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Cycles et chemins dans les digraphes, idéaux monomiaux et partitions des nombres entiers

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These de Doctorat de Mathématiques

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Abstract

In this thesis, we work in two directions, both concern problems in combinatorics.

The first direction is related to the study of an important invariant of oriented graphs which is the chromatic number. Given a graph with large chromatic number, it is natural to ask whether it must contain a cycle or path with particular properties. We are interested in studying such question in directed graphs. More precisely, we are interested in the study of the existence of special oriented paths or cycles in digraphs with bounded chromatic numbers.

Cohen et al. [21] conjectured that for every oriented cycle C , there exists a constant $f(C)$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C and they proved this for cycles of the form $C(1, 1, 1, 1)$. In this direction, we prove that if D is a digraph having a spanning out-tree T with no subdivisions of $C(k_1, 1, 1, 1)$, then the chromatic number of D is at most $18k_1$.

We are interested in the study of an analogous problem about paths in digraphs, more precisely in studying the existence of paths of three blocks of the form $P(1, k, 1)$. We show that every $\frac{3n^2}{4}$ -chromatic digraph contains any path of three blocks of length $n - 1$. Indeed, we prove that any $(2n - 5)$ -chromatic digraph contains a path of the form $P(1, n - 3, 1)$. Moreover, we show that every $(n + 1)$ -chromatic digraph contains a subdivision of $P(1, n - 3, 1)$.

The second direction concerns the study of identities of integer partitions using algebraic and combinatorial tools. Two of the most famous identities of integer partitions were found by Rogers and Ramanujan. We prove two new partition identities dual to those. These new identities are inspired by a correspondence between three kinds of objects: neighborly partitions, monomial ideals, and some infinite graphs.

For $i \in \{1, 2\}$ and for a positive integer n , we call neighborly partitions of n the set $\mathcal{N}_i(n)$ of partitions λ of n which satisfy the following properties:

1. For every part λ_j of λ , there exists $l \in \mathbf{N}_{>0}$, $l \neq j$ such that: $|\lambda_l - \lambda_j| \leq 1$.
2. Each part of λ can occur at most twice.

3. For every $l \in \mathbf{N}$, $\lambda_l \geq 3 - i$ (i.e. for $i = 1$, there are no parts equal to 1).

The terminology *neighborly* is inspired by the property 1. which says that for every part λ_j of λ , there is a *neighbor* part λ_l of λ which is equal or at a distance 1 to λ_j . With a neighborly partition λ , we associate a graph G_λ as follows: The set $V(G_\lambda)$ of vertices of G_λ is in bijection with the set of parts of λ ; if h is a part of λ of multiplicity 1 (that is, it occurs as a part only once), the associated vertex is called x_h ; if h is a part of λ of multiplicity 2, the vertices associated with the two equal parts are named respectively x_h and y_h . The set $E(G_\lambda)$ of edges of G_λ is given by

$$E(G_\lambda) = \{(x_{h+1}, x_h), (x_l, y_l), \text{ for every } x_{h+1}, x_h, x_l, y_l \in V(G_\lambda)\}.$$

Let λ be a neighborly partition. We define the signature $\delta(\lambda)$ of λ by:

$$\delta(\lambda) = \sum_H (-1)^{|E(H)|},$$

where H ranges over the vertex spanning subgraphs of G_λ , which have no isolated vertices.

Let n be a positive integer; for $i \in \{1, 2\}$, let $\mathcal{R}_i(n)$ be the set of partitions of n whose parts are larger or equal to $3 - i$, distinct and congruent to 0 or $\pm(i) \pmod{5}$. The new identities we have found are given by the following theorem:

Theorem 1. *For $i \in \{1, 2\}$, we have the identities*

$$\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|} = \prod_{j \geq 3-i, j \equiv 0, \pm i \pmod{5}} (1 - q^j).$$

where $|\lambda|$ is the sum of parts of λ .

In this direction, we also study a family of ideals related to the jet schemes of the double point $\text{Spec}K[x]/x^2$ and conclude some results concerning the generating series of some special type of integer partitions.

Résumé

Dans cette thèse, nous travaillons dans deux directions, toutes deux concernant des problèmes de combinatoire.

La première direction est liée à l'étude d'un invariant important des graphes orientés qui est le nombre chromatique. Étant donné un graphe dont le nombre chromatique est assez grand, il est naturel de se demander s'il doit contenir un cycle ou un chemin avec des propriétés particulières. Nous nous sommes intéressés à étudier une telle question pour les graphes orientés. Plus précisément, nous nous intéressons à l'étude de l'existence de certains chemins et cycles orientés dans les digraphes avec nombres chromatiques bornés.

Nous désignons par $C(k_1, k_2, \dots, k_n)$ le cycle orienté formé de n blocs de longueurs k_1, k_2, \dots , et k_n respectivement. Cohen et al. a conjecturé que pour tout cycle orienté C , il existe une constante $f(C)$ telle que tout digraphe fort de nombre chromatique au moins $f(C)$ contient une subdivision de C et ils l'ont prouvé pour des cycles de la forme $C(1, 1, 1, 1)$. Dans cette direction, nous montrons que si D est un digraphe ayant un arbre extérieur couvrant T sans subdivisions de $C(k_1, 1, 1, 1)$, alors le nombre chromatique de D est au plus $18k_1$.

Nous nous intéressons à l'étude d'un problème analogue de chemins dans les digraphes, plus précisément à l'étude de l'existence de chemins de trois blocs de la forme $P(r, k, l)$. Nous montrons que tout digraphe $\frac{3n^2}{4}$ -chromatique contient n'importe quel chemin de trois blocs de longueur $n - 1$. En effet, nous prouvons que tout digraphe $(2n - 5)$ -chromatique contient un chemin de la forme $P(1, n - 3, 1)$. De plus, nous montrons que tout digraphe $(n + 1)$ -chromatique contient une subdivision de $P(1, n - 3, 1)$.

La deuxième direction concerne l'étude des identités des partitions entières à l'aide d'outils algébriques et combinatoires. Deux des identités les plus célèbres des partitions entières ont été trouvées par Rogers et Ramanujan. Nous prouvons deux nouvelles identités de partitions duales de celles-ci. Ces nouvelles identités s'inspirent d'une correspondance entre trois types d'objets: les partitions voisines, les idéaux monomiaux et certains graphes infinis.

Pour $i \in \{1, 2\}$ et pour un entier positif n , l'ensemble $\mathcal{N}_i(n)$ de partitions voisines de n est par définition l'ensemble de partitions λ de n qui vérifient les propriétés suivantes:

1. Pour toute partie λ_j de λ , il existe $l \in \mathbf{N}_{>0}$, $l \neq j$ tel que : $|\lambda_l - \lambda_j| \leq 1$.
2. Chaque partie de λ peut apparaître au plus deux fois.
3. Pour chaque $l \in \mathbf{N}$, $\lambda_l \geq 3 - i$ (c'est-à-dire que pour $i = 1$, il n'y a pas de parties égales à 1).

La terminologie *voisine* est inspirée de la propriété 1. qui dit que pour chaque partie λ_j de λ , il existe une partie *voisine* λ_l de λ qui est égale ou à une distance 1 de λ_j . A une partition voisine λ , on associe un graphe G_λ comme suit : L'ensemble $V(G_\lambda)$ des sommets de G_λ est en bijection avec l'ensemble des parties de λ ; si h est une partie de λ de multiplicité 1 (c'est-à-dire qu'il n'apparaît comme partie qu'une seule fois), le sommet associé est appelé x_h ; si h est une partie de λ de multiplicité 2, les sommets associés aux deux parties égales sont nommés respectivement x_h et y_h . L'ensemble $E(G_\lambda)$ des arêtes de G_λ est donnée par

$$E(G_\lambda) = \{(x_{h+1}, x_h), (x_l, y_l), \text{ pour tout } x_{h+1}, x_h, x_l, y_l \in V(G_\lambda)\}.$$

Soit λ une partition voisine. On définit la signature $\delta(\lambda)$ de λ par :

$$\delta(\lambda) = \sum_H (-1)^{|E(H)|},$$

où H varie dans l'ensemble des sous-graphes H de G_λ tel que $V(H) = V(G_\lambda)$ et qui n'ont pas de sommets isolés.

Soit n un entier positif; pour $i \in \{1, 2\}$, soit $\mathcal{R}_i(n)$ l'ensemble des partitions de n dont les parties sont supérieures ou égales à $3 - i$, distinctes et congruentes à 0 ou $\pm(i) \pmod{5}$. Les nouvelles identités que nous avons trouvées sont données par le théorème suivant :

Theorem 2. *Pour $i \in \{1, 2\}$, nous avons les identités*

$$\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|} = \prod_{j \geq 3-i, j \equiv 0, \pm i \pmod{5}} (1 - q^j).$$

où $|\lambda|$ est la somme des parties de λ .

Toujours en rapport avec la théorie des partitions, nous étudions une famille d'idéaux définissant les espaces de jets du point double $\text{Spec}K[x]/(x^2)$ et concluons quelques résultats concernant les séries génératrice d'un certain type particulier de partitions entières.

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Chapter 1

Introduction (English Version)

In this thesis, we work in two directions, both concern problems in combinatorics. The first direction is related to the study of an important invariant of oriented graphs which is the chromatic number. More precisely, we are interested in the study of the existence of special oriented paths or cycles in digraphs with bounded chromatic numbers. The second direction concerns the study of identities of integer partitions using algebraic and combinatorial tools. Two of the most famous identities of integer partitions were found by Rogers and Ramanujan. We prove two new partition identities dual to those. These new identities are inspired by a correspondence between three kinds of objects: neighborly partitions, monomial ideals, and some infinite graphs. In this direction, we also study a family of ideals related to the jet schemes of the double point $\text{Spec}K[x]/x^2$ and conclude some results concerning the generating series of some special type of integer partitions.

Before going deep into the details, we will present in Chapters 3 and 4 basic definitions, standard notations, and preliminary results that will be significant for our research. The first one will introduce the readers to the fundamentals and preliminaries in graph theory and the second will give an outlook over the fundamentals of partition theory and its relation with algebraic objects.

Paths and Cycles in Digraphs with Large Chromatic Number:

To start with Direction 1, we answer the following question for particular cases: Given an n -chromatic digraph D , can we guarantee the presence of a copy of a path or a cycle of certain properties in D ?

In Chapter 5, we will study the existence of oriented cycles of four blocks in digraphs with large chromatic number. A digraph or an oriented graph D is said to be strong if for any two vertices u and v of D , there is a directed path from u to v . Given an oriented cycle C , a block is a maximal directed subpath of C . We denote by $C(k_1, k_2, \dots, k_n)$ the oriented cycle formed of n blocks of lengths k_1, k_2, \dots , and k_n respectively. We are interested in the study

of the existence of oriented cycles in strong digraphs. In [15], Bondy proved that every strong digraph D contains a directed cycle of length at least $\chi(D)$. A subdivision of a digraph F is a digraph F' obtained by replacing each arc (x, y) of F by an xy -directed path of length at least 1. Since any directed cycle of length at least k can be seen as a subdivision of the directed cycle C_k of length k , Cohen et al. [21] conjectured that Bondy's theorem can be extended to all oriented cycles: For every oriented cycle C , there exists a constant $f(C)$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C . In fact, Cohen et al. [21] proved this conjecture in their article for cycles with two blocks. In particular, they showed that the chromatic number of strong digraphs with no subdivisions of two blocks cycles $C(k_1, k_2)$ is bounded from above by $O((k_1 + k_2)^4)$. More recently, Kim et al. [42] improved this upper bound to $O((k_1 + k_2)^2)$. In [39], El Joubbeh solved this conjecture for Hamiltonian digraphs, and demonstrated a linear version by showing that any $3n$ -chromatic Hamiltonian digraph contains a subdivision for any oriented cycle of order n . Furthermore, Cohen et al. confirmed in [21] the above conjecture for a particular case of cycles with four blocks by proving that for D a strong digraph with no subdivisions of four blocks cycles $C(1, 1, 1, 1)$, we have $\chi(D) \leq 24$. In [6] Al Mniny conjectured that for every positive integers k_1, k_2, k_3 and k_4 , there is an integer $g(k_1, k_2, k_3, k_4)$ such that every strong digraph with no subdivisions of $C(k_1, k_2, k_3, k_4)$ has chromatic number at most $g(k_1, k_2, k_3, k_4)$. In the same paper, Al Mniny proved this conjecture for digraphs containing spanning out-tree when $k_2 = k_3 = k_4 = 1$ for $g(k_1, 1, 1, 1) = 8^3 k_1$. In Chapter 5, we are going to improve the bound established in [6] by proving that if D is a digraph having a spanning out-tree T with no subdivisions of $C(k, 1, 1, 1)$, then the chromatic number of D is at most $18k$. In our proof, we use for the first time in such investigations what is called a wheel in the study of the chromatic number of such digraphs. A wheel is a graph made up of a chordless cycle and a vertex adjacent to at least three vertices of the cycle, such vertex is called universal.

Chapter 6 will be devoted for answering the main question arised above for oriented paths with three blocks. Recall that a block of a path in a digraph is a maximal directed subpath, with the length of a path is the number of its edges. Following the result we obtained concerning the cycles of four blocks of the form $C(k, 1, 1, 1)$, we can easily see that any $18k$ -chromatic digraph contains a path of three blocks of the form $P(1, k, 1)$, for k a positive integer. Indeed, Gallai–Roy's celebrated theorem [30, 53] states that every digraph with chromatic number at least n contains a directed path of length $n - 1$. El-Sahili and Kouider [24] showed that every digraph with chromatic $n + 1$ contains every path of order n with two blocks. However, Addario et.al.[57] refined this result for any digraph D with chromatic number at least n . For the case of oriented paths with three blocks, Burr's result [19], guarantees the existence of any path of three blocks of length $n - 1$ in any digraph

with chromatic number at least $(n - 1)^2$. In Chapter 6, we show that every $\frac{3n^2}{4}$ -chromatic digraph contains any path of three blocks of length $n - 1$. El Joubbeh [40] improved this result by demonstrating that every digraph with chromatic number at least $4.6n$ contains any path with three blocks of length $n - 1$. Recently Tarhini found a better linear bound for the chromatic number of digraphs that contains no path with three blocks. Moreover, Mortada et.al proved in [48] that any $(n + 1)$ -chromatic digraph contains a three blocks path of length $n - 1$, in which two consecutive blocks are of length 1 each. For k and l two positive integers, Tarhini et. al [55] proved the existence of a $P(k, 1, l)$ of length $n - 1$ in any $(2n + 2)$ -chromatic digraph D and they proved that the existence of such an oriented path is ensured in any n -chromatic digraph that contains a hamiltonian directed path. Recently, Tarhini improved her result in [55] by proving that any $(2n - 3)$ -chromatic digraph contains any path $P(k, 1, l)$ of length $n - 1$. In Chapter 6, we prove that any $(2k + 1)$ -chromatic digraph contains a $P(1, k, 1)$, for k a positive integer. Moreover, we show that this becomes $k + 4$ if we search for the existence of a subdivision of $P(1, k, 1)$.

Identities of Integer Partitions:

Moving to Direction 2 of our research, we begin with Chapter 7 by giving new integer partition identities which are dual to the so called Rogers-Ramanujan identities. An integer partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \dots, \lambda_r$ which sums up to n . The λ_i are called the parts of λ and r its size. Partition theorists are more concerned in the study of restricted partitions with certain conditions over the size of the partition or the parts themselves. An axial problem in the theory of integer partitions is the study of partition identities, this is about finding two types of partitions A and B such that for n any positive integer, the number of partitions of type A of n is equal to the number of partitions of type B of n . Among the most famous partition identities, we have the Rogers-Ramanujan identities:

Theorem 3. *(Rogers-Ramanujan identities) Let n be a positive integer. For $i \in \{1, 2\}$, let $\mathcal{T}_i(n)$ be the set of partitions of n without equal nor consecutive parts and the part 1 appears at most $i - 1$ times. Let $\mathcal{E}_i(n)$ be the number of partitions of n into parts congruent to $\pm(2 + i) \pmod{5}$. Then we have*

$$|\mathcal{T}_i(n)| = |\mathcal{E}_i(n)|.$$

The notation $|A|$ in the theorem stands for the cardinal of a set A .

Example 4. $\mathcal{E}_2(5) = \{4 + 1, 1 + 1 + 1 + 1 + 1\}$ and $\mathcal{T}_2(5) = \{5, 4 + 1\}$

These identities appear in many fields other than combinatorics, such as statistical mechanics, number theory, representation theory, algebraic geometry, probability theory or commutative algebra [1, 2, 3, 4, 5, 8, 10, 12, 17, 18,

29, 31, 34].

The main goal of Chapter 7 is to prove two new identities which are in some sense dual to the Rogers-Ramanujan identities. We begin by introducing the notions appearing in these new identities; neighborly partitions, signature of a neighbourly partition:

For $i \in \{1, 2\}$ and for a positive integer n , we call neighborly partitions of n the set $\mathcal{N}_i(n)$ of partitions λ of n which satisfy the following properties:

1. For every part λ_j of λ , there exists $l \in \mathbf{N}_{>0}$, $l \neq j$ such that: $|\lambda_l - \lambda_j| \leq 1$.
2. Each part of λ can occur at most twice.
3. For every $l \in \mathbf{N}$, $\lambda_l \geq 3 - i$ (i.e. for $i = 1$, there are no parts equal to 1).

The terminology *neighborly* is inspired by the property 1. which says that for every part λ_j of λ , there is a *neighbor* part λ_l of λ which is equal or at a distance 1 to λ_j . With a neighborly partition λ , we associate a graph G_λ as follows: The set $V(G_\lambda)$ of vertices of G_λ is in bijection with the set of parts of λ ; if h is a part of λ of multiplicity 1 (that is, it occurs as a part only once), the associated vertex is called x_h ; if h is a part of λ of multiplicity 2, the vertices associated with the two equal parts are named respectively x_h and y_h . The set $E(G_\lambda)$ of edges of G_λ is given by

$$E(G_\lambda) = \{(x_{h+1}, x_h), (x_l, y_l), \text{ for every } x_{h+1}, x_h, x_l, y_l \in V(G_\lambda)\}.$$

Example 5. For the neighborly partition $\lambda = 2 + 1 + 1$, G_λ is given by $V(G_{2+1+1}) = \{x_2, x_1, y_1\}$, $E(G_{2+1+1}) = \{(x_2, x_1), (x_1, y_1)\}$ and it has the following shape

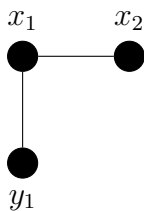


Figure 1.1: The graph G_{2+1+1}

A subgraph H of G_λ is said to be vertex-spanning if $V(H) = V(G_\lambda)$; it is said to be without isolated vertices if any vertex of H is an endpoint of some edge in $E(H)$. Let λ be a neighborly partition. We define the signature $\delta(\lambda)$ of λ by:

$$\delta(\lambda) = \sum_H (-1)^{|E(H)|},$$

where H ranges over the vertex spanning subgraphs of G_λ , which have no isolated vertices and $|E(H)|$, as mentioned before, is the cardinal of $E(H)$. Let n be a positive integer; for $i \in \{1, 2\}$, let $\mathcal{R}_i(n)$ be the set of partitions of n whose parts are larger or equal to $3 - i$, distinct and congruent to 0 or $\pm(i) \pmod{5}$. The new identities we have found are given by the following theorem:

Theorem 6. *Let n be a positive integer. For $i \in \{1, 2\}$, we have the identities*

$$\sum_{\lambda \in \mathcal{N}_i(n)} \delta(\lambda) = \sum_{\lambda \in \mathcal{R}_i(n)} (-1)^{\text{size}(\lambda)}.$$

By considering the generating sequence of both sides of Theorem 6, we find the following equivalent statement:

Corollary 7. *For $i \in \{1, 2\}$, we have the identities*

$$\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|} = \prod_{j \geq 3-i, j \equiv 0, \pm i \pmod{5}} (1 - q^j).$$

where $|\lambda|$ is the sum of parts of λ .

Chapter 8 focuses on the investigation of the Hilbert series and Gröbner basis of the family of ideals defining the jet schemes of a double point $D = \text{Spec}K[x]/x^2$, for K a field of characteristic zero. Recall that the $(m - 1)$ -jet scheme of X is defined as the space of maps: $\text{Spec}K[t]/t^m \rightarrow X$. In the case of the double point, such a map is defined by a polynomial: $x(t) = x_0 + x_1 t + \dots + x_{m-1} t^{m-1}$, where $x(t)^2 \equiv 0 \pmod{t^m}$. By expanding this equation, we get a set of polynomials in the polynomial ring $R_m = K[x_0, \dots, x_{m-1}]$ defined as: $f_k = \sum_{i=0}^{k-1} x_i x_{k-1-i}$, for $k = 1, \dots, m$. The ideal I_m of R_m generated by f_1, \dots, f_m is the defining ideal of the $(m - 1)$ -jet scheme of the double point.

Assigning to each variable x_i the weight $(i, 1)$, we induce a bigrading on R_m/I_m as $\oplus_{i,j \geq 1} (R_m/I_m)_{i,j}$. Define the bigraded Hilbert series of R_m/I_m to be the series in variables q and t given by:

$$H_m(q, t) = \sum_{i,j \geq 0} \dim_k (R_m/I_m)_{i,j} q^i t^j$$

In Chapter 8, following [11] we study the initial terms of the Gröbner basis of I_m with respect to the reverse lexicographic order and we draw for this consequences concerning the generating series of some partition identities:

Definition 8. *Let n be a positive integer. Let λ be a partition of n . Denote $SC(\lambda)$ to be the pair of the smallest consecutive or equal parts of λ . For*

$SC(\lambda) = (i, j)$, denote by $|SC(\lambda)|$ the sum $i + j$. Define the k -value of λ as follows: $k(\lambda) = |SC(\lambda)| - |\lambda| - 1$. For a partition λ with no consecutive or equal parts we consider $|k(\lambda)| = +\infty$.

We call Δ_m -partitions of n the set $\Delta_m(n)$ of partitions λ of n with parts less than or equal to m such that for any $k(\lambda)$ parts $i_1 \leq \dots \leq i_{k(\lambda)}$ of λ less than or equal to $\lfloor \frac{m-3k-1}{2} \rfloor + 1 + 2(k-1)$ we have $|i_t - i_{t+1}| \leq 1$ for some t . Denote by $\Delta_m(n, p)$ the set of all the Δ_m -partitions of n with size p .

Definition 9. Define the generating series of $\Delta_m(n, p)$ to be the two variable power series given by:

$$\mathbf{H}_m(q, t) = \sum_{n, p \geq 0} \Delta_m(n, p) q^n t^p$$

Theorem 10. Let $n \geq 1$ be an integer. The generating series of the Δ_m -partitions is given by:

$$\mathbf{H}_m(q, t) = H_m(q, qt) = \sum_{p=0}^{\infty} \frac{\binom{h(m,p)+1}{p}_q q^{p^2} t^p}{(1 - q^{m+1-h(m,p)}t) \dots (1 - q^m t)}$$

where $h(m, p) = \lfloor \frac{m-p}{2} \rfloor$ and $\binom{a}{b}_q = \frac{(1-q) \dots (1-q^a)}{(1-q) \dots (1-q^b) (1-q) \dots (1-q^{a-b})}$

Keywords— Oriented cycle, Oriented path, Strong Digraph, Chromatic number, Rogers-Ramanujan identities, Integer partitions, Neighborly partitions, Monomial ideals, Simple graphs, Hilbert series.

