



Stochastic PDEs, finite-volume schemes and applications in mechanics

Caroline Bauzet

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en vue de l'obtention du diplôme d'habilitation à diriger des recherches

Discipline : Mathématiques

EDP stochastiques, schémas volumes-finis et applications à la mécanique

par

Caroline BAUZET

Soutenu publiquement le 18 décembre 2023, après avis des rapporteurs :

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Membre invité : M. Thierry **GALLOUËT**.

Remerciements

Je tiens tout d’abord à exprimer ma profonde gratitude et mes sincères remerciements à Thierry Gallouët et Guy Vallet pour tout ce que j’ai pu apprendre auprès d’eux ces dix dernières années, à travers nos collaborations et nos échanges. Leur disponibilité, leur écoute, leur confiance, leur soutien, leurs conseils et leurs encouragements m’ont permis de faire mon bonhomme de chemin au pays des Équations aux Dérivées Partielles Stochastiques, et de m’y être suffisamment aventurée pour avoir une histoire à raconter dans ce manuscrit. Durant ce “voyage”, je n’ai jamais eu honte de leur demander mon chemin quand je me sentais perdue et pour cette bienveillance (en plus des réponses apportées), j’aimerais les remercier particulièrement.

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Pour finir, merci au mathématicien norvégien du XIXème siècle Axel Thue pour la citation suivante, illustrant parfaitement la motivation qui m’a animée pour obtenir les résultats présentés dans ce manuscrit :

“Il n’est pas nécessaire qu’un problème de maths ait des applications pratiques pour qu’il soit intéressant et il peut être très agréable pour l’esprit d’essayer de résoudre des questions apparemment futiles.”

*Les personnes visées se reconnaîtront en page suivante.

1. Celle qui me faisait sécher mes cours de salsa pour un mojito de plus.
2. Celle qui pourrait porter un tee-shirt avec la mention “j’peux pas, j’ai montagne”.
3. Celui qui aurait pu être à l’origine de la citation “Je suis dans l’armée, je tiens pas un stand de crêpes”.
4. Celui à qui je dois mes airs hispaniques.
5. Celle qui m’appelle “ma p’tite Caro”.
6. Ceux qui m’appellent “tatie Caro”.
7. Celle dont le régime alimentaire est compris par peu de personnes (dont je ne fais pas partie).
8. Celles qui finissent le Rinquinquin et laissent les petits poneys s’échapper.
9. Ceux qui ne jurent que par l’apéritif du roy.
10. Celui qui sait réparer presque tout, sauf peut-être les spoils de série.
11. Ceux qui m’appellent “tata Caro”.
12. Celle qu’il faut maintenant booker trois mois à l’avance pour un simple dîner entre filles.
13. Celle qui m’a appris à me servir de ma perceuse et construire un dressing.
14. Celle qui m’a embarquée dans la mission “Chili Con Caro”.
15. Celui qui partage (en plus de tout le reste) de façon inespérée, ma folie des chats.
16. Ceux qui m’appellent “Britney”.
17. Celui qui m’appelle “Carolin” depuis toujours.
18. Celle dont le rire ne s’invente pas.
19. Celui que j’appelle “mon petit crapaud”.
20. Celui qui m’appellerait probablement “maman” s’il savait parler.
21. Celle qui m’appelle “Caly”.
22. Celles avec qui je partage beaucoup, mais certainement pas la bosse des maths, si tant est qu’elle existe.
23. Celle qui me chauffe une salle “à la française”.
24. Celle dont il ne faut pas toucher les pattes arrières.
25. Celle que j’appelais “ma doudou”.

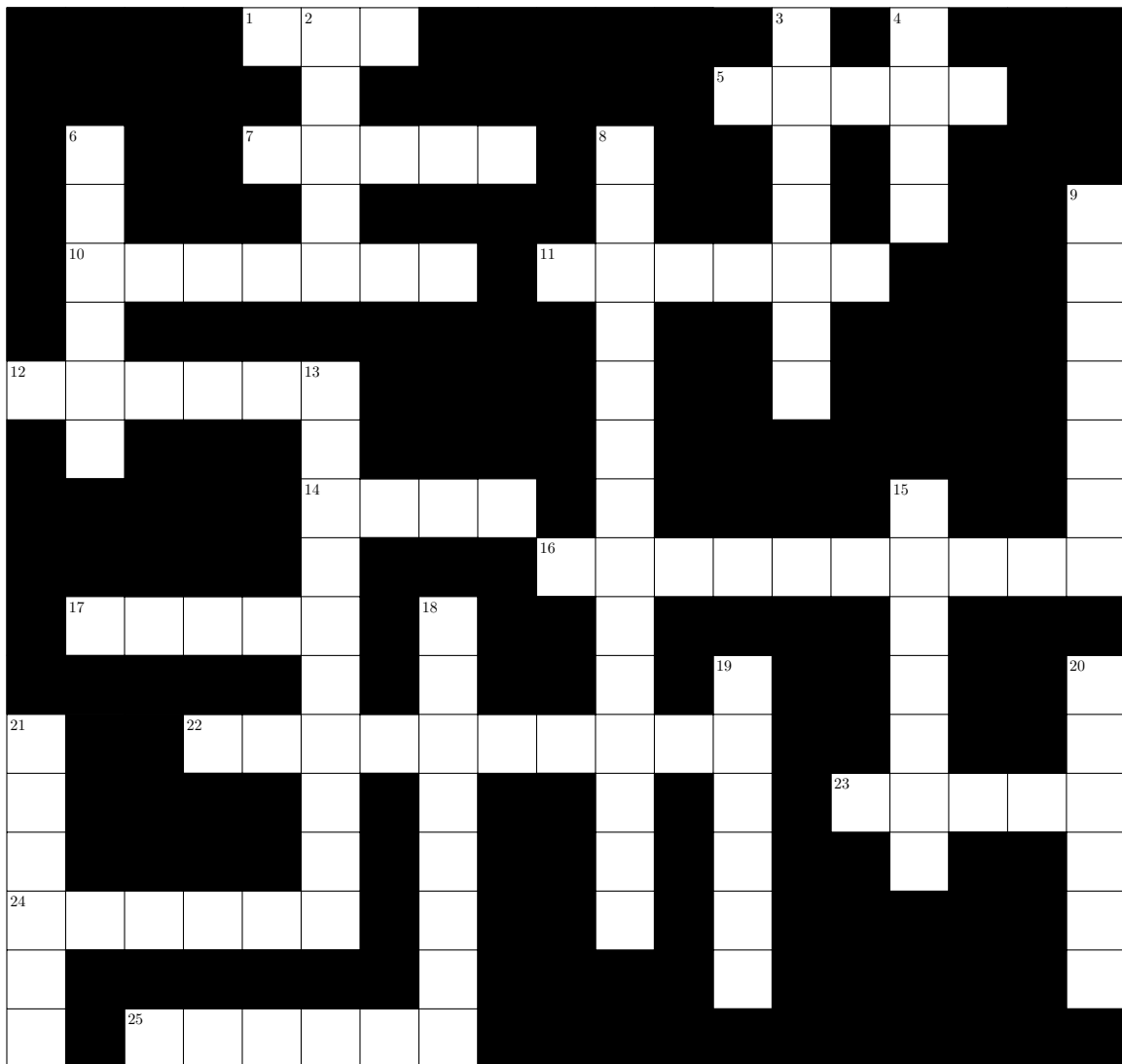


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Introduction

Les équations aux dérivées partielles (EDP) interviennent dans de nombreux domaines que ce soit pour décrire des problèmes issus de la physique, de la biologie, de la médecine, des sciences humaines et sociales ou encore pour répondre à des questions environnementales. Depuis de nombreuses années, ces équations sont étudiées dans le cadre déterministe. Il s'agit alors de modèles à une échelle macroscopique à travers un comportement moyen de type homogénéisation en faisant une hypothèse d'homogénéité spatiale ou *via* une modélisation numérique ignorant certains effets, parfois non négligeables, issus d'échelles spatiales inférieures à celle du maillage. Cependant, les questionnements récents provenant des domaines précédemment cités mettent en évidence la nécessité de prendre en compte de nouvelles problématiques : effets d'interactions inter-échelles, prise en compte des échelles non représentées, fluctuations microscopiques, propriétés méconnues, nouvelle complexité des phénomènes rencontrés, incertitude dans le choix du modèle, données incomplètes ou entachées d'erreurs, influence des effets stochastiques ou aléatoires...

Une façon d'aborder ces problématiques est de considérer des modèles basés sur l'utilisation d'EDP stochastiques, *i.e.* des EDP avec une source pilotée par une intégrale stochastique de type Itô ou Stratonovich. Contrairement aux équations différentielles stochastiques utilisées depuis de nombreuses années pour la modélisation dans divers domaines (équations de Langevin en physique, modèle de Black-Scholes en finance...), les EDP stochastiques sont encore mal comprises et de nombreuses questions se posent : type d'équation restreint, peu de modèles et d'approches numériques... Le développement d'outils méthodologiques, tant d'un point de vue théorique que numérique apparaît comme un objectif prioritaire pour appréhender de façon efficace l'utilisation de modèles incluant de telles composantes stochastiques.

Dans ce contexte, l'étude d'EDP stochastiques est un sujet actuellement en plein essor de par l'enrichissement que celles-ci apportent aux modèles existants (voir par exemple [BBNP14],[Hai09], [Par07], [Par21] ou encore [LPS14] pour des aspects numériques). On peut en effet noter l'intérêt grandissant de la communauté mathématique pour l'étude de ce type d'équations ces dernières années. À ce titre, le moteur de recherche documentaire MathSciNet de l'American Mathematical Society affiche plus de 4000 citations pour le code 60H15 correspondant aux EDP stochastiques, et plus de la moitié de ces citations sont issues après 2010. Enfin, l'Union Mathématique Internationale a récompensé Martin Hairer de la Médaille Fields en 2014 pour ses travaux sur ces équations, première médaille décernée sur cette thématique.

Je présente dans ce manuscrit les quelques pierres que j'ai apportées à l'édifice commun sur les EDP stochastiques ces dix dernières années, et celles que j'envisage de déposer pour les dix prochaines. Plus précisément, je me suis concentrée jusqu'à présent sur l'analyse d'un point de vue théorique et numérique de problèmes non-linéaires perturbés stochastiquement au sens d'Itô. Les bruits que j'ai considérés étaient uniquement temporels : il aurait été intéressant d'envisager conjointement une dépendance spatiale (ce qui a été fait par certains auteurs par la suite), mais je n'ai pas choisi d'aller dans cette direction car le cas temporel me semblait être à lui seul un beau challenge. L'approche mathématique choisie dans mes recherches a consisté à adapter et combiner des techniques connues en analyse des EDP avec des outils de calcul stochastique et de théorie des probabilités. Les équations que j'ai étudiées ont été de deux types : d'un côté des lois de conservation scalaires (d'ordre un puis deux) pour lesquelles je me suis attachée à la construction et l'analyse de convergence de schémas numériques de type volumes-finis, et de l'autre des équations paraboliques liées à la modélisation en mécanique du solide pour lesquelles j'ai regardé des questions d'existence et d'unicité de solutions. Ces deux axes de recherche ont été menés en parallèle avec des membres de divers laboratoires de recherche en France, en Italie et en Allemagne, impliquant au total une partie du travail doctoral de trois étudiants. Ils ont par ailleurs donné lieu à neuf pub-

lications et deux actes de congrès dans des revues internationales à comité de lecture, ainsi qu’une prépublication soumise.

Le manuscrit est organisé comme suit : une synthèse de mes activités scientifiques est d’abord présentée. Sont listés dans cette dernière mes publications, les encadrements doctoraux que j’ai effectués, mes collaborations ainsi que les financements associés, le nombre et la nature de mes communications, les événements que j’ai organisés et enfin l’ensemble des jurys, conseils ou comités auxquels j’ai pris part. Ensuite, le corps du manuscrit (en anglais) est consacré à une description détaillée et chronologique des résultats, et des cheminements qui m’y ont menée, obtenus depuis ma thèse. Ce dernier se décline en trois parties* correspondant aux différents thèmes que j’ai abordés avec mes trois groupes de collaborateurs :

- **Partie 1** : *Schémas volumes-finis pour des lois de conservation scalaires du 1er ordre forcées par un bruit de type multiplicatif*. Sont présentées dans cette partie mes premières contributions sur la discrétisation espace-temps par des méthodes volumes-finis de problèmes hyperboliques avec perturbation stochastique. Elles ont été réalisées au cours de mon post-doctorat au sein de l’Institut de Mathématiques de Marseille (I2M) et de mes premières années d’enseignant-chercheur au sein du Laboratoire de Mécanique et d’Acoustique (LMA). Fruits d’une collaboration suivie avec J. CHARRIER et T. GALLOUËT (tous deux membres de l’I2M), elles comprennent également le travail doctoral de V. CASTEL que nous co-encadrions ensemble. À l’époque, il y avait très peu de résultats d’existence et d’unicité de solution pour ces lois de conservation scalaires hyperboliques avec un forçage stochastique, surtout dans le cas multidimensionnel. À ce titre, le premier résultat complet sur l’étude théorique du problème de Cauchy (cas multidimensionnel, bruit général) n’avait été proposé qu’en 2010 par A. DEBUSSCHE et J. VOVELLE [DV10], et ce par le biais d’une approche par formulation cinétique. Par ailleurs, tout restait à faire en termes d’approximation conjointe espace-temps. Rappelons que dans le cas déterministe, ces équations étaient usuellement discrétisées par des méthodes de type volumes-finis. Regarder comment adapter ces techniques au cas stochastique était pour moi une direction de recherche toute indiquée après avoir travaillé durant ma thèse sur les aspects théoriques de ces équations d’une part (en utilisant une approche entropique), et sur leur discrétisation temporelle d’autre part (par des méthodes de time-splitting d’opérateur). À travers quatre publications dont j’expose les résultats dans cette première partie [BCG16a, BCG16b, BCG17, BCC20], nous avons su répondre aux questions suivantes :
 - Chapitre I : Comment construire un premier schéma volumes-finis pour discrétiser notre problème hyperbolique stochastique dans le cas multidimensionnel ? Ce schéma converge-t-il vers la solution qui nous intéresse ? Peut-on conserver ce résultat de convergence pour une classe plus générale de schémas ? Peut-on considérer un terme de flux avec une dépendance espace-temps ?
 - Chapitre II : Comment prendre en compte des conditions de bord de type Dirichlet non-homogènes dans notre approximation volumes-finis ?
 - Chapitre III : Est-il possible de montrer la convergence du schéma numérique sans disposer d’un cadre théorique nous donnant existence et unicité de la solution physiquement admissible ? Peut-on relever le challenge technique d’une dépendance espace-temps du terme de flux la plus générale possible ?
- **Partie 2** : *Problèmes paraboliques stochastiques et mécanique du solide*. Je traite ici de mes travaux en lien avec la modélisation en mécanique du solide auxquels quatre publications sont associées [BBB⁺17, MLB18, BLM19, BLMZ21]. Ceux-ci ont principalement vu le jour grâce à de fructueux temps d’échange avec un membre de mon laboratoire (F. LEBON) et une collaboratrice italienne (G. BONFANTI de l’Université de Brescia). Ces travaux m’ont permis d’élargir mes thématiques de recherche afin que ces dernières s’inscrivent dans celles de l’équipe “Matériaux et Structures” dont je fais partie depuis mon recrutement au LMA. Notons que s’ajoutent à la liste des personnes impliquées dans une partie de ces travaux, deux collaboratrices étrangères (E. BONETTI de l’Université de Milan et A. ZIMMERMANN à l’époque en poste à l’Université de Duisburg-Essen), mon directeur de thèse (G. VALLET de l’Université de Pau) et également A.A. MAITLO, alors en thèse au LMA sous la direction de F. LEBON et moi-même.
 - Dans le Chapitre IV, je reviens sur les résultats obtenus dans [BBB⁺17]. Ces derniers portaient sur l’analyse mathématique d’une équation de type Allen-Cahn avec contrainte et forcée par une composante stochastique, pour laquelle nous nous sommes attachés à montrer l’existence et l’unicité d’une solution. Usuellement, une telle équation est utilisée pour décrire divers phénomènes physiques, en particulier ceux concernant les phénomènes de changements de phase. Dans notre contexte, l’idée était de modéliser l’évolution de l’endommagement d’un matériau au cours du temps. L’ajout d’une perturbation stochastique était quant à lui motivé par la prise en compte des changements à l’échelle microscopique de la structure du matériau comme par exemple la formation de cavités au cours de l’endommagement.

*Les trois parties peuvent être lues indépendamment les unes des autres car les notations communes sont volontairement répétées.

- L'article [MLB18] correspond au premier travail doctoral d'A.A MAITLO. L'objectif de cet article était de proposer un modèle d'interface collée prenant en compte de l'endommagement non local et des contraintes unilatérales (conditions de type Signorini). La méthodologie proposée consistait à étudier une structure composite endommagée, composée de deux adhérents et d'un adhésif mince (avec un paramètre d'épaisseur ϵ). L'adhésif, fissuré au niveau microscopique, était soumis à deux régimes, l'un en traction dans lequel le matériau était considéré comme mou (par rapport aux adhérents), et l'autre en compression dans lequel le matériau était considéré comme rigide. À l'aide d'une analyse asymptotique formelle par rapport au petit paramètre ϵ (méthode dite des développements asymptotiques raccordés) et en supposant des conditions de liaisons parfaites entre les adhérents et l'adhésif, le modèle d'interface a ensuite été dérivé. Je ne commenterai pas dans ce manuscrit de tels résultats, les ayant suivis avec un regard très extérieur.
- Dans le Chapitre V sont détaillés les travaux développés dans [BLM19, BLMZ21]. L'idée initiale était de s'intéresser à des systèmes d'EDP stochastiques doublement non-linéaires appliqués à des phénomènes de transitions de phase irréversibles, étudiés dans le cas déterministe par une de mes collaboratrices (G. BONFANTI, [BFL00]). Nous avons en tête la modélisation du processus de solidification de la colle dans un tube induite par des actions thermiques et par des mouvements à l'échelle microscopique, en ajoutant de plus une condition d'irréversibilité sur le processus de solidification (porté par une contrainte dans l'équation). L'analyse du système forcé par une composante stochastique demandant beaucoup de technicité pour un travail doctoral, nous avons préféré commencer par étudier une version régularisée du système en question. Cela nous a conduit à l'étude d'un couplage entre une équation de type Barenblatt avec forçage stochastique et d'une équation de la chaleur avec terme source aléatoire.
- **Partie 3 : Approximation volumes-finis de problèmes paraboliques forcés par un bruit de type multiplicatif.** Dans cette dernière partie sont exposées mes investigations récentes sur la discrétisation espace-temps par des méthodes volumes-finis (schémas TPFA[†]) de problèmes paraboliques avec perturbation stochastique. Elles ont été réalisées en collaboration avec F. NABET (École polytechnique), A. ZIMMERMANN (maintenant au TU Clausthal) et concernent une partie du travail doctoral de K. SCHMITZ (en thèse sous la direction d'A. ZIMMERMANN). Elles ont donné lieu à une publication [BNSZ23a], deux actes de congrès à comité de lecture [BN20, BNSZ23b] et un article soumis [BSZ23]. La motivation initiale de tous ces travaux était de proposer un schéma numérique pour approcher la solution du problème d'Allen-Cahn stochastique avec contrainte que j'avais étudié dans [BBB⁺17], afin d'en rendre l'analyse plus complète. Voulant mettre à profit et approfondir mes connaissances en méthodes volumes-finis, il m'a semblé naturel de me tourner vers ce type d'approximation, plutôt que de m'essayer aux méthodes éléments-finis (plus communément employées sur des problèmes paraboliques stochastiques). Ignorant tout des techniques de discrétisation de l'opérateur laplacien, il m'a semblé essentiel de commencer par étudier la chose sur un problème simplifié, et mon regard s'est porté vers l'équation de la chaleur stochastique.
 - Dans le Chapitre VI sont présentées les études [BN20, BNSZ23a] accomplies sur l'approximation de cette fameuse équation de la chaleur avec un forçage stochastique de type multiplicatif (*i.e.* avec un bruit dépendant de l'inconnue du problème, et ce de façon linéaire ou non). Pour des raisons techniques que j'explique en détail dans le Chapitre VI, il nous a fallu utiliser des outils assez complexes (au regard de la "simplicité" de l'équation) classiquement employés par la communauté probabiliste, pour arriver à faire converger notre schéma vers la solution souhaitée, et ce dans le cas d'un bruit multiplicatif non-linéaire.
 - Le Chapitre VII répond quant à lui à une question que je n'arrivais pas à me sortir de la tête, suite à la finalisation de nos travaux précédents : est-il possible de montrer la convergence du schéma volumes-finis pour l'équation de la chaleur stochastique sans invoquer les grands théorèmes de théorie des probabilités ? Et il en engendre une seconde : est-il possible de généraliser une telle méthode à un problème de diffusion-convection avec un terme de réaction non linéaire ? Les résultats exposés dans ce dernier chapitre (issus de [BNSZ23b, BSZ23]) me plaisent beaucoup, d'abord parce qu'ils sont obtenus en mélangeant nombre de techniques que j'ai pu apprendre, employer puis partager avec mes divers groupes de collaborations et aussi parce qu'à mon sens, ils scellent d'une certaine façon l'unité de ces dix années de recherche.

Le présent manuscrit s'achève ensuite par la présentation des perspectives de recherche que j'envisage à plus ou moins long terme, dans la continuité des trois principaux thèmes exposés précédemment.

[†]Two-Point Flux Approximation.

Synthèse des activités scientifiques

Postes occupés et diplôme

- 2014 Maître de conférences. Aix-Marseille Université et Laboratoire de Mécanique et d'Acoustique (LMA).
- 2013 Postdoctorat. Institut de Mathématiques de Marseille (I2M). Durée : 1 an. Encadrant : T. Gallouët.
- 2013 Thèse de doctorat. Université de Pau et des Pays de l'Adour. Directeur de thèse : G. Vallet.

Distinctions

- 2021 Titulaire de la PEDR. Durée : 4 ans.
- 2021 CRCT au sein du LMA accordé par Aix-Marseille Université au titre de mon retour de congé maternité. Équipe *Matériaux et Structures*. Durée : 6 mois.
- 2020 Délégation CNRS au sein de l'Institut de Mathématiques de Marseille. Équipe *Analyse Appliquée*. Durée : 6 mois.
- 2017 Titulaire de la PEDR. Durée : 4 ans.

Liste des travaux

► Pré-publication (1)

- Convergence of a TPFA scheme for a diffusion-convection equation with a multiplicative stochastic noise, with K. SCHMITZ and A. ZIMMERMANN, 43 pages, soumis en avril 2023.

► Articles publiés dans des revues internationales à comités de lecture (17)

- Convergence of a finite-volume scheme for a heat equation with a multiplicative Lipschitz noise, with F. NABET, K. SCHMITZ and A. ZIMMERMANN, *Mathematical Modelling and Numerical Analysis*, 2023.
- Well-posedness result for a system of random heat equation coupled with a multiplicative stochastic Barenblatt equation, with F. LEBON, A.A. MAITLO and A. ZIMMERMANN, *Stochastic Analysis and Applications*, 2021.
- Existence and uniqueness result for an hyperbolic scalar conservation law with a stochastic force using a finite volume approximation, with V. CASTEL and J. CHARRIER, *Journal of Hyperbolic Differential Equations*, 2020.
- The Neumann problem for a Barenblatt equation with a multiplicative stochastic force and a nonlinear source term, with F. LEBON and A.A. MAITLO, *Nonlinear Differential Equations and Applications*, 2019.
- A multi-scale model of soft imperfect interface with non-local damage, with F. LEBON and A.A. MAITLO, *Journal of Multiscale Modelling*, 2018.

- A global existence and uniqueness result for a stochastic Allen-Cahn equation with constraint, with E. BONETTI, G. BONFANTI, F. LEBON and G. VALLET, *Mathematical Methods in the Applied Sciences*, 2017.
- Numerical approximation of stochastic conservation laws on bounded domains, with J. CHARRIER and T. GALLOUËT, *Mathematical Modelling and Numerical Analysis*, 2017.
- Convergence of monotone finite volume schemes for hyperbolic scalar conservation laws with a multiplicative noise, with J. CHARRIER and T. GALLOUËT, *Stochastic Partial Differential Equations: Analysis and Computations*, 2016.
- Convergence of flux-splitting finite volume schemes for hyperbolic scalar conservation laws with a multiplicative stochastic perturbation, with J. CHARRIER and T. GALLOUËT, *Mathematics of Computation*, 2016.
- Time-splitting approximation of the Cauchy problem for stochastic conservation, *Mathematics and Computers in Simulation*, 2015.
- A degenerate parabolic-hyperbolic Cauchy problem with a stochastic force, with G. VALLET and P. WITTBOLD, *Journal of Hyperbolic Differential Equations*, 2015.
- The Dirichlet problem for a conservation law with a multiplicative stochastic perturbation, with G. VALLET and P. WITTBOLD, *Journal of Functional Analysis*, 2014.
- On a time-splitting method for a stochastic conservation law with Dirichlet boundary condition and numerical experiments, *Journal of Evolution Equations*, 2014.
- On the stochastic $p(t,x)$ Laplace equation, with G. VALLET, P. WITTBOLD and A. ZIMMERMANN, *Stochastic Partial Differential Equations: Analysis and Computations*, 2013.
- The Cauchy problem for a conservation law with a multiplicative stochastic perturbation, with G. VALLET and P. WITTBOLD, *Journal of Hyperbolic Differential Equations*, 2012.
- On a class of quasilinear Barenblatt equations, with J. GIACOMONI and G. VALLET, *Real Academia des Ciencias de Zaragoza*, 2012.
- On abstract Barenblatt equations, with G. VALLET, Special Issue in Honor of Professor Jesus Ildefonso Diaz's 60th birthday, *Differential Equations & Applications*, 2011.

► Articles publiés dans des actes de congrès à comités de lecture (3)

- Finite Volume Approximations for Non-Linear Parabolic Problems with Stochastic Forcing, with F. NABET, K. SCHMITZ and A. ZIMMERMANN, *Proceedings of the 10th International Symposium of Finite Volumes for Complex Applications (FVCA X)*, to appear.
- Convergence of a finite-volume scheme for a heat equation with a multiplicative stochastic force, with F. NABET, *Proceedings of the 9th International Symposium of Finite Volumes for Complex Applications (FVCA IX)*, 2020.
- Method for solving a stochastic conservation law, *Proceedings of the Twelfth International Conference Zaragoza-Pau on Mathematics*, Jaca, 2013.

Encadrements

- 2023 **Co-encadrement du stage de M2 et de la thèse** de C. Sultan (Aix-Marseille Université et LMA).
 Titre : *Propagation of P-SV seismic waves through horizontally layered random medium*.
 Co-encadrant (LMA) : R. Cottreau (50%).
 Financement : Bourse de l'École doctorale 353 (Sciences pour l'Ingénieur : Mécanique, Physique, Micro et Nanoélectronique).
- 2022 **Accueil au LMA** de K. Schmitz, doctorante en 3ème année à l'Université de Duisburg-Essen (Allemagne) sous la direction de P. Wittbold et A. Zimmermann.
 Titre du projet : *Stochastic models in solid mechanics : theory and numerical approximation*.
 Financement : Procope Mobility Program (Campus France).
 Durée : 1 mois.

- 2017 **Co-encadrement de la thèse** de A.A. Maitlo (Aix-Marseille Université et LMA) soutenue en 2020.
 Titre : *Modélisation d'interfaces imparfaites linéaires et non linéaires et prise en compte d'effets stochastiques*.
 Co-encadrant (LMA) : F. Lebon (50%).
Financement : Higher Education Commission from Pakistan Ministry et Ministère Français des Affaires Étrangères et Européennes.
 Publications : 3.
 Devenir : Assistant Professor at QUAID-E-AWAM UNIVERSITY, Pakistan.
- 2016 **Co-encadrement du stage de M2 et de la thèse** de V. Castel (Aix-Marseille Université et I2M).
 Titre : *Approximation volumes finis d'EDP stochastiques*.
 Co-encadrants (I2M) : J. Charrier (33%) et T. Gallouët (34%).
Financement : Bourse présidentielle (Aix-Marseille Université).
 Publication : 1.
Thèse abandonnée par le candidat en septembre 2018 au profit de son poste d'agrégé dans le secondaire.

Partenariats

► Co-auteurs*

- **Elena Bonetti**, Professeur associé au sein de l'Université de Milan et du Département de Mathématiques F. Enriques en Italie. Elle est spécialiste de l'analyse de systèmes d'EDP (existence et unicité de solutions, dépendance continue par rapport aux données, régularité, comportement en temps long...) utilisés pour modéliser des phénomènes de changement/séparation de phases, d'endommagement de matériaux élastiques et visco-élastiques, des problèmes en plasticité...
- **Giovanna Bonfanti**, Professeur associé au sein de l'Université de Brescia et du Département d'Ingénierie Civile, Architecture, Territoire, Environnement et Mathématiques en Italie. Elle est spécialiste de l'analyse de systèmes d'EDP (existence, unicité, régularité et analyse asymptotique des solutions) qui décrivent des phénomènes de transition de phase, des problèmes d'endommagement et aussi des problèmes de contact avec adhérence et frottement.
- **Julia Charrier**, Maître de conférences au sein d'Aix-Marseille Université et de l'Institut de Mathématiques de Marseille (I2M, UMR CNRS 7373). Elle est spécialiste des méthodes numériques déterministes et probabilistes pour des EDP/EDS à coefficients aléatoires, elle s'intéresse aussi aux méthodes numériques pour les lois de conservation scalaires stochastiques.
- **Thierry Gallouët**, Professeur des Universités au sein d'Aix-Marseille Université et de l'Institut de Mathématiques de Marseille (I2M, UMR CNRS 7373). Il est spécialiste, entre autres choses, des lois de conservation, tant d'un point de vue théorique que numérique par des méthodes de type volumes-finis. Il s'intéresse à présent également à l'analyse numérique des EDP stochastiques.
- **Frédéric Lebon**, Professeur au sein d'Aix-Marseille Université et du Laboratoire de Mécanique et d'Acoustique (LMA, UMR CNRS 7031). Il est spécialiste des questions de modélisation en mécanique des solides et des structures, en particulier tout ce qui concerne les problèmes de contact et d'interface mais plus largement en mécanique non régulière.
- **Flore Nabet**, Maître de conférences au sein de l'École polytechnique et du Centre de Mathématiques Appliquées (CMAP, UMR CNRS 7641). Elle est spécialiste de l'étude de schémas numériques pour des EDP déterministes de types elliptiques ou paraboliques, aussi bien d'un point de vue théorique (existence de solutions au problème discret, convergence des solutions approchées, estimations d'erreur,...) que pratique (implémentations, tests numériques,...).
- **Kerstin Schmitz**, étudiante en thèse sous la direction de P. Wittbold et A. Zimmermann à l'Université Duisburg-Essen en Allemagne. Elle travaille sur des EDP stochastiques en p-laplacien d'une part et sur l'approximation volumes finis d'EDP de type parabolique avec bruit multiplicatif d'autre part.
- **Guy Vallet**, Professeur des Universités au sein de l'Université de Pau et des Pays de l'Adour et du Laboratoire de Mathématiques et de leurs Applications de Pau (LMAP, UMR CNRS 5142). Il est spécialiste de problèmes d'évolutions non linéaires de type parabolique (dégénéré changeant de type comme l'équation des milieux poreux, monotone singulier/dégénéré comme le p-Laplacien, pseudo-monotone ou encore non local comme le Laplacien

*J'ai listé uniquement les co-auteurs des travaux réalisés après ma thèse. Les doctorants que j'ai encadrés ne sont pas re-listés ici.

fractionnaire) et de type hyperbolique d'ordre un. Il travaille principalement sur l'analyse mathématique de ces problèmes en déterministe comme en stochastique.

- **Aleksandra Zimmermann**, Professeur au sein du TU Clausthal et de l'Institut de Mathématiques, Clausthal-Zellerfeld en Allemagne. Elle est spécialiste de l'analyse d'EDP non linéaires, plus particulièrement des questions d'existence et d'unicité de la solution de problèmes pseudo-monotones de type p -Laplacien avec perturbation stochastique et d'équations non linéaires elliptiques et paraboliques dans les espaces d'Orlicz avec données L^1 .

► Groupes de recherche (7)

2023 **Porteuse** d'un projet PHC Procope Plus (Campus France).

Titre : *Modèles stochastiques appliqués à la mécanique du solide et des fluides*.

Porteuse allemande : A. Zimmermann.

Durée : 1 an.

2023 **Membre** d'un projet PHC Procope (Campus France).

Titre : *Finite volume Approximation of Stochastic Scalar COnservation Laws (FIASSCOL)*.

Porteuses : F. Nabet et A. Zimmermann.

Durée : 2 ans.

2020 **Porteuse** d'un projet PHC Procope (Campus France).

Titre : *EDP Stochastiques : Modélisation Et Approximation (ESMEA)*.

Porteuse allemande : A. Zimmermann.

Durée : 2 ans.

2019 **Membre** d'un projet ANR.

Titre : *SIMulation ALéatoire en dimension INfinie (SIMALIN)*.

Porteur : L. Goudenège.

Durée : 4 ans.

2018 **Membre** d'un GDR ECO-Math.

Titre : *Modèles mathématiques et méthodes numériques en mécanique des solides*.

Durée : 2 ans.

2015 **Membre** d'un projet Institut Carnot ISIFoR

Titre : *Stochastic Models And Scale CHanging (SMASCH)*.

Porteur du projet : G. Vallet.

Durée : 2 ans.

2014 **Membre** d'un projet PHC Galilée (Campus France)

Titre : *Modélisation de la détérioration des pierres monumentales dans un but de conservation et de promotion du patrimoine historique*.

Porteurs du projet : E. Bonetti et F. Lebon.

Durée : 2 ans.

► Accueil de chercheurs (11)

08/23 **G. Bonfanti**, Université de Brescia (Italie), 1 semaine.

Financement : Projet Procope Plus.

08/23 **N. Sapountzoglou, K. Schmitz et A. Zimmermann**, TU Clausthal (Allemagne), 1 semaine.

Financement : Projet Procope Plus.

10/22 **F. Nabet**, CMAP, École polytechnique, 2 jours.

10/22 **A. Zimmermann**, Université de Duisburg-Essen (Allemagne), 2 semaines.

Titre : *Modèles stochastiques en mécanique du solide*.

Financement : AAP IMI* Collaborations internationales.

*Appel à projet de l'Institut Mécanique et Ingénierie, Aix-Marseille Université.

04/22 **F. Nabet**, CMAP, École polytechnique, 3 jours.

03/22 **K. Schmitz** et **A. Zimmermann**, Université de Duisburg-Essen (Allemagne), 1 semaine.

12/21 **G. Bonfanti**, Université de Brescia (Italie), 1 semaine.

11/21 **K. Schmitz** et **A. Zimmermann**, Université de Duisburg-Essen (Allemagne), 4 jours.

Financement : PHC Procope ESMEA.

11/20 **G. Bonfanti**, Université de Brescia (Italie), 1 mois.

Titre : *Problèmes d'endommagement couplé de volume et de surface d'un matériau : modélisation, analyse mathématique et applications.*

Financement : Campagne FIR Aix-Marseille Université (accueil de scientifiques étrangers).

03/20 **A. Zimmermann**, Université de Duisburg-Essen (Allemagne), 5 jours.

Financement : PHC Procope ESMEA.

11/19 **F. Nabet**, CMAP, École polytechnique, 3 jours.

► Séjours invités (7)

2020 Fédération de Mathématiques Centrale Supélec, Gif-sur-Yvette, invitée par L. Goudenège. Durée : 3 jours.

Financement : ANR Simalin.

2019 Tata Institute of Fundamental Research Centre For Applicable Mathematics (TIFR CAM), Bangalore, Inde, invitée par I. Biswas. Durée : 10 jours.

Financement : TIFR CAM.

2018 Institute of Mathematics of the Romanian Academy, Bucarest, Roumanie, invitée par L. Badea. Durée : 5 jours.

Financement : GDR ECO-Math.

2017 Dipartimento di Ingegneria Civile, Architettura, Territorio, Ambiente e di Matematica, Università di Brescia, Italie, invitée par G. Bonfanti. Durée : 5 jours.

Financement : INdAM (Istituto Nazionale di Alta Matematica).

2016 Institut Élie Cartan, Université de Lorraine, invitée par M. Dozzi. Durée : 2 jours.

2015 Istituzioni di Matematica, Università di Milano, Italie, invitée par E. Bonetti. Durée : 5 jours.

Financement : PHC Galilée.

2015 Istituzioni di Matematica, Università di Milano, Italie, invitée par E. Bonetti. Durée : 3 jours.

Financement : PHC Galilée.

Diffusion des travaux

- Communications orales dans des conférences nationales : 6.
- Communications orales dans des conférences internationales : 10.
- Séminaires et groupes de travail à l'étranger : 4.
- Séminaires et groupes de travail en France : 19.
- Posters dans des conférences nationales et internationales : 6.

Organisation d'évènements

- 2023 **Co-organisatrice** du Workshop *Stochastic models in mechanics: theoretical and numerical aspects*, Marseille, avec F. Lebon et A. Zimmermann. (<https://wksp-stochmod-lma.sciencesconf.org>).
Financement : Ambassade de France à Berlin (projet Procope Plus), Federal Ministry of Education and Research, TU Clausthal, Laboratoire de Mécanique et d'Acoustique et Projet ANR-19-CE40-0016 (SIMALIN).
- 2021 **Co-organisatrice** du Workshop *Numerical Analysis of Stochastic Partial Differential Equations*, CIRM, Marseille, avec C-E Bréhier, J. Charrier, L. Goudenège et A. Richard (<https://conferences.cirm-math.fr/2408.html>).
Financement : ANR SIMALIN.
- 2019 **Co-organisatrice** de la journée *Analyse Appliquée & LMA*, avec B. Lombard, LMA, Marseille (<https://journee-aa-lma.sciencesconf.org>).
Financement : LMA.
- 2015 **Co-organisatrice** des *Journées thématiques sur les lois de conservation stochastiques : théorie, analyse numérique et applications*, avec J. Charrier et T. Gallouët, FRUMAM, Marseille (<https://journées-lcs.sciencesconf.org>).
Financement : GDR MoMaS.
- 2015 **Co-organisatrice** du Mini-Symposium *Numerical methods for stochastic partial differential equations*, Congrès MAMERN VI, Pau, avec J. Charrier.
- 2012 **Co-organisatrice** du séminaire des doctorants du LMAP, Université de Pau et des Pays de l'Adour.

Participation à des jurys, comités et conseils

- 2023 **Membre du jury de thèse** de A. Messaoudi, Aix-Marseille Université.
Titre : *Radiative transfer models for acoustic waves in a randomly fluctuating medium with boundaries : boundary effects and weak localization*.
Directeurs de thèse : R. Cotteneau (LMA) et C. Gomez (I2M).
- 2023 **Membre du comité de sélection** pour un poste d'ATER (Section CNU 60).
École Centrale Marseille, affectation au LMA.
- 2023 **Membre du comité de sélection** pour un poste de Maître de Conférences (Section CNU 60).
Aix-Marseille Université (IUT), affectation au LMA ou à l'IUSTI.
- 2020 **Membre suppléante** au CNU 26. Participation à 5 conseils depuis 2020 : PEDR (1), Qualifications (1), Promotions (2) et RIPEC (1).
- 2020 **Membre du jury de thèse** de Y. Tahraoui, Université de Pau et des Pays de l'Adour.
Titre : *Problèmes paraboliques à contraintes, déterministes et stochastiques*.
Directeurs de thèse : A. Mokrane et G. Vallet.
- 2019 **Membre du jury de thèse** de S. Martel, École des Ponts ParisTech.
Titre : *Theoretical and numerical analysis of invariant measures of viscous stochastic scalar conservation laws*.
Directeurs de thèse : S. Boyaval et J. Reygner.
- 2018 **Membre élue** du conseil de laboratoire du LMA, de mars 2018 à décembre 2023.
- 2016 **Membre du comité de sélection** pour un poste de Maître de Conférences (Section CNU 26).
Aix-Marseille Université, affectation à l'Institut de Mathématiques de Marseille.

Introduction (english version)

Partial differential equations (PDEs) are used in many fields, whether to describe problems in physics, biology, medicine, human and social sciences, or to address environmental questions. For many years, these equations have been studied within a deterministic framework. This has involved studies on a macroscopic scale, using average behavior of homogenization type, under the assumption of spatial homogeneity, or using numerical modeling that ignores certain effects, sometimes non-negligible, arising from spatial scales below those of the mesh. However, recent developments in the above-mentioned fields highlight the need to take new issues into account: effects of inter-scale interactions, consideration of unrepresented scales, microscopic fluctuations, unknown properties, new complexity of the phenomena encountered, uncertainty in the choice of model, incomplete or distorted data, influence of stochastic or random effects ...

One way of approaching these problems is to consider models based on the use of stochastic PDEs, *i.e.* PDEs with a source driven by a stochastic integral of the Itô or Stratonovich type. Unlike stochastic differential equations, which have been used for many years for modeling in a variety of fields (Langevin equations in physics, Black-Scholes model in finance...), stochastic PDEs are still poorly understood, and many questions arise: restricted equation type, few models and numerical approaches... The development of methodological tools, both from a theoretical and numerical point of view, appears to be a priority objective if we want to make an effective use of models including such stochastic components.

In this context, the study of stochastic PDEs is currently a fast-growing topic, due to the enrichment they bring to existing models (see, for example, [BBNP14],[Hai09], [Par07], [Par21] or even [LPS14] for numerical aspects). In recent years, the mathematical community has shown growing interest in the study of this type of equation. For example, the American Mathematical Society's search engine MathSciNet displays over 4,000 citations for code 60H15 corresponding to stochastic PDEs, and more than half of these are from after 2010. Finally, the International Mathematical Union awarded Martin Hairer the Fields Medal in 2014 for his work on these equations, the first medal awarded on this theme.

In this manuscript, I present the few stones I have contributed to the common edifice on stochastic PDEs over the last ten years, and those I plan to lay down for the next ten. More specifically, I have so far concentrated on the theoretical and numerical analysis of stochastically perturbed nonlinear problems in Itô's sense. Let me mention that the noises I used were purely temporal: it would have been interesting to consider spatial dependence at the same time (as some authors have done subsequently), but I didn't choose to go in that direction, as the temporal case alone already seemed quite challenging. The mathematical approach chosen in my research has consisted in adapting and combining well-known techniques in PDEs analysis with tools from stochastic calculus and probability theory. The equations I studied were of two types: on the one hand, scalar conservation laws of order one and then two, for which I focused on the construction and convergence analysis of finite-volume numerical schemes, and on the other, parabolic equations linked to solid mechanics modeling, for which I looked at questions of existence and uniqueness of solutions. These two lines of research were carried out in parallel with members of various research laboratories in France, Italy and Germany, involving in total part of the doctoral work of three students. They also gave rise to nine publications and two conference proceedings in international peer-reviewed journals, as well as one submitted pre-publication.

The manuscript is organized as follows: first, a summary of my scientific activities is presented. This lists my publications, doctoral supervisions, collaborations and associated funding, the number and nature of talks I have given, events I have organized, and all the juries, councils or committees I have taken part in. Then, the body of the manuscript (in English)

is devoted to a detailed, chronological description of the results (and the paths that led me there) obtained after my PhD thesis. The latter is divided into three parts* corresponding to the different themes I have tackled with my three groups of collaborators:

- **Part 1:** *Finite-volume schemes for first-order scalar conservation laws with a multiplicative stochastic force.* This section presents my first contributions to the space-time discretization of stochastically perturbed hyperbolic problems, using finite-volume methods. They were carried out during my PostDoc at the Institut de Mathématiques de Marseille (I2M) and my first years as associate professor at the Laboratoire de Mécanique et d'Acoustique (LMA). From ongoing collaboration with J. CHARRIER and T. GALLOUËT (both members of I2M), they also include the doctoral researches of V. CASTEL, which we co-supervised together. At the time, there were very few existence and uniqueness results of solution for these hyperbolic scalar conservation laws with stochastic forcing, especially in the multidimensional case. For instance, the first complete result on the theoretical study of the Cauchy problem (multidimensional case, general noise) was only proposed in 2010 by A. DEBUSSCHE and J. VOVELLE [DV10], using an approach by kinetic formulation. Moreover, everything remained to be done in terms of joint space-time approximation. Remind that in the deterministic case, these equations were usually discretized using finite-volume methods. After having worked on the theoretical aspects of these equations during my PhD thesis (using an entropic approach) and their temporal discretization (through time-splitting methods), looking at how these techniques could be adapted to the stochastic case was an ideal research topic for me. Through four publications whose results are presented in this first part [BCG16a, BCG16b, BCG17, BCC20], we have been able to answer the following questions:
 - Chapter I: How can we construct a finite-volume scheme to discretize our stochastic hyperbolic problem in the multidimensional case? Does this scheme converge towards the solution we are interested in? Can we preserve this convergence result for a more general class of schemes? Can we consider a flux term with a spatial and temporal dependence?
 - Chapter II: How can we take non-homogeneous Dirichlet-type boundary conditions in our finite-volume approximation into account?
 - Chapter III: Is it possible to show the convergence of our scheme without a theoretical framework giving us the existence and uniqueness of the physically admissible solution? Can we raise the technical challenge of the most general possible space-time dependence of the flux term?
- **Part 2:** *Stochastic parabolic problems and solid mechanic.* Here, I discuss my contribution in solid mechanics modeling, which is the subject of four publications [BBB⁺17, MLB18, BLM19, BLMZ21]. These works were mainly the result of fruitful exchanges with a member of my laboratory (F. LEBON) and an Italian collaborator (G. BONFANTI from the University of Brescia). It has enabled me to broaden my research themes so that they fit in with those of the "Materials and Structures" team, of which I have been a member since my recruitment to the LMA. Note that two foreign collaborators have also been involved in part of these papers (E. BONETTI from the University of Milan and A. ZIMMERMANN then at the University of Duisburg-Essen), additionally my PhD thesis supervisor (G. VALLET from the University of Pau) and also A.A. MAITLO, then a PhD student at the LMA under the supervision of F. LEBON and myself.
 - In Chapter IV, I return to the results obtained in [BBB⁺17]. They concerned the mathematical analysis of an Allen-Cahn type equation, constrained by a subdifferential term and forced by a stochastic component, for which we showed the existence and uniqueness of a solution. Usually, such an equation is used to describe various physical phenomena, in particular those involving phase changes. In our context, the idea was to model the evolution of material damage over time. The addition of a stochastic perturbation was motivated by the need to take into account microscopic changes in the structure of the material, such as the formation of cavities during damage.
 - The article [MLB18] corresponds to A.A. MAITLO's first doctoral work. The aim of this paper was to propose a model of bonded interface including nonlocal damage and unilateral constraints (Signorini's conditions). The idea was to derive it from a nonlocal damaged mechanical model of composite structure made by two adherents and a thin adhesive (with a thickness parameter ϵ). The adhesive which was cracked at microscopic level, was subjected to two regimes, one in traction in which the material was considered soft, and one in compression, in which the bulk parameter was taken as hard. The model of interface was then obtained by asymptotic analysis with respect to the small parameter ϵ , assuming perfect bonding conditions between the adherents and the adhesive. I will not comment on such results in this manuscript, having followed them with a very outsider's eye.

*The three parts can be read independently of each other, as common notations are deliberately repeated.

