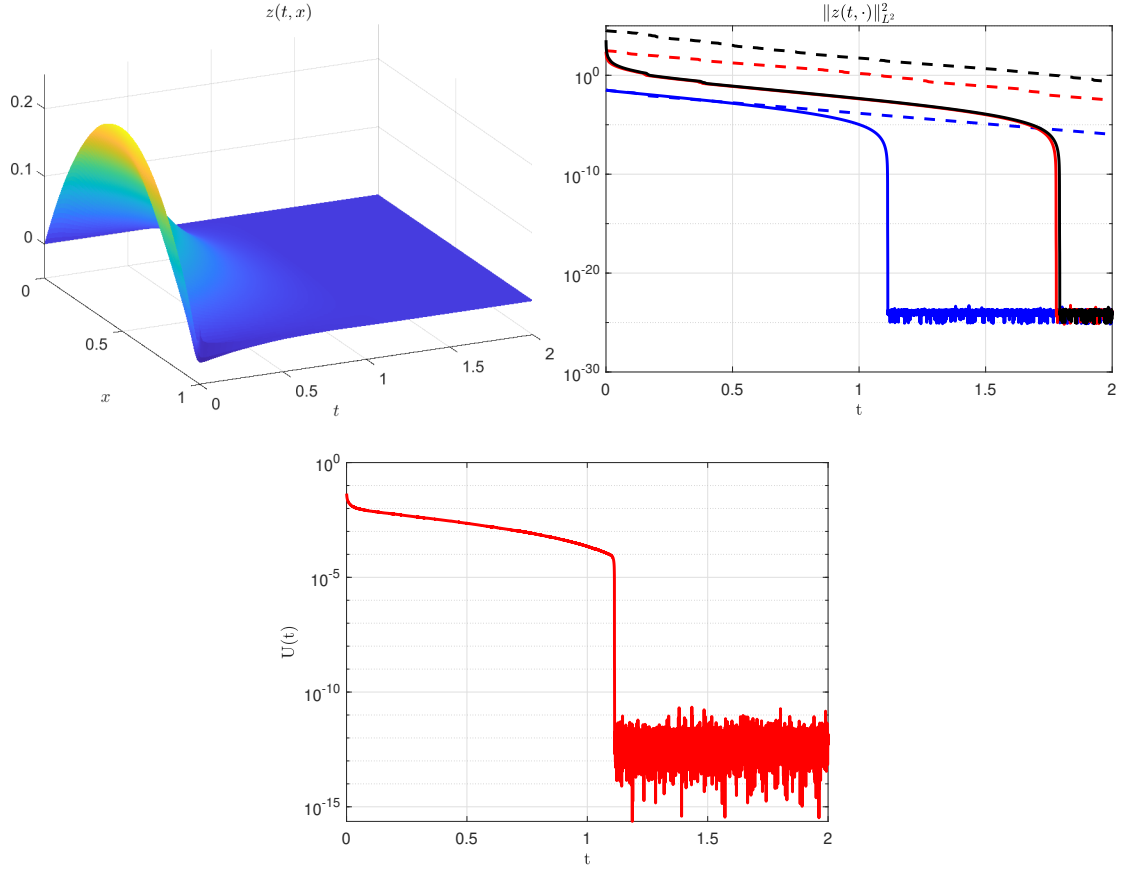


Figure 5.2: On top left, the evolution of the state  $z(t, x)$  of the closed-loop system (5.1)-(5.4) with the control  $U(t)$  in (5.20) with  $\mathcal{K}(V(z(t, \cdot))) = c_1 V(z(t, \cdot))^\alpha + c_2 V(z(t, \cdot))^\beta$  (whose time evolution is given at the bottom) for the initial condition  $z(t_0, x) = x - x^2$ . On top right, the evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2$  of the closed-loop system (5.1)-(5.4) in a logarithmic scale in a blue line for the initial condition  $z(t_0, x) = x - x^2$ , in a red line for  $z(t_0, x) = 100(x - x^2)$ , and in a black line for  $z(t_0, x) = 1000(x - x^2)$ , where we used the nonlinear control  $U(t)$  given in (5.20) and we took  $\mathcal{K}(V(z(t, \cdot))) = c_1 V(z(t, \cdot))^\alpha + c_2 V(z(t, \cdot))^\beta$  to get **FxTS** as shown in solid lines. Then, we took  $\mathcal{K}(V(z(t, \cdot))) = c_1 V(z(t, \cdot))$  to get exponential stability shown in dashed lines.

follow the lines of Section 5.3, where we can start by computing the time derivative of  $V$  given in (5.5) along the solutions of (5.1)-(5.4) and then by integrating by parts twice, we recover

$$\frac{dV(z(t, \cdot))}{dt} \leq (\sigma^2 + 2\lambda)V(z(t, \cdot)) + 2e^\sigma z_x(t, 1)z(t, 1) - \sigma e^\sigma |z(t, 1)|^2, \quad (5.31)$$

$$\begin{aligned} &= (\sigma^2 + 2\lambda)V(z(t, \cdot)) + 2e^\sigma z_x(t, 1) \left( \tilde{U}(t) + d(t) - \tilde{M} \operatorname{sign}(z_x(t, 1)) \right) \\ &\quad - \sigma e^\sigma \left( \tilde{U}(t) + d(t) - \tilde{M} \operatorname{sign}(z_x(t, 1)) \right)^2, \end{aligned} \quad (5.32)$$

By expanding this last inequality and using the fact that  $\tilde{U}(t)$  is assumed to satisfy<sup>2</sup>, in addition,  $\text{sign}(\tilde{U}(t)) = -\text{sign}(z_x(t, 1))$ , we recover (5.15) which yields

$$\begin{aligned} \frac{dV(z(t, \cdot))}{dt} &\leq (\sigma^2 + 2\lambda)V(z(t, \cdot)) + 2e^\sigma z_x(t, 1)\tilde{U}(t) - \sigma e^\sigma |\tilde{U}(t)|^2 + 2e^\sigma z_x(t, 1)d(t) \\ &\quad - 2e^\sigma \tilde{M}|z_x(t, 1)| - 2\sigma e^\sigma \tilde{U}(t)d(t) + 2\sigma e^\sigma \tilde{M} \text{sign}(z_x(t, 1))\tilde{U}(t) \\ &\quad + 2\sigma e^\sigma \tilde{M} \text{sign}(z_x(t, 1))d(t) - \sigma e^\sigma \tilde{M}^2 - \sigma e^\sigma |d(t)|^2, \end{aligned} \quad (5.33)$$

$$\begin{aligned} &\leq (\sigma^2 + 2\lambda)V(z(t, \cdot)) + 2e^\sigma z_x(t, 1)\tilde{U}(t) - \sigma e^\sigma |\tilde{U}(t)|^2 + 2e^\sigma |z_x(t, 1)||d(t)| \\ &\quad - 2e^\sigma \tilde{M}|z_x(t, 1)| + 2\sigma e^\sigma |\tilde{U}(t)||d(t)| + 2\sigma e^\sigma \tilde{M} \text{sign}(z_x(t, 1))\tilde{U}(t) \\ &\quad + 2\sigma e^\sigma \tilde{M}|d(t)| - \sigma e^\sigma \tilde{M}^2 - \sigma e^\sigma |d(t)|^2, \end{aligned} \quad (5.34)$$

$$\begin{aligned} &= (\sigma^2 + 2\lambda)V(z(t, \cdot)) + 2e^\sigma z_x(t, 1)\tilde{U}(t) - \sigma e^\sigma |\tilde{U}(t)|^2 + 2e^\sigma |z_x(t, 1)||d(t)| \\ &\quad - 2e^\sigma \tilde{M}|z_x(t, 1)| + 2\sigma e^\sigma |\tilde{U}(t)||d(t)| - 2\sigma e^\sigma \tilde{M}|\tilde{U}(t)| + 2\sigma e^\sigma \tilde{M}|d(t)| \\ &\quad - \sigma e^\sigma \tilde{M}^2 - \sigma e^\sigma |d(t)|^2, \end{aligned} \quad (5.35)$$

$$\begin{aligned} &= (\sigma^2 + 2\lambda)V(z(t, \cdot)) + 2e^\sigma z_x(t, 1)\tilde{U}(t) - \sigma e^\sigma |\tilde{U}(t)|^2 - \sigma e^\sigma \left[ |d(t)| - \tilde{M} \right]^2 \\ &\quad + 2e^\sigma \left[ |z_x(t, 1)| + \sigma |\tilde{U}(t)| \right] \left[ |d(t)| - \tilde{M} \right], \end{aligned} \quad (5.36)$$

$$\begin{aligned} &\leq -2e^\sigma \left[ \frac{\sigma}{2} |\tilde{U}(t)|^2 - z_x(t, 1)\tilde{U}(t) - \mathcal{B}(V(z(t, \cdot))) \right] - \mathcal{K}(V(z(t, \cdot))) \\ &\quad + 2e^\sigma \left[ |z_x(t, 1)| + \sigma |\tilde{U}(t)| \right] \left[ |d(t)| - \tilde{M} \right], \end{aligned} \quad (5.37)$$

where  $\mathcal{B}(\cdot)$  is given as in (5.8) and  $\tilde{U}(t)$  is chosen to satisfy  $\frac{\sigma}{2}\tilde{U}(t)^2 - z_x(t, 1)\tilde{U}(t) - \mathcal{B}(V(z(t, \cdot))) = 0$ , we get

$$\tilde{U}(t) = \frac{z_x(t, 1) - \text{sign}(z_x(t, 1))\sqrt{z_x(t, 1)^2 + 2\sigma\mathcal{B}(V(z(t, \cdot)))}}{\sigma}, \quad (5.38)$$

in order to obtain

$$\frac{dV(z(t, \cdot))}{dt} \leq -\mathcal{K}(V(z(t, \cdot))) + 2e^\sigma \left[ |z_x(t, 1)| + \sigma |\tilde{U}(t)| \right] \left[ |d(t)| - \tilde{M} \right], \quad \forall \sigma > 0; \forall \tilde{M} > 0. \quad (5.39)$$

As we can observe, the last inequality ensures the attenuation of the control-matched disturbance  $d(t)$  or the rejection when  $\sup_{t \geq t_0} |d(t)| \leq M$  for some upper-bound  $M > 0$ .

Notice that

$$\tilde{U}(t) = -\text{sign}(z_x(t, 1)) \left( \frac{\sqrt{z_x(t, 1)^2 + 2\sigma\mathcal{B}(V(z(t, \cdot)))} - \sqrt{z_x(t, 1)^2}}{\sigma} \right), \quad (5.40)$$

and that  $\text{sign}(\tilde{U}(t)) = -\text{sign}(z_x(t, 1))$  as we assumed. Moreover,  $\tilde{U}(t)$  stays bounded for all  $t \geq t_0$ .

## 5.7 Limitation of our approach

Despite the advantages of this approach, it is not straightforward to determine how to adapt it to the problem of finite/fixed-time *estimation* when for instance  $z_x(t, 1)$  is measured beforehand. In fact, to achieve this goal, the most logical step to do is to consider an observer of the form

$$\begin{cases} \hat{z}_t(t, x) = \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) - P_1(x, z_x(t, 1) - \hat{z}_x(t, 1)), & t \geq t_0, x \in [0, 1], \end{cases} \quad (5.41)$$

$$\begin{cases} \hat{z}(t, 0) = -P_2(z_x(t, 1) - \hat{z}_x(t, 1)), & t \geq t_0, \end{cases} \quad (5.42)$$

$$\begin{cases} \hat{z}(t, 1) = U(t) - P_3(z_x(t, 1) - \hat{z}_x(t, 1)) & t \geq t_0, \end{cases} \quad (5.43)$$

<sup>2</sup>see (5.40)

which yields an error system of the form

$$\begin{cases} \tilde{z}_t(t, x) = \tilde{z}_{xx}(t, x) + \lambda \tilde{z}(t, x) + P_1(x, \tilde{z}_x(t, 1)), & t \geq t_0, x \in [0, 1], \\ \tilde{z}(t, 0) = P_2(\tilde{z}_x(t, 1)), & t \geq t_0, \\ \tilde{z}(t, 1) = P_3(\tilde{z}_x(t, 1)), & t \geq t_0, \end{cases} \quad (5.44)$$

$$\tilde{z}(t, 0) = P_2(\tilde{z}_x(t, 1)), \quad t \geq t_0, \quad (5.45)$$

$$\tilde{z}(t, 1) = P_3(\tilde{z}_x(t, 1)), \quad t \geq t_0, \quad (5.46)$$

for some observer-gain functions  $P_1(\cdot)$ ,  $P_2(\cdot)$ , and  $P_3(\cdot)$ . The next step is to use the Lyapunov function  $V$  given in (5.5) to recover some inequality of the form

$$\frac{dV(\tilde{z}(t, \cdot))}{dt} \leq [cV(\tilde{z}(t, \cdot)) + \mathcal{K}(V(\tilde{z}(t, \cdot))) + Q_1(\tilde{z}_x(t, 1))] - \mathcal{K}(V(\tilde{z}(t, \cdot))), \quad (5.47)$$

for some positive coefficient  $c$  and some polynomial  $Q_1(\cdot)$  given in terms of  $P_1(\cdot)$ ,  $P_2(\cdot)$ , and  $P_3(\cdot)$  and satisfying  $Q_1(0) = 0$ . From this last inequality, we need to ensure that  $cV(\tilde{z}(t, \cdot)) + \mathcal{K}(V(\tilde{z}(t, \cdot))) + Q_1(\tilde{z}_x(t, 1)) = 0$  to recover inequality (2.115) (i.e.  $\frac{d}{dt}V(z(t, \cdot)) \leq -\mathcal{K}(V(z(t, \cdot)))$ ). To achieve this, we need to express  $Q_1(\cdot)$  (in turn  $P_1(\cdot)$ ,  $P_2(\cdot)$ , and  $P_3(\cdot)$ ) in terms of  $\tilde{z}(t, \cdot)$ . However, this not possible as  $Q_1(\cdot)$  (in turn  $P_1(\cdot)$ ,  $P_2(\cdot)$ , and  $P_3(\cdot)$ ) should depend only on  $\tilde{z}_x(t, 1)$  by definition.

## Conclusion

In this chapter, we treated the problem of nonlinear boundary stabilization, with any predefined type of convergence, for a class of 1D reaction-diffusion systems. To achieve this, we used the "spatially weighted  $L^2$ -norm" as a Lyapunov functional candidate  $V$ . By taking the time derivative of this functional along the trajectories of the reaction-diffusion PDE  $\frac{d}{dt}V(z(t, \cdot))$ , we noticed that we got an inequality that relates  $\frac{d}{dt}V(z(t, \cdot))$  to a second-degree polynomial involving the control  $U(t)$  subtracted by the term  $\mathcal{K}(V(z(t, \cdot)))$  (for any continuous function  $\mathcal{K}$  such that  $\mathcal{K}(0) = 0$ ). By computing the root of this polynomial, we managed to design a nonlinear control  $U(t)$  and to obtain the inequality  $\frac{d}{dt}V(z(t, \cdot)) \leq -\mathcal{K}(V(z(t, \cdot)))$ . Using this last inequality, for well-chosen expressions of  $\mathcal{K}$ , we ensured finite/fixed-time stabilization of the reaction-diffusion PDE. The present chapter did not study the existence/uniqueness issues of the solutions of the closed-loop system. To this purpose, ideas contained in [198], [199] can be used. However, the obtained stability estimates will certainly help the analysis.



## Part II

# Time-varying feedbacks for prescribed-time stabilization in infinite-dimensional settings





# Prescribed-time stabilization of LTI systems with distributed input delay

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## Contents

<b>6.1</b>	<b>Introduction</b>	<b>107</b>
<b>6.2</b>	<b>Problem statement</b>	<b>108</b>
<b>6.3</b>	<b>Prescribed-time predictor-based control: an ODE-PDE setting and Reduction-based and backstepping-forwarding transformations</b>	<b>108</b>
6.3.1	Scalar case	109
6.3.2	General case	116
<b>6.4</b>	<b>Simulations</b>	<b>125</b>

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This chapter deals with the problem of prescribed-time stabilization of controllable linear systems with distributed input delay. We model the input delay as a transport PDE and reformulate the original problem as a cascade ODE-PDE system while accounting for the infinite dimensionality of the actuator. We build on reduction-based and backstepping-forwarding transformations to convert the system into a target system having the prescribed-time stability property. Then, we prove the bounded invertibility of the transformations, and hence we show that the prescribed-time stability property is preserved into the original problem. To better illustrate the ideas of this approach, we focus first on the scalar case. Then, we give a sketch of the main lines for the general case. To this end, we choose the ODE dynamics of the target system to be a Linear Time-Varying system so that we can rely on recent developments which include a polynomial-based Vandermonde matrix and the generalized Laguerre polynomials that allow a compact formulation for the stability analysis. A simulation example is presented to illustrate the obtained results.

## 6.1 Introduction

In this chapter, we combine the ideas of [200], [129], and [176] to handle the problem of stabilization of LTI systems in the presence of distributed input delay. We model the input delay as a transport PDE and reformulate the original problem as a cascade ODE-PDE system while accounting for the infinite dimensionality of the actuator. We build on the reduction-based and backstepping-forwarding transformations to transform the system into a target system having the prescribed-time stability property. We relate back to such property through a suitable study of the bounded invertibility of the aforementioned transformations. The resulting predictor-like feedback is made up of time-varying gains. To better illustrate the ideas of this approach, we focus first on the scalar case. Then, we give the detailed steps for the general case.

This chapter is organized as follows. In Section 6.2, we introduce the studied system (an LTI system with distributed input delay). In Section 6.3, we use an ODE-PDE setting and suitable transformations to come up with a prescribed-time predictor-based controller. We focus on the scalar case to better communicate the main ideas of our approach. The generalization to the general case is provided in Section 6.3.2. In Section 6.4 we consider a numerical example to illustrate the results.

## 6.2 Problem statement

We consider the following controllable linear system with distributed input delay as stated in [200]:

$$\dot{z}(t) = Az(t) + \int_0^D B(D - \sigma)U(t - \sigma)d\sigma, \quad t \geq t_0, \quad (6.1)$$

where  $z(t) \in \mathbb{R}^n$  is the system state,  $U(\theta) \in \mathbb{R}$  for  $\theta \in [t - D, t]$  is the actuator state,  $U(t) \in \mathbb{R}$  is the control input,  $D > 0$  is a known constant delay and  $A, B$  are the system matrix and input vector of appropriate dimensions, respectively. The input vector  $B(\cdot)$  is a continuous real-valued vector function defined on  $[0, D]$ .

The objective of this chapter is to design predictor-like feedback achieving prescribed time stability (**PTS**) in light of Definitions 2.1 and 2.3, but whose specific notion is adapted to the current problem (which is infinite-dimensional) as we will see in Section 6.3.1.2 and Section 6.3.2.3. In order to better communicate the key ideas in our approach, we are going to deal first with the analysis and design for a simple linear scalar equation with distributed input delay.

## 6.3 Prescribed-time predictor-based control: an ODE-PDE setting and Reduction-based and backstepping-forwarding transformations

We reformulate the system (6.1) into a cascade ODE-PDE setting (i.e. cascade linear hyperbolic PDE with an LTI system) introduced in [53], [200], and that employed a backstepping-forwarding transformation and a reduction-based change of variable. As in [145], [176], the main idea of our approach is to transform the original system into a target system that is **PTS** (in an appropriate sense in light of Definitions 2.1 and 2.3) and that we choose to satisfy the property of convergence in a prescribed time  $t_0 + D + T$ . Here,  $T$  is fixed a priori,  $D$  is the known input delay, and for simplicity of notations, we take the initialization time  $t_0 = 0$ .

As previously mentioned, in an attempt to better illustrate the key ideas of our method, we first deal with a scalar linear system with distributed input delay. The generalization to the  $n$ -dimensional case (i.e. LTI systems of the form (6.1)) follows the same strategy as the one we used in the scalar case (which represents the heart of this chapter contribution) as well as some suitable changes of variables and some transformations in the framework of linear time-varying systems.

### 6.3.1 Scalar case

Let us consider the following scalar control system:

$$\dot{z}(t) = az(t) + b \int_0^D U(t - \sigma) d\sigma, \quad t \geq 0. \quad (6.2)$$

which is a particular case of (6.1) in the problem statement, with the state  $z \in \mathbb{R}$ , the delay  $D \geq 0$ , and the distributed control input  $\int_0^D U(t - \sigma) d\sigma$  which is delayed by  $\sigma \in [0, D]$  units of time. The input initial condition is  $U(t) = 0$  for all  $t \in [-D, 0)$ .

Following [200], the system (6.2) can be rewritten as ODE-PDE system:

$$\left\{ \begin{array}{ll} \dot{z}(t) = az(t) + b \int_0^D u(t, \sigma) d\sigma, & t \geq 0, \sigma \in [0, D], \\ u_t(t, x) = u_x(t, x), & t \geq 0, x \in [0, D], \\ u(t, D) = U(t), & t \geq 0, \end{array} \right. \quad (6.3)$$

$$u_t(t, x) = u_x(t, x), \quad t \geq 0, x \in [0, D], \quad (6.4)$$

$$u(t, D) = U(t), \quad t \geq 0, \quad (6.5)$$

where  $u(t, \cdot)$  is the transport PDE state whose solution is given by  $u(t, x) = U(t + x - D) = u(t - \sigma)$ , where  $\sigma = D - x$ . We aim at stabilizing (6.3)-(6.5) (in turn (6.2)) in a prescribed time  $D + T$ .

#### 6.3.1.1 Reduction-based and backstepping-forwarding transformation

We consider the following reduction-based change of variables:

$$\bar{z}(t) = z(t) + \int_0^D q(\sigma) u(t, \sigma) d\sigma. \quad (6.6)$$

where  $q(\cdot)$  and its domain are yet to be characterized while meeting the property  $q(0) = 0$ . In addition, we consider the following backstepping-forwarding transformation:

$$\omega(t, x) = u(t, x) - \Gamma(t, x) \bar{z}(t), \quad (6.7)$$

where the function  $\Gamma$  is time-varying. The inverse transformation is given as follows:

$$u(t, x) = \omega(t, x) + \Gamma(t, x) \bar{z}(t). \quad (6.8)$$

Under (6.6) and (6.7), we want to transform (6.3) into the following target system:

$$\left\{ \begin{array}{ll} \dot{\bar{z}}(t) = -\gamma_2(t) \bar{z}(t), & t \geq 0, \\ \omega_t(t, x) = \omega_x(t, x), & t \geq 0, x \in [0, D], \\ \omega(t, D) = 0, & t \geq 0, \end{array} \right. \quad (6.9)$$

$$\omega_t(t, x) = \omega_x(t, x), \quad t \geq 0, x \in [0, D], \quad (6.10)$$

$$\omega(t, D) = 0, \quad t \geq 0, \quad (6.11)$$

where  $\omega : [0, \infty) \times [0, D] \rightarrow \mathbb{R}$  is the transport PDE state and  $\gamma_2$  is given as in (2.53) by:

$$\gamma_2(t) = \frac{\gamma_{2,0}^2 T^2}{(T - t)^2}. \quad (6.12)$$

In order to characterize the transformation (6.7), let us compute its time and spatial derivatives.

$$\omega_t(t, x) = u_t(t, x) - \Gamma_t(t, x)\bar{z}(t) - \Gamma(t, x)\dot{\bar{z}}(t), \quad (6.13)$$

$$\begin{aligned} &= u_t(t, x) - \Gamma_t(t, x)z(t) - \Gamma_t(t, x) \int_0^D q(\sigma)u(t, \sigma)d\sigma \\ &\quad - \Gamma(t, x)\dot{z}(t) - \Gamma(t, x) \int_0^D q(\sigma)u_t(t, \sigma)d\sigma, \end{aligned} \quad (6.14)$$

$$\begin{aligned} &= u_t(t, x) - \Gamma_t(t, x)z(t) - \Gamma_t(t, x) \int_0^D q(\sigma)u(t, \sigma)d\sigma \\ &\quad - a\Gamma(t, x)z(t) - b\Gamma(t, x) \int_0^D u(t, \sigma)d\sigma - \Gamma(t, x) \int_0^D q(\sigma)u_t(t, \sigma)d\sigma. \end{aligned} \quad (6.15)$$

Next, by integrating the last term by parts, we get

$$\begin{aligned} \omega_t(t, x) &= u_t(t, x) - \Gamma_t(t, x)z(t) - a\Gamma(t, x)z(t) - \Gamma_t(t, x) \int_0^D q(\sigma)u(t, \sigma)d\sigma \\ &\quad - b\Gamma(t, x) \int_0^D u(t, \sigma)d\sigma + \Gamma(t, x) \int_0^D q'(\sigma)u(t, \sigma)d\sigma - \Gamma(t, x)q(D)u(t, D). \end{aligned} \quad (6.16)$$

Next, from (6.7) at  $x = D$  and after that using (6.6), we recover that

$$u(t, D) = \Gamma(t, D)\bar{z}(t), \quad (6.17)$$

$$= \Gamma(t, D)z(t) + \Gamma(t, D) \int_0^D q(\sigma)u(t, \sigma)d\sigma. \quad (6.18)$$

Thus, the previous equality becomes:

$$\omega_t(t, x) = u_t(t, x) + F(t, D, x)z(t) + \int_0^D G(t, D, x, \sigma)u(t, \sigma)d\sigma, \quad (6.19)$$

where

$$F(t, D, x) = -\Gamma_t(t, x) - \left(a + \Gamma(t, D)q(D)\right)\Gamma(t, x), \quad (6.20)$$

and

$$G(t, D, x, \sigma) = -\left(\Gamma_t(t, x) + \Gamma(t, D)q(D)\Gamma(t, x)\right)q(\sigma) + \Gamma(t, x)\left(q'(\sigma) - b\right). \quad (6.21)$$

On the other hand, the space derivative of  $\omega(t, x)$  is given by

$$\omega_x(t, x) = u_x(t, x) - \Gamma_x(t, x)\bar{z}(t), \quad (6.22)$$

$$= u_x(t, x) - \Gamma_x(t, x) \left( z(t) + \int_0^D q(\sigma)u(t, \sigma)d\sigma \right). \quad (6.23)$$

Subtracting (6.23) from (6.19), we get

$$\omega_t(t, x) - \omega_x(t, x) = u_t(t, x) - u_x(t, x) + \int_0^D (G(t, D, x, \sigma) + \Gamma_x(t, x)q(\sigma))u(t, \sigma)d\sigma \quad (6.24)$$

$$+ (F(t, D, x) + \Gamma_x(t, x))z(t). \quad (6.25)$$

Following the standard approach to finding the kernel equations, we can prove that the time-varying function  $\Gamma$  and the function  $q$  satisfy the following PDE system:

$$\begin{cases} q'(\sigma) + aq(\sigma) = b, & (6.26) \\ \Gamma_x(t, x) - \Gamma_t(t, x) = (a + \Gamma(t, D)q(D))\Gamma(t, x), & (6.27) \end{cases}$$

where  $q$  and  $\Gamma$  are defined on the domains, respectively  $\mathcal{T}_q : \{\sigma : 0 \leq \sigma \leq D\}$  and  $\mathcal{T}_\Gamma : \{(t, x) : 0 \leq x \leq D, 0 \leq t < T + x - D\}$ .