Chapter 2

Introduction (Version Française)

Dans cette thèse, nous travaillons dans deux directions, toutes deux concernant des problèmes de combinatoire. La première direction est liée à l'étude d'un invariant important des graphes orientés qui est le nombre chromatique. Plus précisément, nous nous intéressons à l'étude de l'existence de certains chemins et cycles orientés dans les digraphes à nombres chromatiques bornés. La deuxième direction concerne l'étude des identités des partitions entières à l'aide d'outils algébriques et combinatoires. Deux des identités les plus célèbres des partitions entières ont été trouvées par Rogers et Ramanujan. Nous prouvons deux nouvelles identités de partition duales de celles-ci. Ces nouvelles identités s'inspirent d'une correspondance entre trois types d'objets : les partitions voisines, les monômes idéaux et certains graphes infinis. Dans cette direction, nous étudions également une famille d'idéaux définissant les schémas de jets du point double $SpecK[x]/x^2$ et concluons quelques résultats concernant la série génératrice d'un certain type particulier de partitions entières.

Avant d'entrer dans le vif du sujet, nous présenterons dans les Chapitres 3 et 4 les définitions de base, les notations standard et les résultats préliminaires qui seront importants pour notre recherche. La première introduira les lecteurs aux fondements et aux préliminaires de la théorie des graphes et la seconde donnera une vue d'ensemble des fondements de la théorie des partitions et de sa relation avec les objets algébriques.

Chemins et Cycles dans les Digraphes à Grand Nombre Chromatique:

Pour commencer la première partie, nous répondons à la question suivante pour des cas particuliers : Étant donné un digraphe n -chromatique D , peut-on garantir la présence d'une copie d'un chemin ou d'un cycle de certaines propriétés dans D ? Nous répondons à cette question dans deux directions

pour certains types de chemins et de cycles.

Dans le chapitre 5, nous allons étudier l'existence de cycles orientés de quatre blocs dans des digraphes à grand nombre chromatique. Un digraphe D est dit fort si pour deux sommets quelconques u et v de D , il existe un chemin orienté de u à v . Étant donné un cycle orienté C , un bloc est un sous-chemin orienté maximal de C . Nous désignons par $C(k_1, k_2, \dots, k_n)$ le cycle orienté formé de n blocs de longueurs k_1, k_2, \dots , et k_n respectivement. Nous nous intéressons à l'étude de l'existence des cycles orientés dans les digraphes forts. Dans [15], Bondy a prouvé que tout digraphe fort D contient un cycle orienté de longueur au moins $\chi(D)$. Une subdivision d'un digraphe F est un digraphe F' obtenu en remplaçant chaque arc (x, y) de F par un chemin dirigé de x à y de longueur au moins 1. Puisque tout cycle dirigé de longueur au moins k peut être vu comme une subdivision du cycle dirigé C_k de longueur k , Cohen et al. [21] ont conjecturé que le théorème de Bondy peut être étendu à tous les cycles orientés : Pour tout cycle orienté C , il existe une constante $f(C)$ telle que tout digraphe fort dont le nombre chromatique est au moins égal à $f(C)$ contient une subdivision de C . En fait, Cohen et al. [21] ont prouvé cette conjecture dans leur article pour les cycles à deux blocs. En particulier, ils ont montré que le nombre chromatique des digraphes forts sans subdivisions de cycles à deux blocs $C(k_1, k_2)$ est limité par le haut par $O((k_1 + k_2)^4)$. Plus récemment, Kim et al. [42] ont amélioré cette limite supérieure à $O((k_1 + k_2)^2)$. Dans [39], El Joubbeh a résolu cette conjecture pour les digraphes hamiltoniens, et a démontré une version plus forte en montrant que tout digraphe hamiltonien chromatique à $3n$ contient une subdivision pour tout cycle orienté d'ordre n . De plus, Cohen et al. dans [21] ont confirmé la conjecture ci-dessus pour le cas des cycles à quatre blocs en prouvant que pour D un digraphe fort sans subdivisions des cycles à quatre blocs $C(1, 1, 1, 1)$, on a $\chi(D) \leq 24$. En [6] Al Mniny a conjecturé que pour tout entier positif k_1, k_2, k_3 et k_4 , il existe un entier $g(k_1, k_2, k_3, k_4)$ tel que tout digraphe fort sans subdivisions de $C(k_1, k_2, k_3, k_4)$ a un nombre chromatique au plus égal à $g(k_1, k_2, k_3, k_4)$. Dans le même article, Al Mniny a prouvé cette conjecture pour les digraphes contenant un arbre sortant spanning lorsque $k_2 = k_3 = k_4 = 1$ pour $g(k_1, 1, 1, 1) = 8^3 k_1$. Dans le chapitre 5, nous allons améliorer la borne établie dans [6] en prouvant que si D est un digraphe ayant un arbre de sortie spanning T sans subdivisions de $C(k, 1, 1, 1)$, alors le nombre chromatique de D est au plus $18k$. Dans notre preuve, nous utilisons pour la première fois dans de telles recherches ce qu'on appelle une roue afin d'étudier le nombre chromatique des digraphes. Une roue est un graphe composé d'un cycle sans corde et d'un sommet adjacent à au moins trois sommets du cycle, un tel sommet est appelé universel.

Le chapitre 6 sera consacré à la réponse à la question principale soulevée ci-dessus pour les chemins orientés à trois blocs. Rappelons qu'un bloc d'un

chemin dans un digraphe est un sous chemin dirigé maximal, la longueur d'un chemin étant le nombre de ses arêtes. Suite au résultat que nous avons obtenu concernant les cycles de quatre blocs de la forme $C(k, 1, 1, 1)$, nous pouvons voir que tout digraphe $18k$ -chromatique contient un chemin de trois blocs de la forme $P(1, k, 1)$, pour k un entier positif. En effet, le célèbre théorème de Gallai-Roy [30, 53] stipule que tout digraphe dont le nombre chromatique est au moins égal à n contient un chemin dirigé de longueur $n - 1$. El-Sahili et Kouider [24] ont montré que tout digraphe de chromatique $n + 1$ contient tout chemin d'ordre n à deux blocs. Cependant, Addario et.al.[57] ont raffiné ce résultat pour tout digraphe D avec un nombre chromatique d'au moins n . Pour le cas des chemins orientés, le résultat de Burr, [19], garantit l'existence de tout chemin de trois blocs de longueur $n - 1$ dans tout digraphe dont le nombre chromatique est au moins $(n - 1)^2$. Au chapitre 6, nous montrons que tout digraphe $\frac{3n^2}{4}$ -chromatique contient un chemin quelconque de trois blocs de longueur $n - 1$. El Joubbeh [40] a amélioré ce résultat en démontrant que tout digraphe de nombre chromatique au moins $4, 6n$ contient tout chemin de trois blocs de longueur $n - 1$. Récemment, Tarhini a trouvé une meilleure borne linéaire pour le nombre chromatique de digraphes qui ne contient aucun chemin avec trois blocs. De plus, Mortada et.al ont prouvé dans [48] que tout digraphe $(n + 1)$ -chromatique contient un chemin à trois blocs de longueur $n - 1$, dans lequel deux blocs consécutifs sont de longueur 1 chacun. Soit k, l deux entiers positifs, Tarhini et. al [55] ont prouvé l'existence d'un $P(k, 1, l)$ de longueur $n - 1$ dans tout digraphe $(2n + 2)$ -chromatique D et ils ont prouvé que l'existence d'un tel chemin orienté est assurée dans tout digraphe n -chromatique qui contient un chemin dirigé hamiltonien. Récemment, Tarhini a amélioré son résultat dans [55] en prouvant que tout digraphe $(2n - 3)$ -chromatique contient tout chemin $P(k, 1, l)$ de longueur $n - 1$. Dans ce chapitre, nous prouvons que tout digraphe chromatique de $2k + 1$ contient un $P(1, k, 1)$, pour k un entier positif. De plus, nous montrons que cela devient $k + 4$ si l'on recherche l'existence d'une subdivision de $P(1, k, 1)$.

Identités des Partitions Entières:

Pour aborder la deuxième partie de notre recherche, nous commençons avec le Chapitre 7 en donnant de nouvelles identités de partition entière qui sont duales aux identités dites de Rogers-Ramanujan. Une partition entière d'un entier positif n est une séquence finie non croissante d'entiers positifs $\lambda_1, \dots, \lambda_r$ dont la somme est égale à n . Les λ_i sont appelés les parties de λ et r sa taille. Les théoriciens des partitions sont plus concernés par l'étude des partitions restreintes avec certaines conditions sur la taille de la partition ou des parties elles-mêmes. Un problème axial dans la théorie des partitions entières est l'étude des identités de partition, il s'agit de trouver deux types de partitions A et B tels que pour n tout entier positif, le nombre de partitions de type A de n est égal au nombre de partitions de type B de n . Parmi les identités de partition les plus célèbres, nous avons les identités de Rogers-Ramanujan :

Theorem 11. (*Identités de Rogers-Ramanujan*) Soit n un entier positif. Pour $i \in \{1, 2\}$, soit $\mathcal{T}_i(n)$ l'ensemble des partitions de n sans parties égales ni consécutives et la partie 1 apparaît au plus $i - 1$ fois. Soit $\mathcal{E}_i(n)$. le nombre de partitions de n en parties congruentes à $\pm(2 + i) \pmod{5}$. On a alors

$$|\mathcal{T}_i(n)| = |\mathcal{E}_i(n)|.$$

La notation $|A|$ dans le théorème représente le cardinal d'un ensemble A .

Example 12. $\mathcal{E}_2(5) = \{4 + 1, 1 + 1 + 1 + 1 + 1\}$ and $\mathcal{T}_2(5) = \{5, 4 + 1\}$

Ces identités apparaissent dans de nombreux domaines autres que la combinatoire, tels que la mécanique statistique, la théorie des nombres, la théorie des représentations, la géométrie algébrique, la théorie des probabilités ou l'algèbre commutative. [1, 2, 3, 4, 5, 8, 10, 12, 17, 18, 29, 31, 34].

L'objectif principal du chapitre 7 est de prouver deux nouvelles identités qui sont, en quelque sorte, duales des identités de Rogers-Ramanujan. Nous commençons par introduire les notions apparaissant dans ces nouvelles identités : partitions voisines, signature d'une partition voisine. Pour $i \in \{1, 2\}$ et pour un entier positif n , on appelle partitions voisines de n l'ensemble $\mathcal{N}_i(n)$ des partitions λ de n qui satisfont aux propriétés suivantes :

1. Pour chaque partie λ_j de λ , il existe $l \in \mathbf{N}_{>0}$, $l \neq j$ tel que $|\lambda_l - \lambda_j| \leq 1$.
2. Chaque partie de λ peut être répétée au maximum deux fois.
3. For every $l \in \mathbf{N}$, $\lambda_l \geq 3 - i$ (c'est-à-dire que pour $i = 1$, il n'y a pas de parties égales à 1).

La terminologie "voisine" est inspirée de la propriété 1. qui dit que pour chaque partie λ_j de λ , il existe une partie λ_l de λ qui est égale ou à une distance de 1 de λ_j . A une partition voisine λ , on associe un graphe G_λ comme suit : L'ensemble $V(G_\lambda)$ des sommets de G_λ est en bijection avec l'ensemble des parties de λ ; si h est une partie de λ de multiplicité 1, le sommet associé est appelé x_h ; si h est une partie de λ de multiplicité 2, les sommets associés aux deux parties égales sont nommés respectivement x_h et y_h . L'ensemble $E(G_\lambda)$ des arêtes de G_λ est donné par

$$E(G_\lambda) = \{(x_{h+1}, x_h), (x_l, y_l), \text{ pour chaque } x_{h+1}, x_h, x_l, y_l \in V(G_\lambda)\}.$$

Example 13. Pour la partition voisine $\lambda = 2 + 1 + 1$, G_λ est donné par $V(G_{2+1+1}) = \{x_2, x_1, y_1\}$, $E(G_{2+1+1}) = \{(x_2, x_1), (x_1, y_1)\}$ et elle a la forme suivante

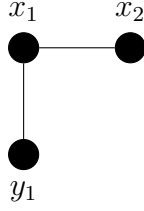


Figure 2.1: Le graphe G_{2+1+1}

On dit d'un sous-graphe H de G_λ qu'il s'étend sur plusieurs sommets si $V(H) = V(G_\lambda)$; on dit qu'il n'a pas de sommets isolés si tout sommet de H est le point final d'une arête dans $E(H)$. Soit λ une partition de voisinage. Nous définissons la signature $\delta(\lambda)$ de λ par:

$$\delta(\lambda) = \sum_H (-1)^{|E(H)|},$$

où H s'étend sur les sous-graphes d'étendue des sommets de G_λ , qui n'ont pas de sommets isolés et $|E(H)|$, comme mentionné précédemment, est le cardinal de $E(H)$.

Soit n un entier positif; pour $i \in \{1, 2\}$, soit $\mathcal{R}_i(n)$ l'ensemble des partitions de n dont les parties sont plus grandes ou égales à $3 - i$, distinctes et congruentes à 0 ou $\pm(i) \pmod{5}$. Les nouvelles identités que nous avons trouvées sont données par le théorème suivant:

Theorem 14. *Soit n un entier positif. Pour $i \in \{1, 2\}$, nous avons les identités suivantes:*

$$\sum_{\lambda \in \mathcal{N}_i(n)} \delta(\lambda) = \sum_{\lambda \in \mathcal{R}_i(n)} (-1)^{\text{taille}(\lambda)}.$$

En considérant la suite génératrice des deux côtés du théorème 14, on trouve l'énoncé équivalent suivant:

Corollary 15. *Pour $i \in \{1, 2\}$, on a:*

$$\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|} = \prod_{j \geq 3-i, j \equiv 0, \pm i \pmod{5}} (1 - q^j).$$

où $|\lambda|$ est la somme des parties de λ .

Le Chapitre 8 est consacré à l'étude de la série de Hilbert et de la base de Grobner de la famille des idéaux définissant les schémas de jets d'un point double $D = \text{Spec}K[x]/x^2$, pour K un corps de caractéristique zéro. Rappelons que le schéma de jets $m - 1$ de X est défini comme l'espace des cartes: $\text{Spec}K[t]/t^m \rightarrow X$. Dans le cas du point double, une telle

carte est définie par un polynôme : $x(t) = x_0 + x_1t + \dots + x_{m-1}t^{m-1}$, où $x(t)^2 \equiv 0 \pmod{t^m}$. En développant cette équation, on obtient un ensemble de polynômes dans l'anneau de polynômes $R_m = K[x_0, \dots, x_{m-1}]$ défini comme :

$f_k = \sum_{i=0}^{k-1} x_i x_{k-1-i}$, for $k = 1, \dots, m$. L'idéal I_m de R_m engendré par f_1, \dots, f_m est l'idéal de définition du schéma à jets de $m - 1$ du point double.

En attribuant à chaque variable x_i le poids $(i, 1)$, on induit un bigrading sur R_m/I_m comme $\bigoplus_{i,j \geq 1} (R_m/I_m)_{i,j}$. Définir la série de Hilbert bigarrée de R_m/I_m comme étant la série en variables q et t donnée par :

$$H_m(q, t) = \sum_{i,j \geq 0} \dim_k (R_m/I_m)_{i,j} q^i t^j$$

Dans le chapitre 8, suivant [11] nous étudions les termes dominants de la base de Gröbner de I_m par rapport à l'ordre lexicographique inverse et nous en tirons des conséquences concernant la série génératrice de certaines identités de partition :

Definition 16. Soit n un entier positif. Soit λ une partition de n . Indique que $SC(\lambda)$ est la paire des plus petites parties consécutives ou égales de λ . Pour $SC(\lambda) = (i, j)$, notons $|SC(\lambda)|$ la somme $i + j$. Définissez la valeur k de λ comme suit : $k(\lambda) = |SC(\lambda)| - |\lambda| - 1$. Pour une partition λ sans parties consécutives ou égales, nous considérons $|k(\lambda)| = +\infty$.

On appelle Δ_m -partitions de n l'ensemble $\Delta_m(n)$ de partitions λ de n avec des parties inférieures ou égales à m tel que pour tout $k(\lambda)$ parties $i_1 \leq \dots \leq i_{k(\lambda)}$ de λ inférieur ou égal à $\lfloor \frac{m - 3k - 1}{2} \rfloor + 1 + 2(k - 1)$ nous avons $|i_t - i_{t+1}| \leq 1$ pour quelques t .

Soit $\Delta_m(n, p)$ l'ensemble de toutes les Δ_m -partitions de n de taille p .

Definition 17. Définissez la série génératrice de $\Delta_m(n, p)$ comme étant la série à deux puissances variables donnée par :

$$\mathbf{H}_m(q, t) = \sum_{n,p \geq 0} \Delta_m(n, p) q^n t^p$$

Theorem 18. Soit $n \geq 1$ un entier. La série génératrice des partitions Δ_m est donnée par :

$$\mathbf{H}_m(q, t) = H_m(q, qt) = \sum_{p=0}^{\infty} \frac{\binom{h(m,p)+1}{p}_q q^{p^2} t^p}{(1 - q^{m+1-h(m,p)}t) \dots (1 - q^m t)}$$

où $h(m, p) = \lfloor \frac{m-p}{2} \rfloor$ et $\binom{a}{b}_q = \frac{(1-q) \dots (1-q^a)}{(1-q) \dots (1-q^b) (1-q) \dots (1-q^{a-b})}$

Mots clefs Cycle orienté, chemin orienté, digraphe fort, nombre chromatique, identités de Rogers-Ramanujan, partitions entières, partitions voisines, idéaux monomiaux, graphes simples, séries de Hilbert.

Chapter 3

Preliminaries of Graph Theory

The introduction will be divided into two parts to acquaint the reader with the terminologies of each research direction we shall represent in this thesis.

3.1 Graphs, Paths, and Cycles

A graph is nothing but a collection of nodes called vertices that might be connected by a line, called an edge. Those mathematical structures first appeared in the paper of Leonhard Euler on the seven bridges of Königsberg[27]. The two enormous islands of Kneiphof and Lomse, which were connected to the two parts of the city's mainland by seven bridges, were part of Königsberg, which was situated on both sides of the Pregel River. The challenge was coming up with a route through the city that would only cross each of those bridges once. Euler noticed that the route chosen inside each land mass is unimportant. The argument was done over the order of the bridges crossed in the route. By removing all features other than the list of land masses and the bridges connecting them, he was able to reformulate the issue in abstract terms and establish the groundwork for graph theory. Each land mass is replaced by an abstract "vertex" or node, and each bridge is replaced by an abstract link, or "edge".

To this day, today's technology may regard graphs as the ultimate abstraction for a variety of real-world issues. Graphs are most useful when they identify arbitrary, important relationships in data that would otherwise be missed. A good example to talk about here is social graphs. Social graphs draw edges between you and the people, places, and things you interact with online. The finest example of using graphs in social relations is perhaps Facebook's Graph API. On the Graph API, everything is a vertex. It includes things like Users, Pages, Places, Groups, Comments, Photos, Photo Albums, Stories, Videos, Notes, Events, and so on. And every link or relationship is considered as an edge. This might include a user publishing a photo or a comment, a user upgrading their relationship status, etc. These sets of vertices

by $E(G)$. For x and y two connected vertices, we write the edge connecting them as (x, y) and we say that x and y are adjacent in G . To simplify, we denote the edge $e = (x, y)$ by xy , and we call x and y the end-vertices of e . If two edges e and f have one common end-vertex, we say that they are adjacent. While if they have the same end-vertices, we say that this is a multiple edge. If an edge e has equal end-vertices, it is said to be a loop. In our work, we will consider only simple graphs, that is with no loops and no multiple edges. The order of a graph G is the cardinality of $V(G)$, denoted by $v(G)$. And the cardinality of $E(G)$, denoted by $e(G)$, is called the size of G .

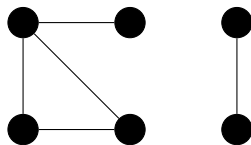


Figure 3.3: A graph G over 6 vertices with 5 edges

3.1.2 Vertex Degree

For a graph G and a vertex x of G , we denote by $N_G(x)$, the set of all vertices of G that are adjacent to x , it is called the set of neighbors of x . The degree of x in G , denoted by $d_G(x)$, is the cardinality of $N_G(x)$. For simplicity, we can omit G from the notation. The maximum degree of a graph G , denoted by $\Delta(G)$ is the maximum of the degrees of the vertices of G . Similarly, the minimum degree of a graph, denoted by $\delta(G)$ is the minimum of the degrees of the vertices of G . G is said to be k -regular graph if $d(x) = k \forall x \in V(G)$. A vertex x is said to be isolated if $d(x) = 0$, that is, it has no neighbors. One of the theorems that will be relevant to mention is the following:

Theorem 19. [27] For a graph G , we have:

$$\sum_{x \in V(G)} d_G(x) = 2e(G)$$

3.1.3 Subgraphs

A subgraph H of a graph G is a graph H whose vertex set and edge set are subsets of those of G . If H is a subgraph of G , then G is said to be a supergraph of H . An induced subgraph of a graph G is a subgraph H of G formed from a subset of the vertices of $V(G)$ and all of the edges in G connecting pairs of vertices in that subset. Formally, consider any graph $G = (V, E)$ and any subset S of V , the graph whose vertex set is S and whose edge set is made up of all of the edges in E that have both ends in S is thus said to be the induced subgraph of G by S and denoted by $G[S]$. A spanning subgraph H of G is a subgraph that satisfies $V(H) = V(G)$, no condition is required over the edges of H .

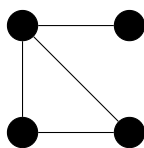


Figure 3.4: This graph H is an induced subgraph of the graph G in Figure 3.3

Actually, we can use set operations to extract from a graph or more a new graph. Specifically, let G be a graph and let S be a subset of the vertices of G , $G - S$ is the induced subgraph of G by $V(G) - S$, that is, the subgraph obtained from G by deleting the vertices in S , and consequently the edges connecting them. Basically, the union of two graphs G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$.

3.1.4 Particular Graphs

We can obtain graphs with certain structures by setting constraints on the set of vertices or edges. Some of those particular graphs will be the heart of our study in Chapters 2 and 3.

- **Complete Graph:** A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. A complete graph over n vertices is denoted by K_n .