- Chapter **V** gives details of the results obtained in [BLM19, BLMZ21]. The initial idea was to focus on double nonlinear stochastic PDEs systems applied to irreversible phase transition phenomena, studied in the deterministic case by one of my collaborators (G. BONFANTI, [BFL00]). We had in mind the modeling of the solidification process of glue in a tube, induced by thermal actions and microscopic movements, with the addition of an irreversibility condition on the solidification process (carried by a constraint in the equation). As the analysis of a system forced by a stochastic component was technically demanding for a PhD thesis, we preferred to begin by studying a regularized version of the system in question. This led us to the study of a coupling between a Barenblatt-type equation with stochastic forcing and a heat equation with random source term.
- **Part 3: Finite-volume approximation of parabolic problems forced by a multiplicative noise.** This final section describes my recent investigations into the space-time discretization by finite-volume methods (TPFA[†] schemes) of parabolic problems with stochastic perturbation. They were carried out in collaboration with F. NABET (École polytechnique), A. ZIMMERMANN (now at TU Clausthal) and concern part of the doctoral work of K. SCHMITZ (in PhD thesis under the supervision of A. ZIMMERMANN). They resulted in one publication [BNSZ23a], two refereed conference proceedings [BN20, BNSZ23b] and one submitted preprint [BSZ23]. The initial motivation for all these studies was to propose a numerical scheme to approximate the solution of the stochastic Allen-Cahn problem with constraint which I studied a few years ago from a theoretical point of view ([BBB⁺17]), in order to make its analysis more complete. Wanting to use and deepen my knowledge of finite-volume methods, it seemed natural to turn to this type of approximation, rather than try my hand at element-finite methods (more commonly used on stochastic parabolic problems). Since I knew nothing about discretization techniques for the Laplacian operator, I thought it essential to start by studying it on a simplified problem, and I turned my attention to the stochastic heat equation.
 - Chapter **VI** presents the studies [BN20, BNSZ23a] carried out on the approximation of this famous heat equation with a multiplicative stochastic forcing, *i.e.* with a noise depending linearly or not on the unknown of the problem. For technical reasons, which I explain in detail in Chapter **VI**, we had to use rather complex tools (compared to the "simplicity" of the equation) which are classically employed by the probabilistic community, to get the convergence of our scheme towards the desired solution, in the case of non-linear multiplicative noise.
 - Chapter **VII** answers a question I couldn't get out of my head following the completion of our previous work: is it possible to show the convergence of the finite-volume scheme for the stochastic heat equation without invoking the great theorems of probability theory? And a second: is it possible to generalize such a method to a diffusion-convection problem with a nonlinear reaction term? The results presented in this final chapter (from [BNSZ23b, BSZ23]) are very pleasing to me, firstly because they have been obtained by mixing a number of techniques that I have been able to learn, use and share with my various collaborating groups, and also because, in my opinion, they seal in a way the unity of these ten years of research.

The present manuscript then concludes with a presentation of my research project, build on the three themes introduced above.

[†]Two-Point Flux Approximation.

Summaries of scientific activities

Positions and diploma

- 2014 Assistant Professor. Aix-Marseille Université and Laboratoire de Mécanique et d'Acoustique (LMA).
- 2013 Postdoctoral Position. Institut de Mathématiques de Marseille (I2M). Duration: 1 year. Supervisor: T. Gallouët.
- 2013 PhD thesis. Université de Pau et des Pays de l'Adour. Supervisor: G. Vallet.

Awards

- 2021 Recipient of the national research and scientific supervision bonus (french name: PEDR). Duration: 4 years.
- 2021 Research leave granted by Aix-Marseille Université for my return from maternity leave, at the LMA. Duration: 6 months.
- 2020 Research leave granted by the CNRS at the I2M in the team: *Analyse Appliquée*. Duration: 6 months.
- 2017 Recipient of the national research and scientific supervision bonus (french name: PEDR). Duration: 4 years.

Scientific results

► Pre-publication (1)

- Convergence of a TPFA scheme for a diffusion-convection equation with a multiplicative stochastic noise, with K. SCHMITZ and A. ZIMMERMANN, 43pages, submitted in April, 2023.

► Publications in international peer-reviewed journals (17)

- Convergence of a finite-volume scheme for a heat equation with a multiplicative Lipschitz noise, with F. NABET, K. SCHMITZ and A. ZIMMERMANN, *Mathematical Modelling and Numerical Analysis*, 2023.
- Well-posedness result for a system of random heat equation coupled with a multiplicative stochastic Barenblatt equation, with F. LEBON, A.A. MAITLO and A. ZIMMERMANN, *Stochastic Analysis and Applications*, 2021.
- Existence and uniqueness result for an hyperbolic scalar conservation law with a stochastic force using a finite volume approximation, with V. CASTEL and J. CHARRIER, *Journal of Hyperbolic Differential Equations*, 2020.
- The Neumann problem for a Barenblatt equation with a multiplicative stochastic force and a nonlinear source term, with F. LEBON and A.A. MAITLO, *Nonlinear Differential Equations and Applications*, 2019.
- A multi-scale model of soft imperfect interface with non-local damage, with F. LEBON and A.A. MAITLO, *Journal of Multiscale Modelling*, 2018.

- A global existence and uniqueness result for a stochastic Allen-Cahn equation with constraint, with E. BONETTI, G. BONFANTI, F. LEBON and G. VALLET, *Mathematical Methods in the Applied Sciences*, 2017.
- Numerical approximation of stochastic conservation laws on bounded domains, with J. CHARRIER and T. GALLOUËT, *Mathematical Modelling and Numerical Analysis*, 2017.
- Convergence of monotone finite volume schemes for hyperbolic scalar conservation laws with a multiplicative noise, with J. CHARRIER and T. GALLOUËT, *Stochastic Partial Differential Equations: Analysis and Computations*, 2016.
- Convergence of flux-splitting finite volume schemes for hyperbolic scalar conservation laws with a multiplicative stochastic perturbation, with J. CHARRIER and T. GALLOUËT, *Mathematics of Computation*, 2016.
- Time-splitting approximation of the Cauchy problem for stochastic conservation, *Mathematics and Computers in Simulation*, 2015.
- A degenerate parabolic-hyperbolic Cauchy problem with a stochastic force, with G. VALLET and P. WITTBOLD, *Journal of Hyperbolic Differential Equations*, 2015.
- The Dirichlet problem for a conservation law with a multiplicative stochastic perturbation, with G. VALLET and P. WITTBOLD, *Journal of Functional Analysis*, 2014.
- On a time-splitting method for a stochastic conservation law with Dirichlet boundary condition and numerical experiments, *Journal of Evolution Equations*, 2014.
- On the stochastic $p(t,x)$ Laplace equation, with G. VALLET, P. WITTBOLD and A. ZIMMERMANN, *Stochastic Partial Differential Equations: Analysis and Computations*, 2013.
- The Cauchy problem for a conservation law with a multiplicative stochastic perturbation, with G. VALLET and P. WITTBOLD, *Journal of Hyperbolic Differential Equations*, 2012.
- On a class of quasilinear Barenblatt equations, with J. GIACOMONI and G. VALLET, *Real Academia des Ciencias de Zaragoza*, 2012.
- On abstract Barenblatt equations, with G. VALLET, Special Issue in Honor of Professor Jesus Ildefonso Diaz's 60th birthday, *Differential Equations & Applications*, 2011.

► Proceedings (3)

- Finite Volume Approximations for Non-Linear Parabolic Problems with Stochastic Forcing, with F. NABET, K. SCHMITZ and A. ZIMMERMANN, *Proceedings of the 10th International Symposium of Finite Volumes for Complex Applications (FVCA X)*, to appear.
- Convergence of a finite-volume scheme for a heat equation with a multiplicative stochastic force, with F. NABET, *Proceedings of the 9th International Symposium of Finite Volumes for Complex Applications (FVCA IX)*, 2020.
- Method for solving a stochastic conservation law, *Proceedings of the Twelfth International Conference Zaragoza-Pau on Mathematics*, Jaca, 2013.

Supervisions of PhD students

2023 **Cédric Sultan**. Aix-Marseille Université and LMA.

Title: *Propagation of P-SV seismic waves through horizontally layered random medium*.

Co-supervisor (LMA): R. Cottreau (50%).

Funding: Bourse de l'École doctorale 353 (Sciences pour l'Ingénieur : Mécanique, Physique, Micro et Nanoélectronique).

2022 Hosting **Kerstin Schmitz** at the LMA, a 3rd-year PhD thesis student at the University of Duisburg-Essen (Germany) under the supervision of P. Wittbold and A. Zimmermann.

Title of the project: *Stochastic models in solid mechanics : theory and numerical approximation*.

Funding: Procope Mobility Program (Campus France).

Duration: 1 month.

- 2017 **Asghar Ali Maitlo**. Aix-Marseille Université and LMA, thesis defended in 2020.
 Title: *Modelisation of imperfect linear and non linear interfaces: consideration of random effects*.
 Co-supervisor (LMA): F. Lebon (50%).
 Funding : Higher Education Commission from Pakistan Ministry and Ministère Français des Affaires Étrangères et Européennes.
 Publications: 3.
 Current position: Assistant Professor at QUAID-E-AWAM UNIVERSITY, Pakistan.
- 2016 **Vincent Castel**. Aix-Marseille Université and I2M.
 Title: *Finite-volume approximations of stochastic PDE*.
 Co-supervisors (I2M): J. Charrier (33%) and T. Gallouët (34%).
 Funding: Presidential fellowship from Aix-Marseille Université.
 Publication: 1.
PhD thesis abandoned by the student in September 2018 in favor of a secondary school teaching position.

Partnerships

► Co-authors*

- **Elena Bonetti**, Associate Professor at Università degli Studi di Milano and Dipartimento di matematica “F. Enriques”, in Italy. She is a specialist in the analysis of systems of PDEs (existence and uniqueness of solutions, continuous dependence results with respect to data, regularity of solutions, long-time behavior...) used to model phase change/separation phenomena, damage to elastic and viscoelastic materials, plasticity problems...
- **Giovanna Bonfanti**, Associate Professor at Università degli Studi di Brescia and Dipartimento di Ingegneria Civile, Architettura, Territorio, Ambiente e di Matematica, in Italy. She is a specialist in the analysis of PDEs systems (existence, uniqueness, regularity and asymptotic analysis of solutions) that describe phase transition phenomena, damage problems and also contact problems with adhesion and friction.
- **Julia Charrier**, Assistant Professor at Aix-Marseille Université and Institut de Mathématiques de Marseille (I2M, UMR CNRS 7373). She is a specialist in deterministic and probabilist numerical methods for PDEs/SDEs with random coefficients. She is also interested in numerical methods for stochastic scalar conservation laws.
- **Thierry Gallouët**, Professor at Aix-Marseille Université and Institut de Mathématiques de Marseille (I2M, UMR CNRS 7373). Among other things, he is a specialist in conservation laws, both theoretically and numerically using finite-volume methods. He is now also interested in the numerical analysis of stochastic partial differential equations.
- **Frédéric Lebon**, Professor at Aix-Marseille Université and Laboratoire de Mécanique et d’Acoustique (LMA, UMR CNRS 7031). He is a specialist in modeling issues in solid and structural mechanics, in particular contact and interface problems, but more broadly in non-regular mechanics.
- **Flore Nabet**, Assistant Professor at École polytechnique and Centre de Mathématiques Appliquées (CMAP, UMR CNRS 7641). She is a specialist in the study of numerical schemes for deterministic PDEs of elliptic or parabolic type, both from a theoretical (existence of solutions to the discrete problem, convergence of approximate solutions, error estimates, etc.) and practical (implementations, numerical tests, etc.) point of view.
- **Kerstin Schmitz**, PhD student (2020-2024) under the supervision of P. Wittbold and A. Zimmermann at Universität Duisburg-Essen, in Germany. She is working on stochastic and deterministic PDEs involving the p-Laplacian operator on the one hand, and on the finite-volume approximation of stochastic parabolic PDEs on the other.
- **Guy Vallet**, Professor at Université de Pau et des Pays de l’Adour and Laboratoire de Mathématiques et de leurs Applications de Pau (LMAP, UMR CNRS 5142). He is a specialist in nonlinear evolution problems of parabolic type (degenerate changing type such as the porous media equation, monotone singular/degenerate such as the p-Laplacian, pseudo-monotone or non-local such as the fractional Laplacian) and of first-order hyperbolic type. He works mainly on the mathematical analysis of these problems in both deterministic and stochastic modes.

*I have listed only the co-authors of work carried out after my PhD thesis. The PhD students I supervised are not listed here.

- **Aleksandra Zimmermann**, Professor at TU Clausthal and Institut für Mathematik, in Germany. She is a specialist in the analysis of nonlinear PDEs, more particularly on questions of existence and uniqueness of solution of pseudo-monotone problems of p-Laplacian type with stochastic perturbation, and of nonlinear elliptic and parabolic equations in Orlicz spaces with data in L^1 .

► Research groups (7)

- 2023 **Co-leader** of a project PHC Procope Plus (Campus France).
 Title: *Stochastic models in mechanics: theoretical and numerical aspects*.
 German co-leader: A. Zimmermann.
 Duration: 1 year.
- 2023 **Member** of a project PHC Procope (Campus France).
 Title: *Finite volume Approximation of Stochastic Scalar Conservation Laws (FIASSCOL)*.
 Leaders: F. Nabet and A. Zimmermann.
 Duration: 2 years.
- 2020 **Co-leader** of a project PHC Procope (Campus France).
 Title: *EDP Stochastiques : Modélisation Et Approximation (ESMEA)*.
 German co-leader: A. Zimmermann.
 Duration: 2 years.
- 2019 **Member** of an ANR project.
 Title: *SIMulation ALéatoire en dimension INfinie (SIMALIN)*.
 Leader: L. Goudenège.
 Duration: 4 years.
- 2018 **Member** of a research group (GDR) ECO-Math.
 Title: *Modèles mathématiques et méthodes numériques en mécanique des solides*.
 Duration: 2 years.
- 2015 **Member** of a project Institut Carnot ISIFoR
 Title: *Stochastic Models And Scale CHanging (SMASCH)*.
 Leader: G. Vallet.
 Duration: 2 years.
- 2014 **Member** of a project PHC Galilée (Campus France)
 Title: *Modélisation de la détérioration des pierres monumentales dans un but de conservation et de promotion du patrimoine historique*.
 Leaders: E. Bonetti and F. Lebon.
 Duration: 2 years.

► Hosting at the LMA (11)

- 08/23 **G. Bonfanti**, Università degli Studi di Brescia (Italy), 1 week.
Funding: Procope Plus project.
- 08/23 **N. Sapountzoglou, K. Schmitz** and **A. Zimmermann**, TU Clausthal (Germany), 1 week.
Funding: Procope Plus project.
- 10/22 **F. Nabet**, CMAP, École polytechnique, 2 days.
- 10/22 **A. Zimmermann**, Universität Duisburg-Essen (Germany), 2 weeks.
 Title: *Modèles stochastiques en mécanique du solide*.
Funding : AAP IMI* Collaborations internationales.
- 04/22 **F. Nabet**, CMAP, École polytechnique, 3 days.
- 03/22 **K. Schmitz** and **A. Zimmermann**, Universität Duisburg-Essen (Germany), 1 week.

*Call for projects from the local mechanics institute.

- 12/21 **G. Bonfanti**, Università degli Studi di Brescia (Italy), 1 week.
- 11/21 **K. Schmitz** and **A. Zimmermann**, Universität Duisburg-Essen (Germany), 4 days.
Funding: PHC Procope ESMEA.
- 11/20 **G. Bonfanti**, Università degli Studi di Brescia (Italy), 1 month.
Title: *Problèmes d'endommagement couplé de volume et de surface d'un matériau : modélisation, analyse mathématique et applications*.
Funding: Campagne FIR Aix-Marseille Université (hosting foreign scientists).
- 03/20 **A. Zimmermann**, Universität Duisburg-Essen (Germany), 5 days.
Funding: PHC Procope ESMEA.
- 11/19 **F. Nabet**, CMAP, École polytechnique, 3 days.

► Research stays (7)

- 2020 Fédération de Mathématiques Centrale Supélec, Gif-sur-Yvette, invited by L. Goudenège. Duration: 3 days.
Funding: ANR Simalin.
- 2019 Tata Institute of Fundamental Research Centre For Applicable Mathematics (TIFR CAM), Bangalore, India, invited by I. Biswas. Duration: 10 days.
Funding: TIFR CAM.
- 2018 Institute of Mathematics of the Romanian Academy, Bucarest, Romania, invited by L. Badea. Duration: 5 days.
Funding: GDR ECO-Math.
- 2017 Dipartimento di Ingegneria Civile, Architettura, Territorio, Ambiente e di Matematica, Università degli Studi di Brescia, Italy, invited by G. Bonfanti. Duration: 5 days.
Funding: INdAM (Istituto Nazionale di Alta Matematica).
- 2016 Institut Élie Cartan, Université de Lorraine, invited by M. Dozzi. Duration: 2 days.
- 2015 Istituzioni di Matematica, Università di Milano, Italy, invited by E. Bonetti. Duration: 5 days.
Funding: PHC Galilée.
- 2015 Istituzioni di Matematica, Università di Milano, Italy, invited by E. Bonetti. Duration: 3 days.
Funding: PHC Galilée.

Communications

- Oral communications in national conferences: 6.
- Oral communications in international conferences: 10.
- Seminars and working groups in France: 19.
- Seminars and working groups abroad: 4.
- Posters in national and international conferences: 6.

Scientific events

- 2023 **Co-organizer** of the workshop *Stochastic models in mechanics: theoretical and numerical aspects*, Marseille, with F. Lebon and A. Zimmermann. (<https://wksp-stochmod-lma.sciencesconf.org>).
Funding: French embassy in Berlin (Procope Plus project), Federal Ministry of Education and Research, TU Clausthal, LMA and ANR project 19-CE40-0016 (SIMALIN).
- 2021 **Co-organizer** of the workshop *Numerical Analysis of Stochastic Partial Differential Equations*, CIRM, Marseille, with C-E Bréhier, J. Charrier, L. Goudenège and A. Richard (<https://conferences.cirm-math.fr/2408.html>).
Funding: ANR SIMALIN.

- 2019 **Co-organizer** of the day *Analyse Appliquée & LMA*, with B. Lombard, LMA, Marseille (<https://journee-aa-lma.sciencesconf.org>).
Funding: LMA.
- 2015 **Co-organizer** of the thematic days *Journées thématiques sur les lois de conservation stochastiques : théorie, analyse numérique et applications*, with J. Charrier and T. Gallouët, FRUMAM, Marseille (<https://journées-lcs.sciencesconf.org>).
Funding: GDR MoMaS.
- 2015 **Co-organizer** of the Mini-Symposium *Numerical methods for stochastic partial differential equations*, Congress MAMERN VI, Pau, with J. Charrier.
- 2012 **Co-organizer** of the PhD students seminar, Université de Pau et des Pays de l'Adour.

Participation in juries, committees and councils

- 2023 **Member of the PhD thesis jury** of A. Messaoudi, Aix-Marseille Université.
Title: *Radiative transfer models for acoustic waves in a randomly fluctuating medium with boundaries: boundary effects and weak localization*.
Supervisors: R. Cotteneau (LMA) et C. Gomez (I2M).
- 2023 **Member of the selection committee** for an ATER position in Mechanics (Section CNU 60).
École Centrale Marseille, assignment to the LMA.
- 2023 **Member of the selection committee** for an associate professor position in Mechanics (Section CNU 60).
Aix-Marseille Université (IUT), assignment to the LMA or to the IUSTI.
- 2020 **Elected member** of the national council of universities (CNU Section 26). Participation in 5 councils since 2020.
- 2020 **Member of the PhD thesis jury** of Y. Tahraoui, Université de Pau et des Pays de l'Adour.
Title: *Problèmes paraboliques à contraintes, déterministes et stochastiques*.
Supervisors: A. Mokrane and G. Vallet.
- 2019 **Member of the PhD thesis jury** of S. Martel, École des Ponts ParisTech.
Title: *Theoretical and numerical analysis of invariant measures of viscous stochastic scalar conservation laws*.
Supervisors: S. Boyaval and J. Reygner.
- 2018 **Elected member** of the LMA Laboratory Council, from March 2018 to December 2023.
- 2016 **Member of the selection committee** for an associate professor position in Mathematics (Section CNU 26).
Aix-Marseille Université, assignment to the I2M.

PART 1

Finite-volume schemes for first-order scalar conservation laws with a multiplicative stochastic force

Chapter I

Convergence of finite-volume schemes for hyperbolic scalar conservation laws with noise

In this chapter are presented my first contributions ([BCG16a] and [BCG16b]) in collaboration with J. CHARRIER and T. GALLOUËT on the numerical approximation of a class of multi-dimensional nonlinear hyperbolic scalar conservation laws with a multiplicative stochastic perturbation. The most general result we obtained concerned the strong convergence of any finite-volume approximate solution constructed by a general monotone flux scheme towards the unique stochastic entropy solution of the considered hyperbolic problem.

I.1 Introduction

Written in a probabilists' way, the stochastic hyperbolic problems we were interested in took the following form:

$$\begin{cases} du + \operatorname{div}_x [\vec{v}f(u)] dt &= g(u) dW & \text{in } \Omega \times \mathbb{R}^d \times (0, T), \\ u(\omega, x, 0) &= u_0(x), & \omega \in \Omega, x \in \mathbb{R}^d, \end{cases} \quad (\text{I.1})$$

where d was a positive integer, $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration* $(\mathcal{F}_t)_{t \geq 0}$, div_x denoted the divergence operator with respect to the space variable (belonging to $\mathbb{R}^d$) and $(W(t))_{t \geq 0}$ was a standard, one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. As mentioned by [Kim06], this problem had to be understood, with differentiation in the sense of distributions, in the following way:

$$\begin{aligned} & \int_{\mathbb{R}^d} u_0(x) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \left(u(\omega, x, t) - \int_0^t g(u(\omega, x, s)) dW(s) \right) \partial_t \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \vec{v}(x, t) f(u(\omega, x, t)) \cdot \nabla_x \varphi(x, t) dx dt = 0, \end{aligned}$$

for almost all ω in Ω and for all test function φ in $\mathcal{D}(\mathbb{R}^d \times [0, T])$. Let me recall that, even in the deterministic case (*i.e.* when $g = 0$), a weak solution to a nonlinear scalar conservation law is not unique in general and so the mathematical stake consists in introducing a selection criterion in order to identify the physical relevant solution. For this reason, we considered in the studies ([BCG16a] and [BCG16b]) presented in this chapter, a stochastic version of the entropy condition proposed by S.N. KRZHKOV in the 70s, the one used in my PhD thesis in collaboration with G. VALLET and P. WITTBOLD [BVW12], and presented in Definition I.3.

By denoting, for any $x \in \mathbb{R}^d$, $|x|$ the euclidian norm of x , we assumed the following hypotheses on the data:

**i.e.* an increasing family (indexed by the time variable $t \in [0, T]$) of σ -algebras.

$\mathcal{H}_1$: $u_0 \in L^2(\mathbb{R}^d)$.

$\mathcal{H}_2$: $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with $f(0) = 0$.

$\mathcal{H}_3$: $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with $g(0) = 0$.

$\mathcal{H}_4$: $\vec{v} \in \mathcal{C}^1(\mathbb{R}^d \times [0, T]; \mathbb{R}^d)$ and $\operatorname{div}_x(\vec{v}(x, t)) = 0, \forall (x, t) \in \mathbb{R}^d \times [0, T]$.

$\mathcal{H}_5$: There exists $V < \infty$ such that $|\vec{v}(x, t)| \leq V, \forall (x, t) \in \mathbb{R}^d \times [0, T]$.

$\mathcal{H}_6$: g is a bounded function.

Remark I.1

- Note that $\mathcal{H}_1$ to $\mathcal{H}_3$ were sufficient to prove the well-posedness of Problem (I.1) (see [BVW12] in the particular case where $\vec{v}$ was a constant vector). Moreover, using the divergence-free property of $\vec{v}$ (stated in $\mathcal{H}_4$), one could assume by convenience that $f(0) = 0$ without loss of generality.
- $g(0) = 0$ was a technical condition coming from [BVW12] and was also used to show stability estimates on the finite-volume approximate solution.
- $\mathcal{H}_6$ was more a technical assumption, it was particularly used to prove the convergence of the numerical scheme.

I.1.1 State of the art

The study of hyperbolic problems with a stochastic force such as (I.1) is a recent subject that has attracted the attention of several research groups, particularly in the last twenty years. Let us go back ten years ago to draw up a state of the art on the subject when I started to look at these issues of numerical approximation. Note that the following studies have all been done in the case of a constant vector $\vec{v}$, i.e. without any spatial or temporal dependence of the flux term. From a theoretical point of view, only few papers have been devoted to the study of the well-posedness of Problem (I.1):

Let me mention the work [FN08], where the authors introduced a notion of strong entropy solution in order to prove the uniqueness of the entropy solution. Using vanishing viscosity and compensated compactness arguments, they established the existence of strong entropy solutions. Note that the use of such a compactness tool reduced their study to the one-dimensional case and to genuinely nonlinear flux functions. The authors concluded to the uniqueness of stochastic entropy solution by comparing a strong entropy solution with a stochastic entropy one.

In the paper [CDK12], the authors proposed a generalization of [FN08] to the multi-dimensional case. They identified a class of nonlinear stochastic balance laws for which uniform spatial BV bound for vanishing viscosity approximations could be achieved. Moreover they established temporal equicontinuity in L^1 of the approximations, uniformly in the viscosity coefficient. They particularly proved that this stochastic problem was well-posed by using a uniform spatial BV-bound.

Using a kinetic formulation, in [DV10] the authors proved the first complete well-posedness result for scalar conservation laws set in a d -dimensional torus (with $d \geq 1$) and driven by a general multiplicative noise. As an extension of this work, [Hof14] presented a Bhatnagar-Gross-Krook-like approximation of this problem. Using the stochastic characteristics method, the author established the existence of an approximate solution and showed its convergence to the kinetic solution of [DV10].

In my PhD thesis, under Assumptions $\mathcal{H}_1$ to $\mathcal{H}_3$ and by the way of Young measure-valued solutions, we proved with my co-authors in [BVW12] a result of existence and uniqueness of the solution to the multi-dimensional Cauchy problem in $L^2(\Omega \times (0, T) \times \mathbb{R}^d)$. Since the method consisted in comparing a weak measure-valued entropy solution to a regular one (the viscous solution in this case) and not to a strong one, we were able to consider very general assumptions on the data.

In [BVW14] we investigated the Dirichlet Problem for (I.1) set in a bounded domain D of $\mathbb{R}^d$ with homogeneous boundary conditions. We proved a result of existence and uniqueness of the stochastic entropy solution by using the concept of measure-valued solutions and Kruzhkov semi-entropy formulations.

Concerning the study of numerical experiments for scalar conservation laws with multiplicative noise, there were also, to my knowledge, few papers and none of them proposed a convergence study for a space and time discretization of the problem. Let me cite the work of [HR91] where a time-discretization of the equation were studied by the use of an operator-splitting method. They presented a result of pathwise convergence to prove the existence of pathwise weak solutions to the Cauchy problem for (I.1) in the one-dimensional case.

In my PhD thesis, by taking advantage of the theoretical framework developed with my co-authors in [BVW14], I proposed a generalization of the work [HR91] in a bounded domain D of $\mathbb{R}^d$ [Bau14]. I proved that the pathwise weak solution obtained in [HR91] was the unique entropy weak solution of the stochastic conservation law and that the whole sequence of approximation given by the time-splitting scheme was converging in $L^p(\Omega \times (0, T) \times D)$ for any finite p . Then, as an extension in [Bau15], I proposed the same type of discretization for the problem set in $\mathbb{R}^d$, by using the study of the continuous case [BVW12]. Note that as the previous authors, the convergence study involved only a time-discretization of the equation.

Let me mention the paper [Krö08] where the author studied well-posedness of a scalar conservation law perturbed by an additive random noise term. In a first part, he proposed a full time-space finite-volume method in one and two spatial dimensions but without any convergence study. In a second part, numerical experiments were realized on a few model problems. Note that the stochastic (Itô) part of the equation was approximated by the Euler-Maruyama method.

Then, in [KR12] the authors were interested in a method of handling the finite-volume schemes for the approximate solution of the Cauchy problem for a hyperbolic balance law with random noise and investigated on a space-discretization of the equation. For a class of strongly monotone numerical fluxes they established the pathwise convergence of a semi-discrete finite-volume solution towards a stochastic entropy solution. The main tool was a stochastic version of the compensated compactness approach. It allowed them to avoid the use of a maximum principle and total-variation estimates but restricted the study to the one-dimensional case and to the use of genuinely nonlinear flux functions.

Naturally, the first research direction I took after my thesis during my post-doctoral with J. CHARRIER and T. GALLOUËT was to introduce a convergence result for a full space-time discretization of Problem (I.1) in order to fill the gap left by the previous authors. The starting point for this first study was the use of the theoretical framework developed in my PhD [BVW12] under Assumptions $\mathcal{H}_1$ to $\mathcal{H}_5$ and with the vector $\vec{v}$ constant. Let me recall it after introducing some notations and make precise the functional setting.

I.1.2 Notations and theoretical framework

Let us assume as in [BVW12], that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_5$ hold and that the vector $\vec{v}$ is constant.

Notations I.2

- For any $R > 0$, $\mathcal{B}(0, R)$ denotes the ball of $\mathbb{R}^d$ centered at 0 of radius R .
- $\mathbb{E}[\cdot]$ denotes the expectation, i.e. the integral over Ω with respect to the probability measure $\mathbb{P}$.
- $\mathcal{D}^+(\mathbb{R}^d \times [0, T])$ denotes the subset of nonnegative elements of $\mathcal{D}(\mathbb{R}^d \times [0, T])$.
- $\mathcal{P}_T$ denotes the predictable σ -field on $(0, T) \times \Omega$, i.e. the σ -field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $(s, t] \times B$ for any $B \in \mathcal{F}_s$, $s, t \in [0, T]$, $s < t$.
- For a given separable Hilbert space X , we denote by $\mathcal{N}_W^2(0, T; X)$ the separable Hilbert space of the predictable X -valued processes. It is the space $L^2((0, T) \times \Omega; X)$ for the product measure $dt \otimes d\mathbb{P}$ on $\mathcal{P}_T$ (see [DPZ92] p.94).
- $\mathcal{A}$ denotes the set of any $\mathcal{C}^3(\mathbb{R})$ convex functions η (called smooth entropies in what follows) such that η' , η'' and η''' are bounded functions.
- $\Phi^\eta : \mathbb{R} \rightarrow \mathbb{R}$ denotes the entropy flux associated with a smooth entropy $\eta \in \mathcal{A}$. It is defined for any $a \in \mathbb{R}$ by

$$\Phi^\eta(a) = \int_0^a \eta'(\sigma) f'(\sigma) d\sigma.$$

- For any $\kappa \in \mathbb{R}$, $|\cdot - \kappa| : \mathbb{R} \rightarrow \mathbb{R}$ denotes a Kruzhkov's entropy.
- For any $\kappa \in \mathbb{R}$, $\Phi(\cdot, \kappa) : \mathbb{R} \rightarrow \mathbb{R}$ denotes the entropy flux associated with Kruzhkov's entropy $|\cdot - \kappa|$. It is defined for any $a \in \mathbb{R}$ by

$$\Phi(a, \kappa) = f(a \top \kappa) - f(a \perp \kappa) = \text{sign}(a - \kappa)(f(a) - f(\kappa)),$$
 where $a \top \kappa$ denotes the maximum of a and κ and $a \perp \kappa$ the minimum of a and κ .
- L_f and L_g denote the Lipschitz constants of f and g .

In [BVW12], we proved that Problem (I.1) admitted a unique stochastic entropy solution in the following sense:

Definition I.3 (Stochastic entropy solution)

A function u of $\mathcal{N}_W^2(0, T; L^2(\mathbb{R}^d)) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))$ is an entropy solution of Problem (I.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$ if it satisfies the following entropy inequalities

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \eta(u(\omega, x, t)) \partial_t \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \Phi^\eta(u(\omega, x, t)) \vec{v} \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \eta'(u(\omega, x, t)) g(u(\omega, x, t)) \varphi(x, t) dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} g^2(u(\omega, x, t)) \eta''(u(\omega, x, t)) \varphi(x, t) dx dt, \end{aligned}$$

for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and for almost all $\omega \in \Omega$.

Remark I.4

Let me emphasize that the entropy inequalities presented in Definition I.3 limited ourselves to the use of smooth entropies $\eta \in \mathcal{A}$. Indeed, we were not able to get, as in the deterministic case, a formulation with Kruzhkov's entropies due to the second order term $\int_0^T \int_{\mathbb{R}^d} g^2(u(\omega, x, t)) \eta''(u(\omega, x, t)) \varphi(x, t) dx dt$ involving the second order derivative of the entropy $\eta \in \mathcal{A}$.

Adapting well-known methods from the deterministic setting, we obtained in a first step, via a vanishing viscosity approximation, existence of measure-valued entropy solutions in the following sense:

Definition I.5 (Measure-valued entropy solution)

A function μ of $\mathcal{N}_W^2(0, T; L^2(\mathbb{R}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d \times (0, 1)))$ is a measure-valued entropy solution of Problem (I.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$ if it satisfies the following inequalities

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta(\mu(\omega, x, t, \beta)) \partial_t \varphi(x, t) d\beta dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \Phi^\eta(\mu(\omega, x, t, \beta)) \vec{v} \cdot \nabla_x \varphi(x, t) d\beta dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mu(\omega, x, t, \beta)) g(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \int_0^1 g^2(\mu(\omega, x, t, \beta)) \eta''(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dt, \end{aligned}$$

for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and for almost all $\omega \in \Omega$.

An adaptation of Kruzhkov's doubling variables technique allowed us in a second step to prove a contraction principle for such measure-valued entropy solutions (the so called "interior Kato's inequality"). As a by-product we deduced existence and uniqueness of the entropy solution of Problem (I.1), as stated in the following theorem from [BVW12]:

Theorem I.6

Under Assumptions $\mathcal{H}_1$ to $\mathcal{H}_5$ in the particular case where the vector $\vec{v}$ is constant, there exists a unique measure-valued entropy solution for Problem (I.1) and this solution is obtained by viscous approximation. Moreover, it is the unique stochastic entropy solution in the sense of Definition I.3.

Remark I.7

Let us mention that the unique stochastic entropy solution of Problem (I.1) given by Theorem I.6 satisfied the initial condition in the following sense: for any compact set $\mathcal{K} \subset \mathbb{R}^d$ (see [BVW12])

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \mathbb{E} \left[\int_{\mathcal{K}} |u(\omega, x, t) - u_0(x)| dx \right] = 0.$$

Remark I.8 (Application to Burgers' equation)

Following [Val08], if it was assumed in addition that $0 \leq u_0(x) \leq 1$ for almost all x in $\mathbb{R}^d$ and that g was compactly supported in $[0, 1]$, it was possible to show that the unique stochastic entropy solution u of Problem (I.1) satisfied $0 \leq u(\omega, x, t) \leq 1$, for almost all (ω, x, t) in $\Omega \times \mathbb{R}^d \times (0, T)$. Indeed, thanks to Itô's formula, this maximum principle was direct for the viscous solution u_ϵ (defined as the unique weak solution of Problem (I.17) introduced further), then it was conserved at the limit for u . This property allowed us to treat the cases where f and g were only locally Lipschitz-continuous. In particular, we could affirmed that all our results (stated in this chapter) remained true for the stochastic Burgers equation (i.e. when $f(u) = u^2$).

Relying on this theoretical framework, our first objective with J. CHARRIER and T. GALLOUËT was to construct a finite-volume scheme and prove its convergence towards a measure-valued entropy solution in the sense of Definition I.5 and invoke Theorem I.6 to conclude. Indeed, once the convergence of the scheme towards a measure-valued entropy solution was proved, the uniqueness result given by Theorem I.6 allowed us to obtain directly the information that the convergence finally held to the unique stochastic entropy solution in the sense of Definition I.3

I.2 Finite-volume framework

I state here the general framework associated with the finite-volume discretization of Problem (I.1) employed in the first part of this manuscript. In order to perform it, we used on the one hand, a uniform discretization of the time interval $[0, T]$, by taking $N \in \mathbb{N}^*$ and defining the time step $k = \frac{T}{N} \in \mathbb{R}_+^*$; and on the other hand by considering an admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ with a space step $h \in \mathbb{R}_+^*$, in the following sense (with associated notations below):

Definition I.9 (Admissible mesh of $\mathbb{R}^d$)

An admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ for the discretization of Problem (I.1) is given by a family of disjoint polygonal connected subsets of $\mathbb{R}^d$ such that $\mathbb{R}^d$ is the union of the closure of the elements of $\mathcal{T}$ (which are called control volumes in the following) and such that the common interface of any two control volumes is included in a hyperplane of $\mathbb{R}^d$. It is assumed that $h = \operatorname{size}(\mathcal{T}) = \sup\{\operatorname{diam}(K), K \in \mathcal{T}\} < \infty$ and that, for some $\bar{\alpha} \in \mathbb{R}_+^*$, we have for any control volume $K \in \mathcal{T}$

$$\bar{\alpha} h^d \leq |K|, \quad \text{and} \quad |\partial K| \leq \frac{1}{\bar{\alpha}} h^{d-1}, \quad \forall K \in \mathcal{T}, \quad (\text{I.2})$$

where $|K|$ (respectively $|\partial K|$) denotes the d -dimensional Lebesgue measure of K (respectively the $(d-1)$ -dimensional Lebesgue measure of ∂K).

Notations I.10

- . $\mathcal{E}$ the set of all the interfaces of the mesh $\mathcal{T}$.
- . $|\sigma|$ the $(d-1)$ -dimensional Lebesgue measure of the interface σ , for any $\sigma \in \mathcal{E}$.
- . $\mathcal{E}_K$ the set of interfaces of the control volume K .
- . $\mathcal{N}(K)$ the set of control volumes neighbors of the control volume K .
- . $\sigma_{K,L}$ the common interface between K and L , for any $L \in \mathcal{N}(K)$.
- . $n_{K,\sigma}$ the unit normal to interface σ , outward to the control volume K , for any $\sigma \in \mathcal{E}_K$ and, by abusing this notation, $n_{K,L}$ will denote the unit normal to $\sigma_{K,L}$, oriented from K to L , for any $L \in \mathcal{N}(K)$.

Using the following discretization of the initial condition:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \forall K \in \mathcal{T}, \quad (\text{I.3})$$

the ideology of our finite-volume method for Problem (I.1) was to approximate its stochastic entropy solution u by a function $u_{\mathcal{T},k} : \Omega \times \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ (called the finite-volume approximation) which was constant on each space-time grid $K \times [nk, (n+1)k]$, $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$:

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n(\omega) \text{ for } \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k]. \quad (\text{I.4})$$

For any $n \in \{0, \dots, N-1\}$, knowing $\{u_K^n(\omega), K \in \mathcal{T}\}$, the discrete unknowns $\{u_K^{n+1}(\omega), K \in \mathcal{T}\}$ were computed thanks to the following equation:

$$\frac{|K|}{k} (u_K^{n+1}(\omega) - u_K^n(\omega)) + \sum_{L \in \mathcal{N}(K)} Q_{K \rightarrow L}^n(\omega) = |K| g(u_K^n(\omega)) \frac{W^{n+1}(\omega) - W^n(\omega)}{k}, \quad (\text{I.5})$$

where for any $n \in \{0, \dots, N-1\}$ and any $K \in \mathcal{T}$, $Q_{K \rightarrow L}^n(\omega)$ denoted the exchange of flux from K towards L over the time interval $[nk, (n+1)k]$ and $W^n(\omega) = W(nk, \omega)$, $\forall n \in \{0, \dots, N\}$. The principle of conservation (mass, energy, depending on the meaning of u) required the relationship

$$Q_{K \rightarrow L}^n(\omega) = -Q_{L \rightarrow K}^n(\omega), \forall K, L \in \mathcal{T}, \forall n \in \{0, \dots, N-1\}, \forall \omega \in \Omega. \quad (\text{I.6})$$

Developing a finite-volume method for Problem (I.1), meant determining a choice of numerical fluxes $Q_{K \rightarrow L}^n$ satisfying at least the conservation condition (I.6).

Remark I.11 (On the measurability of the finite-volume approximate solution)

Using properties of the Brownian motion, for all K in $\mathcal{T}$ and all n in $\{0, \dots, N-1\}$, u_K^n was naturally $\mathcal{F}_{nk}$ -measurable and so, as an elementary process adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, $u_{\mathcal{T},k}$ was predictable with values in $L^2(\mathbb{R}^d)$.

I.3 Numerical schemes studied

Chronologically, here are the different numerical fluxes and their associated schemes we studied with my co-authors under Assumptions $\mathcal{H}_1$ to $\mathcal{H}_6$ in [BCG16a] and [BCG16b]. For each of them, we proved that the generated sequence of finite-volume approximations $(u_{\mathcal{T},k})_{\mathcal{T},k}$ defined by (I.3)-(I.4)-(I.5) converged to the unique stochastic entropy solution of Problem (I.1) in the sense of Definition I.3. The proof of this convergence result (stated in Theorem I.16) will be commented in the next section.

I.3.1 Upwind scheme

In the particular case where the function f is non-decreasing and the vector $\vec{v}$ is constant, the numerical fluxes $Q_{K \rightarrow L}^n(\omega)$ for the upwind scheme are defined as follows

$$Q_{K \rightarrow L}^n(\omega) = |\sigma_{K,L}| (\vec{v} \cdot n_{K,L}) f(u_{\sigma_{K,L}}^n(\omega))$$

where $u_{\sigma_{K,L}}^n(\omega)$ denotes the upstream value at time nk with respect to $\sigma_{K,L}$, equal to $u_K^n(\omega)$ if $\vec{v} \cdot n_{K,L} \geq 0$ and to $u_L^n(\omega)$ if $\vec{v} \cdot n_{K,L} < 0$.

I.3.2 Flux-splitting schemes

By noting that every Lipschitz-continuous function can be decomposed as the sum of a non-decreasing function f_1 and a non-increasing one f_2 , the construction of a flux-splitting scheme is done as follows: associate to the non-decreasing part f_1 upwind numerical fluxes, and downwind numerical fluxes to the non-increasing part f_2 . This leads to the following numerical fluxes for the flux-splitting schemes (by assuming again that the vector $\vec{v}$ is constant):

$$Q_{K \rightarrow L}^n(\omega) = |\sigma_{K,L}| \left[(\vec{v} \cdot n_{K,L})^+ (f_1(u_K^n(\omega)) + f_2(u_L^n(\omega))) - (\vec{v} \cdot n_{K,L})^- (f_1(u_L^n(\omega)) + f_2(u_K^n(\omega))) \right],$$

where for any $a \in \mathbb{R}$, $a^- = -\min(a, 0)$ and $a^+ = \max(a, 0)$.

Here are some classical examples of flux-splitting schemes:

Examples I.12

- The most simple example corresponds to the case where the flux function f is monotone. Indeed, if f is non-decreasing, then $f_1 = f$ and $f_2 = 0$ and this leads to an upwind scheme. Otherwise, if f is non-increasing, then $-f$ fits into the previous case.
- The Engquist-Osher scheme concerns Lipschitz-continuous function f being additionally convex or concave. In this case either f is monotone and it comes down to the previous case, or f' vanishes in a unique interval of $\mathbb{R}$. In the second case $\mathbb{R}$ is the union of two intervals and f is monotone on each of them, which leads to a natural splitting.
- The modified Lax-Friedrichs scheme in the sense of [EGH00], whose generalization in the case of an hyperbolic system (also known as Rusanov scheme), corresponds to a decomposition of the flux in the following way: $f = f_1 + f_2$ where $\forall x \in \mathbb{R}$, $f_1(x) = \frac{f(x)}{2} + Dx$ and $f_2(x) = \frac{f(x)}{2} - Dx$, with $2D \geq L_f$.

I.3.3 Monotone flux schemes

To give the definition of monotone flux schemes, I first need to introduce the notion of monotone numerical flux:

Definition I.13 (Monotone numerical flux)

We say that a function $F \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ is a monotone numerical flux if it satisfies the following properties:

- **Monotonicity** : For any $b \in \mathbb{R}$, $a \mapsto F(a, b)$ is non-decreasing and for any $a \in \mathbb{R}$, $b \mapsto F(a, b)$ is non-increasing.
- **Lip-diaq regularity** : There exists $F_1, F_2 > 0$ such that for any $a, b \in \mathbb{R}$

$$|F(b, a) - F(a, a)| \leq F_1 |a - b| \text{ and } |F(a, b) - F(a, a)| \leq F_2 |a - b|.$$

- **Consistency** : For any $a \in \mathbb{R}$, $F(a, a) = f(a)$.

Once a monotone numerical flux F in the sense of Definition I.13 is considered, the associated monotone numerical scheme can be defined by considering

$$Q_{K \rightarrow L}^n(\omega) = |\sigma_{K,L}| \left\{ v_{K,L}^n F(u_K^n(\omega), u_L^n(\omega)) - v_{L,K}^n F(u_L^n(\omega), u_K^n(\omega)) \right\}$$

where, by denoting $d\gamma$ the $(d-1)$ -dimensional Lebesgue measure

$$v_{K,L}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\vec{v}(x, t) \cdot n_{K,L})^+ d\gamma(x) dt,$$

$$v_{L,K}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\vec{v}(x, t) \cdot n_{L,K})^+ d\gamma(x) dt = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\vec{v}(x, t) \cdot n_{K,L})^- d\gamma(x) dt,$$

with for any $a \in \mathbb{R}$, $a^- = -\min(a, 0)$ and $a^+ = \max(a, 0)$.

Let me give some examples of monotone flux schemes:

Examples I.14

- Any flux-splitting scheme : by considering the following decomposition of the flux $f = f_1 + f_2$ with f_1 non-decreasing and f_2 non-increasing, set for any $a, b \in \mathbb{R}$

$$F_{FS}(a, b) = f_1(a) + f_2(b).$$

- The modified Lax-Friedrichs scheme with a parameter $D \geq 2L_f$: for any $a, b \in \mathbb{R}$,

$$F_{LF}^D(a, b) = \frac{f(a) + f(b)}{2} + D(a - b). \quad (\text{I.7})$$

- The Godunov scheme defined for any $a, b \in \mathbb{R}$ as follows

$$F_G(a, b) = \begin{cases} \min_{s \in [a, b]} f(s) & \text{if } a \leq b \\ \max_{s \in [b, a]} f(s) & \text{if } a \geq b. \end{cases} \quad (\text{I.8})$$

Note that the convergence result we obtained for a class of monotone flux schemes in [BCG16b] was entirely based on the following strong property satisfied by any monotone numerical flux F , whose proof can be found in [CH00]:

Lemma I.15 (Decomposition of a monotone numerical flux)

Any monotone numerical flux F in the sense of Definition I.13 can be written as a convex combination of a Godunov flux and a modified Lax-Friedrichs flux. Precisely, for any $a, b \in \mathbb{R}$ there exists $\theta(a, b) \in [0, 1]$ such that

$$F(a, b) = \theta(a, b)F_G(a, b) + (1 - \theta(a, b))F_{LF}^D(a, b),$$

where F_G is a Godunov monotone flux defined by (I.8) and F_{LF}^D is a modified Lax-Friedrichs flux defined by (I.7) with parameter $D = \max(F_1, F_2)$, where F_1 and F_2 are the Lip-diac coefficients of F .

Lemma I.15 is very powerful in the sense that it allowed us to generalize to a class of monotone schemes some results that we only knew to show (*a priori*) for Godunov and modified Lax-Friedrichs schemes.

I.4 Main result and overview of the proof

In [BCG16a] and [BCG16b], we proved for all the numerical schemes presented in the previous section the following convergence result:

Theorem I.16 (Convergence of the scheme)

Assume that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_6$ hold. Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes in the sense of Definition I.9 such that the mesh size h_m tends to 0 when m tends to $+\infty$. Let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$ and set for any $m \in \mathbb{N}$, $k_m = \frac{T}{N_m}$ the time step. For a fixed $m \in \mathbb{N}$, let $u_{\mathcal{T}_m, k_m}$ be the finite-volume approximate solution defined by (I.3)-(I.4)-(I.5). Then $(u_{\mathcal{T}_m, k_m})_{m \in \mathbb{N}}$ converges to the unique stochastic entropy solution of Problem (I.1) in the sense of Definition I.3, strongly in $L^p(\Omega \times \mathcal{K} \times (0, T))$ for any $1 \leq p < 2$ and any compact set $\mathcal{K}$ of $\mathbb{R}^d$, and weakly in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$, as m tends to $+\infty$ and $\frac{k_m}{h_m}$ tends to 0.

