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THÈSE DE DOCTORAT

DE
L'UNIVERSITÉ D'ANGERS

SOUS LE SCEAU DE
LA COMUE ANGERS-LE MANS

ÉCOLE DOCTORALE N° 641

*Mathématiques et Sciences et Technologies du numérique
de l'Information et de la Communication*

Spécialité : Mathématiques et leurs Interactions

Par

Thomas CHOUTEAU

**Determinantal point processes related to the discrete Painlevé II
hierarchy and the Pearcey process**

Thèse présentée et soutenue à ANGERS, le 20 Octobre 2023

Unité de recherche : LAREMA, UMR 6093, ANGERS

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RÉSUMÉ

Processus ponctuels déterminantaux associés à la hiérarchie discrète de Painlevé II et au processus de Pearcey

Cette thèse vise à présenter des résultats nouveaux concernant le processus de Pearcey et la hiérarchie discrète de Painlevé II. Ces processus ont tous deux des interprétations en termes de processus ponctuels déterminantaux et de problèmes de Riemann-Hilbert. Après avoir rappelé les résultats classiques sur les processus ponctuels et les problèmes de Riemann-Hilbert, nous présentons, d'une part la hiérarchie discrète de Painlevé II, son lien avec les partitions aléatoires d'entier et les polynômes orthogonaux, d'autre part le processus de Pearcey et sa fonction génératrice. Concernant la hiérarchie discrète de Painlevé II, cette thèse est l'occasion de présenter une nouvelle paire de Lax associée à cette hiérarchie d'équations discrètes. Ce résultat s'appuie sur le lien entre partitions aléatoires d'entier, déterminants de Toeplitz et polynômes orthogonaux. De cette paire de Lax il est alors possible de déterminer un opérateur récursif permettant de calculer la N -ième équation de la hiérarchie. Des résultats concernant les limites continues et les asymptotiques des membres de la hiérarchie sont aussi présentés. En ce qui concerne le processus de Pearcey, via l'étude d'un problème de Riemann-Hilbert associé à un opérateur intégrable, nous présentons une formule "à la Tracy-Widom" pour décrire la fonction génératrice du processus. Cette formule est reliée à des équations différentielles et aux dérivées partielles couplées dont on détermine les asymptotiques des solutions.

ABSTRACT

Determinantal Point Processes related to the discrete Painlevé II hierarchy and the Pearcey process

The aim of this thesis is to present new results concerning the Pearcey process and the discrete Painlevé II hierarchy. These processes both have interpretations in terms of determinantal point processes and Riemann-Hilbert problems. After recalling classical results on point processes and Riemann-Hilbert problems, we present, on one hand the discrete Painlevé II hierarchy, its link with random integer partitions and orthogonal polynomials, and on the other hand the Pearcey process and its generating function. Concerning the discrete Painlevé II hierarchy, this thesis presents a new Lax pair associated with this hierarchy of discrete equations. This result is based on the link between random integer partitions, Toeplitz determinants and orthogonal polynomials. From this Lax pair it is then possible to determine a recursive operator for calculating the N -th equation of the hierarchy. Results concerning the continuous limits and asymptotics of hierarchy members are also presented. For the Pearcey process, via the study of a Riemann-Hilbert problem associated to an integrable operator, we present “a Tracy-Widom like formula” to describe the generating function of the process. This formula is linked to coupled differential and partial differential equations, whose solution asymptotics are computed.

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THIBAUT - Ah j'ai pas la ref.

THOMAS - Je me doute. Attends, je te montre.

Thomas montre une vidéo sur son ordinateur à Thibault. Ils reprennent ensuite la discussion.

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INTRODUCTION

1.1 Une petite histoire de processus ponctuels

1.1.1 Le processus de Pearcey

L'intégrale de Pearcey (voir Figure 1.1)

$$\text{Pe}(x, y) := \int_{-\infty}^{+\infty} e^{i(t^4 + xt^2 + yt)} dt, \quad (1.1)$$

fut introduite par le physicien Trevor Pearcey en 1946 pour l'étude du champ électromagnétique proche du point de rebroussement d'une caustique [Pea46].

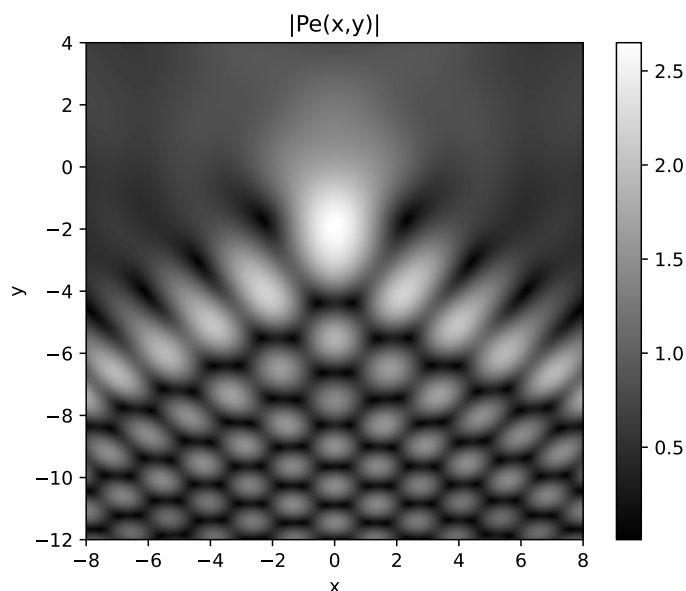


FIGURE 1.1 – La valeur absolue de l'intégrale de Pearcey en fonction de ses deux paramètres x et y

Elle fait partie d'une classe d'intégrales dites de diffraction (voir [BH10] par exemple) apparaissant dans la théorie des catastrophes introduite par René Thom [Tho72], branche de la théorie des bifurcations. Une classification des catastrophes établit qu'il existe sept formes de catastrophes possibles dépendant d'au plus quatre paramètres (voir par exemple [Tho83, Chapitre 5]). Deux d'entre elles sont respectivement caractérisées par l'intégrale de Pearcey (une variable et deux paramètres) et la fonction d'Airy. L'intégrale de Pearcey revêt un rôle important dans divers domaines de la physique et des mathématiques, notamment en optique [BK96], en mécanique quantique, en acoustique [Mar87] ou encore pour l'étude asymptotique d'intégrales oscillatoires [OD00].

Il est possible de définir un processus ponctuel déterminantal [Mac75] à partir de l'intégrale de Pearcey : le processus de Pearcey. Il s'agit d'un processus ponctuel déterminantal universel associé à l'opérateur de Pearcey \mathbb{K}_{Pe} dont le noyau est donné par

$$K_{Pe}(x, y; \tau) := \frac{1}{(2i\pi)^2} \int_{\Sigma} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu, \quad (1.2)$$

où pour $(x, \tau) \in \mathbb{R}^2$, $\theta_x(\mu) := \frac{\mu^4}{4} - \frac{\tau\mu^2}{2} - x\mu$ et $\Sigma := \Sigma_- \cup \Sigma_+$ est décrit dans la Figure 1.2.

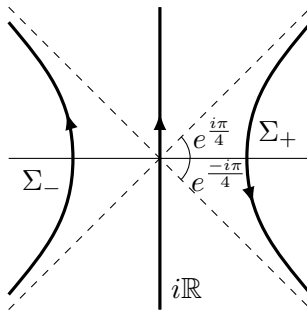


FIGURE 1.2 – Contours pour l'opérateur \mathbb{K}_{Pe}

Ce processus ponctuel apparaît dans l'étude de plusieurs modèles aléatoires, ce qui illustre son universalité.

Le premier modèle pour lequel le processus de Pearcey a été étudié était un modèle de matrices aléatoires introduit en 1998 par Brézin et Hikami [BH98a, BH98b].

La théorie des matrices aléatoires a vu le jour dans les années 1920 [Wis28] et a connu un essor important dans les années 1950 [Dys53, Wig55, Wig58] en physique afin de modéliser des noyaux d'atomes lourds pour l'étude du spectre de l'Hamiltonien caractérisant le noyau. Wigner a notamment montré que le spectre des matrices aléatoires qu'il considérait, lorsque la taille des matrices tend vers l'infini, tend à devenir continu (après un changement d'échelle). Les valeurs propres sont alors presque sûrement incluses dans un support borné et leur distribution est donnée par la loi du demi-cercle de Wigner dont la densité est définie par

$$f(x) := \begin{cases} \frac{2}{\pi} \sqrt{1-x^2} & , x \in [-1, 1], \\ 0 & , \text{sinon.} \end{cases} \quad (1.3)$$

Le modèle de matrices aléatoires présenté par Brézin et Hikami consistait à étudier le spectre de matrices aléatoires munies d'une mesure dépendant d'une source externe (une matrice H_0 fixé). Pour être précis, considérons une matrice $H := H_0 + V$ de taille $N \times N$ où V est une matrice hermitienne aléatoire gaussienne. La mesure de probabilité considérée est alors donnée par

$$P(H) = \frac{1}{Z_N} e^{-\frac{N}{2} \text{Tr}(H^2 - 2H_0H)}, \quad (1.4)$$

où Z_N est une constante de normalisation.

Lorsque H_0 est la matrice nulle, on retrouve l'exemple classique de l'Ensemble Gaussien Unitaire (GUE). L'étude de ce modèle de matrices aléatoires dont la distribution des valeurs propres, lorsque la taille des matrices aléatoires tend vers l'infini en considérant un bon changement d'échelle, est caractérisée par la loi du demi-cercle (1.3) de Wigner (voir Figure 1.3).

L'étude des valeurs propres de matrices du GUE est alors décrite par deux processus ponctuels déterminantaux : les processus du noyau sinus et du noyau d'Airy. Le processus du noyau sinus décrit la densité des valeurs propres pour un intervalle inclus dans le support de la loi du demi-cercle et suffisamment éloigné du bord du support. Cette densité correspond à l'étude pour le « *bulk* » [MG60]. L'étude des valeurs propres proches du bord du support (« *edge* ») est, quand à elle, décrite par le processus du noyau d'Airy [For93], ce qui permet entre autre d'étudier la distribution de la plus grande des valeurs propres d'une matrice aléatoire. La probabilité de trou pour le processus du noyau sinus et la distribution de la plus large des particules (plus grande valeur propre) pour le processus du noyau

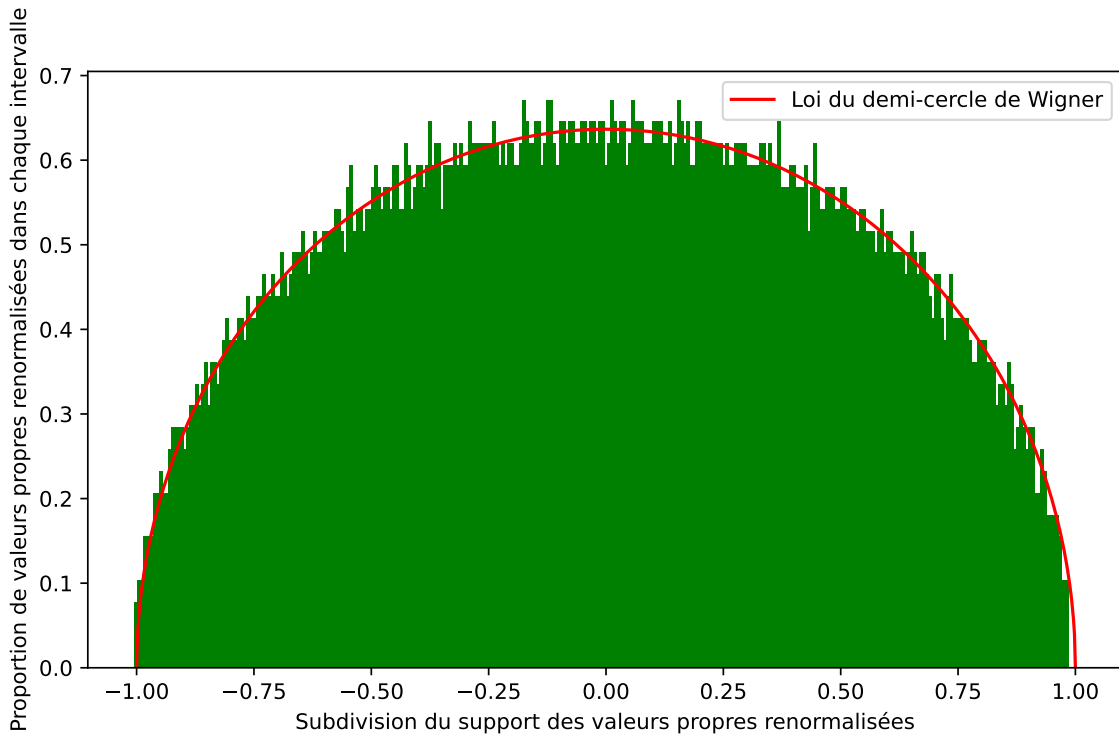


FIGURE 1.3 – Densité des valeurs propres d’une matrice aléatoire hermitienne gaussienne de taille 3000×3000 .

d’Airy s’expriment toutes deux en fonction de solutions aux équations de Painlevé [Pai00]. La probabilité de trou pour le modèle limite dans le cadre du noyau sinus (« *bulk* ») est alors caractérisée par l’équation de Painlevé V [JMMS80]. Concernant le noyau d’Airy, Tracy et Widom montrèrent en 1994 que la distribution de la plus grande des valeurs propres d’une matrice aléatoire du GUE est liée à l’équation de Painlevé II [TW94b]. Les processus du noyau d’Airy et du noyau sinus n’apparaissent pas que dans le cadre du modèle du GUE dans l’étude des matrices aléatoires. On peut par exemple citer les articles [TV11, TV10] dans lesquels des résultats d’universalité sur les statistiques locales des valeurs propres de matrices de Wigner sont donnés. Ces résultats assurent que si les entrées de la matrice sont indépendantes et en fixant les quatre premiers moments alors à nouveau les processus ponctuels à l’intérieur du support et au bord du support sont décrits par les processus du noyau sinus et du noyau d’Airy.

Lorsque la matrice H_0 est non-nulle, par exemple une matrice prenant ± 1 comme valeurs propres avec autant de valeurs propres positives que négatives, la distribution des valeurs propres présente alors une forme différente que le cas précédent. La densité des valeurs

propres se concentre maintenant en deux points distincts de part et d'autre de zéro. Il s'agit du cas tangent où la mesure limite passe d'un support à un intervalle à un support avec deux intervalles disjoints. Lorsque la taille de la matrice tend vers l'infini, la densité prend la forme de deux demi-cercles tangents (voir Figure 1.4). Au point de tangence, on observe un point de rebroussement (ou cusp) et le processus ponctuel dans un voisinage de ce point et pour un bon changement d'échelle est alors décrit par un processus différent de ceux du noyau sinus et du noyau d'Airy. Proche du point de rebroussement, le processus est décrit par le processus de Pearcey. Ce comportement, lorsque la taille des matrices tend vers l'infini, est universel puisqu'il ne dépend pas, dans la limite d'échelle, de la distribution des valeurs propres de la source externe à condition qu'elles soient supportées dans deux intervalles disjoints.

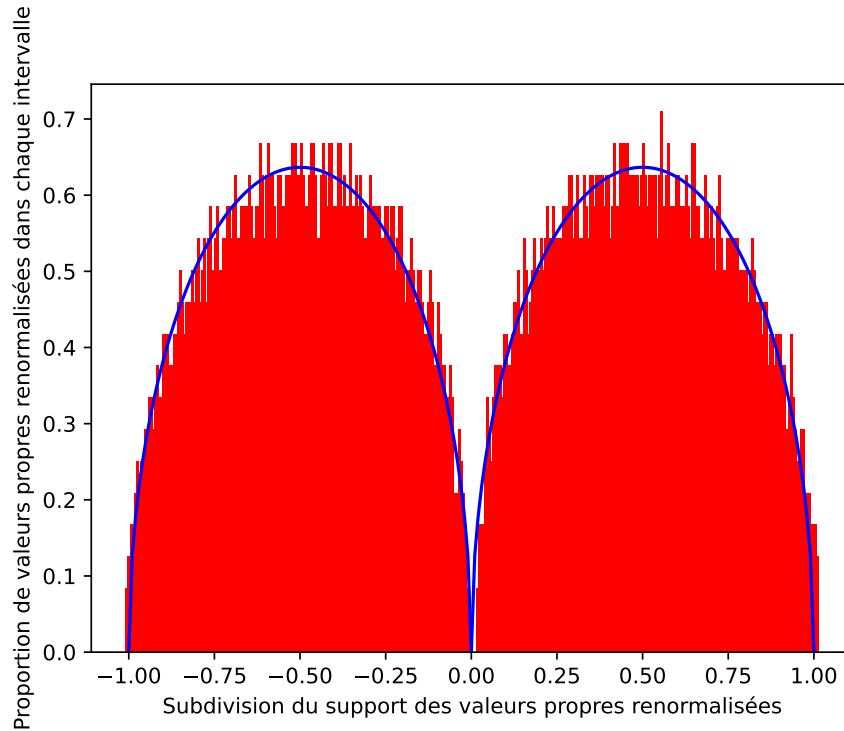


FIGURE 1.4 – Densité des valeurs propres d'une matrice aléatoire avec source externe H_0 de taille 2000×2000 .

Le modèle suivant en lien avec le processus de Pearcey concerne des ponts browniens. Le pont brownien est un mouvement brownien conditionné à débuter et à terminer respectivement en un point W_0 et en W_1 fixés. Les premières observations de mouvements browniens

datent de la fin du XVIII-ème siècle et son nom fait référence à Robert Brown [Bro28] à la suite de travaux d’observation au microscope de particules à l’intérieur de grains de pollen après éclatement de ce dernier. Brown y mentionne notamment des travaux passés de John Turberville Needham [Nee45] qui aurait, lui aussi, observé un mouvement « chaotique » de particules dans des grains de pollen. Par la suite, ce mouvement a été étudié par plusieurs scientifiques [Bac00, Ein06, Wie23] afin d’en donner une description/définition précise. Étudiant le mouvement brownien, Dyson définit un nouveau type de gaz de Coulomb [Dys62a] : N particules chargées soumises à des forces de répulsion et dont les trajectoires décrivent des mouvements browniens. Il établit un lien entre le spectre de matrices hermitiennes de taille $N \times N$ et ce modèle de gaz de Coulomb. Dans une série de trois articles Aptekarev, Bleher et Kuijlaars [BK04, ABK05, BK06] étudient l’Ensemble Gaussien Unitaire avec source externe discuté précédemment. Ils établissent notamment un lien entre ce modèle de matrices aléatoires et un modèle de ponts browniens. Ce nouveau modèle est décrit comme suit. Ils considèrent $2N$ ponts browniens non standards), commençant tous en 0 à $t = 0$ et dont la moitié est attachée en 1 et l’autre en -1 à $t = 1$ (voir Figure 1.5 (gauche)).

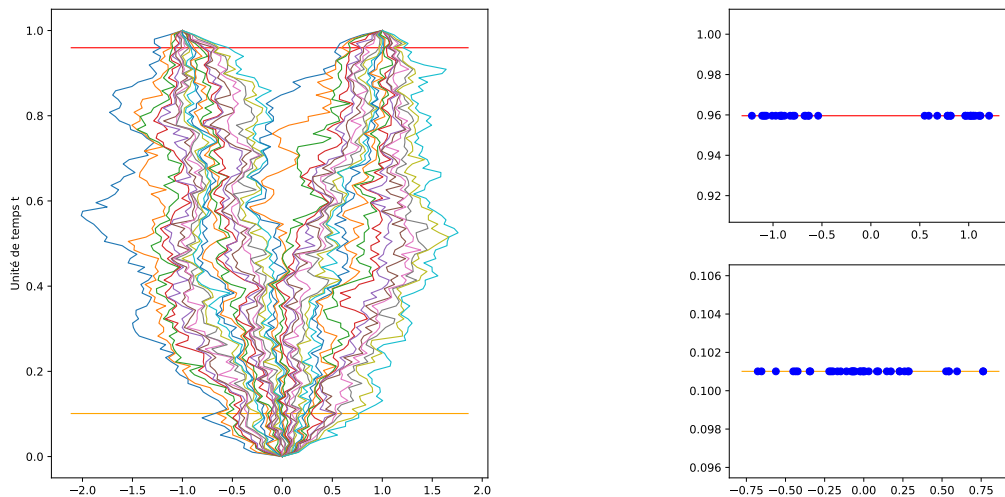


FIGURE 1.5 – A gauche, simulation de 40 ponts browniens non standards ne s’intersectant pas. A droite, répartition des points des ponts browniens a un instant t_1 proche de 1 (haut) et un instant t_0 proche de 0 (bas).

A chaque $t \in (0, 1)$ fixé, on associe une distribution de points sur l'axe réel correspondant au $2N$ positions occupées par les ponts browniens à l'instant t . On obtient alors pour chaque $t \in (0, 1)$ un processus ponctuel à $2N$ particules. Pour ce modèle, si t_0 est proche de 0, les $2N$ particules sont supportées dans un même intervalle autour de 0 (voir Figure 1.5 (bas droite)). Au contraire, si t_1 est proche de 1, les $2N$ particules sont désormais supportées dans deux intervalles disjoints, l'un autour de -1 , l'autre autour de 1 (voir Figure 1.5 (haut droite)).

Lorsque le nombre de ponts browniens tend vers l'infini (voir Figure 1.6), on observe toujours cette différence pour les supports des particules lorsque t est proche de 0 ou de 1. Il existe alors un temps critique t_c pour lequel, si $t < t_c$ (respectivement $t > t_c$) alors les particules du processus ponctuel au temps t sont supportées dans un même intervalle (respectivement dans deux intervalles disjoints). On observe de plus que, lorsque le nombre de ponts browniens tend vers l'infini, une forme limite de cardioïde se dessine et que presque sûrement tous les ponts browniens sont à l'intérieur du cardioïde.

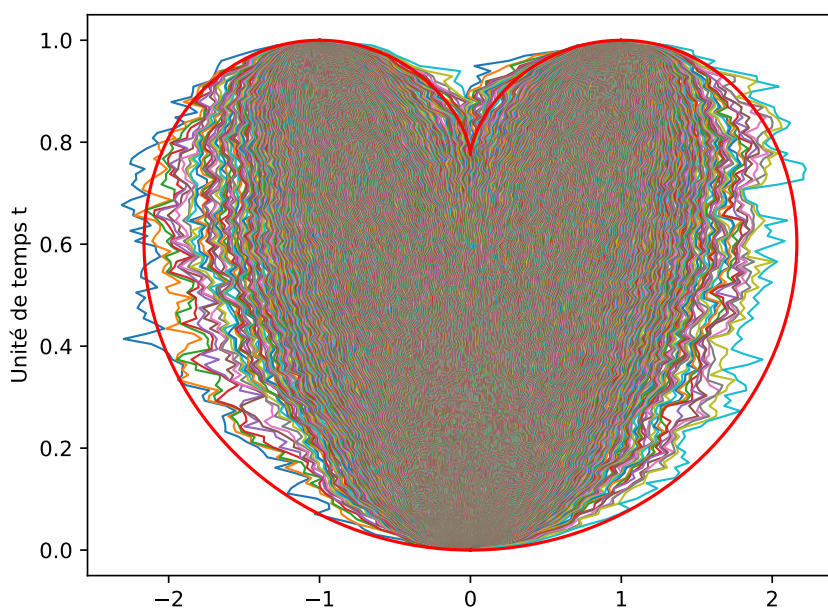


FIGURE 1.6 – Simulation de 2000 ponts browniens non standards ne s'intersectant pas et enveloppe limite

Dans le processus limite, le temps critique t_c est lié au point de rebroussement du cardioïde ayant pour coordonnées $(0, t_c)$. Pour t proche du temps critique t_c le processus ponctuel est alors décrit par le processus de Pearcey. Tout comme pour le modèle de matrices aléatoires avec source externe, le processus de Pearcey ne décrit pas uniquement le processus ponctuel proche du « cusp » du cardioïde dans le cas symétrique introduit par Bléher et Kuijlaars. Des modèles similaires de N ponts browniens ne s'intersectant pas, dont pN , $p \in (0, 1)$, (respectivement $(1 - p)N$) ponts browniens terminent en 0 (respectivement en $a\sqrt{N}$), présentent à nouveau, lorsque N tend vers l'infini, une forme limite avec un point de rebroussement et le processus ponctuel proche de ce point est décrit par le processus de Pearcey (voir par exemple [ADvM09]). Ce comportement montre une fois de plus l'universalité du processus de Pearcey.

Un dernier modèle aléatoire pour lequel le processus de Pearcey apparaît est un modèle de partitions planes aléatoires introduit par Okounkov et Reshetikhin [OR05]. Ce modèle de partition aléatoire est lié aux processus de Schur et à des problèmes de pavage par des losanges. Tout comme pour le modèle de matrices aléatoires avec source externe de Brézin et Hikami, la forme limite pour l'étude des partitions aléatoires conduit aussi aux processus du noyau sinus et du noyau d'Airy.

L'étude des modèles précédemment discutés a mené à plusieurs équations différentielles ou aux dérivées partielles satisfaites par le déterminant de Fredholm associé à l'opérateur de Pearcey $\mathbb{K}_{\mathbb{P}_e}$ dont le noyau est défini par l'équation (1.2). En étudiant le modèle de ponts browniens non-standards, Tracy et Widom [TW06] dérivèrent une équation aux dérivées partielles pour le processus de Pearcey étendu (à plusieurs temps). A partir du modèle de matrices aléatoires avec source externe et en étudiant les équations bilinéaires de Hirota et la fonction tau associé à l'équation KP, Adler et van Moerbeke [AvM07] obtinrent une équation aux dérivées partielles non-linéaires d'ordre quatre pour le processus de Pearcey. Enfin Bertola et Cafasso [BC12] redécrivèrent les mêmes équations obtenus par Adler et van Moerbeke pour le processus de Pearcey et en introduisirent une nouvelle. Ces équations aux dérivées partielles permettent entre autre une meilleure compréhension du processus ponctuel associé au noyau de Pearcey.

1.1.2 Autour des équations de Painlevé

L'étude des modèles décrits précédemment mènent à plusieurs processus ponctuels déterminantaux dont les processus du noyau sinus et du noyau d'Airy. Ces derniers sont tous deux liés aux équations de Painlevé [Pai00], une classification des équations différentielles

d'ordre deux possédant la propriété de Painlevé : équation dont les seules singularités mobiles de la solution générale sont des pôles. Une singularité d'une solution d'une équation différentielle désigne ici un point proche duquel il existe plusieurs détermination de la solution. Ces singularités sont dites mobiles si elles dépendent des conditions initiales de l'équation différentielle.

La seconde moitié du XIX-ème siècle voit plusieurs mathématiciens s'intéresser à la question de déterminer/classifier les équations différentielles dont les solutions admettent des points critiques fixes (ne dépendant pas des conditions initiales). Divers éléments de réponses sont d'abord proposés pour les équations d'ordre un (voir par exemple [Fuc84, Poi85, Pic89]). Suite aux travaux de Picard, cette problématique s'est étendue aux équations différentielles d'ordre deux et l'on doit la classification de ces équations à trois mathématiciens : Painlevé [Pai00], Fuchs [Fuc07] et Gambier [Gam10]. Par la suite les équations d'ordres trois et supérieurs ont aussi été étudiées [Cha11, Gar12].

Lors de la classification des équations d'ordre deux par Painlevé, Gambier et Fuchs, cinquante équations (listées par exemple dans [Inc56]) satisfaisant la propriété de Painlevé furent classifiées. Parmi les cinquante équations listées, les solutions générales de six d'entre elles (voir la liste ci-dessous, équations (1.5)) conduisent à de nouvelles fonctions spéciales : les transcendentes de Painlevé. Quant aux quarante-quatre autres équations, leur solution s'expriment soit en fonction de fonctions spéciales déjà connues soit en fonction des transcendentes de Painlevé.

La liste des six équations de Painlevé est la suivante.

$$\begin{aligned}
\frac{d^2y}{dt^2} &= 6y^2 + t & (PI) \\
\frac{d^2y}{dt^2} &= 2y^3 + ty + \alpha & (PII) \\
ty \frac{d^2y}{dt^2} &= t \left(\frac{dy}{dt} \right)^2 - y \frac{dy}{dt} + \delta t + \beta y + \alpha \frac{y^3}{t} + \gamma \frac{y^4}{t} & (PIII) \\
y \frac{d^2y}{dt^2} &= \frac{1}{2} \left(\frac{dy}{dt} \right)^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \frac{3}{2}y^4 & (PIV) \\
\frac{d^2y}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1} \right) \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1} & (PV) \\
\frac{d^2y}{dt^2} &= \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left(\frac{dy}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \\
&\quad \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2} \right) & (PVI)
\end{aligned} \tag{1.5}$$

Les équations de Painlevé apparaissent dans plusieurs domaines de la physique et des mathématiques par exemple en physique statistique dans l'étude du modèle d'Ising bidimensionnel [BMW73, MTW77, JMMS80], en lien avec les polynômes orthogonaux (voir par exemple [VA18]), ou bien des modèles de croissance aléatoire [For03].

L'équation de Painlevé qui va nous intéresser par la suite est la deuxième équation de Painlevé (PII dans équation (1.5)) et la hiérarchie d'équations qui y est associée et plus précisément l'analogue discret de cette hiérarchie d'équations. Il est connu que l'équation de Painlevé est une réduction auto-similaire de l'équation aux dérivées partielles de Korteweg–de Vries modifiée (équation (1.6))

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2 \frac{\partial u}{\partial x} = 0 \quad (\text{mKdV}). \quad (1.6)$$

L'équation de Korteweg-de Vries modifiée (1.6) est liée à l'équation de Korteweg-de Vries (1.7) via une transformation de Miura [Miu68] : $v := u_x - u^2$ où u_x est la dérivée partielle par rapport à la variable x de u .

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + 6v \frac{\partial v}{\partial x} = 0 \quad (\text{KdV}) \quad (1.7)$$

L'équation de KdV (1.7) fut introduite par Korteweg et de Vries [KdV95] en 1895 dans le cadre de l'étude de vagues en faible profondeur. Il s'agit d'une équation intégrable au sens où elle possède une paire de Lax (voir par exemple [Lax68]). L'équation de KdV (1.7) (respectivement modifiée de KdV (1.6)) est la première équation de la hiérarchie de KdV (respectivement de l'équation modifiée de KdV) que l'on peut construire en considérant l'algèbre des opérateurs pseudo-différentiels [Dic03, MJD00] ou bien les opérateurs de récursion de Lenard [Lax76]. Les équations de ces deux hiérarchies sont liées par la transformation de Miura précédemment présentée.

La construction de la hiérarchie de Painlevé II [Jos04] repose sur la réduction auto-similaire de l'équation de Painlevé II et de l'équation modifiée de KdV. En effet, partant de la N -ième équations de la hiérarchie de KdV modifiée et en considérant une réduction auto-similaire de celle-ci, il est possible de construire une équation différentielle d'ordre $2N$. La hiérarchie d'équations ainsi construite est appelée hiérarchie de Painlevé II. Cette hiérarchie d'équation apparaît notamment en physique lors de l'étude de modèle de fermions [DMS18]. Il existe aussi plusieurs analogues de cette hiérarchie (vectorielle [CT23a], opératorielle [Tar21], intégral-différentielle [BCT22, Kra21] et même discrète [CJ99a]).

Comme énoncé précédemment, cette thèse vise à présenter de nouveaux résultats sur la hiérarchie discrète de Painlevé II.

Les équations de Painlevé (1.5) admettent des analogues discrets (voir par exemple [RGH91]). Il s'agit d'équations discrètes non-linéaires d'ordre deux. Elles sont considérées comme des analogues discrets des équations de Painlevé au sens où il est possible d'obtenir les équations de Painlevé comme limite continue de celles-ci.

Au début des années 2000, divers travaux (voir par exemple [AvM03, Bai03, Bor03]) menèrent à l'équation discrète de Painlevé II et établirent un lien avec les déterminants de Toeplitz. En effet si D_n désigne le déterminant d'une matrice de Toeplitz de taille $(n+1) \times (n+1)$ associé au symbole $\varphi(z) = \exp(\theta(z+z^{-1}))$ alors D_n satisfait l'équation suivante :

$$\frac{D_{n-1}D_{n+1}}{D_n^2} = 1 - x_n^2, \quad (1.8)$$

où x_n satisfait l'équation discrète de Painlevé II :

$$nx_n + \theta_1(1 - x_n^2)(x_{n+1} + x_{n-1}). \quad (1.9)$$

Dans [Bor03], Borodin étudia des probabilités discrètes de trou exprimées comme déterminants de Fredholm d'opérateurs discrets (noyau de Bessel discret et noyau hypergéométrique ${}_2F_1$). L'étude de problèmes de Riemann-Hilbert associé à ces opérateurs discrets permit d'exprimer ces déterminants de Fredholm en fonction de solutions des équations discrètes de Painlevé II (pour le noyau discret de Bessel) et Painlevé V (pour le noyau hypergéométrique ${}_2F_1$). Adler et van Moerbeke [AvM03] obtinrent la deuxième équation discrète de Painlevé II lors de l'étude d'intégrale sur le groupe unitaire en considérant le réseau de Toeplitz et l'algèbre de Virasoro associé à ce réseau. Finalement, l'approche de Baik utilisa des méthodes de Riemann-Hilbert pour l'étude de polynômes orthogonaux sur le cercle unité [Bai03]. Il montra que les termes constants de la famille polynômes orthogonaux construits satisfaisait une relation de récurrence d'ordre deux décrite par l'équation discrète de Painlevé II. En considérant, le symbole $\varphi(z) = \exp(\theta_1(z+z^{-1}) + \theta_2(z^2+z^{-2}))$, Adler et van Moerbeke obtinrent une équation discrète d'ordre quatre analogue d'ordre supérieur de l'équation de Painlevé II. Cet équation correspond à la seconde de la hiérarchie de Painlevé II introduite par Cresswell et Joshi [CJ99a]. La définition de cette hiérarchie discrète d'équation vient de la compati-

lité d'un système de deux équations matricielles de taille 2×2 définit comme suit :

$$\Phi_{n+1}(z; N) = L_n(z; N)\Phi_n(z; N), \quad \frac{\partial}{\partial z}\Phi_n(z; N) = M_n(z; N)\Phi_n(z; N), \quad (1.10)$$

où les matrices $L_n(z; N), M_n(z; N)$ dépendent des variables z et n et aussi d'un autre paramètre N entier fixé. La N -ième équation de la hiérarchie de Painlevé II est alors obtenue en considérant la condition de compatibilité pour le système d'équations (1.10). Il est alors possible de calculer la N -ième équation de la hiérarchie de manière récursive.

1.2 Contributions originales

Avec ce panorama en tête, ce travail de thèse a eu pour visée d'explorer deux directions. La première concerne la fonction génératrice associée au processus de Pearcey avec pour résultat une formule à la « Tracy-Widom » [TW94b] pour la fonction génératrice du processus. Plus précisément, il est montré que le logarithme de la fonction génératrice du processus satisfait une équation différentielle d'ordre deux impliquant deux fonctions p et q liées par des équations couplées différentielles ordinaires et aux dérivées partielles.

La deuxième direction concerne la hiérarchie de Painlevé II discrète [CJ99a]. En s'inspirant des travaux de Baik sur les polynômes orthogonaux sur le cercle unité [Bai03], nous avons obtenu une nouvelle paire de Lax pour cette hiérarchie discrète et proposé une procédure récursive afin de calculer la N -ième équation de la hiérarchie. De plus, nous avons montré que ces équations apparaissent bien comme limite continue pour les équations de la hiérarchie de Painlevé II.

1.2.1 Résultats originaux sur la fonction génératrice du processus de Pearcey [Cho23]

La fonction génératrice associée au processus ponctuel déterminantal du noyau de Pearcey est définie par :

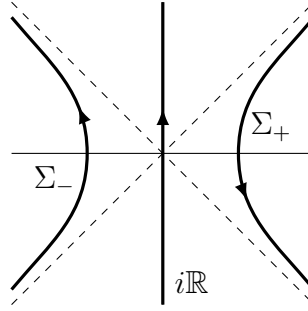
$$F(a, \tau, k) := \mathbb{E} \left(\prod_{j=1}^{N-1} (1 - k_j)^{\sharp(a_j, a_{j+1})} \right) = \det \left(1 - \sum_{j=1}^{N-1} k_j \chi_{(a_j, a_{j+1})} \mathbb{K}_{\text{Pe}} \right) \quad (1.11)$$

où $N > 1$ est un entier, $a := (a_1, \dots, a_N)$ tel que $a_1 < \dots < a_N$, $k := (k_0, k_1, \dots, k_{N-1}, k_N)$ tel que $k_0 = k_N = 0$, $k_j \in [0, 1]$, $j = \llbracket 1, N-1 \rrbracket$ et $k_j \neq k_{j+1}$, $j = \llbracket 0, N-1 \rrbracket$ et \sharp_I est la variable aléatoire comptant le nombre de points du processus dans l'intervalle I . Dans le deuxième et le troisième termes dans la définition de $F(a, \tau, k)$, l'espérance \mathbb{E} correspond à l'espérance du processus ponctuel et la notation \det désigne un déterminant de Fredholm (voir par exemple [BDS16, Appendix A] pour la définition d'un déterminant de Fredholm).

Dans la définition précédente \mathbb{K}_{Pe} désigne l'opérateur de Pearcey ayant pour noyau :

$$K_{\text{Pe}}(x, y; \tau) := \frac{1}{(2i\pi)^2} \int_{\Sigma} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu \quad (1.12)$$

où $(x, \tau) \in \mathbb{R}^2$, $\theta_x(\mu) := \frac{\mu^4}{4} - \frac{\tau\mu^2}{2} - x\mu$ et $\Sigma := \Sigma_- \cup \Sigma_+$ est comme suit :



L'expression de $F(a, \tau, k)$ comme déterminant de Fredholm permet, après réécriture de ce déterminant de Fredholm comme celui d'un opérateur intégrable au sens de [IKS90], d'étudier un problème de Riemann-Hilbert. Il est alors possible de déterminer un système d'équations différentielles pour la solution de ce problème de Riemann-Hilbert pour les variables τ , μ variable spectrale du problème de Riemann-Hilbert et s un paramètre de translation. La compatibilité de ces équations différentielles mènent au résultat suivant.