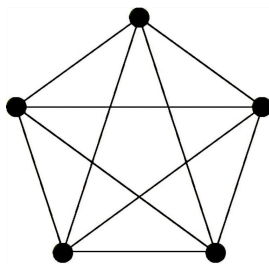


Figure 3.5: The complete graph K_5 over 5 vertices

- **Walk:** A walk is a sequence of alternating vertices and edges of the form $v_1e_1v_2e_2 \dots v_{n-1}e_{n-1}v_n$, where $e_i = v_iv_{i+1}$ for $i = 1, \dots, n - 1$. If $v_1 = v_n$, we call this a closed walk. The length of a closed walk is the number of edges in this walk. If this number is odd, we call it an odd closed walk, otherwise, we say that it is an even closed walk.

- **Path:** For a graph P over the n vertices v_1, \dots, v_n , P is said to be a path of length $n - 1$ if $E(P) = \{v_i v_{i+1}, 1 \leq i \leq n - 1\}$. Usually, we write this graph as $P = v_1 \dots v_n$ and we set v_1 and v_n to be the end-vertices of P . It is clear that $e(P) = n - 1$, which is the length of P denoted by $l(P)$. We say that P is a $v_1 v_n$ -path and for any two vertices v_i and v_j in $V(P)$, we denote by $P_{[v_i, v_j]}$ the subpath of P with end-vertices v_i and v_j , that is $P_{[v_i, v_j]} = v_i v_{i+1} \dots v_j$. Given a graph G , a hamiltonian path P of G is a spanning path in G , that is $V(P) = V(G)$.

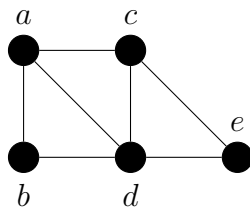


Figure 3.6: A graph G with a hamiltonian path $P = bdace$ of length 4.

- **Connected Graph:** A graph G is said to be connected if for any two vertices x and y of G , G contains, as a subgraph, a path P of end-vertices x and y . Else, we say that G is disconnected. A connected component of G is a connected subgraph H of G that is not a part of a larger connected subgraph of G .
- **Cycle:** A cycle is a path whose end-vertices are equal. To simplify, a graph is said to be a cycle, denoted by $C = v_1 \dots v_n v_1$, if $V(C) = v_1, \dots, v_n$ and $E(C) = \{v_i v_{i+1}, 1 \leq i \leq n - 1\} \cup \{v_1 v_n\}$. For simplicity we might write C as $v_1 \dots v_n$. The length of C , denoted by $l(C)$, is the number of the edges in C . If C is a cycle with odd length, we say that C is an odd cycle. Else, it is called an even cycle. If n is even we say that C is an even cycle, else it is called an odd cycle. Similar to paths, for P the subpath of C of end-vertices v_i and v_j , we write P as $C_{[v_i, v_j]}$. For a graph G , define the girth of G , denoted by $g(G)$, to be the length of the shortest cycle in G . A chord of a cycle C is an edge not in the edge set of C whose endpoints lie in the vertex set C . A cycle with no chord is said to be chordless.

Given a graph G , a hamiltonian cycle C of G is a spanning cycle in G . A hamiltonian graph is a graph that contains a hamiltonian cycle. However, if a graph G has no cycles as subgraphs, we say that G is acyclic, or often a forest. A connected forest is called a tree.

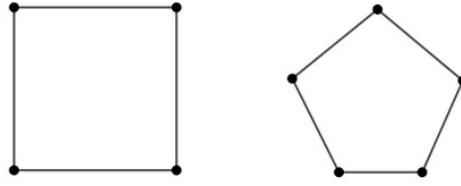


Figure 3.7: Even and Odd cycles of lengths 4 and 5 respectively.

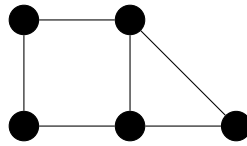


Figure 3.8: An example of a Hamiltonian graph

A lemma that will be useful in the sequel is the following:

Lemma 20. *If G is a graph that contains an odd closed walk, then G contains an odd cycle.*

- **Bipartite Graph:** A bipartite graph G is a graph for which we can partition $V(G)$ into two disjoint sets A and B and for which the edges of G can only occur between a vertex in A and another in B . We usually write $G = (A, B, E)$. We say that A and B are stable sets of G ; that is $E(G[A]) = E(G[B]) = \phi$.

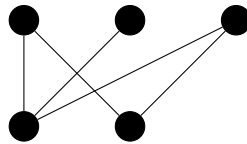


Figure 3.9: An example of a bipartite graph

The following is a relevant theorem that we will use in our study over cycles in directed graphs:

Theorem 21. *{König, 1920} A graph G over at least two vertices is a bipartite graph if and only if it contains no odd cycles.*

Proof. \Rightarrow The first direction is quite easy. Indeed, let $G = (A, B, E)$ be a bipartite graph that contains a cycle $C = v_1 \dots v_n v_1$, with $v_1 \in A$ without loss of generality. Every edge of C has one end in A and the other in B . Thus, when starting with v_1 in A , to end up at v_1 in A , we have to pass an even number of edges. Hence, C is an even cycle.

\Leftarrow Let $G = (V, E)$ be a graph with no odd cycles. We will assume

without loss of generality that G is connected. Let v_0 be a vertex in G . We will partition V to A and B as follows:

$$A = \{v \in V / : \text{the shortest path between } v \text{ and } v_0 \text{ is of even length}\}$$

$$B = \{v \in V / : \text{the shortest path between } v \text{ and } v_0 \text{ is of odd length}\}$$

We claim that A is stable. Suppose for contradiction that A contains an edge xy for $x, y \in A$. Then the union between the shortest path between x and v_0 , the shortest path between y and v_0 , and the edge xy is an odd closed walk. Hence G contains an odd cycle, contradiction. Similarly we can see that B is a stable set of G . Therefore, G is a bipartite graph. \square

- **Wheel** A wheel is a graph made up of a chordless cycle and a vertex adjacent to at least three vertices of the cycle, such vertex is called universal.

3.1.5 Hyper-Graphs

A hyper-graph is a graph generalization in which one edge can connect any collection of vertices. In contrast, an edge in a graph connects exactly two vertices. Basically, a set V , whose components are referred to as vertices, and a family E of subsets of V , referred to as edges or hyper-edges, together create a hyper-graph. It is denoted by $H = (V, E)$. The concept of a hyper-graph is a variant of the familiar concepts of a complex, a block design and a network.

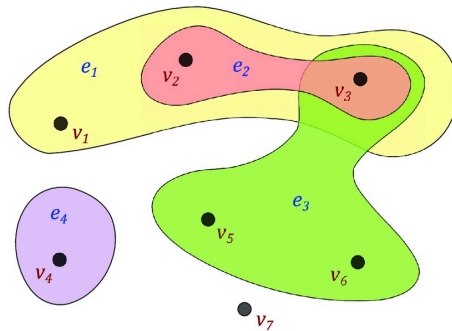


Figure 3.10: An example of a hyper graph with 7 vertices and 4 hyper-edges

3.2 Digraphs

A digraph is an orientation of a graph. Indeed, we have seen that a graph can be a representation of a data network as social relationships on Facebook.

Some relationships can be double-sided, as Instagram. It is a two-way connection between two people, X and Y, in which X can choose whether or not to follow Y, and vice versa. Even though X followed Y, unless Y followed X back, he will have no access on the latter's data.

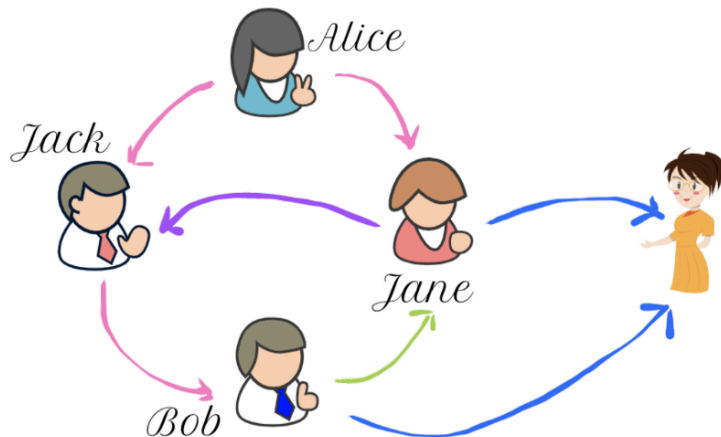


Figure 3.11: An example of relations of Instagram represented as a directed graph

3.2.1 Basic Definitions

A digraph D is shortly a directed graph. It is made up of two sets: $V(D)$ the set of vertices, and $A(D)$ the set of arcs such that each arc $a \in A(D)$ is an ordered pair (u, v) of vertices of D . We say in this case that a is directed from u to v , with u the tail and v the head of a . We often write V and A for simplicity. We denote by $v(D)$ and $a(D)$ the cardinals of V and A respectively. For two adjacent vertices x and y , we denote by xy the arc between x and y if its orientation is unknown. Considering a digraph D and ignoring the directions of its arcs, we obtain a graph which we call the underlying graph of D and will be denoted by $G(D)$.

As we defined before a sub graph of a graph, we can define a sub digraph of a digraph with the same terminologies. As previously specified, we can also define a bipartite digraph to be a digraph whose underlying graph is bipartite.

3.2.2 Vertex Degree

Consider a digraph D and let $a = (u, v)$ be an arc of D oriented from u to v , then we say that u is an in-neighbor of v and that v is an out-neighbor of u . For a vertex v in D , we denote by $N_D^+(v)$ the set of all out-neighbors of v in D and by $N_D^-(v)$ the set of all in-neighbors of v in D . The cardinal of $N_D^+(v)$

will be called the out-degree of v and denoted by $d_D^+(v)$. Similarly, we define the in-degree of v , denoted by $d_D^-(v)$, to be the cardinal of $N_D^-(v)$. For v a vertex of D satisfying $d_D^+(v) = 0$, we say that v is a sink of D . However, if $d_D^-(v) = 0$, we say that v is a source of D .

Euler's identity[27] for graphs and vertex-degrees can be extended to digraphs as the following:

Theorem 22. *Let D be a digraph, we have:*

$$\sum_{v \in V} d_D^+(v) = \sum_{v \in V} d_D^-(v) = a(D)$$

Define the maximum out-degree of D , denoted by $\Delta^+(D)$, to be the maximum of $d_D^+(v)$, for all $v \in D$. Define the minimum out-degree of D , denoted by $\delta^+(D)$, to be the minimum of $d_D^+(v)$, for all $v \in D$. Similarly we can define the maximum in-degree and the minimum in-degree of D , denoted by $\Delta^-(D)$ and $\delta^-(D)$ respectively.

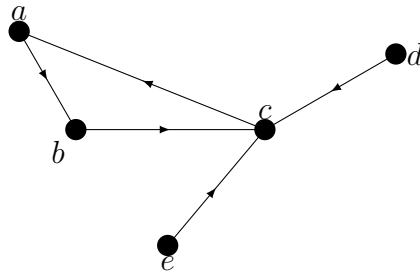


Figure 3.12: A digraph D over 5 vertices.
Notice that $d_D^+(c) = 1$ and $d_D^-(c) = 3$.

3.2.3 Particular Digraphs

- **Tournaments:** A tournament T is shortly an orientation of a complete graph. For a tournament over n -vertices, we write T_n .

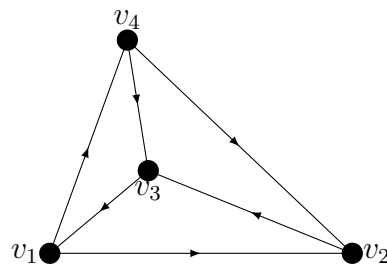


Figure 3.13: A tournament T_4 over 4 vertices

- **Oriented Paths:** An orientation of a path is said to be an oriented path. The length of an oriented path P is exactly the length of $G(P)$. For $P = v_1 \dots v_n$ an oriented path, we denote by $P_{[v_i, v_j]}$ the oriented subpath of P $v_i v_{i+1} \dots v_j$, for $1 \leq i \leq j \leq n$. An arc (v_i, v_j) of P is said to be forward if $j = i + 1$. Else it is said to be backward. If all the arcs of P have the same orientation, we say that P is a directed path. For instance, for an oriented path P over the vertex-set $V(P) = \{v_1, \dots, v_n\}$, we say that P is a directed path if $A(P) = \{(v_i, v_{i+1}), 1 \leq i \leq n - 1\}$. In this case, we say that P is a $v_1 v_n$ -directed path with v_1 its origin and v_n its terminus. A spanning oriented path of a digraph D is called hamiltonian.

A block of an oriented path P is a maximal directed subpath of P . It is said to be a forward block, respectively backward block, if all its arcs are forward in P , respectively backward. We denote by $P^+(k_1, \dots, k_s)$ an oriented path of s -consecutive blocks starting by a forward block of length k_1 . However, we denote by $P^-(k_1, \dots, k_s)$ an oriented path of s -consecutive blocks starting by a backward block of length k_1 . And we call this a path with s -blocks or an s -block path. An anti-directed path is an oriented path of the form $P(1, 1, \dots, 1)$.

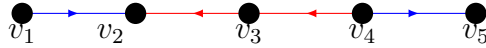


Figure 3.14: An example of an oriented path $P^+(1, 2, 1)$

- **Oriented Cycles:** The notion of cycles in graphs can be extended to digraphs. An oriented cycle is simply an orientation of a cycle. Given an oriented cycle C , a block is a maximal directed subpath of C . We denote by $C^+(k_1, k_2, \dots, k_n)$ the oriented cycle formed of n blocks of lengths k_1, k_2, \dots , and k_n respectively, starting by k_1 forward arcs. Conversely, we can define $C^-(k_1, k_2, \dots, k_n)$ to be the oriented cycle formed of n blocks of lengths k_1, k_2, \dots , and k_n respectively, starting by k_1 backward arcs. The length of C , denoted by $l(C)$, is the sum of lengths of all its blocks. Moreover, for a block B directed from x to y of an oriented cycle C , we call x (resp. y) the source (resp. the sink) of B , and we write B as $C_{[x,y]}$. An oriented cycle of arcs with the same direction is called a directed cycle or a circuit. An oriented cycle whose all blocks are each of length 1 is called an anti-directed cycle.

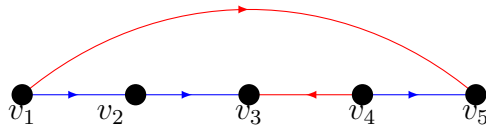


Figure 3.15: An example of an oriented cycle $C = C^+(2, 1, 1, 1)$

A hamiltonian oriented cycle is an orientation of a hamiltonian cycle. If a digraph D contains a hamiltonian oriented cycle, we say that D is hamiltonian. However, if D contains no oriented cycles, we say that it is acyclic or an oriented forest.

- **Subdivision of a Digraph:** A subdivision of a digraph D is a digraph D' obtained by replacing each arc (x, y) of D by an xy -directed path of length at least 1.



Figure 3.16: $P = P^+(1, 3, 1)$ is a subdivision of $P^+(1, 2, 1)$

- **Strong Digraphs:** A digraph D is said to be strong or strongly connected if for any two vertices u and v of D , D contains a uv -directed path.
- **Oriented Forests:** An oriented forest is a digraph that contains no oriented cycles. An orientation of a tree is said to be an oriented tree. An out-tree T is an oriented tree with one and only one vertex whose in-degree is equal to zero, called the source or root of T , and the rest of the vertices have in-degree equal to 1. Inversely, an in-tree is an oriented tree T with one vertex of out-degree zero, called the sink of T , and the rest of the vertices are of out-degree 1. Similarly, we can define an in-forest and an out-forest.

Let T be a spanning out-tree of a digraph D rooted at r . For a vertex x of D , there is a unique rx -directed path in T , denoted by $T_{[r,x]}$. The level of x with respect to T , denoted by $l_T(x)$, is the length of this path. For a non-negative integer i , set $L_i(T)$ to be the set of all the vertices having a level i in T . For $x \in V(T)$, the ancestors of x are the vertices that belong to $T_{[r,x]}$. For an ancestor y of x , we will write $y \leq_T x$ and we will denote by $T_{[y,x]}$ the directed path in T from y to x . For two vertices x and y of T , the least common ancestor, z , of x and y is the common ancestor of x and y having the highest level in T . Let D' be a sub-digraph of a digraph D . Let $x \in V(D')$, x is said to be a minimal in D' for \leq_T if $\forall v \in V(D')$ satisfying $v \leq_T x$, we have $x \leq_T v$. Moreover, x is said to be a maximal in D' for \leq_T if $\forall v \in V(D')$ satisfying $x \leq_T v$, we have $v \leq_T x$.

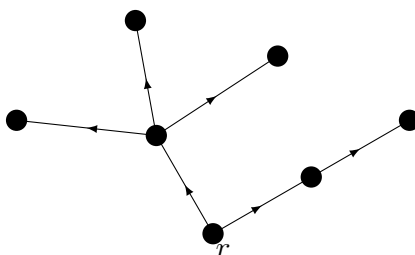


Figure 3.17: An out-tree T rooted at r .

An arc (x, y) of D is said to be forward (resp. backward) with respect to T if $l_T(x) < l_T(y)$ (resp. $l_T(x) \geq l_T(y)$). For two adjacent vertices x and y , we denote by xy the arc between x and y if its orientation is unknown.

A maximal out-tree T of D is a spanning out-tree for which for any backward arc with respect to T , say (x, y) , there exists a yx -directed path in T . We can easily see that for a maximal out-tree T of a digraph D , $L_i(T)$ is stable in D for all $i \geq 0$. We can also notice that every strong digraph D has a spanning out-tree. Indeed, suppose not and let T be an out-tree of D with maximal number of vertices. Then there exists a vertex $x \in V(D) - V(T)$. Since D is strong then there exists a vertex $y \in V(T)$ such that $(y, x) \in A(D)$. Let $T' = T \cup (y, x)$, an out-tree of D with $V(T') > V(T)$, a contradiction. Moreover, in [6], Al Mniny proved that any digraph D that has a spanning out-tree admits a maximal out-tree. Consequently, every strong digraph admits a maximal out-tree.

3.3 Coloring

The initial graph coloring results deal entirely with planar graphs in the form of map coloring. Francis Guthrie proposed the four color problem while attempting to color a map of England, finding that four colors were sufficient to color the map so that no parts sharing a common border received the same color.

Here is yet another example of graph coloring. A company produces n chemicals C_1, C_2, \dots, C_n . When these chemicals are brought into touch with one another, some chemical pairings that are incompatible with one another might explode. The factory wants to partition its warehouse into segments so that incompatible chemicals can be kept in separate areas as a precaution. How many segments should the warehouse be divided into at the very least? We will transform this problem into a graph problem. We obtain a graph G on the vertex set $\{v_1, \dots, v_n\}$ by joining v_i and v_j if and only if the chemicals C_i and C_j are incompatible. We will assign the vertices of G colors. What is the minimal number of colors assigned to the vertices of G so that no two

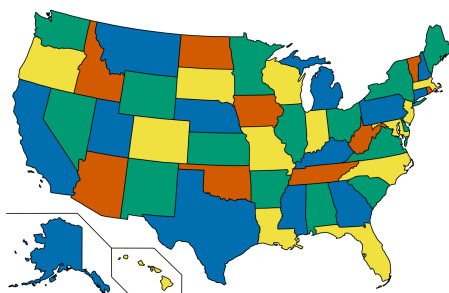


Figure 3.18: A four-coloring of a map of the United States.

adjacent vertices are assigned the same color?

In the sequel we will give the terminology of graph coloring. The same terminology is applicable for digraphs.

3.3.1 Basic Definition

A graph coloring is a labeling of the vertices. In other words, a coloring of a graph G using k colors or labels is a mapping ϕ from the set of vertices of G , $V(G)$, to the set of colors $\{1, 2, \dots, k\}$. If this coloring satisfies that no adjacent vertices are assigned the same color, we call this a k -proper coloring of G and we say that G is k -colorable. The chromatic number of a graph G , denoted by $\chi(G)$, is the minimal number of colors for which such an assignment is possible. A digraph is k -chromatic if its chromatic number equals k .

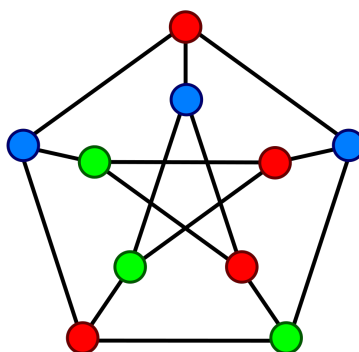


Figure 3.19: A proper coloring of a graph using 3 colors, the minimum number of colors possible.

The following proposition shows the relevance between the chromatic number and the maximum degree of a graph:

Proposition 23. *For any graph G , we have:*

$$\chi(G) \leq \Delta(G) + 1$$

According to Brooks [16], if G is a connected graph which is neither a complete graph nor an odd cycle, then we have $\chi(G) \leq \Delta(G)$.

An intriguing result over the chromatic number of the union of two digraphs is the following:

Theorem 24. [21] *Consider any two digraphs D_1 and D_2 , we have:*

$$\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2).$$

3.3.2 Critical Graphs

A graph G is said to be k -critical if $\chi(G) = k$ and $\chi(G - v) = k - 1$ for every vertex $v \in G$, for k a positive integer. We can see that every k -chromatic graph G contains a k -critical subgraph H . Indeed, if G is not k -critical, we can proceed by deletion of vertices until we obtain such a subgraph.

The following is a relevant proposition for our study over chromatic number and paths in digraphs that we will see later:

Proposition 25. *Let G be a k -critical graph, for k a positive integer, then $d_G(v) \geq k - 1$ for every $v \in V(G)$.*

Proof. Let G be a k -critical graph. Suppose for the contrary that G has a vertex v with a degree at most $k - 2$. Let $H = G - v$, as G is k -critical, we have that $\chi(H) = k - 1$. Let us consider ϕ to be a $(k - 1)$ -proper coloring of H with the color set $\{1, \dots, k - 1\}$. As v has at most $k - 2$ neighbors, let us say without loss of generality that those neighbors are assigned by ϕ colors from the set $\{1, \dots, k - 2\}$. We will extend ϕ to a coloring of G , ϕ' , by labeling v by the color $k - 1$. This is a $(k - 1)$ -proper coloring of G , a contradiction. \square

Chapter 4

Preliminaries of Integer Partitions and Commutative Algebra

In this chapter, we represent the terminology and definitions concerning the second direction of our study.

4.1 Partition Theory

The theory of number partitions is an intriguing field of number theory. Leonard Euler introduced the notion of partitions in the 18th century. Although the theory of partition had been explored and debated by many other notable mathematicians such as Gauss, Jacobi, Schur, McMahon, and Andrews, among others, Ramanujan's collaboration with Prof. G.H. Hardy created a revolutionary change in the field of partition theory of numbers.

4.1.1 Basic Definitions

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_r$ that sum up to n . The λ_i are called the parts of λ . For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n , we often write $|\lambda|$ for the sum of parts of λ , which is exactly n . Set r to be the size or length of λ , denoted by $l(\lambda)$. We usually write $\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \dots)$, such that f_i is the number of parts of λ that are equal to i .

Define the partition function $p(n)$ to be the number of integer partitions of n . By convention, $p(0) = 1$. Denote by P the set of all integer partitions.