I propose in this section to present the ideas developed in [BCG16a] with my co-authors for upwind scheme to obtain such a convergence result, in the particular case where the vector $\vec{v}$ was constant. I will comment throughout the text how we generalized them in [BCG16b] to a class of monotone numerical fluxes, with in addition a spatial and temporal dependence of the flux function $\vec{v}f$ supported by the vector $\vec{v}$.

Remark I.17

The main difficulty of these works was to choose suitable tools of the finite-volume framework compatible with the stochastic one and the restrictions brought by the noise. There were essentially three main constraints to keep in mind:

- Firstly, the use of Kruzhkov's entropies was out of range since the stochastic version of the entropy inequalities contained a new term involving the second order derivative of the entropy η (see Definition I.3). Although this new term was nonnegative, it was unfortunately not in the good side of the inequality and we were unable to remove it from the formulation. This point restricted the available technics of the deterministic finite-volume framework to the one involving smooth entropies.
- Secondly, due to the construction of the Itô integral, an explicit discretization of the noise term was our only option.
- Thirdly, since the increments of the Brownian motion are not $L^\infty(\Omega)$, even if $u_0 \in L^\infty(\mathbb{R}^d)$, a $L^\infty(\Omega \times \mathbb{R}^d \times (0, T))$ bound for the finite-volume approximate solution was not possible, see Remark I.20 below.

In the sequel, let us assume that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_6$ hold, that the function f is non-decreasing and that the vector $\vec{v}$

is constant. Recall that the approximate upwind finite-volume approximation $u_{\mathcal{T},k} : \Omega \times \mathbb{R}^d \times [0, T)$ was defined by

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n(\omega) \text{ for } \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k), \quad (\text{I.9})$$

where the discrete unknowns u_K^n (note that I will omit for convenience in the sequel the random dependence), were computed as follows: for any $K \in \mathcal{T}$, any $n \in \{0, \dots, N-1\}$

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad (\text{I.10})$$

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_K} |\sigma| \vec{v} \cdot n_{K,\sigma} f(u_\sigma^n) = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}, \quad (\text{I.11})$$

where $W^n := W(nk) \forall n \in \{0, \dots, N\}$ and u_σ^n denoted the upstream value at time nk with respect to σ . More precisely, if σ was the interface between the control volumes K and L , u_σ^n was equal to u_K^n if $\vec{v} \cdot n_{K,\sigma} \geq 0$ and to u_L^n if $\vec{v} \cdot n_{K,\sigma} < 0$. Note that since $\text{div}_x(\vec{v}) = 0$, relation (I.11) could be rewritten as follows:

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_K} |\sigma| \vec{v} \cdot n_{K,\sigma} (f(u_\sigma^n) - f(u_K^n)) = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}.$$

Moreover, playing with the sign of $\vec{v} \cdot n_{K,\sigma}$, a third equivalent formulation of (I.11) was given by

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{\substack{\sigma \in \mathcal{E}_K \\ \sigma = \sigma_{K,L}}} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- (f(u_K^n) - f(u_L^n)) = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k},$$

where $(\vec{v} \cdot n_{K,\sigma})^- = -\min(\vec{v} \cdot n_{K,\sigma}, 0)$.

Using the fact that the initial condition of our problem u_0 was in $L^2(\mathbb{R}^d)$, we firstly derived stability estimates satisfied by the finite-volume approximation $(u_{\mathcal{T},k})_{\mathcal{T},k}$ by multiplying (I.11) by u_K^n . The key ingredients for obtaining such estimates were the properties of the increments of the Brownian motion $W^{n+1} - W^n$ for any $n \in \{0, \dots, N-1\}$ and the monotony of the function f which allowed us to apply the following technical lemma (from [EGH00]):

Lemma I.18

Let $\mathcal{G} : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone Lipschitz-continuous function with a Lipschitz constant $C_{\mathcal{G}} > 0$. Then:

$$\left| \int_c^d \mathcal{G}(t) - \mathcal{G}(c) dt \right| \geq \frac{1}{2C_{\mathcal{G}}} (\mathcal{G}(d) - \mathcal{G}(c))^2, \forall c, d \in \mathbb{R}.$$

Note that in the general case, Lemma I.18 could also be applied by using the monotony of the numerical fluxes F instead of f .

Proposition I.19 ($L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))$ estimate for $(u_{\mathcal{T},k})_{\mathcal{T},k}$)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the Courant-Friedrichs-Levy (CFL) condition

$$k \leq \frac{\bar{\alpha}^2 h}{L_f V}. \quad (\text{I.12})$$

Let $u_{\mathcal{T},k}$ be the finite-volume approximate solution defined by (I.9), (I.10) and (I.11). Then we have the bound:

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{R}^d))} \leq e^{L_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)}.$$

Remark I.20 (Why a $L^\infty(\Omega \times \mathbb{R}^d \times (0, T))$ estimate for $(u_{\mathcal{T},k})_{\mathcal{T},k}$ was out of reach?)

As mentioned in Remark I.8, if one was concerned by the modeling of fluid flow in porous media and that the stochastic entropy solution u had to be a saturation, one could prove that $0 \leq u \leq 1$ as soon as $0 \leq u_0 \leq 1$ and $\text{supp } g \subset [0, 1]$.

Note that if we assumed that $u_0 \in L^\infty(\mathbb{R}^d)$ with $0 \leq u_0 \leq 1$, this bound did not hold for the approximate solution $u_{\mathcal{T},k}$, and this approximation was even unbounded in $L^\infty(\Omega \times \mathbb{R}^d \times (0, T))$. This was due to the fact that the increments of the Brownian motion were not bounded. To illustrate it, we gave in [BCG16a] the following example:

“Let $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$ be given by the finite-volume scheme (I.9), (I.10) and (I.11) with $d = 1$, $\vec{v} = 1$, $f(x) = x$, $g(x) = x(1-x)1_{[0,1]}(x)$, $0 < \epsilon < 1$ and assume that for all $K \in \mathcal{T}$, $u_K^0 = 1 - \epsilon$. Then

$$u_K^1 = 1 - \epsilon + g(1 - \epsilon)W^1.$$

Denote by $a = g(1 - \epsilon) > 0$. Since $W^1 \sim \mathcal{N}(0, k)$, $\mathbb{P}(W^1 > \frac{\epsilon}{a}) > 0$ and so $\mathbb{P}(u_K^1 > 1) > 0$, and we can even prove that u_K^1 does not belong to $L^\infty(\Omega)$.”

Following the study of deterministic case as performed in [EGH00], additional estimates satisfied by the set of discrete unknowns $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$ were required. Such estimates, called “weak BV estimates” by the authors, played a crucial role to justify the convergence of the scheme. More precisely, they were used to prove that the difference between the numerical entropy fluxes and the entropy fluxes Φ^η at $u_{\mathcal{T},k}$ converged towards 0 as the space and time discretization parameters tended to 0. The idea of the proof to get these “weak BV estimates” was in some ways quite similar to that of Proposition I.19 and was obtained under a more restrictive CFL condition, see (I.13) below.

Proposition I.21 (Weak BV estimates)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying, for some $\xi \in (0, 1)$, the CFL condition

$$k \leq \frac{(1 - \xi)\bar{\alpha}^2 h}{L_f V}. \quad (\text{I.13})$$

Let $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$ be given by (I.10) and (I.11). Then the following hold:

1. There exists $C_{BV,1} \in \mathbb{R}_+^*$, only depending on T, u_0, ξ, L_f and L_g such that

$$\sum_{n=0}^{N-1} k \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma = \sigma_{K,L}}} |\sigma| |\vec{v} \cdot n_{K,\sigma}| \mathbb{E} \left[(f(u_K^n) - f(u_L^n))^2 \right] \leq C_{BV,1}.$$

2. Let $R > 0$ be such that $h < R$. Then there exists $C_{BV,2} \in \mathbb{R}_+^*$, only depending on $R, d, T, \bar{\alpha}, u_0, \xi, L_f$ and L_g such that

$$\sum_{n=0}^{N-1} k \sum_{\substack{\sigma \in \mathcal{E}^R \\ \sigma = \sigma_{K,L}}} |\sigma| |\vec{v} \cdot n_{K,\sigma}| \mathbb{E} \left[|f(u_K^n) - f(u_L^n)| \right] \leq C_{BV,2} h^{-1/2},$$

where $\mathcal{E}^R$ denotes the set of interfaces of $\mathcal{T}_R = \{K \in \mathcal{T} \text{ such that } K \subset \mathcal{B}(0, R)\}$.

Remark I.22 (Extension of Weak BV estimates to any monotone numerical flux F)

Let me mention that the extension of these Weak BV estimates to a class of monotone numerical fluxes F in the sense of Definition I.13 was technical but made use of the same arguments. Indeed, the monotony of f which was one of the key point (since it allowed us to apply Lemma I.18), was translated on the monotone numerical fluxes. Furthermore, the time and space dependence of the flux function through the vector $\vec{v}$ required a certain technicality (as in the deterministic case) but did not pose a substantive problem. In such a general case, under the following reinforced CFL condition: (by denoting F_1 and F_2 the Lip-dia coefficients of the numerical flux F)

$$k \leq (1 - \xi) \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)}, \quad (\text{I.14})$$

for some $\xi \in (0, 1)$, the corresponding Weak BV estimates had the following form:

- There exists $\tilde{C}_{BV,1} \in \mathbb{R}_+^*$, only depending on T, u_0, ξ, F_1, F_2 and L_g such that

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ F(u_K^n, u_L^n) - f(u_K^n) \right\}^2 + v_{L,K}^n \left\{ F(u_L^n, u_K^n) - f(u_K^n) \right\}^2 \right] \leq \tilde{C}_{BV,1}.$$

- There exists $\tilde{C}_{BV,2} \in \mathbb{R}_+^*$, only depending on $R, d, T, \bar{\alpha}, u_0, \xi, F_1, F_2$ and L_g such that

$$\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathfrak{T}_R^n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{u_L^n \leq c \leq d \leq u_K^n} (F(d, c) - f(d)) + \max_{u_L^n \leq c \leq d \leq u_K^n} (F(d, c) - f(c)) \right\} \right. \\ \left. + v_{L,K}^n \left\{ \max_{u_L^n \leq c \leq d \leq u_K^n} (f(d) - F(c, d)) + \max_{u_L^n \leq c \leq d \leq u_K^n} (f(c) - F(c, d)) \right\} \right] \leq \tilde{C}_{BV,2} h^{-1/2},$$

where

$$\mathcal{T}_R = \{K \in \mathcal{T} \text{ such that } K \subset \mathcal{B}(0, R)\} \text{ and } \mathfrak{T}_R^n = \{(K, L) \in \mathcal{T}_R^n \text{ such that } L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}.$$

Remark I.23

Note that the CFL Conditions (I.12) and (I.13) were exactly the same as in the deterministic case (that is $g = 0$). In particular, the stochastic noise did not interfere with obtaining the associated estimates. Let me precise that in the deterministic case, Condition (I.13) for some $\xi \in (0, 1)$ was sufficient to show the convergence of $(u_{\mathcal{T},k})_{\mathcal{T},k}$ to the unique entropy solution of the associated hyperbolic problem. However, in the stochastic case this condition did not seem to be sufficient to show the convergence of the scheme, that is why we assumed the stronger assumption $k/h \rightarrow 0$ as $h \rightarrow 0$ in the statement of Theorem I.16. This hypothesis on k/h was perhaps technical, and was used for deriving discrete entropy estimates satisfied by the discrete unknowns u_K^n and consequently by $u_{\mathcal{T},k}$. With respect to the usual deterministic CFL Condition (I.13), it was a relatively weak hypothesis since k/h could go to 0 as slowly as wanted. Unfortunately we were not able to see, through numerical simulations for example, if this condition was necessary or not.

At first glance, the stability estimates on $(u_{\mathcal{T},k})_{\mathcal{T},k}$ stated in Proposition I.19 only provided (up to a subsequence) weak convergence for $(u_{\mathcal{T},k})_{\mathcal{T},k}$. Moreover, due to the nonlinearity of f and g , compactness arguments were needed in order to identify the weak limit of the associated nonlinear sequences $(f(u_{\mathcal{T},k}))_{\mathcal{T},k}$ and $(g(u_{\mathcal{T},k}))_{\mathcal{T},k}$. Additionally, these arguments had to be compatible with the random variable. The concept of Young measures was appropriate and the technique was based on the notion of narrow convergence of Young measures (or entropy processes), referring to [Bal00] and [EGH95]. Precisely, we had the following convergence result:

Proposition I.24 (Convergence of $(u_{\mathcal{T},k})_{\mathcal{T},k}$)

Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes in the sense of Definition I.9 such that the mesh size h_m tends to 0 when m tends to $+\infty$. Let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$ and set for any $m \in \mathbb{N}$, $k_m = \frac{T}{N_m}$ the time step. For a fixed $m \in \mathbb{N}$, let $u_{\mathcal{T}_m, k_m}$ be the finite-volume approximate solution defined by (I.9), (I.10) and (I.11). Assume that for any $m \in \mathbb{N}$, $(\mathcal{T}_m, k_m)$ satisfies the CFL Condition (I.12). Then, (up to a subsequence denoted in the same way) $(u_{\mathcal{T}_m, k_m})_{m \in \mathbb{N}}$ converges in the sense of Young measures to an “entropy process” denoted by μ in $L^2(\Omega \times \mathbb{R}^d \times (0, T) \times (0, 1))$.

Using in what follows (for the sake of simplicity) the notations $\mathcal{T} = \mathcal{T}_m$, $h = \text{size}(\mathcal{T}_m)$, $k = k_m$ and $N = N_m$, the previous convergence result was a direct consequence of the boundedness of $(u_{\mathcal{T},k})_{\mathcal{T},k}$ independently of the time and space discretization parameters given by Proposition I.19. Precisely, given a Carathéodory function* $\Psi : \Omega \times \mathbb{R}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ such that $(\Psi(\cdot, u_{\mathcal{T},k}))_{\mathcal{T},k}$ was uniformly integrable, we had:

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \Psi(\omega, x, t, u_{\mathcal{T},k}(\omega, x, t)) dx dt \right] \rightarrow \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 \Psi(\omega, x, t, \mu(\omega, x, t, \beta)) d\beta dx dt \right] \text{ as } h, k \rightarrow 0. \quad (\text{I.15})$$

Let me recall that a sequence $(\psi_n)_{n \geq 0}$ of functions $\psi_n : \Omega \times \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}$ is said to be uniformly integrable (see

*For almost any $(\omega, x, t) \in \Omega \times \mathbb{R}^d \times (0, T)$ the function $\nu \mapsto \Psi(\omega, x, t, \nu)$ is continuous and for all $\nu \in \mathbb{R}$, the function $(\omega, x, t) \mapsto \Psi(\omega, x, t, \nu)$ is measurable.

[BVW12, Section A.3.2]) if it satisfies the following properties:

- $(\psi_n)_{n \geq 0}$ is bounded in $L^1(\Omega \times \mathbb{R}^d \times (0, T))$.
- $(\psi_n)_{n \geq 0}$ is equi-integrable, that is to say that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable set $\mathcal{A}$ of $\Omega \times \mathbb{R}^d \times (0, T)$ satisfying $(\mathcal{L}^{d+1} \otimes \mathbb{P})(\mathcal{A}) \leq \delta$, we have for any $n \in \mathbb{N}$,

$$\int_{\mathcal{A}} |\psi_n(\omega, x, t)| dx dt d\mathbb{P} \leq \varepsilon$$

(where $\mathcal{L}^{d+1}$ is the $d + 1$ -dimensional Lebesgue measure).

- For any $\varepsilon > 0$ there exists a measurable set $\mathcal{A}_\varepsilon$ of $\Omega \times \mathbb{R}^d \times (0, T)$ with $(\mathcal{L}^{d+1} \otimes \mathbb{P})(\mathcal{A}_\varepsilon) < \infty$ and such that for any $n \in \mathbb{N}$

$$\int_{\mathcal{A}_\varepsilon} |\psi_n(\omega, x, t)| dx dt d\mathbb{P} \leq \varepsilon.$$

Remark I.25 (On the measurability of the entropy process μ)

Since the sequence $(u_{\mathcal{T},k})_{\mathcal{T},k}$ was bounded in the Hilbert space $\mathcal{H}_W^2(0, T; L^2(\mathbb{R}^d))$, by identification one showed that it converged weakly in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$ towards $\int_0^1 \mu(\cdot, \beta) d\beta$, so that $\int_0^1 \mu(\cdot, \beta) d\beta$ was a predictable process with values in $L^2(\mathbb{R}^d)$. The interesting point was the measurability of μ with respect to all its variables (ω, x, t, β) . Revisiting the work [Pan96] with the σ -field $\mathcal{P}_T \otimes \mathcal{L}(\mathbb{R}^d)$, it was possible to affirm (see [BVW12, Section A.3.3]) that μ was measurable for the σ -field $\mathcal{P}_T \otimes \mathcal{L}(\mathbb{R}^d \times (0, 1))$, thus $\mu \in \mathcal{H}_W^2(0, T; L^2(\mathbb{R}^d \times (0, 1)))$.

Remark I.26 (On the $L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d \times (0, 1)))$ regularity of μ)

Since the sequence of approximate solutions $(u_{\mathcal{T},k})_{\mathcal{T},k}$ was bounded in $L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))$ according to Proposition I.19, we could show by following [BVW12] that $\mu \in L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d \times (0, 1)))$.

Note that if we were able to prove that such a limit μ was a measure-valued entropy solution of Problem (I.1) in the sense of Definition I.5, then we could affirm the completeness of the proof of Theorem I.16. Indeed, knowing this information about μ in hand, the reduction result of [BVW12] stated in Theorem I.6 then allowed us to affirm firstly that μ was the unique stochastic entropy solution of Problem (I.1) in the sense of Definition I.3, hence its independence with respect to the variable $\beta \in (0, 1)$ and we then denoted it by u . Secondly, we could deduce that all the sequence $(u_{\mathcal{T},k})_{\mathcal{T},k}$ converged strongly in $L^1(\Omega \times \mathcal{K} \times (0, T))$ towards u , for any compact set $\mathcal{K}$ of $\mathbb{R}^d$. And thirdly, owing to the boundedness of $(u_{\mathcal{T},k})_{\mathcal{T},k}$ in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$, Vitali's theorem ([Dro01, Corollaire 1.3.3]) then allowed us to conclude that we had strong convergence in $L^p(\Omega \times \mathcal{K} \times (0, T))$ for any $1 \leq p < 2$ and any compact set $\mathcal{K}$ of $\mathbb{R}^d$, as stated in Theorem I.16.

Our strategy was then to exploit the continuous framework developed in [BVW12]. Since μ possessed the regularities required by Definition I.5, it then remained to show that it satisfied the following entropy inequalities:

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta(\mu(\omega, x, t, \beta)) \partial_t \varphi(x, t) d\beta dx dt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \Phi^\eta(\mu(\omega, x, t, \beta)) \vec{v} \cdot \nabla_x \varphi(x, t) d\beta dx dt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mu(\omega, x, t, \beta)) g(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dW(t) \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \int_0^1 g^2(\mu(\omega, x, t, \beta)) \eta''(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dt, \end{aligned} \quad (\text{I.16})$$

for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and for almost all ω in Ω .

In order to prove the convergence of $(u_{\mathcal{T},k})_{\mathcal{T},k}$ towards a measure-valued entropy solution in the sense of Definition I.5, we naturally thought of showing firstly that $u_{\mathcal{T},k}$ satisfied itself (in some sense) entropy estimates. Recall that in the deterministic case, it is classical (starting from (I.9), (I.10) and (I.11) with $g = 0$), to obtain discrete entropy estimates satisfied by the discrete unknowns u_K^n ($K \in \mathcal{T}$ and $n \in \{0, \dots, N-1\}$) and then continuous entropy estimates satisfied by the associated finite-volume approximation, even in the general case with monotone numerical flux F in the sense of

Definition I.13. This is due to the fact that monotone flux schemes are consistent in the finite-volume sense *i.e.* L^∞ -stable and “consistent with the entropy inequalities”, both under a CFL-type condition (see [EGH00]). Precisely, the obtention of discrete entropy estimates is quite immediate using for all $\kappa \in \mathbb{R}$ the famous Kruzhkov’s entropy-flux pairs $(|\cdot - \kappa|, \Phi(\cdot, \kappa))$ for which the numerical entropy flux $G_F(\cdot, \cdot, \kappa) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$G_F(a, b, \kappa) = F(a \top \kappa, b \top \kappa) - F(a \perp \kappa, b \perp \kappa), \quad \forall (a, b) \in \mathbb{R}^2.$$

As explained in the presentation of the theoretical framework, Kruzhkov’s entropies were not allowed in the stochastic case so that we had to consider techniques compatible with smooth entropies η belonging to $\mathcal{A}$. In the study of the continuous case ([BVW12]), let me remind that entropy inequalities satisfied by the unique stochastic entropy solution of Problem (I.1) were derived from the study of the following viscous parabolic regularization (with a parameter $\epsilon > 0$):

$$\begin{cases} du_\epsilon + \operatorname{div}_x(\vec{v}f(u_\epsilon))dt - \epsilon \Delta u_\epsilon dt &= g(u_\epsilon)dW & \text{in } \Omega \times \mathbb{R}^d \times (0, T), \\ u_\epsilon(\omega, x, 0) &= u_0(x), & \omega \in \Omega, x \in \mathbb{R}^d. \end{cases} \quad (\text{I.17})$$

Indeed, they were obtained by applying Itô’s formula (see [DPZ92]) to the functional $\mathcal{F} : (t, v) \in [0, T] \times \mathbb{R} \mapsto \eta(v) \in \mathbb{R}$, $\eta \in \mathcal{A}$ and the viscous approximation u_ϵ . In our discrete case, the application of such an Itô’s formula to $u_{\mathcal{T},k}$ was out of range due to its lack of continuity in time. For this reason, we introduced a time-continuous approximate solution

$$\bar{u}_{\mathcal{T},k} : \Omega \times \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R},$$

constructed from $u_{\mathcal{T},k}$. Precisely, for any $K \in \mathcal{T}$ and any $n \in \{0, \dots, N-1\}$, we firstly defined the stochastic process $\bar{u}_K^n$ from $\Omega \times [nk, (n+1)k]$ to $\mathbb{R}$ by

$$\begin{aligned} \bar{u}_K^n(\omega, s) &= u_K^n(\omega) + \frac{s - nk}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- \left(f(u_\sigma^n(\omega)) - f(u_K^n(\omega)) \right) + g(u_K^n(\omega))(W(s, \omega) - W(nk, \omega)) \\ &= u_K^n(\omega) + \int_{nk}^s \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- \frac{f(u_\sigma^n(\omega)) - f(u_K^n(\omega))}{|K|} dt + \int_{nk}^s g(u_K^n(\omega)) dW(t), \end{aligned} \quad (\text{I.18})$$

for any $(\omega, s) \in \Omega \times [nk, (n+1)k]$. Note that in this way, $\bar{u}_K^n(\omega, nk) = u_K^n(\omega)$ and $\bar{u}_K^n(\omega, (n+1)k) = u_K^{n+1}(\omega)$. And from the stochastic process $\bar{u}_K^n$, we secondly defined

$$\bar{u}_{\mathcal{T},k}(\omega, x, t) = \bar{u}_K^n(\omega, t), \quad \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k]. \quad (\text{I.19})$$

In order to insure the convergence of $\bar{u}_{\mathcal{T},k}$ and $u_{\mathcal{T},k}$ towards the same possible limit (in a weak or strong sense, whatever), we firstly estimated their difference in the $L^2(\Omega \times \mathbb{R}^d \times (0, T))$ -norm. This is stated in the following proposition:

Proposition I.27

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the CFL Condition (I.13). Let $\bar{u}_{\mathcal{T},k}$ be the time-continuous approximate solution defined by (I.18) and (I.19), and $u_{\mathcal{T},k}$ be the finite-volume approximate solution defined by (I.9), (I.10) and (I.11). Then

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{R}^d \times (0, T))}^2 \leq k L_g^2 \|u_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{R}^d \times (0, T))}^2 + h \frac{(1 - \xi)^2 \bar{\alpha}^2}{V L_f^2} C_{BV,1},$$

where the constant $C_{BV,1}$ comes from the Weak BV estimates stated in Proposition I.21.

The proof of this result was mainly based on the weak BV estimates and for this reason, it was easily generalized to the case of monotone numerical fluxes, and as previously the space-time dependence of $\vec{v}$ did not pose a fundamental problem. All the necessary tools needed to understand how we derived the following discrete entropy estimates satisfied by the discrete unknowns have been now introduced:

Proposition I.28 (Discrete entropy estimates)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$, let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step and assume that

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then, for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and $\mathbb{P}$ -a.s. in Ω :

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi^\eta(u_K^n) \vec{v} \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dx dW(t) \\ & + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \\ & \geq \mathcal{R}^{h,k}, \end{aligned} \tag{I.20}$$

where for any $\mathbb{P}$ -measurable set A , $\mathbb{E}[\mathbb{1}_A \mathcal{R}^{h,k}] \rightarrow 0$ as $(h, \frac{k}{h}) \rightarrow (0, 0)$.

The proof of this proposition was separated in two steps: in the first one we showed that Inequality (I.20) held for a convenient rest $\mathcal{R}^{h,k}$ and in the second step, we proved that $\mathbb{E}[\mathbb{1}_A \mathcal{R}^{h,k}] \rightarrow 0$ as $(h, \frac{k}{h}) \rightarrow (0, 0)$. I propose here to comment briefly how we obtained (I.20). In more detail, I will explain on the one hand the origin of the unusual assumption on $\frac{k}{h}$ and on the other hand why the explicit knowledge of the expression of the employed numerical fluxes was helpful. Let me remark that by assuming $k/h \rightarrow 0$ as $h \rightarrow 0$, we could then suppose that the CFL Condition (I.13) held for some $\xi \in (0, 1)$. In this manner, the estimates given by Proposition I.19 and Proposition I.21 were available.

By considering an entropy $\eta \in \mathcal{A}$, a test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and a $\mathbb{P}$ -measurable set A , we thus had existence of a constant $R > h$ such that $\text{supp} \varphi \subset \mathcal{B}(0, R - h) \times [0, T]$ and we were then able to define the following subset of $\mathcal{T}$

$$\mathcal{T}_R = \{K \in \mathcal{T} \text{ such that } K \subset \mathcal{B}(0, R)\}$$

which took into account the compact support in space of the test function φ . The application of Itô's formula to the process $\bar{u}_K^n$ defined by (I.18) and the functional $\mathcal{F} : (t, v) \in [0, T] \times \mathbb{R} \mapsto \eta(v) \in \mathbb{R}$ on the interval $[nk, (n+1)k]$ gave $\mathbb{P}$ -a.s. in Ω

$$\begin{aligned} \eta(\bar{u}_K^n((n+1)k)) &= \eta(\bar{u}_K^n(nk)) + \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma}) - \frac{f(u_\sigma^n) - f(u_K^n)}{|K|} dt \\ &+ \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) g(u_K^n) dW(t) \\ &+ \frac{1}{2} \int_{nk}^{(n+1)k} \eta''(\bar{u}_K^n(t)) g^2(u_K^n) dt. \end{aligned} \tag{I.21}$$

After multiplying (I.21) by $\int_K \varphi(x, nk) dx$ and summing for all $K \in \mathcal{T}_R$ and $n \in \{0, \dots, N-1\}$, we arrived at

$$\mathcal{A}^{h,k} = \mathcal{B}^{h,k} + \mathcal{C}^{h,k} + \mathcal{D}^{h,k},$$

where $\mathbb{P}$ -a.s. in Ω

$$\begin{aligned}\mathcal{A}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} [\eta(u_K^{n+1}) - \eta(u_K^n)] \int_K \varphi(x, nk) dx, \\ \mathcal{B}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- (f(u_\sigma^n) - f(u_K^n)) dt \frac{1}{|K|} \int_K \varphi(x, nk) dx, \\ \mathcal{C}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \eta'(\bar{u}_K^n(t)) g(u_K^n) dW(t) \int_K \varphi(x, nk) dx, \\ \mathcal{D}^{h,k} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \eta''(\bar{u}_K^n(t)) g^2(u_K^n) dt \int_K \varphi(x, nk) dx.\end{aligned}$$

Let me note that $\mathcal{A}^{h,k}$ already appeared in (I.20). Additionally, one can easily see the correspondence between the terms $\mathcal{C}^{h,k}$ and $\mathcal{D}^{h,k}$ and those in (I.20) involving respectively g and g^2 . Their difference two by two gave rise to the existence of residues denoted by $\mathcal{R}_\mathcal{C}^{h,k}$ and $\mathcal{R}_\mathcal{D}^{h,k}$ contained in the rest $\mathcal{R}^{h,k}$. Let me simply note that Assumption $\mathcal{H}_6$ on the boundedness of g was required to show that $\mathbb{E}[\mathbb{1}_A \mathcal{R}_\mathcal{C}^{h,k}]$ and $\mathbb{E}[\mathbb{1}_A \mathcal{R}_\mathcal{D}^{h,k}]$ tended to 0. This restriction on g might come from the lack of a $L^\infty(\Omega \times \mathbb{R}^d \times (0, T))$ estimate on $u_{\mathcal{T},k}$.

For the term $\mathcal{B}^{h,k}$ involving the numerical fluxes, there was much more work to do. For this reason, I focus in what follows on this term which was responsible for the sign of the discrete entropy estimates (I.20) and the assumption $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$. Our idea was to decompose $\mathcal{B}^{h,k}$ in the following way

$$\mathcal{B}^{h,k} = \tilde{\mathcal{B}}^{h,k} - \hat{\mathcal{B}}^{h,k} + \tilde{\mathcal{B}}^{h,k} - \hat{\mathcal{B}}^{h,k} + \hat{\mathcal{B}}^{h,k} - \overline{\mathcal{B}}^{h,k} + \overline{\mathcal{B}}^{h,k}, \quad (\text{I.22})$$

where

$$\begin{aligned}\tilde{\mathcal{B}}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} k \eta'(u_K^n) \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- (f(u_\sigma^n) - f(u_K^n)) \frac{1}{|K|} \int_K \varphi(x, nk) dx \\ \hat{\mathcal{B}}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} k \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- [\Phi^\eta(u_\sigma^n) - \Phi^\eta(u_K^n)] \frac{1}{|K|} \int_K \varphi(x, nk) dx, \\ \overline{\mathcal{B}}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi^\eta(u_K^n) \vec{v} \cdot \nabla_x \varphi(x, nk) dx dt,\end{aligned}$$

and to show that on the one hand $\mathbb{E}[\mathbb{1}_A |\tilde{\mathcal{B}}^{h,k} - \mathcal{B}^{h,k}|]$ and $\mathbb{E}[\mathbb{1}_A |\hat{\mathcal{B}}^{h,k} - \overline{\mathcal{B}}^{h,k}|]$ tended to 0 and on the other hand that $\mathbb{P}$ -a.s. in Ω , $\tilde{\mathcal{B}}^{h,k} - \hat{\mathcal{B}}^{h,k} \leq 0$. Roughly speaking, $\tilde{\mathcal{B}}^{h,k}$ and $\mathcal{B}^{h,k}$ were very similar in the sense that $\mathcal{B}^{h,k}$ was obtained by replacing in $\tilde{\mathcal{B}}^{h,k}$ the term $\bar{u}_K^n(t)$, $t \in (nk, (n+1)k)$ by u_K^n (in other words, the replacement at the discrete scale of $\bar{u}_{\mathcal{T},k}$ by $u_{\mathcal{T},k}$), $\hat{\mathcal{B}}^{h,k}$ contained the “upwind” numerical entropy flux associated with the entropy-flux pair (η, Φ^η) and $\overline{\mathcal{B}}^{h,k}$ was exactly the desired contribution of the entropy flux Φ^η at $u_{\mathcal{T},k}$ in (I.20).

Firstly, note that the constraint $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$ was required to show the convergence towards 0 of $\mathbb{E}[\mathbb{1}_A |\tilde{\mathcal{B}}^{h,k} - \mathcal{B}^{h,k}|]$. Indeed, the study of this quantity led to the evaluation of

$$\left| \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \mathbb{1}_A \int_{nk}^{(n+1)k} \{ \eta'(\bar{u}_K^n(s)) - \eta'(u_K^n) \} ds \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\vec{v} \cdot n_{K,\sigma})^- (f(u_\sigma^n) - f(u_K^n)) \frac{1}{|K|} \int_K \varphi(x, nk) dx \right] \right|$$

which contained (in some manner) at the discrete level the difference $\bar{u}_{\mathcal{T},k} - u_{\mathcal{T},k}$. Unfortunately, the control on this difference given by Proposition I.27 did not seem to be enough to compensate the contribution of the numerical flux for which we only had the Weak BV estimates stated in Proposition I.21. Our computations led us to the boundedness of $\mathbb{E}[\mathbb{1}_A |\tilde{\mathcal{B}}^{h,k} - \mathcal{B}^{h,k}|]$ by $\frac{k}{h}$ multiplied by a constant independent of the discretization parameters.

Secondly, in order to show that $\mathbb{P}$ -a.s. in Ω , $\tilde{\mathcal{B}}^{h,k} - \hat{\mathcal{B}}^{h,k} \leq 0$, we clearly exploited the fact that we considered an upwind scheme. Indeed, in this particular case we then had an explicit expression for the “upwind” numerical entropy

flux associated with the entropy-flux pair (η, Φ^η) which was contained in the term $\hat{\mathcal{B}}^{h,k}$. This was crucial for us because since f and η' were non-decreasing, this allowed us to state that $\mathbb{P}$ -a.s. in Ω

$$\eta'(u_K^n)[f(u_\sigma^n) - f(u_K^n)] - [\Phi^\eta(u_\sigma^n) - \Phi^\eta(u_K^n)] = \int_{u_K^n}^{u_\sigma^n} (\eta'(u_K^n) - \eta'(s))f'(s) ds \leq 0, \quad (\text{I.23})$$

and thus we obtained the following “sign property”, $\mathbb{P}$ -a.s in Ω :

$$\tilde{\mathcal{B}}^{h,k} - \hat{\mathcal{B}}^{h,k} \leq 0. \quad (\text{I.24})$$

Remark I.29 (Extension of (I.24) to a monotone numerical flux F)

The obtention of (I.24) in [BCG16b] was not as obvious and direct for a general monotone numerical flux F in the sense of Definition I.13, due to the fact that we worked with smooth entropies $\eta \in \mathcal{A}$. Note that in this general case, the corresponding terms $\tilde{\mathcal{B}}^{h,k}$ and $\hat{\mathcal{B}}^{h,k}$ in the splitting (I.22) were given by

$$\begin{aligned} \tilde{\mathcal{B}}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta'(u_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ \hat{\mathcal{B}}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n G_F^\eta(u_K^n, u_L^n) - v_{L,K}^n G_F^\eta(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \end{aligned}$$

and the problem was to construct for F a numerical entropy flux $G_F^\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ associated with the entropy-flux pair (η, Φ^η) . To overcome this difficulty, we took advantage of the decomposition property satisfied by F , the one stated in Lemma I.15. As a consequence, we obtained the following decomposition for G_F^η :

$$G_F^\eta(a, b) = \theta(a, b)G_G^\eta(a, b) + (1 - \theta(a, b))G_{LF}^\eta(a, b), \forall (a, b) \in \mathbb{R}^2, \quad (\text{I.25})$$

where G_G^η and G_{LF}^η denoted respectively the “Godunov” and “modified Lax-Friedrichs” numerical entropy fluxes associated with the entropy-flux pair (η, Φ^η) and $\theta(a, b) \in [0, 1]$ was given by Lemma I.15. Using this, if we were able to show the “sign property” respectively for the modified Lax-Friedrichs couple (F_{LF}^D, G_{LF}^η) and Godunov’s one (F_G, G_G^η) , then the result would hold true for (F, G_F^η) .

The “modified Lax-Friedrichs” numerical entropy flux associated with the entropy-flux pair (η, Φ^η) was defined by

$$G_{LF}^\eta(a, b) = \frac{\Phi^\eta(a) + \Phi^\eta(b)}{2} + D(\eta(a) - \eta(b)), \forall (a, b) \in \mathbb{R}^2.$$

Proving the “sign property” for upwind scheme allowed us to extend it naturally to any flux-splitting schemes since they were equivalent to “upwinding” on the non-decreasing part of the flux and “downwinding” on the non-increasing one. In that manner, the case of modified Lax-Friedrichs numerical fluxes was resolved. It then remained to prove it for Godunov numerical fluxes. In this case, we were led to analyze the sign of terms similar to the left hand side of (I.23). Precisely:

$$\eta'(u_K^n)[F_G(u_K^n, u_L^n) - f(u_K^n)] - [G_G^\eta(u_K^n, u_L^n) - \Phi^\eta(u_K^n)] \quad (\text{I.26})$$

$$\text{and } \eta'(u_K^n)[F_G(u_L^n, u_K^n) - f(u_K^n)] - [G_G^\eta(u_L^n, u_K^n) - \Phi^\eta(u_K^n)]. \quad (\text{I.27})$$

For all $(a, b) \in \mathbb{R}^2$ by denoting $s(a, b) \in [a \perp b, a \top b]$ a real such that $F_G(a, b) = f(s(a, b))$, we could define G_G^η the “Godunov” numerical entropy flux associated with the entropy-flux pair (η, Φ^η) by

$$G_G^\eta(a, b) = \Phi^\eta(s(a, b)), \forall (a, b) \in \mathbb{R}^2.$$

Using this explicit expression of G_G^η , the control of the signs of (I.26) and (I.27) were handled with the same kind of tricks as for obtaining (I.23). Let me mention that as previously, the space-time dependence of $\vec{v}$ only brought technical difficulties which were managed as in the deterministic case.

Thirdly, note that due to the divergence theorem, $\overline{\mathcal{B}}^{h,k}$ could be rewritten as

$$\overline{\mathcal{B}}^{h,k} = k \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \vec{v} \cdot n_{K,\sigma} \varphi(x, nk) d\gamma(x) \Phi^\eta(u_K^n).$$

Using this, the fact that $\operatorname{div} \vec{v} = 0$ and that $\hat{\mathcal{B}}^{h,k}$ contained the “upwind” numerical entropy flux, we were able to decompose $\hat{\mathcal{B}}^{h,k}$ in the following explicit manner:

$$\hat{\mathcal{B}}^{h,k} = \overline{\mathcal{B}}^{h,k} + \mathcal{R}_{B,1}^{h,k} + \mathcal{R}_{B,2}^{h,k} \quad (\text{I.28})$$

where by denoting x_σ the center of the edge σ

$$\begin{aligned} \mathcal{R}_{B,1}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} k \sum_{\sigma \in \mathcal{E}_K} |\sigma| \vec{v} \cdot n_{K,\sigma} (\Phi^\eta(u_K^n) - \Phi^\eta(u_\sigma^n)) \left\{ \frac{1}{|K|} \int_K \varphi(x, nk) dx - \varphi(x_\sigma, nk) \right\}, \\ \text{and } \mathcal{R}_{B,2}^{h,k} &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}_R} \sum_{\sigma \in \mathcal{E}_K} \left(|\sigma| \vec{v} \cdot n_{K,\sigma} \varphi(x_\sigma, nk) - \int_{\sigma} \vec{v} \cdot n_{K,\sigma} \varphi(x, nk) d\gamma(x) \right) \Phi^\eta(u_K^n). \end{aligned}$$

Invoking Weak BV estimates stated in Proposition I.21, we were able to show that $\mathbb{E} [\mathbb{1}_A |\mathcal{R}_{B,1}^{h,k}|]$ tended to 0 as $(h, k) \rightarrow (0, 0)$. At last, to prove that $\mathbb{E} [\mathbb{1}_A |\mathcal{R}_{B,2}^{h,k}|]$ also went to 0, we used the regularity of the test function φ . Finally, we obtained that

$$\mathbb{E} [\mathbb{1}_A |\hat{\mathcal{B}}^{h,k} - \overline{\mathcal{B}}^{h,k}|] \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0). \quad (\text{I.29})$$

Remark I.30 (Extension of (I.29) to any monotone numerical flux F)

When we considered in [BCG16b] a general monotone numerical flux F in the sense of Definition I.13, the corresponding terms $\hat{\mathcal{B}}^{h,k}$ and $\overline{\mathcal{B}}^{h,k}$ in the splitting (I.22) were given by (using notations of Remark I.29)

$$\begin{aligned} \hat{\mathcal{B}}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n G_F^\eta(u_K^n, u_L^n) - v_{L,K}^n G_F^\eta(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ \overline{\mathcal{B}}^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \Phi^\eta(u_K^n) \vec{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt. \end{aligned}$$

To control the associated rest $\mathcal{R}_{B,1}^{h,k}$ in the decomposition (I.28) we were led in particular to the evaluation of the following quantities

$$G_F^\eta(u_K^n, u_L^n) - \Phi^\eta(u_K^n), G_F^\eta(u_K^n, u_L^n) - \Phi^\eta(u_L^n), G_F^\eta(u_L^n, u_K^n) - \Phi^\eta(u_L^n) \text{ and } G_F^\eta(u_L^n, u_K^n) - \Phi^\eta(u_K^n).$$

Using (I.25) and the explicit knowledge of $G_{LF}^\eta(a, b)$ and $G_G^\eta(a, b)$, we were able to conclude, by treating separately modified Lax-Friedrichs' and Godunov's cases that $\mathbb{E} [\mathbb{1}_A |\mathcal{R}_{B,1}^{h,k}|]$ tended to 0 as $(h, k) \rightarrow (0, 0)$ thanks to Weak BV estimates. At last, studying the analogous of $\mathcal{R}_{B,2}^{h,k}$ in the general case was not a big issue, it was just more technical due to the space-time dependence of $\vec{v}$.

Owing to the regularity of the test function φ , Proposition I.28 led us naturally to the obtention of the following continuous entropy estimates on the finite-volume approximation $u_{\mathcal{T},k}$:

Proposition I.31 (Continuous entropy estimates on the finite-volume approximation)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step and assume that

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then, for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and for almost all ω in Ω :

$$\begin{aligned} & \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \eta(u_{\mathcal{T},k}(\omega, x, t)) \varphi_t(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \Phi^\eta(u_{\mathcal{T},k}(\omega, x, t)) \vec{v} \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k}(\omega, x, t)) g(u_{\mathcal{T},k}(\omega, x, t)) \varphi(x, t) dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \eta''(u_{\mathcal{T},k}(\omega, x, t)) g^2(u_{\mathcal{T},k}(\omega, x, t)) \varphi(x, t) dx dt \\ & \geq \tilde{\mathcal{R}}^{h,k}(\omega), \end{aligned} \tag{I.30}$$

where for any $\mathbb{P}$ -measurable set A , $\mathbb{E}[\mathbf{1}_A \tilde{\mathcal{R}}^{h,k}] \rightarrow 0$ as $(h, \frac{k}{h}) \rightarrow (0, 0)$.

To conclude, it remained to pass to the limit in (I.30) and this was possible thanks to Proposition I.24. Indeed, seeing each integrand of (I.30) as a Carathéodory function Ψ at $u_{\mathcal{T},k}$ with $(\Psi(\cdot, u_{\mathcal{T},k}))_{\mathcal{T},k}$ uniformly integrable, we obtained thanks to (I.15) that the entropy process μ satisfied the entropy inequalities (I.16) required by Definition I.5. Using the reduction result of [BVW12] stated in Theorem I.6, the proof of Theorem I.16 was then complete.

Remark I.32 (Extension of the continuous framework for a time and space dependent flux function)

Let me mention that the reduction result stated in Theorem I.6 was only valid for a constant vector $\vec{v}$. Since it was a key argument to conclude to the convergence of our finite-volume schemes, we extended this existence and uniqueness result to the case of a time and space dependent flux function $\vec{v}f$ (through the vector $\vec{v}$) in the appendix of [BCG16b]. Following [BVW12], we studied the parabolic regularization (I.17) (with $\vec{v}$ non constant) to prove existence and uniqueness of the stochastic entropy solution for Problem (I.1) in the sense of Definition I.3.

Chapter II

Numerical approximation of stochastic conservation laws on bounded domains

A natural question that came up after the studies presented in Chapter I was the numerical consideration of boundary conditions for stochastic hyperbolic problems. This chapter is then devoted to the presentation of a finite-volume method for the discretization of scalar conservation laws with a multiplicative stochastic force defined on a bounded domain D of $\mathbb{R}^d$ with non-homogeneous Dirichlet boundary conditions and a given initial data in $L^\infty(D)$. This survey was published in a joint work with J. CHARRIER and T. GALLOUËT [BCG17] and addressed in the literature the first convergence result for a space-time discretization of a first-order stochastic conservation law on bounded domain.

II.1 Introduction

Our aim in [BCG17] was to construct a finite-volume scheme to approximate the following nonlinear scalar conservation law with a stochastic multiplicative force, posed over a bounded domain:

$$\begin{cases} du + \operatorname{div}_x [\vec{v}f(u)] dt &= g(u) dW & \text{in } \Omega \times D \times (0, T), \\ u(\omega, x, 0) &= u_0(x), & \omega \in \Omega, x \in D, \\ u(\omega, x, t) &= u^b(x, t), & \omega \in \Omega, x \in \partial D, t \in (0, T), \end{cases} \quad (\text{II.1})$$

where D was a polygonal subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) with boundary ∂D , $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$, div_x denoted the divergence operator with respect to the space variable (which belonged to D) and $(W(t))_{t \geq 0}$ was a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

As mentioned in Chapter I, even in the deterministic case, a weak solution to a nonlinear scalar conservation law is not unique in general when the problem is posed on the whole space. Since this specificity still holds in a bounded domain, the mathematical challenge also consists in introducing a selection criterion in order to identify a unique solution. In [BCG17] we considered a stochastic version of the entropy condition proposed by F. OTTO in his PhD (see [Ott96]) to take into account our non-homogeneous Dirichlet boundary conditions.

We assumed the following hypotheses:

$$\mathcal{H}_1: u_0 \in L^\infty(D).$$

$$\mathcal{H}_2: u^b \in L^\infty(\partial D \times (0, T)).$$

$$\mathcal{H}_3: f: \mathbb{R} \rightarrow \mathbb{R} \text{ is a Lipschitz-continuous function with } f(0) = 0.$$

$\mathcal{H}_4$: $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function.

$\mathcal{H}_5$: $\vec{v} : D \times [0, T] \rightarrow \mathbb{R}^d$ is a Lipschitz-continuous function and satisfies $\operatorname{div}_x[\vec{v}(x, t)] = 0, \forall (x, t) \in D \times [0, T]$.

$\mathcal{H}_6$: There exists $V < \infty$ such that $|\vec{v}(x, t)| \leq V, \forall (x, t) \in D \times [0, T]$.

$\mathcal{H}_7$: g is a bounded function.

Remark II.1 (On these assumptions)

- . $\mathcal{H}_1$ to $\mathcal{H}_6$ were used to prove the well-posedness of Problem (II.1). Note that, as it was classically done for hyperbolic scalar conservation laws, for convenience we could assume that $f(0) = 0$ without loss of generality.
- . The study presented in this chapter could be extended to the case $\operatorname{div}_x[\vec{v}(x, t)] \neq 0$ (providing additional technical difficulties) following for example the work [CH00] in the deterministic case.
- . As in Chapter I, $\mathcal{H}_7$ was a technical and sufficient assumption used to show the convergence of the finite-volume scheme, which probably compensated for the lack of a $L^\infty(\Omega \times D \times (0, T))$ estimate on the sequence of approximate solutions $(u_{\mathcal{T},k})_{\mathcal{T},k}$.

The plan of our reasoning was the following one: proving as in Chapter I convergence of our finite-volume approximation towards a stochastic measure-valued entropy solution and exploiting reduction result for such solutions from the theoretical framework to ensure the convergence towards the unique stochastic entropy solution. Our starting point was then to find such a well-posedness result for Problem (II.1). Let me state it after introducing some notations and make precise the functional setting. Let me mention that most of them are similar to those presented in Chapter I, but to make Chapter II self-contained, I will repeat them here.