Théorème 1.1. *Soit $s \in \mathbb{R}$, $F(a + s, \tau, k)$ définie comme en (5.1) où $a + s = (a_1 + s, \dots, a_N + s)$, alors*

$$\frac{\partial^2}{\partial s^2} \log(F(a + s, \tau, k)) = p^T(s)q(s), \quad (1.13)$$

où $(p(s), q(s)) = (p(s, \tau, a, k), q(s, \tau, a, k))$ sont des vecteurs de taille N satisfaisant : un système d'équations différentielles couplées du troisième ordre

$$\begin{cases} \partial_{sss} p^T + 3(\partial_s p^T) q p^T + 3p^T q (\partial_s p^T) - \tau \partial_s p^T + p^T D_{s,a} = 0, \\ \partial_{sss} q + 3(\partial_s q) p^T q + 3q p^T (\partial_s q) - \tau \partial_s q - D_{s,a} q = 0, \end{cases} \quad (1.14)$$

où $D_{s,a} := \text{Diag}(a_1 + s, \dots, a_N + s)$

et un système d'équations de la chaleur non-linéaires couplées

$$\begin{cases} -\frac{1}{2}\partial_{ss}p^T - \partial_\tau p^T = p^T q p^T, \\ -\frac{1}{2}\partial_{ss}q + \partial_\tau q = q p^T q. \end{cases} \quad (1.15)$$

Après un changement de variable pour la variable spectrale et une déformation du contour du problème de Riemann-Hilbert, il est possible de déterminer l'asymptotique de $p(s)$ et $q(s)$ lorsque $s \rightarrow \infty$. On obtient alors la proposition suivante.

Proposition 1.2. *Soient $p = (p_i)_{i \in \llbracket 1, N \rrbracket}$ et $q = (q_i)_{i \in \llbracket 1, N \rrbracket}$ satisfaisant les équations (1.13), (1.14) et (1.15).*

Alors : $p_i(s) \sim \sqrt{k_i - k_{i-1}}P(a_i + s)$ et $q_i(s) \sim \sqrt{k_i - k_{i-1}}Q(a_i + s)$ lorsque $s \rightarrow \infty$ où

$$\begin{aligned} Q(s) &:= \frac{1}{2i\pi} \int_{i\mathbb{R}} e^{-\frac{\mu^4}{4} + \tau \frac{\mu^2}{2} + s\mu} d\mu & \text{et } Q'''(s) - \tau Q'(s) &= sQ(s), \\ P(s) &:= \frac{1}{2i\pi} \int_{\Sigma} e^{\frac{\mu^4}{4} - \tau \frac{\mu^2}{2} - s\mu} d\mu & \text{et } P'''(s) - \tau P'(s) &= -sP(s). \end{aligned} \quad (1.16)$$

De plus lorsque $s \rightarrow \infty$:

$$Q(s) = \sqrt{\frac{2}{3\pi}} s^{-1/3} e^{-\frac{3}{8}s^{4/3} - \frac{\tau}{4}s^{2/3} + \frac{\tau^2}{6}} \cos\left(\frac{3}{4} \sin\left(\frac{2\pi}{3}\right) s^{4/3} - \frac{\tau}{2} \sin\left(\frac{2\pi}{3}\right) s^{2/3} - \frac{\pi}{6}\right) + o_{s \rightarrow \infty}(1), \quad (1.17)$$

$$P(s) = \sqrt{\frac{2}{3\pi}} s^{-1/3} e^{\frac{3}{8}s^{4/3} + \frac{\tau}{4}s^{2/3} - \frac{\tau^2}{6}} \cos\left(\frac{3}{4} \sin\left(\frac{2\pi}{3}\right) s^{4/3} - \frac{\tau}{2} \sin\left(\frac{2\pi}{3}\right) s^{2/3} - \frac{\pi}{6}\right) + o_{s \rightarrow \infty}(1). \quad (1.18)$$

Les deux résultats précédents permettent une meilleure compréhension de la fonction génératrice du processus de Pearcey. Les équations couplées (1.14) n'ont été découvertes que récemment pour le processus de Pearcey (pour la distribution de la "plus grande particule" du processus de Pearcey [KZ22] et pour la fonction génératrice du processus de Pearcey [Cho23]). En ce qui concerne les asymptotiques de P et Q (??), elles jouent le même rôle que la fonction d'Airy dans l'étude du noyau d'Airy et de l'équation de Painlevé II. Celles-ci fixent des comportements au bord pour $p(s)$ et $q(s)$ assurant ainsi l'unicité de la solution du système d'équation (1.14).

Dans un second temps, ces résultats présentent un intérêt par la manière dont on les a prouvés. En effet les raisonnements utilisés peuvent être reproduits pour étudier des généralisations d'ordre supérieur de l'opérateur de Pearcey (voir par exemple [KZ22]).

L'étude de ces opérateurs permettrait par exemple d'introduire une hiérarchie d'équations couplées associée à l'opérateur de Pearcey et ses analogues d'ordre supérieur.

1.2.2 Résultats originaux sur la hiérarchie discrète de Painlevé II et les polynômes orthogonaux sur le cercle unité [CT23b]

On introduit $\varphi(z) := \exp\left(\sum_{i=1}^N \frac{\theta_i}{i} (z + z^{-1})\right)$ (θ_i constantes réelles et $N \in \mathbb{N}_{\geq 1}$) généralisation d'ordre supérieur du symbole définissant D_n dans l'équation (1.8). Ce même symbole a été étudié récemment dans [BBW21] pour l'étude de mesures de Schur multi-critiques. Le n -ième déterminant de Toeplitz associé à ce symbole est alors :

$$D_n^N := D_n = \det\left((\varphi_{i-j})_{0 \leq i, j \leq n}\right) \quad (1.19)$$

où φ_k est le k -ième coefficient de Fourier de φ .

L'étude du problème de Riemann-Hilbert pour les polynômes orthogonaux sur le cercle unité ([Bai03] for $N = 1$, [BDS16][Chapter 5]) par rapport à la mesure positive $\varphi(z)dz$ permet de déduire un système d'équations discrètes et différentielles pour Ψ solution du problème de Riemann-Hilbert. Ce système d'équations est alors lié, via une transformation explicite, au système d'équations (4.16) : paire de Lax pour la hiérarchie de Painlevé II discrète. Ainsi ce nouveau système d'équation est bien une paire de Lax pour la hiérarchie de Painlevé II discrète.

La suite de ce travail consiste à déterminer une manière récursive pour construire la N -ième équation discrète de la hiérarchie discrète. Pour cela, on s'appuie sur la compatibilité du système précédemment décrit. On construit alors un opérateur récursif L que l'on itérera N fois. Cet opérateur récursif est différent de celui décrit par Cresswell et Joshi pour définir la N -ième équation discrète de la hiérarchie de Painlevé II [CJ99a]. Notre résultat se présente de la manière suivante :

Théorème 1.3. *Pour $N \geq 1$ entier fixé, le déterminant de Toeplitz D_n (4.3), où $n \geq 1$, associé au symbole $\varphi(z)$ satisfait l'équation suivante :*

$$\frac{D_n D_{n-2}}{D_{n-1}^2} = 1 - x_n^2, \quad (1.20)$$

et x_n résout l'équation discrète non-linéaire d'ordre $2N$

$$nx_n + \left(-v_n - v_n \text{Perm}_n + 2x_n \Delta^{-1} (x_n - (\Delta + I)x_n \text{Perm}_n)\right) L^N(0) = 0, \quad (1.21)$$

avec L l'opérateur de récursion défini comme suit

$$L(u_n) := \left(x_{n+1} \left(2\Delta^{-1} + I\right) \left((\Delta + I)x_n \text{Perm}_n - x_n\right) + v_{n+1} (\Delta + I) - x_n x_{n+1}\right) u_n,$$

$v_n := 1 - x_n^2$, Δ l'opérateur de différence $\Delta : u_n \rightarrow u_{n+1} - u_n$ et Perm_n la transformation $\mathbb{C} \left[(x_j)_{j \in \llbracket 0, 2n \rrbracket} \right]$ agissant par permutation des indices de la manière suivante

$$\begin{aligned} \text{Perm}_n : \quad \mathbb{C} \left[(x_j)_{j \in \llbracket 0, 2n \rrbracket} \right] &\longrightarrow \mathbb{C} \left[(x_j)_{j \in \llbracket 0, 2n \rrbracket} \right] \\ P \left((x_{n+j})_{-n \leq j \leq n} \right) &\longmapsto P \left((x_{n-j})_{-n \leq j \leq n} \right). \end{aligned}$$

L'équation (1.21) est la N -ième équation de la hiérarchie discrète de Painlevé II.

Finalement, en paramétrant les θ_i en fonction de n comme dans [BBW21], nous calculons la limite continue des équations pour x_n et obtenons les équations de la hiérarchie de Painlevé II.

Ce résultat apporte une nouvelle construction de la hiérarchie discrète de Painlevé II et enrichit nos connaissances sur ce sujet. En étudiant la paire de Lax que nous avons obtenu, nous avons défini un opérateur récursif L différent de celui présenté initialement par Cresswell et Joshi pour définir la hiérarchie discrète de Painlevé II.

Dans les travaux de Cresswell et Joshi ainsi que de Adler et van Moerbeke, les calculs de limites continues pour les équations discrètes de la hiérarchie n'étaient faits que pour les premiers termes de la hiérarchie. Nous présentons ici une preuve permettant de conclure que les équations de la hiérarchie de Painlevé II apparaissent comme limite continue des équations de la hiérarchie discrète de Painlevé II.

Enfin, nous apportons aussi un élément de réponse quant aux asymptotiques des x_n en conjecturant, via une approche numérique, que ceux-ci se comportent comme des fonctions de Bessel multi-variées.

1.3 Organisation du manuscrit

Le manuscrit sera principalement composé de deux parties.

La première aura pour but d'introduire les objets et concepts mathématiques dont

nous aurons besoin pour établir les résultats présentés dans la section précédente. Cette partie concerne les chapitres 2 et 3.

Le chapitre 2 concerne les processus ponctuels et plus précisément les processus ponctuels déterminantaux. Un des objectifs de ce chapitre sera de donner une intuition ainsi qu'une description précise de ce qu'est un processus ponctuel. Dans un second temps, on s'attachera à présenter les processus ponctuels déterminantaux, l'un des objets centraux de cette thèse. Ce chapitre aura aussi pour vocation d'introduire les objets et propriétés d'intérêt pour la suite de cette thèse : les probabilités de trou, la distribution de la plus grande particule ou encore la fonction génératrice d'un processus. Tout au long de ce chapitre nous illustrerons les notions présentées en considérant des exemples classiques de processus ponctuels tels que le processus de Poisson, l'Ensemble Gaussien Unitaire et le processus du noyau d'Airy.

Dans un second temps, le Chapitre 3 introduit l'outil central de mes travaux de thèse : les problèmes de Riemann-Hilbert (RHP). Dans la suite de cette thèse (chapitres 4 et 5), l'étude de problèmes de Riemann-Hilbert permettra de présenter des résultats nouveaux sur les processus de Pearcey et sur la hiérarchie discrète de Painlevé II. L'objectif du chapitre est donc de décrire les problèmes de Riemann-Hilbert. Nous préciserons, entre autres, comment résoudre ces problèmes dans certains cas. On illustrera cette notion avec l'exemple du problème de Riemann-Hilbert pour les polynômes orthogonaux sur la droite réelle et son lien avec l'équation discrète de Painlevé I. Enfin, on terminera ce chapitre en présentant la notion d'opérateurs intégrables, des opérateurs auxquels il est possible d'associer un RHP et pour lesquels les solutions du problème de Riemann-Hilbert et la résolvante de l'opérateur intégrable (si elles existent) sont reliées par une relation.

La seconde partie de cette thèse sera consacrée à la preuve des divers résultats présentés dans la Section 1.2 sur les contributions originales ainsi qu'à la présentation de perspectives de recherche dans la continuité de cette thèse. Elle comporte les chapitres 4, 5 et 6.

Le chapitre 4 aura pour objectif de présenter les résultats décrits dans la sous-section 1.2.2. Nous expliquerons comment, partant de l'étude de déterminants de Toeplitz en lien avec des partitions aléatoires d'entiers, il est possible de lier ces partitions aléatoires avec les polynômes orthogonaux sur le cercle unité. Nous obtiendrons une nouvelle paire de Lax pour la hiérarchie discrète de Painlevé II de l'étude du problème de Riemann-Hilbert pour les polynômes orthogonaux sur le cercle unité. Nous expliquerons aussi comment calculer la limite continue des équations discrètes de la hiérarchie pour obtenir les équations

(continues) de la hiérarchie de Painlevé II. Ce chapitre reprendra l'article [CT23b].

Dans le chapitre 5, il sera question du processus de Pearcey et plus précisément de la fonction génératrice de ce processus. On présentera ce processus ponctuel dont on a déjà discuté dans cette introduction puis nous étudierons un problème de Riemann-Hilbert associé au processus. L'étude de ce RHP avec des méthodes standards permettra de dériver une paire de Lax pour le problème de Riemann-Hilbert dont on déduira deux systèmes d'équations différentielles et aux dérivées partielles couplées. Ces systèmes d'équations seront mis en relation avec la fonction génératrice du processus afin d'obtenir une formule « à la Tracy-Widom ». Ce chapitre suivra l'article [Cho23].

Enfin, le chapitre 6 sera un chapitre d'ouverture où je présenterai les travaux sur lesquels je suis en train de travailler et les questions que j'aimerais étudier par la suite.

DETERMINANTAL POINT PROCESSES

As we mentioned in the introduction, point processes model many phenomena in physics and mathematics. While studying interacting particles such as fermions [Mac75], Macchi introduced the concept of fermionic point process (referred to as determinantal point process in this work). In these processes, the points of the random configuration are subject to repulsive forces. In quantum mechanics, the Pauli exclusion principle [Pau25] states that, in a quantum system, two or more identical particles (like fermions, for instance: particles with half-integer spins) cannot occupy the same quantum state at the same time. This property is associated with the repulsion property of determinantal point processes. Recently, models of fermions in non-harmonic traps were studied [DMS18] and were found to be related to higher-order Airy kernel point process. Other mathematical models, such as random matrix models (see for instance [GMW98] for a survey on the subject from the quantum physics point of view) or random growth models [FS11, Fer08] are also linked to determinantal point process. The first appearance of random matrices in physics dates back to nuclear physics. In the early 1950s, random matrices were studied [Dys53, Wig55] in order to model certain heavy nuclei. The study of the spectrum of these random matrices, whose size tends to be large, provided a better understanding of these nuclei. The eigenvalues of random matrices give rise to a random configuration of points, forming point processes. Some ensembles of random matrices, like the Gaussian Unitary Ensemble for instance, are even related to determinantal point processes.

The primary objective of this chapter is to introduce the general theory of point processes and, more precisely, of determinantal ones. In fact, this thesis aims to present two results related to determinantal point processes. The first one concerns the generating function of the Pearcey process, and the second result relates to the hierarchy of discrete Painlevé II equation, both with interpretations in terms of determinantal point processes.

This Chapter is organized into three sections.

In the first one, we start with an informal discussion to give an intuition about the notion of point processes, then we introduce the definitions specific to this concept, such as the

space of counting measures and the associated σ -algebra. The notion of point processes will be illustrated using the example of the Poisson point process, a classical point process. The second section focuses on the theory of determinantal point processes. After defining the correlation functions of a point process and providing an example for the Poisson process from the first section, we will specifically define what constitutes a determinantal point process and present a classical example of determinantal point process: the Gaussian Unitary Ensemble.

Finally, in the third section, we will describe objects related to determinantal point process and give properties to describe these objects in terms of Fredholm determinants of certain operators connected to the correlation functions of the process. The Airy kernel point process will be introduced, and quantities of interest for the associated determinantal point process will be presented.

This chapter is mainly inspired by the following references: [BDS16, Chapter 11], [Joh06] and [Sos00].

2.1 Counting measures space and point processes

The aim of this first section is to introduce the notion of point processes, to illustrate this notion and to define the factorial moment measure and correlation functions associated to a simple point process. This latter notion will be crucial in the next section when describing determinantal point processes. With this objective in mind, one has to introduce correlation functions associated to a point process. As we will see in what follows, the correlation functions associated to a point process (if they exist) describe the probability to have, in a union of infinitesimal intervals, exactly one particle in each intervals. Then the correlation function of a point process describes the local distribution of particles of the process.

2.1.1 Description of point processes

The purpose of this part is to describe point processes. Hence, we begin with a brief paragraph to provide an intuition of what a point process is.

Consider a non-decreasing real-valued function $x(t)$, where $t > 0$, which represents, for instance, the position of a particle on the real line subjected to a certain physical phenomenon. Let $(t_i)_{i \in I} \in \mathbb{R}_{>0}^I$ with I being a countable set, forming a non-decreasing

sequence of times. Then the set of points $X := (x_i)_{i \in I} = (x(t_i))_{i \in I}$ corresponds to a collection of points on the real line. X represents a configuration of points in \mathbb{R} , and there are several different ways to describe it. For example, we can consider the collection of intervals $\mathcal{X} := (x_i, x_{i+1})$ or the following counting measure $\xi := \sum_{i \in I} \delta_{x_i}$ or the non-decreasing integer-valued function $N : x \rightarrow \sum_{i \in I} \mathbb{1}_{(x_i, \infty)}(x)$. If the behavior of the particle is deterministic, the points x_i on the real line are fixed. However, if the particle's position at time t_i is not deterministic and is instead governed by a probability law on the real line at each time t_i , then the configuration of points X becomes random. Similarly \mathcal{X} , ξ or N are now random objects, and the probability of obtaining a specific random configuration of points X represents a point process on the real line.

With this image in mind, we can formally define what a point process is. In the preceding discussion, we mention four different objects to illustrate the concept of a point process. In what follows, we will focus on point process described using the set of counting measures. Nevertheless, it is possible to adapt the notions described above with the point of view of random configuration of points, random collection of intervals or random non-decreasing integer valued function.

First we define the set of counting measures on which we will define point processes.

Definition 2.1. *Let $\mathcal{N}(\mathbb{R})$ be the set of locally finite measures defined as*

$$\mathcal{N}(\mathbb{R}) := \{\xi \text{ counting measure on } \mathbb{R} \text{ and s.t. } \forall B \text{ bounded Borel set, } \xi(B) < \infty\}.$$

Remark 2.2. *One can replace \mathbb{R} in the previous definition and in what follows by any complete metric space Λ . This generalization would not have been possible if we had instead described the models with collections of intervals or non-decreasing integer-valued functions, which are closely related to the order relation on \mathbb{R} .*

Our goal is to define a probability on the set of counting measures $\mathcal{N}(\mathbb{R})$ in order to discuss random counting measures. To achieve this, we need to define a σ -algebra. We introduce the following one.

Definition 2.3. *Let $n \in \mathbb{N}$ and $B \subset \Lambda$ be a bounded Borel set.*

Introduce $\mathcal{C}_B^n := \{\xi \in \mathcal{N}(\mathbb{R}) | \xi(B) = n\}$ and define \mathcal{B} the σ -algebra generated by the family $(\mathcal{C}_B^n)_{n \in \mathbb{N}, B \subset \mathbb{R}}$.

We choose this σ -algebra because, in what follows, most of the proofs of propositions and theorems require conditioning with respect to the number of particles in a given bounded Borel set. Hence, this choice of σ -algebra is a natural and practical one.

Now that we have introduced the σ -algebra \mathcal{B} , we can define a probability measure on the measurable space $(\mathcal{N}(\mathbb{R}), \mathcal{B})$, i.e a point process.

Definition 2.4. *A point process on \mathbb{R} is a probability measure \mathbb{P} on $(\mathcal{N}(\mathbb{R}), \mathcal{B})$.*

Let B be a bounded Borel set on \mathbb{R} , and define $\#_B$ the random variable which counts the number of points of the process in B . One can write $\#_B$ as

$$\#_B := \sum_{i=1}^{\xi(B)} \delta_{x_i}, \quad (2.1)$$

where $x_i \in B$.

The following is a very simple example of point processes.

Example 2.5. Let $(\mathcal{N}(\mathbb{R}), \mathcal{B})$ be a measurable space (see Definitions 2.1 and 2.3). Consider the following probability \mathbb{P} on $(\mathcal{N}(\mathbb{R}), \mathcal{B})$ satisfying for any bounded Borel set B :

- $\mathbb{P}(\xi(B) = 2) = 1$, if $0 \in B$,
- $\mathbb{P}(\xi(B) = i) = 0$, if $0 \in B$ and $i \neq 2$,
- $\mathbb{P}(\xi(B) = i) = 0$, if $0 \notin B$ and for any $i \in \mathbb{N}$.

This choice of \mathbb{P} defines a point process on \mathbb{R} . However, this point process is not particularly interesting since it is, in a sense, deterministic. Indeed, this point process is supported on the measure $2\delta_0$.

In what follows, point processes will be simple. A point process is simple if there is almost surely at most one particle at each position, i.e., for any $x \in \mathbb{R}$, $\mathbb{P}(\#_{\{x\}} \leq 1) = 1$. In the previous example, the point process was not simple.

According to Definition 2.3 of the σ -algebra \mathcal{B} , the distribution of the point process is entirely determined by the family of probabilities $(\mathbb{P}(\{\#_B = n\}))_{n \in \mathbb{N}, B \subset \mathbb{R}}$, where $\#_B$ is defined in equation (2.1), and B is any bounded Borel set. We will see later (Theorem 2.15) that one can consider only void probabilities $\mathbb{P}(\{\#_B = 0\})$ to describe a simple point process. The idea behind this result is that it is equivalent to knowing the localization of particles and knowing where the process does not have particles.

The point process described in the previous example was a straightforward example of point processes. Now, we introduced one of the most universal point processes: the Poisson point process.

Example 2.6 (Poisson point process). Let $\nu > 0$, then the Poisson point process has the following distribution:

$$\mathbb{P}(\#_B = n) = \frac{(\nu|B|)^n}{n!} \exp(-\nu|B|), \quad (2.2)$$

where $n \in \mathbb{N}$, $B \subset \mathbb{R}$ is a bounded Borel set, and $|B| := \lambda(B)$ with λ being the Lebesgue measure on \mathbb{R} .

The Poisson point process is a simple point process.

2.1.2 Factorial moment measure

In the previous part, we described point processes on \mathbb{R} . In this part we will describe how, starting from a point process on \mathbb{R} , one can extend it into a process on \mathbb{R}^n . This extension allows us to introduce a measure, related to a point process, on Borel sets of \mathbb{R}^n . This measure can be expressed in terms of the expectation with respect to the probability of the point process and in terms of factorials of the number of points in Borel. Therefore, it is called factorial moment measure.

First, we define a random variable X_n from $(\mathcal{N}(\mathbb{R}), \mathcal{B}, \mathbb{P})$ to $(\mathcal{N}(\mathbb{R}^n), \mathcal{B}_n)$, where \mathcal{B}_n is the product σ -algebra with respect to \mathcal{B} .

Proposition 2.7. *Let \mathbb{P} be a point process on $\mathcal{N}(\mathbb{R})$, and define*

$$X_n(\xi) := \sum_{x_{i_1} \neq \dots \neq x_{i_n}} \delta_{(x_{i_1}, \dots, x_{i_n})}, \quad (2.3)$$

where $\xi := \sum \delta_{x_i}$, and the sum is over n -tuple of points of the process ξ .

Then X_n is a random variable from $(\mathcal{N}(\mathbb{R}), \mathcal{B}, \mathbb{P})$ to $(\mathcal{N}(\mathbb{R}^n), \mathcal{B}_n)$, where \mathcal{B}_n is the product σ -algebra with respect to \mathcal{B} .

Proof. Let $m \in \mathbb{N}$ and $B_1, \dots, B_n \subset \mathbb{R}$. We show that the set

$$X_n^{-1}\{\xi^{(n)} \in \mathcal{N}(\Lambda^n) : \xi^{(n)}(B_1 \times \dots \times B_n) = m\} = \{\xi \in \mathcal{N}(\Lambda) : X_n(\xi)(B_1 \times \dots \times B_n) = m\}$$

is included in \mathcal{B} .

One can prove that it is enough to consider the case where B_i 's satisfies either $B_i = B_j$ either $B_i \cap B_j = \emptyset$ for $i \neq j$. Indeed it is always possible to decompose $B_1 \times \dots \times B_n$ into a countable union and intersection of disjoint Borel sets. Therefore, consider B_1, \dots, B_n

such that $B_i = B_j$ or $B_i \cap B_j = \emptyset$ for $i \neq j$.

Since for any $\sigma \in \mathcal{S}^n$, $X_n(\xi)(B_1 \times \dots \times B_n) = X_n(\xi)(B_{\sigma(1)} \times \dots \times B_{\sigma(n)})$ according to the definition of X_n (see equation (2.3)). Then one can rearrange sets in the Cartesian product, and we can consider that

$$B_1 \times \dots \times B_n = A_1^{n_1} \times \dots \times A_p^{n_p}$$

with A_i and A_j disjoint sets for $i \neq j$ and $n_1 + \dots + n_p = n$.

Then, for this choice of B_i 's, we have

$$X_n(\xi)(B_1 \times \dots \times B_n) = \sum_{i_1 \neq \dots \neq i_n} \chi_{A_1}(x_{i_1}) \times \dots \times \chi_{A_1}(x_{i_{n_1}}) \times \dots \times \chi_{A_p}(x_{i_n}), \quad (2.4)$$

where χ_A is the characteristic function of the set A , and x_i 's are points of the process, i.e, $\xi = \sum_i \delta_{x_i}$.

Since A_i 's are pairwise disjoint, $\chi_{A_i}(x)\chi_{A_j}(x) = 0$ for any $x \in \mathbb{R}$ and any $i \neq j$. Then, it is possible to decompose the sum in the right-hand side of equation (2.4) into a product of sum over A_i 's. It reads as:

$$X_n(\xi)(B_1 \times \dots \times B_n) = \prod_{i=1}^p \sum_{j_1 \neq \dots \neq j_{n_i}} \chi_{A_i}(x_{j_1}) \times \dots \times \chi_{A_i}(x_{j_{n_i}}).$$

Fix $1 \leq i \leq p$, we want to compute

$$\sum_{j_1 \neq \dots \neq j_{n_i}} \chi_{A_i}(x_{j_1}) \times \dots \times \chi_{A_i}(x_{j_{n_i}}).$$

To do this, we count the number of n_i -tuples of elements such that the product is equal to 1. For the first term, we have $\xi(A_i)$ possibilities, for the second $\xi(A_i) - 1$ since we do not allow two elements x_j to be equal because the point process is simple and so on.

Hence,

$$\sum_{j_1 \neq \dots \neq j_{n_i}} \chi_{A_i}(x_{j_1}) \times \dots \times \chi_{A_i}(x_{j_{n_i}}) = \xi(A_i)(\xi(A_i) - 1) \dots (\xi(A_i) - n_i + 1),$$

and it is possible to rewrite $X_n(\xi)(B_1 \times \dots \times B_n)$ as

$$X_n(\xi)(B_1 \times \dots \times B_n) = \prod_{i=1}^p \xi(A_i)(\xi(A_i) - 1) \dots (\xi(A_i) - n_i + 1). \quad (2.5)$$

But $X_n(\xi)(B_1 \times \dots \times B_n) = m$, then there exist m_1, \dots, m_p integers such that $m_1 m_2 \dots m_p = m$ and for any $1 \leq i \leq p$,

$$\xi(A_i)(\xi(A_i) - 1) \dots (\xi(A_i) - n_i + 1) = m_i.$$

We therefore deduce, for $1 \leq i \leq p$, $\xi(A_i) = m'_i$, and $\xi \in \bigcap_{1 \leq i \leq p} \mathcal{C}_{A_i}^{m'_i}$.

Then

$$X_n^{-1}\{\xi^{(n)} \in \mathcal{N}(\Lambda^n) : \xi^{(n)}(B_1 \times \dots \times B_n) = m\}$$

is included in \mathcal{B} , which implies that X_n is a random variable. □

Let \mathbb{P} be a point process on $\mathcal{N}(\mathbb{R})$. According to Proposition 2.7, X_n is a random variable, which allows us to extend the point process \mathbb{P} to a point process on $\mathcal{N}(\mathbb{R}^n)$. Let $\mathbb{P}^{(n)}$ denote this new point process. $\mathbb{P}^{(n)}$ is defined as the push-forward of \mathbb{P} with respect to X_n .

Definition 2.8. Define a measure M_n on \mathbb{R}^n as:

$$M_n(A) := \mathbb{E}(X_n(\xi)(A)), \tag{2.6}$$

where \mathbb{E} denotes the expectation with respect to \mathbb{P} , and A is a bounded Borel set in \mathbb{R}^n .

$M_n(A)$ corresponds to the expected number of n -tuples of distinct points of the process in A . For $n = 1$, $M_1(A) = \mathbb{E}(X_1(\xi)(A)) = \mathbb{E}(\sharp_A)$ is the expected number of points of the process in the Borel set A .

According to the Definition 2.8 of M_n , for A_1, A_2, \dots, A_l disjoint bounded Borel sets in \mathbb{R} and for n_1, \dots, n_l integers such that $\sum n_i = n$, one can express M_n (see the equation (2.5)) as

$$M_n(A_1^{n_1} \times \dots \times A_l^{n_l}) = \mathbb{E} \left(\prod_{i=1}^l \xi(A_i)(\xi(A_i) - 1) \dots (\xi(A_i) - n_i + 1) \right). \tag{2.7}$$

M_n is then called factorial moment measure for $(\mathcal{N}(\mathbb{R}), \mathcal{B}, \mathbb{P})$.

2.1.3 Correlation functions of a point process

The purpose of this subsection is to introduce correlation functions of a point process. In statistical mechanics, correlation functions characterize the state of a system of microscopic variables at different positions and quantify how these variables are related.

Definition 2.9. A non-negative function ρ_n (if it exists) is called the n -th correlation function of the factorial moment measure M_n (or of the process) if M_n is absolutely continuous with respect to λ^n , the Lebesgue measure on \mathbb{R}^n , such that

$$M_n(A_1 \times \dots \times A_n) = \int_{A_1 \times \dots \times A_n} \rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n)$$

for any Borel sets $A_1, \dots, A_n \subset \mathbb{R}$.

Consider a point process \mathbb{P} on the real line with a factorial moment measure M_n that is absolutely continuous with respect to the Lebesgue measure. Let y_1, \dots, y_n be n distinct points on \mathbb{R} . If $Y_i = [y_i, y_i + \Delta y_i]$ are infinitesimal intervals, because of the Lebesgue differentiation theorem,

$$\rho_n(y_1, \dots, y_n) = \lim_{\Delta y_i \rightarrow 0} \frac{1}{\Delta y_1 \dots \Delta y_n} \int_{Y_1 \times \dots \times Y_n} \rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n). \quad (2.8)$$

But M_n is absolutely continuous with respect to the Lebesgue measure. Then

$$\rho_n(y_1, \dots, y_n) = \lim_{\Delta y_i \rightarrow 0} \frac{1}{\Delta y_1 \dots \Delta y_n} M_n(Y_1 \times \dots \times Y_n). \quad (2.9)$$

The y_i 's are distinct, then for Δy_i small enough, the Y_i 's are disjoint, and there is at most one particle (the point process is simple) in each Y_i . Then (see equation (2.7)),

$$M_n(Y_1 \times \dots \times Y_n) = \mathbb{E}(\xi(Y_1) \dots \xi(Y_n)) \quad (2.10)$$

where $\xi(Y_1) \dots \xi(Y_n)$ equals 1 if there is one particle in each Y_i else 0. Then

$$\mathbb{E}(\xi(Y_1) \dots \xi(Y_n)) = \mathbb{E}(\mathbf{1}_{\{\xi(Y_i)=1\}}) = \mathbb{P}(\text{there is exactly one particle in each } Y_i) \quad (2.11)$$

Then

$$\rho_n(y_1, \dots, y_n) = \lim_{\Delta y_i \rightarrow 0} \frac{\mathbb{P}(\text{there is exactly one particle in each } Y_i)}{\Delta y_1 \dots \Delta y_n}. \quad (2.12)$$

Thus the correlation function of a simple point process describes the density of particles of the process.

Example 2.10 (Poisson point process). Consider the Poisson point process defined as in Example 2.6. The correlation function for the Poisson point process is $\rho_n(x_1, \dots, x_n) = \nu^n$. Indeed, as we mentioned before, for Δy_i small enough, Y_i are disjoint. Then $(\#_{Y_i})_{1 \leq i \leq n}$ is

a family of independent random variables, and one can write equation (2.12) as

$$\rho_n(y_1, \dots, y_n) = \prod_{i=1}^n \lim_{\Delta y_i \rightarrow 0} \frac{\mathbb{P}(\#_{Y_i} = 1)}{\Delta y_i} = \prod_{i=1}^n \lim_{\Delta y_i \rightarrow 0} \frac{\nu \Delta y_i \exp(-\nu \Delta y_i)}{\Delta y_i} = \nu^n, \quad (2.13)$$

where the first equality is due to the independence of $\#_{Y_i}$, and the second equality comes from the equation (2.2).

2.2 Determinantal point process

The goal of this section is mainly to define determinantal point processes, which are point processes whose correlation functions are expressed in terms of determinant of matrices. These types of point processes have found applications in various mathematical models, including integer partitions [Oko01], associated Young diagrams and Maya diagrams, random permutations, random matrices [Joh06], aztec diamonds [Joh02], (T)ASEP ((Totally) Asymmetric Simple Exclusion Process) [BO17] and more.

We describe some of these point processes. Let's begin with the integer partition model. Consider a fixed integer $n \in \mathbb{N}_{>0}$. A partition of the integer n is a non-increasing sequence of integers $(\lambda_1, \lambda_2, \dots, \lambda_p, 0, 0, \dots)$ such that $\sum_{1 \leq i \leq p} \lambda_i = n$. A geometric representation of such partitions is through Young diagrams. Consider a partition $(\lambda_1, \lambda_2, \dots, \lambda_p, 0, 0, \dots)$ and construct p rows of blocks, where the first row has λ_1 blocks, the second one has λ_2 blocks, and so on. Then glue these rows of blocks together to form a Young diagram (see Figure 2.2 for an example of Young diagram associated to the partition $(4, 4, 3, 1, 1, 0, \dots)$). The Plancherel measure can be defined on the set of Young diagrams of size n . By Poissonizing this measure, one can obtain a measure on the set of all Young diagrams (without a fixed size), leading to a point process. Indeed, to each integer partition $(\lambda_1, \lambda_2, \dots, \lambda_p, 0, 0, \dots)$ (or equivalently, Young diagram), one can associate the following counting measure: $\sum \delta_{\lambda_i - i + \frac{1}{2}}$. Alternatively, one can consider the Maya diagram which can be described as follows: consider empty sites on all integers plus a half, and place particles (represented by red dots in Figure 2.2) on site at location $\lambda_i - i + \frac{1}{2}$.

The point process described by this model is determinantal. This model has been found to be related to the Tracy-Widom distribution. Furthermore, for the generalization of the Poissonized-Plancherel measure (Schur measure), it has been shown to be related to higher order analogues of the Tracy-Widom distribution [BBW21, KZ21]. The Tracy-Widom distribution and its higher order analogues are known to be connected with the

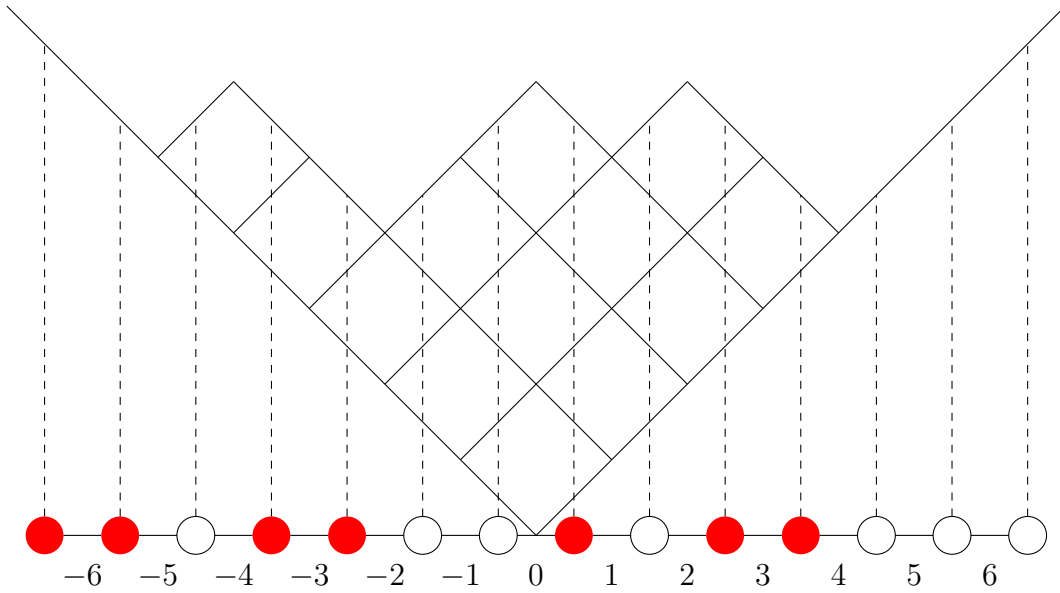


Figure 2.1 – Correspondence between integer partitions, Young diagrams and Maya diagrams: example with the partition $(4, 4, 3, 1, 1, 0, \dots)$

Gaussian Unitary Ensemble which is a determinantal point process.

The model of determinantal point process we discussed above is connected to the GUE-Tracy-Widom distribution. Therefore, after defining correlation functions and determinantal point process, we will conclude this section by illustrating the notion of a determinantal point process using the example of the Gaussian Unitary Ensemble. We will describe this model and express its correlation functions in terms of determinants of matrices.