Example 26. • $p(1)=1$ as 1 has only one integer partition which is 1.

- $p(4)=5$ as 4 has the following integer partitions:

4

3+1

2+2=2²

2 + 1 + 1 = 1²2

1 + 1 + 1 + 1 = 1⁴

For $\lambda = (\lambda_1, \dots, \lambda_r)$ a partition, we may define a new partition $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ such that λ'_i is the number of parts of λ that are greater than or equal to i . Such a partition is called the conjugate of λ .

Example 27. Let $\lambda = (8, 6, 6, 5, 1)$ be a partition of 26. The conjugate of λ is

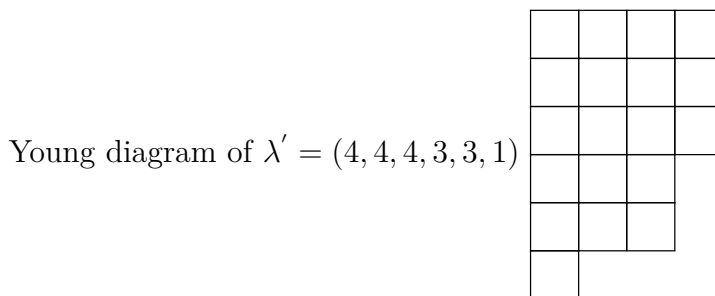
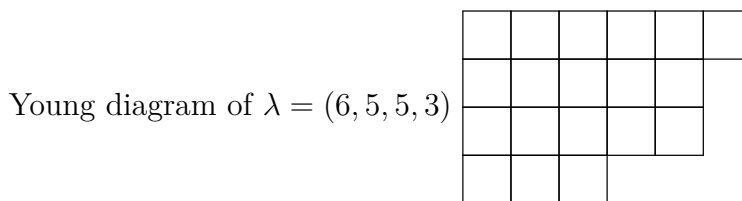
$\lambda' = (5, 4, 4, 4, 4, 3, 1, 1)$.

4.1.2 Graphical Representation of Partitions

An effective elementary device to study partitions is the graphical representation.

Let λ be an integer partition of a positive integer n . A Young diagram $[\lambda]$ of λ is a collection of n 1×1 squares (i, j) on a square grid \mathbf{Z}^2 , with $1 \leq i \leq l(\lambda)$ and $1 \leq j \leq \lambda_i$.

The conjugate Young diagram $[\lambda']$ is the reflection of $[\lambda]$ through the line $i = j$. And it is obvious that applying the conjugate twice returns us to the initial partition λ .



A partition identity is an equality carried on between the number of integer partitions of n that satisfies a condition A and the number of integer

partitions of n that satisfies a condition B , which works for all positive integers n .

The following is the first identity of integer partitions which we will expose the reader to and will be demonstrated using the Young diagram:

Theorem 28. *Let n and m be two positive integers. The number of integer partitions of n with at most m parts is equal to the number of integer partitions of n with parts that are less than or equal to m .*

Proof. In fact, by mapping each partition onto its conjugate, we may establish a one-to-one relationship between the two classes of partitions. This mapping is a bijection, and we can see from the Young diagram that the condition "at most m parts" changes to the condition "no part exceeds m " and vice versa. \square

Example 29. *Let us consider the partitions of 6, "into at most 2 parts": (6), (5, 1), (4, 2), and (3, 3). And then consider the partitions of 6 "into parts less than or equal to 2": (1, 1, 1, 1, 1, 1), (2, 1, 1, 1), (2, 2, 1, 1) and (2, 2, 2), which are the conjugates of the latter respectively.*

4.1.3 Generating Series of Integer Partitions

Definition 30. *The generating function $f(q)$ of the sequence a_0, a_1, \dots is the power series $f(q) = \sum_{n \geq 0} a_n q^n$.*

It is sufficient to think of $f(q)$ as a formal power series in q for many of the problems we will encounter.

Definition 31. *Define the generating series of one variable of the partition function to be the power series given by:*

$$P(q) = \sum_{n \geq 0} p(n)q^n$$

Theorem 32. *The generating series of the partition function is given by:*

$$P(q) = \frac{1}{\prod_{n \in \mathbf{N}} (1 - q^n)}$$

Proof. Indeed, we can write the right-sided product as the following:

$$\begin{aligned} & \frac{1}{\prod_{n \in \mathbf{N}} (1 - q^n)} \\ &= \prod_{n \in \mathbf{N}} (1 + q^n + q^{2n} + q^{3n} + \dots) \\ &= (1 + q + q^2 + q^3 + \dots)(1 + q^2 + q^4 + q^6 + \dots)(1 + q^3 + q^6 + q^9 + \dots) \dots \\ &= \sum_{a_1 \geq 0} \sum_{a_2 \geq 0} \sum_{a_3 \geq 0} \dots q^{1a_1 + 2a_2 + 3a_3 + \dots} \end{aligned}$$

Observe that the exponent of q in this series corresponds to the partition $(1^{a_1}2^{a_2}3^{a_3}\dots)$. Hence q^n will occur once in this sum for each partition of n . Therefore, we get that $P(q) = \frac{1}{\prod_{n \in \mathbf{N}} (1 - q^n)}$. \square

Definition 33. Let n be a positive integer. Define $P_{n,k}$ to be the set of integer partitions of n with at most k parts, and denote by $p_k(n)$ its cardinal.

Considering the conjugate mapping, we can find a bijection between $P_{n,k}$ and the set of partitions of n with all parts are less than or equal to k . Hence and following the same reasoning as the previous theorem, we can see that the generating series of $p_k(n)$ is given as follows:

$$P_k(q) = \sum_{n \in \mathbf{N}} p_k(n)q^n = \frac{1}{\prod_{n \leq k} (1 - q^n)}$$

Definition 34. Let n be a positive integer. Define (O, n) to be the set of integer partitions of n with only odd parts, and denote by $p(O, n)$ its cardinal. Similarly, define (D, n) to be the set of integer partitions of n with distinct parts, and denote by $p(D, n)$ its cardinal.

One of the most interesting identities of integer partitions given by Euler is the following:

Theorem 35. (Euler)

Let n be any positive integer, we have:

$$p(O, n) = p(D, n)$$

Proof. For the sake of proving this identity, we will prove an identity over the generating series of either partition function.

First, let us notice that if we follow the same reasoning as theorem 26, we find that:

$$\sum_{n \geq 0} p(O, n)q^n = \frac{1}{\prod_{n \geq 0} (1 - q^{2n+1})}$$

Similarly, we have:

$$\sum_{n \geq 0} p(D, n)q^n = \prod_{n \geq 1} (1 + q^n)$$

Noticing that:

$$\prod_{n \geq 1} (1 + q^n) = \prod_{n \geq 1} \frac{(1 - q^{2n})}{(1 - q^n)} = \prod_{n \geq 1} (1 - q^{2n-1})^{-1}$$

we get that:

$$\sum_{n \geq 0} p(O, n)q^n = \sum_{n \geq 0} p(D, n)q^n$$

By equating the coefficients of q^n for $n \geq 1$ in both power series, we get Euler's identity. \square

We can also define the generating series of the partition function of two variables taking into consideration the sizes of the partitions.

Definition 36. *Define the generating series of the partition function of two variables to be the power series given by:*

$$P(q, t) = \sum_{\lambda \in P} q^{|\lambda|} t^{l(\lambda)}$$

4.1.4 Rogers-Ramanujan Identities

The Rogers-Ramanujan identities apply to fundamental hypergeometric series and integer partitions. Leonard James Rogers originally found and established the identities in 1894, which were later rediscovered (without proof) by Srinivasa Ramanujan before 1913. Rogers-Ramanujan identities continue to be one of the most intriguing episodes in the history of partitions. These identities appear in many fields other than combinatorics, such as statistical mechanics, number theory, representation theory, algebraic geometry, probability theory or commutative algebra [1, 2, 3, 4, 5, 8, 10, 12, 17, 18, 29, 31, 34].

Theorem 37. *(Rogers-Ramanujan Identities)*

Let n be a positive integer. For $i \in \{1, 2\}$, let $\mathcal{T}_i(n)$ be the set of partitions of n without equal nor consecutive parts and the part 1 appears at most $i - 1$ times. Let $\mathcal{E}_i(n)$ be the number of partitions of n into parts congruent to $\pm(2 + i) \pmod{5}$. Then we have

$$|\mathcal{T}_i(n)| = |\mathcal{E}_i(n)|.$$

The notation $|A|$ in the theorem stands for the cardinal of a set A .

Example 38. $\mathcal{E}_2(4) = \{4, 1 + 1 + 1 + 1\}$ and $\mathcal{T}_2(4) = \{4, 3 + 1\}$

Note that it is not difficult to obtain the generating series of the second object, namely:

$$\begin{aligned} \text{For } i = 1, & \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+2})(1 - q^{5j+3})} \\ \text{For } i = 2, & \prod_{j=0}^{\infty} \frac{1}{(1 - q^{5j+1})(1 - q^{5j+4})} \end{aligned}$$

4.2 Commutative Algebra

For the sake of establishing a common language, the following introduces some notations and elementary definitions.

4.2.1 Basic Definitions

A ring is an abelian group $(R, +)$ over which we can define a multiplication operation:

$(a, b) \rightarrow ab$ and an identity element 1, satisfying the following for every a, b , and c in R :

$$\begin{aligned} a(bc) &= (ab)c && \text{associativity} \\ a(b+c) &= ab+ac \\ (b+c)a &= ba+ca && \text{distributivity} \\ 1a &= a1 = a && \text{identity} \end{aligned}$$

A ring R is said to be commutative if $ab = ba$ for all $a, b \in R$. Nearly, we will deal with commutative rings in this thesis and we shall omit the adjective. An element $u \in R$ is said to be invertible if there exists an element $v \in R$ so that $uv = 1$. Such v is unique and is denoted by u^{-1} , and called the inverse of u . A field is a ring in which every non-zero element is invertible. We write $\mathbf{Z}, \mathbf{Q}, \mathbf{R}$ and \mathbf{C} respectively for the ring of integers, and fields of rational, real, and complex numbers. The characteristic of a ring R , often denoted $\text{char}(R)$, is defined to be the smallest positive integer n so that $n \cdot 1 = 0$. If such n doesn't exist, we say that R is of characteristic zero.

An ideal I of a commutative ring R is an additive subgroup such that $rs \in I$ for all $r \in R$ and $s \in I$. An ideal is said to be generated by $S \subset R$ if every element $t \in I$ can be written in the form:

$$t = \sum_1^n r_i s_i \text{ for } r_i \in R \text{ and } s_i \in S$$

We usually write $I = \langle S \rangle$. By convention, the ideal generated by the empty set is 0. An ideal generated by only one element is said to be principal. I is said to be finitely generated if it is generated by a finite set of elements. For I and J two ideals of R , define the ideal quotient of I by J , denoted by $(I : J)$, to be the ideal of R given by:

$$(I : J) = \{f \in R / : fJ \subset I\}$$

If A and B are two rings, then the direct product of A and B , denoted by $A \times B$, is the ring of ordered pairs (a, b) with $a \in A$ and $b \in B$ defined with the following operations:

$$\begin{aligned} (a, b) + (a', b') &= (a + a', b + b') \\ (a, b)(a', b') &= (aa', bb') \end{aligned}$$

A ring homomorphism or a ring map, or shortly homomorphism, from a ring R to a ring S , is a homomorphism of abelian groups that preserves multiplication and associates the identity of R to the identity of S . A subring of R is a subset of R closed under addition, subtraction, and multiplication, and contains the identity of R . The commutative algebra over a commutative ring R is a commutative ring S together with a homomorphism of rings $\phi : R \rightarrow S$. A subring S' of S that includes the image of R is said to be a subalgebra of S . An interesting example of an R -algebra is the polynomial ring $S = R[x_1, \dots, x_r]$ in finitely many variables.

Let k be a commutative ring, and let us consider the polynomial ring over k in the variables x_1, \dots, x_r , denoted by $k[x_1, \dots, x_r]$. The elements of k are generally referred to as scalars. A monomial is a product of variables, it has a degree equal to the number of those factors considering the repetitions. The unique monomial of degree 0 is 1. A term is a product of a scalar with a monomial. Every polynomial f can be uniquely written as a finite sum of non-zero terms. It is said to be homogeneous if all those monomials have the same degree.

For R a ring and I an ideal of R , we can define an equivalence relation \sim on R as follows:

$$a \sim b \text{ if and only if } a - b \in I$$

For $a \in R$, the equivalence class of a in R is given by:

$$[a] = a + I = \{a + r / : r \in I\}$$

The set of all those equivalence classes form a ring accompanied with the following operations:

$$\begin{aligned} (a + I) + (b + I) &= (a + b) + I \\ (a + I)(b + I) &= (ab) + I \end{aligned}$$

It is called the quotient ring or factor ring and denoted by R/I .

For a ring R , an R -module M is an abelian group with a map: $R \times M \rightarrow M$ satisfying the following for every $r, s \in R$, $m, n \in M$:

$$\begin{aligned} r(sm) &= (rs)m \\ r(m + n) &= rm + rn \\ (r + s)m &= rm + sm \\ 1m &= m \end{aligned}$$

The most intriguing examples of R -modules are the ideals of R and their corresponding quotient rings.

If M and N are two R -modules, define the direct sum of M and N to be the

module $M \oplus N$ given by $\{(m, n) / : m \in M, n \in N\}$ with $r(m, n) = (rm, rn)$ for $r \in R, m \in M$, and $n \in N$. A free R -module is a module that is isomorphic to a direct sum of copies of R . The direct sum of n copies of R , denoted by R^n , is a free R -module with basis $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$.

4.2.2 Graded Rings

We will apply grading on rings, namely polynomial rings, as well as the objects we will analyze as ideals and quotient rings. The main idea behind this tool is to understand the properties of a graded object X by seeing it as a direct sum of vector spaces and analyzing the attributes of each vector space.

A graded ring is a ring R together with a direct sum decomposition of abelian groups:

$$R = R_0 \oplus R_1 \oplus R_2 \dots$$

where

$$R_i R_j \subset R_{i+j}, \text{ for } i, j \geq 0$$

A quite fascinating example is the graded polynomial ring:

Example 39. We will introduce a grading over the polynomial ring $R = k[x_1, \dots, x_n]$, for k a field. Set $\deg(x_i) = 1$ for all i . The degree of a monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is $\alpha_1 + \dots + \alpha_n$. Now consider R_i to be the k -vector space of homogeneous polynomials of degree i with $R_0 = k$.

An element $a \in R$ is said to be homogeneous if $a \in R_i$, for some i . An ideal I of R is said to be homogeneous if it is generated by homogeneous elements. For $f \in R$, we can uniquely write f as:

$$f = f_0 + f_1 + f_2 + \dots \text{ with } f_i \in R_i$$

where the f_i are called the homogeneous components of f .

Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring, a graded module M over R is a module that satisfies:

$$M = \bigoplus_{i \in \mathbf{N}} M_i$$

for the M_i are abelian groups satisfying $R_i M_j \subset M_{i+j}$, for all i, j .

Considering a grading over a ring R , we may measure its dimension by measuring the dimensions of its graded components. Let M be a finitely generated graded module over $k[x_1, \dots, x_n]$. The Hilbert-Poincaré function of M is given as follows:

$$H_M(s) := \dim_k M_s$$

Define also the Hilbert-Poincaré series to be given by the following q -series:

$$Hilb_M(q) = \sum_{i \in \mathbf{N}} H_M(i) q^i$$

Example 40. • Consider the polynomial ring of one variable x over the field \mathbf{C} . We can write $\mathbf{C}[x]$ as the direct sum of the \mathbf{C} -vector spaces of homogeneous polynomials of degree i , $i \in \mathbf{N}$. Hence:

$$\text{Hilb}_{\mathbf{C}[x]}(q) = 1 + q + q^2 + q^3 + \dots = (1 - q)^{-1}$$

- Let k be a field and consider the polynomial ring over k with variables $\{x_i\}_{i \in \mathbf{N}}$ for which we will assign each x_i a weight i . We will assign this polynomial ring a grading following the decomposition into k -vector spaces of homogeneous polynomials of weight i , denoted by R_i , $i \in \mathbf{N}$. Notice here that R_i is generated by polynomials of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots$ with weight i , that is $1\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = i$, which is a partition of i . Therefore, there is a one-to-one correspondence between the generators of R_i and the partitions of i . Hence we can find that $\dim_K R_i = p(i)$. And so the Hilbert series of this polynomial ring is given by:

$$\text{Hilb}_{K[x_1, x_2, \dots]}(q) = \sum_{i \in \mathbf{N}} p(i)q^i = \prod_{i \geq 1} (1 - q^i)^{-1}$$

The following lemma would be a great exercise to familiarize the readers with the Hilbert Poincaré series:

Lemma 41. (Greuel 2002) Consider the polynomial ring $R = K[x_1, \dots, x_n]$ and let I be a homogeneous ideal of R . Let $f \in R$ be a homogeneous polynomial of degree d . We have:

$$\text{Hilb}_{R/I}(q) = \text{Hilb}_{R/(I, f)}(q) + q^d \text{Hilb}_{R/(I, f)}(q)$$

Example 42. Consider the graded polynomial ring $R = \mathbf{C}[x, y, z]$ for which x, y , and z are each assigned a weight 1. Let I be the homogeneous ideal generated by $\{xy, xz\}$.

$$\begin{aligned} \text{Hilb}_{R/I}(q) &= \text{Hilb}_{R/(I, x)}(q) + q \text{Hilb}_{R/(I, x)}(q) \\ &= \text{Hilb}_{\mathbf{C}[y, z]}(q) + q \text{Hilb}_{\mathbf{C}[x]}(q) \\ &= \frac{1}{(1 - q)^2} + q \frac{1}{(1 - q)} = \frac{1 + q - q^2}{(1 - q)^2} \end{aligned}$$

Many generating series of partitions satisfying some conditions can be seen as Hilbert series of some graded rings. Let K be a field of characteristic zero and consider R to be the graded polynomial ring over the variables $\{x_i\}_{i \in \mathbf{N}}$ considering the grading following the decomposition into k -vector spaces of homogeneous polynomials of weight i , denoted by R_i . For instance, let h be a positive integer; consider the set $D(h)$ of partitions of h into distinct parts. Denote by D the set of all partitions with distinct parts. Let $I_D \in R$ be the

ideal generated by x_i^2 , $i \geq 1$. Note that a monomial $M \in R$, belongs to I_D if and only if it is divisible by one of its generators, say x_i^2 , for some i positive integer. In particular, for a monomial $M_\lambda \in R_h$ associated to a partition λ of h , $M_\lambda \in I_D$ if and only if $\lambda \notin D$. By the definition of a quotient ring, we then have:

$$\lambda \in D \text{ if and only if } M_\lambda \neq 0 \text{ in } R/I_D$$

So we have the equality:

$$\sum_{h \geq 0} |D(h)|q^h = \text{Hilb}_{R/I_D}(q)$$

Now, as the generators of I_D form a regular sequence [49], and since x_i^2 has the weight $2i$, we have:

$$\text{Hilb}_{R/I_D}(q) = \prod_{i \geq 1} \frac{(1 - q^{2i})}{(1 - q^i)} = \prod_{i \geq 1, i \text{ odd}} \frac{1}{(1 - q^i)}$$

Notice that the right sided series is the generating series of partitions with only odd parts. Then we obtain the identity obtained by Euler in theorem 28. In a similar way, we can obtain that the Hilbert series of the quotient ring:

$$\frac{K[x_i, i \geq 1]}{\langle x_i^2, x_i x_{i+1}, i \geq 1 \rangle}$$

is the generating series of the partitions with neither equal nor consecutive parts, which is the type of partitions that appears in Rogers-Ramanujan identity.

4.2.3 Gröbner Basis

Let K be a field and let R be the polynomial ring over K defined over the variables x_1, \dots, x_n . Let M be a free R -module with basis. A monomial ordering on M is a total ordering $<$ defined over the monomial of M so that for any monomials a, b and c with $a \neq 1$, we have $ab < ac$ whenever $b < c$.

For m a polynomial in M and $<$ a monomial ordering over M , define the initial term of m , denoted by $in_{<}(m)$ to be the greatest term of m with respect to $<$. Define the initial ideal of M , denoted by $in_{<}(M)$, to be the monomial submodule generated by the monomials $in_{<}(m)$ for all $m \in M$.

The following will be a crucial lemma to compute the Hilbert series of the quotient ring $R/in_{<}(M)$:

Lemma 43 (Greuel[33]). *Let $<$ be a graded monomial ordering on the polynomial ring R and let $M \subset R$ be a homogeneous ideal and consider its initial ideal $in_{<}(M)$ with respect to $<$. Then:*

$$\text{Hilb}_{R/M}(q) = \text{Hilb}_{R/in_{<}(M)}(q)$$

Consider a free module M with basis. A Gröbner basis of M with respect to an order $<$ is a set of elements $g_1, \dots, g_s \in M$ such that if N is the submodule of M generated by g_1, \dots, g_s , then $in_{<}(g_1), \dots, in_{<}(g_s)$ generate $in_{<}(M)$.

Part I
Cycles and Paths in Digraphs
with Large Chromatic Number

Chapter 5

Four Blocks Cycles

5.1 Introduction

Since the graph coloring was first defined, it was accompanied with many questions. How to determine the chromatic number of a given graph? How to find a graph coloring with as few colors as possible? How does the chromatic number relate to other graph invariants as edges, vertex degree, sub graphs, girth?

Following the definition of the chromatic number we may obtain the following upper bound:

Proposition 44. *For a graph G with m edges, the chromatic number of G satisfies:*

$$\chi(G) \leq \sqrt{2m + \frac{1}{4}} + \frac{1}{2}$$

Proof. Let ϕ be a proper coloring of G with exactly $\chi(G)$ colors. Then G has at least one edge between any two color classes. Thus $m \geq \frac{1}{2}\chi(G)(\chi(G) - 1)$. Solving this inequality for $\chi(G)$, we obtain the assertion claimed. \square

One simple way to color a graph G with not too many colors is the greedy algorithm: Set $V(G) = \{v_1, \dots, v_n\}$, color the vertices one by one in order assigning to v_i the smallest integer not yet assigned to one of its already colored neighbors. Hence we can see that $\chi(G) \leq \Delta(G) + 1$.

Brooks [16] studied the chromatic number for connected graphs rather than complete graphs or odd cycles:

Theorem 45 (Brooks[16]). *Let G be a connected graph. If G is neither complete graph nor odd cycle, then:*

$$\chi(G) \leq \Delta(G)$$

As we have seen, a graph G with large chromatic number must have a large maximum degree, trivially at least $\chi(G) - 1$. What can we say about such graph? Can we guarantee the existence of a complete graph with large order as a subgraph of G ? The answer comes to be No.