**Proposition 6.1**

The system (6.26)-(6.27) has well-posed  $C^\infty$  solutions on  $\mathcal{T}_q$  and  $\mathcal{T}_\Gamma$ , given by

$$q(\sigma) = \frac{b}{a}(1 - e^{-a\sigma}), \quad (6.28)$$

$$\Gamma(t, x) = -\frac{a(a + \gamma_2(t + x - D))}{b(1 - e^{-aD})} e^{\gamma_{2,0}T(\sqrt{\gamma_2(t)} - \sqrt{\gamma_2(t+x-D)})}, \quad (6.29)$$

where  $\gamma_2$  is defined by (6.12).

*Proof.* From (6.26), since  $q(0) = 0$ , we find  $q(\sigma) = \frac{b}{a}(1 - e^{-a\sigma})$ . Concerning (6.27), we know, thanks to the chosen  $\bar{z}$ -dynamics of target system (6.9)-(6.11), that

$$-\gamma_2(t)\bar{z}(t) = \dot{\bar{z}}(t) = \dot{z}(t) + \int_0^D q(\sigma)u_t(t, \sigma)d\sigma, \quad (6.30)$$

$$= az(t) + b \int_0^D u(t, \sigma)d\sigma - \int_0^D q'(\sigma)u(t, \sigma)d\sigma + q(D)u(t, D), \quad (6.31)$$

$$= az(t) + a \int_0^D q(\sigma)u(t, \sigma)d\sigma + q(D)\Gamma(t, D)\bar{z}(t), \quad (6.32)$$

$$= a\bar{z}(t) + q(D)\Gamma(t, D)\bar{z}(t), \quad (6.33)$$

$$= (a + q(D)\Gamma(t, D))\bar{z}(t), \quad (6.34)$$

which leads to

$$-\gamma_2(t) = a + q(D)\Gamma(t, D). \quad (6.35)$$

Moreover, by using (6.35) alongside the following change of variables:

$$\Gamma(t, x) = e^{\int_0^x \gamma_2(s)ds} \bar{\Gamma}(t, x), \quad (6.36)$$

in (6.27), we get

$$\bar{\Gamma}_x(t, x) - \bar{\Gamma}_t(t, x) = 0, \quad (6.37)$$

whose solution is obtained by the method of characteristics as follows:

$$\bar{\Gamma}(t, x) = \bar{\Gamma}(t + x - D, D) = e^{-\int_0^{t+x-D} \gamma_2(s)ds} \Gamma(t + x - D, D). \quad (6.38)$$

As a result,

$$\Gamma(t, x) = \Gamma(t + x - D, D) e^{-\int_t^{t+x-D} \gamma_2(s)ds}, \quad (6.39)$$

$$= \Gamma(t + x - D, D) e^{\gamma_{2,0}T(\sqrt{\gamma_2(t)} - \sqrt{\gamma_2(t+x-D)})}. \quad (6.40)$$

The only thing left is to compute  $\Gamma(t + x - D, D)$ . To do this, we replace (6.28) in (6.35),

$$\Gamma(t, D) = \frac{-a - \gamma_2(t)}{q(D)} = \frac{-a - \gamma_2(t)}{\frac{b}{a}(1 - e^{-aD})}, \quad (6.41)$$

which finally yields

$$\Gamma(t, x) = \frac{-a - \gamma_2(t + x - D)}{\frac{b}{a}(1 - e^{-aD})} e^{\gamma_{2,0}T(\sqrt{\gamma_2(t)} - \sqrt{\gamma_2(t+x-D)})}. \quad (6.42)$$

□

### Prescribed-time predictor-based control

From (6.7) and (6.6), at  $x = D$ , and using (6.28), (6.29) the boundary control is then,

$$U(t) = \frac{-(a + \gamma_2(t))}{\frac{b}{a}(1 - e^{-aD})} \left( z(t) + \frac{b}{a} \int_0^D (1 - e^{-a\sigma}) u(t, \sigma) d\sigma \right), \quad (6.43)$$

where  $U(t + \sigma - D) = u(t, \sigma)$ .

Equivalently, we have:

$$U(t) = \frac{-(a + \gamma_2(t))}{\frac{b}{a}(1 - e^{-aD})} \left( z(t) + \frac{b}{a} \int_0^D (1 - e^{-a\sigma}) U(t + \sigma - D) d\sigma \right). \quad (6.44)$$

#### 6.3.1.2 Stability analysis

We first study the stability of the target system and then we establish the bounded invertibility of the transformations by a suitable norm equivalence.

##### Lemma 6.1

Let  $\gamma_{2,0} > 0$ . Let  $T > 0$  be fixed and  $D > 0$  be a known delay. Then, the dynamics of (6.9) satisfies the following estimates:<sup>1</sup>

$$|\bar{z}(t)|^2 \leq \eta_{\bar{z}} e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} |\bar{z}_0|^2, \quad (6.45)$$

for any  $\bar{z}_0 \in \mathbb{R}$  and for all  $t \in [0, T)$ , where  $\eta_{\bar{z}} = e^{2T\gamma_{2,0}^2}$ . In particular,  $|\bar{z}(t)|^2 \rightarrow 0$  as  $t \rightarrow T$  and  $|\bar{z}(t)| \equiv 0$  for  $t \geq T$ .

Moreover, the transport PDE  $\omega$  of target system (6.9)-(6.11) is **FxTS** i.e. for any  $\omega(0, x) \in L^2(0, D)$ , it holds that  $\|\omega(t, \cdot)\|_{L^2(0, D)} \equiv 0$  for all  $t \geq D$ .

*Proof.* From (6.9), we recover that

$$\bar{z}(t) = \bar{z}_0 e^{-\int_0^t \gamma_2(s) ds}, \text{ for any } t \in [0, T), \quad (6.46)$$

and

$$\bar{z}(t) = 0 \text{ for any } t \geq T. \quad (6.47)$$

Therefore, (6.46) leads to,

$$|\bar{z}(t)|^2 \leq \eta_{\bar{z}} e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} |\bar{z}_0|^2, \quad (6.48)$$

with  $\eta_{\bar{z}} = e^{2T\gamma_{2,0}^2}$ . Then  $|\bar{z}(t)| \rightarrow 0$  as  $t \rightarrow T$ . On the other hand, by the method of the characteristics, the solution of (6.10)-(6.11) is **FxTS**. Indeed,  $\omega(t, x) = \omega_0(t + x - D)$  for  $t \leq D - x$  and  $\omega(t, x) = 0$  for  $t \geq D - x$ . Hence, we can conclude that  $\|\omega(t, \cdot)\|_{L^2(0, D)} \equiv 0$  for all  $t \geq D$ .  $\square$

##### Proposition 6.2

For the transformations (6.6) and (6.7), the following estimates hold:

$$\|u(t, \cdot)\|_{L^2(0, D)}^2 \leq 2\|\omega(t, \cdot)\|_{L^2(0, D)}^2 + 2\|\Gamma(t, \cdot)\|_{L^2(0, D)}^2 |\bar{z}(t)|^2, \quad (6.49)$$

---

<sup>1</sup> $e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}}$  is a monotonically decreasing smooth "bump-like" function having the property  $e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} \equiv 0, \forall t \geq T$  (see e.g. [201]).

and

$$|z(t)|^2 \leq \left(2 + 4\|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 \|q\|_{L^2(0,D)}^2\right) |\bar{z}(t)|^2 + 4\|q\|_{L^2(0,D)}^2 \|\omega(t, \cdot)\|_{L^2(0,D)}^2. \quad (6.50)$$

*Proof.* On one hand, from (6.7), we have

$$|u(t, x)| \leq |\omega(t, x)| + |\Gamma(t, x)| |\bar{z}(t)|, \quad (6.51)$$

then

$$|u(t, x)|^2 \leq 2|\omega(t, x)|^2 + 2|\Gamma(t, x)|^2 |\bar{z}(t)|^2, \quad (6.52)$$

from which (6.49) is obtained.

On the other hand, by the Cauchy-Schwarz inequality, we get

$$|z(t)| \leq |\bar{z}(t)| + \int_0^D |q(\sigma)| |u(t, \sigma)| d\sigma, \quad (6.53)$$

$$\leq |\bar{z}(t)| + \|q\|_{L^2(0,D)} \|u(t, \cdot)\|_{L^2(0,D)}. \quad (6.54)$$

Using Young's inequality, we obtain

$$|z(t)|^2 \leq 2|\bar{z}(t)|^2 + 2\|q\|_{L^2(0,D)}^2 \|u(t, \cdot)\|_{L^2(0,D)}^2, \quad (6.55)$$

which combined with (6.49) leads to (6.50).  $\square$

**Lemma 6.2**

Let  $\Gamma$  be given by (6.29). Then, the following holds true:

$$\lim_{t \rightarrow D+T} \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} = 0. \quad (6.56)$$

*Proof.* Let  $I(t) = \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}}$ . Using (6.27), we obtain

$$I(t) = \int_0^D e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} |\Gamma(t, x)|^2 dx, \quad (6.57)$$

$$= \int_0^D e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} \frac{a^2(a+\gamma_2(t+x-D))^2}{b^2(1-e^{-aD})^2} e^{2\gamma_{2,0}T(\sqrt{\gamma_2(t)}-\sqrt{\gamma_2(t+x-D)})} dx, \quad (6.58)$$

$$\leq \int_0^D \frac{2a^2(a^2 + \gamma_2(t+x-D)^2)}{b^2(1-e^{-aD})^2} e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} dx, \quad (6.59)$$

$$= 2\delta \int_0^D (a^2 + \gamma_2(t+x-D)^2) e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} dx, \quad (6.60)$$

$$\leq F_1(t-D) + F_2(t-D), \quad (6.61)$$

where  $\delta = \frac{a^2}{b^2(1-e^{-aD})^2}$ , and

$$F_1(t-D) = 2\delta a^2 \int_0^D e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} dx, \quad (6.62)$$

$$F_2(t-D) = 2\delta \int_0^D \gamma_2(t+x-D)^2 e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} dx. \quad (6.63)$$



It can be shown that  $F_1(t - D)$  and  $F_2(t - D)$  converge to zero in time  $D + T$ . Then,  $I(t) \rightarrow 0$  as  $t \rightarrow D + T$ . Let us study first  $F_2(t - D)$ . We introduce the following change of variables in (6.63)

$$s = 2\gamma_{2,0}T\sqrt{\gamma_2(t + x - D)} = \frac{2(\gamma_{2,0}T)^2}{T - t - x + D}, \quad (6.64)$$

we have

$$\frac{ds}{dx} = \frac{2(\gamma_{2,0}T)^2}{(T - t - x + D)^2} = \frac{s^2}{2(\gamma_{2,0}T)^2}, \quad (6.65)$$

then, (6.63) becomes

$$F_2(t - D) = 2\delta \int_{2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2\gamma_{2,0}T\sqrt{\gamma_2(t)}} \frac{s^4}{(2\gamma_{2,0}T)^4} \frac{2(\gamma_{2,0}T)^2}{s^2} e^{-s} ds, \quad (6.66)$$

$$= \frac{\delta}{(2\gamma_{2,0}T)^2} \int_{2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2\gamma_{2,0}T\sqrt{\gamma_2(t)}} s^2 e^{-s} ds, \quad (6.67)$$

$$= \frac{\delta}{(2\gamma_{2,0}T)^2} [-(s^2 + 2s + 2)e^{-s}]_{2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2\gamma_{2,0}T\sqrt{\gamma_2(t)}}. \quad (6.68)$$

Hence, we obtain,

$$F_2(t - D) \leq \frac{\delta}{(2\gamma_{2,0}T)^2} \left( 4\gamma_{2,0}^2 T^2 \gamma_2(t - D)^2 + 4\gamma_{2,0}T\sqrt{\gamma_2(t - D)} + 2 \right) e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}, \quad (6.69)$$

and  $F_2(t - D) \rightarrow 0$  as  $t \rightarrow D + T$ .

On the other hand, using the previous change of variable (6.64) in (6.62), we obtain:

$$F_1(t - D) = 2\delta(a\gamma_{2,0}T)^2 \int_{2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2\gamma_{2,0}T\sqrt{\gamma_2(t)}} \frac{e^{-s}}{s^2} ds, \quad (6.70)$$

$$\leq 2\delta(a\gamma_{2,0}T)^2 \int_{2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{\infty} \frac{e^{-s}}{s^2} ds. \quad (6.71)$$

The form of (6.71) allows to use the following generalized exponential integral:

$$E_n(v) = v^{n-1} \int_v^{\infty} \frac{e^{-s}}{s^n} ds, \quad v > 0, n \in \mathbb{N}. \quad (6.72)$$

Moreover, the following inequality holds [202, Section 2]:

$$0 < \frac{e^{-v}}{2+v} \leq E_2(v) \leq \frac{e^{-v}}{1+v}. \quad (6.73)$$

Then, we obtain that

$$F_1(t - D) = 2\delta(a\gamma_{2,0}T)^2 \left[ \frac{E_2(2\gamma_{2,0}T\sqrt{\gamma_2(t-D)})}{(2\gamma_{2,0}T\sqrt{\gamma_2(t-D)})} \right], \quad (6.74)$$

$$\leq 2\delta(a\gamma_{2,0}T)^2 \left[ \frac{e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}}{(2\gamma_{2,0}T\sqrt{\gamma_2(t-D)})(1+2\gamma_{2,0}T\sqrt{\gamma_2(t-D)})} \right]. \quad (6.75)$$

Hence, from (6.75), we get that  $F_1(t - D) \rightarrow 0$  as  $t \rightarrow D + T$ . Finally, we conclude from (6.61) that  $I(t) \rightarrow 0$  as  $t \rightarrow D + T$ .  $\square$

### Proposition 6.3

Let  $q$  be given by (6.28). Then, the following inequality holds:

$$|\bar{z}_0|^2 \leq 2(1 + \|q\|_{L^2(0,D)}^2)(|z_0|^2 + \|u(0, \cdot)\|_{L^2(0,D)}^2). \quad (6.76)$$

*Proof.* By replacing  $t = 0$  in (6.6) and by the Cauchy-Schwarz inequality, we get

$$|\bar{z}_0| \leq |z_0| + \int_0^D |q(\sigma)| |u(0, \sigma)| d\sigma, \quad (6.77)$$

$$\leq |z_0| + \|q\|_{L^2(0,D)} \|u(0, \cdot)\|_{L^2(0,D)}. \quad (6.78)$$

Next, by Young's inequality, we obtain

$$|\bar{z}_0|^2 \leq 2|z_0|^2 + 2\|q\|_{L^2(0,D)}^2 \|u(0, \cdot)\|_{L^2(0,D)}^2, \quad (6.79)$$

from which (6.76) is deduced.  $\square$

**Theorem 6.1**

Let  $\gamma_2$  be given by (6.12). Let  $D > 0$ ,  $\gamma_{2,0} > 0$ ,  $T > 0$  fixed. Let  $\Gamma$  be given by (6.29). Then, the solution of the closed-loop system (6.3)-(6.5) with prescribed-time predictor-based control (6.43) is **PTS** in the following sense: for any  $z_0 \in \mathbb{R}$  and  $u(0, \cdot) \in L^2(0, D)$ , the quantity  $|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2$  remains bounded for  $t \in [0, \max\{D, T\}]$ , and for all  $t \in [\max\{D, T\}, D + T)$ , the following estimate holds:

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq \eta_{\bar{z}} M e^{-2\gamma_{2,0} T \sqrt{\gamma_2(t)}} \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 \left( |z_0|^2 + \|u(0, \cdot)\|_{L^2(0,D)}^2 \right), \quad (6.80)$$

with  $M = 4(1 + 2\|q\|_{L^2(0,D)}^2)(1 + \|q\|_{L^2(0,D)}^2)$ . In particular,

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \rightarrow 0 \text{ as } t \rightarrow D + T, \quad (6.81)$$

Moreover,  $|U(t)| \rightarrow 0$  as  $t \rightarrow T$ .

*Proof.* By Proposition 6.2 we have for all  $t \in [0, D + T)$

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq M_1 \|\omega(t, \cdot)\|_{L^2(0,D)}^2 + M_2(t) |\bar{z}(t)|^2, \quad (6.82)$$

with  $M_1 = 2 + 4\|q\|_{L^2(0,D)}^2$  and  $M_2(t) = 2 + M_1 \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2$ .

**Case 1:** Assume that  $T \geq D$ . By the fact that  $\|\omega(t, \cdot)\|_{L^2(0,D)} \equiv 0$  as  $t \geq D$ , it holds that for any  $t \in [D, D + T)$ ,

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq M_2(t) |\bar{z}(t)|^2, \quad (6.83)$$

$$= 2|\bar{z}(t)|^2 + M_1 \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 |\bar{z}(t)|^2. \quad (6.84)$$

By Lemma 6.1, we have  $|\bar{z}(t)| \equiv 0$  for  $t \geq T$ , and it holds for  $t \in [T, D + T)$ ,

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq M_1 \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 |\bar{z}(t)|^2, \quad (6.85)$$

$$\leq \eta_{\bar{z}} M_1 e^{-2\gamma_{2,0} T \sqrt{\gamma_2(t)}} \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 |\bar{z}_0|^2. \quad (6.86)$$

Notice that  $\|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 |\bar{z}(t)|^2$  does not vanish when  $t \rightarrow T$ , even though  $\bar{z}(t)$  vanishes. This is because, the rate of the growth-in-time of  $\Gamma(t, \cdot)$  is the same the rate of the decreasing-time of  $\bar{z}$ , on the interval  $[0, T]$ .

**Case 2:** Assume that  $T \leq D$ . Then, using the fact that  $|\bar{z}(t)| \equiv 0$  for  $t \geq T$ , then it holds for  $t \in [T, D + T)$ ,

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq M_1 \left( \|\omega(t, \cdot)\|_{L^2(0,D)}^2 + \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 |\bar{z}(t)|^2 \right). \quad (6.87)$$

Using the fact that  $\|\omega(t, \cdot)\|_{L^2(0,D)} \equiv 0$  as  $t \geq D$ , the inequality (6.86) holds for  $t \in [D, D+T]$ .

Now, from Proposition 6.3, we obtain

$$|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq \eta_{\bar{z}} M e^{-2\gamma_{2,0}T\sqrt{\gamma_2(t)}} \|\Gamma(t, \cdot)\|_{L^2(0,D)}^2 (|z_0|^2 + \|u(0, \cdot)\|_{L^2(0,D)}^2), \quad (6.88)$$

where  $M = M_1(2 + 2\|q\|_{L^2(0,D)}^2)$ . We finally obtain by Lemma 6.2, that  $|z(t)|^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \rightarrow 0$ , as  $t \rightarrow D+T$ . It remains to show that  $|U(t)| \rightarrow 0$ . Indeed, from the transformation (6.7) it holds

$$|U(t)| = |\Gamma(t, D)| |\bar{z}(t)| \leq \frac{a|a + \gamma_2(t)|}{b|1 - e^{-a}|} \sqrt{\eta_{\bar{z}}} |\bar{z}_0| e^{-\gamma_{2,0}T\sqrt{\gamma_2(t)}}, \quad (6.89)$$

from which we can conclude that  $|U(t)| \rightarrow 0$  as  $t \rightarrow T$ .  $\square$

### 6.3.2 General case

In this section, we present the design of the predictor-feedback prescribed-time stabilization of the general case (6.1). Let us consider the cascade ODE-PDE formulation of (6.1), i.e.

$$\begin{cases} \dot{z}(t) = Az(t) + \int_0^D B(\sigma)u(t, \sigma)d\sigma, & t \geq t_0, \sigma \in [0, D], & (6.90) \\ u_t(t, x) = u_x(t, x), & t \geq t_0, x \in [0, D], & (6.91) \\ u(t, D) = U(t), & t \geq t_0. & (6.92) \end{cases}$$

Let  $\bar{B}_D = \int_0^D e^{-A(D-y)} B(y)dy$ . We assume that the pair  $(A, \bar{B}_D)$  is controllable:

#### Assumption 6.1

The controllability matrix  $\mathbf{C}_D = [\bar{B}_D, A\bar{B}_D, \dots, A^{n-1}\bar{B}_D]$  is of full rank  $n$ .

#### 6.3.2.1 Reduction-based and backstepping-forwarding transformations

We consider the following reduction-based change of variables inspired from [200]:

$$\bar{z}(t) = P \left( z(t) + \int_0^D \int_0^\sigma e^{-A(\sigma-y)} B(y)dy u(t, \sigma)d\sigma \right), \quad (6.93)$$

where

$$P = [g^\top, (gA)^\top, \dots, (gA^{n-1})^\top]^\top, \quad (6.94)$$

and  $g$  the  $n$ -th row of the matrix  $\mathbf{C}_D^{-1}$  (existence of the inverse comes from Assumption 6.1).

In addition, consider the following backstepping-forwarding transformation

$$\omega(t, x) = u(t, x) - \Gamma(t, x)^\top \bar{z}(t), \quad (6.95)$$

where  $\Gamma(t, x)$  is a space and time-varying vector function such that

$$\Gamma(t, x)^\top \bar{z}(t) = \sum_{i=1}^n \Gamma_i(t, x) \bar{z}_i(t). \quad (6.96)$$

The inverse transformation is given as follows:

$$u(t, x) = \omega(t, x) + \Gamma(t, x)^\top \bar{z}(t). \quad (6.97)$$

Then, the system (6.90)-(6.92) is mapped into the following target system:

$$\begin{cases} \dot{\bar{z}}(t) = C(t)\bar{z}(t), & t \geq t_0, \end{cases} \quad (6.98)$$

$$\begin{cases} \omega_t(t, x) = \omega_x(t, x), & t \geq t_0, x \in [0, D], \end{cases} \quad (6.99)$$

$$\begin{cases} \omega(t, D) = 0, & t \geq t_0, \end{cases} \quad (6.100)$$

where  $C(t)$  is a companion canonical matrix, i.e.

$$C(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -p_0(t) & -p_1(t) & \cdots & -p_{n-1}(t) \end{bmatrix}, \quad (6.101)$$

where the functions  $p_{i-1}, i \in \{1, \dots, n\}$  are defined as in [145], [176] by

$$p_0(t) = \bar{\sigma}_n(r_1, \dots, r_n) \gamma_2(t)^n, \quad (6.102)$$

and for  $j = 1, \dots, n-1$ ,

$$p_j(t) = \frac{(\sqrt{\gamma_2(t)})^{n-j}}{(\gamma_{2,0}T)^{n-j}} \sum_{k=j}^n (-1)^{k-j} \bar{\sigma}_{n-k}(r_1, \dots, r_n) \binom{k-1}{j-1} \frac{k!}{j!} (\gamma_{2,0}T\sqrt{\gamma_2(t)})^{n-k}, \quad (6.103)$$

where  $\gamma_2(\cdot)$  is given in (6.12), the coefficients  $r_1, \dots, r_n$  are positive reals such that  $r_i \neq r_j$  for  $i \neq j$  in the range of  $n$ , and the polynomials  $\bar{\sigma}_{n-k}(\cdot)$  are the elementary symmetric polynomials defined by

$$\bar{\sigma}_0(r_1, \dots, r_n) = 1, \quad (6.104)$$

$$\bar{\sigma}_k(r_1, \dots, r_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} r_{i_1} r_{i_2} \dots r_{i_k}, \quad k \in \{1, \dots, n-1\}, \quad (6.105)$$

$$\bar{\sigma}_n(r_1, \dots, r_n) = \prod_{i=1}^n r_i, \quad (6.106)$$

$$\bar{\sigma}_k(r_1, \dots, r_n) = 0, \quad k \in \{n+1, n+2, \dots\}. \quad (6.107)$$

Similar computations to the scalar case prove that the PDE equation of transformation (6.95) is as follows:

$$\Gamma_x(t, x) - \Gamma_t(t, x) = C(t)^\top \Gamma(t, x), \quad (6.108)$$

where  $\Gamma$  is defined on  $\mathcal{T}_\Gamma$ .