II.1.1 Notations and theoretical framework

Notations II.2

- . $\mathbb{E}[\cdot]$ denotes the expectation, i.e. the integral over Ω with respect to the probability measure $\mathbb{P}$.
- . $\mathcal{D}^+(\mathbb{R}^d \times [0, T])$ denotes the subset of nonnegative elements of $\mathcal{D}(\mathbb{R}^d \times [0, T])$.
- . $\mathcal{P}_T$ denotes the predictable σ -field on $(0, T) \times \Omega$, i.e. the σ -field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $(s, t] \times B$ for any $B \in \mathcal{F}_s, s, t \in [0, T], s < t$.
- . For a given separable Hilbert space X , we denote by $\mathcal{N}_W^2(0, T; X)$ the separable Hilbert space of the predictable X -valued processes. It is the space $L^2((0, T) \times \Omega; X)$ for the product measure $dt \otimes d\mathbb{P}$ on $\mathcal{P}_T$ (see [DPZ92] p.94).
- . $\mathcal{A}$ denotes the set of nonnegative convex functions η_κ (called smooth entropies in what follows) belonging to $\mathcal{C}^{2,1}(\mathbb{R})$, such that η_κ admits 0 as a minimum which is reached at a unique point $\kappa \in \mathbb{R}$. We also suppose that η'_κ and η''_κ are bounded functions.
- . $\Phi^{\eta_\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ denotes the entropy flux associated with a smooth entropy $\eta_\kappa \in \mathcal{A}$. It is defined for any $a \in \mathbb{R}$ by

$$\Phi^{\eta_\kappa}(a) = \int_\kappa^a \eta'_\kappa(\sigma) f'(\sigma) d\sigma.$$

- . L_f and L_g denote the Lipschitz constants of f and g .

Our idea was to adapt to the stochastic case the work [Vov02], which established the convergence of finite-volume monotone schemes for deterministic scalar conservation laws on bounded domains. This work used the concept of entropy solution introduced in [Ot96] for Dirichlet boundary conditions. Such a notion of solution was well-suited for numerical approximation (see [Vov02]) and was additionally equivalent to the Bardos-Leroux-Nédélec (BLN) concept of solution in the case where the solution was of bounded variation. Under Assumptions $\mathcal{H}_1$ to $\mathcal{H}_6$, we adapted this notion of solution to the stochastic case and this led us to the following definition:

Definition II.3 (Stochastic entropy solution)

A function u of $\mathcal{N}_W^2(0, T; L^2(D)) \cap L^\infty(0, T; L^2(\Omega \times D))$ is an entropy solution of Problem (II.1) with the initial condition $u_0 \in L^\infty(D)$ and the boundary condition $u^b \in L^\infty(\partial D \times (0, T))$, if it satisfies the following entropy inequalities

$$\begin{aligned} 0 \leq & \int_D \eta_\kappa(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_D \eta_\kappa(u(\omega, x, t)) \partial_t \varphi(x, t) dx dt \\ & + \int_0^T \int_D \Phi^{\eta_\kappa}(u(\omega, x, t)) \vec{v}(x, t) \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_D \eta'_\kappa(u(\omega, x, t)) g(u(\omega, x, t)) \varphi(x, t) dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_D g^2(u(\omega, x, t)) \eta''_\kappa(u(\omega, x, t)) \varphi(x, t) dx dt \\ & + L_f V \int_0^T \int_{\partial D} \varphi(x, t) \eta_\kappa(u^b(x, t)) d\gamma(x) dt, \end{aligned}$$

for any entropy $\eta_\kappa \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and for almost all $\omega \in \Omega$.

For technical reasons, as in Chapter I and as in [Vov02] for the deterministic case, we also needed to consider a more general notion of solution namely the concept of stochastic measure-valued entropy solution:

Definition II.4 (Stochastic measure-valued entropy solution)

A function μ of $\mathcal{N}_W^2(0, T; L^2(D \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times D \times (0, 1)))$ is a measure-valued entropy solution of Problem (II.1) with the initial condition $u_0 \in L^\infty(D)$ and the boundary condition $u^b \in L^\infty(\partial D \times (0, T))$, if it satisfies the following entropy inequalities

$$\begin{aligned} 0 \leq & \int_D \eta_\kappa(u_0) \varphi(x, 0) dx + \int_0^T \int_D \int_0^1 \eta_\kappa(\mu(\omega, x, t, \beta)) \partial_t \varphi(x, t) d\beta dx dt \\ & + \int_0^T \int_D \int_0^1 \Phi^{\eta_\kappa}(\mu(\omega, x, t, \beta)) \vec{v}(x, t) \cdot \nabla_x \varphi(x, t) d\beta dx dt \\ & + \int_0^T \int_D \int_0^1 \eta'_\kappa(\mu(\omega, x, t, \beta)) g(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_D \int_0^1 g^2(\mu(\omega, x, t, \beta)) \eta''_\kappa(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dt \\ & + L_f V \int_0^T \int_{\partial D} \varphi(x, t) \eta_\kappa(u^b(x, t)) d\gamma(x) dt, \end{aligned}$$

for any entropy $\eta_\kappa \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and for almost all $\omega \in \Omega$.

We derived a well-posedness theory for Problem (II.1) by combining arguments from [KN16a] and [DV10] in the following manner. Firstly, according to the uniqueness and reduction result of [KN16a], we had existence of a unique generalized kinetic solution which was actually a kinetic solution to the first order stochastic conservation law (II.1). Secondly, using the same arguments as in [DV10], we could show that a kinetic solution was an entropy solution and *vice versa*. Then, by exploiting the equivalence between the notions of measure-valued entropy solution and generalized kinetic solution, we arrived at the following result:

Theorem II.5

Under assumptions $\mathcal{H}_1$ to $\mathcal{H}_6$ there exists a unique measure-valued entropy solution for Problem (II.1). Moreover, it is the unique stochastic entropy solution in the sense of Definition II.3.

II.2 Finite-volume framework

In the sequel, let me assume that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_7$ hold, consider an admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ in the sense of Definition I.9, a monotone numerical flux F in the sense of Definition I.13, and $N \in \mathbb{N}^*$ defining a time step $k = \frac{T}{N} \in \mathbb{R}_+^*$. We approximated in [BCG17] the unique stochastic entropy solution u of Problem (II.1) by a function $u_{\mathcal{T},k} : \Omega \times D \times [0, T) \rightarrow \mathbb{R}$ which was constant on each space-time grid $K \times [nk, (n+1)k)$, $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$:

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n(\omega) \text{ for } \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k), \quad (\text{II.2})$$

where the set $\{u_K^0, K \in \mathcal{T}\}$ was given by the mean value of the initial condition on each control volume:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \forall K \in \mathcal{T}. \quad (\text{II.3})$$

For any $n \in \{0, \dots, N-1\}$, knowing $\{u_K^n, K \in \mathcal{T}\}$, the discrete unknowns $\{u_K^{n+1}, K \in \mathcal{T}\}$ were computed thanks to the following equation*:

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_K} |\sigma| \left\{ v_{K,\sigma}^{n,+} F(u_K^n, u_{K,\sigma}^n) - v_{K,\sigma}^{n,-} F(u_{K,\sigma}^n, u_K^n) \right\} = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}, \quad (\text{II.4})$$

where for any $\sigma \in \mathcal{E}_K$ and any $n \in \{0, \dots, N-1\}$, by denoting for any $a \in \mathbb{R}$, $a^- = -\min(a, 0)$ and $a^+ = \max(a, 0)$,

$$\begin{aligned} v_{K,\sigma}^{n,+} &= \frac{1}{k|\sigma|} \int_{nk}^{(n+1)k} \int_{\sigma} (\vec{v}(x, t) \cdot n_{K,\sigma})^+ d\gamma(x) dt, \\ v_{K,\sigma}^{n,-} &= \frac{1}{k|\sigma|} \int_{nk}^{(n+1)k} \int_{\sigma} (\vec{v}(x, t) \cdot n_{K,\sigma})^- d\gamma(x) dt, \\ u_{K,\sigma}^n &= \begin{cases} u_L^n & \text{if } \sigma = \sigma_{K,L}, \\ u_{\sigma}^{b,n} & \text{if } \sigma \in \mathcal{E}^b, \end{cases} = \frac{1}{k|\sigma|} \int_{nk}^{(n+1)k} \int_{\sigma} u^b(x, t) d\gamma(x) dt, \\ W^n &= W(nk), \end{aligned}$$

with $\mathcal{E}^b = \{\sigma \in \mathcal{E} : |\sigma \cap \partial D| > 0\}$ the set of boundary interfaces.

Notations II.6

When $\sigma = \sigma_{K,L}$, we denoted $v_{K,L}^{n,+} = v_{K,\sigma_{K,L}}^{n,+}$ and $v_{K,L}^{n,-} = v_{K,\sigma_{K,L}}^{n,-}$, and using these we had $v_{L,K}^{n,+} = v_{K,L}^{n,-}$.

Remark II.7 (Consequence of the divergence free property of $\vec{v}$)

Note that for any interface $\sigma \in \mathcal{E}$

$$v_{K,\sigma}^{n,+} - v_{K,\sigma}^{n,-} = \frac{1}{k|\sigma|} \int_{nk}^{(n+1)k} \int_{\sigma} \vec{v}(x, t) \cdot n_{K,\sigma} d\gamma(x) dt \text{ and } v_{K,\sigma}^{n,+} + v_{K,\sigma}^{n,-} = \frac{1}{k|\sigma|} \int_{nk}^{(n+1)k} \int_{\sigma} |\vec{v}(x, t) \cdot n_{K,\sigma}| d\gamma(x) dt.$$

Moreover, since $\operatorname{div}_x [\vec{v}(x, t)] = 0$ for any $(x, t) \in D \times [0, T]$, we had

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| (v_{K,\sigma}^{n,+} - v_{K,\sigma}^{n,-}) = 0. \quad (\text{II.5})$$

Remark II.8 (Other written of the scheme)

By denoting for any $\sigma \in \mathcal{E}$, $K \in \mathcal{T}$ and $n \in \{0, \dots, N-1\}$

$$F_{K,\sigma}^n(a, b) = |\sigma| \left\{ v_{K,\sigma}^{n,+} F(a, b) - v_{K,\sigma}^{n,-} F(b, a) \right\},$$

*by omitting for convenience the random dependence which appears from u_K^1 and comes from $W^1(\omega) - W^0(\omega)$.

as a consequence of (II.5) we obtained that:

$$\forall a \in \mathbb{R}, \forall K \in \mathcal{T}, \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^n(a, a) = 0,$$

which allowed us to rewrite the numerical scheme (II.4) in the following way :

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_K} \left\{ F_{K,\sigma}^n(u_K^n, u_{K,\sigma}^n) - F_{K,\sigma}^n(u_K^n, u_K^n) \right\} = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}. \quad (\text{II.6})$$

Remark II.9 (On the measurability of the approximate finite-volume solution)

Let me mention that using properties of the Brownian motion, for all K in $\mathcal{T}$ and all n in $\{0, \dots, N-1\}$, u_K^n was $\mathcal{F}_{nk}$ -measurable and so, as an elementary process adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, $u_{\mathcal{T},k}$ was predictable with values in $L^2(D)$.

II.3 Main result and overview of the proof

In [BCG17] we proved the following convergence result:

Theorem II.10 (Convergence of the scheme)

Assume that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_7$ hold. Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes in the sense of Definition I.9 such that the mesh size h_m tends to 0 when m tends to $+\infty$. Let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$ and set for any $m \in \mathbb{N}$, $k_m = \frac{T}{N_m}$ the time step. For a fixed $m \in \mathbb{N}$, let $u_{\mathcal{T}_m, k_m}$ be the finite-volume approximation defined by the monotone finite-volume scheme (II.2), (II.3) and (II.4). Then $(u_{\mathcal{T}_m, k_m})_{m \in \mathbb{N}}$ converges to the unique stochastic entropy solution of (II.1) in the sense of Definition II.3, strongly in $L^p(\Omega \times D \times (0, T))$ for any $1 \leq p < 2$ and weakly in $L^2(\Omega \times D \times (0, T))$, as m tends to $+\infty$ and $\frac{k_m}{h_m}$ tends to 0.

The plan of the proof to show such a convergence result was the same as the one exposed in Chapter I. Firstly, under a classical deterministic CFL condition, we derived a $L^\infty(0, T; L^2(\Omega \times D))$ stability estimate satisfied by the finite-volume approximation $(u_{\mathcal{T},k})_{\mathcal{T},k}$. Secondly, under a more restrictive CFL condition, we obtained the famous crucial Weak BV estimates for the set of discrete unknowns $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$. Thirdly, the introduction of a time-continuous interpolation $\bar{u}_{\mathcal{T},k}$ allowed us to get continuous entropy estimates on $u_{\mathcal{T},k}$. Note that as in Chapter I, we took again advantage of the convex decomposition of any monotone numerical flux F by Godunov and modified Lax-Friedrichs numerical fluxes to obtain such entropy estimates. Fourthly, using compactness arguments provided by Young-measure theory, we passed to the limit in the continuous entropy estimates satisfied by the finite-volume approximation $u_{\mathcal{T},k}$. This allowed us to prove its convergence towards a measure-valued entropy solution of Problem (II.1) in the sense of Definition II.4 at the cost of the assumption $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$. To conclude the proof of Theorem II.10, we invoked the reduction result stated in Theorem II.5. Let me review these main steps of the proof by starting with the stability estimates on the sequence of finite-volume approximate solutions $(u_{\mathcal{T},k})_{\mathcal{T},k}$:

Proposition II.11 ($L^\infty(0, T; L^2(\Omega \times D))$ estimate for $(u_{\mathcal{T},k})_{\mathcal{T},k}$)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the Courant-Friedrichs-Levy (CFL) condition

$$k \leq \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)}. \quad (\text{II.7})$$

Let $u_{\mathcal{T},k}$ be the finite-volume approximate solution defined by (II.2), (II.3) and (II.4). Then we have the following bound

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times D))} \leq e^{L_g^2 T} \left(\|u_0\|_{L^2(D)}^2 + 2T|D|g^2(0) + V(F_1 + F_2) \|u^b\|_{L^2((0,T) \times \partial D)}^2 \right)^{1/2}.$$

where F_1 and F_2 are the Lip-dia coefficients of the numerical flux F .

Note that as a consequence of this stability result, we obtained the $L^2(\Omega \times D \times (0, T))$ boundedness of $(u_{\mathcal{T},k})_{\mathcal{T},k}$. The idea of the proof was the same as the one of Proposition I.19 in Chapter I. Namely exploiting the properties of the increments of the Brownian motion and the monotony of the numerical fluxes F in combinaison with the technical Lemma I.18 on monotone functions. Let me mention that we had double work in bounded domain since for the terms containing the numerical fluxes, we separated the contributions of boundary's edges from those of the interior ones using the following result from [Vov02]:

Lemma II.12 (Gathering by edges)

Let $n \in \mathbb{N}$ and $\rho^n : \mathcal{T} \times \mathcal{E} \rightarrow \mathbb{R}$ be a mapping such that[†] $\rho_{K,\sigma_{K,L}}^n = -\rho_{L,\sigma_{K,L}}^n$ if $u_K^n = u_L^n$. Then we have

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \rho_{K,\sigma}^n = \sum_{\sigma \in \mathcal{E}^b} \rho_{K,\sigma}^n + \sum_{(K,L) \in \mathfrak{T}_n} (\rho_{K,\sigma_{K,L}}^n + \rho_{L,\sigma_{K,L}}^n), \quad (\text{II.8})$$

where $\mathfrak{T}_n = \{(K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}$.

The use of Lemma II.12 and a technical extension of the Weak BV estimates in $\mathbb{R}^d$ (presented in Chapter I) to our bounded domain D led us to:

Proposition II.13 (Weak BV estimates)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the CFL condition

$$k \leq (1 - \xi) \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)}, \quad (\text{II.9})$$

for some $\xi \in (0, 1)$.

Let $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$ be given by (II.3) and (II.4). Then the following hold:

1. There exists $\bar{C}_{BV,1} \in \mathbb{R}_+^*$, only depending on $T, |D|, u_0, u^b, \xi, F_1, F_2, L_g$ and $g(0)$ such that

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| E \left[v_{K,\sigma}^{n,+} \{F(u_K^n, u_{K,\sigma}^n) - f(u_K^n)\}^2 + v_{K,\sigma}^{n,-} \{F(u_{K,\sigma}^n, u_K^n) - f(u_K^n)\}^2 \right] \leq \bar{C}_{BV,1}.$$

2. There exists $\bar{C}_{BV,2} \in \mathbb{R}_+^*$, only depending on $T, |D|, \bar{\alpha}, u_0, u^b, \xi, F_1, F_2, L_g$ and $g(0)$ such that

$$\begin{aligned} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathfrak{T}_n} |\sigma_{K,L}| E \left[v_{K,L}^{n,+} \left\{ \max_{u_L^n \leq c \leq d \leq u_K^n} (F(d, c) - f(d)) + \max_{u_L^n \leq c \leq d \leq u_K^n} (F(d, c) - f(c)) \right\} \right. \\ \left. + v_{K,L}^{n,-} \left\{ \max_{u_L^n \leq c \leq d \leq u_K^n} (f(d) - F(c, d)) + \max_{u_L^n \leq c \leq d \leq u_K^n} (f(c) - F(c, d)) \right\} \right] \leq \bar{C}_{BV,2} h^{-1/2}, \end{aligned}$$

where

$$\mathfrak{T}_n = \{(K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}.$$

Arguing as in Chapter I, we arrived at the following first convergence result for the finite-volume approximation $u_{\mathcal{T},k}$:

Proposition II.14 (Convergence of $(u_{\mathcal{T},k})_{\mathcal{T},k}$)

Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes in the sense of Definition I.9 such that the mesh size h_m tends to 0 when m tends to $+\infty$. Let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$ and set for any $m \in \mathbb{N}$, $k_m = \frac{T}{N_m}$. For a fixed $m \in \mathbb{N}$, let $u_{\mathcal{T}_m, k_m}$ be the finite-volume approximate solution defined by (II.2), (II.3) and (II.4). Assume that for any $m \in \mathbb{N}$, $(\mathcal{T}_m, k_m)$ satisfies the CFL Condition (II.7). Then, (up to a subsequence denoted in the same way) $(u_{\mathcal{T}_m, k_m})_{m \in \mathbb{N}}$ converges in the sense of Young measures to an “entropy process” denoted by μ in $L^2(\Omega \times D \times (0, T) \times (0, 1))$.

[†]For any $(K, \sigma) \in \mathcal{T} \times \mathcal{E}$, we denote by convenience $\rho^n(K, \sigma) = \rho_{K,\sigma}^n$.

Remark II.15 (Additional regularity and measurability of the limit μ)

Let me mention that we were able to prove as specified in Chapter I (Remarks I.25 and I.26) that μ was an element of $\mathcal{N}_W^2(0, T; L^2(D \times (0, 1)))$ and that it also belonged to $L^\infty(0, T; L^2(\Omega \times D \times (0, 1)))$.

For the same technical reasons as explained in Chapter I, we introduced in an intermediate way, a time-continuous finite-volume approximation $\bar{u}_{\mathcal{T},k}$ defined on $\Omega \times D \times [0, T)$ by

$$\bar{u}_{\mathcal{T},k}(\omega, x, t) = \bar{u}_K^n(\omega, t), \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k) \quad (\text{II.10})$$

where for any $K \in \mathcal{T}$ and any $n \in \{0, \dots, N-1\}$, $\bar{u}_K^n : \Omega \times [nk, (n+1)k] \rightarrow \mathbb{R}$ was defined by

$$\begin{aligned} \bar{u}_K^n(s) &= u_K^n - \frac{s - nk}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| \left\{ v_{K,\sigma}^{n,+}(F(u_K^n, u_{K,\sigma}^n) - f(u_K^n)) - v_{K,\sigma}^{n,-}(F(u_{K,\sigma}^n, u_K^n) - f(u_K^n)) \right\} \\ &\quad + \int_{nk}^s g(u_K^n) dW(t). \end{aligned} \quad (\text{II.11})$$

The study of the difference between $\bar{u}_{\mathcal{T},k}$ and $u_{\mathcal{T},k}$ in the $L^2(\Omega \times D \times (0, T))$ -norm led us to the following estimate, thanks to the Weak BV estimates stated in Proposition II.13:

Proposition II.16

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the CFL Condition (II.9). Let $\bar{u}_{\mathcal{T},k}$ be the time-continuous approximate solution defined by (II.10)-(II.11), and $u_{\mathcal{T},k}$ be the finite-volume approximation defined by (II.2), (II.3) and (II.4). Then there exists $C_{TA} \in \mathbb{R}_+^*$ depending only on $T, |D|, V, L_g, g(0), F_1, F_2, \bar{\alpha}, u_0$ and u^b such that*

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 \leq C_{TA}(h + k).$$

We now had all the necessary tools to derive discrete entropy estimates satisfied by the discrete unknowns:

Proposition II.17 (Discrete entropy estimates)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^$, let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step and assume that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then for any entropy $\eta_\kappa \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and $\mathbb{P}$ -a.s. in Ω :

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta_\kappa(u_K^{n+1}) - \eta_\kappa(u_K^n)) \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K \Phi^{\eta_\kappa}(u_K^n) \vec{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K \eta'_\kappa(u_K^n) g(u_K^n) \varphi(x, nk) dx dW(t) \\ & + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K \eta''_\kappa(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \\ & + L_f V \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}^b} \int_{nk}^{(n+1)k} \int_\sigma \eta_\kappa(u^b(x, t)) \varphi(x, nk) d\gamma(x) dt \\ & \geq \mathcal{R}^{h,k}, \end{aligned} \quad (\text{II.12})$$

where for any $\mathbb{P}$ -measurable set A , $\mathbb{E}[\mathbb{1}_A \mathcal{R}^{h,k}] \rightarrow 0$ as $(h, \frac{k}{h}) \rightarrow (0, 0)$.

As explained in Chapter I, the obtention of discrete entropy estimates such as (II.12) was not an easy task since we worked with smooth entropies $\eta_\kappa \in \mathcal{A}$. To overcome this difficulty, we exploited the decomposition property of any monotone

numerical flux F stated in Lemma I.15. Such a property allowed us to restrict our study to Godunov and modified Lax-Friedrichs numerical fluxes. Arguing as in Chapter I and owing to the “Gathering by edges” Lemma II.12, we were able to prove that (II.12) held. Let me precise that the fact that we were in a bounded domain added new terms to manage for which the explicit knowledge of the numerical entropy fluxes denoted $G_F^{\eta_\kappa}$ and associated with an entropy-flux pair $(\eta_\kappa, \Phi^{\eta_\kappa})$ was also crucial. Indeed, at some point we were led to prove the following inequality $\mathbb{P}$ almost surely in Ω :

$$v_{K,\sigma}^{n,+} G_F^{\eta_\kappa}(u_K^n, u_\sigma^{b,n}) - v_{K,\sigma}^{n,-} G_F^{\eta_\kappa}(u_\sigma^{b,n}, u_K^n) + L_f V \eta_\kappa(u_\sigma^{b,n}) \geq 0,$$

and without the explicit decomposition $G_F^{\eta_\kappa}(a, b) = \theta(a, b) G_G^{\eta_\kappa}(a, b) + (1 - \theta) G_{LF}^{\eta_\kappa}(a, b)$, with $G_G^{\eta_\kappa}$ and $G_{LF}^{\eta_\kappa}$ being respectively Godunov and modified Lax-Friedrichs numerical entropy fluxes associated with an entropy-flux pair $(\eta_\kappa, \Phi^{\eta_\kappa})$, then we were in a dead end. Finally, note that the condition on $\frac{k}{h}$ was still required to prove the convergence of $\mathbb{E}[\mathbb{1}_A \mathcal{R}^{h,k}]$ towards 0, for any $\mathbb{P}$ -measurable set A .

Owing to the regularity of the test function φ , Proposition II.17 led us naturally to the obtention of the following continuous entropy estimates satisfied by the finite-volume approximate solution $u_{\mathcal{T},k}$:

Proposition II.18 (Continuous entropy estimates on the finite-volume approximation)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$, let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step and assume that

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then, for any entropy $\eta_\kappa \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and for almost all ω in Ω :

$$\begin{aligned} & \int_D \eta_\kappa(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_D \eta_\kappa(u_{\mathcal{T},k}(\omega, x, t)) \varphi_t(x, t) dx dt \\ & + \int_0^T \int_D \Phi^{\eta_\kappa}(u_{\mathcal{T},k}(\omega, x, t)) \vec{v}(x, t) \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_D \eta'_\kappa(u_{\mathcal{T},k}(\omega, x, t)) g(u_{\mathcal{T},k}(\omega, x, t)) \varphi(x, t) dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_D \eta''_\kappa(u_{\mathcal{T},k}(\omega, x, t)) g^2(u_{\mathcal{T},k}(\omega, x, t)) \varphi(x, t) dx dt \\ & + L_f V \int_0^T \int_{\partial D} \varphi(x, t) \eta_\kappa(u^b(x, t)) d\gamma(x) dt \\ & \geq \tilde{\mathcal{R}}^{h,k}(\omega), \end{aligned} \tag{II.13}$$

where for any $\mathbb{P}$ -measurable set A , $\mathbb{E}[\mathbb{1}_A \tilde{\mathcal{R}}^{h,k}] \rightarrow 0$ as $(h, k/h) \rightarrow (0, 0)$.

Passing to the limit in (II.13) thanks to Young-measures, we concluded that the limit μ provided by Proposition II.14 was a measure-valued entropy solution in the sense of Definition II.4. Thanks to Theorem II.5, μ was hence the unique stochastic entropy solution in the sense of Definition II.3, did not depend on the additional variable $\beta \in (0, 1)$ and therefore we denoted it by u . Hence, all the sequence of approximate solution $(u_{\mathcal{T},k})_{\mathcal{T},k}$ converged to u strongly in $L^1(\Omega \times D \times (0, T))$. In addition, since $(u_{\mathcal{T},k})_{\mathcal{T},k}$ was bounded in $L^2(\Omega \times D \times (0, T))$, we obtained that the convergence finally held strongly in $L^p(\Omega \times D \times (0, T))$ for any $1 \leq p < 2$ ([Dro01, Corollaire 1.3.3]).

Remark II.19

A posteriori, I would say that taking into account the boundary conditions in the numerical scheme didn't pose any particular technical difficulty. In fact, I think that the main problems linked to the numerical analysis were solved with the analysis of the finite-volume scheme in $\mathbb{R}^d$, but we had to be sure of this when carrying out this study. On the other hand, the essential information we needed to conclude that our scheme was converging, was to be sure of having a result (in the continuous case) for reducing the measure-valued entropy solution in the sense of Definition II.4 to an entropy solution in the sense of Definition II.3 (as stated in Theorem II.5).

Chapter III

Existence and uniqueness result for stochastic conservation laws using a finite-volume scheme

In this chapter is presented my last contribution on the numerical approximation of stochastic hyperbolic problems, which is published in a joint paper with V. CASTEL and J. CHARRIER [BCC20]. The objective of this work was threefold. Firstly, to propose the study of multi-dimensional nonlinear scalar conservation laws forced by a multiplicative stochastic noise with a more general time and space dependent flux-function than the ones considered in the literature. The underlying motivation was to take up this technical challenge in the context of V. CASTEL's PhD thesis. Secondly, to propose an original approach to get both the existence and the uniqueness of the stochastic entropy solution of our problem by using the finite-volume approximation, instead of the one coming from the viscous parabolic regularization (I.17). Thirdly to address an important preliminary work to the establishment of strong error estimates.

III.1 Introduction

In [BCC20], we were interested in the Cauchy problem for the following multi-dimensional nonlinear stochastic problem:

$$\begin{cases} du + \operatorname{div}_x [\vec{f}(\cdot, \cdot, u)] dt &= g(u) dW & \text{in } \Omega \times \mathbb{R}^d \times (0, T) \\ u(\omega, x, 0) &= u_0(x), & \omega \in \Omega, x \in \mathbb{R}^d, \end{cases} \quad (\text{III.1})$$

where d was a positive integer, $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$, div_x denoted the divergence operator with respect to the space variable (which belonged to $\mathbb{R}^d$) and $(W(t))_{t \geq 0}$ was a standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

By denoting for any $x_0 \in \mathbb{R}^d$, $|x_0|$ its euclidian norm and for any $R > 0$, $\mathcal{B}(x_0, R)$ the ball of $\mathbb{R}^d$ centered at x_0 of radius R , we assumed the following hypotheses on the data:

$$\mathcal{H}_1: u_0 \in L^2(\mathbb{R}^d).$$

$$\mathcal{H}_2: g : \mathbb{R} \rightarrow \mathbb{R} \text{ is a Lipschitz-continuous and bounded function such that } g(0) = 0.$$

$$\mathcal{H}_3: \vec{f} \in \mathcal{C}^1(\mathbb{R}^d \times [0, T] \times \mathbb{R}, \mathbb{R}^d) \text{ and there exists a constant } C_{\vec{f}} > 0 \text{ such that for any } (x, t, v) \in \mathbb{R}^d \times [0, T] \times \mathbb{R},$$

$$\left| \frac{\partial \vec{f}}{\partial v}(x, t, v) \right| \leq C_{\vec{f}}.$$

$\mathcal{H}_4$: For all $x \in \mathbb{R}^d$, there exist two reals $C_{\vec{f}}^x > 0$ and $R^x > 0$ such that for all $(t, v) \in [0, T] \times \mathbb{R}$ and for all $y \in \mathcal{B}(x, R^x)$,

$$|\vec{f}(x, t, v) - \vec{f}(y, t, v)| \leq C_{\vec{f}}^x |x - y|.$$

$\mathcal{H}_5$: There exists a real $C_{\vec{f}}^T > 0$ such that for all $(x, v) \in \mathbb{R}^d \times \mathbb{R}$ and for all $(t, t') \in [0, T]^2$,

$$|\vec{f}(x, t, v) - \vec{f}(x, t', v)| \leq C_{\vec{f}}^T |t - t'|.$$

$\mathcal{H}_6$: For all $(x, t, v) \in \mathbb{R}^d \times [0, T] \times \mathbb{R}$, $\operatorname{div}_x [\vec{f}(x, t, v)] = \sum_{i=1}^d \frac{\partial f_i}{\partial x_i}(x_1, \dots, x_d, t, v) = 0$.

$\mathcal{H}_7$: For all $x \in \mathbb{R}^d$, there exist two reals $\tilde{C}_{\vec{f}}^x > 0$ and $\tilde{R}^x > 0$ such that for all $(t, v) \in [0, T] \times \mathbb{R}$ and for all $y \in \mathcal{B}(x, \tilde{R}^x)$,

$$\left| \frac{\partial \vec{f}}{\partial v}(x, t, v) - \frac{\partial \vec{f}}{\partial v}(y, t, v) \right| \leq \tilde{C}_{\vec{f}}^x |x - y|.$$

Remark III.1 (On these assumptions)

- Note that, as it was classically done for hyperbolic scalar conservation laws, for convenience we were able to assume that for all (x, t) in $\mathbb{R}^d \times [0, T]$, $\vec{f}(x, t, 0) = \vec{0}$ without loss of generality thanks to $\mathcal{H}_6$.
- The existence and uniqueness result stated in this chapter might hold without assuming the boundedness of g (for example by adapting [BVW12] and [BCG16b]), but we used this assumption to get the convergence of the numerical approximation.
- As mentioned in the previous chapter, the results of [BCC20] could be extended to the “divergence free” case i.e. when $\operatorname{div}_x [\vec{f}(x, t, v)] \neq 0$, at the cost of new technical difficulties, following for example [CH00] in the deterministic case.
- By assuming additionally that g had compact support and $u_0 \in L^\infty(\mathbb{R}^d)$, it was possible to prove that the unique stochastic entropy solution of Problem (III.1) was also bounded. Thus, all the results of [BCC20] were still valid by assuming only local bounds with respect to the third variable v in Assumptions $\mathcal{H}_3$, $\mathcal{H}_4$ and $\mathcal{H}_5$ (i.e. if the constants $C_{\vec{f}}$, $C_{\vec{f}}^x$ and $C_{\vec{f}}^T$ depended on v). In particular, it enabled us to treat a larger class of flux functions such as Burgers’ one.

III.1.1 Aim of the study

The joint work [BCC20] presented in this chapter extended the results obtained in [BCG16b] and detailed in Chapter I, to a more general flux function $(x, t, v) \mapsto \vec{f}(x, t, v)$. More importantly, it provided a new proof of uniqueness which did not rely on the use of viscous parabolic approximations at all. Indeed, let me recall that in [BCG16b], we generalized the existence and uniqueness result proposed in [BVW12] through the study of a viscous parabolic approximation denoted u_ϵ , unique weak solution of Problem (I.17). I would like to emphasize on the fact that having this theoretical background was essential for us to show the convergence of our finite-volume approximation towards the unique stochastic entropy solution of the associated problem.

In [BCC20], both existence and uniqueness of a solution were established by using only the numerical approximation. To do so, we proposed a new way to adapt Kruzhkov’s doubling variables technique to the stochastic framework. The idea was to compare two measure-valued entropy solutions, knowing that one of the two was the weak limit of our finite-volume approximation. I propose to expose in this chapter the main features of this comparaison technique. Note that the obtained estimates were (and still are) expected to constitute a crucial step towards the establishment of error estimates (as in the deterministic case, see [EGH00] for instance).

III.1.2 Notations and concept of solution

I define in this section some notations, make precise the functional setting and give an adaptation of the definitions of stochastic entropy and measure-valued entropy solutions (introduced in Chapter I) to our more general time and space dependent flux function $\vec{f}$.

Notations III.2

- For any $R > 0$, $\mathcal{B}(0, R)$ denotes the ball of $\mathbb{R}^d$ centered at 0 of radius R .
- For $p = 1, d$ or $d + 1$, $\|\cdot\|_\infty$ denotes the $L^\infty(\mathbb{R}^p)$ norm.
- $\mathbb{E}[\cdot]$ denotes the expectation, i.e. the integral over Ω with respect to the probability measure $\mathbb{P}$.
- $\mathcal{D}^+(\mathbb{R}^d \times [0, T])$ denotes the subset of non-negative elements of $\mathcal{D}(\mathbb{R}^d \times [0, T])$.
- $\mathcal{P}_T$ denotes the predictable σ -field on $(0, T) \times \Omega$, i.e. the σ -field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $(s, t] \times B$ for any $B \in \mathcal{F}_s$, $s, t \in [0, T]$, $s < t$.
- For a given separable Hilbert space X , we denote by $\mathcal{N}_W^2(0, T; X)$ the separable Hilbert space of the predictable X -valued processes. It is the space $L^2((0, T) \times \Omega; X)$ for the product measure $dt \otimes d\mathbb{P}$ on $\mathcal{P}_T$ (see [DPZ92] p.94).
- $\mathcal{A}$ denotes the set of any $\mathcal{C}^3(\mathbb{R})$ convex functions η (called smooth entropies in what follows) such that η' , η'' and η''' are bounded functions.
- For any smooth entropy $\eta \in \mathcal{A}$ we introduce the function $\vec{\Phi}^\eta : \mathbb{R}^d \times [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^d$ defined by

$$\vec{\Phi}^\eta(x, t, a, b) = \int_b^a \eta'(\sigma - b) \frac{\partial \vec{f}}{\partial v}(x, t, \sigma) d\sigma,$$

for all $(x, t, a, b) \in \mathbb{R}^d \times [0, T] \times \mathbb{R}^2$.

- For any $\kappa \in \mathbb{R}$, $|\cdot - \kappa| : \mathbb{R} \rightarrow \mathbb{R}$ denotes a Kruzhkov's entropy.
- For any $\kappa \in \mathbb{R}$, $\vec{\Phi}(\cdot, \cdot, \cdot, \kappa) : \mathbb{R}^d \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^d$ denotes the entropy flux associated with Kruzhkov's entropy $|\cdot - \kappa|$. It is defined for any $(x, t, a) \in \mathbb{R}^d \times [0, T] \times \mathbb{R}$ by

$$\vec{\Phi}(x, t, a, \kappa) = \vec{f}(x, t, a \top \kappa) - \vec{f}(x, t, a \perp \kappa) = \text{sgn}(a - \kappa) [\vec{f}(x, t, a) - \vec{f}(x, t, \kappa)],$$

where $a \top \kappa$ denotes the maximum of a and κ and $a \perp \kappa$ the minimum of a and κ .

- L_g denotes the Lipschitz constant of g .
- Assumptions $\mathcal{H}_4$ and $\mathcal{H}_7$ give us for any $R > 0$ the existence of two constants $C_{\vec{f}}^R > 0$ and $\tilde{C}_{\vec{f}}^R > 0$ such that for all $(t, v) \in [0, T] \times \mathbb{R}$ and for all $(x, y) \in \mathcal{B}(0, R)^2$,

$$|\vec{f}(x, t, v) - \vec{f}(y, t, v)| \leq C_{\vec{f}}^R |x - y| \text{ and } \left| \frac{\partial \vec{f}}{\partial v}(x, t, v) - \frac{\partial \vec{f}}{\partial v}(y, t, v) \right| \leq \tilde{C}_{\vec{f}}^R |x - y|.$$

Using these notations, we were interested in the following concepts of solutions:

Definition III.3 (Stochastic entropy solution)

A function u of $\mathcal{N}_W^2(0, T; L^2(\mathbb{R}^d)) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))$ is an entropy solution of Problem (III.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$, if it satisfies the following entropy inequalities

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \eta(u(\omega, x, t)) \partial_t \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \vec{\Phi}^\eta(x, t, u(\omega, x, t), 0) \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \eta'(u(\omega, x, t)) g(u(\omega, x, t)) \varphi(x, t) dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} g^2(u(\omega, x, t)) \eta''(u(\omega, x, t)) \varphi(x, t) dx dt, \end{aligned}$$

for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and for almost all $\omega \in \Omega$.

Definition III.4 (Measure-valued entropy solution)

A function μ of $\mathcal{N}_W^2(0, T; L^2(\mathbb{R}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d \times (0, 1)))$ is a measure-valued entropy solution of Problem (III.1) with the initial condition $u_0 \in L^2(\mathbb{R}^d)$, if it satisfies the following inequalities

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta(\mu(\omega, x, t, \beta)) \partial_t \varphi(x, t) d\beta dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \vec{\Phi}^\eta(x, t, \mu(\omega, x, t, \beta), 0) \cdot \nabla_x \varphi(x, t) d\beta dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'(\mu(\omega, x, t, \beta)) g(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dW(t) \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \int_0^1 g^2(\mu(\omega, x, t, \beta)) \eta''(\mu(\omega, x, t, \beta)) \varphi(x, t) d\beta dx dt, \end{aligned}$$

for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$ and for almost all $\omega \in \Omega$.

III.2 Finite-volume framework

In the sequel, assume that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_7$ hold, consider an admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ in the sense of Definition I.9 and $N \in \mathbb{N}^*$ defining a time step $k = \frac{T}{N} \in \mathbb{R}_+^*$. As an extension of Definition I.13 which was written in the particular case where $\vec{f}(x, t, v) = \vec{v}(x, t) f(v)$, I define here a family of monotone numerical fluxes associated with our general flux function $\vec{f}$.

Definition III.5 (Family of monotone numerical fluxes associated with a flux function $\vec{f}$)

A family of functions $(F_{K,L}^n)_{n \in \mathbb{N}, K \in \mathcal{T}, L \in \mathcal{N}(K)}$ from $\mathbb{R}^2$ to $\mathbb{R}$ is said to be a family of monotone numerical fluxes associated with a flux function $\vec{f}$, if there exist $F_1 > 0$ and $F_2 > 0$ such that for any $n \in \mathbb{N}$, any $K \in \mathcal{T}$ and any $L \in \mathcal{N}(K)$, $F_{K,L}^n$ satisfies the following properties:

- . **Monotonicity** : For any $(a, b) \in \mathbb{R}^2$, $v \mapsto F_{K,L}^n(v, b)$ is non-decreasing and $v \mapsto F_{K,L}^n(a, v)$ is non-increasing.
- . **Lip-diaq regularity** : For any $(a, b) \in \mathbb{R}^2$, the following inequalities hold
$$|F_{K,L}^n(b, a) - F_{K,L}^n(a, a)| \leq F_1 |b - a| \text{ and } |F_{K,L}^n(a, b) - F_{K,L}^n(a, a)| \leq F_2 |b - a|.$$
- . **Conservativity** : For any $(a, b) \in \mathbb{R}^2$, $F_{K,L}^n(a, b) = -F_{L,K}^n(b, a)$.

. **Consistency** : For any $v \in \mathbb{R}$, $F_{K,L}^n(v, v) = f_{K,L}^n(v)$,

where by denoting γ the $(d-1)$ -dimensional Lebesgue measure, $f_{K,L}^n : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f_{K,L}^n(v) = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \vec{f}(x, t, v) \cdot n_{K,L} d\gamma(x) dt, \forall v \in \mathbb{R}.$$

As in the works exposed in Chapter I, we approximated the stochastic entropy solution u of Problem (III.1) by a function $u_{\mathcal{T},k} : \Omega \times \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ which was constant on each space-time grid $K \times [nk, (n+1)k)$, $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$:

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n(\omega) \text{ for } \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k). \quad (\text{III.2})$$

The set $\{u_K^0, K \in \mathcal{T}\}$ was given by the mean value of the initial condition on each control volume:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \forall K \in \mathcal{T}. \quad (\text{III.3})$$

Once a family of numerical fluxes $(F_{K,L}^n)_{n \in \mathbb{N}, K \in \mathcal{T}, L \in \mathcal{N}(K)}$ in the sense of Definition III.5 was chosen, for any $n \in \{0, \dots, N-1\}$, knowing $\{u_K^n, K \in \mathcal{T}\}$, the discrete unknowns $\{u_K^{n+1}, K \in \mathcal{T}\}$ were computed thanks to the following equation, $\mathbb{P}$ -a.s. in Ω

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k}, \quad (\text{III.4})$$

where $W^n = W(nk)$ for $n \in \{0, \dots, N\}$.

Remark III.6 (Consequence of the divergence free property of $\vec{f}$)

Note that Assumption $\mathcal{H}_6$ on f implied that: $\forall s \in \mathbb{R}, \forall n \in \{0, \dots, N-1\}, \forall K \in \mathcal{T}, \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(s, s) = 0$.

III.3 Main result and overview of the proof

In [BCC20], our main (two-in-one) result was the obtention of the existence and uniqueness of the stochastic entropy solution of Problem (III.1) by using exclusively our finite-volume approximate solution $u_{\mathcal{T},k}$. Namely, we proved a theoretical result for the continuous problem thanks to our numerical scheme:

Theorem III.7 (Existence, uniqueness of the entropy solution and convergence of the scheme)

Assume that Assumptions $\mathcal{H}_1$ to $\mathcal{H}_7$ hold. Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes in the sense of Definition I.9 such that the mesh size h_m tends to 0 when m tends to $+\infty$. Let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$ and set for any $m \in \mathbb{N}$, $k_m = \frac{T}{N_m}$ the time step. For a fixed $m \in \mathbb{N}$, let $u_{\mathcal{T}_m, k_m}$ be the finite-volume approximation defined by the finite-volume scheme (III.2)-(III.3)-(III.4). Then Problem (III.1) admits a unique stochastic entropy solution u in the sense of Definition III.3, and the sequence of finite-volume approximation $(u_{\mathcal{T}_m, k_m})_{m \in \mathbb{N}}$ converges to this solution, strongly in $L^p(\Omega \times \mathcal{K} \times (0, T))$ for any $1 \leq p < 2$ and any compact set $\mathcal{K}$ of $\mathbb{R}^d$, and weakly in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$, as m tends to $+\infty$ and $\frac{k_m}{h_m}$ tends to 0.

The proof of this main result was decomposed in two steps:

- In the first one, we obtained the convergence of the finite-volume approximate solution $(u_{\mathcal{T},k})_{\mathcal{T},k}$ towards a measure-valued entropy solution (denoted by μ) of Problem (III.1) in the sense of Definition III.4. Note that this first step was a technical generalization of the arguments exposed in Chapter I to the case of a general time-space dependent flux function f : derivation of stability estimates on the sequence $(u_{\mathcal{T},k})_{\mathcal{T},k}$, obtention of Weak BV estimates satisfied by the set of discrete unknowns $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$, introduction of a time-continuous finite-volume approximation $\bar{u}_{\mathcal{T},k}$, obtention of continuous entropy estimates satisfied by $u_{\mathcal{T},k}$ by using a convex decomposition of our monotone numerical flux $F_{K,L}^n$ by a Godunov's part and a modified Lax-Friedrichs' one, and then passage to the limit in these estimates thanks to Young measures' theory.

- In the second step, we proved a reduction result on the obtained measure-valued entropy solution μ . Precisely, we showed its uniqueness by adapting Kruzhkov's doubling variables technique to the stochastic case, taking inspiration from ideas developed in the continuous case with G. VALLET and P. WITTBOLD in [BVW12]. In such a study, the idea was to compare any measure-valued entropy solution satisfying Definition III.4 to the viscous parabolic approximation u_ϵ , unique weak solution of (I.17). In [BCC20], we then revisited this proof by comparing any measure-valued entropy solution ν satisfying Definition III.4 to the limit μ of our sequence of finite-volume approximate solution $(u_{\mathcal{T},k})_{\mathcal{T},k}$. This comparison principle led us to prove that $\mu = \nu$, and then standards arguments allowed us to conclude that μ did not depend on the parameter $\beta \in (0, 1)$ and that it was hence the unique stochastic entropy solution of Problem (III.1) in the sense of Definition III.3.

Let us review in what follows these two steps.

III.3.1 Step 1 : convergence of $(u_{\mathcal{T},k})_{\mathcal{T},k}$ towards a measure-valued entropy solution

The results and proofs of this section followed closely the work [BCG16b] presented in Chapter I and were obtained by adapting some technics from [CH00] (in the deterministic case) to handle the general time-space dependence of the flux function $\vec{f}$. Let me mention that in order to prepare Kruzhkov's doubling variables technique, contrary to [BCG16b], we needed in [BCC20] in the discrete and continuous entropy estimates to know explicitly the dependence of the constants appearing in the majorations of terms involving the smooth entropy $\eta \in \mathcal{A}$ and the test function φ .

Proposition III.8 ($L^\infty(0, T; L^2(\Omega \times \mathbb{R}^d))$ estimate for $(u_{\mathcal{T},k})_{\mathcal{T},k}$)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the Courant-Friedrichs-Levy (CFL) condition:

$$k \leq \frac{\bar{\alpha}^2 h}{F_1 + F_2}. \quad (\text{III.5})$$

Let $u_{\mathcal{T},k}$ be the finite-volume approximate solution defined by (III.2)-(III.3)-(III.4). Then, we have the following bound:

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{R}^d))} \leq e^{L_g^2 T/2} \|u_0\|_{L^2(\mathbb{R}^d)}.$$

Proposition III.9 (Weak BV estimates)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ satisfying the CFL condition:

$$k \leq (1 - \xi) \frac{\bar{\alpha}^2 h}{F_1 + F_2}, \text{ for some } \xi \in (0, 1). \quad (\text{III.6})$$

Let $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$ be given by (III.3) and (III.4). Then the following hold:

1. There exists $C_{BV,1} \in \mathbb{R}_+^*$, only depending on T, u_0, ξ, F_1, F_2 and L_g such that:

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[\left(F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right)^2 \right] \leq C_{BV,1}.$$

2. Let $R > 0$ be such that $h < R$. Then, there exists $C_{BV,2} \in \mathbb{R}_+^*$, only depending on $R, d, T, \bar{\alpha}, u_0, \xi, F_1, F_2$ and L_g such that:

$$\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_R^n} |\sigma_{K,L}| \mathbb{E} \left[\max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(d, d) \right) + \max_{u_L^n \leq c \leq d \leq u_K^n} \left(F_{K,L}^n(d, c) - F_{K,L}^n(c, c) \right) \right] \leq C_{BV,2} h^{-1/2},$$

where

$$\mathcal{I}_R^n = \left\{ (K, L) \in \mathcal{T}_R^2 \text{ such that } L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n \right\} \text{ and } \mathcal{T}_R = \left\{ K \in \mathcal{T} \text{ such that } K \subset \mathcal{B}(0, R) \right\}.$$

3. Let $R > 0$ be such that $h < R$. Then there exists $C_{BV,3} \in \mathbb{R}_+^*$, only depending on $R, d, T, \bar{\alpha}, u_0, \xi, F_1, F_2$ and L_g such that:

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} |K| \times \mathbb{E} \left[|u_K^{n+1} - u_K^n| \right] \leq C_{BV,2} \times h^{-1/2} + C_{BV,3} \times k^{-1/2}.$$

Remark III.10

Comparing with the previous chapters, I have additionally stated here a “time”-Weak BV estimate (point 3.) which was useful to control exactly the dependence with respect to k and h of terms involving the difference “ $\eta(u_K^{n+1}) - \eta(u_K^n)$ ” for Kruzhkov’s doubling variables technique and thus crucial for the preparation of error estimates.