Theorem 46 (Erdős [26]). *For every two positive integers g and k , there exists a graph G for which $g(G) > g$ and $\chi(G) > k$.*

Can we answer this question for another graph classes? More specifically, given a graph with large chromatic number, it is natural to ask whether it must contain a cycle with particular properties. König showed that a 2-colorable graph has no cycles of odd length. Erdős and Hajnal [25] supported this by proving that every graph with chromatic number at least k contains an odd cycle of length at least k . Further results on graphs with prescribed lengths of cycles have been obtained [36, 41, 43, 45, 60].

What can we say if we considered the analogous problem for directed graphs?

Indeed, for the case of tournaments, Camion [20] showed that a tournament has a directed hamilton cycle if and only if it is strong. Rosenfeld in [52] conjectured that any oriented cycle of order n is contained in any tournament of order n , provided n is large enough. This has been verified for cycles with a block of length $n - 1$ by Grünbaum, for alternating cycles by Rosenfeld [52] and Thomassen [58], and for cycles with just two blocks by Benhocine and Wojda [13].

In this chapter, we are interested in the study of the existence of oriented cycles in strong digraphs in specific. In [15], Bondy proved that every strong digraph D contains a directed cycle of length at least $\chi(D)$. Since any directed cycle of length at least k can be seen as a subdivision of the directed cycle C_k of length k , Cohen et al. [21] conjectured that Bondy's theorem can be extended to all oriented cycles:

Conjecture 47. (Cohen et al. [21]). *For every oriented cycle C , there exists a constant $f(C)$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C .*

In fact, Cohen et al. [21] proved this conjecture in their article for cycles with two blocks. In particular, they showed that the chromatic number of strong digraphs with no subdivisions of two blocks cycles $C(k_1, k_2)$ is bounded from above by $O((k_1 + k_2)^4)$. More recently, Kim et al. [42] improved this upper bound to $O((k_1 + k_2)^2)$. In [39], El Joubbeh solved this conjecture for Hamiltonian digraphs, and demonstrated a stronger version by showing that any $3n$ -chromatic Hamiltonian digraph contains a subdivision for any

oriented cycle of order n . Furthermore, Cohen et al. in [21] confirmed the above conjecture for the case of some cycles with four blocks.

Theorem 48. *Let D be a strong digraph with no subdivisions of four blocks cycles $C(1, 1, 1, 1)$, then $\chi(D) \leq 24$.*

In [6], Al Mniny proved conjecture 47 for oriented cycles of four blocks of the form $C(k, 1, 1, 1)$ for $f(C) = 8^3k$.

In this Chapter, we improve the bound established in [6] by proving that if D is a digraph having a spanning out-tree T with no subdivisions of $C(k_1, 1, 1, 1)$, then the chromatic number of D is at most $18k_1$.

In our proof, we use for the first time in such investigations the wheels in order to study the chromatic number of the digraphs. Recall that a wheel is a graph made up of a chordless cycle and a vertex adjacent to at least three vertices of the cycle, such vertex is called universal. In [59], Thomassen et al. proved that every graph having no wheel as a subgraph is 3-colorable. This result play an important role in our improvement.

5.2 Coloring Digraphs without Four Blocks Cycles

Given an oriented cycle C , a block is a maximal directed subpath of C . We denote by $C(k_1, k_2, \dots, k_n)$ the oriented cycle formed of n blocks of lengths k_1, k_2, \dots , and k_n respectively. Moreover, for a block B of an oriented cycle C with ends x and y , if B is directed from x to y we denote by x (resp. y) the source (resp. the sink) of B , and we write B as $C_{[x,y]}$. In general, for an oriented cycle C with s blocks, we will denote by B_i the blocks of C , such that B_{2i} has ends x_{i+1} and y_i for all $1 \leq i \leq \frac{s}{2} - 1$ and B_{2i+1} has ends x_{i+1} and y_{i+1} for all $0 \leq i \leq \frac{s}{2} - 1$.

Now we are ready to prove our theorem:

Let k be a positive integer and let D be a digraph with a spanning out-tree. D admits a maximal out-tree, say T .

We used the same partition of D introduced by Al Mniny[6]. However we found less bounds for the chromatic number of each part, sometimes by using cycles and sometimes by using the wheels that facilitated our study of such digraph.

For $i = 0, \dots, k - 1$, let $V_i := \cup_{\alpha \geq 0} L_{i+\alpha k}(T)$. Define D_i to be the subdigraph of D induced by V_i , and then partition the arcs of D_i as follows:

$$\begin{aligned} A_1 &:= \{(x, y) | x \leq_T y\}; \\ A_2 &:= \{(x, y) | y \leq_T x\}; \\ A_3 &:= A(D_i) \setminus (A_1 \cup A_2). \end{aligned}$$

For $0 \leq i \leq k - 1$ and $j = 1, 2, 3$, let D_i^j be the spanning subdigraph of D_i

whose arc-set is A_j .

Let C be a cycle of D of s blocks, say B_1, \dots, B_s with $s \geq 4$. We will use the same notations for the ends of the blocks of C as introduced above such that the x_i 's (resp. y_i 's) are the sources (resp. sinks) of the blocks of C . Without loss of generality, we will suppose that x_1 is minimal in $\{x_i\}_{1 \leq i \leq s/2}$ for \leq_T . Let $0 \leq i \leq k-1$, C is said to be a mixed cycle if all its blocks are induced by arcs in D_i^1 except for the block B_1 that contains a vertex $z_1 \neq y_1$ such that the arcs of $C_{[x_1, z_1]}$ belong to T and the arcs of $C_{[z_1, y_1]}$ belong to D_i^1 . B_1 is called the mixed block of C .

Remark that x_1 is the smallest for \leq_T in C .

Lemma 49. *If D contains a mixed cycle then it contains a subdivision of $C(k, 1, 1, 1)$.*

Proof. Let C be a mixed cycle of D . We will proceed by induction on the number of blocks of C , s .

For $s = 4$, if $l(C_{[x_1, z_1]}) \geq k$ then C is a subdivision of $C(k, 1, 1, 1)$. Else let z be the minimal for \leq_T in $C - C_{[x_1, z_1]}$. We will study three cases depending on the position of z on C . Indeed, if $z \in C_{]x_1, y_1[}$ then we replace in C , $C_{]z_1, z[}$ by $T_{]z_1, z[}$ and get a subdivision of $C(k, 1, 1, 1)$. Else if $z \in C_{]x_1, y_2[}$ then $C_{]z_1, y_1[} \cup C_{[x_2, y_1]} \cup C_{[x_2, y_2]} \cup T_{]z_1, z[} \cup C_{[z, y_2]}$ is a subdivision of $C(k, 1, 1, 1)$. Finally if $z = x_2$, in this case we will introduce z' to be a minimal for \leq_T in $C - C_{[x_1, z_1]} - \{x_2\}$. If $z' \in C_{[x_2, y_1]}$ or $C_{[x_2, y_2]}$ then replace in C , $C_{[x_2, z']}$ by $T_{[x_2, z']}$ and get a subdivision of $C(k, 1, 1, 1)$. Else if $z' \in C_{]x_1, y_2[}$ then replace in C , $C_{]x_2, y_2[} \cup C_{]z', y_2[}$ by $T_{[x_2, z']}$ and get a subdivision of $C(k, 1, 1, 1)$. The only case left to study is if $z' \in C_{]z_1, y_1[}$, then replace in C , $C_{]x_2, y_1[} \cup C_{]z', y_1[}$ by $T_{[x_2, z']}$ and get a subdivision of $C(k, 1, 1, 1)$.

Suppose it is true up to $s-2$ and let's prove it for s , $s \geq 6$. Notice that $z_1 \leq_T x_i$ for all $1 \leq i \leq \frac{s}{2}$. Let $i \neq 1$ be the integer such that $l_T(x_i)$ is minimal. If $i \geq 3$, then $C_{]z_1, y_1[} \cup C_{[x_2, y_1]} \cup \dots \cup C_{[x_i, y_{i-1}]} \cup T_{]z_1, x_i[}$ contains a mixed cycle of D of blocks less than or equal to $s-2$. Otherwise $i = 2$, we can get a mixed cycle of D of blocks less than or equal to $s-2$ by replacing in C , $C_{]z_1, y_1[} \cup C_{[x_2, y_1]}$ by $T_{]z_1, x_2[}$. In both cases, using the induction hypothesis, we get that D contains a subdivision of $C(k, 1, 1, 1)$. \square

Consequently, if D contains a cycle C of s blocks, $s \geq 4$, whose all arcs are in D_i^1 , $0 \leq i \leq k-1$, then D contains a subdivision of $C(k, 1, 1, 1)$.

Let C be a cycle of D of s blocks, say B_1, \dots, B_s . We will use the same notations for the ends of the blocks of C as introduced above such that the y_i 's (resp. x_i 's) are the sources (resp. sinks) of the blocks of C . Let $0 \leq i \leq k-1$, C is said to be back-mixed if $s \geq 6$ and all its blocks are induced by arcs in D_i^2 except for B_s that contains a vertex z_1 such that the arcs of $C_{[z_1, x_1]}$ belong to T , the arcs of $C_{[y_{\frac{s}{2}}, z_1]}$ belong to D_i^2 , $z_1 \neq y_{\frac{s}{2}}$, and z_1 is a minimal in C for

\leq_T . B_s is called the back-mixed block of C .

Remark that x_1 is the smallest for \leq_T in $C - C_{[z_1, x_1]}$.

Lemma 50. *If D contains a back-mixed cycle then it contains a subdivision of $C(k, 1, 1, 1)$.*

Proof. Let C be a back-mixed cycle of D . We will proceed by induction on the number of blocks of C , s . Notice that $x_1 \leq_T x_i$ for all $1 \leq i \leq \frac{s}{2}$.

For $s = 6$. Let $i \neq 1$ be the integer such that $l_T(x_i)$ is minimal. If $i = 2$, then $C_{[y_2, x_2]} \cup C_{[y_2, x_3]} \cup C_{[y_3, x_3]} \cup C_{[y_3, x_1]} \cup T_{[x_1, x_2]}$ contains a subdivision of $C(k, 1, 1, 1)$. Else $i = 3$, then $C_{[y_1, x_1]} \cup C_{[y_1, x_2]} \cup C_{[y_2, x_2]} \cup C_{[y_2, x_3]} \cup T_{[x_1, x_3]}$ contains a subdivision of $C(k, 1, 1, 1)$.

Suppose it is true up to $s - 2$, and let's prove it for s , $s \geq 8$. Let $i \neq 1$ be the integer such that $l_T(x_i)$ is minimal. If $i > 3$, then $C_{[y_1, x_1]} \cup C_{[y_1, x_2]} \cup \dots \cup C_{[y_{i-1}, x_i]} \cup T_{[x_1, x_i]}$ contains a back-mixed cycle of D with blocks less than or equal to $s - 2$, and so D contains a subdivision of $C(k, 1, 1, 1)$. Else if $i = 2$, then replace in C , $C_{]y_1, x_1[} \cup C_{]y_1, x_2[}$ by $T_{[x_1, x_2]}$, this contains a back-mixed cycle of D of blocks less than or equal $s - 2$, and so D contains a subdivision of $C(k, 1, 1, 1)$. Finally if $i = 3$, notice that $C_{[y_1, x_1]} \cup C_{[y_1, x_2]} \cup C_{[y_2, x_2]} \cup C_{[y_2, x_3]} \cup T_{[x_1, x_3]}$ contains a subdivision of $C(k, 1, 1, 1)$. \square

Consequently, if D contains a cycle C of s blocks, $s \geq 6$, whose all arcs are in D_i^2 , $0 \leq i \leq k - 1$, then D contains a subdivision of $C(k, 1, 1, 1)$.

Let C be a cycle of D of 4 blocks, say B_1, \dots, B_4 . We will use the same notations for the ends of the blocks of C as introduced above such that the y_i 's (resp. x_i 's) are the sources (resp. sinks) of the blocks of C . C is said to be a bad 4-blocks cycle if it is either a $C(1, 1, 1, 1)$ (named bad of type 1) or a $C(2, 1, 1, 1)$ such that $B_1 = (y_1, z_1) \cup (z_1, x_1)$ and $x_1 \leq_T x_2 \leq_T z_1 \leq_T y_2 \leq_T y_1$ (named bad of type 2). Else C is said to be a good 4-blocks cycle.

Lemma 51. *Let C be a cycle with 4-blocks in D_i^2 , $0 \leq i \leq k - 1$. If D contains no subdivision of $C(k, 1, 1, 1)$, then C is a bad cycle.*

Proof. Suppose to the contrary that C is a good 4-blocks cycle. Indeed, consider without loss of generality x_1 to be a minimal in C for \leq_T . Let z minimal in $C - \{x_1\}$ for \leq_T . If $z \in C_{]y_i, x_1[}$ for $i = 1$ or 2 then replace in C , (z, x_1) by $T_{[x_1, z]}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $z = x_2$. Let z' be a minimal in $C - \{x_1, x_2\}$ for \leq_T . If $z' \in C_{]y_i, x_2[}$ for $i = 1$ or 2 then replace (z', x_2) by $T_{[x_2, z']}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Notice also that y_1 and y_2 can not be both minimal in $C - \{x_1, x_2\}$ for \leq_T , since else C is bad of type 1, a contradiction. Moreover if without loss of generality y_1 is minimal in $C - \{x_1, x_2\}$ for \leq_T , then let z'' be a maximal in $C - \{y_2\}$ for \leq_T such that $z'' \leq_T y_2$. If $z'' \in C_{]y_2, x_i[}$ for $i = 1$ or 2 then replace in C , (y_2, z'') by $T_{[z'', y_2]}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $z'' = y_1$ or $z'' = x_2$ and so C is bad of type 1, a contradiction. Hence $z' \notin C_{]y_i, x_2[}$

for $i = 1, 2$. If without loss of generality $z' \in C_{]y_1, x_1[}$, let z'' be a minimal in $C - \{x_1, x_2, z'\}$ for \leq_T . If $z'' = y_1$ then replace in C , (y_1, z') by $T_{[z', y_1]}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else if $z'' \in C_{]y_i, x_2[}$ for $i = 1, 2$ then $C_{[z', x_1]} \cup T_{[z', z'']} \cup C_{[y_2, z'']} \cup C_{[y_2, x_1]}$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else if $z'' \in C_{]y_2, x_1[}$ then replace in C , $C_{[z', x_1]} \cup C_{[z'', x_1]}$ by $T_{[z', z'']}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else if $z'' \in C_{]y_1, z'[}$ then $C_{[z', x_1]} \cup T_{[z', z'']} \cup C_{[y_1, z'']} \cup C_{[y_1, x_2]} \cup T_{[x_1, x_2]}$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $z'' = y_2$ then let z''' be a maximal in $C - \{y_1\}$ with $z''' \leq_T y_1$. Notice that z''' can not be y_2 since else C is bad of type 2, where $x_1 \leq_T x_2 \leq_T z' \leq_T y_2 \leq_T y_1$, a contradiction. Hence $z''' \in C_{]y_1, x_2[}$ or $C_{]y_1, z'[}$ and so replace in C , (y_1, z''') by $T_{[z''', y_1]}$ and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. □

Lemma 52. *If D has no subdivision of $C(k, 1, 1, 1)$ then $\chi(D_i^3) \leq 2$, for all $0 \leq i \leq k - 1$.*

Proof. We claim that the underlying graph of D_i^3 is bipartite. Indeed, suppose to the contrary that D_i^3 contains an odd cycle $C = x_1 \dots x_t$. Without loss of generality, suppose that x_1 is with minimal level for T in C . Now we will study two cases:

Case 1: Neither x_t is ancestor of x_2 nor x_2 is ancestor of x_t .

In this case $l_T(x_t) = l_T(x_2)$, since else let y be their least common ancestor, so $T_{[y, x_2]} \cup T_{[y, x_t]} \cup (x_1, x_t) \cup (x_1, x_2)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Thus $t \geq 5$. Similarly, we can see that both $T_{[y, x_t]}$ and $T_{[y, x_2]}$ have lengths less than k .

Notice that x_2 is an ancestor of x_{t-1} . If not, let z be the least common ancestor of x_2 and x_{t-1} . If $x_1 \notin T_{[z, x_{t-1}]}$ then $T_{[z, x_{t-1}]} \cup T_{[z, x_2]} \cup (x_1, x_2) \cup (x_1, x_t) \cup x_{t-1}x_t$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else, $T_{[y, x_t]} \cup x_{t-1}x_t \cup T_{[y, x_2]} \cup (x_1, x_2) \cup T_{[x_1, x_{t-1}]}$ is subdivision of $C(k, 1, 1, 1)$, a contradiction.

As well, x_t is an ancestor of x_3 . Thus, (x_2, x_3) and (x_t, x_{t-1}) are arcs in D_i^3 . Then $T_{[x_t, x_3]} \cup T_{[x_2, x_{t-1}]} \cup (x_2, x_3) \cup (x_t, x_{t-1})$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction.

Case 2: Without loss of generality, suppose that $x_2 \leq_T x_t$.

Let i the smallest integer greater than 2 satisfying $l_T(x_i) > l_T(x_{i-1})$. We will study the following cases:

If $i = 3$. Consider the least common ancestor of x_1 and x_3 , y , then $T_{[y, x_1]} \cup (x_1, x_t) \cup T_{[y, x_3]} \cup (x_2, x_3) \cup T_{[x_2, x_t]}$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Notice that D_i^3 has no path of type $P(1, 2)$ satisfying the same properties of the $x_t x_1 x_2 x_3$ which is a path of type $P(1, 2)$ with $x_2 \leq_T x_t$, since else we can find similarly a subdivision of $C(k, 1, 1, 1)$ in D , a contradiction.

If $i = t$, let z the least common ancestor of x_1 and x_{t-2} , then $T_{[z, x_1]} \cup (x_1, x_t) \cup T_{[z, x_{t-2}]} \cup (x_{t-1}, x_{t-2}) \cup (x_{t-1}, x_t)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction.

If $4 \leq i < t$. Then $l_T(x_i) > l_T(x_{i-2})$, since else we can consider the least

common ancestor of x_1 and x_i to find a subdivision of $C(k, 1, 1, 1)$, a contradiction. Hence $l_T(x_i) > l_T(x_{i-2})$. Notice that $x_{i-2} \leq_T x_i$, since else let y be their least common ancestor, and so $T_{[y, x_{i-2}]} \cup T_{[y, x_i]} \cup (x_{i-1}, x_i) \cup (x_i, x_{i-2})$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Note that here $i = 4$, since else one can notice that $(x_{i-1}, x_i) \cup (x_{i-1}, x_{i-2}) \cup (x_{i-2}, x_{i-3})$ is a path $P(1, 2)$ as in case $i = 3$, a contradiction. Notice that neither $x_4 \leq_T x_t$ nor $x_t \leq_T x_4$, since else we can combine the directed path in T between x_4 and x_t and the oriented path in C between x_4 and x_t and find a subdivision of $C(k, 1, 1, 1)$, a contradiction. Moreover, using the least common ancestor of x_1 and x_3 , we can prove that $l_T(x_4) = l_T(x_t)$ and the length of the directed paths in T from their least common ancestor to each is less than k . Now, using the least common ancestor of x_4 and x_t we can show similarly that $l_T(x_1) = l_T(x_3)$. Notice that $t \geq 7$.

Denote by y the least common ancestor of x_5 and x_t . Notice here that neither x_1 nor x_3 is ancestor of x_t .

Also $y \neq x_t$, since else we can find a subdivision of $C(k, 1, 1, 1)$, a contradiction. Besides, $y \neq x_5$, since else we can use the least common ancestor of x_1 and x_3 to find a subdivision of $C(k, 1, 1, 1)$, a contradiction.

We claim also that neither x_1 nor x_3 is ancestor of x_5 . Indeed, if $x_1 \leq_T x_5$, we'll consider the least common ancestor of x_4 and x_t to find a subdivision of $C(k, 1, 1, 1)$, a contradiction. Similarly, if $x_3 \leq_T x_5$, we'll consider the least common ancestor of x_4 and x_t and the least common ancestor of x_1 and x_3 to find a subdivision of $C(k, 1, 1, 1)$, a contradiction.

Hence denote by z the least common ancestor of x_1 and x_3 . If $y \notin T_{[z, x_i]}$, $i = 1, 3$, then $T_{[z, x_3]} \cup (x_3, x_4) \cup x_4 x_5 \cup T_{[y, x_5]} \cup T_{[y, x_t]} \cup T_{[z, x_1]} \cup (x_1, x_t)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else, y is in particular the least common ancestor of x_2 and x_5 and $l(T_{[y, x_2]}) \geq k$, then $T_{[y, x_2]} \cup T_{[y, x_5]} \cup x_5 x_4 \cup (x_3, x_4) \cup (x_3, x_2)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction.

Hence D_i^3 contains no odd cycle then $\chi(D_i^3) \leq 2$.

□

Now we are ready to prove our main result:

Theorem 53. *Let k be a positive integer and D a digraph with a spanning out-tree with no subdivisions of $C(k, 1, 1, 1)$ then the chromatic number of D is at most $18k$.*

Proof. Let D be a digraph with a spanning out-tree with no subdivisions of $C(k, 1, 1, 1)$. Let T be a maximal out-tree of D . We will consider the same partition for the arcs of D used in the beginning of this section.

Claim 1: $\chi(D_i^1) \leq 3$, for all $0 \leq i \leq k - 1$.

Proof. Suppose to the contrary that $\chi(D_i^1) > 3$, then D_i^1 contains a wheel of cycle say C and a universal vertex x . C can't be a cycle of 4 blocks or more, since else D contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then C

is a cycle of 2 blocks. Denote by z_1 (resp. z_2) the vertex of C with in-degree (resp. out-degree) zero. We will reach a contradiction finding a subdivision of $C(k, 1, 1, 1)$ by studying the order of the levels of x and its neighbors. Denote by x_1, x_2, x_3 three neighbors of x on C such that $l_T(x_1) < l_T(x_2) < l_T(x_3)$. If $l_T(x) < l_T(x_2)$, then since $z_1 \notin \{x_2, x_3\}$, $C_{[z_1, x_2]} \cup (x, x_2) \cup (x, x_3) \cup C_{[z_1, x_3]}$ is cycle with 4 blocks in D_i^1 and so D contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $l_T(x_2) < l_T(x)$. If x_1 and x_2 belong to the same block on C , then $(C - C_{[x_1, x_2]}) \cup (x_1, x) \cup (x_2, x)$ is a cycle with 4 blocks in D_i^1 and so D contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else $C_{[x_1, z_2]} \cup C_{[x_2, z_2]} \cup (x_2, x) \cup (x_1, x)$ is a cycle with 4 blocks in D_i^1 and so D contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. \square

Claim 2: $\chi(D_i^2) \leq 3$, for all $0 \leq i \leq k - 1$.

Proof. Suppose to the contrary that $\chi(D_i^2) > 3$, then D_i^2 contains a wheel of cycle say C and a universal vertex x . Notice that D_i^2 contains no good 4-blocks cycle or a cycle with 6 blocks or more, since else D contains a subdivision of $C(k, 1, 1, 1)$, a contradiction. In particular, C can't be neither a good 4-blocks cycle nor a cycle with 6 blocks or more.