2.2.1 Definition of a determinantal point process

In this subsection, we describe what a determinantal point process is. In Chapters 4 and 5, we will study Toeplitz determinants for a family of orthogonal polynomial on the unit circle and the generating function associated to the Pearcey process, both of which are related to determinantal point processes.

A point process is determinantal if its correlation functions are of a certain form.

Definition 2.11. *A point process is said to be determinantal if its correlation functions exist and if there exists $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ such that $\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j)_{1 \leq i, j \leq n})$ for*

all $x_1, \dots, x_n \in \mathbb{R}$ and $n \geq 1$.

K is called the correlation kernel of the determinantal point process.

A determinantal point process is always simple.

When the correlation functions of a determinantal point process exist, it is normal to ask about their uniqueness and how they characterize the determinantal point process. This question was first discussed by Lenard [Len73, Len75a, Len75b], and then by Soshnikov [Sos00]. Lenard presented a necessary and sufficient condition for the existence of a point process with given correlation functions (see Theorem 1 in [Sos00]). Then Soshnikov introduced the following theorem to characterize determinantal point processes.

Theorem 2.12 (Theorem 3 in [Sos00]). *Consider \mathcal{K} an Hermitian locally trace class operator on $L^2(\mathbb{R})$, i.e. an operator with kernel k satisfying $\forall(x, y) \in \mathbb{R}^2, k(x, y) = \overline{k(y, x)}$ and such that for any compact $B \subset \mathbb{R}$, $\mathcal{K}\chi_B$ is trace class. Then \mathcal{K} defines a determinantal point process if and only if $0 \leq \mathcal{K} \leq 1$. If the corresponding determinantal point process exists, then it is unique.*

2.2.2 The Gaussian Unitary Ensemble: a determinantal point process

In the introduction, we discussed two random matrix models: one with an external source and the other one known as the Gaussian Unitary Ensemble (GUE). These two matrix models are associated with determinantal point processes. Now, our focus will be on presenting the GUE model.

In the introduction, we discussed two random matrix models: one with external source and the other one known as the Gaussian Unitary Ensemble (GUE). These two matrix models are associated to determinantal point processes and the purpose of the next discussion is to present the GUE model.

Denote by $\mathcal{H}_N = \{H \in \mathcal{M}_N(\mathbb{C}), H = H^*\}$ the set of Hermitian matrices. This vector space has dimension N^2 as a real vector space, consisting of the N real diagonal entries and the $(N^2 - N)/2$ real and imaginary parts of the upper triangle entries. In the Gaussian Unitary Ensemble, a matrix $H \in \mathcal{H}_N$ is chosen such that its real diagonal coefficients H_{jj} and real H_{jk}^R and imaginary H_{jk}^I parts of upper triangle ($j < k$) coefficients are independent and H_{jj} is distributed (respectively H_{jk}^R and H_{jk}^I are) as real mean 0 variance 1 (respectively $\frac{1}{2}$) Gaussian random variable.

The probability distribution on \mathcal{H}_N is given by:

$$P^{(N)}(H)dH := \frac{1}{Z_N} \exp\left(-\operatorname{Tr}(H^2)\right) dH = \frac{1}{Z_N} \exp\left(-\operatorname{Tr}(H^2)\right) \prod_{j=1}^N dH_{jj} \prod_{j<k} dH_{jk}^R dH_{jk}^I, \quad (2.14)$$

where Z_N is a normalization constant. The term unitary refers to the fact that the distribution is invariant under conjugation with a unitary matrix.

The eigenvalues of Hermitian matrices are real numbers, so studying the distribution of GUE eigenvalues gives rise to a random configuration of points on the real line, i.e. a point process on the real line.

Theorem 2.13 (Theorem 3.3.1 [Meh04]). *The joint probability density function for the eigenvalues of matrices from the GUE is given by:*

$$P_{N,2}(x_1, \dots, x_N) = C_{N,2} \exp\left(-\sum_{j=1}^N x_j^2\right) \prod_{j<k} |x_j - x_k|^2, \quad (2.15)$$

with $C_{N,2}$ a normalization constant.

A consequence of Theorem 2.13 is that a.s., the eigenvalues of the GUE are all distinct.

From the equation (3) in [Dys62b], the n -point correlation function for this point process is given by:

$$\rho_{N,2}(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} P_{N,2}(x_1, \dots, x_n, \xi_{n+1}, \dots, \xi_N) \lambda(d\xi_{n+1}) \dots \lambda(d\xi_N). \quad (2.16)$$

For the GUE, the joint probability density function for the eigenvalues can be rewritten in terms of a determinant of a matrix whose coefficients are Hermite's polynomials, then one can express the correlation functions of the point process as determinants which means that the point process is determinantal. The correlation function is then

$$\rho_{N,2}(x_1, \dots, x_n) = \det(K_N(x_i, x_j))_{i,j=1,\dots,n}, \quad (2.17)$$

where $K_N(x, y) := \sum_{k=0}^{N-1} \frac{1}{2^k k! \sqrt{\pi}} H_k(x) H_k(y) \exp\left(-\frac{x^2 + y^2}{2}\right)$ with H_j 's Hermite's polynomials.

2.3 Quantities of interest for determinantal point processes

In the previous section, we introduced determinantal point processes and illustrated them with the example of GUE matrices. Now, we will introduce the main quantities of interest associated with determinantal point processes, which will be relevant throughout this thesis.

In the previous section, we introduced determinantal point processes and illustrated them with the example of GUE matrices. In what follows we will introduce the main quantities of interest associated with determinantal point process, which will be relevant throughout this thesis. These quantities include the gap probability, the last particle distribution, and the generating function of a point process. One of the remarkable properties of determinantal point processes is that under certain conditions on the correlation kernel K , the previously mentioned quantities can be expressed in terms of Fredholm determinants of trace-class operators. This section is also a good opportunity to introduce Tracy-Widom's result, that we mentioned several times previously.

2.3.1 Description of gap probability, last particle distribution and generating function of a determinantal point process

We will present the main quantities of interest for the study of determinantal point processes.

Proposition 2.14 (Proposition 2.2 [Joh06]). *Consider a point process such that all correlation functions exist.*

Let $\phi : \mathbb{R} \rightarrow \mathbb{C}$ be a function bounded, measurable, and compactly supported in a Borel set B . If

$$\sum_{n=0}^{\infty} \frac{\|\phi\|_{\infty}^n}{n!} M_n(B^n) < \infty, \quad (2.18)$$

then

$$\mathbb{E} \left(\prod_i (1 + \phi(x_i)) \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^n \phi(x_i) \rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n), \quad (2.19)$$

where for $\xi = \sum_i \delta_{x_i} \in \mathcal{N}(\mathbb{R})$, $\prod_i (1 + \phi(x_i)) = \prod_i^{\xi(B)} (1 + \phi(x_i))$.

Let B be a bounded Borel set, then the previous proposition with $\phi = -\chi_B$ reads as

$$\mathbb{E} \left(\prod_i (1 - \chi_B(x_i)) \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \prod_{i=1}^n \rho_n(x_1, \dots, x_n) \lambda(dx_1) \dots \lambda(dx_n). \quad (2.20)$$

But, if $x \in B$, $1 - \chi_B(x) = 0$, otherwise, $1 - \chi_B(x) = 1$. Hence the left-hand side in the previous equation can be rewritten as \mathbb{P} (no particle in B), which is called the gap probability of the point process. Gap probabilities characterize simple point processes.

Theorem 2.15 (Theorem 9.2.XII (Rényi, 1967; Mönch, 1971) [DVJ08]). *A simple point process is entirely determined by its gap probabilities (or void probability) $\mathbb{P}(\#_B = 0)$ for any B bounded Borel set.*

Remark 2.16. *The previous theorem was first proved by Alfred Rényi [Rén67] in the case of the Poisson point process. The proof of Theorem 2.15 is due to Mönch [Mö71], who showed that the Poisson hypothesis is not required. Finally, studying the class of completely monotone set functions, Kurtz [Kur75] characterized gap probabilities (or zero probability functions or void probabilities or avoidance functions) of point processes.*

The left-hand side in equation (2.19) can be rewritten as the Fredholm determinant of a certain operator. Let $K(x, y)$ be the kernel of a determinantal point process. Consider K a kernel which induces an integral operator \mathbb{K} on $L^2(\mathbb{R})$ defined by:

$$\mathbb{K}f(x) = \int_{\mathbb{R}} K(x, y)f(y)\lambda(dy) \quad (2.21)$$

Proposition 2.17 (Proposition 2.9 [Joh06]). *Consider a determinantal point process on \mathbb{R} with a Hermitian correlation kernel K , i.e., $\forall(x, y) \in \mathbb{R}^2, K(x, y) = \overline{K(y, x)}$.*

Let B be a bounded Borel set on \mathbb{R} . If K induces a trace-class operator on $L^2(B, dx)$, is continuous on $B \times B$ and satisfies

$$\int_B K(x, x)\lambda(dx) < \infty, \quad (2.22)$$

then for all $\phi \in L^\infty(\mathbb{R})$ compactly supported in B , one has $\mathbb{E}(\prod_i(1 + \phi(x_i))) = \det(1 + \mathbb{K}\phi)_{L^2(B)}$ where the right-hand side is the Fredholm determinant of $\mathbb{K}\phi$.

Moreover, if K defines a trace-class operator on $L^2((t, \infty))$ for all t in \mathbb{R} and

$$\int_t^\infty K(x, x)\lambda(dx) < \infty, \quad (2.23)$$

then the point process has a largest particle ξ_{\max} and $\mathbb{P}(\xi_{\max} \leq t) = \det(1 - \mathbb{K})_{L^2((t, \infty))}$, where $\det(1 - \mathbb{K})_{L^2((t, \infty))}$ denotes the Fredholm determinant of the operator $1 - \mathbb{K}$,

$$\det(1 - \mathbb{K})_{L^2((t, \infty))} := \prod_{n \geq 1} (1 - \lambda_n(\mathbb{K}))$$

with $\{\lambda_n(\mathbb{K})\}$ the spectrum of \mathbb{K} .

Another important quantity is the generating function of a determinantal point process. It is defined as in the following theorem.

Theorem 2.18 (Theorem 2 [Sos00]). *Let $(\mathcal{N}(\mathbb{R}), \mathcal{B}, \mathbb{P})$ be a determinantal point process with the kernel K . For any finite number of disjoint bounded Borel sets $B_j \subset \mathbb{R}, j = 1, \dots, N$, the generating function of the probability distribution of $\sharp B_j$ is given by*

$$\mathbb{E} \left(\prod_{j=1}^{N-1} z_j^{\sharp B_j} \right) = \det \left(1 + \chi_B \sum_{j=1}^{N-1} (z_j - 1) K \chi_{B_j} \right), \quad (2.24)$$

where $B = \cup_{j=1}^{N-1} B_j$.

From the generating function, one can compute the joint probability of the occupancy numbers of particles as follows:

$$\mathbb{P} \left(\bigcap_{j=1}^{N-1} \sharp B_j = m_j \right) = \frac{1}{m_1! \dots m_{N-1}!} \frac{\partial^{m_1 + \dots + m_{N-1}}}{\partial z_1^{m_1} \dots \partial z_{N-1}^{m_{N-1}}} \mathbb{E} \left(\prod_{j=1}^{N-1} z_j^{\sharp B_j} \right)_{\vec{z}=(0, \dots, 0)}, \quad (2.25)$$

where $\vec{z} = (z_1, \dots, z_{N-1})$. See [Joh01] for a similar computation.

2.3.2 Quantities of interest for the Airy kernel point process

We finish this section by presenting the Airy kernel point process in order to illustrate the previous theorems.

In section 2.2.2, we introduced the Gaussian Unitary Ensemble. Its correlation functions ρ_n are expressed in terms of Hermite polynomials (see equation (2.17) and the definition of the correlation kernel $K_N(x, y)$ above). Because of the Christoffel-Darboux formula, one can write $K_N(x, y)$ as:

$$K_N(x, y) = \left(\frac{N}{2} \right)^{\frac{1}{2}} \frac{\phi_N(x) \phi_{N-1}(y) - \phi_{N-1}(x) \phi_N(y)}{x - y}, \quad (2.26)$$

where $\phi_j(x) := \left(\frac{1}{2^j j! \sqrt{\pi}}\right)^{\frac{1}{2}} H_j(x) \exp\left(-\frac{x^2}{2}\right)$.

The kernel K_N depends on N , the size of the matrices, and a natural question is what is the behavior of eigenvalues of these random matrices as the size of the matrices tends to infinity. As mentioned in the introduction, this behavior is described by the Wigner semi-circle law, and for a certain rescaling, the edge behavior of eigenvalues is described by the Airy kernel point process. More precisely, (see [For93]) for any x, y fixed in \mathbb{R} ,

$$\lim_{N \rightarrow \infty} \frac{1}{2^{1/2} N^{1/6}} K_N \left(\sqrt{2N} + \frac{x}{2^{1/2} N^{1/6}}, \sqrt{2N} + \frac{y}{2^{1/2} N^{1/6}} \right) = K_{\text{Airy}}(x, y), \quad (2.27)$$

where

$$K_{\text{Airy}}(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}, \quad (2.28)$$

with Ai being the Airy function. Written in this form, the Airy kernel is called integrable in the sense of IIKS (Its, Izergin, Korepin, Slavnov) [IIKS90].

Since the Airy function Ai is real-valued and because of the symmetric role played by x and y in the definition of $K_{\text{Airy}}(x, y)$, K_{Airy} is Hermitian. Moreover, for any bounded Borel set B , K_{Airy} is continuous on $B \times B$. Finally, K_{Airy} is a trace-class operator. To see this, we can use the Airy equation $\text{Ai}''(x) - x\text{Ai}(x) = 0$. By integrating by parts, one can express K_{Airy} as

$$K_{\text{Airy}}(x, y) = \int_0^\infty \text{Ai}(x+t)\text{Ai}(y+t)dt \quad (2.29)$$

Then the integral operator associated with the kernel K_{Airy} can be written as a composition of two Hilbert-Schmidt operator with kernel $A(x, t) := \text{Ai}(x+t)$. Hence, it is trace-class on $L^2(B)$ for any bounded Borel set B and on $L^2((t, \infty))$ for any $t \in \mathbb{R}$.

According to Proposition 2.17, the gap probability and the last particle distribution are expressed in terms of Fredholm determinant of the operator \mathbb{K}_{Airy} . Let B be a bounded Borel set and $t \in \mathbb{R}$. Then

$$\mathbb{P}(\text{no particle in } B) = \det(1 - \mathbb{K}_{\text{Airy}})_{L^2(B)}, \quad (2.30)$$

$$\mathbb{P}(\xi_{\max} \leq t) = \det(1 - \mathbb{K}_{\text{Airy}})_{L^2((t, \infty))}. \quad (2.31)$$

Similarly, from Theorem 2.18, the generating function for the determinantal point process

is given by

$$\mathbb{E} \left(\prod_{j=1}^{N-1} z_j^{\#B_j} \right) = \det \left(1 + \chi_B \sum_{j=1}^{N-1} (z_j - 1) K_{\text{Airy}} \chi_{B_j} \right)_{L^2(\mathbb{R})}, \quad (2.32)$$

where $B = \cup_{j=1}^{N-1} B_j$ with B_j 's bounded Borel sets.

All these quantities were studied [TW94b, CD18] and shown to be related to the Painlevé II equation. For the largest particle distribution, the result is due to Tracy and Widom and is called the Tracy-Widom distribution.

Theorem 2.19 ([TW94b]). *Let $s \in \mathbb{R}$ and \mathbb{K}_{Airy} be the operator with kernel K_{Airy} as in equation (2.28). Then the largest particle distribution for the Airy kernel point process is*

$$\mathbb{P}(\xi_{\max} \leq s) = \exp \left(- \int_s^\infty (x - s) q(x)^2 dx \right), \quad (2.33)$$

where q is the Hastings Mc-Leod [HM80] solution to the Painlevé II equation:

$$q''(x) = xq(x) + 2q(x)^3 \text{ and } q(x) \sim \text{Ai}(x), \quad x \rightarrow \infty. \quad (2.34)$$

An alternative way of writing the equation (2.33) is as follows:

$$\frac{d^2}{ds^2} \log \left(\det (1 - \mathbb{K}_{\text{Airy}})_{L^2((s, \infty))} \right) = -q(s)^2 \quad (2.35)$$

with q as in equation (2.34).

Similar equations to the equation (2.35) appear in the study of other determinantal point processes. The goal of the Chapter 4 (respectively Chapter 5) will be to present a discrete analogue of the equation (2.35) for gap probabilities expressed in terms of Toeplitz determinants and related to the discrete Painlevé II hierarchy (respectively an analogue formula for the generating function associated with the Pearcey process).

We finish this section by mentioning other results related to the Airy kernel point process. Recent work has extended the result in Theorem 2.19 to other cases.

Studying the higher order Airy kernels, Cafasso, Claeys and Girotti [CCG21] introduced a higher-order analogue of the Tracy-Widom distribution and related it to the Painlevé II hierarchy [Jos04]. This result reads as follows. Let n be a positive integer, and consider the higher-order Airy kernel operator $\mathbb{K}_{\text{Ai}}^{(n)}$ with kernel

$$K_{\text{Ai}}^{(n)}(x, y) := \int_{\mathbb{R}} \text{Ai}_n(x+t) \text{Ai}_n(y+t) dt \quad (2.36)$$

where $\text{Ai}_n(x) := \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^{2n+1}}{2n+1} + tx\right) dt$ (for $n = 1$, one recovers the usual Airy kernel).

This operator is related to a determinantal point process (because of Theorem 2.12), and it admits a largest particle. The distribution of the largest particle is given by the Fredholm determinant $F_n(s) := \det\left(1 - \mathbb{K}_{\text{Ai}}^{(n)}\right)_{L^2(s, \infty)}$, and the following holds.

The logarithm of $F_n(s)$ satisfies

$$\frac{d^2}{ds^2} \log(F_n(s)) = -q^2((-1)^{n+1}s), \quad (2.37)$$

where q satisfies the n -th equation of the Painlevé II hierarchy with asymptotic

$$q((-1)^{n+1}s) = \mathcal{O}\left(\exp\left(-Cs^{\frac{2n+1}{2n}}\right)\right) \quad (2.38)$$

for some $C > 0$. The equation (2.37) can be rewritten as

$$F_n(s) = \exp\left(-\int_s^\infty (x-s)q((-1)^{n+1}x)^2 dx\right). \quad (2.39)$$

Another possible generalization of the Airy process is to study multiplicative statistics associated with it.

Recently the question of multiplicative statistics associated with determinantal point process was raised. For the Airy kernel, this generalization was first studied by Amir, Corwin and Quastel in [ACQ11]. They deformed the Airy kernel the following way.

With σ in the class of function smooth except at a finite number of points at which it has bounded jumps and which approaches 0 at $-\infty$ and 1 at ∞ , consider the kernel K^σ defined as

$$K^\sigma(x, y) := \int_{-\infty}^\infty \sigma(t) \text{Ai}(x+t) \text{Ai}(y+t) dt. \quad (2.40)$$

Then the following holds:

Proposition 2.20 (Proposition 5.2 [ACQ11]). *For K^σ defined as above, let $q_t(s)$ be the solution to*

$$\frac{d^2}{ds^2} q_t(s) = \left(s + t + 2 \int_{-\infty}^\infty \sigma'(r) q_r(s)^2 dr\right) q_t(s), \quad (2.41)$$

where $q_t(s) \sim \text{Ai}(s+t)$ as $s \rightarrow \infty$. Then

$$L(\sigma) = \det(1 - K^\sigma)_{L^2((s, \infty))} = \exp\left(-\int_s^\infty (x-s) \int_{-\infty}^\infty \sigma'(t) q_t(s)^2 dt dx\right). \quad (2.42)$$

This proposition is a generalization of Theorem 2.19 in the sense that if $\sigma = \chi_{(s,\infty)}$, then the previous result reduces to the one of Tracy and Widom. The equation (2.41) is called the integro-differential Painlevé II equation. There are other works on multiplicative statistics for the Airy process. We can mention, for instance, the works on free Fermions at finite temperature of Dean et al. [DLDMS15] and on the Schur Process of Betea and Bouttier [BB19], in which the integro-differential Painlevé II equation appears.

Recently, multiplicative statistics for the higher-order Airy kernel were also studied and proved to be in relation with a hierarchy of integro-differential equations [Kra21, BCT22] related to the Painlevé II hierarchy.

RIEMANN-HILBERT PROBLEMS

This chapter is dedicated to the notion of Riemann-Hilbert problem, a useful concept for the study of determinantal point processes and their relation to integrable equations. In the preceding chapter, we discussed the notion of point processes and more precisely of determinantal ones. Some of the quantities presented in Chapter 2 were expressed in terms of the Fredholm determinant of the integral operator associated with the correlation kernel. In this chapter, we will introduce Riemann-Hilbert problems (noted as RHP hereafter) and explain how to connect a Fredholm determinant of an integrable operator with a solution of a RHP.

Riemann-Hilbert problem were first mentioned in the study of the surjectivity of the monodromy map in the theory of Fuchsian systems, also known as the twenty-first Hilbert's problem. Riemann's name is attached to Hilbert's because of his idea that an analytic function could be entirely defined by its singularities and monodromy properties. A first solution to this problem was given by Plemelj [Ple08] in 1908. He reduced RHP to a boundary value problem and provided a solution in terms of a Cauchy transform operator. Subsequently, Plemelj's work was influential in the development of RHP. In this thesis we will consider a simpler version of RHP, which consists of a matrix valued factorization problem. From now and throughout the rest of the thesis, we will refer to this factorization problem as the Riemann-Hilbert problem, distinct from the original one on Fuchsian systems.

Although the original problem presented by Hilbert concerned Fuchsian linear differential equations, Riemann-Hilbert problems also have links with various fields of physics and mathematics, particularly in integrable systems. They have been applied and developed in various areas, including the analysis of quantum field and statistical physics models [CK00]; quantum and classical dynamical systems and the theory of integrable nonlinear partial differential equations [KS97] and ordinary differential equations.

The main focus of this thesis is the last area mentioned above. Indeed, studying Riemann-Hilbert problem for the Pearcey process (chapter 5) and the orthogonal polynomials on

the unit circle (Chapter 4), we intend to introduce non-linear ODE and PDE related to the generating function for the Pearcey process and a hierarchy of discrete equations to describe Toeplitz determinants in connection with the discrete gap probabilities for random partitions for a certain choice of Schur measures.

The Chapter is organized into three sections. In the first section, we will describe Riemann-Hilbert problems, boundary value problems on the complex plane. To solve such a problem, we will introduce the Cauchy transform as suggested by Plemelj's work. We will discuss the properties of this integral transformation and explain how to solve a scalar Riemann-Hilbert problem using it. Additionally, for Riemann-Hilbert problems ($n \times n$ matrix valued, $n \geq 2$), we will present a necessary and sufficient condition for the existence of a solution to a RHP.

In a second time, we will study orthogonal polynomials to illustrate the concept of RHP. We will begin by recalling the definition of orthogonal polynomials on the real line and some of their basic properties. Subsequently, we will introduce the Riemann-Hilbert problem associated with these orthogonal polynomials, as first stated by Fokas, Its and Kitaev [FIK90, FIK91]. For this RHP, it is possible to compute the exact solution in terms of orthogonal polynomials and Cauchy transforms and to prove the uniqueness of the solution. Moreover, we will study this RHP for a weight specialization and establish a connection with the discrete Painlevé I equation. This Riemann-Hilbert problem will provide us with a suitable example to present some classical techniques and methods that will be useful throughout this thesis. Similar methods will be employed in Chapter 4 for the study of orthogonal polynomials on the unit circle.

The last section is dedicated to integrable operators and, more precisely, integrable operators in the sense of IKS [IKS90]. We will show how these operators can be linked to a RHP. In chapter 5, we will study the generating function of the Pearcey process via a RHP in connection with an integrable operator.

The literature on Riemann-Hilbert problem is extensive, and we principally used the chapter 5 of [BDS16] and the articles [DIZ97] and [Its11] to write this chapter.

3.1 Presentation of Riemann-Hilbert problem and first properties

We begin this chapter with the presentation of the Riemann-Hilbert problem. We also describe an integral transformation: the Cauchy transform, which is an important tool for solving the RHP.

3.1.1 Definition of Riemann-Hilbert problems

To define a Riemann-Hilbert problem, one has to consider an oriented contour Σ on the complex plane \mathbb{C} and a jump matrix J . In what follows, the contours we consider are finite unions of simple smooth curves on \mathbb{C} with only a finite number of self-intersection. The limits as $z' \in \mathbb{C}$ tends to $z \in \Sigma$ are important in order to describe a Riemann-Hilbert problem. The limit approaching the contour from the left-hand side (respectively right-hand side) by going through the contour will be denoted by the symbol $+$ (respectively $-$).

The Figure 3.1.1 gives an example of how the contours are and how we write limits approaching the contours.

The definition of a Riemann-Hilbert problem is the following one.

Definition 3.1. *Let Σ and J be an oriented contour on \mathbb{C} and a matrix $J \in \mathcal{M}_n(\mathbb{C})$ as described in the previous paragraph.*

We search a matrix-valued function m such that:

- (1) $m : \mathbb{C} \setminus \Sigma \rightarrow \mathcal{G}l_n(\mathbb{C})$ is analytic,
- (2) $m_+(z) = m_-(z)J(z)$, $z \in \Sigma$, where m is continuous up to boundary of the contours, and $m_{\pm}(z)$ are non-tangential limits approaching z from the left($+$) or the right($-$) for almost every $z \in \Sigma$,
- (3) m has the following asymptotic

$$m(z) \sim I_n \text{ as } |z| \rightarrow \infty. \tag{3.1}$$

In summary, a Riemann-Hilbert problem for an oriented contour Σ and a jump matrix J consists of finding a factorization $m_-^{-1}m_+$ of the jump matrix J on Σ , which admits an analytic continuation for m_+ (respectively m_-) on the left-hand side (respectively right-hand side) of the contour Σ and which is normalized at infinity.

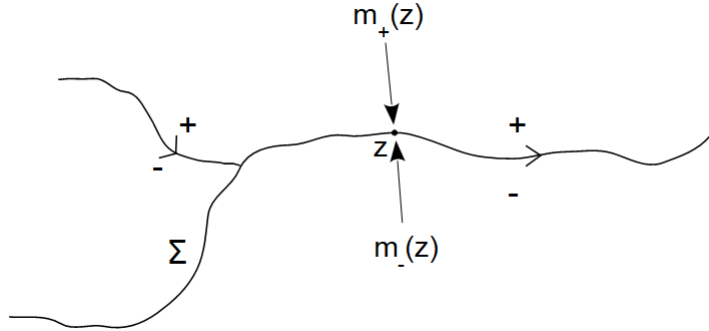


Figure 3.1 – An example of simple contour for a Riemann-Hilbert problem

3.1.2 Cauchy transform and Plemelj formula: existence of solutions to RHP

A natural question when considering a Riemann-Hilbert problem is the existence of a solution to this problem and, if one exists, whether it is unique. One of the first answers to the existence of a solution to a Riemann-Hilbert problem was given by Plemelj (1908). By studying the Cauchy transform, he provided a solution to a scalar version of the RHP. Indeed, the Cauchy transform (as we will see in equation (3.4)) satisfies a boundary condition on the contour on which the transformation is defined, and it exactly solves the factorization for the RHP (see Example 3.4). This condition is known as Sokhotski–Plemelj theorem since it was first proved by Sokhotski (1868) and rediscovered by Plemelj. This property makes the Cauchy transform an important tool for solving a Riemann-Hilbert problem.

We describe this integral transform and present some of its properties.

Definition 3.2. *Let Σ be an oriented contour on \mathbb{C} and $h \in L^p(\Sigma)$ with $1 \leq p < \infty$, for $z \in \mathbb{C} \setminus \Sigma$, the Cauchy transform of h is defined as*

$$(Ch)(z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{h(s)}{s - z} ds. \quad (3.2)$$

Proposition 3.3 (Section 5.5 in [BDS16]). *The Cauchy operator C satisfies the following properties. Let $1 < p < \infty$ and $h \in L^p(\Sigma)$.*

(1) Ch is an analytic function on $\mathbb{C} \setminus \Sigma$.

(2) Ch admits the following limit as $|z| \rightarrow \infty$, $\lim_{|z| \rightarrow \infty} Ch(z) = 0$ for $z \in \mathbb{C} \setminus \Sigma$.

Denote by C_{\pm} the (non-tangential) limiting operators defined on Σ as

$$(C_{\pm}h)(z) := \lim_{z' \rightarrow z} Ch(z'), \quad (3.3)$$

with z' in the \pm side of Σ . Then

(3) $C_{\pm} : L^p(\Sigma) \rightarrow L^p(\Sigma)$ is a continuous operator.

(4) The operator C_+ and C_- satisfy the following two equations

$$C_+ - C_- = 1 \quad (3.4)$$

and

$$C_+ + C_- = iH \quad (3.5)$$

where H denotes the operator of Hilbert transform

$$Hh(z) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{\{s \in \Sigma, |s-z| > \epsilon\}} \frac{h(s)}{z-s} ds, \quad z \in \Sigma.$$

The equation (3.4) will play an important role in expressing solutions of a Riemann-Hilbert problem.

Example 3.4 (Plemelj formula). Let \mathbb{R} with standard orientation be the contour, and J be a smooth positive function with the asymptotic $J(z) = 1 + o(z^{-a})$ as $z \rightarrow |\infty|$ with $a > 0$.

Then, for $p > \max(1, \frac{1}{a})$, according to the Proposition 3.3, $\log(J) \in L^p(\Sigma)$, and m defined by

$$m(z) := e^{(C \log J)(z)}$$

is analytic on $\mathbb{C} \setminus \mathbb{R}$ and tends to one as $|z| \rightarrow \infty$.

Moreover, because of equation (3.4), m satisfies the following jump on the contour:

$$m_+(z) = e^{(C_+ \log J)(z)} = e^{(C_- \log J)(z) + \log J(z)} = m_-(z)J(z). \quad (3.6)$$

Thus, m is a solution of the Riemann-Hilbert problem on \mathbb{R} with the jump J .

The previous example corresponds to a scalar Riemann-Hilbert problem, and because of the Plemelj formula, one can construct a solution to a scalar Riemann-Hilbert problem. When J is the jump matrix of a RHP with $J \in \mathcal{M}_n(\mathbb{C})$ and $n \geq 2$, the Cauchy transform still plays an important role in solving a Riemann-Hilbert problem. Even if the equation (3.6) no longer holds, it is still possible to solve a RHP in some cases. The following proposition gives a necessary and sufficient condition for a RHP to admit a solution.

Proposition 3.5. *Let Σ be an oriented contour and $J \in \mathcal{GL}_n(\mathbb{C})$. If μ is a solution of $\mu = I + C_+(\mu(I - J^{-1}))$, then*

$$m(z) = I + \int_{\Sigma} \frac{\mu(s)(I - J^{-1}(s))}{s - z} ds \quad (3.7)$$

is a solution of RHP(Σ, J).

Reciprocally, if m is a solution of RHP(Σ, J), then m_+ satisfies

$$m_+ = I + C_+(m_+(I - J^{-1})). \quad (3.8)$$

From the previous proposition, we see that the resolution of RHP(Σ, J) is equivalent to the invertibility of the operator $1 - C_+(\cdot(I - J^{-1}))$.

Application 3.6. *Consider the Riemann-Hilbert problem with contour \mathbb{R} oriented from $-\infty$ to $+\infty$ and jump matrix $J(z) = \begin{pmatrix} 1 & a(z) \\ 0 & 1 \end{pmatrix}$ with $a \in \mathcal{S}(\mathbb{R})$.*

Define $\mu(z) = \begin{pmatrix} 1 & C_+(a)(z) \\ 0 & 1 \end{pmatrix}$, the matrix-valued function μ is a solution of $\mu = I + C_+(\mu(I - J^{-1}))$. Then, from the proposition 3.5,

$$m(z) = I_2 + \int_{\Sigma} \frac{\mu(s)(I - J^{-1}(s))}{s - z} ds = \begin{pmatrix} 1 & C(a)(z) \\ 0 & 1 \end{pmatrix} \quad (3.9)$$

is a solution of RHP(Σ, J).

3.2 A Riemann-Hilbert problem associated to orthogonal polynomials and the discrete Painlevé I equation

In the previous chapter, we discussed the example of the Gaussian Unitary ensemble. We saw that this determinantal point process is linked to Hermite polynomials (see equation (2.17)), which are orthogonal polynomials on the real line with respect to the weight $\exp(-x^2)$. Actually, orthogonal polynomials play an important role in the study of integrable system, random matrices or Painlevé equations (see for instance [Bai03, Meh04, VA18]).

In this section we will describe what orthogonal polynomials are, then we will explain how to relate them to a Riemann-Hilbert problem and how to solve this Riemann-Hilbert problem. Finally, for a special choice of weight for orthogonal polynomials, we will compute a system of equations satisfied by the solution of the RHP. This system of equations will be called Lax pair for the discrete Painlevé I equation.

This section mainly follows [Its11].

3.2.1 Orthogonal polynomials on the real line

The aim of this subsection is to introduce orthogonal polynomials on the real line.

Let μ be a positive measure on \mathbb{R} , and assume that μ has finite moments to all orders. Consider the family of orthonormal polynomials $\{P_n(z)\}_{n \in \mathbb{N}}$ with respect to μ , i.e., the family of polynomials $P_n(z) = \gamma_n^{(n)} z^n + \gamma_{n-1}^{(n)} x^{n-1} + \dots + \gamma_0^{(n)}$ satisfying

$$\int_{\mathbb{R}} P_n(z) P_m(z) \mu(z) dz = \delta_{nm}, \forall (n, m) \in \mathbb{N}^2, \quad (3.10)$$

where δ_{nm} is the Kronecker delta. Define $\{\pi_n(z)\}_{n \in \mathbb{N}}$ as the family of monic orthogonal polynomials, $\pi_n(z) := \frac{1}{\gamma_n^{(n)}} P_n(z)$.

As we will see in Chapter 4, orthogonal polynomials are related to some determinants as Toeplitz or Hankel determinants. For orthogonal polynomials on the real line, one has the following results.

The moments of the measure μ and the coefficients of the polynomials $P_n(z)$ are linked

via a Hankel determinant.

$$P_n(z) = \frac{1}{\sqrt{H_n H_{n-1}}} \det \begin{pmatrix} \int s^0 d\mu(s) & \int s^1 d\mu(s) & \cdots & \int s^n d\mu(s) \\ \int s^1 d\mu(s) & \int s^2 d\mu(s) & \cdots & \int s^{n+1} d\mu(s) \\ \vdots & \vdots & \cdots & \vdots \\ \int s^{n-1} d\mu(s) & \int s^n d\mu(s) & \cdots & \int s^{2n-1} d\mu(s) \\ 1 & z & \cdots & z^n \end{pmatrix} = \sqrt{\frac{H_{n-1}}{H_n}} (z^n + \dots) \quad (3.11)$$

where H_n is the Hankel determinant for the moments of the measure μ

$$H_n := \det \left(\int s^{i+j} d\mu(s) \right)_{0 \leq i, j \leq n} = \det \begin{pmatrix} \int s^0 d\mu(s) & \int s^1 d\mu(s) & \cdots & \int s^n d\mu(s) \\ \int s^1 d\mu(s) & \int s^2 d\mu(s) & \cdots & \int s^{n+1} d\mu(s) \\ \vdots & \vdots & \ddots & \vdots \\ \int s^n d\mu(s) & \int s^{n+1} d\mu(s) & \cdots & \int s^{2n} d\mu(s) \end{pmatrix}. \quad (3.12)$$

3.2.2 A Riemann-Hilbert problem for orthogonal polynomials on the real line

This subsection is devoted to presenting the Riemann-Hilbert problem associated with orthogonal polynomials. This Riemann-Hilbert problem was first introduced by Fokas, Its and Kitaev [FIK90, FIK91] for the study of the matrix model approach to 2D quantum gravity. Let $w \in \mathcal{S}(\mathbb{R})$ be a non-negative function, and consider the measure μ on \mathbb{R} absolutely continuous with respect to the Lebesgue measure, defined as $d\mu(z) := w(z)dz$.

Define the jump matrix J as $J(z) := \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}$ where $z \in \mathbb{R}$.

Finally, consider the contour $\Sigma = \mathbb{R}$ oriented from $-\infty$ to $+\infty$. We introduce the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 3.7 (RHP for OPRL (Orthogonal Polynomials on the Real Line)). *Consider $n \in \mathbb{N}$ and search a function $m_n : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{M}_2(\mathbb{C})$ satisfying:*

- (1) m_n is analytic for every $z \in \mathbb{C} \setminus \mathbb{R}$;
- (2) m_n has continuous boundary values $m_{n,\pm}$ as z approaches non-tangentially the contour either from the left or from the right, and they are related for all $z \in \mathbb{R}$ through

$$m_{n,+}(z) = m_{n,-}(z)J(z),$$

(3) m_n is normalized at ∞ as

$$m_n(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \rightarrow I_2, \text{ as } |z| \rightarrow \infty$$

Proposition 3.8. *If $RHP(\mathbb{R}, J)$ has a solution, then it is unique.*

Remark 3.9. *The proof of this proposition can be adapted to other Riemann-Hilbert problems to prove the uniqueness of the solution to a RHP. The important arguments in the next proof are that the determinant of the jump matrix J and the asymptotic for m_n as $|z| \rightarrow \infty$ are both equal to one.*

Proof. Let m_n be a solution of $RHP(\mathbb{R}, J)$. From the first item in RHP 3.7, m_n is analytic on $\mathbb{C} \setminus \mathbb{R}$, then $\det m_n(z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. Moreover, according to the definition of J , $\forall z \in \mathbb{R}$, $\det J(z) = 1$. Hence $\det m_{n,+} = \det m_{n,-} \det J = \det m_{n,-}$. By applying Morera's theorem, we conclude that $\det m_n$ is entire.

Finally,

$$m_n(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \rightarrow I_2,$$

therefore $\det m_n(z) \rightarrow 1$ and $\det m_n$ is bounded on \mathbb{C} .