For the same technical reasons as explained in Chapter I, we introduced in an intermediate way, a time-continuous finite-volume approximation $\bar{u}_{\mathcal{T},k}$ defined on $\Omega \times \mathbb{R}^d \times [0, T)$ by

$$\bar{u}_{\mathcal{T},k}(\omega, x, t) = \bar{u}_K^n(\omega, t), \omega \in \Omega, x \in K \text{ and } t \in [nk, (n+1)k], \quad (\text{III.7})$$

where for any $K \in \mathcal{T}$ and any $n \in \{0, \dots, N-1\}$, $\bar{u}_K^n : \Omega \times [nk, (n+1)k] \rightarrow \mathbb{R}$ was defined by (using Remark III.6)

$$\bar{u}_K^n(s) = u_K^n - \frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ F_{K,L}^n(u_K^n, u_L^n) - F_{K,L}^n(u_K^n, u_K^n) \right\} + g(u_K^n)(W(s) - W^{nk}). \quad (\text{III.8})$$

Remark III.11 (On the role of $\bar{u}_{\mathcal{T},k}$ for the uniqueness proof)

Note that $\bar{u}_{\mathcal{T},k}$ also played a crucial role in the adaptation of Kruzhkov’s doubling variables technique presented in the next subsection. Indeed, as explained above, we revisited in [BCC20] ideas developed in [BVW12] and so $u_{\mathcal{T},k}$ played the role of the viscous parabolic approximation u_ϵ in [BVW12]. At some point, Itô’s derivation formula was employed on u_ϵ and for this reason $\bar{u}_{\mathcal{T},k}$ entered again in scene to compensate the lack of regularity in time of $u_{\mathcal{T},k}$.

Thanks to Weak BV estimates, we arrived at the following control of the error between the approximations $\bar{u}_{\mathcal{T},k}$ and $u_{\mathcal{T},k}$, with a sharper estimate than the one given in Chapter I (Proposition I.27) since the bound was equal, up to a constant, to $h + k$:

Proposition III.12

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step satisfying the CFL Condition (III.6). Let $\bar{u}_{\mathcal{T},k}$ be the time-continuous finite-volume approximate solution defined by (III.7) and (III.8), and $u_{\mathcal{T},k}$ be the finite-volume approximate solution defined by (III.2)-(III.3)-(III.4). Then, there exists a constant $C_{TA} \in \mathbb{R}_+^*$ depending only on $T, L_g, F_1, F_2, \bar{\alpha}, u_0$ and ξ such that:

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 \leq C_{TA} k.$$

I do not detail again how we obtained the following discrete and continuous entropy estimates since it was a rather technical generalization of Chapter I. On the other hand, I would like to emphasize the care we took to make explicit the control of the rests with respect to the parameters k, h , the entropy $\eta \in \mathcal{A}$ and the test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T))$:

Proposition III.13 (Discrete entropy estimates)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step satisfying the CFL Condition (III.6). Then, there exists a constant C_{DE} depending only on $R, d, \bar{\alpha}, u_0, \xi, F_1, F_2, C_{\bar{f}}^R, C_{\bar{f}}^T, \|g\|_\infty$ and L_g such that for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and $\mathbb{P}$ -a.s in Ω :

$$\begin{aligned}
& - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \frac{\eta(u_K^{n+1}) - \eta(u_K^n)}{k} \varphi(x, nk) dx dt \\
& + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \vec{\Phi}^\eta(x, t, u_K^n, 0) \cdot \nabla_x \varphi(x, nk) dx dt \\
& + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dx dW(t) \\
& + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_R} \int_{nk}^{(n+1)k} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \\
& \geq \tilde{\mathcal{R}}^{h,k,\eta,\varphi},
\end{aligned} \tag{III.9}$$

where for any $\mathbb{P}$ -measurable set A :

$$\begin{aligned}
& \left| \mathbb{E} \left[\mathbf{1}_A \tilde{\mathcal{R}}^{h,k,\eta,\varphi} \right] \right| \\
& \leq C_{DE} \left\{ \|\eta''\|_\infty \|\varphi\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty h^{1/2} + (\|\partial_t \varphi\|_\infty \|\eta'\|_\infty + \|\varphi\|_\infty \|\eta''\|_\infty) \times k^{1/2} + \|\varphi\|_\infty \|\eta'''\|_\infty k \right\}.
\end{aligned}$$

Proposition III.14 (Continuous entropy estimates on the finite-volume approximation)

Let $\mathcal{T}$ be an admissible mesh in the sense of Definition I.9, $N \in \mathbb{N}^*$, $T \in \mathbb{R}_+^*$ and let $k = \frac{T}{N} \in \mathbb{R}_+^*$ be the time step such that the CFL Condition (III.6) holds for some $\xi \in (0, 1)$. Then, there exists a constant C_{CE} , which depends only on $R, d, \bar{\alpha}, u_0, \xi, F_1, F_2, C_{\bar{f}}^R, C_{\bar{f}}^T, \|g\|_\infty$ and L_g , such that for any entropy $\eta \in \mathcal{A}$, any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$ and for almost all ω in Ω :

$$\begin{aligned}
& \int_{\mathbb{R}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^d} \eta(u_{\mathcal{T},k}(\omega, x, t)) \partial_t \varphi(x, t) dx dt \\
& + \int_0^T \int_{\mathbb{R}^d} \vec{\Phi}^\eta(x, t, u_{\mathcal{T},k}(\omega, x, t), 0) \cdot \nabla_x \varphi(x, t) dx dt \\
& + \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k}(\omega, x, t)) g(u_{\mathcal{T},k}(\omega, x, t)) \varphi(x, t) dx dW(t) \\
& + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \eta''(u_{\mathcal{T},k}(\omega, x, t)) g^2(u_{\mathcal{T},k}(\omega, x, t)) \varphi(x, t) dx dt \\
& \geq \mathcal{R}^{h,k,\eta,\varphi}(\omega),
\end{aligned} \tag{III.10}$$

where for any $\mathbb{P}$ -measurable set A :

$$\begin{aligned}
& \left| \mathbb{E} \left[\mathbf{1}_A \mathcal{R}^{h,k,\eta,\varphi} \right] \right| \leq C_{CE} \left\{ \|\eta''\|_\infty \|\varphi\|_\infty \frac{k^{1/2}}{h^{1/2}} + \|\eta'\|_\infty \|\nabla_x \varphi\|_\infty h^{1/2} + (\|\partial_t \varphi\|_\infty \|\eta'\|_\infty + \|\varphi\|_\infty \|\eta''\|_\infty) \times k^{1/2} \right. \\
& \quad \left. + (\|\eta'''\|_\infty \|\varphi\|_\infty + \|\eta'\|_\infty \|\partial_t \nabla_x \varphi\|_\infty + \|\eta''\|_\infty \|\partial_t \varphi\|_\infty) \times k + \int_{\mathcal{B}(0,R)} |u_0(x) - u_{\mathcal{T},0}(x)| dx \right\},
\end{aligned}$$

with $u_{\mathcal{T},0} : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $u_{\mathcal{T},0}(x) = u_K^0$ if $x \in K$.

Proposition III.8 allowed us to pass to the limit in (III.10) and we arrived at the following convergence result for $u_{\mathcal{T},k}$:

Proposition III.15 (Convergence of $(u_{\mathcal{T},k})_{\mathcal{T},k}$)

Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes in the sense of Definition I.9 such that the mesh size h_m tends to 0 when m tends to $+\infty$. Let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$ and set for any $m \in \mathbb{N}$, $k_m = \frac{T}{N_m}$ the time step. For a fixed $m \in \mathbb{N}$, let $u_{\mathcal{T}_m, k_m}$ be the finite-volume approximate solution defined by (III.2)-(III.3) and (III.4). Then, (up to a subsequence denoted in the same way) $(u_{\mathcal{T}_m, k_m})_{m \in \mathbb{N}}$ converges in the sense of Young measures to a measure-valued entropy solution μ of Problem (III.1) in the sense of Definition III.4 as m tends to $+\infty$ and $\frac{k_m}{h_m}$ goes to 0.

III.3.2 Step 2 : reduction result

Our objective in this second step was to prove the uniqueness of the measure-valued entropy solution μ obtained by Proposition III.15. The idea to get such a uniqueness result was to adapt the method of doubling variables introduced by S.N. KRUSHKOV in the deterministic case to the stochastic one. Let me recall briefly the outline of this deterministic technique, which is based on the use of Kruzhkov's entropy-flux pairs $(|\cdot - \kappa|, \vec{\Phi}(\cdot, \cdot, \cdot, \kappa))$, $\kappa \in \mathbb{R}$. Using these pairs, the comparison of two generic measure-valued entropy solutions $\nu, \tilde{\nu}$ by the so called "Kruzhkov's doubling variables technique" leads to the following Local Kato's inequality:

$$\begin{aligned} 0 \leq & \int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \tilde{\nu}(x, t, \beta)| \partial_t \varphi(x, t) d\alpha d\beta dx dt \\ & + \int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 \vec{\Phi}(x, t, \nu(x, t, \alpha), \tilde{\nu}(x, t, \beta)) \cdot \nabla_x \varphi(x, t) d\alpha d\beta dx dt, \end{aligned} \quad (\text{III.11})$$

for any test function $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times [0, T])$. By exploiting the finite propagation speed property for conservation laws with Lipschitz-continuous flux function, it is classical to prove that* (for an arbitrary $R > 0$)

$$\int_0^T \int_{\mathcal{B}(0, R + C_F T)} \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \tilde{\nu}(x, t, \beta)| d\alpha d\beta dx dt = 0.$$

Then, since R is arbitrary large, one concludes that on the one hand $\nu(x, t, \alpha) = \tilde{\nu}(x, t, \beta)$ for almost all (x, t, α, β) in $\mathbb{R}^d \times (0, T) \times (0, 1)^2$, which proves the uniqueness of the measure-valued entropy solution ν . On the other hand, it is possible to repeat argumentations of Kruzhkov's doubling variables by comparing ν with itself, and this leads for almost all $(x, t, \alpha) \in \mathbb{R}^d \times (0, T) \times (0, 1)$ to

$$\nu(x, t, \alpha) = \tilde{\nu}(x, t, \beta), \text{ for almost all } \beta \in (0, 1).$$

Thus defining u from $\mathbb{R}^d \times (0, T)$ to $\mathbb{R}$ by $u(x, t) = \int_0^1 \nu(x, t, \beta) d\beta$, one obtains $u(x, t) = \nu(x, t, \alpha)$ for almost all (x, t, α) in $\mathbb{R}^d \times (0, T) \times (0, 1)$, which proves the independence of ν with respect to its third variable in $(0, 1)$ and at last existence and uniqueness of the entropy solution.

Unfortunately, as explained in Chapter I, the use of Kruzhkov's entropy was out of range in the stochastic case and additionally the comparison of two generic measure-valued entropy solutions did not seem to be successful (see [BVW12]). For this last reason, such a uniqueness result was obtained in [BVW12] and [BCG16b] by comparing the viscous parabolic solution u_ϵ of (I.17) to a measure-valued entropy solution ν of Problem (III.1) satisfying Definition III.4. In [BCC20], we revisited this proof by replacing in the comparison technique u_ϵ by our finite-volume approximate solution $u_{\mathcal{T},k}$.

To do so, we firstly introduced a family of smooth entropies approaching Kruzhkov's entropies:

*The idea is to consider a particular choice of φ in link with the properties of the flux function $\vec{f}$, see [EGH00] for example.

Definition III.16 (Remarkable smooth entropy)

Set $\delta > 0$ and define the function $\eta_\delta: \mathbb{R} \rightarrow \mathbb{R}_+$ by $\eta_\delta(0) = 0$ and:

$$\eta'_\delta(a) = \begin{cases} 1 & \text{if } a > \delta \\ \sin\left(\frac{\pi}{2\delta}a\right) & \text{if } |a| \leq \delta \\ -1 & \text{if } a < -\delta. \end{cases}$$

Secondly, we introduced several mollifier sequences:

Definition III.17 (Triplet of mollifier sequences)

Let $\rho, \bar{\rho} \in \mathcal{D}(\mathbb{R})$ and $\tilde{\rho} \in \mathcal{D}(\mathbb{R}^d)$ be such that:

- $\text{Supp}(\rho) \subset [-1; 1]$, $\forall a \in \mathbb{R}, \rho(a) \geq 0$, $\rho(-a) = \rho(a)$ and $\int_{\mathbb{R}} \rho(a) da = 1$.
- $\text{Supp}(\bar{\rho}) \subset [-2; 0]$, $\forall t \in \mathbb{R}, \bar{\rho}(t) \geq 0$, $\bar{\rho}(-t-1) = \bar{\rho}(t-1)$ and $\int_{\mathbb{R}} \bar{\rho}(t) dt = 1$.
- $\text{Supp}(\tilde{\rho}) \subset \mathcal{B}(0, 1)$, $\forall x \in \mathbb{R}^d, \tilde{\rho}(x) \geq 0$, $\tilde{\rho}(-x) = \tilde{\rho}(x)$ and $\int_{\mathbb{R}^d} \tilde{\rho}(x) dx = 1$.

Set $l, p, q \in \mathbb{N}^*$. We then define a triplet $(\rho_l, \bar{\rho}_p, \tilde{\rho}_q)$ of mollifiers sequences $\rho_l, \bar{\rho}_p: \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{\rho}_q: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\rho_l(a) = l\rho(la), \bar{\rho}_p(t) = p\rho(pt) \text{ and } \tilde{\rho}_q(x) = q^d \tilde{\rho}(qx), \text{ for any } a \in \mathbb{R}, t \in \mathbb{R}, x \in \mathbb{R}^d.$$

Setting $\kappa \in \mathbb{R}$, $\delta \in \mathbb{R}_+^*$, $l, p, q \in \mathbb{N}^*$, η_δ a remarkable smooth entropy in the sense of Definition III.16, $(\rho_l, \bar{\rho}_p, \tilde{\rho}_q)$ a triplet of mollifier sequences in the sense of Definition III.17 and $\varphi \in \mathcal{D}^+(\mathbb{R}^d \times (0, T))$ satisfying for some $R > 0$ $\text{Supp}(\varphi) \subset \mathcal{B}(0, R) \times [0, T)$, we then defined, for any fixed $(y, s) \in \mathbb{R}^d \times (0, T)$, the test function $\tilde{\Psi}_{y,s}: \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}$ by

$$\tilde{\Psi}_{y,s}(x, t) = \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y), \forall (x, t) \in \mathbb{R}^d \times (0, T).$$

In the same manner, for fixed $(x, t) \in \mathbb{R}^d \times (0, T)$ we introduced the test function $\Psi_{x,t}: \mathbb{R}^d \times (0, T) \rightarrow \mathbb{R}$ defined by

$$\Psi_{x,t}(y, s) = \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y), \forall (y, s) \in \mathbb{R}^d \times (0, T).$$

Considering, ν a measure-valued entropy solution in the sense of Definition III.4, fixing $(y, s) \in \mathbb{R}^d \times (0, T)$ and writing entropy inequalities satisfied by ν with the entropy $\eta_\delta(\cdot - \kappa)$ and the test function $\tilde{\Psi}_{y,s}$, then after multiplying the obtained inequality by $\rho_l(u_{\mathcal{T},k}(y, s) - \kappa)$, taking the expectation and integrating with respect to (κ, y, s) on $\mathbb{R} \times \mathbb{R}^d \times (0, T)$ we arrived at[†]

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \geq 0 \tag{III.12}$$

[†]Note that to get this inequality, we used Fubini's theorem and the fact that $\vec{\Phi}^{\eta_\delta}(\cdot - \kappa)(\cdot, \cdot, \cdot, 0)$ was equal, up to a constant, to $\vec{\Phi}^{\eta_\delta}(\cdot, \cdot, \cdot, \kappa)$.

where by denoting $Q = \mathbb{R}^d \times (0, T)$ and omitting the random variable:

$$\begin{aligned}
A_1 &= \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, t, \alpha) - \kappa) \partial_t \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx dt d\kappa dy ds \right], \\
A_2 &= \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\nu(x, t, \alpha) - \kappa) \varphi(x, t) \bar{\rho}'_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx dt d\kappa dy ds \right], \\
A_3 &= \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \tilde{\Phi}^{\eta_\delta}(x, t, \nu(x, t, \alpha), \kappa) \cdot \nabla_x \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx dt d\kappa dy ds \right], \\
A_4 &= \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \tilde{\Phi}^{\eta_\delta}(x, t, \nu(x, t, \alpha), \kappa) \cdot \nabla \tilde{\rho}_q(x-y) \varphi(x, t) \bar{\rho}_p(t-s) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx dt d\kappa dy ds \right], \\
A_5 &= \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta'_\delta(\nu(x, t, \alpha) - \kappa) g(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) d\alpha dx dW(t) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\kappa dy ds \right], \\
A_6 &= \frac{1}{2} \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta''_\delta(\nu(x, t, \alpha) - \kappa) g^2(\nu(x, t, \alpha)) \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx dt d\kappa dy ds \right], \\
A_7 &= \mathbb{E} \left[\int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(u_0(x) - \kappa) \varphi(x, 0) \bar{\rho}_p(-s) \tilde{\rho}_q(x-y) \rho_l(u_{\mathcal{T},k}(y, s) - \kappa) d\alpha dx d\kappa dy ds \right].
\end{aligned}$$

After that, by fixing $(x, t, \alpha) \in \mathbb{R}^d \times (0, T) \times (0, 1)$ and considering the test function $\Psi_{x,t}$ in the continuous entropy estimates (III.10) satisfied by $u_{\mathcal{T},k}$, multiplying it by $\rho_l(\nu(x, t, \alpha) - \kappa)$, taking the expectation and then integrating it with respect to (κ, x, t, α) on $\mathbb{R} \times \mathbb{R}^d \times (0, T) \times (0, 1)$ we obtained:

$$B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 \geq \mathcal{R}, \quad (\text{III.13})$$

where:

$$\begin{aligned}
B_1 &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \partial_s \varphi(x, t) \times \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_2 &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, t) \times -\bar{\rho}'_p(t-s) \times \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_3 &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \tilde{\Phi}^{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), \kappa) \cdot \nabla_y \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_4 &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \varphi(x, t) \bar{\rho}_p(t-s) \tilde{\Phi}^{\eta_\delta}(y, s, u_{\mathcal{T},k}(y, s), \kappa) \cdot \left(-\nabla \bar{\rho}_m(x-y) \right) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_5 &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \eta'_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, t) g(u_{\mathcal{T},k}(y, s)) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) dy dW(s) \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa dx dt d\alpha \right], \\
B_6 &= \frac{1}{2} \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_Q \eta''_\delta(u_{\mathcal{T},k}(y, s) - \kappa) \varphi(x, t) g^2(u_{\mathcal{T},k}(y, s)) \bar{\rho}_p(t-s) \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy ds d\kappa dx dt d\alpha \right], \\
B_7 &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \int_{\mathbb{R}^d} \eta_\delta(u_0(y) - \kappa) \varphi(x, t) \bar{\rho}_p(t) \tilde{\rho}_q(x-y) \rho_l(\nu(x, t, \alpha) - \kappa) dy d\kappa dx dt d\alpha \right], \\
\mathcal{R} &= \mathbb{E} \left[\int_0^1 \int_Q \int_{\mathbb{R}} \mathcal{R}^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}} \rho_l(\nu(x, t, \alpha) - \kappa) d\kappa d\alpha dx dt \right].
\end{aligned}$$

Firstly, since the test function φ depended only on the variables x and t , we noticed that $B_1 = B_3 = 0$, and due to the disjoint supports of φ and $\bar{\rho}_p$ we also had $B_7 = 0$. Secondly, since the term $\mathcal{R}^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}}$ was given by Proposition III.14, we then had a majoration for any $(x, t) \in \mathbb{R}^d \times (0, T)$ and for any $\mathbb{P}$ -measurable set A of $\left| \mathbb{E} \left[\mathbf{1}_A \mathcal{R}^{h,k,\eta_\delta(\cdot-\kappa), \Psi_{x,t}} \right] \right|$. Using the explicit expressions of our mollifier sequences, it was possible to improve it and to show the existence of a constant $\mathcal{C} > 0$ independent of the parameters h, k, δ, p, q, l such that

$$|\mathcal{R}| \leq \mathcal{C} \left(\frac{pq^d}{\delta} \sqrt{\frac{k}{h}} + q^{d+1} \sqrt{h} + p^2 q^d \sqrt{k} + \frac{pq^d \sqrt{k}}{\delta} + \frac{pq^d k}{\delta^2} + p^2 k q^d \left(q + \frac{1}{\delta} \right) + \int_{\mathcal{B}(0,R)} |u_{\mathcal{T},0}(z) - u_0(z)| dz \right). \quad (\text{III.14})$$

The addition of Inequalities (III.12) and (III.13) combined with the previous remarks, led us to the following

$$A_1 + (A_2 + B_2) + A_3 + (A_4 + B_4) + (A_5 + B_5) + (A_6 + B_6) + A_7 \geq \mathcal{R}, \quad (\text{III.15})$$

and owing to the even property of η_δ and ρ_l , Fubini's theorem allowed us to affirm that $A_2 + B_2 = 0$. Then, our aim was to pass to the limit in (III.15) as h, k, δ go to 0 and p, q, l tend to $+\infty$ in order to recover a stochastic version of the "Local Kato inequality" (III.11). As mentioned in the study of the continuous case [BVW12] the order of the passages to the limit was crucial. Let me recall that in [BVW12], we only had to handle the parameters δ, p, q, l and ϵ , where $\epsilon > 0$ was the parameter of the viscous parabolic approximation u_ϵ solution of (I.17), which was converging towards μ . In that case, we arrived at the following functional order for the passages to the limit:

1. $p \rightarrow +\infty$ (Convolution in time).
2. $l \rightarrow +\infty$ (Trick to overcome measurability restriction of the stochastic integral).
3. $\delta \rightarrow 0$ (Approximation of Kruzhkov's entropy).
4. $\epsilon \rightarrow 0$ (Approximation of μ).
5. $q \rightarrow +\infty$ (Convolution in space).

Note that one of the key points in [BVW12] was to study the sum of the stochastic terms corresponding to $A_5 + B_5 + A_6 + B_6$ as an indissociable block, what we mimicked in [BCC20]. Since the role of u_ϵ was played by $u_{\mathcal{T},k}$ in [BCC20], the parameter ϵ was then "replaced" by the time step k and the mesh size h , which could not be chosen independently. Moreover we realized that they could no longer be chosen independently of the parameter of convolution in time p . Finally, still in comparison with [BVW12], note that we had additionally to handle in [BCC20] the rest $\mathcal{R}$, which was fortunately precisely bounded by (III.14). By taking into account all these constraints, we proposed in [BCC20] the following choice: passing firstly to the limit simultaneously with respect to h, k, p , and then following the order suggested in [BVW12]: secondly l , thirdly δ and fourthly q .

In order to do that, we considered $(h_r)_{r \in \mathbb{N}}$ a sequence of mesh sizes such that $\lim_{r \rightarrow \infty} h_r = 0$. Then we defined a sequence of time steps $(k_r)_{r \in \mathbb{N}}$ by $k_r = (h_r)^\zeta$ and a sequence of parameters $(p_r)_{r \in \mathbb{N}}$ for the convolution in time by $p_r = (h_r)^{-\theta}$ with $\zeta, \theta \in \mathbb{N}^*$ to be chosen carefully. A first constraint on the integer ζ was to respect the sufficient condition $(h_r, k_r/h_r) \rightarrow (0, 0)$ as $r \rightarrow +\infty$ to prove the convergence of $\left| \mathbb{E} \left[\mathbb{1}_A \mathcal{R}^{h,k,\eta_\delta(\cdot-\kappa),\Psi_{x,t}} \right] \right|$ towards 0. Additionally, the evaluation of the rest $\mathcal{R}$ and the difference between each term $A_1, A_3, A_4 + B_4, A_5 + B_5 + A_6 + B_6$ and A_7 and their respective limits led us in particular to the simultaneous control of the following ratios involving p_r, h_r and k_r :

$$\frac{\sqrt{k_r}}{\sqrt{h_r}}, p_r \sqrt{k_r}, p_r k_r, \frac{p_r}{h_r^2} \left(k_r + \frac{2}{p_r} \right) \sqrt{k_r + \frac{2}{p_r}}, k_r \sqrt{p_r}, p_r \frac{\sqrt{k_r}}{\sqrt{h_r}}, p_r^2 \sqrt{k_r}, p_r \sqrt{k_r}, p_r^2 k_r.$$

In order to stabilize the most explosive terms $\frac{p_r}{h_r^2} \left(k_r + \frac{2}{p_r} \right) \sqrt{k_r + \frac{2}{p_r}}$ and $p^2 \sqrt{k}$ of the previous list, we arrived at the following choice : $\zeta = 21$ and $\theta = 5$. Then, at the cost of many technical calculations (written in details in [BCC20]), we proved successively that:

$$\begin{aligned} \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_1 &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \mu(x, t, \beta)| \partial_t \varphi(x, t) d\alpha d\beta dx dt \right], \\ \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_2 + B_2 &= 0, \\ \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_3 &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 \vec{\Phi}(x, t, \nu(x, t, \alpha), \mu(x, t, \beta)) \cdot \nabla_x \varphi(x, t) d\alpha d\beta dx dt \right], \\ \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_4 + B_4 &= 0, \\ \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_5 + B_5 + A_6 + B_6 &= 0, \\ \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} A_7 &= 0, \\ \lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} \mathcal{R} &= 0. \end{aligned}$$

For instance, let us focus on the term A_1 to give an idea of the technical nature of the proof. The study of such a term (and others too) was handled by decomposing it in the following manner:

$$A_1 = (A_1 - A_{1,1}) + (A_{1,1} - A_{1,2}) + (A_{1,2} - A_{1,3}) + (A_{1,3} - A_{1,4}) + (A_{1,4} - A_{1,5}) + A_{1,5},$$

where:

$$\begin{aligned} A_{1,1} &= E \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(\nu(x, t, \alpha) - \kappa) \partial_t \varphi(x, t) \tilde{\rho}_q(x - y) \rho_l(u_{\mathcal{T},k}(y, t) - \kappa) d\alpha dx dt d\kappa dy \right], \\ A_{1,2} &= E \left[\int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)) \partial_t \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right], \\ A_{1,3} &= E \left[\int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} \int_0^1 |\nu(x, t, \alpha) - u_{\mathcal{T},k}(y, t)| \partial_t \varphi(x, t) \tilde{\rho}_q(x - y) d\alpha dx dt dy \right], \\ A_{1,4} &= E \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 |\nu(x, t, \alpha) - u_{\mathcal{T},k}(x, t)| \partial_t \varphi(x, t) d\alpha dx dt \right], \\ \text{and } A_{1,5} &= E \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \mu(x, t, \beta)| \partial_t \varphi(x, t) d\beta d\alpha dx dt \right]. \end{aligned}$$

The successive controls of $|A_1 - A_{1,1}|$, $|A_{1,1} - A_{1,2}|$, $|A_{1,2} - A_{1,3}|$, $|A_{1,3} - A_{1,4}|$ and $|A_{1,4} - A_{1,5}|$ with respect to the parameters r, l, δ , and q (by applying in particular (I.15) to $A_{1,4}$) allowed us to show that

$$\lim_{q \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{l \rightarrow +\infty} \lim_{r \rightarrow +\infty} |A_1 - A_{1,5}| \leq 0,$$

and we obtained the desired result of convergence for A_1 . Finally, we arrived at the following stochastic version of the “Local Kato inequality” (III.11)

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 |\nu(x, t, \alpha) - \mu(x, t, \beta)| \partial_t \varphi(x, t) d\alpha d\beta dx dt \right] \\ &\quad + \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_0^1 \vec{\Phi}(x, t, \nu(x, t, \alpha), \mu(x, t, \beta)) \cdot \nabla_x \varphi(x, t) d\alpha d\beta dx dt \right], \end{aligned}$$

which was very similar to the deterministic version (III.11) since the only difference was that the expectation was applied. Finally, we concluded with the same reasoning as in the deterministic case, with the same particular choice for the test function φ .

PART 2

Stochastic parabolic problems and solid mechanic

Chapter IV

Damage problems in continuum media under stochastic dynamics

In this chapter is presented a published joint work in collaboration with E. BONETTI, G. BONFANTI, F. LEBON and G. VALLET [BBB⁺17]. We were interested in the analysis of a time noise-driven scalar Allen-Cahn equation modeling the evolution of damage in continuum media in the presence of stochastic dynamics. The nonlinear character of the equation was mainly due to a multivoque maximal monotone operator representing a constraint on the damage variable which was forced to take physically admissible values. By a “Moreau-Yosida” approximation and a time-discretization procedure, we proved a result of global-in-time existence and uniqueness of the solution to the stochastic problem.

IV.1 Introduction

We considered in [BBB⁺17] a real $T > 0$, a smooth bounded domain D of $\mathbb{R}^d$ with $d \geq 1$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$, and a standard, one-dimensional Brownian motion $(W(t))_{t \geq 0}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. For a given initial condition u_0 , by denoting $\mathbf{n}$ the outward unit normal vector to ∂D , we were interested in finding a pair (u, ξ) satisfying the following stochastic problem:

$$\begin{cases} du + (\xi - \Delta u) dt &= (w_s(u) + f) dt + h(u) dW \text{ and } \xi \in \partial I_{[0,1]}(u) & \text{in } \Omega \times D \times (0, T), & \text{(IV.1a)} \\ u(\omega, x, 0) &= u_0(x) & \omega \in \Omega, x \in D, & \text{(IV.1b)} \\ \nabla u \cdot \mathbf{n} &= 0 & \text{in } \Omega \times \partial D \times (0, T). & \text{(IV.1c)} \end{cases}$$

We assumed the following hypotheses on the data:

$$\mathcal{H}_1: u_0 \in H^1(D).$$

$$\mathcal{H}_2: 0 \leq u_0(x) \leq 1, \text{ for almost all } x \in D.$$

$$\mathcal{H}_3: h : \mathbb{R} \rightarrow \mathbb{R} \text{ is a Lipschitz-continuous function with a Lipschitz constant } L_h \text{ and satisfying } h(0) = h(1) = 0.$$

$$\mathcal{H}_4: w_s : \mathbb{R} \rightarrow [0, +\infty) \text{ is a Lipschitz-continuous function with a Lipschitz constant } L_{w_s} \text{ and satisfying } w_s(0) = 0.$$

$$\mathcal{H}_5: f \in \mathcal{N}_W^2(0, T; L^2(D))^{\dagger}.$$

[†]As mentioned in Part I, for a given separable Hilbert space X we denote by $\mathcal{N}_W^2(0, T; X)$ the separable Hilbert space of the predictable X -valued processes endowed with the norm $\|\phi\|_{\mathcal{N}_W^2(0, T; X)}^2 := \mathbb{E} \left[\int_0^T \|\phi\|_X^2 dt \right]$ (see [DPZ92] p.94).

Remark IV.1

The subdifferential $\partial I_{[0,1]}$ represented a physical constraint on u which was forced to take values in the interval $[0, 1]$. More precisely, $I_{[0,1]} : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ was defined by

$$I_{[0,1]}(v) = \begin{cases} 0 & \text{if } v \in [0, 1], \\ +\infty & \text{else.} \end{cases}$$

For any $v \in [0, 1]$, it resulted that (see e.g. [Br 73])

$$\partial I_{[0,1]}(v) = \begin{cases} \{0\} & \text{if } v \in]0, 1[, \\ \mathbb{R}^- & \text{if } v = 0, \\ \mathbb{R}^+ & \text{if } v = 1. \end{cases}$$

Using this, (IV.1a) could also be written in the following way:

$$w_s(u) + f - \partial_t \left(u - \int_0^\cdot h(u) dW \right) + \Delta u \in \partial I_{[0,1]}(u) \text{ in } \Omega \times D \times (0, T),$$

where the stochastic integral was understood in the sense of It  and Δ denoted the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary condition.

IV.1.1 Motivation and state of the art

Equation (IV.1a) is known in the literature as an Allen-Cahn type equation with constraint and can be used to describe several physical phenomena, like phase transitions. In [BBB⁺17], we dealt with the analysis of this equation, having in mind the evolution of damage in continuum media. Thus, we assumed that u represented a damage parameter, *i.e.* the local proportion of active cohesive bonds in the micro-structure of a material. In this direction, the function f on the right hand side of (IV.1a) stood for an external source of damage (mechanical or chemical). With this interpretation, we included in the model a constraint forcing u to take values in the interval $[0, 1]$, so that $u = 1$ meant that the material was completely undamaged, $u = 0$ that it was completely damaged while $u \in (0, 1)$ described an intermediate situation. The physical constraint was ensured by the presence of a sub-differential graph, *i.e.* a multivoque maximal monotone operator. In addition, we included in the equation a multiplicative time noise whose diffusion coefficient h depended on the damage parameter itself. From a physical point of view, the presence of a random process reflected the fact that the phenomenon of damage was related to microscopic changes in the structure and configuration of the material lattice as a consequence of breaking bonds and the formation of cavities and voids. These phenomena were clearly related to stochastic processes occurring at a microscopic level (as introduced in Ising materials), which we aimed to take into account in a macroscopic description.

When we looked at the study of Problem (IV.1), the literature on the deterministic Allen-Cahn equation was very rich, also including the presence of non-smooth (monotone) operators (see, among the others, [AC79, CMZ11, CGPGS13, DPKPW10, War96]), and the stochastic case was provided by a smaller quantity of surveys (see [AKM16, RNT12] and the references therein). Let me underline that, in these contributions the sub-differential operator was replaced by smooth nonlinearities (possibly with a prescribed growth condition), as double-well potentials. Furthermore, the study of stochastic partial differential inclusions in a rather general situation was carried out in [BR97a, BR97b, Ras96], as well as questions concerning transition semigroup and invariant measures in [BDP05].

IV.1.2 Derivation of the model

As an equation describing a phase transition process (as damage), (IV.1a) could be recovered as a balance law referring to the theory introduced in [Fr 02]. Such a theory relied on a generalized version of the principle of virtual powers where micro-forces and micro-motions responsible for the phase transition were included. Concerning this approach, we quoted the papers [BBR08, BBR07, BBR12, BS04, BSS05] where volume and surface damage models were deduced and analytically investigated. According to Fr mond's theory, let me detail how did we derive the System (IV.1). To this aim, we introduced as state variables of the model the damage parameter u and its gradient ∇u . Thus, the free energy functional read

$$\Psi(u, \nabla u) = \hat{w}_s(u) + \frac{1}{2} |\nabla u|^2 + I_{[0,1]}(u), \quad (\text{IV.2})$$

where the indicator function of the interval $[0, 1]$ restricted the domain of the free energy to the (physically) admissible values for u . Moreover, the function $\hat{w}_s$ was related to the internal cohesion of the material and was allowed to depend on the damage parameter itself: denoting by $w_s := -\hat{w}'_s$, we could suppose that w_s was a positive function, vanishing once $u = 0$ (the cohesion was null in the case of complete damage). Then, we introduced the dissipation of the system by a pseudo-potential of dissipation “à la Moreau”. As dissipative variable we considered the time derivative of $z = u - \int_0^t h(u) dW$. Thus, the resulting dissipation functional was

$$\Phi(z_t) = \frac{1}{2} |z_t|^2. \quad (\text{IV.3})$$

Note that, once $h = 0$, we recovered exactly the choice used for deterministic damage models (rate-dependent), like e.g. in [BBR08, BBR07, BSS05]. Neglecting acceleration effects, the generalized principle of virtual powers (in the case when no macroscopic displacements were considered) led us to the balance law

$$B - \operatorname{div} \mathbf{H} = f, \quad (\text{IV.4})$$

supplemented by a natural boundary condition $\mathbf{H} \cdot \mathbf{n} = 0$. B and $\mathbf{H}$ were microscopic forces and stresses acting in the material specified by the following constitutive laws

$$B = \frac{\partial \Psi}{\partial u} + \frac{\partial \Phi}{\partial z_t}, \quad \mathbf{H} = \frac{\partial \Psi}{\partial \nabla u}. \quad (\text{IV.5})$$

Hence, (IV.1a) and (IV.1c) easily followed combining (IV.2)-(IV.3) with (IV.4)-(IV.5).

IV.1.3 Concept of solution

The concept of solution we were interested in for Problem (IV.1) was the following one:

Definition IV.2

Any pair (u, ξ) with u belonging to $\mathcal{N}_W^2(0, T; H^1(D)) \cap L^\infty(0, T; L^2(\Omega; H^1(D))) \cap L^2(\Omega; \mathcal{C}(0, T; L^2(D)))$, $0 \leq u \leq 1$ and ξ in $\mathcal{N}_W^2(0, T; L^2(D))$ is a solution to Problem (IV.1) if for almost all t in $(0, T)$, for almost all ω in Ω and for any φ in $H^1(D)$

$$\begin{aligned} & \int_D \partial_t \left(u(\omega, x, t) - \int_0^t h(u(\omega, x, s)) dW(s) \right) \varphi(x) dx + \int_D \nabla u(\omega, x, t) \cdot \nabla \varphi(x) dx + \int_D \xi(\omega, x, t) \varphi(x) dx \\ &= \int_D (w_s(u(\omega, x, t)) + f(\omega, x, t)) \varphi(x) dx, \end{aligned}$$

with $\xi \in \partial I_{[0,1]}(u)$ and $\mathbb{P}$ -a.s. in Ω , $\lim_{t \rightarrow 0} u(\cdot, t) = u_0(\cdot)$ in $L^2(D)$.

IV.2 Main result and overview of the proof

The main result of [BBB⁺17] was the following one:

Theorem IV.3

Under hypotheses $\mathcal{H}_1$ - $\mathcal{H}_5$ there exists a unique pair (u, ξ) solution to Problem (IV.1) in the sense of Definition IV.2.

Let me present the main steps of the proof of Theorem IV.3. Firstly, we introduced a family of approximating problems involving a regularization of the subdifferential $\partial I_{[0,1]}$. More precisely, setting $\epsilon > 0$, we considered the following stochastic problem:

$$\begin{cases} du_\epsilon + (\psi_\epsilon(u_\epsilon) - \Delta u_\epsilon) dt &= (w_s(u_\epsilon) + f) dt + h(u_\epsilon) dW & \text{in } \Omega \times D \times (0, T), & (\text{IV.6a}) \\ u_\epsilon(\omega, x, 0) &= u_0(x) & \omega \in \Omega, x \in D, & (\text{IV.6b}) \\ \nabla u_\epsilon \cdot \mathbf{n} &= 0 & \text{in } \Omega \times \partial D \times (0, T), & (\text{IV.6c}) \end{cases}$$

where ψ_ϵ denoted the Moreau-Yosida approximation of $\partial I_{[0,1]}$ (see e.g. [Bar76, Bré73]), defined for all $v \in \mathbb{R}$ by

$$\psi_\epsilon(v) = -\frac{(v)^-}{\epsilon} + \frac{(v-1)^+}{\epsilon} = \begin{cases} \frac{v}{\epsilon} & \text{if } v \leq 0 \\ 0 & \text{if } v \in [0, 1] \\ \frac{v-1}{\epsilon} & \text{if } v \geq 1, \end{cases}$$

where $v^- = -\min(v, 0)$ and $v^+ = \max(v, 0)$. From classical results in the literature (based on a semi-implicit in time discretization scheme [DPZ92, PR07]), we proved for Problem (IV.6) existence and uniqueness of a solution u_ϵ in the following sense:

Definition IV.4

A stochastic process u_ϵ of $L^2(\Omega; \mathcal{C}(0, T; L^2(D))) \cap L^\infty(0, T; L^2(\Omega; H^1(D))) \cap \mathcal{N}_W^2(0, T; H^1(D))$ such that $\partial_t(u_\epsilon - \int_0^\cdot h(u_\epsilon) dW)$ and Δu_ϵ belong to $L^2(\Omega \times D \times (0, T))$, is a solution to the stochastic Problem (IV.6) if for almost all t in $(0, T)$ and for almost all ω in Ω , the following variational formulation holds for any $\varphi \in H^1(D)$

$$\begin{aligned} & \int_D \partial_t \left(u_\epsilon(\omega, x, t) - \int_0^t h(u_\epsilon(\omega, x, s)) dW(s) \right) \varphi(x) dx + \int_D \nabla u_\epsilon(\omega, x, t) \cdot \nabla \varphi(x) dx \\ & + \int_D \psi_\epsilon(u_\epsilon(\omega, x, t)) \varphi(x) dx = \int_D (w_s(u_\epsilon(\omega, x, t)) + f(\omega, x, t)) \varphi(x) dx, \end{aligned}$$

and $\mathbb{P}$ -a.s. in Ω , $\lim_{t \rightarrow 0} u_\epsilon(\cdot, t) = u_0(\cdot)$ in $L^2(D)$.

Secondly, we derived successively the following boundedness results independently of the parameter $\epsilon > 0$:

Proposition IV.5

- $(u_\epsilon)_{\epsilon > 0}$ is bounded in $L^\infty(0, T; L^2(\Omega \times D)) \cap \mathcal{N}_W^2(0, T; H^1(D))$.
- $(\psi_\epsilon(u_\epsilon))_{\epsilon > 0}$ is bounded in $\mathcal{N}_W^2(0, T; L^2(D))$.
- $(u_\epsilon)_{\epsilon > 0}$ is bounded in $L^\infty(0, T; L^2(\Omega; H^1(D)))$.
- $(\partial_t(u_\epsilon - \int_0^\cdot h(u_\epsilon) dW))_{\epsilon > 0}$ and $(\Delta u_\epsilon)_{\epsilon > 0}$ are bounded in $L^2(\Omega \times D \times (0, T))$.
- $\left(\frac{u_\epsilon^-}{\epsilon}\right)_{\epsilon > 0}$ and $\left(\frac{(u_\epsilon - 1)^+}{\epsilon}\right)_{\epsilon > 0}$ are bounded in $L^2(\Omega \times D \times (0, T))$.

These stability estimates led us thirdly to the existence of the following weak and strong limits:

Proposition IV.6

There exist $u \in \mathcal{N}_W^2(0, T; H^1(D)) \cap L^\infty(0, T; L^2(\Omega; H^1(D))) \cap L^2(\Omega; \mathcal{C}(0, T; L^2(D)))$, ξ, ξ_1, ξ_2 in $\mathcal{N}_W^2(0, T; L^2(D))$ and $\bar{w}_s, \bar{h}$ in $\mathcal{N}_W^2(0, T; H^1(D))$ such that up to subsequences denoted in the same way, the following convergences hold as $\epsilon \rightarrow 0$

- $u_\epsilon \rightarrow u$ weakly in $\mathcal{N}_W^2(0, T; H^1(D))$.
- $u_\epsilon \rightarrow u$ weakly- $\star$ in $L^\infty(0, T; L^2(\Omega; H^1(D)))$.
- $h(u_\epsilon) \rightarrow \bar{h}$ weakly in $L^2(\Omega \times (0, T); H^1(D))$.
- $u_\epsilon - \int_0^\cdot h(u_\epsilon) dW \rightarrow u - \int_0^\cdot \bar{h} dW$ weakly in $L^2(\Omega; H^1(D \times (0, T)))$.
- $w_s(u_\epsilon) \rightarrow \bar{w}_s$ weakly in $L^2(\Omega \times (0, T); H^1(D))$.
- $\psi_\epsilon(u_\epsilon) \rightarrow \xi$ weakly in $L^2(\Omega \times D \times (0, T))$.
- $-\frac{u_\epsilon^-}{\epsilon} \rightarrow \xi_1$ weakly in $L^2(\Omega \times D \times (0, T))$, with $\xi_1 \leq 0$.
- $\frac{(u_\epsilon - 1)^+}{\epsilon} \rightarrow \xi_2$ weakly in $L^2(\Omega \times D \times (0, T))$, with $\xi_2 \geq 0$.
- $u_\epsilon(T) \rightarrow u(T)$ weakly in $L^2(\Omega \times D)$.
- $u_\epsilon^- \rightarrow 0$ and $(u_\epsilon - 1)^+ \rightarrow 0$ strongly in $L^2(\Omega \times D \times (0, T))$.