If C is a cycle with 2 blocks, then denote by z_1 (resp. z_2) the vertex of C with out-degree (resp. in-degree) zero. We will reach a contradiction finding a subdivision of $C(k, 1, 1, 1)$ by studying the order of the levels of x and its neighbors. Let x_1, x_2, x_3 to be three neighbors of x on C $/:$ $l_T(x_1) < l_T(x_2) < l_T(x_3)$. If $l_T(x) < l_T(x_1)$, then $T_{[x_1, x_2]} \cup C_{[x_2, x_3]} \cup (x_1, x) \cup (x_3, x)$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction. Else if $l_T(x) > l_T(x_3)$. If x_1 and x_2 belong to the same block on C , then $T[x_2, x_3] \cup C_{[x_2, x_1]} \cup (x, x_1) \cup (x, x_3)$ is a subdivision of $C(k, 1, 1, 1)$. Else $T[x_2, x_3] \cup C_{[x_2, z_1]} \cup C_{[x_1, z_1]} \cup (x, x_1) \cup (x, x_3)$ is a subdivision of $C(k, 1, 1, 1)$. Else if $l_T(x_1) < l_T(x) < l_T(x_2)$, then let z to be of minimal level in T $/:$ $z \in C$ and $x \leq_T z$. It is clear that $z \neq z_2$. So replace by $T_{[x, z]} \cup (x, x_1)$, $C_{[z, x_1]}$ (resp. $C_{[z, z_1]} \cup C_{[x_1, z_1]}$) if x_1 and z belong to the same block on C (resp. if x_1 and z belong to different blocks on C), and get a subdivision of $C(k, 1, 1, 1)$, a contradiction. Then $l_T(x_2) < l_T(x) < l_T(x_3)$, let z to be of minimal level in T $/:$ $z \in C$ and $x \leq_T z$. If $z \neq z_2$, we will proceed as the case before. Else, $z = z_2$ and so $z = z_2 = x_3$. Let z' in C of maximal level in T such that $z' \leq_T x$. Then $(x_3, x) \cup T[z', x] \cup C_{[x_3, z_1]} \cup C_{[z', z_1]}$ is a subdivision of $C(k, 1, 1, 1)$, a contradiction.

Hence C is a bad 4-blocks cycle. We will study the following cases:

If $C=C(2, 1, 1, 1)$ where $x_1 \leq_T x_2 \leq_T z_1 \leq_T y_2 \leq_T y_1$, then since x has at least 3 neighbors in C which is of length 5, then we can see that there are two adjacent vertices of C which are both neighbors of x . Notice that whenever two adjacent vertices of C , a and b , are both neighbors of x then replace in C , the arc (a, b) by the arcs ax and bx and so we get in D_i^2 a cycle with 6 blocks or a good 4-blocks cycle, a contradiction.

If $C = C(1, 1, 1, 1)$ with y_1 and y_2 are not ancestors one of the other, if x_1

and y_1 are both neighbors of x then replace in C , (y_1, x_1) by $\{xx_1\} \cup \{xy_1\}$, we will get a $C(2, 1, 1, 1)$ of good type in D_i^2 , a contradiction. Then x_2 and y_2 are both neighbors of x . Then replace in C , (y_2, x_2) by $\{xx_2\} \cup \{xy_2\}$, and so we will get a $C(2, 1, 1, 1)$ of good type in D_i^2 , a contradiction.

If $C = C(1, 1, 1, 1)$ with $x_1 \leq_T x_2 \leq_T y_2 \leq_T y_1$, we claim that x_1 and y_1 can't be both neighbors of x , since else replace in C , (y_1, x_1) by $\{xx_1\} \cup \{xy_1\}$. We will either get a $C(2, 1, 1, 1)$ of good type in D_i^2 , a contradiction, or we will get a bad 4-blocks cycle of type 2, $C(2, 1, 1, 1)$, and so in this case we have $T_{[x_2, x]} \cup (y_1, x) \cup (y_1, x_1) \cup (y_2, x_1) \cup (y_2, x_2)$ is a subdivision of $C(k, 1, 1, 1)$ in D , a contradiction. Then x_2 and y_2 are both neighbors of x and so replace in C , (y_2, x_2) by $\{xx_2\} \cup \{xy_2\}$. We will either get a $C(2, 1, 1, 1)$ of good type in D_i^2 , a contradiction, or we will get a bad 4-blocks cycle of type 2, $C(2, 1, 1, 1)$, in D_i^2 and in this case replace in C , (y_1, x_2) by $T_{[y_1, x]} \cup (x, x_2)$ and get a subdivision of $C(k, 1, 1, 1)$ in D , a contradiction.

If $C = C(1, 1, 1, 1)$ with $x_1 \leq_T x_2 \leq_T y_1 \leq_T y_2$, we proceed similarly as the case before. \square

With the fact that $D_i = D_i^1 \cup D_i^2 \cup D_i^3$ then $\chi(D_i) \leq 2.3.3 = 18$ for all $i \in \{0, \dots, k-1\}$. Consequently, as $V(D_i)$, $0 \leq i \leq k-1$ form a partition of $V(D)$, we obtain a proper $18k$ -coloring of D by giving to each D_i 18 distinct colors. This implies the hoped result. \square

Chapter 6

Three Blocks Paths

6.1 Introduction

This chapter is devoted to study the existence of oriented paths in digraphs. Digraphs considered here have no loops or multiple edges.

Problem: Given a digraph D with chromatic number n , what kind of oriented paths can we find in D ?

Many answers were given for the case of tournaments. Grünbaum proved in [35] that, with three exceptions, every tournament contains an antidirected hamiltonian path. In 1972, Rosenfeld [51] gave an easier proof of a stronger result: in a tournament on at least 9 vertices, each vertex is the origin of an antidirected hamiltonian path. He also conjectured: there is an integer $N > 7$ such that every tournament on n vertices, $n \geq N$, contains any orientation of the hamiltonian path. Rosenfeld's conjecture was verified by Thomason, who proved in [56] that N exists and is less than 2^{128} . In [37], Havet and Thomassé proved that any n -tournament contains any oriented path of length $n - 1$ except in three cases: the directed 3-cycle, the regular tournament on five vertices, and the Paley tournament on seven vertices; in these cases D contains no antidirected path of length $n - 1$. Further results about paths in tournaments was established in [7, 28, 54].

In general, for any digraph D of chromatic number n , the situation is quite different. To start with Gallai–Roy's celebrated theorem [30, 53], they proved that every digraph with chromatic number at least n contains a directed path of length $n - 1$:

Theorem 54. *Every digraph D contains a directed path with $\chi(G)$ vertices.*

Proof. Let P be the longest directed path of D and denote by k its order. Consider a maximal acyclic subdigraph D' of D . D' contains no directed path with length greater than $k - 1$. Construct a coloring c of D using k colors so that each vertex v is assigned the color $c(v)$ which is the number of

vertices of a longest directed path in D' starting at v . We claim that this is a proper coloring.

Indeed, let us consider any arc $(u, v) \in A(D)$. If $(u, v) \in A(D')$, let P be a longest directed path that starts at v and ends at a vertex $w \in D'$. As D' contains no cycle then $u \notin V(P)$. Now consider the directed path starting at u , $(u, v) \cup P$, and so $c(u) > c(v)$. Else if $(u, v) \notin A(D')$, then $D' \cup (u, v)$ contains a directed cycle. As D' is maximally acyclic, D' contains a vu -directed path P . Let Q be a longest directed path in D' starting at u . Because D' is acyclic then $V(P) \cup V(Q) = \{u\}$. Thus $P \cup Q$ is a directed path in D' starting at v and so $c(v) > c(u)$.

In both cases $c(u) \neq c(v)$ for any arc (u, v) of D . Hence c is a proper coloring with k colors. Hence D contains a directed path with $\chi(G)$ vertices. \square

Moving to paths with two blocks, El-Sahili and Kouider [24] showed that every digraph with chromatic $n + 1$ contains every path of order n with two blocks. However, Addario et.al.[57] refined this result for any digraph D with chromatic number at least n . For the case of oriented paths with three blocks, Burr's result [19] guarantees the existence of any path of three blocks of length $n - 1$ in any digraph with chromatic number at least $(n - 1)^2$.

In this Chapter, we show that every $\frac{3n^2}{4}$ -chromatic digraph contains any path of three blocks of length $n - 1$. El Joubbeh [40] improved this result later by demonstrating that every digraph with chromatic number at least $4.6n$ contains any path with three blocks of length $n - 1$. Recently, Tarhini found a better linear bound for the chromatic number of digraphs that contains no path with three blocks. Moreover, Mortada et.al proved in [48] that any $(n + 1)$ -chromatic digraph contains a three blocks path of length $n - 1$, in which two consecutive blocks are of length 1 each. For k and l two positive integers, Tarhini et. al [55] proved the existence of a $P(k, 1, l)$ of length $n - 1$ in any $(2n + 2)$ -chromatic digraph D and they proved that the existence of such an oriented path is ensured in any n -chromatic digraph that contains a hamiltonian directed path. Recently, Tarhini improved her result in [55] by proving that any $(2n - 3)$ -chromatic digraph contains any path $P(k, 1, l)$ of length $n - 1$.

In this Chapter, we are interested in the study of three blocks paths of the form $P(1, k, 1)$. Indeed, we prove that any $(2n - 5)$ -chromatic digraph contains a path of the form $P(1, n - 3, 1)$. Moreover, we show that every $(n + 1)$ -chromatic digraph contains a subdivision of $P(1, n - 3, 1)$.

6.2 Paths with Three Blocks

The aim of the work done in this section is to find a bound for the chromatic number of digraphs with no paths with three blocks. Using our method, there

is no hope to get an optimal value for $\chi(D)$ and all we look for is to give a bound better than that given by Burr [19]. We follow in this section the same reasoning as El Sahili in [23].

Let $f(n)$ be the smallest positive integer such that any $f(n)$ -chromatic digraph contains a $P(k, l, r)$, for k, l , and r three positive integers that sum up to $n-1$. It follows from the result of Burr [19] that $f(n) \leq (n-1)^2$. In this section we will improve this bound to $\frac{3n^2}{4}$. Given an oriented path P , we denote by \bar{P} the path obtained from P by reversing the directions of all the arcs of P .

The following lemma will be relevant for the proof of our result:

Lemma 55. *Let G be a graph containing no K_{2n+1} , $n \geq 2$. Suppose that we can orient G in such a way that each vertex has out-degree at most n , then $\chi(G) \leq 2n$.*

Define the sequence $g(m, i)$ for $m \geq 4$ and $0 \leq i \leq \frac{m}{2} - 1$ by:

$$g(m, i) = g(m-1, i-1) + 2(m-3) \text{ for } m \geq 5 \text{ and } i \geq 1$$

$$g(m, 0) = m \text{ and } g(4, 1) = 4$$

Lemma 56. *Any $g(m, i)$ -chromatic digraph D contains any paths $P(k, m-1-k-i, i)$ for k a positive integer less than $m-1$.*

Proof. We will proceed by induction on i . For $i = 0$, the case is solved by Addario et al. [57]. For $i = 1$ and $m = 4$ see [22] and [48].

Let $i \geq 1$ and $m \geq 5$. Let D be a $g(m, i)$ -chromatic digraph with no $P(k, m-1-k-i, i)$. Let H be the sub digraph of D induced by the vertices of out-degree in D at least $m-2$. Set H' to be the sub digraph induced by $V(D-H)$. We claim that H contains no path of the form $P(k, m-1-k-i, i-1)$. Suppose to the contrary that H contains a path $Q = P(k, m-1-k-i, i-1)$ and let v be the end vertex of Q in the block of length $i-1$. Since $d^+(v) \geq m-2$, then there exists an out-neighbor u of v outside Q . Notice so that $Q \cup (v, u)$ is a $P(k, m-1-k-i, i)$, a contradiction. And so, following the induction hypothesis, we get that $\chi(H) < g(m-1, i-1)$. Besides, following the previous lemma, we can find that $\chi(H') \leq 2(m-3)$, since else H' contains a tournament of order $2m-5$ and so by [56] it contains a $P(k, m-1-k-i, i)$. But, since $\chi(D) \leq \chi(H) + \chi(H')$, we get that $\chi(D) < g(m, i)$, a contradiction. \square

Similarly we can prove the following lemma:

Lemma 57. *Any $g(m, i)$ -chromatic digraph D contains any paths $\bar{P}(k, m-1-k-i, i)$ for k a positive integer less than $m-1$.*

When looking deep on the sequence $g(m, i)$, we can see that it is increasing with respect to i .

Lemma 58. *Let $m \geq 4$, $g(m, i)$ is an increasing sequence with respect to i .*

Proof. We argue by induction on i .

For $i = 1$, if $m = 4$, then $g(4, 1) = 4 \geq g(4, 0)$. Else, we have:

$$g(m, 1) = g(m - 1, 0) + 2(m - 3) = m - 1 + 2(m - 3) \geq m = g(m, 0)$$

Now suppose it is true for i , that is $g(m, i) \geq g(m, i - 1)$ for all $m \geq 4$.

We have:

$$g(m, i + 1) = g(m - 1, i) + 2(m - 3) \geq g(m - 1, i - 1) + 2(m - 3) = g(m, i)$$

Hence, $g(m, i)$ is increasing with respect to i . \square

Now we are ready to give the quadratic bound we have found for $f(m)$, for m a positive integer.

Theorem 59. *Every $\frac{3m^2}{4}$ -chromatic digraph contains a three blocks path of length $m - 1$.*

Proof. Following that every $g(m, i)$ -chromatic digraph contains a $P(k, m - 1 - k - i, i)$ and a $\bar{P}(k, m - 1 - k - i, i)$ for any k a positive integer less than $m - 1$, we can notice that $f(m)$ is less than or equal $g(m, \frac{m}{2} - 1)$ for m even and less than or equal $g(m, \frac{m-3}{2})$ for m odd. Now we will bound those two terms.

For m even, we have:

$$\begin{aligned} & g\left(m, \frac{m}{2} - 1\right) \\ &= g\left(m - 1, \left(\frac{m}{2} - 1\right) - 1\right) + 2(m - 3) \\ &= g\left(m - 2, \left(\frac{m}{2} - 1\right) - 2\right) + 2(m - 3) + 2((m - 1) - 3) \\ &= g\left(m - \left(\frac{m}{2} - 2\right), 1\right) + 2(m - 3) + \dots + 2\left(m - \left(\frac{m}{2} - 3\right) - 3\right) \\ &= g\left(m - \left(\frac{m}{2} - 1\right), 0\right) + 2(m - 3) + \dots + 2\left(m - \left(\frac{m}{2} - 2\right) - 3\right) \\ &= g\left(\frac{m + 2}{2}, 0\right) + 2 \sum_{k=0}^{\frac{m}{2}-2} (m - k - 3) \\ &= \frac{3}{4}m^2 - 3m + 5 \end{aligned}$$

For m odd, we have:

$$\begin{aligned}
& g\left(m, \frac{m-3}{2}\right) \\
&= g\left(m-1, \left(\frac{m-3}{2}\right) - 1\right) + 2(m-3) \\
&= g\left(m-2, \left(\frac{m-3}{2}\right) - 2\right) + 2(m-3) + 2((m-1) - 3) \\
&= g\left(m - \left(\frac{m-3}{2} - 1\right), 1\right) + 2(m-3) + \dots + 2\left(m - \left(\frac{m-5}{2} - 1\right) - 3\right) \\
&= g\left(m - \left(\frac{m-3}{2} - 1\right) - 1, 0\right) + 2(m-3) + \dots + 2\left(m - \left(\frac{m-5}{2}\right) - 3\right) \\
&= g\left(\frac{m+3}{2}, 0\right) + 2 \sum_{k=0}^{\frac{m-5}{2}} (m-k-3) \\
&= \frac{3}{4}m^2 - \frac{7}{2}m + \frac{27}{4}
\end{aligned}$$

Therefore, $f(m) \leq \frac{3m^2}{4}$, for all m . □

6.3 Subdivision of Paths with Three Blocks

This section is devoted to study the existence of a subdivision of $P(1, k, 1)$ in digraphs, for k a positive integer.

Theorem 60. *Any $(k+4)$ -chromatic digraph D contains a path of the form $P(1, l, 1)$ for some $l \geq k$.*

Proof. Let D be a $(k+4)$ -chromatic digraph. Suppose to the contrary that D contains no $P(1, l, 1)$ for all $l \geq k$. Suppose without loss of generality that D is $k+4$ -critical, and so $d(v) \geq k+3$ for all $v \in V(D)$.

Note that D contains a $P(k, 1)$. Let $P = x_1x_2\dots x_{s+1}y$ be a path of the form $P(s, 1)$ such that $s \geq k$ and s is maximal. Due to the maximality of s , any in-neighbor of x_1 must belong to P . Besides, all the out-neighbors of x_1 belong to P , since else if there exists $z \in N^+(x_1) - P$, so $(x_1, z) \cup P$ is a $P(1, s, 1)$ for $s \geq k$, a contradiction.

Let i be the minimal integer greater than or equal to 3 such that $x_i \in N(x_1)$. Note that $|P_{[x_i, x_{s+1}]}| \geq |N(x_1) - \{x_2, y\}| \geq k+1$. If $(x_1, x_i) \in E(D)$ then the path $(x_1, x_2) \cup (x_1, x_i) \cup P_{[x_i, x_{s+1}]} \cup (y, x_{s+1})$ is a $P(1, l, 1)$ for $l \geq k$, a contradiction. Else $(x_i, x_1) \in E(D)$ and so the path $(x_i, x_1) \cup P_{[x_i, x_{s+1}]} \cup (y, x_{s+1})$ is a $P(1, l, 1)$ for $l \geq k$, a contradiction. □

Conjecture 61. *Any $(k+3)$ -chromatic digraph D contains a $P(1, l, 1)$ for some $l \geq k$.*

6.4 Paths with Three Blocks of the form $P(1,k,1)$

The problem is quite different when we search for an exact copy of $P(1, k, 1)$ in a digraph D . Indeed, let D be a $(3k+4)$ -chromatic digraph with no $P(1, k, 1)$, k is a positive integer. Let U be the set of all origins of any $P(k, 1)$ in D . It is clear that $U \neq \phi$. If all the vertices in U has an out-degree in D less than $k+1$ then, by lemma 2.1, $\chi(U) \leq 2k+2$. Then $\chi(D-U) \geq k+2$ and so $D-U$ contains a $P(k, 1)$, a contradiction. Hence there exists $u \in U$ such that $d^+(u) > k+1$. Let $P_u(k, 1)$ be a path of the form $P(k, 1)$ in D with origin u . Let $z \in N^+(u) - P_u(k, 1)$ and so the path $P_u(k, 1) \cup (u, z)$ is $P(1, k, 1)$, contradiction. Hence every $(3k+4)$ -chromatic digraph contains a $P(1, k, 1)$. In the sequel, we improve this to $2k+1$.

Theorem 62. *Any $(2k+1)$ -chromatic digraph contains a $P(1, k, 1)$, k is a positive integer.*

Proof. Let D be a digraph with $\chi(D) = 2k+1$ with $k \geq 2$ positive integer. Suppose without loss of generality that D is critical. We argue by the way of contradiction assuming that D has no $P(1, k, 1)$. Let D_1 be the subdigraph of D induced by the vertices of D with $d_D^-(v) \geq k+1$ and D_2 the subdigraph of D induced by $V(D) - V(D_1)$. Note that all the vertices in D_2 have an out-degree in D is greater than or equal k . We either have $\chi(D_2) \geq k$ or $\chi(D_1) \geq k+2$. Suppose first that $\chi(D_2) \geq k$. Let P be a directed path of maximal length in D_2 . It's clear that $l(P) \geq k-1$.

Claim 1: $l(P) \geq k$.

Proof. Suppose to the contrary that $l(P) = k-1$. Set $P = x_1 \dots x_k$. Note that $N_{D_2}^+(x_k) \subset P$ then x_k has at least two out-neighbors in D_1 , say a_1 and a_2 . Since $d^+(x_1) \geq k$, let v an out-neighbor of x_1 outside P . If $v \notin \{a_1, a_2\}$ then $N^-(a_i) = P \cup \{v\}$ for $i = 1, 2$, since else let $z \in N^-(a_i) - (P \cup \{v\})$ then $(z, a_i) \cup (x_k, a_i) \cup P \cup (x_1, v)$ is a $P(1, k, 1)$, contradiction. But now $(v, a_2) \cup (x_k, a_2) \cup P \cup (x_1, a_1)$ is a $P(1, k, 1)$, contradiction. Thus $v \in \{a_1, a_2\}$. Without loss of generality consider that $v = a_2$. Note that $N^-(a_1) = P \cup \{v\}$ and $N^+(x_1) \subseteq P \cup \{v\}$. Since $(a_2, a_1) \in D$ and $d^-(a_2) \geq k+1$, then a_2 has an in-neighbor outside $P \cup \{a_1\}$, say x . Set $(x, a_2) \cup (x_k, a_2) \cup P \cup (x_1, a_1)$ is a $P(1, k, 1)$, contradiction. \square

Set then $P = x_1 \dots x_{r+k}$ with $r \geq 1$ and $P_1 = x_r \dots x_{r+k}$.

Claim 2: $N^+(x_{r+k}) \cap D_1 = \phi$.

Proof. Suppose to the contrary that $N^+(x_{r+k}) \cap D_1 \neq \phi$. If $|N^+(x_{r+k}) \cap D_1| \geq 2$. Then a contradiction can be reached using the same argumentation as claim 1. Then $|N^+(x_{r+k}) \cap D_1| = 1$. Let $a_1 \in N^+(x_{r+k}) \cap D_1$. We claim that $x_r \in N^-(a_1)$. Suppose not, then there exists $w \in N^-(a_1) - P_1$. If $(x_{r+k-1}, a_1) \in E(D)$, then $N^+(x_r) = [x_{r+1}, x_{r+k}]$ and so $(x_r, x_{r+2}) \in E(D)$. Consequently, $(x_r, x_{r+1}) \cup (x_r, x_{r+2}) \cup [x_{r+2}, x_{r+k}] \cup (x_{r+k}, a_1) \cup (w, a_1)$

is a $P(1, k, 1)$, contradiction. Hence $(x_{r+k-1}, a_1) \notin E(D)$ and so there exists $w' \in N^-(a_1) - P_1 - \{w\}$. In this case $N^+(x_{r+1}) = [x_{r+2}, x_{r+k}] \cup \{a_1\}$. And so $N^+(x_r) = [x_{r+1}, x_{r+k}]$ in particular $(x_r, x_{r+2}) \in E(D)$, contradiction as before. Hence $x_r \in N^-(a_1)$. Consequently, $N^+(x_{r+1}) = [x_{r+2}, x_{r+k}] \cup \{a_1\}$. In particular $(x_{r+1}, x_{r+k}) \in E(D)$, but $N^+(x_{r+k}) \geq k$ then there exists $x_j \in N^+(x_{r+k})$ for $1 \leq j < r$ so $r \geq 2$. As before $x_{r-1} \in N^-(a_1)$. Then x_{r+k} has at least two out-neighbors in $[x_1, x_r[$, let x_i be one of them with i maximal, then $(x_{i-1}, x_i) \cup [x_{r+1}, x_{r+k}] \cup (x_{r+k}, x_i) \cup (x_{r+1}, a_1)$ is a $P(1, k, 1)$, contradiction. \square

Claim 3: If $(x_{r+k}, x_r) \in E(D)$ then $(x_{r+k}, x_1) \in E(D)$.