Liouville's theorem then assures us that $\det m_n$ is constant, and given the limiting behavior when $|z| \rightarrow \infty$, $\det m_n \equiv 1$.

It is now assumed that there is a second solution \tilde{m}_n to $RHP(\mathbb{R}, J)$. Because of the previous discussion on $\det(m_n)$, $m_n(z)$ is invertible for all $z \in \mathbb{C}$.

We then define $p(z) = \tilde{m}_n(z)m_n(z)^{-1}$.

The function p is analytic on $\mathbb{C} \setminus \Sigma$.

Moreover, $p_+(z) = \tilde{m}_{n,+}(z)m_{n,+}(z)^{-1} = (\tilde{m}_{n,-}(z)J(z))(J(z)^{-1}m_{n,-}(z)^{-1}) = p_-(z)$. The same arguments as before with Morera's theorem and Liouville's theorem lead to $p \equiv I_2$.

Then, if $RHP(\mathbb{R}, J)$ has a solution, it is unique. \square

We have just proved that if the Riemann-Hilbert problem 3.7 has a solution, then this solution is unique. It remains to verify that this RHP has a solution. Actually, the Riemann-Hilbert problem 3.7 has a solution, and it is expressed in terms of orthogonal polynomials π_n .

Theorem 3.10 (Theorem 5.7 [BDS16]). *The solution m_n to $RHP(\mathbb{R}, J)$ exists, is unique*

and given by:

$$m_n(z) = \begin{pmatrix} \pi_n(z) & (C(\pi_n w))(z) \\ -2i\pi \left(\gamma_{n-1}^{(n-1)}\right)^2 \pi_{n-1}(z) & -2i\pi \left(\gamma_{n-1}^{(n-1)}\right)^2 (C(\pi_{n-1} w))(z) \end{pmatrix}$$

Proof. One can find the proof of this theorem in [BDS16] before Theorem 5.7. Here we give an intuition of how to prove this theorem.

As stated in the previous subsection, the Cauchy operator is a central tool and plays an important role in solving Riemann-Hilbert problems. In this example, the jump matrix J_m is an upper triangular matrix. Then, from the jump we see directly that elements in the first column have to be continuous on the whole complex plane (no jump on \mathbb{R}), and that elements in the second column have to satisfy

$$m_{i2,+}(z) - m_{i2,-}(z) = w(z)m_{i1,-}(z) = w(z)m_{i1}(z), \quad (3.13)$$

with m_{ij} element in i -th row and j -th column of the matrix m .

From the previous equation, we deduce that if m_{i1} is known, then, because of the equation (3.4), m_{i2} has to be the Cauchy transform of $w m_{i1}$.

Finally expanding $\frac{1}{z-s}$ as $|z| \rightarrow \infty$ in the expression of the Cauchy transform of $w m_{i1}$, and because of the asymptotic behavior for the Riemann-Hilbert problem, one can conclude that m_{11} (respectively m_{21}) is a polynomial of degree n (respectively of degree $n-1$) orthogonal to all monomials of degree less than n (respectively less than $n-1$). \square

3.2.3 A Lax pair for the discrete Painlevé I equation

In this subsection, we study the previous Riemann-Hilbert problem specifying the weight $w(z)$ for the measure. Let N be a positive integer. The study of the previous Riemann-Hilbert problem with the weight $\exp\left(-N\left(\frac{t}{4}z^4 + \frac{a}{2}z^2\right)\right)$ leads to the discrete Painlevé I equation.

To obtain the discrete Painlevé I equation, one has to modify the Riemann-Hilbert problem into one with a constant jump. We define $\Psi(n, z) := m_n(z) \exp\left(-\frac{N}{2}\left(\frac{t}{4}z^4 + \frac{a}{2}z^2\right)\sigma_3\right)$

with $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ being a Pauli matrix, and m_n being the solution of $RHP(\mathbb{R}, J_m)$.

Ψ has a jump on \mathbb{R} because of the one of m_n , and one can easily check that this jump is

constant on \mathbb{R} and given by

$$\Psi(n, z)_+ = \Psi(n, z)_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (3.14)$$

Then $\Psi(n, z)$ satisfies the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 3.11 (RHP for Ψ). *Consider $n \in \mathbb{N}$ and search a function $\Psi(n, \cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{M}_2(\mathbb{C})$ satisfying:*

- (1) $\Psi(n, \cdot)$ is analytic for every $z \in \mathbb{C} \setminus \mathbb{R}$;
- (2) $\Psi(n, \cdot)$ has continuous boundary values $\Psi(n, \cdot)_\pm$ while approaching non-tangentially the contour either from the left or from the right, and they are related for all $z \in \mathbb{R}$ through

$$\Psi(n, z)_+ = \Psi(n, z)_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

- (3) $\Psi(n, \cdot)$ is normalized at ∞ as

$$\Psi(n, z) = \left(I_2 + \frac{m_1(n)}{z} + \mathcal{O}(1/z^2) \right) \exp \left(-\frac{N}{2} \left(\frac{t}{4} z^4 + \frac{a}{2} z^2 \right) \sigma_3 \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \text{ as } |z| \rightarrow \infty$$

The Riemann-Hilbert problem for Ψ has a constant jump on the contour, and this form for the jump will provide us a way to compute differential and difference equations for Ψ .

Proposition 3.12. *Let Ψ be the solution to Riemann-Hilbert problem 3.11.*

Then Ψ satisfies the system of equation

$$\Psi(n+1, z) = U(n, z)\Psi(n, z) \quad (3.15)$$

$$\frac{\partial}{\partial z} \Psi(n, z) = A(n, z)\Psi(n, z) \quad (3.16)$$

where $U(n, z) = \begin{pmatrix} z & 2i\pi (\gamma_{n-1}^{(n-1)})^2 \\ -\frac{1}{2i\pi (\gamma_{n+1}^{(n+1)})^2} & 0 \end{pmatrix}$ and $A(n, z) = z^3 A_3 + z^2 A_2 + z A_1 + A_0$

with $A_3 = \frac{-tN}{2} \sigma_3$ and $A_2 = -tN \begin{pmatrix} 0 & 2i\pi (\gamma_{n-1}^{(n-1)})^2 \\ -\frac{1}{2i\pi (\gamma_n^{(n)})^2} & 0 \end{pmatrix}$

Proof. Let Ψ be the solution to Riemann-Hilbert problem 3.11. In the proof of Proposition 3.8, we showed that $\det m_n \equiv 1$, then, by definition of Ψ , $\det \Psi \equiv 1$, and for all $z \in \mathbb{C}$, $\Psi(n, z)$ is invertible.

Define $U(n, z) := \Psi(n+1, z)\Psi(n, z)^{-1}$. From the first assumption on Ψ in RHP 3.11, U is analytic on $\mathbb{C} \setminus \mathbb{R}$. Because of the constant jump, $\Psi(n, z)$ and $\Psi(n+1, z)$ have the same jump, which implies that $U(n, z)$ has no jump on \mathbb{R} and is continuous on \mathbb{C} . Again, Morera's theorem assures us that $U(n, z)$ is entire.

We compute the asymptotic of U as $|z| \rightarrow \infty$ from the one of Ψ . We obtain

$$U(n, z) = \left(I_2 + \frac{m_1(n+1)}{z} + \mathcal{O}(1/z^2) \right) \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \left(I_2 - \frac{m_1(n)}{z} + \mathcal{O}(1/z^2) \right). \quad (3.17)$$

Then from Liouville's theorem, $U(n, z)$ is a polynomial of degree 1.

$$U(n, z) = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + m_1(n+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} m_1(n) = \begin{pmatrix} z & 2i\pi (\gamma_{n-1}^{(n-1)})^2 \\ -\frac{1}{2i\pi (\gamma_{n+1}^{(n+1)})^2} & 0 \end{pmatrix}. \quad (3.18)$$

Similarly, one can show that $A(n, z) := \frac{\partial}{\partial z} \Psi(n, z)\Psi(n, z)^{-1}$ is well defined, is entire and is a polynomial of degree 3 given by

$$A(n, z) = z^3 A_3 + z^2 A_2 + z A_1 + A_0, \quad (3.19)$$

where $A_3 = \frac{-tN}{2} \sigma_3$ and $A_2 = -tN \begin{pmatrix} 0 & 2i\pi (\gamma_{n-1}^{(n-1)})^2 \\ -\frac{1}{2i\pi (\gamma_n^{(n)})^2} & 0 \end{pmatrix}$. □

Proposition 3.13. *The system of equations (3.15) and (3.16) is a Lax pair for the discrete Painlevé I equation.*

More precisely, define $R_n := \begin{pmatrix} \gamma_{n-1}^{(n-1)} \\ \gamma_n^{(n)} \end{pmatrix}^2$. From the compatibility condition between matrices A and U , one proves that R_n satisfies the discrete Painlevé I equation

$$\frac{n}{N} = aR_n + tR_n (R_{n-1} + R_n + R_{n+1}). \quad (3.20)$$

In the previous proof, coefficients of degree 1 and 0 of $A(n, z)$ are not known. One can compute these two matrices recursively from the compatibility condition of the system of

equations (3.15) and (3.16). Indeed, on one hand,

$$\frac{\partial}{\partial z}\Psi(n+1, z) = \frac{\partial}{\partial z}(U(n, z)\Psi(n, z)) = \left(\frac{\partial}{\partial z}U(n, z) + U(n, z)A(n, z)\right)\Psi(n, z). \quad (3.21)$$

On the other hand,

$$\frac{\partial}{\partial z}\Psi(n+1, z) = A(n+1, z)\Psi(n+1, z) = A(n+1, z)U(n, z)\Psi(n, z). \quad (3.22)$$

But $\Psi(n, z)$ is invertible for all $z \in \mathbb{C}$. Then, the compatibility for the system of equations (3.15), (3.16) reads as

$$\frac{\partial}{\partial z}U(n, z) = A(n+1, z)U(n, z) - U(n, z)A(n, z). \quad (3.23)$$

The equation (3.23) is called the zero curvature equation for the system of equations (3.15), (3.16).

From this equation, one obtains recursive equations for A_i and one can compute A_1 and A_0 from these recursive equations. The equation (3.23) leads to an over-determined system of equations, and the remaining equation is the following one.

$$\frac{-1}{N} + a(R_{n+1} - R_n) + t(R_{n+1}R_{n+2} + R_{n+1}^2 - R_{n-1}R_n - R_n^2) = 0, \quad (3.24)$$

where $R_n := \left(\frac{\gamma_{n-1}^{(n-1)}}{\gamma_n^{(n)}}\right)^2$.

Integrating the previous equation leads to the discrete Painlevé I equation,

$$\frac{n}{N} = aR_n + tR_n(R_{n-1} + R_n + R_{n+1}). \quad (3.25)$$

According to the equation (3.11) and to the definition of R_n , one can express R_n in function of the matrix moment Hankel determinants H_n ,

$$R_n = \frac{H_{n-2}H_n}{H_{n-1}^2}. \quad (3.26)$$

Hence the computations of these Hankel determinants reduce to a non-linear recurrence relation of order five.

3.3 Integrable operators and Riemann-Hilbert problems

The goal of this section is to describe what integrable operators in the sense of IKS (for Its, Izergin, Korepin and Slavnov) [IKS90] are. This notion will be important for the study of the generating function for the Pearcey process because we will study the Riemann-Hilbert problem associated with an integrable operator in chapter 5.

3.3.1 Description of integrable operators and associated resolvent operators

Let Σ be an oriented contour in \mathbb{C} . We introduce the definition of an integrable operator, which is an integral operator with a kernel of a certain form.

Definition 3.14. *An operator $\mathbb{K} : L^2(\Sigma) \rightarrow L^2(\Sigma)$ is said to be integrable if its kernel is of the form*

$$K(x, y) = \frac{f^T(x)g(y)}{x - y} \text{ where } f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, g = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}, \text{ with } f, g \in (L^\infty(\Sigma))^N. \quad (3.27)$$

We already introduced two kernels which are integrable in the sense of IKS. In the previous chapter, the kernel $K_N(x, y)$ in the equation (2.26) is integrable. Indeed, defining

$$f(x) := \left(\frac{N}{2}\right)^{\frac{1}{2}} \begin{pmatrix} \Phi_N(x) \\ \Phi_{N-1}(x) \end{pmatrix} \text{ and } g(y) := \begin{pmatrix} \Phi_{N-1}(y) \\ -\Phi_N(y) \end{pmatrix}, \quad (3.28)$$

the kernel K_N can be written as

$$K_N(x, y) = \frac{f^T(x)g(y)}{x - y}, \quad (3.29)$$

where f and g are both in $L^\infty(\mathbb{R})$ because of the definition of $\Phi_j(x)$ in the equation (2.26).

The second integrable kernel we discussed previously is the one of the Airy kernel defined in equation (2.28). Introducing:

$$f(x) := \begin{pmatrix} \text{Ai}(x) \\ \text{Ai}'(x) \end{pmatrix} \text{ and } g(y) := \begin{pmatrix} \text{Ai}'(y) \\ -\text{Ai}(y) \end{pmatrix}, \quad (3.30)$$

we see that \mathbb{K}_{Airy} is integrable in the sense of IKS. The form of this kernel is similar to the one of $K_N(x, y)$.

In these two examples of integrable operators, the vectors f and g describing the kernel are of size two. It is not always the case, as suggested by the Definition 3.14. Indeed, in chapter 5, we will study an integrable operator with a kernel defined by vectors of size $N + 1$, where N is a positive integer.

A quantity of interest in the study of integrable operators is the resolvent operator defined as $\mathbb{L} := (1 - \mathbb{K})^{-1} - 1 = \mathbb{K}(1 - \mathbb{K})^{-1}$ if $1 - \mathbb{K}$ is invertible.

Lemma 3.15 (Lemma 2.8 [DIZ97]). *If \mathbb{K} is an integrable operator and $1 - \mathbb{K}$ is invertible then the resolvent operator $\mathbb{L} := \mathbb{K}(1 - \mathbb{K})^{-1}$ is integrable and its kernel is given by*

$$L(x, y) = \frac{F^T(x)G(y)}{x - y} \text{ où } F = \begin{pmatrix} (1 - \mathbb{K})^{-1}f_1 \\ \vdots \\ (1 - \mathbb{K})^{-1}f_N \end{pmatrix}, G = \begin{pmatrix} (1 - \mathbb{K}^T)^{-1}g_1 \\ \vdots \\ (1 - \mathbb{K}^T)^{-1}g_N \end{pmatrix} \quad (3.31)$$

3.3.2 Riemann-Hilbert problems associated to integrable operators

We now establish the relation between integrable operators and Riemann-Hilbert problems. We first introduce the Riemann-Hilbert problem associated with integrable operators. Then we establish the link between solutions to the RHP for integrable operators and the resolvent of these operators.

Riemann-Hilbert Problem 3.16 (RHP for the study of integrable operators). *Let \mathbb{K} be an integrable operator (see Definition 3.14) with kernel*

$$K(x, y) = \frac{f^T(x)g(y)}{x - y}.$$

Search a function $m : \mathbb{C} \setminus \Sigma \rightarrow \mathcal{M}_N(\mathbb{C})$ satisfying:

- (1) m is analytic for every $z \in \mathbb{C} \setminus \Sigma$;
- (2) m has continuous boundary values m_{\pm} while approaching non-tangentially the contour either from the left or from the right, and they are related for all $z \in \Sigma$ through

$$m(z)_+ = m(z)_- J_m(z),$$

with $J_m(z) = I_N - 2i\pi \frac{f(z)g^T(z)}{1 + i\pi f^T(z)g(z)}$;

(3) m is normalized at ∞ as

$$m(z) \sim I_N, \text{ as } |z| \rightarrow \infty.$$

Remark 3.17. Sometimes an additional hypothesis on f and g is required. One asks f and g to satisfy

$$f^T(z)g(z) = \sum_{k=1}^N f_k(z)g_k(z) = 0. \quad (3.32)$$

With this additional hypothesis, the jump of RHP 3.16 becomes $J_m(z) = I_N - 2i\pi f(z)g^T(z)$.

The Airy kernel in the IKS form satisfies this hypothesis.

The following theorem implies a relation between RHP for integrable operators and their resolvent operator.

Theorem 3.18 (Theorem 5.21 [BDS16], Lemma 2.12 [DIZ97]). *Let \mathbb{K} be an integrable operator with kernel*

$$K(x, y) = \frac{f^T(x)g(y)}{x - y}.$$

If the operator $1 - \mathbb{K}$ is invertible, then the RHP(Σ, J_m) has a solution m unique, and the resolvent operator \mathbb{L} of \mathbb{K} and the solution m to RHP(Σ, J_m) are linked through the equation:

$$\begin{cases} F = \frac{m_+ f}{1 - i\pi f^T g} \\ G = \frac{\left((m_+)^T\right)^{-1} g}{1 + i\pi f^T g} \end{cases}$$

Remark 3.19. *In Lemma 2.12 in [DIZ97], the result is stated with the assumption that f and g satisfy $f^T(z)g(z) = 0$.*

In the first section on Riemann-Hilbert problems, we finished the discussion by stating that solving a RHP was equivalent to the invertibility of the operator $1 - C_+ (\cdot (I - J^{-1}))$. According to the previous theorem, the invertibility of the operator $1 - \mathbb{K}$ assures the existence and uniqueness of the solution of the Riemann-Hilbert problem for integrable operators.

THE DISCRETE PAINLEVÉ II HIERARCHY

This chapter summarizes the article “Recursion relation for Toeplitz determinants and the discrete Painlevé II hierarchy” [CT23b] that I co-authored with Sofia Tarricone and which is published in *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*.

4.1 Presentation of the discrete Painlevé II equation and statement of the results

Let us consider the symbol $\varphi(z) = e^{w(z)}$, with

$$w(z) := v(z) + v(z^{-1}) \quad \text{and} \quad v(z) := \sum_{j=1}^N \frac{\theta_j}{j} z^j, \quad (4.1)$$

for θ_j being real constants and $N \geq 1$ natural number. The n -th Toeplitz matrix associated to this symbol and denoted by $T_n(\varphi)$ is a square $(n+1)$ -dimensional matrix which entries are given by

$$T_n(\varphi)_{i,j} := \varphi_{i-j}, \quad i, j = 0, \dots, n. \quad (4.2)$$

Here for every $k \in \mathbb{Z}$, φ_k is the k -th Fourier coefficient of $\varphi(z)$, namely

$$\varphi_k = \int_{-\pi}^{\pi} e^{-ik\beta} \varphi(e^{i\beta}) \frac{d\beta}{2\pi},$$

so that $\sum_{k \in \mathbb{Z}} \varphi_k z^k = \varphi(z)$. Notice that, even though it is not emphasized in our notation, the functions φ_k and thus the Toeplitz matrix $T_n(\varphi)$ explicitly depend on the natural parameter N which enters in the definition of $v(z)$ in equation (4.1).

In the present work, it is indeed the dependence on this parameter N that we want to study. In particular, we show that the Toeplitz determinants associated to $T_n(\varphi)$, naturally

defined as

$$D_n^N := D_n = \det(T_n(\varphi)), \tag{4.3}$$

are related to some solutions of a discrete version of the Painlevé II hierarchy, indexed over the parameter N (the dependence on N is dropped in the rest of the paper). Our interest in these Toeplitz determinants comes from their appearance in the recent paper [BBW21]. The authors there consider some probability measures on the set of integer partitions called *multicritical* Schur measures, which are a particular case of Schur measures introduced by Okounkov in [Oko01]. They are generalizations of the classical Poissonized Plancherel measure and they are defined as

$$\mathbb{P}(\{\lambda\}) = Z^{-1} s_\lambda[\theta_1, \dots, \theta_N]^2, \quad \text{with } Z = \exp\left(\sum_{i=1}^N \frac{\theta_i^2}{i}\right). \tag{4.4}$$

Here $s_\lambda[\theta_1, \dots, \theta_N]$ denotes a Schur symmetric function indexed by a partition λ that can be expressed as $s_\lambda[\theta_1, \dots, \theta_N] = \det_{i,j} h_{\lambda_i - i + j}[\theta_1, \dots, \theta_N]$ where $\sum_{k \geq 0} h_k z^k = \exp\left(\sum_{i=1}^N \frac{\theta_i}{i} z^i\right)$. In [BBW21], the authors first used the term *multicritical* to underline that they obtained a different limiting edge behavior for these Schur measures compared to the classical case of the Poissonized Plancherel measure ($N = 1$) which is characterised by the Tracy-Widom GUE distribution. For more details, we remind to their Theorem 1 or our discussion in the paragraph *Continuous limit* below, for instance see equation (4.23) where the higher order Tracy-Widom distributions appear.

In this setting, denoting by $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ a generic integer partition and by $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \dots \geq 0)$ its conjugate partition (namely such that $\lambda'_j = |\{i : \lambda_i \geq j\}|$), major quantities of interest of the model are, for any given $n \in \mathbb{N}$,

$$r_n := \mathbb{P}(\lambda_1 \leq n) \quad \text{and} \quad q_n := \mathbb{P}(\lambda'_1 \leq n), \tag{4.5}$$

that are often called discrete gap probabilities as random partitions have a natural interpretation in terms of random configuration of points on the set of semi-integers. Indeed, associating the set $\{\lambda_i - i + 1/2\} \subset \mathbb{Z} + \frac{1}{2}$ to a partition λ (see [Oko01]), r_n and q_n can be expressed in terms of a Fredholm determinant of a discrete kernel which corresponds to the gap probability in the determinantal point process defined through the same kernel. According to Geronimo-Case/Borodin-Okounkov formula [BO00], there is a relation between this Fredholm determinant and the Toeplitz determinant D_n and this implies that r_n and q_n (up to a constant factor) are Toeplitz determinants. It leads to (for instance

[BBW21], Proposition 6 and 7):

$$q_n = e^{-\sum_{j=1}^N \theta_j^2/j} D_{n-1}. \quad (4.6)$$

For r_n instead, one should define $\tilde{\theta}_i = (-1)^{i-1} \theta_i$ and by taking $\tilde{w}(z) = \tilde{v}(z) + \tilde{v}(z^{-1})$ where $\tilde{v}(z)$ is nothing than $v(z)$ with θ_i replaced by $\tilde{\theta}_i$ as given above, the Toeplitz determinant \tilde{D}_n associated to the symbol $\tilde{\varphi}(z) = e^{\tilde{w}(z)}$ would give the analogue formula

$$r_n = e^{-\sum_{j=1}^N \tilde{\theta}_j^2/j} \tilde{D}_{n-1}.$$

Notice that in the simplest case, when $N = 1$, the quantities r_n and q_n coincide. Moreover, thanks to Schensted's theorem [Sch61], they are also equal to the discrete probability distribution function of the length of the longest increasing subsequence of random permutations of size m , with m distributed as a Poisson random variable.

In the case $N = 1$, the relation of these quantities with the theory of discrete Painlevé equations was shown two decades ago independently and through very different methods by Borodin [Bor03], Baik [Bai03], Adler and Van Moerbeke [AvM03] and Forrester and Witte [FW06]¹. In particular they all proved that for every $n \geq 1$, the following chain of equalities holds

$$\frac{D_n D_{n-2}}{D_{n-1}^2} = \frac{q_{n+1} q_{n-1}}{q_n^2} = \frac{r_{n+1} r_{n-1}}{r_n^2} = 1 - x_n^2, \quad (4.7)$$

where x_n solves the second order nonlinear difference equation

$$\theta_1(x_{n+1} + x_{n-1})(1 - x_n^2) + nx_n = 0, \quad (4.8)$$

with certain initial conditions. Equation (4.8) is a particular case of the so called discrete Painlevé II equation [RGH91], a *discrete analogue* of the classical second order ODE known as the Painlevé II equation [Pai00]. This means that performing some continuous limit of equation (4.8) one gets back the Painlevé II equation. The Painlevé II equations, discrete and continuous ones, depend in general on an additional constant term $\alpha \in \mathbb{R}$. In the present work we consider the discrete Painlevé II equation and its hierarchy in the homogeneous case where $\alpha = 0$. Its continuous limit will correspond as well to the case $\alpha = 0$.

1. They obtained an analogue of equation (4.7) for Toeplitz determinant associated to symbols which are not necessarily positive or even real valued.

Remark 4.1. *The homogeneous Painlevé II equation admits a famous solution [HM80], called the Hastings-McLeod solution, found by requiring a specific boundary condition at ∞ . In parallel, one might wonder what is the large n behavior of the solution x_n of the discrete Painlevé II equation (4.8). Its behavior is expressed in terms of the Bessel functions. First, this is suggested by the following heuristic arguments. Because of the definition of r_n (4.5), as $n \rightarrow \infty$, r_n tends to one and according to the equation (4.7), x_n tends to zero. Then for large n , the nonlinear term in equation (4.8) is small compared to the linear ones and the equation (4.8) reduces to the equation*

$$\theta_1(x_{n+1} + x_{n-1}) + nx_n = 0$$

which indeed admits $J_{-n}(2\theta_1)$, the Bessel function of the first kind of order $-n$, as a solution.

The claim is confirmed by a result of the recent work [CR23]. The authors there studied the finite temperature deformation for the discrete Bessel point process. The Fredholm determinant of the finite temperature discrete Bessel kernel they studied depends on a function σ . In the case when $\sigma = 1_{\mathbb{Z}_+}$ (the characteristic function of the set of positive half integers), the Fredholm determinant is then equal to r_n . Then from equations (1.33) and (1.36) of Theorem III in [CR23] together with equation (4.7), one can deduce that for n large $x_n^2 \sim J_n(2\theta_1)^2$ and, because of the previous discussion, one can conclude

$$x_n \sim J_{-n}(2\theta_1) = (-1)^n J_n(2\theta_1),$$

see also Figure 4.1.

For $N > 1$, Adler and van Moerbeke presented in [AvM03], a generalization of equation (4.7) by proving that x_n satisfies some recurrence relation written in terms of the Toeplitz lattice Lax matrices. The main result of our work is a recurrence relation for x_n defined via a N -times iterating discrete operator which establishes the link with the discrete Painlevé II hierarchy [CJ99a]. The precise result is stated as below.

Theorem 4.2. *For any fixed $N \geq 1$, for the Toeplitz determinants D_n (4.3), $n \geq 1$ associated to the symbol $\varphi(z)$ (4.1), we have*

$$\frac{D_n D_{n-2}}{D_{n-1}^2} = 1 - x_n^2 \tag{4.9}$$

where x_n solves the $2N$ order nonlinear difference equation

$$nx_n + \left(-v_n - v_n \text{Perm}_n + 2x_n \Delta^{-1} (x_n - (\Delta + I)x_n \text{Perm}_n)\right) L^N(0) = 0 \quad (4.10)$$

where L is a discrete recursion operator defined as

$$L(u_n) := \left(x_{n+1} \left(2\Delta^{-1} + I\right) \left((\Delta + I)x_n \text{Perm}_n - x_n\right) + v_{n+1} (\Delta + I) - x_n x_{n+1}\right) u_n. \quad (4.11)$$

Here $v_n := 1 - x_n^2$, Δ denotes the difference operator

$$\Delta : u_n \rightarrow u_{n+1} - u_n$$

and Perm_n is the transformation of the space $\mathbb{C} \left[(x_j)_{j \in [[0, 2n]]} \right]$ acting by permuting indices in the following way

$$\begin{aligned} \text{Perm}_n : \quad \mathbb{C} \left[(x_j)_{j \in [[0, 2n]]} \right] &\longrightarrow \mathbb{C} \left[(x_j)_{j \in [[0, 2n]]} \right] \\ P \left((x_{n+j})_{-n \leq j \leq n} \right) &\longmapsto P \left((x_{n-j})_{-n \leq j \leq n} \right). \end{aligned} \quad (4.12)$$

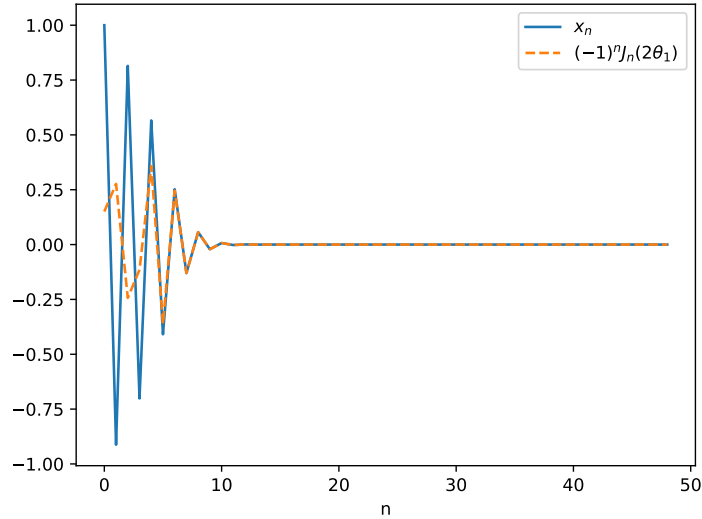


Figure 4.1 – For $N = 1$, the graphs of x_n and $(-1)^n J_n(2\theta_1)$ in function of n for $\theta_1 = 3$.

Remark 4.3. According to equation (4.10) and the definition of the operator L (4.11)

we need to perform discrete integrations to compute the N -th equation of the discrete Painlevé II hierarchy. It is always possible to accomplish this discrete integration. The operator Δ^{-1} , inverse of the difference operator Δ , is applied to $(\Delta + I)x_n \text{Perm}_n - x_n$ and it is possible to write this operator as a derivative. Indeed

$$(\Delta + I)x_n \text{Perm}_n - x_n = \Delta x_n \text{Perm}_n + (\text{Perm}_n - I)x_n.$$

The first term on the right hand side is a derivative and because of the definition of Perm_n , the second term can be expressed as a derivative.

Equation (4.10), together with the definition of the recursion operator L in (4.11), of the quantity v_n and of the transformation Perm_n in (4.12) is indeed the N -th member of the discrete Painlevé II hierarchy. The first equations of the hierarchy read as

$$N = 1 : \quad nx_n + \theta_1(x_{n+1} + x_{n-1})(1 - x_n^2) = 0, \quad (4.13)$$

$$N = 2 : \quad nx_n + \theta_1(1 - x_n^2)(x_{n+1} + x_{n-1}) \\ + \theta_2(1 - x_n^2)(x_{n+2}(1 - x_{n+1}^2) + x_{n-2}(1 - x_{n-1}^2) - x_n(x_{n+1} + x_{n-1})^2) = 0 \quad (4.14)$$

$$N = 3 : \quad nx_n + \theta_1(1 - x_n^2)(x_{n+1} + x_{n-1}) \quad (4.15) \\ + \theta_2(1 - x_n^2)(x_{n+2}(1 - x_{n+1}^2) + x_{n-2}(1 - x_{n-1}^2) - x_n(x_{n+1} + x_{n-1})^2) \\ + \theta_3(1 - x_n^2)(x_n^2(x_{n+1} + x_{n-1})^3 + x_{n+3}(1 - x_{n+2}^2)(1 - x_{n+1}^2) + x_{n-3}(1 - x_{n-2}^2)(1 - x_{n-1}^2)) \\ + \theta_3(1 - x_n^2)(-2x_n(x_{n+1} + x_{n-1})(x_{n+2}(1 - x_{n+1}^2) + x_{n-2}(1 - x_{n-1}^2)) - x_{n-1}x_{n-2}^2(1 - x_{n-1}^2)) \\ + \theta_3(1 - x_n^2)(-x_{n+1}x_{n+2}^2(1 - x_{n+1}^2) - x_{n+1}x_{n-1}(x_{n+1} + x_{n-1})) = 0,$$

with the first one coinciding with the discrete Painlevé II equation (4.8). Computations with the operator (4.11) introduced in Theorem 4.2 for $N = 1$ and 2 are done in Example 4.35.

Remark 4.4. *The same heuristic argument used in Remark 4.1 applies also when $N > 1$ (since x_n still tends to zero as $n \rightarrow \infty$), thus suggesting that the N -th equation of the discrete Painlevé II hierarchy reduces to a linear discrete equation for large n . For $N = 2$ and 3, the reduced equations are*

$$N = 2 : \quad nx_n + \theta_1(x_{n+1} + x_{n-1}) + \theta_2(x_{n+2} + x_{n-2}) = 0,$$

$$N = 3 : \quad nx_n + \theta_1(x_{n+1} + x_{n-1}) + \theta_2(x_{n+2} + x_{n-2}) + \theta_3(x_{n+3} + x_{n-3}) = 0.$$

Similar recurrence relations appeared in [DCL⁺92] for the multivariable generalized Bessel functions (GBFs). These generalized Bessel functions were discussed in [Oko01, KZ21] in the context of Schur measures for random partitions and generalizations of the previous recurrence equations were introduced (in particular, see equation 3.2b in [KZ21]). We denote by $J_n^{(N)}(\xi_1, \dots, \xi_N)$ a N -variable GBFs of order n . In [DCL⁺92], $J_n^{(N)}(\xi_1, \dots, \xi_N)$ is defined as a discrete convolution product of N Bessel functions. In particular, if $j_n^{(k)}(\xi)$ is the n^{th} Fourier coefficient of the function $\beta \rightarrow e^{2i\xi \sin(k\beta)}$ then

$$J_n^{(N)}(\xi_1, \dots, \xi_N) := j_n^{(N)}(\xi_N) * j_n^{(N-1)}(\xi_{N-1}) * \dots * j_n^{(1)}(\xi_1)(n),$$

where $*$ denotes the discrete convolution.

In the case $N = 1$, the symbol we considered was $\varphi(e^{i\beta}) = e^{\theta_1(e^{i\beta} + e^{-i\beta})} = e^{2\theta_1 \cos(\beta)}$ and the large n asymptotic behavior of x_n was given by $J_{-n}(2\theta_1)$ which is the n^{th} Fourier coefficient of the function $\beta \rightarrow e^{\theta_1(e^{i\beta} - e^{-i\beta})}$ up to a constant $(-1)^n$.

For $N > 1$, the symbol is $\varphi_N(e^{i\beta}) = \prod_{k=1}^N e^{\frac{\theta_k}{k}(e^{ik\beta} + e^{-ik\beta})} = \prod_{k=1}^N e^{2\frac{\theta_k}{k} \cos(k\beta)}$. Then, we conjecture that the large n asymptotic behavior of $x_n^{(N)}$ would be given by the n^{th} Fourier coefficient of $\beta \rightarrow \prod_{k=1}^N e^{\frac{(-1)^{k+1}\theta_k}{k}(e^{ik\beta} - e^{-ik\beta})}$ which is precisely $J_n^{(N)}(\xi_1, \dots, \xi_N)$ up to some constant and proper rescaling :

$$x_n^{(N)} \sim (-1)^n J_n^{(N)} \left(\left((-1)^i \frac{2}{i} \theta_i \right)_{1 \leq i \leq N} \right),$$

see also Figure 4.2.

Remark 4.5. Notice that for $N = 1, 2$ the equations (4.13) and (4.14) coincide with the ones found in [AvM03]. Also notice that in the physics literature, Periwal and Schewitz [PS90] found similar discrete equations for $N = 1, 2$ (with different coefficients sign) in the context of unitary matrix models and used their solutions to evaluate the behavior of some typical integrals in the large dimensional limit passing through the continuous limit of their discrete equations. For $N = 1$, the discrete Painlevé II equation was also found in [His96] as a particular case of the string equation for the full unitary matrix model, i.e. for $w(z) = \theta_1 z + \theta_{-1} z^{-1}$. The dependence in $\theta_{\pm 1}$ of x_n (and some other x_n^*) was also studied there and it produced some evolution equations related, after some change of variables, to the two-dimensional Toda equations. This would suggest that for the general case $N > 1$, the dependence of x_n on times $\theta_1, \dots, \theta_N$ would be related to the one-dimensional Toda hierarchy (see also [Oko01]).

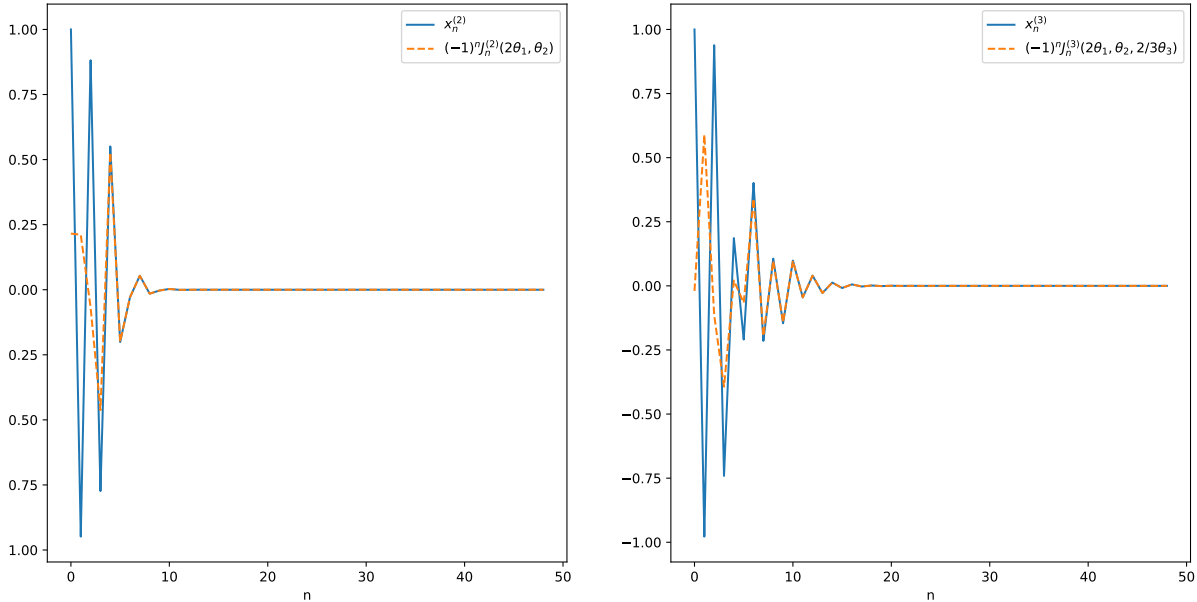


Figure 4.2 – The graphs of $x_n^{(N)}$ and $(-1)^n J_n^{(N)} \left((\theta_i)_{1 \leq i \leq N} \right)$ (for $N = 2$ on left and $N = 3$ on the right) in function of n for $\theta_1 = 3, \theta_2 = 1.2$ and $\theta_3 = 2.6$.