Remark IV.7

Note that by definition of ψ_ϵ , $\psi_\epsilon(u_\epsilon) = -\frac{u_\epsilon^-}{\epsilon} + \frac{(u_\epsilon - 1)^+}{\epsilon}$, and so by uniqueness of weak limits $\xi = \xi_1 + \xi_2$. Additionally, since $\psi_\epsilon(u_\epsilon)u_\epsilon$ could be rewritten as

$$\psi_\epsilon(u_\epsilon)u_\epsilon = \frac{u_\epsilon^-}{\epsilon} \times u_\epsilon^- + \frac{(u_\epsilon - 1)^+}{\epsilon} \times ((u_\epsilon - 1)^+ + 1),$$

we obtained that

$$\mathbb{E} \left[\int_0^T \int_D \psi_\epsilon(u_\epsilon(\omega, x, t)) u_\epsilon(\omega, x, t) dx ds \right] \rightarrow \mathbb{E} \left[\int_0^T \int_D \xi_2(\omega, x, t) dx ds \right] \text{ as } \epsilon \rightarrow 0.$$

Using these results, by passing to the limit in the variational formulation satisfied by the sequence $(u_\epsilon)_{\epsilon>0}$, we proved for almost all t in $(0, T)$, for almost all ω in Ω and for any φ in $H^1(D)$ the following equality

$$\begin{aligned} & \int_D \partial_t \left(u(\omega, x, t) - \int_0^t \bar{h}(\omega, x, s) dW(s) \right) \varphi(x) dx + \int_D \nabla u(\omega, x, t) \cdot \nabla \varphi(x) dx + \int_D \xi(\omega, x, t) \varphi(x) dx \\ &= \int_D (\bar{w}_s(\omega, x, t) + f(\omega, x, t)) \varphi(x) dx. \end{aligned}$$

In particular, we deduced that u was a continuous $L^2(D)$ -valued predictable process satisfying for any t in $[0, T]$

$$u(., t) = u(0) + \int_0^t (\Delta u(., s) - \xi(., s) + \bar{w}_s(., s) + f(., s)) ds + \int_0^t \bar{h}(., s) dW(s),$$

in $L^2(D)$ and $\mathbb{P}$ -a.s. in Ω , where Δ denoted the Laplace operator on $H^1(D)$ associated with the formal Neumann boundary condition. To conclude the proof of the existence result stated in Theorem IV.3, it remained to identify the weak limits ξ , $\bar{w}_s$ and $\bar{h}$ coming from the nonlinear terms. To do so, we applied a classical technique in the study of parabolic SPDEs involving exponential weighted in time norms. As I will discuss an adaptation of this technique to the case of finite-volume approximations for a nonlinear parabolic problem in Chapter VII, I give here only the main steps of this method. Setting $\alpha > 0$ and, by applying Itô's formula to the process u_ϵ and the functional $\mathcal{F} : [0, T] \times L^2(D) \rightarrow \mathbb{R}$ defined for any $(t, v) \in [0, T] \times L^2(D)$ by $\mathcal{F}(t, v) = \frac{1}{2} e^{-\alpha t} \|v\|_{L^2(D)}^2$, we arrived at (after taking the expectation and by omitting in what follows the random variable)

$$\begin{aligned} & e^{-\alpha T} \frac{1}{2} \mathbb{E} [\|u_\epsilon(T)\|_{L^2(D)}^2] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |\nabla u_\epsilon(x, s)|^2 dx ds \right] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} u_\epsilon(x, s) \psi_\epsilon(u_\epsilon(x, s)) dx ds \right] \\ &+ \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left(\frac{\alpha}{2} |u_\epsilon(x, s)|^2 - u_\epsilon(x, s) w_s(u_\epsilon(x, s)) - \frac{1}{2} h^2(u_\epsilon(x, s)) \right) dx ds \right] \\ &= \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} f(x, s) u_\epsilon(x, s) dx ds \right] + \frac{1}{2} \|u_0\|_{L^2(D)}^2. \end{aligned} \tag{IV.7}$$

By passing to the superior limit with respect to ϵ in (IV.7), we obtained using Remark IV.7

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \left(e^{-\alpha T} \frac{1}{2} \mathbb{E} [\|u_\epsilon(T)\|_{L^2(D)}^2] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |\nabla u_\epsilon(x, s)|^2 dx ds \right] \right) + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \xi_2(x, s) dx ds \right] \\ &+ \liminf_{\epsilon \rightarrow 0} \left(\mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left(\frac{\alpha}{2} |u_\epsilon(x, s)|^2 - u_\epsilon(x, s) w_s(u_\epsilon(x, s)) - \frac{1}{2} h^2(u_\epsilon(x, s)) \right) dx ds \right] \right) \\ &\leq \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} f(x, s) u(x, s) dx ds \right] + \frac{1}{2} \|u_0\|_{L^2(D)}^2. \end{aligned} \tag{IV.8}$$

Using the Lipchitz nature of h and w_s , we proved that

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} \left(\mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left(\frac{\alpha}{2} |u_\epsilon(x, s)|^2 - u_\epsilon(x, s) w_s(u_\epsilon(x, s)) - \frac{1}{2} h^2(u_\epsilon(x, s)) \right) dx ds \right] \right) \\ & \geq \liminf_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |u_\epsilon(x, s) - u(x, s)|^2 \left(\frac{\alpha}{2} - L_{w_s} - \frac{L_h^2}{2} \right) dx ds \right] + \mathbb{E} \left[\int_0^T \int_D \frac{\alpha}{2} e^{-\alpha s} |u(x, s)|^2 dx ds \right] \\ & \quad + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left\{ \frac{1}{2} h^2(u(x, s)) - u(x, s) \bar{w}_s(x, s) - h(u(x, s)) \bar{h}(x, s) \right\} dx ds \right]. \end{aligned}$$

Then, by injecting this in (IV.8) and by choosing $\alpha > 0$ such that $\frac{\alpha}{2} - L_{w_s} - \frac{L_h^2}{2} \geq 0$, we arrived at

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \left(e^{-\alpha T} \frac{1}{2} \mathbb{E} [\|u_\epsilon(T)\|_{L^2(D)}^2] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |\nabla u_\epsilon(x, s)|^2 dx ds \right] \right) + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \xi_2(x, s) dx ds \right] \\ & \quad + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left\{ \frac{\alpha}{2} |u(x, s)|^2 + \frac{1}{2} h^2(u(x, s)) - u(x, s) \bar{w}_s(x, s) - h(u(x, s)) \bar{h}(x, s) \right\} dx ds \right] \\ & \leq \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} f(x, s) u(x, s) dx ds \right] + \frac{1}{2} \|u_0\|_{L^2(D)}^2. \end{aligned} \quad (\text{IV.9})$$

Besides, by applying Itô's formula to the process u and the previous functional $\mathcal{F}$, we got, after taking the expectation

$$\begin{aligned} & e^{-\alpha T} \frac{1}{2} \mathbb{E} [\|u(T)\|_{L^2(D)}^2] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |\nabla u(x, s)|^2 dx ds \right] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \xi(x, s) u(x, s) dx ds \right] \\ & \quad + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left\{ \frac{\alpha}{2} |u(x, s)|^2 - u(x, s) \bar{w}_s(x, s) - \frac{1}{2} \bar{h}^2(x, s) \right\} dx ds \right] \\ & = \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} f(x, s) u(x, s) dx ds \right] + \frac{1}{2} \|u_0\|_{L^2(D)}^2. \end{aligned}$$

And by injecting it in (IV.9) we obtained

$$\begin{aligned} & \limsup \left(e^{-\alpha T} \frac{1}{2} \mathbb{E} [\|u_\epsilon(T)\|_{L^2(D)}^2] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |\nabla u_\epsilon(x, s)|^2 dx ds \right] \right) \\ & \quad + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left\{ \xi_2(x, s) - \xi(x, s) u(x, s) + \frac{1}{2} (h(u(x, s)) - \bar{h}(x, s))^2 \right\} dx ds \right] \\ & \leq e^{-\alpha T} \frac{1}{2} \mathbb{E} [\|u(T)\|_{L^2(D)}^2] + \mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} |\nabla u(x, s)|^2 dx ds \right]. \end{aligned}$$

Thanks to the properties of the weak convergence and of the superior and inferior limit, we arrived at

$$\mathbb{E} \left[\int_0^T \int_D e^{-\alpha s} \left\{ \xi_2(x, s) - \xi(x, s) u(x, s) + \frac{1}{2} (h(u(x, s)) - \bar{h}(x, s))^2 \right\} dx ds \right] \leq 0.$$

Thanks to Proposition IV.6 and Remark IV.7, we had that almost everywhere in $\Omega \times D \times (0, T)$, $0 \leq u \leq 1$ and that $\xi = \xi_1 + \xi_2$ with $\xi_1 \leq 0$ and $\xi_2 \geq 0$. Then $\xi_2 - \xi u = (1 - u)\xi_2 - \xi_1 u \geq 0$ and a first consequence of the previous inequality was that $\bar{h} = h(u)$. A second consequence was that $\xi_2 - \xi u = 0$ and since:

- if $u = 0$ then $\xi_2 = 0$. So $\xi = \xi_1 \leq 0$ and in this way $\xi \in \mathbb{R}^-$,
- if $u = 1$ then $\xi_1 = 0$. So $\xi = \xi_2 \geq 0$ and in this way $\xi \in \mathbb{R}^+$,
- if $0 < u < 1$, then $\xi_1 = \xi_2 = 0$ and $\xi = 0$,

and finally $\xi \in \partial I_{[0,1]}(u)$. After that, we proved the strong convergence of $(u_\epsilon)_\epsilon$ towards u in $L^2((0, T) \times \Omega; H^1(D))$, and by using the Lipschitz property of w_s we deduced that $\bar{w}_s = w_s(u)$ and that the convergence of $(w_s(u_\epsilon))_{\epsilon > 0}$ towards $w_s(u)$ held strongly in $L^2(\Omega \times D \times (0, T))$. At last, the uniqueness of the pair of solution was derived in a very standard way by the use of Itô's derivation formula with the previous functional $\mathcal{F}$.

Chapter V

Phase change models: consideration of temperature and random effects

In this chapter are presented two contributions around Barenblatt's type equation with a stochastic force, written in collaboration with A.A. MAITLO during his PhD thesis, F. LEBON and A. ZIMMERMANN [BLM19, BLMZ21]. These studies represented intermediate and preliminary works in view to aboard applications in phase transition phenomena through the study of stochastic nonlinear evolution systems as proposed in the deterministic case in [BFL00], including for example irreversible phase changes such as solidification of glue, phenomenon of cooking an egg,... In [BLM19], we investigated the Neumann problem for a Barenblatt equation with a multiplicative stochastic force and a non linear source term. Through a time-discretization of the equation and thanks to results on maximal monotone operators, we were able to handle the triple non-linearity of the equation and prove the well-posedness of the problem. Then in [BLMZ21] we extended this theoretical result by coupling such a stochastic Barenblatt equation with the classical heat equation in view to take into account temperature effects. I expose in this chapter main ideas employed in [BLMZ21] to study from a mathematical point of view the coupling system. The content of [BLM19] will not be presented since the argumentation to handle the stochastic Barenblatt equation developed in such a paper was used identically in [BLMZ21].

V.1 Introduction

We considered in [BLMZ21] a real $T > 0$, a smooth bounded domain D of $\mathbb{R}^d$ with $d \geq 1$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$, and a standard, one-dimensional Brownian motion $(W(t))_{t \geq 0}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. For given initial conditions χ_0, ϑ_0 , by denoting $\mathbf{n}$ the outward unit normal vector to ∂D , we were interested in finding a pair (ϑ, χ) satisfying a system composed by a heat equation with a random source term (V.1a), which was coupled with a Barenblatt's one perturbed in the sense of Itô (V.1b). Firstly, we considered an additive stochastic force:

$$\left\{ \begin{array}{ll} \partial_t \vartheta - \Delta \vartheta + \partial_t \left(\chi - \int_0^\cdot h dW \right) = 0 & \text{in } \Omega \times D \times (0, T), \\ \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot h dW \right) \right) - \Delta \chi = \vartheta & \text{in } \Omega \times D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ and } \nabla \chi \cdot \mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0, T), \\ \vartheta(\omega, x, 0) = \vartheta_0(x) \quad \text{and} \quad \chi(\omega, x, 0) = \chi_0(x) & \omega \in \Omega, x \in D, \end{array} \right. \quad \begin{array}{l} \text{(V.1a)} \\ \text{(V.1b)} \\ \text{(V.1c)} \\ \text{(V.1d)} \end{array}$$

and secondly a multiplicative one:

$$\begin{cases} \partial_t \vartheta - \Delta \vartheta + \partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) dW \right) = 0 & \text{in } \Omega \times D \times (0, T), & (\text{V.2a}) \\ \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) dW \right) \right) - \Delta \chi = \vartheta & \text{in } \Omega \times D \times (0, T), & (\text{V.2b}) \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ and } \nabla \chi \cdot \mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0, T), & (\text{V.2c}) \\ \vartheta(\omega, x, 0) = \vartheta_0(x) \text{ and } \chi(\omega, x, 0) = \chi_0(x) & \omega \in \Omega, x \in D. & (\text{V.2d}) \end{cases}$$

We assumed the following hypotheses on the data:

$$(\mathcal{A}_1) \quad h \in \mathcal{N}_W^2(0, T; H^1(D))^\dagger.$$

$$(\mathcal{A}_2) \quad \tilde{\alpha} = I_d + \alpha, \text{ where } I_d : \mathbb{R} \rightarrow \mathbb{R} \text{ is the identity function and } \alpha : \mathbb{R} \rightarrow \mathbb{R} \text{ is a Lipschitz continuous, coercive and non-decreasing function such that } \alpha(0) = 0.$$

$$(\mathcal{A}_3) \quad \vartheta_0, \chi_0 \in H^1(D).$$

$$(\mathcal{A}_4) \quad \mathcal{H} : H^1(D) \rightarrow H^1(D) \text{ is a Lipschitz continuous mapping.}$$

V.1.1 Notations and concepts of solution

Throughout this chapter, the following notations will be used:

Notations V.1

- $\mathbb{E}[\cdot]$ denotes the expectation, i.e., the integral over Ω with respect to the probability measure $\mathbb{P}$.
- $L_\alpha > 0$ denotes the Lipschitz constant of α .
- $C_\alpha > 0$ denotes the coerciveness constant of α which satisfies for any u, v in $\mathbb{R}$,
$$(\alpha(u) - \alpha(v))(u - v) \geq C_\alpha(u - v)^2.$$
- $L_{\mathcal{H}} > 0$ denotes the Lipschitz constant of $\mathcal{H}$.

Let me introduce the concept of solutions we were interested in for System (V.1) and System (V.2):

Definition V.2

Any pair of predictable processes $(\vartheta, \chi) \in (\mathcal{N}_W^2(0, T; H^1(D)))^2$ such that $\vartheta \in L^2(\Omega; H^1(D \times (0, T)))$ and $\partial_t(\chi - \int_0^\cdot h dW) \in L^2(\Omega \times D \times (0, T))$, is a solution of System (V.1) if for almost all t in $(0, T)$, for almost all ω in Ω , the following system of variational formulations holds, for any $\varphi \in H^1(D)$,

$$\begin{aligned} \int_D \partial_t \vartheta(\omega, x, t) \varphi(x) dx + \int_D \partial_t \left(\chi(\omega, x, t) - \int_0^t h(\omega, x, s) dW(s) \right) \varphi(x) dx + \int_D \nabla \vartheta(\omega, x, t) \cdot \nabla \varphi(x) dx &= 0 \\ \int_D \tilde{\alpha} \left(\partial_t \left(\chi(\omega, x, t) - \int_0^t h(\omega, x, s) dW(s) \right) \right) \varphi(x) dx + \int_D \nabla \chi(\omega, x, t) \cdot \nabla \varphi(x) dx &= \int_D \vartheta(\omega, x, t) \varphi(x) dx, \end{aligned}$$

and $\mathbb{P}$ -a.s. in Ω , $\lim_{t \rightarrow 0} \vartheta(\cdot, t) = \vartheta_0(\cdot)$ and $\lim_{t \rightarrow 0} \chi(\cdot, t) = \chi_0(\cdot)$ in $H^1(D)$.

[†]As mentioned in Part I, for a given separable Hilbert space X , we denote by $\mathcal{N}_W^2(0, T; X)$ the separable Hilbert space of predictable X -valued processes endowed with the norm $\|\phi\|_{\mathcal{N}_W^2(0, T; X)}^2 = \mathbb{E} \left[\int_0^T \|\phi\|_X^2 dt \right]$ (see [DPZ92] p.94).

Definition V.3

Any pair of predictable processes $(\vartheta, \chi) \in (\mathcal{N}_W^2(0, T; H^1(D)))^2$ such that $\vartheta \in L^2(\Omega, H^1(D \times (0, T)))$ and $\partial_t(\chi - \int_0^\cdot \mathcal{H}(\chi) dW) \in L^2(\Omega \times D \times (0, T))$, is a solution of System (V.2) if for almost all t in $(0, T)$, for almost all ω in Ω , the following system of variational formulations holds, for any $\varphi \in H^1(D)$,

$$\begin{aligned} & \int_D \partial_t \vartheta(\omega, x, t) \varphi(x) dx + \int_D \partial_t \left(\chi(\omega, x, t) - \int_0^t \mathcal{H}(\chi(\omega, x, s)) dW(s) \right) \varphi(x) dx + \int_D \nabla \vartheta(\omega, x, t) \cdot \nabla \varphi(x) dx = 0 \\ & \int_D \tilde{\alpha} \left(\partial_t \left(\chi(\omega, x, t) - \int_0^t \mathcal{H}(\chi(\omega, x, s)) dW(s) \right) \right) \varphi(x) dx + \int_D \nabla \chi(\omega, x, t) \cdot \nabla \varphi(x) dx = \int_D \vartheta(\omega, x, t) \varphi(x) dx, \\ & \text{and } \mathbb{P}\text{-a.s. in } \Omega, \lim_{t \rightarrow 0} \vartheta(\cdot, t) = \vartheta_0(\cdot) \text{ and } \lim_{t \rightarrow 0} \chi(\cdot, t) = \chi_0(\cdot) \text{ in } H^1(D). \end{aligned}$$

V.1.2 State of the art

In the deterministic case, *i.e.*, when $h = 0$ in (V.1) and $\mathcal{H} = 0$ in (V.2), one application of such nonlinear evolution system is the description of phase transition phenomena, including irreversible phase changes (for instance, solidification of glue, cooking an egg,...) see [BFL00] for further details. Let me mention that equations of the form “ $\beta(\partial_t u) - \Delta u = 0$ ”, with β a non-decreasing function, called “Barenblatt’s equations” in the sequel, were initially studied by G.I. BARENBLATT for the theory of fluids in elasto-plastic porous medium [Bar96], under the assumption that the porous medium was irreversibly deformable. After that, intensive studies was carried out on this type of equations, see, *e.g.*, [HV94, Igb96, KPV91] for more details. Moreover, this type of equations appeared in various applications: irreversible phase change modeling [Pta07], reaction-diffusion with absorption problems in Biochemistry [Pta07], irreversible damage and fracture evolution analysis [BBLR17, BBL19, OMDL16] and recently in constrained stratigraphic problems in Geology [AGLV06a, AGLV06b, AGMV09, AGV03, Val03].

Concerning the study of Barenblatt equations with a stochastic force term, a few papers were written when we started to look at this kind of problems. To the best of our knowledge, none of them proposed the study of the coupling with a random heat equation. We mentioned [ASV11], where the authors were interested in a Barenblatt’s equation with stochastic coefficients. In [BGV12], we proposed with J. GIACOMONI and G. VALLET an existence and uniqueness result for a stochastic Barenblatt’s equation under Dirichlet boundary conditions in the case of additive and multiplicative Itô type noise (namely, equations (V.1b) and (V.2b) with $\vartheta = 0$). After that, we investigated on well-posedness theory for stochastic abstract problems of Barenblatt’s type in [BV11]. Then, we proposed with F. LEBON and A.A. MAITLO in [BLM19] an extension of [BGV12], by considering Neumann boundary conditions and additionally the presence of a Lipschitz-continuous nonlinear source term β :

$$\begin{cases} \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) dW \right) \right) - \Delta \chi + \beta(\chi) = 0 & \text{in } \Omega \times D \times (0, T), & \text{(V.3a)} \\ \nabla \chi \cdot \mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0, T), & \text{(V.3b)} \\ \chi(\omega, x, 0) = \chi_0(x) & \omega \in \Omega, x \in D. & \text{(V.3c)} \end{cases}$$

Taking inspiration from techniques developed in my previous collaboration [BGV12], we proved in [BLM19] the well-posedness of Problem (V.3).

V.1.3 Aim of the study

In the study of composite or bonded structures, temperature effects in the evolution of damage at the interface can not be ignored, it is even a fundamental coupling [BBL19, ZC14]. Additionally, it cannot be denied that stochastic and random effects are also important from a modeling point of view in order to take into account several phenomena such as microscopic fluctuations, random forcing, effects of interscale interactions... In this direction, we had in mind the introduction of a stochastic force in the deterministic nonlinear evolution system studied by one of our collaborator G. BONFANTI in [BFL00]. Precisely, she was interested in finding a pair (ϑ, χ) solution of the following system posed on $D \times (0, T)$:

$$\begin{cases} \partial_t \vartheta + \partial_t \chi - \Delta \vartheta = 0, & \text{(V.4a)} \\ \partial_t \chi - \Delta \chi + \partial I_{[0, +\infty[}(\partial_t \chi) + \partial I_{[0, 1]}(\chi) \ni \vartheta, & \text{(V.4b)} \end{cases}$$

completed by homogeneous Neumann boundary conditions on the boundary of D and a given initial data. System (V.4) featured severe analytical difficulties due to the presence of multivalued operators rendering the constraints on the internal

variables (accounting also for irreversible evolutions). For these reasons, in the framework of A.A. MAITLO's PhD, we decided (as a preliminary work) to start with the study of a regularized version of System (V.4) under a stochastic force (of additive and then multiplicative type), by considering Systems (V.1) and (V.2), having in mind that α would play the role of the Moreau-Yosida approximation of $\partial I_{[0,+\infty[}$.

V.2 Main results and overview of the proofs

Let me now state the main results we obtained in [BLMZ21]:

Theorem V.4

Under Assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, there exists a unique pair (ϑ, χ) solution of System (V.1) in the sense of Definition V.2.

We also proved that the unique solution of System (V.1) satisfied the following stability result, asserting the continuous dependence of the solution with respect to the integrand h in the stochastic noise:

Proposition V.5

Consider $h, \hat{h}$ in $\mathcal{N}_W^2(0, T; H^1(D))$ and denote by (ϑ, χ) and $(\hat{\vartheta}, \hat{\chi})$ the associated solutions to the System (V.1) in the sense of Definition V.2 with the respective set of data (ϑ_0, χ_0, h) and $(\vartheta_0, \chi_0, \hat{h})$. Then, there exists a constant $C_\alpha^T > 0$, which only depends on T , L_α and C_α such that, for any t in $[0, T]$

$$\begin{aligned} & \mathbb{E} [\|(\vartheta - \hat{\vartheta})(t)\|_{L^2(D)}^2] + \mathbb{E} [\|\nabla(\vartheta - \hat{\vartheta})(t)\|_{L^2(D)}^2] + \frac{1}{4} \mathbb{E} [\|(\chi - \hat{\chi})(t)\|_{L^2(D)}^2] + \frac{1}{4} \mathbb{E} [\|\nabla(\chi - \hat{\chi})(t)\|_{L^2(D)}^2] \\ & \leq C_\alpha^T \left(\|h - \hat{h}\|_{L^2(\Omega \times D \times (0, t))}^2 + \|\nabla(h - \hat{h})\|_{L^2(\Omega \times D \times (0, t))}^2 \right). \end{aligned}$$

Proposition V.5 allowed us to extend our well-posedness result to the multiplicative case:

Theorem V.6

Under Assumptions $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$, there exists a unique pair (ϑ, χ) solution of System (V.2) in the sense of Definition V.3.

The sketches of the proofs of these results were the following ones. Firstly, we were focused on the study of the additive case, namely in proving the existence of a couple (ϑ, χ) solution of System (V.1). To do so, the idea was to use a semi-implicit time discretization scheme. After deriving stability estimates satisfied by the time approximations of the couple (ϑ, χ) , our aim was to pass to the limit on the obtained discrete system with respect to the time step. Such estimates allowed us to extract weakly converging subsequences, but this mode of convergence was not enough *a priori* to identify the weak limit of the sequence approximating the nonlinear term $\tilde{\alpha}(\partial_t(\chi - \int_0^\cdot h dW))$. Note that due to the random variable, classical deterministic tools of compactness (like the use of compact injections) were out of range with respect to our knowledge in probability theory. Indeed, we could have imagined trying to identify this limit by using the stochastic compactness argument which, in few word, appeals among other things to Prokhorov's theorem, Skorokhod's representation theorem and Gyöngy-Krylov arguments (see Chapter VI for more details on this method), but at this time, I did not know these techniques at all, and more honestly I did not have the courage to go and look at how to use them. Fortunately, the monotone nature of $\tilde{\alpha}$ allowed us (as we did in [BGV12] and [BLM19]) to invoke the so called “Minty-trick” from the theory on maximal monotone operators (see [Lio69]), and consequently to identify such a weak limit. Secondly, we proved uniqueness of the solution (ϑ, χ) to System (V.1) by the use of classical energy estimates well known for the heat equation and adapted to the random case. This allowed us to show additionally at the limit on the time discretization parameter that the couple (ϑ, χ) depended continuously on the data as stated in Proposition V.5. Thirdly, exploiting this dependence result, we were able to extend, thanks to a fixed point argument, our result of existence and uniqueness to the multiplicative case, as announced in Theorem V.6. Note that one of the key argument of this work was the consideration in the first place of the additive case. Indeed, we were then able to reduce the study of equations involving $\chi - \int_0^\cdot h dW$ (in a continuous or discrete form) to the case of a heat equation with a random second member by the change of variable $U = \chi - \int_0^\cdot h dW$. For example, Equation (V.1b) was rewritten as

$$\partial_t U - \Delta U = \vartheta - \alpha(\partial_t U) + \int_0^\cdot \Delta h dW \text{ in } \Omega \times D \times (0, T),$$

and by seeing $\alpha(\partial_t U)$ as a given data, we could invoke existing results in the literature on non-homogeneous heat equation with a random source term: $\partial_t U - \Delta U = g$ with $g = \vartheta - \alpha(\partial_t U) + \int_0^\cdot \Delta h dW$, and derived useful estimates on U .

PART 3

Finite-volume approximation of parabolic problems forced by a multiplicative noise

Chapter VI

Convergence of a finite-volume scheme for a multiplicative stochastic heat equation

In this chapter are presented my first contributions ([BN20] and [BNSZ23a]) in collaboration with F. NABET, K. SCHMITZ and A. ZIMMERMANN on the discretization by finite-volume schemes of a particular class of stochastic parabolic problems. More precisely, we were interested in the approximation of heat equations set in a bounded domain of $\mathbb{R}^2$ with Neumann boundary conditions and forced by a Lipschitz continuous multiplicative noise in the sense of Itô. Note that such a noise was itself carried by a function g which was on the one hand linear in [BN20], and on the other hand more general in [BNSZ23a]. In both cases, we considered a discretization which was semi-implicit in time and a two-point flux approximation scheme (TPFA) in space and proved the convergence of the finite-volume approximate solution towards the unique variational solution of the considered problem. Since the introductive study [BN20] can be seen as a particular case of [BNSZ23a], I will present in this chapter this last contribution.

VI.1 Introduction

We considered in [BNSZ23a] a bounded, open, connected and polygonal subset Λ of $\mathbb{R}^2$, $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$ and $(W(t))_{t \geq 0}$ a standard, one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ and a real $T > 0$. We were interested in the following nonlinear stochastic heat equation under Neumann boundary conditions:

$$\begin{cases} du - \Delta u \, dt &= g(u) \, dW & \text{in } \Omega \times (0, T) \times \Lambda, \\ u(0, \cdot) &= u_0 & \text{in } \Omega \times \Lambda, \\ \nabla u \cdot \mathbf{n} &= 0, & \text{on } \Omega \times (0, T) \times \partial\Lambda, \end{cases} \quad (\text{VI.1})$$

where $\mathbf{n}$ denoted the unit normal vector to $\partial\Lambda$ outward to Λ . We assumed the following hypotheses on the data:

$\mathcal{H}_1$: $u_0 \in L^2(\Omega; H^1(\Lambda))$ is $\mathcal{F}_0$ -measurable.

$\mathcal{H}_2$: $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with Lipschitz constant $L_g \geq 0$.

Remark VI.1

1. In [BN20], we assumed that g was defined for any $v \in \mathbb{R}$ by $g(v) = \lambda v$, for a given $\lambda \in \mathbb{R}$ and that u_0 belonged to $L^2(\Lambda)$. In the presence of a nonlinear integrand g in the multiplicative noise, we assumed in [BNSZ23a] a more restrictive regularity for the initial condition with respect to the space variable. In my opinion, this was due to the way we employed tools from probability's theory to prove the convergence of the finite-volume scheme. Namely, we will see later that a tightness property on $L^2(0, T; L^2(\Lambda))$ of some sequence of laws generated by our finite-volume approximation was used, and such a property was (in part) derived from the estimate (VI.8),

itself due to the $H^1(\Lambda)$ regularity in space of u_0 . Without this assumption on u_0 , I am not claiming that it is not possible to prove the convergence of the scheme, only that I don't know how to recover the tightness property on $L^2(0, T; L^2(\Lambda))$, and so to prove the convergence of the scheme with the tools presented in what follows. For instance, I will give a hint in Chapter VII to obtain the convergence (with a totally different technique) by assuming $u_0 \in L^2(\Omega; L^2(\Lambda))$ (see Remark VII.13).

2. Note that Assumption $\mathcal{H}_2$ implied

$$|g(r)|^2 \leq C_{L_g}(1 + |r|^2) \quad (\text{VI.2})$$

for all $r \in \mathbb{R}$ and a constant $C_{L_g} \geq 0$ only depending on the Lipschitz constant $L_g \geq 0$ of g and on $g(0)$. In particular, our scheme applied for square integrable, additive noise with appropriate measurability assumptions.

3. Although the hyperbolic case was treated for any space dimension $d \geq 1$, we noticed (with my co-authors) that elliptic and parabolic equations were mainly discretized in the deterministic case for dimensions 1, 2 or 3. We believed that this was mainly related to the discretization of the Laplace operator which required the construction of a second mesh on which a “weak” gradient existed, and beyond dimension 3, it was delicate to imagine such a mesh. Since the case $d > 3$ did not seem to be treated in the deterministic case, we did not answer ourselves on a possible extension in that direction. From the point of view of applications and implementations, the cases $d = 2$ and $d = 3$ were most interesting, this was the reason why we did not mention the case $d = 1$. Additionally, in order to present a work that could be addressed to both the probabilistic and the PDE community, we chose to place ourselves in dimension 2 since this was the case that allowed us to appropriate and adapt deterministic discretization techniques. For our future works, we plan to pay attention to develop an efficient notation in order to address the dimensions $d = 2$ and 3 simultaneously.

VI.1.1 Theoretical framework

It is well-known that the theoretical framework associated with Problem (VI.1) is widely established in the literature. Indeed, it is possible to find many existence and uniqueness results for various concepts of solutions associated with this problem such as mild solutions, variational solutions, pathwise solutions and weak solutions, see e.g. [Par75], [KR81], [DPZ92], [LR15] and [Par21]. In [BNSZ23a], we were interested in the concept of solution as defined below, which we called a variational solution:

Definition VI.2 (Variational solution)

A variational solution to Problem (VI.1) is an $(\mathcal{F}_t)_{t \geq 0}$ -adapted stochastic process

$$u \in L^2(\Omega; \mathcal{C}([0, T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0, T; H^1(\Lambda)))$$

such that for all $t \in [0, T]$,

$$u(t) - u_0 - \int_0^t \Delta u(s) ds = \int_0^t g(u(s)) dW(s)$$

in $L^2(\Lambda)$ and $\mathbb{P}$ -a.s in Ω , where Δ denotes the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary condition.

VI.1.2 State of the art

The study of numerical schemes for stochastic partial differential equations (SPDEs) has attracted a lot of attention in the last decades and there exists an extensive literature on this topic. A list of references for the numerical analysis of SPDEs and an overview of the state of the art is given in [DP09], [ACQS20], [ZK17] and [OPW22].

Concerning the theoretical and numerical study of stochastic heat equations, semigroup techniques may be used to construct mild solutions (see, e.g., [DPZ92]). However, from the point of view of applications and mathematical modeling, it is often interesting to consider first-order perturbations of the stochastic heat equation and more complicated, nonlinear second order operators, such as the p -Laplacian or the porous medium operator. For these nonlinear SPDEs, the semigroup approach is not available and variational techniques have been developed in [Par75], [KR81] and [LR15].

In the numerical analysis of variational solutions to parabolic SPDEs, spatial discretization of finite-element type have been frequently used (see, e.g., [BBNP14], [GM05], [BHL21] and the references therein). Let me mention the recent work of [DGL22], where the authors investigated the convergence of a large class of numerical schemes for the stochastic transient Leray-Lions equation with multiplicative noise. They used a generic GDM framework (Gradient Discretisation Method) which covers various schemes, including finite element, some finite-volumes method, discontinuous Galerkin, mass-lumped finite elements... On the other hand, for stochastic scalar conservation laws, finite-volume schemes have been studied in [BCG16b], [BCG16a], [BCG17], [FGH18], [Maj18], [DV18], [BCC20] and [DV20]. To the best of our knowledge, there were no results on TPFA schemes for parabolic SPDEs when we started working in this direction.

VI.1.3 Aim of the study

In the contribution presented in this chapter ([BNSZ23a]), we wanted to extend the finite-volume approximation results obtained in the hyperbolic case to the stochastic heat equation with Lipschitz continuous multiplicative noise. Having applications to nonlinear operators and also to degenerate parabolic-hyperbolic problems with stochastic force in mind for the future, we proposed a method for the convergence of the scheme which didn't rely on mild solutions and results from semigroup theory. The main technical challenge was the nonlinear multiplicative nature of the noise. Indeed, from the stability estimates satisfied by the finite-volume approximate solution, we get (up to subsequences) weak convergence results in several functional spaces and unfortunately this mode of convergence was not enough to identify the weak limit of the nonlinear term in the stochastic integral. For this reason, we had to propose a compactness argument compatible with the random variable and the passage to the limit in the finite-volume scheme.

VI.2 Finite-volume framework

VI.2.1 Admissible finite-volume meshes and notations

In order to perform a finite-volume approximation of the variational solution of Problem (VI.1) on $[0, T] \times \Lambda$ we needed first of all to set a choice for the temporal and spatial discretization. For the time discretization, by setting $N \in \mathbb{N}^*$, we defined a fixed time step $\Delta t = \frac{T}{N}$ and divided the interval $[0, T]$ in $0 = t_0 < t_1 < \dots < t_N = T$ equidistantly with $t_n = n\Delta t$ for all $n \in \{0, \dots, N\}$. For the space discretization, referring to [EGH00], we considered finite-volume admissible meshes in the sense of the following definition.

Definition VI.3 (Admissible finite-volume mesh)

An admissible finite-volume mesh $\mathcal{T}$ of Λ (see Fig. VI.1) is given by a family of open, polygonal, and convex subsets K , called control volumes of $\mathcal{T}$, satisfying the following properties:

- $\bar{\Lambda} = \bigcup_{K \in \mathcal{T}} \bar{K}$.
- If $K, L \in \mathcal{T}$ with $K \neq L$ then $K \cap L = \emptyset$.
- If $K, L \in \mathcal{T}$, with $K \neq L$ then either the one-dimensional Lebesgue measure of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L}$ is the edge of the mesh denoted $\sigma = K|L$ separating the control volumes K and L .
- To each control volume $K \in \mathcal{T}$, we associate a point $x_K \in \bar{K}$ (called the center of K) such that: If $K, L \in \mathcal{T}$ are two neighbouring control volumes the straight line between the centers x_K and x_L is orthogonal to the edge $\sigma = K|L$.

Note that, in comparison with the hyperbolic case presented in Part I, the definition of admissible finite-volume mesh was slightly different in the parabolic one since we imposed an orthogonality condition for any neighboring control volumes.

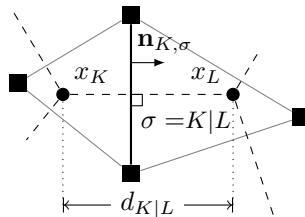
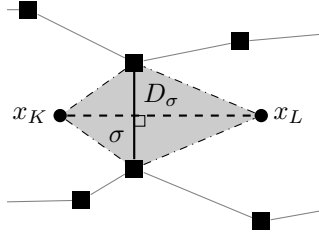


Figure VI.1: Notations of the mesh $\mathcal{T}$ associated with Λ

For a given admissible finite-volume mesh $\mathcal{T}$ of Λ , we used the following notations.

Notations VI.4

- $h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K) : K \in \mathcal{T}\}$, the mesh size.
- $d_h \in \mathbb{N}$ denotes the number of control volumes $K \in \mathcal{T}$ with $h = \text{size}(\mathcal{T})$.
- λ_2 denotes the two-dimensional Lebesgue measure.
- $\mathcal{E}$ is the set of the edges of the mesh $\mathcal{T}$ and we define $\mathcal{E}_{\text{int}} := \{\sigma \in \mathcal{E} : \sigma \not\subseteq \partial\Lambda\}$, $\mathcal{E}_{\text{ext}} := \{\sigma \in \mathcal{E} : \sigma \subseteq \partial\Lambda\}$.
- For $K \in \mathcal{T}$, $\mathcal{E}_K$ is the set of edges of K and $m_K := \lambda_2(K)$.
- Let $K, L \in \mathcal{T}$ be two neighbouring control volumes. For $\sigma = K|L \in \mathcal{E}_{\text{int}}$, let m_σ be the length of σ and $d_{K|L}$ the distance between x_K and x_L .
- For neighbouring control volumes $K, L \in \mathcal{T}$, we denote the unit vector on the edge $\sigma = K|L$ pointing from K to L by $\mathbf{n}_{K,\sigma}$.
- For $\sigma = K|L \in \mathcal{E}_{\text{int}}$, the diamond D_σ (see Fig. VI.2) is the open quadrangle whose diagonals are the edge σ and the segment $[x_K, x_L]$. For $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$, we define $D_\sigma := K$. Then, $\bar{\Lambda} = \bigcup_{\sigma \in \mathcal{E}} \bar{D}_\sigma$.
- $m_{D_\sigma} = \lambda_2(D_\sigma)$ is the two-dimensional Lebesgue measure of the diamond D_σ . Note that, for all $\sigma \in \mathcal{E}_{\text{int}}$, $m_{D_\sigma} = \frac{m_\sigma d_{K|L}}{2}$.
- $\mathcal{N}$ is the maximum of edges incident to any vertex.

Figure VI.2: Notations on a diamond cell D_σ for $\sigma \in \mathcal{E}_{\text{int}}$

Using these notations, we introduced a positive number

$$\text{reg}(\mathcal{T}) = \max \left(\mathcal{N}, \max_{\substack{K \in \mathcal{T} \\ \sigma \in \mathcal{E}_K}} \frac{\text{diam}(K)}{d(x_K, \sigma)} \right) \quad (\text{VI.3})$$

measuring the regularity of a given mesh and used to perform the convergence analysis of finite-volume schemes. Let me mention that this number was supposed to be uniformly bounded when the mesh size tended to 0 for the convergence results to hold.

VI.2.2 Discrete unknowns and piecewise constant functions

Setting $N \in \mathbb{N}^*$, defining $\Delta t = \frac{T}{N}$ and considering $\mathcal{T}$ an admissible finite-volume mesh of Λ in the sense of Definition VI.3 with a mesh size h , for $n \in \{0, \dots, N-1\}$ given, the idea of a finite-volume scheme for the approximation of Problem (VI.1) was to associate to each control volume $K \in \mathcal{T}$ and time t_n a discrete unknown value denoted $u_K^n \in \mathbb{R}$, expected to be an approximation of $u(t_n, x_K)$, where u was the variational solution of (VI.1). Before presenting the numerical scheme satisfied by the discrete unknowns $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$, let me introduce some general notations.

For any arbitrary vector $(w_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ we defined a piecewise constant function $w_h^n : \Lambda \rightarrow \mathbb{R}$ by

$$w_h^n(x) := \sum_{K \in \mathcal{T}} w_K^n \mathbb{1}_K(x), \quad \forall x \in \Lambda.$$

Note that since the mesh $\mathcal{T}$ was fixed, by the continuous mapping defined from $\mathbb{R}^{d_h}$ to $L^2(\Lambda)$ by

$$(w_K^n)_{K \in \mathcal{T}} \mapsto \sum_{K \in \mathcal{T}} \mathbb{1}_K w_K^n,$$

we were able to consider the space $\mathbb{R}^{d_h}$ as a finite-dimensional subspace of $L^2(\Lambda)$ and to naturally identify the function and the vector

$$w_h^n \equiv (w_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}.$$

Then, knowing for all $n \in \{0, \dots, N\}$ the function w_h^n , we were also able to define the following piecewise constant functions in time and space $w_{h,N}^r, w_{h,N}^l : [0, T] \times \Lambda \rightarrow \mathbb{R}$ by

$$\begin{aligned} w_{h,N}^r(t, x) &:= \sum_{n=0}^{N-1} w_h^{n+1}(x) \mathbb{1}_{[t_n, t_{n+1})}(t) \text{ if } t \in [0, T) \text{ and } w_{h,N}^r(T, x) := w_h^N(x), \\ w_{h,N}^l(t, x) &:= \sum_{n=0}^{N-1} w_h^n(x) \mathbb{1}_{[t_n, t_{n+1})}(t) \text{ if } t \in (0, T] \text{ and } w_{h,N}^l(0, x) := w_h^0(x). \end{aligned} \quad (\text{VI.4})$$

Remark VI.5

The superscripts r and l in (VI.4) did not refer to the continuity properties of the associated functions (which could have been chosen either càdlàg or càglàd). The difference was that in our case, the finite volume approximate solution $u_{h,N}^l$ was adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, whereas $u_{h,N}^r$ was not.

As for the piecewise constant function in space, since $\mathcal{T}$ and N were fixed, by the continuous mapping defined from $\mathbb{R}^{d_h \times N}$ to $L^2(0, T; L^2(\Lambda))$ by

$$(w_K^n)_{\substack{n \in \{0, \dots, N-1\} \\ K \in \mathcal{T}}} \mapsto \sum_{\substack{K \in \mathcal{T} \\ n \in \{0, \dots, N-1\}}} \mathbb{1}_K \mathbb{1}_{[t_n, t_{n+1})} w_K^n,$$

the space $\mathbb{R}^{d_h \times N}$ could be considered as a finite-dimensional subspace of $L^2(0, T; L^2(\Lambda))$ and we were able to do the identifications

$$\begin{aligned} w_{h,N}^l &\equiv (w_K^n)_{\substack{n \in \{0, \dots, N-1\} \\ K \in \mathcal{T}}} \in \mathbb{R}^{d_h \times N}, \\ w_{h,N}^r &\equiv (w_K^{n+1})_{\substack{n \in \{0, \dots, N-1\} \\ K \in \mathcal{T}}} \in \mathbb{R}^{d_h \times N}. \end{aligned}$$

VI.2.3 Discrete norms

For a fixed $n \in \{0, \dots, N-1\}$, an arbitrary vector $(w_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ naturally identified with the piecewise constant function in space $w_h^n \equiv (w_K^n)_{K \in \mathcal{T}}$, we employed the following discrete norms:

Definition VI.6 (Discrete L^2 -norm)

We define the L^2 -norm of $w_h^n \in \mathbb{R}^{d_h}$ as follows

$$\|w_h^n\|_{L^2(\Lambda)} = \left(\sum_{K \in \mathcal{T}} m_K |w_K^n|^2 \right)^{\frac{1}{2}}.$$

Definition VI.7 (Discrete H^1 -seminorm)

We define the H^1 -seminorm of $w_h^n \in \mathbb{R}^{d_h}$ as follows

$$|w_h^n|_{1,h} := \left(\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |w_K^n - w_L^n|^2 \right)^{\frac{1}{2}}.$$

I have now introduced all the necessary definitions and notations to present the finite-volume scheme studied in [BNSZ23a]. This is the aim of the next subsection.

VI.2.4 The finite-volume scheme

Firstly, we defined the vector $u_h^0 \equiv (u_K^0)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ by the discretization of the initial condition u_0 of Problem (VI.1) over each control volume:

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}. \quad (\text{VI.5})$$

For this given initial $\mathcal{F}_0$ -measurable random vector $u_h^0 \in \mathbb{R}^{d_h}$, the finite-volume scheme we proposed read as:

For any $n \in \{0, \dots, N-1\}$, knowing $u_h^n \equiv (u_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$, we searched for $u_h^{n+1} \equiv (u_K^{n+1})_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ such that, for almost every $\omega \in \Omega$, the vector u_h^{n+1} was solution to the following random equations

$$\frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) = \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n), \quad \forall K \in \mathcal{T}, \quad (\text{VI.6})$$

where for any $n \in \{0, \dots, N\}$, $W^n = W(t_n)$.

Remark VI.8

1. The second term on the left-hand side of (VI.6) is the classical two-point flux approximation of the Laplace operator obtained formally by integrating such an operator on each control volume $K \in \mathcal{T}$, then applying the Gauss-Green theorem to the term $\int_K \Delta u(t_{n+1}, x) dx$ and finally combining Taylor expansions of the function $u(t_{n+1}, \cdot)$ at the points x_K and x_L together with the orthogonality condition on the mesh (see [EGH00, Section 10] for more details).
2. Note that the time-implicit discretization of the Laplace operator has several analytic advantages. First of all, calculations for the obtention of stability estimates are simplified. Secondly, it avoids the need to use a CFL-condition. Last but not least, for more general nonlinear operators such as the p -Laplace operator, an implicit time discretization is more appropriate. However, an explicit time discretization of the noise is crucial and can not be omitted due to the non-anticipative character of the Itô stochastic integral.

Contrary to the hyperbolic case presented in Part 1, the discretization presented here was not fully explicit and we thus had to prove the well-posedness of the scheme. By using the theorem of Stampacchia (see e.g. [Bré11, Theorem 5.6]) we arrived at the following result:

Proposition VI.9 (Existence and uniqueness of a discrete solution)

Assume that hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. Let $\mathcal{T}$ be an admissible finite-volume mesh of Λ in the sense of Definition VI.3 with a mesh size h and $N \in \mathbb{N}^*$. Then, there exists a unique solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ to Problem (VI.6) associated with the initial vector u_h^0 defined by (VI.5). Additionally, for any $n \in \{0, \dots, N\}$, u_h^n is a $\mathcal{F}_{t_n}$ -measurable random vector.

The solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ of the scheme (VI.5)-(VI.6) was then used to build the right and left finite-volume approximations $u_{h,N}^r$ and $u_{h,N}^l$ defined by (VI.4) for the variational solution u of Problem (VI.1).

VI.3 Main result and overview of the proof

Our main result in [BNSZ23a] was to propose a finite-volume scheme for the approximation of the unique variational solution in the sense of Definition VI.2 and to show its stochastically strong convergence by passing to the limit with respect to the time and space discretization parameters. This is stated in the following convergence result:

Theorem VI.10 (Convergence of the scheme)

Assume that hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible finite-volume meshes of Λ in the sense of Definition VI.3 such that the mesh size h_m tends to 0, and let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence of positive numbers which tends to $+\infty$. For a fixed $m \in \mathbb{N}$, let u_{h_m, N_m}^r and u_{h_m, N_m}^l be respectively the right and left in time finite-volume approximations defined by (VI.4), (VI.5)-(VI.6) with $\mathcal{T} = \mathcal{T}_m$ and $N = N_m$. Then $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$ converge strongly in $L^p(\Omega; L^2(0, T; L^2(\Lambda)))$ for any $p \in [1, 2)$ to the variational solution of Problem (VI.1) in the sense of Definition VI.2.

For $m \in \mathbb{N}^*$, let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes of Λ in the sense of Definition VI.3 such that the mesh size h_m tends to 0 when m tends to $+\infty$ and let $(N_m)_m \subseteq \mathbb{N}^*$ be a sequence with $\lim_{m \rightarrow +\infty} N_m = +\infty$ and $\Delta t_m := \frac{T}{N_m}$.

For the sake of simplicity, I will use in what follows the notations $\mathcal{T} = \mathcal{T}_m$, $h = \text{size}(\mathcal{T}_m)$, $\Delta t = \Delta t_m$ and $N = N_m$. The outline of the proof of Theorem VI.10 was the following one. As in the hyperbolic case, we started by deriving stability estimates satisfied by the sequences of finite-volume approximations: by multiplying (VI.6) by u_K^{n+1} and using the same argumentations as in the hyperbolic case to handle the stochastic noise, we firstly obtained the following boundedness result:

Proposition VI.11 (Bounds on the discrete solutions)

There exists a constant $C_1 > 0$, depending only on u_0 , C_{L_g} , $|\Lambda|$ and T such that

$$\mathbb{E} [\|u_h^n\|_{L^2(\Lambda)}^2] + \mathbb{E} \left[\sum_{k=0}^{n-1} \|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2 \right] + \Delta t \sum_{k=0}^{n-1} \mathbb{E} [u_h^{k+1}|_{1,h}^2] \leq C_1, \quad \forall n \in \{1, \dots, N\}.$$

As a direct consequence of this proposition, we derived the following bounds on the right and left finite-volume approximations $(u_{h,N}^l)_{h,N}$ and $(u_{h,N}^r)_{h,N}$ defined by (VI.4).

Lemma VI.12

The sequences $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$ are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, independently of the discretization parameters $N \in \mathbb{N}^$ and $h \in \mathbb{R}_+^*$.*

Lemma VI.13

There exist a constant $C_2 > 0$ depending only on u_0 , C_{L_g} , $|\Lambda|$ and T and a constant $C_3 > 0$ additionally depending on the mesh regularity $\text{reg}(\mathcal{T})$ (defined by (VI.3)), such that

$$\int_0^T \mathbb{E} [|u_{h,N}^r(t)|_{1,h}^2] dt \leq C_2 \quad (\text{VI.7})$$

and

$$\int_0^T \mathbb{E} [|u_{h,N}^l(t)|_{1,h}^2] dt \leq C_3. \quad (\text{VI.8})$$

Using this we obtained (up to subsequences) weak convergence results for $(u_{h,N}^l)_{h,N}$ and $(u_{h,N}^r)_{h,N}$ towards the same stochastic process u belonging to $L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ and the existence of a predictable stochastic process g_u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that for all $t \in [0, T]$,

$$u(t) = u_0 + \int_0^t \Delta u(s) ds + \int_0^t g_u(s) dW(s) \quad (\text{VI.9})$$

in $L^2(\Lambda)$ and $\mathbb{P}$ -a.s in Ω , where Δ denoted the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary condition. The technical challenge consisted in the identification of the weak limit g_u of $(g(u_{h,N}^l))_{h,N}$, namely how to prove that it was equal to $g(u)$? In [BN20] we took advantage of the linearity of g to identify directly g_u as $g(u)$ whereas in [BNSZ23a] such a direct reasoning was not conclusive.