Proof. Suppose to the contrary that $(x_{r+k}, x_1) \notin E(D)$. Since $(x_{r+k}, x_r) \in E(D)$ then $N^+(x_{r+1}) \subseteq [x_{r+2}, x_{r+k}] \cup \{x_{r-1}\}$, since else we get a $P(1, k, 1)$, a contradiction. But $d^+(x_{r+1}) \geq k$, then $N^+(x_{r+1}) = [x_{r+2}, x_{r+k}] \cup \{x_{r-1}\}$. In particular, $(x_{r+1}, x_{r+k}) \in E(D)$. Since $d^+(x_{r+k}) \geq k$, then there exists $1 < i < r - 1$ such that $x_i \in N^+(x_{r+1})$. Hence $(x_{i-1}, x_i) \cup [x_{r+1}, x_{r+k}] \cup (x_{r+k}, x_i) \cup (x_{r+1}, x_{r-1})$ is $P(1, k, 1)$, a contradiction. \square

Now if $|N^+(x_{r+k}) \cap [x_2, x_r[| \geq 3$, then we get $P(1, k, 1)$, contradiction. If $|N^+(x_{r+k}) \cap [x_2, x_r[| = 2$ then so $(x_{r+k}, x_1) \in E(D)$, since else we get that $(x_{r+k}, x_{r+1}) \in E(D)$, but since $d^+(x_{r+1}) \geq k$ then x_{r+1} has at least two out-neighbors outside P_1 which gives a contradiction as before. If $|N^+(x_{r+k}) \cap [x_2, x_r[| \leq 1$, then $(x_{r+k}, x_1) \in E(D)$. Hence x_i can play the role of x_{r+k} for every $x_i \in V(P)$. We claim that $N^-(x_i) \subset P$ for every $x_i \in V(P)$. Otherwise, without loss of generality suppose that there exists $z \in N^-(x_{r+k}) - P$. $z \in D_1$, since else $(z, x_{r+k}) \cup (x_{r+k}, x_1) \cup [x_1, x_{r+k-1}]$ is a directed path in D_2 longer than P , a contradiction. Since $N^+(x_r) \subseteq P$ then $N^+(x_r) = [x_{r+1}, x_{r+k}]$, otherwise we get a $P(1, k, 1)$, in particular, (x_r, x_{r+2}) and $(x_r, x_{r+k}) \in E(D)$. But $d^+(x_{r+k}) \geq k$, then there exists $x_j \in N^+(x_{r+k})$ such that $1 < j < r$. Hence $(x_r, x_{r+1}) \cup (x_r, x_{r+2}) \cup [x_{r+2}, x_{r+k}] \cup (x_{r+k}, x_j) \cup (x_{j-1}, x_j)$ is a $P(1, k, 1)$, a contradiction. Hence $N^-(x_i) \subset P$ for every $x_i \in V(P)$. Since also $N^+(x_i) \subset P$ and D is critical then $V(D) = V(D_2) = V(P)$. Since $d^+(x_i) \geq k$ and $d^-(x_i) \leq k$ for every $x_i \in V(P)$, then $d^+(x_i) = d^-(x_i) = k$ for every $x_i \in V(P)$ and so D is a $2k$ -regular digraph. Since $\chi(D) = 2k + 1$, then D is a $2k + 1$ -tournament and so it contains a $P(1, k, 1)$, a contradiction.

Consequently, $\chi(D_2) < k$ and so $\chi(D_1) \geq k + 2$, then let $P = z_1 \dots z_{k+r}$ be a directed path in D_1 with maximal length in $(r \geq 2)$ and let $P_2 = z_1 \dots z_{k+1}$.

Claim 4: $N^-(z_1) \cap D_2 = \emptyset$.

Proof. Suppose to the contrary that $N^-(z_1) \cap D_2 \neq \emptyset$. Let $a_1 \in D_2 \cap N^-(z_1)$. Since $d^-(z_k) \geq k + 1$, then there exists $v \in N^-(z_k) - [z_1, z_{k+1}] - \{a_1\}$. Note that $d^+(a_1) \geq k$. If there exists $w \in N^+(a_1) - [z_1, z_k] - \{v\}$, we get a $P(1, k, 1)$, a contradiction. Hence $N^+(a_1) \subseteq ([z_1, z_k] \cup \{v\})$. If $(a_1, z_2) \in E(D)$ then $N^-(z_{k+1}) = [z_1, z_k] \cup \{a_1\}$ since else we get a $P(1, k, 1)$, a contradiction. But $N^+(a_1) \subseteq ([z_1, z_k] \cup \{v\})$, a contradiction. And so $(a_1, z_3) \in E(D)$.

Since $d^-(z_{k+2}) \geq k + 1$ and $z_{k+2} \notin N^+(a_1)$, then there exists $x \in N^-(z_{k+2}) - ([z_3, z_k] \cup \{a_1, z_1\})$. Hence $(x, z_{k+2}) \cup [z_3, z_{k+2}] \cup (a_1, z_3) \cup (a_1, z_1)$ is a $P(1, k, 1)$, contradiction. \square

Since $d^-(z_1) \geq k + 1$ then there exists $z_{t_1}, z_{t_2} \in N^-(z_1)$ such that $k < t_1 < t_2 < k + r$. Also since $d^-(z_k) \geq k + 1$ then $|N^-(z_k) - [z_1, z_{k-1}]| \geq 2$. Set $I = N^-(z_k) - [z_1, z_{k-1}]$. $I \subseteq \{z_{t_1}, z_{t_1+1}\} \cap \{z_{t_2}, z_{t_2+1}\}$ since else we get a $P(1, k, 1)$, a contradiction. But $|I| \geq 2$, and so $|\{z_{t_1}, z_{t_1+1}\} \cap \{z_{t_2}, z_{t_2+1}\}| \geq 2$ then $\{t_1, t_1 + 1\} = \{t_2, t_2 + 1\}$, a contradiction. \square

Part II
**Neighborly Partitions and New
Identities of Integer Partitions**

Chapter 7

Neighborly Partitions and the numerators of Rogers-Ramanujan Identities

7.1 Introduction

Among the most famous and ubiquitous formulas involving q -series, we find the Rogers-Ramanujan identities:

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q) \dots (1-q^k)} = \frac{\prod_{j \geq 1, j \equiv 0, \pm 2 \pmod{5}} (1-q^j)}{\prod_{j \geq 1} (1-q^j)}, \quad (7.1)$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(1-q) \dots (1-q^k)} = \frac{\prod_{j \geq 2, j \equiv 0, \pm 1 \pmod{5}} (1-q^j)}{\prod_{j \geq 2} (1-q^j)}. \quad (7.2)$$

On the left side of the identities, the term corresponding to $k = 0$ is taken to be 1. Usually the right side of this identity is written after the obvious simplifications which allow to put 1 in the numerators. We chose this form, because the numerators play an important role in this chapter. As revealed by MacMahon [44], these identities can be stated in the realm of the theory of partitions: recall first that an integer partition of a positive integer n is simply a decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_r)$, such that $|\lambda| := \lambda_1 + \dots + \lambda_r = n$. The λ_i 's are called the parts of λ and $r =: \text{size}(\lambda)$ is its size; see [9, 14] for more about the theory of partitions.

Theorem 63. (*Rogers-Ramanujan identities*) *Let n be a positive integer. For $i \in \{1, 2\}$, let $\mathcal{T}_i(n)$ be the set of partitions of n without equal nor consecutive parts and the part 1 appears at most $i - 1$ times. Let $\mathcal{E}_i(n)$ be the number of partitions of n into parts congruent to $\pm(2 + i) \pmod{5}$. Then we have*

$$|\mathcal{T}_i(n)| = |\mathcal{E}_i(n)|.$$

The notation $|A|$ in the theorem stands for the cardinal of a set A . Observe that the sums side in the q -series identities corresponds to the generating series of the $|\mathcal{T}_i(n)|$ (first for $i = 2$ and then for $i = 1$) and the products side corresponds to the generating series of the $|\mathcal{E}_i(n)|$. These identities appear in many fields other than combinatorics, such as statistical mechanics, number theory, representation theory, algebraic geometry, probability theory or commutative algebra [1, 2, 3, 4, 5, 8, 10, 12, 17, 18, 29, 31, 34].

The main goal of this chapter is to prove two identities which are in some sense dual to the Rogers-Ramanujan identities. We begin by introducing the notions appearing in these new identities: neighborly partitions, signature of a neighbourly partition.

Definition 64. For $i \in \{1, 2\}$ and for a positive integer n , we call neighborly partitions of n the set $\mathcal{N}_i(n)$ of partitions λ of n which satisfy the following properties:

1. For every part λ_j of λ , there exists $l \in \mathbf{N}_{>0}$, $l \neq j$ such that $|\lambda_l - \lambda_j| \leq 1$.
2. The multiplicity of any part of λ is at most 2 (i.e. there is no part of λ which is repeated more than twice).
3. For every $l \in \mathbf{N}$, $\lambda_l \geq 3 - i$ (i.e. for $i = 1$, there are no parts equal to 1).

The terminology *neighborly* is inspired by the property 1. which says that for every part λ_j of λ , there is a *neighbor* part λ_l of λ which is equal or at a distance 1 to λ_j . As an example, the integer 4 has 5 partitions:

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Then the neighborly partitions are

$$\mathcal{N}_1(4) = \{2 + 2\}, \quad \mathcal{N}_2(4) = \{2 + 2, 2 + 1 + 1\}.$$

With a neighborly partition λ , we associate a graph G_λ as follows: The set $V(G_\lambda)$ of vertices of G_λ is in bijection with the set of parts of λ ; if h is a part of λ of multiplicity 1, the associated vertex is called x_h ; if h is a part of λ of multiplicity 2, the vertices associated with the two equal parts are named respectively x_h and y_h . The set $E(G_\lambda)$ of edges of G_λ is given by

$$E(G_\lambda) = \{(x_{h+1}, x_h), (x_l, y_l), \text{ for every } x_{h+1}, x_h, x_l, y_l \in V(G_\lambda)\}.$$

For example $V(G_{2+1+1}) = \{x_2, x_1, y_1\}$, $E(G_{2+1+1}) = \{(x_2, x_1), (x_1, y_1)\}$ and G_{2+1+1} has the following shape

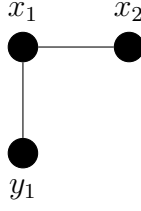


Figure 7.1: The graph G_{2+1+1}

A subgraph H of G_λ is said to be vertex-spanning if $V(H) = V(G_\lambda)$; it is said to be without isolated vertices if any vertex of H is an endpoint of some edge in $E(H)$.

Definition 65. Let λ be a neighborly partition. We define the signature $\delta(\lambda)$ of λ by

$$\delta(\lambda) = \sum_H (-1)^{|E(H)|},$$

where H ranges over the vertex spanning subgraphs of G_λ , which have no isolated vertices and $|E(H)|$, as mentioned before, is the cardinal of $E(H)$.

We now are ready to state the main theorem. Let n be a positive integer; for $i \in \{1, 2\}$, let $\mathcal{R}_i(n)$ be the set of partitions of n whose parts are larger or equal to $3 - i$, distinct and congruent to 0 or $\pm(i) \pmod{5}$.

Theorem 66. [47] Let n be a positive integer. For $i \in \{1, 2\}$, we have the identities

$$\sum_{\lambda \in \mathcal{N}_i(n)} \delta(\lambda) = \sum_{\lambda \in \mathcal{R}_i(n)} (-1)^{\text{size}(\lambda)}.$$

Example 67. 1. For $n = 6$ and $i = 1$, we have $\mathcal{N}_1(6) = \{3 + 3\}$ and $\mathcal{R}_1(6) = \{6\}$. The graph G_{3+3} has the following shape



Figure 7.2: The graph G_{3+3}

and it has no (strict) subgraphs which are vertex-spanning without isolated vertices. Hence, we have

$$\sum_{\lambda \in \mathcal{N}_1(6)} \delta(\lambda) = \delta(3+3) = (-1)^1 = \sum_{\lambda \in \mathcal{R}_1(6)} (-1)^{\text{size}(\lambda)} = (-1)^{\text{size}(6)} = (-1)^1.$$

2. For $n = 6$ and $i = 2$, we have $\mathcal{N}_2(6) = \{3+3, 3+2+1, 2+2+1+1\}$ and $\mathcal{R}_2(6) = \emptyset$. The graphs corresponding to the partitions $3+3, 3+2+1$ have no (strict) vertex-spanning subgraphs without isolated vertices; hence we have $\delta(3+3) = (-1)^1$ and $\delta(3+2+1) = (-1)^2$. The graph $G_{2+2+1+1}$ has the shape

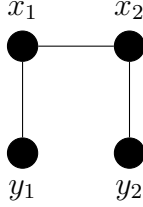
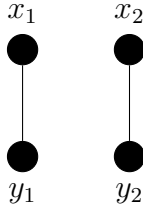


Figure 7.3: The graph $G_{2+2+1+1}$

and beside $G_{2+2+1+1}$ itself, its only vertex-spanning subgraph without isolated vertices is



Hence, we have $\delta(2+2+1+1) = (-1)^3 + (-1)^2 = 0$ and

$$\begin{aligned} \sum_{\lambda \in \mathcal{N}_2(6)} \delta(\lambda) &= \delta(3+3) + \delta(3+2+1) + \delta(2+2+1+1) = -1 + 1 + 0 \\ &= 0 = \sum_{\lambda \in \mathcal{R}_2(6)} (-1)^{\text{size}(\lambda)}. \end{aligned}$$

For $i \in \{1, 2\}$, We set $\mathcal{N}_i = \cup_{n \in \mathbb{N}_{>0}} \mathcal{N}_i(n)$. By considering the generating sequence of both sides of Theorem 66, we find the following equivalent statement:

Corollary 68. For $i \in \{1, 2\}$, we have the identities

$$\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|} = \prod_{j \geq 3-i, j \equiv 0, \pm i \pmod{5}} (1 - q^j).$$

One observes that the right side of the identities in corollary 68 corresponds to the numerators of the identities (7.1) and (7.2). As we will show later, the left side of the identities in the theorem is related to the left side of the identities (7.1) and (7.2). Here we use the Rogers-Ramanujan identities to prove theorem 66. The proof of the main theorem is actually a proof that the Rogers-Ramanujan identities are equivalent to corollary 68; in particular, a direct proof (i.e. which does not use the Rogers-Ramanujan identities) of theorem 66 would also give a proof of the Rogers-Ramanujan identities.

7.2 Strategy and Proof of the Main Result

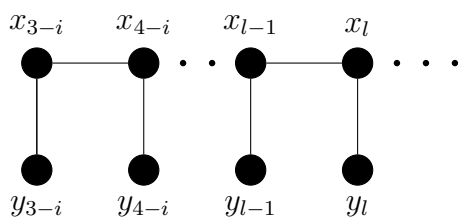
We will divide the scheme of the proof into three steps which are the subject of the three subsections of this section.

1. For $i \in \{1, 2\}$, we interpret the q -series $\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|}$ (see corollary 68) in terms of a generating series S_i "counting" the subgraphs of an infinite (simple) graph G_i^∞ .
2. For $i \in \{1, 2\}$, we give a formula relating S_i to the Hilbert series \mathcal{H}_i of a graded ring which is the quotient of polynomial ring with a (infinite) countable number of variables: In this step, we use the notion of edge ideals, and the result that we prove is valid for any edge ideal.
3. For $i \in \{1, 2\}$, we describe a formula relating \mathcal{H}_i in terms of the generating series of $|\mathcal{T}_i(n)|$ (see theorem 63). Then we use the Rogers-Ramanujan identities to obtain the main results.

All the graphs that we will consider are simple, i.e they do not have more than one edge between any two vertices and no edge starts and ends at the same vertex.

7.2.1 Neighborly partitions and enumerating subgraphs

For $i \in \{1, 2\}$, we begin by considering the infinite graph G_i^∞ whose set of vertices is $V(G_i^\infty) = \{x_j, y_j; j \in \mathbb{N}, j \geq 3 - i\}$ and whose set of edges is $E(G_i^\infty) = \{(x_j, x_{j+1}), (x_j, y_j); j \geq 3 - i\}$. Notice that for a neighborly partition $\lambda \in \mathcal{N}_i$, the graph G_λ is an *induced* subgraph of G_i^∞ which has no isolated vertices. So G_i^∞ has the following shape



Recall that a subgraph H is said to be induced if an edge of G is an edge of H whenever its endpoints are vertices of H . In particular, such a subgraph is completely determined by its vertices. Conversely any induced subgraph of G_i^∞ , $i = 1, 2$, without isolated vertices is of the type G_λ for some $\lambda \in \mathcal{N}_i$.

Definition 69. Let G be a simple graph, let $V(G) = \{v_j, j \in I\}$ be its set of vertices that we assume countable. We call the multivariable subgraph enumerating series of G the series in the variables $(v_j)_{j \in I}$ and which is defined by

$$S_G(\mathbf{v}, z) = \sum_H \left(\prod_{v_j \in V(H)} v_j \right) z^{|E(H)|},$$

where H ranges over finite subgraphs of G without isolated vertices and where we denoted by \mathbf{v} the multivariable $(v_j)_{j \in I}$.

Note that, since we are only considering finite subgraphs, the monomials in the multivariable subgraph enumerating series make intervene a finite number of variables. We will express the series corollary 68 in terms of the multivariable subgraph enumerating series of $G_i^\infty, i = 1, 2$. Recall the vertices v_j of $G_i^\infty, i = 1, 2$ belong to $\{x_j, y_j; j \in \mathbf{N}, j \geq 3 - i\}$. Denote by \mathbf{x} the multivariable $(x_j)_{j \geq 3-i}$ and by \mathbf{y} the multivariable $(y_j)_{j \geq 3-i}$.

Lemma 70. For $i \in \{1, 2\}$, set $S_i(\mathbf{v}, z) := S_{G_i^\infty}(\mathbf{v}, z) = S_i(\mathbf{x}, \mathbf{y}, z)$. Let $S_i^w(q, z)$ be the series which is obtained from $S_i(\mathbf{x}, \mathbf{y}, z)$ by substituting x_j and y_j by q^j . We have

$$\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|} = S_i^w(q, -1).$$

Proof. For $i \in \{1, 2\}$, we have

$$\begin{aligned} S_i(\mathbf{v}, z) &= \sum_H \left(\prod_{v_j \in V(H)} v_j \right) z^{|E(H)|} \\ &= \sum_{V \subset V(G_i^\infty)} \left(\sum_{\{H; V(H)=V\}} \left(\prod_{v_j \in V} v_j \right) z^{|E(H)|} \right) = \sum_{V \subset V(G_i^\infty)} \left(\prod_{v_j \in V} v_j \right) \left(\sum_{\{H; V(H)=V\}} z^{|E(H)|} \right), \end{aligned}$$

where H ranges over finite subgraphs of G_i^∞ without isolated vertices and V over the sets of vertices of such subgraphs H . So we have gathered the subgraphs which have the same set of vertices V together. The induced subgraph of G_i^∞ whose set of vertices is such a V is of the form G_λ for some $\lambda \in \mathcal{N}_i$. In particular, if we replace in $\prod_{v_j \in V} v_j$ the x_j 's and the y_j 's by q^j , we obtain $q^{|\lambda|}$. Moreover if we set $\varphi(z) = \sum_{\{H; V(H)=V\}} z^{|E(H)|}$, we obtain by the definition of the signature that $\delta(\lambda) = \varphi(-1)$ and the lemma follows. \square

7.2.2 Multigraded Hilbert series of Edge ideals and enumerating subgraphs

Let G be a simple graph, let $V(G) = \{v_j, j \in I\}$ be its set of vertices that we assume countable or finite. Let $E(G)$ be set of edges of G that we also assume countable or finite (this is sufficient for our purposes). Let \mathbf{K} be a field of zero characteristic. We consider the ring of polynomials $A = \mathbf{K}[\mathbf{v}] = \mathbf{K}[v_j, j \in I]$ whose variables range in the set of vertices $V(G)$. With the graph

G we associate its edge ideal which is a square-free monomial ideal generated by

$$\mathcal{I}(G) = \langle v_j v_l \mid (v_j, v_l) \in E(G) \rangle.$$

We are interested in the multigraded Hilbert series which counts the monomials in the quotient ring of the polynomial ring by an edge ideal.

Definition 71. *Let G be a simple graph as above. The multigraded Hilbert series of $A/\mathcal{I}(G)$ is the series \mathcal{H}_G in the variables $v_j, j \in I$ which is defined by*

$$\mathcal{H}_{A/\mathcal{I}(G)}(\mathbf{v}) = \mathcal{H}_G(\mathbf{v}) = \sum_{M \notin \mathcal{I}(G)} M,$$

where M ranges over the monomials of A (making intervene a finite number of variables) which are not in the ideal $\mathcal{I}(G)$.

Note that the multigraded Hilbert series of $A/\mathcal{I}(G)$ "counts" the monomials in A which are not zero in $A/\mathcal{I}(G)$. We have the following formula expressing the multigraded Hilbert series in terms of the multivariable subgraph enumerating series.