The first construction of a discrete Painlevé II hierarchy in [CJ99a] used the integrability property of the continuous one, in the following sense. It is well known that the classical Painlevé II equation admits an entire hierarchy of higher order analogues. Indeed, this equation can be obtained as a self-similarity reduction of the modified KdV equation. Thus, the higher order members of the Painlevé II hierarchy are but analogue self-similarity reductions of the corresponding higher order members of the modified KdV hierarchy (see e.g. [FN80]). In some way, this implies that the Lax representation of the KdV hierarchy in terms of isospectral deformations becomes for the Painlevé II hierarchy a Lax representation in terms of isomonodromic deformations [CJM06].

In [CJ99a] then, the discrete Painlevé II hierarchy is defined as the compatibility condition of a sort of “discretization” of the Lax representation of the Painlevé II hierarchy. In particular, they considered the compatibility condition of a linear 2×2 matrix-valued system of the following type

$$\Phi_{n+1}(z) = L_n(z)\Phi_n(z), \quad \frac{\partial}{\partial z}\Phi_n(z) = M_n(z)\Phi_n(z), \quad (4.16)$$

where the coefficients $L_n(z), M_n(z)$ are explicit matrix-valued rational function in z , depending on $x_\ell, \ell = n + N, \dots, n - N$, in some recursive (on N) way. This allows the authors there to compactly write the N -th discrete Painlevé II equation using some recursion operators. The linear system that we obtain in Proposition 4.17 and that encodes our hierarchy as written in (4.10) is mapped into the one of [CJ99a] through an explicit transformation, as shown in Propostion 4.24, thus implying that (4.10) is indeed the same discrete Painlevé II hierarchy.

Continuous limit The aim of this paragraph is to explain heuristically the reason why our result given in Theorem 4.2 can be considered as the discrete analogue of the generalized Tracy–Widom formula for higher order Airy kernels (namely the result contained in Theorem 1.1 of [CCG21], case $\tau_i = 0$).

For $N = 1$, Borodin in [Bor03] already pointed out that formula (4.7) with (4.8) can be seen as a discrete analogue of the classical Tracy-Widom formula for the GUE Tracy-Widom distribution [TW94b, TW94a]. In other words, he described how to pass from the left to the right in the picture below

$$\begin{array}{ccc}
 \text{“Discrete case”} & \xrightarrow{\text{Baik-Deift-Johansson}} & \text{“Continuous case”} \\
 \frac{D_n D_{n-2} - D_{n-1}^2}{D_{n-1}^2} = -x_n^2, & & \frac{d^2}{dt^2} \log \det(1 - \mathcal{K}_{\text{Ai}}|_{(t, +\infty)}) = -u^2(t), \\
 \text{with } nx_n + \theta(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0. & & \text{with } u''(t) = 2u^3(t) + tu(t), u(t) \underset{t \rightarrow \infty}{\sim} \text{Ai}(t).
 \end{array}$$

where $\text{Ai}(t)$ denotes the classical Airy function and \mathcal{K}_{Ai} denotes the integral operator acting on $L^2(\mathbb{R})$ through the Airy kernel. This connection was achieved by using the scaling limit computed by Baik, Deift and Johansson in [BDJ99] for the distribution of the first part of partitions in the Poissonized Plancherel random partition model (which is recovered in Theorem 1 of [BBW21] for $N = 1$). In some way, as emphasized by Borodin, their result not only assures the existence of a limiting function for the D_n , in this case $D(t) = \det(1 - \mathcal{K}_{\text{Ai}}|_{(t, +\infty)})$, for a certain continuous variable t . It also encodes already how the discrete function x_n , should be rescaled in terms of a differentiable function $u(t)$ to get back, from the recursion relation for D_n , the Tracy-Widom formula.

To generalize this result for the case $N > 1$, we proceed by adapting the method used by Borodin in [Bor03] for $N = 1$ to the higher order cases, using the scaling proposed in

[BBW21]² for the multicritical case (notice that their n corresponds to our N), instead of the Baik-Deift-Johansson's one that only holds for $N = 1$.

We recall that D_n is the Toeplitz determinant associated to the symbol $\varphi(z)$ (4.1) (which depends on $\theta_i, i = 1, \dots, N$ and thus on N). In the following discussion we write explicitly the dependence on the family of parameters $(\theta_i), i = 1, \dots, N$ of $D_n = D_n(\theta_i)$, $x_n = x_n(\theta_i)$, $r_n = r_n(\theta_i)$ and $q_n = q_n(\theta_i)$. Consider equation (4.9) written in terms of the Toeplitz determinants $D_n(\theta_i)$ in this way

$$\frac{D_{n-2}(\theta_i)D_n(\theta_i) - D_{n-1}^2(\theta_i)}{D_{n-1}^2(\theta_i)} = -x_n^2(\theta_i). \quad (4.17)$$

From the equation (4.6), this previous equation can be expressed in terms of $q_n(\theta_i)$ defined as (4.5). It becomes

$$\frac{q_{n-1}(\theta_i)q_{n+1}(\theta_i) - q_n^2(\theta_i)}{q_n^2(\theta_i)} = -x_n^2(\theta_i). \quad (4.18)$$

According to Lemma 8 in [BBW21], with the change of parameters $\tilde{\theta}_i = (-1)^{i-1}\theta_i$, we have $q_n(\theta_i) = r_n(\tilde{\theta}_i)$. Thus equation (4.18) now reads as

$$\frac{r_{n-1}(\tilde{\theta}_i)r_{n+1}(\tilde{\theta}_i) - r_n^2(\tilde{\theta}_i)}{r_n^2(\tilde{\theta}_i)} = -x_n^2(\theta_i). \quad (4.19)$$

Following the scaling limit described in Theorem 1 of [BBW21], we define the following scaling for the discrete variable n

$$n = b\theta + t\theta^{\frac{1}{2N+1}}d^{-\frac{1}{2N+1}}, \quad \iff \quad t = (n - b\theta)\theta^{-\frac{1}{2N+1}}d^{\frac{1}{2N+1}} \quad (4.20)$$

with b, d defined as

$$b = \frac{N+1}{N}, \quad d = \binom{2N}{N-1}$$

and choose $\tilde{\theta}_i$ (respectively θ_i) all proportional to $\theta = \tilde{\theta}_1 = \theta_1$ in the following way

$$\tilde{\theta}_i = (-1)^{i-1} \frac{(N-1)!(N+1)!}{(N-i)!(N+i)!} \theta, \quad i = 1, \dots, N,$$

respectively

$$\theta_i = \frac{(N-1)!(N+1)!}{(N-i)!(N+i)!} \theta, \quad i = 1, \dots, N. \quad (4.21)$$

2. Up to the correction of the typo $d \rightarrow d^{-1}$ in their statement of Theorem 1.

Now recall the definition of $r_n(\tilde{\theta}_i)$ (4.5) in function of $\mathbb{P} = \mathbb{P}_{\tilde{\theta}_i}$ (see equation (5.35) for the definition of \mathbb{P} and the dependence on the family of parameters (θ_i)). From the previous scaling it is now possible to express $r_n(\tilde{\theta}_i)$ in function of t and θ

$$r_n(\tilde{\theta}_i) = \mathbb{P}_{\tilde{\theta}_i} \left(\frac{\lambda_1 - b\theta}{(\theta d^{-1})^{\frac{1}{2N+1}}} \leq t \right) \quad (4.22)$$

and according to Theorem 1 of [BBW21], the limiting behavior of the probability distribution function of λ_1 in this setting is given by

$$\lim_{\theta \rightarrow +\infty} r_n(\tilde{\theta}_i) = \lim_{\theta \rightarrow +\infty} \mathbb{P}_{\tilde{\theta}_i} \left(\frac{\lambda_1 - b\theta}{(\theta d^{-1})^{\frac{1}{2N+1}}} \leq t \right) = F_N(t), \text{ with } F_N(t) = \det(1 - \mathcal{K}_{\text{Ai}_{2N+1}}|_{(t, \infty)}) \quad (4.23)$$

where $\mathcal{K}_{\text{Ai}_{2N+1}}$ is the integral operator acting with higher order Airy kernel (see for instance equation (2.7) in [BBW21]).

As we did for $r_n(\tilde{\theta}_i)$ in equation (4.22), we express $r_{n+1}(\tilde{\theta}_i)$ and $r_{n-1}(\tilde{\theta}_i)$ in function of t and θ .

$$r_{n\pm 1}(\tilde{\theta}_i) = \mathbb{P}_{\tilde{\theta}_i} \left(\frac{\lambda_1 - b\theta}{(\theta d^{-1})^{\frac{1}{2N+1}}} \leq t \pm (\theta d^{-1})^{-\frac{1}{2N+1}} \right)$$

With this discussion and this scaling for n , (θ_i) and $(\tilde{\theta}_i)$, we deduce that

$$-\lim_{\theta \rightarrow +\infty} \frac{x_n^2(\theta_i)}{(\theta d^{-1})^{-\frac{2}{2N+1}}} = \lim_{\theta \rightarrow +\infty} \frac{r_{n-1}(\tilde{\theta}_i)r_{n+1}(\tilde{\theta}_i) - r_n^2(\tilde{\theta}_i)}{(\theta d^{-1})^{-\frac{2}{2N+1}} r_n^2(\tilde{\theta}_i)} = \frac{d^2}{dt^2} \log F_N(t)$$

where the first equality comes from equation (4.19) and the second from equation (4.23).

From now on we drop the dependence on $\theta_i, i = 1, \dots, N$ in the notation. The previous equation suggests that, in order to be consistent with Theorem 1.1 of [CCG21], the discrete function x_n appearing in formula (4.17) in the scaling (4.20) for n and (4.21) for (θ_i) limit should be

$$-x_n^2 \sim -(\theta)^{-\frac{2}{2N+1}} d^{\frac{2}{2N+1}} u^2(t)$$

with $u(t)$ solution of the N -th equation of the Painlevé II hierarchy. This can be proved directly by computing the scaling limit of the equations of the discrete Painlevé II hierarchy

we found for x_n in Theorem 4.2. Indeed, for every fixed N , we write x_n as

$$x_n = (-1)^n \theta^{-\frac{1}{2N+1}} d^{\frac{1}{2N+1}} u(t) \tag{4.24}$$

with $u(t)$ a smooth function of the variable t defined as in equation (4.20). Now recall that x_n solves the discrete equation (4.10) of order $2N$ for every $N \geq 1$. The continuous limit of the discrete equations of the hierarchy (4.10), under the definition of x_n (4.24) and the scaling of the parameters θ_i as (4.21), gives the equations of the classical Painlevé II hierarchy. For any fixed N the computation should be done in the same way: consider the N -th discrete equation of the hierarchy (4.10) and replace each θ_i with the values given in formula (4.21). Then substitute x_n with the definition in (4.24) and for $\theta \rightarrow +\infty$ compute the asymptotic expansion of every term $x_{n+K} \propto u(t + K\theta^{-\frac{1}{2N+1}} d^{\frac{1}{2N+1}})$, $K = -N, \dots, N$ appearing in the discrete equation. The coefficient of θ^{-1} resulting after this procedure coincides indeed with the N -th equation of the Painlevé II hierarchy. For $N = 1, 2, 3$ the computations are explicitly done in the Appendix 4.4.

Remark 4.6. *It is worthy to mention that in [CCG21], the authors also consider a generalization of the Fredholm determinant $F_N(t)$, recalled here in (4.23), depending on additional parameters τ_i . Those are related to solutions of the general Painlevé II hierarchy, which depends as well on the τ_i . With the scaling as in [BBW21] for the θ_i 's, the continuous limit for our discrete equations leads to the Painlevé II hierarchy with $\tau_i = 0$ for all i . This is consistent with the fact that the limiting behavior in [BBW21], written here in equation (4.23), involves indeed the Fredholm determinant $F_N(t)$ corresponding to $\tau_i = 0$ for all i (the same already appeared in [DMS18]).*

Methodology and outline The rest of the work is devoted to prove Theorem 4.2. In order to do so, we introduce the classical Riemann-Hilbert characterization [BDS16] of the family of orthogonal polynomials on the unit circle (OPUC for brevity) with respect to a measure defined by the symbol $\varphi(z)$. Classical results from orthogonal polynomials theory allow to achieve almost directly formula (4.17) where x_n is defined as the constant term of the n -th monic orthogonal polynomial of the family. The Riemann-Hilbert problem for the OPUC is then used to deduce a linear system of the same type of (4.16) which is proven to be in relation with the Lax pair introduced by Cresswell and Joshi [CJ99a] for the discrete Painlevé II hierarchy. This is done in Section 4.2. The explicit computation of the Lax pair together with the construction of the recursion operator and the hierarchy

for x_n as written in (4.10) are done in Section 4.3.

4.2 OPUC: the Riemann-Hilbert approach and a discrete Painlevé II Lax pair

In this section we introduce the relevant family of orthogonal polynomials on the unit circle. We recall some of their properties and their Riemann–Hilbert characterization. Afterward we derive a Lax pair associated to the Riemann-Hilbert problem and establish the relation with the Lax pair for discrete Painlevé II hierarchy (4.16) introduced by Cresswell and Joshi [CJ99a]. The proofs of the results for orthogonal polynomials stated in here can be found in the classical reference [BDS16].

We denote by S^1 the unit circle in \mathbb{C} counterclockwise oriented. We consider the following positive measure on S^1 (absolutely continuous w.r.t. the Lebesgue measure there)

$$d\mu(\beta) = \frac{e^{w(e^{i\beta})}}{2\pi} d\beta, \quad (4.25)$$

where the function $w(z)$ for any $z \in \mathbb{C}$ is given as in equation (4.1). The family of orthogonal polynomials on the unit circle (OPUC) w.r.t. the measure (4.25) is defined as the collection of polynomials $\{p_n(z)\}_{n \in \mathbb{N}}$ written as

$$p_n(z) = \kappa_n z^n + \dots \kappa_0, \quad \kappa_n > 0 \quad (4.26)$$

and such that the following relation holds for any indices k, h

$$\int_{-\pi}^{\pi} \overline{p_k(e^{i\beta})} p_h(e^{i\beta}) \frac{d\mu(\beta)}{2\pi} = \delta_{k,h}.$$

The family of monic orthogonal polynomials $\{\pi_n(z)\}$ associated to the previous ones is defined in analogue way, so that $p_n(z) = \kappa_n \pi_n(z)$.

4.2.1 Toeplitz determinants related to OPUC

We recall that $\varphi(z) = e^{w(z)}$, $z \in S^1$ with $w(z)$ defined as in (4.1) and that we defined $D_n := \det(T_n(\varphi))$ (by convention $D_{-1} = 1$) to be the n -Toeplitz determinant associated to the symbol φ (see equations (4.2),(4.3)). Because $\varphi(z)$ is a real nonnegative function,

$D_n \in \mathbb{R}_{>0}$.

Proposition 4.7. *If $\varphi(z)$ is a real nonnegative function, we have that*

$$p_\ell(z) = \frac{1}{\sqrt{D_\ell D_{\ell-1}}} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-\ell+1} & \varphi_{-\ell} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-\ell+2} & \varphi_{-\ell+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{\ell-1} & \varphi_{\ell-2} & \cdots & \varphi_0 & \varphi_{-1} \\ 1 & z & \cdots & z^{\ell-1} & z^\ell \end{pmatrix}, \quad \ell \geq 0. \quad (4.27)$$

Proof. The proof is similar to the one for the orthogonal polynomials on the real line, that can be found e.g. in [Dei99], equation (3.5) and following discussion. \square

Corollary 4.8. *The ratio of two consecutive Toeplitz determinants is expressed as*

$$\frac{D_{\ell-1}}{D_\ell} = \kappa_\ell^2, \quad \ell \geq 0. \quad (4.28)$$

Proof. Thanks to formula (4.27), we have that

$$p_\ell(z) = \frac{1}{\sqrt{D_\ell D_{\ell-1}}} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-\ell+1} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-\ell+2} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{\ell-1} & \varphi_{\ell-2} & \cdots & \varphi_0 \end{pmatrix} z^\ell + \cdots = \sqrt{\frac{D_{\ell-1}}{D_\ell}} z^\ell + \cdots,$$

and by definition $p_\ell(z) = \kappa_\ell \pi_\ell(z)$ with the latter being the ℓ -th monic orthogonal polynomial on S^1 . Thus formula (4.28) follows. \square

4.2.2 Riemann-Hilbert problem associated to OPUC

The family $\{\pi_n\}$ of orthogonal polynomials has a well known characterization in terms of a 2×2 dimensional Riemann-Hilbert problem, also depending on $n \geq 0$.

Riemann-Hilbert Problem 4.9. *The function $Y(z) := Y(n, \theta_j; z) : \mathbb{C} \rightarrow \text{GL}(2, \mathbb{C})$ has the following properties:*

- (1) $Y(z)$ is analytic for every $z \in \mathbb{C} \setminus S^1$;

(2) $Y(z)$ has continuous boundary values $Y_{\pm}(z)$ while approaching non-tangentially S^1 either from the left or from the right, and they are related for all $z \in S^1$ through

$$Y_+(z) = Y_-(z)J_Y(z), \quad \text{with } J_Y(z) = \begin{pmatrix} 1 & z^{-n}e^{w(z)} \\ 0 & 1 \end{pmatrix};$$

(3) $Y(z)$ is normalized at ∞ as

$$Y(z) \sim \left(I + \sum_{j=1}^{\infty} \frac{Y_j(n, \theta_j)}{z^j} \right) z^{n\sigma_3}, \quad z \rightarrow \infty,$$

where σ_3 denotes the Pauli's matrix $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

It is known from [BDJ99] that the above Riemann-Hilbert problem, for each $n \geq 0$, admits a unique solution which is explicitly written in terms of the family $\{\pi_n(z)\}$. Before stating the result, we introduce the following notation. For every polynomial $q(z)$, $z \in \mathbb{C}$, its reverse polynomial $q^*(z)$ is defined as the polynomial of the same degree such that

$$q^*(z) := z^n \overline{q(\bar{z}^{-1})}.$$

For every ($L^p(S^1)$) function $f(y)$, its Cauchy transform $\mathcal{C}f(z)$ is defined for any $z \notin S^1$ as

$$(\mathcal{C}f(y))(z) := \frac{1}{2\pi\mathfrak{B}} \int_{S^1} \frac{f(y)}{y-z} dy.$$

Remark 4.10. Notice that the results in [BDJ99] for the Riemann–Hilbert characterization a family of orthogonal polynomials on the unit circle are a sort of extension of the results known from [FIK90, FIK91] for the case of orthogonal polynomials on the real line.

Theorem 4.11. For every $n \geq 0$, the Riemann-Hilbert problem 4.9 admits a unique solution $Y(z)$ that is written as

$$Y(z) = \begin{pmatrix} \pi_n(z) & \mathcal{C}\left(y^{-n}\pi_n(y)e^{w(y)}\right)(z) \\ -\kappa_{n-1}^2\pi_{n-1}^*(z) & -\kappa_{n-1}^2\mathcal{C}\left(y^{-n}\pi_{n-1}^*(y)e^{w(y)}\right)(z) \end{pmatrix}. \quad (4.29)$$

Moreover, $\det(Y(z)) \equiv 1$.

Proof. See Lemma 4.1 in [BDJ99]. □

The solution $Y(z)$ has a symmetry which will be very useful in the following section.

Corollary 4.12. *The unique solution $Y(z)$ of the Riemann-Hilbert problem 4.9 is such that*

$$Y(z) = \sigma_3 Y(0)^{-1} Y(z^{-1}) z^{n\sigma_3} \sigma_3, \quad (4.30)$$

$$Y(z) = \overline{Y(\bar{z})}. \quad (4.31)$$

Proof. See Proposition 5.12 in [BDS16]. □

Notice that the factor $Y(0) = Y(n, \theta_j; 0)$ appearing in equation (4.30) has a very explicit form, by equation (4.29). This will be useful in the following sections.

Lemma 4.13. *For every $n \geq 0$ we have*

$$Y(0) = Y(n, \theta_j; 0) = \begin{pmatrix} x_n & \kappa_n^{-2} \\ -\kappa_{n-1}^2 & x_n \end{pmatrix}, \quad (4.32)$$

where we denoted with $x_n := \pi_n(0)$ and κ_n is defined as in equation (4.26). Moreover, we have

$$\frac{\kappa_{n-1}^2}{\kappa_n^2} = 1 - x_n^2 \quad (4.33)$$

and we have $x_n \in \mathbb{R}$.

Proof. The first column of $Y(n; 0)$ directly follows from the evaluation in $z = 0$ of $Y(n; z)$ as given in equation (4.29). Indeed $Y^{11}(n; 0) = \pi_n(0)$ and $Y^{21}(n; 0) = -\kappa_{n-1}^2 \pi_{n-1}^*(0)$ but we observe that

$$\pi_{n-1}^*(0) = z^{n-1} \overline{\pi_{n-1}(\bar{z}^{-1})} \Big|_{z=0} = z^{n-1} (z^{-(n-1)} + \dots + \overline{\pi_{n-1}(0)}) \Big|_{z=0} = 1.$$

Thus we conclude that $Y^{21}(n; 0) = -\kappa_{n-1}^2$. For what concerns the second column of $Y(n; 0)$, we first find the (2, 2)-entry. This is indeed easily deduced from the symmetry given in (4.30). In the limit for $z \rightarrow \infty$ it gives

$$Y(n; 0) = \sigma_3 Y^{-1}(n; 0) \sigma_3,$$

thus $Y^{22}(n; 0) = Y^{11}(n; 0) = \pi_n(0)$. Finally for the entry (1, 2) of $Y(n; 0)$, we compute it

explicitly using the orthonormality property of the polynomials $p_m(z)$

$$\begin{aligned} Y^{12}(n; 0) &= \frac{1}{2\pi i} \int_{S^1} \frac{\pi_n(s) s^{-n} w(s)}{s} ds = \int_{-\pi}^{\pi} \pi_n(e^{i\theta}) \overline{e^{in\theta}} w(e^{i\theta}) \frac{d\theta}{2\pi} = \frac{1}{\kappa_n^2} \int_{-\pi}^{\pi} p_n(e^{i\theta}) \overline{p_n(e^{i\theta})} w(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= \frac{1}{\kappa_n^2}. \end{aligned}$$

Equation (4.33) comes from the fact that $\det(Y(n, \theta_j; z)) = 1$ identically in z and so in particular for $z = 0$ by writing $Y(n, \theta_j; 0)$ as in equation (4.32), relation (4.33) is obtained.

Finally, the fact that x_n is real follows from the entry (1, 1) of equation (4.31) together with equation (4.29). \square

At this point, we are already able to express the ratio of Toeplitz determinants in terms of the constant term of the monic orthogonal polynomials, as follows.

Corollary 4.14. *For every $n \geq 1$, the Toeplitz determinants D_n satisfy the recursion relation*

$$\frac{D_{n-2} D_n}{D_{n-1}^2} = 1 - x_n^2. \quad (4.34)$$

Proof. Putting together equation (4.33) with equation (4.28) (for two consecutive integers) we obtain the recursion relation (4.34). \square

We emphasize again that the symbol $\varphi(z)$ actually depends on the natural parameter N , so the Toeplitz determinants $D_n, n \geq 1$ (4.3) do as well as $x_n = \pi_n(0), n \geq 1$ do (since it is the constant coefficient of the n -th monic OPUC w.r.t. the N -depending measure (4.25), (4.1)). The N -dependence of the latter will be emphasized in the following section, where x_n is proved to be a solution of the N -th higher order generalization of the discrete Painlevé II equation.

We consider now the following matrix-valued function

$$\Psi(n, \theta_j; z) := \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(n, \theta_j; z) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} e^{w(z) \frac{\sigma_3}{2}}. \quad (4.35)$$

Thanks to the properties of $Y(z; n, \theta_j)$ from the RH problem 4.9 one can prove that $\Psi(n, \theta_j; z)$ satisfies the following RH problem.

Riemann-Hilbert Problem 4.15. *The function $\Psi(z) := \Psi(n, \theta_j; z) : \mathbb{C} \rightarrow \text{GL}(2, \mathbb{C})$ has the following properties:*

- (1) $\Psi(z)$ is analytic for every $z \in \mathbb{C} \setminus \{S^1 \cup \{0\}\}$;
(2) $\Psi(z)$ has continuous boundary values $\Psi_{\pm}(z)$ while approaching non-tangentially S^1 either from the left or from the right, and they are related for all $z \in S^1$ through

$$\Psi_+(z) = \Psi_-(z)J_0, \quad J_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad (4.36)$$

- (3) $\Psi(z)$ has asymptotic behavior near 0 given by

$$\Psi(z) \sim \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(0) \left(I + \sum_{j=1}^{\infty} z^j \tilde{Y}_j(n) \right) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} e^{w(z)\frac{\sigma_3}{2}}, \quad z \rightarrow 0; \quad (4.37)$$

- (4) $\Psi(z)$ has asymptotic behavior near ∞ given by

$$\Psi(z) \sim \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} \left(I + \sum_{j=1}^{\infty} \frac{Y_j(n)}{z^j} \right) \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix} e^{w(z)\frac{\sigma_3}{2}}, \quad |z| \rightarrow \infty. \quad (4.38)$$

Proposition 4.16. *The function $\Psi(n, \theta_j; z)$ defined in (4.35) solves the Riemann–Hilbert problem 4.15.*

Proof. The analyticity condition and the asymptotic expansions at $0, \infty$ given in (4.37), (4.38) follows directly from the definition (4.35) and the fact that $Y(z)$ solves the RH problem 4.9. Condition (4.36) follows from direct computation

$$\begin{aligned} \Psi(z)_+ &= \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y_+(z) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} e^{w(z)\frac{\sigma_3}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y_-(z) J_Y(z) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} e^{w(z)\frac{\sigma_3}{2}} \\ &= \Psi_-(z) \begin{pmatrix} 1 & 0 \\ 0 & z^{-n} \end{pmatrix} e^{-w(z)\frac{\sigma_3}{2}} \begin{pmatrix} 1 & z^{-n}e^{w(z)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} e^{w(z)\frac{\sigma_3}{2}} \\ &= \Psi_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

□

4.2.3 A linear differential system for $\Psi(z)$

From the solution of the Riemann-Hilbert problem 4.15 we deduce the following equations (in the following we omit in Ψ the dependence on θ_j that should be considered only

as parameters and not actual variables like n, z).

Proposition 4.17. *We have*

$$\Psi(n+1; z) = U(n; z)\Psi(n; z), \quad \partial_z \Psi(n; z) = T(n; z)\Psi(n; z), \quad (4.39)$$

with

$$U(n; z) := \begin{pmatrix} z + x_n x_{n+1} & -x_{n+1} \\ -(1 - x_{n+1}^2)x_n & 1 - x_{n+1}^2 \end{pmatrix} = \sigma_+ z + U_0(n), \quad (4.40)$$

where $\sigma_+ := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$T(n; z) := T_1(n)z^{N-1} + T_2(n)z^{N-2} + \dots + T_{2N+1}(n)z^{-N-1} = \sum_{k=1}^{2N+1} T_k z^{N-k}, \quad (4.41)$$

where

$$T_1(n) = \frac{\theta_N}{2} \sigma_3. \quad (4.42)$$

Remark 4.18. *The coefficient $(T_i(n))_{2 \leq i \leq 2N+1}$ defined in equation (4.41) will be computed in the section 4.3.*

Proof. We first prove the first equation. We start by defining the quantity

$U(n; z) := \Psi(n+1; z)\Psi^{-1}(n; z)$. Since the jump condition for $\Psi(z)$ (4.36) is independent of n , $U(n; z)$ is analytic everywhere. Plugging in equation (4.38) we have the expansion at ∞

$$U(n; z) = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_{n+1}^{-2} \end{pmatrix} \left(I + \frac{Y_1(n+1)}{z} + \mathcal{O}(z^{-2}) \right) z^{(n+1)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} z^{-n\sigma_3} \left(I - \frac{Y_1(n)}{z} + \mathcal{O}(z^{-2}) \right) \\ \times \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix},$$

from which we deduce that $U(n; z)$ is a polynomial in z of degree 1, by Liouville theorem. Moreover its matrix-valued coefficient are written as

$$U(n; z) = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \kappa_{n+1}^{-2} \end{pmatrix} Y(n+1; 0) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y^{-1}(n; 0) \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix}}_{=U_0(n)}.$$

Doing the computation and using equation (4.32) we obtain

$$U_0(n) = \begin{pmatrix} Y^{11}(n+1;0)Y^{22}(n;0) & -\kappa_n^2 Y^{11}(n+1;0)Y^{12}(n;0) \\ \kappa_{n+1}^{-2} Y^{21}(n+1;0)Y^{22}(n,0) & -Y^{21}(n+1;0)Y^{12}(n;0) \end{pmatrix} = \begin{pmatrix} x_{n+1}x_n & -x_{n+1} \\ -(1-x_{n+1}^2)x_n & 1-x_{n+1}^2 \end{pmatrix}.$$

For what concerns the second equation, we define $T(n; z) := \partial_z \Psi(n; z) \Psi^{-1}(n; z)$. From the asymptotic behavior of $\Psi(n; z)$ at 0 and ∞ we can deduce that $T(n; z)$ is a meromorphic function in z with behavior at ∞ described by

$$T(n; z) \sim \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} \left(I + \frac{Y_1(n)}{z} + O(z^{-2}) \right) \frac{V'(z)}{2} \sigma_3 \left(I - \frac{Y_1(n)}{z} + O(z^{-2}) \right) \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix}$$

(polynomial behavior of degree $N - 1$) while at 0 its behavior is described by

$$T(n; z) \sim \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(n, 0) \left(I + \tilde{Y}_1(n)z + O(z^2) \right) \frac{-V'(z^{-1})}{2z^2} \sigma_3 \left(I - \tilde{Y}_1(n)z + O(z^2) \right) \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix},$$

i.e. there is a pole of order $N + 1$. In conclusion we can write

$$T(n; z) = \frac{\theta_N}{2} \sigma_3 z^{N-1} + T_2(n) z^{N-2} + \cdots + T_{2N+1}(n) z^{-N-1}.$$

□

Moreover, thanks to the symmetry for the solution of the Riemann–Hilbert problem $Y(z)$ stated in (4.30), we have that the coefficient matrix $T(n; z)$ satisfies a symmetry property.

Proposition 4.19. *$T(n; z)$ has the following symmetry*

$$T(n; z^{-1}) = -z^2 \left(K(n) T(n; z) K(n)^{-1} - n z^{-1} I_2 \right), \quad (4.43)$$

$$\text{with } K(n) := \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(n; 0) \sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix}.$$

Remark 4.20. *Notice that for all n , the matrix $K(n)$ is s.t. $K(n)^{-1} = K(n)$ since we have the identity $x_n^2 + \frac{\kappa_{n-1}^2}{\kappa_n^2} = 1$.*

Proof. On the one hand,

$$\partial_z(\Psi(n; z^{-1})) = -\frac{1}{z^2}T(n; z^{-1})\Psi(n; z^{-1}).$$

On the other hand, using the symmetry (4.30) for Y we deduce the following symmetry for Ψ : $\Psi(n; z^{-1}) = z^{-n} \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(0)\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix} \Psi(n; z)\sigma_3$.

This previous equation leads to

$$\partial_z(\Psi(n; z^{-1})) = z^{-n} \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(0)\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix} \partial_z \Psi(n; z)\sigma_3 - nz^{-1}\Psi(n; z^{-1}).$$

Then

$$T(n; z^{-1}) = -z^2 \left(\begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(0)\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix} T(n; z) \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} \sigma_3 Y(0)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^2 \end{pmatrix} - nz^{-1}I_2 \right).$$

□

The symmetry (4.43) reflects on the coefficients $T_k(n), k = 1, \dots, 2N + 1$ as written below.

Corollary 4.21. *The coefficients $T_k(n), k = 1, \dots, 2N + 1$ satisfy*

$$T_j(n) = -K(n)T_{2N+2-j}(n)K(n)^{-1}, \quad j = 1, \dots, N \quad (4.44)$$

$$T_{N+1}(n) = -K(n)T_{N+1}(n)K(n)^{-1} + nI_2. \quad (4.45)$$

Proof. Indeed, by replacing the exact shape of $T(n; z)$ in equation (4.43) we have

$$\begin{aligned} \sum_{k=1}^{2N+1} T_k(n)z^{-N+k} = T(n; z^{-1}) &= -z^2 \left(\sum_{k=1}^{2N+1} KT_k(n)K^{-1}z^{N-k} - nz^{-1}I_2 \right) \\ &= - \sum_{k=1}^{2N+1} KT_k(n)K^{-1}z^{N+2-k} + nzI_2 \\ &= - \sum_{j=1}^{2N+1} KT_{2N+2-j}(n)K^{-1}z^{-N+j} + nzI_2 \end{aligned}$$

so looking at the powers z^{-N+j} for $j = 1, \dots, N$ we get equation (4.44) and for $j = N + 1$ we get equation (4.45). □

Notice first that from equations (4.44) if the first $N+1$ coefficients of $T(n; z)$ are known, then we can obtain the remaining ones. Second, notice that the coefficient $T_{N+1}(n)$ plays an important role since it solves an equation, the one given in (4.45).

4.2.4 Relation with the Cresswell-Joshi Lax pair

To conclude this section, we describe how the Lax pair (4.39) is related with the one of the discrete Painlevé II hierarchy (4.16) originally introduced by Cresswell and Joshi in [CJ99a] as follows.

Definition 4.22. *A Lax pair for the discrete Painlevé II hierarchy is given by a pair of matrices $(L_n(z), M_n(z))$, defining the coefficients of a discrete-differential system for a matrix-valued function $\Phi(n; z)$, such as*

$$\Phi(n+1; z) = \begin{pmatrix} z & x_n \\ x_n & 1/z \end{pmatrix} \Phi(n; z) = L_n(z) \Phi(n; z), \quad (4.46)$$

$$\frac{\partial}{\partial z} \Phi(n; z) = M_n(z) \Phi(n; z), \quad (4.47)$$

with the property that $M_n(z) = \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & -A_n(z) \end{pmatrix}$ with A_n, B_n and C_n are rational in z (and depending also on N).

Remark 4.23. *Specifically, in section 3.1 of [CJ99a], the authors proved that the compatibility condition of the system of equations (4.46) and (4.47) defines the coefficients of the matrix $M_n(z)$, leaving in turns only one discrete equation of order $2N$ for x_n . This is defined as the N -th member of the discrete Painlevé II hierarchy.*

We establish now a link between this Lax Pair and the system (4.39) we obtained starting from the OPUC. We define

$$\Phi(n; z) := \sigma_3 \begin{pmatrix} z^{-n+3/2} & 0 \\ 0 & z^{-n+1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_{n-1} & 1 \end{pmatrix} \Psi(n-1; z^2).$$

Proposition 4.24. $\Phi(n; z)$ defined as above satisfies the system of equations (4.46),(4.47).

Proof. First we compute the discrete equation for $\Phi(n; z)$. From the definition, we have

$$\Phi(n+1; z) = \sigma_3 \begin{pmatrix} z^{-n+1/2} & 0 \\ 0 & z^{-n-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_n & 1 \end{pmatrix} \Psi(n; z^2).$$

According to equation (4.39)

$$\begin{aligned} \Phi(n+1; z) &= \sigma_3 \begin{pmatrix} z^{-n+1/2} & 0 \\ 0 & z^{-n-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_n & 1 \end{pmatrix} U(n-1; z^2) \Psi(n-1; z^2) \\ &= \sigma_3 \begin{pmatrix} z^{-n+1/2} & 0 \\ 0 & z^{-n-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_n & 1 \end{pmatrix} U(n-1; z^2) \begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix} \begin{pmatrix} z^{n-3/2} & 0 \\ 0 & z^{n-1/2} \end{pmatrix} \sigma_3 \Phi(n; z) \\ &= \begin{pmatrix} z & x_n \\ x_n & 1/z \end{pmatrix} \Phi(n; z). \end{aligned}$$

Now we compute the derivative with respect to z .

Defining $M_n(z) := \left(\frac{\partial}{\partial z} \Phi(n; z) \right) \Phi(n; z)^{-1}$, similar computations lead to

$$\begin{aligned} M_n(z) &= z^{-1} \sigma_3 \begin{pmatrix} -n+3/2 & 0 \\ 0 & -n+1/2 \end{pmatrix} \sigma_3 + 2z \sigma_3 \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_{n-1} & 1 \end{pmatrix} T(n-1; z^2) \\ &\quad \times \begin{pmatrix} 1 & 0 \\ x_{n-1} & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \sigma_3. \quad (4.48) \end{aligned}$$

We need to prove two things: first the trace of $M_n(z)$ is null and then entries of $M_n(z)$ are rational in z .

For the trace of $M_n(z)$ we use the fact that $\text{Tr}(T(n; z)) = nz^{-1}$. Then

$$\text{Tr}(M_n(z)) = (-2n+2)z^{-1} + 2z \text{Tr}(T(n-1; z^2)) = 0.$$

From the expression of $T(n; z)$ (4.41) and the equation (4.48) we conclude entries of $M_n(z)$ are rational in z . \square

4.3 From the Lax Pair to the discrete Painlevé II hierarchy

In this section we study the compatibility condition associated to the linear system (4.39). This first allows us to reconstruct completely the matrix $T(n; z)$ and then to obtain an explicit $2N$ order discrete equation for x_n which corresponds to equation (4.10).

4.3.1 The symmetry in the compatibility condition

We study the consequences of the symmetry (4.43) for the matrix $T(n; z)$ on the compatibility condition for the Lax pair introduced in Proposition 4.17. More precisely we show that, thanks to the symmetry (4.43) the compatibility condition contains an overdetermined system of equations.