**Proposition 6.4**

The system (6.108) has a well-posed  $C^\infty$  solution on  $\mathcal{T}_\Gamma$ , given by

$$\Gamma(t, x) = V(t)^{-\top} \mathbf{D}(t)^{-1} \mathbf{D}(t+x-D) V(t+x-D)^\top \Gamma(t+x-D, D), \quad (6.109)$$

where

$$\mathbf{D}(s) = \text{diag} \left[ e^{-r_1 \gamma_{2,0} T \sqrt{\gamma_2(s)}}, \dots, e^{-r_n \gamma_{2,0} T \sqrt{\gamma_2(s)}} \right], \quad (6.110)$$

$$\Gamma_i(t+x-D, D) = a_{i-1} - p_{i-1}(t+x-D), \quad i \in \{1, \dots, n\}, \quad (6.111)$$

$p_{i-1}$  defined by (6.102)-(6.103), and  $V$  is polynomial-based Vandermonde matrix given as follows:

$$V(t) = \begin{bmatrix} 1 & \cdots & 1 \\ (\delta^0(-r_1\gamma_2))(t) & \cdots & (\delta^0(-r_n\gamma_2))(t) \\ \vdots & \ddots & \vdots \\ (\delta^{n-2}(-r_1\gamma_2))(t) & \cdots & (\delta^{n-2}(-r_n\gamma_2))(t) \end{bmatrix}, \quad (6.112)$$

with

$$(\delta^k(-r_i\gamma_2))(t) = \frac{-r_i\gamma_2(t)(\sqrt{\gamma_2(t)})^k}{(\gamma_{2,0}T)^k} k! L_k^{(1)}\left(r_i\gamma_{2,0}T\sqrt{\gamma_2(t)}\right), \quad k \in \{0, \dots, n-2\}, \quad (6.113)$$

where  $L_k^{(1)}(\cdot)$  are the generalized Laguerre polynomials (see [145], [176], [203]).

*Proof.* Using the following change of variable:

$$\bar{\Gamma}(t, x) = V(t)^\top \Gamma(t, x), \quad (6.114)$$

we get

$$\bar{\Gamma}_x(t, x) - \bar{\Gamma}_t(t, x) = D_g(t)\bar{\Gamma}(t, x), \quad (6.115)$$

where

$$D_g(t) = \text{diag}[-r_1\gamma_2(t), -r_2\gamma_2(t), \dots, -r_n\gamma_2(t)], \quad (6.116)$$

is a diagonal matrix such that  $V$  satisfies (see [145], [176], [203]):

$$V(t)D_g(t) = C(t)V(t) - \dot{V}(t). \quad (6.117)$$

From (6.115)-(6.117), we have

$$\bar{\Gamma}_{i,x}(t, x) - \bar{\Gamma}_{i,t}(t, x) = -r_i c(t)\bar{\Gamma}_i(t, x), \quad (6.118)$$

for all  $i \in \{1, \dots, n\}$ . The solution is given by,

$$\bar{\Gamma}_i(t, x) = e^{r_i\gamma_{2,0}T\sqrt{\gamma_2(t)}} e^{-r_i\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} \bar{\Gamma}_i(t+x-D, D), \quad (6.119)$$

similar to (6.29) in the scalar case. As a result,

$$\bar{\Gamma}(t, x) = \mathbf{D}(t)^{-1} \mathbf{D}(t+x-D) \bar{\Gamma}(t+x-D, D), \quad (6.120)$$

where  $\mathbf{D}(t)$  is given in (6.110). Returning to the original variable  $\Gamma$ , we get

$$V(t)^\top \Gamma(t, x) = \mathbf{D}(t)^{-1} \mathbf{D}(t+x-D) V(t+x-D)^\top \Gamma(t+x-D, D), \quad (6.121)$$

then

$$\Gamma(t, x) = V(t)^{-\top} \mathbf{D}(t)^{-1} \mathbf{D}(t+x-D) V(t+x-D)^\top \Gamma(t+x-D, D). \quad (6.122)$$

Now, let us calculate  $\Gamma(t, D)$ . By computing the time derivative of (6.93) and by integration by parts, we get

$$\dot{\bar{z}}(t) = P \left( \dot{z}(t) + \int_0^D \int_0^\sigma e^{-A(\sigma-y)} B(y) dy u_t(t, \sigma) d\sigma \right), \quad (6.123)$$

$$\begin{aligned} &= P \left( Az(t) + \int_0^D B(\sigma) u(t, \sigma) d\sigma + \int_0^D \int_0^\sigma A e^{-A(\sigma-y)} B(y) dy u(t, \sigma) d\sigma \right. \\ &\quad \left. - \int_0^D B(\sigma) u(t, \sigma) d\sigma + \left[ \int_0^\sigma e^{-A(\sigma-y)} B(y) dy u(t, \sigma) \right]_0^D \right), \end{aligned} \quad (6.124)$$

$$= P \left( AP^{-1} \bar{z}(t) + \int_0^D e^{-A(D-y)} B(y) dy u(t, D) \right), \quad (6.125)$$

$$= PAP^{-1} \bar{z}(t) + P \bar{B}_D u(t, D). \quad (6.126)$$

Moreover, using the fact that  $u(t, D) = \Gamma(t, D)^\top \bar{z}(t)$  in (6.126), we obtain

$$C(t)\bar{z}(t) = \dot{\bar{z}}(t) \tag{6.127}$$

$$= A_c \bar{z}(t) + B_c u(t, D) \tag{6.128}$$

$$= A_c \bar{z}(t) + B_c \Gamma(t, D)^\top \bar{z}(t), \tag{6.129}$$

where  $A_c = PAP^{-1}$  is a Companion canonical matrix, i.e.

$$A_c(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \tag{6.130}$$

and  $B_c = P\bar{B}_D = [0, \dots, 0, 1]^\top$  (see [204, Section 3]). Finally, we obtain,

$$B_c \Gamma(t, D)^\top = C(t) - A_c, \tag{6.131}$$

and

$$\Gamma_i(t, D) = -(p_{i-1}(t) - a_i). \tag{6.132}$$

This concludes the proof.  $\square$

### 6.3.2.2 Prescribed-time predictor-based control

From (6.95), at  $x = D$ , and using (6.93), the boundary control is given as follows:

$$U(t) = \Gamma(t, D)^\top P \left( z(t) + \int_0^D \int_0^\sigma e^{A(y-\sigma)} B(y) dy u(t, \sigma) d\sigma \right), \tag{6.133}$$

where  $U(t + \sigma - D) = u(t, \sigma)$ . Equivalently, we have:

$$U(t) = \Gamma(t, D)^\top P \left( z(t) + \int_0^D \int_0^\sigma e^{-A(\sigma-y)} B(y) dy U(t + \sigma - D) d\sigma \right). \tag{6.134}$$

### 6.3.2.3 Stability result

As in the scalar case, we start by studying the stability of the target system and then we establish the bounded invertibility of the transformations by a suitable norm equivalence.

#### Proposition 6.5

*There exists a polynomial function  $R_1(\cdot)$  in terms of  $\gamma_2(t+x-D)$  such that: for all  $t \in [D, D+T]$  the following estimate holds true:*

$$|\Gamma(t, x)^\top \bar{z}(t)|^2 \leq \bar{\eta}_{\bar{z}} R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_2, 0 T \sqrt{\gamma_2(t+x-D)}} \|\bar{z}(0)\|_2^2. \tag{6.135}$$

*Proof.* Let us consider the following change of variables

$$\bar{z}(t) = V(t) \tilde{z}(t). \tag{6.136}$$

Using this change of variables and using (6.98), we get

$$C(t)\bar{z}(t) = \dot{\tilde{z}}(t), \quad (6.137)$$

$$= \dot{V}(t)\tilde{z}(t) + V(t)\dot{\tilde{z}}(t), \quad (6.138)$$

from which we recover that

$$C(t)V(t)\tilde{z}(t) = \dot{V}(t)\tilde{z}(t) + V(t)\dot{\tilde{z}}(t), \quad (6.139)$$

and thus, using (6.117), we get

$$\dot{\tilde{z}}(t) = D_g(t)\tilde{z}(t), \quad (6.140)$$

Solving this last equation, we obtain

$$\tilde{z}(t) = \mathbf{D}(t)\mathbf{D}(0)^{-1}\tilde{z}(0). \quad (6.141)$$

From Proposition 6.4, we have

$$|\Gamma(t, x)^\top \bar{z}(t)|^2 = |\Gamma(t+x-D, D)^\top V(t+x-D)\mathbf{D}(t+x-D)\mathbf{D}(t)^{-1}V(t)^{-1}\bar{z}(t)|^2. \quad (6.142)$$

Using (6.136), we get

$$|\Gamma(t, x)^\top \bar{z}(t)|^2 = |\Gamma(t+x-D, D)^\top V(t+x-D)\mathbf{D}(t+x-D)\mathbf{D}(t)^{-1}\tilde{z}(t)|^2. \quad (6.143)$$

Using (6.141) followed by (6.136) for  $t = 0$ , we get

$$|\Gamma(t, x)^\top \bar{z}(t)|^2 = |\Gamma(t+x-D, D)^\top V(t+x-D)\mathbf{D}(t+x-D)\mathbf{D}(t)^{-1}\mathbf{D}(t)\mathbf{D}(0)^{-1}\tilde{z}(0)|^2, \quad (6.144)$$

$$= |\Gamma(t+x-D, D)^\top V(t+x-D)\mathbf{D}(t+x-D)\mathbf{D}(0)^{-1}V(0)^{-1}\tilde{z}(0)|^2. \quad (6.145)$$

Using the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |\Gamma(t, x)^\top \bar{z}(t)|^2 &\leq \|\Gamma(t+x-D, D)\|_2^2 \|V(t+x-D)\|_2^2 \|\mathbf{D}(t+x-D)\|_2^2 \|\mathbf{D}(0)^{-1}\|_2^2 \\ &\quad \times \|V(0)^{-1}\|_2^2 \|\tilde{z}(0)\|_2^2, \end{aligned} \quad (6.146)$$

$$= \bar{\eta}_{\tilde{z}} \|\Gamma(t+x-D, D)\|_2^2 \|V(t+x-D)\|_2^2 \|\mathbf{D}(t+x-D)\|_2^2 \|\tilde{z}(0)\|_2^2, \quad (6.147)$$

$$\leq \bar{\eta}_{\tilde{z}} \|\Gamma(t+x-D, D)\|_2^2 \|V(t+x-D)\|_2^2 e^{-2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} \|\tilde{z}(0)\|_2^2, \quad (6.148)$$

where  $\bar{\eta}_{\tilde{z}} = \|\mathbf{D}(0)^{-1}\|_2^2 \|V(0)^{-1}\|_2^2$ .

Next, using the fact that for any  $s \in \mathbb{R}^n$ :  $\|s\|_2 \leq \|s\|_F$  (where  $\|\cdot\|_F$  is the Frobenius norm), we get

$$|\Gamma(t, x)^\top \bar{z}(t)|^2 \leq \bar{\eta}_{\tilde{z}} \|\Gamma(t+x-D, D)\|_2^2 \|V(t+x-D)\|_F^2 e^{-2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} \|\tilde{z}(0)\|_2^2. \quad (6.149)$$

Now, looking at the components of  $\Gamma$  and  $V$ , we can easily see that

$$\|\Gamma(t+x-D, D)\|_2^2 \|V(t+x-D)\|_F^2 = R_1(\sqrt{\gamma_2(t+x-D)}), \quad (6.150)$$

for some polynomial function  $R_1(\cdot)$  in terms of  $\sqrt{\gamma_2(t+x-D)}$ . Then

$$|\Gamma(t, x)^\top \bar{z}(t)|^2 \leq \bar{\eta}_{\tilde{z}} R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t+x-D)}} \|\tilde{z}(0)\|_2^2. \quad (6.151)$$

□

**Proposition 6.6**

For the transformations (6.93) and (6.95), the following estimates hold:

$$\|u(t, \cdot)\|_{L^2(0,D)}^2 \leq 2\|\omega(t, \cdot)\|_{L^2(0,D)}^2 + 2\|\Gamma(t, \cdot)^\top \bar{z}(t)\|_{L^2(0,D)}^2, \quad (6.152)$$

and

$$\|z(t)\|_2^2 \leq N_1 \|\bar{z}(t)\|_2^2 + M_1 \|u(t, \cdot)\|_{L^2(0,D)}^2, \quad (6.153)$$

where  $N_1 = 2\|P^{-1}\|_2$  and  $M_1 = 2\|e^{A\cdot}\|_{L^2(0,D)}^2 \|e^{-A\cdot}\|_{L^2(0,D)}^2 \|B(\cdot)\|_{L^2(0,D)}^2$ .

*Proof.* On one hand, from (6.95), we have

$$|u(t, x)| \leq |\omega(t, x)| + |\Gamma(t, x)^\top \bar{z}(t)|, \quad (6.154)$$

then by Young's inequality, we get

$$|u(t, x)|^2 \leq 2|\omega(t, x)|^2 + 2|\Gamma(t, x)^\top \bar{z}(t)|^2, \quad (6.155)$$

from which (6.152) is obtained.

On the other hand, by applying the Cauchy-Schwarz inequality on (6.93), we get

$$\|z(t)\|_2 \leq \left\| P^{-1} \bar{z}(t) - \int_0^D \int_0^\sigma e^{-A(\sigma-y)} B(y) u(t, \sigma) dy d\sigma \right\|_2, \quad (6.156)$$

$$\leq \|P^{-1} \bar{z}(t)\|_2 + \int_0^D \int_0^D \|e^{Ay}\|_2 \|e^{-A\sigma}\|_2 \|B(y)\|_2 |u(t, \sigma)| dy d\sigma, \quad (6.157)$$

$$= \|P^{-1} \bar{z}(t)\|_2 + \left( \int_0^D \|e^{Ay}\|_2 \|B(y)\|_2 dy \right) \left( \int_0^D \|e^{-A\sigma}\|_2 |u(t, \sigma)| d\sigma \right), \quad (6.158)$$

$$\leq \|P^{-1}\|_2 \|\bar{z}(t)\|_2 + \|e^{A\cdot}\|_{L^2(0,D)} \|e^{-A\cdot}\|_{L^2(0,D)} \|B(\cdot)\|_{L^2(0,D)} \|u(t, \cdot)\|_{L^2(0,D)}. \quad (6.159)$$

Using Young's inequality, we obtain

$$\|z(t)\|_2^2 \leq 2\|P^{-1}\|_2^2 \|\bar{z}(t)\|_2^2 + 2\|e^{A\cdot}\|_{L^2(0,D)}^2 \|e^{-A\cdot}\|_{L^2(0,D)}^2 \|B(\cdot)\|_{L^2(0,D)}^2 \|u(t, \cdot)\|_{L^2(0,D)}^2, \quad (6.160)$$

$$= N_1 \|\bar{z}(t)\|_2^2 + M_1 \|u(t, \cdot)\|_{L^2(0,D)}^2, \quad (6.161)$$

which leads to (6.153).  $\square$

**Lemma 6.3**

Let  $\Gamma$  be given by (6.29). Then, the following property holds true:

$$\lim_{t \rightarrow D+T} \int_0^D R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t+x-D)}} dx = 0. \quad (6.162)$$

*Proof.* Let  $I(t) = \int_0^D R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t+x-D)}} dx$  and let  $\kappa = \deg(R)$ . Notice that  $I(t) = \sum_{i=0}^{\kappa} p_i J_i(t)$  with  $\{p_i = R^{(i)}(0)\}_{\{i=0, \dots, \kappa\}}$  are the polynomial coefficients of  $R_1(\cdot)$  and

$$J_i(t) = \int_0^D \left( \sqrt{\gamma_2(t+x-D)} \right)^i e^{-2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t+x-D)}} dx. \quad (6.163)$$

Let us consider the following change of variables:

$$s = 2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t+x-D)}, \quad (6.164)$$

$$= \frac{2r_{\min} \gamma_{2,0}^2 T^2}{D+T-t-x}, \quad (6.165)$$



from which we recover,

$$\frac{ds}{dx} = \frac{2r_{\min}\gamma_{2,0}^2 T^2}{(D+T-t-x)^2}, \quad (6.166)$$

$$= \frac{s^2}{2r_{\min}\gamma_{2,0}^2 T^2}, \quad (6.167)$$

By using this change of variables, we obtain

$$J_j(t) = \frac{2r_{\min}\gamma_{2,0}^2 T^2}{(2r_{\min}\gamma_{2,0}T)^j} \int_{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t)}} s^{j-2} e^{-s} ds, \quad (6.168)$$

$$= \gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-j} \int_{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t)}} s^{j-2} e^{-s} ds. \quad (6.169)$$

Case 1: Let  $i \in \{2, 3, \dots, p\}$  (i.e.  $i-2 \geq 0$ ). Then, by using multiple integrations by parts, we obtain,

$$J_i(t) = \gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i} \int_{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t)}} s^{i-2} e^{-s} ds, \quad (6.170)$$

$$= \gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i} \left[ -(i-2)! \sum_{k=0}^{i-2} \frac{s^k}{k!} e^{-s} \right]_{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t)}} \quad (6.171)$$

$$\leq (i-2)! \gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i} \left( \sum_{k=0}^{i-2} \frac{(2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)})^k}{k!} \right) e^{-2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}. \quad (6.172)$$

Note that proving (6.171) is straightforward by induction.

Case 2: Let  $i \in \{0, 1\}$  (i.e.  $i-2 < 0$ ). Then, by using the generalized exponential, defined as

$$E_n(r) = r^{n-1} \int_r^\infty \frac{e^{-s}}{s^n} ds, \quad r > 0, n \in \mathbb{N}^*, \quad (6.173)$$

and its property [202, Section 2]:

$$\frac{e^{-r}}{n+r} \leq E_n(r) \leq \frac{e^{-r}}{n-1+r}, \quad (6.174)$$

we obtain,

$$J_i(t) = \gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i} \int_{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t)}} \frac{e^{-s}}{s^{2-i}} ds, \quad (6.175)$$

$$\leq \gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i} \int_{2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}^{+\infty} \frac{e^{-s}}{s^{2-i}} ds, \quad (6.176)$$

$$:= \frac{\gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i}}{(2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)})^{1-i}} E_{2-i}(2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}), \quad (6.177)$$

$$\leq \frac{\gamma_{2,0}T(2r_{\min}\gamma_{2,0}T)^{1-i} e^{-2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}}}{\left(1-i+2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}\right) \left(2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t-D)}\right)^{1-i}}. \quad (6.178)$$

Finally, by using (6.172) from Case 1 and (6.178) from Case 2, we recover,

$$I(t) \leq \bar{R} \left( 2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t-D)} \right) e^{-2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t-D)}}, \quad (6.179)$$

where

$$\bar{R}(s) = \gamma_{2,0} T \left( \sum_{i=0}^1 \frac{c_i (2r_{\min} \gamma_{2,0} T)^{1-i}}{(1-i+s)s^{1-i}} + \sum_{i=2}^p c_i (2r_{\min} \gamma_{2,0} T)^{1-i} (i-2)! \sum_{k=0}^{i-2} \frac{s^k}{k!} \right).$$

Hence, we conclude that  $I(t) \rightarrow 0$  as  $t \rightarrow D+T$ .

□

**Proposition 6.7**

The following estimate holds:

$$\|\bar{z}(t_0)\|_2^2 \leq N_0 \|z(t_0)\|_2^2 + M_0 \|u(t_0, \cdot)\|_{L^2(0,D)}^2, \quad (6.180)$$

for any  $t_0 \geq 0$  where  $N_0 = 2\|P\|_2^2$  and  $M_0 = 2\|P\|_2^2 \|e^{-A \cdot}\|_{L^2(0,D)}^2 \|e^{A \cdot}\|_{L^2(0,D)}^2 \|B(\cdot)\|_{L^2(0,D)}^2$ .

*Proof.* By replacing  $t = t_0$  in (6.93) and by the Cauchy-Schwarz inequality and the Minkowski inequality, we get

$$\|\bar{z}(t_0)\|_2 = \left\| P \left( z(t_0) + \int_0^D \int_0^\sigma e^{-A(\sigma-y)} B(y) dy u(t_0, \sigma) d\sigma \right) \right\|_2, \quad (6.181)$$

$$\leq \|P\|_2 \left( \|z(t_0)\|_2 + \left\| \int_0^D \int_0^\sigma e^{-A(\sigma-y)} B(y) dy u(t_0, \sigma) d\sigma \right\|_2 \right), \quad (6.182)$$

$$\leq \|P\|_2 \left( \|z(t_0)\|_2 + \int_0^D \int_0^\sigma \|e^{-A\sigma}\|_2 \|e^{Ay}\|_2 \|B(y)\|_2 dy |u(t_0, \sigma)| d\sigma \right), \quad (6.183)$$

$$\leq \|P\|_2 \left( \|z(t_0)\|_2 + \left( \int_0^D \|e^{-A\sigma}\|_2 |u(t_0, \sigma)| d\sigma \right) \left( \int_0^D \|e^{Ay}\|_2 \|B(y)\|_2 dy \right) \right), \quad (6.184)$$

$$\leq \|P\|_2 \left( \|z(t_0)\|_2 + \|e^{-A \cdot}\|_{L^2(0,D)} \|u(t_0, \cdot)\|_{L^2(0,D)} \|e^{A \cdot}\|_{L^2(0,D)} \|B(\cdot)\|_{L^2(0,D)} \right). \quad (6.185)$$

Next, by Young's inequality, we obtain

$$\|\bar{z}(t_0)\|_2^2 \leq 2\|P\|_2^2 \|z(t_0)\|_2^2 + 2\|P\|_2^2 \|e^{-A \cdot}\|_{L^2(0,D)}^2 \|u(t_0, \cdot)\|_{L^2(0,D)}^2 \|e^{A \cdot}\|_{L^2(0,D)}^2 \|B(\cdot)\|_{L^2(0,D)}^2, \quad (6.186)$$

$$\leq N_0 \|z(t_0)\|_2^2 + M_0 \|u(t_0, \cdot)\|_{L^2(0,D)}^2, \quad (6.187)$$

from which (6.180) is deduced. □

**Theorem 6.2**

Let  $\gamma_2$  be given by (6.12) and let  $r_{\min} = \min_{i \in \{1, \dots, n\}} \{r_i\}$  with  $r_i > 0$  involved in (6.102)-(6.103). Let  $D > 0$ ,  $\gamma_{2,0} > 0$ ,  $T > 0$  fixed. Let  $\gamma$  be given by (6.109). Then, the solution of the closed-loop system (6.90)-(6.92) with prescribed-time predictor-based control (6.133) is **PTS** in the following sense: for any  $z_0 \in \mathbb{R}^n$  and  $u(0, \cdot) \in L^2(0, D)$ , the quantity  $\|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2$  remains bounded for  $t \in [0, \max\{D, T\}]$ , and for all  $t \in [\max\{D, T\}, D+T]$ , the following estimate holds:

$$\begin{aligned} \|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 &\leq M \int_0^D R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_{2,0} T \sqrt{\gamma_2(t+x-D)}} dx \\ &\quad \times \left( \|z_0\|_2^2 + \|u(0, \cdot)\|_{L^2(0,D)}^2 \right), \end{aligned} \quad (6.188)$$

where

$$M = 2(M_1 + 1)\bar{\eta}_{\bar{z}} \max(N_0, M_0). \quad (6.189)$$

In particular,  $\|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \rightarrow 0$  as  $t \rightarrow D + T$ .

Moreover,  $|U(t)| \rightarrow 0$  as  $t \rightarrow T$ .

*Proof.* From Proposition 6.6, we have

$$\|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \leq N_1 \|\bar{z}(t)\|_2^2 + (M_1 + 1) \|u(t, \cdot)\|_{L^2(0,D)}^2, \quad (6.190)$$

$$\leq N_1 \|\bar{z}(t)\|_2^2 + 2(M_1 + 1) \left( \|\omega(t, \cdot)\|_{L^2(0,D)}^2 + \|\Gamma(t, \cdot)^\top \bar{z}(t)\|_{L^2(0,D)}^2 \right), \quad (6.191)$$

where  $N_1 = 2\|P^{-1}\|_2$  and  $M_1 = 2\|e^{A\cdot}\|_{L^2(0,D)}^2 \|e^{-A\cdot}\|_{L^2(0,D)}^2 \|B(\cdot)\|_{L^2(0,D)}^2$ .