Lemma 72. *Let G be a simple graph as above. We have*

$$\mathcal{H}_G(\mathbf{v}) = \frac{S_G(\mathbf{v}, -1)}{\prod_{j \in I} (1 - v_j)}.$$

Proof. First one remarks that (one may think of the one variable and then two variables cases to be convinced)

$$\mathcal{H}_A(\mathbf{v}) = \frac{1}{\prod_{j \in I} (1 - v_j)}.$$

Now recall that a monomial belongs to a monomial ideal if and only if it is divisible by at least one of its generators. Let $M = \{m_1, m_2, \dots\}$ be the set of generators of $\mathcal{I}(G)$ which are in bijection with the edges of G . We have the formula

$$\begin{aligned} \mathcal{H}_{A/\mathcal{I}(G)}(\mathbf{v}) &= \mathcal{H}_A(\mathbf{v}) - \sum_{m \in M} m \mathcal{H}_A(\mathbf{v}) \\ &+ \sum_{\{m_{j_1}, m_{j_2}\} \subset M} \text{lcm}(m_{j_1}, m_{j_2}) \mathcal{H}_A(\mathbf{v}) + \dots + \\ &(-1)^k \sum_{\{m_{j_1}, m_{j_2}, \dots, m_{j_k}\} \subset M} \text{lcm}(m_{j_1}, m_{j_2}, \dots, m_{j_k}) \mathcal{H}_A(\mathbf{v}) + \dots = \\ &\mathcal{H}_A(\mathbf{v}) \left[1 - \sum_{m \in M} m + \sum_{\{m_{j_1}, m_{j_2}\} \subset M} \text{lcm}(m_{j_1}, m_{j_2}) + \dots + \right. \\ &\left. (-1)^k \sum_{\{m_{j_1}, m_{j_2}, \dots, m_{j_k}\} \subset M} \text{lcm}(m_{j_1}, m_{j_2}, \dots, m_{j_k}) + \dots \right], \end{aligned} \quad (7.3)$$

where lcm stands for the least common multiple. The formula expresses that the (non-zero) monomials $A/\mathcal{I}(G)$ are the monomials in A except those which are divisible by one of the m'_j s. When taking out the monomials which are divisible by one of the m_j (which is expressed by $-\sum_{m \in M} m \mathcal{H}_A(\mathbf{v})$), we take out twice those which are divisible at the same time by some m_{j_1} and m_{j_2} (i.e. by $lcm(m_{j_1}, m_{j_2})$); so we need to add once those which are divisible by both (this is expressed by adding $\sum_{\{m_{j_1}, m_{j_2}\} \subset M} lcm(m_{j_1}, m_{j_2}) \mathcal{H}_A(\mathbf{v})$) hence adding twice those which are divisible by three and so on: this is simply the inclusion exclusion-principle since the monomials in $\mathcal{I}(G)$ are those which belong to the union of the ideals generated by the m'_j s. Now remark that choice of k monomials $m_{j_1}, m_{j_2}, \dots, m_{j_k}$ corresponds to the choice of k edges of G , i.e. a subgraph of G having k edges and the variables appearing in the associated monomial (which are square free, the graph G being simple) correspond to vertices of this subgraph; hence we have

$$\begin{aligned} S_G(\mathbf{v}, -1) &= 1 - \sum_{m \in M} m + \dots + (-1)^k \sum_{\{m_{j_1}, m_{j_2}, \dots, m_{j_k}\} \subset M} lcm(m_{j_1}, m_{j_2}, \dots, m_{j_k}) + \dots \end{aligned}$$

and the proposition follows. □

Remark 73. 1. Note that the formula (7.3) in lemma 72 can also be seen as a direct application of the Taylor resolution (see [50]) of the monomial ideal $\mathcal{I}(G)$.

2. A variant of the formula in lemma 72 exists in the literature, often for special gradings [32]; however, this exact statement will be needed in the next subsection and the proof makes the paper self-contained.

7.2.3 Proof of the main result

For $i = 1, 2$ we now consider the \mathbf{K} -algebra $\mathcal{P}_i := K[x_j, y_j, j \geq 3 - i] / \mathcal{I}(G_i^\infty)$ which, by definition of the ideal $\mathcal{I}(G_i^\infty)$ is equal to

$$\frac{\mathbf{K}[x_j, y_j, j \geq 3 - i]}{\langle x_j y_j, x_j x_{j+1}, j \geq 3 - i \rangle}.$$

By giving to x_j and y_j , for $j \geq 3 - i$, the weight j , the algebra \mathcal{P} inherits the structure of a graded algebra, i.e. we can write

$$\mathcal{P}_i = \bigoplus_{h \in \mathbf{Z}_{\geq 0}} \mathcal{P}_{i,h}$$

where the $\mathcal{P}_{i,h}$'s are additive groups satisfying $\mathcal{P}_{i,h} \cdot \mathcal{P}_{i,h'} \subset \mathcal{P}_{i,h+h'}$. By abuse of notation, the $\mathcal{P}_{i,h}$ is a \mathbf{K} -vector space generated by all the monomials which

does not belong to $\mathcal{I}(G_i^\infty)$ and which have weight equal to h . The Hilbert-Poincaré series of the graded algebra \mathcal{P}_i is by definition

$$\mathcal{HP}_{\mathcal{P}_i}(q) := \sum_{h \in \mathbf{Z}_{\geq 0}} \dim_{\mathbf{K}} \mathcal{P}_{i,h} q^h.$$

One notices that the monomials in the variables x_j, y_j which are of weight h are exactly those such that when we replace x_j and y_j by q^j , we obtain q^h : for instance, $x_j x_{j'}$ is of weight $j + j'$ and the mentioned substitution gives $q^{j+j'}$. Hence, if we set $\mathcal{H}_{\mathcal{P}_i}^w(q)$ to be the series which is obtained from $\mathcal{H}_{G_i^\infty}(\mathbf{v})$ (see Definition 71) by substituting x_j and y_j by q^j we find

$$\mathcal{HP}_{\mathcal{P}_i}(q) = \mathcal{H}_{\mathcal{P}_i}^w(q).$$

Applying lemma 72 and lemma 70 we deduce the following:

Proposition 74. *We have*

$$\mathcal{HP}_{\mathcal{P}_i}(q) = \frac{\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|}}{\prod_{j \geq 3-i} (1 - q^j)^2}.$$

To make the link with Rogers-Ramanujan identities we will consider polarization of monomial ideals: this is a procedure that allows to associate with a monomial ideal a squarefree monomial ideal in a polynomial ring which has more variables. The interesting fact is that invariants of both ideals are very related. The procedure amounts to replace any power of the type x^e (in some polynomial ring having x as a variable) by the monomial $xy_1 \dots y_{e-1}$ where the y_i are variables in a new polynomial ring extending the variables in the original polynomial ring. By applying this process to the generators of a monomial ideal, we obtain a squarefree monomial ideal. The important examples for us are the ideals

$$\mathcal{I}(G_i^\infty) = \langle x_j y_j, x_j x_{j+1}, j \geq 3 - i \rangle \subset \mathbf{K}[x_j, y_j, j \geq 3 - i], \text{ for } i = 1, 2.$$

These are the polarization (this is why we called our ring above \mathcal{P}_i) of the ideals

$$\langle x_j^2, x_j x_{j+1}, j \geq 3 - i \rangle \subset \mathbf{K}[x_j, j \geq 3 - i],$$

where the grading is induced from giving to x_j the weight j . We have the following (Corollary 1.6.3 in [38], see also [46, 50]):

$$\mathcal{HP}_{\mathcal{P}_i}(q) = \frac{\mathcal{HP}_{\mathbf{K}[x_j, j \geq 3-i] / \langle x_j^2, x_j x_{j+1}, j \geq 3-i \rangle}(q)}{\prod_{j \geq 3-i} (1 - q^j)}. \quad (7.4)$$

Note that, in *loc. cit.*, the proof is given for finitely generated ideals. But this extends easily to our situation. Indeed, the \mathbf{K} -algebra

$$R := \mathbf{K}[x_j, j \geq 3 - i] / \langle x_j^2, x_j x_{j+1}, j \geq 3 - i \rangle$$

is the inductive limit of the \mathbf{K} -finitely generated algebra

$$R_n = \mathbf{K}[x_j, 3 - i \leq j \leq n + 1] / \langle x_j^2, x_j x_{j+1}, 3 - i \leq j \leq n \rangle.$$

Moreover, since the weights of the variables x_j are growing we can see that

$$\mathcal{HP}_{R_n}(q) = \mathcal{HP}_R(q) \text{ modulo } q^n.$$

This implies

$$\lim_{n \rightarrow +\infty} \mathcal{HP}_{R_n}(q) = \mathcal{HP}_R(q),$$

and that the case of finitely generated ideals gives the equality (7.4).

Now everything is settled down for the proof of Corollary 68 (which is equivalent to Theorem 66).

Proof. On one hand, from Proposition 74 and the equality (7.4) we obtain that for $i \in \{1, 2\}$ we have

$$\frac{\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|}}{\prod_{j \geq 3-i} (1 - q^j)^2} = \frac{\mathcal{HP}_{\mathbf{K}[x_j, j \geq 3-i] / \langle x_j^2, x_j x_{j+1}, j \geq 3-i \rangle}(q)}{\prod_{j \geq 3-i} (1 - q^j)}. \quad (7.5)$$

On the other hand, the homogeneous components of weight h of

$$R = \frac{\mathbf{K}[x_j, j \geq 3 - i]}{\langle x_j^2, x_j x_{j+1}, j \geq 3 - i \rangle}$$

is generated by the monomials $x_{i_1} \dots x_{i_r}$ such that $i_1 + \dots + i_r = h$ and which are not divisible by neither x_j^2 nor $x_j x_{j+1}$, for any $j \in \mathbf{N}$. So the data of such a monomial is equivalent to the data of a partitions of h without equal nor consecutive parts and the part 1 appears at most $i - 1$ times (because of the condition $j \geq 3 - i$ in the indices of the variables x_j). Hence we have

$$\mathcal{HP}_R(q) = \sum_{h \geq 0} |\mathcal{T}_i(h)| q^h,$$

which is the left side of the identities (7.1) (for $i = 2$) and of (7.2) (for $i = 1$). From the equalities (7.5), (7.1) and (7.2), we get that for $i \in \{1, 2\}$, we have:

$$\frac{\sum_{\lambda \in \mathcal{N}_i} \delta(\lambda) q^{|\lambda|}}{\prod_{j \geq 3-i} (1 - q^j)} = \frac{\prod_{j \geq 3-i, j \equiv 0, \pm i \pmod{5}} (1 - q^j)}{\prod_{j \geq 3-i} (1 - q^j)},$$

and the theorem follows. □

Chapter 8

Δ_m -Partitions and Jet Schemes of the Double Point

8.1 Introduction

In this chapter, for a positive integer m we study the family of ideals defining the jet scheme of the double point $\text{Spec}K[x]/x^2$ (K being an algebraically closed field of characteristic 0) and we conclude some results concerning the generating series of a special type of integer partitions that we will introduce below.

As we mentioned before, the $(m-1)$ -jet scheme of X is the space of maps of the form: $\text{Spec}K[t]/t^m \rightarrow X$. Such a map is determined by the data $(x_0, \dots, x_{m-1}) \in K$ for which $x(t)^2 \equiv 0 \pmod{t^m}$, where $x(t) = x_0 + x_1t + \dots + x_{m-1}t^{m-1}$. When expanding this equation, we get a set of polynomials of the form:

$$f_1 = x_0^2, f_2 = 2x_0x_1, \dots, f_m = \sum_{i=0}^{m-1} x_i x_{m-1-i}$$

Let R_m be the polynomial ring over the field K defined over the variables x_0, \dots, x_{m-1} . The ideal defining the $(m-1)$ -jet scheme of X in $\text{Spec}R_m$ is the ideal $I_m \subset R_m$ generated by f_1, \dots, f_m . We will endow R_m with a double grading by assigning to each x_i the bigrading $(i, 1)$, for $0 \leq i \leq m-1$. Notice that the ideal I_m is homogeneous for both grading.

Define the bigraded Hilbert series of R_m/I_m to be the series in $\mathbf{Z}[[q, t]]$ given by:

$$H_m(q, t) = \sum_{i, j \geq 0} \dim_K(R_m/I_m)_{i, j} q^i t^j$$

Gorsky et. al. [11] found an explicit formula for this Hilbert series:

Theorem 75 ([11]). *The Hilbert series $H_m(q, t)$ is given by the following*

explicit formula:

$$H_m(q, t) = \sum_{p=0}^{\infty} \frac{\binom{h(m,p)+1}{p}_q q^{p(p-1)} t^p}{(1 - q^{m-h(m,p)} t) \dots (1 - q^{m-1} t)}$$

where $h(m, p) = \lfloor \frac{m-p}{2} \rfloor$ and $\binom{a}{b}_q = \frac{(1-q) \dots (1-q^a)}{(1-q) \dots (1-q^b) (1-q) \dots (1-q^{a-b})}$

In the following section, we determine the initial ideal of I_m for the reverse lexicographic order and we conclude the generating series of a new type of integer partitions.

Definition 76. Let n be a positive integer. Let λ be a partition of n . Denote $SC(\lambda)$ to be the pair of the smallest consecutive or equal parts of λ . For $SC(\lambda) = (i, j)$, denote by $|SC(\lambda)|$ the sum $i + j$. Define the k -value of λ as follows: $k(\lambda) = |SC(\lambda)| - |\lambda| - 1$. For a partition λ with no consecutive or equal parts we consider $|k(\lambda)| = +\infty$.

We call Δ_m -partitions of n the set $\Delta_m(n)$ of partitions λ of n with parts less than or equal to m such that for any $k(\lambda)$ parts $i_1 \leq \dots \leq i_{k(\lambda)}$ of λ less than or equal to $\lfloor \frac{m - 3k - 1}{2} \rfloor + 1 + 2(k - 1)$ we have $|i_t - i_{t+1}| \leq 1$ for some t . Denote by $\Delta_m(n, p)$ the set of all the Δ_m -partitions of n with size p .

Definition 77. Define the generating series of $\Delta_m(n, p)$ to be the two variable power series given by:

$$\mathbf{H}_m(q, t) = \sum_{n, p \geq 0} \Delta_m(n, p) q^n t^p$$

Theorem 78. Let $n \geq 1$ be an integer. The generating series of the Δ_m -partitions is given by:

$$\mathbf{H}_m(q, t) = H_m(q, qt) = \sum_{p=0}^{\infty} \frac{\binom{h(m,p)+1}{p}_q q^{p^2} t^p}{(1 - q^{m+1-h(m,p)} t) \dots (1 - q^m t)}$$

where $h(m, p) = \lfloor \frac{m-p}{2} \rfloor$ and $\binom{a}{b}_q = \frac{(1-q) \dots (1-q^a)}{(1-q) \dots (1-q^b) (1-q) \dots (1-q^{a-b})}$

8.2 The Initial Ideal of I_m and the Δ_m -Partitions

Let $R_m = K[x_0, \dots, x_{m-1}]$ and let $I_m \subset R_m$ be the ideal generated by f_1, \dots, f_m , as defined above. In a paper of Gorsky et. al [11], they found a recursive definition for the Gröbner basis of I_m with respect to the reverse lexicographic order. Following their work, we will compute the generators of the initial ideal of I_m .

Order the monomials in R_m following the reverse lexicographic order, that is: $x^\alpha < x^\beta$ if and only if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and the rightmost entry of $\alpha - \beta$ is negative. The following two operators were defined in [11] in the sake of finding the Gröbner basis G_m of I_m with respect to the reverse lexicographic order.

Define a ring homomorphism $S : R_m \rightarrow R_{m+1}$ by the equation $S(x_i) = x_{i+1}$. This homomorphism is called the shift operator. Define the modified shift operator \tilde{S} over I_m as follows:

For any element $p \in I_m$ of the form $p = \sum_{i=1}^m \phi_i f_i$, define $\tilde{S}(p)$ as:

$$\tilde{S}(p) = \sum_{i=1}^m S(\phi_i) f_{i+2}$$

Theorem 79 ([11]). *Let*

$$G_1 = \{f_1\} \subset R_1, \quad G_2 = \{f_1, f_2\} \subset R_2$$

and recursively define the sets G_m , $m \geq 3$ as follows:

$$G_m = x_0 S^2(G_{m-3}) \cup \{f_1, f_2\} \cup \tilde{S}(G_{m-2}).$$

Then G_m is a Gröbner basis for I_m .

In the following, we will determine the initial ideal of I_m .

Lemma 80. *The initial ideal $L(I_m)$ with respect to the reverse lexicographic order is generated by the following monomials:*

$$x_j x_i x_{i_1} \dots x_{i_k}$$

such that $j = i$ or $i + 1$, $k = i + j - m + 1$, $0 \leq i_1 \leq \lfloor \frac{m - 3k - 1}{2} \rfloor$ and

$$i_{l-1} + 2 \leq i_l \leq \lfloor \frac{m - 3k - 1}{2} \rfloor + 2(l - 1)$$

for which $x_{i_1} = \dots = x_{i_k} = 1$ if $k \leq 0$

Proof. We will proceed by induction. For $m = 1$ or 2 , we can easily check that it is true.

Suppose it is true up to $m - 1$, let's prove it for m .

The initial terms of $\tilde{S}(G_{m-2})$, which are the initial terms of G_{m-2} shifted once by S have the following form:

$$S(x_j x_i x_{i_1} \dots x_{i_k})$$

such that $j = i$ or $i + 1$, $k = i + j - m + 2 + 1$, $0 \leq i_1 \leq \lfloor \frac{m - 2 - 3k - 1}{2} \rfloor$

and $i_{l-1} + 2 \leq i_l \leq \lfloor \frac{m - 2 - 3k - 1}{2} \rfloor + 2(l - 1)$

That is:

$$\begin{aligned}
& x_{j+1}x_{i+1}x_{i_1+1}\dots x_{i_k+1} \\
& \text{such that } j = i \text{ or } i + 1, k = (i + 1) + (j + 1) - m + 1, \\
& 1 \leq i_1 + 1 \leq \lfloor \frac{m - 3k - 1}{2} \rfloor \text{ and} \\
& i_{l-1} + 2 + 1 \leq i_l + 1 \leq \lfloor \frac{m - 2 - 3k - 1}{2} \rfloor + 2(l - 1) + 1
\end{aligned}$$

That is:

$$\begin{aligned}
& x_{j'}x_{i'}x_{i'_1}\dots x_{i'_k} \\
& \text{such that } j = i \text{ or } i + 1, k = i' + j' - m + 1, 1 \leq i'_1 \leq \lfloor \frac{m - 3k - 1}{2} \rfloor \text{ and} \\
& i_{l-1} + 2 \leq i'_l \leq \lfloor \frac{m - 3k - 1}{2} \rfloor + 2(l - 1)
\end{aligned}$$

The initial terms of $x_0S^2(G_{m-3})$ have the following form:

$$\begin{aligned}
& x_{j+2}x_{i+2}x_0x_{i_1+2}\dots x_{i_k+2} \\
& \text{such that } j = i \text{ or } i + 1, k = i + j - m + 3 + 1 \text{ and} \\
& i_{l-1} + 2 + 2 \leq i_l + 2 \leq \lfloor \frac{m - 3 - 3k - 1}{2} \rfloor + 2(l - 1) + 2
\end{aligned}$$

That is:

$$\begin{aligned}
& x_{j'}x_{i'}x_{i'_1}x_{i'_2}\dots x_{i'_{k'}} \\
& \text{such that } j' = i' \text{ or } i' + 1, k' = k + 1 = i' + j' - m + 1, i'_1 = 0 \text{ and} \\
& i'_{l-1} + 2 \leq i'_l \leq \lfloor \frac{m - 3(k + 1) - 1}{2} \rfloor + 2(l - 1)
\end{aligned}$$

That is:

$$\begin{aligned}
& x_{j'}x_{i'}x_{i'_1}x_{i'_2}\dots x_{i'_{k'}} \\
& \text{such that } j' = i' \text{ or } i' + 1, i' \leq 2(m - 1)/3, k' = k + 1 = i' + j' - m + 1, \\
& i'_1 = 0 \text{ and } i'_{l-1} + 2 \leq i'_l \leq \lfloor \frac{m - 3k' - 1}{2} \rfloor + 2(l - 1)
\end{aligned}$$

That is:

$$\begin{aligned}
& x_{j'}x_{i'}x_{i'_1}x_{i'_2}\dots x_{i'_{k'}} \\
& \text{such that } j' = i' \text{ or } i' + 1, k' = i' + j' - m + 1, i'_1 = 0 \text{ and} \\
& i'_{l-1} + 2 \leq i'_l \leq \lfloor \frac{m - 3k' - 1}{2} \rfloor + 2(l - 1)
\end{aligned}$$

□

Following from [33], the Hilbert series of the quotient ring R_m/I_m is equal to that of the quotient ring $R_m/L(I_m)$.

Denote by R'_m the polynomial ring in K over the variables x_1, \dots, x_m . This ring is nothing but the image of R_m by the shift operator S . Similarly, denote by I'_m the ideal of R'_m generated by $S(f_i)$, $1 \leq i \leq m$. As in the case of R_m , we consider $R'_m = \oplus R'_{i,j}$. Notice that the image by S of a monomial

$x^\alpha = x_{i_1}^{\alpha_1} \dots x_{i_j}^{\alpha_j}$ of R_m of degree j and weight $i = \sum_{t=1}^{t=j} i_t \alpha_t$ is given by $S(x^\alpha) = x_{i_1+1}^{\alpha_1} \dots x_{i_j+1}^{\alpha_j}$ which is a monomial in R'_m of degree j also but of weight $i + j$. Hence, we can see that $R_{i,j}$ is isomorphic to $R'_{i+j,j}$, and so $\text{Hilb}_{R'_m/I'_m}(q, t) = \text{Hilb}_{R_m/I_m}(q, qt)$.

In the following theorem, we reveal a one to one correspondance between the monomials generating R'_m/I'_m and the Δ_m -partitions:

Theorem 81. *Let $n \geq 1$ be an integer. The generating series of the Δ_m -partitions is given by:*

$$\mathbf{H}_m(q, t) = H_m(q, qt) = \sum_{p=0}^{\infty} \frac{\binom{h(m,p)+1}{p}_q q^{p^2} t^p}{(1 - q^{m+1-h(n,p)t}) \dots (1 - q^{mt})}$$

where $h(m, p) = \lfloor \frac{m-p}{2} \rfloor$ and $\binom{a}{b}_q = \frac{(1-q) \dots (1-q^a)}{(1-q) \dots (1-q^b)(1-q) \dots (1-q^{a-b})}$

Proof. Notice first that the initial ideal of I'_m is generated by:

$$x_j x_i x_{i_1} \dots x_{i_k}$$

such that $j = i$ or $i + 1$, $k = i + j - m - 1$, $1 \leq i_1 \leq \lfloor \frac{m - 3k - 1}{2} \rfloor + 1$ and

$$i_{l-1} + 2 \leq i_l \leq \lfloor \frac{m - 3k - 1}{2} \rfloor + 1 + 2(l - 1)$$

We will prove a one-to one correspondance between the monomial generating R'_m/I'_m and the Δ_m partitions.

Consider a partition λ of n that corresponds to a monomial generating R'_m/I'_m and consider j, i, i_1, \dots, i_k to be parts of λ such that (j, i) is a consecutive or equal pair and $i_1 \leq i_2 \dots \leq i_k \leq \frac{m - 3k - 1}{2} + 1 + 2(k - 1)$ with $k = i + j - m - 1$. If $i_k \geq i_{k-1} + 2 \geq \dots i_1 + 2(k - 1)$ then λ corresponds to a monomial in I'_m , a contradiction. Then we have $|i_t - i_{t+1}| \leq 1$ for some t .

Moreover, for any consecutive or equal pair (i', j') smaller than (i, j) , λ has no parts of the form $j_1, \dots, j_{k'}$ satisfying $k' = i' + j' - m - 1$, $1 \leq j_1 \leq \lfloor \frac{m - 3k' - 1}{2} \rfloor + 1$ and $j_{l-1} + 2 \leq j_l \leq \lfloor \frac{m - 3k' - 1}{2} \rfloor + 1 + 2(l - 1)$. For the contrary, suppose that there exist such $j_1, \dots, j_{k'}$, so for some $1 \leq t \leq k'$. We have i_1 is strictly greater than $\lfloor \frac{m - 3k' - 1}{2} \rfloor + 1 + 2(t - 1)$ and so $k < k'$, a contradiction.

Hence we can deduce a one to one correspondence between the generators of R'_m/I'_m and the Δ_m -partitions. And so the theorem yields.

□

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