We recall that the compatibility condition reads as

$$\sigma_+ - T(n+1; z)U(n; z) + U(n; z)T(n; z) = 0, \quad (4.49)$$

where we have to replace $U(n; z)$ as in (4.40) and $T(n; z)$ as

$$T(n; z) = \sum_{k=1}^{N+1} T_k(n)z^{N-k} + \sum_{k=N+2}^{2N+1} -K(n)T_{2N+2-k}(n)K(n)^{-1}z^{N-k}, \quad (4.50)$$

and with the coefficient $T_{N+1}(n)$ satisfying equation (4.45).

Lemma 4.25. *The compatibility condition (4.49), for $U(n; z), T(n; z)$ as described above, corresponds to the following system*

$$\begin{aligned} T_1(n+1)\sigma_+ - \sigma_+T_1(n) &= 0, \\ T_{j+1}(n+1)\sigma_+ - \sigma_+T_{j+1}(n) + T_j(n+1)U_0(n) - U_0(n)T_j(n) &= \sigma_+\delta_{j,N}, \quad j = 1, \dots, N, \\ T_{N+1}(n) &= -K(n)T_{N+1}(n)K(n)^{-1} + nI_2. \end{aligned}$$

Proof. The compatibility condition (4.49), after replacing $U(n; z), T(n; z)$ of the prescribed form, involves powers of z from N to $-N-1$. Imposing that the coefficients

of each of these powers of z is identically zero, we obtain the following equations

$$z^N : T_1(n+1)\sigma_+ - \sigma_+T_1(n) = 0 \quad (4.51)$$

$$z^{N-j}, j = 1, \dots, N :$$

$$T_{j+1}(n+1)\sigma_+ - \sigma_+T_{j+1}(n) + T_j(n+1)U_0(n) - U_0(n)T_j(n) = \sigma_+\delta_{j,N} \quad (4.52)$$

$$z^{-1} : T_{N+1}(n+1)U_0(n) - U_0(n)T_{N+1}(n) - K(n+1)T_N(n+1)K(n+1)^{-1}\sigma_+ \\ + \sigma_+K(n)T_N(n)K(n)^{-1} = 0 \quad (4.53)$$

$$z^{N-j}, j = N+2, \dots, 2N :$$

$$-K(n+1)T_{2N+1-j}(n+1)K(n+1)^{-1}\sigma_+ + \sigma_+K(n)T_{2N+1-j}(n)K(n)^{-1} + U_0(n)K(n) \\ \times T_{2N+2-j}(n)K(n)^{-1} - K(n+1)T_{2N+2-j}(n+1)K(n+1)^{-1}U_0(n) = 0 \quad (4.54)$$

$$z^{-N-1} : -K(n+1)T_1(n+1)K(n+1)^{-1}U_0(n) + U_0(n)K(n)T_1(n)K(n)^{-1} = 0. \quad (4.55)$$

With the change of indices $2N+1-j = k$, $\iff k = 2N+1-j = N-1, \dots, 1$, the equation (4.54) becomes:

$$-K(n+1)T_k(n+1)K(n+1)^{-1}\sigma_+ + \sigma_+K(n)T_k(n)K(n)^{-1} - K(n+1)T_{k+1}(n+1) \\ \times K(n+1)^{-1}U_0(n) + U_0(n)K(n)T_{k+1}(n)K(n)^{-1} = 0. \quad (4.56)$$

We now show that equations (4.53), (4.54), (4.55) are equivalent to the first ones (4.51), (4.52) thanks to the symmetry of the coefficients $T_k(n)$ given in (4.44) together with the equation for $T_{N+1}(n)$, already obtained in (4.45).

To start with, we notice the following relations

$$\tilde{U}_0(n) := K(n+1)^{-1}U_0(n)K(n) = \sigma_+,$$

and

$$\tilde{\sigma}(n) := K(n+1)^{-1}\sigma_+K(n) = U_0(n),$$

deduced by using multiple times relation (4.33), namely $x_n^2 + \frac{\kappa_{n-1}^2}{\kappa_n^2} = 1$.

- 1) Let us consider first the equation (4.55) obtained from the coefficient of the term z^{-N-1} . Multiplying by $K(n+1)^{-1}$ to the left and by $K(n)$ to the right, we obtain

$$-T_1(n+1)\tilde{U}_0(n) + \tilde{U}_0(n)T_1(n) = 0,$$

that is exactly (4.51).

- 2) Let us consider now equations (4.56), obtained from the coefficients of the term z^{N-j} , $j = N+2, \dots, 2N$. By multiplying by $K(n+1)^{-1}$ to the left and by $K(n)$ to the right as before, we obtain the equations for $k = N-1, \dots, 1$

$$-T_k(n+1)\tilde{\sigma}(n) + \tilde{\sigma}(n)T_k(n) - T_{k+1}(n+1)\tilde{U}_0(n) + \tilde{U}_0(n)T_{k+1}(n) = 0$$

which is exactly equation (4.52) for $j = 1, \dots, N-1$.

- 3) The last equation is (4.53) obtained from the coefficient of the term z^{-1} . We multiply, again, by $K(n+1)^{-1}$ to the left and by $K(n)$ to the right, and we get

$$K(n+1)^{-1}T_{N+1}(n+1)K(n+1)\tilde{U}_0(n) - \tilde{U}_0(n)K(n)^{-1}T_{N+1}(n)K(n) - T_N(n+1)\tilde{\sigma}(n) + \tilde{\sigma}(n)T_N(n) = 0,$$

and then we replace the symmetry for the term $T_{N+1}(n)$ namely the equation (4.45) (that indeed it has not be used until now)

$$-T_{N+1}(n+1)\tilde{U}_0(n) + \tilde{U}_0(n)T_{N+1}(n) + \tilde{U}_0(n) - T_N(n+1)\tilde{\sigma}(n) + \tilde{\sigma}(n)T_N(n) = 0.$$

And this is again exactly equation (4.52), for $j = N$.

Thus the compatibility condition (4.49) is reduced to the equations in the statement, namely equations (4.51), (4.52), (4.45). \square

Now, we use equations (4.51), (4.52) together with the initial condition for $T_1(n)$ given in (4.42), to recursively find the coefficients $T_k(n)$, for $k = 1, \dots, N+1$, in terms of the $x_{n\pm j}$, $j = 1, \dots, N$. With the coefficients $T_k(n)$ computed in such a way, the symmetry for $T_{N+1}(n)$, i.e. equation (4.45), once $T_{N+1}(n)$ is determined, provides an actual discrete equation for x_n of order $2N$, that is what we call the higher order analogue of the discrete Painlevé II equation (that coincide for $N = 1, 2$ to the ones already appeared in literature [AvM03, Bor03, CJ99a]).

4.3.2 The recursion

In this subsection we explain how equations (4.51), (4.52) resulting from the compatibility condition (4.49) can be used to find recursively (in k) all the coefficients $T_k(n)$, $k = 1, \dots, N + 1$ of $T(n; z)$.

Lemma 4.26. *For every $i = 1, \dots, N$, starting from the initial condition (4.42) $T_1(n) = \frac{\theta_N}{2}\sigma_3$, we have*

$$\begin{aligned} T_{i+1,12}(n) &= x_{n+1}(2\Delta^{-1} + I) \left(\frac{x_{n+1}}{v_{n+1}} T_{i,21}(n+1) - x_n T_{i,12}(n) \right) + v_{n+1} T_{i,12}(n+1) - x_n x_{n+1} T_{i,12}(n), \\ T_{i+1,21}(n+1) &= x_n v_{n+1} (2\Delta^{-1} + I) \left(\frac{x_{n+1}}{v_{n+1}} T_{i,21}(n+1) - x_n T_{i,12}(n) \right) + v_{n+1} T_{i,21}(n) \\ &\quad - x_n x_{n+1} T_{i,21}(n+1), \\ T_{i+1,11}(n) &= -T_{i+1,22}(n) + n\delta_{i,N} = \Delta^{-1} \left(\frac{-x_{n+1}}{v_{n+1}} T_{i+1,21}(n+1) + x_n T_{i+1,12}(n) \right) + n\delta_{i,N}, \end{aligned}$$

where

$$\Delta : T_i(n) \rightarrow T_i(n+1) - T_i(n), \quad (4.57)$$

$$v_n := 1 - x_n^2, \quad (4.58)$$

Proof. We rewrite equations (4.51), (4.52) for $i = 1, \dots, N$, entry by entry. For the first one, we have

$$\begin{cases} T_{1,11}(n+1) - T_{1,11}(n) = 0 \\ T_{1,12}(n) = T_{1,21}(n+1) = 0 \end{cases}$$

This is satisfied by $T_1(n)$ given in (4.42). For the second one, for any $1 \leq i \leq N$ we have the four equations:

$$\begin{aligned} T_{i+1,11}(n+1) - T_{i+1,11}(n) &= -T_{i,11}(n+1)x_n x_{n+1} + T_{i,12}(n+1)(1 - x_{n+1}^2)x_n \\ &\quad + x_n x_{n+1} T_{i,11}(n) - x_{n+1} T_{i,21}(n) + \delta_{i,N}, \end{aligned}$$

$$T_{i+1,12}(n) = -x_{n+1} T_{i,11}(n+1) + T_{i,12}(n+1)(1 - x_{n+1}^2) - x_n x_{n+1} T_{i,12}(n) + x_{n+1} T_{i,22}(n),$$

$$T_{i+1,21}(n+1) = -T_{i,21}(n+1)x_n x_{n+1} + T_{i,22}(n+1)x_n(1 - x_{n+1}^2) - T_{i,11}(n)x_n(1 - x_{n+1}^2) + (1 - x_{n+1}^2)T_{i,21}(n),$$

$$0 = T_{i,21}(n+1)x_{n+1} - T_{i,22}(n+1)(1 - x_{n+1}^2) - x_n(1 - x_{n+1}^2)T_{i,12}(n) + T_{i,22}(n)(1 - x_{n+1}^2).$$

Using the notations introduced in (4.57), (4.58), the previous equations become: $1 \leq$

$i \leq N$:

$$\Delta T_{i+1,11}(n) = -x_n x_{n+1} \Delta T_{i,11}(n) + x_n v_{n+1} T_{i,12}(n+1) - x_{n+1} T_{i,21}(n) + \delta_{i,N}, \quad (4.59)$$

$$T_{i+1,12}(n) = -x_{n+1} T_{i,11}(n+1) + v_{n+1} T_{i,12}(n+1) - x_n x_{n+1} T_{i,12}(n) + x_{n+1} T_{i,22}(n), \quad (4.60)$$

$$T_{i+1,21}(n+1) = -x_n x_{n+1} T_{i,21}(n+1) + x_n v_{n+1} T_{i,22}(n+1) - x_n v_{n+1} T_{i,11}(n) + v_{n+1} T_{i,21}(n), \quad (4.61)$$

$$v_{n+1} \Delta T_{i,22}(n) = x_{n+1} T_{i,21}(n+1) - x_n v_{n+1} T_{i,12}(n). \quad (4.62)$$

From these equations, we see that in order to obtain the diagonal terms, there is a “discrete integration” to perform, while the off-diagonal terms are directly determined from the previous ones. Moreover, we can rewrite the four equations as only two equations involving only the off-diagonal terms. Indeed, because of $\text{Tr}(T(n; z)) = nz^{-1}$, $T_{i,11}(n, z) = -T_{i,22}(n, z)$ for $1 \leq i \leq N$. Thus (4.62) can be written as

$$v_{n+1} \Delta T_{i,11}(n) = -x_{n+1} T_{i,21}(n+1) + x_n v_{n+1} T_{i,12}(n).$$

Formally, $1 \leq i \leq N$

$$T_{i,11}(n) = -T_{i,22}(n) = \Delta^{-1} \left(\frac{-x_{n+1}}{v_{n+1}} T_{i,21}(n+1) + x_n T_{i,12}(n) \right), \quad (4.63)$$

which still holds for $i = N + 1$ up to adding the “constant” n on the right hand side. Using this in (4.60) and (4.61), we obtain:

$$T_{i+1,12}(n) = x_{n+1} (2\Delta^{-1} + I) \left(\frac{x_{n+1}}{v_{n+1}} T_{i,21}(n+1) - x_n T_{i,12}(n) \right) + v_{n+1} T_{i,12}(n+1) - x_n x_{n+1} T_{i,12}(n),$$

$$T_{i+1,21}(n+1) = x_n v_{n+1} (2\Delta^{-1} + I) \left(\frac{x_{n+1}}{v_{n+1}} T_{i,21}(n+1) - x_n T_{i,12}(n) \right) + v_{n+1} T_{i,21}(n) - x_n x_{n+1} T_{i,21}(n+1).$$

□

We notice that, defining the discrete recursion operator

$$\mathcal{L} \begin{pmatrix} u_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1}(2\Delta^{-1} + I) \begin{pmatrix} \frac{x_{n+1}}{v_{n+1}}y_n - x_n u_n \\ v_{n+1} \end{pmatrix} + (v_{n+1}(\Delta + I) - x_n x_{n+1})u_n \\ x_n v_{n+1}(2\Delta^{-1} + I) \begin{pmatrix} \frac{x_{n+1}}{v_{n+1}}y_n - x_n u_n \\ v_{n+1} \end{pmatrix} + (v_{n+1}(\Delta + I)^{-1} - x_n x_{n+1})y_n \end{pmatrix}, \quad (4.64)$$

we can rewrite the two equations for the off-diagonal entries of $T_i(n)$ obtained above as

$$\begin{pmatrix} T_{i+1,12}(n) \\ T_{i+1,21}(n+1) \end{pmatrix} = \mathcal{L} \begin{pmatrix} T_{i,12}(n) \\ T_{i,21}(n+1) \end{pmatrix}, \quad 1 \leq i \leq N. \quad (4.65)$$

And, recursively we obtain

$$\begin{pmatrix} T_{N+1,12}(n) \\ T_{N+1,21}(n+1) \end{pmatrix} = \mathcal{L}^N \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.66)$$

This procedure allows to construct the whole matrix $T(n; z)$, starting from the initial condition $T_1(n) = \frac{\theta_N}{2}\sigma_3$ and iterating the operator \mathcal{L} we obtain off diagonal terms of $T(n; z)$ and compute diagonal one with equation (4.63). Below we implemented this method to find the matrix $T(n; z)$ in the first few cases $N = 1, 2$.

Example 4.27. In the case $N = 1$, the matrix $T(n; z) = T_1(n) + T_2(n)z^{-1} + T_3(n)z^{-2}$. Knowing $T_1(n)$, we only have to find $T_2(n)$ using the recurrence relation given from the compatibility i.e. equations (4.59), (4.60), (4.61) for $i = 1$. Since: $T_{1,12}(n) = T_{1,21}(n) = 0$, and $T_{1,11}(n) = \theta_N/2 = -T_{1,22}(n)$ we have

$$T_{2,11}(n) = n,$$

$$T_{2,12}(n) = -x_{n+1}(T_{1,11}(n+1) + T_{1,11}(n)) = -\theta_1 x_{n+1},$$

$$T_{2,21}(n+1) = x_n v_{n+1}(T_{1,22}(n+1) + T_{1,22}(n)) = -\theta_1 x_n v_{n+1},$$

and $T_{2,22}(n) = n - T_{2,11}(n) = 0$. Moreover the symmetry which reflects terms of $T(n; z)$ two by two gives $T_3(n) = -K(n)T_1(n)K(n)$. Thus the Lax matrix for $N = 1$ is

$$T(n; z) = \frac{\theta_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} n & -\theta_1 x_{n+1} \\ -\theta_1 v_n x_{n-1} & 0 \end{pmatrix} + \frac{\theta_1}{z^2} \begin{pmatrix} \frac{1}{2} - x_n^2 & x_n \\ v_n x_n & x_n^2 - \frac{1}{2} \end{pmatrix}.$$

Example 4.28. In the case $N = 2$, the matrix $T(n; z) = T_1(n)z + T_2(n) + T_3(n)z^{-1} + T_4(n)z^{-2} + T_5(n)z^{-3}$. This time we have to find $T_2(n)$ (that will be almost the same as before) and also $T_3(n)$ using the recurrence relation given from the compatibility i.e. equations (4.59), (4.60), (4.61) for $i = 1$ and 2. First we find $T_2(n)$ ($i = 1$ above), we have

$$T_{2,11}(n) = \frac{\theta_1}{2},$$

$$T_{2,12}(n) = -x_{n+1}(T_{1,11}(n+1) + T_{1,11}(n)) = -\theta_2 x_{n+1},$$

$$T_{2,21}(n+1) = x_n v_{n+1}(T_{1,22}(n+1) + T_{1,22}(n)) = -\theta_2 x_n v_{n+1},$$

and $T_{2,22}(n) = -T_{2,11} = -\frac{\theta_1}{2}$. Then we consider the equation for $i = 2$ and we find $T_3(n)$. We have

$$\Delta T_{3,11}(n) = x_n v_{n+1}(-\theta_2 x_{n+2}) - x_{n+1}(-\theta_2 x_{n-1} v_n) + 1 \implies T_{3,11}(n) = n - \theta_2 x_{n-1} x_{n+1} v_n,$$

$$T_{3,12}(n) = -\theta_1 x_{n+1} - \theta_2 (v_{n+1} x_{n+2} - x_n x_{n+1}^2),$$

$$T_{3,21}(n+1) = (-\theta_1 x_n - \theta_2 (v_n x_{n-1} - x_n^2 x_{n+1})) v_{n+1},$$

and $T_{3,22}(n) = n - T_{3,11}(n) = \theta_2 x_{n-1} x_{n+1} v_n$. Finally, we take $T_4(n) = -K(n)T_2(n)K(n)$ and $T_5(n) = -K(n)T_1(n)K(n)$. Thus the Lax matrix for $N = 2$ is

$$\begin{aligned} T(n; z) = & z \frac{\theta_2}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{\theta_1}{2} & -\theta_2 x_{n+1} \\ -\theta_2 x_{n-1} v_n & -\frac{\theta_1}{2} \end{pmatrix} + \\ & \frac{1}{z} \begin{pmatrix} n - \theta_2 x_{n-1} x_{n+1} v_n & -\theta_1 x_{n+1} - \theta_2 (v_{n+1} x_{n+2} - x_n x_{n+1}^2) \\ (-\theta_1 x_{n-1} - \theta_2 (v_{n-1} x_{n-2} - x_n x_{n-1}^2)) v_n & \theta_2 x_{n-1} x_{n+1} v_n \end{pmatrix} \\ & + \frac{1}{z^2} \begin{pmatrix} -\theta_2 v_n (x_n x_{n-1} + x_n x_{n+1}) + \frac{\theta_1}{2} (v_n - x_n^2) & -\theta_2 (v_n x_{n-1} + x_n^2 x_{n+1}) \\ -\theta_2 (v_n x_{n+1} + x_n^2 x_{n-1}) v_n & \theta_2 v_n (x_n x_{n-1} + x_n x_{n+1}) - \frac{\theta_1}{2} (v_n - x_n^2) \end{pmatrix} \\ & + \frac{\theta_2}{z^3} \begin{pmatrix} \frac{1}{2} - x_n^2 & x_n \\ v_n x_n & x_n^2 - \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Now that we have reconstructed the whole matrix $T(n; z)$ in terms of $x_{n\pm j}$, $j = -N, \dots, N$ we are left with the equation that $T_{N+1}(n)$ has to satisfy, namely (4.45). We now show that actually this coincide with only one scalar equation in $T_{N+1,12}$ and $T_{N+1,21}$. Indeed, entry by entry it reads as the following system of four equations. From

the off-diagonal entries

$$\begin{aligned} v_n T_{N+1,12}(n) &= x_n (T_{N+1,11}(n) - T_{N+1,22}(n)) - T_{N+1,21}(n), \\ v_n T_{N+1,21}(n) &= x_n v_n (T_{N+1,11}(n) - T_{N+1,22}(n)) - v_n^2 T_{N+1,12}(n). \end{aligned} \quad (4.67)$$

and from the diagonal entries

$$\begin{aligned} n - (1 + x_n^2) T_{N+1,11}(n) - v_n T_{N+1,22}(n) + x_n T_{N+1,21}(n) + x_n v_n T_{N+1,12}(n) &= 0, \\ n - (1 + x_n^2) T_{N+1,22}(n) - v_n T_{N+1,11}(n) - x_n T_{N+1,21}(n) - x_n v_n T_{N+1,12}(n) &= 0. \end{aligned}$$

We notice first that the four above equations are all the same. The first and the second equations are the same up to a multiplication by v_n . Using the relation $T_{N+1,11}(n) + T_{N+1,22}(n) = n$ we can rewrite the third and the fourth equations and obtain the same equation up to a sign. Finally multiplying by x_n the first equation and using the relation $T_{N+1,11}(n) + T_{N+1,22}(n) = n$ we obtain the third one. Thus from now on we will refer only to (4.67), as for the remaining equation.

Using equation (4.62) and $\text{Tr}(T(n; z)) = nz^{-1}$, we express equation (4.67) in function of $T_{N+1,12}(n)$ and $T_{N+1,21}(n)$. Consider equation (4.67), with the identity $\text{Tr}(T_{N+1}(n)) = n$, it is rewritten as

$$v_n T_{N+1,12}(n) = x_n (n - 2T_{N+1,22}(n)) - T_{N+1,21}(n).$$

Equation (4.62) holds also for $i = N + 1$. It means it is possible to replace $T_{N+1,22}(n)$ in the previous equation and obtain

$$nx_n - v_n T_{N+1,12}(n) - T_{N+1,21}(n) - 2x_n \Delta^{-1} \left(-x_n T_{N+1,12}(n) + \frac{x_{n+1}}{v_{n+1}} (\Delta + I) T_{N+1,21}(n) \right) = 0. \quad (4.68)$$

4.3.3 The relation between $T_{i,12}(n)$ and $T_{i,21}(n)$

The previous equation (4.68) depends on $T_{N+1,12}(n)$ and $T_{N+1,21}(n)$. The aim of this part is to establish a connection between $T_{i,12}(n)$ and $T_{i,21}(n)$ to rewrite equation (4.68) just in function of $T_{N+1,12}(n)$.

To accomplish this, we study the compatibility condition of $C(n; z) := T(n; z)^2$ and $U(n; z)$.

$C(n; z)$ is rational in z with a pole of order $-2N - 2$ at 0. We write $C(n; z)$ as:

$$C(n; z) = \sum_{i=1}^{4N+1} C_i(n) z^{2N-1-i} \quad (4.69)$$

with

$$C_i(n) := \sum_{j=1}^i T_j(n) T_{i+1-j}(n) \quad (4.70)$$

where $C_1(n) = \frac{\theta_N^2}{4} I_2$.

In what follows we will need this lemma:

Lemma 4.29. *Diagonal coefficients of $C_i(n)$ defined as in (4.70) satisfy the following equation:*

$$\begin{aligned} \forall 1 \leq i \leq N, C_{i,11}(n) &= C_{i,22}(n), \\ C_{N+1,11}(n) &= n\theta_N + C_{N+1,22}(n). \end{aligned}$$

Proof. We express $C_{i,11}(n)$ in function of $T_{i,kj}(n)$. With the equation (4.70)

$$C_{i,11}(n) = \sum_{j=1}^i T_{j,11}(n) T_{i+1-j,11}(n) + T_{j,12}(n) T_{i+1-j,21}(n).$$

Then, the sum index change $j = i - k + 1$ leads to

$$C_{i,11}(n) = \sum_{k=1}^i T_{i-k+1,11}(n) T_{k,11}(n) + T_{i-k+1,12}(n) T_{k,21}(n).$$

Finally with the relation $\text{Tr}(T(n; z)) = nz^{-1}$,

— if $1 \leq i \leq N$,

$$C_{i,11}(n) = \sum_{k=1}^i T_{i-k+1,22}(n) T_{k,22}(n) + T_{k,21}(n) T_{i-k+1,12}(n) = C_{i,22}(n).$$

— if $i = N + 1$,

$$C_{N+1,11}(n) = -2nT_{1,22}(n) + \sum_{k=1}^{N+1} T_{N-k+2,22}(n) T_{k,22}(n) + T_{k,21}(n) T_{N-k+2,12}(n) = n\theta_N + C_{N+1,22}(n).$$

□

We deduce the compatibility condition for C and U from the one for T and U .

Lemma 4.30. $C(n; z)$ (4.69) and $U(n; z)$ (4.40) satisfy the following compatibility condition:

$$C(n+1; z)U(n; z) - U(n; z)C(n; z) = T(n+1; z)\sigma_+ + \sigma_+T(n; z). \quad (4.71)$$

Proof. Multiplying on the left (respectively on the right) equation (4.49) by $T(n+1; z)$ (respectively $T(n; z)$) and summing these two equations leads to the result. \square

The left (respectively right) hand side of the equation in the previous lemma is an expression in powers of z from z^{2N-1} to z^{-2N-2} (respectively from z^{N-1} to z^{-N-1}). This equation leads to recursive equation for $C_i(n)$. We consider only expression in powers of z from z^{2N-1} to z^{N-1} .

According to (4.49) and (4.71), $\forall 1 \leq i \leq N$, $C_i(n)$ and $T_i(n)$ satisfy the same recursive equation (see equations (4.59),..., (4.62)). For $i = N+1$, the equation is a bit different. The term with $\delta_{i,N}$ is now multiplied by θ_N .

From these equations we deduce the following result.

Proposition 4.31. Let $C_i(n)$ be as in (4.70). Then $\forall 1 \leq i \leq N$, $C_i(n) = \alpha_i I_2$ and $C_{N+1}(n) = \theta_N n \sigma_+ + \alpha_{N+1} I_2$.

Proof. We prove Proposition 4.31 by induction.

For $i = 1$, we already know $C_1(n) = \frac{\theta_N^2}{4}$.

Suppose $C_i(n) = \alpha_i I_2$ for $i \leq N-1$.

$C_{i+1}(n)$ satisfies the following equations:

$$\begin{aligned} \Delta C_{i+1,11}(n) &= -x_n x_{n+1} \Delta C_{i,11}(n) + x_n v_{n+1} C_{i,12}(n+1) - x_{n+1} C_{i,21}(n) + \theta_N \delta_{i,N}, \\ C_{i+1,12}(n) &= -x_{n+1} C_{i,11}(n+1) + v_{n+1} C_{i,12}(n+1) - x_n x_{n+1} C_{i,12}(n) + x_{n+1} C_{i,22}(n), \\ C_{i+1,21}(n+1) &= -x_n x_{n+1} C_{i,21}(n+1) + x_n v_{n+1} C_{i,22}(n+1) - x_n v_{n+1} C_{i,11}(n) + v_{n+1} C_{i,21}(n). \end{aligned}$$

Using induction hypothesis,

$$\begin{aligned} \Delta C_{i+1,11}(n) &= -0 \cdot x_n x_{n+1} + 0 \cdot x_n v_{n+1} - 0 \cdot x_{n+1} + \theta_N \delta_{i,N} = \theta_N \delta_{i,N}, \\ C_{i+1,12}(n) &= -x_{n+1} \alpha_i + 0 \cdot v_{n+1} - 0 \cdot x_n x_{n+1} + x_{n+1} \alpha_i = 0, \\ C_{i+1,21}(n+1) &= -0 \cdot x_n x_{n+1} + x_n v_{n+1} \alpha_i - x_n v_{n+1} \alpha_i + 0 \cdot v_{n+1} = 0. \end{aligned}$$

From the first equation we conclude $C_{i+1,11}(n) = \alpha_{i+1}$ if $i \leq N-1$ (respectively $C_{N+1,11}(n) = \theta_N n + \alpha_{N+1}$ if $i = N$) and according to Lemma 4.29 $C_{i+1,22}(n) = \alpha_{i+1}$ (respectively $C_{N+1,22}(n) = \alpha_{N+1}$) which concludes the proof. \square

From equation (4.70) and Proposition 4.31, we obtain

$$\theta_N T_{i,11}(n) = \alpha_i - \sum_{j=2}^{i-1} T_{j,11}(n) T_{i-j+1,11}(n) + T_{j,12}(n) T_{i-j+1,21}(n), \quad (4.72)$$

$$\theta_N T_{N+1,11}(n) = n\theta_N + \alpha_{N+1} - \sum_{j=2}^N T_{j,11}(n) T_{N-j+2,11}(n) + T_{j,12}(n) T_{N-j+2,21}(n). \quad (4.73)$$

With all this discussion on $C(n; z)$ it is now possible to prove the following proposition.

Proposition 4.32. *The following holds: $\forall 1 \leq i \leq N+1$, $T_{i,11}(n)$, $T_{i,12}(n)$ and $T_{i,21}(n)$ are polynomials in x_{n+j} 's. Moreover the following symmetries hold:*

$\exists (Q_{i,n}((u_{n+j})_{1-i \leq j \leq i-1}), P_{i,n}((u_{n+j})_{1-i \leq j \leq i-1}))$ polynomials in u_{n+j} 's such that,

$$T_{i,11}(n) = Q_{i,n}((x_{n+j})_{1-i \leq j \leq i-1}) = Q_{i,n}((x_{n-j})_{1-i \leq j \leq i-1}),$$

$$T_{i,12}(n) = P_{i,n}((x_{n+j})_{1-i \leq j \leq i-1})$$

and

$$T_{i,21}(n) = v_n P_{i,n}((x_{n-j})_{1-i \leq j \leq i-1}).$$

Proof. We prove this proposition by strong induction.

For $i = 1$, $T_1(n) = \frac{\theta_N}{2} \sigma_3$, then defining $Q_{1,n}(u_n) := \frac{\theta_N}{2}$, $P_{1,n}(u_n) := 0$; $T_{1,11}(n) = Q_{1,n}(x_n)$, $T_{1,12}(n) = P_{1,n}(x_n)$ and $T_{1,21}(n) = v_n P_{1,n}(x_n)$.

Now suppose the property true for all $j \in [[1, i]]$ with $i \leq N$ and let $(Q_{j,n}, P_{j,n})_{j \leq i}$ be polynomials in x_{n+j} 's satisfying the property.

According to (4.72) (and (4.73) for $i = N$) and strong induction hypothesis, $T_{i+1}(n)$ is a polynomial in x_{n+j} 's and the invariance when you exchange x_{n+j} by x_{n-j} holds.

Because of equation (4.60) (respectively equation (4.61)) and of induction hypothesis, there exists

$P_{i+1,n}((u_{n+j})_{-i \leq j \leq i})$ (respectively $\tilde{P}_{i+1,n}((u_{n+j})_{-i \leq j \leq i})$) a polynomial such that

$$T_{i+1,12}(n) = P_{i+1,n}((x_{n+j})_{-i \leq j \leq i}),$$

respectively

$$T_{i+1,21}(n) = \tilde{P}_{i+1,n}((x_{n+j})_{-i \leq j \leq i}).$$

Now we establish the link between $P_{i+1,n}$ and $\tilde{P}_{i+1,n}$. According to equation (4.60) and the relation $\text{Tr}(T(n; z)) = nz^{-1}$:

$$\begin{aligned} P_{i+1,n}((x_{n+j})_{j=-i}^i) &= -x_{n+1}Q_{i,n+1}((x_{n+j})_{j=-i}^{i-2}) + v_{n+1}P_{i,n+1}((x_{n+j})_{j=-i}^{i-2}) - x_n x_{n+1}P_{i,n}((x_{n+j})_{j=1-i}^{i-1}) \\ &\quad - x_{n+1}Q_{i,n}((x_{n+j})_{j=1-i}^{i-1}). \end{aligned}$$

Then

$$\begin{aligned} v_n P_{i+1,n}((x_{n-j})_{j=-i}^i) &= v_n (-x_{n-1}Q_{i,n-1}((x_{n-j})_{j=-i}^{i-2}) + v_{n-1}P_{i,n-1}((x_{n-j})_{j=-i}^{i-2}) \\ &\quad - x_n x_{n-1}P_{i,n}((x_{n-j})_{j=1-i}^{i-1}) - x_{n-1}Q_{i,n}((x_{n-j})_{j=1-i}^{i-1})). \end{aligned}$$

From induction hypothesis and $\text{Tr}(T(n; z)) = nz^{-1}$:

$$v_n P_{i+1,n}((x_{n-j})_{j=-i}^i) = -x_{n-1}v_n T_{i,11}(n-1) + v_n T_{i,21}(n-1) + x_{n-1}x_n T_{i,21}(n) + x_{n-1}v_n T_{i,22}(n).$$

According to equation (4.61),

$$v_n P_{i+1,n}((x_{n-j})_{j=-i}^i) = T_{i+1,21}(n+1).$$

Then

$$v_n P_{i+1,n}((x_{n-j})_{j=-i}^i) = \tilde{P}_{i+1,n}((x_{n+j})_{-i \leq j \leq i})$$

and this concludes the proof. \square

Define $\mathbb{C}[(x_j)_{j \in [[0, 2n]]}]$ and the transformation

$$\begin{aligned} \text{Perm}_n : \quad \mathbb{C}[(x_j)_{j \in [[0, 2n]]}] &\longrightarrow \mathbb{C}[(x_j)_{j \in [[0, 2n]]}] \\ P((x_{n+j})_{-n \leq j \leq n}) &\longmapsto P((x_{n-j})_{-n \leq j \leq n}) \end{aligned}$$

. From the previous proposition

$$T_{i,21}(n) = v_n \text{Perm}_n(T_{i,12}(n)). \quad (4.74)$$

Remark 4.33. As a consequence of the Proposition 4.32 the equation (4.67) is a poly-

mial in x_{n+j} 's and is invariant when you apply $Perm_n$ to this equation because $Perm_n^2 = Id$ and

$$Perm_n v_n = v_n Perm_n.$$

We use the link we established in Proposition 4.32 between $T_{i,12}(n)$ and $T_{i,21}(n)$ to rewrite the operator \mathcal{L} (4.64) as a scalar operator:

$$L(u_n) := \left(x_{n+1} \left(2\Delta^{-1} + I \right) \left((\Delta + I) x_n Perm_n - x_n \right) + v_{n+1} (\Delta + I) - x_n x_{n+1} \right) u_n. \quad (4.75)$$

Finally, collecting all the results from the previous sections, we state and proof the following theorem.

Theorem 4.34. *The system (4.39), with $T(n; z)$ of the form (4.50) and coefficient $T_{N+1}(n)$ satisfying the symmetry condition (4.45), is a Lax pair for the N -th higher order discrete Painlevé II equation and the equation is given by the expression:*

$$n x_n + \left(2x_n \Delta^{-1} (x_n - (\Delta + I) x_n Perm_n) - v_n - v_n Perm_n \right) T_{N+1,12}(n) = 0, \quad (4.76)$$

where $T_{N+1,12}(n) = L^N(0)$ with L as in (4.75).

Proof. Replacing $T_{N+1,21}(n)$ with the relation (4.74), equation (4.68) now reads as

$$n x_n + \left(2x_n \Delta^{-1} (x_n - (\Delta + I) x_n Perm_n) - v_n - v_n Perm_n \right) T_{N+1,12}(n) = 0.$$

Equations (4.65) and (4.66) with the relation (4.74) reduce to

$$T_{i+1,12}(n) = L(T_{i,12}(n)) \quad \text{and} \quad T_{N+1,12}(n) = L^N(0),$$

which concludes the proof. □

The next two examples explain for $N = 1, 2$ how to compute explicitly equation (4.76).

Example 4.35. Using the expression defined in Theorem 4.34, we compute the first equation (4.13) and the second (4.14).

For $N = 1$: First we compute $T_{2,12}(n)$ with the operator L (4.75).

$$T_{2,12}(n) = 2x_{n+1} \Delta^{-1}(0) = -\theta_1 x_{n+1}$$

where $-\theta_1/2$ is the integration constant.

Replacing $T_{2,12}(n)$ in equation (4.76),

$$nx_n + v_n\theta_1(x_{n+1} + x_{n-1}) + 2x_n\Delta^{-1}(\theta_1x_nx_{n+1} - \theta_1x_nx_{n-1}) = 0.$$

Then

$$(n + \alpha)x_n + \theta_1v_n(x_{n+1} + x_{n-1}) = 0.$$

This equation is the same as equation (4.13) if we choose the integration constant α to be zero.

For $N = 2$: We compute $T_{3,12}(n)$. Computations are the same for $T_{2,12}(n)$ except for the integration constant, $T_{2,12}(n) = -\theta_2x_{n+1}$.

$$T_{3,12}(n) = L(T_{2,12}(n)) = (x_nx_{n+1}^2 - v_{n+1}x_{n+2})\theta_2 + x_{n+1}(2\Delta^{-1} + I)(-\theta_2x_nx_{n+1} + \theta_2x_nx_{n+1})$$

$$\text{Then } T_{3,12}(n) = \theta_2(x_nx_{n+1}^2 - v_{n+1}x_{n+2}) - \theta_1x_{n+1}.$$

Replacing $T_{3,12}(n)$ in equation (4.76),

$$(n + \alpha)x_n + \theta_2v_n(v_{n+1}x_{n+2} + v_{n-1}x_{n-2} - x_n(x_{n+1} + x_{n-1})^2) + \theta_1v_n(x_{n+1} + x_{n-1}) = 0$$

which is the same equation as (4.14).

We finally conclude the work by noticing that Theorem 4.34 together with Corollary 4.14 give the proof of Theorem 4.2.

Remark 4.36. *In our setting, the fixed $N \geq 1$ define the order $(2N)$ of the discrete equation solved by x_n , the quantity related to the Toeplitz determinants D_n . An alternative approach could be to leave N variate and consider it as a second discrete variable for x_n . In effect, this is done in [HW96], where the authors consider orthogonal polynomials on the real line, w.r.t. a weight $\rho(\lambda; N)d\lambda$ and where the dependence on an integer parameter N is such that $\rho(\lambda; N + 1) = \lambda\rho(\lambda; N)$. In this case the relevant quantities to consider (related to the Hankel determinants) are the coefficients of the three terms recurrence relation satisfied by these polynomials. The authors there proved that these quantities solve (up to some change of variables) the discrete-time Toda molecule equation, a coupled system of discrete equations in the two variables n, N . The result deeply relies on the quasi-periodic condition satisfied by the weight ρ . Back to our setting, the measure we have for*

our orthogonal polynomials on the unit circle is such that

$$d\mu(\lambda; N + 1) = e^{\sum_{j=1}^{N+1} \frac{\theta_j}{j} (e^{i\lambda_j} + e^{-i\lambda_j})} \frac{d\lambda}{2\pi} = e^{\frac{\theta_{N+1}}{N+1} (e^{i\lambda(N+1)} + e^{-i\lambda(N+1)})} d\mu(\lambda; N).$$

This relation does not seem as promising as the one for ρ for the study of the N -dependence, but it is another point that we could further investigate.