Next, using Proposition 6.5, we obtain

$$\begin{aligned} \|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 &\leq N_1 \|\bar{z}(t)\|_2^2 + 2(M_1 + 1) \|\omega(t, \cdot)\|_{L^2(0,D)}^2 + 2(M_1 + 1) \bar{\eta}_{\bar{z}} \\ &\quad \times \int_0^D R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_2, 0 T \sqrt{\gamma_2(t+x-D)}} dx \|\bar{z}(0)\|_2^2. \end{aligned} \quad (6.192)$$

Finally, by Proposition 6.7, we get

$$\begin{aligned} \|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 &\leq N_1 \|\bar{z}(t)\|_2^2 + 2(M_1 + 1) \|\omega(t, \cdot)\|_{L^2(0,D)}^2 + 2(M_1 + 1) \bar{\eta}_{\bar{z}} \\ &\quad \times \int_0^D R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_2, 0 T \sqrt{\gamma_2(t+x-D)}} dx \\ &\quad \times \left( N_0 \|z(0)\|_2^2 + M_0 \|u(0, \cdot)\|_{L^2(0,D)}^2 \right), \end{aligned} \quad (6.193)$$

where  $N_0 = 2\|P\|_2^2$  and  $M_0 = 2\|P\|_2^2 \|e^{-A\cdot}\|_{L^2(0,D)}^2 \|e^{A\cdot}\|_{L^2(0,D)}^2 \|B(\cdot)\|_{L^2(0,D)}^2$ .

As in Theorem 6.1, we need to discuss the case  $T \geq D$  and the case  $T \leq D$ , after that we use the fact that  $\|\omega(t, \cdot)\|_{L^2(0,D)} \equiv 0$  when  $t \geq D$  and the fact that  $\|\bar{z}(t)\|_2 \equiv 0$  when  $t \geq T$  to conclude that in both cases the following inequality holds:

$$\begin{aligned} \|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 &\leq M \int_0^D R_1(\sqrt{\gamma_2(t+x-D)}) e^{-2r_{\min} \gamma_2, 0 T \sqrt{\gamma_2(t+x-D)}} dx \\ &\quad \times \left( N_0 \|z(0)\|_2^2 + M_0 \|u(0, \cdot)\|_{L^2(0,D)}^2 \right), \end{aligned}$$

with  $M = 2(M_1 + 1)\bar{\eta}_{\bar{z}} \max(N_0, M_0)$ . Finally, by using Lemma 6.3, we conclude that  $\|z(t)\|_2^2 + \|u(t, \cdot)\|_{L^2(0,D)}^2 \rightarrow 0$  as  $t \rightarrow D + T$ . Moreover, from (6.133) and using the Cauchy-Schwartz inequality and the Minkowski inequality, we get

$$|U(t)| = \|B_c U(t)\|_2, \quad (6.194)$$

$$= \|B_c \Gamma(t, D)^\top \bar{z}(t)\|_2, \quad (6.195)$$

$$= \|(C(t) - A_c) \bar{z}(t)\|_2, \quad (6.196)$$

$$\leq \|C(t) - A_c\|_2 \|\bar{z}(t)\|_2, \quad (6.197)$$

$$(6.198)$$

Looking at the components of  $C(t)$  and using [176, Lemma 2], we can prove that

$$|U(t)| \leq R_2(\sqrt{\gamma_2(t)})e^{-2r_{\min}\gamma_{2,0}T\sqrt{\gamma_2(t)}}\|z(0)\|_2 \quad (6.199)$$

for some positive polynomial function  $R_2(\cdot)$ . In particular, we have  $U(t) \rightarrow 0$  as  $t \rightarrow T$ .  $\square$

## 6.4 Simulations

We consider a scalar linear equation with distributed input delay (6.2), with  $a = 2$ ,  $b = 0.5$ ,  $\gamma_{2,0} = 2$  and  $D = 1$ s. We fix  $T = 4$ s. Numerical simulations were done by discretizing the cascade ODE-PDE system (6.3) and making use of transformation (6.7). Figure 6.1 shows on the top right, the evolution of  $z(t)$  the solution of the closed-loop system (6.3), in blue using the prescribed-time control (6.43), and in red dashed line using the following predictor feedback for exponential stabilization (see [193], [200]),

$$U(t) = k \left( z(t) + b \int_0^D \int_0^\sigma e^{a(y-\sigma)} dy u(t, \sigma) d\sigma \right), \quad (6.200)$$

where, we chose  $k = -18.5$  such that  $a + b \times k < 0$ . On the top left it shows the evolution of  $u(t, x)$  the solution of the transport PDE. Finally, it shows on the bottom the evolution of the  $L^2$ -norm of the closed-loop system (plotted in logarithmic scale) with the prescribed-time control  $U(t)$  given in (6.43) in solid lines and using the predictor feedback (6.200) for exponential stabilization in dashed lines, for 3 different initial conditions  $z_0 = 1, 100z_0$ , and  $10000z_0$ . We can observe that under the prescribed-time control  $U(t)$ , no matter what initial condition we take, the convergence to the origin is always ensured at time  $D + T = 5$ s.

## Conclusion

In this chapter, we extended the existing results of prescribed-time delay-compensation and stabilization of LTI systems with input delay to distributed input delay. The main ideas were developed first for a scalar LTI system with distributed input delay and after that generalized to the  $n$ th-dimensional LTI case. The prescribed-time predictor feedback design was achieved based on the backstepping approach using a time-varying backstepping-forwarding transformation and a reduction-based change of variables.

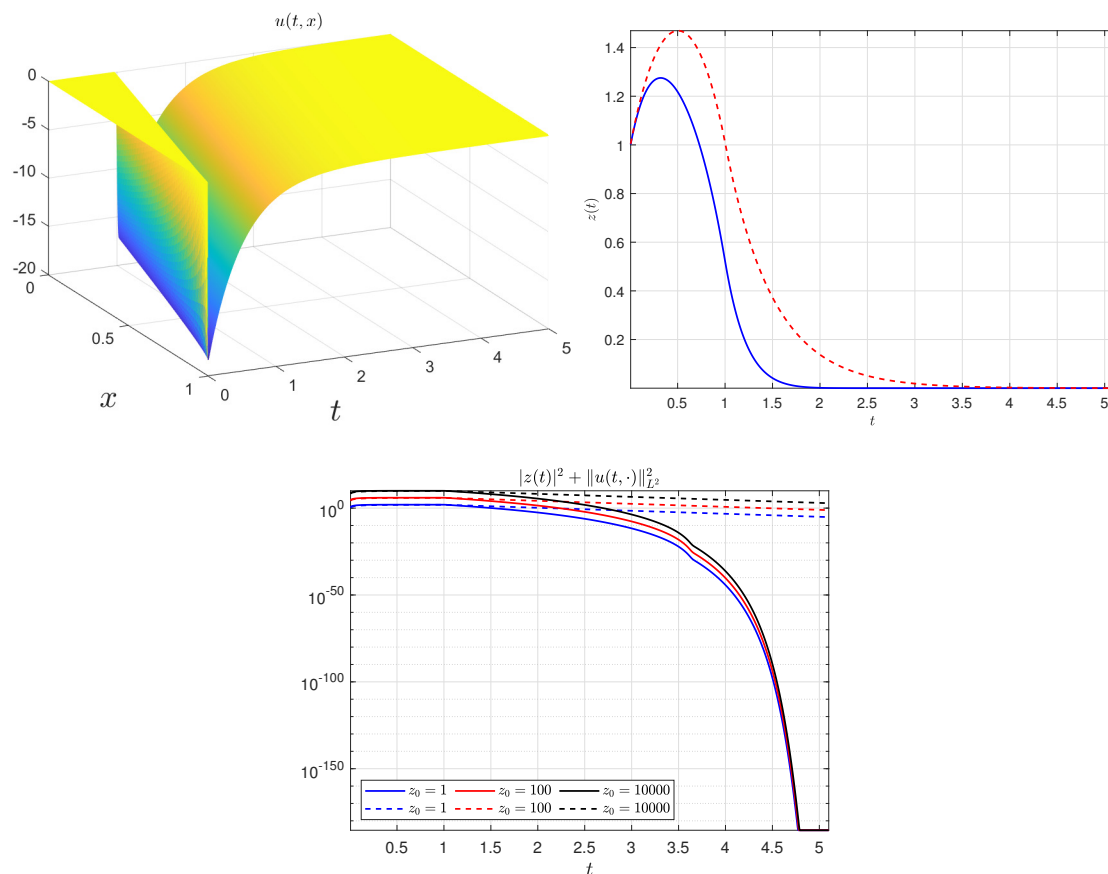


Figure 6.1: On the top right, the evolution of the solution of the closed-loop system (6.3) (logarithmic scale) in blue line using the prescribed-time predictor-based control (6.43) with delay  $D = 1$ s, settling time  $T = 4$ s and initial condition  $z_0 = 1$ , and in red dashed line using the predictor feedback (6.200) for exponential stabilization. On the top left, the numerical solution of the transport PDE  $u(t, x)$ . On the bottom, The evolution of the  $L^2$ -norm of the closed-loop system (6.3) (logarithmic scale) in blue solid lines using the prescribed-time predictor-based control (6.43) and in dashed lines using the classical predictor feedback (6.200) for exponential stabilization, with delay  $D = 1$ s, prescribed time  $T = 4$ s and for 3 initial conditions  $z_0 = 1$ ,  $100z_0$ , and  $10000z_0$ .

# Prescribed-time boundary output-feedback stabilization of a class of reaction-diffusion with input delay

## Contents

<b>7.1</b>	<b>Introduction</b>	<b>128</b>
<b>7.2</b>	<b>Problem statement</b>	<b>129</b>
<b>7.3</b>	<b>Prescribed-time boundary stabilization by full-state feedback in the delay-free case</b>	<b>130</b>
<b>7.4</b>	<b>Prescribed-time stabilization by full-state feedback for the input delay case</b>	<b>131</b>
7.4.1	Time-varying infinite-dimensional backstepping transformation	131
7.4.2	Target System	134
7.4.3	Time-varying infinite-dimensional inverse transformation	135
7.4.4	Prescribed-time predictor-based controller	135
7.4.5	Stability analysis	136
<b>7.5</b>	<b>Prescribed-time output boundary feedback stabilization in the delay-free case</b>	<b>139</b>
7.5.1	Observer design	140
7.5.2	Control design	140
<b>7.6</b>	<b>Prescribed-time stabilization by output feedback for the input delay case</b>	<b>141</b>
7.6.1	Time-varying infinite-dimensional backstepping transformation	141
7.6.2	Target System	142
7.6.3	Time-varying infinite-dimensional inverse transformation	142
7.6.4	Prescribed-time predictor-based output controller	143
7.6.5	Stability analysis	144
<b>7.7</b>	<b>Numerical results</b>	<b>148</b>

In this chapter, we consider a 1D reaction-diffusion system with boundary input delay and propose a general method for studying the problem of prescribed-time boundary stabilization. To achieve this objective, we first reformulate the system as a PDE-PDE cascade system (i.e., a cascade of a linear transport partial differential equation (PDE) with a linear reaction-diffusion PDE), where the transport equation represents the effect of the input delay. We then apply a time-varying infinite-dimensional backstepping transformation to convert the cascade system into a prescribed-time stable (PTS) target system. The stability analysis is conducted on the target

system, and the desired stability property is transferred back to the closed-loop system using the inverse transformation. The effectiveness of the proposed approach is demonstrated through numerical simulations.

## 7.1 Introduction

In this chapter, we address the problem of achieving prescribed-time boundary output-feedback stabilization for a class of linear 1D reaction-diffusion partial differential equations (PDEs) with input delay. We propose a novel approach, inspired by the employment of state predictions, as represented by [69, Chapter 11] to solve this problem. This approach is an advantageous alternative but requires radical advancements to be adjusted from nonlinear ODEs to linear PDEs. Our contribution builds upon the results of [163] and [166] for the case of delayed input and extends the results of [146] for the delay-dependent case to output-feedback stabilization.

The main idea of our approach is to transform the original PDE system into a PDE-PDE cascade system and then apply a time-varying backstepping transformation. Unlike [146] - where both the parabolic and hyperbolic parts of the cascade system are transformed using two different invertible backstepping transformations - we only transform the hyperbolic PDE state. This transformation leads to a stable target system that ensures the desired prescribed-time convergence. Finally, by inverse transformation, we transfer the stability property and the desired Non-Asymptotic convergence back to the original closed-loop system. Moreover, unlike [146], our resulting predictor-based controller does not depend on a spatial derivative of the state and overcomes also the issue of incompatibility of boundary conditions of kernel equations (which arises when considering point-wise damping term in the design) as observed in [146]. In the course of developing a methodology alternative to [145], [146], [166], we develop a number of technical innovations, located throughout the chapter and its appendices, and usable in future works on predictor-based prescribed-time stabilization.

This chapter is organized as follows. In Section 7.2, we introduce the unstable 1D reaction-diffusion system with boundary input delay. In Section 7.4, we focus on the problem of prescribed-time stabilization of the original system using full-state feedback control, where we start by reformulating the system as a PDE-PDE cascade system; we use an invertible backstepping transformation to link the cascade systems to some well-chosen prescribed-time stable target systems. We perform a stability analysis on the target systems. Then, by inverse transformation, we establish the boundedness of the state of the original systems and their convergence to the origin in a prescribed time using a suitable norm equivalence. In Section 7.4.4, we switch to the problem of prescribed-time output-feedback stabilization, where we adapt our approach to taking into account the dynamics of the proposed observer system. In Section 7.7, we consider a numerical example to illustrate the main results.

## 7.2 Problem statement

Let us consider the following reaction-diffusion equation with a constant reaction term and a known constant boundary input delay  $D > 0$ :

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \in [t_0, t_0 + D + T], x \in [0, 1], & (7.1) \\ z(t, 0) = 0, & t \in [t_0, t_0 + D + T], & (7.2) \\ z(t, 1) = U(t - D), & t \in [t_0, t_0 + D + T], & (7.3) \\ y(t) = z_x(t, 1), & t \in [t_0, t_0 + D + T], & (7.4) \\ z(t_0, x) = z_0(x), & x \in [0, 1], & (7.5) \end{cases}$$

with the initial time  $t_0 \geq 0$ , the reaction term  $\lambda \in \mathbb{R}$ , the state  $z(t, x) \in \mathbb{R}$ , the control  $U(t) \in \mathbb{R}$ , with the following initial condition:  $U(t_0 + s) = 0$  for all  $s \in [-D, 0]$ , the collocated output  $y(t) \in \mathbb{R}$ , and the initial condition  $z_0 \in L^2(0, 1)$ .

Our main goal is to design a predictor-based output-feedback controller achieving prescribed-time stabilization of the closed-loop system (7.1)-(7.5) in the following sense: there exist a class  $\mathcal{KL}$  function  $\beta$  and a function  $\mu : [t_0, t_0 + D + T] \rightarrow \mathbb{R}_+$ , where  $\mu$  tends to infinity as  $t$  goes to  $t_0 + D + T$ , such that for any initial condition and for all  $t \in [t_0, t_0 + D + T]$ , the following estimate holds:  $\|z(t, \cdot)\|_{L^2(0,1)} \leq \beta\left(\|z_0\|_{L^2(0,1)}, \mu(t - t_0 - D)\right)$ .

In order to solve this problem we follow three steps:

- **Step 1: Prescribed-time full-state feedback stabilization:** we start by solving the problem of prescribed-time stabilization by full-state feedback of the closed-loop system (7.1)-(7.5). The idea is to first represent the input delay as a linear transport PDE (inspired from [205]), so that the system (7.1)-(7.5) is rewritten as a parabolic-transport hyperbolic PDE-PDE cascade system. Then, we propose a novel infinite-dimensional transformation (inspired from [69, Chapter 11, page 171]) to transform the resulting cascade system into a suitable target system that is prescribed-time stable. The target system is chosen to be a parabolic-transport hyperbolic PDE-PDE cascade system with diffusion dynamics being exactly as in (7.1) but whose Dirichlet right boundary condition is given not only in terms of a boundary term of transport PDE which vanishes after delay  $D$  but includes also a feedback term which renders the diffusion PDE converging to zero after the delay. Such a feedback is borrowed from [163]. Finally, the stability property is transferred to (7.1)-(7.5) via the inverse transformation.
- **Step 2: Prescribed-time observer design:** We employ the prescribed-time observer design from [166] to estimate the states of (7.1)-(7.5) in a prescribed time.
- **Step 3: Prescribed-time output-feedback stabilization:** Finally, we combine the designed prescribed-time observer with the full-state control to ensure output-feedback stabilization in a prescribed time of the closed-loop system (7.1)-(7.5).

**Remark 7.1**



*Extension of our result to the case of input-output delay*

$$\begin{cases} \bar{z}_t(t, x) = \bar{z}_{xx}(t, x) + \lambda \bar{z}(t, x), & t \in [t_0, t_0 + D_1 + D_2 + T), \quad x \in [0, 1], & (7.6) \\ \bar{z}(t, 0) = 0, & t \in [t_0, t_0 + D_1 + D_2 + T), & (7.7) \\ \bar{z}(t, 1) = U(t - D_1), & t \in [t_0, t_0 + D_1 + D_2 + T), & (7.8) \\ y(t) = \bar{z}_x(t - D_2, 1), & t \in [t_0, t_0 + D_1 + D_2 + T), & (7.9) \end{cases}$$

is straightforward by using the following change of variables  $z(t, x) = \bar{z}(t - D_2, x)$ , as it allows to obtain a system of the form of (7.1)-(7.5) (with  $D = D_1 + D_2$ ) for which we can apply our approach to ensure prescribed-time stabilization.

**Remark 7.2**

Note that in our approach, it is necessary to have a prescribed-time boundary controller for the delay-free case of system (7.1)-(7.5).

### 7.3 Prescribed-time boundary stabilization by full-state feedback in the delay-free case

Before presenting our approach, let us briefly summarize the main results of [163] on prescribed-time boundary stabilization of system (7.1)-(7.5) when  $D = 0$ .

Consider the following blow-up function:

$$\gamma_m(t - \bar{t}_0) := \frac{\gamma_{m,0}^m T^m}{(\bar{t}_0 + T - t)^m}, \quad (7.10)$$

for  $m \in \mathbb{N}^*$ , defined for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$  where  $T > 0$  is a priori fixed.

We recall the following time-varying boundary control:

$$U(t) = \int_0^1 K(1, s, t - \bar{t}_0) z(t, s) ds, \quad (7.11)$$

where  $K$  is given explicitly in [163, Lemma 1] by

$$K(x, s, t - \bar{t}_0) = -\frac{1}{2} \gamma_2(t - \bar{t}_0) s \sum_{n=0}^{\infty} \frac{\left(\sqrt{\gamma_2(t - \bar{t}_0)}\right)^n}{(n+1)!} \left(\frac{x^2 - s^2}{4T\gamma_{2,0}}\right)^n L_n^{(1)}\left(-T\gamma_{2,0}\sqrt{\gamma_2(t - \bar{t}_0)}\right), \quad (7.12)$$

where  $L_n^{(1)}(\cdot)$  are the generalized Laguerre polynomials. In addition, (7.12) can be simplified using the first-order modified Bessel function  $I_1(\cdot)$  to get,

$$K(x, s, t - \bar{t}_0) = -\gamma_2(t - \bar{t}_0) s e^{\frac{\sqrt{\gamma_2(t - \bar{t}_0)}(x^2 - y^2)}{4T\gamma_{2,0}}} \frac{I_1\left(\sqrt{\gamma_2(t - \bar{t}_0)}(x^2 - s^2)\right)}{\sqrt{\gamma_2(t - \bar{t}_0)}(x^2 - s^2)}, \quad (7.13)$$

for  $(x, s, t) \in \mathcal{T} := \{(x, s, t) \in [0, 1]^2 \times [\bar{t}_0, \bar{t}_0 + T) : s \leq x\}$ .

Therefore, the closed-loop system (7.1)-(7.5) with  $D = 0$  and under the time-varying boundary feedback (7.11)-(7.13) is prescribed-time stable (PTS) [163, Theorem 3] provided that

$$2\gamma_{2,0}T > 1. \quad (7.14)$$

#### 7.4. Prescribed-time stabilization by full-state feedback for the input delay case 131

More precisely, there exist positive constants  $c_k > 0$ ,  $c_l > 0$ , and  $M > 0$  such that for any initial condition  $z(\bar{t}_0, \cdot)$  at an initial time  $\bar{t}_0$ , we have

$$\|z(t, \cdot)\|_{L^2(0,1)} \leq \xi_1(t - \bar{t}_0) \|z(\bar{t}_0, \cdot)\|_{L^2(0,1)}, \quad (7.15)$$

where

$$\xi_1(t - \bar{t}_0) := M \left( c_l \gamma_2(t - \bar{t}_0) e^{-\alpha_0 \sqrt{\gamma_2(t - \bar{t}_0)}} + e^{-\gamma_{2,0} T \sqrt{\gamma_2(t - \bar{t}_0)}} \right), \quad (7.16)$$

with  $M := e^{(\gamma_{2,0}^2 + \lambda)T} \left( 1 + e^{\frac{1}{4T} + \gamma_{2,0} c_k} \right) > 0$  and  $\alpha_0 := \frac{4\gamma_{2,0}^2 T^2 - 1}{4T\gamma_{2,0}} > 0$ .

Furthermore,

$$|U(t)| \leq \xi_2(t - \bar{t}_0) \|z(\bar{t}_0, \cdot)\|_{L^2(0,1)}, \quad (7.17)$$

with

$$\xi_2(t - \bar{t}_0) := c_l M \gamma_2(t - \bar{t}_0) e^{-\alpha_0 \sqrt{\gamma_2(t - \bar{t}_0)}}. \quad (7.18)$$

In particular,  $\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$ , and  $|U(t)| \rightarrow 0$  when  $t \rightarrow \bar{t}_0 + T$ .

##### Remark 7.3

To achieve exponential stabilization, it is sufficient to replace the control gain  $K$  by

$$K_{exp}(x, s) = -(\lambda + \lambda_0) s \frac{I_1(\sqrt{(\lambda + \lambda_0)(x^2 - s^2)})}{\sqrt{(\lambda + \lambda_0)(x^2 - s^2)}}, \quad (7.19)$$

for  $\lambda_0 \geq 0$  and  $(x, s) \in \mathcal{T} := \{(x, s) \in [0, 1]^2 : s \leq x\}$ .

## 7.4 Prescribed-time stabilization by full-state feedback for the input delay case

Let us now consider the PDE-PDE cascade representation of (7.1)-(7.5),

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \in [t_0, t_0 + D + T), \quad x \in [0, 1], & (7.20) \\ z(t, 0) = 0, & t \in [t_0, t_0 + D + T), & (7.21) \\ z(t, 1) = v(t, 0), & t \in [t_0, t_0 + D + T), & (7.22) \\ v_t(t, y) = v_y(t, y), & t \in [t_0, t_0 + D + T), \quad y \in [0, D], & (7.23) \\ v(t, D) = U(t), & t \in [t_0, t_0 + D + T). & (7.24) \end{cases}$$

with  $(t, x, y) \in [t_0, t_0 + D + T) \times [0, 1] \times [0, D]$ , and  $v(t, \cdot)$  is the transport PDE state whose solution is given by

$$v(t, y) = \begin{cases} 0, & t_0 \leq t + y \leq t_0 + D, \\ U(t + y - D), & t + y \geq t_0 + D. \end{cases} \quad (7.25)$$

Note that the control's initial condition is taken as  $U(t_0 + s) = 0$  for all  $s \in [-D, 0]$ .

### 7.4.1 Time-varying infinite-dimensional backstepping transformation

Inspired by [69, Chapter 11, page 171], we propose the following time-varying infinite-dimensional backstepping transformation:

$$\omega(t, y) = v(t, y) - \mathcal{F}(t + y - t_0 - D, \varphi(t, \cdot, y)), \quad (7.26)$$

where  $\mathcal{F}$  is given by

$$\mathcal{F}(t + y - t_0 - D, \varphi(t, \cdot, y)) := \int_0^1 K(1, s, t + y - t_0 - D)\varphi(t, s, y)ds \quad (7.27)$$

with  $K$  defined as in (7.13) (with  $\bar{t}_0 = t_0 + D$ ), and the predictor  $\varphi$  is chosen to satisfy  $\varphi(t, x, y) = z(t + y, x)$  which means it is the solution of

$$\begin{cases} \varphi_y(t, x, y) = \varphi_{xx}(t, x, y) + \lambda\varphi(t, x, y), & (7.28) \\ \varphi(t, 0, y) = 0, & (7.29) \\ \varphi(t, 1, y) = v(t, y), & (7.30) \\ \varphi(t, x, 0) = z(t, x). & (7.31) \end{cases}$$

with  $(t, x, y) \in \{(t, x, y) \in [t_0, t_0 + D + T) \times [0, 1] \times [0, D] : t + y \in [t_0 + D, t_0 + D + T)\}$ .