To overcome this difficulty, we decided to use compactness tool given by probability theory, the so-called *stochastic compactness argument* see, e.g., [BFH18]. I give here a brief summary of how we employed this method.

- We firstly proved, thanks to many technical lemmas (involving uniform control of time and space translate of our finite-volume sequences), bounds on the Gagliardo seminorms of the approximate solutions. More precisely, we showed, for any fixed $\alpha \in (0, \frac{1}{2})$, independently of the discretization parameters $N \in \mathbb{N}^*$ and $h \in \mathbb{R}_+^*$, the boundedness of $(u_{h,N}^l)_{h,N}$ in $L^2(\Omega; \mathcal{W})$, where

$$\mathcal{W} = L^2(0, T; W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0, T; L^2(\Lambda)),$$

with

$$W^{\alpha,2}(\Lambda) = \left\{ \varphi \in L^2(\Lambda) : \left(\int_{\Lambda} \int_{\Lambda} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2+2\alpha}} dx dy \right)^{\frac{1}{2}} < \infty \right\}$$

$$\text{and } W^{\alpha,2}(0, T; \Lambda) = \left\{ \phi \in L^2(0, T; L^2(\Lambda)) : \left(\int_0^T \int_0^T \frac{\|\phi(t) - \phi(s)\|_{L^2(\Lambda)}^2}{|t - s|^{1+2\alpha}} dt ds \right)^{\frac{1}{2}} < \infty \right\}.$$

Let me mention that some of the estimates leading to the boundedness of $(u_{h,N}^l)_{h,N}$ in $L^2(\Omega; \mathcal{W})$ came from (VI.8), itself obtained thanks to the $L^2(\Omega; H^1(\Lambda))$ regularity of u_0 .

- Secondly, using the compact embedding of $\mathcal{W}$ in $L^2(0, T; L^2(\Lambda))$ ([FG95, Theorem 2.1]), we derived the tightness property of the sequences of laws $(\mathbb{P} \circ (u_{h,N}^l)^{-1})_{h,N}$ on $L^2(0, T; L^2(\Lambda))$. This allowed us to get, by the theorem of Prokhorov ([Bil99, Theorem 5.1]), up to subsequences denoted in the same manner, convergence in law of $(u_{h,N}^l)_{h,N}$ towards a probability measure μ_{∞}^1 defined on $L^2(0, T; L^2(\Lambda))$. Then, by considering the sequence of random vectors $(Y_{h,N})_{h,N}$ defined by

$$Y_{h,N} = (u_{h,N}^l, u_{h,N}^r - u_{h,N}^l, W, u_{h,N}^0)$$

with values in

$$\mathcal{X} = L^2(0, T; L^2(\Lambda)) \times L^2(0, T; L^2(\Lambda)) \times \mathcal{C}([0, T]) \times L^2(\Lambda),$$

we got the existence of a not relabeled subsequence of $(Y_{h,N})_{h,N}$ converging in law, towards a probability measure μ_{∞} on $\mathcal{X}$ with marginal laws $\mu_{\infty}^1, \delta_0, \mathbb{P} \circ W^{-1}, \mathbb{P} \circ (u_0)^{-1}$.

- Thirdly, the Skorokhod representation theorem ([Bil99, Theorem 6.7]) applied to $(Y_{h,N})_{h,N}$ enabled us to obtain existence of a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y'_{h,N} = (v_{h,N}, z_{h,N}, B_{h,N}, v_{h,N}^0), u_{\infty}, W_{\infty}, v_0$ both defined on $(\Omega', \mathcal{A}', \mathbb{P}')$, with $Y'_{h,N}, W_{\infty}, v_0$ having respectively the same laws as $Y_{h,N}, W, u_0$, with in addition $\mathbb{P}' \circ (u_{\infty})^{-1} = \mu_{\infty}^1$, and such that

$$\begin{aligned} (v_{h,N})_{h,N} &\text{ converged towards } u_{\infty} \text{ strongly in } L^2(0, T; L^2(\Lambda)), \mathbb{P}'\text{-a.s. in } \Omega', \\ (z_{h,N})_{h,N} &\text{ converged towards } 0 \text{ strongly in } L^2(0, T; L^2(\Lambda)), \mathbb{P}'\text{-a.s. in } \Omega', \\ (B_{h,N})_{h,N} &\text{ converged towards } W_{\infty} \text{ strongly in } \mathcal{C}([0, T]), \mathbb{P}'\text{-a.s. in } \Omega', \\ (v_{h,N}^0)_{h,N} &\text{ converged towards } v_0 \text{ strongly in } L^2(\Lambda), \mathbb{P}'\text{-a.s. in } \Omega'. \end{aligned}$$

We were also able to prove (thanks to equality in law), for any fixed parameters $N \in \mathbb{N}^*$ and $h \in \mathbb{R}_+^*$, in a first time that $v_{h,N}$ and $z_{h,N}$ had the same piecewise constant structure as $u_{h,N}^l$ and $u_{h,N}^r - u_{h,N}^l$ respectively, and in a second time that there were in fact finite-volume functions. This led us to show that for any $n \in \{0, \dots, N\}$ and any $K \in \mathcal{T}$, v_K^{n+1} satisfied the semi-implicit equation

$$\frac{m_K}{\Delta t} (v_K^{n+1} - v_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_{\sigma}}{d_K |L|} (v_K^{n+1} - v_L^{n+1}) - \frac{m_K}{\Delta t} g(v_K^n) (B_{h,N}(t_{n+1}) - B_{h,N}(t_n)) = 0, \quad (\text{VI.10})$$

$\mathbb{P}'$ -a.s. in Ω' , (by denoting $v_{h,N}^l \equiv (v_K^n)_{n \in \{0, \dots, N-1\}, K \in \mathcal{T}}$). Additionally, since $\mathbb{P}' \circ (v_{h,N}^l)^{-1} = \mathbb{P} \circ (u_{h,N}^l)^{-1}$, from Lemma VI.12, it followed the existence of a constant $C' \geq 0$ such that

$$\mathbb{E}' \left[\|v_{h,N}^l\|_{L^2(0, T; L^2(\Lambda))}^2 \right] \leq C', \quad (\text{VI.11})$$

and by repeating on (VI.10) the arguments used for the obtention of Proposition VI.11, we obtained the existence of a constant $C'_1 \geq 0$ such that

$$\mathbb{E}' \left[\int_0^T |v_{h,N}^r|_{1,h}^2 dt \right] + \mathbb{E}' \left[\sum_{n=0}^{N-1} \|v_h^{n+1} - v_h^n\|_{L^2(\Lambda)}^2 \right] \leq C'_1. \quad (\text{VI.12})$$

- Fourthly, we were concerned about the identification of the nature of the stochastic processes $(B_{h,N}(t))_{t \in [0, T]}$ and $(W_{\infty}(t))_{t \in [0, T]}$. Adapting ideas from [BFH18, OPW22], we defined for any $t \in [0, T]$, $\mathcal{F}_t^{h,N}$ to be the smallest sub- σ -field of $\mathcal{A}'$ generated by $v_{h,N}^0$ and $B_{h,N}(s)$ for $0 \leq s \leq t$, and then we constructed $(\mathfrak{F}_t^{h,N})_{t \in [0, T]}$,

the right-continuous, $\mathbb{P}'$ -augmented filtration of $(\mathcal{F}_t^{h,N})_{t \in [0,T]}$. We then proved the adaptativity of $(v_{h,N}^l)_{h,N}$ to $(\mathfrak{F}_t^{h,N})_{t \in [0,T]}$, and, using successively [RS06, Proposition 3], [DM80, p.75], [DPZ14, Theorem 3.11], we identified $(B_{h,N}(t))_{t \in [0,T]}$ as a Brownian motion with respect to $(\mathfrak{F}_t^{h,N})_{t \in [0,T]}$. This stage allowed us to give a sense to the stochastic integral $\int_0^\cdot g(v_{h,N}^l(s, \cdot)) dB_{h,N}(s)$ coming from the discrete term “ $g(v_K^n)(B_{h,N}(t_{n+1}) - B_{h,N}(t_n))$ ” in the new numerical scheme (VI.10). In the same spirit, we wanted to show that the stochastic process $(W_\infty(t))_{t \in [0,T]}$ was a Brownian motion, and that a filtration might be chosen in order to have compatibility of u_∞ with stochastic integration in the sense of Itô with respect to $(W_\infty(t))_{t \in [0,T]}$. Since u_∞ was a random variable taking values in $L^2(0, T; L^2(\Lambda))$, $u_\infty(t, \cdot)$ was only defined for a.e. $t \in [0, T]$, and then the construction of an appropriate filtration induced (in part) by u_∞ became delicate. For this reason, we proposed a right-continuous, $\mathbb{P}'$ -augmented filtration $(\mathfrak{F}_t^\infty)_{t \in [0,T]}$ induced by $v_0, W_\infty(s)$ and $\int_0^s u_\infty(r) dr$, $0 \leq s \leq t$. After that, we proved the strong convergence of the sequence $(B_{h,N})_{h,N}$ towards W_∞ in $L^2(\Omega, \mathcal{C}([0, T]))$, then that $(W_\infty(t))_{t \in [0,T]}$ was a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t \in [0,T]}$ and at last that u_∞ admitted a $(\mathfrak{F}_t^\infty)_{t \in [0,T]}$ -predictable $d\mathbb{P}' \otimes dt$ representative. Finally, we had all the necessary tools to give a sense to the stochastic integral $\int_0^\cdot g(u_\infty(s, \cdot)) dW_\infty(s)$, and to prove the convergence of the nonlinear term $\int_0^\cdot g(v_{h,N}^l(s, \cdot)) dB_{h,N}(s)$ towards the latter, strongly in $L^2(0, T; L^2(\Lambda))$ and $\mathbb{P}'$ -a.s. in Ω' . To do this, we used on the one hand, the convergence in probability of $(B_{h,N})_{h,N}$ towards W_∞ in $\mathcal{C}([0, T])$, on the other hand the convergence in probability of $(g(u_{h,N}^l))_{h,N}$ towards $g(u_\infty)$ in $L^2(0, T; L^2(\Lambda))$, and [DGHT11, Lemma 2.1] gave us directly the desired result.

- Fifthly, we proved thanks to (VI.11), (VI.12) and Vitali's theorem ([Dro01, Corollaire 1.3.3]), the strong convergence in $L^p(\Omega'; L^2(0, T; L^2(\Lambda)))$ for any $p \in [1, 2)$ of $v_{h,N}^l$ and $v_{h,N}^r$ towards u_∞ and of $\int_0^\cdot g(v_{h,N}^l(s, \cdot)) dB_{h,N}(s)$ towards $\int_0^\cdot g(u_\infty(s, \cdot)) dW_\infty(s)$, as well as the following regularities:

$$u_\infty \in L^2(\Omega'; L^2(0, T; H^1(\Lambda))) \text{ and } \int_0^\cdot g(u_\infty(s, \cdot)) dW_\infty(s) \in L^2(\Omega'; \mathcal{C}([0, T]; L^2(\Lambda))).$$

Multiplying the new numerical scheme (VI.10) by $\mathbb{1}_A \xi(t) \varphi(x_K)$, with $A \in \mathcal{A}'$, $\xi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\xi(T) = 0$, and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ satisfying $\nabla \varphi \cdot \mathbf{n} = 0$ on $\partial\Lambda$, summing over each control volume $K \in \mathcal{T}$, integrating over each time interval $[t_n, t_{n+1}]$, then summing over $n = 0, \dots, N-1$ and finally taking the expectation, we were then able to prove by passing to the limit in the obtained equality the following:

$$\begin{aligned} & \mathbb{E}' \left[\mathbb{1}_A \int_0^T \int_\Lambda \left(u_\infty(t, x) - \int_0^t g(u_\infty(s, x)) dW_\infty(s) \right) \xi'(t) \varphi(x) dx dt \right] \\ & + \mathbb{E}' \left[\mathbb{1}_A \int_\Lambda v_0(x) \xi(0) \varphi(x) dx \right] = \mathbb{E}' \left[\mathbb{1}_A \int_0^T \int_\Lambda \nabla u_\infty(t, x) \cdot \nabla \varphi(x) \xi(t) dx dt \right]. \end{aligned} \quad (\text{VI.13})$$

Owing to the density of the set $\{\varphi \in \mathcal{D}(\mathbb{R}^2) \mid \nabla \varphi \cdot \mathbf{n} = 0 \text{ on } \partial\Lambda\}$ in $H^1(\Lambda)$ given by [Dro02, Theorem 1.1], we obtained that (VI.13) applied to all $\varphi \in H^1(\Lambda)$. On the one hand, by denoting the dual space of $H^1(\Lambda)$ by $H^1(\Lambda)^*$, we deduced from (VI.13) that $\frac{d}{dt} \left(u_\infty(t) - \int_0^t g(u_\infty) dW_\infty \right)$ belonged to $L^2(\Omega'; L^2(0, T; H^1(\Lambda)^*))$, and on the other hand, from the Lipschitz continuity of g combined with the chain rule for Sobolev functions, it followed that $g(u_\infty) \in L^2(\Omega'; L^2(0, T; H^1(\Lambda)))$ and thanks to [PR07, Lemma 2.4.1],

$$\nabla \left(\int_0^t g(u_\infty) dW_\infty \right) = \int_0^t g'(u_\infty) \nabla u_\infty dW_\infty,$$

hence $u_\infty - \int_0^\cdot g(u_\infty) dW_\infty \in L^2(\Omega'; L^2(0, T; H^1(\Lambda)))$. From [Rou13, Lemma 7.3], we were able to affirm that $u_\infty - \int_0^\cdot g(u_\infty(s, \cdot)) dW_\infty(s)$ belonged to $L^2(\Omega'; \mathcal{C}([0, T]; L^2(\Lambda)))$, and therefore u_∞ too since we already knew that $\int_0^\cdot g(u_\infty(s, \cdot)) dW_\infty(s)$ was an element of such a space. Finally, with the help of all this new information, we proved that u_∞ satisfied for all $t \in [0, T]$,

$$u_\infty(t) - v_0 - \int_0^t \Delta u_\infty ds = \int_0^t g(u_\infty) dW_\infty \quad \text{in } L^2(\Lambda) \text{ and } \mathbb{P}'\text{-a.s. in } \Omega',$$

where Δ denoted the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary condition. Note that at this stage, we thus showed that our initial finite-volume scheme converged to a martingale solution of (VI.1), i.e., the stochastic basis (composed of a probability space, a filtration and a Brownian motion) was not *a priori* given, but part of the solution.

- The sixth and final stage consisted in proving a result of pathwise uniqueness of solutions to (VI.1): roughly speaking, martingale solutions of (VI.1) on a joint stochastic basis and with respect to the same initial datum coincided. This, together with a classical argument of Gyöngy and Krylov* [GK96, Lemma 1.1] allowed us to deduce convergence in probability of our finite-volume approximation $(u_{h,N}^l)_{h,N}$ towards an element $\tilde{u}$ in $L^2(0, T; L^2(\Lambda))$. Then, up to a not relabeled subsequence, we got $\mathbb{P}$ -a.s convergence of $(u_{h,N}^l)_{h,N}$ towards $\tilde{u}$. Combining this with boundedness result on $(u_{h,N}^l)_{h,N}$ in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ stated in Lemma VI.12, we could affirm thanks again to [Dro01, Corollaire 1.3.3], that the convergence held strongly in $L^p(\Omega; L^2(0, T; L^2(\Lambda)))$ for $1 \leq p < 2$. Additionally, we deduced the identification $\tilde{u} = u$, where u was the weak limit of $(u_{h,N}^l)_{h,N}$ in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, satisfying in particular (VI.9). At last to complete the proof of Theorem VI.10, it remained to prove that u was the unique variational solution of Problem (VI.1) in the sense of Definition VI.2. This last point was immediate by passing to the limit in the initial numerical scheme (VI.6), since the nonlinear nature of g was no longer a problem.

Remark VI.14 (Improved convergence)

Note that since our finite-volume approximations were bounded in $L^\infty(0, T; L^2(\Omega; L^2(\Lambda)))$, using again Vitali's theorem [Dro01, Corollaire 1.3.3], it is possible (which we did not do in [BNSZ23a]) to improve the spaces in which their convergences are stated in Theorem VI.10: namely that the convergences of $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$ towards u hold strongly in $L^p(0, T; L^2(\Omega; L^2(\Lambda)))$ for any finite p .

Remark VI.15

Let me mention that all this argumentation was quite technical and required a fine knowledge of probability theory to approach an equation which at first sight seemed quite simple to study. On the one hand, the use of such a probabilistic artillery would allow us to consider in the future the finite-volume approximation of more complex stochastic parabolic problems, involving for example nonlinear second order operators. On the other hand, I would like to mention that after the publication of [BNSZ23a], I felt frustrated that we did not propose a simpler technique avoiding the use of “stochastic compactness method”. For this reason, I stuck to my first idea when we started working on this project with F. NABET, K. SCHMITZ and A. ZIMMERMANN, namely: how to adapt a famous technique for parabolic SPDE (see [DPZ92, PR07] for example) which allows to show the existence of variational solutions through a temporal discretization of the equation by using exponential weighted in time norm? This technique is presented in Chapter VII, for a convection-diffusion problem with multiplicative noise including the particular case of the stochastic heat equation studied in the present chapter.

*If for any arbitrary pair of subsequences $(u_{\tilde{h}_k, \tilde{N}_k}^l)_{\tilde{h}_k, \tilde{N}_k}$, $(u_{\hat{h}_k, \hat{N}_k}^l)_{\hat{h}_k, \hat{N}_k}$ of $(u_{h,N}^l)_{h,N}$, there exists a joint subsequence $(u_{h_k, \tilde{N}_k}^l, u_{h_k, \hat{N}_k}^l)_k$ converging in law to a probability measure η on $L^2(0, T; L^2(\Lambda))^2$ supported on the diagonal $\{(x, y) \in L^2(0, T; L^2(\Lambda))^2 \mid x = y\}$, then $(u_{h,N}^l)_{h,N}$ converges in probability towards u in $L^2(0, T; L^2(\Lambda))$.

Chapter VII

TPFA scheme for nonlinear diffusion-convection problems with a stochastic force

In this chapter are presented accepted and submitted contributions [BNSZ23b, BSZ23] in collaboration with F. NABET, K. SCHMITZ and A. ZIMMERMANN on the finite-volume approximation of non-linear parabolic problems set in a bounded domain of $\mathbb{R}^2$, under homogeneous Neumann boundary conditions and with a multiplicative noise in the sense of Itô. The considered discretization was semi-implicit in time and two-point flux in space. We proved its well-posedness in [BNSZ23b] and then, the aim of [BSZ23] was to address the convergence analysis of such a finite-volume scheme. For technical reasons, we established this convergence proof in the particular case of a linear convection term “ $\operatorname{div}_x(\mathbf{v}u)$ ”. By adapting well-known methods for the time-discretization of stochastic PDEs involving weighted in time norm (as presented in Chapter IV), we showed that the associated finite-volume approximation converged strongly towards the unique variational solution of the continuous problem.

VII.1 Introduction

Considering $T > 0$ a real, Λ a bounded, open, connected, and polygonal set of $\mathbb{R}^2$, $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$ and a standard, one-dimensional Brownian motion $(W(t))_{t \geq 0}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$, we were interesting in the following non-linear parabolic problem forced by a multiplicative stochastic noise:

$$\begin{cases} du - \Delta u \, dt + \operatorname{div}_x(\mathbf{v}f(u)) \, dt &= g(u)dW + \beta(u) \, dt & \text{in } \Omega \times (0, T) \times \Lambda, \\ u(0, \cdot) &= u_0 & \text{in } \Omega \times \Lambda, \\ \nabla u \cdot \mathbf{n} &= 0, & \text{on } \Omega \times (0, T) \times \partial\Lambda, \end{cases} \quad (\text{VII.1})$$

where div_x denoted the divergence operator with respect to the space variable and $\mathbf{n}$ the unit normal vector to $\partial\Lambda$ outward to Λ . We assumed the following hypotheses on the data:

$\mathcal{A}_1$: $u_0 \in L^2(\Omega; H^1(\Lambda))$ is $\mathcal{F}_0$ -measurable.

$\mathcal{A}_2$: $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function.

$\mathcal{A}_3$: $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with $\beta(0) = 0$.

$\mathcal{A}_4$: $\mathbf{v} \in \mathcal{C}^1([0, T] \times \bar{\Lambda}; \mathbb{R}^2)$, $\operatorname{div}_x(\mathbf{v}(t, x)) = 0$ for all $(t, x) \in [0, T] \times \Lambda$ and $\mathbf{v}(t, x) \cdot \mathbf{n}(x) = 0$ for all $(t, x) \in [0, T] \times \partial\Lambda$.

$\mathcal{A}_5$: $f : \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing Lipschitz-continuous function with $f(0) = 0$.

Remark VII.1

Note that the $H^1(\Lambda)$ regularity in space of the initial condition u_0 was only used in [BSZ23] to obtain the estimates (VII.8) and (VII.10) on the discrete H^1 semi-norm of the sequences $(u_{h,N}^l)_{h,N}$ and $(g(u_{h,N}^l))_{h,N}$. I give a hint in the Remark VII.13 that would weaken this hypothesis, unlike the previous chapter on the stochastic heat equation where I have no idea with regard to the technique employed.

I introduce in the next subsections some notations and make precise the functional setting.

VII.1.1 Notations

- $L_\beta \geq 0$ the Lipschitz constant of β .
- $L_g \geq 0$ the Lipschitz constant of g .
- $L_f \geq 0$ the Lipschitz constant of f .
- $C_{L_g} \geq 0$ a constant only depending on L_g and $g(0)$, satisfying for all $r \in \mathbb{R}$

$$|g(r)|^2 \leq C_{L_g}(1 + |r|^2). \quad (\text{VII.2})$$

- $\mathbb{E}[\cdot]$ denotes the expectation, i.e. the integral over Ω with respect to the probability measure $\mathbb{P}$.

Remark VII.2

As mentioned in Chapter VI, the existence of the constant C_{L_g} was given by Assumption $\mathcal{A}_2$. It allowed us to apply our scheme for square integrable, additive noise with appropriate measurability assumptions.

VII.1.2 Concept of solution and main result

The theoretical framework associated with Problem (VII.1) has been well established in the literature for decades. Indeed, we can find many existence and uniqueness results for various concepts of solutions associated with this problem such as mild solutions, variational solutions, pathwise solutions and weak solutions, see, e.g., [DPZ92] and [LR15]. In [BNSZ23b, BSZ23], we were interested in the concept of solution as defined below, called a variational solution in what follows:

Definition VII.3 (Variational solution)

A stochastic process u in $\mathcal{N}_W^2(0, T; L^2(\Lambda))^\dagger$ is a variational solution to Problem (VII.1) if it belongs to

$$L^2(\Omega; \mathcal{C}([0, T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0, T; H^1(\Lambda)))$$

and satisfies, for all $t \in [0, T]$,

$$u(t) - u_0 - \int_0^t \Delta u(s) ds + \int_0^t \operatorname{div}_x(\mathbf{v}(s, \cdot) f(u(s))) ds = \int_0^t g(u(s)) dW(s) + \int_0^t \beta(u(s)) ds$$

in $L^2(\Lambda)$ and $\mathbb{P}$ -a.s. in Ω , where Δ denotes the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary condition.

Existence, uniqueness and regularity of this variational solution is well-known in the literature, see, e.g., [Par75], [Par21], [KR81], [LR15].

VII.1.3 Aim of the studies

As far as we knew, stochastic problems of type (VII.1) had not been studied from a numerical point of view when we started working on them. In [BNSZ23b, BSZ23], our aim was to fill the gap left in the literature by proposing the analysis of a space and time-discretization of the non-linear diffusion-convection equation forced by a stochastic noise and under homogeneous Neumann boundary conditions (VII.1). The added value comparing to existing results was twofold:

[†] As mentioned in Part I, for a given separable Hilbert space X we denote by $\mathcal{N}_W^2(0, T; X)$ the separable Hilbert space of the predictable X -valued processes endowed with the norm $\|\phi\|_{\mathcal{N}_W^2(0, T; X)}^2 := \mathbb{E} \left[\int_0^T \|\phi\|_X^2 dt \right]$ (see [DPZ92] p.94).

- Firstly, the taking into account of a convection term $\operatorname{div}_x(\mathbf{v}f(u))$ which was very interesting from a modeling point of view and opened the door to many extensions (such as Stefan's problem, porous medium equation...).
- Secondly, the fact that we avoided the use of technical tools from the probabilist framework for the passage to the limit in the non-linear terms (such as the theorem of Prokhorov, Skorokhod's representation theorem, concept of martingale solution, Gyöngy-Krylov argument...) as we did in our previous work [BNSZ23a] exposed in Chapter VI for the stochastic heat equation (corresponding to Problem (VII.1) with $\mathbf{v} = \mathbf{0}$ and $\beta = 0$).

VII.1.4 The finite-volume scheme

Under the finite-volume notations introduced in Chapter VI, we firstly defined the vector $u_h^0 \equiv (u_K^0)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ by the discretization of the initial condition u_0 of Problem (VII.1) over each control volume:

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}. \quad (\text{VII.3})$$

The finite-volume scheme we proposed read, for this given initial $\mathcal{F}_0$ -measurable random vector $u_h^0 \in \mathbb{R}^{d_h}$, as follows: for any $n \in \{0, \dots, N-1\}$, knowing $u_h^n \equiv (u_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$, we searched for $u_h^{n+1} \equiv (u_K^{n+1})_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ such that, $\mathbb{P}$ -a.s. in Ω , the vector u_h^{n+1} was a solution to the following random equations

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} f(u_\sigma^{n+1}) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ &= \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}, \end{aligned} \quad (\text{VII.4})$$

where, the $(d-1)$ -dimensional Lebesgue measure was denoted by γ ,

$$v_{K,\sigma}^{n+1} = \frac{1}{\Delta t |\sigma|} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(t, x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) dt,$$

and u_σ^{n+1} denoted the upstream value at time t_{n+1} with respect to σ defined as follows: if $\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K$ was the interface between the control volumes K and L (i.e. $\sigma = K|L$), u_σ^{n+1} was equal to u_K^{n+1} if $v_{K,\sigma}^{n+1} \geq 0$ and to u_L^{n+1} if $v_{K,\sigma}^{n+1} < 0$. Note also that $W^{n+1} - W^n$ denoted the increments of the Brownian motion between t_{n+1} and t_n :

$$W^{n+1} - W^n = W(t_{n+1}) - W(t_n) \text{ for } n \in \{0, \dots, N-1\}.$$

Remark VII.4 (Equivalent writtens of the scheme)

Using the divergence-free property of $\mathbf{v}$ (i.e. $\operatorname{div}_x(\mathbf{v}(t, x)) = 0$ for all $(t, x) \in [0, T] \times \Lambda$), we could equivalently rewrite the scheme (VII.4) in the following way:

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} (f(u_\sigma^{n+1}) - f(u_K^{n+1})) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ &= \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}. \end{aligned} \quad (\text{VII.5})$$

Moreover, using (VII.5) and the fact that* $v_{K,\sigma}^{n+1} = (v_{K,\sigma}^{n+1})^+ - (v_{K,\sigma}^{n+1})^-$, another equivalent formulation of the scheme (VII.4) was given by

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (v_{K,\sigma}^{n+1})^- (f(u_K^{n+1}) - f(u_L^{n+1})) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ &= \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}. \end{aligned} \quad (\text{VII.6})$$

*For any $a \in \mathbb{R}$, $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$.

In [BNSZ23b], for a given initial vector u_h^0 defined by (VII.3), we proved the existence of a discrete solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ to Problem (VII.4) thanks to [Lio69, Lemma 4.3], and we derived its uniqueness by adapting arguments from [EGH00, Proposition 26.1]:

Proposition VII.5 (Existence and uniqueness of a discrete solution)

Assume that hypotheses $\mathcal{A}_1$ to $\mathcal{A}_5$ hold. Let $\mathcal{T}$ be an admissible finite-volume mesh of Λ in the sense of Definition VI.3 with a mesh size h and $N \in \mathbb{N}^$. Then, there exists a unique solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ to Problem (VII.4) associated with the initial vector u_h^0 defined by (VII.3). Additionally, for any $n \in \{0, \dots, N\}$, u_h^n is a $\mathcal{F}_{t_n}$ -measurable random vector.*

The solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ of the scheme (VII.3)-(VII.4) was then used to build the right and left finite-volume approximations $u_{h,N}^r$ and $u_{h,N}^l$ defined by (VI.4) for the variational solution u of Problem (VII.1).

VII.2 Main result and overview of the proof

The main result of [BSZ23] was to show the strong convergence of our finite-volume approximations towards the unique variational solution of Problem (VII.1) in the sense of Definition VII.3. For technical reasons in the identification of the limit coming from the discretization of the convection term “ $\operatorname{div}_x(\mathbf{v}f(u))$ ”, we proved such a convergence result in the particular case where f was equal to the identity function. Using the same finite-volume notations as in Chapter VI, our main result was:

Theorem VII.6 (Convergence of the scheme)

Assume that hypotheses $\mathcal{A}_1$ to $\mathcal{A}_4$ hold and that f is the identity function. Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible finite-volume meshes of Λ in the sense of Definition VI.3 such that the mesh size h_m tends to 0 and let $(N_m)_{m \in \mathbb{N}} \subset \mathbb{N}^$ be a sequence of positive numbers which tends to infinity. For a fixed $m \in \mathbb{N}$, let u_{h_m, N_m}^r and u_{h_m, N_m}^l be respectively the right and left in time finite-volume approximations defined by (VI.4), (VII.3)-(VII.4) with $\mathcal{T} = \mathcal{T}_m$ and $N = N_m$. Then $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$ converge strongly in $L^p(0, T; L^2(\Omega; L^2(\Lambda)))$ (for any finite p) to the variational solution of Problem (VII.1) in the sense of Definition VII.3.*

The main steps of the proof of Theorem VII.6 were the following ones. We started by the derivation of stability estimates in suitable functional spaces satisfied by the sequences $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$, $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$, $(g(u_{h_m, N_m}^l))_{m \in \mathbb{N}}$, $(\beta(u_{h_m, N_m}^r))_{m \in \mathbb{N}}$ and $(f(u_{h_m, N_m}^r))_{m \in \mathbb{N}}$. These estimates allowed us to extract weakly converging subsequences and to pass to the limit in the numerical scheme. Firstly, we obtained the existence of an Itô stochastic process u , the weak limit (up to subsequences) of $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$. Secondly, by taking advantage of the use of exponential weighted in time norm (as employed in Chapter IV) and adapting it to our TPFA scheme, we proved (in the particular case where f was equal to the identity function) that this convergence finally held strongly. Thirdly, using this last information, we were able to identify weak limits coming from the non-linear terms $(g(u_{h_m, N_m}^l))_{m \in \mathbb{N}}$ and $(\beta(u_{h_m, N_m}^r))_{m \in \mathbb{N}}$. Fourthly, we concluded that u was the unique variational solution of (VII.1) in the sense of Definition VII.3 and fifthly that subsequences were not needed anymore. Let me review these stages under assumptions $\mathcal{A}_1$ to $\mathcal{A}_5$ in order to understand why the nonlinearity of the convection term “ $\operatorname{div}_x(\mathbf{v}f(u))$ ” is (up to now) an issue.

For $m \in \mathbb{N}^*$, let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes of Λ in the sense of Definition VI.3 such that the mesh size h_m tends to 0 when m tends to $+\infty$, and let $(N_m)_{m \in \mathbb{N}} \subseteq \mathbb{N}^*$ be a sequence with $\lim_{m \rightarrow +\infty} N_m = +\infty$ and set $\Delta t_m := \frac{T}{N_m}$. For the sake of simplicity, let me use in what follows the notations $\mathcal{T} = \mathcal{T}_m$, $h = \operatorname{size}(\mathcal{T}_m)$, $\Delta t = \Delta t_m$ and $N = N_m$. In [BNSZ23b, BSZ23], after multiplying the numerical scheme (VII.6) with u_K^{n+1} , combining arguments on monotone functions from the hyperbolic case to handle the discretization of the convection term (see Lemma I.18 from Chapter I) with the ones developed for the stochastic heat equation in Chapter VI, we arrived at:

Proposition VII.7 (Bounds on the discrete solutions)

There exists a constant $C_0 > 0$, depending only on u_0 , C_{L_g} , L_β , $|\Lambda|$ and T such that for any $N \in \mathbb{N}^$ large enough and any $h \in \mathbb{R}_+^*$*

$$\mathbb{E} \left[\|u_h^n\|_{L^2(\Lambda)}^2 \right] + \mathbb{E} \left[\sum_{k=0}^{n-1} \|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2 \right] + \Delta t \sum_{k=0}^{n-1} \mathbb{E} \left[|u_h^{k+1}|_{1,h}^2 \right] \leq C_0, \quad \forall n \in \{1, \dots, N\}.$$

Thanks to Proposition VII.7 and the Lipschitz nature of g , β and f , we obtained successively the following bounds on the right and left finite-volume approximations defined by (VI.4):

Lemma VII.8

The sequences $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$ are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}^*$ and $h \in \mathbb{R}_+^*$. Additionally, $(u_{h,N}^l)_{h,N}$ is bounded in $\mathcal{N}_W^2(0, T; L^2(\Lambda))$.

Lemma VII.9

There exist a constant $C_1 \geq 0$ depending only on u_0 , C_{L_g} , L_β , $|\Lambda|$, and T , and a constant $C_2 \geq 0$ additionally depending on the mesh regularity $\text{reg}(\mathcal{T})$ (defined by (VI.3)), such that

$$\int_0^T \mathbb{E} [|u_{h,N}^r(t)|_{1,h}^2] dt \leq C_1 \quad (\text{VII.7})$$

and

$$\int_0^T \mathbb{E} [|u_{h,N}^l(t)|_{1,h}^2] dt \leq C_2. \quad (\text{VII.8})$$

Lemma VII.10

The sequences $(g(u_{h,N}^r))_{h,N}$, $(g(u_{h,N}^l))_{h,N}$, $(\beta(u_{h,N}^r))_{h,N}$, $(\beta(u_{h,N}^l))_{h,N}$, $(f(u_{h,N}^r))_{h,N}$, and $(f(u_{h,N}^l))_{h,N}$ are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}^*$ and $h \in \mathbb{R}_+^*$.

Lemma VII.11

There exists a constant $C_3 \geq 0$ depending only on u_0 , L_g , L_β , L_f , $|\Lambda|$, and T , and a constant $C_4 \geq 0$ additionally depending on the mesh regularity $\text{reg}(\mathcal{T})$ (defined by (VI.3)), such that

$$\int_0^T \mathbb{E} [|g(u_{h,N}^r(t))|_{1,h}^2] dt + \int_0^T \mathbb{E} [|f(u_{h,N}^r(t))|_{1,h}^2] dt \leq C_3 \quad (\text{VII.9})$$

and

$$\int_0^T \mathbb{E} [|g(u_{h,N}^l(t))|_{1,h}^2] dt + \int_0^T \mathbb{E} [|f(u_{h,N}^l(t))|_{1,h}^2] dt \leq C_4. \quad (\text{VII.10})$$

Using all these estimates, we obtained the following weak convergence results:

Proposition VII.12 (Extraction of weakly converging subsequences)

Up to subsequences denoted in the same way, the following holds:

- There exists $u \in L^2(\Omega; L^2(0, T; H^1(\Lambda))) \cap \mathcal{N}_W^2(0, T; L^2(\Lambda))$ such that

$$u_{h,N}^l \rightarrow u \text{ and } u_{h,N}^r \rightarrow u \text{ both weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

- There exists g_u in $L^2(\Omega; L^2(0, T; H^1(\Lambda))) \cap \mathcal{N}_W^2(0, T; H^1(\Lambda))$ such that

$$g(u_{h,N}^r) \rightarrow g_u \text{ and } g(u_{h,N}^l) \rightarrow g_u \text{ both weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

- There exists β_u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that

$$\beta(u_{h,N}^r) \rightarrow \beta_u \text{ and } \beta(u_{h,N}^l) \rightarrow \beta_u \text{ both weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

- There exists f_u in $L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ such that

$$f(u_{h,N}^r) \rightarrow f_u \text{ and } f(u_{h,N}^l) \rightarrow f_u \text{ both weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

Remark VII.13

The fact that g_u belonged to $\mathcal{N}_W^2(0, T; H^1(\Lambda))$ was one of the key points to prove Proposition VII.14. In [BSZ23], we showed it by using the estimate (VII.10) on $\int_0^T \mathbb{E} [|g(u_{h,N}^l(t))|_{1,h}^2] dt$, which resulted from the $L^2(\Omega; H^1(\Lambda))$ regularity of the initial condition u_0 . A posteriori, I think that this hypothesis could be relaxed by assuming only a $L^2(\Omega; L^2(\Lambda))$ regularity for u_0 . Indeed, in my opinion, it should be possible to recover the information “ g_u in $\mathcal{N}_W^2(0, T; H^1(\Lambda))$ ” by using the estimate (VII.9) on $\int_0^T \mathbb{E} [|g(u_{h,N}^r(t))|_{1,h}^2] dt$ and the fact that $(g(u_{h,N}^l))_{h,N}$ is bounded in $\mathcal{N}_W^2(0, T; L^2(\Lambda))$, both only requiring u_0 to be in $L^2(\Omega; L^2(\Lambda))$.

Then, our aim was to use these weak convergence results in order to pass to the limits in our finite-volume scheme. To do so, we set a $\mathbb{P}$ -measurable set $A \in \mathcal{A}$, a temporal test function $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and a spatial test function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\nabla \varphi \cdot \mathbf{n} = 0$ on $\partial\Lambda$. For $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$ and $t \in [t_n, t_{n+1}]$, after multiplying (VII.5) with $\mathbb{1}_A \xi(t) \varphi(x_K)$, then summing over each control volume $K \in \mathcal{T}$, integrating over each time interval $[t_n, t_{n+1}]$, and finally summing over $n = 0, \dots, N-1$, we obtained, after taking the expectation:

$$S_{1,m} + S_{2,m} + S_{3,m} + S_{4,m} = S_{5,m}, \quad (\text{VII.11})$$

where

$$\begin{aligned} S_{1,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbb{1}_A \xi(t) m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} \varphi(x_K) dt \right] \\ S_{2,m} &= -\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbb{1}_A \xi(t) m_K g(u_K^n) \frac{W^{n+1} - W^n}{\Delta t} \varphi(x_K) dt \right] \\ S_{3,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbb{1}_A \xi(t) \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \varphi(x_K) dt \right] \\ S_{4,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbb{1}_A \xi(t) \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} (f(u_\sigma^{n+1}) - f(u_K^{n+1})) \varphi(x_K) dt \right] \\ S_{5,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbb{1}_A \xi(t) m_K \beta(u_K^{n+1}) \varphi(x_K) dt \right]. \end{aligned}$$

After studying separately the limit as m goes to $+\infty$ of each of these terms, using in particular the density of the set $\{\varphi \in \mathcal{D}(\mathbb{R}^2) \mid \nabla \varphi \cdot \mathbf{n} = 0 \text{ on } \partial\Lambda\}$ in $H^1(\Lambda)$ given by [Dro02, Theorem 1.1], and arguing as in [BNSZ23b] to prove path's continuity of u , we arrived at:

Proposition VII.14

The weak limit u of our finite-volume approximate solutions $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$ introduced in Proposition VII.12 has $\mathbb{P}$ -a.s. continuous paths with values in $L^2(\Lambda)$ and satisfies for all $t \in [0, T]$,

$$u(t) = u_0 + \int_0^t \Delta u(s) ds - \int_0^t \operatorname{div}_x (\mathbf{v}(s, \cdot) f_u(s)) ds + \int_0^t g_u(s) dW(s) + \int_0^t \beta_u(s) ds,$$

in $L^2(\Lambda)$ and $\mathbb{P}$ -a.s. in Ω , where Δ denotes the Laplace operator on $H^1(\Lambda)$ associated with the formal Neumann boundary condition.

Additionally, we proved thanks to Itô's formula that such a weak limit u also satisfies, for any parameter $c > 0$, a stochastic energy equality:

Lemma VII.15 (Stochastic energy equality)

For any $c > 0$, the stochastic process u introduced in Proposition VII.12 satisfies the following stochastic energy equality:

$$\begin{aligned}
& e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] + 2 \int_0^t e^{-cs} \mathbb{E} [\|\nabla u(s)\|_{L^2(\Lambda)}^2] ds \\
& - 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x) f_u(s, x)) \cdot \nabla u(s, x) dx \right] ds \\
& = \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] - c \int_0^t e^{-cs} \mathbb{E} [\|u(s)\|_{L^2(\Lambda)}^2] ds + \int_0^t e^{-cs} \mathbb{E} [\|g_u(s)\|_{L^2(\Lambda)}^2] ds \\
& + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta_u(s, x) u(s, x) dx \right] ds, \quad \forall t \in [0, T].
\end{aligned} \tag{VII.12}$$

To conclude, it remained to identify the weak limits g_u , β_u and f_u . Our idea was to adapt a standard technique in the time-approximation of parabolic SPDEs involving weighted in time norm. In that direction, we had to construct, for our finite-volume approximations $u_{h,N}^r$ and $u_{h,N}^l$, a discrete version of the stochastic energy equality (VII.12), where the key argument was the consideration of a clever parameter $c > 0$. Namely, by choosing $c > 0$ depending only on L_g and L_β such that for any N big enough

$$\begin{aligned}
& \int_0^t e^{-cs} \mathbb{E} [\|g(u_{h,N}^r(s)) - g(u(s))\|_{L^2(\Lambda)}^2] ds - ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} [\|u_{h,N}^r(s) - u(s)\|_{L^2(\Lambda)}^2] ds \\
& + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\beta(u_{h,N}^r(s, x)) - \beta(u(s, x)))(u_{h,N}^r(s, x) - u(s, x)) dx \right] ds \leq 0,
\end{aligned}$$

we arrived, from the numerical scheme (VII.6) multiplied by $\Delta t e^{-ct_n} u_K^{n+1}$, at the following inequality

$$\begin{aligned}
& \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|u_{h,N}^r(s)\|_{1,h}^2] ds dt \\
& \leq \int_0^T \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g(u_{h,N}^r(s, x)) g(u(s, x)) dx \right] ds dt \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s))\|_{L^2(\Lambda)}^2] ds dt \\
& - 2ce^{-c\Delta t} \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} u_{h,N}^r(s, x) u(s, x) dx \right] ds dt \\
& + ce^{-c\Delta t} \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|u(s)\|_{L^2(\Lambda)}^2] ds dt + T (e^{c\Delta t} - 1) \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] \\
& + T\Delta t \left((1 + 2L_\beta) \|u_{h,N}^r\|_{L^\infty(0,T;L^2(\Omega;L^2(\Lambda)))}^2 + L_g^2 \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] + 2cL_\beta \|u_{h,N}^r\|_{L^2(\Omega;L^2(0,T;L^2(\Lambda)))}^2 \right) \\
& + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s, x)) u(s, x) dx \right] ds dt \\
& + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u(s, x)) (u_{h,N}^r(s, x) - u(s, x)) dx \right] ds dt.
\end{aligned} \tag{VII.13}$$

Note that as for the obtention of stability estimates stated in Proposition VII.7, the contribution of the discretization of the flux term disappeared here due to its sign. By passing to the superior limit in (VII.13) we obtained

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt + 2 \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] dt - c \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g_u(s, x) g(u(s, x)) dx \right] ds dt - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s))\|_{L^2(\Lambda)}^2] ds dt \\
& + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta_u(s, x) u(s, x) dx \right] ds dt.
\end{aligned}$$

Then, thanks to the stochastic energy equality (VII.12) we arrived at

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt + 2 \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|\nabla u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& - 2 \int_0^T e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x) f_u(s, x)) \cdot \nabla u(s, x) dx \right] ds \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g_u(s)\|_{L^2(\Lambda)}^2] ds dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g_u(s, x) g(u(s, x)) dx \right] ds dt \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s))\|_{L^2(\Lambda)}^2] ds dt,
\end{aligned}$$

which yielded to

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt + 2 \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|\nabla u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& - 2 \int_0^T e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x) f_u(s, x)) \cdot \nabla u(s, x) dx \right] ds \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s)) - g_u(s)\|_{L^2(\Lambda)}^2] ds dt.
\end{aligned}$$

In order to pursue our reasoning, we needed to control the inferior limit on the left hand side of this last inequality. For this reason, we adapted to the evolutionary in time and stochastic case a result from [HM01] (Lemma 2.2):

Lemma VII.16

For any $c > 0$, the stochastic process u introduced in Proposition VII.12 satisfies the following inequality:

$$\int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} |\nabla u(s, x)|^2 dx \right] ds dt \leq \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt. \quad (\text{VII.14})$$

Using this lemma, we obtained

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt + \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s)) - g_u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt - 2 \int_0^T e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x) f_u(s, x)) \cdot \nabla u(s, x) dx \right] ds. \quad (\text{VII.15})
\end{aligned}$$

By assuming that f was the identity function, thus $f_u = u$ and by the divergence free property of $\mathbf{v}$, the following held

$$2 \int_0^T e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x) u(s, x)) \cdot \nabla u(s, x) dx \right] ds = 0.$$

In this particular case, by weak convergence of $(u_{h,N}^r)_m$ towards u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, the following inequality was always true

$$\int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt \leq \liminf_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt,$$

and by injecting it in (VII.15), this allowed us to conclude firstly that $g_u = g(u)$, secondly that $(u_{h,N}^r)_m, (u_{h,N}^l)_m$ converged strongly to u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, that $\beta_u = \beta(u)$ and at last that u was the unique variational solution of Problem (VII.1) in the sense of Definition VII.3. The improved convergences of the finite-volume approximations $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$ in the spaces $L^p(0, T; L^2(\Omega; L^2(\Lambda)))$ (for any finite p) was due to their boundedness in $L^\infty(0, T; L^2(\Omega; L^2(\Lambda)))$ combined with Vitali's theorem [Dro01, Corollaire 1.3.3].

Remark VII.17

Note that for the moment, we don't know how to conclude with this technique if f is nonlinear since the presence of $-2 \int_0^t e^{-cs} \mathbb{E} \left[\int_\Lambda (\mathbf{v}(s, x) f_u(s, x)) \cdot \nabla u(s, x) dx \right] ds$ in (VII.15) is an issue.

Perspectives (english version)

In conclusion, since my arrival in Marseille, I have built up a network of collaborations between France, Italy and Germany, which has enabled me to actively investigate the themes presented in this manuscript: on the one hand, around the finite-volume approximation of first and second-order scalar conservation laws forced by multiplicative noise, and on the other, on questions of theoretical analysis for stochastic parabolic problems applied to solid mechanics. The resulting work naturally opens the door to research directions that will enable me to perpetuate these collaborations, on the one hand, and to supervise doctoral work on my own, on the other. Below, I outline the main directions of my scientific project: firstly, the ones in line with the contributions presented in Parts 1, 2 and 3 respectively, and then those at the interface between Parts 1 and 3, and 2 and 3. Here I use the same notation as before: let $(W(t))_{t \geq 0}$ be a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, D a smooth bounded domain of $\mathbb{R}^d$ ($d \geq 1$), Λ a bounded, open, connected and polygonal subset of $\mathbb{R}^d$ ($1 \leq d \leq 3$) and $g : \mathbb{R} \rightarrow \mathbb{R}$ a Lipschitz-continuous function.