4.4 The continuous limit

This appendix contains further computations for the continuous limit of the equations of the discrete Painlevé II hierarchy (4.10) in the first cases $N = 1, 2, 3$. To obtain it, we follow the scaling limit given in Theorem 1 of [BBW21] as already recalled in the Introduction.

The case $N = 1$. Notice that in this case we recover the same computation done in [Bor03], Chapter 9. We consider equation (4.13) written as

$$x_{n+1} + x_{n-1} + \frac{nx_n}{\theta_1(1 - x_n^2)} = 0$$

in which the only parameter appearing is $\theta_1 = \theta$. Following the scaling limit of Theorem 1 [BBW21], in the case $N = 1$, we have

$$b = 2, \quad d = 1 \quad \text{and} \quad x_n = (-1)^n \theta^{-\frac{1}{3}} u(t) \quad \text{with} \quad t = (n - 2\theta) \theta^{-\frac{1}{3}}.$$

Now, for $\theta \rightarrow +\infty$ we compute

$$\begin{aligned} x_{n\pm 1} &\sim (-1)^{n+1} \theta^{-\frac{1}{3}} u(t \pm \theta^{-\frac{1}{3}}) \\ &\sim (-1)^{n+1} \theta^{-\frac{1}{3}} \left(u(t) \pm \theta^{-\frac{1}{3}} u'(t) + \frac{\theta^{-\frac{2}{3}}}{2} u''(t) + O(\theta^{-1}) \right), \end{aligned}$$

that gives

$$x_{n+1} + x_{n-1} \sim (-1)^{n+1} 2\theta^{-\frac{1}{3}} u(t) + (-1)^{n+1} \theta^{-1} u''(t) + O(\theta^{-1}).$$

The other term appearing in the discrete Painlevé II equation gives instead

$$\begin{aligned} \frac{nx_n}{\theta_1(1-x_n^2)} &\sim (2\theta + t\theta^{\frac{1}{3}})(-1)^n\theta^{-\frac{1}{3}}u(t)\theta^{-1} \left(1 + \theta^{-\frac{2}{3}}u^2(t) + O(\theta^{-1})\right) \\ &\sim (-1)^n2\theta^{-\frac{1}{3}}u(t) + (-1)^n\theta^{-1} \left(tu(t) + 2u^3(t)\right) + O(\theta^{-1}). \end{aligned}$$

Thus equation (4.8) in this scaling limit gives at the first order (coefficient of θ^{-1}) the second order differential equation

$$u''(t) - tu(t) - 2u^3(t) = 0,$$

which coincides indeed with the Painlevé II equation.

The case $N = 2$. We consider equation (4.14), with the parameters θ_1, θ_2 rescaled as $\theta_1 = \theta, \theta_2 = \frac{\theta}{4}$. It reads as

$$\frac{nx_n}{(1-x_n^2)} + \theta(x_{n+1} + x_{n-1}) + \frac{\theta}{4} \left(x_{n+2}(1-x_{n+1}^2) + x_{n-2}(1-x_{n-1}^2) - x_n(x_{n+1} + x_{n-1})^2\right) = 0 \quad (4.77)$$

and this time we consider the following scaling limit (case $N = 2$ of Theorem 1 in [BBW21])

$$b = \frac{3}{2}, \quad d = 4 \quad \text{and} \quad x_n = (-1)^n\theta^{-\frac{1}{5}}4^{\frac{1}{5}}u(t), \quad \text{with} \quad t = \left(n - \frac{3}{2}\theta\right)\theta^{-\frac{1}{5}}4^{\frac{1}{5}}.$$

For $\theta \rightarrow +\infty$, similar computations gives the fourth order differential equation

$$tu(t) + 6u(t)^5 - 10u(t)u'(t)^2 - 10u(t)^2u''(t) + u'''(t) = 0$$

which corresponds to the second equation of the Painlevé II hierarchy. Detailed computations to obtain certain terms from the previous equation are given below. We begin with the expansion of the first term in equation (4.77):

$$\begin{aligned} \frac{nx_n}{(1-x_n^2)} &\sim \left(\frac{3}{2}\theta + 4^{-\frac{1}{5}}\theta^{\frac{1}{5}}t\right) (-1)^n\theta^{-\frac{1}{5}}4^{\frac{1}{5}}u(t) \left(1 + 4^{\frac{2}{5}}\theta^{-\frac{2}{5}}u^2(t) + 4^{\frac{4}{5}}\theta^{-\frac{4}{5}}u^4(t) + O(\theta^{-1})\right) \\ &\sim (-1)^n \left(\frac{3}{2}4^{\frac{1}{5}}\theta^{\frac{4}{5}}u(t) + \frac{3}{2}4^{\frac{3}{5}}\theta^{\frac{3}{5}}u(t)^3 + tu(t) + 6u(t)^5 + O\left(\theta^{-\frac{1}{5}}\right)\right). \end{aligned}$$

Computing expansions of $x_{n\pm 1}, x_{n\pm 2}$ as $\theta \rightarrow \infty$, we obtain:

$$\begin{aligned}
 x_{n\pm 1} &\sim (-1)^{n+1} 4^{\frac{1}{5}} \theta^{-\frac{1}{5}} u(t \pm 4^{\frac{1}{5}} \theta^{-\frac{1}{5}}) \\
 &\sim (-1)^{n+1} 4^{\frac{1}{5}} \theta^{-\frac{1}{5}} \left(u(t) \pm 4^{\frac{1}{5}} \theta^{-\frac{1}{5}} u'(t) + \frac{4^{\frac{2}{5}} \theta^{-\frac{2}{5}}}{2} u''(t) \pm \frac{4^{\frac{3}{5}} \theta^{-\frac{3}{5}}}{6} u'''(t) + \frac{4^{\frac{4}{5}} \theta^{-\frac{4}{5}}}{24} u''''(t) + O(\theta^{-1}) \right) \\
 x_{n\pm 2} &\sim (-1)^n 4^{\frac{1}{5}} \theta^{-\frac{1}{5}} u(t \pm 2\theta^{-\frac{1}{5}} 4^{\frac{1}{5}}) \\
 &\sim (-1)^n 4^{\frac{1}{5}} \theta^{-\frac{1}{5}} \left(u(t) \pm 4^{\frac{1}{5}} 2\theta^{-\frac{1}{5}} u'(t) + 4^{\frac{7}{5}} \theta^{-\frac{2}{5}} u''(t) \pm \frac{4^{\frac{8}{5}} 2\theta^{-\frac{3}{5}}}{3} u'''(t) + \frac{4^{\frac{9}{5}} \theta^{-\frac{4}{5}}}{3} u''''(t) + O(\theta^{-1}) \right)
 \end{aligned}$$

that gives for the second term of equation (4.77)

$$\theta(x_{n+1} + x_{n-1}) \sim (-1)^{n+1} \left(4^{\frac{1}{5}} 2\theta^{\frac{4}{5}} u(t) + 4^{\frac{3}{5}} \theta^{\frac{2}{5}} u''(t) + \frac{1}{3} u''''(t) + O(\theta^{-\frac{1}{5}}) \right).$$

Some linear and non-linear terms appear with the expansion of the third term of equation (4.77). The linear one is:

$$\frac{\theta}{4}(x_{n+2} + x_{n-2}) \sim (-1)^n \left(4^{\frac{1}{5}} \theta^{\frac{4}{5}} \frac{1}{2} u(t) + 4^{\frac{3}{5}} \theta^{\frac{2}{5}} u''(t) + \frac{4}{3} u''''(t) + O(\theta^{-\frac{1}{5}}) \right).$$

Non-linear ones are:

$$\begin{aligned}
 \frac{\theta}{4} x_n (x_{n+1} + x_{n-1})^2 &\sim (-1)^n u(t) \left(4^{\frac{3}{5}} \theta^{\frac{2}{5}} u(t)^2 + 4u(t)u''(t) + O(\theta^{-\frac{1}{5}}) \right) \\
 \frac{\theta}{4} x_{n\pm 2} x_{n\pm 1}^2 &\sim (-1)^n \left(4^{-\frac{2}{5}} \theta^{\frac{2}{5}} u(t)^3 \pm 4^{\frac{4}{5}} \theta^{\frac{1}{5}} u(t)^2 u'(t) + 3u(t)^2 u''(t) + 5u(t)u'(t)^2 \right).
 \end{aligned}$$

From these computations, we see that we recover exactly

$$tu(t) + 6u(t)^5 - 10u(t)u'(t)^2 - 10u(t)^2 u''(t) + u''''(t) = 0.$$

The case $N = 3$. We consider equation (4.15) with the parameters $\theta_1, \theta_2, \theta_3$ rescaled as $\theta_1 = \theta, \theta_2 = \frac{2\theta}{5}, \theta_3 = \frac{\theta}{15}$ and rewritten as

$$\begin{aligned} & \frac{nx_n}{\theta(1-x_n^2)} + (x_{n+1} + x_{n-1}) + \frac{2}{5} \left(x_{n+2}(1-x_{n+1}^2) + x_{n-2}(1-x_{n-1}^2) - x_n(x_{n+1} + x_{n-1})^2 \right) \\ & + \frac{1}{15} \left(x_n^2(x_{n+1} + x_{n-1})^3 + x_{n+3}(1-x_{n+2}^2)(1-x_{n+1}^2) + x_{n-3}(1-x_{n-2}^2)(1-x_{n-1}^2) \right) \\ & + \frac{1}{15} \left(-2x_n(x_{n+1} + x_{n-1})(x_{n+2}(1-x_{n+1}^2) + x_{n-2}(1-x_{n-1}^2)) - x_{n-1}x_{n-2}^2(1-x_{n-1}^2) \right) \\ & + \frac{1}{15} \left(-x_{n+1}x_{n+2}^2(1-x_{n+1}^2) - x_{n+1}x_{n-1}(x_{n+1} + x_{n-1}) \right) = 0. \end{aligned}$$

Finally, we consider the following scaling limit (case $N = 3$ of Theorem 1 in [BBW21])

$$b = \frac{4}{3}, \quad d = 15 \quad \text{and} \quad x_n = (-1)^n \theta^{-\frac{1}{7}} 15^{\frac{1}{7}} u(t) \quad \text{with} \quad t = \left(n - \frac{4}{3}\theta \right) \theta^{-\frac{1}{7}} 15^{\frac{1}{7}}.$$

Again, for $\theta \rightarrow +\infty$ the asymptotic expansion of the equation above results at the first order (coefficient of θ^{-1}) into the sixth order differential equation

$$\begin{aligned} & tu(t) + 20u(t)^7 - 140u(t)^3u'(t)^2 - 70u(t)^4u''(t) + 70u'(t)^2u''(t) + 42u(t)u''(t)^2 + 56u(t)u'(t)u'''(t) \\ & + 14u(t)^4u''(t) - u''''''(t) = 0 \end{aligned}$$

which corresponds to the third equation in the Painlevé II hierarchy.

Remark 4.37. Computations for $N = 2$ and $N = 3$ were performed with Maple/Mathematica. Files are available on demand.

THE GENERATING FUNCTION FOR THE PEARCEY PROCESS

This chapter is inspired by my work “A Riemann-Hilbert approach to the study of the generating functions associated to the Pearcey process” [Cho23] published in *Mathematical Physics, Analysis and Geometry*.

5.1 Presentation of the Pearcey process and statement of the results

The Pearcey process is a universal determinantal point process associated with the Pearcey kernel (see (5.12) for the definition of the kernel). It first appeared in the spectral analysis of random matrices with external sources [BH98b]. If $2N$ is the size of the matrices, the Pearcey process describes the behavior, when $N \rightarrow \infty$, of eigenvalues near a point where the density of states admits a cusp-like gap. Another model linked with the Pearcey process is the one of $2N$ 1-dimension non-intersecting Brownian motions [BK06] starting from 0 at $t = 0$ and ending at $\pm a$ (half particles at a and the others at $-a$) at time $t = 1$. For this model, in the large N -limit, there exists a time t_c such that, for $t < t_c$, the distribution of particles is supported in an interval, and for $t > t_c$, it splits in two disjoint intervals. The distribution of particles of the Brownian motion for t close to t_c is described by the Pearcey process. Another example that reveals the universality of the Pearcey process is the one of random skew plane partition [OR05]. Studying the limit shape associated with this model leads to different processes: Beta process, extended Airy process, and extended Pearcey process.

As known for the Airy process [TW94b, AvM05], it is possible to express probabilities associated with the Pearcey process in terms of some partial differential equations by studying the Fredholm determinant of the Pearcey kernel operator (5.12). Studying the

Brownian motion model at several times and deriving the kernel associated to the Pearcey process, Tracy and Widom [TW06], obtained partial differential equations for the distribution associated to the Pearcey process. Using KP-tau functions and Hirota bilinear equations for the study of random matrices with external sources, Adler and van Moerbeke [AvM07] introduced a non-linear fourth order PDE for the Pearcey process. With Riemann-Hilbert methods Bertola and Cafasso [BC12] obtained the same PDE as Adler and van Moerbeke and introduced a new one for the gap probability of the Pearcey process.

The aim of this chapter is to present a Tracy-Widom formula for the generating function of the Pearcey process linked with two vector valued functions satisfying a system of coupled non-linear third order differential equations and of non-linear heat equation.

Let $N > 1$ be an integer, $\vec{\mathbf{a}} := (a_1, \dots, a_N)$ with $a_1 < \dots < a_N$ and $\vec{\mathbf{k}} = (k_0, k_1, \dots, k_{N-1}, k_N)$ such that $k_0 = k_N = 0$, $k_j \in [0, 1]$, $j = 1, \dots, N - 1$ and $k_j \neq k_{j+1}$, $j = 0, \dots, N - 1$.

Consider the Pearcey process associated to the kernel K_P (5.12) (which depends on an additional positive parameter τ) and define its generating function

$$F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) := \mathbb{E} \left(\prod_{j=1}^{N-1} (1 - k_j)^{\sharp(a_j, a_{j+1})} \right) \quad (5.1)$$

where $\sharp I$ is the random variable counting the number of points of the process in the interval I .

This generating function is well-defined for $(a_i)_{1 \leq i \leq N}$ finites.

Computing derivatives with respect to k_j 's of $F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}})$ allows us to express the joint probability of occupancy numbers of particles as follows:

$$\mathbb{P} \left(\bigcap_{j=1}^{N-1} \sharp(a_j, a_{j+1}) = m_j \right) = \frac{(-1)^{m_1 + \dots + m_{N-1}}}{m_1! \dots m_{N-1}!} \frac{\partial^{m_1 + \dots + m_{N-1}}}{\partial k_1^{m_1} \dots \partial k_{N-1}^{m_{N-1}}} F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) \Big|_{\vec{\mathbf{k}}=(1, \dots, 1)} \quad (5.2)$$

(see for instance [Joh01] for a similar computation).

Remark 5.1. Since for all $B \subset \mathbb{R}$ bounded Borel set $\sharp B < \infty$ almost surely, $F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) > 0$.

Indeed, with Jensen's inequality,

$$F(\vec{\mathbf{a}}, \tau, \vec{\mathbf{k}}) \geq \exp \left(\sum_{j=1}^{N-1} \log(1 - k_j) \mathbb{E}(\#(a_j, a_{j+1})) \right)$$

and $\forall 1 \leq j \leq N - 1, \log(1 - k_j) \mathbb{E}(\#(a_j, a_{j+1})) > -\infty$.

Charlier and Moreillon studied the generating function of the Pearcey process [CM23] on intervals of the form $(-x_j, x_j)$. They considered a parameter r of dilatation of their intervals and obtained an expression for the generating function of the Pearcey process linked with a Hamiltonian. Recently, Kimura and Zahabi [KZ22] introduced higher order Pearcey kernels and obtained a Hamiltonian structure for the level spacing distribution by studying a different Lax Pair for the Pearcey process on intervals of the form $[-s, s]$.

In our work, we follow a different way. We consider the generating function (5.1) not necessarily on symmetric intervals, and we use a parameter s of translation of intervals instead of dilatation. This work is inspired by works of Claeys and Doeraene [CD18] and Charlier and Doeraene [CD19] where they studied the generating function for the Airy process and for the Bessel process. Recently Cafasso and Tarricone [CT23a] obtained a Tracy-Widom type formula for the generating function associated to the higher order Airy process.

The generating function associated to the Pearcey process satisfies the following:

Theorem 5.2. *Let $s \in \mathbb{R}$, $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ be as (5.1) where $\vec{\mathbf{a}} + s = (a_1 + s, \dots, a_N + s)$, then*

$$\frac{\partial^2}{\partial s^2} \log \left(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) \right) = p^T(s)q(s) \quad (5.3)$$

with $(p(s), q(s)) = (p(s, \tau, \vec{\mathbf{a}}, \vec{\mathbf{k}}), q(s, \tau, \vec{\mathbf{a}}, \vec{\mathbf{k}}))$ vectors of size N satisfying:

i- a coupled third order differential equation

$$\begin{cases} \partial_{sss} p^T + 3(\partial_s p^T) q p^T + 3p^T q (\partial_s p^T) - \tau \partial_s p^T + p^T D_{s, \vec{\mathbf{a}}} = 0 \\ \partial_{sss} q + 3(\partial_s q) p^T q + 3q p^T (\partial_s q) - \tau \partial_s q - D_{s, \vec{\mathbf{a}}} q = 0 \end{cases} \quad (5.4)$$

where $D_{s, \vec{\mathbf{a}}} := \text{Diag}(a_1 + s, \dots, a_N + s)$ and

ii- a coupled non-linear heat equation

$$\begin{cases} -\frac{1}{2} \partial_{ss} p^T - \partial_\tau p^T = p^T q p^T \\ -\frac{1}{2} \partial_{ss} q + \partial_\tau q = q p^T q \end{cases} \quad (5.5)$$

Moreover $p_i(s) \sim \sqrt{k_i - k_{i-1}}P(a_i + s)$ and $q_i(s) \sim \sqrt{k_i - k_{i-1}}Q(a_i + s)$ as $s \rightarrow \infty$ where

$$\begin{aligned} Q(s) &:= \frac{1}{2i\pi} \int_{i\mathbb{R}} e^{-\frac{\mu^4}{4} + \tau\frac{\mu^2}{2} + s\mu} d\mu & \text{and } Q'''(s) - \tau Q'(s) = sQ(s) \\ P(s) &:= \frac{1}{2i\pi} \int_{\Sigma} e^{\frac{\mu^4}{4} - \tau\frac{\mu^2}{2} - s\mu} d\mu & \text{and } P'''(s) - \tau P'(s) = -sP(s) \end{aligned} \quad (5.6)$$

Remark 5.3. Defining $u(s, \tau) := \log \left(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) \right)$, from equations (5.4) and (5.5) we obtain that u satisfies

$$\frac{\partial^2}{\partial \tau^2} u(s, \tau) + \frac{1}{2} \left(\frac{\partial^2}{\partial s^2} u(s, \tau) \right)^2 + \frac{1}{12} \frac{\partial^4}{\partial s^4} u(s, \tau) - \frac{1}{3} \tau \frac{\partial^2}{\partial s^2} u(s, \tau) = 0 \quad (5.7)$$

In fact, if we define $v(s, \tau) := \frac{\partial^2}{\partial s^2} u(s, \tau)$, according to the equation 5.3, $v(s, \tau) = p^T(s)q(s)$. Differentiating v with respect to τ and using equation (5.5) to express $\partial_\tau p^T$ and $\partial_\tau q$, we obtain:

$$\partial_\tau v = \frac{1}{2} \partial_s \left(p^T \partial_s q - \partial_s p^T q \right) \quad (5.8)$$

Differentiating a second time v with respect to τ (again using equation (5.5)) yields to

$$\partial_{\tau\tau} v = \frac{1}{2} \left(2v \partial_s v + \frac{1}{2} \left(p^T \partial_{sss} q + \partial_{sss} p^T q \right) - \frac{1}{2} \partial_s \left(\partial_s p^T \partial_s q \right) \right) \quad (5.9)$$

Recall $v(s, \tau) = p^T(s)q(s)$, then differentiating three times with respect to s the following equation holds:

$$\partial_s \left(\partial_s p^T \partial_s q \right) = \frac{1}{3} \left(\partial_{sss} v - \left(p^T \partial_{sss} q + \partial_{sss} p^T q \right) \right) \quad (5.10)$$

Replacing $\partial_s \left(\partial_s p^T \partial_s q \right)$ in equation (5.9) and using equation (5.4) we obtain:

$$\partial_{\tau\tau} v = \partial_s \left(-v \partial_s v - \frac{1}{12} \partial_{sss} v + \frac{1}{3} \tau \partial_s v \right). \quad (5.11)$$

Replacing v by $\partial_{ss} u$ and integrating twice with respect to s we prove u satisfies equation (5.7).

Equation (5.7) was already known for the gap probability for the Pearcey process (see for instance equation (1.10) in [ACvM12]).

The rest of this chapter is organized as follows: in Section 5.2 we establish a link between $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and a Fredholm determinant of an integrable operator in the sense of IKS [IKS90] and introduce the Riemann-Hilbert problem associated to this in-

tegrable operator. In Section 5.3 we present a Lax pair associated to the Riemann-Hilbert problem for the Pearcey kernel and obtain a coupled vector differential equation with respect to the variable s and a coupled non-linear heat equation with respect to s and τ for elements of the Riemann-Hilbert problem. Finally, in Section 5.4 we compute the logarithmic derivative of $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ with respect to s and prove Theorem 5.2 using the results of Sections 5.2 and 5.3.

5.2 From generating function of the Pearcey process to a Riemann-Hilbert Problem

In this section we establish a link between the generating function $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of an integrable operator and introduce the Riemann-Hilbert Problem associated to this integrable operator.

The Pearcey kernel operator

The Pearcey process is a determinantal point process on \mathbb{R} associated to the Pearcey kernel operator.

For $(x, \tau) \in \mathbb{R}^2$ define $\theta_x(\mu) := \frac{\mu^4}{4} - \frac{\tau\mu^2}{2} - x\mu$ and introduce the Pearcey Kernel

$$K_P(x, y; \tau) := \frac{1}{(2i\pi)^2} \int_{\Sigma} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu \quad (5.12)$$

where $\Sigma := \Sigma_- \cup \Sigma_+$ is as in the Figure 5.1:

Introduce $\mathbb{K}_P : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ with kernel K_P . According to Theorem 2 of [Sos00], if s and a_i 's are all finite, then the generating function $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of the operator \mathbb{K}_P are linked in the following way:

$$F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) = \det \left(1 - \sum_{j=1}^{N-1} k_j \chi_{(a_j+s, a_{j+1}+s)} \mathbb{K}_P \right) \quad (5.13)$$

where χ_I is the characteristic function of the interval I .

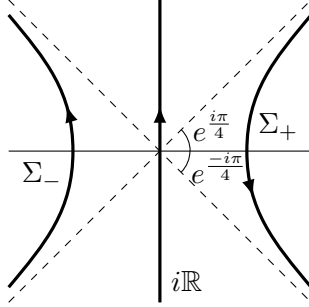


Figure 5.1 – The contour for the operator \mathbb{K}_P

From Pearcey kernel operator to an integrable operator

We link the Fredholm determinant of the operator \mathbb{K}_P with the one of an integrable operator in the sense of [IIKS90] so that we study $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ with Riemann-Hilbert problem associated to integrable operator.

Define

$$\tilde{f}(\mu) = \frac{1}{2i\pi} \begin{pmatrix} e^{\frac{1}{2}\theta_0(\mu)} \chi_{\Sigma}(\mu) \\ \sqrt{k_1 - k_0} e^{-\frac{1}{2}\theta_0(\mu) + (a_1+s)\mu} \chi_{i\mathbb{R}}(\mu) \\ \vdots \\ \sqrt{k_N - k_{N-1}} e^{-\frac{1}{2}\theta_0(\mu) + (a_N+s)\mu} \chi_{i\mathbb{R}}(\mu) \end{pmatrix} \quad \text{and} \quad (5.14)$$

$$\tilde{g}(\mu) = \begin{pmatrix} e^{-\frac{1}{2}\theta_0(\mu)} \chi_{i\mathbb{R}}(\mu) \\ \sqrt{k_1 - k_0} e^{\frac{1}{2}\theta_0(\mu) - (a_1+s)\mu} \chi_{\Sigma}(\mu) \\ \vdots \\ \sqrt{k_N - k_{N-1}} e^{\frac{1}{2}\theta_0(\mu) - (a_N+s)\mu} \chi_{\Sigma}(\mu) \end{pmatrix}, \quad (5.15)$$

where \tilde{f} and \tilde{g} are vectors of size $(N + 1) \times 1$ partitioned in a block 1×1 and a block $N \times 1$. This decomposition will be useful because in what follows we will study a Riemann-Hilbert problem of size $(N + 1) \times (N + 1)$ and matrices will be partitioned in four blocks of size 1×1 , $1 \times N$, $N \times 1$ and $N \times N$.

Define $\mathbb{K} : L^2(\Sigma \cup i\mathbb{R}) \rightarrow L^2(\Sigma \cup i\mathbb{R})$ integrable operator in the sense of [IIKS90] with kernel

$$K(u, v) = \frac{\tilde{f}^T(u) \tilde{g}(v)}{u - v} \quad (5.16)$$

We establish the relation between $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of the operator \mathbb{K} as follow:

Proposition 5.4. *Let $F(\vec{\mathbf{a}} + s, \vec{\mathbf{k}})$ be as in Theorem 5.2 and \mathbb{K} be as in (5.16). Then*

$$F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}) = \det(1 - \mathbb{K})_{L^2(\Sigma \cup i\mathbb{R})}$$

Proof. The idea of the proof is to compose the operator \mathbb{K} with multiplication operator and Fourier operator so that its Fredholm determinant will not change and to obtain an operator defined on $L^2(\mathbb{R})$ with kernel $\sum_{j=1}^{N-1} k_j \chi_{(a_j+s, a_{j+1}+s)}(x) K_P(x, y)$.

The proof of this proposition is similar to the one of Theorem 4.1 in [BC12] and can be adapted as follows.

Since $i\mathbb{R}$ and Σ are disjoint, one can decompose $L^2(\Sigma \cup i\mathbb{R})$ as $L^2(i\mathbb{R}) \oplus L^2(\Sigma)$ and write the following equality using matrix notation:

$$\det(1 - \mathbb{K})_{L^2(\Sigma \cup i\mathbb{R})} = \det \left(1 - \begin{pmatrix} 0 & \sum_{j=1}^N \mathcal{G}_j \\ \mathcal{F} & 0 \end{pmatrix} \right)_{L^2(i\mathbb{R}) \oplus L^2(\Sigma)} = \det \left(1 - \sum_{j=1}^N \mathcal{G}_j \circ \mathcal{F} \right)_{L^2(i\mathbb{R})}$$

where \mathcal{F} and \mathcal{G}_j are defined below with \mathcal{G}_j depending on k_j 's

$$\begin{aligned} \mathcal{F} &: L^2(i\mathbb{R}) \longrightarrow L^2(\Sigma) \\ f &\longmapsto \frac{e^{\frac{1}{2}\theta_0(\lambda)}}{2i\pi} \int_{\Sigma} \frac{e^{-\frac{1}{2}\theta_0(\mu)}}{\lambda - \mu} f(\mu) d\mu \end{aligned}$$

$$\begin{aligned} \mathcal{G}_j &: L^2(\Sigma) \longrightarrow L^2(i\mathbb{R}) \\ g &\longmapsto (k_j - k_{j-1}) \frac{e^{-\frac{1}{2}\theta_0(\xi) + \xi(a_j+s)}}{2i\pi} \int_{\Sigma} \frac{e^{\frac{1}{2}\theta_0(\lambda) - \lambda(a_j+s)}}{\xi - \lambda} g(\lambda) d\lambda \end{aligned}$$

Then composing with a multiplication operator \mathcal{M} and using a Fourier operator \mathcal{T} (for example as in [Gir14]) it is possible to relate Fredholm determinants of \mathbb{K} and \mathbb{K}_P . Here the Fourier composition allows to go from an operator on $L^2(i\mathbb{R})$ to one on $L^2(\mathbb{R})$.

With $\mathcal{M} := e^{-\frac{1}{2}\theta_0(\mu)}$ and

$$\begin{aligned} \mathcal{T} &: L^2(i\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \\ f &\longmapsto \frac{1}{\sqrt{2i\pi}} \int_{i\mathbb{R}} e^{-\xi x} f(\xi) d\xi \end{aligned}$$

$\mathcal{T} \circ \mathcal{M}^{-1} \circ \mathcal{G}_j \circ \mathcal{F} \circ \mathcal{M} \circ \mathcal{T}^{-1}$ has kernel

$$\begin{aligned} \mathcal{L}_j(x, y) &= \frac{k_j - k_{j-1}}{(2i\pi)^2} \int_{i\mathbb{R}} e^{\xi(a_j+s-x)} \int_{i\mathbb{R}} \int_{\Sigma} \frac{e^{\theta_{a_j+s}(\lambda) - \theta_y(\mu)}}{(\xi - \lambda)(\lambda - \mu)} d\lambda d\mu \frac{d\xi}{2i\pi} \\ \mathcal{L}_j(x, y) &= \begin{cases} \frac{k_j - k_{j-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_+} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu, & x > a_j + s \\ -\frac{k_j - k_{j-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_-} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu, & x < a_j + s \end{cases} \end{aligned}$$

If $x < a_1 + s$, then $x < a_j + s$ for all j and $\sum_{j=1}^N \mathcal{T} \circ \mathcal{M}^{-1} \circ \mathcal{G}_j \circ \mathcal{F} \circ \mathcal{M} \circ \mathcal{T}^{-1}$ has kernel

$$-\frac{\sum_{j=1}^N k_j - k_{j-1}}{(2i\pi)^2} \int_{\Sigma_-} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu = -\frac{k_N - k_0}{(2i\pi)^2} \int_{\Sigma_-} \int_{i\mathbb{R}} \frac{e^{\theta_x(\mu) - \theta_y(\lambda)}}{(\lambda - \mu)} d\lambda d\mu = 0$$

The same holds for $x > a_N + s$.

If $x \in (a_j + s, a_{j+1} + s)$, then

$$\begin{aligned} \sum_{j=1}^N \mathcal{L}_j(x, y) &= \frac{\sum_{\ell=1}^j k_\ell - k_{\ell-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_+} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu \\ &\quad - \frac{\sum_{\ell=j+1}^N k_\ell - k_{\ell-1}}{(2i\pi)^2} \int_{i\mathbb{R}} \int_{\Sigma_-} \frac{e^{\theta_x(\lambda) - \theta_y(\mu)}}{(\mu - \lambda)} d\lambda d\mu \\ &= k_j \chi_{(a_j+s, a_{j+1}+s)}(x) K_P(x, y) \end{aligned}$$

This concludes the proof. □

We can study $\det(1 - \mathbb{K})_{L^2(\Sigma \cup i\mathbb{R})}$ with the theory of Riemann-Hilbert problems associated to an integrable operator. We describe the Riemann-Hilbert problem associated to \mathbb{K} in what follows.

Riemann-Hilbert Problem associated to Pearcey kernel operator

In the Proposition 5.4, we proved a relation between $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and the Fredholm determinant of the IKS-integrable operator \mathbb{K} .

In the Subsection 3.3.2 of Chapter 3, we described how to associate a Riemann-Hilbert problem to an integrable operator in the sense of IKS. According to the proposition 3.16, we associate the following Riemann-Hilbert problem to the operator \mathbb{K} and then to the

generating function $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$.

The contours for the Riemann-Hilbert problem associated to $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ is $\Sigma \cup i\mathbb{R}$ oriented as in the previous figure.

For the jump matrix, introduce

$$f(\mu) := \begin{pmatrix} -\sqrt{k_1 - k_0} e^{-\theta_{a_1+s}(\mu)} \\ \vdots \\ -\sqrt{k_N - k_{N-1}} e^{-\theta_{a_N+s}(\mu)} \end{pmatrix} \chi_{i\mathbb{R}}(\mu) \text{ and}$$

$$g(\mu) := \begin{pmatrix} -\sqrt{k_1 - k_0} e^{\theta_{a_1+s}(\mu)} \\ \vdots \\ -\sqrt{k_N - k_{N-1}} e^{\theta_{a_N+s}(\mu)} \end{pmatrix} \chi_{\Sigma}(\mu)$$

Riemann-Hilbert Problem 5.5 (RHP for Γ). *We consider the Riemann-Hilbert problem with contours $\Sigma \cup i\mathbb{R}$ and jump matrix $J(\mu; \vec{\mathbf{a}}, \tau, s) := \left(\begin{array}{c|c} 1 & g^T(\mu) \\ \hline f(\mu) & I_n \end{array} \right) = I_{N+1} - 2i\pi \tilde{f}(\mu) \tilde{g}^T(\mu)$ where $I_k \in M_k(\mathbb{C})$ is the identity matrix.*

We search a matrix valued function $\Gamma(\mu) = \Gamma(\mu; \vec{\mathbf{a}}, \tau, s)$ such that:

- $\Gamma : \mathbb{C} \setminus \Sigma \cup i\mathbb{R} \rightarrow \mathcal{GL}_{N+1}(\mathbb{C})$ is analytic
- $\Gamma_+(\mu) = \Gamma_-(\mu)J(\mu)$, $\mu \in \Sigma \cup i\mathbb{R}$ where Γ is continuous up to boundary of the contours and $\Gamma_{\pm}(\mu)$ are non-tangential limits approaching μ from left(+) or right(-).
-

$$\Gamma(\mu) = I_{N+1} + \sum_{j \geq 1} \frac{\Gamma_j}{\mu^j} = I_{N+1} + \frac{1}{\mu} \left(\begin{array}{c|c} -\delta(\vec{\mathbf{a}}, \tau, s) & p^T(\vec{\mathbf{a}}, \tau, s) \\ \hline q(\vec{\mathbf{a}}, \tau, s) & \Delta(\vec{\mathbf{a}}, \tau, s) \end{array} \right) + \mathcal{O}(\mu^{-2}) \text{ as } \mu \rightarrow \infty \quad (5.17)$$

Remark 5.6. *If Γ satisfies the previous RHP, then $\det(\Gamma)$ is entire and according to the asymptotic $\det(\Gamma) \equiv 1$. Since $\det(\Gamma) \equiv 1$, the previous RHP has an unique solution (if it exists), $\text{Tr}(\Gamma_1) = 0$ and $\delta = \text{Tr}(\Delta)$.*

5.3 Study of the Riemann-Hilbert Problem associated to the Pearcey kernel operator

In this section, we obtain a Lax Pair associated to the previous RHP which leads to a system of coupled vector differential equation for p^T and q (see equation (5.17) for the definition of p^T and q). Studying the derivative with respect to τ yields to a coupled non-linear heat equation satisfied by p^T and q .

A Lax Pair for Ψ

Let Γ be a solution of RHP 5.5 and T be the gauge transformation:

$$T(\mu) := e^{\frac{1}{N+1} \sum_{j=1}^N \theta_{a_j+s}(\mu)} \text{Diag} \left(1, e^{-\theta_{a_1+s}(\mu)}, \dots, e^{-\theta_{a_N+s}(\mu)} \right).$$

Defining $\Psi(\mu) := \Gamma(\mu)T(\mu)$, the following result holds for Ψ .

Proposition 5.7 (Lax pair for Ψ). *If Γ is solution of RHP 5.5 then Ψ satisfy a system of partial differential equation.*

$$\begin{cases} \partial_s \Psi(\mu) = A(\mu) \Psi(\mu) \\ \partial_\mu \Psi(\mu) = B(\mu) \Psi(\mu) \\ \partial_\tau \Psi(\mu) = C(\mu) \Psi(\mu) \end{cases} \quad (5.18)$$

where

$$A(\mu) = \mu A_1 + A_0 = \frac{\mu}{N+1} \left(\begin{array}{c|c} -N & (0) \\ \hline (0) & I_N \end{array} \right) + \left(\begin{array}{c|c} 0 & p^T \\ \hline -q & (0) \end{array} \right) \quad (5.19)$$

$$B(\mu) = \mu^3 \tilde{B}_3 + \mu^2 B_2 + \mu(\tilde{B}_1 + B_1) + \tilde{B}_0 + B_0 \quad (5.20)$$

with B_j 's depending on Γ_j 's and $\tilde{B}_3 = -A_1$, $\tilde{B}_1 = \tau A_1$

$$\tilde{B}_0 = sA_1 + \frac{1}{N+1} \text{Diag} \left(-\sum_{j=1}^N a_j, Na_1 - \sum_{j \neq 1} a_j, \dots, Na_N - \sum_{j \neq N} a_j \right)$$

and

$$C(\mu) = \mu^2 \tilde{C}_2 + \mu C_1 + C_0 = \frac{\mu^2}{2} A_1 + \mu C_1 + C_0 \quad (5.21)$$

with C_1 and C_0 depending on Γ_j 's.

Proof. It is easy to show that if Γ satisfies the previous *RHP* then Ψ satisfies a *RHP* with jump matrix which does not depend on $\vec{\mathbf{a}}, \tau, s$ and μ (the jump only depends on $k_j, j = 0, \dots, N$).

Then Ψ and $\partial_s \Psi$ have the same jump on the contours. From this fact we deduce that $A(\mu) := \partial_s \Psi(\mu) \Psi(\mu)^{-1}$ is entire. Using Liouville's theorem with asymptotics of Ψ and $\partial_s \Psi$ from their *RHP* we conclude A is a polynomial of degree 1 in μ . More precisely, we compute A and obtain:

$$A(\mu) = \mu A_1 + A_0 = \frac{\mu}{N+1} \left(\begin{array}{c|c} -N & (0) \\ \hline (0) & I_N \end{array} \right) + \left(\begin{array}{c|c} 0 & p^T \\ \hline -q & (0) \end{array} \right)$$

Using the same method we conclude that $B(\mu) := \partial_\mu \Psi(\mu) \Psi(\mu)^{-1}$ (respectively $C(\mu) := \partial_\tau \Psi(\mu) \Psi(\mu)^{-1}$) is a polynomial of degree 3 (respectively degree 2). We will not do all computations for B and C with the asymptotic as we did for A : we precise what we will do.