Notice that  $\varphi$  can be computed explicitly (see [206, Chapter 3, page 266]) as follows:

$$\varphi(t, x, y) = 2 \sum_{n=1}^{+\infty} \sin(n\pi x) e^{(\lambda - n^2\pi^2)y} \left[ \int_0^1 \sin(n\pi s) z(t, s) ds + n\pi(-1)^{n+1} \int_0^y e^{-(\lambda - n^2\pi^2)\tau} v(t, \tau) d\tau \right], \quad (7.32)$$

$$\begin{aligned} &= \int_0^1 \left[ 2 \sum_{n=1}^{+\infty} \sin(n\pi x) \sin(n\pi s) e^{(\lambda - n^2\pi^2)y} \right] z(t, s) ds \\ &\quad + \int_0^y \left[ 2\pi \sum_{n=1}^{+\infty} n(-1)^{n+1} \sin(n\pi x) e^{(\lambda - n^2\pi^2)(y-\tau)} \right] v(t, \tau) d\tau, \end{aligned} \quad (7.33)$$

Now, by substituting (7.32) in (7.26), we obtain,

$$\omega(t, y) = v(t, y) - \int_0^1 K(1, x, t + y - t_0 - D)\varphi(t, x, y)dx, \quad (7.34)$$

$$\begin{aligned} &= v(t, y) - \int_0^1 K(1, x, t + y - t_0 - D) \left( \int_0^1 \left[ 2 \sum_{n=1}^{+\infty} \sin(n\pi x) \sin(n\pi s) e^{(\lambda - n^2\pi^2)y} \right] z(t, s) ds \right. \\ &\quad \left. + \int_0^y \left[ 2\pi \sum_{n=1}^{+\infty} n(-1)^{n+1} \sin(n\pi x) e^{(\lambda - n^2\pi^2)(y-\tau)} \right] v(t, \tau) d\tau \right) dx, \end{aligned} \quad (7.35)$$

$$\begin{aligned} &= v(t, y) - \int_0^1 \left[ 2 \sum_{n=1}^{+\infty} \sin(n\pi s) e^{(\lambda - n^2\pi^2)y} \int_0^1 K(1, x, t + y - t_0 - D) \sin(n\pi x) dx \right] z(t, s) ds \\ &\quad + \int_0^y \left[ 2\pi \sum_{n=1}^{+\infty} n(-1)^{n+1} e^{(\lambda - n^2\pi^2)(y-\tau)} \int_0^1 K(1, x, t + y - t_0 - D) \sin(n\pi x) dx \right] v(t, \tau) d\tau, \end{aligned} \quad (7.36)$$

$$:= v(t, y) - \int_0^1 \bar{\gamma}_2(y, s, t + y - t_0 - D) z(t, s) ds + \int_0^y \bar{q}(y - \tau, t + y - t_0 - D) v(t, \tau) d\tau, \quad (7.37)$$

where the new kernels  $\bar{\gamma}$  and  $\bar{q}$  are given by

$$\bar{\gamma}(y, s, t + y - t_0 - D) := 2 \sum_{n=1}^{+\infty} e^{(\lambda - n^2\pi^2)y} \sin(n\pi s) \int_0^1 K(1, x, t + y - t_0 - D) \sin(n\pi x) dx, \quad (7.38)$$

#### 7.4. Prescribed-time stabilization by full-state feedback for the input delay case 133

and

$$\bar{q}(y - \tau, t + y - t_0 - D) := -2\pi \sum_{n=1}^{+\infty} e^{(\lambda - n^2 \pi^2)(y - \tau)} n(-1)^{n+1} \int_0^1 K(1, x, t + y - t_0 - D) \sin(n\pi x) dx, \quad (7.39)$$

**Remark 7.4**

To obtain exponential stabilization of (7.1)-(7.5), it is sufficient to replace the transformation (7.26) by

$$\omega(t, y) = v(t, y) - \mathcal{F}_{exp}(\varphi(t, \cdot, y)), \quad (7.40)$$

where  $\varphi$  is generated as before, i.e., from (7.28)-(7.31), and  $\mathcal{F}_{exp}$  is defined as,

$$\mathcal{F}_{exp}(\varphi(t, \cdot, y)) := \int_0^1 K_{exp}(1, s) \varphi(t, s, y) ds \quad (7.41)$$

with the gain  $K_{exp}$  given as in (7.19). Moreover, using (7.32), the transformation (7.26) can be simplified as follows:

$$\omega(t, y) = v(t, y) - \int_0^1 K_{exp}(1, x) \varphi(t, x, y) dx, \quad (7.42)$$

$$= v(t, y) - \int_0^1 K_{exp}(1, x) \left( \int_0^1 \left[ 2 \sum_{n=1}^{+\infty} \sin(n\pi x) \sin(n\pi s) e^{(\lambda - n^2 \pi^2)y} \right] z(t, s) ds \right. \quad (7.43)$$

$$\left. + \int_0^y \left[ 2\pi \sum_{n=1}^{+\infty} n(-1)^{n+1} \sin(n\pi x) e^{(\lambda - n^2 \pi^2)(y - \tau)} \right] v(t, \tau) d\tau \right) dx, \quad (7.44)$$

$$= v(t, y) - \int_0^1 \left[ 2 \sum_{n=1}^{+\infty} \sin(n\pi s) e^{(\lambda - n^2 \pi^2)y} \int_0^1 K_{exp}(1, x) \sin(n\pi x) dx \right] z(t, s) ds \quad (7.45)$$

$$+ \int_0^y \left[ 2\pi \sum_{n=1}^{+\infty} n(-1)^{n+1} e^{(\lambda - n^2 \pi^2)(y - \tau)} \int_0^1 K_{exp}(1, x) \sin(n\pi x) dx \right] v(t, \tau) d\tau, \quad (7.46)$$

$$:= \int_0^1 \bar{\gamma}_{exp}(y, s) z(t, s) ds + \int_0^y \bar{q}_{exp}(y - \tau) v(t, \tau) d\tau, \quad (7.47)$$

where the kernels  $\bar{\gamma}_{exp}$  and  $\bar{q}_{exp}$  are given by,

$$\bar{\gamma}_{exp}(y, s) := 2 \sum_{n=1}^{+\infty} e^{(\lambda - n^2 \pi^2)y} \sin(n\pi s) \int_0^1 K_{exp}(1, x) \sin(n\pi x) dx, \quad (7.48)$$

and

$$\bar{q}_{exp}(y - \tau) := -2\pi \sum_{n=1}^{+\infty} n(-1)^{n+1} e^{(\lambda - n^2 \pi^2)(y - \tau)} \int_0^1 K_{exp}(1, x) \sin(n\pi x) dx, \quad (7.49)$$

hence, recovering the kernels of the backstepping transformation obtained in [205] for  $\lambda_0 = 0$ .

### 7.4.2 Target System

Next, using (7.26), we transform the system (7.20)-(7.24) into the following target system:

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \in [t_0, t_0 + D + T], x \in [0, 1], & (7.50) \\ z(t, 0) = 0, & t \in [t_0, t_0 + D + T], & (7.51) \\ z(t, 1) = \omega(t, 0) + \mathcal{F}(t - t_0 - D, z(t, \cdot)), & t \in [t_0, t_0 + D + T], & (7.52) \\ \omega_t(t, y) = \omega_y(t, y), & t \in [t_0, t_0 + D + T], y \in [0, D], & (7.53) \\ \omega(t, D) = 0, & t \in [t_0, t_0 + D + T], & (7.54) \end{cases}$$

where  $\omega : [t_0, t_0 + D + T] \times [0, D] \rightarrow \mathbb{R}$  is the transport PDE state, and

$$\mathcal{F}(t - t_0 - D, z(t, \cdot)) = \int_0^1 K(1, s, t - t_0 - D) z(t, s) ds. \quad (7.55)$$

The transformation is realized by using the fact that  $\varphi(t, x, y) = z(t + y, x)$  for all  $(t, x, y) \in [t_0, t_0 + D + T] \times [0, 1] \times [0, D]$ , and noticing that the time-varying transformation (7.26) satisfies (7.53) (i.e.  $\omega_t(t, y) = \omega_y(t, y)$ ). Indeed, we have

$$\omega_t(t, y) = v_t(t, y) - \frac{\partial \mathcal{F}(t + y - t_0 - D, \varphi(t, \cdot, y))}{\partial t}, \quad (7.56)$$

$$= v_y(t, y) - \int_0^1 K_t(1, s, t + y - t_0 - D) \varphi(t, s, y) ds \quad (7.57)$$

$$- \int_0^1 K(1, s, t + y - t_0 - D) \varphi_t(t, s, y) ds, \quad (7.58)$$

$$= v_y(t, y) - \int_0^1 K_y(1, s, t + y - t_0 - D) \varphi(t, s, y) ds \quad (7.59)$$

$$- \int_0^1 K(1, s, t + y - t_0 - D) \varphi_y(t, s, y) ds, \quad (7.60)$$

$$= v_y(t, y) - \frac{\partial \mathcal{F}(t + y - t_0 - D, \varphi(t, \cdot, y))}{\partial y}, \quad (7.61)$$

$$= \omega_y(t, y) \quad (7.62)$$

where we have used the fact that:

$$K_t(1, s, t + y - t_0 - D) = \frac{\partial(t + y)}{\partial t} \frac{\partial K(1, s, t + y - t_0 - D)}{\partial(t + y)} \quad (7.63)$$

$$= \frac{\partial K(1, s, t + y - t_0 - D)}{\partial(t + y)} \quad (7.64)$$

$$= \frac{\partial(t + y)}{\partial y} \frac{\partial K(1, s, t + y - t_0 - D)}{\partial(t + y)} \quad (7.65)$$

$$= K_y(1, s, t + y - t_0 - D), \quad (7.66)$$

#### Remark 7.5

It is important to highlight the key feature of the chosen target system (7.50)-(7.54): when  $t \geq t_0 + D$ ,  $\omega(t, 0)$  vanishes in (7.52); then, the resulting target system (7.50)-(7.52) with time-varying feedback  $U(t) = \mathcal{F}(t - t_0 - D, z(t, \cdot))$  converges in prescribed-time  $t_0 + D + T$  to zero, in the light of the results introduced in Section 7.3.

### 7.4.3 Time-varying infinite-dimensional inverse transformation

The inverse transformation is given by

$$v(t, y) = \omega(t, y) + \mathcal{F}(t + y - t_0 - D, \psi(t, \cdot, y)), \quad (7.67)$$

with

$$\mathcal{F}(t + y - t_0 - D, \psi(t, \cdot, y)) := \int_0^1 K(1, s, t + y - t_0 - D) \psi(t, s, y) ds, \quad (7.68)$$

which, similarly as in (7.27),  $\psi$  is chosen to satisfy  $\psi(t, x, y) = z(t + y, x)$ . Consequently,  $\psi$  is the solution to

$$\begin{cases} \psi_y(t, x, y) = \psi_{xx}(t, x, y) + \lambda \psi(t, x, y), & (7.69) \\ \psi(t, 0, y) = 0, & (7.70) \end{cases}$$

$$\begin{cases} \psi(t, 1, y) = \omega(t, y) + \mathcal{F}(t + y - t_0 - D, \psi(t, \cdot, y)), & (7.71) \\ \psi(t, x, 0) = z(t, x). & (7.72) \end{cases}$$

Similarly to the direct transformation, we recover (7.23) (i.e.  $v_t(t, y) = v_y(t, y)$ ) from the inverse transformation.

### 7.4.4 Prescribed-time predictor-based controller

Using (7.26) at  $y = D$ , we recover the expression of the boundary control  $U(t)$ ,

$$U(t) = v(t, D) = \mathcal{F}(t - t_0, \varphi(t, \cdot, D)) = \int_0^1 K(1, s, t - t_0) \varphi(t, s, D) ds, \quad (7.73)$$

where  $\varphi$  is the solution of (7.28)-(7.31) and  $K$  is given in (7.13). Equivalently, using (7.67) at  $y = D$ ,

$$U(t) = v(t, D) = \mathcal{F}(t - t_0, \psi(t, \cdot, D)), \quad (7.74)$$

where  $\psi$  is the solution of (7.69)-(7.72).

From (7.38), the expression of the control  $U(t)$  in (7.73) can be simplified as follows:

$$U(t) = \int_0^1 \bar{\gamma}(s, D, t - t_0) z(t, s) ds + \int_0^D \bar{q}(D - \tau, t - t_0) v(t, \tau) d\tau, \quad (7.75)$$

where  $\bar{\gamma}$  and  $\bar{q}$  are given in (7.38)-(7.39).

#### Remark 7.6

In light of Remark 7.4, the expression of the control  $U_{exp}(t)$  that achieves exponential stabilization of (7.1)-(7.5) is given as follows:

$$U_{exp}(t) = \int_0^1 \bar{\gamma}_{exp}(s, D) z(t, s) ds + \int_0^D \bar{q}_{exp}(D - \tau) v(t, \tau) d\tau, \quad (7.76)$$

where  $\bar{\gamma}_{exp}$  and  $\bar{q}_{exp}$  are given in (7.48) and (7.49), respectively.

### 7.4.5 Stability analysis

In this subsection, we start by performing the stability analysis on the target system (7.50)-(7.54). Then, by the inverse transformation (7.69)-(7.72) we establish the boundedness of the state of the original system (7.20)-(7.24) and its convergence to zero in a prescribed time using a suitable norm equivalence.

**Proposition 7.1**

Let  $\gamma_{2,0}$  satisfy (7.14). There exists a polynomial function  $Q(\cdot)$  in terms of  $\gamma_2(t+y-t_0-D)$  such that, for all  $t \in [t_0+D, t_0+D+T)$ :

$$\int_0^D \xi_2(t+y-t_0-D)^2 dy \leq Q\left(2\alpha_0\sqrt{\gamma_2(t-t_0-D)}\right) e^{-2\alpha_0\sqrt{\gamma_2(t-t_0-D)}}. \quad (7.77)$$

where  $\xi_2$  is defined in (7.18).

*Proof.* Let  $A$  be defined as follows:

$$A = \int_0^D \xi_2(t+y-t_0-D)^2 dy, \quad (7.78)$$

$$:= \frac{c_l M}{(2\alpha_0)^4} \int_0^D (2\alpha_0)^4 \gamma_2(t+y-t_0-D)^2 e^{-2\alpha_0\sqrt{\gamma_2(t+y-t_0-D)}} dy, \quad (7.79)$$

$$= \frac{c_l M}{(2\alpha_0)^4} \int_0^D (2\alpha_0\sqrt{\gamma_2(t+y-t_0-D)})^4 e^{-2\alpha_0\sqrt{\gamma_2(t+y-t_0-D)}} dy. \quad (7.80)$$

where  $M$ ,  $c_k$ , and  $\alpha_0$  are all defined in (7.16).

Next, let us consider the following change of variables:

$$s = 2\alpha_0\sqrt{\gamma_2(t+y-t_0-D)}, \quad (7.81)$$

$$= \frac{2\alpha_0\gamma_{2,0}T}{(t_0+D+T-t-y)}, \quad (7.82)$$

from which we recover,

$$\frac{ds}{dy} = \frac{2\alpha_0\gamma_{2,0}T}{(t_0+D+T-t-y)^2}, \quad (7.83)$$

$$= \frac{2\alpha_0}{\gamma_{2,0}T} \gamma_2(t+y-t_0-D), \quad (7.84)$$

$$= \frac{s^2}{2\alpha_0\gamma_{2,0}T}. \quad (7.85)$$

Now, by using (7.82)-(7.85) in (7.78) (along with  $dy = 2\alpha_0\gamma_{2,0}T\frac{ds}{s^2}$ ), we obtain

$$A = \frac{\gamma_{2,0}c_lMT}{8\alpha_0^3} \int_{2\alpha_0\sqrt{\gamma_2(t-t_0-D)}}^{2\alpha_0\sqrt{\gamma_2(t-t_0)}} s^2 e^{-s} ds \quad (7.86)$$

Finally, by integrating by parts twice, we recover

$$A = \frac{\gamma_{2,0}c_lMT}{8\alpha_0^3} \left[ -(s^2 + 2s + 2)e^{-s} \right]_{2\alpha_0\sqrt{\gamma_2(t-t_0-D)}}^{2\alpha_0\sqrt{\gamma_2(t-t_0)}}, \quad (7.87)$$

$$= -Q\left(2\alpha_0\sqrt{\gamma_2(t-t_0)}\right) e^{-2\alpha_0\sqrt{\gamma_2(t-t_0)}} + Q\left(2\alpha_0\sqrt{\gamma_2(t-t_0-D)}\right) e^{-2\alpha_0\sqrt{\gamma_2(t-t_0-D)}}, \quad (7.88)$$

$$\leq Q\left(2\alpha_0\sqrt{\gamma_2(t-t_0-D)}\right) e^{-2\alpha_0\sqrt{\gamma_2(t-t_0-D)}}, \quad (7.89)$$

#### 7.4. Prescribed-time stabilization by full-state feedback for the input delay case 137

where  $Q(s) = \frac{\gamma_{2,0}c_1MT}{8\alpha_0^3}(s^2 + 2s + 2)$ .

This concludes the proof.  $\square$

##### Proposition 7.2

For the transport PDE  $v(t, x)$  satisfying (7.23), the following estimates holds for  $t \in [t_0 + D, t_0 + D + T)$ :

$$\|v(t, \cdot)\|_{L^2(0,D)} \leq \xi_3(t - t_0 - D)\|z(t_0 + D, \cdot)\|_{L^2(0,1)}. \quad (7.90)$$

where

$$\xi_3(t - t_0 - D) := Q\left(2\alpha_0\sqrt{\gamma_2(t - t_0 - D)}\right)^{1/2} e^{-\alpha_0\sqrt{\gamma_2(t - t_0 - D)}}. \quad (7.91)$$

where  $Q(\cdot)$  is given as in Proposition 7.1. In particular, it holds  $\|v(t, \cdot)\|_{L^2(0,D)} \rightarrow 0$  for all  $t \rightarrow t_0 + D + T$ .

*Proof.* From (7.67) and using the fact that  $\omega(t, y) = 0, \forall (t, y) \in [t_0 + D, t_0 + D + T) \times [0, D]$ , we recover that

$$v(t, y) = \mathcal{F}(t + y - t_0 - D, \psi(t, \cdot, y)). \quad (7.92)$$

Next, by squaring the previous equality, using the fact that  $\psi(t, \cdot, y) = z(t + y, \cdot)$ , and using (7.17) for  $\bar{t}_0 := t_0 + D$ , we get,

$$|v(t, y)|^2 \leq \xi_2(t + y - t_0 - D)^2 \|z(t_0 + D, \cdot)\|_{L^2(0,1)}^2. \quad (7.93)$$

Now, by integrating from 0 to  $D$  with respect to  $y$  and using (7.77) in Proposition 7.1, we obtain,

$$\|v(t, \cdot)\|_{L^2(0,D)}^2 \leq \int_0^D \xi_2(t + y - t_0 - D)^2 dy \|z(t_0 + D, \cdot)\|_{L^2(0,1)}^2, \quad (7.94)$$

$$\leq \xi_3(t - t_0 - D)^2 \|z(t_0 + D, \cdot)\|_{L^2(0,1)}^2, \quad (7.95)$$

where

$$\xi_3(t - t_0 - D) = Q\left(2\alpha_0\sqrt{\gamma_2(t - t_0 - D)}\right)^{1/2} e^{-\alpha_0\sqrt{\gamma_2(t - t_0 - D)}}, \quad (7.96)$$

Now, by passing to the square roots, we recover (7.90). In particular, we can clearly see that  $\xi_3(t - t_0 - D) \rightarrow 0$  as  $t \rightarrow t_0 + D + T$ . As a result, we obtain that  $\|v(t, \cdot)\|_{L^2(0,D)} \rightarrow 0$  when  $t \rightarrow t_0 + D + T$ .  $\square$

Let us now introduce our first main result.

##### Theorem 7.1

Let  $\gamma_{2,0}$  be chosen such that (7.14) is ensured. Let  $T > 0$ ,  $D > 0$  and  $t_0 \geq 0$ . Then, the solution of the closed-loop system (7.20)-(7.24) with the prescribed-time time-varying controller (7.73) (or (7.74)) is prescribed-time stable in the following sense: For any initial condition  $z_0$ , the quantities  $\|z(t, \cdot)\|_{L^2(0,1)}$  and  $\|v(t, \cdot)\|_{L^2(0,D)}$  remain bounded for all  $t \in [t_0, t_0 + D]$ ; and for all  $t \in [t_0 + D, t_0 + D + T)$ , the following norm  $I(t) = \|z(t, \cdot)\|_{L^2(0,1)} + \|v(t, \cdot)\|_{L^2(0,D)}$  satisfies,

$$I(t) \leq \mathcal{B}_D \xi_4(t - t_0 - D) \|z_0\|_{L^2(0,1)}, \quad (7.97)$$

with  $\xi_4 = \xi_1 + \xi_3$  and  $\mathcal{B}_D = 2e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \frac{1}{2\sqrt{\pi t_0}} \right)$ . In particular,  $I(t) \rightarrow 0$  as  $t \rightarrow t_0 + D + T$  and  $|U(t)| \rightarrow 0$  as  $t \rightarrow t_0 + T$ .



*Proof.* • Boundedness of the norm  $\|z(t, \cdot)\|_{L^2(1,0)}$  in  $[t_0, t_0 + D]$ :

Using the fact that  $v(t, 0) = U(t - D) = 0$  for all  $t \in [t_0, t_0 + D]$ , the solution of (7.20)-(7.24) is given explicitly from [206, Chapter 3] by

$$z(t, x) = 2 \sum_{n=1}^{+\infty} e^{(\lambda - n^2 \pi^2)t} \sin(n\pi x) \int_0^1 \sin(n\pi y) z_0(y) dy. \quad (7.98)$$

Then, we have

$$|z(t, x)| \leq 2 \sum_{n=1}^{+\infty} e^{(\lambda - n^2 \pi^2)t} |\sin(n\pi x)| \int_0^1 |\sin(n\pi y)| |z_0(y)| dy, \quad (7.99)$$

$$\leq 2 \int_0^1 |z_0(y)| dy e^{\lambda t} \sum_{n=1}^{+\infty} e^{-n^2 \pi^2 t}. \quad (7.100)$$

Next, using the Cauchy-Schwartz inequality, we obtain

$$|z(t, x)| \leq 2 \|z_0\|_{L^2(0,1)} e^{\lambda t} \sum_{n=1}^{+\infty} e^{-n^2 \pi^2 t}. \quad (7.101)$$

Now, using the fact that  $t \in (t_0, t_0 + D]$  and the fact that  $\sum_{n=1}^{+\infty} e^{-n^2 \pi^2 t}$  is a convergent series, we obtain

$$|z(t, x)| \leq 2 \|z_0\|_{L^2(0,1)} e^{\lambda(t_0+D)} \sum_{n=1}^{+\infty} e^{-n^2 \pi^2 t_0}, \quad (7.102)$$

$$\leq 2 \|z_0\|_{L^2(0,1)} e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \int_1^{+\infty} e^{-x^2 \pi^2 t_0} dx \right), \quad (7.103)$$

$$\leq 2 \|z_0\|_{L^2(0,1)} e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \int_0^{+\infty} e^{-x^2 \pi^2 t_0} dx \right), \quad (7.104)$$

$$= 2 \|z_0\|_{L^2(0,1)} e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \frac{1}{2\sqrt{\pi t_0}} \right). \quad (7.105)$$

Finally, by squaring and integrating with respect to the variable  $x$  from 0 to 1, we get, for all  $t \in [t_0, t_0 + D]$

$$\|z(t, \cdot)\|_{L^2(0,1)} \leq \mathcal{B}_D \|z_0\|_{L^2(0,1)}, \quad (7.106)$$

with  $\mathcal{B}_D := 2 \|z_0\|_{L^2(0,1)} e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \frac{1}{2\sqrt{\pi t_0}} \right)$ .

• PTS of the closed-loop system of (7.20)-(7.24):

Using (7.15) and (7.90) from Proposition 7.2, we have

$$I(t) = \|z(t, \cdot)\|_{L^2(0,1)} + \|v(t, \cdot)\|_{L^2(0,D)}, \quad (7.107)$$

$$\leq \xi_1(t - t_0 - D) \|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \xi_3(t - t_0 - D) \|z(t_0 + D, \cdot)\|_{L^2(0,1)}, \quad (7.108)$$

$$= \xi_4(t - t_0 - D) \|z(t_0 + D, \cdot)\|_{L^2(0,1)}, \quad (7.109)$$

for all  $t \in [t_0 + D, t_0 + D + T]$ , where  $\xi_4(\cdot) := \xi_1(\cdot) + \xi_3(\cdot)$ .