Finite-volume approximation of first-order scalar conservation laws forced by a multiplicative noise: limit of the study, error estimate and random flux

Following our initial work on the space-time discretization by finite-volume methods of stochastic scalar conservation laws of the type (I.1) presented in Part 1, several numerical studies have been carried out around this theme: numerical simulations of the stochastic Burgers equation [ABGH15], convergence of finite-volume schemes with monotone numerical fluxes using a kinetic approach and by considering an infinite dimensional noise (with additionally a space dependence in the stochastic integrand) [DV18, DV20], convergence of a Godunov scheme in the case of multi-dimensional noise [FGH18], convergence of finite-difference schemes and then flux-splitting's one with Lévy noise [KMV16, Maj18], time-splitting operator's method with semi-linear noise [KS18] or finite-volume approximation of the invariant measure in the viscous case [BMR22]. Other related studies include the development of Galerkin-Discontinuous methods [LST20], spatial and then temporal semi-discretization of fractional stochastic conservation laws [KV23, BM], and temporal, spatial and then spatio-temporal approximations of conservation laws with rough fluxes [GPS16, Hof16, HKRS20]. In view of the recent bibliography on the subject, in addition to the many questions that the numerical implementation of these stochastic hyperbolic problems would raise, the research directions that interest me are the following:

- Firstly, there is the question of the necessity of the “ $k = o(h)$ ” hypothesis, linking the discretization parameters in time k and space h , on which the convergence of all the finite-volume schemes I have studied in the hyperbolic framework is based. Indeed, in a later work (consisting of two joint publications [DV18, DV20]), the authors showed the convergence of a finite-volume scheme with monotone numerical fluxes under a classical CFL assumption of the type “ $k = \mathcal{O}(h)$ ”. To do this, they employed a kinetic approach and a mode of convergence that is weak (in the probabilistic sense) with respect to the random variable (convergence in law). This raises the question of whether, in our case, the assumption “ $k = o(h)$ ” can also be relaxed to “ $k = \mathcal{O}(h)$ ”? Or is it required because of the entropy approach? The compactness tool used? Or for some other reason we haven't yet identified?
- Secondly, the question of estimating the approximation error committed by our numerical scheme is obviously a point I wish to investigate. The comparison presented in Chapter III between the finite-volume approximation $(u_{\mathcal{T},k})_{\mathcal{T},k}$ of Problem (III.1), defined by (III.2)-(III.3)-(III.4), and a measure-valued entropy solution of the same problem (in the sense of Definition III.4), is an important preliminary step towards obtaining such estimates. The

idea would be to quantify the strong error (in the probabilistic sense [BHS18]) by controlling, for any compact K of $\mathbb{R}^d \times [0, T]$, a term of the type

$$\mathbb{E} \left[\left(\int_K |u_{\mathcal{T},k}(x, t) - u(x, t)| dx dt \right)^2 \right],$$

where u denotes the unique entropy solution of Problem (III.1) in the sense of Definition III.3. To do this, I have in mind to relate the results of Chapter III to the deterministic techniques developed in [CH00], where error estimates for second-order schemes were obtained using smooth entropies.

- Thirdly, the inclusion of random effects in the flux term (excluding "rough fluxes") is a question I was often asked when I gave oral presentations on this topic, and which I find relevant in terms of applications. From a theoretical point of view, I think that the extension of the result of existence and uniqueness of an entropy solution obtained in [BCG16b] would fit well, and could be the subject of a Master thesis. Considering a flux of the type

$$\mathbb{R}^d \times [0, T] \times \Omega \times \mathbb{R} \ni (x, t, \omega, u) \mapsto \vec{v}(x, t, \omega) f(u) \in \mathbb{R}^d,$$

where f would satisfy the same assumptions as in [BCG16b] (stated in Chapter I), while the regularity of the random vector $\vec{v} : \mathbb{R}^d \times [0, T] \times \Omega \rightarrow \mathbb{R}^d$ would remain to be defined. On the numerical analysis side, I expect the work involved in constructing the numerical fluxes and analyzing the scheme to be more substantial.

Applications in solid mechanics: consideration of non-local effects and irreversibility condition

Following on from the work presented in Part 2, discussions are underway with my German and Italian colleagues on these two directions:

- The first is a generalization of the study [BBB⁺17] based on the stochastic Allen-Cahn equation with constraint, presented in Chapter IV. For this purpose, we have in mind, together with K. SCHMITZ and A. ZIMMERMANN, to consider a p -Laplacian operator (with $p \geq 2$) instead of the standard Laplacian in System (IV.1), which gives:

$$du + (\xi - \Delta_p u) dt = (w_s(u) + f) dt + g(u) dW \quad \text{and} \quad \xi \in \partial I_{[0,1]}(u) \text{ in } \Omega \times (0, T) \times D,$$

for an initial data $u_0 \in W^{1,p}(D)$ and where $\Delta_p : u \mapsto \operatorname{div}_x(|\nabla u|^{p-2} \nabla u)$ is associated with homogeneous Neumann boundary conditions. The underlying idea is to take into account more general non-local effects in the damage process, as proposed in [RR14] for another form of equation. Note that the monotonicity techniques I used for the standard Laplacian (which corresponds to the case $p = 2$) in [BBB⁺17] are *a priori* compatible with this p -Laplacian operator, which suggests that their adaptation to this more general case is conceivable. Thanks to the expertise of my German collaborators on this operator, my fine analysis of the stochastic Allen-Cahn equation with constraint, and the financial support we have available until the end of 2025 to meet (PHC Procope France-Allemagne by Campus France), we shouldn't have to wait too long to obtain original results on this theme.

- The second follows on from a reflection initiated with G. BONFANTI and R. ROSSI in 2017, during a visit with my colleague F. LEBON at the University of Brescia. The idea was to include stochastic effects in a deterministic phase change model studied by G. BONFANTI [BFL00]:

$$\begin{cases} \partial_t \vartheta + \partial_t \chi - \Delta \vartheta = 0 & \text{in } D \times (0, T), \\ \partial_t \chi - \Delta \chi + \partial I_{[0,+\infty]}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni \vartheta & \text{in } D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ and } \nabla \chi \cdot \mathbf{n} = 0 & \text{in } \partial D \times (0, T), \end{cases}$$

where the unknowns ϑ and χ satisfied initial conditions at $t = 0$. When ϑ denotes the relative temperature of a two-phase material (located in D) and χ denotes the *phase-field* (e.g. the local proportion of one phase), this doubly non-linear system can describe a large class of phase transition including irreversible phase changes (for instance, solidification of glue, cooking an egg, ...). The authors' idea (and therefore ours) was to focus on the glue solidification process in a tube. Noting χ the volume fraction of glue in the solid state, assuming the absence of air bubbles in our tube, the fraction in the liquid state is therefore given by $1 - \chi$. This type of system then describes the progressive transformation of the glue from the liquid to the solid state, induced by thermal actions carried

by the unknown ϑ and by movements at the microscopic scale, adding moreover a condition of irreversibility on the solidification process *via* the term $\partial I_{[0,+\infty[}(\partial_t \chi)$. Initially, we wanted to translate into our model the random influence of the glue solidification rate $\partial_t \chi$ on temperature variations ϑ by considering the following system:

$$\begin{cases} \partial_t \left(\vartheta - \int_0^\cdot g(\partial_t \chi) dW \right) + \partial_t \chi - \Delta \vartheta = 0 & \text{in } \Omega \times D \times (0, T), \\ \partial_t \chi - \Delta \chi + \partial I_{[0,+\infty[}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni \vartheta & \text{in } \Omega \times D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ and } \nabla \chi \cdot \mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0, T), \end{cases}$$

with a contribution of $\partial_t \chi$ in the stochastic integral and given initial conditions. After several reflections on this system, we are short of ideas on how to control all these nonlinearities on χ and $\partial_t \chi$, our aim being to propose an existence and uniqueness result of solutions. We had in mind to use a “à la Moreau-Yosida” regularization of the subdifferentials $\partial I_{[0,+\infty[}$ and $\partial I_{[0,1]}$ as we had done in [BBB⁺17], but already the study of the regularized system poses difficulties. For these reasons, we decided to finally introduce stochasticity into the model by considering the case (which seems more reasonable to us) of random variations in the governing equation of the temperature ϑ , *i.e.* without involving $\partial_t \chi$ in our noise. More precisely, we are currently interested in the following problem:

$$\begin{cases} \partial_t \left(\vartheta - \int_0^\cdot g(\vartheta) dW \right) + \partial_t \chi - \Delta \vartheta = 0 & \text{in } \Omega \times D \times (0, T), \\ \partial_t \chi - \Delta \chi + \partial I_{[0,+\infty[}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni \vartheta & \text{in } \Omega \times D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ and } \nabla \chi \cdot \mathbf{n} = 0 & \text{in } \Omega \times \partial D \times (0, T), \end{cases}$$

with given initial conditions. To make progress on this project, we realized that we need time to meet: our skills are highly complementary on this subject, making it very difficult for each of us to operate on our own. With this in mind, we regularly respond to calls for projects to obtain funding for trips between France and Italy, and are considering taking part in the 2024 campaign of the Institut de Mécanique et d’Ingénierie (on which my laboratory depends) for a post-doctoral fellowship.

Finite-volume schemes for stochastic parabolic PDEs: non-linear convection, non-linear diffusion and Stefan’s problem

The research presented in Part 3 was carried out with the idea of serving as preliminary work to propose a finite-volume approximation of the following diffusion-convection equation:

$$du - \Delta \varphi(u) dt + \operatorname{div}_x (\mathbf{v} f(u)) dt = g(u) dW + \beta(u) dt \quad \text{in } \Omega \times (0, T) \times \Lambda,$$

under homogeneous boundary conditions (Neumann’s or Dirichlet’s type) on $\partial \Lambda$, with a given initial condition at $t = 0$ and where formally $\Delta \varphi(u) = \operatorname{div}_x (\varphi'(u) \nabla u)$. By considering $\varphi, f, \beta : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz-continuous functions and $\mathbf{v} \in \mathcal{C}^1([0, T] \times \bar{\Lambda}; \mathbb{R}^d)$ a divergence free vector, the study of the following three specific cases could be used as a basis for a PhD thesis:

- **First case:** $\varphi(u) = u$. Under Neumann homogeneous boundary conditions, this is a generalization of the convergence result presented in Chapter VII to the case of a non-linear flux. Remember that the technical challenge lies solely in the non-linearity of the flux $\mathbf{v} f$ for which we are unable to identify the weak limit of the corresponding discrete term. For this reason, for the moment we only have a convergence result in the case where the function f is linear: either because the compactness tool used is insufficient or badly employed, or because the chosen scheme does not return enough information, or for some other reason not yet understood. I then see two options to consider for dealing with the non-linear case: to adapt the probabilistic techniques used in Chapter VI on the stochastic heat equation to show the convergence of the TPFA scheme introduced in Chapter VII, or to propose a DDFV[†] scheme to approximate our problem. The added value of such a scheme is that it allows, among other things, to define a discrete gradient DDFV as well as a discrete divergence in duality with the latter. I would like to see whether this property is sufficient to generalize the technique for identifying weak limits coming from non-linear terms (based on the use of weighted exponentials) presented in Chapter VII. Initially, looking at how to adapt these DDFV methods to the simpler case of the stochastic heat equation (VI.1) could be the subject of a Master thesis research project co-supervised with F. NABET, which is a specialist in this type of schemes in the deterministic case.

[†]Discrete Duality Finite Volume. See for example [ABH07, Kre11].

- **Second case:** φ is a nondecreasing function (but φ is not necessarily increasing, it can be constant on an interval which has a positive Lebesgue measure) and $f \equiv \beta \equiv 0$. In this case, the problem is a “degenerate” parabolic one and is known in the deterministic case (*i.e.* when $g \equiv 0$) as “Stefan’s problem”. It is a simplified mathematical model describing for example the evolution of the boundary between two phases of a material undergoing a phase change (with homogeneous Dirichlet boundary conditions, see [EG23]). This is a problem that we wished to analyze with J. CHARRIER and T. GALLOUËT as part of V. CASTEL’s PhD thesis (2016-2018), and which first motivated the numerical study carried out on the stochastic heat equation. In collaboration with F. NABET, K. SCHMITZ and A. ZIMMERMANN, our approach would be to generalize the convergence work outlined in Chapter VI on the stochastic heat equation (VI.1) for a TPFA scheme, again using probabilistic compactness methods, in which my German collaborators are experts. Note that for this we were inspired by the proof of convergence of the scheme developed in [EGH00] for this famous Stefan’s problem. This used in particular estimates on the time and space translations of the discrete version of $\varphi(u)$. A very recent preprint has been published on the subject [DKL23], where the authors have shown a general result on the convergence of a large class of numerical schemes, belonging to the “Gradient Discretization Method” (GDM) class, to a martingale[†] solution of the stochastic Stefan problem. It would be interesting to see if we can show in the case of a TPFA scheme (which does not belong to the class of classical GDMs treated in [DKL23], but to a particular non-symmetric version [DE17]), a strong convergence result to a variational solution in $L^p(\Omega; L^2(0, T; L^2(\Lambda)))$, with $p \in [1, 2)$, as we did in [BNSZ23a] (see Chapter VI).
- **Third case:** φ increasing and f linear. Under suitable boundary conditions, the idea of this last case is to see if the techniques developed in Chapter VII with a TPFA scheme are still valid in the case of a non-degenerating and non-linear diffusion $\Delta\varphi(u)$, so that the considered problem always remains parabolic. A recent work has been addressed in that direction in [BGV23], where the authors investigated what they called the “singular-degenerate parabolic case” (which corresponds to $\varphi(u) = |u|^{p-2}u$ with $p > 1$ and $f \equiv \beta \equiv 0$). They proved the convergence of a fully discrete numerical approximation (using finite-elements in space) towards the very weak solution of their problem, in particular by taking advantage of the monotonicity of the mapping in the diffusion term. In our case, a combination with some arguments employed in the theoretical analysis of the continuous case (see for instance [Val08] in a more general framework) would then perhaps be necessary. If this first approach does not work, using and generalizing (to the convection term) the “stochastic compactness” arguments employed for the study of the previous case would be another possible angle of attack.

The ideal would be to accompany the study of each of these cases with numerical simulations.

Finite-volume approximation of scalar conservation laws with a stochastic force and a degenerate diffusion

A research direction naturally linking Parts 1 and 3 of this manuscript, as well as a work carried out within the framework of my PhD thesis in collaboration with G. VALLET and P. WITTBOLD [BVW15], concerns the fully numerical approximation of the Cauchy problem for the following degenerate-hyperbolic parabolic equation:

$$du - \Delta\varphi(u) dt + \operatorname{div}_x(\vec{v}f(u)) dt = g(u)dW \quad \text{in } \Omega \times (0, T) \times \mathbb{R}^d,$$

with $\vec{v} \in \mathbb{R}^d$, $\varphi, f : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz-continuous functions, an initial condition $u_0 \in L^2(\mathbb{R}^d)$, and where formally $\Delta\varphi(u) = \operatorname{div}_x(\varphi'(u)\nabla u)$. We assume further that the function φ is non-decreasing, but not increasing so that the above equation degenerates into a problem exhibiting locally hyperbolic behaviour. As with a scalar conservation law of order 1 of type (I.1), there is in general no uniqueness of weak solutions for this kind of degenerate problem, and so one of the mathematical challenges consists in introducing a criterion for selecting the physically admissible solution. Theoretical work has been done in this direction *via* a kinetic approach for another form of degeneracy of the diffusion operator ([Hof13, DHV16]), and we have proposed in [BVW15] an existence and uniqueness result for the stochastic entropy solution of the above problem. Studies of time-splitting methods have been addressed in the literature, for approximating the unique kinetic solution [KN16b] and the entropy’s one [DJ20].

Using the numerical results developed in Part 1 for the hyperbolic case and in Part 3 for the parabolic one, the idea I have in mind would be to construct a sequence of finite-volume approximations $(v_{\mathcal{T},k})_{\mathcal{T},k}$, and to show its convergence towards the unique entropy solution obtained by artificial viscosity in [BVW15]. There are at least two difficulties with this study: the first is that, for the same reasons as explained in Part 1, the use of Kruzhkov’s entropies is out of reach in the stochastic

[†]A martingale solution is a triplet $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}), \tilde{u})$, formed by a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *i.e.* a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ equipped with a filtration $\tilde{\mathbb{F}}$, a Brownian motion $\tilde{W}$ adapted to $\tilde{\mathbb{F}}$ and the variational solution $\tilde{u}$ of a Stefan’s problem perturbed by $\tilde{W}$ and defined on such a new probability space.

case. We therefore first need to construct approximate entropy inequalities satisfied by our sequence $(v_{\mathcal{T},k})_{\mathcal{T},k}$ using smooth entropies: a possible approach would be to follow the ideas developed in [EGHM02] in the deterministic case. After overcoming this first difficulty, the second arises, which did not exist in the hyperbolic work presented in Part 1: the absence of a notion of measure-valued entropy solution for such a degenerate problem, which will make the convergence analysis of the numerical scheme more difficult. As a reminder, in the hyperbolic case, we had a definition (and existence) of a measure-valued entropy solution (Definition I.5) and a theoretical result of “reduction of the measure-valued entropy solution” (Theorem I.6), which allowed us to conclude directly after having only shown the convergence of the sequence of finite-volume approximations to a measure-valued entropy solution. Since we have no such result for the degenerate case, we have to find another way of proving the convergence of our sequence $(v_{\mathcal{T},k})_{\mathcal{T},k}$ to the unique entropy solution given by [BVW15].

One possible approach would be to use the numerical scheme directly to show the existence and uniqueness of this entropy solution, as we did in the work [BCC20] presented in Chapter III. To this end, remind that we adapted Kruzhkov’s doubling variables technique by comparing any measure-valued entropy solution μ (satisfying Definition I.5) with our volume-finite approximation $u_{\mathcal{T},k}$ constructed using a scheme with monotone numerical fluxes. In [BVW15], it was by comparing two parabolic* approximations u_ϵ and u_θ obtained by artificial viscosity, that we proved our existence and uniqueness result of an entropy solution. In the degenerate case we are interested in, one could imagine mixing these two approaches, by comparing the finite-volume approximation $v_{\mathcal{T},k}$ with a parabolic one u_ϵ .

Numerical approximation of the stochastic Allen-Cahn equation with constraint

I conclude by presenting a research perspective linking Parts 2 and 3. After the theoretical study of the stochastic Allen-Cahn equation with constraint (IV.1) presented in Part 2, a natural question that arises is that of numerical analysis. In the light of the parabolic work carried out in Part 3, the idea would be to propose a TPFA scheme not to approximate Problem (IV.1), but its regularized version (IV.6), where the sub-differential $\partial I_{[0,1]}$ is formally replaced by a monotone “à la Moreau-Yosida” approximation ψ_ϵ , with $\epsilon > 0$ the regularization parameter:

$$du_\epsilon + (\psi_\epsilon(u_\epsilon) - \Delta u_\epsilon) dt = (w_s(u_\epsilon) + f) dt + g(u_\epsilon) dW \text{ in } \Omega \times (0, T) \times D.$$

The construction and stability analysis of the scheme could be done initially by applying to the above regularized version, the methods developed in Chapter VII for Problem (VII.1):

$$du - \Delta u dt + \operatorname{div}_x (\mathbf{v} f(u)) dt = \beta(u) dt + g(u) dW \text{ in } \Omega \times (0, T) \times \Lambda.$$

Indeed, the regularized problem can be seen as a special case of (VII.1) by taking in (VII.1): $f \equiv 0$ and $\beta \equiv w_s + f - \psi_\epsilon$. In a second step, it would be a matter of choosing a clever relation between the regularization parameter $\epsilon > 0$ of the subdifferential and the discretization steps in time and space (respectively N and h) of the finite-volume scheme, for a simultaneous passage to the limit on ϵ , N and h , in order to recover at the limit the solution of the stochastic Allen-Cahn Problem (IV.1). Finally, the question of numerical simulation around this problem is part of future work in collaboration with S. DUMONT and F. NABET within the framework of a PhC Procope project (Campus France) led by the latter in 2024.

*solutions of $du - \Delta \varphi(u) dt - \lambda \Delta u dt + \operatorname{div}_x (\mathbf{v} f(u)) dt = g(u) dW$ in $\Omega \times (0, T) \times \mathbb{R}^d$ respectively for $\lambda = \epsilon$ and $\lambda = \theta$.

Perspectives

En conclusion, j’ai tissé depuis mon arrivée à Marseille un réseau de collaborations entre la France, l’Italie et l’Allemagne qui m’a permis d’investiguer activement les thèmes présentés dans ce manuscrit : d’un côté autour de l’approximation volumes-finis de lois de conservation scalaires d’ordre un puis deux forcées par un bruit multiplicatif, et de l’autre sur des questions d’analyse théorique pour des problèmes paraboliques stochastiques appliqués à la mécanique du solide. Les travaux qui en découlent ouvrent naturellement la porte à des directions de recherche qui me permettront d’une part de pérenniser ces collaborations, et d’autre part d’encadrer par moi-même des travaux doctoraux. Je détaille ci-après les grandes lignes de mon projet scientifique : d’abord les perspectives s’inscrivant dans la lignée des travaux présentés respectivement dans les Parties 1, 2 et 3, et ensuite celles situées à l’interface entre les Parties 1 et 3 puis 2 et 3. Je reprends ici les notations employées précédemment : $(W(t))_{t \geq 0}$ un mouvement brownien de dimension 1 défini sur un espace de probabilités $(\Omega, \mathcal{F}, \mathbb{P})$, D un ouvert régulier et borné de $\mathbb{R}^d$ ($d \geq 1$), Λ un ouvert borné, connexe et polygonal de $\mathbb{R}^d$ ($1 \leq d \leq 3$) et $g : \mathbb{R} \rightarrow \mathbb{R}$ une fonction lipschitzienne.

Approximation volumes-finis de lois de conservation scalaires stochastiques du 1^{er} ordre : limites de l’étude, estimations d’erreur et flux aléatoire

Suite à nos premiers travaux sur la discrétisation espace-temps par des méthodes volumes-finis de lois de conservation scalaires stochastiques du type (I.1) présentés en Partie 1, plusieurs études numériques ont été menées autour de cette thématique : simulations numériques de l’équation de Burgers stochastique [ABGH15], convergence de schémas volumes-finis à flux numériques monotones par une approche cinétique dans le cas d’un bruit de dimension infinie (avec une dépendance spatiale dans l’intégrande stochastique) [DV18, DV20], convergence d’un schéma de Godunov dans le cas d’un bruit multi-dimensionnel [FGH18], convergence de schémas différences finies puis flux-splitting avec bruit de Lévy [KMV16, Maj18], time-splitting d’opérateur avec bruit semi-linéaire [KS18] ou encore approximation volumes-finis de la mesure invariante dans le cas visqueux [BMR22]. Remarquons que des études connexes au sujet ont également été réalisées : développement de méthodes de type Galerkin-Discontinu [LST20], semi-discrétisation en espace puis en temps de lois de conservation fractionnaires stochastiques [KV23, BM], ou encore approximations temporelles, spatiales puis spatio-temporelles pour des lois de conservation avec flux stochastiques (“rough fluxes”) [GPS16, Hof16, HKRS20]. Au regard de la bibliographie récente sur le sujet, en plus des nombreuses questions que soulèveraient l’implémentation numérique de ces problèmes hyperboliques stochastiques, les directions de recherche qui m’intéressent sont les suivantes.

- Premièrement, se pose la question de la nécessité de l’hypothèse “ $k = o(h)$ ”, liant les paramètres de discrétisation en temps k et en espace h , sur laquelle repose la convergence de tous les schémas volumes-finis que j’ai étudiés en hyperbolique. En effet, dans un travail postérieur (formé de deux publications conjointes [DV18, DV20]), les auteurs ont montré la convergence d’un schéma volumes-finis à flux numériques monotones sous une hypothèse de CFL classique du type “ $k = \mathcal{O}(h)$ ”. Pour cela, ces derniers ont employé une approche cinétique et un mode de convergence faible (au sens probabiliste) par rapport à la variable de probabilité (convergence en loi). On peut alors se demander si dans notre cas, l’hypothèse “ $k = o(h)$ ”, peut être également relaxée en “ $k = \mathcal{O}(h)$ ”. Ou bien est-elle requise à cause de l’approche entropique ? De l’outil de compacité employé ? Ou encore pour une autre raison que nous n’avons pas encore identifiée ?
- Deuxièmement, la question de l’estimation de l’erreur d’approximation commise par notre schéma numérique est

de façon évidente un point que je souhaite investiguer. Le travail de comparaison présenté dans le Chapitre III entre l'approximation volumes-finis $(u_{\mathcal{T},k})_{\mathcal{T},k}$ du Problème (III.1), définie par (III.2)-(III.3)-(III.4), et une solution mesure entropique de ce même problème (au sens de la Définition III.4), constitue une étape préliminaire importante pour l'obtention de telles estimations. L'idée serait d'arriver à quantifier l'erreur forte (au sens probabiliste [BHS18]) en contrôlant, pour tout compact K de $\mathbb{R}^d \times [0, T]$, un terme du type

$$\mathbb{E} \left[\left(\int_K |u_{\mathcal{T},k}(x, t) - u(x, t)| dx dt \right)^2 \right],$$

u étant l'unique solution entropique du Problème (III.1) dans le sens de la Définition III.3. Pour cela, il faudrait mettre en lien les résultats du Chapitre III avec les techniques déterministes développées dans [CH00], où des estimations d'erreurs pour des schémas d'ordre 2 ont été obtenues en employant des entropies régulières.

- Troisièmement, la prise en compte d'effets aléatoires dans le terme de flux (hors "rough fluxes") est une question que l'on m'a souvent posée lorsque j'ai présenté oralement les travaux de cette première partie. D'un point de vue théorique, je pense que l'extension du résultat d'existence et d'unicité d'une solution entropique obtenu dans [BCG16b] s'adapterait bien et pourrait faire l'objet d'un stage de Master 2. En considérant un flux du type

$$\mathbb{R}^d \times [0, T] \times \Omega \times \mathbb{R} \ni (x, t, \omega, u) \mapsto \vec{v}(x, t, \omega) f(u) \in \mathbb{R}^d,$$

où f satisfait les mêmes hypothèses que dans [BCG16b] (énoncées au Chapitre I), la régularité du vecteur aléatoire $\vec{v} : \mathbb{R}^d \times [0, T] \times \Omega \rightarrow \mathbb{R}^d$ resterait quant à elle à définir. En revanche, du côté de l'analyse numérique, je pense que le travail de construction des flux numériques et d'analyse du schéma sera plus conséquent.

Applications en mécanique du solide : prise en compte d'effets non-locaux et condition d'irréversibilité

Dans la continuité des travaux présentés en Partie 2, des réflexions avec mes collaboratrices allemandes et italiennes sont engagées sur les pistes suivantes :

- La première porte sur une généralisation de l'étude [BBB⁺17] faite autour de l'équation d'Allen-Cahn stochastique avec contrainte, présentée au Chapitre IV. Pour cela, nous avons en tête avec K. SCHMITZ et A. ZIMMERMANN, de considérer un p -laplacien (avec $p \geq 2$) à la place du laplacien standard dans le Système (IV.1), ce qui donne :

$$du + (\xi - \Delta_p u) dt = (w_s(u) + f) dt + g(u) dW \text{ et } \xi \in \partial I_{[0,1]}(u) \text{ dans } \Omega \times (0, T) \times D,$$

pour une donnée initiale $u_0 \in W^{1,p}(D)$ et où $\Delta_p : u \mapsto \operatorname{div}_x(|\nabla u|^{p-2} \nabla u)$ est associé à des conditions de Neumann homogènes sur ∂D . L'idée sous-jacente étant de prendre en compte des effets non-locaux plus généraux dans le processus d'endommagement, comme proposé dans [RR14] pour une autre forme d'équation. Notons que les techniques de monotonie que j'ai utilisées pour le laplacien standard (qui correspond au cas $p = 2$) dans [BBB⁺17] sont *a priori* compatibles avec cet opérateur p -laplacien, ce qui laisse à penser que leur adaptation dans ce cas plus général est envisageable. Grâce à l'expertise de mes collaboratrices allemandes sur cet opérateur, de mon analyse fine de l'équation d'Allen-Cahn stochastique avec contrainte, et des moyens financiers dont nous disposons jusque fin 2025 pour nous rencontrer (PHC Procope France-Allemagne par Campus France), l'obtention de résultats originaux sur cette thématique ne devrait pas trop se faire attendre.

- La deuxième fait suite à une réflexion lancée avec G. BONFANTI et R. ROSSI en 2017, lors d'une visite avec mon collègue F. LEBON à l'Université de Brescia. L'idée était d'inclure des effets stochastiques dans un modèle déterministe de changement de phases étudié par G. BONFANTI [BFL00] :

$$\begin{cases} \partial_t \vartheta + \partial_t \chi - \Delta \vartheta = 0 & \text{dans } \Omega \times D \times (0, T), \\ \partial_t \chi - \Delta \chi + \partial I_{[0,+\infty]}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni \vartheta & \text{dans } \Omega \times D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ et } \nabla \chi \cdot \mathbf{n} = 0 & \text{sur } \Omega \times \partial D \times (0, T), \end{cases}$$

où les inconnues ϑ et χ satisfaisaient des conditions initiales en $t = 0$ et sous des conditions de Neumann homogènes au bord de l'ouvert $D \subset \mathbb{R}^d$ considéré. Ce système, doublement non linéaire, peut-être appliqué à diverses situations physiques mais l'idée des auteurs (et donc la nôtre) était de se concentrer sur un phénomène

de transition de phase irréversible : celui du processus de solidification de la colle dans un tube. En notant χ la fraction de volume de colle à l'état solide, en supposant l'absence de bulle d'air dans notre tube, celle à l'état liquide est donc donnée par $1 - \chi$. Ce type de système décrit alors la transformation progressive de la colle de l'état liquide vers l'état solide induite par des actions thermiques portées par l'inconnue ϑ et par des mouvements à l'échelle microscopique, en ajoutant de plus une condition d'irréversibilité sur le processus de solidification *via* le terme $\partial I_{[0,+\infty[}(\partial_t \chi)$. Initialement, nous voulions traduire dans notre modèle l'influence aléatoire de la vitesse de solidification de la colle $\partial_t \chi$ sur les variations de la température ϑ en considérant le problème suivant :

$$\begin{cases} \partial_t \left(\vartheta - \int_0^\cdot g(\partial_t \chi) dW \right) + \partial_t \chi - \Delta \vartheta = 0 & \text{dans } \Omega \times D \times (0, T), \\ \partial_t \chi - \Delta \chi + \partial I_{[0,+\infty[}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni \vartheta & \text{dans } \Omega \times D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ et } \nabla \chi \cdot \mathbf{n} = 0 & \text{dans } \Omega \times \partial D \times (0, T), \end{cases}$$

avec une contribution de $\partial_t \chi$ dans l'intégrale stochastique. Après plusieurs réflexions autour de ce système, nous manquons d'idées pour arriver à contrôler toutes ces non-linéarités sur χ et $\partial_t \chi$, notre objectif étant de proposer un résultat d'existence et d'unicité de solutions. Nous avons en tête d'utiliser une régularisation "à la Moreau-Yosida" des sous-différentiels $\partial I_{[0,+\infty[}$ et $\partial I_{[0,1]}$ comme nous avons fait dans [BBB⁺17], mais déjà l'étude du système régularisé pose des difficultés. Pour ces raisons, nous avons décidé d'introduire finalement de la stochasticité dans le modèle en considérant le cas (qui nous semble plus raisonnable) de variations aléatoires de la température ϑ du système, c'est à dire sans faire intervenir $\partial_t \chi$ dans notre bruit. Plus précisément, on s'intéresse au problème suivant :

$$\begin{cases} \partial_t \left(\vartheta - \int_0^\cdot g(\vartheta) dW \right) + \partial_t \chi - \Delta \vartheta = 0 & \text{dans } \Omega \times D \times (0, T), \\ \partial_t \chi - \Delta \chi + \partial I_{[0,+\infty[}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni \vartheta & \text{dans } \Omega \times D \times (0, T), \\ \nabla \vartheta \cdot \mathbf{n} = 0 \text{ et } \nabla \chi \cdot \mathbf{n} = 0 & \text{dans } \Omega \times \partial D \times (0, T). \end{cases}$$

Pour avancer sur ce projet, notre constat est que nous avons besoin de temps de rencontre : nos compétences étant fortement complémentaires sur ce sujet, il nous est en effet très difficile de fonctionner chacun de notre côté. Dans cette optique nous répondons régulièrement à des appels à projets pour obtenir des moyens financiers pour des voyages entre la France et l'Italie et envisageons de participer à la campagne 2024 de l'Institut de Mécanique et d'Ingénierie (dont dépend mon laboratoire) pour l'obtention d'une bourse postdoctorale.

Schémas volumes-finis pour des EDP paraboliques stochastiques : les cas d'une convection ou d'une diffusion non-linéaire et celui du problème de Stefan

Les recherches présentées dans la Partie 3 ont été menées dans l'idée de servir de travaux préliminaires pour proposer une approximation volumes-finis de l'équation de diffusion-convection suivante :

$$du - \Delta \varphi(u) dt + \operatorname{div}_x (\mathbf{v} f(u)) dt = g(u) dW + \beta(u) dt \quad \text{dans } \Omega \times (0, T) \times \Lambda,$$

sous des conditions aux limites homogènes (de type Neumann ou Dirichlet) au bord de Λ , une condition initiale en $t = 0$ à définir et où formellement $\Delta \varphi(u) = \operatorname{div}_x (\varphi'(u) \nabla u)$. En considérant $\varphi, f, \beta : \mathbb{R} \rightarrow \mathbb{R}$ des fonctions lipchitziennes et $\mathbf{v} \in \mathcal{C}^1([0, T] \times \bar{\Lambda}; \mathbb{R}^d)$ une fonction vectorielle à divergence nulle, l'étude des trois cas particuliers suivants pourrait servir de support à un sujet de thèse :

- **Premier cas** : $\varphi(u) = u$. Sous des conditions de Neumann homogènes au bord, il s'agit de la généralisation au cas d'un flux non-linéaire du résultat de convergence présenté au Chapitre VII. Rappelons que le challenge technique réside uniquement dans la non-linéarité du flux pour lequel nous n'arrivons pas à identifier la limite du terme discret correspondant. Pour cette raison, nous avons pour le moment seulement un résultat de convergence dans le cas où la fonction f est linéaire : soit parce que l'outil de compacité utilisé n'est pas suffisant ou mal employé, soit parce que le schéma choisi ne restitue pas assez d'informations, soit pour une autre raison non identifiée pour le moment. Je vois alors deux pistes à envisager pour traiter le cas non-linéaire : adapter les techniques probabilistes utilisées au Chapitre VI sur l'équation de la chaleur stochastique pour faire converger le schéma TPFA introduit au Chapitre VII, ou bien proposer un schéma DDFV[†] pour approcher notre problème. La valeur ajoutée d'un tel schéma est qu'il permet, entre autres choses, de définir un gradient discret DDFV ainsi qu'une divergence discrète en dualité avec

[†]Discrete Duality Finite Volume. Voir par exemple [ABH07, Kre11].

ce dernier. J'aimerais voir si cette propriété est suffisante pour généraliser la technique d'identification des limites faibles (basée sur l'utilisation d'exponentielles à poids) présentée au Chapitre VII. Regarder dans un premier temps comment adapter ces méthodes DDFV au cas plus simple de l'équation de la chaleur stochastique (VI.1), pourrait faire l'objet d'un sujet de recherche de Master 2 en co-encadrement avec F. NABET, spécialiste de ce type de schéma dans le cas déterministe.

- **Deuxième cas :** φ croissante avec φ' pouvant s'annuler (sur un intervalle de mesure non nulle) et $f \equiv \beta \equiv 0$. Dans ce cas, le problème considéré devient parabolique dégénéré, et est connu dans le cas déterministe (*i.e.* quand $g \equiv 0$) sous le nom de problème de Stefan. C'est un modèle mathématique simplifié permettant par exemple de décrire l'évolution de la frontière entre deux phases d'un matériau subissant un changement de phase (sous des conditions de Dirichlet homogènes au bord, voir par exemple [EG23]). Il s'agit d'un problème que nous souhaitons analyser avec J. CHARRIER et T. GALLOUËT dans le cadre de la thèse de V. CASTEL (2016-2018), et qui avait motivé en premier lieu l'étude numérique faite autour de l'équation de la chaleur stochastique. En collaboration avec F. NABET, K. SCHMITZ et A. ZIMMERMANN, notre approche consisterait à généraliser le travail de convergence exposé au Chapitre VI sur l'équation de la chaleur stochastique (VI.1) pour un schéma TPFA, en utilisant de nouveau les méthodes de compacité probabilistes, dont mes collaboratrices allemandes sont expertes. Notons que pour cela, nous nous étions inspirées de la preuve de convergence du schéma développée dans [EGH00] pour ce fameux problème de Stefan. Celle-ci faisait appel en particulier à des estimations sur les translations en temps et en espace de la version discrète de $\varphi(u)$. Notons qu'un preprint très récent est sorti sur le sujet [DKL23], où les auteurs ont montré un résultat général sur la convergence d'une large classe de schémas numériques, appartenant à la classe des "Gradient Discretization Method" (GDM), vers une solution martingale* du problème de Stefan stochastique. Il serait intéressant de regarder si on peut montrer dans le cas d'un schéma TPFA (n'appartenant pas à la classe des GDM traités dans [DKL23], mais à une version non-symétrique particulière [DE17]), un résultat de convergence forte vers une solution variationnelle dans $L^p(\Omega; L^2(0, T; L^2(\Lambda)))$, avec $p \in [1, 2)$, comme nous l'avons fait dans [BNSZ23a] (voir Chapitre VI).
- **Troisième cas :** φ strictement croissante et f linéaire. Sous des conditions au bord à définir, l'idée de ce dernier cas est de voir si les techniques développées au Chapitre VII avec un schéma TPFA restent toujours valables dans le cas d'une diffusion non-linéaire $\Delta\varphi(u)$ ne dégénérant pas, de sorte que le problème considéré reste toujours parabolique. Un récent travail est paru sur ce thème [BGV23], les auteurs ont étudié l'approximation numérique d'une classe d'équations "dégénérées paraboliques singulières" (correspondant à $\varphi(u) = |u|^{p-2}u$ avec $p > 1$ et $f \equiv \beta \equiv 0$). Ils ont montré la convergence d'un schéma espace-temps (en utilisant une discrétisation éléments finis en espace) vers la solution faible de leur problème, en s'appuyant en particulier sur la monotonie de la fonction dans le terme de diffusion. Dans notre cas, une combinaison avec certains arguments employés dans l'analyse théorique du problème continu (voir par exemple [Val08] dans un cadre plus général) serait alors peut-être nécessaire. Si cette première approche ne fonctionne pas, employer et généraliser (au terme de convection) les méthodes de compacité stochastique proposées pour l'étude du cas précédent serait un autre angle d'attaque envisageable.

Notons que l'idée serait d'accompagner l'étude de chacun de ces cas par des simulations numériques.

Approximation volumes-finis de lois de conservation scalaires avec diffusion dégénérée et forçage stochastique

Une direction de recherche mettant en lien naturellement les Parties 1 et 3 de ce manuscrit, ainsi qu'un travail réalisé dans le cadre de ma thèse en collaboration avec G. VALLET et P. WITTBOLD [BVW15], porte sur l'approximation numérique du problème de Cauchy pour l'équation parabolique dégénéré-hyperbolique suivante :

$$du - \Delta\varphi(u) dt + \operatorname{div}_x(\vec{v}f(u)) dt = g(u)dW \quad \text{dans } \Omega \times (0, T) \times \mathbb{R}^d,$$

avec $\vec{v} \in \mathbb{R}^d$, $\varphi, f : \mathbb{R} \rightarrow \mathbb{R}$ des fonctions lipschitziennes, une donnée initiale $u_0 \in L^2(\mathbb{R}^d)$, et où formellement $\Delta\varphi(u) = \operatorname{div}_x(\varphi'(u)\nabla u)$. On suppose de plus que la fonction φ est croissante, mais pas strictement croissante de sorte que l'équation ci-dessus dégénère en un problème faisant apparaître localement un comportement hyperbolique. Comme pour une loi de conservation scalaire d'ordre 1 de type (I.1), il n'y a en général pas unicité des solutions faibles pour ce genre de problèmes dégénérés, et c'est pourquoi le défi mathématique consiste à introduire un critère permettant de

*Une solution martingale est un triplet $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}), \tilde{u})$, formé d'une base stochastique $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *i.e.* un espace de probabilité $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ muni d'une filtration $\tilde{\mathbb{F}}$, d'un mouvement brownien $\tilde{W}$ adapté à $\tilde{\mathbb{F}}$ et de la solution variationnelle $\tilde{u}$ d'un problème de Stefan bruité par $\tilde{W}$ et posé sur ce nouvel espace de probabilité.

sélectionner la solution physiquement admissible. Des travaux théoriques ont été effectués dans ce sens par le biais d’une approche cinétique pour une autre forme de dégénérescence de l’opérateur de diffusion ([Hof13, DHV16]), et nous avons proposé dans [BVW15] un résultat d’existence et d’unicité de la solution entropique stochastique du problème ci-dessus. Du côté de l’analyse numérique, des méthodes de time-splitting d’opérateur ont été proposées pour approcher la solution cinétique dans [KN16b] et entropique dans [DJ20].

En s’appuyant sur les résultats numériques développés en Partie 1 dans le cas hyperbolique et en Partie 3 pour le cas parabolique, l’idée ici serait de construire une suite d’approximations volumes-finis $(v_{\mathcal{T},k})_{\mathcal{T},k}$, et de montrer sa convergence vers l’unique solution entropique obtenue par viscosité artificielle dans [BVW15]. Au moins deux difficultés se présentent pour cette étude : la première est que, pour les mêmes raisons que celles exposées dans la Partie 1, l’utilisation des entropies de Kruzhkov est hors de portée dans le cas stochastique. Il faut donc d’abord arriver à construire des inégalités d’entropie approchées satisfaites par notre suite $(v_{\mathcal{T},k})_{\mathcal{T},k}$ employant des entropies régulières : une piste potentielle serait de suivre des idées développées dans [EGHM02] dans le cas déterministe. Une fois cette première difficulté surmontée arrivera la seconde, que nous n’avons pas dans les travaux en hyperbolique exposés en Partie 1, à savoir : l’absence de notion de solution mesure entropique pour un tel problème dégénéré, ce qui va rendre plus ardue l’analyse de convergence du schéma numérique. Pour mémoire, nous disposons en hyperbolique d’une définition de solution mesure entropique (Définition I.5) et d’un résultat théorique de “réduction de la solution mesure entropique” (Théorème I.6), lesquels nous permettaient de conclure directement après avoir seulement montré la convergence de la suite d’approximations volumes-finis vers une solution mesure entropique. Ne possédant pas de tels résultats dans le cas dégénéré, il faut trouver une autre manière pour prouver la convergence de notre suite $(v_{\mathcal{T},k})_{\mathcal{T},k}$ vers l’unique solution entropique donnée par [BVW15]. Une piste possible serait d’utiliser directement le schéma numérique pour montrer l’existence et l’unicité de cette solution entropique, comme nous l’avons fait dans le travail [BCC20] exposé au Chapitre III. Nous avons pour cela adapté la technique de dédoublement de variables de Kruzhkov en comparant une solution mesure entropique quelconque μ (satisfaisant la Définition I.5) avec notre approximation volumes-finis $u_{\mathcal{T},k}$ construite à l’aide d’un schéma à flux monotones. Dans [BVW15], c’est en comparant deux approximations paraboliques* u_ϵ et u_θ obtenues par viscosité artificielle, que nous avons démontré notre résultat d’existence et d’unicité de solution entropique. Dans le cas dégénéré qui nous intéresse, on pourrait imaginer mélanger ces deux approches, en comparant l’approximation volumes-finis $v_{\mathcal{T},k}$ avec une approximation parabolique u_ϵ .

Schéma volumes-finis pour l’équation d’Allen-Cahn stochastique avec contrainte

Je termine par la présentation d’une perspective de recherche mettant en lien les Parties 2 et 3. Après l’étude théorique faite sur l’équation d’Allen-Cahn stochastique avec contrainte (IV.1) exposée dans la Partie 2, une question naturelle qui se pose est celle de l’analyse numérique. Au regard des travaux menés en parabolique dans la Partie 3, l’idée serait de proposer un schéma TPFA non pas pour approcher le Problème (IV.1), mais sa version régularisée (IV.6), où le sous-différentiel $\partial I_{[0,1]}$ est formellement remplacé par une approximation “à la Moreau-Yosida” monotone ψ_ϵ , avec $\epsilon > 0$ le paramètre de régularisation:

$$du_\epsilon + (\psi_\epsilon(u_\epsilon) - \Delta u_\epsilon) dt = (w_s(u_\epsilon) + f) dt + g(u_\epsilon) dW \text{ dans } \Omega \times (0, T) \times D.$$

La construction et l’analyse de stabilité du schéma pourrait se faire dans un premier temps en appliquant à la version régularisée ci-dessus les méthodes développées au Chapitre VII pour le Problème (VII.1):

$$du - \Delta u dt + \operatorname{div}_x (\mathbf{v} f(u)) dt = \beta(u) dt + g(u) dW \text{ dans } \Omega \times (0, T) \times \Lambda.$$

En effet, (IV.6) peut-être vue comme un cas particulier de (VII.1) en prenant dans (VII.1) : $f \equiv 0$ et $\beta \equiv w_s + f - \psi_\epsilon$. Dans un second temps, il s’agirait de choisir une relation astucieuse entre le paramètre de régularisation $\epsilon > 0$ du sous-différentiel et les pas de discrétisation en temps et en espace (respectivement N et h) du schéma volumes-finis, pour un passage à la limite simultané sur ϵ , N et h , afin de retrouver à la limite la solution du problème d’Allen-Cahn stochastique (IV.1). Enfin, la question des simulations numériques autour de ce problème fait partie d’un travail futur en collaboration avec S. DUMONT et F. NABET dans le cadre d’un projet PhC Procope (Campus France) porté par cette dernière en 2024.

*solutions de $du - \Delta \varphi(u) dt - \lambda \Delta u dt + \operatorname{div}_x (\mathbf{v} f(u)) dt = g(u) dW$ dans $\Omega \times (0, T) \times \mathbb{R}^d$ respectivement pour $\lambda = \epsilon$ et $\lambda = \theta$.

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EDP stochastiques, schémas volumes-finis et applications à la mécanique

Résumé : Sont présentées dans ce manuscrit les activités de recherche que j'ai menées depuis mon arrivée à Marseille il y a dix ans suite à mon doctorat. Ces dernières s'inscrivent dans le domaine mathématique de l'analyse théorique et de l'approximation numérique d'EDP stochastiques. Plus précisément, elles concernent l'étude de problèmes non-linéaires perturbés par une intégrale stochastique d'Itô dépendant ou non de la solution, on parle alors de bruit multiplicatif ou additif. L'approche mathématique choisie dans mes recherches consiste à adapter et combiner les techniques connues en analyse des EDP avec des outils de théorie des probabilités et de calcul stochastique. Les EDP stochastiques que j'ai étudiées ont été de deux types : d'un côté des lois de conservation scalaires de type hyperbolique puis parabolique pour lesquelles je me suis attachée à la construction et l'analyse de convergence de schémas numériques de type volumes-finis, et de l'autre des équations paraboliques liées à la modélisation en mécanique du solide pour lesquelles j'ai regardé des questions d'existence et d'unicité de solution.

Stochastic PDEs, finite-volume schemes and applications in mechanics

Abstract: In this manuscript are presented the research activities I have carried out since my arrival in Marseille ten years ago following my PhD thesis. They are part of the mathematical field of theoretical analysis and numerical approximation of stochastic partial differential equations (SPDEs). More precisely, they concern the study of non-linear problems perturbed by a stochastic integral of Itô's type depending or not of the solution, commonly referred to as multiplicative (respectively additive) noise. The approach employed here is based on the use and combination of techniques from PDE's analysis, probability theory and stochastic calculus. Two types of SPDEs were studied: on the one hand, scalar conservation laws of hyperbolic and then parabolic type for which construction and convergence analysis of finite-volume schemes have been developed, and on the other hand parabolic equations related to solid mechanics modeling for which well-posedness issues have been investigated.