Computations with the asymptotic involve T , its partial derivative with respect to μ (respectively τ) and $(\Gamma_j)_{j \geq 1}$. For now we will only compute terms in B (respectively C) which does not depend on $(\Gamma_j)_{j \geq 1}$.

We start with B and write it as:

$$B(\mu) = \mu^3 \tilde{B}_3 + \mu^2 B_2 + \mu(\tilde{B}_1 + B_1) + \tilde{B}_0 + B_0$$

where only $(B_j)_{j \in \{0,1,2\}}$ (respectively $(\tilde{B}_j)_{j \in \{0,1,3\}}$) depends (respectively does not depend) on $(\Gamma_j)_{j \geq 1}$.

$$B(\mu) = (\Gamma_\mu T + \Gamma T_\mu) T^{-1} \Gamma^{-1} = \Gamma_\mu \Gamma^{-1} + \Gamma T_\mu T^{-1} \Gamma^{-1}$$

$\Gamma_\mu \Gamma^{-1} = \mathcal{O}(\mu^{-2})$ as $\mu \rightarrow \infty$, then

$$B(\mu) \sim \left(I_{N+1} + \frac{\Gamma_1}{\mu} + \mathcal{O}(\mu^{-2}) \right) T_\mu T^{-1} \left(I_{N+1} - \frac{\Gamma_1}{\mu} + \mathcal{O}(\mu^{-2}) \right)$$

$B(\mu) = T_\mu T^{-1} + \sum_{j=0}^2 \mu^j B_j$ because of Liouville's theorem. We compute $T_\mu T^{-1}$ and obtain:

$$\tilde{B}_3 = -A_1, \quad \tilde{B}_1 = \tau A_1$$

$$\tilde{B}_0 = sA_1 + \frac{1}{N+1} \text{Diag} \left(-\sum_{j=1}^N a_j, Na_1 - \sum_{j \neq 1} a_j, \dots, Na_N - \sum_{j \neq N} a_j \right)$$

Similarly $C(\mu) = T_\tau T^{-1} + \sum_{j=0}^1 \mu^j C_j$ and $T_\tau T^{-1} = \frac{\mu^2}{2} A_1$.

□

Remark 5.8. For δ , p^T and q as in (5.17) the following holds: $\partial_s \delta = -p^T q$.

Actually, computing the term in $1/\mu$ in the asymptotic of A , we obtain $\partial_s \Gamma_1 + [\Gamma_2, A_1] - [\Gamma_1, A_1] \Gamma_1$. Then, because of Liouville's theorem this term is 0 and the block 1×1 on the diagonal block matrix leads to the equation. This equation will be useful later when we will compute the logarithmic derivative for $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$.

Proposition 5.9. Let p^T and q be as in (5.17). Then they satisfy the following coupled vector 3^{rd} order differential equation and non-linear coupled heat equation:

$$\begin{cases} \partial_{sss} p^T + 3(\partial_s p^T) q p^T + 3p^T q (\partial_s p^T) - \tau \partial_s p^T + p^T D_{s, \vec{\mathbf{a}}} = 0 \\ \partial_{sss} q + 3(\partial_s q) p^T q + 3q p^T (\partial_s q) - \tau \partial_s q - D_{s, \vec{\mathbf{a}}} q = 0 \end{cases} \quad (5.22)$$

$$\begin{cases} -\frac{1}{2} \partial_{ss} p^T - \partial_\tau p^T = p^T q p^T \\ -\frac{1}{2} \partial_{ss} q + \partial_\tau q = q p^T q \end{cases} \quad (5.23)$$

Proof. The compatibility condition for the Lax pair of Ψ leads to the equation

$$\partial_s B - \partial_\mu A = [A, B] \quad (5.24)$$

We use the same approach as in [WE07] and [CT23a]. For this proof, we change the notation to write an element in a matrix in order to make it easier to read. We write B_j^{ki} blocs of matrix B_j where B_j^{11} is a scalar, B_j^{12} is a row of size N , B_j^{21} a column of size N and B_j^{22} a $N \times N$ matrix. Then (5.24) gives a polynomial equation in μ and we obtain an equation for every monomial. This leads to the following equations:

$$B_2^{12} = -p^T, \quad B_2^{21} = q \quad (5.25)$$

$$\begin{cases} \partial_s B_j^{11} = p^T B_j^{21} + B_j^{12} q \\ \partial_s B_j^{12} = -B_{j-1}^{12} \delta_{j \neq 0} + p^T B_j^{22} - B_j^{11} p^T + \tau p^T \delta_{j,1} + p^T D_{s, \vec{\alpha}} \delta_{j,0} \\ \partial_s B_j^{21} = B_{j-1}^{12} \delta_{j \neq 0} - q B_j^{11} + B_j^{22} q + \tau q \delta_{j,1} + D_{s, \vec{\alpha}} q \delta_{j,0} \\ \partial_s B_j^{22} = -q B_j^{12} - B_j^{21} p^T \end{cases} \quad (5.26)$$

where $D_{s, \vec{\alpha}} = \text{Diag}(a_1 + s, \dots, a_N + s)$ and $\delta_{i,j}$ is the Kronecker delta.

We define formally the operator ∂_s^{-1} such that $\partial_s^{-1} \partial_s = 1$. From the first and the last equation of (5.26) we obtain $B_j^{11} = \partial_s^{-1} (p^T B_j^{21} + B_j^{12} q)$ and $B_j^{22} = -\partial_s^{-1} (q B_j^{12} + B_j^{21} p^T)$.

With $j = 2$,

$$B_2^{11} = \partial_s^{-1}(0) = c_2^{11}, \quad B_2^{22} = \partial_s^{-1}((0)) = c_2^{22}$$

with c_2^{11} and c_2^{22} independent of s . Actually with the asymptotics we obtain: $B_2 = -A_0$. Then $c_0^{11} = 0$ and $c_0^{22} = (0)$. Same for B_1 , even if it depends on Γ_2 , diagonal terms only depend on Γ_1 . The asymptotic leads to

$$B_1^{11} = p^T q, \quad B_1^{22} = -qp^T$$

Using second and third equations of (5.26) we compute B_1^{12} and B_1^{21} with $j = 2$ then B_0^{12} and B_0^{21} with $j = 1$.

$$\begin{aligned} B_1^{12} &= \partial_s p^T, \quad B_1^{21} = \partial_s q \\ B_0^{12} &= -\partial_{ss} p^T - 2p^T q p^T + \tau p^T, \quad B_0^{21} = \partial_{ss} q + 2q p^T q - \tau q \end{aligned}$$

Using first and last equations of (5.26) with $j = 0$ and integrating, we compute B_0^{11} and B_0^{22} .

$$\begin{aligned} B_0^{11} &= p^T (\partial_s q) - (\partial_s p^T) q \\ B_0^{22} &= q (\partial_s p^T) - (\partial_s q) p^T \end{aligned}$$

Finally, it remains two equations (second and third of (5.26) with $j = 0$). Replacing B_0^{ki} in these equations we obtain a system of equations satisfied by p^T and q :

$$\begin{cases} \partial_{sss} p^T + 3(\partial_s p^T) q p^T + 3p^T q (\partial_s p^T) - \tau \partial_s p^T + p^T D_{s, \vec{\alpha}} = 0 \\ \partial_{sss} q + 3(\partial_s q) p^T q + 3q p^T (\partial_s q) - \tau \partial_s q - D_{s, \vec{\alpha}} q = 0 \end{cases} \quad (5.27)$$

Studying the compatibility condition of A and C the same way as for A and B , p^T and q

satisfied a coupled non-linear heat equation:

$$\begin{cases} -\frac{1}{2}\partial_{ss}p^T - \partial_\tau p^T = p^T q p^T \\ -\frac{1}{2}\partial_{ss}q + \partial_\tau q = q p^T q \end{cases} \quad (5.28)$$

□

In Appendix A of [KZ22], doing formal computation on a semi infinite interval, Kimura and Zahabi obtained a scalar version of the system of coupled differential equation (5.27). Similar equations as in Proposition 5.9 appeared in the study of the limiting one-point distribution of periodic TASEP [BLS22]. The authors obtained coupled mKdV equations and coupled non-linear heat equations. Combining these two equations, they proved the second log-derivative of the Fredholm determinant they studied satisfies the second Kadomtsev-Petviashvili equation. It is possible to do a similar computation and to obtain a PDE for the second log-derivative of the Fredholm determinant of \mathbb{K} . The next section will be partially devoted to this computation.

5.4 The logarithmic derivative of F and proof of Theorem 5.2

Now that we have obtained a Lax pair for $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ and deduced coupled ODEs and PDEs for $p(s)$ and $q(s)$, we conclude the proof of Theorem 5.2. It remains to obtain the equation (5.3) that links $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ with $p(s)$ and $q(s)$ and to compute the asymptotics of $p(s)$ and $q(s)$ as $s \rightarrow \infty$.

From Remark 5.1 and Proposition 5.4, the Fredholm determinant of $(1 - \mathbb{K})$ is different from zero and $1 - \mathbb{K}$ is invertible.

Defining

$$\tilde{F} := (1 - \mathbb{K})^{-1} \tilde{f}, \quad \text{with } \tilde{f} \text{ as in equation (5.14),} \quad (5.29)$$

if we write $\tilde{F} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_N \end{pmatrix}$ and $\tilde{g} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}$ (where \tilde{g} is defined in equation (5.15)), we

have the following result.

Lemma 5.10. *Let Δ be as in (5.17) and F_i 's, g_i 's as above. Then:*

$$\Delta = \int_{\Sigma \cup i\mathbb{R}} \begin{pmatrix} F_1(\mu) \\ \vdots \\ F_N(\mu) \end{pmatrix} (g_1(\mu), \dots, g_N(\mu)) d\mu$$

Proof. According to the theory of Riemann-Hilbert problem, since $(1 - \mathbb{K})$ is invertible, the resolvent of \mathbb{K} and the unique solution to RHP 5.5 are linked and $\tilde{F} = \Gamma_+ \tilde{f}$.

With this last equality we obtain:

$$\Gamma(\xi) = I_{N+1} - \int_{\Sigma \cup i\mathbb{R}} \frac{\tilde{F}(\mu) \tilde{g}^T(\mu)}{\mu - \xi} d\mu$$

Expanding $\frac{1}{\mu - \xi}$ we express Γ_1 in function of \tilde{F} and \tilde{g} .

$$\Gamma_1 = \int_{\Sigma \cup i\mathbb{R}} \tilde{F}(\mu) \tilde{g}^T(\mu) d\mu$$

According to this previous equality and because of the decomposition by blocks of Γ_1 in (5.5), with

$$\tilde{F} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_N \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}, \quad (5.30)$$

$$\Delta = \int_{\Sigma \cup i\mathbb{R}} \begin{pmatrix} F_1(\mu) \\ \vdots \\ F_N(\mu) \end{pmatrix} (g_1(\mu), \dots, g_N(\mu)) d\mu.$$

□

Proposition 5.11. *Let $F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})$ be as in Theorem 5.2 and δ as in (5.17). The following holds:*

$$\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = -\delta$$

Proof.

$$\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = \frac{\partial}{\partial s} \log(\det(1 - \mathbb{K})) = \frac{\partial}{\partial s} \text{Tr}(\log(1 - \mathbb{K})) = -\text{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K})$$

Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\Sigma \cup i\mathbb{R})$.

$$\mathrm{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K}) = \sum_{n \in \mathbb{N}} \langle (1 - \mathbb{K})^{-1} \partial_s \mathbb{K} e_n, e_n \rangle$$

But $\partial_s \mathbb{K}$ has kernel $\chi_{i\mathbb{R}}(u) \tilde{f}^T(u) \tilde{g}(v) \chi_\Sigma(v)$. Then

$$\mathrm{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K}) = \sum_{n \in \mathbb{N}} \langle (1 - \mathbb{K})^{-1} (\chi_{i\mathbb{R}} \tilde{f}^T), e_n \rangle \langle \chi_\Sigma \tilde{g}, e_n \rangle$$

According to (5.30), definitions of \tilde{f} (5.14), \tilde{g} (5.15) and \tilde{F} (5.29),

$$\mathrm{Tr}((1 - \mathbb{K})^{-1} \partial_s \mathbb{K}) = \sum_{n \in \mathbb{N}} (\langle F_1, e_n \rangle, \dots, \langle F_N, e_n \rangle) \begin{pmatrix} \langle g_1, e_n \rangle \\ \vdots \\ \langle g_N, e_n \rangle \end{pmatrix} = \mathrm{Tr}(\Delta)$$

The last equation is a consequence of Lemma 5.10 and the fact that e_n is an orthonormal basis. Finally, because of $-\delta + \mathrm{Tr}(\Delta) = \mathrm{Tr}(\Gamma_1) = 0$ (see Remark 5.6),

$$\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = -\delta$$

□

We use the previous proposition and the discussion on the Lax Pair to prove Theorem 5.2.

Proof of Theorem 5.2. Using the previous proposition we derive $\frac{\partial}{\partial s} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}}))$ with respect to s .

$$\frac{\partial^2}{\partial s^2} \log(F(\vec{\mathbf{a}} + s, \tau, \vec{\mathbf{k}})) = -\partial_s \delta = p^T(s)q(s) \quad (5.31)$$

because $\partial_s \delta = -p^T q$ (see Remark 5.8). From the Proposition 5.9 p^T and q satisfy equations (5.4) and (5.5).

It remains to compute the asymptotics for $p_i(s)$ and $q_i(s)$ as $s \rightarrow \infty$.

Introduce the function X defined by $X(\mu) := \Gamma(s^{1/3}\mu)$. $X(\mu)$ has jumps on $\Sigma \cup i\mathbb{R}$ of the

form $\left(\frac{1}{f(s^{1/3}\mu)} \middle| \frac{g^T(s^{1/3}\mu)}{I_n} \right)$ where

$$f(s^{1/3}\mu) = \begin{pmatrix} -\sqrt{k_1 - k_0} e^{-s^{4/3} \left(\frac{\mu^4}{4} - \frac{\tau\mu}{2s^{2/3}} - \left(1 + \frac{a_1}{s}\right)\mu \right)} \\ \vdots \\ -\sqrt{k_N - k_{N-1}} e^{-s^{4/3} \left(\frac{\mu^4}{4} - \frac{\tau\mu}{2s^{2/3}} - \left(1 + \frac{a_N}{s}\right)\mu \right)} \end{pmatrix} \chi_{i\mathbb{R}}(\mu) \text{ and}$$

$$g(s^{1/3}\mu) = \begin{pmatrix} -\sqrt{k_1 - k_0} e^{s^{4/3} \left(\frac{\mu^4}{4} - \frac{\tau\mu}{2s^{2/3}} - \left(1 + \frac{a_1}{s}\right)\mu \right)} \\ \vdots \\ -\sqrt{k_N - k_{N-1}} e^{s^{4/3} \left(\frac{\mu^4}{4} - \frac{\tau\mu}{2s^{2/3}} - \left(1 + \frac{a_N}{s}\right)\mu \right)} \end{pmatrix} \chi_{\Sigma}(\mu)$$

According to Figure 5.2, one can deform the contour $\Sigma \cup i\mathbb{R}$ into $\tilde{\Sigma} \cup i\mathbb{R}$ so that the jump on $i\mathbb{R}$ and Σ_+ tends to the identity matrix I_{N+1} as $s \rightarrow \infty$. Such a deformation for the jump on the contour Σ_- is not possible.

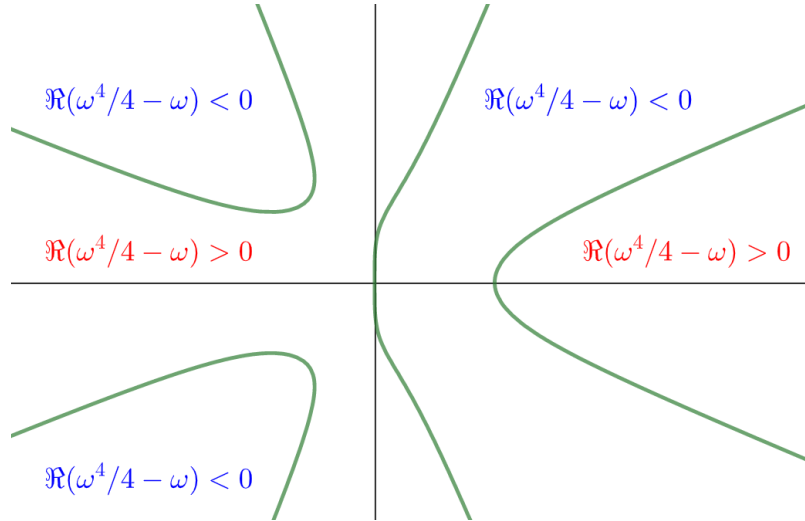


Figure 5.2 – The sign of $\Re(\omega^4/4 - \omega)$

Then X satisfies the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 5.12 (RHP for X). *We search a matrix valued function $X(\mu) = X(\mu; \vec{a}, \tau, s)$ such that:*

- $X : \mathbb{C} \setminus \tilde{\Sigma} \cup i\mathbb{R} \rightarrow \mathcal{G}\ell_{N+1}(\mathbb{C})$ is analytic

- $X_+(\mu) = X_-(\mu)J(s^{1/3}\mu)$, $\mu \in \tilde{\Sigma} \cup i\mathbb{R}$ where X is continuous up to boundary of the contours and $X_{\pm}(\mu)$ are non-tangential limits approaching μ from left(+) or right(-).

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$$X(\mu) = I_{N+1} + \sum_{j \geq 1} \frac{X_j}{\mu^j} \text{ as } \mu \rightarrow \infty \quad (5.32)$$

We want to study the asymptotic of X as $s \rightarrow \infty$ by applying the small norm theorem (see for instance Theorem 5.1.5 in [Its11]).

Lemma 5.13. *Let X satisfies the Riemann-Hilbert Problem 5.12. Then, as $s \rightarrow \infty$,*

$$X(\mu, s) \rightarrow I_{N+1} + T(\mu) \quad (5.33)$$

where T is a upper triangular matrix with zero diagonal of the form $T(\mu) = \left(\begin{array}{c|c} 0 & t(\mu) \\ \hline 0_N & 0_{N \times N} \end{array} \right)$ with 0_N and $0_{N \times N}$ the zero matrix of size $N \times 1$ and $N \times N$ and t a matrix of size $1 \times N$.

Proof. We introduce the lower and upper triangular matrices with zero diagonal

$$T_-(\mu) = \tilde{f}(\mu)\tilde{g}^T(\mu)\chi_{i\mathbb{R}}(\mu) \text{ and } T_+(\mu) = \tilde{f}(\mu)\tilde{g}^T(\mu)\chi_{\Sigma}(\mu)$$

and the function \tilde{X} solution of the following Riemann-Hilbert problem:

- $\tilde{X} : \mathbb{C} \setminus \tilde{\Sigma} \rightarrow \mathcal{G}\ell_{N+1}(\mathbb{C})$ is analytic
- $\tilde{X}_+(\mu) = \tilde{X}_-(\mu) \left(I_{N+1} - 2i\pi T_+(s^{1/3}\mu) \right)$, $\mu \in \tilde{\Sigma}$ where \tilde{X} is continuous up to boundary of the contours and $\tilde{X}_{\pm}(\mu)$ are non-tangential limits approaching μ from left(+) or right(-).

—

$$\tilde{X}(\mu) = I_{N+1} + \mathcal{O}(\mu^{-1}) \text{ as } \mu \rightarrow \infty$$

Define $\mathcal{X}(\mu) := X(\mu)\tilde{X}(\mu)^{-1}$, the function \mathcal{X} satisfies a Riemann-Hilbert problem on the contours $i\mathbb{R}$ with the jump matrix $I_{N+1} - 2i\pi T_-(s^{1/3}\mu)$. The matrix $T_-(s^{1/3}\mu)$ satisfies: let $C := \sup_{i \in \{1, \dots, N\}} \frac{|\sqrt{k_i - k_{i-1}}|}{2\pi}$ then

$$\|T_-(s^{1/3}\mu)\|_{L^\infty(i\mathbb{R})} = \sup_{\mu \in i\mathbb{R}} C e^{-s^{4/3} \left(\frac{\mu^4}{4} - \frac{\tau\mu^2}{2s^{2/3}} - \left(1 + \frac{a_i}{s}\right)\mu \right)} \rightarrow 0 \text{ and}$$

$\|T_-(s^{1/3}\mu)\|_{L^2(i\mathbb{R})} \rightarrow 0$ exponentially fast as $s \rightarrow \infty$. We conclude from the small norm theorem that $\mathcal{X} \sim I_{N+1}$ as $s \rightarrow \infty$ which implies $X \sim I_{N+1} + T$ as $s \rightarrow \infty$ where T is an upper triangular matrix with zero diagonal of the form $T(\mu) = \begin{pmatrix} 0 & t(\mu) \\ 0_N & 0_{N \times N} \end{pmatrix}$ with 0_N and $0_{N \times N}$ the zero matrix of size $N \times 1$ and $N \times N$ and t a matrix of size $1 \times N$.

□

From Lemma 5.13 we deduce asymptotics for p_i and q_i ($i \in \llbracket 1, N \rrbracket$) and we finish the proof of Theorem 5.2.

As for Γ_1 in the proof of Lemma 5.10, one can express $X_1(s)$ as $\int_{\tilde{\Sigma} \cup i\mathbb{R}} X_-(\mu) \tilde{f}(s^{1/3}\mu) \tilde{g}^T(s^{1/3}\mu) d\mu$. Moreover $\Gamma_1(s)$ and $X_1(s)$ satisfy the following relation, $X_1(s) = s^{-1/3}\Gamma_1(s)$. Then

$$\begin{aligned} q_i(s) &= \Gamma_1(s)_{i+1,1} = s^{1/3} X_1(s)_{i+1,1} = \left(s^{1/3} \int_{\tilde{\Sigma} \cup i\mathbb{R}} X_-(\mu) \tilde{f}(s^{1/3}\mu) \tilde{g}^T(s^{1/3}\mu) d\mu \right)_{i+1,1} \\ &\sim \frac{\sqrt{k_i - k_{i-1}}}{2i\pi} \int_{i\mathbb{R}} e^{-\frac{\mu^4}{4} + \tau \frac{\mu^2}{2} + (a_i + s)\mu} d\mu = \sqrt{k_i - k_{i-1}} Q(a_i + s) \end{aligned} \quad (5.34)$$

where Q satisfies the differential equation $Q'''(s) - \tau Q'(s) = sQ(s)$.

$$\begin{aligned} p_i(s) &= \Gamma_1(s)_{1,i+1} = s^{1/3} X_1(s)_{1,i+1} = \left(s^{1/3} \int_{\tilde{\Sigma} \cup i\mathbb{R}} X_-(\mu) \tilde{f}(s^{1/3}\mu) \tilde{g}^T(s^{1/3}\mu) d\mu \right)_{1,i+1} \\ &\sim \frac{\sqrt{k_i - k_{i-1}}}{2i\pi} \int_{\Sigma} e^{\frac{\mu^4}{4} - \tau \frac{\mu^2}{2} - (a_i + s)\mu} d\mu = \sqrt{k_i - k_{i-1}} P(a_i + s) \end{aligned} \quad (5.35)$$

where P satisfies the differential equation $P'''(s) - \tau P'(s) = -sP(s)$. □

We now compute the asymptotics for P and Q as $s \rightarrow \infty$ in order to conclude the proof of Theorem 5.2.

Proposition 5.14. *Let P and Q defined respectively as in equations (5.35) and (5.34). Then, as $s \rightarrow \infty$:*

$$Q(s) \sim_{s \rightarrow \infty} \sqrt{\frac{2}{3\pi}} s^{-1/3} e^{-\frac{3}{8}s^{4/3} - \frac{\tau}{4}s^{2/3} + \frac{\tau^2}{6}} \cos\left(\frac{3}{4} \sin\left(\frac{2\pi}{3}\right) s^{4/3} - \frac{\tau}{2} \sin\left(\frac{2\pi}{3}\right) s^{2/3} - \frac{\pi}{6}\right) \quad (5.36)$$

$$P(s) \sim_{s \rightarrow \infty} \sqrt{\frac{2}{3\pi}} s^{-1/3} e^{\frac{3}{8}s^{4/3} + \frac{\tau}{4}s^{2/3} - \frac{\tau^2}{6}} \cos\left(\frac{3}{4} \sin\left(\frac{2\pi}{3}\right) s^{4/3} - \frac{\tau}{2} \sin\left(\frac{2\pi}{3}\right) s^{2/3} - \frac{\pi}{6}\right) \quad (5.37)$$

Proof. By definition,

$$Q(s) = \frac{1}{2i\pi} \int_{i\mathbb{R}} e^{-\frac{\mu^4}{4} + \tau \frac{\mu^2}{2} + s\mu} d\mu$$

The derivative with respect to μ of $\theta_s(\mu)$ is $\frac{d}{ds}\theta_s(\mu) = \mu^3 - \tau\mu - s$ whose roots are

$$\mu_k := \mu_k(s, \tau) = j^k \sqrt[3]{\frac{1}{2} \left(s + \sqrt{s^2 - \frac{2^2\tau^3}{3^3}} \right)} + j^{-k} \sqrt[3]{\frac{1}{2} \left(s - \sqrt{s^2 - \frac{2^2\tau^3}{3^3}} \right)} \quad (5.38)$$

where $j := e^{\frac{2\pi}{3}}$ and $k \in \{0, 1, 2\}$. We only consider saddle points for $k = 1$ and 2 .

Denote by μ_* either μ_1 or μ_2 . One can deform the contour $i\mathbb{R}$ into a contour \mathcal{C}_Q so that it passes through μ_1 and μ_2 and the following holds $\Re(\theta_s(\mu) - \theta_s(\mu_*)) < 0$.

See Figure 5.3 for the study of the sign of $\Re(\theta_s(\mu) - \theta_s(\mu_*))$ for $s = 100$ and $\tau = 1$. As $s \rightarrow \infty$ and τ is fixed, the algebraic curve $\Re(\theta_s(\mu) - \theta_s(\mu_*)) = 0$ keeps a similar shape and it is always possible to deform $i\mathbb{R}$ into \mathcal{C}_Q .

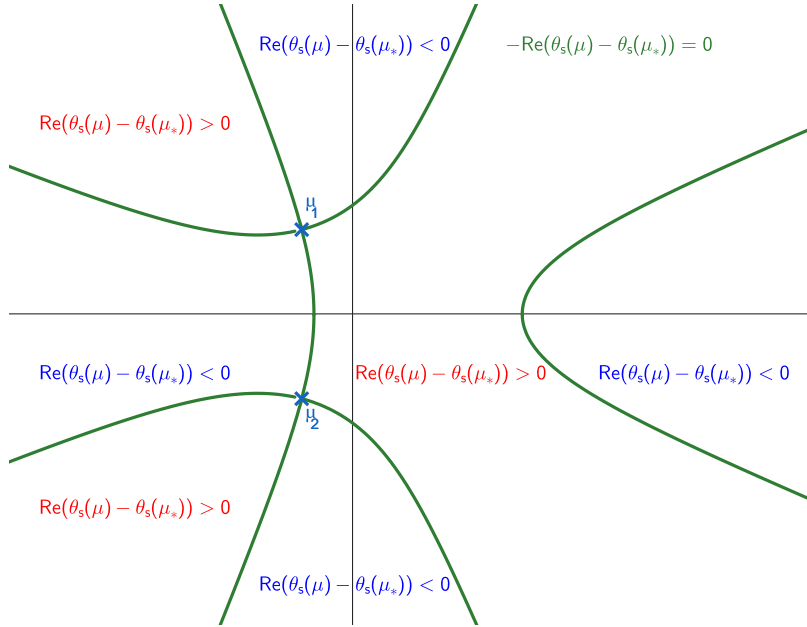


Figure 5.3 – The sign of $\Re(\theta_s(\mu) - \theta_s(\mu_*))$ for $\tau = 1$ and $s = 100$

Then

$$\begin{aligned} Q(s) &= \frac{e^{-\theta_s(\mu_*)}}{2i\pi} \int_{\mathcal{C}_Q} e^{-(\theta_s(\mu) - \theta_s(\mu_*))} d\mu \\ &= \frac{e^{-\theta_s(\mu_*)}}{2i\pi} \int_{\mathcal{C}_Q} e^{-\left(\frac{3}{2}\mu_*^2(\mu - \mu_*)^2 \left(1 - \frac{\tau}{3\mu_*^2} + \frac{2}{3\mu_*}(\mu - \mu_*) + \frac{1}{6\mu_*^2}(\mu - \mu_*)^2\right)\right)} d\mu \end{aligned} \quad (5.39)$$

and one can approximate Q as $s \rightarrow \infty$ by expanding μ_* and by approximating the integral.

It follows

$$Q(s) \sim \sqrt{\frac{2}{3\pi}} s^{-1/3} e^{-\frac{3}{8}s^{4/3} - \frac{\tau}{4}s^{2/3} + \frac{\tau^2}{6}} \cos\left(\frac{3}{4} \sin\left(\frac{2\pi}{3}\right) s^{4/3} - \frac{\tau}{2} \sin\left(\frac{2\pi}{3}\right) s^{2/3} - \frac{\pi}{6}\right) \quad (5.40)$$

The same method leads to the asymptotic for P as $s \rightarrow \infty$. □

ONGOING PROJECTS AND FUTURE DIRECTIONS

We conclude this thesis with a discussion of ongoing projects and future directions based on the work presented previously in Chapters 4 and 5.

6.1 Ongoing projects

Orthogonal polynomials on the real line and the discrete Painlevé I hierarchy:

Sofia Tarricone and I initiated a discussion with Giorgio Gubbiotti on the discrete Painlevé I hierarchy. This hierarchy of discrete equations was originally introduced by Cresswell and Joshi [CJ99b] as a compatibility condition for a scalar Lax pair. The discrete Painlevé I equation also arises in the study of orthogonal polynomials on the real line as, described in subsection 3.2.3. Specifically, the square of the quotient of two consecutive leading terms of orthogonal polynomials with respect to the weight $\exp\left(-N\left(\frac{t}{4}z^4 + \frac{a}{2}z^2\right)\right)$ satisfies the discrete Painlevé I equation. This result was obtained by considering the Riemann-Hilbert problem for orthogonal polynomials on the real line and by computing a 2×2 matrix valued Lax pair. Considering weights of the form $\exp\left(-N\sum_{i=1}^n \frac{\theta_i}{2^i} z^{2^i}\right)$, we expect to present a new Lax pair for the discrete Painlevé I hierarchy.

An Aztec diamond model for the Pearcey Process:

I am currently working on a project with Dan Betea, Mattia Cafasso and Killian Raschel on a “biased” domino tiling of the Aztec diamond. The tilings of the Aztec diamond are related to interlacing random partitions and to Maya diagrams. The main idea is to introduce a new model where the Pearcey process describes a determinantal point process in the neighborhood of a triple critical point of the arctic curve in the large N limit, where $2N$ is the size of the Aztec diamond. The study of the limit curve leads to a parametrized curve with a cusp-like

singularity. We hope to prove that the limiting kernel close to this cusp is given by the Pearcey kernel.

6.2 Future directions

A model for higher order Pearcey operator: Continuing the work on the Aztec diamond, I plan to investigate the possibility of the arctic curve of the Aztec diamond admitting a critical point of order $2n + 1$ (where n is an integer). If such a singularity exists, I hope to be able to express the point process close to this singularity in terms of a determinant, behaving as a higher analogue of Pearcey kernel (similar to the odd p -Airy kernel in [KZ22]) in the N large limit.

Afterward, I aim to study the generating function associated with such a determinantal point process, as done in [Cho23], to obtain a Lax pair and derive higher analogues of equations (5.3) and (5.4). The idea is to present a hierarchy of coupled equations for the Pearcey kernel and higher analogues.

Discrete equation for the Pearcey process and higher order operator: Kimura and Zahabi [KZ22] studied random partitions with respect to Schur measure and introduced the p -Airy kernel, where even p corresponds to higher-order Airy kernels, while odd p corresponds to the Pearcey kernel ($p = 3$ and higher-order versions). Gap probabilities for these operators, expressed as Fredholm determinants, yield a Hamiltonian system. From this Hamiltonian, they expressed Fredholm determinants as higher Tracy-Widom distributions for even p and as a scalar version of the system of equations (5.4) in Theorem 5.2 ($p = 3$) and higher analogue for $p = 5, 7$.

As for the discrete Painlevé II hierarchy, by studying bi-orthogonal polynomial on the unit circle [FW06], I hope to obtain a discrete analogue of coupled equations (5.4), and eventually, to obtain higher analogues of these equations. This work should be conducted alongside the research on the higher-order Pearcey kernel, since I hope that these discrete equations will have continuous limit equations associated with the Pearcey kernel and higher-order generalizations.

Study of the higher order Airy kernel: I plan to study a Riemann-Hilbert problem associated with the higher-order Airy kernels. The starting point for this study is a matrix valued $2n \times 2n$ Riemann-Hilbert problem. The higher-order Airy kernel has been

studied using various RHP methods (Fourier transforms [CCG21], operator-valued RHP [BCT22]). The method I am interested in has not been explored for now, and I expect it to lead to the Painlevé II hierarchy. Furthermore, I hope this approach will allow the study of multiplicative statistics of the higher-order Airy kernel. By deforming the kernel, I expect to relate the RHP with the KdV-hierarchy or the Gelfand-Dickey hierarchy and eventually to re-derive the integro-differential Painlevé II hierarchy. Moreover, I expect to establish a connection with the KPZ equation. In [CC22], by studying Riemann-Hilbert problem, the authors established a connection between the Airy process and the lower tail of the KPZ equation (see also [ACQ11] and [BG16] for the connection between the KPZ equation and the Airy process).

Conclusion

By pursuing these ongoing projects and future directions, we aim to contribute to the advancement of mathematics in various areas, ranging from orthogonal polynomials and discrete Painlevé equations to determinantal processes and the KPZ universality class. These endeavors hold the promise of revealing new connections and unveiling new mathematical structures in mathematical physics.

PUBLICATIONS

Title: A Riemann Hilbert approach to the study of the generating function associated to the Pearcey process

Journal: Mathematical Physics, Analysis and Geometry

Abstract: Using Riemann-Hilbert methods, we establish a Tracy-Widom like formula for the generating function of the occupancy numbers of the Pearcey process. This formula is linked to a coupled vector differential equation of order three. We also obtain a non linear coupled heat equation. Combining these two equations we obtain a PDE for the logarithm of the generating function of the Pearcey process.

Link: <https://doi.org/10.1007/s11040-023-09455-8>

Title: Recursion relation for Toeplitz determinants and the discrete Painlevé II hierarchy

Journal: SIGMA. Symmetry, Integrability and Geometry. Methods and Applications

Abstract: Solutions of the discrete Painlevé II hierarchy are shown to be in relation with a family of Toeplitz determinants describing certain quantities in multicritical random partitions models, for which the limiting behavior has been recently considered in the literature. Our proof is based on the Riemann-Hilbert approach for the orthogonal polynomials on the unit circle related to the Toeplitz determinants of interest. This technique allows us to construct a new Lax pair for the discrete Painlevé II hierarchy that is then mapped to the one introduced by Cresswell and Joshi.

Link: <https://doi.org/10.3842/SIGMA.2023.030>

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Titre : Processus ponctuels déterminantaux associés à la hiérarchie discrète de Painlevé II et au processus de Pearcey

Mot clés : Processus ponctuels déterminantaux, Problèmes de Riemann-Hilbert, Processus de Pearcey, Polynômes Orthogonaux sur le cercle unité, Systèmes intégrables

Résumé : Cette thèse vise à présenter des résultats nouveaux concernant le processus de Pearcey et la hiérarchie discrète de Painlevé II. Ces processus ont tous deux des interprétations en termes de processus ponctuels déterminantaux et de problèmes de Riemann-Hilbert. Après avoir rappelé les résultats classiques sur les processus ponctuels et les problèmes de Riemann-Hilbert, nous présentons, d'une part la hiérarchie discrète de Painlevé II, son lien avec les partitions aléatoires d'entier et les polynômes orthogonaux, d'autre part le processus de Pearcey et sa fonction génératrice. Concernant la hiérarchie discrète de

Painlevé II, cette thèse est l'occasion de présenter une nouvelle paire de Lax associée à cette hiérarchie d'équations discrètes. Ce résultat s'appuie sur le lien entre partitions aléatoires d'entier, déterminants de Toeplitz et polynômes orthogonaux. De cette paire de Lax il est alors possible de déterminer un opérateur récursif permettant de calculer la N -ième équation de la hiérarchie. En ce qui concerne le processus de Pearcey, via l'étude d'un problème de Riemann-Hilbert associé à un opérateur intégrable, nous présentons une formule "à la Tracy-Widom" pour décrire la fonction génératrice du processus.

Title: Determinantal Point Processes related to the discrete Painlevé II hierarchy and the Pearcey process

Keywords: Determinantal point processes, Riemann-Hilbert Problems, Pearcey process, Orthogonal polynomials on the unit circle, Integrable systems

Abstract: The aim of this thesis is to present new results concerning the Pearcey process and the discrete Painlevé II hierarchy. These processes both have interpretations in terms of determinantal point processes and Riemann-Hilbert problems. After recalling classical results on point processes and Riemann-Hilbert problems, we present, on one hand the discrete Painlevé II hierarchy, its link with random integer partitions and orthogonal polynomials, and on the other hand the Pearcey process and its generating function. Concerning the discrete Painlevé II hierarchy,

this thesis presents a new Lax pair associated with this hierarchy of discrete equations. This result is based on the link between random integer partitions, Toeplitz determinants and orthogonal polynomials. From this Lax pair it is then possible to determine a recursive operator for calculating the N -th equation of the hierarchy. For the Pearcey process, via the study of a Riemann-Hilbert problem associated to an integrable operator, we present "a Tracy-Widom like formula" to describe the generating function of the process.