Next, using inequality (7.106) for  $t = t_0 + D$ , we recover (7.97). In particular, we have that  $I(t) \rightarrow 0$  when  $t \rightarrow t_0 + D + T$ . Furthermore, we deduce that  $\|z(t, \cdot)\|_{L^2(0,1)}$  is bounded for all

## 7.5. Prescribed-time output boundary feedback stabilization in the delay-free case 189

$t \in [t_0, t_0 + D + T)$ .

- Boundedness of  $\|v(t, \cdot)\|_{L^2(0,D)}$  in  $[t_0, t_0 + D]$ :

Notice that  $v$  is given in  $[t_0, t_0 + D] \times [0, D]$  by

$$v(t, y) = \begin{cases} 0, & t \in [t_0, t_0 + D - y], \\ U(t + y - D), & t \in [t_0 + D - y, t_0 + D]. \end{cases} \quad (7.110)$$

Thus, for all  $(t, y) \in [t_0, t_0 + D] \times [0, D]$ , we have

$$|v(t, y)| \leq |U(t + y - D)|. \quad (7.111)$$

Next, by squaring the previous inequality, integrating with respect to  $y$  from 0 to  $D$ , and using the Cauchy-Schwartz inequality, we get

$$\|v(t, \cdot)\|_{L^2(0,D)}^2 \leq \int_0^D |U(t + y - D)|^2 dy, \quad (7.112)$$

$$= \int_0^D |\mathcal{F}(t + y - t_0 - D, \psi(t, \cdot, y))|^2 dy, \quad (7.113)$$

$$= \int_0^D \left( \int_0^1 K(1, s, t + y - t_0 - D) \psi(t, s, y) ds \right)^2 dy, \quad (7.114)$$

$$\leq \int_0^D \|K(1, \cdot, t + y - t_0 - D)\|_{L^2(0,1)}^2 \|\psi(t, \cdot, y)\|_{L^2(0,1)}^2 dy, \quad (7.115)$$

$$= \int_0^D \|K(1, \cdot, t + y - t_0 - D)\|_{L^2(0,1)}^2 \|z(t + y, \cdot)\|_{L^2(0,1)}^2 dy. \quad (7.116)$$

Seeing that  $\|z(t, \cdot)\|_{L^2(0,1)}$  is bounded for all  $t \in [t_0, t_0 + D + T)$  and that  $\|K(1, \cdot, t + y - t_0 - D)\|_{L^2(0,1)}$  is bounded in  $[t_0, t_0 + D]$ , we deduce that  $\|v(t, \cdot)\|_{L^2(0,D)}$  is bounded for all  $t \in [t_0, t_0 + D]$ .

- Convergence of the control to the origin in a prescribed time:

From the equations (7.74) and (7.93), we have for all  $t \geq t_0 + D$

$$|U(t)| = |\mathcal{F}(t - t_0, \psi(t, \cdot, D))| \quad (7.117)$$

$$\leq \xi_2(t - t_0) \|z(t_0 + D, \cdot)\|_{L^2(0,1)}. \quad (7.118)$$

In particular, it is clear that  $U(t) \rightarrow 0$  when  $t \rightarrow t_0 + T$ . This concludes the proof.  $\square$

## 7.5 Prescribed-time output boundary feedback stabilization in the delay-free case

As before, let us first summarize the main results of [166] on prescribed-time output boundary feedback stabilization of the delay-free case system (7.1)-(7.5) when  $D = 0$ .

### 7.5.1 Observer design

Assume that  $D = 0$ . The following observer system was proposed in [166]:

$$\begin{cases} \hat{z}_t(t, x) = \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) + P(x, t - \bar{t}_0, T) \\ \quad \times [z_x(t, 1) - \hat{z}_x(t, 1)], & t \in [t_0, t_0 + D + T), x \in [0, 1], & (7.119) \\ \hat{z}(t, 0) = 0, & t \in [t_0, t_0 + D + T), & (7.120) \\ \hat{z}(t, 1) = U(t), & t \in [t_0, t_0 + D + T), & (7.121) \\ \hat{z}(\bar{t}_0, x) = \hat{z}_0(x), & x \in [0, 1], & (7.122) \end{cases}$$

with observer gain  $P$  given by

$$\begin{aligned} P(x, t - \bar{t}_0, T) := & -\frac{\gamma_3(t - \bar{t}_0)}{2\gamma_{3,0}^2} x \sum_{n=0}^{+\infty} \left( \frac{\gamma_3(t - \bar{t}_0)^{\frac{1}{3}}}{4T\gamma_{3,0}^{\frac{1}{3}}} \right)^n \frac{(-1 - x^2)^n}{(n+1)!} \sum_{j=0}^n \sum_{k=0}^j \frac{1}{j!} \binom{j}{k} \\ & \times \binom{n+2+k}{n-j} \left( \frac{-T\gamma_3(t - \bar{t}_0)^{\frac{2}{3}}}{2\gamma_{3,0}} \right)^j, \end{aligned} \quad (7.123)$$

and  $\gamma_3$  defined in (7.10). The observer state  $\hat{z}(t, \cdot)$  converges to  $z(t, \cdot)$  within the prescribed terminal time  $\bar{t}_0 + T$  provided that

$$\gamma_{3,0}T > \sqrt[3]{4}. \quad (7.124)$$

More precisely, there exist a positive constant  $\alpha_1$  and a positive polynomial function  $Q_1(\cdot)$  in terms of  $\gamma_2(t - \bar{t}_0)$  such that, for any initial conditions  $z(\bar{t}_0, \cdot)$  and  $\hat{z}(\bar{t}_0, \cdot)$ , the following inequality holds for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$ :

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \zeta_1(t - \bar{t}_0) \|z(\bar{t}_0, \cdot) - \hat{z}(\bar{t}_0, \cdot)\|_{L^2(0,1)}, \quad (7.125)$$

where

$$\zeta_1(t - \bar{t}_0) := Q_1(\alpha_1 \gamma_2(t - \bar{t}_0)) e^{-\alpha_1 \gamma_2(t - \bar{t}_0)}. \quad (7.126)$$

In particular,  $\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$  as  $t \rightarrow \bar{t}_0 + T$ .

### 7.5.2 Control design

We recall the following time-varying boundary output control:

$$U(t) := \int_0^1 K(1, s, t - \bar{t}_0) \hat{z}(t, s) ds, \quad (7.127)$$

where the control gain  $K$  is as in (7.13) (subject to (7.14)) and  $\hat{z}(t, \cdot)$  is generated from (7.119)-(7.122).

Using the control (7.127), the closed-loop PDE system (7.1)-(7.5) is prescribed-time stable in the following sense: there exist two positive constants  $\alpha_2$  and  $\alpha_3$  and two positive polynomial functions  $Q_2(\cdot)$  and  $Q_3(\cdot)$  defined in terms of  $\gamma_1(t - \bar{t}_0)$  such that for any for any initial conditions  $z(\bar{t}_0, \cdot)$  and  $\hat{z}(\bar{t}_0, \cdot)$  at initial time  $\bar{t}_0$ , the following inequality holds:

$$\|z(t, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \zeta_2(t - \bar{t}_0) (\|z(\bar{t}_0, \cdot)\|_{L^2(0,1)} + \|\hat{z}(\bar{t}_0, \cdot)\|_{L^2(0,1)}), \quad (7.128)$$

for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$ , where

$$\zeta_2(t - \bar{t}_0) := Q_2(\alpha_2 \gamma_1(t - \bar{t}_0)) e^{-\alpha_2 \gamma_1(t - \bar{t}_0)}. \quad (7.129)$$

Furthermore, we have

$$|U(t)| \leq \zeta_3(t - \bar{t}_0) \|\hat{z}(\bar{t}_0, \cdot)\|_{L^2(0,1)}, \quad (7.130)$$

for all  $t \in [\bar{t}_0, \bar{t}_0 + T)$ , with

$$\zeta_3(t - \bar{t}_0) := Q_3(\alpha_3 \gamma_1(t - \bar{t}_0)) e^{-\alpha_3 \gamma_1(t - \bar{t}_0)}. \quad (7.131)$$

In particular,  $\|\hat{z}(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$ ,  $\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow 0$ , and  $|U(t)| \rightarrow 0$  when  $t \rightarrow \bar{t}_0 + T$ .

## 7.6 Prescribed-time stabilization by output feedback for the input delay case

Following the same lines of Section 7.4, let us adapt our approach to design an observed-based control of (7.73) for (7.1)-(7.5) and its PDE-PDE cascade representation (7.20)-(7.24). We propose the following observer for (7.1)-(7.5):

$$\left\{ \begin{array}{l} \hat{z}_t(t, x) = \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) \\ \quad + P(x, t - t_0, D + T) \\ \quad \times [z_x(t, 1) - \hat{z}_x(t, 1)], \quad t \in [t_0, t_0 + D + T), \quad x \in [0, 1], \quad (7.132) \\ \hat{z}(t, 0) = 0, \quad t \in [t_0, t_0 + D + T), \quad (7.133) \\ \hat{z}(t, 1) = U(t - D), \quad t \in [t_0, t_0 + D + T), \quad (7.134) \end{array} \right.$$

with the observer gain  $P$  is given as in (7.123) where we replace  $T$  by  $D + T$  in the expression of  $\gamma_3(t - t_0)$  to ensure that the convergence of  $\hat{z}(t, \cdot)$  to  $z(t, \cdot)$  is achieved in  $t_0 + D + T$  instead of  $t_0 + T$ . Note that (7.134) can be always expressed using  $v(t, \cdot)$  (i.e.  $\hat{z}(t, 1) = v(t, 0)$ ) as it was done in (7.22).

### Remark 7.7

As our goal is to design an output-feedback control  $U(t)$  for (7.20)-(7.24), we do not need to estimate the dynamics of (7.23)-(7.24) because it is expressed in terms of the control  $U(t)$  which, in turn, is in terms of the observed state  $\hat{z}(t, \cdot)$ . Consequently, it is clear that, if  $\gamma_{3,0}$  satisfies the following condition:

$$\gamma_{3,0}(D + T) > \sqrt[3]{4}, \quad (7.135)$$

then, from (7.125), the following inequality holds:

$$\|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \zeta_1(t - t_0 - D) \|z_0 - \hat{z}_0\|_{L^2(0,1)}, \quad (7.136)$$

for  $t \in [t_0, t_0 + D + T)$  where  $z_0 = z(t_0, \cdot)$  and  $\hat{z}_0 = \hat{z}(t_0, \cdot)$ . In particular,  $\|z(t, \cdot)\|_{L^2(0,1)} \rightarrow \|\hat{z}(t, \cdot)\|_{L^2(0,1)}$  as  $t \rightarrow t_0 + D + T$ .

### 7.6.1 Time-varying infinite-dimensional backstepping transformation

As in Subsection 7.4.1, we consider the following time-varying infinite-dimensional backstepping transformation:

$$\omega(t, y) = v(t, y) - \mathcal{F}(t + y - t_0 - D, \hat{\varphi}(t, \cdot, y)), \quad (7.137)$$

where  $\mathcal{F}$  has the same structure as in (7.27), i.e.,

$$\mathcal{F}(t+y-t_0-D, \hat{\varphi}(t, \cdot, y)) := \int_0^1 K(1, s, t+y-t_0-D) \hat{\varphi}(t, s, y) ds \quad (7.138)$$

and  $\hat{\varphi}$  is chosen to satisfy  $\hat{\varphi}(t, x, y) = \hat{z}(t+y, x)$ , and therefore, is the solution of the following parabolic PDE:

$$\begin{cases} \hat{\varphi}_y(t, x, y) = \hat{\varphi}_{xx}(t, x, y) + \lambda \hat{\varphi}(t, x, y) + P(x, t+y-t_0, D+T) \\ \quad \times [\varphi_x(t, 1, y) - \hat{\varphi}_x(t, 1, y)], & (7.139) \\ \hat{\varphi}(t, 0, y) = 0, & (7.140) \\ \hat{\varphi}(t, 1, y) = v(t, y), & (7.141) \\ \hat{\varphi}(t, x, 0) = \hat{z}(t, x). & (7.142) \end{cases}$$

with  $\varphi$  being generated from (7.28)-(7.31).

## 7.6.2 Target System

Using (7.137), we transform respectively (7.20)-(7.24) and (7.132)-(7.134) into the two following target systems:

$$\begin{cases} z_t(t, x) = z_{xx}(t, x) + \lambda z(t, x), & t \in [t_0, t_0 + D + T], \quad x \in [0, 1], & (7.143) \\ z(t, 0) = 0, & t \in [t_0, t_0 + D + T], & (7.144) \\ z(t, 1) = \omega(t, 0) + \mathcal{F}(t-t_0-D, \hat{z}(t, \cdot)), & t \in [t_0, t_0 + D + T], & (7.145) \\ \omega_t(t, y) = \omega_y(t, y), & t \in [t_0, t_0 + D + T], \quad y \in [0, D], & (7.146) \\ \omega(t, D) = 0, & t \in [t_0, t_0 + D + T], & (7.147) \end{cases}$$

and

$$\begin{cases} \hat{z}_t(t, x) = \hat{z}_{xx}(t, x) + \lambda \hat{z}(t, x) \\ \quad + P(x, t-t_0, D+T) \\ \quad \times [z_x(t, 1) - \hat{z}_x(t, 1)], & t \in [t_0, t_0 + D + T], \quad x \in [0, 1], & (7.148) \\ \hat{z}(t, 0) = 0, & t \in [t_0, t_0 + D + T], & (7.149) \\ \hat{z}(t, 1) = \omega(t, 0) + \mathcal{F}(t-t_0-D, \hat{z}(t, \cdot)), & t \in [t_0, t_0 + D + T]. & (7.150) \end{cases}$$

where  $\omega : [t_0, t_0 + D + T] \times [0, D] \rightarrow \mathbb{R}$  is the transport PDE state.

Note that using the fact that  $\varphi(t, x, y) = z(t+y, x)$  and  $\hat{\varphi}(t, x, y) = \hat{z}(t+y, x)$  for all  $(t, x, y) \in [t_0, t_0 + D + T] \times [0, 1] \times [0, D]$ , it is clear that (7.137) verifies (7.146)-(7.147).

## 7.6.3 Time-varying infinite-dimensional inverse transformation

The inverse transformation is given by,

$$v(t, y) = \omega(t, y) + \mathcal{F}(t+y-t_0-D, \hat{\psi}(t, \cdot, y)), \quad (7.151)$$

where

$$\mathcal{F}(t+y-t_0-D, \hat{\psi}(t, \cdot, y)) := \int_0^1 K(1, s, t+y-t_0-D) \hat{\psi}(t, s, y) ds \quad (7.152)$$

where  $\hat{\psi}$  is the solution of

$$\begin{cases} \hat{\psi}_y(t, x, y) = \hat{\psi}_{xx}(t, x, y) + \lambda \hat{\psi}(t, x, y) + P(x, t + y - t_0, D + T) \\ \quad \times [\psi_x(t, 1, y) - \hat{\psi}_x(t, 1, y)], & (7.153) \\ \hat{\psi}(t, 0, y) = 0, & (7.154) \\ \hat{\psi}(t, 1, y) = \omega(t, y) + \mathcal{F}(t + y - t_0 - D, \hat{\psi}(t, \cdot, y)), & (7.155) \\ \hat{\psi}(t, x, 0) = \hat{z}(t, x). & (7.156) \end{cases}$$

and  $\psi$  is generated from

$$\begin{cases} \psi_y(t, x, y) = \psi_{xx}(t, x, y) + \lambda \psi(t, x, y), & (7.157) \\ \psi(t, 0, y) = 0, & (7.158) \\ \psi(t, 1, y) = \omega(t, y) + \mathcal{F}(t + y - t_0 - D, \hat{\psi}(t, \cdot, y)), & (7.159) \\ \psi(t, x, 0) = z(t, x). & (7.160) \end{cases}$$

Similarly, from the inverse transformation (7.151), we recover (7.23)-(7.24).

#### 7.6.4 Prescribed-time predictor-based output controller

As in Subsection 7.4.4, we recover the expression of the boundary control  $U(t)$  as follows:

$$U(t) = v(t, D) = \mathcal{F}(t - t_0, \hat{\varphi}(t, \cdot, D)) := \int_0^1 K(1, s, t - t_0) \hat{\varphi}(t, s, D) ds, \quad (7.161)$$

from (7.137) at  $y = D$ , where  $\hat{\varphi}$  is generated from (7.28)-(7.31) and  $K$  is given in (7.13). Likewise from (7.151) at  $y = D$ , we can get

$$U(t) = v(t, D) = \mathcal{F}(t - t_0, \hat{\psi}(t, \cdot, D)), \quad (7.162)$$

where  $\hat{\psi}$  is generated from (7.153)-(7.156).

#### Remark 7.8

To achieve exponential output-feedback stabilization, it is sufficient to replace the control gain  $K$  by (7.19) as in Remark 7.4 and the observer gain  $P$  by

$$P_{exp}(x) = -(\lambda + \lambda_0)x \frac{I_1(\sqrt{(\lambda + \lambda_0)(1 - x^2)})}{\sqrt{(\lambda + \lambda_0)(1 - x^2)}}, \quad (7.163)$$

for  $\lambda_0 \geq 0$  and  $x \in [0, 1]$ . The expression of the exponential predictor-based output-feedback controller  $U_{exp}(t)$  is then given by

$$U_{exp}(t) = \int_0^1 K_{exp}(1, s) \hat{\varphi}(t, s, D) ds, \quad (7.164)$$

or

$$U_{exp}(t) = \int_0^1 K_{exp}(1, s) \hat{\psi}(t, s, D) ds, \quad (7.165)$$

with  $\hat{\varphi}$  and  $\hat{\psi}$  are respectively generated from (7.139)-(7.142) and (7.153)-(7.156) using the observer gain  $P_{exp}(x)$  (7.163).

### 7.6.5 Stability analysis

In this subsection, we start by performing a stability analysis on the target system (7.143)-(7.147). Then, by the inverse transformation (7.151), we establish the boundedness of the state of the original system (7.20)-(7.24) and its convergence to zero in a prescribed time using a suitable norm equivalence.

**Proposition 7.3**

Let  $\gamma_{2,0}$  and  $\gamma_{3,0}$  satisfy (7.14) and (7.135) respectively. Then, there exists a rational function  $Q_4(\cdot)$  in terms of  $\gamma_1(t - t_0 - D)$  such that,

$$\int_0^D \zeta_3(t + y - t_0 - D)^2 dy \leq Q_4(\alpha_3 \gamma_1(t - t_0 - D)) e^{-\alpha_3 \gamma_1(t - t_0 - D)}. \quad (7.166)$$

where  $\gamma_1(\cdot)$  and  $\zeta_3(\cdot)$  given in (7.10) and (7.131) respectively.

*Proof.* Let  $B$  be defined as follows,

$$B = \int_0^D \zeta_3(t + y - t_0 - D)^2 dy, \quad (7.167)$$

$$:= \int_0^D Q_3(\alpha_3 \gamma_1(t + y - t_0 - D)) e^{-\alpha_3 \gamma_1(t + y - t_0 - D)} dy, \quad (7.168)$$

where  $\zeta_3$  is given in (7.129).

Next, let us consider the following change of variables:

$$\begin{aligned} s &= \alpha_3 \gamma_1(t + y - t_0 - D), \\ &= \frac{\alpha_3 \gamma_{1,0} T}{(t_0 + D + T - t - y)}, \end{aligned} \quad (7.169)$$

from which we recover,

$$\begin{aligned} \frac{ds}{dy} &= \frac{\alpha_3 \gamma_{1,0} T}{(t_0 + D + T - t - y)^2}, \\ &= \frac{\alpha_3}{\gamma_{1,0} T} \gamma_2(t + y - t_0 - D), \\ &= \frac{s^2}{\alpha_3 \gamma_{1,0} T}. \end{aligned} \quad (7.170)$$

Now, by using (7.169)-(7.170) in (7.168) (along with  $dy = \alpha_3 \gamma_{1,0} T \frac{ds}{s^2}$ ), we obtain

$$B = \alpha_3 \gamma_{1,0} T \int_{\alpha_3 \gamma_1(t - t_0 - D)}^{\alpha_3 \gamma_1(t - t_0)} \frac{Q_3(s)}{s^2} e^{-s} ds. \quad (7.171)$$

By noticing that the polynomial function  $Q_3(\cdot)$  (of degree  $p \in \mathbb{N}^*$ ) can be expressed as  $Q_3(s) = \sum_{i=0}^p c_i s^i$  with some positive coefficients  $c_i > 0$ , we get,

$$B = \alpha_3 \gamma_{1,0} T \sum_{i=0}^p c_i \int_{\alpha_3 \gamma_1(t - t_0 - D)}^{\alpha_3 \gamma_1(t - t_0)} s^{i-2} e^{-s} ds. \quad (7.172)$$

Now, let us calculate each sub-integral of the previous expression. To do that, let us consider the following two cases:

Case 1: Let  $i \in \{2, 3, \dots, p\}$  (i.e.  $i - 2 \geq 0$ ). Then, by using multiple integrations by parts, we obtain,

$$\int_{\alpha_3 \gamma_1(t-t_0-D)}^{\alpha_3 \gamma_1(t-t_0)} s^{i-2} e^{-s} ds = \left[ -(i-2)! \sum_{j=0}^{i-2} \frac{s^j}{j!} e^{-s} \right]_{\alpha_3 \gamma_1(t-t_0-D)}^{\alpha_3 \gamma_1(t-t_0)} \quad (7.173)$$

$$\leq (i-2)! \sum_{j=0}^{i-2} \frac{(\alpha_3 \gamma_1(t-t_0-D))^j}{j!} e^{-\alpha_3 \gamma_1(t-t_0-D)}. \quad (7.174)$$

Note that proving (7.173) is straightforward by induction.

Case 2: Let  $i \in \{0, 1\}$  (i.e.  $i - 2 < 0$ ). Then, by using the generalized exponential, defined as

$$E_n(r) = r^{n-1} \int_r^\infty \frac{e^{-s}}{s^n} ds, \quad r > 0, n \in \mathbb{N}^*, \quad (7.175)$$

and its property [202, Section 2]:

$$\frac{e^{-r}}{n+r} \leq E_n(r) \leq \frac{e^{-r}}{n-1+r}, \quad (7.176)$$

we obtain,

$$\int_{\alpha_3 \gamma_1(t-t_0-D)}^{\alpha_3 \gamma_1(t-t_0)} \frac{e^{-s}}{s^{2-i}} ds \leq \int_{\alpha_3 \gamma_1(t-t_0-D)}^{+\infty} \frac{e^{-s}}{s^{2-i}} ds \quad (7.177)$$

$$:= \frac{1}{(\alpha_3 \gamma_1(t-t_0-D))^{1-i}} E_{2-i}(\alpha_3 \gamma_1(t-t_0-D)), \quad (7.178)$$

$$\leq \frac{e^{-\alpha_3 \gamma_1(t-t_0-D)}}{(1-i + \alpha_3 \gamma_1(t-t_0-D))(\alpha_3 \gamma_1(t-t_0-D))^{1-i}}. \quad (7.179)$$

Finally, by using (7.174) from Case 1 and (7.179) from Case 2, we recover,

$$B \leq Q_4(\alpha_3 \gamma_1(t-t_0-D)) e^{-\alpha_3 \gamma_1(t-t_0-D)}, \quad (7.180)$$

where

$$Q_4(s) = \alpha_3 \gamma_{1,0} T \left( \sum_{i=0}^1 \frac{c_i}{(1-i+s)s^{1-i}} + \sum_{i=2}^p c_i (i-2)! \sum_{j=0}^{i-2} \frac{s^j}{j!} \right). \quad (7.181)$$

This concludes the proof.  $\square$

#### Proposition 7.4

For the transport PDE  $v(t, x)$  satisfying (7.23), there exists a positive polynomial function  $Q_4(\cdot)$  in terms of  $\gamma_1(t-t_0-D)$  such that the following estimate holds for  $t \in [t_0 + D, t_0 + D + T]$ :

$$\|v(t, \cdot)\|_{L^2(0,D)} \leq \zeta_4(t-t_0-D) (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)}), \quad (7.182)$$

where

$$\zeta_4(t-t_0-D)^2 := Q_4(\alpha_3 \gamma_1(t+y-t_0-D)) e^{-\alpha_3 \gamma_1(t+y-t_0-D)}, \quad (7.183)$$

with  $\gamma_1(\cdot)$  given in (7.10). In particular, it holds  $\|v(t, \cdot)\|_{L^2(0,D)} \rightarrow 0$  for all  $t \rightarrow t_0 + D + T$ .



*Proof.* Let  $t \in [t_0 + D, t_0 + D + T)$ . Then, from (7.151) and using the fact that  $\omega(t, y) = 0, \forall (t, y) \in [t_0 + D, t_0 + D + T) \times [0, D]$ , we recover

$$v(t, y) = \mathcal{F}(t + y - t_0 - D, \hat{\psi}(t, \cdot, y)). \quad (7.184)$$

Next, by squaring the previous equality and using the fact that  $\hat{\psi}(t, \cdot, y) = \hat{z}(t + y, \cdot)$  and using (7.130) for  $t_0 := t_0 + D$ , we obtain,

$$|v(t, y)|^2 \leq \zeta_3(t + y - t_0 - D)^2 (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)})^2. \quad (7.185)$$

Now, by integrating from 0 to  $D$  with respect to  $y$  and using (7.166) in Proposition 7.3, we obtain,

$$\|v(t, \cdot)\|_{L^2(0,D)}^2 \leq \int_0^D \zeta_3(t + y - t_0 - D)^2 dy (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)})^2, \quad (7.186)$$

$$\leq \zeta_4(t - t_0 - D)^2 (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)})^2. \quad (7.187)$$

where

$$\zeta_4(t - t_0 - D)^2 := Q_4(\alpha_3 \gamma_1(t - t_0 - D)) e^{-\alpha_3 \gamma_1(t - t_0 - D)}. \quad (7.188)$$

Finally, by passing to the square roots,

$$\|v(t, \cdot)\|_{L^2(0,D)} \leq \zeta_4(t - t_0 - D) (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)}). \quad (7.189)$$

In particular, we get  $\|v(t, \cdot)\|_{L^2(0,D)} \rightarrow 0$  as  $t \rightarrow t_0 + D + T$ . □

Let us now give our second main result,

**Theorem 7.2**

Let  $\gamma_{2,0} > 0$ , and  $\gamma_{3,0} > 0$  be chosen such that (7.14) and (7.135) are ensured. Let  $T > 0$ ,  $D > 0$  and  $t_0 \geq 0$ . Then, the solution of the closed-loop system (7.20)-(7.24) with the observer (7.132)-(7.134) and the prescribed-time time-varying output control (7.161) (or (7.162)) is prescribed-time stable in the following sense: For any initial conditions  $z_0$  and  $\hat{z}_0$ , the quantities  $\|z(t, \cdot)\|_{L^2(0,1)}$ ,  $\|\hat{z}(t, \cdot)\|_{L^2(0,1)}$ , and  $\|v(t, \cdot)\|_{L^2(0,D)}$  remain bounded for all  $t \in [t_0, t_0 + D]$ ; and for all  $t \in [t_0 + D, t_0 + D + T)$ , the following norm  $I(t) = \|z(t, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t, \cdot)\|_{L^2(0,1)} + \|v(t, \cdot)\|_{L^2(0,D)}$  satisfies,

$$I(t) \leq L_D \zeta_5(t - t_0 - D) (\|z_0\|_{L^2(0,1)} + \|\hat{z}_0\|_{L^2(0,1)}), \quad (7.190)$$

where  $\zeta_5(\cdot) := \zeta_2(\cdot) + \zeta_4(\cdot)$ ,  $L_D := (\mathcal{B}_D + \bar{\mathcal{B}}_D) > 0$ ,  $\mathcal{B}_D := 2e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \frac{1}{2\sqrt{\pi t_0}} \right)$  and  $\bar{\mathcal{B}}_D := \mathcal{B}_D + \sup_{s \in [t_0, t_0+D]} \zeta_1(s - t_0 - D)$ . In particular,  $I(t) \rightarrow 0$  as  $t \rightarrow t_0 + D + T$  and  $|U(t)| \rightarrow 0$  as  $t \rightarrow t_0 + T$ .

*Proof.* • Boundedness of the two norms  $\|z(t, \cdot)\|_{L^2(1,0)}$  and  $\|\hat{z}(t, \cdot)\|_{L^2(1,0)}$  in  $[t_0, t_0 + D]$ :

As in the proof of Theorem 7.1, it is clear that:

$$\|z(t, \cdot)\|_{L^2(0,1)} \leq \mathcal{B}_D \|z_0\|_{L^2(0,1)}, \quad (7.191)$$

$$\leq \mathcal{B}_D (\|z_0\|_{L^2(0,1)} + \|\hat{z}_0\|_{L^2(0,1)}), \quad (7.192)$$

in  $[t_0, t_0 + D]$ , with  $\mathcal{B}_D = 2e^{\lambda(t_0+D)} \left( e^{-\pi^2 t_0} + \frac{1}{2\sqrt{\pi t_0}} \right)$ .

Moreover, using (7.136) and (7.192), we obtain

$$\|\hat{z}(t, \cdot)\|_{L^2(0,1)} \leq \|z(t, \cdot)\|_{L^2(0,1)} + \|z(t, \cdot) - \hat{z}(t, \cdot)\|_{L^2(0,1)}, \quad (7.193)$$

$$\leq \mathcal{B}_D (\|z_0\|_{L^2(0,1)} + \|\hat{z}_0\|_{L^2(0,1)}) + \zeta_1(t - t_0 - D) \|z_0 - \hat{z}_0\|_{L^2(0,1)}, \quad (7.194)$$

$$\leq \bar{\mathcal{B}}_D (\|z_0\|_{L^2(0,1)} + \|\hat{z}_0\|_{L^2(0,1)}), \quad (7.195)$$

with  $\bar{\mathcal{B}}_D := \mathcal{B}_D + \sup_{s \in [t_0, t_0+D]} \zeta_1(s - t_0 - D)$ .

- PTS of the closed-loop system of (7.20)-(7.24):

Using (7.128) and (7.182) from Proposition 7.4, we have

$$I(t) = \|z(t, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t, \cdot)\|_{L^2(0,1)} + \|v(t, \cdot)\|_{L^2(0,D)}, \quad (7.196)$$

$$\begin{aligned} &\leq \zeta_2(t - t_0 - D) (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)}) \\ &\quad + \zeta_4(t - t_0 - D) (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)}), \end{aligned} \quad (7.197)$$

$$= \zeta_5(t - t_0 - D) (\|z(t_0 + D, \cdot)\|_{L^2(0,1)} + \|\hat{z}(t_0 + D, \cdot)\|_{L^2(0,1)}), \quad (7.198)$$

for all  $t \in [t_0 + D, t_0 + D + T)$ , where  $\zeta_5(\cdot) := \zeta_2(\cdot) + \zeta_4(\cdot)$ . Next, using (7.192) and (7.195) for  $t = t_0 + D$ , we recover (7.190). In particular, we have that  $I(t) \rightarrow 0$  when  $t \rightarrow t_0 + D + T$ . Furthermore, we deduce that  $\|z(t, \cdot)\|_{L^2(0,1)}$  and  $\|\hat{z}(t, \cdot)\|_{L^2(0,1)}$  are bounded for all  $t \in [t_0, t_0 + D + T)$ .

- Boundedness of  $\|v(t, \cdot)\|_{L^2(0,D)}$  in  $[t_0, t_0 + D]$ :

Notice that  $v$  is given in  $[t_0, t_0 + D] \times [0, D]$  by

$$v(t, y) = \begin{cases} 0, & t \in [t_0, t_0 + D - y], \\ U(t + y - D), & t \in [t_0 + D - y, t_0 + D]. \end{cases} \quad (7.199)$$

Thus, for all  $(t, y) \in [t_0, t_0 + D] \times [0, D]$ , we have

$$|v(t, y)| \leq |U(t + y - D)|. \quad (7.200)$$

Next, by squaring the previous inequality, integrating with respect to  $y$  from 0 to  $D$ , and passing to the square roots, we get

$$\|v(t, \cdot)\|_{L^2(0,D)}^2 \leq \int_0^D |U(t + y - D)|^2 dy, \quad (7.201)$$

$$= \int_0^D |\mathcal{F}(t + y - t_0 - D, \hat{\psi}(t, \cdot, y))|^2 dy, \quad (7.202)$$

$$= \int_0^D \left( \int_0^1 K(1, s, t + y - t_0 - D) \hat{\psi}(t, s, y) ds \right)^2 dy, \quad (7.203)$$

$$\leq \int_0^D \|K(1, \cdot, t + y - t_0 - D)\|_{L^2(0,1)}^2 \|\hat{\psi}(t, \cdot, y)\|_{L^2(0,1)}^2 dy, \quad (7.204)$$

$$= \int_0^D \|K(1, \cdot, t + y - t_0 - D)\|_{L^2(0,1)}^2 \|\hat{z}(t + y, \cdot)\|_{L^2(0,1)}^2 dy. \quad (7.205)$$

Seeing that  $\|\hat{z}(t, \cdot)\|_{L^2(0,1)}$  is bounded for all  $t \in [t_0, t_0 + D + T)$  and that  $\|K(1, \cdot, t - t_0 - D)\|_{L^2(0,1)}$  is bounded in  $[t_0, t_0 + D]$ , we deduce that  $\|v(t, \cdot)\|_{L^2(0,D)}$  is bounded for all  $t \in [t_0, t_0 + D]$ .

- Convergence of the control to the origin in a prescribed time:

From the expressions (7.162) and (7.185), we have for all  $t \geq t_0 + D$

$$|U(t)| = |\mathcal{F}(t - t_0, \hat{\psi}(t, \cdot, D))|, \tag{7.206}$$

$$\leq \zeta_3(t - t_0) \|z(t_0 + D, \cdot)\|_{L^2(0,1)}. \tag{7.207}$$

In particular, it is clear that  $U(t) \rightarrow 0$  when  $t \rightarrow t_0 + T$ . This concludes the proof.  $\square$

## 7.7 Numerical results

In this section, we give numerical simulations for the closed-loop system (7.20)-(7.24) with prescribed-time predictor-based output-feedback controller  $U(t)$  given in (7.161) used to attain prescribed-time stabilization. We take the delay  $D = 0.5\text{s}$ , the reaction coefficient  $\lambda = 11$ , the initial time  $t_0 = 0$ , the prescribed time  $T = 1\text{s}$  and the initial conditions:  $z_0(x) = 10(x - x^2)$ ,  $\hat{z}_0(x) = 0$ ,  $v_0(y) = 0$ . We take the initial conditions of the blow-up functions  $\gamma_2$  and  $\gamma_3$ , given in (7.10), respectively as  $\gamma_2(0) = 3.3$  and  $\gamma_3(0) = 2.2$ .

For the numerical simulations, we implement an implicit Euler scheme for the parabolic subsystems combined with the two-step Lax–Wendroff method for the hyperbolic subsystems. The discretization with respect to space and time is done with steps  $\Delta x = 10^{-2}$ ,  $\Delta y = 2 \times 10^{-3}$ ,  $\Delta t = 10^{-3}$  for the interval  $[t_0, 0.7(t_0 + D + T))$  and  $\Delta t = 5 \times 10^{-5}$  for the interval  $[0.7(t_0 + D + T), t_0 + D + T)$ .

Figure 7.1 shows on the top left the evolution of  $z(t, x)$ , the state of reaction-diffusion PDE (7.20)-(7.22), on the top right the evolution of  $\hat{z}(t, x)$ , the observer state of reaction-diffusion PDE (7.132)-(7.134), on the bottom left the evolution of  $v(t, y)$ , the state of the transport PDE (7.23)-(7.24) and on the bottom right the evolution of  $\omega(t, y)$  the state of the transport PDE (7.53)-(7.54), with the prescribed-time predictor-based output controller  $U(t)$  given in (7.161). Figure 7.2 shows in a logarithmic scale the evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2 + \|\hat{z}(t, \cdot)\|_{L^2(0,1)}^2 + \|v(t, \cdot)\|_{L^2(0,D)}^2$  of the closed-loop system (7.20)-(7.24) with the prescribed-time predictor-based output-feedback controller  $U(t)$  given in (7.161) for the initial condition  $10z_0(x)$  and for 3 different delays:  $D = 0.5\text{s}$ ,  $D = 0.6\text{s}$ , and  $D = 0.7\text{s}$ . As it can be observed, the norm of the closed-loop system (7.20)-(7.24) converges to the origin in a prescribed time equal to  $t_0 + D + T$  no matter what delay we take.

Finally, in the upper left and right of Figure 7.3, we give a comparison between the norm of the closed-loop system (7.20)-(7.24), for a delay  $D = 0.5\text{s}$ , using the prescribed-time predictor-based output-feedback controller  $U(t)$  given in (7.161) (in red solid line) and the same norm using the exponential predictor-based controller  $U_{\text{exp}}(t)$  given in (7.164) with the constant-gain  $\lambda + \lambda_0 = 11$  involved in the control and observer kernels  $K_{\text{exp}}$  and  $P_{\text{exp}}$  given in Remark 7.8 (in black dashed line). In particular, we compare the case with a higher constant-gain  $\lambda + \lambda_0 = 28$  (in blue dashed line). On the bottom of Figure 7.3, we give a comparison between the controllers  $U(t)$  (in red solid line) and  $U_{\text{exp}}(t)$  with the gain  $\lambda + \lambda_0 = 11$  (in black dashed line) and with a higher gain  $\lambda + \lambda_0 = 28$  (in blue dashed line).

As it can be observed, at the delay  $D = 0.5\text{s}$ , the norm of the closed-loop system (7.20)-(7.24) using the exponential controller  $U_{\text{exp}}(t)$  (with the higher gain  $\lambda + \lambda_0 = 28$ ) exhibits the "peaking phenomenon" [207] (see the blue dashed curve in the upper left plot of Figure 7.3). After the delay  $D = 0.5\text{s}$ , the norm outpaces the same norm using the prescribed-time controller  $U(t)$ . However, as time progresses, the curves of the two norms cross, and from then on the norm of the closed-loop system (7.20)-(7.24) using the prescribed-time controller  $U(t)$  outperforms the same norm

using the exponential controller  $U_{\text{exp}}(t)$ . This is due to the fact that the exponential controller starts with an aggressive control effort, because of the high gain  $\lambda + \lambda_0$ , but with time its effort diminishes (see bottom left of Figure 7.3). In contrast, the prescribed-time controller  $U(t)$  starts with a moderate effort to avoid peaking and then gradually increases its control effort towards the end of the simulation to ensure that the convergence is completed in the prescribed time (see the upper left and bottom plots of Figure 7.3).

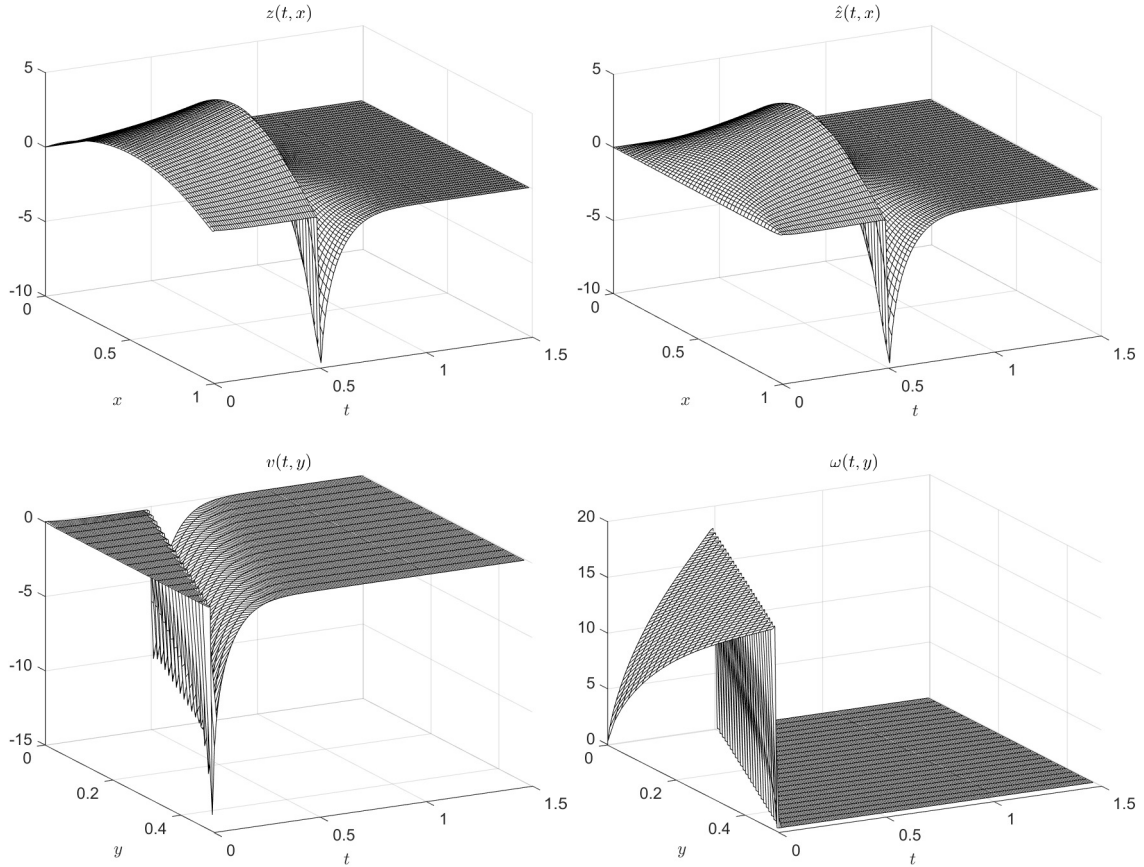


Figure 7.1: On the top left: the evolution of the state  $z(t, x)$  of the parabolic PDE (7.20)-(7.22) for the initial condition  $\hat{z}(t_0, x) = 10(x - x^2)$ ,  $x \in [0, 1]$ . On the top right: the evolution of the observer state  $\hat{z}(t, x)$  of the parabolic PDE (7.132)-(7.134) for the initial condition  $\hat{z}(t_0, x) = 0$ ,  $x \in [0, 1]$ . On the bottom left: the evolution of  $v(t, y)$  the state of the hyperbolic PDE (7.23)-(7.24) with prescribed-time predictor-based controller  $U(t)$  in (7.161) for the initial condition  $v(t_0, y) = 0$ ,  $y \in [0, D]$ , with a delay  $D = 0.5\text{s}$ . On the bottom right: the evolution of  $\omega(t, y)$  the state of the transport PDE (7.53)-(7.54).

## conclusion

In this chapter, we dealt with the problem of output-feedback stabilization in **prescribed time** of a linear 1D reaction-diffusion PDE with boundary input delay. To apply this approach, the PDE was rewritten into a cascade of the parabolic reaction-diffusion PDE with a hyperbolic PDE. Using an invertible time-varying infinite-dimensional backstepping transformation coupled with advanced

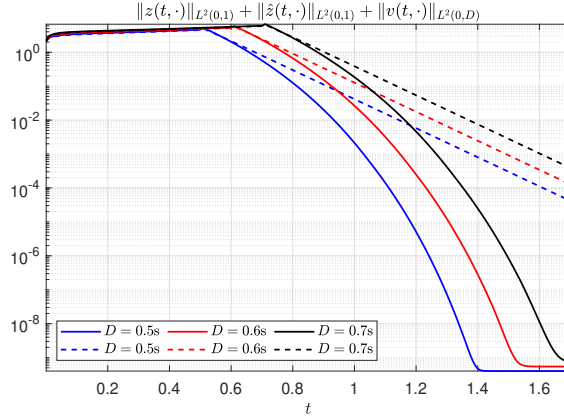


Figure 7.2: The evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2 + \|\hat{z}(t, \cdot)\|_{L^2(0,1)}^2 + \|v(t, \cdot)\|_{L^2(0,D)}^2$  of the closed-loop system (7.20)-(7.24) with prescribed-time predictor-based controller  $U(t)$  (7.161) in solid lines and with exponential predictor-based controller  $U_{\text{exp}}(t)$  (7.76) for  $\lambda_0 = 0$  in dashed lines (logarithmic scale), for the initial conditions  $z(t_0, x) = 10(x - x^2)$ ,  $\hat{z}(t_0, x) = 0$ , and  $v(t_0, y) = 0$ , and for 3 different delays:  $D = 0.5\text{s}$  in blue;  $D = 0.6\text{s}$  in red;  $D = 0.7\text{s}$  in black.

predictor-based concepts, we transformed the PDE-PDE unstable system into a well-chosen **PTS** target system. Using the inverse transformation we ensured the desired prescribed-time stability property for the original system.

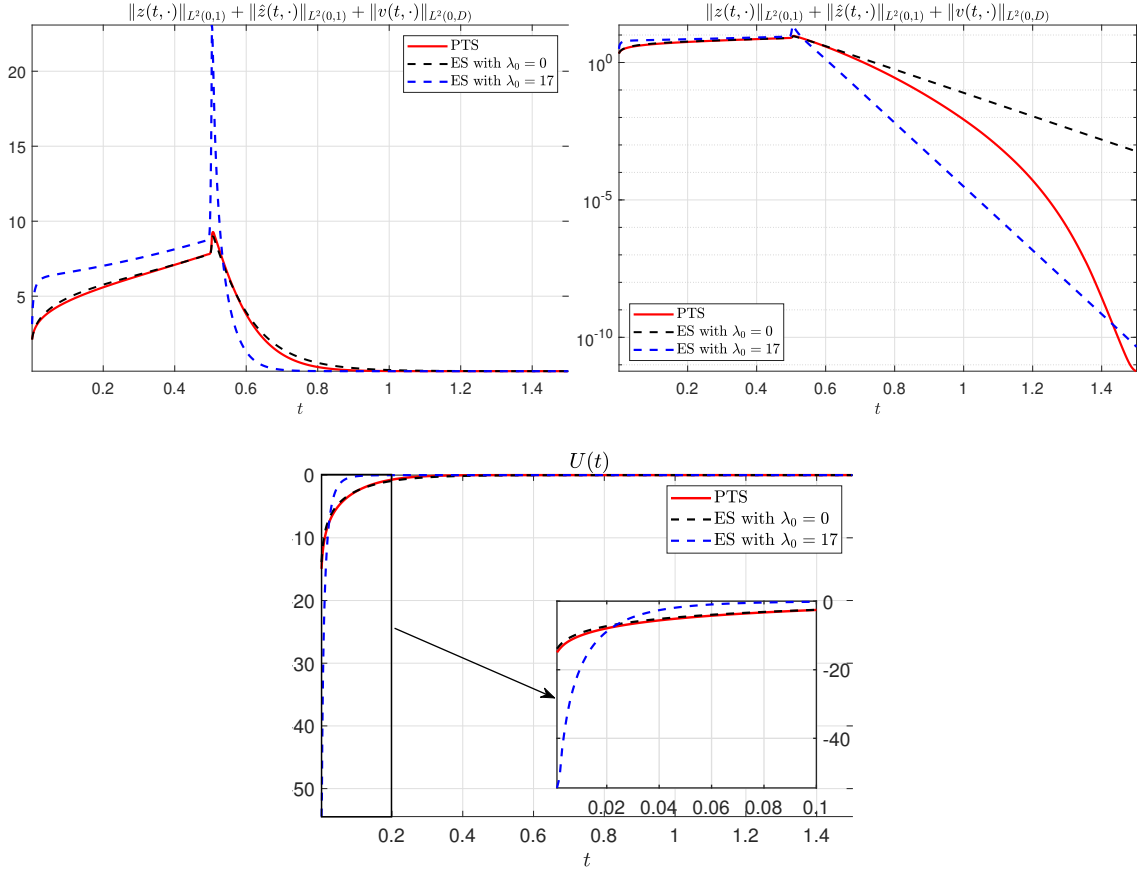


Figure 7.3: On the top: The evolution of the norm  $\|z(t, \cdot)\|_{L^2(0,1)}^2 + \|\hat{z}(t, \cdot)\|_{L^2(0,1)}^2 + \|v(t, \cdot)\|_{L^2(0,D)}^2$  of the closed-loop system (7.20)-(7.24) with prescribed-time predictor-based controller  $U(t)$  (7.161) in red solid line and with exponential predictor-based controller  $U_{\text{exp}}(t)$  (7.164) for  $\lambda_0 = 0$  in black dashed line and  $\lambda_0 = 17$  in blue dashed line (normal scale on the left and logarithmic scale on the right), for the initial conditions  $z(t_0, x) = 10(x - x^2)$ ,  $\hat{z}(t_0, x) = 0$ , and  $v(t_0, y) = 0$ , and for a delay  $D = 0.5$ s. On the bottom, the evolution of the applied prescribed-time predictor-based controller  $U(t)$  (7.161) in red solid line along with the evolution of the exponential predictor-based controller  $U_{\text{exp}}(t)$  (7.164) for  $\lambda_0 = 0$  in black dashed line and  $\lambda_0 = 17$  in blue dashed line.



# Conclusion and perspectives

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## Conclusion

In this thesis, we have investigated the problems of stabilization and estimation in finite, fixed, and prescribed time for two distinct classes of infinite-dimensional systems starting with linear time-invariant (LTI) systems subject to pointwise/distributed input/sensor delays and ending up with reaction-diffusion PDEs systems with boundary control. To deal with the first class of systems, we extended the well-developed finite-dimensional tools (homogeneity-based and time-varying tools) - applied for LTI systems - to fit into the infinite-dimensional settings; In the process, we improved some of the classical infinite-dimensional methods including state prediction methods and the backstepping approach for PDEs. For the second class of systems, we presented a novel approach, inspired by classical ideas in relation to *Control Lyapunov Functions* (CLF). This approach not only helps in finite/fixed-time boundary state-dependent feedback stabilization but can be potentially extended to attenuation/rejection of control matched disturbances, finite/fixed-time tracking, and Non-Asymptotic boundary stabilization of reaction-diffusion with a nonlinear reaction term. However, extensions of this approach to observer design are not straightforward, and the problem of the well-posedness of the closed-loop system is challenging due to the nonlinearity of the boundary control as well as its non-differentiability at zero. Finally, we proposed a novel generic approach, extending the previously used infinite-dimensional techniques of state predictions and PDE backstepping method to the more challenging problem of prescribed-time output-feedback boundary stabilization of reaction-diffusion PDEs.

More precisely, the main contributions of this thesis are as follows:

- In Chapter 3, we dealt with the problem of finite-time estimation of linear time-invariant (LTI) systems, in the observable form, with delayed output. The main ideas relied on rewriting the system into an ODE-PDE cascade setting, where the PDE part modeled the effect of the delay on the output. The nonlinear gains were designed such that the error observer system is either **FTS** or **FxTS**. To achieve this, we used the backstepping approach where we chose a suitable nonlinear target system satisfying a chosen finite/fixed-time convergence property. Finally, we used the invertibility of the backstepping transformation to pass this property to the error system.
- In Chapter 4, we solved the problem of finite/fixed-time stabilization of a chain of integrators with input delay. The chain of integrators was rewritten into an ODE-PDE setting, where the PDE part modeled the effect of the delay on the input. To solve this problem, we extended the well-developed finite-dimensional homogeneity-based tools to fit into the infinite-dimensional settings and combined them with the PDE backstepping approach. The predictor-based controller was designed using a nonlinear infinite-dimensional backstepping transformation that linked the ODE-PDE setting to the well-chosen finite/fixed-time stable target system.



The finite/fixed-time stability property was transferred back to the original system by the inverse transformation and using  $\mathcal{GKL}$  functions.

- In Chapter 5, we treated the problem of nonlinear boundary stabilization, with any predefined type of convergence, for a class of 1D reaction-diffusion systems. To achieve this, we used the “spatially weighted  $L^2$ -norm” as a Lyapunov functional candidate  $V$ . By taking the time derivative of this functional along the trajectories of the reaction-diffusion PDE  $\frac{d}{dt}V(z(t, \cdot))$ , we noticed that we got an inequality that relates  $\frac{d}{dt}V(z(t, \cdot))$  to a second-degree polynomial involving the control  $U(t)$  subtracted by the term  $\mathcal{K}(V(z(t, \cdot)))$  (for any continuous function  $\mathcal{K}$  such that  $\mathcal{K}(0) = 0$ ). By computing the root of this polynomial, we managed to design a nonlinear control  $U(t)$  and to obtain the inequality  $\frac{d}{dt}V(z(t, \cdot)) \leq -\mathcal{K}(V(z(t, \cdot)))$ . Using this last inequality, for well-chosen expressions of  $\mathcal{K}$ , we ensured finite/fixed-time stabilization of the reaction-diffusion PDE. The present chapter did not study the existence/uniqueness issues of the solutions of the closed-loop system. To this purpose, ideas contained in [198], [199] can be used. However, the obtained stability estimates will certainly help the analysis.
- In Chapter 6, we extended the existing results of prescribed-time delay-compensation and stabilization of LTI systems with input delay to distributed input delay. The main ideas were developed first for a scalar LTI system with distributed input delay and after that generalized to the  $n$ th-dimensional LTI case. The prescribed-time predictor feedback design was achieved based on the backstepping approach using a time-varying backstepping-forwarding transformation and a reduction-based change of variables.
- Finally, in Chapter 7, we extended the results of the previously presented delay compensation techniques from nonlinear ODEs with input delay to linear PDEs with boundary input delay. Using the developed approach, we tackled the problem of output-feedback stabilization in **prescribed time** of a linear 1D reaction-diffusion PDE with boundary input delay. To apply this approach, the PDE was rewritten into a cascade of the parabolic reaction-diffusion PDE with a hyperbolic PDE. Using an invertible time-varying infinite-dimensional backstepping transformation coupled with advanced predictor-based concepts, we transformed the PDE-PDE unstable system into a well-chosen **PTS** target system. Using the inverse transformation we ensured the desired prescribed-time stability property for the original system.

## Perspectives

In conclusion, this thesis has made significant progress in studying the Non-Asymptotic stabilization and estimation problems of some classes of infinite-dimensional systems. However, it is important to note that there are still many open problems worth exploring. We list some of the possible directions that can be further studied in the future.

- In light of the results obtained in Chapter 4, it is natural to consider extending the results of Chapter 6 from prescribed-time stabilization to finite/fixed-time stabilization of LTI systems with distributed delay. Afterward, one can try to combine the state-dependent tools, developed for instance in Chapter 4, with time-varying tools to the challenging problem of the “robustification” of the predictor-based prescribed-time controllers, with respect to external disturbances, for the distributed delay case as well as for the pointwise delay case studied in [176], following the same lines of [136].

- An interesting direction, would be to extend the results of Chapters 3, 4, 6 and 7 to more complex interconnected dynamics namely nonlinear ODEs or linear parabolic PDEs that are interconnected with hyperbolic quasilinear/semilinear PDEs with non-local terms for which we can also use the results in [147].
- It could be also interesting to continue working on ODEs with input/output delay but consider other types of delays including stochastic, time-variant, state-dependent or input-dependent delays, where the lastly mentioned type of delays is defined implicitly through an integral of the past input values. This type of delay arises for instance in microfluidic processes involving the Zweifach-Fung effect (see [208], [209]).
- Another interesting application would be to address the design and analysis of multi-variable finite/fixed-time extremum seeking for static maps subject to arbitrarily long time delays using the techniques developed in Chapter 4 and extending the results [210] to delay-dependent case inspired by [211].
- Finally, it could be interesting to investigate in depth the Lyapunov-based method introduced in Chapter 5, solve its issue and limitations, and extend it to more general complex high-order PDEs starting with more general classes of nonlinear reaction-diffusion-advection systems. One can also try to apply this approach for global stabilization by boundary feedback of nonlinear parabolic PDEs that may blow up in the absence of the control. One could also attempt to mix the results of Chapter 7 with the finite/fixed-time controller introduced in Chapter 5 to deal with the problem of finite/fixed-time delay compensation for these classes of PDEs.



# Appendix

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Due to the difficulty of constructing Lyapunov functions for nonlinear systems, especially in the framework of Non-Asymptotic stability, alternative techniques such as homogeneity theory can be valuable tools for stability analysis. Homogeneity theory is a technique that allows local properties of a system to be extended globally. It is used to prove stability without the need for a Lyapunov function. It is based on the idea that the behavior of a nonlinear system can be characterized by its degree of homogeneity.

## A.1 Standard homogeneity

Let us start by recalling the notion of standard homogeneity,

**Definition A.1** ([212, Section 17])

Let  $n$  and  $m$  be two positive integers. If a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  has the following property:

$$f(\lambda z) = \lambda^k f(z), \quad (\text{A.1})$$

for all  $z \in \mathbb{R}^n$ , all real number  $\lambda \neq 0$  and some real number  $k \in \mathbb{R}$ , then  $f$  is said to be a homogeneous function of degree  $k$ .

**Example A.1**

All linear functions are homogeneous of degree 1.

**Example A.2**

The scalar function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(z) = -\text{sign}(z), \quad (\text{A.2})$$

is homogeneous of degree 0.

**Example A.3**

The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(z_1, z_2) = \begin{cases} \frac{z_1^{\alpha_1} + z_2^{\alpha_1}}{z_1^{\alpha_2} + z_2^{\alpha_2}}, & \text{if } z_1^{\alpha_2} + z_2^{\alpha_2} \neq 0, \\ 0, & \text{else,} \end{cases} \quad (\text{A.3})$$

is homogeneous of degree  $\alpha_1 - \alpha_2$ . Notice that  $f$  is discontinuous if  $\alpha_1 \leq \alpha_2$ .

**Theorem A.1** ( see [212], [213])

If the system (2.1) is homogeneous and if the origin of (2.1) is locally asymptotically stable (**AS**), then it is globally **AS**.

**Theorem A.2** ([214])

If the system (2.1) is homogeneous, then the origin of (2.1) is globally AS if and only if there exists a homogeneous and continuous function  $V$ , of class  $C^1$  on  $\mathbb{R}^n \setminus \{0\}$ , such that

- $V(0) = 0$  and  $V(z) > 0$  for all  $z \in \mathbb{R}^n \setminus \{0\}$  ( $V$  is positive-definite);
- $\dot{V}(z) < 0$  all  $z \in \mathbb{R}^n \setminus \{0\}$  ( $\dot{V}$  is negative-definite);
- $V(z) \rightarrow +\infty$  as  $\|z\|_2 \rightarrow +\infty$  ( $V$  is radially unbounded).

## A.2 Weighted Homogeneity

In the late 1950s, weighted Homogeneity appeared for the first time in V. I. Zubov's paper [70]. In 1986, this concept reappeared in H. Hermes' paper [215], and later on in many other papers including [216], [217]. Weighted homogeneity enlarges the concept of standard homogeneity by allowing the multiplicative factor  $\lambda$  to have different powers for each coordinate. Therefore, the dilation is different from the one used in the standard case.

Now, let us introduce the following definitions:

**Definition A.2**

Given some vector of weights  $\bar{r} = [\bar{r}_1, \dots, \bar{r}_n] \in \mathbb{R}_+^n$ , the following continuous mapping:

$$z \mapsto \|z\|_{\bar{r}, \rho} := \left( \sum_{i=1}^n |z_i|^{\rho/\bar{r}_i} \right)^{1/\rho}, \quad (\text{A.4})$$

where  $\rho \geq \bar{r}_{\max} := \max_{1 \leq j \leq n} \bar{r}_j$  ( $\bar{r}_{\min} := \min_{1 \leq j \leq n} \bar{r}_j$ ) is called the  $\bar{r}$ -homogeneous norm. When the value of  $\rho$  is omitted (i.e. when the following notation  $\|z\|_{\bar{r}}$  is used to represent the  $\bar{r}$ -homogeneous norm), it means that  $\rho = \prod_{i=1}^n \bar{r}_i$ .

**Definition A.3**

A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $\bar{r}$ -homogeneous of degree  $k \in \mathbb{R}$  if

$$V(\lambda^{\bar{r}_1} z_1, \dots, \lambda^{\bar{r}_n} z_n) = \lambda^k V(z_1, \dots, z_n), \quad \forall z \in \mathbb{R}^n, \forall \lambda > 0. \quad (\text{A.5})$$

**Example A.4**

The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(z_1, z_2) = z_1^2 + z_2^3, \quad (\text{A.6})$$

is  $[2, \frac{4}{3}]$ -homogeneous of degree 4 since

$$f(\lambda^2 z_1, \lambda^{\frac{4}{3}} z_2) = \lambda^4 (z_1^2 + z_2^3) = \lambda^4 f(z_1, z_2); \quad (\text{A.7})$$

Moreover,  $f$  is also  $[1, \frac{2}{3}]$ -homogeneous of degree 2. This shows that the homogeneity degree and weights are not unique.

**Definition A.4**

A vector field  $\mathcal{K} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be  $\bar{r}$ -homogeneous of degree  $\nu \in \mathbb{R}$  if

$$\mathcal{K}(\lambda^{\bar{r}_1} z_1, \dots, \lambda^{\bar{r}_n} z_n) = \lambda^\nu \Lambda_{\bar{r}}(\lambda) \mathcal{K}(z_1, \dots, z_n), \quad \forall z \in \mathbb{R}^n, \quad \forall \lambda > 0, \quad (\text{A.8})$$

where the matrix  $\Lambda_{\bar{r}}(\lambda) = \text{diag}[\lambda^{\bar{r}_1}, \dots, \lambda^{\bar{r}_n}]$  is called the dilation matrix associated to the vector of weights  $\bar{r}$ .

**Remark A.1**

Notice that in the previous Definition, saying that the vector field  $\mathcal{K} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $\bar{r}$ -homogeneous of degree  $\nu$  is equivalent to say that the functions  $\mathcal{K}_i$  are  $\bar{r}$ -homogeneous of degree  $\nu + \bar{r}_i$ , for each  $i \in \{1, \dots, n\}$ .

**Example A.5**

The vector field  $\mathcal{K} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$\mathcal{K}(z_1, z_2) = [\text{sign}(z_1), \text{sign}(z_2)], \quad (\text{A.9})$$

is  $[1, 1]$ -homogeneous of degree  $-1$  since for  $i \in \{1, 2\}$

$$\mathcal{K}_i(\lambda z_1, \lambda z_2) = [\text{sign}(z_1), \text{sign}(z_2)], \quad (\text{A.10})$$

$$= \lambda^1 \lambda^{-1} \mathcal{K}_i(\lambda z_1, \lambda z_2). \quad (\text{A.11})$$

**Remark A.2**

Notice that there is an incompatibility between the standard homogeneity and the weighted homogeneity concepts when the weights are all taken to be equal to 1. For instance, by looking at the previous example, we can spot this incompatibility since the vector field  $\mathcal{K}$  is homogeneous of degree 0 with respect to the standard homogeneity given in Definition A.1 but as we proved is  $[1, 1]$ -homogeneous of degree  $-1$  with respect to the weighted homogeneity.

**Theorem A.3** ([74], [218], [219])

If the system (2.1) is  $\bar{r}$ -homogeneous of degree  $\nu$  and **AS** at the origin, then it is

1. globally **ES** at the origin if  $\nu = 0$ ,
2. globally **FTS** at the origin if  $\nu < 0$ ,
3. globally nearly **FxTS** if  $\nu > 0$ .

Another fundamental result in the study of finite-time stability has been proven by Malkin in 1959 [66] and Krasovskii in 1963 [67] for standard homogeneous systems and it was generalized to weighted homogeneous systems in [74] by Bhat and Bernstein.

**Theorem A.4** ([74])

Let  $\mathcal{K}_1, \dots, \mathcal{K}_p$  be continuous homogeneous vector fields of degrees  $k_1 < k_2 < \dots < k_p$  and denote  $\mathcal{K} = \mathcal{K}_1 + \dots + \mathcal{K}_p$ . Assume moreover that  $\mathcal{K}(0) = 0$ . If the origin is globally **AS** under  $\mathcal{K}_1$  then the origin is locally **AS** under  $\mathcal{K}$ . Moreover, if the origin is **FTS** under  $\mathcal{K}_1$  then the origin is **FTS** under  $\mathcal{K}$ .

The next result is central and provides the existence of homogeneous Lyapunov functions for globally asymptotically stable systems:

**Theorem A.5** ([220])

If the system (2.1) is  $\bar{r}$ -homogeneous, then the origin is globally AS if and only if there exists a homogeneous and continuously differentiable function  $V$  on  $\mathbb{R}^n \setminus \{0\}$ , such that

- $V(0) = 0$  and  $V(z) > 0$  in  $\mathbb{R}^n \setminus \{0\}$ ;
- $\dot{V}(z) < 0$  in  $\mathbb{R}^n \setminus \{0\}$ ;
- $V(z) \rightarrow +\infty$  as  $\|z\|_2 \rightarrow +\infty$ .

### A.3 Linear homogeneity

Homogeneity is a symmetry: scaling time and state leave the dynamical system unchanged. This property can be checked via some algebraic relations to be tested. Moreover, conventional (Euler)  $\subset$  weighted  $\subset$  linear  $\subset$  geometric (coordinate-free) homogeneity ( $\subset$  means "is a subclass of"). Let us recall definitions of linear homogeneity: Let us define a more general notion of dilation than weighted dilation (see [168] for more details).

**Definition A.5** (see [168])

A mapping  $\mathbf{d}: \mathbb{R} \rightarrow \mathcal{M}_{n,n}(\mathbb{R})$  is called a dilation in  $\mathbb{R}^n$  if it satisfies

- Group property:  $\mathbf{d}(0) = \bar{I}_n$ ,  $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s)$ ,  $\forall t, s \in \mathbb{R}$ .
- Continuity property:  $\mathbf{d}$  is continuous, i.e.  $\forall t > 0, \forall \varepsilon > 0, \exists \gamma > 0$ :

$$|s - t| < \gamma \Rightarrow \|\mathbf{d}(s) - \mathbf{d}(t)\|_2 \leq \varepsilon,$$

- Limit property:  $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)z\|_2 = 0$  and  $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)z\|_2 = +\infty$  uniformly on the unit sphere.

**Remark A.3**

Note that we can recover:

1. standard dilation when  $\mathbf{d}(s) = e^s I$ ;
2. weighted dilation when  $\mathbf{d}(s) = \text{diag}[e^{r_1 s}, \dots, e^{r_n s}]$ ,  $r_i > 0$ ,  $\forall i \in \{1, \dots, n\}$ .

Canonical homogeneous norm plays a central role but it requires the dilation to be monotone. This issue is recalled as follows:

**Definition A.6** (see [168])

The dilation  $\mathbf{d}$  is monotone in  $\mathbb{R}^n$  if  $\|\mathbf{d}(s)\|_2 < 1$ ,  $\forall s < 0$ . Then, the continuous mapping  $z \mapsto \|z\|_{\mathbf{d}} = e^{s_z}$  where  $s_z \in \mathbb{R} : \|\mathbf{d}(-s_z)z\|_2 = 1$ , is called the canonical homogeneous norm. The homogeneous unit sphere is then defined as  $S_{\mathbf{d}} = \{z \in \mathbb{R}^n : \|z\|_{\mathbf{d}} = 1\}$ .

**Remark A.4**

The canonical homogeneous norm is positive definite,  $\mathbf{d}$ -homogeneous of degree 1 (see below for the definition of  $\mathbf{d}$ -homogeneity).

**Definition A.7** (see [168])

The dilation  $\mathbf{d}$  is a linear dilation if and only if  $\mathbf{d}(s) = e^{G_{\mathbf{d}} s}$  where  $G_{\mathbf{d}} \in \mathcal{M}_n(\mathbb{R})$  is anti-Hurwitz<sup>1</sup>.

<sup>1</sup>all eigenvalues have positive real parts; this is the origin of  $\dot{z} = -G_{\mathbf{d}} z$  is GAS.

For any linear monotone dilation in  $\mathbb{R}^n$ , the canonical homogeneous norm is continuous on  $\mathbb{R}^n$  and locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$ . Moreover, it is differentiable on  $\mathbb{R}^n \setminus \{0\}$  provided that  $\|\cdot\|$  is induced by  $P > 0, PG_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0$ , (see [168]):

$$\frac{\partial \|z\|_{\mathbf{d}}}{\partial z} = \|z\|_{\mathbf{d}} \frac{z^T \mathbf{d}^T (-\ln \|z\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|z\|_{\mathbf{d}})}{z^T \mathbf{d}^T (-\ln \|z\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z}. \quad (\text{A.12})$$

The notion of  $\mathbf{d}$ -homogeneous function or vector fields is given below.

**Definition A.8**

Let  $\mathbf{d}$  be a linear dilation. A function  $\phi$  or a vector field  $\mathcal{K}$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\kappa \in \mathbb{R}$  if and only if for all  $s \in \mathbb{R}$  we have:

$$\phi(\mathbf{d}(s)z) = e^{\kappa s} \phi(z), \quad (\text{A.13})$$

$$\mathbf{d}(-s)\mathcal{K}(\mathbf{d}(s)z) = e^{\kappa s}\mathcal{K}(z). \quad (\text{A.14})$$

These concepts simplify the finite/fixed-time stability analysis: if the origin of a  $\mathbf{d}$ -homogeneous system of degree  $\kappa \in \mathbb{R}$  is globally **AS**, then if  $\kappa < 0$  it is also globally **FTS**,  $\kappa = 0$  it is globally **ES**,  $\kappa > 0$  it is also nearly **FxTS** (a prerequisite for **FxTS**). The next definition is adapted from [85].

**Definition A.9**

The vector field  $\mathcal{K} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has a  $(\mathbf{d}_a, \kappa_a, \mathcal{K}_a)$ -homogeneous approximation at  $a \in \{0, \infty\}$  if and only if there exists a real constant  $\kappa_a$  such that

$$\lim_{s \rightarrow \log(a)} \sup_{\|z\|_2=1} \|e^{-\kappa_a s} \mathbf{d}_a(-s)f(\mathbf{d}_a(s)z) - f_a(z)\|_2 = 0. \quad (\text{A.15})$$

Notice that the approximation at 0 is useful for finite-time stability, the approximation at  $\infty$  is useful for nearly fixed-time stability, and both approximations are useful for fixed-time stability. From [85, Theorem 2.20], one can deduce:

**Lemma A.1**

Assume that (2.1) is globally **AS**.

- If  $\mathcal{K}$  has a  $(\mathbf{d}_0, \kappa_0, \mathcal{K}_0)$ -homogeneous approximation at 0 with  $\mathcal{K}_0$  globally **AS** and  $\kappa_0 < 0$ , then the origin of (2.1) is globally **FTS**.
- In addition, if  $\mathcal{K}$  has a  $(\mathbf{d}_\infty, \kappa_\infty, \mathcal{K}_\infty)$ -homogeneous approximation at  $\infty$  with  $\mathcal{K}_\infty$  globally **AS** and  $\kappa_\infty > 0$  then the origin of (2.1) is globally **FxTS**. And then there exists a bi-limit homogeneous Lyapunov function for (2.1).





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**Titre :** Contrôle et estimation en temps fini de certaines classes d'EDP.

**Résumé :** L'objectif principal de cette thèse est d'explorer et d'analyser les problèmes d'estimation et de stabilisation non-asymptotique (en temps fini, fixe, et prescrit) de certaines classes de systèmes de dimension infinie, notamment les systèmes linéaires invariants en temps avec retards (ponctuels ou distribués) d'entrée ou de sortie et les équations aux dérivées partielles (EDP) de type réaction-diffusion. Comme les résultats existants sur ces classes de systèmes sont peu nombreux, nous commençons par revoir les concepts et les résultats sur les outils non asymptotiques pour les systèmes de dimension finie. Ensuite, nous étendons ces outils pour les systèmes de dimension infinie. Dans ce contexte, nous commençons par le problème de compensation, en temps fini/fixe, des retards d'entrée et de sortie pour les systèmes LTI en utilisant la méthode du *backstepping* pour les EDP (avec des transformations inversibles non-linéaires et/ou variant en temps). Pour appliquer cette approche, nous reformulons le système considéré en une cascade de système EDO-EDP où la partie EDP est une équation de transport hyperbolique qui modélise l'effet du retard sur l'entrée/sortie. Ensuite, nous traitons le problème de la stabilisation frontière en temps fini/fixe d'une classe des EDP de réaction-diffusion. À notre connaissance, ce problème est resté ouvert dans la littérature pendant une période considérable. Nous abordons ce problème complexe à l'aide de méthodes classiques liées aux *Fonctions de Lyapunov de Contrôle (CLF)*. Nous donnons quelques indications sur l'extension de cette approche au problème de stabilisation entrée-état (ISS) et au problème du suivi en temps fini/fixe pour les EDP de réaction-diffusion. Nous soulignons les limitations de notre méthode pour la conception des observateurs. Enfin, nous abordons le problème de la compensation, en temps prescrit, des retards d'entrée des systèmes de réaction-diffusion par une commande par retour d'état basée sur un observateur en utilisant la méthode du *backstepping* pour les EDP. Ce problème est difficile, car il nécessite de traiter la conception des observateurs et des contrôleurs avec des gains variant en temps qui tendent vers l'infini lorsque le temps se rapproche du temps prescrit de convergence.

**Mots clés :** stabilité en temps fini, estimation des états, stabilisation, systèmes LTI avec retards, équations aux dérivées partielles, Approche basée sur les fonctions de Lyapunov, Backstepping.

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**Title:** Finite-time control and estimation of some classes of PDEs.

**Abstract:** This Ph.D. thesis is devoted to the problems of non-asymptotic (finite, fixed, prescribed-time) estimation and stabilization of some classes of infinite-dimensional systems, namely LTI systems subject to input/sensor (pointwise or distributed) delays and reaction-diffusion PDEs. As the existing results on these classes of systems are few, we begin by reviewing relevant concepts and results on non-asymptotic tools (including homogeneity-based tools and time-varying tools) for finite-dimensional systems. Afterward, we extend these tools to infinite-dimensional settings. Firstly, we start with the problem of input and sensor delay compensation in finite/fixed/prescribed time of LTI systems where we use the so-called *backstepping* approach for PDEs (with some nonlinear and/or time-varying invertible transformations). To apply this approach, we reformulate the considered LTI system into a cascade ODE-PDE system where the PDE part is a hyperbolic transport equation that models the effect of the delay on the input/output. Secondly, we consider the problem of boundary state-dependent finite/fixed-time stabilization of reaction-diffusion PDEs. To the best of our knowledge, this problem has remained open in the literature for a considerable long time. We tackle this challenging problem using classical methods related to *Control Lyapunov Functions CLF*. We provide some hints on how we to extend this approach to input-to-state stabilization and non-asymptotic tracking problem for reaction-diffusion PDEs. We point out the limitations of our approach to observer design. Finally, we tackle the problem of input delay compensation of reaction-diffusion systems in prescribed time by output feedback using the *backstepping* approach. This problem is challenging, as one deals with observer and control designs with some time-varying gains that go to infinity when the time gets closer to the prescribed time of convergence.

**Keywords:** finite-time stability, states estimation, stabilization, LTI systems with delays, partial differential equations, Lyapunov-based approach, Backstepping.